

Gallai-Ramsey number for K_5

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Abstract

Given a graph H , the k -colored Gallai Ramsey number $gr_k(K_3 : H)$ is defined to be the minimum integer n such that every k -coloring of the edges of the complete graph on n vertices contains either a rainbow triangle or a monochromatic copy of H . Fox et al. [J. Fox, A. Grinshpun, and J. Pach. The Erdős-Hajnal conjecture for rainbow triangles. *J. Combin. Theory Ser. B*, 111:75–125, 2015.] conjectured the value of the Gallai Ramsey numbers for complete graphs. Recently, this conjecture has been verified for the first open case, when $H = K_4$.

In this paper we attack the next case, when $H = K_5$. Surprisingly it turns out, that the validity of the conjecture depends upon the (yet unknown) value of the Ramsey number $R(5, 5)$. It is known that $43 \leq R(5, 5) \leq 48$ and conjectured that $R(5, 5) = 43$ [B.D. McKay and S.P. Radziszowski. Subgraph counting identities and Ramsey numbers. *J. Combin. Theory Ser. B*, 69:193–209, 1997]. If $44 \leq R(5, 5) \leq 48$, then Fox et al.’s conjecture is true and we present a complete proof. If, however, $R(5, 5) = 43$, then Fox et al.’s conjecture is false, meaning that at least one of these two conjectures must be false. For the case when $R(5, 5) = 43$, we show lower and upper bounds for the Gallai Ramsey number $gr_k(K_3 : K_5)$.

1 Introduction

Given a graph G and a positive integer k , the k -color Ramsey number $r_k(G)$ is the minimum number of vertices n such that every k -coloring of the edges of K_N for $N \geq n$ must contain a monochromatic copy of G . We refer to [11] for a dynamic survey of known Ramsey numbers. As a restricted version of the Ramsey number, the k -color Gallai-Ramsey number $gr_k(K_3 : G)$ is defined to be the minimum integer n such that every k -coloring of the edges of K_N for

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$N \geq n$ must contain either a rainbow triangle or a monochromatic copy of G . We refer to [3] for a dynamic survey of known Gallai-Ramsey numbers. In particular, the following was recently conjectured for complete graphs.

Conjecture 1 ([2]). *For $k \geq 1$ and $p \geq 3$,*

$$gr_k(K_3 : K_p) = \begin{cases} (r(p) - 1)^{k/2} + 1 & \text{if } k \text{ is even,} \\ (p - 1)(r(p) - 1)^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

The case where $p = 3$ was actually verified in 1983 by Chung and Graham [1]. A simplified proof was given by Gyárfás et al. [6].

Theorem 1 ([1]). *For $k \geq 1$,*

$$gr_k(K_3 : K_3) = \begin{cases} 5^{k/2} + 1 & \text{if } k \text{ is even,} \\ 2 \cdot 5^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

The next case, where $p = 4$, was proven in [7].

Theorem 2. *For $k \geq 1$,*

$$gr_k(K_3 : K_4) = \begin{cases} 17^{k/2} + 1 & \text{if } k \text{ is even,} \\ 3 \cdot 17^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Our main result is to essentially prove Conjecture 1 in the case where $p = 5$. This result is particularly interesting since $r(K_5, K_5)$ is still not known. Let $R = r(K_5, K_5) - 1$ and note that the known bounds on this Ramsey number give us $42 \leq R \leq 47$.

Theorem 3. *For any integer $k \geq 2$,*

$$gr_k(K_3 : K_5) = \begin{cases} R^{k/2} + 1 & \text{if } k \text{ is even,} \\ 4 \cdot R^{(k-1)/2} + 1 & \text{if } k \text{ is odd} \end{cases}$$

unless $R = 42$, in which case we have

$$\begin{cases} gr_k(K_3 : K_5) = 43 & \text{if } k = 2, \\ 42^{k/2} + 1 \leq gr_k(K_3 : K_5) \leq 43^{k/2} + 1 & \text{if } k \geq 4 \text{ is even,} \\ 169 \cdot 42^{(k-3)/2} + 1 \leq gr_k(K_3 : K_5) \leq 4 \cdot 43^{(k-1)/2} + 1 & \text{if } k \geq 3 \text{ is odd.} \end{cases}$$

Theorem 3 is proven in Section 4. Note that if $R = 43$, then Theorem 3 implies that Conjecture 1 is false.

Also recall the following well known conjecture about the sharp value for the 2-color Ramsey number of K_5 .

Conjecture 2 ([10]). *$R(K_5, K_5) = 43$.*

By Theorem 3, it turns out that at least one of Conjecture 1 or Conjecture 2 must be false.

In order to prove Theorem 3, we actually prove a more refined version, stated in Theorem 4. Note that Theorem 3 follows from Theorem 4 by setting $r = k$, $s = 0$ and $t = 0$.

To simplify the notation, we let c_1 denote the case where r, s, t are all even, c_2 denote the case where r, s are both even and t is odd, and so on for c_3, \dots, c_{11} .

Theorem 4. *For nonnegative integers r, s, t , let $k = r + s + t$. Then*

$$gr_k(K_3 : rK_5, sK_4, tK_3) = \begin{cases} R^{r/2} \cdot 17^{s/2} \cdot 5^{t/2} + 1 & \text{if } r, s, t \text{ are even, } (c_1) \\ 2 \cdot R^{r/2} \cdot 17^{s/2} \cdot 5^{(t-1)/2} + 1 & \text{if } r, s \text{ are even, and } t \text{ is odd, } (c_2) \\ 3 \cdot R^{r/2} \cdot 17^{(s-1)/2} + 1 & \text{if } r \text{ is even, } s \text{ is odd, and } t = 0, (c_3) \\ 4 \cdot R^{(r-1)/2} + 1 & \text{if } r \text{ is odd, and } s = t = 0, (c_4) \\ 8 \cdot R^{r/2} \cdot 17^{(s-1)/2} \cdot 5^{(t-1)/2} + 1 & \text{if } r \text{ is even, and } s, t \text{ are odd, } (c_5) \\ 13 \cdot R^{(r-1)/2} \cdot 17^{s/2} \cdot 5^{(t-1)/2} + 1 & \text{if } r, t \text{ are odd, and } s \text{ is even, } (c_6) \\ 16 \cdot R^{r/2} \cdot 17^{(s-1)/2} \cdot 5^{(t-2)/2} + 1 & \text{if } r, t \text{ are even, } t \geq 2, \text{ and } s \text{ is odd, } (c_7) \\ 24 \cdot R^{(r-1)/2} \cdot 17^{(s-1)/2} \cdot 5^{t/2} + 1 & \text{if } r, s \text{ are odd, and } t \text{ is even, } (c_8) \\ 26 \cdot R^{(r-1)/2} \cdot 17^{s/2} \cdot 5^{(t-2)/2} + 1 & \text{if } r \text{ is odd, } s \text{ is even, } t \geq 2 \text{ is even, } (c_9) \\ 48 \cdot R^{(r-1)/2} \cdot 17^{(s-1)/2} \cdot 5^{(t-1)/2} + 1 & \text{if } r, s, t \text{ are odd, } (c_{10}) \\ 72 \cdot R^{(r-1)/2} \cdot 17^{(s-2)/2} + 1 & \text{if } r \text{ is odd, } t = 0, \text{ and } s \geq 2 \text{ is even. } (c_{11}) \end{cases}$$

For ease of notation, let $g(r, s, t)$ be the value of $gr_k(K_3 : rK_5, sK_4, tK_3)$ claimed above. Also, for each i with $1 \leq i \leq 11$, let $g_i(r, s, t) = g(r, s, t) - 1$ in the case where (c_i) holds.

2 Preliminaries

In this section, we recall some known results and provide several helpful lemmas that will be used in the proof. First we state the main tool for looking at colored complete graphs with no rainbow triangle.

Theorem 5 ([4]). *In any coloring of a complete graph containing no rainbow triangle, there exists a nontrivial partition of the vertices (called a Gallai-partition) such that there are at most two colors on the edges between the parts and only one color on the edges between each pair of parts.*

In light of this result, a colored complete graph with no rainbow triangle is called a *Gallai coloring* and the partition resulting from Theorem 5 is called a *Gallai partition*.

Next recall some useful Ramsey numbers.

Theorem 6 ([5]).

$$R(K_3, K_5) = 14.$$

Theorem 7 ([9]).

$$R(K_4, K_5) = 25.$$

Also a general lower bound for Gallai-Ramsey numbers, a special case of the main result in [8]. We will present a more refined construction later for the purpose of proving Theorem 4.

Lemma 1 ([8]). *For a connected complete graph H of order n and an integer $k \geq 2$, we have*

$$gr_k(K_3 : H) \geq \begin{cases} (R(H, H) - 1)^{k/2} + 1 & \text{if } k \text{ is even,} \\ (n - 1) \cdot (R(H, H) - 1)^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

We next present several tables of values which concisely capture computations that will be used throughout the proof. Each cell contains the ratio of the corresponding type $g(r_1, s_1, t_1)$ in relation to the order of the whole graph $g(r, s, t)$ in the given case. For example, the top left cell of Table 1 contains the value of the ratio

$$\frac{g(r, s, t - 1)}{g(r, s, t)}$$

in the case (c_1) .

Each row of the following tables represents a case (perhaps with some sub-cases) and each column represents a Type, one of the referenced inequalities listed above it. In some cells containing two values, these values correspond to the extra assumptions listed in the far right column. The cases marked with $-$ do not occur because of base assumptions. The maximum value in each column yields an upper bound on the ratio for that type over all the cases, and these are displayed in Inequalities (1)-(22).

Table 1 contains the case analysis for the following inequalities:

$$\text{Type T1: } \frac{g(r, s, t-1)}{g(r, s, t)} \leq \frac{1}{2}, \quad (1)$$

$$\text{Type T2: } \frac{g(r, s, t-2)}{g(r, s, t)} \leq \frac{1}{5}, \quad (2)$$

$$\text{Type T3: } \frac{g(r, s-1, t+1)}{g(r, s, t)} \leq \frac{2}{3}, \quad (3)$$

$$\text{Type T4: } \frac{g(r, s-1, t)}{g(r, s, t)} \leq \frac{1}{3}, \quad (4)$$

$$\text{Type T5: } \frac{g(r, s-1, t-1)}{g(r, s, t)} \leq \frac{1}{8}, \quad (5)$$

$$\text{Type T6: } \frac{g(r, s-2, t+2)}{g(r, s, t)} \leq \frac{13}{36}. \quad (6)$$

Case	(1)T1	(2)T2	(3)T3	(4)T4	(5)T5	(6)T6	
(c_1)	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{8}{17}$	$\frac{3}{17}$ $\frac{16}{85}$	$\frac{8}{85}$	$\frac{5}{17}$	$t = 0$ $t \geq 2$
(c_2)	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{8}{17}$	$\frac{4}{17}$	$\frac{3}{34}$ $\frac{8}{85}$	$\frac{5}{17}$	$t = 1$ $t \geq 3$
(c_3)	—	—	$\frac{2}{15}$	$\frac{1}{3}$	—	$\frac{16}{51}$	
(c_4)	—	—	—	—	—	—	
(c_5)	$\frac{3}{8}$ $\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{5}{17}$	$t = 1$ $t \geq 3$
(c_6)	$\frac{4}{13}$ $\frac{2}{5}$ $\frac{72}{221}$	$\frac{1}{5}$	$\frac{24}{65}$	$\frac{48}{221}$	$\frac{24}{221}$	$\frac{5}{17}$	$t = 1$ $s = 0$ $t \geq 3$ $t = 1$ $s > 2$
(c_7)	$\frac{1}{2}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{5}{16}$	$\frac{1}{8}$	$\frac{5}{17}$	
(c_8)	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{13}{24}$	$\frac{3}{17}$ $\frac{13}{60}$	$\frac{13}{120}$	$\frac{5}{17}$	$t = 0$ $t \geq 2$
(c_9)	$\frac{1}{2}$	$\frac{2}{13}$	$\frac{24}{221}$	$\frac{60}{221}$	—	$\frac{5}{17}$	
(c_{10})	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{13}{120}$	$\frac{13}{48}$	$\frac{8}{85}$ $\frac{13}{120}$ $\frac{3}{4}$ $\frac{34}{12}$	$\frac{5}{17}$	$s, t \geq 3$ $s = 1$ $t > 3$ $s \geq 3$ $t = 1$ $s = t = 1$
(c_{11})	—	—	$\frac{2}{3}$	$\frac{1}{3}$	—	$\frac{13}{36}$	
Max	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{8}$	$\frac{13}{36}$	

Table 1: Types T1 - T6.

