Verified Operational Semantics in Logical Form: A Lean 4 Formalization of the OSLF Algorithm (DRAFT --- February 23, 2026)

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Abstract

We present a comprehensive Lean 4 formalization of Operational Semantics in Logical Form (OSLF), the algorithm that mechanically derives spatial-behavioral type systems from rewrite rules. Our formalization spans 41,700+ lines across 118 Lean files (89 in the core OSLF tree, 29 in the process-calculi layer), with 0 sorries and 0 custom axioms across the entire pipeline. The formalization covers: (i) MeTTaIL, a meta-language for defining process calculi with capture-avoiding substitution and a totalized pattern matcher; (ii) the ϱ-calculus with full reduction semantics, structural congruence, and modal type system; (iii) the abstract OSLF framework instantiated for *six* languages (ϱ-calculus, lambda calculus, Petri nets, TinyML, MeTTaMinimal, and MeTTaFull), each with *fully proven* Galois connections $\Diamond \dashv \blacksquare$; (iv) executable rewrite engines (specialized and generic), including premise-aware rewriting with pluggable relation environments, proven sound with respect to declarative reduction specifications; (v) a *constructor category* built from sort-crossing constructors with a `SubobjectFibration` and `ChangeOfBase`, connecting to the GSLT categorical infrastructure; (vi) *derived typing rules* where the modal operator ($\Diamond$ or $\blacksquare$) assigned to each constructor is determined automatically by its position in the constructor category; (vii) a *presheaf-primary categorical lift* with interface-selected base categories, representable-fiber bridges, and graph-object reduction semantics; (viii) a *Beck–Chevalley analysis* of substitution as change-of-base, with a proven counterexample showing the strong condition fails and concrete representable/graph square theorems; (ix) a MeTTa Core interpreter specification with confluence and progress; and (x) *paper-parity endpoints* providing a unified canonical package (`coreMain_paper_parity_full_package`) that bundles Theorem 1 forward substitutability, full presheaf Grothendieck category, scoped/full comparison, TOGL graph-modal bridge, and internal-language bridge for any language instance. All Galois connections, the type soundness theorem, the engine soundness theorem, the constructor fibration, the derived typing rules, the Beck–Chevalley analysis, and the π-to-ϱ encoding all carry zero sorries.

1 Introduction

The OSLF algorithm [2] takes a rewrite system as input and produces a spatial-behavioral type system as output. The core insight is that every reduction relation induces a pair of adjoint modal operators:

$$\Diamond\varphi = \{p \mid \exists q.\ p \rightsquigarrow q \ \wedge\ q \in \varphi\} \quad \text{(step-future / possibly)} \tag{1}$$
$$\blacksquare\varphi = \{q \mid \forall p.\ p \rightsquigarrow q \ \Rightarrow\ p \in \varphi\} \quad \text{(step-past / rely)} \tag{2}$$

and that $\Diamond \dashv \blacksquare$ forms a Galois connection. Combined with the spatial decomposition from parallel composition, this yields a type system where types are "behavioral neighborhoods" and typing is substitutability under bisimulation.

Previous treatments of OSLF were paper-only. We give the first machine-checked formalization, connecting:

- a *generic* abstract framework (any rewrite system),
- a *concrete* ϱ-calculus instance with all rules proven,
- a *categorical* derivation via a constructor category with fibered change-of-base and derived typing rules,
- an *executable* reduction engine proven sound w.r.t. the spec, and
- *six language instances* validating the full pipeline.

Contributions.

1. A Lean 4 formalization of the OSLF algorithm as an abstract structure (`RewriteSystem` → `OSLFTypeSystem`) and its full ϱ-calculus instance (`rhoOSLF`) with a proven Galois connection.
2. A categorical proof that the Galois connection arises from the adjoint triple $\exists_f \dashv f^* \dashv \forall_f$ applied to the reduction span, with the result shown equal to the direct ϱ-calculus modalities.
3. A *constructor category* built from sort-crossing constructors of any `LanguageDef`, with a `SubobjectFibration` and `ChangeOfBase` connecting to the GSLT infrastructure.
4. *Derived typing rules*: the modal operator ($\Diamond$/$\blacksquare$/id) assigned to each constructor is determined automatically by its classification (quoting/reflecting/neutral), and the assignment is proven correct for the ϱ-calculus.
5. A *Beck–Chevalley analysis* of the COMM rule as change-of-base along the substitution map, with a proven counterexample showing the GSLT strong Beck–Chevalley condition does not hold for the constructor fibration, plus representable-fiber and graph-object square theorems consumed by checker-facing corollaries.

6. Executable reduction engines handling COMM, DROP, and context descent, with machine-checked soundness theorems.

7. Premise-aware declarative reduction relations (`DeclReducesWithPremises`) independent of the engine, with proven soundness and completeness of the generic premise-aware engine.

8. *Six* OSLF instantiations validating generality: ϱ-calculus, lambda calculus, Petri nets, and TinyML (a multi-sort CBV λ-calculus with booleans, pairs, and thunks), plus MeTTaMinimal and MeTTaFull state-machine clients.

9. MeTTaIL: a meta-language for defining process calculi, with totalized pattern matching, 29 proven theorems about capture-avoiding substitution, and a complete ϱ-calculus language definition.

10. A verified bounded model checker for OSLF formulas, with support for predecessor-based $\blacksquare$ checking and proven soundness.

11. A MeTTa Core interpreter specification with progress, confluence, and barbed bisimulation properties (97 proven theorems, 0 sorries).

12. A dedicated sorry-free core entrypoint (`CoreMain.lean`) and machine-readable FULL tracker (`Framework/FULLStatus.lean`) for review.

13. A *paper-parity canonical package* (`coreMain_paper_parity_full_package`) unifying Theorem 1 (substitutability), the full presheaf Grothendieck category, scoped/full comparison, TOGL graph-modal bridge, and internal-language bridge into a single endpoint.

14. A *strict paper-claim tracker* (`Framework/PaperClaimTracker.lean`) mapping 27 verifiable claims from three source papers to concrete Lean theorem names, with 25 fully proven and 2 assumption-scoped.

15. *Assumption-necessity counterexamples* (`Framework/AssumptionNecessity.lean`) proving that the `types.Nonempty` guard on Π/Σ formation and the `hClosed` fragment-closure hypothesis cannot be discharged in general.

16. *Robustness canaries* (`Framework/PaperParityCanaries.lean`) instantiating the canonical package on ϱ-calculus and λ-calculus, plus a negative canary showing `hClosed` does real work.

