Stockholm
University

On linear graph invariants related to Ramsey and edge numbers

or how I learned to stop worrying and love the alien invasion

Oliver Krüger

Doctoral Thesis in Mathematics at Stockholm University, Sweden 2019

On linear graph invariants related to Ramsey and edge numbers

or how I learned to stop worrying and love the alien invasion

Oliver Krüger

Academic dissertation for the Degree of Doctor of Philosophy in Mathematics at Stockholm University to be publicly defended on Monday 16 December 2019 at 10.00 in sal 14, hus 5, Kräftriket, Roslagsvägen 101.

Abstract

In this thesis we study the Ramsey numbers, R(l,k), the edge numbers, e(l,k;n) and graphs that are related to these. The edge number e(l,k;n) may be defined as the least natural number m for which all graphs on n vertices and less than m edges either contains a complete subgraph of size l or an independent set of size k. The Ramsey number R(l,k) may then be defined as the least natural number n for which $e(l,k;n) = \infty$.

In Paper I, IV and V we study strict lower bounds for e(l,k;n). In Paper I we do this in the case where l = 3 by, in particular, showing $e(G) \geq (1/3)(17n(G) - 35\alpha(G) - N(C_4;G))$ for all triangle-free graphs G, where $N(C_4;G)$ denotes the number of cycles of length 4 in G. In Paper IV we describe a general method for generating similar inequalities to the one above but for graphs that may contain triangles, but no complete subgraphs of size 4. We then show a selection of the inequalities we get from the computerised generation. In Paper V we study the inequality

$e(G) \geq (1/2)(\mathrm{ceil}((7l - 2)/2)n(G) - l\,\mathrm{floor}((5l + 1)/2)\alpha(G))$

for $l \geq 2$, and examine the properties of graphs G without cliques of size l+1 such that G is minimal with respect to the above inequality not holding, and show for small l that no such graphs G can exist.

In Paper II we study constructions of graphs G such that e(G) - e(3,k;n) is small when $n \leq 3.5(k-1)$. We employ a description of some of these graphs in terms of 'patterns' and a recursive procedure to construct them from the patterns. We also present the result of computer calculations where we actually have performed such constructions of Ramsey graphs and compare these lists to previously computed lists of Ramsey graphs.

In Paper III we develop a method for computing, recursively, upper bounds for Ramsey numbers R(l,k). In particular the method uses bounds for the edge numbers e(l,k;n). In Paper III we have implemented this method as a computer program which we have used to improve several of the best known upper bounds for small Ramsey numbers R(l,k).

Keywords: *Ramsey number, edge number, minimal Ramsey graph, independence number, clique number, Turán's theorem, crochet pattern, H13-pattern, linear graph invariant, triangle-free graph.*

Stockholm 2019
http://urn.kb.se/resolve?urn=urn:nbn:se:su:diva-174786

ISBN 978-91-7797-905-0
ISBN 978-91-7797-906-7

Stockholm University

Department of Mathematics

Stockholm University, 106 91 Stockholm

ON LINEAR GRAPH INVARIANTS RELATED TO RAMSEY AND EDGE NUMBERS

Oliver Krüger

Stockholm University

On linear graph invariants related to Ramsey and edge numbers

or how I learned to stop worrying and love the alien invasion

Oliver Krüger

ISBN print 978-91-7797-905-0
ISBN PDF 978-91-7797-906-7

Printed in Sweden by Universitetsservice US-AB, Stockholm 2019

Till far - för att du
definierade primtalen.

Abstract

In this thesis we study the Ramsey numbers, $R(\ell,k)$, the edge numbers, $e(\ell,k;n)$ and graphs that are related to these. The edge number $e(\ell,k;n)$ may be defined as the least natural number m for which all graphs on n vertices and less than m edges either contains a complete subgraph of size ℓ or an independent set of size k. The Ramsey number $R(\ell,k)$ may then be defined as the least natural number n for which $e(\ell,k;n)=\infty$.

In Paper I, IV and V we study strict lower bounds for $e(\ell,k;n)$. In Paper I we do this in the case where $\ell=3$ by, in particular, showing $e(G)\geq \frac{1}{3}\left(17n(G)-35\alpha(G)-\mathrm{N}(C_4;G)\right)$ for all triangle-free graphs G, where $\mathrm{N}(C_4;G)$ denotes the number of cycles of length 4 in G. In Paper IV we describe a general method for generating similar inequalities to the one above but for graphs that may contain triangles, but no complete subgraphs of size 4. We then show a selection of the inequalities we get from the computerised generation. In Paper V we study the inequality

$$e(G)\geq \frac{1}{2}\left(\left\lceil\frac{7\ell-2}{2}\right\rceil n(G)-\ell\left\lfloor\frac{5\ell+1}{2}\right\rfloor\alpha(G)\right)$$

for $\ell\geq 2$, and examine the properties of $K_{\ell+1}$-free graphs G such that G is minimal with respect to the above inequality *not* holding, and show for small ℓ that no such graphs G can exist.

In Paper II we study constructions of graphs G such that $e(G)-e(3,k;n)$ is small and $n\leq 3.5(k-1)$. We employ a description of some of these graphs in terms of 'patterns' and a recursive procedure to construct them from the patterns. We also present the result of computer calculations where we actually have preformed such constructions of Ramsey graphs and compare these lists to previously computed lists of Ramsey graphs.

In Paper III we develop a method for computing, recursively, upper bounds for Ramsey numbers $R(\ell,k)$. In particular the method uses bounds for the edge numbers $e(\ell,k;n)$. In Paper III we have implemented this method as a computer program which we have used to improve several of the best known upper bounds for small Ramsey numbers $R(\ell,k)$.

Sammanfattning

I denna avhandling studerar vi Ramseytalen, $R(\ell,k)$, kanttalen, $e(\ell,k;n)$ och grafer som är relaterade till dessa. Kanttalet $e(\ell,k;n)$ kan definieras som det minsta naturliga talet m för vilket alla grafer med n hörn och färre än m kanter antingen innehåller en komplett delgraf av storlek ℓ eller en oberoende mängd av storlek k. Ramseytalet $R(\ell,k)$ kan då definieras som det minsta talet n för vilket $e(\ell,k;n) = \infty$.

I den första, fjärde och sista artikeln (I, IV och V) studerar vi strikta undre gränser för $e(\ell,k;n)$. I den första artikeln gör vi det speciellt för $\ell = 3$ genom att visa att $e(G) \geq \frac{1}{3}\left(17n(G) - 35\alpha(G) - \mathrm{N}(C_4;G)\right)$ för alla triangelfria grafer G, där $\mathrm{N}(C_4;G)$ är antalet cykler av längd 4 i G. I den fjärde artikeln beskriver vi en allmän metod för att, med hjälp av datorberäkningar, framställa liknande olikheter som den ovan fast för grafer som kan ha trianglar, men inte har några kompletta delgrafer av storlek 4. Vi visar sedan ett urval av de olikheter som vi får från datorberäkningarna. I den sista artikeln studerar vi istället olikheten

$$e(G) \geq \frac{1}{2}\left(\left\lceil \frac{7\ell-2}{2} \right\rceil n(G) - \ell \left\lfloor \frac{5\ell+1}{2} \right\rfloor \alpha(G)\right),$$

för $\ell \geq 2$, och undersöker egenskaper för de $K_{\ell+1}$-fria grafer G med egenskapen att de är minimala med avseendet på att ovanstående olikhet *inte* uppfylls, samt visar att för små värden på ℓ så finns inga sådana grafer G.

I den andra artikeln studerar vi konstruktioner av grafer G så att $e(G) - e(3,k;n)$ är litet och $n \leq 3.5(k-1)$. Vi använder ett sätt att beskriva somliga av dessa i termer av ”mönster” och en rekursiv procedur för att konstruera de eftersökta graferna från mönstren. Vi redogör också för resultat av datorberäkningar där vi har framställt Ramseygrafer enligt dessa konstruktioner och jämför dessa listor av grafer med befintliga listor från beräkningar av Ramseygrafer.

I den tredje artikeln utvecklar vi en metod för att beräkna, rekursivt, övre gränser för Ramseytal $R(\ell,k)$. Metoden utnyttjar bland annat gränser på kanttalen $e(\ell,k;n)$. Vi har i artikeln implementerat denna metod och utnyttjar ett datorprogram för att förbättra de kända övre gränserna på flertalet små Ramseytal $R(\ell,k)$.

List of Papers

The following papers, referred to in the text by their Roman numerals, have papers that are based on them included in this thesis. For details on how the published versions differ from the manuscripts see Section 1.8.

PAPER I: **An invariant for minimum triangle-free graphs**
O. KRÜGER, Australas. J. Combin. **74** (2019), 371–388.

PAPER II: **A computerised classification of some almost minimal triangle-free Ramsey graphs**
O. KRÜGER, *eprint arXiv:1710.06644* [math.CO], 18 Oct 2017.

PAPER III: **An improved method for recursively computing upper bounds for two-colour Ramsey numbers**
O. KRÜGER, *eprint arXiv:1804.00322* [math.CO], 1 Apr 2018.
(Accepted for publication in Utilitas Math.)

PAPER IV: **Searching for non-negative invariants for K_4-free graphs**
O. KRÜGER, *manuscript*, 2019.

PAPER V: **On invariants related to edge numbers of $K_{\ell+1}$-free graphs**
O. KRÜGER, *manuscript*, 2019.

Versions of Papers I and II were also included in:
O. KRÜGER, *On minimal triangle-free Ramsey graphs*, Licentiate Thesis in Mathematics, Stockholm University, Jan. 2018.

Acknowledgements

I would here like to generalise Alfréd Rényi's assertion that 'a mathematician is a machine for turning coffee into theorems' to the following statement.

A mathematician is a machine for turning beans into theorems.

By this statement I also would like to acknowledge everyone who has provided me with beans, of coffee or other variety, during the last five years. Foremost I thank every person that is a member, or has ever been a member, of the Matlag. I would also like to thank everyone who has provided me with espresso, or beans for making espresso.

As all members of the Matlag have been thanked it may seem redundant to thank the member who has been most significant for me again. It is however impossible to show too much, or even enough, gratitude to my supervisor Jörgen Backelin. Your guidance has been invaluable.

I would also like to thank all the wonderful friends and colleagues I have had the privilege to meet during my time as a Ph.D. student at Stockholm University. Thank you, tack, grazie, danke, obrigado, bedankt, gracias. You are all great! (Yes, you are included!)