Table 2 contains the case analysis for the following inequalities:

$$\text{Type T7: } \frac{g(r, s-2, t+1)}{g(r, s, t)} \leq \frac{13}{72}, \quad (7)$$

$$\text{Type T8: } \frac{g(r, s-2, t)}{g(r, s, t)} \leq \frac{1}{17}, \quad (8)$$

$$\text{Type T9: } \frac{g(r-1, s+1, t)}{g(r, s, t)} \leq \frac{3}{4}, \quad (9)$$

$$\text{Type T10: } \frac{g(r-1, s+1, t-1)}{g(r, s, t)} \leq \frac{17}{48}, \quad (10)$$

$$\text{Type T11: } \frac{g(r-1, s, t+1)}{g(r, s, t)} \leq \frac{5}{13}, \quad (11)$$

$$\text{Type T12: } \frac{g(r-1, s, t)}{g(r, s, t)} \leq \frac{5}{26}. \quad (12)$$

Case	(7)	(8)	(9)	(10)	(11)	(12)	
(c_1)	$\frac{2}{17}$	$\frac{1}{17}$	$\frac{24}{R}$	$\frac{48}{5R}$	$\frac{13}{R}$	$\frac{26}{5R}$	
(c_2)	$\frac{5}{34}$	$\frac{1}{17}$	$\frac{24}{R}$	$\frac{48}{5R}$	$\frac{13}{R}$	$\frac{26}{5R}$	
(c_3)	$\frac{8}{51}$	$\frac{1}{17}$	$\frac{24}{R}$	—	—	—	
(c_4)	—	—	$\frac{3}{4}$	—	—	—	
(c_5)	$\frac{2}{17}$	$\frac{1}{17}$	$\frac{221}{8R}$	$\frac{9}{R}$ $\frac{221}{20R}$	$\frac{15}{R}$	$\frac{6}{R}$	$t = 1$ $t \geq 3$
(c_6)	$\frac{2}{17}$	$\frac{1}{17}$	$\frac{8}{13}$	$\frac{3}{13}$ $\frac{16}{65}$	$\frac{5}{13}$	$\frac{2}{13}$	$t = 1$ $t \geq 3$
(c_7)	$\frac{5}{34}$	$\frac{1}{17}$	$\frac{221}{8R}$	$\frac{221}{16R}$	$\frac{15}{R}$	$\frac{15}{2R}$	
(c_8)	$\frac{2}{17}$	$\frac{1}{17}$	$\frac{17}{24}$	$\frac{17}{60}$	$\frac{1}{3}$	$\frac{2}{15}$	
(c_9)	$\frac{5}{34}$	$\frac{1}{17}$	$\frac{8}{13}$	$\frac{4}{13}$	$\frac{5}{13}$	$\frac{5}{26}$	
(c_{10})	$\frac{5}{34}$	$\frac{1}{17}$	$\frac{17}{24}$	$\frac{17}{48}$	$\frac{1}{15}$	$\frac{1}{6}$	
(c_{11})	$\frac{13}{72}$	$\frac{1}{18}$	$\frac{17}{24}$	—	—	—	$s = 2$
Max	$\frac{13}{72}$	$\frac{1}{17}$	$\frac{3}{4}$	$\frac{17}{48}$	$\frac{5}{13}$	$\frac{5}{26}$	

Table 2: Types T7 - T12.

Table 3 contains the case analysis for the following inequalities:

$$\text{Type T13: } \frac{g(r-1, s, t-1)}{g(r, s, t)} \leq \frac{1}{13}, \quad (13)$$

$$\text{Type T14: } \frac{g(r-1, s-1, t+2)}{g(r, s, t)} \leq \frac{5}{24}, \quad (14)$$

$$\text{Type T15: } \frac{g(r-1, s-1, t+1)}{g(r, s, t)} \leq \frac{1}{9}, \quad (15)$$

$$\text{Type T16: } \frac{g(r-1, s-1, t)}{g(r, s, t)} \leq \frac{1}{24}, \quad (16)$$

$$\text{Type T17: } \frac{g(r-2, s+2, t)}{g(r, s, t)} \leq \frac{18}{R}, \quad (17)$$

$$\text{Type T18: } \frac{g(r-2, s+1, t+1)}{g(r, s, t)} \leq \frac{12}{R}. \quad (18)$$

Case	(13)	(14)	(15)	(16)	(17)	(18)	
(c_1)	$\frac{13}{5R}$	$\frac{120}{17R}$	$\frac{48}{17R}$	$\frac{24}{17R}$	$\frac{17}{R}$	$\frac{8}{R}$	
(c_2)	$\frac{13}{5R}$ $\frac{36}{17R}$ $\frac{2}{R}$	$\frac{120}{17R}$	$\frac{60}{17R}$	$\frac{24}{17R}$	$\frac{17}{R}$	$\frac{8}{R}$	$t \geq 1$ $t = 1, s \geq 2$ $t - 1 = s = 0$
(c_3)	—	$\frac{120}{17R}$ $\frac{20}{3R}$	$\frac{13}{3R}$	$\frac{24}{17R}$ $\frac{4}{3R}$	$\frac{17}{R}$	$\frac{34}{3R}$	$s \geq 3$ $s = 1$
(c_4)	—	—	—	—	$\frac{18}{R}$	$\frac{12}{R}$	
(c_5)	$\frac{3}{R}$	$\frac{65}{8R}$	$\frac{13}{4R}$	$\frac{13}{8R}$	$\frac{17}{R}$	$\frac{85}{8R}$	
(c_6)	$\frac{1}{13}$	$\frac{40}{221}$	$\frac{16}{221}$	$\frac{8}{221}$	$\frac{17}{R}$	$\frac{120}{13R}$	
(c_7)	$\frac{3}{R}$	$\frac{65}{8R}$	$\frac{65}{16R}$	$\frac{13}{8R}$	$\frac{17}{R}$	$\frac{85}{8R}$	
(c_8)	$\frac{15}{13}$	$\frac{24}{40}$	$\frac{12}{20}$	$\frac{24}{8}$	$\frac{17}{R}$	$\frac{24}{20}$	
(c_9)	$\frac{1}{13}$	$\frac{221}{221}$	$\frac{221}{221}$	$\frac{221}{221}$	$\frac{17}{R}$	$\frac{13}{13R}$	
(c_{10})	$\frac{1}{15}$ $\frac{1}{16}$	$\frac{5}{24}$	$\frac{5}{48}$	$\frac{1}{24}$	$\frac{17}{R}$	$\frac{221}{24R}$	$t \geq 1$ $t = 1$
(c_{11})	—	$\frac{5}{24}$	$\frac{1}{9}$	$\frac{1}{24}$	$\frac{17}{R}$	$\frac{34}{3R}$	
Max	$\frac{1}{13}$	$\frac{5}{24}$	$\frac{1}{9}$	$\frac{1}{24}$	$\frac{18}{R}$	$\frac{12}{R}$	

Table 3: Types T13 - T18.

Table 4 contains the case analysis for the following inequalities:

$$\text{Type T19: } \frac{g(r-2, s+1, t)}{g(r, s, t)} \leq \frac{6}{R}, \quad (19)$$

$$\text{Type T20: } \frac{g(r-2, s, t+2)}{g(r, s, t)} \leq \frac{13}{2R}, \quad (20)$$

$$\text{Type T21: } \frac{g(r-2, s, t+1)}{g(r, s, t)} \leq \frac{13}{4R}, \quad (21)$$

$$\text{Type T22: } \frac{g(r-2, s, t)}{g(r, s, t)} \leq \frac{1}{R}. \quad (22)$$

Case	(19)	(20)	(21)	(22)	
	$\frac{16}{5R}$	$\frac{5}{R}$	$\frac{2}{R}$	$\frac{1}{R}$	$t \geq 2$
	$\frac{3}{R}$				$t = 0$
(c_1)	$\frac{3}{R}$	$\frac{5}{R}$	$\frac{2}{R}$	$\frac{1}{R}$	
(c_2)	$\frac{4}{R}$	$\frac{5}{R}$	$\frac{5}{2R}$	$\frac{1}{R}$	
(c_3)	$\frac{17}{3R}$	$\frac{16}{3R}$	$\frac{8}{3R}$	$\frac{1}{R}$	
(c_4)	$\frac{6}{R}$	$\frac{13}{2R}$	$\frac{13}{4R}$	$\frac{1}{R}$	
(c_5)	$\frac{17}{4R}$	$\frac{5}{R}$	$\frac{2}{R}$	$\frac{1}{R}$	
(c_6)	$\frac{48}{13R}$	$\frac{5}{R}$	$\frac{2}{R}$	$\frac{1}{R}$	
(c_7)	$\frac{85}{16R}$	$\frac{3}{R}$	$\frac{1}{2R}$	$\frac{1}{R}$	
(c_8)	$\frac{221}{60R}$	$\frac{5}{R}$	$\frac{2}{R}$	$\frac{1}{R}$	$t \geq 2$
	$\frac{3}{R}$				$t = 0$
(c_9)	$\frac{60}{13R}$	$\frac{5}{R}$	$\frac{1}{2R}$	$\frac{1}{R}$	
(c_{10})	$\frac{221}{48R}$	$\frac{5}{R}$	$\frac{1}{2R}$	$\frac{1}{R}$	
(c_{11})	$\frac{17}{3R}$	$\frac{221}{36R}$	$\frac{221}{72R}$	$\frac{1}{R}$	
Max	$\frac{6}{R}$	$\frac{13}{2R}$	$\frac{13}{4R}$	$\frac{1}{R}$	

Table 4: Types T19 - T22.

Next we provide several lemmas specific to the proof of Theorem 3 but first some definitions.

We call a part X of a Gallai partition *free*, if it contains neither red nor blue edges. We call a part *red* (*blue*) if it contains red (respectively blue) edges, but no red (blue) copy of a K_3 , and no blue (red) edges. Note that these notations do not characterize all parts since clearly a part X might fall into none of these categories.

Let H be a Gallai colored complete graph where red and blue are the colors appearing on edges of the reduced graph. We call such a graph (or part of the partition) H a (R_i, B_j) -graph if it contains neither a red copy of K_i nor a blue copy of K_j .

Let $w_{i,j}(H) = \frac{|H|}{|G|}$ be the *weight* of H as a subgraph of an (R_i, B_j) -graph G . For convenience, when a part A of a Gallai partition of H is assumed, let H_R (and H_B) denote the sets of vertices in $H \setminus A$ with all red (respectively blue) edges to A .

For (R_3, B_3) -graphs, we get the following.

Lemma 2. *Let H be an (R_3, B_3) -graph, whose parts are either free, red, or blue. Then $w_{5,5}(H) \leq \frac{6.5}{R}$.*

Proof. In order to avoid a red or blue triangle, the graph H has $t \leq 5 = R(3, 3) - 1$ parts. If all parts are free, then $w_{5,5}(H) \leq \frac{t}{R} \leq \frac{5}{R}$ by Inequality (22). Suppose H has a red part (and note that a symmetric argument also works for a blue part). Then H_R is empty and H_B contains no blue edges. If all parts of H_B are free, then $w_{5,5}(H_B) \leq \frac{2}{R}$ by Inequality (22) since there can be at most two parts in H_B (with all red edges in between them). On the other hand, if H_B contains a red part, then $w_{5,5}(H_B) \leq \frac{3.25}{R}$ by Inequality (21) since H consists of two red parts joined by blue edges. In either case, we have $w_{5,5}(H) \leq \frac{6.5}{R}$. \square

For (R_3, B_4) -graphs, we get the following.

Lemma 3. *Let H be a (R_3, B_4) -graph, whose parts are either free, red, or blue. Then*

- (i) $w_{5,5}(H) \leq \frac{9.75}{R}$, and furthermore
- (ii) if H contains no red part, then $w_{5,5}(H) \leq \frac{9.5}{R}$.

Proof. Since H is an (R_3, B_4) -graph, H has $t \leq 8 = R(3, 4) - 1$ parts. If all parts are free, then $w_{5,5}(H) \leq \frac{t}{R} \leq \frac{8}{R}$ by Inequality (22). Suppose first that H contains no red parts. Let X_1 be a blue part, so $w_{5,5}(X_1) = \frac{3.25}{R}$ by Inequality (21). Then H_R is an (R_2, B_4) -graph and H_B is an (R_3, B_2) -graph. Similar arguments as in the proof of Lemma 2 lead to

$$w_{5,5}(H_B) \leq \max \left\{ 3 \cdot \frac{1}{R}, \frac{3.25}{R} + \frac{1}{R} \right\} = \frac{4.25}{R}$$

and

$$w_{5,5}(H_R) \leq 2 \cdot \frac{1}{R} = \frac{2}{R}$$

since there is no red part. This gives

$$w_{5,5}(H) \leq \frac{1}{R} (3.25 + 4.25 + 2) = \frac{9.5}{R}.$$

Now suppose that H contains a red part X_1 . Then H_R is empty and H_B is an (R_3, B_3) -graph. By Inequality (21) and Lemma 2, we obtain

$$w_{5,5}(H) \leq w_{5,5}(X_1) + w_{5,5}(H_B) \leq \frac{1}{R} (3.25 + 6.5) = \frac{9.75}{R}.$$

□

The sharpness of Lemma 3 is given by the following examples:

- (1) Three red parts joined by blue edges
- (2) Two blue parts joined by red edges, which are joined by blue edges with a red part

For (R_5, B_3) -graphs, we get the following.