Scope Boundary and Non-Claims. This draft reports what is machine-checked in the current Lean corpus; it is not a claim that every construction discussed across the broader literature folders has been fully formalized end-to-end. In particular, the cited OSLF and Native Type Theory papers explicitly list larger future directions that remain outside the current core claim surface, including:

- full internal natural-transformation presentation of modalities ($\Omega \to \Omega$ endomorphisms) over the presheaf route,
- larger extensions such as stochastic and quantum variants listed as future-work directions in the source papers.

Note: the Grothendieck construction for Native Type Theory, the internal language (frame-derived $\to/\neg$, $\top/\bot/\wedge/\vee$), and the TOGL graph-modal bridge are now formalized as theorem-level endpoints bundled in `coreMain_paper_parity_full_package`; the Π/Σ type formation is assumption-scoped (`types.Nonempty` guard, proven necessary). Accordingly, this document distinguishes *verified core endpoints* from *research-frontier extensions*, and the strict paper-alignment status is tracked explicitly in `Mettapedia/OSLF/Framework/FULLStatus.lean`.

2 Background: The ϱ-Calculus

The reflective higher-order calculus [1] extends the π-calculus with:

- *Quoting*: any process P can be turned into a name $@P$ (name = quoted process).
- *Dequoting*: $*x$ recovers the process quoted by name x.
- *No built-in names*: all names arise from quoting, giving the calculus a reflective character.

The reduction rules are:

$$\text{COMM:} \quad \{n!(q) \mid \mathbf{for}(x \leftarrow n)\{p\} \mid \mathit{rest}\} \rightsquigarrow \{p[@q/x] \mid \mathit{rest}\} \tag{3}$$

$$\text{DROP:} \quad *(@P) \rightsquigarrow P \tag{4}$$

plus structural congruence (11 rules) and contextual reduction under parallel composition.

3 Formalization Architecture

3.1 Module Structure

The formalization is organized in seven directories plus standalone files:

Component	Lines	Sorries	Content
MeTTaIL/	3,592	0	Meta-language AST, substitution, matching, declarative r
MeTTaCore/	3,839	0	Interpreter specification
Framework/	14,708	0	Abstract OSLF + categorical bridge + 6 instances
OSLF/RhoCalculus/	60	0	OSLF ϱ-calculus facades
NativeType/	1,444	0	Native type construction + Grothendieck
Formula.lean	1,204	0	Verified bounded model checker
Main/CoreMain/SpecIndex	2,001	0	Focused OSLF re-exports + spec index
Languages/PiCalculus/	8,768	0	π-calculus + ϱ-encoding
Languages/RhoCalculus/	4,848	0	Concrete ϱ-calculus + engine
Total	**41,729**	**0**	**0 sorries, 0 axioms**

Operational entrypoints (`CoreMain.lean` and `Main.lean`) remain on the core boundary, with process-calculus facades exposed under `Mettapedia/Languages/ProcessCalculi*.lean`. The $\pi \to \varrho$ encoding (forward simulation, backward normalization, weak bisimilarity, and encoding morphism) is fully proven with 0 sorries; for detailed status see `AIrxiv/airxiv_2026-02-19_pi_rho_embedding_status.tex`.

3.2 Canonical Endpoint Map

For review and downstream reuse, the canonical API surface is concentrated in `Mettapedia/OSLF/CoreMain.lean` and `Mettapedia/OSLF/Framework/PiRhoCanonicalBridge.lean`. The exact endpoint names are:

CoreMain-facing entrypoints

- `piRho_coreMain_canonical_contract_end_to_end`
- `coreMain_piRho_canonical_contract`
- `coreMain_piRho_contract_projection_api`
- `coreMain_nativeType_piOmega_translation_endpoint`
- `coreMain_nativeType_piOmegaProp_translation_endpoint`
- `coreMain_nativeType_piProp_colax_rules_endpoint`
- `coreMain_nativeType_full_presheaf_morphism_endpoint`
- `coreMain_nativeType_scoped_full_constructor_comparison_package`
- `coreMain_category_topos_package`
- `coreMain_topos_internal_language_bridge_package`
- `coreMain_togl_graph_modal_bridge_package`
- `coreMain_section12_worked_examples`

- coreMain_dependent_parametric_generated_typing
- coreMain_paper_parity_full_package
- coreMain_paper_parity_canonical_package

Bridge-level canonical contracts

- PiRhoCoreMainCanonicalContract
- piRho_coreMain_predDomain_endpoint
- piRho_coreMain_predDomain_transfer_bundle_end_to_end
- piRho_coreMain_predDomain_reachable_state_end_to_end
- reachableCoreStarFiniteSubrelation
- reachableDerivedStarFiniteSubrelation
- hm_converse_rhoCoreCanonicalRel
- hm_converse_rhoDerivedCanonicalRel
- ToposTOGLBridge.topos_internal_language_bridge_package
- ToposTOGLBridge.togl_graph_modal_bridge_package
- ToposTOGLBridge.graphChainN_iff_relCompN
- ToposTOGLBridge.diamondIterN_iff_graphChainN
- ToposTOGLBridge.togl_internal_graph_correspondence_layer
- NativeType.fullPresheafGrothendieckCategory
- NativeType.scoped_full_scoped_obj_roundtrip
- AssumptionNecessity.types_nonempty_necessary_for_piSigma
- AssumptionNecessity.hClosed_necessary_for_fragment
- AssumptionNecessity.not_global_hImageFinite_rhoCoreStarRel

The `Framework/` directory (14,708 lines, 32 files) is the largest component. Key files:

File	Lines	Content
PiRhoCanonicalBridge.lean	1,876	$\pi \to \varrho$ canonical bridge
CategoryBridge.lean	1,603	Presheaf topos, Yoneda, Ω classifier
BeckChevalleyOSLF.lean	1,233	Substitution as change-of-base
TinyMLInstance.lean	758	CBV λ-calculus with booleans/pairs/thunks
PLNSelectorLanguageDef.lean	723	PLN selector as `LanguageDef`
MeTTaMinimalInstance.lean	700	MeTTa state-machine client
ConstructorCategory.lean	460	Sort quiver + free category
TypeSynthesis.lean	441	`langOSLF` pipeline
PLNSelectorGSLT.lean	373	PLN selector rules as OSLF/GSLT
DerivedTyping.lean	346	Generic typing rules from constructor category
MeTTaFullInstance.lean	346	Full MeTTa language definition
ModalEquivalence.lean	353	Constructor change-of-base $\leftrightarrow$ modalities
GeneratedTyping.lean	474	`GenHasType` typing rules
SynthesisBridge.lean	294	Three-layer bridge
LanguageMorphism.lean	506	Language translation infrastructure
ConstructorFibration.lean	251	`SubobjectFibration` + `ChangeOfBase`
DerivedModalities.lean	250	Adjoint triple derivation
PetriNetInstance.lean	233	Petri net OSLF instance
LambdaInstance.lean	218	Lambda calculus OSLF instance
RewriteSystem.lean	196	Abstract OSLF input/output
RhoInstance.lean	134	ϱ-calculus instance
ToposTOGLBridge.lean	767	TOGL graph-modal + internal language
ToposReduction.lean	401	Graph-object reduction semantics
FULLStatus.lean	554	Machine-readable paper-parity tracker
AssumptionNecessity.lean	279	Assumption-necessity counterexamples
PaperClaimTracker.lean	156	Strict 27-claim paper-claim tracker
PaperParityCanaries.lean	104	Robustness canaries (positive + negative)

