Verified Operational Semantics in Logical Form: A Lean 4 Formalization of the OSLF Algorithm (DRAFT --- February 16, 2026)

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Abstract

We present a comprehensive Lean 4 formalization of Operational Semantics in Logical Form (OSLF), the algorithm that mechanically derives spatial-behavioral type systems from rewrite rules. Our formalization spans 22,300+ lines across 58 Lean files, with 0 sorries across the entire core pipeline (29 sorries remain only in auxiliary π-to-ϱ encoding correctness proofs). The formalization covers: (i) MeTTaIL, a meta-language for defining process calculi with capture-avoiding substitution and a totalized pattern matcher; (ii) the ϱ-calculus with full reduction semantics, structural congruence, and modal type system; (iii) the abstract OSLF framework instantiated for *four* languages (ϱ-calculus, lambda calculus, Petri nets, and TinyML), each with *fully proven* Galois connections $\Diamond \dashv \blacksquare$; (iv) executable rewrite engines (specialized and generic), including premise-aware rewriting with pluggable relation environments, proven sound with respect to declarative reduction specifications; (v) a *constructor category* built from sort-crossing constructors with a `SubobjectFibration` and `ChangeOfBase`, connecting to the GSLT categorical infrastructure; (vi) *derived typing rules* where the modal operator ($\Diamond$ or $\blacksquare$) assigned to each constructor is determined automatically by its position in the constructor category; (vii) a *presheaf-primary categorical lift* with interface-selected base categories, representable-fiber bridges, and graph-object reduction semantics; (viii) a *Beck–Chevalley analysis* of substitution as change-of-base, with a proven counterexample showing the strong condition fails and concrete representable/graph square theorems; and (ix) a MeTTa Core interpreter specification with confluence and progress. All Galois connections, the type soundness theorem, the engine soundness theorem, the constructor fibration, the derived typing rules, and the Beck–Chevalley analysis carry zero sorries.

1 Introduction

The OSLF algorithm [?] takes a rewrite system as input and produces a spatial-behavioral type system as output. The core insight is that every reduction relation induces a pair of adjoint modal operators:

$$\Diamond\varphi = \{p \mid \exists q.\ p \rightsquigarrow q \ \wedge\ q \in \varphi\} \quad \text{(step-future / possibly)} \tag{1}$$

$$\blacksquare\varphi = \{q \mid \forall p.\ p \rightsquigarrow q \ \Rightarrow\ p \in \varphi\} \quad \text{(step-past / rely)} \tag{2}$$

and that $\Diamond \dashv \blacksquare$ forms a Galois connection. Combined with the spatial decomposition from parallel composition, this yields a type system where types are "behavioral neighborhoods" and typing is substitutability under bisimulation.

Previous treatments of OSLF were paper-only. We give the first machine-checked formalization, connecting:

- a *generic* abstract framework (any rewrite system),
- a *concrete* ϱ-calculus instance with all rules proven,
- a *categorical* derivation via a constructor category with fibered change-of-base and derived typing rules,
- an *executable* reduction engine proven sound w.r.t. the spec, and
- *four language instances* validating the full pipeline.

Contributions.

1. A Lean 4 formalization of the OSLF algorithm as an abstract structure (`RewriteSystem` $\rightarrow$ `OSLFTypeSystem`) and its full ϱ-calculus instance (`rhoOSLF`) with a proven Galois connection.
2. A categorical proof that the Galois connection arises from the adjoint triple $\exists_f \dashv f^* \dashv \forall_f$ applied to the reduction span, with the result shown equal to the direct ϱ-calculus modalities.
3. A *constructor category* built from sort-crossing constructors of any `LanguageDef`, with a `SubobjectFibration` and `ChangeOfBase` connecting to the GSLT infrastructure.
4. *Derived typing rules*: the modal operator ($\Diamond$/$\blacksquare$/id) assigned to each constructor is determined automatically by its classification (quoting/reflecting/neutral), and the assignment is proven correct for the ϱ-calculus.
5. A *Beck–Chevalley analysis* of the COMM rule as change-of-base along the substitution map, with a proven counterexample showing the GSLT strong Beck–Chevalley condition does not hold for the constructor fibration, plus representable-fiber and graph-object square theorems consumed by checker-facing corollaries.
6. Executable reduction engines handling COMM, DROP, and context descent, with machine-checked soundness theorems.

