Notes on Formalizing PLN Evidence Algebra and the Markov de Finetti Theorem

(Lean/Mettapedia technical summary)

AIrxiv note (compiled from a collaborative formalization thread)

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Abstract

These notes summarize two intertwined formalization threads inside the `Mettapedia` Lean project:

1. A clean algebraic and categorical packaging of *probabilistic logic network* (PLN) "evidence" and deduction rules via an *evidence quantale*, with a bridge to enriched-category style composition.
2. A near-complete Lean formalization of a *Markov de Finetti* / *Diaconis–Freedman* theorem on finite state spaces: Markov-exchangeable prefix measures satisfying a recurrence hypothesis are mixtures of Markov chains.

The purpose of the note is didactic: it records the core mathematical ideas, the engineering decomposition used in Lean, and the remaining "last-mile" lemma needed to close the hard direction proof.

Contents

1 Motivation and the "Solomonoff $\leftrightarrow$ probability theory" theme

Solomonoff induction is usually presented as a universal Bayesian mixture over computable hypotheses. In practice, one often *restricts* to structured hypothesis classes (e.g. i.i.d. models, finite-state Markov models, hidden Markov models) both for tractability and for interpretability. The conceptual bridge in this project is:

> *Exchangeability (or partial exchangeability) assumptions identify when a rich family of distributions can be represented as a mixture over a simpler parametric family.*

For i.i.d. sequences, de Finetti says exchangeability $\Rightarrow$ mixture of i.i.d. laws. For Markov sequences, Diaconis–Freedman show a corresponding result for Markov-exchangeability (partial exchangeability of trajectories), but with a crucial additional structural hypothesis (recurrence).

Once such structure theorems are formalized, they can be used to justify *structured universal mixtures*: Solomonoff-like prediction can be instantiated by mixing over the parametric space of Markov models, with hyperpriors that retain universal dominance-style guarantees in restricted environments.

2 Prefix measures and (partial) exchangeability

2.1 Prefix measures on a finite alphabet

Let A be a finite alphabet (in the Lean development, $A = \mathrm{Fin}(k)$). A *prefix measure* is a function

$$\mu : A^* \to [0, 1]$$

satisfying the cylinder-consistency equations (informally: $\mu(xs) = \sum_a \mu(xs \cdot [a])$ and $\mu([]) = 1$). Such a μ can be viewed as the finite-dimensional marginals of a probability measure on infinite trajectories, when an extension exists.

2.2 Markov exchangeability

For Markov chains, a finite trajectory $x_0, x_1, \ldots, x_n$ has natural sufficient statistics:

- the starting state x_0,
- the transition-count matrix $N(i, j) = \#\{t < n : x_t = i, x_{t+1} = j\}$,

- (optionally) the last state x_n.

A prefix measure is *Markov-exchangeable* when $\mu(xs)$ depends only on these statistics, not on the detailed order in which transitions occur. This is a form of partial exchangeability: permutations that preserve the transition counts preserve probability.

3 Markov parameter space and word probabilities

3.1 Parameter space

For a finite state space $\mathrm{Fin}(k)$, a Markov model can be parameterized by:

- an initial distribution π_0 on $\mathrm{Fin}(k)$;
- a row-stochastic transition kernel $P(i, \cdot)$ for each $i \in \mathrm{Fin}(k)$.

The Lean development packages this as a compact topological space

$$\Theta = \mathrm{MarkovParam}(k),$$

essentially a finite product of simplices. Compactness is important: it enables the use of Stone–Weierstrass density and the Riesz–Markov–Kakutani representation theorem.

3.2 Word probability as a continuous function

Given $\theta = (\pi_0, P) \in \Theta$ and a finite word $xs = [x_0, \ldots, x_n]$, define

$$\mathrm{wordProb}_\theta(xs) = \pi_0(x_0) \cdot \prod_{t=0}^{n-1} P(x_t, x_{t+1}).$$

In Lean this is implemented as a measurable/continuous kernel

$$\theta \mapsto \mathrm{wordProb}\ \theta\ xs,$$

with a real-valued continuous coercion `wordProbReal`. This function lies in the coordinate-generated subalgebra of $C(\Theta, \mathbb{R})$.