3.3 Abstract OSLF Framework

The abstract layer defines two key structures:

Listing 1: The OSLF input and output (RewriteSystem.lean)

```
structure RewriteSystem where
  Sorts    : Type*
  procSort : Sorts
  Term     : Sorts -> Type*
  Reduces  : Term procSort -> Term procSort -> Prop

structure OSLFTypeSystem where
  Sorts    : Type*
  procSort : Sorts
  Term     : Sorts -> Type*
  Pred     : Sorts -> Type*
  [frame   : (S : Sorts) -> Frame (Pred S)]
  satisfies  : (S : Sorts) -> Term S -> Pred S -> Prop
  diamond    : Pred procSort -> Pred procSort
```

```
  box         : Pred procSort -> Pred procSort
  galois      : GaloisConnection diamond box
```

3.4 The Galois Connection

The central theorem: $\Diamond \dashv \blacksquare$.

Theorem 1 (Galois Connection, 0 sorries)**.** *For all predicates φ, ψ on ϱ-calculus processes:*

$$\Diamond\varphi \leq \psi \iff \varphi \leq \blacksquare\psi$$

where $\Diamond\varphi(p) = \exists q.\ p \rightsquigarrow q \wedge \varphi(q)$ and $\blacksquare\psi(q) = \forall p.\ p \rightsquigarrow q \rightarrow \psi(p)$.

In Lean:

Listing 2: The Galois connection (Reduction.lean)

```
theorem galois_connection :
    GaloisConnection possiblyProp relyProp := by
  intro phi psi
  constructor
  . intro h q hrely p hred
    exact h (hrely p hred)
  . intro h p hposs
    exact h.2 p hposs.1 hposs.2
```

3.5 Categorical Derivation via Adjoint Triples

The Galois connection arises from the general theory of change-of-base along a span.

Definition 2 (Reduction Span)**.** *A span $\mathcal{S} \xleftarrow{\text{src}} E \xrightarrow{\text{tgt}} \mathcal{S}$ where E is the set of reduction edges,* src *extracts the source, and* tgt *extracts the target.*

From any such span we derive three operations on predicates:

$$f^*(\psi)(e) = \psi(\text{tgt}(e)) \quad \text{(pullback)} \tag{5}$$

$$\exists_f(\varphi)(q) = \exists e.\ \text{tgt}(e) = q \wedge \varphi(e) \quad \text{(direct image)} \tag{6}$$

$$\forall_f(\varphi)(q) = \forall e.\ \text{tgt}(e) = q \rightarrow \varphi(e) \quad \text{(universal image)} \tag{7}$$

Theorem 3 (Derived Galois, 0 sorries)**.** *For any `ReductionSpan`, the composition $\Diamond = \exists_{\text{src}} \circ \text{tgt}^*$ and $\blacksquare = \forall_{\text{tgt}} \circ \text{src}^*$ form a Galois connection. Furthermore, for the ϱ-calculus span, the derived operators equal the concrete `possiblyProp` and `relyProp`.*

Listing 3: Derived modalities equal concrete (DerivedModalities.lean)

```
theorem derived_diamond_eq_possiblyProp :
    derivedDiamond rhoSpan = possiblyProp := ...
```

```
theorem derived_box_eq_relyProp :
    derivedBox rhoSpan = relyProp := ...

theorem rho_galois_from_span :
    GaloisConnection (derivedDiamond rhoSpan)
                     (derivedBox rhoSpan) :=
  derived_galois rhoSpan
```

4 Constructor Category and Fibration

A key contribution of this formalization is the *constructor category*: a non-discrete category built from the sort-crossing constructors of any `LanguageDef`, replacing the discrete `Discrete R.Sorts` from the earlier categorical lift.

4.1 Sort Quiver and Free Category

Given a `LanguageDef`, we extract the *unary sort-crossing constructors*: grammar rules with exactly one `.simple` parameter whose base sort differs from the constructor's output sort. These become the arrows of a quiver on the language's sorts.

Listing 4: Constructor category (ConstructorCategory.lean)

```
-- Sort type: valid sort names
def LangSort (lang : LanguageDef) :=
  { s : String // s IN lang.types }

-- Sort-crossing arrows
structure SortArrow (lang : LanguageDef)
    (dom cod : LangSort lang) where
  label : String
  valid : (label, dom.val, cod.val) IN unaryCrossings lang

-- Free category: paths of sort-crossing arrows
inductive SortPath (lang : LanguageDef)
    : LangSort lang -> LangSort lang -> Type where
  | nil  : SortPath lang s s
  | cons : SortPath lang s t -> SortArrow lang t u
         -> SortPath lang s u
```

For the ϱ-calculus: 2 objects (Proc, Name), 2 arrows (`NQuote`: Proc $\to$ Name, `PDrop`: Name $\to$ Proc), and composites `PDrop` $\circ$ `NQuote` and `NQuote` $\circ$ `PDrop`.

Each arrow has a *semantic function* `arrowSem`: wrapping a pattern in the constructor's `.apply` node (e.g., $p \mapsto$ `NQuote`(p)). This extends to paths via `pathSem`, with a proven composition law `pathSem_comp`.

A *universal property* (free category lifting) is proven: any assignment of objects and arrows to a target category $\mathcal{C}$ lifts uniquely to a functor `liftFunctor`, with uniqueness proven in `lift_map_unique`.

4.2 SubobjectFibration and ChangeOfBase

Over the constructor category we build a fibration and change-of-base (`Framework/ConstructorFibration.lean`, 251 lines, 0 sorries):

Listing 5: Constructor fibration (ConstructorFibration.lean)

```
-- Each sort has fiber Pattern -> Prop (a Frame)
def constructorFibration (lang : LanguageDef) :
    SubobjectFibration (ConstructorObj lang) where
  Sub   := fun _ => Pattern -> Prop
  frame := fun _ => Pi.instFrame

-- Full change-of-base with proven adjunctions
def constructorChangeOfBase (lang : LanguageDef) :
    ChangeOfBase (constructorFibration lang) where
  pullback f        := pb (pathSem lang f)
  directImage f     := di (pathSem lang f)
  universalImage f  := ui (pathSem lang f)
  direct_pullback_adj f := di_pb_adj (pathSem lang f)
  pullback_universal_adj f := pb_ui_adj (pathSem lang f)
  ...
```

The adjunctions $\exists_f \dashv f^* \dashv \forall_f$ are proven (not axiomatized), following from the generic `di_pb_adj` / `pb_ui_adj` in `DerivedModalities.lean`.

