Stix Two Math

# Towards a Complete Formalization of PLN

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**Abstract**

We present $\nu$PLN, a reformulation of Probabilistic Logic Networks (PLN) grounded in Solomonoff Universal Induction. By treating PLN truth values as posterior distributions over a global probability space, we derive the standard PLN rules from first principles and establish a convergence theorem analogous to that of Solomonoff induction. The key insight is that de Finetti's representation theorem justifies evidence counts as sufficient statistics, providing a principled foundation for PLN's truth value calculus.

## 1 Introduction

The goal is similar to Solomonoff Universal Induction [7], that is we want to approach a first order (unknown but computable) distribution $\mu$ given observations, using a second order (uncomputable but known) distribution $\nu$, called the Universal Distribution. In Solomonoff Induction, observations are bit strings produced by a Turing machine[1]. In PLN however, observations are outcomes from an indexed boolean random variable, representing the outputs of evaluating a predicate on some inputs. Such predicate is called the *observable predicate*. In practice PLN allows multiple observable predicates however one can assume one predicate without loss of generality. Indeed, to emulate multiple predicates, one can introduce an extra component in the predicate's domain to "select" the predicate of interest. Also, since it is observed by a random variable, such predicate is not necessarily deterministic (though it could be). As such, one may think of the observable predicate as being a program drawn from a certain Probabilistic Programming Language. In the following section we formally define the above.

Because this reformulation of PLN departs somewhat from the definition of PLN in the PLN book [4], we give it a new name, $\nu$PLN.

[*] This document extends Nil Geisweiller's original draft, completing the derivations and adding the convergence theorem.

[1] Note that even though the sample space of $\nu$ is made of deterministic Turing machines, $\nu$ can approximate any computable distribution $\mu$ (thus non-deterministic) by maintaining an ensemble of such Turing machines.

# 2 Definitions

In its original form, PLN purposely avoids relying on an underlying global probability distribution. I am not against this in principle. I will simply admit that I cannot conceive a complete definition of PLN that does not rely on such global probability distribution. I would also point out that a publication released after the PLN book by the principal authors of the PLN book, Ben Goertzel and Matt Iklé, very much aligns with the idea of a global probability distribution [3], and was in fact a great source of inspiration for writing this very document. The next subsection is dedicated to define the global probability distribution which $\nu$PLN is intended to derive from.

## 2.1 Global Probability Distribution

Let $(\Omega, \mathcal{F}, \nu)$ be a probability space such that

- $\mathcal{F}$ is the event space, a $\sigma$-algebra on $\Omega$.
- $\nu : \mathcal{F} \rightarrow [0, 1]$ is a universal distribution, further defined below.
- $\Omega$, the set of possible worlds, is the sample space associated to $\nu$, such that each element $\hat{\omega} \in \Omega$ contains
  1. a probabilistic predicate $\hat{\mu} \in \mathcal{L}$ over a domain $\mathcal{D}$, described in a probabilistic programming language $\mathcal{L}$,
  2. a mapping of $\hat{\mu}$ from $\mathcal{D}$ to Boolean. For instance if $\mathcal{D}$ is $\mathbb{N}$, then a possible mapping could be $(\hat{\mu}\ 0) = \texttt{True}$, $(\hat{\mu}\ 1) = \texttt{False}$, $(\hat{\mu}\ 2) = \texttt{True}, \ldots$, corresponding so far to a predicate indicating the evenness of a natural number. Note that the world $\hat{\omega}$ contains the entire, potentially infinite, mapping, even though in reality an observer can only have access to a finite subset of it.

An observer lives in a particular world $\omega \in \Omega$, called the *true world*, which includes the *true generator* or *true predicate*, $\mu$, and the *true history*, which is the complete mapping of evaluations of $\mu$ over the domain $\mathcal{D}$. Note that since $\mu$ is probabilistic, it does not deterministically determine the history, thus the true history is just one possible history among an infinity of histories compatible with $\mu$. Note that throughout the document $\omega$ and $\mu$ refer to the true world and true generator respectively, which is to be contrasted with $\hat{\omega}$ and $\hat{\mu}$ which refer to arbitrary elements of $\Omega$ and $\mathcal{L}$ respectively. Although when it is clear that the definitions are generic and apply to any arbitrary world and the true world alike, we will use $\omega$ and $\mu$ as well. Since $\mu$ is a probabilistic predicate, its type signature cannot merely be

$$\mu : \mathcal{D} \rightarrow \text{Bool}$$

where $\text{Bool} = \{\texttt{False}, \texttt{True}\}$. To capture its probabilistic nature we give it the following type signature

$$\mu : \mathcal{D} \rightarrow \Omega \rightarrow \text{Bool}$$

in a curried fashion. Meaning that given its argument, it produces a boolean random variable. Therefore the observable predicate can be viewed as an indexed boolean random variable. Of course, $\mu$ never gets to be evaluated on a different world than the one it belongs to, thus we can write $(\mu\ a)$ while meaning $(\mu\ a\ \omega)$, but we still need to keep the $\Omega$ argument in order to reason about possible worlds since the true world is unknown. Also, in cases where the domain $\mathcal{D}$ can be decomposed into multiple components, a curried notation will be used interchangeably. So instead of

$$\mu : (\mathcal{D}_1 \times \cdots \times \mathcal{D}_n) \rightarrow \Omega \rightarrow \text{Bool}$$

the following

$$\mu : \mathcal{D}_1 \rightarrow \cdots \rightarrow \mathcal{D}_n \rightarrow \Omega \rightarrow \text{Bool}$$

will be used. This will be convenient to express inheritance relationships between partially applied predicates. As already hinted, the application of $\mu$ to an input $x$ is denoted with the traditional functional programming style

$$(\mu\ x)$$

Thus if the domain is decomposed into subdomains $\mathcal{D}_1$ to $\mathcal{D}_n$, applying $\mu$ to all its inputs $x_1$ to $x_n$ will be denoted

$$(\mu\ x_1\ \ldots\ x_n)$$

Likewise for the $\omega$ argument

$$(\mu\ x_1\ \ldots\ x_n\ \omega)$$

Such functional programming notation is used for $\mu$ because it is currying-friendly. For the rest, we keep using the traditional mathematical function application style, such as

$$\nu(E)$$

denoting the application of the probability distribution $\nu$ to the event $E$. In case $\mu$ is known to be deterministic, $\Omega$ could potentially be dropped, but that is not going to be our working assumption for now. Additionally to the functional programming style, we may use parametric notation, so for instance instead of

$$(\mu\ x_1\ x_2)$$

we may write

$$(\mu_{x_1}\ x_2)$$

where $x_1$ is viewed as a parameter and $x_2$ is viewed as the argument of $\mu_{x_1}$. With that, let us now define key random variables to access $\Omega$:

- $M : \Omega \rightarrow \mathcal{L}$ with measurable space $(\mathcal{L}, \mathcal{F}_\mathcal{L})$, where $\mathcal{L}$ is a certain probabilistic programming language and $\mathcal{F}_\mathcal{L}$ is a $\sigma$-algebra on $\mathcal{L}$. Thus, $M$ takes a world $\hat{\omega} \in \Omega$ and outputs the probabilistic program $\hat{\mu} \in \mathcal{L}$ generating that world. Note that this random variable is inaccessible from an observer within that world. An observer within that world only has access to a finite record of evaluations of $\hat{\mu}$. However, this random variable is important to reason about multiple worlds, we suspect it is particularly important for higher order reasoning.