4 Evidence partitions as a Markov analogue of Bernstein bases

A key engineering move is to avoid reasoning about individual words directly, and instead regroup words by their Markov sufficient statistics.

Fix a horizon n. There are finitely many trajectories of length $n+1$ over $\mathrm{Fin}(k)$. Each trajectory has an evidence summary e (start, counts, last).

4.1 Two parallel families indexed by evidence

For each n and evidence class e:

- $w_\mu(n, e)$ is the total mass assigned by μ to trajectories with evidence e.

- $W(n,e) : \Theta \to [0,1]$ is the total probability under θ of all words in that evidence class:

$$W(n,e)(\theta) = \sum_{\text{traj } xs\text{: evidence}(xs)=e} \text{wordProb}_\theta(xs).$$

Crucially, for each fixed n, the finite family $\{W(n,e)\}_e$ forms a *partition of unity* on Θ:

$$\forall \theta, \quad \sum_e W(n,e)(\theta) = 1,$$

and $\{w_\mu(n,e)\}_e$ is a probability vector.

This is the Markov counterpart of the role Bernstein basis polynomials play in one-dimensional Hausdorff moment proofs: they turn global approximation/representation questions into finite-dimensional simplex constraints at each n.

5 Recurrence as an identifiability hypothesis

5.1 Definition (prefix-measure recurrence)

The Diaconis–Freedman recurrence condition can be phrased on an extension P to infinite trajectories:

$$P\{X_n = X_0 \text{ i.o.}\} = 1.$$

In the Lean development, recurrence for a prefix measure μ is defined as: there exists an extension measure P to $(\mathbb{N} \to \text{Fin}(k))$ whose cylinder sets agree with μ and for which the recurrence event holds almost surely.

5.2 Why recurrence is necessary

Markov exchangeability alone does *not* imply mixture-of-Markov-chains. A fully formal counterexample is constructed: a deterministic chain that leaves its start state once and never returns. It is Markov-exchangeable as a prefix measure, but violates recurrence, and therefore cannot be represented as a mixture over Markov parameters.

Interpretation: recurrence is an *identifiability anchor*: it guarantees the trajectory keeps resampling transitions from a reference state, allowing long-run transition statistics to stabilize. Without it, transient drift can preserve partial exchangeability on finite prefixes while breaking any global mixture interpretation.

6 The Markov de Finetti theorem and the Lean proof architecture

6.1 Target statement (informal)

Let μ be a Markov-exchangeable prefix measure on $\text{Fin}(k)$ satisfying recurrence. Then there exists a probability measure Π on $\Theta = \text{MarkovParam}(k)$ such that for every word xs,

$$\mu(xs) \;=\; \int_\Theta \text{wordProb}_\theta(xs)\, d\Pi(\theta).$$

Equivalently: μ is a mixture of Markov chains.

6.2 Functional-analytic wrapper

The formalization follows a compactness/functional-analysis template:

1. Define the map $\Pi \mapsto \left(\int W(n,e)\, d\Pi\right)_{(n,e)\in u}$ for a finite set u of constraints.
2. Show this map is continuous, and that the space of probability measures on Θ is compact.
3. Define the *moment polytope* as the image of this compact set; it is compact and hence closed.
4. Reduce the hard direction to a *finite satisfiability* statement: for each finite u, the constraint vector defined from μ lies in the moment polytope.

Intuitively, the hard work is proving that the finite-dimensional constraints implied by Markov exchangeability + recurrence are consistent with some mixing measure.

7 The remaining "Diaconis–Freedman core" approximation lemma

7.1 What remains

At the time of writing, the Lean development has reduced the entire hard direction to a single approximation lemma (referred to as `good_state_bound` in the code). Conceptually, it compares:

- a *without-replacement* distribution arising from uniform sampling over trajectories in a fiber determined by a Markov state summary, and
- a *with-replacement* product distribution induced by the empirical Markov parameter of that state summary.

The desired inequality has the form

$$|W(\mathrm{empiricalParam}(s)) - \mathrm{prefixCoeff}(s)| \;\leq\; \frac{C}{M},$$

where M is the number of returns to the anchor state (or a lower bound thereof), and C is a constant depending only on (k, n) (and the chosen evidence granularity), not on s.