Key proven properties:

- **Pullback functoriality**: $(f \circ g)^* = g^* \circ f^*$ (from `pathSem_comp`), $id^*(\varphi) = \varphi$.
- **Frame morphism**: $f^*(\varphi \wedge \psi) = f^*(\varphi) \wedge f^*(\psi)$ and $f^*(\top) = \top$ (both by `rfl`).
- **Monotonicity** of all three operations (from adjunctions).

4.3 Modal Equivalence

The file `Framework/ModalEquivalence.lean` (353 lines) connects the constructor change-of-base to the OSLF modalities:

Theorem 4 (Modal = Change-of-Base, 0 sorries)**.** *The* OSLF *modalities are Set-level change-of-base along the reduction span:*

$$\Diamond_{lang} = \exists_{\mathrm{src}} \circ \mathrm{tgt}^* \qquad \textit{(definitional)}$$
$$\blacksquare_{lang} = \forall_{\mathrm{tgt}} \circ \mathrm{src}^* \qquad \textit{(definitional)}$$

For the ϱ-calculus, this gives the *typing actions*:

- `NQuote` (Proc $\to$ Name): $\varphi \mapsto \Diamond\varphi$ ("can reduce to φ")
- `PDrop` (Name $\to$ Proc): $\alpha \mapsto \blacksquare\alpha$ ("all predecessors satisfy α")

The composite `PDrop` $\circ$ `NQuote` gives $\blacksquare \circ \Diamond$ and `NQuote` $\circ$ `PDrop` gives $\Diamond \circ \blacksquare$. The typing action Galois connection $\Diamond \dashv \blacksquare$ is proven as an instance of the general language Galois connection.

4.4 Derived Typing Rules

The file `Framework/DerivedTyping.lean` (346 lines, 0 sorries) derives typing rules generically from the constructor category structure.

Each sort-crossing arrow is automatically classified:

Listing 6: Constructor classification (DerivedTyping.lean)

```
inductive ConstructorRole where
  | quoting    -- domain = procSort: introduces diamond
  | reflecting -- codomain = procSort: introduces box
  | neutral    -- neither: identity

def classifyArrow (lang : LanguageDef) (procSort : String)
    (arr : SortArrow lang dom cod) : ConstructorRole :=
  if dom.val = procSort then .quoting
  else if cod.val = procSort then .reflecting
  else .neutral
```

Theorem 5 (Classification Correctness, 0 sorries)**.** *For the ϱ-calculus:*

- `NQuote` *is classified as* quoting*; its typing action equals* $\Diamond$*.*
- `PDrop` *is classified as* reflecting*; its typing action equals* $\blacksquare$*.*

The `DerivedHasType` judgment provides a generic typing rule for unary sort-crossing constructors: apply the constructor's typing action ($\Diamond$/$\blacksquare$/id) to the argument's predicate, then tag the result at the output sort.

4.5 Beck–Chevalley for Substitution

The file `Framework/BeckChevalleyOSLF.lean` (1,233 lines, 0 sorries) analyzes the COMM rule's substitution $p[@q/x]$ as a change-of-base map.

Definition 6 (COMM Substitution Map)**.** σ_q *: Pattern* $\to$ *Pattern defined by* $\sigma_q(\mathit{pBody}) = \mathit{openBVar}\ 0\ (\mathtt{NQuote}(q))\ \mathit{pBody}$.

This induces the adjoint triple $\exists_{\sigma_q} \dashv \sigma_q^* \dashv \forall_{\sigma_q}$ via the same `pb`/`di`/`ui` infrastructure, and the modal+substitution Galois connections compose:

Theorem 7 (Composed Galois, 0 sorries)**.** $\Diamond \circ \exists_{\sigma_q} \dashv \sigma_q^* \circ \blacksquare$

The COMM rule's type preservation (`comm_preserves_type` from `Soundness.lean`) is re-expressed categorically as a pullback inequality:

Theorem 8 (Substitutability as Pullback, 0 sorries)**.** *For any typing context* Γ*, type* τ*, variable* x*, value* q*, and type* σ *with* $\Gamma \vdash q : \sigma$*:*

$$\mathtt{typedAt}(\Gamma[x \mapsto \sigma], \tau) \ \leq\ \sigma_q^*(\mathtt{typedAt}(\Gamma, \tau))$$

The COMM substitution map factors through the constructor semantics: $\sigma_q(p) = \mathit{openBVar}\ 0\ (\mathit{pathSem}\ \mathtt{nquoteMor}\ q)\ p$.

Theorem 9 (Strong Beck–Chevalley Fails, 0 sorries)**.** *The GSLT's universal Beck–Chevalley condition $f^* \circ \exists_g = \exists_{\pi_1} \circ \pi_2^*$ does* ***not*** *hold for the constructor fibration.*

Concretely, for the commuting square $\mathtt{PDrop} \circ \mathtt{NQuote} = \mathtt{PDrop} \circ \mathtt{NQuote}$*:*

$$\mathtt{PDrop}^*(\exists_{\mathtt{PDrop}}(\top))(\mathtt{fvar}\, x) = \top$$

but

$$\exists_{\mathtt{NQuote}}(\mathtt{NQuote}^*(\top))(\mathtt{fvar}\, x) = \bot$$

because $\mathtt{NQuote}(q) \neq \mathtt{fvar}\, x$ *for all* q*.*

The counterexample is proven by exhibiting a concrete witness at `fvar` "x": the LHS is inhabited by $\langle \mathtt{fvar}\ x, \mathit{rfl}, \top \rangle$ while the RHS requires some p with $\mathtt{NQuote}(p) = \mathtt{fvar}\ x$, which is impossible since $\mathtt{NQuote}(p) = \mathtt{.apply}$ "NQuote" $[p]$.

5 Executable Rewrite Engine

A formalization that only *specifies* reduction is incomplete for verification of actual implementations. We provide `reduceStep`, a computable function that enumerates all one-step reducts, proven sound w.r.t. the propositional `Reduces`.

5.1 Engine Design

Listing 7: The executable engine (Engine.lean)

```
def reduceStep (p : Pattern) (fuel : Nat := 100)
    : List Pattern :=
  match fuel with
  | 0 => []
  | fuel + 1 =>
    match p with
    | .collection .hashBag elems none =>
      let commReducts := findAllComm elems
      let parReducts :=
        reduceElemsAux (reduceStep . fuel) elems
        |>.map fun (i, elem') =>
          .collection .hashBag (elems.set i elem') none
      commReducts ++ parReducts
    | .apply "PDrop" [.apply "NQuote" [inner]] =>
      [inner]
    | _ => []
```

The engine handles COMM (all output-input pairs on matching channels), DROP ($*(@P) \rightsquigarrow P$), and PAR (recursive reduction under parallel composition). Non-deterministic races produce multiple reducts in the output list.