To my family, thank you for always being there to listen to me lament my existence and provide me with cobeans (in mathematics the prefix 'co-' may be used for something that is a counterpart or dual).

Lastly, I thank the people in and around Sol Invictus. You have made my life bearable by distracting me from mathematics when I most needed to be distracted.

Contents

General Introduction

In this chapter we give a short introduction to the topic of this Ph.D. thesis and motivate the questions investigated. We introduce classical two-colour Ramsey theory for graphs and explain how this is related to edge numbers and edge-minimal graphs, which are the main objects we study throughout this thesis. We also give an introduction to each of the five papers (subsequent chapters) of this thesis.

The majority of this chapter is copied, verbatim, from the introduction of the author's licentiate thesis [18]. For full details on the relationship between this thesis and other works published by the author see Section 1.8.

We begin with the following quote, which relates the subtitle of this thesis to its contents, attributed to Paul Erdös.

> *'[Suppose] aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.'* [1]

1.1 Ramsey theory on graphs

Graphs in this thesis will always be simple undirected finite graphs. In this thesis we will generally only consider graphs up to isomorphism, we will therefore employ the usual convention to say 'graph' where we more formally mean an isomorphism class of graphs. We define the *clique number*, $\omega(G)$, of a graph G to be the maximum size of a complete subgraph of G, and the *independence number*, $\alpha(G)$, to be the maximum size of an independent set in G.

One of the most fundamental questions of two-colour Ramsey theory on graphs can be formulated as follows: What is the least natural number n such that every graph on n vertices either contains a clique of size ℓ or an independent set of size k? A particular consequence of a theorem from 1930 by F. Ramsey in [30], concerning a problem in formal logic, is that there exist such a number, denoted $R(\ell,k)$, for each natural $\ell,k \geq 1$. The numbers $R(\ell,k)$ are called the (two-colour) *Ramsey numbers*.

For most values of ℓ,k it is a difficult problem to determine the Ramsey number $R(\ell,k)$. In particular, the only $\ell,k \geq 3$ for which we know the exact

[1] Quoted as it appears in by Ronald L. Graham and Joel H. Spencer, in *Scientific American* vol. 263, no. 1 (July 1990), pp. 112–117.

value of $R(\ell,k)$ is when $\{\ell,k\} \in \{\{3,k\} \mid 3 \leq k \leq 9\} \cup \{\{4\},\{4,5\}\}$. Determining these values is work done by several different authors over several decades in the last century [8–10; 14–16; 24; 25]. For all other values of ℓ and k we have only some upper and lower bounds on the Ramsey numbers, although some asymptotics are known. For a survey on the topic of these, and similar, Ramsey numbers which is regularly updated with new results see [27].

To determine exact values and better bounds on Ramsey numbers it is sometimes useful to know about the structure of graphs G which have n vertices but for which $\omega(G) < \ell$ and $\alpha(G) < k$, where we have fixed ℓ, n and k. Such graphs only exist when $n < R(\ell,k)$, and often they exhibit some common properties. Such graphs are called *Ramsey* $(\ell,k;n)$*-graphs*, or sometimes just $(\ell,k;n)$-graphs. All graphs G are $(\ell,k;n)$-graphs for some choice of ℓ, k and n, but the ones we are most interested in are those where n is close to the upper bound $R(\ell,k)-1$. Note that the existence of an $(\ell,k;n)$-graph shows, in particular, that $R(\ell,k) \geq n+1$, since $R(\ell,k)$ can be thought of as the least number n such that there are no $(\ell,k;n)$-graphs.

Example 1.1.1. The cycle of length 5, C_5, has clique number 2 and independence number 2. It is therefore a Ramsey $(3,3;5)$-graph. This implies that $R(3,3) \geq 6$. It is not difficult to show that there are no Ramsey $(3,3;6)$-graphs, and therefore $R(3,3) = 6$.

In fact, one can show that C_5 is the only Ramsey $(3,3;5)$-graph since every other graph on five vertices either contains a clique of size 3 or an independent set of size 3.

One property of Ramsey $(\ell,k;n)$-graphs we are particularly interested in is the number of edges. We now define one of the main objects we investigate in this thesis. Om this thesis these numbers ar also used as a tool for bounding Ramsey numbers.

Definition 1.1.1. The *minimum edge number* $e(\ell,k;n)$ is defined as the least number of edges an $(\ell,k;n)$-graph can have, i.e.

$$e(\ell,k;n) = \min\{e(G) \mid G \text{ is an } (\ell,k;n)\text{-graph}\}.$$

We have seen in Example 1.1.1 that C_5 is the only $(3,3;5)$-graph and therefore we have $e(3,3;5) = 5$. We may also observe that one can express the Ramsey numbers in terms of the minimum edge numbers as

$$R(\ell,k) = \min\{n \in \mathbb{N} \mid e(\ell,k;n) = \infty\}.$$

This means that a complete determination of the minimum edge numbers would in particular yield a complete determination of the Ramsey numbers. But

k	3	4	5	6	7	8	9	10	11	12	13	14	15
$R(3,k) \geq$	6	9	14	18	23	28	36	40	47	53	60	67	74
$R(3,k) \leq$	6	9	14	18	23	28	36	42	50	59	68	77	87

Table 1.1: Best known bounds on the Ramsey numbers $R(3,k)$ for small k.

since computing the Ramsey numbers is notoriously difficult any effort to completely establish the edge numbers seems futile. However, the minimum edge numbers have been used to compute some small Ramsey numbers.

In Paper I and Paper II of this thesis we study the minimum edge numbers of the form $e(3,k;n)$, which are related to the Ramsey numbers $R(3,k)$. It is only prudent to here mention that the asymptotic behaviour of the Ramsey numbers $R(3,k)$ is quite well known as $k \to \infty$ (unlike $R(\ell,k)$ for $\ell \geq 4$), since we have the following theorem by J. H. Kim.

Theorem 1.1.1 (Kim [17])**.**

$$R(3,k) = \Theta\left(\frac{k^2}{\log k}\right).$$

Despite this, not much is known about the exact value of the Ramsey numbers $R(3,k)$. For $k \leq 15$ we only have the upper and lower bounds given in Table 1.1, for references see [27]. The exact value of $R(3,k)$ is unknown for $k \geq 10$.

Ramsey $(\ell,k;n)$-graphs G with the property that $e(G) = e(\ell,k;n)$ are said to be *minimal Ramsey graphs*. We say that a graph G is *triangle-free* if it contains no complete subgraph of size 3, i.e. if $\omega(G) < 3$. For any set of vertices $S \subseteq V(G)$ we define the *closed neighbourhood*, $N[S]$, to be the vertex set $S \cup N(S)$, where $N(S) = \bigcup_{v \in S} N(v)$ is the union of the neighbourhoods of vertices in S. When there is no ambiguity we may drop the use of set parenthesis in this and other notation and write e.g. $N[v_1,v_2]$ instead of $N[\{v_1,v_2\}]$. For any vertex $v \in V(G)$ we define the *neighbourhood removed subgraph*, G_v, to be the graph obtained by removing v and all its neighbours (and the edges incident to those vertices) from G. In other words G_v is the induced subgraph on $V(G) \setminus N[v]$. More generally we may, for a set $S \subseteq V(G)$, define the graph G_S to be the induced subgraph on $V(G) \setminus N[S]$.

The number of neighbours of a vertex v is will be called the *valency* of v, and a vertex with valency 1 will be called monovalent, valency 2 called bivalent and so on. The valency of v will be denoted by $d(v)$. We will denote the minimum and maximum valency of a graph G by $\delta(G)$ and $\Delta(G)$, respectively.

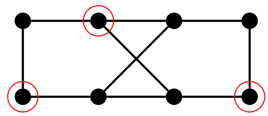

Figure 1.1: The graph Ch_3, which is a Ramsey $(3,4;8)$-graph. The circled vertices form an independent set of maximal size.

We illustrate some of the main ideas used to compute small Ramsey numbers, by utilising bounds on even smaller Ramsey numbers and minimal Ramsey graphs in Example 1.1.2.

Example 1.1.2. We will in this example show that $R(3,4) = 9$, $R(3,5) = 14$ and $R(3,6) = 18$.

We first observe that if G is a Ramsey $(3,\ell;n)$-graph then G_v is a Ramsey $(3,\ell-1;n-d(v)-1)$-graph for every vertex v, since G_v contains no triangles when G is triangle-free, any independent set in G_v can be extended to an independent set in G by adding v to it, and if G has n vertices then $|V(G_v)| = |V(G)\setminus N[v]| = n-d(v)-1$.

We begin by showing $R(3,4) = 9$. Note that the graph in Figure 1.1, which in this thesis is denoted by Ch_3, is a Ramsey $(3,4;8)$-graph. This implies, in particular, that $R(3,4) \geq 9$. One approach to showing that, in fact, $R(3,4) = 9$ is to assume that G is a Ramsey $(3,4;9)$-graph. Thus G_v has to be a Ramsey $(3,3;8-d(v))$-graph for all $v \in V(G)$. By $R(3,3) = 6$, as we have seen in Example 1.1.1, we must have $d(v) \geq 3$ for all $v \in V(G)$. Furthermore, the neighbourhood $N(v)$ of every vertex v forms an independent set in G since G is triangle-free. This means that $|N(v)| < 4$ for all $v \in V(G)$. Hence, G has to be 3-regular, which is impossible since we can not have a graph on an odd number of vertices where every vertex has odd valency. This contradiction proves that there are no $(3,4;9)$-graphs and therefore $R(3,4) = 9$.

Now we will show that $R(3,5) = 14$. The graph which is denoted H_{13} in this thesis, and is illustrated in Figure 1.9, is a Ramsey $(3,5;13)$-graph, which shows that $R(3,5) \geq 14$. Assume, now, that G is a Ramsey $(3,5;14)$-graph. For every vertex $v \in V(G)$ we have that G_v is a $(3,4;13-d(v))$-graph. By $R(3,4) = 9$ we get $d(v) \geq 5$ for all $v \in V(G)$, but then $N(v)$ is an independent set of size 5 or G contains a triangle. This is clearly a contradiction and thus $R(3,5) = 14$.

