ξPLN: A Correct and Complete Foundation for Probabilistic Logic Networks
(DRAFT)

Zar Codex CLI (OpenAI GPT-5.2)

February 4, 2026

Abstract

Probabilistic Logic Networks (PLN) couples logical structure with uncertain inference. In practice, however, many PLN presentations mix distinct layers: (i) a probabilistic semantic model, (ii) an evidence/weight plumbing model, and (iii) operational truth-value views (strength, confidence, interval bounds). This leads to "semantic drift": algebraic operators are treated as if they were the logic itself, and link-level inference attempts to propagate correlation-sensitive quantities without carrying the information required to do so correctly.

We present ξPLN, a complete PLN foundation designed to be (a) mathematically correct and (b) extensible to tractable sublayers. The core move is to make the complete layer a *posterior-state calculus*: the proof object is a revisable world-model posterior state, and all link/event truth values are obtained by querying that state. We give a reference complete instance as a Dirichlet posterior over complete worlds (`JointEvidence`), prove that evidence extraction commutes with revision, and formalize a no-go theorem showing that complete link evidence cannot, in general, be computed from local per-link evidence alone. We then explain how Bayesian-network-style sublayers arise as restricted but tractable world-model classes, keeping PLN's operational rules as compiled tactics with explicit assumptions.

1 Motivation and Context

PLN, as described by Goertzel, Iklé et al. [1], represents uncertain judgments using truth values such as *strength* (a probability-like estimate) and *confidence* (a reliability-like quantity). Modern *nuPLN* work (Nil Geisweiller, draft and v1) makes the key step explicit: a complete grounding requires a global probabilistic semantics—a world model with a joint distribution over possible worlds and a Bayesian update story.

Our experience formalizing PLN in Lean is that correctness failures almost always come from mixing layers:

- treating operational truth-value operators (interval bounds, confidence heuristics) as if they were the logic; or
- trying to compute complete posteriors for derived links from local link truth values without carrying the correlation information needed to do so.

ξPLN is a response: it isolates a *complete* core calculus and makes all fast PLN rules live as derived/compiled approximations relative to declared model classes.

2 Three Semantic Layers (and Why They Must Not Be Conflated)

We distinguish three layers.

2.1 Layer 1: World-model posterior states (complete semantics)

The complete layer carries a posterior state in some explicit class of world models. Conceptually: *a distribution or structured distribution* that can answer any query by conditioning/marginalization (or by exact inference within the chosen model class).

In the Lean prototype, this layer is represented by `JointEvidence`:

$$\texttt{JointEvidence}(n) := \texttt{Fin}(2^n) \to \mathbb{R}^{\infty}_{\geq 0}$$

interpreted as Dirichlet pseudo-counts over the 2^n complete worlds for n propositional atoms. Revision is pointwise addition of pseudo-counts.

2.2 Layer 2: Evidence algebra (quantale / Heyting structure)

PLN uses an evidence carrier `Evidence` as a compositional algebra of observations. In our formalization, `Evidence` is the 2D count object (n^+, n^-) with:

- parallel aggregation $\oplus$ ("revision"): add counts componentwise;
- order (information ordering): coordinatewise $\leq$;
- rich lattice structure (complete lattice; Heyting operators exist).

This layer is *not* itself the complete probabilistic semantics; it is an algebraic semantics for evidence aggregation and logical structure. It is particularly important for FO/HO extensions, where truth values are naturally Heyting-valued (cf. the SatisfyingSet construction in the codebase).

2.3 Layer 3: Operational views (strength, weight, confidence, bounds)

Strength/weight/confidence/intervals are *views* extracted from evidence. They are indispensable operationally, but they should not be used as if they were complete semantic state. ξPLN adopts the mantra:

> Probability is what you query; evidence is what you carry.

3 ξPLN Core: World-model Revision + Query

The complete core calculus is intentionally simple: it is the calculus of revising posterior states plus a query interface.