- $D_{x\in\mathcal{D}} : \Omega \to$ Bool, a Boolean random variable indexed by values in $\mathcal{D}$. Unlike $M$, $D_{x\in\mathcal{D}}$ is at least partially accessible from an observer within that world. Meaning, such observer can gather data for a finite subset $\mathcal{S}$ of $\mathcal{D}$. In this case $D_\mathcal{S}$ represents a finite family of Boolean random variables, corresponding the set of accessible observations. $M$ and $D_{x\in\mathcal{D}}$ are related by the following equality

  $$(\hat{\mu}\ x\ \hat{\omega}) = (D_x\ \hat{\omega})$$

  where $\hat{\omega} \in \Omega$ such that $(M\ \hat{\omega}) = \hat{\mu}$. Or simply, in curried fashion

  $$(\hat{\mu}\ x) = D_x$$

Due to the equality above, the distribution of observations is entirely determined by a model $\hat{\mu}$. In other words, it suffices to define a distribution over $\mathcal{L}$, the prior distribution over possible models, to define $\nu$ (as far as $M$ and $D_x$ are concerned anyway). Then, relating observations to models can be done using regular Bayesian inference. The prior is defined by

$$\nu(M \in L)$$

where $L \in \mathcal{F}_\mathcal{L}$. Note how it is expressed in terms of elements of $\mathcal{F}_\mathcal{L}$, instead of elements of $\mathcal{L}$. It is because, for the purpose of recovering PLN with the Bayesian approach, $\mathcal{L}$ needs to be continuous.[2] I will explain this in detail, but for now let us use this notation to formulate Bayes' theorem

$$\nu(M \in L|D_\mathcal{S}) = \frac{\nu(M \in L) \times \nu(D_\mathcal{S}|M \in L)}{\nu(D_\mathcal{S})}$$

where $\mathcal{S}$ is a finite subset of $\mathcal{D}$. For the sake of simplicity, we may drop $M$ since it is the only random variable that ranges over $\mathcal{L}$, and simply write

$$\nu(L|D_\mathcal{S}) = \frac{\nu(L) \times \nu(D_\mathcal{S}|L)}{\nu(D_\mathcal{S})}$$

An example of prior will be given in Section 4.1, but in general it can be viewed as a parameter of $\nu$PLN. That is, given a certain prior of $\nu$ over $\mathcal{L}$, one can derive a certain flavor of $\nu$PLN. Since $(\mu\ x) = D_x$, a history data point will be represented as $(\mu\ x) =$ `True` or $(\mu\ x) =$ `False`, and the usage of the random variable $D_x$, and especially $D_\mathcal{S}$, will be reserved in the formulation of $\nu$ to refer to the observed history.

[2] This requirement is sufficient but not necessary. The Beta posterior can also arise from countable mixtures via Beta-Bernoulli conjugacy and de Finetti's representation theorem for exchangeable sequences, without requiring a continuous $\mathcal{L}$.

## 2.2 PLN Global Distribution vs Solomonoff Induction

Note that unlike with Solomonoff Universal Induction, $\hat{\mu}$ represents a probabilistic predicate rather than a computable probability function calculating the probability of any event, although the latter can be derived from the former.

The *true distribution* in Solomonoff Induction corresponds to the *real predicate* here. We prefer to use the word *real* when denoting objective reality, rather than *true* because *true predicate* could be understood as a predicate that always outputs true.

# 3 PLN and the Global Distribution

In Section 4 we proceed to derive PLN rules from the global probability distribution introduced in Section 2.1, but for now let us recall important notions of PLN and how to relate them to the global distribution defined in Section 2.1.

## 3.1 Concepts vs Predicates

The PLN book describes respectively the notions of *concepts* and *predicates*, and their respective associated relationships *inheritance* and *implication.* As explained in Section 2.6 *Higher-Order Logical Relationships* of the PLN book, there is a perfect isomorphism between concepts and inheritance on one side and predicates and implication on the other side. Concerned with conciseness, we will pick a side, the predicate side, and essentially forget about the other side, the concept side, for the rest of this document. But before we do so let us recall what is a concept, the inheritance between two concepts, and the isomorphism between concepts and predicates.

### 3.1.1 Concepts and Inheritance

A concept is a fuzzy (or, as I prefer to say, probabilistic, for reasons I will explain in Section 3.3) set, and the *extensional* inheritance between two concepts is a probabilitized subset relationship. Originally, in the PLN book, the inheritance relationship is defined as an explicit mixture of extensional and intensional inheritances. We will show however that they are in fact both the same thing, the extensional inheritance is a way to approach inheritance solely via *induction*, and intensional inheritance is a way to approach inheritance solely via *abduction.*[3] Any one side, extensional or intensional, is good enough to define the other, so let us pick one, the extensional side, and define inheritance with it. The extensional inheritance between two concepts can be viewed as a probabilitized subset relationship. It allows to express things like "most members of a

[3] This characterization via induction/abduction is non-standard. A more precise formulation, developed in [3], shows that extensional inheritance emerges as a special case of intensional inheritance when concept properties are singletons. The two are unified through mutual information: $P(W|F) = P(W) \cdot 2^{I(F;W)}$.

set are also members of another set". For instance one could express in PLN that 90% of birds fly, with

$$(\texttt{Inheritance Bird Fly}) \doteq 0.9$$

Such knowledge might have been obtained by observing a finite sample of birds and whether or not they fly. Meaning there could be an uncertainty on the 90% itself. To represent such uncertainty PLN uses a second order distribution, in this case a Beta distribution as it is an ideal choice to represent the posterior of the parameter of a Bernoulli distribution given observations. Under this assumption only two numbers are required to determine the parameters, $\alpha$ and $\beta$, of the associated Beta distribution. A *truth value* called *simple truth value* was created for this purpose and is thus described by two numbers: a strength (a proxy for a probability estimate) and a confidence over this strength from which the $\alpha$ and $\beta$ parameters of the Beta distribution can be recovered. For instance, given a simple truth value one may express that 90% of birds fly with a confidence of 0.99

$$(\texttt{Inheritance Bird Fly}) \ \doteq <0.9, 0.99>$$

where 0.9 is the strength and 0.99 is the confidence. The confidence is a value between 0 and 1 that actually encodes the sample size that was used to obtained the strength. The higher the sample size, the higher the confidence. More information about that will be provided in Section 3.3 but for now let us just leave it at that as it is enough for what we are concerned with in this Section which is the isomorphism between concepts and predicates.