In the final part of this example we show that $R(3,6) = 18$. Radziszowski and Kreher have shown, in [28], that there are seven Ramsey $(3,6;17)$-graphs. One of these graphs can be formed by taking the disjoint union of H_{13} (with vertices labelled as in Definition 1.2.4) and C_4 with vertices $V(C_4) = \{c_0, c_1, c_2, c_4\}$, then adding the edges c_0x_1, c_0x_5, c_1x_3, c_1x_9, c_2x_0, c_2x_2, c_2x_6, c_3x_4, c_3x_8 and

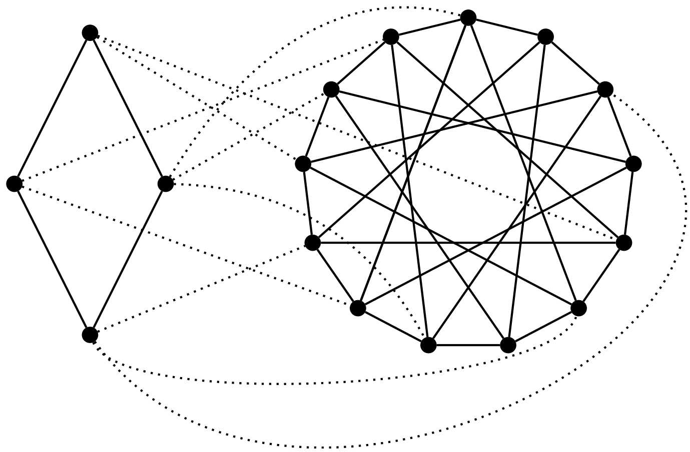

Figure 1.2: Example of a Ramsey $(3,6;17)$-graph. Dotted lines indicate the edges that are added to the disjoint union of C_4 and H_{13} in the definition.

c_3x_{11} to the edge set. The resulting graph is illustrated in Figure 1.2 and shows, in particular, that $R(3,6) \geq 18$.

Assume there is some Ramsey $(3,6;18)$-graph G. For every $v \in V(G)$ have that G_v is a $(3,5;17-d(v))$-graph and therefore, by $R(3,5) = 14$, we have $\delta(G) \geq 4$. Since $N(v)$ is independent for all $v \in V(G)$ we also have $\Delta(G) \leq 5$. Suppose v is a tetravalent vertex in G. The graph G_v has to be (isomorphic to) H_{13} since it can be shown that H_{13} is the unique Ramsey $(3,5;13)$-graph. In particular, H_{13} is a minimal Ramsey graph. But H_{13} is 4-regular and $e(3,5;13) = 26$ so if we have at least 14 edges between $N(v)$ and G_v we would get a vertex of valency at least 6. Thus we have at most one pentavalent neighbour of every tetravalent vertex in G. On the other hand if the number of edges between $N(v)$ and G_v is 13 (or 12) then a neighbour of v would be tetravalent with at least two pentavalent neighbours. This show that there are no tetravalent vertices in G, i.e. G has to be 5-regular. This means that $e(G_v) = e(G) - 25 = 45 - 25 = 20$, which is equal to the minimum edge number $e(3,5;12) = 20$. There is a unique minimal Ramsey $(3,5;12)$-graph, which is denoted by BC_4 in this thesis and is illustrated in Figure 1.6. Hence $G_v \cong BC_4$ for all $v \in V(G)$. Note that in BC_4 every cycle of length 4 contains at least two tetravalent vertices. Fix some $v \in V(G)$, and note that some neighbour, w, of v is adjacent to only trivalent vertices of $G_v \cong BC_4$. The neighbours of w in G_v forms an independent set of trivalent vertices in G_v. It is easily seen by looking at the independent sets of size 4 among trivalent vertices in BC_4 that this implies the existence of a cycle of length 4 containing only trivalent vertices in G_w, contradicting $G_w \cong BC_4$. This contradiction shows that there are no $(3,6;18)$-graphs and therefore $R(3,6) = 18$.

One more example, using more immediately lower bounds on the edge numbers, can be seen in Example 1.5.1. There we show, using the results from Paper III, that $R(5,8) \leq 215$.

The previous example illustrates a very special case of a more general philosophy which may be used to determine upper bounds on Ramsey numbers using minimum edge numbers and characterisations of minimal Ramsey graphs. The principle behind this is to assume that we have a $(\ell,k;n_0)$-graph G for some n_0 that we desire to show is an upper bound on $R(\ell,k)$. This graph must then contain other Ramsey graphs as subgraphs. The structure of these subgraphs can then be used to derive a contradiction, which shows that $R(\ell,k) \leq n_0$. These techniques have been employed by Kalbfleich [15] to compute $R(3,6)$, by Graver and Yackel [8] to compute $R(3,7)$, by Grinstead and Roberts [10] to compute $R(3,8)$ and $R(3,9)$. This has also been employed successfully by Radziszowski and Kreher to compute several upper bounds for $R(3,k)$.

Furthermore, McKay and Radziszowski used refinements of this technique to compute $R(4,5)$ in [25]. These, and similar, considerations also play an important role in the recent proof that $R(5,5) \leq 48$ by Angeltveit and McKay, see [1].

Having a good understanding of the minimum edge numbers and the minimal Ramsey graphs appear to be a very useful tool for improving upper bounds on the Ramsey numbers.

We will use the principle which the above example illustrates heavily in Paper III of this thesis to improve the upper bounds on many of the values $R(\ell,k)$.

1.2 Linear invariants bounding the edge numbers of triangle-free graphs

We can express a trivial lower bound on the minimum edge numbers $e(3,k+1;n)$ as $e(3,k+1;n) \geq n-k$ or equivalently as the proposition that for every triangle-free graph G, we have that $e(G) \geq n(G) - \alpha(G)$. If we write this as $e(G) - n(G) + \alpha(G) \geq 0$ we get the first inequality in a series of similar inequalities, which are

$$
\begin{aligned}
t_1(G) &:= e(G) - n(G) + \alpha(G) \geq 0,\\
t_2(G) &:= e(G) - 3n(G) + 5\alpha(G) \geq 0,\\
t_3(G) &:= e(G) - 5n(G) + 10\alpha(G) \geq 0, \text{and}\\
t_4(G) &:= e(G) - 6n(G) + 13\alpha(G) \geq 0.
\end{aligned}
$$

These inequalities have been shown to hold for all triangle-free graphs by Radziszowski and Kreher in [28; 29].

Note that if $G \cong G_1 + G_2$, where the right hand side means taking the disjoint union of the two graphs G_1 and G_2, we have $t_i(G_1+G_2) = t_i(G_1)+t_i(G_2)$ for all the invariants t_i listed above. This is the characterising property of what we call *linear graph invariants*. Note that each of the inequalities $t_i(G) \geq 0$ gives bounds on the minimum edge numbers. Furthermore, determining what triangle-free graphs that achieve these inequalities with equality determines some minimal Ramsey graphs.

In particular, the triangle-free graphs G such that $t_1(G) = 0$ are exactly those that consist of isolated vertices and isolated edges, i.e. they are disjoint unions of complete graphs on one and two vertices, respectively. These are precisely the minimal Ramsey $(3,k+1;n)$-graphs for $k < n \leq 2k$. Similarly, the graphs for which $t_2(G) = 0$ are minimal Ramsey $(3,k+1;n)$-graphs for $2k < n \leq 5k/2$, the graphs for which $t_3(G) = 0$ are minimal $(3,k+1;n)$-graphs for $5k/2 \leq n \leq 3k$ and lastly those graphs for which $t_4(G) = 0$ are minimal $(3,k+1;n)$-graphs for $3k \leq n \leq 13k/4 - 1$ or $n = 13k/4$ when $k \equiv 0 \mod 4$ (see [4]).

1.2.1 Extremal graphs with respect to the linear inequalities

We here give a brief description of the minimal Ramsey graphs which are such that they satisfy the inequalities involving linear graph invariants listed above with equality. Note that if G_1, G_2 are two graphs such that $t_i(G_1) = t_i(G_2) = 0$ for some i, then by linearity of the invariant t_i we have that also $t_i(G_1+G_2) = 0$. So to characterise the graphs G for which $t_i(G) = 0$ it is enough to characterise the *connected* graphs with this property. All the graphs that are introduced here also play important roles in Paper I and Paper II.

Example 1.2.1. There are only two connected triangle-free graphs G such that $t_1(G) = 0$, namely $G \cong K_1$ (for which $e(G) = 0$, $n(G) = 1$ and $\alpha(G) = 1$) and $G \cong K_2$ (for which $e(G) = 1$, $n(G) = 2$ and $\alpha(G) = 1$). One way to see that there are no other connected triangle-free graphs G for which $t_1(G) = 0$ is the following. Note that for $v \in V(G)$ we have that $e(G_v) = e(G) - \sum_{w \in N(v)} d(w)$, $n(G_v) = n(G) - d(v) - 1$ and $\alpha(G_v) \leq \alpha(G) - 1$. This means that we get

$$t_1(G_v) = e(G_v) - n(G_v) + \alpha(G_v) \leq t_1(G) + d(v) - \sum_{w \in N(v)} d(w).$$

In particular, if v is assumed to be of minimal valency in the graph we get that $t_1(G_v) \leq -d(v)^2 + d(v)$. It is now clear that if we (inductively) assume that $t_1(G_v) \geq 0$, then $d(v) \leq 1$ and if $d(v) = 1$ then the neighbour of v is also

Figure 1.3: The two connected triangle-free graphs which have t_1-value 0.

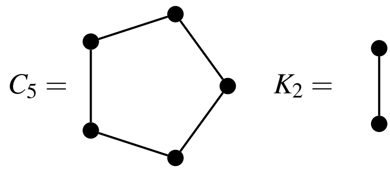

Figure 1.4: The two connected triangle-free graphs which have t_2-value 0.

monovalent. This means that either v is an isolated vertex (it forms a K_1) or it is one of the vertices in a K_2.

Example 1.2.2. Note that $t_2(K_2) = 0$, and that $t_2(C_5) = 0$ since $e(C_5) = n(C_5) = 5$ and $\alpha(C_5) = 2$. The graphs K_2 and C_5 are the only two connected triangle-free graphs with t_2-value 0. We can see this in a like manner as we did for the graphs with t_1-value 0. We observe that we here must have

$$t_2(G_v) \leq t_2(G) - \sum_{w \in N(v)} d(w) + 3d(v) - 2. \tag{1.1}$$

If we inductively have that $t_2(G_v) \geq 0$ and $t_2(G_v) = 0 \Rightarrow G_v \in \{K_2, C_5\}$ it is easy to see from inequality (1.1) that we must have minimum valency 1 or 2, and in both cases G must be regular. Furthermore if $d(v) = 2$ and the graph is 2-regular, then we get that $t_2(G_v) = 0$ and therefore $G_v \cong K_2$, whence $G \cong C_5$.