3.1 World-model interface

In Lean, we isolate a minimal interface `WorldModel` (posterior state + query projections):

```
inductive PLNQuery (Atom : Type*) where
  | prop : Atom → PLNQuery Atom
  | link : Atom → Atom → PLNQuery Atom

class WorldModel (State : Type*) (Query : Type*) [EvidenceType State]
    where
  evidence : State → Query → Evidence
  evidence_add :   W  W q, evidence ( W +  W) q = evidence  W q + evidence
       W q
```

Here `EvidenceType` means: revision is an additive commutative monoid on `State`. Standard PLN event/link queries are represented by `PLNQuery Atom`; all truth-value quantities are derived by mapping extracted `Evidence` to strengths/weights. See `Mettapedia/Logic/PLNWorldModel.lean`.

3.2 Reference complete instance: Dirichlet over worlds

For finite propositional scope (atoms `Fin n`), we instantiate `WorldModel` with the Dirichlet world-table `JointEvidence`:

- `evidence(E, prop(A))` sums world counts where an atom is `true` vs `false`;
- `evidence(E, link(A,B))` sums world counts where `A` is true and `B` true vs false.

Crucially, extraction commutes with revision:

> revising joint evidence and then extracting a query is equal to extracting and revising at the query-evidence level.

Formally, the Lean prototype proves:

$$\mathrm{evidence}(E_1 + E_2, q) = \mathrm{evidence}(E_1, q) \oplus \mathrm{evidence}(E_2, q)$$

for all queries q. See `Mettapedia/Logic/PLNJointEvidence.lean`.

3.3 Probability views

From `Evidence` we obtain posterior-mean probabilities (improper-prior strength):

$$P(A) = \frac{\#(A)}{\#(\top)}, \qquad P(B \mid A) = \frac{\#(A \wedge B)}{\#(A)}$$

where $\#(\cdot)$ is a world-count sum in `JointEvidence`. The Lean file `Mettapedia/Logic/PLNJointEvidenceProba` proves the exact ratio forms for these views.

4 A No-Go Theorem: "Complete" Link Inference Cannot Be Local

One might hope for a sequent-calculus-like system that takes local per-link evidence (`Evidence` for A, B, C, $A \Rightarrow B$, $B \Rightarrow C$) and computes complete evidence for $A \Rightarrow C$. ξPLN asserts that this is impossible in general without extra assumptions: correlations live in the joint state.

Theorem 1 (No local complete deduction rule)**.** *There is no function*

$$f : \texttt{Evidence}^5 \to \texttt{Evidence}$$

that, for all joint evidence states E, computes the exact link evidence for $A \Rightarrow C$ from only the local premises `Evidence`(A), `Evidence`(B), `Evidence`(C), `Evidence`$(A \Rightarrow B)$, `Evidence`$(B \Rightarrow C)$.

The Lean proof constructs two different joint evidence states on three atoms that agree on all these premises but disagree on the conclusion `linkEvidence(A,C)`. See `Mettapedia/Logic/PLNJointEvidenceNoGo.lean`.

Interpretation. This theorem is the formal core of "you must carry correlations". It does not make fast PLN useless; it tells us exactly what fast PLN *cannot* claim: completeness for arbitrary world models without structural assumptions.

5 Tractable Sublayers: Bayesian Networks as Restricted World Models

The complete world-table `JointEvidence` is exponential in n. To scale, we restrict the world-model class while preserving correctness *relative to that class.*

The next natural step is a Bayesian-network-style world model:

- a DAG structure specifying factorization;
- local conditional tables with Dirichlet evidence (counts) per CPT row;
- revision = add local Dirichlet counts (conjugate update);
- query = exact inference by variable elimination / junction tree, with complexity controlled by treewidth.

Lean status. We have implemented the *evidence plumbing* part of this sublayer:

- a Boolean BN CPT query type `CPTQuery` (node + parent configuration),
- a CPT posterior state `CPTState` storing `Evidence` per CPT entry, and
- an additive projection from `JointEvidence` to CPT evidence by marginalization (summing compatible worlds), which commutes with revision.