Lemma 4. *Let H be an (R_5, B_3) -graph, whose parts are either free, red or blue. Then*

- (i) $w_{5,5}(H) \leq \frac{13}{R}$ if H contains only free parts;
- (ii) $w_{5,5}(H) \leq \frac{13}{R}$ if H contains at least one blue part;
- (iii) $w_{5,5}(H) \leq \frac{12.25}{R}$ if H contains exactly one red part;
- (iv) $w_{5,5}(H) \leq \frac{14.5}{R}$ if H contains at least two red parts but no two red parts joined by blue edges;
- (v) $w_{5,5}(H) \leq \frac{13.5}{R}$ if H contains exactly two red parts and they are joined by blue edges;
- (vi) $w_{5,5}(H) \leq \frac{16.25}{R}$.

Proof. Since H is an (R_5, B_3) -graph, H has $t \leq 13 = R(5, 3) - 1$ parts. If all parts are free, then $w_{5,5}(H) \leq \frac{t}{R} \leq \frac{13}{R}$ by Inequality (22). This proves (i) and means that we may assume that H contains at least one red or blue part.

Suppose first that H contains a blue part X_1 , so $w_{5,5}(X_1) \leq \frac{3.25}{R}$ by Inequality (21). Then H_B is empty and H_R is an (R_4, B_3) -graph. By Lemma 3 (and symmetry of red and blue) we obtain

$$w_{5,5}(H) = w_{5,5}(X_1) + w_{5,5}(H_R) \leq \frac{1}{R} (3.25 + 9.75) = \frac{13}{R}.$$

This shows (ii) and means that we may assume H contains no blue parts for the remainder of the proof.

Suppose next that H contains exactly one red part X_1 . Then H_R has at most $R(3, 3) - 1 = 5$ parts, H_B has at most $R(5, 2) - 1 = 4$ parts, and each of these parts must be free. By Inequalities (21) and (22), this gives

$$w_{5,5}(H) \leq \frac{1}{R} (3.25 + 5 + 4) = \frac{12.25}{R},$$

confirming (iii).

Next suppose there are at least two red parts X_1 and X_2 but no two red parts joined by blue edges. With only red edges between the red parts, there can only be two such parts. Then H_R (with respect to X_1) contains only free parts other than X_2 and H_B also contains only free parts. As in (iii), H_R has at most $R(3, 3) - 1 = 5$ parts, H_B has at most $R(5, 2) - 1 = 4$ parts. By Inequalities (21) and (22), this gives

$$\begin{aligned} w_{5,5}(H) &= w_{5,5}(X_1) + w_{5,5}(X_2) + w_{5,5}(H_R \setminus X_2) + w_{5,5}(H_B) \\ &\leq \frac{1}{R} (3.25 + 3.25 + 4 + 4) \\ &= \frac{14.5}{R}, \end{aligned}$$

confirming (iv).

Next suppose there are exactly two red parts X_1 and X_2 and they are joined by blue edges. Then H_R (with respect to X_1) contains only free parts and at most $R(3, 3) - 1 = 5$ of them and H_B contains only free parts other than X_2 but no blue edges at all so there can be at most 3 total parts in H_B . By Inequalities (21) and (22), this means

$$\begin{aligned} w_{5,5}(H) &= w_{5,5}(X_1) + w_{5,5}(H_R) + w_{5,5}(X_2) + w_{5,5}(H_B \setminus X_2) \\ &\leq \frac{1}{R} (3.25 + 5 + 3.25 + 2) \\ &= \frac{13.5}{R}, \end{aligned}$$

confirming (v).

Finally let X_1 be a red part. Then H_B contains no blue edges so it contains at most 4 free parts, one red part and at most 2 free parts, or two red parts. By Inequalities (21) and (22), this means that

$$w_{5,5}(H_B) \leq \max \left\{ \frac{4}{R}, \frac{3.25 + 2}{R}, \frac{2 \cdot 3.25}{R} \right\} = \frac{6.5}{R}.$$

On the other side, H_R contains no red or blue triangle so it has at most 5 free parts, one red part and at most 2 free parts, or two red parts (joined by blue edges). By Inequalities (21) and (22), this means that

$$w_{5,5}(H_R) \leq \max \left\{ \frac{5}{R}, \frac{3.25 + 2}{R}, \frac{2 \cdot 3.25}{R} \right\} = \frac{6.5}{R}.$$

Finally, we have

$$w_{5,5}(H) \leq w_{5,5}(X_1) + w_{5,5}(H_B) + w_{5,5}(H_R) \leq \frac{3.25 + 6.5 + 6.5}{R} = \frac{16.25}{R},$$

confirming (vi). \square

Lemma 5. *Let H be an (R_3, B_3) -graph, whose parts are either free, red or blue. Then $w_{4,5}(H) \leq \frac{2}{9}$.*

Proof. To avoid a red or blue triangle, H must have at most $R(3, 3) - 1 = 5$ parts. If all these parts are free, then by Inequality (16), we have $w_{4,5}(H) \leq \frac{5}{24}$. If H has a red or blue part X_1 , say red, then H_R is empty and H_B contains no blue edges. If all parts in H_B are free, then $w_{4,5}(H_B) \leq \frac{1}{12}$ by Inequality (16). On the other hand, if H_B contains a red part, then $w_{4,5}(H_B) \leq \frac{1}{9}$ by Inequality (15). In either case, we have $w_{4,5}(H) \leq \frac{2}{9}$. \square

Lemma 6. *Let H be an (R_3, B_4) -graph, whose parts are either free, red or blue. Then*

- (i) $w_{4,5}(H) \leq \frac{25}{72}$ if H contains exactly two blue parts, no red part, and the reduced graph of H is the unique 2-coloring of K_5 containing no monochromatic triangle (see Figure 1), or
- (ii) $w_{4,5}(H) \leq \frac{1}{3}$ otherwise.

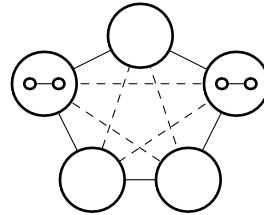


Figure 1: The structure of H with solid edges being blue and dashed edges being red

Proof. This proof is very similar to the proof of Lemma 3. Since H is an (R_3, B_4) -graph, it must have $t \leq 8 = R(3, 4) - 1$ parts. If all parts are free, then by Inequality (16), we have $w_{4,5}(H) \leq \frac{t}{24} \leq \frac{8}{24} = \frac{1}{3}$.

Suppose first that H contains no red parts but does contain at least one blue part. Let X_1 be a blue part, so $w_{4,5}(X_1) = \frac{1}{9}$ by Inequality (15). Then H_R is an (R_2, B_4) -graph and H_B is an (R_3, B_2) -graph. Since H_B contains no blue edge and there are no red parts, H_B must contain at most 2 parts and these must be free so $w_{4,5}(H_B) \leq \frac{2}{24}$ by Inequality (16). Since H_R contains no red edge, there can be either at most three parts in H_R that are all free (with all blue edges in

between them), or one blue part and one free part. By Lemmas (15) and (16), this means that

$$w_{4,5}(H_R) \leq \max \left\{ 3 \cdot \frac{1}{24}, \frac{1}{9} + \frac{1}{24} \right\} = \frac{11}{72}.$$

Putting these together, we get

$$w_{4,5}(H) \leq \frac{1}{72} (8 + 6 + 11) = \frac{25}{72}.$$

In fact, this bound is only achievable if H_R contains one blue part and one free part since otherwise $w_{4,5}(H_R) \leq \frac{1}{8} = \frac{9}{72}$ meaning that $w_{4,5}(H) \leq \frac{23}{72} < \frac{1}{3}$. Therefore, in order for $w_{4,5}(H) > \frac{1}{3}$, H must have the structure pictured in Figure 1.

Finally suppose H contains a red part X_1 . Then H_R is empty and H_B is an (R_3, B_3) -graph. By Inequality (15) and Lemma 5, we obtain

$$w_{4,5}(H) \leq w_{4,5}(X_1) + w_{4,5}(H_B) \leq \frac{1}{9} + \frac{2}{9} = \frac{1}{3}.$$

□

Lemma 7. *Let H be an (R_5, B_3) -graph, whose parts are either free, red or blue. Then*

- (i) $w_{4,5}(H) \leq \frac{5}{9}$ if H consists of 5 red parts and the reduced graph of H is the unique 2-coloring of K_5 with no monochromatic triangle, or
- (ii) $w_{4,5}(H) \leq \frac{39}{72}$ otherwise.

Proof. Since H is an (R_5, B_3) -graph, there are at most $R(5, 3) - 1 = 13$ parts in H . If all these parts are free, then $w_{4,5}(H) \leq \frac{13}{24} = \frac{39}{72}$ by Inequality (16).

First suppose H contains a blue part X_1 . Then H_B is empty and H_R is an (R_4, B_3) -graph so by Inequality (15) and Lemma 6, we get

$$w_{4,5}(H) \leq w_{4,5}(X_1) + w_{4,5}(H_R) \leq \frac{1}{9} + \frac{25}{72} = \frac{33}{72}.$$

We may therefore assume H contains no blue part.

If H contains 5 red parts as described in the statement then clearly $w_{4,5}(H) \leq \frac{5}{9}$ so suppose this is not the case. That is, suppose H contains a red part X_1 and at least one free part, so H_B (defined in terms of X_1) contains no blue edges and H_R is an (R_3, B_3) -graph. Then H_B either contains at most 4 parts that are all free (with all red edges in between them), or one red part and two free parts, or two red parts. By Inequalities (15) and (16), this means

$$w_{4,5}(H_B) \leq \max \left\{ \frac{4}{24}, \frac{1}{9} + \frac{2}{24}, \frac{2}{9} \right\} = \frac{2}{9}.$$

If H_B is not two red parts, then $w_{4,5}(H_B) \leq \frac{14}{72}$. This together with Lemma 5 gives

$$w_{4,5}(H) \leq w_{4,5}(X_1) + w_{4,5}(H_B) + w_{4,5}(H_R) \leq \frac{1}{9} + \frac{14}{72} + \frac{2}{9} = \frac{38}{72}.$$

Thus, suppose H_B has two red parts (so $w_{4,5}(H_B) \leq \frac{2}{9}$) and H_R does not have two red parts. By the proof of Lemma 5, we have $w_{4,5}(H_R) \leq \frac{5}{24}$. Putting these together, we get

$$w_{4,5}(H) \leq w_{4,5}(X_1) + w_{4,5}(H_B) + w_{4,5}(H_R) \leq \frac{1}{9} + \frac{2}{9} + \frac{5}{24} = \frac{39}{72}.$$

□

3 Three Colors

In this section, we discuss a lower bound example that leads to a counterexample to either Conjecture 1 or Conjecture 2.

Lemma 8. *There exists a 3-colored copy of K_{169} which contains no rainbow triangle and no monochromatic copy of K_5 .*

Proof. Let G_{rb} be a sharpness example on 13 vertices for the Ramsey number $R(K_3, K_5) = 14$ say using colors red and blue respectively. Such an example as G_{rb} is 4-regular in red and 8-regular in blue. Similarly, let G_{rg} be a copy of the same graph with all blue edges replaced by green edges. We construct the desired graph G by making 13 copies of each vertex in G_{rb} and for each set of copies (corresponding to a vertex), insert a copy of G_{rg} . If an edge uv in G_{rb} is red (respectively blue), then all edges in G between the two inserted copies of G_{rg} corresponding to u and v are colored red (respectively blue). Then G contains no rainbow triangle by construction but also contains no monochromatic K_5 . Since $|G| = 169$, this provides the desired example. □

Note that if $R(K_5, K_5) = 43$ so $R = 42$, then Conjecture 1 claims that $gr_3(K_3 : K_5) = 169$ but this example refutes this claim. On the other hand, if $R(K_5 : K_5) > 43$, then the conjecture holds for K_5 , as proven in Section 4 below.