The general idea outlined above to determine the graphs with t_i-value 0 for $i = 1$ and $i = 2$, in Examples 1.2.1 and 1.2.2, is also important in the proof of the main theorem of Paper I. However, there we work with a slightly more complicated invariant where the inductive step is far less obvious - which seems to require a significantly more technical approach.

The characterisation of the (connected) triangle-free graphs for which the t_3-value is 0 was done by Radziszowski and Kreher in [29]. These graphs are the ones defined in the following two definitions.

The first class of graphs (which appear in [3, Chapter 3] and [4] as *chains* denoted by Ch_k, in [29] as F_k and in [13] as H_k) that we need is defined as follows.

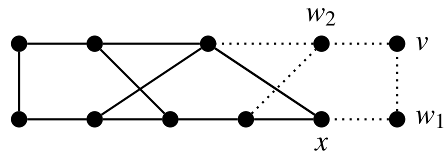

Figure 1.5: The graph Ch_4, the non-dotted edges are edges of a Ch_3-subgraph where $(Ch_4)_v \cong Ch_3$, with vertices named as in Definition 1.2.1

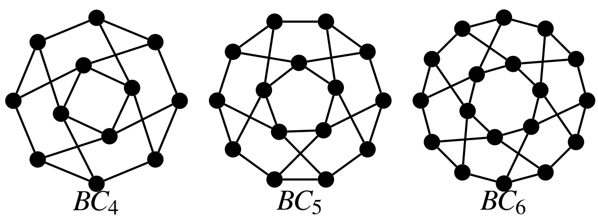

Figure 1.6: Examples of bicycle graphs BC_k for $k = 4, 5, 6$.

Definition 1.2.1. Let Ch_2 be a cycle of length 5. We recursively define Ch_{k+1} for $k \geq 2$. Let $x \in V(Ch_k)$ be some bivalent vertex. Let $V(Ch_{k+1}) = V(Ch_k) \,\dot\cup\, \{v, w_1, w_2\}$ and $E(Ch_{k+1}) = E(Ch_k) \cup \{vw_1, vw_2, w_1x\} \cup \{w_2y \mid y \in N(x)\}$.

We have illustrated $Ch_2 = C_5$ and Ch_3 in Figures 1.4 and 1.1, respectively. In Figure 1.5 we illustrate the graph Ch_4 and highlight how this graph may be constructed recursively from the graph Ch_3.

Our second class of extremal graphs that we will define here are the so called *bicycles* BC_k. The graphs BC_k also appear with this notation in [2, p. 20] and [4] or as *extended k-chains* (in [13], denoted E_k) and G_k (in [29]).

Definition 1.2.2. Let BC_k, $k \geq 4$, be a graph consisting of an induced cycle on vertices $c_1, c_2 \ldots, c_{2k}$ and one induced cycle on vertices $d_1, d_2 \ldots, d_k$. Connect the cycles by edges d_ic_{2i-2} and d_ic_{2i+1} for $i \in \{1, \ldots, k\}$, taking indices modulo $2k$ for c_i's and modulo k for d_i's.

Apart from the two infinite classes of minimal Ramsey graphs defined in Definitions 1.2.1 and 1.2.2 the following three graphs play crucial roles in this thesis. Two of them are 3-regular graphs, denoted $(2C_7)_{2i}$ and W_5, which may be defined as follows.

Definition 1.2.3. Let $V((2C_7)_{2i}) = \{a_0, a_1, \ldots, a_6\} \cup \{b_0, b_1, \ldots, b_6\}$ and the edges of $(2C_7)_{2i}$ be such that both $a_0, a_1, \ldots, a_6$ and $b_0, b_1, \ldots, b_6$ form cycles of length 7 in $(2C_7)_{2i}$. Connect these two cycles by adding an edge b_ia_{2i} for all $i \in \{0, 1, \ldots, 6\}$, taking indices modulo 7.

Let $V(W_5) = \{a_0, a_1\} \cup \{b_0, \dots, b_3\} \cup \{c_0, \dots, c_7\}$ and the edges of W_5 be such that a_0a_1 is an edge, $b_0, \dots, b_4$ are independent and $c_0, \dots, c_7$ form a cycle of length 8. Add edges b_ia_i for $i \in \{1,2,3,4\}$ taking a_i-indices modulo 2. Also add edges b_ic_{2i} and b_ic_{2i+3} for $i \in \{1,2,3,4\}$ taking indices modulo 8.

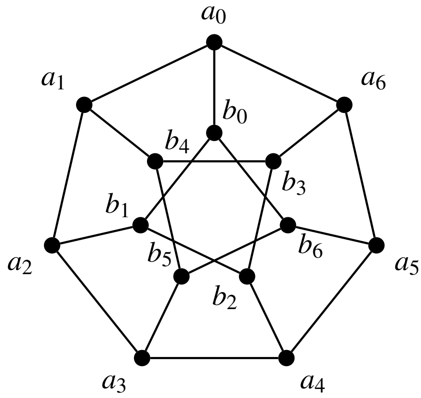

Figure 1.7: The graph $(2C_7)_{2i}$.

The graph $(2C_7)_{2i}$ is also known as a generalised Petersen graph, variously denoted $\mathrm{GP}(7,2)$ or $\mathrm{P}(7,2)$. It has been illustrated in Figure 1.7, and the graph W_5 is illustrated in Figure 1.8.

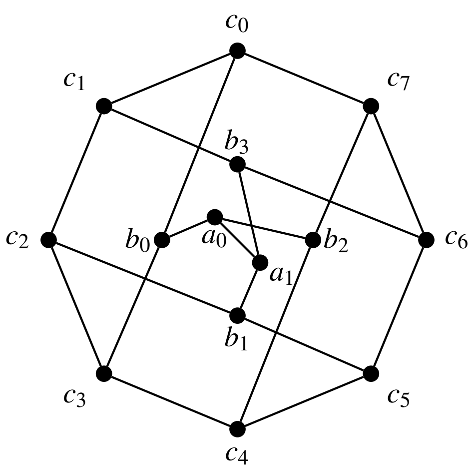

Figure 1.8: The graph W_5.

The final graph which we will define here is called H_{13}, and is of particular

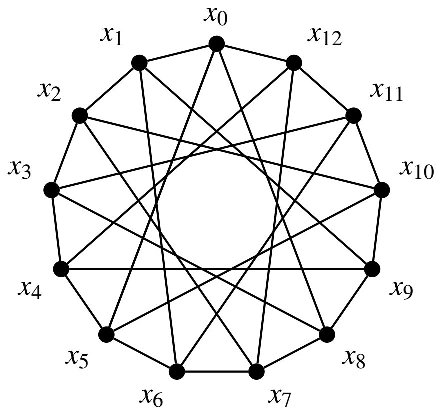

Figure 1.9: The graph H_{13}.

importance in Paper II. It is the unique minimal $(3,5;13)$-graph (which has 26 edges). It is also illustrated in Figure 1.9.

Definition 1.2.4. Let $V(H_{13}) = \{x_0, x_1, \ldots, x_{12}\}$ and let the edges of H_{13} be such that $x_ix_j \in E(H_{13})$ if and only if $i - j \equiv k \mod 13$ for some $k \in \{\pm 1, \pm 5\}$.

Since H_{13} has 26 edges, 13 vertices and independence number 4 we have $t_4(H_{13}) = 0$.

As we have mentioned above we have that the graphs Ch_k, have t_3-value 0 for all $k \geq 2$ and the graphs BC_k have t_3-value 0 for all $k \geq 4$. It has been shown by Radziszowski and Kreher in [29] that these are in fact the *only* connected triangle-free graphs G for which $t_3(G) = 0$. It was suggested by Radziszowski and Kreher that the bicycles BC_k, and the regular graph H_{13} could be the only graphs G for which $t_4(G) = 0$. This was later proved to be the case by Backelin, see e.g. [4, Theorem 3].

1.3 A new invariant and its extremal graphs

In a similar vein as the results concerning the non-negativity of the linear invariants defined in the previous section, we define, in Paper I, a linear invariant

$$\nu(G) := 3e(G) - 17n(G) + 35\alpha(G) + \mathrm{N}(C_4; G),$$

where $\mathrm{N}(C_4; G)$ denotes the number of cycles of length 4 in the graph G. The main theorem of Paper I shows that this linear invariant is non-negative for all triangle-free graphs G, and characterises the graphs which have ν-value 0.

Theorem 1.3.1 (Theorem 2.1.1, Paper I). *If G is a triangle-free graph then $\nu(G) \geq 0$, and if $\nu(G) = 0$, with G connected, then $G \in \{W_5, (2C_7)_{2i}\} \cup \{Ch_k \mid k \geq 2\} \cup \{BC_k \mid k \geq 5\}$.*

This theorem implies, in particular, that $e(C_{\leq 4}, k+1; n) \geq \frac{17}{3}n - \frac{13}{3}k$, where $e(C_{\leq 4}, k+1; n)$ is the least number of edges possible in a graph containing no cycle of length at most 4, on n vertices and with independence number at most k. This is a variation on the ordinary minimum edge numbers $e(\ell, k+1; n)$ where we instead of restricting the clique number of the graphs to be less than ℓ, we make restrict the girth to be at least 5.

The theorem also has interesting consequences for Ramsey $(3, k+1; n)$-graphs. It shows that if G is a $(3, k+1; n)$-graph on $\frac{17}{3}n - \frac{35}{3}k - \frac{m}{3}$ edges for some $m \geq 0$ then G contains at least m cycles of length 4. This gives us information about the structure of minimal Ramsey $(3, k+1; n)$-graphs for some k and n, and graphs that are in some sense 'close' to being minimal Ramsey graphs, by having few, but not necessarily the least possible number of, edges.

Another consequence of Theorem 2.1.1 of Paper I is that it improves the best known lower bound on the *independence ratio* $\mathrm{i}(G) := \alpha(G)/n(G)$ in some special cases. A general result by Hopkins and Staton in 1982, in [11], implies that $\mathrm{i}(G) \geq \frac{7}{23}$ for all graphs G of girth at least 5 and average valency 4. Their result has later been improved on by Lichiardopol in 2004 (see [21]). It follows from Theorem 2.1.1 of Paper I that $\mathrm{i}(G) \geq \frac{11}{35}$ for all graphs of girth at least 5 and average valency 4. This improves the bound by Hopkins and Staton and also improves the bounds given by Lichiardopol for graphs of maximum valency 4 and girth 5, 6 and 7. For graphs of girth at least 8 Lichiardopol provides a better lower bound on the independence ratio than $\frac{11}{35}$.