7. Premise-aware declarative reduction relations (`DeclReducesWithPremises`) independent of the engine, with proven soundness and completeness of the generic premise-aware engine.

8. *Six* OSLF instantiations validating generality: ϱ-calculus, lambda calculus, Petri nets, and TinyML (a multi-sort CBV λ-calculus with booleans, pairs, and thunks), plus MeTTaMinimal and MeTTaFull state-machine clients.

9. MeTTaIL: a meta-language for defining process calculi, with totalized pattern matching, 29 proven theorems about capture-avoiding substitution, and a complete ϱ-calculus language definition.

10. A verified bounded model checker for OSLF formulas, with support for predecessor-based $\blacksquare$ checking and proven soundness.

11. A MeTTa Core interpreter specification with progress, confluence, and barbed bisimulation properties (97 proven theorems, 0 sorries).

12. A dedicated sorry-free core entrypoint (`CoreMain.lean`) and machine-readable FULL tracker (`Framework/FULLStatus.lean`) for review.

2 Background: The ϱ-Calculus

The reflective higher-order calculus [**?**] extends the π-calculus with:

- *Quoting*: any process P can be turned into a name $@P$ (name = quoted process).
- *Dequoting*: $*x$ recovers the process quoted by name x.
- *No built-in names*: all names arise from quoting, giving the calculus a reflective character.

The reduction rules are:

$$\text{COMM:} \quad \{n!(q) \mid \mathbf{for}(x \leftarrow n)\{p\} \mid \mathit{rest}\} \rightsquigarrow \{p[@q/x] \mid \mathit{rest}\} \tag{3}$$

$$\text{DROP:} \quad *(@P) \rightsquigarrow P \tag{4}$$

plus structural congruence (11 rules) and contextual reduction under parallel composition.

3 Formalization Architecture

3.1 Module Structure

The formalization is organized in seven directories plus standalone files:

Directory	Lines	Sorries	Content
`MeTTaIL/`	2,929	0	Meta-language AST, substitution, matching, declarative reduction
`MeTTaCore/`	2,946	0	Interpreter specification
`Framework/`	4,400	0	Abstract OSLF + categorical bridge + 4 instances
`RhoCalculus/`	3,893	0	Concrete ϱ-calculus + engine
`PiCalculus/`	6,582	29	π-calculus + ϱ-encoding
`NativeType/`	263	0	Native type construction
`Formula.lean`	582	0	Verified bounded model checker
`Main.lean`	387	0	Focused OSLF re-exports
Total	**22,320**	**29**	**Core: 0 sorries**

All 29 remaining sorries are in the π-to-ϱ encoding correctness proofs (`PiCalculus/RhoEncodingCorrectnes` with one major theorem (Proposition 2: substitution invariance) already fully proven. The entire core OSLF pipeline—MeTTaIL, Framework, RhoCalculus, Formula, and MeTTa Core—carries **0 sorries and 0 axioms**. Operational entrypoints (`CoreMain.lean` and `Main.lean`) are kept on this core boundary; process-calculus facades are exposed separately under `Mettapedia/Languages/ProcessCalculi*.lean`.