4 Proof of Theorem 4 (and Theorem 3)

Note that the lower bound for Theorem 3 follows from Lemma 1 and was also presented in [2] but the lower bound for Theorem 4 must be more detailed.

Proof. For the lower bounds, use the following constructions. For all constructions, we start with an i -colored base graph G_i (constructed below) and inductively suppose we have constructed an i -colored graph G_i containing no rainbow

triangle and no appropriately colored monochromatic cliques. For each two unused colors requiring a K_5 , we construct G_{i+2} by making R copies of G_i , adding all edges in between the copies to form a blow-up of a sharpness example for $r(K_5, K_5)$ on R vertices. For each two unused colors requiring a K_4 , we construct G_{i+2} by making 17 copies of G_i , adding all edges in between the copies to form a blow-up of a sharpness example for $r(K_4, K_4)$ on 17 vertices. For each two unused colors requiring a K_3 , we construct G_{i+2} by making 5 copies of G_i , adding all edges in between the copies to form a blow-up of the sharpness example for $r(K_3, K_3)$ on 5 vertices.

The base graphs for this construction are constructed by case as follows. For Case (c_1) , the base graph G_0 is a single vertex. For Case (c_2) , the base graph G_1 is a monochromatic copy of K_2 . For Case (c_3) , the base graph G_1 is a monochromatic copy of K_3 . For Case (c_4) , the base graph G_1 is a monochromatic K_4 . For Case (c_5) , the base graph G_2 is a sharpness example on 8 vertices for $r(K_3, K_4) = 9$. For Case (c_6) , the base graph G_2 is a sharpness example on 13 vertices for $r(K_3, K_5) = 14$. For Case (c_7) , the base graph G_3 is two copies of a sharpness example for $r(K_3, K_4) = 9$ with all edges in between the copies having a third color. For Case (c_8) , the base graph G_2 is a sharpness example on 24 vertices for $r(K_4, K_5) = 25$. For Case (c_9) , the base graph G_3 is two copies of a sharpness example on 13 vertices for $r(K_3, K_5) = 14$. For Case (c_{10}) , the base graph G_3 is two copies of a sharpness example on 24 vertices for $r(K_4, K_5) = 25$ with all edges in between the copies having a third color. For Case (c_{11}) , the base graph G_3 is three copies of a sharpness example on 24 vertices for $r(K_4, K_5) = 25$ with all edges in between the copies having a third color. These base graphs and the corresponding completed constructions contain no rainbow triangle and no appropriately colored monochromatic cliques.

For the upper bound, let G be a Gallai coloring of K_n where n is given in the statement. We prove this result by induction on $3r + 2s + t$, meaning that it suffices to either reduce the order of a desired monochromatic subgraph or eliminate a color. Consider a Gallai partition of G and let q be the number of parts in this partition. Choose such a partition so that q is minimized.

Claim 1. *We may assume that $q \geq 4$.*

Proof. For a contradiction, suppose $q \leq 3$. If $q = 3$, then the reduced graph is a 2-colored triangle, which contains two edges of the same color. This means that there is a bipartition of the vertices so that all edges in between have one color, contradicting the minimality of q . Thus, assume $q = 2$. Let red be the color between the two sets, A and B .

First suppose that red is among the last t colors, so we hope to find a red triangle. To avoid a red triangle, there must be no red edges within A or B . By induction on $3r + 2s + t$ and using Inequality (1), we get

$$|G| = |A| + |B| \leq 2[g(r, s, t - 1)] = g(r, s, t) < |G|,$$

a contradiction.

Next suppose that red is among the middle s colors, so we hope to find a red K_4 . To avoid a red K_4 , only one of A or B can have any red edges. Suppose A is allowed to have red edges so B is not. Then observe that A cannot contain a red triangle as this would also create a red K_4 . Thus, by induction on $3r+2s+t$ and using Inequalities (3) and (4) respectively, we get

$$|G| = |A| + |B| \leq g(r, s-1, t+1) + g(r, s-1, t) < g(r, s, t) < |G|,$$

a contradiction.

Finally suppose red is among the first r colors, so we hope to find a red K_5 . Supposing that the red clique number within A is at least as large as the red clique number within B , we get the following requirements:

- A contains no red K_4 , and
- if A contains a red K_3 , then B contains no red edges.

These leave only two options:

1. A and B both may contain red edges but no red K_3 , or
2. A contains a red K_3 (but no red K_4) and B contains no red edges.

For the first option, we remove 1 from r but add 1 to t within both A and B . By induction on $3r+2s+t$ and using Inequality (11), we get

$$|G| = |A| + |B| \leq 2g(r-1, s, t+1) < g(r, s, t) < |G|,$$

a contradiction.

For the second option, we remove 1 from r in both A and B but add 1 to s in A . By induction on $3r+2s+t$ and using Inequalities (9) and (12), we get

$$|G| = |A| + |B| \leq g(r-1, s+1, t) + g(r-1, s, t) < g(r, s, t) < |G|,$$

a contradiction. This completes the proof of Claim 1. \square

Let D be the reduced graph of the Gallai partition, with vertices w_i corresponding to parts G_i of the partition. Let V_r denote the set of vertices in D whose corresponding sets in the partition contain at least one red edge and let V_b denote the set of vertices in D whose corresponding sets in the partition contain at least one blue edge. Let $p_2 = |V_r \cap V_b|$ be the number of parts containing at least one red and at least one blue edge, $p_1 = |V_r \Delta V_b|$ be the number of parts containing at least one red edge or at least one blue edge but not both, and $p_0 = |V(D) \setminus (V_r \cup V_b)|$ be the number of parts with no red or blue edges.

For each vertex $w_i \in D$, let $d_r(w_i)$ and $d_b(w_i)$ denote its red and blue degrees respectively within D . Then $d_r(w_i) + d_b(w_i) = q - 1$ for all i . By the choice of the Gallai partition with the smallest number of parts, the following fact is immediate.

Fact 1. *For all $w_i \in V(D)$, we have $d_r(w_i), d_b(w_i) \geq 1$.*

To avoid a monochromatic copy of K_5 , the following facts follow immediately from the relevant definitions.

Fact 2. If $w_i \in D$ is in a red $K_4 \subseteq D$, then $w_i \notin V_r$. If $w_i \in D$ is in a blue $K_4 \subseteq D$, then $w_i \notin V_b$.

Fact 3. For all i ,

$$d_r(w_i) \leq 24, \text{ and } d_b(w_i) \leq 24.$$

If a vertex $w_i \in D$ has at least $r(3, 5) = 14$ incident edges in red (in D), then the neighborhood contains either a red K_3 or a blue K_5 . Certainly the latter is not an option so the former must occur, meaning that w_i is contained in a red K_4 within D . By Fact 2, we get the following fact.

Fact 4. If $d_r(w_i) \geq 14$, then $w_i \notin V_r$. If $d_b(w_i) \geq 14$, then $w_i \notin V_b$.

The remainder of the proof is broken into cases based on where red and blue fall in the list of colors relative to the first r colors, the middle s colors, and the last t colors.

Case 1. Both red and blue occur within the last t colors.

In this case, the graph G contains no red or blue triangle. Since $r(K_3, K_3) = 6$, we find that $4 \leq q \leq 5$. By Fact 1, for every i , it follows that G_i contains no red or blue edge. This means that every G_i is colored with at most $k - 2$ colors with

$$|G_i| \leq g(r, s, t - 2).$$

By induction and Inequality (2),

$$|G| = \sum_{i=1}^q |G_i| \leq 5g(r, s, t - 2) \leq g(r, s, t) < |G|,$$

a contradiction, completing the proof of Case 1.

Case 2. Red is among the middle s colors while blue is among the last t colors.

In this case, the graph G contains no red K_4 and no blue triangle. Since $r(K_4, K_3) = 9$, we find that $4 \leq q \leq 8$. By Fact 1, for every i , it follows that G_i contains no blue edge and no red triangle.

If G_i contains no red edges for some i , then G_i is colored with at most $k - 2$ colors with $|G_i| \leq g(r, s - 1, t - 1)$ so by induction and Inequality (5),

$$|G_i| \leq g(r, s - 1, t - 1) \leq \frac{1}{8}g(r, s, t). \quad (23)$$

Next if G_i contains at least one red edge, then by Fact 1, there can be no red triangle in G_i so $|G_i| \leq g(r, s - 1, t)$. Therefore, by the induction hypothesis and Inequality (4), we have

$$|G_i| \leq g(r, s - 1, t) \leq \frac{1}{3}g(r, s, t). \quad (24)$$

By Inequalities (23) and (24), we get the key inequality

$$|G| \leq \left(p_1 \frac{1}{3} + p_0 \frac{1}{8} \right) g(r, s, t). \quad (25)$$

This means that as long as we can show

$$\left(p_1 \frac{1}{3} + p_0 \frac{1}{8} \right) \leq 1, \quad (26)$$

then we obtain a contradiction by showing $|G| \leq g(r, s, t)$. The remainder of this case can be concluded by establishing Inequality (26), which follows by the same argument as used in the corresponding case of [7].

Case 3. Both red and blue occur within the middle s colors.

In this case, the graph G contains no red or blue K_4 and cases (c_4) and (c_9) cannot occur since $s \geq 2$. Since $r(K_4, K_4) = 18$, we find that $4 \leq q \leq 17$. First some bounds on the orders of the parts G_i , leading to a counterpart of Inequality 25.

First suppose G_i contains no red and no blue edges. Then by induction and Inequality (8), imply that

$$|G_i| \leq g(r, s - 2, t) = \frac{1}{17}g(r, s, t). \quad (27)$$

Next suppose G_i contains no blue edges but contains some red edges. Then by induction and Inequality (7), we get

$$|G_i| \leq g(r, s - 2, t + 1) \leq \frac{13}{72}g(r, s, t). \quad (28)$$

Finally suppose G_i contains both red and blue edges. Then by induction and Inequality (6), we get

$$|G_i| \leq g(r, s - 2, t + 2) = \frac{13}{36}g(r, s, t). \quad (29)$$

Combining Inequalities (27), (28), and (29), we obtain the key inequality

$$|G| \leq \left(p_2 \frac{13}{36} + p_1 \frac{13}{72} + p_0 \frac{1}{17} \right) g(r, s, t). \quad (30)$$

As in Case 2, if we can show that

$$p_2 \frac{13}{36} + p_1 \frac{13}{72} + p_0 \frac{1}{17} \leq 1, \quad (31)$$

then we will arrive at a contradiction that $|G| \leq g(r, s, t)$. Thus, for the remainder of the proof of this case, it suffices to show Inequality (31).

Next we derive several facts. Within the red neighborhood of some vertex w_i in R , there can be no red triangle since otherwise we would have a red K_4 in G . There can also be no blue K_4 within this neighborhood so that means the red neighborhood of w_i (and similarly the blue neighborhood) has at most $r(4, 3) - 1 = 8$ vertices. Formally, we obtain the following fact.

Fact 5. For all $w_i \in V(R)$, we have $d_r(w_i), d_b(w_i) \leq 8$.

If a vertex $w_i \in R$ is contained in a red (or blue) triangle, then the part G_i cannot contain any red (respectively blue) edges to avoid creating a red (respectively blue) copy of K_4 . The following fact is then immediate.

Fact 6. If w_i is in a red triangle in R , then $w_i \notin V_r$. Similarly if w_i is in a blue triangle in R , then $w_i \notin V_b$.

If $d_r(w_i) \geq 4$ for some $w_i \in V(R)$, then the red neighborhood of w_i certainly must contain at least one red edge since otherwise, if all edges were blue, we would have a blue K_4 . Thus w_i is in a red triangle in R . A similar observation holds with the roles of red and blue switched. Thus from Fact 6, we obtain the following fact.

Fact 7. If $d_r(w_i) \geq 4$ then $w_i \notin V_r$, and if $d_b(w_i) \geq 4$ then $w_i \notin V_b$.

If two parts G_i and G_j each contain at least one red edge, say e_i and e_j respectively, then the edge $w_i w_j$ in R cannot be red since otherwise the subgraph induced on the vertices of $e_i \cup e_j$ is a red K_4 . Thus, we obtain the following fact.

Fact 8. The subgraph induced on V_r is a blue clique and the subgraph induced on V_b is a red clique.

Next, we prove three helpful claims about the values of p_0, p_1 , and p_2 .

Claim 2. $p_2 = |V_r \cap V_b| \leq 1$ and if $p_2 = 1$, then $q \leq 7$.

Proof. If we have $w_i, w_j \in V_r \cap V_b$, then by Fact 8, $w_i, w_j \in V_r$ implies that the edge $w_i w_j$ is blue in R , while $w_i, w_j \in V_b$ implies that $w_i w_j$ is red, a contradiction.