1.4 Characterising the minimal (or almost minimal) Ramsey graphs

In Paper II we aim to give a description of minimal Ramsey $(3, k+1; n)$-graphs by a form of 'pattern' that was introduced by J. Backelin in [3, Chapter 6]. The main objective of Paper II is to count the number of graphs that are either minimal Ramsey graphs or are close to being minimal Ramsey graphs (i.e. have slightly more edges than $e(3, k+1; n)$) that can be described by these patterns. In several cases we then compare this to counts of Ramsey graphs that have been computed elsewhere. The moral of Paper II is that many, and in some cases almost all, minimal and almost minimal Ramsey graphs can be described by this kind of pattern when $n \leq 3.5k$.

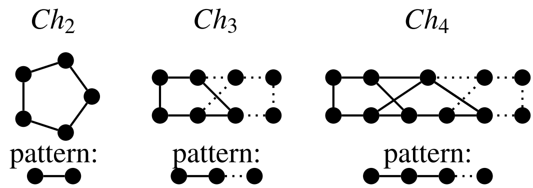

Figure 1.10: Chains Ch_k and their corresponding patterns. The dotted edges indicate in what way we consider Ch_{k-1} to be a neighbourhood removed subgraph of Ch_k, and the corresponding structure in the patterns.

The main philosophy behind the procedure for constructing the graphs that are enumerated Paper II is that we start with some Ramsey minimal graph, in our case K_2 or H_{13}. We then build larger graphs from these by adding in each step only a few edges, but a 'comparatively large' number of vertices, and simultaneously only increasing the independence number by 1. In Paper II this is formalised. We there work with a partially directed graph (that may also include one hyperedge), called a *pattern* (or H_{13}*-pattern*) which encodes exactly how we add vertices and edges, starting with either the graph K_2 or H_{13}. The idea behind this comes from the observation that many of the minimal Ramsey graphs we have seen in fact can be formed this way. We will not go into the details in this introduction, for which we refer to Paper II. We will however give some examples of patterns of graphs.

Example 1.4.1. The graphs Ch_k are constructed from Ch_2 in the way described in Definition 1.2.1. For each $k \geq 3$ we have that $(Ch_k)_v = Ch_{k-1}$ where $v \in V(Ch_k)$ is a bivalent vertex. This may be encoded as a pattern by saying that the pattern for the graph Ch_k is a path of length k. That this pattern has the path of length $k-1$ as a subpattern corresponds precisely to the fact that $(Ch_k)_v = Ch_{k-1}$. Some examples of chains and their corresponding patterns are given in Figure 1.10.

Example 1.4.2. The bicycles, BC_k, correspond to patterns that are cycles of length k. The fact that cycles of length k contains paths as length $k-1$ corresponds precisely to the fact that $(BC_k)_v = Ch_{k-1}$ for all trivalent vertices $v \in V(BC_k)$. We can see examples of this correspondence in Figure 1.11.

The idea described in Paper II is generalising the above way of constructing minimal Ramsey graphs by extensions of smaller minimal Ramsey graphs. The generalised version of patterns does not only describe chains and cycles (that may be constructed recursively from K_2) but also H_{13} and graphs that may be constructed from H_{13} in a special way. We give an example of such a graph,

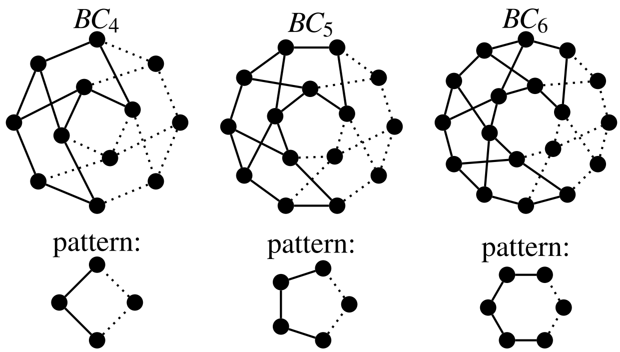

Figure 1.11: Examples of bicycle graphs BC_k and their patterns. The dotted edges illustrate how the paths form subpatterns and the chains form subgraphs in a corresponding way.

with its corresponding pattern, Figure 1.12. For the details on this procedure see Paper II.

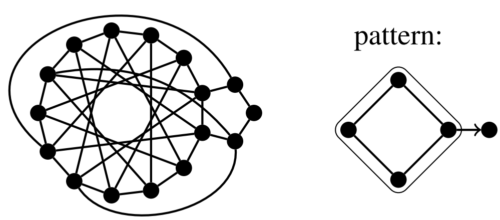

Figure 1.12: The $(3,6;16)$-graph (left) on 33 edges obtained from the H_{13}-pattern (right).

The main result of the Paper II is the enumeration, and computer construction of the graphs that can be described using these so called H_{13}-patterns. This can then be compared to previous calculations that enumerate all (minimal) Ramsey graphs, done in particular by McKay [23], Goedgebeur and Radziszowski [7; 26], and Radziszowski and Kreher [28; 29], combined with Backelin, e.g. in [3, Chapter 6] and [4, Theorem 3]. Some of these works (in particular those by Radziszowski and Kreher and those by Backelin) contain general characterisations of minimal Ramsey graphs which, a fortiori, imply that all minimal Ramsey $(3,k+1;n)$-graphs for which $n \leq 13k/4-1$ must be patterned. The more interesting part of the results in the computations of Paper

II are therefore where $n > 13k/4 - 1$ or where we have Ramsey $(3, k+1; n)$-graphs that are not minimal.

1.5 Improved upper bounds for Ramsey numbers

Recall that we have defined the minimum edge numbers $e(\ell, k; n)$ to be the least number of edges a Ramsey $(\ell, k; n)$-graph can have. Similarly we have the following.

Definition 1.5.1. The *maximum edge number* $E(\ell, k; n)$ is defined as the greatest number of edges an $(\ell, k; n)$-graph can have, i.e.

$$E(\ell, k; n) = \max\{e(G) \mid G \text{ is an } (\ell, k; n)\text{-graph}\}.$$

Note for example that $e(3,3;5) = E(3,3;5) = 5$ since, as we have seen in Example 1.1.1, C_5 is the *only* Ramsey $(3,3;5)$-graph. In general $e(\ell, k; n)$ and $E(\ell, k; n)$ are the greatest and least numbers, respectively, for which

$$e(\ell, k; n) \leq e(G) \leq E(\ell, k; n)$$

for all Ramsey $(\ell, k; n)$-graphs G.

In Paper III of this thesis we employ the idea of using lower bounds on $e(\ell, k; n)$, and upper bounds on $E(\ell, k; n)$ to create an algorithm for computing upper bounds of Ramsey numbers $R(\ell_1, k_1)$, recursively, from upper bounds of $R(\ell_2, k_2)$, where $\ell_2 < \ell_1$ or $k_2 < k_1$.

The method described is a refinement of a method developed in by Huang et al. in [12]. We also show how, in particular, this method may be used to improve several of the best known upper bounds for Ramsey numbers $R(\ell, k)$ where $4 \leq \ell \leq 10$ and $5 \leq k \leq 15$.

The two main theorems of Paper III are the following.

Theorem 1.5.1 (Theorem 4.2.1, Paper III)**.** *If* $p \leq R(\ell, k) - 1$, $\alpha \geq R(\ell-2, k) - 1$, $\beta \geq R(\ell, k-2) - 1$, $\gamma \geq R(\ell-1, k) - 1$ *and* $\delta \geq R(\ell, k-1) - 1$, *then*

$$(p-1)(p-2) \leq \max_{d \in [p-1-\delta, \gamma]} 2\binom{p-d-1}{2} + 2\Delta(\ell, k, p, d) + 3d(p-d-1),$$

where

$$\Delta(\ell, k, p, d) = E(\ell-1, k; d) - e(\ell, k-1; p-d-1).$$

Theorem 1.5.2 (Theorem 4.2.2, Paper III)**.** *Let $p, \alpha, \beta, \gamma, \delta$ be as in Theorem 1.5.1. Then*

$$e(\ell,k;p) \geq \max\left\{\frac{p(p-\delta-1)}{2}, \frac{A-\sqrt{A^2-B}}{12}\right\}$$
$$E(\ell,k;p) \leq \min\left\{\frac{p\gamma}{2}, \frac{A+\sqrt{A^2-B}}{12}\right\},$$

where

$$A = (\alpha - \beta + 3(p-1))p, \quad B = 12p^2(p-1)(p-\beta-2).$$

We will now show how these theorems may be employed to compute upper bounds for Ramsey numbers. We will exemplify this by showing that $R(5,8) \leq 215$, which is a small improvement of the bound $R(5,8) \leq 216$ which appears in [27].

Example 1.5.1. From the upper bounds in [27] we have $R(3,8) \leq 28$, $R(5,6) \leq 87$ and $R(4,8) \leq 84$. Furthermore, if we use the method described in [12], one can easily show that $R(5,7) \leq 142$.

Therefore, if we define $\alpha = 27$, $\beta = 86$, $\gamma = 83$ and $\delta = 141$ these numbers satisfy the premises of Theorem 1.5.1 when $\ell = 5$ and $k = 8$.

Now, note that if $R(5,8) \geq 216$ then the inequality in the statement of Theorem 1.5.1,

$$(p-1)(p-2) \leq \max_{d\in[p-1-141,83]} 2\binom{p-d-1}{2} + 2\Delta(5,8,p,d) + 3d(p-d-1), \tag{1.2}$$

would hold for $p = 215$. We want to show that this is not the case to conclude that $R(5,8) \leq 215$. Inserting this value of p into equation (1.2) we get the inequality

$$45582 \leq \max_{d\in[73,83]} 2\binom{214-d}{2} + 2\Delta(5,8,215,d) + 3d(214-d), \tag{1.3}$$

where $\Delta(5,8,215,d) = E(4,8;d) - e(5,7;214-d)$. Note that we want to compute upper bounds for $\Delta(5,8,215,d)$ where $d \in [73,83]$, i.e. we want bounds for $E(4,8;73) - e(5,7,141), E(4,8,74) - e(5,7,140), \ldots, E(4,8;83) - e(5,7,131)$. To obtain these we will use Theorem 1.5.2.