5.2 Soundness

Theorem 10 (Engine Soundness, 0 sorries)**.** *Every reduct computed by* `reduceStep` *corresponds to a valid* `Reduces`*:*

$$q \in \texttt{reduceStep}(p, \mathit{fuel}) \implies \exists d : p \rightsquigarrow q$$

The proof proceeds by case analysis:

- **COMM**: Each output-input pair is located by `findAllComm`, whose specification is proven to identify valid COMM redex positions. The soundness chain is: list permutation (`perm_extract_two`) → structural congruence (`par_perm`) → `Reduces.comm` → `Reduces.equiv`.
- **DROP**: Direct pattern match yields `Reduces.drop`.
- **PAR**: The `reduceElemsAux_spec` lemma extracts the sub-element index and recursive reduct. List decomposition via `List.set_eq_take_append_cons_drop` connects `List.set` to the $\mathit{before} +\!\!+ [p] +\!\!+ \mathit{after}$ form required by `Reduces.par_any`.

6 Generic MeTTaIL Rewrite Framework

The specialized ϱ-calculus engine is lifted to a language-parametric framework. Given any `LanguageDef` (a list of sorts, constructors, equations, and rewrite rules), the generic engine automatically:

1. matches concrete terms against rule LHS patterns (multiset matching with rest variables),
2. produces variable bindings via alpha-renaming for binders,
3. applies bindings to rule RHS patterns (including substitution evaluation), and
4. iterates under congruence (subterm rewriting inside parallel compositions).

Listing 8: Premise-aware generic rewrite step (Engine.lean)

```
structure RelationEnv where
  tuples : String -> List Pattern -> List (List Pattern)

def applyRuleWithPremisesUsing
    (relEnv : RelationEnv) (lang : LanguageDef)
    (rule : RewriteRule) (term : Pattern) : List Pattern :=
  (matchPattern rule.left term).flatMap fun bs =>
    (applyPremisesWithEnv relEnv lang rule.premises bs).map fun bs'
        =>
      applyBindings bs' rule.right

def rewriteStepWithPremisesUsing
    (relEnv : RelationEnv) (lang : LanguageDef) (term : Pattern) :
        List Pattern :=
  lang.rewrites.flatMap fun rule => applyRuleWithPremisesUsing
      relEnv lang rule term
```

Multiset matching. The key algorithmic contribution is `matchBag`: for a collection pattern with n elements and an optional rest variable, it enumerates all ways to match the n pattern elements against term elements (backtracking search over permutations), binding unmatched elements to the rest variable.

Declarative reduction. The files `MeTTaIL/DeclReduces.lean` and `MeTTaIL/DeclReducesWithPremises.lean` provide declarative inductive reduction relations independent of the executable engine. The generic premise-aware engine is proven both *sound* and *complete* with respect to the premise-aware specification (0 sorries).

7 MeTTaIL and MeTTa Core

7.1 MeTTaIL: The Meta-Language

MeTTaIL ("MeTTa Intermediate Language") defines the AST shared by all process calculi in the formalization, using locally nameless representation. The core type is `Pattern` with 7 constructors:

Listing 9: Pattern AST — locally nameless (Syntax.lean)

```
inductive Pattern where
  | bvar       : Nat -> Pattern          -- bound variable (de
     Bruijn)
  | fvar       : String -> Pattern       -- free variable /
     metavariable
  | apply      : String -> List Pattern -> Pattern
  | lambda     : Pattern -> Pattern      -- binder (no name)
  | multiLambda: Nat -> Pattern -> Pattern
  | subst      : Pattern -> Pattern -> Pattern
  | collection : CollType -> List Pattern
                  -> Option String -> Pattern
```

Key proven properties of substitution (29 theorems, 0 sorries):

- `subst_empty`: empty substitution is the identity;
- `subst_fresh`: substitution on a fresh variable is the identity;
- `commSubst`: the COMM-rule substitution $p[@q/x]$ respects freshness.

7.2 MeTTa Core Interpreter

The `MeTTaCore/` directory provides a complete interpreter specification for Hyperon Experimental MeTTa (3,839 lines, 0 sorries), including:

- `Atom`: the universal term type with `DecidableEq`;
- `Bindings`: variable resolution with merge and transitive lookup;
- `MeTTaState`: the 4-register $\langle i, k, w, o\rangle$ machine;

- `PatternMatch`: bidirectional unification;
- `MinimalOps`: grounded operations ($+, -, *, /, <$, etc.);
- `RewriteRules`: equation-driven rewriting;
- `Atomspace`: multiset-based knowledge base with query operations;
- `Properties`: progress, confluence, and barbed bisimulation.

8 Type Soundness

Theorem 11 (Substitutability, 0 sorries)**.** *If P and Q are bisimilar processes, then they have the same native types:*

$$P \sim Q \implies \forall \tau.\, P : \tau \iff Q : \tau$$

Theorem 12 (Type Preservation, 0 sorries)**.** *The COMM and DROP rules preserve types:*

$$\Gamma \vdash P : \tau \ \wedge \ P \rightsquigarrow Q \implies \Gamma \vdash Q : \tau$$

Both theorems are proven in `RhoCalculus/Soundness.lean` with a 10-constructor typing judgment `HasType` and explicit typing contexts.

9 ϱ-Calculus Instance

The ϱ-calculus is formalized with full reduction semantics (COMM, DROP), structural congruence (11 rules), and multi-step reduction. The OSLF algorithm derives its spatial-behavioral type system, with the Galois connection $\Diamond \dashv \blacksquare$ proven in `RhoCalculus/Soundness.lean` (0 sorries).

10 Type Synthesis: `LanguageDef` $\to$ `OSLFTypeSystem`

The culmination of the formalization is the *type synthesis pipeline*: given any `LanguageDef`, mechanically produce a full `OSLFTypeSystem` with a **proven Galois connection**.

10.1 The Pipeline

The pipeline proceeds in five steps, all implemented in `Framework/TypeSynthesis.lean`:

1. `langReduces lang p q` wraps the executable engine: $q \in$ `rewriteWithContext` *lang p*.
2. `langRewriteSystem lang` assembles a `RewriteSystem`.
3. `langSpan lang` builds the reduction span (edges = reductions).

4. `langDiamond`/`langBox` derive modal operators via `derivedDiamond`/`derivedBox` from the adjoint triple.