See `Mettapedia/Logic/PLNBayesNetWorldModel.lean`.

We have also begun the "fast rule exactness" bridge in the simplest nontrivial case: `Mettapedia/Logic/PLNBayesNetFastRules.lean` sets up the chain BN $A \to B \to C$, proves the required sink-factorization lemmas, and starts deriving the screening-off hypotheses needed to apply `PLNDerivation.pln_deduction_from_total_probability`.

Exact BN query answering (variable elimination / junction tree) is the next step: it will provide tractable evaluation of probabilities for larger query languages, while preserving correctness relative to the declared BN model class.

PLN spirit preserved. Classic PLN link rules (deduction/abduction/induction) become:

- compiled tactics that propose BN queries, or
- conditionally-sound lemmas under explicit assumptions (e.g. conditional independence),

rather than pretending to be globally complete link-level operators.

6 Knuth–Skilling and Heyting Foundations

Knuth–Skilling foundations of inference [2] provide axioms for valuation schemes on logical lattices, extending Cox-style plausibility calculi. In ξPLN, this plays two roles:

1. *Probability as a valuation/view.* At the world-model layer, probabilities arise as valuations on the Boolean algebra of subsets of worlds. This is the classical Kolmogorov story [3] instantiated for finite spaces, and it underwrites the ratio theorems proved in `PLNJointEvidenceProbability`.

2. *Evidence as Heyting-valued semantics.* At the evidence/algebraic layer, `Evidence` forms a rich (non-Boolean) lattice. K&S-style valuation rules persist but Boolean equalities can weaken to inequalities; see `Mettapedia/Logic/EvidenceIntuitionist`

An important cautionary fact—the "totality gate"—is already formalized: because `Evidence` has incomparable elements, it admits no faithful order-embedding into the reals. Thus one should not expect a single real-valued map to capture all evidence information. See `Mettapedia/Logic/PLN_KS_Bridge.lean`.

7 Roadmap and Deliverables

The core Lean artifacts supporting ξPLN are:

- `Mettapedia/Logic/PLNWorldModel.lean`: the world-model interface and derived views;
- `Mettapedia/Logic/PLNJointEvidence.lean`: the reference complete instance and revision-commutes-with-extraction;
- `Mettapedia/Logic/PLNBayesNetWorldModel.lean`: BN-style CPT evidence states and the additive projection from `JointEvidence`;
- `Mettapedia/Logic/PLNBayesNetFastRules.lean`: chain BN infrastructure for proving fast-rule exactness under BN assumptions;
- `Mettapedia/Logic/PLNJointEvidenceProbability.lean`: posterior-mean probability views (ratio theorems);
- `Mettapedia/Logic/PLNJointEvidenceNoGo.lean`: the local-completeness no-go theorem.

Next implementation milestones:

1. extend the Bayesian-network sublayer with exact query answering (variable elimination / junction tree);
2. connect fast PLN rules as compiled tactics/lemmas relative to that BN class;
3. extend query language beyond atoms to formulas/events, enabling FO/HO semantics via SatisfyingSets while keeping probability views as world-model queries.

8 Conclusion

ξPLN formalizes the slogan "pass distributions, not just truth values" in a way that is precise enough to support soundness/completeness claims and flexible enough to support tractable sublayers. The complete layer is not a competing "second PLN"; it is the reference semantics that operational PLN rules approximate, under explicit assumptions, within declared model classes.

References

[1] Ben Goertzel, Matthew Iklé, Izabela Freire Goertzel, and Ari Heljakka. *Probabilistic Logic Networks: A Comprehensive Framework for Uncertain Inference*. Springer, New York, 2009.

[2] Kevin H. Knuth and John Skilling. Foundations of inference. *Axioms*, 1(1):38–73, 2012. arXiv:1008.4831.

[3] Andrey Nikolaevich Kolmogorov. *Grundbegriffe der Wahrscheinlichkeitsrechnung*. Springer, Berlin, 1933. English translation: Foundations of the Theory of Probability, Chelsea, 1956.