We want to apply Theorem 1.5.2 for $\ell = 4$ and $k = 8$ to get upper bounds for $E(4,8,d)$. This means that the values of α, β, γ and δ comes from the bounds $R(2,8) = 8, R(4,6) \leq 41, R(3,8) = 28$ and $R(4,7) = 23$, respectively. Hence, we get that in Theorem 1.5.2 we have

$$A_d = 3d(d-12), \text{ and } B_d = 12d^2(d-1)(d-42).$$

Computing $E(4,8;d) \leq \left\lfloor \min\{d \cdot 83/2, (A_d + \sqrt{A_d^2 - B_d})/12\} \right\rfloor$ for $d \in [73,83]$ we get the upper bounds listed in Table 1.2.

d	73	74	75	76	77	78	79	80	81	82	83
$E(4,8;d) \leq$	985	999	1012	1026	1039	1053	1066	1080	1093	1107	1120

Table 1.2: Upper bounds for $E(4,8;d)$ for $d \in [73,83]$.

Similarly, we want to apply Theorem 1.5.2 for $\ell = 5$ and $k = 7$ for getting lower bounds on $e(5,7;214-d)$, where $d \in [73,83]$. Note that in this case we get the corresponding values of α, β, γ and δ from $R(3,7) = 23$, $R(5,5) \leq 48$, $R(4,7) \leq 61$ and $R(5,6) \leq 87$, respectively. In Theorem 1.5.2 we therefore want to use

$$A_d = (d-214)(3d-614), \text{ and } B_d = 12(d-214)^2(d-213)(d-165).$$

Computing $e(5,7;214-d) \geq \left\lceil \max\{(214-d)(127-d)/2, (A_d - \sqrt{A_d^2 - B_d})/12\} \right\rceil$ for $d \in [73,83]$ we get the lower bounds listed in Table 1.3.

d	73	74	75	76	77	78	79	80	81	82	83
$e(5,7;214-d) \geq$	4192	4069	3953	3844	3740	3641	3545	3452	3362	3275	3190

Table 1.3: Lower bounds for $e(5,7;214-d)$ for $d \in [73,83]$.

Using these values we may now find upper bounds of $\Delta(5,8,215,d)$ for $d \in [73,83]$. If $\Delta'(5,8,215,d)$ is the value of the upper bounds listed in Table 1.2 minus the lower bounds in Table 1.3, for each $d \in [73,83]$, then we have that the right hand side of the inequality (1.3) does not exceed

$$t_d := 2\binom{214-d}{2} + 2\Delta'(5,8,215,d) + 3d(214-d).$$

Explicitly computing this for each value $d \in [73,83]$ we get the values listed in Table 1.4.

d	73	74	75	76	77	78	79	80	81	82	83
t_d	44205	44400	44575	44734	44877	45008	45127	45238	45337	45428	45509

Table 1.4: Values for t_d where $d \in [73,83]$.

Note, in particular, that the maximum of the values listed in Table 1.4 is 45509 and therefore the right hand side of the inequality (1.3) evaluates to at most 45509, which is less than the left hand side. This shows that our choice of $p = 215$, is not valid and whence $R(5,8) \leq 215$.

To conclude this example we note that if we try to do the same procedure as described above, but starting with the assumption that $p = 214$, we instead get that the left hand side in inequality (1.2) becomes 45156 while the right hand side evaluates to at most 45166. Therefore, Theorem 1.5.1 and 1.5.2 are *not* sufficient to also prove $R(5,8) \leq 214$.

In practice, to use this method for computing upper bounds for Ramsey numbers is quite tedious if done by hand. Therefore we have, in Paper III, employed a simple algorithm – implemented as a computer program – for doing the necessary, simple but tedious, calculations. The source code of this computer program appears as an appendix to Paper III, and is also available for scrutiny online at [20].

A selection of the resulting improvements, compared to the best published bounds (see e.g. [27]) are listed in the Table 1.5 with boldface.

ℓ \ k	5	6	7	8	9	10	11	12	13	14	15
5	48	87	**142**	**215**	316	442	**629**	**846**	**1102**	**1442**	**1832**
6		165	298	495	780	1171	**1782**	**2549**	**3526**	**4927**	**6614**
7			**539**	**1029**	**1711**	**2775**	**4518**	**6821**	**10017**	**14841**	**20928**
8				**1865**	**3576**	**6061**	**10297**	**16777**	**25933**	**40140**	**59916**
9					**6582**	**12643**	**22161**	**38000**	**62763**	**100614**	**157549**
10						**23327**	**45488**	**80231**	**139767**	**236772**	**385139**

Table 1.5: Upper bounds of $R(\ell,k)$. Boldface indicates improved bounds (compared to [27]).

1.6 Finding similar invariants of K_4-free graphs

In Paper I and Paper II we are concerned with triangle-free graphs, and some linear invariants that are related to them. In Paper IV and Paper V we try, in two different ways, to develop similar theories but for graphs with higher clique number than 2. In Paper IV we allow triangles in the graphs, but no complete subgraph of size 4.

In Paper IV we develop a method for finding linear graph invariants, similar to the ones we have studied for triangle-free graphs that may be non-negative for all K_4-free graphs. We use some computer calculations to find a list of invariants that are interesting in the sense that both a counterexample to their non-negativity, and a proof of their non-negativity would be useful within the theory of K_4-free graphs.

We also prove, in general, that some of the invariants we find by these computer calculations indeed are non-negative for K_4-free graphs.

We define P as the set of points $(k_1(G), k_2(G), k_3(G), \alpha(G))$, in the lattice $\mathbb{Z}^4$, where $k_i(G)$ denotes the number of complete subgraphs of size i in G where

G ranges over all K_4-free graphs. A *hyperplane* in $\mathbb{R}^4$ is a set of points $\mathbf{x} \in \mathbb{R}^4$ that satisfy an equation $(a,b,c,d)\mathbf{x} = 0$, where $(a,b,c,d) \in \mathbb{R}^4 \setminus \{\mathbf{0}\}$. Similarly, a *half-space* is a set of points $\mathbf{x} \in \mathbb{R}^4$ that satisfy an inequality $(a,b,c,d)\mathbf{x} \geq 0$. For a set of lattice points $S \subseteq \mathbb{Z}^4$ we say that a half-space, defined by the inequality $(a,b,c,d)\mathbf{x} \geq 0$, is a *bounding half-space* if $\dim(\text{span}\{S \cap \{\mathbf{x} \in \mathbb{R}^4 \mid (a,b,c,d)\mathbf{x} = 0\}\}) = 3$.

Note that an inequality of the form

$$ak_1(G) + bk_2(G) + ck_3(G) + d\alpha(G) \geq 0, \tag{1.4}$$

which holds for all K_4-free graphs corresponds to a half-space which contains all of P. A *convex cone* is an intersection of finitely many half-spaces. We would like to determine a convex cone containing P.

Example 1.6.1. We know that $k_1(G) - \alpha(G) \geq 0$ for *all* graphs G, in particular for all K_4-free graphs G. Therefore the half-space

$$\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^4 \mid (1,0,0,-1)\mathbf{x} \geq 0\}$$

contains P.

$\mathcal{H}$ is however not a bounding half-space for P since the only set of points $\mathbf{x} \in P$ such that $(1,0,0,-1)\mathbf{x}$ are those that correspond to graphs, G, where $k_1(G) = \alpha(G)$, i.e. graphs without any edges. Those graphs are just disjoint unions of vertices and the corresponding points of these graphs in P all lie on a line (a subspace of dimension 1).

Furthermore, we do not want the convex cone to contain unnecessarily many points of $\mathbb{Z}^4 \setminus P$. To this end we would like the half-spaces which define the convex cone to be bounding half-spaces. Note that it is not clear, a priori, that there even exists a convex cone for P such that every defining half-space is a bounding half-space, because we may need an infinite number of bounding half-spaces.

It is not trivial to determine bounding half-spaces for the set P since we do not have any explicit description of its points. In Paper IV we develop a method where instead of P we consider a finite subset $P' \subseteq P$, generated from a database of 'interesting' K_4-free graphs. For P' it is then possible to determine a convex cone which contains it and such that every half-space is bounding for P'.

In fact, in Paper IV, we restrict the set P' even further to a set of points P'', defined to be the only points of P' that are relevant for finding bounding half-spaces corresponding to inequalities $(a,b,c,d)\mathbf{x} \geq 0$ where $a \leq 0$, and $b,c,d \geq 0$. Note that this is analogous to the inequalities $t_1(G), t_2(G), t_3(G), t_4(G) \geq 0$ that we discussed in Section 1.2.

Using computer calculations for finding bounding half-spaces for a convex cone containing a finite number of lattice points we then determine a list of potential inequalities of the form in (1.4) that we hypothesise hold for all K_4-free graphs. Some of the inequalities we generate in this way seem to be a consequence of how we restrict from P to P'', while others are more likely to correspond to bounding half-spaces also for P. In particular we prove, in Paper IV, that $(a,b,c,d)\mathbf{x} \geq 0$ corresponds to a bounding half-space for P if

$$\begin{aligned}(a,b,c,d) \in \{&(-3,1,1,5),\\ &(-5,1,0,12),\\ &(-5,1,2,10),\\ &(-6,1,2,13)\}.\end{aligned}$$

1.7 Invariants for $K_{\ell+1}$-free graphs

Finally, in Paper V, we want to generalise the sort of invariant we study in several of the previous papers of this thesis to other classes of graphs. Instead of fixing a number ℓ, such as $\ell = 2$ in Paper I and II or $\ell = 3$ in Paper IV, and looking at $K_{\ell+1}$-free graphs we, in Paper V, consider $K_{\ell+1}$-free graphs for general $\ell \geq 1$.

Perhaps the most fundamental theorem of extremal graph theory is what is known as Mantel's theorem (see [22]), which states that the complete bipartite graph $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ is the triangle-free graph on n vertices with the most number of edges. This was generalised, in [33], by Turán who proved the following theorem.

We define the *Turán graph*, $T(n,k)$, as the n-vertex graph obtained by partitioning the vertex set $V(T(n,k)) = \{1,2,\dots,n\}$ into k disjoint sets $U_1,U_2,\dots,U_k$ of as equal sizes as possible, and prescribing that $vw \in E(T(n,k))$ if and only if v and w belong to different sets of the partition. Note in particular $T(n,2) = K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$.

Theorem 1.7.1 (Turán's theorem, 1941). *If G is a K_{k+1}-free graph with n vertices then $e(G) \leq e(T(n,k))$, with equality if and only if $G \cong T(n,k)$.*

Example 1.7.1. If G is a triangle-free (i.e. K_3-free) graph on n vertices then we have that $e(G) \leq e(T(n,2)) = e(K_{\lceil n/2\rceil,\lfloor n/2\rfloor})$. For example we have that every triangle-free graph on $n = 7$, $n = 8$, $n = 9$ and $n = 10$ vertices has at most 12, 16, 20 and 25 edges, respectively.