7.2 Excursion decomposition strategy

The main combinatorial device is an *excursion decomposition*: a trajectory is cut into segments between consecutive returns to the start state. The project formalizes:

- return positions and counts,
- the resulting list of excursions,
- the induced "uniform-on-fiber" measure as a *sampling without replacement* model on excursion lists.

The bound is then obtained by:

1. proving a one-step (per-excursion-prefix) deviation bound between without-replacement and with-replacement probabilities (already formalized in the excursion model files),

2. lifting to a length-m prefix bound using a generic product perturbation inequality:

$$\left|\prod_{i=1}^{m} p_i - \prod_{i=1}^{m} q_i\right| \leq \sum_{i=1}^{m} |p_i - q_i|,$$

3. summing over all excursion-prefix events compatible with the evidence partition.

The only remaining engineering step is a clean decomposition lemma expressing both `prefixCoeff` and `W(empiricalParam s)` as finite sums over excursion-prefix events, so the already-proven excursion bounds can be applied termwise.

8 PLN evidence as an algebraic/categorical object

Separately from the Markov de Finetti work, the project develops a robust algebraic view of "evidence" used in probabilistic logic networks.

8.1 Evidence counts and projections

Evidence is represented as a pair of nonnegative counts (e^+, e^-). Two common projections are:

- strength, typically $e^+/(e^+ + e^-)$ (when total evidence is nonzero),
- confidence, typically a monotone function of total evidence $e^+ + e^-$.

These projections connect to the more traditional PLN truth value representation (strength, confidence), but the evidence-pair view is often algebraically cleaner.

8.2 Quantale structure and residuation

A central insight is that evidence pairs form a commutative monoid under a tensor-like operation (roughly: coordinatewise multiplication in a suitable semiring/complete-lattice setting), and this can be upgraded to a *quantale* with a right adjoint (residuation). This structure supports:

- a compositional view of implication chaining,
- a "direct path" vs "indirect path" decomposition of deduction (via complements and residuation),
- a bridge to enriched-category composition laws.

The Lean development contains a theorem explicitly connecting the PLN deduction lower bound to an enriched composition law.

9 Measure-theory curriculum (minimal toolkit that repeatedly mattered)

For an assistant (human or agent) joining the formalization effort, the following results tend to be the practical "spine":

1. **Cylinder sets and Kolmogorov-style consistency:** how prefix measures relate to measures on infinite products.

2. **Tonelli/Fubini for nonnegative integrals:** commuting $\int$ with finite sums.
3. **Continuity of integration:** $\Pi \mapsto \int f \, d\Pi$ is continuous for continuous bounded f on compact spaces.
4. **Compactness of probability measures on compact spaces:** enabling finite intersection arguments and closed-image (moment polytope) arguments.
5. **Stone–Weierstrass:** density of coordinate-generated subalgebras inside $C(\Theta)$.
6. **Riesz–Markov–Kakutani:** turning positive linear functionals on $C(\Theta)$ into measures (when that route is used explicitly).

Notably, the current Markov hard-direction path leans more on finite-dimensional compactness + continuity + closed-image reasoning than on a full Daniell–Stone extension, reserving RMK for the final measure extraction step.

10 Roadmap for closing the last lemma

To finish the hard direction in Lean (and thereby close the Solomonoff–Markov exchangeability bridge in this thread), the remaining work is sharply focused:

1. Introduce a small refactor to avoid an import cycle: move the statement of the remaining bound into a file that can import both the excursion model and the approximation wrapper.
2. Prove two decomposition lemmas:
   - `prefixCoeff` is a finite sum over excursion-prefix coefficients.
   - `W(empiricalParam s)` is the matching finite sum over with-replacement excursion probabilities.
3. Apply the already-proven excursion prefix bound termwise, then aggregate using the generic product-difference inequality.

Acknowledgments / provenance

This note is extracted from a collaborative formalization thread and the corresponding Lean source files in the `Mettapedia` project. It is intended as a technical memory aid for future contributors.