### 3.1.2 Isomorphism between Concepts and Predicates

To every concept one can associate a predicate and vice versa. To go from concept to predicate one can use the *indicator function*, and to go from predicate to concept one can use the *satisfying set*. These notions are well known for crisp predicates and sets and thus will not be detailed any further here. The only difference in PLN is that concepts are probabilistic, meaning that a probability (or potentially a second order probability) can be attached to the membership of an element to a concept. Likewise, predicates are probabilistic in the sense that a (second order) probability can be attached to the evaluation of an argument to a predicate. As one would expect the isomorphism also applies between the inheritance relationship on the concept side and the implication relationship on the predicate side. So for instance, on the predicate side one can express the same inheritance relationship between birds and fly as follows

$$(\texttt{Implication IsBird DoesFly}) \ \doteq <0.9, 0.99>$$

where `IsBird` and `DoesFly` have been obtained by taking the indicator functions of `Bird` and `Fly`. As mentioned earlier, we will drop the concept side and only focus on the predicate side for the remaining of the document.

### 3.2 Relating Predicates to $\mu$

As explained in Section 2.1 there is only one global predicate, $\mu$. To manipulate multiple predicates we can simply introduce a additional parameter in the domain of $\mu$ indicating predicates. Let us call this subdomain $\mathcal{P}$ which ranges over symbols representing predicates, in this case $\mu$ may have the type signature

$$\mu : \mathcal{P} \to \mathcal{D} \to \Omega \to \text{Bool}$$

So for instance to express that a cat is a mammal, traditionally represented as

$$\mathit{Cat} \Rightarrow \mathit{Mammal}$$

one may use the following parametric notation of $\mu$

$$\mu_{\mathit{Cat}} \Rightarrow \mu_{\mathit{Mammal}}$$

Likewise, to represent the evaluation of the predicate *Cat* over a certain cat instance, $\mathit{cat}_{123}$, one may use

$$(\mu_{\mathit{Cat}}\ \mathit{cat}_{123})$$

corresponding to the traditional representation

$$(\mathit{Cat}\ \mathit{cat}_{123})$$

where $\mathit{cat}_{123} \in \mathcal{D}$.

### 3.3 Truth Value as Posterior Distribution

A PLN truth value is not merely a point estimate of probability; it encodes an entire posterior distribution over possible probability values. This is the key to understanding why PLN uses two numbers (strength and confidence) rather than one.

The justification for this encoding comes from de Finetti's representation theorem [2]. For an infinite exchangeable sequence of binary observations, de Finetti showed that such a sequence can be represented as a mixture of i.i.d. Bernoulli sequences. A crucial consequence is that the *counts* of positive and negative observations, denoted $(n^+, n^-)$, are *sufficient statistics*. That is, given the counts, the order in which observations arrived is irrelevant for inference.

This result has profound implications for PLN. It means that evidence counts $(n^+, n^-)$ contain all the information needed for Bayesian updating. When we assume a Beta prior $\text{Beta}(\alpha, \beta)$ over the unknown probability parameter $p$, the posterior after observing $n^+$ positive and $n^-$ negative outcomes is:

$$\text{Beta}(\alpha + n^+, \beta + n^-)$$

The PLN simple truth value $<s, c>$ encodes this posterior as follows:

- The *strength* $s$ is the posterior mean:

  $$s = \frac{\alpha + n^+}{\alpha + \beta + n^+ + n^-}$$

  Under the common choice of a uniform prior ($\alpha = \beta = 1$) or Jeffreys prior ($\alpha = \beta = \frac{1}{2}$), this simplifies approximately to $s \approx \frac{n^+}{n^+ + n^-}$ for large sample sizes.

- The *confidence* $c$ encodes the sample size relative to a constant $\kappa$ (often called the "lookahead" or "weight of prior"):

  $$c = \frac{n^+ + n^-}{n^+ + n^- + \kappa}$$

  As the sample size grows, confidence approaches 1. When no observations have been made, confidence is 0.

The choice of $\kappa$ reflects how quickly we gain confidence from observations. A smaller $\kappa$ means faster convergence to high confidence; a larger $\kappa$ means more conservatism.

I would emphasize that this is not merely a convenient encoding—it is the *correct* encoding given the assumptions of exchangeability and Beta-Bernoulli conjugacy. The fact that PLN arrived at this encoding through practical considerations, while Bayesian probability theory derives it from first principles, is a reassuring convergence.

### 3.4 Statement versus Judgement

Like in NAL, a PLN *statement* designates a logical statement without truth value, such as

$$P \Rightarrow Q$$

While a PLN *judgment* designates a PLN statement with a truth value attached to it, such as

$$P \Rightarrow Q \ \doteq <\!0.9, 0.8\!>$$

## 4 Deriving PLN Rules

Our goal here is to derive every PLN rule in the PLN book from the global distribution that has been defined in Section 2.1. By doing so we hope not only to provide a clear unambiguous definition for each rule, but also an ideal to approach as using a global distribution should give us the means to derive a convergence theorem à la Solomonoff.

## 4.1 Predicate Direct Introduction

This rule is meant to calculate the truth value corresponding to a Predicate from direct observations. Its truth value can be understood as the marginal probability of a predicate to be true, irrespective of the inputs. It is a subcase of the Implication Direct Introduction presented in Section 4.2 where the implicant is the *Universal Predicate* (a predicate that is always true), but is described here as its own rule to simplify the presentation. Indeed, it should be easier to understand the Implication Direct Introduction rule after understanding the Predicate Direct Introduction rule.

To derive the predicate direct introduction rule we consider the posterior probability of a Bernoulli process with parameter $p$ given all available observations of the predicate in question. Let us begin with $\mu$ itself. In order to formulate this posterior we need to assume that $\mathcal{L}$ has a Bernoulli sampler in its set of operators. Let $\mathcal{B}$ be the subset of programs of $\mathcal{L}$ consisting of a single Bernoulli call, thus comprised of the following programs

```
(Bernoulli p)
```

where `p` is the parameter of the Bernoulli distribution ranging over $[0, 1]$. Such probabilistic program, when executed, outputs `True` with a probability of `p` or `False` with a probability of $1 - p$. Let $\mathcal{F}_\mathcal{B}$ be a Borel $\sigma$-algebra on $\mathcal{B}$. Let us assume that $\mathcal{F}_\mathcal{B}$ is a subfield of $\mathcal{F}_\mathcal{L}$. We claim that if $\mu$ is restricted to the programs in $\mathcal{B}$ then the posterior of `p` corresponds to the truth value of $\mu$, expressed as follows

$$\nu(L_\mathcal{B}|D_\mathcal{S}) = \frac{\nu(L_\mathcal{B}) \times \nu(D_\mathcal{S}|L_\mathcal{B})}{\nu(D_\mathcal{S})}$$

where $L_\mathcal{B} \in \mathcal{F}_\mathcal{B}$.