The unique graphs that achieve these maximum number of edges are illustrated in Figure 1.13.

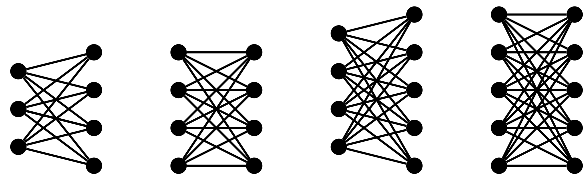

Figure 1.13: The Turán graphs $T(7,2)$, $T(8,2)$, $T(9,2)$ and $T(10,2)$, respectively.

We may reformulate Turán's theorem by considering the complement graph instead. For the complement graph Turán's theorem states that if G is a graph without an independent set of size $k+1$ on n vertices then $e(G) \geq e(\overline{T(n,k)})$. Note that the complement graphs of the Turán graphs, $\overline{T(n,k)}$, are disjoint unions of k complete graphs of as equal sizes as possible.

We, for example, have $\overline{T(7,2)} \cong K_3 + K_4$, $\overline{T(8,2)} \cong 2K_4$, $\overline{T(9,2)} \cong K_4 + K_5$ and $\overline{T(10,2)} \cong 2K_5$.

The independence number of $\overline{T(n,k)}$ is just $\alpha(\overline{T(n,k)}) = \lfloor n/k \rfloor$. Therefore, among all graphs without an independent set of size $k+1$ and with independence number at most $\lfloor n/k \rfloor$, the graphs $\overline{T(n,k)}$ has the least number of edges possible. The question we study in Paper V is what happens if we restrict the independence number further. We study the minimum number of edges among all $K_{\ell+1}$-free graphs without an independent set of size $k+1$. This is just the minimum edge numbers $e(\ell+1, k+1; n)$. From Turán's theorem we get that $e(\ell+1, k+1; n) \geq n(n-k)/(2k)$, but we are interested in whether we can improve this bound if we force ℓ to be strictly less than $\lfloor n/k \rfloor = \alpha(\overline{T(n,k)})$.

Example 1.7.2. Suppose $\ell = 4$ and $k = 2$, we then get that $e(5,3;7) = e(\overline{T(7,2)}) = 9$, $e(5,3;8) = e(\overline{T(8,2)}) = 12$. But note that $\omega(\overline{T(9,2)}) = \omega(\overline{T(10,2)}) = 5$, so these do not satisfy the condition that the clique number is less than $\ell + 1 = 5$.

Every triangle-free graph on 9 (or 10) vertices has at least 16 (or 20) edges, by Turán's theorem, and so does in particular every triangle-free graph with clique number at most 4. Hence we *do* get the weaker statements $e(5,3;9) \geq 16$ and $e(5,3;10) \geq 20$ immediately from Turán's theorem. But what are the actual values of $e(5,3;9)$ and $e(5,3;10)$? Can we get some stronger bound than the bound we get from Turán's theorem?

As a special case of the results presented in Paper V we have that $2e(5,k+1;n) \geq 13n - 42k$. This bound implies that $e(5,3;9) \geq 19$ and $e(5,3;10) \geq 25$. So, in this case we are able to improve on the bounds given by Turán's theorem.

Moreover, we have that the graph illustrated in Figure 1.14, which is denoted $C_{(5,4/2)}$, has independence number 2, clique number 4, 25 edges and therefore $e(5,3;10) = 25$.

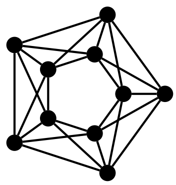

Figure 1.14: The graph $C_{(5,4/2)}$, with 25 edges.

In order to state the main hypothesis we work with in Paper V we define the following graphs.

Definition 1.7.1. Let $\ell \geq 2$. The *butterfly graph* $C_{(5,\ell/2)}$ is the graph formed by a vertex set

$$V(C_{(5,\ell/2)}) = A_1 \mathbin{\dot\cup} A_2 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} A_5,$$

such that $|A_1| = |A_3| = |A_4| = \lfloor \frac{\ell}{2} \rfloor$ and $|A_2| = |A_5| = \lceil \frac{\ell}{2} \rceil$. Moreover, there is an edge in $C_{(5,\ell/2)}$ between $v \in A_i$ and $w \in A_j$ if $v \neq w$ and the indices i and j, taken modulo 5, differ by at most 1.

A special case of the butterfly graphs first appear (with that name) in [5] as graphs which show that the bound $\alpha(G) \geq 2n(G)/(\Delta(G)+\omega(G)+1)$ is sharp. They are also considered, in full generality, in [34].

Example 1.7.3. If $\ell = 2$ then the graph $C_{(5,2/2)}$ has a vertex set that is a disjoint union of five singleton sets, $|A_1| = |A_2| = \cdots = |A_5| = 1$. The vertex in A_1 is adjacent to the vertices in A_5 and A_2, the vertex in A_3 is adjacent to the vertices in A_2 and A_4 and the vertex in A_4 is adjacent to the vertex in A_5. There are no other adjacencies than these. Hence, $C_{(5,2/2)}$ is just a cycle of length 5, i.e. $C_{(5,2/2)} = C_5$.

The graphs $C_{(5,\ell/2)}$ for $2 \leq \ell \leq 5$ have been illustrated in Figure 1.15.

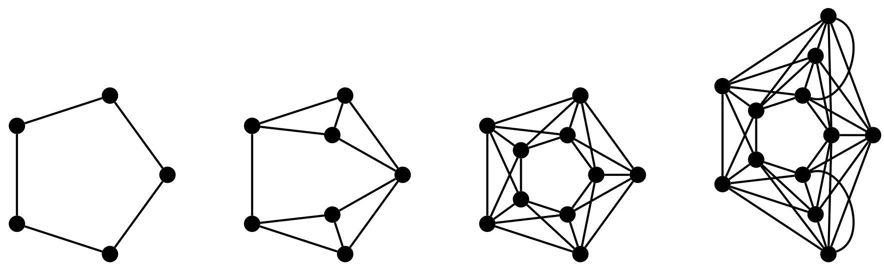

Figure 1.15: The graphs $C_{(5,2/2)}$, $C_{(5,3/2)}$, $C_{(5,4/2)}$ and $C_{(5,5/2)}$, respectively.

The main hypothesis we work with in Paper V is the following, where we define the linear invariants i_ℓ by

$$i_\ell(G) = 2e(G) - \left\lceil \frac{7\ell - 2}{2} \right\rceil n(G) + \ell \left\lfloor \frac{5\ell + 1}{2} \right\rfloor \alpha(G).$$

Hypothesis 1.7.1. *For all $K_{\ell+1}$-free graphs G we have $i_\ell(G) \geq 0$ and if $3 \neq \ell \geq 1$ then $i_\ell(G) = 0$ implies that every connected component of G is either K_ℓ or $C_{(5,\ell/2)}$.*

That $i_\ell(G) \geq 0$ for all $K_{\ell+1}$-free graphs is equivalent to a pair of conjectures of Weigand, which appear in [34]. Note that the coefficients of $e(G), n(G)$ and $\alpha(G)$ in the definition of $i_\ell(G)$ have been chosen in such a way to make $i_\ell(K_\ell) = i_\ell(C_{(5,\ell/2)}) = 0$. The hypothesis then states that K_ℓ and $C_{(5,\ell/2)}$ are the two connected graphs that minimise i_ℓ over all $K_{\ell+1}$-free graphs.

Example 1.7.4.

- If $\ell = 1$ the hypothesis just states that for all K_2-free graphs G (graphs without any edges) we have $2e(G) - 3n(G) + 3\alpha(G) \geq 0$. This is trivially true, since in all such graphs $\alpha(G) = n(G)$. Moreover, the only connected K_2-free graph is K_1 so the second part of the hypothesis is trivial as well.

- If $\ell = 2$ the hypothesis states that $i_2(G) = 2e(G) - 6n(G) + 10\alpha(G) \geq 0$ for all triangle-free graphs G, with equality if and only if every component is either a K_2 or a $C_{(5,2/2)} \cong C_5$. That this holds was shown by Radziszowski and Kreher in [28].

- The case when $\ell = 3$ is somewhat special since the second part of the hypothesis does not apply for this value of ℓ. In the hypothesis we get the inequality $i_3(G) = 2e(G) - 10n(G) + 24\alpha(G) \geq 0$. This inequality does in fact hold for all K_4-free graphs, which was shown by Fraugnaugh and Locke in [6], but there is one additional connected graph (to the two in the hypothesis) that satisfies this with equality. The fact that this exceptional graph, $\mathcal{W}_{8;1,2}$ which is illustrated in Figure 1.16, is the only exception to the statement in the hypothesis was shown by Backelin in [2, Theorem 17].

- Weigand (see [34]) has verified that $i_\ell(G) \geq 0$ for all $K_{\ell+1}$-free graphs G for $\ell \leq 12$ and for $\ell = 14$. Weigand does however not classify the graphs for which we have i_ℓ-value 0.

In Paper V we develop a theory related to to Hypothesis 1.7.1 and, in particular, we prove several interesting lemmas about the invariants i_ℓ.

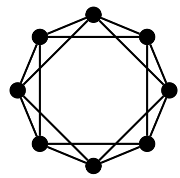

Figure 1.16: The graph $\mathcal{W}_{8;1,2}$.

For example, we show that a minimal counterexample G to Hypothesis 1.7.1, must be such that

$$\ell + 2 \leq \delta(G) \leq \frac{3\ell - 1}{2},$$

if ℓ is odd and

$$\ell + 3 \leq \delta(G) \leq \frac{3\ell}{2} - 2,$$

if ℓ is even. This has the particular consequence that we know that the hypothesis holds for all graphs with maximum valency at most $\ell + 2$ (or at most $\ell + 3$ if ℓ is even). More generally, the hypothesis holds for all *d-degenerate graphs* (i.e. graphs such that every subgraph contains a vertex of valency of at most d), where $d = 2\lfloor \ell/2 \rfloor + 3$.

The concept of *bounded valency Ramsey-numbers*, $R_m(\ell, k)$, were introduced by Staton in [31] (see also [32]). They are defined as follows. Let $n = R_m(\ell, k)$ be the smallest positive integer such that every graph on n vertices with maximum valency at most m either contains a clique of size ℓ or an independent set of size k. Note that $R_m(\ell, k) \leq R(\ell, k)$ for all positive integers m and that if m is large enough (depending on ℓ and k) then $R_m(\ell, k) = R(\ell, k)$.