Now suppose $p_0 = 1$ and, for a contradiction, that $q \geq 8$. If $w_1 \in V_r \cap V_b$, then there are at least 4 other vertices, say $W = \{w_2, w_3, w_4, w_5\}$ with all one color, say red, on edges to w_1 . Since $w_1 \in V_r \cap V_b$ and to avoid a red K_4 , all edges between vertices in W must be blue, forming a blue K_4 for a contradiction. \square

Claim 3. $|V_r| + |V_b| \leq 4$.

Proof. Suppose first that there is a vertex $w_i \in V_r \cap V_b$. Then by Fact 6, w_i is contained in neither a red triangle nor a blue triangle within R . By Fact 8, any vertex of $V_r \setminus \{w_i\}$ must be a blue neighbor of w_i in R , and since the blue neighborhood of w_i induces a red clique in R , again Fact 8 implies that there can only be at most one vertex in $V_r \setminus \{w_i\}$. This means that $|V_r| \leq 2$, and similarly, $|V_b| \leq 2$.

Thus, we may assume $V_r \cap V_b = \emptyset$. We next claim that $|V_r| \leq 3$ and $|V_b| \leq 3$. If $|V_r| \geq 4$, then by Fact 8, the subgraph of R induced on the vertices of V_r contains a blue K_4 , a contradiction. Thus $|V_r| \leq 3$, and symmetrically $|V_b| \leq 3$.

Now suppose that $|V_r| = |V_b| = 3$. If there exists a vertex $w_i \in V_r$ with at least two red neighbors in V_b , then by Fact 8, w_i is in a red triangle in R , and this contradicts Fact 6. Thus, there can be at most one red edge from each

vertex in V_r to V_b , and similarly, at most one blue edge from each vertex in V_b to V_r , for a total of at most 6 edges. But R has 9 edges between V_r and V_b , a contradiction. Finally suppose $|V_r| = 3$ and $|V_b| = 2$. Then again, there can be at most one red edge from each vertex of V_r to V_b , and at most one blue edge from each vertex of V_b to V_r , for a total of at most 5 edges, while R has 6 edges between V_r and V_b , another contradiction. Symmetrically we cannot have $|V_r| = 2$ and $|V_b| = 3$, thus completing the proof of Claim 3. \square

Claim 4. *If either $|V_r| \geq 2$ or $|V_b| \geq 2$, then*

- (a) $q \leq 10$,
- (b) *If $q = 10$, then $p_2 = 0$ and $p_1 = 2$, and*
- (c) *If $q = 9$, then $p_1 + p_2 \leq 3$.*

In particular, if $|V_r| + |V_b| \geq 3$, then either $|V_r| \geq 2$ or $|V_b| \geq 2$ so this claim may be applied.

Proof. Working under the assumption that $|V_r| + |V_b| \geq 3$, without loss of generality, we may assume $|V_r| \geq 2$. Suppose G_1 and G_2 each contain at least one red edge. By Fact 8, all edges from G_1 to G_2 must be blue. For i with $1 \leq i \leq 2$, define

$$\begin{aligned} R_i &= \{j \mid j \geq 3, \text{ and } G_j \text{ is joined to } G_i \text{ by red edges} \\ &\quad \text{and to } G_{3-i} \text{ by blue edges}\} \\ R_{1,2} &= \{j \mid j \geq 3, \text{ and } G_j \text{ is joined to } G_i \text{ and } G_{3-i} \text{ by red edges}\} \\ B &= \{j \mid j \geq 3, \text{ and } G_j \text{ is joined to } G_i \text{ and } G_{3-i} \text{ by blue edges}\} \end{aligned}$$

If $j_1, j_2 \in R_i$, then G_{j_1} and G_{j_2} are joined by blue edges to avoid a red K_4 . Suppose that $|R_i| \geq 3$ for some i , say $|R_1| \geq 3$ with $\{j_1, j_2, j_3\} \subseteq R_1$. Then there is a blue K_4 with vertices chosen from $G_2, G_{j_1}, G_{j_2}, G_{j_3}$, a contradiction. This means that $|R_i| \leq 2$ for each i with $1 \leq i \leq 2$.

If $j_1, j_2 \in R_i \cup R_{1,2}$, then again G_{j_1} and G_{j_2} are joined by blue edges to avoid a red K_4 . Then to avoid a blue K_4 , it is clear that $|R_{1,2}| + |R_i| \leq 3$ for each i with $1 \leq i \leq 2$. Exchanging the roles of the colors, it is also clear that $|B| \leq 3$.

Then if:

- $|R_{1,2}| = 0$, then $|R_{1,2}| + |R_1| + |R_2| + |B| \leq 0 + 2 \cdot 2 + 3 = 7$,
- $|R_{1,2}| = 1$, then $|R_{1,2}| + |R_1| + |R_2| + |B| \leq 1 + 2 \cdot 2 + 3 = 8$,
- $|R_{1,2}| = 2$, then $|R_{1,2}| + |R_1| + |R_2| + |B| \leq 2 + 2 \cdot 1 + 3 = 7$,
- $|R_{1,2}| = 3$, then $|R_{1,2}| + |R_1| + |R_2| + |B| \leq 3 + 2 \cdot 0 + 3 = 6$,

so in every case, $q = 2 + |R_{1,2}| + |R_1| + |R_2| + |B| \leq 2 + 8 = 10$, completing the proof of (a).

If $q = 10$, then we must have $|R_{1,2}| + |R_1| + |R_2| + |B| = 1 + 2 \cdot 2 + 3 = 8$ so $|R_{1,2}| = 1$, $|R_1| = |R_2| = 2$ and $|B| = 3$. By the observations above, all edges between pairs of parts with indices in B are red, meaning that each of these parts is in a red triangle in R . Similarly, all edges between pairs of parts with indices in $R_{1,2} \cup R_i$ are blue for each i with $1 \leq i \leq 2$, meaning that each of these parts is in a blue triangle in R . Thus, for all j with $3 \leq j \leq 10$, we have G_j contains no red or blue edges. Similarly, with $R_i = 2$ for each i , there can be no blue edges in either G_1 or G_2 . This means that $p_2 = 0$ and $p_1 = 2$, completing the proof of (b).

Finally suppose $q = 9$. What remains of the proof of Claim 4, we break into cases based on the value of $|R_{1,2}|$.

If $|R_{1,2}| = 0$, then $|R_1| = |R_2| = 2$ and $|B| = 3$. As in the case when $q = 10$, G_1 and G_2 each contain no blue edges and for all j with $3 \leq j \leq 10$, G_j must contain no red or blue edges. Thus, $p_2 = 0$ and $p_1 = 2$.

If $|R_{1,2}| = 1$, then it is possible that either $|R_1| = 1$ or $|R_2| = 1$, say $|R_1| = 1$. Then the set G_j corresponding to R_1 can have blue edges but all other sets G_j with $j \geq 3$ must have no blue and no red edges. Thus, $p_1 + p_2 \leq 3$. On the other hand, if $|R_1| = |R_2| = 2$, then it is possible that $|B| = 2$. Then at most one of the sets G_j corresponding to B can have red edges but all other sets G_j with $j \geq 3$ must have no blue and no red edges. Thus, $p_1 + p_2 \leq 3$.

If $|R_{1,2}| = 2$, then $|R_1| = |R_2| = 1$ and $|B| = 3$. Each of the sets with indices in $R_1 \cup R_2 \cup R_{1,2}$ is contained in a blue triangle in R , meaning that for all j with $3 \leq j \leq 10$, the set G_j contains no red or blue edges. Each set G_1 and G_2 may contain blue edges or not but in either case, $p_1 + p_2 = 2$, completing the proof of (c) and Claim 4. \square

We now consider subcases based on the value of q .

Subcase 3.1. $13 \leq q \leq 17$.

By Fact 5, we have $d_r(w_i), d_b(w_i) \leq 8$ so this means that $d_b(w_i), d_r(w_i) \geq 4$ for all $w_i \in V(R)$. This means that G_i contains no red or blue edges for all i . Thus $p_2 = p_1 = 0$, $p_0 = q$, and

$$p_2 \frac{16/3}{17} + p_1 \frac{8/3}{17} + p_0 \frac{1}{17} = \frac{q}{17} \leq 1,$$

as required for Inequality (31).

Subcase 3.2. $4 \leq q \leq 10$.

By Claim 2, we have $p_2 \leq 1$. First suppose $p_2 = 1$. Then if $q \geq 8$, every vertex $w_i \in V(R)$ must have at least 4 edges in one color and, by Fact 7, every set G_i is missing either red or blue, contradicting the assumption that $p_2 = 1$. Thus,

we have $4 \leq q \leq 7$. By Claim 3, since $p_2 = 1$, we have $p_1 = |V_r| + |V_b| - 2p_2 \leq 2$. Thus,

$$p_2 \frac{13}{36} + p_1 \frac{13}{72} + p_0 \frac{1}{17} \leq 1 \cdot \frac{13}{36} + 2 \cdot \frac{13}{72} + (q-3) \cdot \frac{1}{17} \leq \frac{1172}{1224} < 1.$$

Next suppose $p_2 = 0$ so by Claim 3, $p_1 \leq 4$. If $q \leq 8$, we get

$$p_2 \frac{13}{36} + p_1 \frac{13}{72} + p_0 \frac{1}{17} \leq 4 \cdot \frac{13}{72} + 4 \cdot \frac{1}{17} = \frac{1172}{1224} < 1.$$

If $q = 10$, then by Claim 4, we have $p_1 = 2$, so

$$p_2 \frac{13}{36} + p_1 \frac{13}{72} + p_0 \frac{1}{17} \leq 2 \cdot \frac{13}{72} + 8 \cdot \frac{1}{17} = \frac{1018}{1224} < 1.$$

If $q = 9$, then by Claim 4, we have $p_1 \leq 3$, so

$$p_2 \frac{13}{36} + p_1 \frac{13}{72} + p_0 \frac{1}{17} \leq 3 \cdot \frac{13}{72} + 6 \cdot \frac{1}{17} = \frac{1095}{1224} < 1.$$

Subcase 3.3. $q \in \{11, 12\}$.

By Claim 4, we see that $|V_r| + |V_b| \leq 2$. By Claim 2, we have $p_2 = 0$ so $p_1 \leq 2$ and

$$p_2 \frac{13}{36} + p_1 \frac{13}{72} + p_0 \frac{1}{17} \leq 2 \cdot \frac{13}{72} + 10 \cdot \frac{1}{17} = \frac{1162}{1224} < 1$$

completing the proof of this subcase, and the proof of Case 3.

Case 4. *Red is among the first r colors while blue is among the last t colors.*

In this case, the graph G contains no red K_5 and no blue triangle. Since $r(K_3, K_5) = 14$, we find that $4 \leq q \leq 13$. By Fact 1, each part of the Gallai partition has both red and blue incident edges in the reduced graph. This means that there can be no red K_4 and no blue edge in any part, leading to the following main subcases.

1. No part has any red edges,
2. There is a part with a red K_3 , and
3. There is a part with red edges but no part has a red K_3 .

We first consider Subcase 1. Since every part G_i contains no red or blue edges, this means that $|G_i| \leq g(r-1, s, t-1)$. By induction and Inequality (13), we get

$$|G| = \sum_{i=1}^q |G_i| \leq \sum_{i=1}^q g(r-1, s, t-1) \leq \frac{q}{13} g(r, s, t) \leq g(r, s, t) < |G|,$$

a contradiction, completing the proof of Subcase 1.

Next we consider Subcase 2. Let G_1 be a part of the Gallai partition containing a red triangle. Partition the remaining vertices of G into G_R and G_B such that G_R contains all vertices in parts having red edges to G_1 and G_B contains all vertices in parts having blue edges to G_1 .

Certainly G_R contains no red edges and no blue triangle and G_B contains no red K_5 and no blue edges. This means that

$$|G_R| \leq g(r-1, s, t) \quad \text{and} \quad |G_B| \leq g(r, s, t-1).$$

Furthermore, since G_1 contains a red triangle but no red K_4 and no blue edges, we get

$$|G_1| \leq g(r-1, s+1, t-1).$$

By considering each of the cases of the statement ((c_1) up to (c_{11})) individually across Types T1, T10, and T12 in Tables 1 and 2, we see that in each of the cases, we have

$$g(r-1, s+1, t-1) + g(r-1, s, t) + g(r, s, t-1) \leq g(r, s, t)$$

except in the two cases (c_7) and (c_{10}). We sharpen the bounds by observing that G_1 contains a red triangle, so every pair of parts in G_R are joined by blue edges. Since G has no blue triangle, this means that G_R must have at most two parts. If G_R has only one part, then by Fact 1, it must have blue edges to some part in G_B , and so cannot contain a blue edge so $|G_R| \leq g(r-1, s, t-1)$. If G_R has two parts, then similarly each cannot contain blue edges meaning that $|G_R| \leq 2g(r-1, s, t-1)$. Then the calculations for these two specific cases become

$$(c_7) : 2\frac{3}{R} + \frac{221}{16R} + \frac{1}{2} < 1, \text{ and}$$

$$(c_{10}) : \frac{2}{15} + \frac{17}{48} + \frac{1}{2} < 1.$$

In either case, $|G| = |G_1| + |G_R| + |G_B| < |G|$, a contradiction.