Now let us make this concrete. Suppose we have observed a finite subset $\mathcal{S} \subset \mathcal{D}$ of evaluations, yielding $n^+$ positive outcomes (where $(\mu\ x) = \texttt{True}$) and $n^-$ negative outcomes (where $(\mu\ x) = \texttt{False}$). Under the assumption that $\mu$ is drawn from $\mathcal{B}$, the likelihood of these observations given parameter $p$ is:

$$\nu(D_\mathcal{S}|p) = p^{n^+}(1-p)^{n^-}$$

If we place a Beta prior $\text{Beta}(\alpha, \beta)$ on $p$, the posterior is $\text{Beta}(\alpha+n^+, \beta+n^-)$ with density:

$$f(p|D_\mathcal{S}) \propto p^{\alpha+n^+-1}(1-p)^{\beta+n^--1}$$

The truth value is then:

- Strength: $s = \frac{\alpha+n^+}{\alpha+\beta+n^++n^-}$ (the posterior mean)
- Confidence: $c = \frac{n^++n^-}{n^++n^-+\kappa}$ (encoding sample size)

Under the Jeffreys prior $\alpha = \beta = \frac{1}{2}$, the strength becomes:

$$s = \frac{\frac{1}{2} + n^+}{1 + n^+ + n^-}$$

which is precisely the Krichevsky-Trofimov estimator used in universal prediction [6]. This is not a coincidence—both PLN and universal prediction are solving the same problem of optimal inference from exchangeable binary data.

## 4.2 Implication Direct Introduction

This rule is not explicitly stated as such in the PLN book but can be derived from iteratively applying induction and revision, and reflects the formula of extensional inheritance/implication given in Section 2.4.1 *The Semantics of Inheritance* of the PLN book, at least the strength part. To obtain the confidence part we assume that the second order distribution is a Beta distribution, like in Section 4.2 *From Imprecise Probabilities to Indefinite Probabilities* of the PLN book. In order to derive a Beta distribution as second order distribution we can assume an underlying Bernoulli process with parameter $p$ with a Beta distribution as prior. The Jeffreys prior where $\alpha = \beta = \frac{1}{2}$ is often the default choice. Given the PLN statement

$$P \Rightarrow Q$$

let us express its truth value as the posterior probability of the parameter $p$ of the underlying Bernoulli process. In order to map a Bernoulli process onto an implication we zoom-in to the data points where $P$ is true.

Concretely, let $\mathcal{S}_P \subseteq \mathcal{S}$ be the subset of observations where predicate $P$ evaluates to true. Within this subset, let:

- $n^+_{PQ}$ = number of cases where both $P$ and $Q$ are true
- $n^-_{PQ}$ = number of cases where $P$ is true but $Q$ is false

The implication $P \Rightarrow Q$ is then treated as a Bernoulli process over the restricted domain $\mathcal{S}_P$, with the same Bayesian update:

- Strength: $s = \frac{\alpha+n^+_{PQ}}{\alpha+\beta+n^+_{PQ}+n^-_{PQ}}$
- Confidence: $c = \frac{n^+_{PQ}+n^-_{PQ}}{n^+_{PQ}+n^-_{PQ}+\kappa}$

Note that the confidence depends only on the number of observations where $P$ is true, not on the total number of observations. This correctly reflects that we can only learn about $P \Rightarrow Q$ from cases where $P$ holds.

## 4.3 Priors

Typical priors for the Beta distribution include:

- **Haldane prior**: $\alpha = \beta = 0$ (improper, maximally uninformative)
- **Jeffreys prior**: $\alpha = \beta = \frac{1}{2}$ (invariant under reparametrization)
- **Bayes/Uniform prior**: $\alpha = \beta = 1$ (flat prior on $[0, 1]$)

The PLN strength formula $s = n^+/(n^+ + n^-)$ corresponds to the Haldane prior. The Jeffreys prior $\alpha = \beta = \frac{1}{2}$ yields the Krichevsky-Trofimov estimator, which has optimal properties for universal prediction. In practice, the choice of prior matters most when sample sizes are small; for large $n^+ + n^-$, all proper priors converge to the same posterior.

## 4.4 Deduction Rule

The PLN deduction rule computes the truth value of $P \Rightarrow R$ given the truth values of $P \Rightarrow Q$ and $Q \Rightarrow R$. Unlike the direct introduction rules which derive truth values from observations, the deduction rule derives truth values from other truth values.

I would emphasize that the deduction formula is not a heuristic—it is derivable from the law of total probability. Let $s_{PQ}$, $s_{QR}$, and $s_{PR}$ denote the strengths of $P \Rightarrow Q$, $Q \Rightarrow R$, and $P \Rightarrow R$ respectively. Let $p_Q$ and $p_R$ denote the marginal probabilities of $Q$ and $R$.

By the law of total probability:

$$P(R|P) = P(R|P,Q) \cdot P(Q|P) + P(R|P,\neg Q) \cdot P(\neg Q|P)$$

Under the conditional independence assumption $P(R|P,Q) \approx P(R|Q)$ and $P(R|P,\neg Q) \approx P(R|\neg Q)$, and using $P(R|\neg Q) = \frac{P(R)-P(Q)\cdot P(R|Q)}{1-P(Q)}$, we obtain:

$$s_{PR} = s_{PQ} \cdot s_{QR} + (1 - s_{PQ}) \cdot \frac{p_R - p_Q \cdot s_{QR}}{1 - p_Q}$$

This is precisely the PLN deduction strength formula. The confidence of the conclusion is typically taken as the minimum of the input confidences, reflecting that a chain is only as strong as its weakest link.

## 4.5 Induction Rule (Source Rule)

The PLN *induction* rule computes the truth value of $A \Rightarrow C$ given $B \Rightarrow A$ and $B \Rightarrow C$. The key insight is that this is simply Bayes' rule followed by deduction.

Given $B \Rightarrow A$ with strength $s_{BA}$, we use Bayes' rule to obtain $A \Rightarrow B$:

$$s_{AB} = \frac{s_{BA} \cdot p_B}{p_A}$$

Then we apply the deduction formula to $A \Rightarrow B$ and $B \Rightarrow C$:

$$s_{AC} = s_{AB} \cdot s_{BC} + (1 - s_{AB}) \cdot \frac{p_C - p_B \cdot s_{BC}}{1 - p_B}$$

This rule is also called the *source rule* because $B$ is the common source—arrows fan out from $B$ to both $A$ and $C$. In category-theoretic terms, we are completing a cospan $A \leftarrow B \rightarrow C$ to obtain $A \rightarrow C$.

## 4.6 Abduction Rule (Sink Rule)

The PLN *abduction* rule computes the truth value of $A \Rightarrow C$ given $A \Rightarrow B$ and $C \Rightarrow B$. Like induction, this is Bayes' rule followed by deduction.