Suppose that G is a $K_{\ell+1}$-free graph, ℓ is odd and that $\Delta(G) \leq \ell + 2$, then we know that the Hypothesis holds for this ℓ. This implies that, since $e(G) \leq \Delta(G)n(G)/2$,

$$(\ell + 2)n(G) - \frac{7\ell - 1}{2} n(G) + \frac{\ell(5\ell + 1)}{2} \alpha(G) \geq 0,$$

which simplifies to the bound

$$n(G) \leq \frac{\ell(5\ell + 1)\alpha(G)}{5(\ell - 1)}.$$

In terms of the bounded valency Ramsey numbers this can be expressed as

$$R_{\ell+2}(\ell + 1, k + 1) \leq \left\lfloor \frac{\ell(5\ell + 1)k}{5(\ell - 1)} + 1 \right\rfloor.$$

Similarly, for even ℓ we get

$$R_{\ell+3}(\ell+1,k+1) \leq \left\lfloor \frac{5\ell^2 k}{5\ell-8} + 1 \right\rfloor.$$

Finally, in Paper V we prove Hypothesis 1.7.1 in some special cases. We indicate how one may show the hypothesis for all even $\ell \leq 14$ and prove it explicitly for $\ell = 16$.

1.8 Comparison between manuscripts in the thesis and other versions

Sections 1.1 through 1.4 of the current chapter are copied, essentially verbatim, from the introduction of the author's licentiate thesis [18]. We will in this section detail how the manuscripts that appear in this thesis correspond to previously available versions.

Other than the differences that are specified in this section some purely cosmetic changes have been made – such as change of fonts, formatting and the use of synonymous terminology (e.g. 'degree' versus 'valency') to make it consistent throughout this thesis. Furthermore, in both Paper I and Paper II the references to the unpublished work [3] have been changed to be more specific with respect to where in [3] they may be found. Also, whenever it is possible we have changed these references to instead refer to an earlier version of the same manuscript, [2], which is available online.

Paper I is an extended manuscript on which the article [19] is based. The main difference is that details of proofs from [18] that do not appear in [19] have been included in Paper I of this thesis for increased readability and self-containment. In [19] the reader is instead referred to the author's licentiate thesis [18] for full proofs. This is, however, entirely undesirable for the present thesis. Here follows a detailed explanation of where details that appear in Paper I have been omitted in [19] from Paper I.

The last paragraph in Section 2.1.1 in Paper I does not appear in [19], but is useful for motivating the introduction of $\mathrm{N}(C_4;\cdot)$ to the uninitiated reader. An example of consequences of Theorem 2.1.1 and clarification of the notation in Section 2.1.4 of Paper I does not appear in [19].

In Section 2.1.4 of Paper I we include (independent) proofs of the following lemmata, since these statements first appear the unpublished works [2–4] by Backelin: Lemma 2.1.3 (appears as [19, Lemma 1.1]), Lemma 2.1.4 (appears as [19, Lemma 1.2]), Lemma 2.1.6 (appears as [19, Lemma 1.4]) and Lemma 2.1.7 (appears as [19, Lemma 1.5]). Furthermore, Corollary 2.1.1 and the definition of an $\mathcal{H}$-avoiding graph from Paper I is omitted in [19]. The proof of [19, Claim 1] is the same as the proof Claim 1 of Paper I, except that some details have been removed making the proof in [19] significantly more difficult to follow. Similarly, some of the details of the proof of parts (vii)-(x) of Lemma 2.2.2 are omitted in the proof of [19, Lemma 2.2] purely for reasons of brevity. For clarity, however, the author has chosen to include these details in Paper I. The same holds for the proofs of [19, Claims 4-10] for which, in [19], the reader is referred to the manuscript which appears in [18], whereas in Paper I of this thesis (where the corresponding statements are Claims 4-17) we include the complete proofs.

Paper II differs from [18, Paper II] mainly in that some errata have been corrected. Throughout Paper II some purely linguistic corrections have been made, and some inconsistencies in use of variable names have been fixed. Errata in the definitions of ‘stitches’ in Section 3.2.2 have also been corrected. Furthermore, Section 3.3 has been expanded upon to include slightly more details on the computer implementation. Lastly, a reference to source code that now is available online has been added to Paper II since the publication of [18, Paper II].

Apart from the purely cosmetic changes (such as replacing the word ‘degree’ with the synonymous word ‘valency’) Paper III only differs from the version that has been accepted for publication in that we have included the source code for the programs used as an appendix to Paper III.

References

[1] V. ANGELTVEIT AND B. D. MCKAY, $R(5,5) \leq 48$, Journal of Graph Theory **89** (2018), 5–13. 23

[2] J. BACKELIN, *Contributions to a Ramsey calculus*, unpublished manuscript version 2007-04-04, `https://www2.math.su.se/~joeb/public/Ramsey/Ramsey.pdf`. Accessed: 2019-10-12. 26, 40, 42

[3] ———, *Contributions to a Ramsey calculus*, unpublished manuscript version 2016-08-18. 25, 29, 31, 42

[4] ———, *Edge number critical triangle free graphs woth low independence number*, arXiv:1309.7874v1 [math.CO] 30 Sep 2013 (2013). 24, 25, 26, 28, 31, 42

[5] S. FAJTLOWICZ, *On the size of independent sets in graphs*, Proc. 9th S-E Conf. on Combinatorics, Graph Theory and Computing, Boca Raton (1978), 269–274. 39

[6] K .L. FRAUGHNAUGH AND S. C. LOCKE, *Lower bounds on size and independence in K_4free graphs*, Journal of Graph Theory **26** (1997), 26:61–71. 40

[7] J. GOEDGEBEUR AND S. P. RADZISZOWSKI, *New computational upper bounds for Ramsey numbers $R(3,k)$*, Electronic Journal of Combinatorics **20** (2013), no. 1. 31

[8] J. E. GRAVER AND J. YACKEL, *Some graph theoretic results associated with Ramsey's theorem*, Journal of Combinatorial Theory **4** (1968), 125–175. 19, 23

[9] R. E. GREENWOOD AND A. M. GLEASON, *Combinatorial relations and chromatic graphs*, Canadian Journal of Mathematics **7** (1955), 1–7.

[10] C. GRINSTEAD AND S. ROBERTS, *On the Ramsey numbers $R(3,8)$ and $R(3,9)$*, Journal of Combinatorial Theory **33** (1982), 27–51. 19, 23

[11] G. HOPKINS AND W. STATON, *Girth and independence ratio*, Canadian Mathematical Bulletin **25** (1982), no. 2, 179–186. 29

[12] Y. HUANG, Y. WANG, W. SHENG, J. YANG, K. ZHANG, AND J. HUANG, *New upper bound formulas with parameters for Ramsey numbers*, Discrete Mathematics **307** (2007), 760–763. 32, 33

[13] C. F. JONES, *Independence in graphs*, Journal of Combinatorial Theory **37** (1984), 254–269. 25, 26

[14] J. G. KALBFLEISCH, *Construction of special edge-chromatic graphs*, Canadian Mathematical Bulletin **8** (1965), 575–584. 19

[15] ______, *Chromatic graphs and Ramsey's theorem*, Ph.D. thesis, University of Waterloo, 1966. 23

[16] G. KÉRY, *On a theorem of Ramsey* (in Hungarian), Matematikai Lapok **15** (1964), 204–224. 19

[17] J. H. KIM, *The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$*, Random Structures & Algorithms **7** (1995), no. 3, 173–207. 20

[18] O. KRÜGER, *On minimal triangle-free Ramsey graphs*, Licentiate Thesis in Mathematics, Stockholm University, Jan. 2018. 18, 42, 43

[19] ______, *An invariant for minimum triangle-free graphs*, The Australasian Journal of Combinatorics **74** (2019), 371–388. 42, 43

[20] ______, *GitHub repository for `ramseyrecursion`*, `https://github.com/tbk1988/ramseyrecursion`, Accessed: 2019-05-21. 35

[21] N. LICHIARDOPOL, *New lower bounds on independence number in triangle-free graphs in terms of order, maximum degree and girth*, Discrete Mathematics **332** (2014), 55–59. 29

[22] W. MANTEL, *Problem 28 (Solution by H. Gouwentak, W. Mantel, J. Teixeira de Mattes, F. Schuh and W. A. Wythoff)*, Wiskundige Opgaven, **10** (1907), 60–61. 37

[23] B. D. MCKAY, *Ramsey graphs*, `http://users.cecs.anu.edu.au/~bdm/data/ramsey.html`, Accessed: 2015-05-06. 31

[24] B. D. MCKAY AND Z. KE MIN, *The value of the Ramsey number $R(3,8)$*, Journal of Graph Theory **16** (1992), 99–105. 19

[25] B. D. MCKAY AND S. P. RADZISZOWSKI, $R(4,5) = 25$, Journal of Graph Theory **19** (1995), 309–322. 19, 23

[26] HOUSE OF GRAPHS, *Minimal Ramsey graphs*, `https://hog.grinvin.org/MinRamsey`, Accessed: 2015-05-06. 31

[27] S. P. RADZISZOWSKI, *Small Ramsey numbers; dynamic survey, revision #15*, Electronic Journal of Combinatorics (2017). 19, 20, 33, 35

[28] S. P. RADZISZOWSKI AND D. L. KREHER, *On (3,k) Ramsey graphs: Theoretical and computational results*, Journal of Combinatorial Mathematics and Combinatorial Computing **4** (1988), 37–52. 21, 24, 31, 40

[29] ———, *Minimum triangle-free graphs*, Ars Combinatoria **31** (1991), 65–92. 24, 25, 26, 28, 31

[30] F. RAMSEY, *On a problem of fromal logic*, Proceedings of the London Mathematical Society **30** (1930), 264–286. 18

[31] W. STATON, *Some Ramsey-type numbers and the independence ratio*, Ph.D. thesis, University of Huston, 1977. 41

[32] ———, *Some Ramsey-type numbers and the independence ratio*, Transactions of the American Mathematical Society **256** (1979), 353–370. 41

[33] P. TURÁN, *On an extremal problem in graph theory*, Matematikai és Fizikai Lapok (in Hungarian) **48** (1941), 436–452. 37

[34] J. WEIGAND, *Size bound and Ramsey-type numbers*, Ph.D. thesis, University of Colorado at Denver, 2006. 39, 40