5. `langOSLF lang` packages everything into an `OSLFTypeSystem`, with the Galois connection proven by `derived_galois`.

Theorem 13 (Automatic Galois Connection, 0 sorries)**.** *For any `LanguageDef` `lang`:*

$$\Diamond_{lang} \dashv \blacksquare_{lang}$$

where $\Diamond_{lang} = \exists_{\mathrm{src}} \circ \mathrm{tgt}^*$ *and* $\blacksquare_{lang} = \forall_{\mathrm{tgt}} \circ \mathrm{src}^*$ *are derived from the reduction span. No manual proof is needed per language.*

11 Language Instances

The pipeline is validated by six language instances of increasing complexity:

11.1 Lambda Calculus (1 sort, 0 crossings)

Untyped lambda calculus with β-reduction (`Framework/LambdaInstance.lean`, 218 lines). One sort (`Term`), two constructors (`App`, `Lam`), one reduction rule. The constructor category is discrete (no sort-crossing constructors). Six executable demos verify β-reduction, multi-step normalization, and formula checking.

11.2 Petri Net (1 sort, 0 crossings)

A simple Petri net with four places and two transitions (`Framework/PetriNetInstance.lean`, 233 lines). Validates that multiset (bag) matching works correctly without any abstraction or substitution machinery. Key properties proven by `native_decide`:

- $\{D\}$ is a dead marking (0 reducts);
- $\{A, B\}$ has exactly 1 reduct via transition T_1.

11.3 ϱ-Calculus (2 sorts, 2 crossings)

The primary instance with sorts Proc and Name, constructors `NQuote`: Proc $\to$ Name and `PDrop`: Name $\to$ Proc. This is the most extensively developed instance, with the full type soundness proof, specialized engine, and Beck–Chevalley analysis (Sections 4–4.5).

11.4 TinyML (2 sorts, 2 crossings, 6 rules)

A call-by-value λ-calculus with booleans, pairs, and thunks (`Framework/TinyMLInstance.lean`, 758 lines, 0 sorries).

Listing 10: TinyML language definition (TinyMLInstance.lean)

```
-- Sorts: Expr (process sort), Val (data sort)
-- Constructors:
--   Expr: App, If, Fst, Snd, Inject(v:Val)
--   Val:  BoolT, BoolF, Lam(^body), PairV, Thunk(e:Expr)
-- Reductions:
--   Beta:     App(Inject(Lam(^body)), Inject(v)) ~> body[v/x]
--   Force:    Inject(Thunk(e)) ~> e
--   IfTrue:   If(Inject(BoolT), t, e) ~> t
--   IfFalse:  If(Inject(BoolF), t, e) ~> e
--   FstPair:  Fst(Inject(PairV(a, b))) ~> Inject(a)
--   SndPair:  Snd(Inject(PairV(a, b))) ~> Inject(b)
```

TinyML mirrors the ϱ-calculus sort structure:

TinyML	**ϱ-Calculus**	**Role**
Expr	Proc	Process sort (carries reduction)
Val	Name	Data sort
`Thunk`: Expr $\to$ Val	`NQuote`: Proc $\to$ Name	Quoting ($\Diamond$)
`Inject`: Val $\to$ Expr	`PDrop`: Name $\to$ Proc	Reflecting ($\blacksquare$)
`Beta`	`COMM`	Main computation rule
`Force`	`DROP`	Quoting/reflecting cancel

The CBV strategy is encoded syntactically: β-reduction requires both the function and argument to be wrapped in `Inject(-)`, ensuring arguments are evaluated to values before substitution.

Key proven results:

- `tinyML_crossings`: exactly 2 sort-crossing constructors (`Inject`, `Thunk`) – `native_decide`.
- `thunk_is_quoting`: `Thunk` classified as quoting; typing action = $\Diamond$.
- `inject_is_reflecting`: `Inject` classified as reflecting; typing action = $\blacksquare$.
- `tinyML_typing_action_galois`: $\Diamond \dashv \blacksquare$ for TinyML typing actions.
- 11 executable demos including a 3-step reduction chain (`force` $\to$ `ifTrue` $\to$ `fstPair`).

11.5 MeTTaMinimal (2 sorts, 1 crossing, 2 rules)

A minimal MeTTa state-machine client (`Framework/MeTTaMinimalInstance.lean`, 700 lines, 0 sorries). One sort crossing (`Eval`: Expr $\to$ State), 2 rules (rewrite and match-fail). Validates the end-to-end path from checker to Beck–Chevalley graph, including `mettaMinimal_checker_to_BC_graph`: the checker's verdicts are compatible with the categorical semantics.

11.6 MeTTaFull (3 sorts, 2 crossings, 4 rules)

A full MeTTa language definition based on the interpreter specification in `MeTTaCore/` (`Framework/MeTTaFullInstance.lean`, 346 lines, 0 sorries). Three sorts (Atom, State, Result), sort-crossing constructors `Interpret`: Atom $\to$ State and `Output`: State $\to$ Result. Four rewrite rules model the interpret–match–rewrite–output cycle. The end-to-end theorem `mettaFull_checker_to_BC_graph` verifies categorical agreement as with MeTTaMinimal.

12 Verified Formula Checker

The file `Formula.lean` provides a verified bounded model checker for OSLF formulas (1,204 lines, 0 sorries). The formula type `OSLFFormula` supports $\Diamond$, $\blacksquare$, $\wedge$, $\vee$, $\implies$, $\top$, $\bot$, and atomic predicates.

Theorem 14 (Checker Soundness, 0 sorries)**.** *If the checker returns* `.sat` *for formula φ at term p, then $p \models \varphi$ in the denotational semantics.*

The checker supports:

- `checkWithPred`: checking with external predecessor functions, enabling bounded $\blacksquare$ verification;
- `aggregateBox`: universal checking of $\blacksquare\varphi$ over a predecessor list, with proven soundness (`aggregateBox_sat`).

13 Categorical Lift and Presheaf Topos

The file `Framework/CategoryBridge.lean` (1,603 lines, 0 sorries) lifts the Set-level OSLF construction into Mathlib's categorical infrastructure, culminating in the presheaf topos $P(T) = \mathbf{Set}^{T^{\mathrm{op}}}$ over the constructor category.

13.1 Modal Adjunction

The predicate type `Pattern → Prop` carries conflicting category instances (`Preorder.smallCategory` vs. `CategoryTheory.Pi`). We introduce a type wrapper `PredLattice` to disambiguate, then lift:

Listing 11: Modal adjunction (CategoryBridge.lean)

```
noncomputable def langModalAdjunction (lang : LanguageDef) :
    (langGaloisL lang).monotone_l.functor |-
    (langGaloisL lang).monotone_u.functor :=
  (langGaloisL lang).adjunction
```

This provides a categorical `Adjunction` between the $\Diamond$ and $\blacksquare$ endofunctors on the predicate preorder category, for any `LanguageDef`.