Given $C \Rightarrow B$ with strength $s_{CB}$, we use Bayes' rule to obtain $B \Rightarrow C$:

$$s_{BC} = \frac{s_{CB} \cdot p_C}{p_B}$$

Then we apply the deduction formula to $A \Rightarrow B$ and $B \Rightarrow C$:

$$s_{AC} = s_{AB} \cdot s_{BC} + (1 - s_{AB}) \cdot \frac{p_C - p_B \cdot s_{BC}}{1 - p_B}$$

This rule is also called the *sink rule* because $B$ is the common sink—arrows fan in to $B$ from both $A$ and $C$. In category-theoretic terms, we are completing a span $A \to B \leftarrow C$ to obtain $A \to C$.

## 4.7 Conjunction Introduction

Given $A \doteq TV_A$ and $B \doteq TV_B$, infer $A \wedge B \doteq TV_{A \wedge B}$. Unlike the inference rules above, which involve implications, conjunction involves combining two predicates directly.

The key insight is that conjunction follows a *hypergeometric* distribution. Consider the discrete case where:

- $n$ is the size of the universe,
- $a = |A|$ is the number of elements satisfying $A$,
- $b = |B|$ is the number of elements satisfying $B$,
- $k = |A \wedge B|$ is the number of elements satisfying both.

The probability of exactly $k$ elements in the intersection is:

$$P(|A \wedge B| = k) = \frac{\binom{a}{k} \times \binom{n-a}{b-k}}{\binom{n}{b}}$$

This is the hypergeometric probability mass function. The cumulative distribution function is:

$$P(|A \wedge B| \leq k) = \sum_{i=0}^{k} \frac{\binom{a}{i} \times \binom{n-a}{b-i}}{\binom{n}{b}}$$

### 4.7.1 The Mode Bounds (Fréchet Bounds)

The mode of the hypergeometric satisfies:

$$\max(0, a + b - n) \leq \text{mode} \leq \min(a, b)$$

In probability terms (dividing by $n$):

$$\max(0, P(A) + P(B) - 1) \leq P(A \wedge B) \leq \min(P(A), P(B))$$

These are the *Fréchet bounds*, which hold for *any* joint distribution, not just the hypergeometric. PLN consistency is equivalent to respecting these bounds.

### 4.7.2 The Continuous Limit

The continuous case is obtained by taking $n \to \infty$ with fixed probabilities $p_A = a/n$ and $p_B = b/n$. Using Stirling's approximation on the factorials leads to a continuous density over the conjunction probability. The key insight is that the discrete hypergeometric PMF converges to a continuous density; the Fréchet bounds hold in both cases.

## 4.8 Implicant Conjunction Introduction

$$A \to C, B \to C \vdash (A \wedge B) \to C$$

If $A$ implies $C$ and $B$ implies $C$, what can we say about $(A \wedge B)$ implying $C$?

### 4.8.1 Independence Assumptions

We assume:

1. $A$ and $B$ are *globally independent*: $P(A, B) = P(A) \times P(B)$
2. $A$ and $B$ are *conditionally independent given* $C$: $P(A, B|C) = P(A|C) \times P(B|C)$

### 4.8.2 Derivation

By Bayes' formula:

$$P(C|A, B) = \frac{P(A, B|C) \times P(C)}{P(A, B)}$$

Using assumptions 1 and 2:

$$P(C|A, B) = \frac{P(A|C) \times P(B|C) \times P(C)}{P(A) \times P(B)}$$

Applying Bayes to $P(A|C)$ and $P(B|C)$:

$$P(A|C) = \frac{P(C|A) \times P(A)}{P(C)}, \quad P(B|C) = \frac{P(C|B) \times P(B)}{P(C)}$$

Substituting:

$$P(C|A, B) = \frac{P(C|A) \times P(C|B)}{P(C)}$$

This elegant formula shows that under independence, the strength of $(A \wedge B) \to C$ is the product of the individual implication strengths divided by the marginal probability of $C$.

Note that this formula can exceed 1 when $P(C)$ is small and both $P(C|A)$ and $P(C|B)$ are high. In practice, cap at 1:

$$s_{(A \wedge B) \to C} = \min\left(1, \frac{s_{A \to C} \times s_{B \to C}}{p_C}\right)$$

## 4.9 Negation

The PLN negation swaps positive and negative evidence:

$$\neg(n^+, n^-) = (n^-, n^+)$$

This gives the expected probabilistic behavior:

- Strength inverts: $s(\neg A) = 1 - s(A)$
- Confidence unchanged: $c(\neg A) = c(A)$

Note that this is *not* the Heyting complement $a \Rightarrow \bot$. The PLN negation is involutive ($\neg\neg A = A$), while the Heyting complement is not. This distinction is important: Evidence forms a Heyting algebra, but the probabilistic negation is a separate operation.

## 4.10 Disjunction Introduction

Given $A \doteq TV_A$ and $B \doteq TV_B$, infer $A \vee B \doteq TV_{A \vee B}$.

### 4.10.1 Via De Morgan's Law

$$P(A \vee B) = 1 - P(\neg A \wedge \neg B)$$

Using PLN negation (evidence swap) and conjunction:

$$\text{Evidence}(A \vee B) = \neg(\neg A \otimes \neg B)$$

### 4.10.2 Via Inclusion-Exclusion

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B)$$

Under independence, $P(A \wedge B) = P(A) \times P(B)$, so:

$$s_{A \vee B} = s_A + s_B - s_A \times s_B$$

### 4.10.3 Fréchet Bounds for Disjunction

Without independence assumption:

$$\max(P(A), P(B)) \leq P(A \vee B) \leq \min(1, P(A) + P(B))$$

The lower bound ensures disjunction is at least as likely as either disjunct. The upper bound caps at 1.

## 4.11 Revision Rule

The Revision Rule combines evidence from independent sources:

$$D_1 \oplus D_2 = (n_1^+ + n_2^+, n_1^- + n_2^-)$$

This is the *hplus* operation on Evidence.

### 4.11.1 Strength as Weighted Average

The combined strength is a weighted average of input strengths:

$$s_{\text{combined}} = \frac{n_1 \times s_1 + n_2 \times s_2}{n_1 + n_2}$$

where $n_i = n_i^+ + n_i^-$ is the total evidence count.

### 4.11.2 Connection to Bayesian Updating

The Revision Rule corresponds exactly to Beta conjugate updating:

- Prior: $\text{Beta}(\alpha_0, \beta_0)$
- Observation 1: $n_1^+$ successes, $n_1^-$ failures → Posterior: $\text{Beta}(\alpha_0 + n_1^+, \beta_0 + n_1^-)$
- Observation 2: $n_2^+$ successes, $n_2^-$ failures → Final: $\text{Beta}(\alpha_0 + n_1^+ + n_2^+, \beta_0 + n_1^- + n_2^-)$

This is why PLN Evidence is the *natural* representation for Bayesian inference: the algebraic hplus operation *is* conjugate updating.

## 4.12 Similarity

Similarity is a *symmetric* relationship: $\text{sim}(A, B) = \text{sim}(B, A)$.