Finally we consider Subcase 3. Let G_1 be a part of the Gallai partition containing at least one red edge (but no red triangle). Again partition the remaining vertices of G into G_R and G_B such that G_R contains all vertices in parts having red edges to G_1 and G_B contains all vertices in parts having blue edges to G_1 .

Then G_R contains no red K_3 and no blue K_3 . Similarly, G_B contains no red K_5 and no blue edges at all. This means that

$$|G_R| \leq g(r-1, s, t+1) \quad \text{and} \quad |G_B| \leq g(r, s, t-1).$$

Furthermore, since G_1 contains red edges but no red triangle and no blue edges, we get

$$|G_1| \leq g(r-1, s, t).$$

By considering each of the cases of the statement ((c_1) up to (c_{11})) individually across Types T1, T11, and T12 in Tables 1 and 2, we see that in each of the

cases, we have

$$\frac{g(r-1, s, t+1) + g(r-1, s, t) + g(r, s, t-1)}{g(r, s, t)} \leq 1$$

except in the two cases (c_7) and (c_9) .

Certainly G_B contains no blue edges so every pair of parts in G_B are joined by red edges. Since G has no red K_5 , there must be at most 4 parts in G_B . By Fact 1, no individual part contains a red K_4 so if there is only one part in G_B , it has order at most $g(r-1, s+1, t-1)$. If a part $G_i \subseteq G_B$ contains a red triangle, then there can only be one other part, which must have no red edge. In this case $|G_B| \leq g(r-1, s+1, t-1) + g(r-1, s, t-1)$. If two parts have red edges (but no red triangle), then these are the only two parts in G_B and $|G_B| \leq 2g(r-1, s, t)$. Finally if one part has red edges (but no red triangle), then there are at most 3 parts in G_B , meaning that $|G_B| \leq g(r-1, s, t) + 2g(r-1, s, t-1)$. Putting these observations together, we see that

$$\begin{aligned} |G_B| &\leq \begin{cases} g(r-1, s+1, t-1), \\ g(r-1, s+1, t-1) + g(r-1, s, t-1), \\ 2g(r-1, s, t), \\ g(r-1, s, t) + 2g(r-1, s, t-1) \end{cases} \\ &\leq g(r-1, s+1, t-1) + g(r-1, s, t-1). \end{aligned}$$

This means that

$$\begin{aligned} |G| &= |G_1| + |G_R| + |G_B| \\ &\leq g(r-1, s, t) + g(r-1, s, t+1) \\ &\quad + g(r-1, s+1, t-1) + g(r-1, s, t-1) \\ &\leq g(r, s, t) \\ &< |G|, \end{aligned}$$

a contradiction, completing the proof of Case 4.

Case 5. *Red is among the first r colors while blue is among the middle s colors.*

In this case, the graph G contains no red K_5 and no blue K_4 . Since $r(K_4, K_5) = 25$, we find that $4 \leq q \leq 24$. We break the proof into subcases based on the red and blue edges that appear within parts of a Gallai partition. These subcases are listed as follows.

- 5.1. No part of the partition contains any red or blue edges.
- 5.2. A part G_1 contains a red copy of K_3 and at least one blue edge.
- 5.3. A part G_1 contains red and blue edges, but no red or blue copy of K_3 .
- 5.4. A part G_1 contains a red copy of K_3 and no blue edges.

5.5. A part G_1 contains blue edges.

5.6. A part G_1 contains red edges but no red copy of K_3 , and no blue edges.

We now consider these subcases.

Subcase 5.1. *No part of the partition contains any red or blue edges.*

If each part G_i of the partition contains no red or blue edges, then this is Type T16 so by Inequality (16), we get $\frac{g(r-1,s-1,t)}{g(r,s,t)} \leq \frac{1}{24}$. This means that

$$\begin{aligned} |G| &= \sum_{i=1}^q |G_i| \\ &\leq \sum_{i=1}^q g(r-1, s-1, t) \\ &\leq \sum_{i=1}^q \frac{1}{24} g(r, s, t) \\ &= \frac{q}{24} g(r, s, t) \\ &\leq g(r, s, t), \end{aligned}$$

a contradiction, completing the proof for Subcase 5.1.

Subcase 5.2. *A part G_1 contains a red copy of K_3 and at least one blue edge.*

Let G_R (and G_B) be the set of vertices with all red (respectively blue) edges to G_1 . Then $|G_B|$ can be bounded from above by $g(r, s-1, t)$. By Inequality (4), we get

$$\frac{g(r, s-1, t)}{g(r, s, t)} \leq \frac{1}{3}.$$

We can also bound $|G_1|$ from above by $g(r-1, s, t+1)$. By Inequality (11), we get

$$\frac{g(r-1, s, t+1)}{g(r, s, t)} \leq \frac{17}{36}.$$

Finally, since G_R contains no red edges, it has at most three parts. No part has a blue K_3 so there can be three parts with no blue edges or one part with blue edges and another with no blue edges. Thus, we obtain

$$\frac{|G_R|}{g(r, s, t)} \leq \max \left\{ 3 \cdot \frac{1}{24}, \frac{1}{9} + \frac{1}{24} \right\} = \frac{11}{72}.$$

Summarizing, we get

$$|G| = |G_1| + |G_R| + |G_B| \leq \left(\frac{17}{36} + \frac{11}{72} + \frac{1}{3} \right) g(r, s, t) = \frac{69}{72} g(r, s, t) < g(r, s, t),$$

a contradiction.

Subcase 5.3. A part G_1 contains red and blue edges, but no red or blue copy of K_3 .

Then G_r is of Type T11 so by Inequality (11), we obtain

$$\frac{g(r-1, s, t+1)}{g(r, s, t)} \leq \frac{17}{36}.$$

Similarly, G_1 is of Type T12 so by Inequality (12), we have

$$\frac{g(r-1, s-1, t+2)}{g(r, s, t)} \leq \frac{2}{9}.$$

Since G_B has no blue edges, it has at most four parts. There can be four parts with no red edges, or one part with red edges and two parts with no red edges, or two parts with a red K_3 in one part and no red edges in the other part, or two parts with red edges, or one part with a red K_3 . This means that

$$\frac{|G_B|}{g(r, s, t)} \leq \max \left\{ 4 \cdot \frac{1}{24}, \frac{1}{9} + \frac{2}{24}, \frac{17}{72} + \frac{1}{24}, 2 \cdot \frac{1}{9}, \frac{17}{72} \right\} g(r, s, t) = \frac{5}{18} g(r, s, t).$$

Summarizing we obtain

$$|G| = |G_1| + |G_R| + |G_B| \leq \left(\frac{2}{9} + \frac{17}{36} + \frac{5}{18} \right) g(r, s, t) = \frac{35}{36} g(r, s, t) < g(r, s, t),$$

a contradiction.

Subcase 5.4. A part G_1 contains a red copy of K_3 and no blue edges.

First a claim that there is only one such part.

Claim 5. At most one part contains a red copy of K_3 .

Proof. Suppose there are two parts G_1 and G_2 containing a red copy of K_3 . To avoid a red copy of K_5 , there must be all blue edges in between G_1 and G_2 . Then G_R , the set of vertices with red edges to G_1 , is an (R_2, B_4) -graph, and G_2 is contained in G_B , the set of vertices with all blue edges to G_1 . We also see that the set of vertices in G_B with all red edges to G_1 , G_{BR} is an (R_2, B_3) -graph, and the set of vertices in G_B with all blue edges to G_2 , G_{BB} is an (R_5, B_2) -graph. We deduce that

$$w(G_R) \leq \max \left\{ 2 \cdot \frac{1}{24}, \frac{1}{9} + \frac{1}{24} \right\} = \frac{11}{72},$$

$$w(G_{BR}) \leq \max \left\{ \frac{1}{9}, 2 \cdot \frac{1}{24} \right\} = \frac{1}{9},$$

$$\begin{aligned}
w(G_{BB}) &\leq \frac{|G_B|}{g(r, s, t)} \\
&\leq \max \left\{ 4 \cdot \frac{1}{24}, \frac{1}{9} + \frac{2}{24}, \frac{17}{72} + \frac{1}{24}, 2 \cdot \frac{1}{9}, \frac{17}{72} \right\} g(r, s, t) \\
&= \frac{5}{18} g(r, s, t).
\end{aligned}$$

We now distinguish two cases. First if G_R contains a blue part, then all edges from this blue part to G_{BB} are red. So G_{BB} contains no red copy of K_4 . Then

$$w(G_{BB}) \leq \max \left\{ \frac{17}{72}, \frac{1}{9} + \frac{1}{24}, 3 \cdot \frac{1}{24} \right\} = \frac{17}{72}.$$

Thus, we obtain

$$\begin{aligned}
|G| &= |G_1| + |G_2| + |G_R| + |G_{BR}| + |G_{BB}| \\
&\leq \left(2 \cdot \frac{5}{26} + \frac{17}{72} + \frac{1}{9} + \frac{17}{72} \right) g(r, s, t) \\
&= \frac{151}{156} g(r, s, t) \\
&< g(r, s, t),
\end{aligned}$$

a contradiction.

If G_{BR} contains no blue part, then $w(G_{BR}) \leq \frac{2}{24}$, so we obtain

$$\begin{aligned}
|G| &= |G_1| + |G_2| + |G_R| + |G_{BR}| + |G_{BB}| \\
&\leq \left(2 \cdot \frac{17}{72} + \frac{11}{72} + \frac{2}{24} + \frac{5}{18} \right) g(r, s, t) \\
&= \frac{71}{72} g(r, s, t) \\
&< g(r, s, t),
\end{aligned}$$

a contradiction. \square

Let G_1 be a part containing a red copy of K_3 . Then G_R is an (R_2, B_4) -graph and G_B is an (R_5, B_3) -graph. We deduce that

$$w(G_R) \leq \max \left\{ \frac{1}{9} + \frac{1}{24}, 3 \cdot \frac{1}{24} \right\} = \frac{11}{72},$$

and $w(G_B) \leq \frac{5}{9}$ by Lemma 7. Thus, we obtain

$$|G| = |G_1| + |G_R| + |G_B| \leq \left(\frac{17}{72} + \frac{11}{72} + \frac{5}{9} \right) g(r, s, t) = \frac{17}{18} g(r, s, t) < g(r, s, t),$$

a contradiction.

For the remaining subcases, we therefore know that each part G_i can contain red (or blue) edges but no red (respectively blue) copy of K_3 , and no blue (respectively red) edges.

Subcase 5.5. *A part G_1 contains a blue edges.*

Suppose G_1 contains a blue edge. Let G_R (or G_B) denote the set of vertices in $G \setminus G_1$ with red (respectively blue) edges to G_1 . Let q_1 be the number of parts of the Gallai partition in G_R and let q_2 be the number of parts of the Gallai partition in G_B . Then

$$q_1 \leq 17 = R(K_4, K_4) - 1$$

since the reduced graph of G_R contains no monochromatic copy of K_4 . Similarly, $q_2 \leq 4$ since G_B contains no blue edges.

For the situation when G_R contains either red or blue edges, we apply Inequality (15) to get $|G_R| \leq g(r-1, s-1, t+1) \leq \frac{1}{9}g(r, s, t)$.

We intend to show that

$$|G| = |G_1| + |G_R| + |G_B| \leq g(r, s, t),$$

which would be a contradiction. This would hold if we could show that

$$|G_1| + |G_B| \leq \frac{1}{3}g(r, s, t) \quad \text{and} \quad |G_R| \leq \frac{2}{3}g(r, s, t)$$

or equivalently if

$$\frac{|G_1| + |G_B|}{g(r, s, t)} \leq \frac{1}{3} \quad \text{and} \quad \frac{|G_R|}{g(r, s, t)} \leq \frac{2}{3}.$$

Certainly there are no blue edges within G_B so every pair of parts in G_B is joined entirely by red edges. Since G contains no red copy of K_5 , there can be at most 4 parts of the Gallai partition in G_B . If two parts have red edges (but certainly no red triangle in this subcase), then these are the only two parts in G_B and if one part has red edges (but no red triangle), then there are at most 3 parts in G_B .