13.2 Presheaf Topos and Yoneda Representables

Following Williams and Stay [3], we construct the presheaf topos $P(T)$ as a lambda-theory with equality:

Listing 12: Presheaf topos (CategoryBridge.lean)

```
def languagePresheafLambdaTheory (lang : LanguageDef) :
    LambdaTheoryWithEquality where
  Obj := Functor (Opposite (ConstructorObj lang)) Type
  -- CCC structure via Mathlib
```

The Yoneda embedding sends each sort s to a representable presheaf $y(s) \in P(T)$. The subobject classifier Ω is constructive via sieves, and the characteristic-map equivalence $\mathrm{Sub}(y(s)) \cong (y(s) \to \Omega)$ is proven (`languageSortFiber_characteristicEquiv`).

A bridge maps Set-level predicates (`Pattern` → `Prop`) into presheaf-level subobjects $\mathrm{Sub}(y(s))$ under a naturality condition (`languageSortFiber_ofPatternPred`). The reduction relation is internalized as a `Subfunctor` of the pair presheaf with graph-object diamond/box characterization (`langDiamondUsing_iff_exists_graphStep`, `langBoxUsing_iff_forall_graphIncoming`).

13.3 Current Scope and the Full Grothendieck Construction

Our presheaf topos construction provides the categorical *home* for OSLF types but does not yet fully internalize the modalities. Specifically:

What is proven. $P(T)$ exists as a CCC with finite limits, subobject classifier, and exponentials (all via Mathlib). Yoneda representables, fiber characterization, and the predicate-to-subfunctor bridge are proven. The reduction relation is internal to $P(T)$ as a subfunctor. Beck–Chevalley instances for specific squares (COMM, substitution) are proven at the subobject level.

What remains. The $\Diamond/\blacksquare$ operators are defined as Set-level endomorphisms on (`Pattern` → `Prop`) and have not yet been lifted to internal natural transformations $\Omega \to \Omega$ in $P(T)$. This would require:

1. *Internalizing the reduction span*: replacing the constant presheaf with a genuine span $y(\mathrm{Proc}) \leftarrow R \to y(\mathrm{Proc})$ in $P(T)$, where the edge set varies with the generalized-element context.
2. *Internal existence/universal image*: defining $\Diamond/\blacksquare$ as $\exists_{\mathrm{src}} \circ \mathrm{tgt}^*$ and $\forall_{\mathrm{tgt}} \circ \mathrm{src}^*$ at the subfunctor level.
3. *Agreement*: proving that the internal modalities, evaluated at representable sorts, reproduce the Set-level modalities.

Additionally, a full Grothendieck construction $NT(T) = \int \mathrm{Sub}$ over $P(T)$ (packaging pairs (P, φ) with $P \in P(T)$ and $\varphi \in \mathrm{Sub}(P)$) would complete the connection to Native Type Theory [3]. Mathlib provides `CategoryTheory.Grothendieck` for $\mathcal{C} \to \mathbf{Cat}$ functors, which could be instantiated with the subobject fibration. Kripke–Joyal forcing semantics, not currently in Mathlib, would be needed to fully connect the internal language of $P(T)$ to OSLF typing judgments.

14 Paper-Parity Endpoints

We provide a *paper-parity canonical package* that bundles the core claims of the three OSLF source papers into a single Lean record type, instantiable on any `LanguageDef`.

Listing 13: Unified paper-parity package (CoreMain.lean)

```
-- The unified endpoint for review:
#check @Mettapedia.OSLF.coreMain_paper_parity_full_package
-- Type: (lang : LanguageDef) -> AtomSem
--      -> ScopedConstructorPred lang -> ...
--      -> CoreMainPaperParityCanonicalPackage lang ...
```

The package contains four milestone bundles:

M1 Full presheaf Grothendieck category for any language (`fullPresheafGrothendieckCategory`).

M2 Scoped/full comparison: roundtrip between `ScopedConstructorPred` and `FullPresheafGrothendieckObj` (`scoped_full_scoped_obj_roundtrip`).

M3 Internal language: frame-derived $\to/\neg$, $\top/\bot/\wedge/\vee$, and assumption-guarded Π/Σ via `topos_full_internal_logic_bridge_package`.

M4 TOGL graph-modal bridge: n-step graph paths $\leftrightarrow$ n-fold relational composition (`graphChainN_iff_relCompN`), modal iteration $\Diamond^n \leftrightarrow$ graph chains (`diamondIterN_iff_graphChainN`), and internal subfunctor correspondence (`togl_internal_graph_correspondence_layer`).

14.1 Traceability: Paper Claims to Lean Theorems

A machine-checked paper-claim tracker (`Framework/PaperClaimTracker.lean`) maps 27 verifiable claims from three source papers to Lean theorem names:

Paper	Loc	Claim	Status
oslf.pdf	Def 1	Rewrite system	proven
oslf.pdf	§3	Step-future $\Diamond$	proven
oslf.pdf	§3	Step-past $\blacksquare$	proven
oslf.pdf	§4	Galois $\Diamond \dashv \blacksquare$	proven
oslf.pdf	§6	OSLF type system output	proven
oslf.pdf	Thm 1	Behavioral $\Rightarrow$ same types (fwd)	proven
oslf.pdf	Thm 1	Full substitutability (image-finite)	assumption-scoped
oslf.pdf	§11	ϱ-calculus instance	proven
oslf.pdf	§11	λ-calculus instance	proven
oslf.pdf	§11	Petri net instance	proven

Paper	Loc	Claim	Status
oslf.pdf	§12.1	Compile-time firewall	proven
oslf.pdf	§12.2	Race detection	proven
oslf.pdf	§12.3	Secrecy	proven
oslf.pdf	§13.1	Dependent/parametric extension	proven
NTT.pdf	§3	Native type = (sort, predicate) pair	proven
NTT.pdf	Thm 23	Full presheaf Grothendieck category	proven
NTT.pdf	Prop 12	Scoped $\leftrightarrow$ full comparison	proven
NTT.pdf	§4	Internal language: $\top/\bot/\wedge/\vee$	proven
NTT.pdf	§4	Frame-derived $\rightarrow/\neg$	proven
NTT.pdf	Prop 19	Π/Σ type formation	assumption-scoped
NTT.pdf	§5	Theory morphism Π/Ω	proven
NTT.pdf	§5	Colax Π/Prop rules	proven
togl.pdf	§2	Graph reduction structure	proven
togl.pdf	§3	Graph-modal bridge	proven
togl.pdf	§4	N-step paths = relational composition	proven
togl.pdf	§4	Modal iteration $\Diamond^n$ = graph chains	proven
togl.pdf	§5	Internal subfunctor correspondence	proven

25 claims are fully proven; 2 are assumption-scoped with necessity counterexamples showing the assumptions cannot be discharged:

- `types_nonempty_necessary_for_piSigma`: for empty families, $\emptyset = \top$ and $\bigsqcup \emptyset = \bot$, so $\top \leq \bot$ fails in any nontrivial Frame.
- `hClosed_necessary_for_fragment`: a seed-restricted fragment is not closed under `ScopedReachable`.
- `not_global_hImageFinite_rhoCoreStarRel`: the core-star endpoint relation is not globally image-finite (via `addRightZeroNest` injection).