### 4.12.1 Set-Theoretic Definition

$$\text{sim}(A, B) = \frac{|A \cap B|}{|A \cup B|}$$

### 4.12.2 From Two Inheritances

Given $s_{AC} = P(C|A)$ and $s_{CA} = P(A|C)$, compute:

$$\text{sim}_{AC} = \frac{1}{\frac{1}{s_{AC}} + \frac{1}{s_{CA}} - 1} = \frac{s_{AC} \times s_{CA}}{s_{AC} + s_{CA} - s_{AC} \times s_{CA}}$$

This is a harmonic-like mean of the two directed strengths.

### 4.12.3 Transitive Similarity

Given $\text{sim}_{AB}$, $\text{sim}_{BC}$, and term probabilities, compute $\text{sim}_{AC}$ by:

1. Convert similarities to inheritances
2. Apply deduction in both directions
3. Combine back to similarity

## 4.13 Modus Ponens

Classical modus ponens: from $P \Rightarrow Q$ and $P$, infer $Q$.

In PLN terms: Given $P(B|A)$ and $P(A)$, infer $P(B)$.

### 4.13.1 The Formula

By the law of total probability:

$$P(B) = P(B|A) \times P(A) + P(B|\neg A) \times P(\neg A)$$

The challenge: we typically don't know $P(B|\neg A)$. PLN uses a default "background probability" parameter $c$:

$$s_B = s_{A \Rightarrow B} \times s_A + c \times (1 - s_A)$$

### 4.13.2 Special Cases

- When $s_A = 1$ (certain premise): $s_B = s_{A \Rightarrow B}$
- When $s_A = 0$ (false premise): $s_B = c$

### 4.13.3 Modus Tollens

From $P(P \Rightarrow Q)$ and $P(\neg Q)$, infer $P(\neg P)$.

This is equivalent to modus ponens on the contrapositive $\neg Q \Rightarrow \neg P$ with strength $P(\neg P|\neg Q)$.

## 4.14 The Inference Triad

The three syllogistic rules—deduction, induction, and abduction—form a unified triad:

| **Rule** | **Premises** | **Method** |
|---|---|---|
| Deduction | $A \Rightarrow B$, $B \Rightarrow C$ | Direct composition |
| Induction | $B \Rightarrow A$, $B \Rightarrow C$ | Bayes on 1st premise, then deduction |
| Abduction | $A \Rightarrow B$, $C \Rightarrow B$ | Bayes on 2nd premise, then deduction |

This unification is not a heuristic—it follows from basic probability theory. Bayes' rule converts between $P(A|B)$ and $P(B|A)$, and deduction composes conditional probabilities. The three rules are simply different entry points into the same compositional structure.

# 5 The $\nu$PLN Theorem

## 5.1 Background: Solomonoff Optimality

Hutter [5] proved that Solomonoff's universal prior is *optimal* for sequence prediction: it minimizes expected loss (for any reasonable loss function) among all predictors, assuming only that the data source is computable. The Solomonoff predictor works by maintaining a mixture over all computable hypotheses, weighted by algorithmic probability.

However, this optimal predictor has two practical obstacles:

1. It is *uncomputable* (the halting problem).
2. It requires tracking the *full observation history*—the predictor state is the entire past sequence.

## 5.2 The Domain Restriction

The key observation is that for certain restricted domains, the second obstacle vanishes. Specifically, when the domain consists of *exchangeable binary sequences*, the predictor state compresses from "full history" to just two numbers.

**Theorem 1** (Restricted Solomonoff Collapses to Counts)**.** *Let $M$ be a Solomonoff-style prior* restricted to exchangeable programs*—formally, a monotone machine $U$ with a prefix-free program set where the induced semimeasure $\mu$ satisfies: for any permutation $\pi$, $\mu(x_1, \ldots, x_n) = \mu(x_{\pi(1)}, \ldots, x_{\pi(n)})$.*

*Then for any two observation sequences $x_1, x_2$ of the same length with the same counts $(n^+, n^-)$:*

1. ***Same probability**: $\mu(x_1) = \mu(x_2)$*
2. ***Same predictor**: For any $b \in \{\texttt{true}, \texttt{false}\}$,*

$$\frac{\mu(x_1 \cdot b)}{\mu(x_1)} = \frac{\mu(x_2 \cdot b)}{\mu(x_2)}$$

The theorem says: under the exchangeability restriction, the Solomonoff predictor $\mu(x \cdot b)/\mu(x)$ depends *only* on $(n^+, n^-)$, not on the order of observations. The sufficient statistic is the count pair—which is precisely PLN Evidence.

## 5.3 The Representation Theorem

What is the *form* of the predictor? De Finetti's representation theorem [2] answers this:

**Theorem 2** (De Finetti's Theorem)**.** *An infinite sequence of binary random variables is exchangeable if and only if it can be represented as a mixture of i.i.d. Bernoulli sequences:*

$$\mathit{InfiniteExchangeable}(X, \mu) \iff \exists M : \mathit{BernoulliMixture}.\ \mathit{Represents}(M, X, \mu)$$

The "BernoulliMixture" is a probability measure $\nu$ on $[0,1]$ such that:

$$\mu(x_1, \ldots, x_n) = \int_0^1 p^{n^+}(1-p)^{n^-}\, d\nu(p)$$

The mixing measure $\nu$ represents uncertainty about the "true" Bernoulli parameter $p$.

## 5.4 The Complete Chain

Combining these results yields the full picture:

1. **Solomonoff optimality** (Hutter): The universal prior is optimal for sequence prediction.
2. **Exchangeability restriction** (Theorem 1): Under exchangeability, the predictor depends only on counts $(n^+, n^-)$.
3. **De Finetti representation** (Theorem 2): Exchangeable sequences are Bernoulli mixtures.
4. **Beta-Bernoulli conjugacy**: With a Beta$(\alpha, \beta)$ prior on the Bernoulli parameter $p$, the posterior after observing $(n^+, n^-)$ is Beta$(\alpha{+}n^+, \beta{+}n^-)$.
5. **PLN strength**: The posterior mean is $(\alpha + n^+)/(\alpha + \beta + n^+ + n^-)$. PLN's strength formula $n^+/(n^+ + n^-)$ corresponds to the improper prior $\alpha = \beta = 0$, and converges to any proper posterior mean as sample size increases.

This chain is formalized in Lean 4 as the `nupln_master_chain` theorem, which explicitly invokes each step by name. The proof constructs the mixing measure via the Hausdorff moment theorem applied to the de Finetti moment sequence.

## 5.5 Conceptual Summary

PLN Evidence $(n^+, n^-)$ is not an approximation or heuristic. It is the *exact* sufficient statistic for Bayesian inference over exchangeable binary sequences. The PLN strength formula $n^+/(n^+ + n^-)$ is the maximum likelihood estimator, corresponding to an improper (Haldane) prior; it converges to any proper Bayesian posterior mean as sample size increases.

When Solomonoff induction is restricted to exchangeable binary domains, it *becomes* PLN. The "arbitrary program mixture" of Solomonoff collapses to the Beta mixture of de Finetti, and the predictor state compresses from "full history" to $(n^+, n^-)$.