This means that

$$\frac{|G_1| + |G_B|}{g(r, s, t)} \leq \max \left\{ \frac{1}{9} + 2 \cdot \frac{1}{9}, \frac{1}{9} + \frac{1}{9} + 2 \cdot \frac{1}{24}, \frac{1}{9} + 4 \cdot \frac{1}{24} \right\} = \frac{3}{9} = \frac{1}{3},$$

as desired.

Within G_R , we first note that there is no red copy of K_4 and no blue copy of K_4 . We may therefore follow along with the proof of Case 3 with the following arguments, cases concerning the possible values of q_1 .

First suppose $q_1 = 17$. If a part in G_2 in G_B contains a red edge, then since $R(K_3, K_4) = 9$, there must be at most 8 parts in G_R with red edges to G_2 and at most 8 parts in G_R with blue edges to G_2 , meaning $q_1 \leq 16$. Thus, no part in G_B has a red edge, so

$$\frac{|G_R|}{g(r, s, t)} + \frac{|G_1| + |G_B|}{g(r, s, t)} \leq \frac{q_1}{24} + \frac{4}{24} \leq \frac{21}{24} < 1,$$

a contradiction. If $13 \leq q_1 \leq 16$, then

$$\frac{|G_R|}{g(r, s, t)} \leq \frac{16}{24} = \frac{2}{3},$$

as claimed.

When $4 \leq q_1 \leq 12$, we apply Claim 4 and note that $p_2 = 0$. Following the proof of Subcase 3.2, we get the following. If $q_1 \leq 8$, then

$$p_1 \cdot \frac{1}{9} + p_0 \cdot \frac{1}{24} \leq \frac{4}{9} + \frac{4}{24} < \frac{2}{3}.$$

If $q_1 = 9$, then

$$p_1 \cdot \frac{1}{9} + p_0 \cdot \frac{1}{24} \leq \frac{3}{9} + \frac{6}{24} = \frac{1}{3} + \frac{1}{4} < \frac{2}{3}.$$

If $q_1 = 10$, then $p_1 \leq 2$ and

$$p_1 \cdot \frac{1}{9} + p_0 \cdot \frac{1}{24} \leq \frac{2}{9} + \frac{8}{24} < \frac{2}{3}.$$

Finally if $11 \leq q_1 \leq 12$, then $p_1 \leq 2$ and

$$p_1 \cdot \frac{1}{9} + p_0 \cdot \frac{1}{24} \leq \frac{2}{9} + \frac{10}{24} = \frac{23}{36} < \frac{2}{3},$$

completing the proof of Subcase 5.5.

For the remaining subcase, all parts are either free or red, and there is at least one red part.

Subcase 5.6. *A part G_1 contains red edges but no red copy of K_3 , and no blue edges.*

Then G_R is an (R_3, B_4) -graph and G_B is an (R_5, B_3) -graph. So by Inequality (15) and Lemmas 6 and 7, we obtain

$$\begin{aligned} |G| &= |G_1| + |G_R| + |G_B| \\ &\leq \max \left\{ \frac{1}{9} + \frac{1}{3} + \frac{5}{9}, \frac{1}{9} + \frac{25}{72} + \frac{39}{72} \right\} \cdot g(r, s, t) \\ &= g(r, s, t) < g(r, s, t) + 1, \end{aligned}$$

a contradiction, unless G_R and G_B are both very specific blow-ups of the unique 2-coloring of K_5 with no monochromatic triangle as in Lemmas 6 and 7.

In order to avoid creating a red K_5 , each part in G_R can have red edges to at most two parts in G_B . This means that there must be at least 15 pairs of parts (one in G_R and one in G_B) with blue edges between them. To avoid creating a blue copy of K_4 , the only way for a part in G_B to have blue edges to three parts in G_R is for all of those blue edges to go to the free parts in G_R . This leaves all red edges from the blue parts in G_R to G_B , making a red copy of K_5 , completing the proof of Subcase 5.6 and Case 5.

Case 6. *Red and blue are both within the first r colors.*

In this case, the graph G contains no red or blue K_5 . Since $r(K_5, K_5) = R+1$, we find that $4 \leq q \leq R$. It turns out that a better bound is almost immediate.

Claim 6. *If red and blue occur within the first r colors, then*

$$q \leq 38.$$

Proof. Suppose, for a contradiction, that $q \geq 39$. By Fact 3, for each vertex $w_i \in D$, we have $d_r(w_i) \leq 24$ and $d_b(w_i) \leq 24$. Since $q = |D| \geq 39$, this means that $d_r(w_i) \geq 14$ and $d_b(w_i) \geq 14$. By Fact 4, we get that $V_r = \emptyset$ and $V_b = \emptyset$, so $p_0 = q$. By induction on the number of colors,

$$|G_i| \leq \frac{1}{R}g(r, s, t)$$

so this means that

$$|G| \leq \frac{q}{R}g(r, s, t) < |G|,$$

a contradiction. \square

We break the remainder of the proof of this case into the following subcases:

1. There is a part G_1 containing a red triangle and a blue triangle but no red or blue copy of K_4 .
2. There is a part G_1 containing a red edge and a blue triangle but no red triangle and no blue copy of K_4 .
3. There is a part G_1 containing a red edge and a blue edge but no red or blue triangle. So G_1 is an (R_3, B_3) -part.
4. There is an (R_2, B_4) -part G_1 .
5. Each part is either a free part or a red part or a blue part.

Subcase 6.1. *There is a part G_1 containing a red triangle and a blue triangle but no red or blue copy of K_4 .*

If we let G_R and G_B be the sets of vertices with all red or respectively blue edges to G_1 , then it is clear that G_R contains no red edges and G_B contains no blue edges. Since all edges between parts in G_R must be blue, there can be at most 4 parts and similar there can be at most 4 parts in G_B . Since G_1 contains a red triangle and a blue triangle but no red or blue copy of K_4 , we see from Inequality (17) that

$$\frac{|G_1|}{|G|} \leq \frac{18}{R}.$$

The orders of G_R and G_B satisfy identical bounds so, by symmetry, we will consider only $|G_R|$.

If G_R contains only one part, this part contains no red edges and perhaps some blue triangles but no blue copy of K_4 (recall that for every part G_i , there exists a part G_j with all blue edges to G_i). By Inequality (19), this means that $|G_R|/|G| \leq \frac{6}{R}$.

If G_R contains two parts, these must have all blue edges between them. Then either one of these parts contains a blue triangle and the other contains no blue edges, or each part contains blue edges but no blue triangle. In the former situation, by Inequalities (22) and (19), we have

$$\frac{|G_R|}{|G|} \leq \frac{6}{R} + \frac{1}{R} = \frac{7}{R}.$$

In the latter situation, by Inequality (20), we have

$$\frac{|G_R|}{|G|} \leq 2 \frac{13}{2 \cdot R} = \frac{13}{R}.$$

If G_R contains three parts, at most one of them can contain any blue edges so, by Inequalities (22) and (21), we have

$$\frac{|G_R|}{|G|} \leq \frac{13}{4 \cdot R} + 2 \frac{1}{R} = \frac{21}{4 \cdot R}.$$

Finally if G_R contains four parts, none of these may contain any blue edges so, by Inequality (22), we have

$$\frac{|G_R|}{|G|} \leq 4 \frac{1}{R} = \frac{4}{R}.$$

Putting these together, we have $\frac{|G_R|}{|G|} \leq \frac{7}{R}$ and so symmetrically we also get $\frac{|G_B|}{|G|} \leq \frac{7}{R}$. These imply that

$$|G| = |G_1| + |G_R| + |G_B| \leq \frac{|G|}{R} (18 + 7 + 7) < |G|,$$

a contradiction, completing the proof of this subcase.

Subcase 6.2. *There is a part G_1 containing a red edge and a blue triangle but no red triangle and no blue copy of K_4 .*

Again let G_R and G_B be the sets of vertices with all red or respectively blue edges to G_1 , so G_R contains no red triangle and G_B contains no blue edges. By Inequality (18), we see that $|G_1| \leq \frac{12}{R}|G|$. From the same argument as in the previous subcase, we see that $|G_B| \leq \frac{7}{R}|G|$. By Inequality (11), we see that $|G_R| \leq \frac{5}{13}|G|$. Putting all these together, we get

$$|G| = |G_1| + |G_B| + |G_R| \leq \left(\frac{12}{R} + \frac{7}{R} + \frac{5}{13} \right) |G| < |G|,$$

a contradiction, completing the proof of this subcase.

Subcase 6.3. *There is an (R_3, B_3) -part G_1 .*

With G_R and G_B being the sets of vertices with red or blue edges respectively to G_1 , we consider several possible situations. We further break into cases based on the surrounding structures.

Subcase 6.3.1. *No other part contains red or blue edges.*

Then since G_B contains no blue triangle and no red copy of K_5 , we see that G_B contains at most $R(K_3, K_5) - 1 = 13$ parts of the Gallai partition and similarly G_R also contains at most 13 parts of the Gallai partition. This means that

$$|G| = |G_1| + |G_B| + |G_R| \leq \frac{|G|}{R} \left(\frac{13}{2} + 13 + 13 \right) = \frac{32.5|G|}{R} < |G|,$$

a contradiction.

Subcase 6.3.2. *There is a (R_2, B_4) -part G_2 .*

In order to avoid creating a blue copy of K_5 , all edges from G_1 to G_2 must be red. Let F_{2B} denote the set of vertices with blue edges to G_2 , let F_{1R} denote any remaining vertices with red edges to G_1 and let F_{1B} denote the set of vertices with blue edges to G_1 . Note that F_{2B} contains no blue edges, F_{1R} contains no red edges since both G_1 and G_2 have all red edges to F_{1R} , and F_{1B} contains no blue triangle and no red copy of K_4 . This means that

$$\begin{aligned} |G| &= |G_1| + |G_2| + |F_{1B}| + |F_{1R}| + |F_{2B}| \\ &\leq \frac{|G|}{R} \left(\frac{13}{2} + 6 + 12 + 7 + 7 \right) \\ &= \frac{38.5|G|}{R} < |G|, \end{aligned}$$

a contradiction.

Subcase 6.3.3. *There is another (R_3, B_3) -part G_2 .*

Without loss of generality, suppose the edges between G_1 and G_2 are red. If we let F_{1R} denote the set of vertices with red edges to G_1 , then all of F_{1R} must have blue edges to G_2 . Let F_{1B} be the remaining vertices, those with blue edges to G_1 . Then F_{1B} contains no blue triangle and F_{1R} contains no blue or red triangle. This means that

$$\begin{aligned} |G| &= |G_1| + |G_2| + |F_{1R}| + |F_{1B}| \\ &\leq \frac{|G|}{R} \left(\frac{13}{2} + \frac{13}{2} + \frac{13}{2} \right) + \frac{5}{13}|G| \\ &\leq \left(\frac{18.5}{R} + \frac{5}{13} \right) |G| < |G|, \end{aligned}$$

a contradiction.

Subcase 6.3.4. *Every other part is either free, red or blue.*

Note that G_R is an (R_3, B_5) -graph and G_B is an (R_5, B_3) -graph. So by Lemma 4, we obtain

$$|G| = |G_1| + |G_R| + |G_B| \leq \left(\frac{13}{2R} |G| + 2 \cdot \frac{16.25}{R} \right) g(r, s, t) = \frac{39}{R} g(r, s, t) < g(r, s, t),$$

a contradiction, completing the proof of this subsubcase.

Subcase 6.4. *There is an (R_2, B_4) -part G_1 .*

With G_R and G_B being the sets of vertices with red or blue edges respectively to G_1 , the graph induced on G_R contains no blue edge. We consider several possible situations and further break into cases based on the surrounding structures.

Subcase 6.4.1. *No part in G_R contains red or blue edges.*

Then since G_B contains no blue edges and no red copy of K_5 , we see that G_B contains at most $R(K_2, K_5) - 1 = 4$ parts of the Gallai partition and similarly G_R also contains at most $R(K_4, K_5) - 1 = 24$ parts of the Gallai partition. This means that

$$|G| = |X_1| + |G_B| + |G_R| \leq \frac{|G|}{R} (6 + 4 + 24) = \frac{34|G|}{R} < |G|,$$

a contradiction.

Subcase 6.4.2. *There is an (R_4, B_2) -part G_2 in G_R .*

Without loss of generality, suppose the edges between G_1 and G_2 are all red. Let F_{1R} be the set of vertices (other than G_2) with red edges to G_1 and let F_{1B} be the set of vertices with blue edges to G_1 . Then F_{1B} contains no blue edges and F_{1R} must have blue edges to G_2 to avoid creating a red copy of K_5 so F_{1R} contains no red or blue copy of K_4 . This means that

$$\begin{aligned} |G| &= |G_1| + |G_2| + |F_{1B}| + |F_{1R}| \\ &\leq \frac{|G|}{R} (6 + 6 + 7 + 18) \\ &= \frac{37}{R} |G| < |G|, \end{aligned}$$

a contradiction.