Robustness canaries (`Framework/PaperParityCanaries.lean`) instantiate the full package on ϱ-calculus and λ-calculus (positive), plus a negative canary demonstrating that the `hClosed` assumption does real work.

15 Status and Roadmap

15.1 Milestones

✓ **Milestone 1**: Executable ϱ-calculus engine. `reduceStep` computes all one-step reducts; `reduceStep_sound` proven with 0 sorries; 6 executable tests pass.

✓ **Milestone 2**: Generic MeTTaIL framework. Language-parametric pattern matching (`Match.lean`) and rewriting (`MeTTaIL/Engine.lean`) for any `LanguageDef`. 8-test agreement suite confirms equivalence with specialized engine.

- ✓ **Milestone 3**: OSLF type synthesis. `langOSLF` mechanically generates an `OSLFTypeSystem` from any `LanguageDef` with auto-proven Galois connection. `GenHasType` provides generated typing rules. Three-layer bridge established.

- ✓ **Milestone 4**: Correctness infrastructure. Totalized `Match.lean` (no `partial def` in core matching). Declarative `DeclReduces` with engine soundness and completeness. Enhanced formula checker with predecessor-based ■ checking.

- ✓ **Milestone 5**: Language instances. Six languages—lambda calculus, Petri nets, ϱ-calculus, TinyML, MeTTaMinimal, and MeTTaFull—each with auto-generated `OSLFTypeSystem` and Galois connection.

- ✓ **Milestone 6**: Presheaf-primary categorical lift. Interface-selected bases (`SortCategoryInterface`) and presheaf-primary predicate fibrations are wired via `CategoryBridge`, including representable-fiber characteristic equivalences and checker-to-fiber soundness.

- ✓ **Milestone 7**: Graph-object BC path and end-to-end clients. `ConstructorCategory`: free category on sort-crossing constructors. `ConstructorFibration`: `SubobjectFibration` + `ChangeOfBase` with proven adjunctions. `ModalEquivalence` and `DerivedTyping`: modal/typing synthesis from constructor structure. `ToposReduction/BeckChevalleyOSLF`: reduction graph objects, explicit substitution squares, and graph-level $\Diamond/\blacksquare$ compatibility consumed by TinyML, MeTTaMinimal, and MeTTa-Full checker-to-semantics theorems. All 0 sorries in this core track.

- ✓ **Milestone 8**: Paper-parity canonical package. `coreMain_paper_parity_full_package` bundles M1 (Grothendieck category), M2 (scoped/full comparison), M3 (internal language), and M4 (TOGL graph-modal bridge) into one endpoint for any `LanguageDef`. 27 paper claims tracked; 25 fully proven, 2 assumption-scoped with proven necessity counterexamples. Robustness canaries on ϱ-calculus and λ-calculus (Section 14).

16 Conclusion

We have presented the first machine-checked formalization of the OSLF algorithm, spanning 41,700+ lines of Lean 4 across 118 files with 0 sorries and 0 custom axioms. The formalization demonstrates that the entire pipeline from rewrite rules to spatial-behavioral types can be made rigorous in a modern proof assistant.

The Galois connection at the heart of OSLF is proven at three levels: (1) directly for the ϱ-calculus, (2) categorically for any reduction span via adjoint composition, and (3) lifted to a Mathlib `Adjunction` between endofunctors on the predicate preorder category. All three levels are shown to agree.

The constructor category (Section 4) provides a genuine categorical backbone: sort-crossing constructors become morphisms, the fibered change-of-base gives the adjoint triple $\exists_f \dashv f^* \dashv \forall_f$ for each constructor, and the modal operators $\Diamond/\blacksquare$ are shown to be the typing actions of quoting/reflecting constructors. The derived typing rules (Section 4.4) make explicit that the assignment `NQuote` $\mapsto \Diamond$, `PDrop` $\mapsto \blacksquare$ is not ad hoc but follows from the constructor's position in the category. The Beck–Chevalley analysis (Section 4.5) shows that while substitution commutes with change-of-base in the COMM-specific case (proven via `comm_preserves_type`), the universal condition fails—a mathematically interesting negative result proven by explicit counterexample.

The presheaf topos $P(T) = \mathbf{Set}^{T^{\mathrm{op}}}$ over the constructor category is constructed with Yoneda representables, subobject classifier, and fiber characterization (Section 13). The full presheaf Grothendieck category for Native Type Theory [3] is now formalized with proven associativity and identity laws, scoped/full comparison roundtrip, and internal-language bridge (Section 14). The remaining frontier item—lifting Set-level modalities into genuine internal $\Omega \to \Omega$ natural transformations in $P(T)$—is precisely characterized in Section 13.3.

The full type synthesis pipeline is validated by six language instances (ϱ-calculus, lambda calculus, Petri nets, TinyML, MeTTaMinimal, and MeTTaFull), each with auto-proven Galois connections. TinyML demonstrates the framework's ability to handle multi-sort CBV languages with binders, conditionals, and sort-crossing constructors, mirroring the ϱ-calculus's NQuote/PDrop structure with Thunk/Inject. The MeTTa instances close the loop from the interpreter specification back to the categorical semantics.

A declarative reduction relation provides an engine-independent specification, and the executable engines are proven both sound and complete with respect to it. All eight *core* implementation milestones are complete and sorry-free. The paper-parity canonical package (Section 14) bundles 27 verifiable claims from three source papers—25 fully proven, 2 assumption-scoped with proven necessity counterexamples—into a single machine-checked endpoint. The strict paper-claim tracker (`Framework/PaperClaimTracker.lean`) and robustness canaries (`Framework/PaperParityCanaries.lean`) provide ongoing regression coverage for the formalization's alignment with the mathematical literature.

References

[1] L. Gregory Meredith and David F. Radestock. A reflective higher-order calculus. *Electronic Notes in Theoretical Computer Science*, 141(5):49–67, 2005.

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[3] Christian Williams and Mike Stay. Native type theory. In *Proceedings of the 4th Annual International Conference on Applied Category Theory (ACT 2021)*, 2021. arXiv:2211.16865.