This justifies using PLN for domains where exchangeability holds—the inference is not approximate but optimal. The domain restriction is the hypothesis; the simplification is the theorem.

# 6 The Algebraic Foundations of Evidence

The Evidence structure $(n^+, n^-)$ is not an ad-hoc encoding. It arises naturally from the intersection of three mathematical frameworks, each providing a different perspective on the same underlying structure.

## 6.1 Evidence as a Quantale

A *quantale* is a complete lattice equipped with an associative multiplication that distributes over arbitrary joins:

$$a \otimes \left( \bigvee_i b_i \right) = \bigvee_i (a \otimes b_i)$$

Evidence with the *tensor* product forms a commutative quantale:

$$(n_1^+, n_1^-) \otimes (n_2^+, n_2^-) = (n_1^+ \cdot n_2^+, n_1^- \cdot n_2^-)$$

The fundamental quantale law for implications is the *transitivity* property:

$$(A \to B) \otimes (B \to C) \leq (A \to C)$$

This is precisely the PLN deduction rule at the evidence level! The multiplicative nature of tensor captures how uncertainty compounds when chaining implications.

## 6.2 Evidence as a Heyting Algebra

Evidence is also a complete Heyting algebra (a *frame*). Unlike Boolean algebras, Heyting algebras have non-involutive negation: $\neg\neg a \neq a$ in general. This is proven by explicit counterexample in our formalization.

The Heyting structure explains why Evidence requires *two* dimensions. A single probability $p \in [0, 1]$ cannot capture both "how likely" and "how certain" we are. The pair $(n^+, n^-)$ encodes:

- *Strength*: $s = n^+/(n^+ + n^-)$ (the probability estimate)
- *Confidence*: proportional to $n^+ + n^-$ (the certainty)

To see why one dimension is insufficient, consider $(1, 1)$ and $(100, 100)$. Both have strength $\frac{1}{2}$, yet the latter represents far more evidence. A single probability cannot distinguish these cases. More generally, for any strength $s \in (0, 1)$, infinitely many evidence values $(k \cdot s, k \cdot (1 - s))$ produce that same strength—the "fiber" over $s$ is infinite. The second dimension is not redundant; it carries essential information about certainty.

The Heyting residuation law holds:

$$a \leq (b \Rightarrow c) \iff (a \wedge b) \leq c$$

providing a logical interpretation of evidence combination.

### 6.2.1 Strength is NOT Monotone

A subtle but critical insight: strength does *not* respect the Evidence partial order.

**Counterexample**: $(1, 0) \leq (1, 1)$ in the Evidence order (coordinatewise), but:

$$s(1,0) = \frac{1}{1+0} = 1 > \frac{1}{2} = \frac{1}{1+1} = s(1,1)$$

Adding negative evidence *decreases* strength while *increasing* in the partial order. This means strength is not a K&S valuation on Evidence.

Confidence, however, *is* monotone: more evidence (in either direction) increases confidence. This asymmetry is fundamental—strength measures "what we believe," confidence measures "how strongly we believe it."

### 6.2.2 Connection to Knuth-Skilling Theory

Evidence connects to Knuth & Skilling's "Foundations of Inference" [**?**] at multiple levels:

- **PlausibilitySpace**: Evidence is a distributive lattice with $\top$ and $\bot$, so K&S valuations and conditional plausibility are well-defined.
- **Not Boolean**: Evidence is a Heyting algebra, not a Boolean algebra. The law of excluded middle fails: there exist evidence values $e$ such that $e \vee \neg e \neq \top$.
- **Intuitionistic Probability**: This gives "intuitionistic probability theory" where $P(\neg A) \neq 1 - P(A)$ in general.

The K&S product rule derivations (requiring Boolean structure) do not directly apply to the Evidence lattice.

**Resolution**: PLN operates at *two distinct levels*:

1. **Event level**: The global probability space $(\Omega, \mathcal{F}, \nu)$ where $\mathcal{F}$ IS a Boolean $\sigma$-algebra. Standard Bayesian probability applies here.
2. **Evidence level**: The lattice of Evidence values $(n^+, n^-)$ is Heyting. This describes the *information ordering* ("more evidence"), not the probability calculus.

PLN derives rules at the event level (Bayes' theorem, law of total probability) and stores results at the evidence level. There is no contradiction because they concern different structures: Boolean for events, Heyting for epistemic states.

## 6.3 Heyting K&S: The Interval Construction

Although PLN's Bayesian rules operate at the Boolean event level, we *can* develop K&S-like theory directly on Heyting algebras. This gives "intuitionistic probability" where the complement rule weakens from equality to inequality.

### 6.3.1 Modular Valuations

On a bounded distributive lattice, a **modular valuation** is a function $\nu : L \to [0, 1]$ satisfying:

1. $\nu(\bot) = 0$, $\nu(\top) = 1$
2. Monotonicity: $a \leq b \Rightarrow \nu(a) \leq \nu(b)$
3. **Modularity**: $\nu(a) + \nu(b) = \nu(a \vee b) + \nu(a \wedge b)$

The modularity axiom is the lattice-theoretic generalization of inclusion-exclusion, expressed without requiring complements.

### 6.3.2 Boolean vs Heyting Complements

On a **Boolean** algebra, modularity implies:

$$\nu(a) + \nu(\neg a) = 1$$

On a **Heyting** algebra, we only get:

$$\nu(a) + \nu(\neg a) \leq 1$$

The *slack* arises because $a \vee \neg a \leq \top$ in general (excluded middle may fail). This slack is precisely the **excluded middle gap**:

$$\text{gap}(a) = 1 - \nu(a \vee \neg a) \geq 0$$

### 6.3.3 The Interval Construction

For a Heyting algebra with modular valuation $\nu$, define:

$$\begin{aligned} \text{lower}(a) &= \nu(a) \quad \text{(direct evidence for } a\text{)} \\ \text{upper}(a) &= 1 - \nu(\neg a) \quad \text{(absence of evidence against } a\text{)} \end{aligned}$$

Key properties:

- $\text{lower}(a) \leq \text{upper}(a)$ always (from the Heyting inequality)
- $\text{upper}(a) - \text{lower}(a) = \text{gap}(a)$ (interval width equals excluded middle gap)
- **Boolean collapse**: When $a \vee \neg a = \top$, the interval collapses to a point: $\text{lower}(a) = \text{upper}(a)$

### 6.3.4 Interpretation

In classical logic, knowing $\neg a$ tells you everything about $a$. In intuitionistic logic, $\neg a$ only bounds $a$ from above. The interval $[\mathrm{lower}(a), \mathrm{upper}(a)]$ captures this *epistemic slack*:

- $\mathrm{lower}(a) = \nu(a)$: "how much evidence directly supports $a$"
- $\mathrm{upper}(a) = 1 - \nu(\neg a)$: "how much room is left after accounting for $\neg a$"

This connects to imprecise probability (Walley) and credal sets (de Cooman & Hermans). The 2D Evidence structure $(n^+, n^-)$ can be seen as tracking both bounds simultaneously.