Subcase 6.4.3. *There is an (R_3, B_2) -part G_2 in G_R .*

Let G_{RR} be the set of vertices in G_R with red edges to G_2 and let G_{RB} be those vertices in G_R with blue edges to G_2 . Then G_B contains no blue edges,

G_{RR} contains no red edges, and G_{RB} contains no red or blue copy of K_4 . This means that

$$\begin{aligned} |G| &= |G_1| + |G_2| + |G_B| + |G_{RR}| + |G_{RB}| \\ &\leq \frac{|G|}{R} \left(6 + \frac{13}{4} + 7 + 7 + 18 \right) \\ &= \frac{41.25}{R} |G| < |G|, \end{aligned}$$

a contradiction.

Subcase 6.4.4. *There is an (R_2, B_3) -part G_2 in G_R .*

Note that we may assume that G_B contains no red triangle since if it did, this structure would be symmetric to the assumed structure considered in Subcase 6.4.2.

Let G_{RR} denote the set of vertices in G_R with red edges to G_2 and let G_{RB} denote the set of vertices in G_R with blue edges to G_2 . Then G_B contains no blue edges and G_{RB} contains no red K_4 and no blue triangle.

Hence G_{RR} is an (R_3, B_5) -graph and G_{RB} is a (R_4, B_3) -graph. Using Lemmas 3 and 4, we obtain

$$\begin{aligned} |G| &= |G_1| + |G_2| + |G_B| + |G_{RR}| + |G_{RB}| \\ &\leq \frac{|G|}{R} \left(6 + \frac{13}{4} + 7 + 16.25 + 9.75 \right) \\ &= \frac{42.25}{R} |G| < |G|, \end{aligned}$$

a contradiction.

Subcase 6.4.5. *There is an (R_2, B_4) -part G_2 .*

Then all edges between G_1 and G_2 are red. Let F_{1B} be the set of vertices with blue edges to G_1 , let F_{RR} be the set of vertices with red edges to both G_1 and G_2 , and let F_{RB} be the set of vertices with red edges to G_1 and blue edges to G_2 . Then F_{1B} contains no blue edges and F_{RB} contains no blue edges and no red K_4 . If F_{RR} contains no blue K_3 , then F_{RR} is an (R_3, B_5) -graph. This means that

$$\begin{aligned} |G| &= |G_1| + |G_2| + |F_{1B}| + |F_{1R}| + |F_{RR}| \\ &\leq \frac{|G|}{R} (6 + 6 + 7 + 16.25) + 6 \\ &= \frac{41.25}{R} |G| < |G|, \end{aligned}$$

a contradiction.

Suppose there is an (R_2, B_4) -part G_3 in F_{RR} . Repeating above arguments leads to

$$\begin{aligned} |G| &= |G_1| + |G_2| + |G_3| + |F_{1B}| + |F_{1R}| + |F_{3R}| + |F_{3B}| \\ &\leq \frac{|G|}{R}(4 \cdot 6 + 2 \cdot 7 + 3.25) \\ &= \frac{41.25}{R}|G| < |G|, \end{aligned}$$

a contradiction.

Subcase 6.5. *Each part is either a free part or a red part or a blue part.*

First suppose that G has exactly one non-free part. Let G_1 be this part containing red edges. Then G_R is an (R_3, B_5) -graph and G_B is an (R_5, B_4) -graph. So we obtain

$$|G| \leq \frac{1}{R} \left(\frac{13}{4} + 13 + 24 \right) = \frac{40.25}{R} \leq \frac{40.25}{43} < 1,$$

a contradiction.

Hence we may assume that G contains at least two non-free parts, say G_1 and G_2 . If G_1 and G_2 both contain red edges, and G_1 and G_2 are joined by blue edges, then we call this a RBR-pair. Analogously, RRR-pairs, RRB-pairs (BBR-pairs), BBB-pairs, BRB-pairs, and BBR-pairs (RBB)-pairs are defined.

Claim 7. *G contains an RRR-pair or a BBB-pair.*

Proof. Suppose not. First assume that G contains a RBR-pair. So let G_1 and G_2 contain red edges and G_1 and G_2 are joined by blue edges. Then G_R is an (R_3, B_5) -graph, G_{BB} is an (R_5, B_3) -graph and G_{BR} is an (R_3, B_4) -graph. Since there is no RRR-pair, both G_R and G_{BR} contain no red parts. Hence $w(G_{BR}) \leq \frac{9.5}{R}$ by Lemma 3 (ii). Now G_R can have at most two blue parts, since otherwise there is a BBB-pair. Now by Lemma 4, we conclude that $w(G_R) \leq \frac{13.5}{R}$. Using the same arguments, we conclude that $w(G_{BB}) \leq \frac{13.5}{R}$. Thus, we obtain

$$|G| \leq \frac{1}{R} \left(2 \cdot \frac{13}{4} + 2 \cdot 13.5 + 9.5 \right) g(r, s, t) = \frac{43}{R} g(r, s, t) \leq g(r, s, t),$$

a contradiction.

Hence we may assume that G contains no RBR-pair, no BRB-pair, but a RRB-pair (BBR-pair). Let G_1 contain red edges, let G_2 contain blue edges and G_1 and G_2 are joined by red edges. Then G_{RR} is an (R_2, B_5) -graph, G_{RB} is an (R_3, B_3) -graph and G_B is an (R_5, B_4) -graph.

By the assumptions there are no red parts in G_{RR}, G_{RB} and G_B , and no blue part in G_{RB} . Furthermore, G_{RR} and G_B have at most one blue part by the assumption. So we conclude that

$$\begin{aligned}
w(G_{RR}) &\leq \max \left\{ 4 \cdot \frac{1}{R}, \frac{13}{4R} + 2 \cdot \frac{1}{R}, 3 \cdot \frac{1}{R} \right\} = \frac{5.25}{R}, \\
w(G_{RB}) &\leq (R(3,3) - 1)) \cdot \frac{1}{R} = \frac{5}{R}, \\
w(G_B) &\leq \max \left\{ 24 \cdot \frac{1}{R}, (24 - 1) \cdot \frac{1}{R} + \frac{13}{4R} \right\} = \frac{26.25}{R}.
\end{aligned}$$

Thus, we obtain

$$|G| \leq \frac{1}{R} \left(2 \cdot \frac{13}{4} + 5.25 + 5 + 26.25 \right) g(r, s, t) = \frac{43}{R} g(r, s, t) \leq g(r, s, t),$$

a contradiction. \square

We now consider a RRR-pair, say G_1 and G_2 each red parts and joined by red edges. Then G_R is an (R_3, B_4) -graph. Suppose first that there is no other red part in G_B . If G_B contains no blue part, then

$$|G| = |G_1| + |G_2| + |G_R| + |G_B| \leq \frac{|G|}{R} (3.25 + 3.25 + 9.75 + 24) = \frac{40.25|G|}{R} < |G|,$$

a contradiction.

Suppose next that there is blue part G_3 in G_B . Let F_2 be the set of parts which are joined by blue edges with G_2 and G_3 , and let F_3 be the set of parts which are joined by blue edges with G_2 and by red edges with G_3 . Then F_2 is an (R_5, B_2) -graph and F_3 is an (R_4, B_4) -graph. Suppose F_3 contains no blue part, then

$$\begin{aligned}
|G| &= |G_1| + |G_2| + |G_R| + |G_3| + |F_2| + |F_3| \\
&\leq \frac{|G|}{R} (3.25 + 3.25 + 9.75 + 3.25 + 4 \cdot 1 + 17) \\
&= \frac{40.5|G|}{R} < |G|,
\end{aligned}$$

a contradiction. Now suppose that F_3 contains a blue part G_4 . Let G_{4R} and G_{4B} denote the parts joined by red or blue edges with G_4 . Then G_{4R} is an (R_3, B_4) -graph and G_{4B} is an (R_4, B_2) -graph. So we obtain

$$\begin{aligned}
|G| &= |G_1| + |G_2| + |G_R| + |G_3| + |F_2| + |G_4| + |G_{4R}| + |G_{4B}| \\
&\leq \frac{|G|}{R} (3.25 + 3.25 + 9.75 + 3.25 + 4 \cdot 1 + 3.25 + 6.5 + 3 \cdot 1) \\
&= \frac{36.25|G|}{R} < |G|,
\end{aligned}$$

a contradiction.

Hence we may assume that there is a red part G_3 in G_B . Let F_1 be the set of parts which are joined by red edges with G_1 and by blue edges with G_2 , let

F_2 be the set of parts which are joined by blue edges with G_2 and G_3 , and let F_3 be the set of parts which are joined by blue edges with G_2 and by red edges with G_3 . Then F_1 is an (R_3, B_4) -graph, F_2 is an (R_5, B_2) -graph, and F_3 is an (R_4, B_4) -graph. By Lemma 3 we have $w(F_i) \leq \frac{9.75}{R}$ for $i = 1, 3$. By Lemma 4, if (i) or (ii) or (iii) holds, then $w(F_2) \leq \frac{13.5}{R}$, so

$$\begin{aligned} |G| &= |G_1| + |G_2| + |G_3| + |F_1| + |F_2| + |F_3| \\ &\leq \frac{|G|}{R} (3 \cdot 3.25 + 2 \cdot 9.75 + 13.5) \\ &= \frac{42.75|G|}{R} < |G|, \end{aligned}$$

a contradiction.

Hence F_2 contains two red parts G_3 and G_4 joined by red edges.

Suppose first that G_1 and G_4 as well as G_2 and G_3 are joined by red edges. Let F_1 be the set of parts which are joined by red edges with G_1 and by blue edges with G_2 and G_4 , let F_2 be the set of parts which are joined by red edges with G_3 and by blue edges with G_2 and G_4 , and let F_3 be the set of parts which are joined by blue edges with G_1 and G_3 . Then F_1 and F_2 are (R_3, B_3) -graphs and F_3 is an (R_5, B_3) -graph.

By Lemmas 2 and 4, we obtain

$$|G| = \sum_{i=1}^4 |G_i| + \sum_{j=1}^3 |F_j| \leq \frac{|G|}{R} (4 \cdot 3.25 + 2 \cdot 6.5 + 16.25) = \frac{42.25|G|}{R} < |G|,$$

a contradiction.

Hence we may assume that G_1 is joined by blue edges with G_3 and G_4 . By Lemma 3, we have $w(F_i) \leq \frac{9.75}{R}$ for $i = 1, 3$. By Lemma 4, if (i) or (ii) or (iii) holds, then $w(F_2) \leq \frac{13.5}{R}$. So we obtain

$$\begin{aligned} |G| &= |G_1| + |G_2| + |G_3| + |F_1| + |F_2| + |F_3| \\ &\leq \frac{|G|}{R} (3 \cdot 3.25 + 2 \cdot 9.75 + 13.5) \\ &= \frac{42.75|G|}{R} < |G|, \end{aligned}$$

a contradiction.

Hence F_2 contains two red parts joined by red edges. By symmetry, replacing G_4 by G_3 , we obtain by Lemma 4 (ii) that G_3 also contains no blue parts.

Now we consider the subgraph H spanned by G_3, G_4, F_2 and F_3 . These parts are all adjacent in blue to G_1 . If $w(H) \leq \frac{26.75}{R}$, then we obtain

$$|G| = |G_1| + |G_2| + |F_1| + |H| \leq \frac{|G|}{R} (2 \cdot 3.25 + 9.75 + 26.75) = \frac{43|G|}{R} \leq |G|,$$

a contradiction.

Next we observe that F_3 is an (R_3, B_3) -graph. If $w(F_3) \leq 4$, then we obtain

$$w(H) \leq \frac{1}{R} (2 \cdot 3.25 + 5 \cdot 3.25 + 4) = \frac{26.75}{R},$$

which gives a contradiction as before.

So we may assume that $w(F_3) > 4$. Now we obtain the following two final cases:

(i) H contains nine red parts.

Since $R(3, 4) = 9$, there is a blue K_4 or a red K_3 leading to a red K_6 , a contradiction.

(ii) H contains eight red parts and a free part. Now contract every red part to a red vertex, we obtain a graph H' with eight red vertices and a vertex. Now $R(3, 4) = 9$ gives a blue K_4 or a red K_3 with at least two red vertices implying that there is a red complete subgraph with at least $2 \cdot 2 + 1 = 5$ vertices, a contradiction, completing the proof of Case 6 and Theorem 4. \square

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