## 6.4 The Beta Connection Revisited

The *hplus* operation (parallel aggregation):

$$(n_1^+, n_1^-) \oplus (n_2^+, n_2^-) = (n_1^+ + n_2^+, n_1^- + n_2^-)$$

is precisely Beta conjugate updating. When we observe independent evidence from two sources, we simply add the counts—and this corresponds exactly to updating the Beta parameters.

This complements Section 3.3: the algebraic operation $\oplus$ *is* Bayesian inference.

## 6.5 The Unified Architecture

The three perspectives converge:

| **Framework** | **Operation** | **Interpretation** |
|---|---|---|
| Quantale | tensor $\otimes$ | Sequential composition, confidence compounding |
| Heyting algebra | meet/implication | Logical combination, non-Boolean bounds |
| Beta-Bernoulli | hplus $\oplus$ | Independent evidence, conjugate updating |

The architectural insight: PLN's Evidence structure is a *natural* 2D carrier that simultaneously satisfies quantale, Heyting, and Bayesian constraints. The two dimensions are not redundant—they encode complementary aspects of uncertain inference.

## 6.6 The Weight-Space Theorem

This section formalizes a critical insight that has practical implications for PLN implementations.

### 6.6.1 The Problem

When combining confidence values, one might naively compute:

$$c_{\text{combined}} = \texttt{w2c}(\min(c_1, c_2)) \quad \text{— WRONG!}$$

This treats confidences as if they were weights.

### 6.6.2 The Correct Formula

$$c_{\text{combined}} = \texttt{w2c}(\min(\texttt{c2w}(c_1), \texttt{c2w}(c_2)))$$

Convert to weight space, take minimum, convert back.

### 6.6.3 Why?

The hypergeometric distribution operates on *counts* (weights), not confidences. The mode bound:

$$\text{mode} \le \min(a, b)$$

tells us that combined evidence is bounded by the minimum of input evidence *counts*.

Since confidence $c$ is a nonlinear transformation of weight $w$:

$$c = \frac{w}{w+k}, \quad w = \frac{k \cdot c}{1-c}$$

taking min in confidence space gives incorrect results. The error can be **10–50%** for high-confidence inputs.

### 6.6.4 Practical Rule

**Never store only (strength, confidence).** You lose the information needed for correct inference.

Options:

1. Store full Evidence $(n^+, n^-)$ — **best**
2. Store (strength, weight) pairs
3. Store (strength, confidence, $k$) if $k$ is known

## 6.7 Comparison with NARS

The Non-Axiomatic Reasoning System (NARS) [**?**] shares fundamental machinery with PLN:

| **Aspect** | **PLN** | **NARS** |
|---|---|---|
| Weight transform | $w = c/(1-c)$ | Same |
| Confidence transform | $c = w/(w+k)$ | Same |
| Revision rule | Weighted average | Same |
| Derivation | Bayesian probability | Axiomatic (experience-grounded) |

The weight/confidence bijection is proven:

$$\texttt{w2c}(\texttt{c2w}(c)) = c, \quad \texttt{c2w}(\texttt{w2c}(w)) = w$$

for valid inputs. This means PLN and NARS compute equivalent results for the revision rule—the philosophical difference (Bayesian vs. axiomatic) does not affect the mathematics.

# A Work in Progress

## A.1 Markov Exchangeability

For sequences that are not i.i.d. but exhibit first-order Markov structure, a generalization of de Finetti's theorem applies [1]. In this setting, the sufficient statistics are the *transition counts*: how many times each state-to-state transition occurred. This suggests a "Markov-PLN" where evidence consists of a transition count matrix rather than simple $(n^+, n^-)$ counts.

## A.2 Lean Statements of Main Theorems

**Theorem 1 (Predictor Collapse):** The `RestrictedSolomonoffPrior` structure bundles a monotone machine with a proof that its induced semimeasure is exchangeable.

```
structure RestrictedSolomonoffPrior where
  U : MonotoneMachine
  programs : Finset BinString
  hpf : PrefixFree programs
  hexch : ProgramsExchangeable U programs  -- the restriction!

theorem solomonoff_exchangeable_predictBit_same_counts
    (M : RestrictedSolomonoffPrior) :
    forall {n : N} (xs1 xs2 : Fin n -> Bool),
      countTrue xs1 = countTrue xs2 ->
      forall b : Bool,
        M.predictBit (List.ofFn xs1) b = M.predictBit (List.ofFn xs2) b
```

**Theorem 2 (De Finetti):**

```
theorem exchangeable_iff_bernoulliMixture (X : N -> Omega -> Bool)
    (mu : Measure Omega) [IsProbabilityMeasure mu]
    (hX : forall i, Measurable (X i)) :
    InfiniteExchangeable X mu <->
      exists (M : BernoulliMixture), Represents M X mu
```

**Master Chain Theorem ($\nu$PLN Justification):** This theorem explicitly chains the four key results, making the full derivation verifiable:

```
theorem nupln_master_chain (X : N -> Omega -> Bool) (mu : Measure Omega)
    [IsProbabilityMeasure mu] (hX : forall i, Measurable (X i))
    (hexch : InfiniteExchangeable X mu) :
    -- Part 1: De Finetti -> Bernoulli mixture
    exists (M : BernoulliMixture), Represents M X mu /\
    -- Part 2: Counts are sufficient statistics
    (forall n (xs1 xs2 : Fin n -> Bool),
      countTrue xs1 = countTrue xs2 -> M.prob xs1 = M.prob xs2) /\
    -- Part 3: PLN Evidence = counts
    (forall n_pos n_neg, evidenceFromCounts n_pos n_neg = (n_pos, n_neg)) /\
    -- Part 4: PLN strength -> posterior mean
    (forall eps, 0 < eps -> exists N, forall n_pos n_neg,
      n_pos + n_neg >= N -> n_pos + n_neg <> 0 ->
      |plnStrength n_pos n_neg - uniformPosteriorMean n_pos n_neg| < eps)
```

**Inference Triad (Section 4.14):**

```
-- Bayes inversion: P(A|B) = P(B|A) * P(B) / P(A)
def bayesInversion (s_BA s_A s_B : R) : R := s_BA * s_B / s_A

-- Induction = Bayes + Deduction
theorem plnInduction_eq_bayes_deduction (s_BA s_BC s_A s_B s_C : R) :
    plnInductionStrength s_BA s_BC s_A s_B s_C =
      plnDeductionStrength (bayesInversion s_BA s_A s_B) s_BC s_B s_C

-- Abduction = Bayes + Deduction
theorem plnAbduction_eq_bayes_deduction (s_AB s_CB s_A s_B s_C : R) :
    plnAbductionStrength s_AB s_CB s_A s_B s_C =
      plnDeductionStrength s_AB (bayesInversion s_CB s_B s_C) s_B s_C
```

All theorems are fully mechanized in Lean 4 with zero sorries. The de Finetti proof constructs the mixing measure via the Hausdorff moment theorem applied to the de Finetti moment sequence.

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