The `Framework/` directory (4,400 lines, 15 files) is the largest component, containing:

File	Lines	Content
`ConstructorCategory.lean`	460	Sort quiver + free category
`TinyMLInstance.lean`	528	CBV λ-calculus with booleans/pairs/thunks
`BeckChevalleyOSLF.lean`	449	Substitution as change-of-base
`DerivedTyping.lean`	346	Generic typing rules from constructor category
`ModalEquivalence.lean`	311	Constructor change-of-base $\leftrightarrow$ modalities
`GeneratedTyping.lean`	294	`GenHasType` typing rules
`SynthesisBridge.lean`	282	Three-layer bridge
`ConstructorFibration.lean`	251	`SubobjectFibration` + `ChangeOfBase`
`DerivedModalities.lean`	250	Adjoint triple derivation
`CategoryBridge.lean`	247	GaloisConnection $\rightarrow$ Adjunction
`PetriNetInstance.lean`	233	Petri net OSLF instance
`LambdaInstance.lean`	218	Lambda calculus OSLF instance
`TypeSynthesis.lean`	201	`langOSLF` pipeline
`RewriteSystem.lean`	196	Abstract OSLF input/output
`RhoInstance.lean`	134	ϱ-calculus instance

3.2 Abstract OSLF Framework

The abstract layer defines two key structures:

Listing 1: The OSLF input and output (RewriteSystem.lean)

```
structure RewriteSystem where
  Sorts    : Type*
  procSort : Sorts
```

```
  Term      : Sorts -> Type*
  Reduces   : Term procSort -> Term procSort -> Prop

structure OSLFTypeSystem where
  Sorts     : Type*
  procSort : Sorts
  Term      : Sorts -> Type*
  Pred      : Sorts -> Type*
  [frame    : (S : Sorts) -> Frame (Pred S)]
  satisfies  : (S : Sorts) -> Term S -> Pred S -> Prop
  diamond    : Pred procSort -> Pred procSort
  box        : Pred procSort -> Pred procSort
  galois     : GaloisConnection diamond box
```

3.3 The Galois Connection

The central theorem: $\Diamond \dashv \blacksquare$.

Theorem 1 (Galois Connection, 0 sorries)**.** *For all predicates φ, ψ on ϱ-calculus processes:*

$$\Diamond\varphi \leq \psi \iff \varphi \leq \blacksquare\psi$$

where $\Diamond\varphi(p) = \exists q.\ p \rightsquigarrow q \wedge \varphi(q)$ and $\blacksquare\psi(q) = \forall p.\ p \rightsquigarrow q \to \psi(p)$.

In Lean:

Listing 2: The Galois connection (Reduction.lean)

```
theorem galois_connection :
    GaloisConnection possiblyProp relyProp := by
  intro phi psi
  constructor
  . intro h q hrely p hred
    exact h (hrely p hred)
  . intro h p hposs
    exact h.2 p hposs.1 hposs.2
```

3.4 Categorical Derivation via Adjoint Triples

The Galois connection arises from the general theory of change-of-base along a span.

Definition 2 (Reduction Span)**.** *A span $\mathcal{S} \xleftarrow{\text{src}} E \xrightarrow{\text{tgt}} \mathcal{S}$ where E is the set of reduction edges,* src *extracts the source, and* tgt *extracts the target.*

From any such span we derive three operations on predicates:

$$f^*(\psi)(e) = \psi(\text{tgt}(e)) \qquad \text{(pullback)} \tag{5}$$

$$\exists_f(\varphi)(q) = \exists e.\ \text{tgt}(e) = q \wedge \varphi(e) \qquad \text{(direct image)} \tag{6}$$

$$\forall_f(\varphi)(q) = \forall e.\ \text{tgt}(e) = q \to \varphi(e) \qquad \text{(universal image)} \tag{7}$$

Theorem 3 (Derived Galois, 0 sorries)**.** *For any* `ReductionSpan`*, the composition* $\Diamond = \exists_{\mathrm{src}} \circ \mathrm{tgt}^*$ *and* $\blacksquare = \forall_{\mathrm{tgt}} \circ \mathrm{src}^*$ *form a Galois connection. Furthermore, for the* ϱ*-calculus span, the derived operators equal the concrete* `possiblyProp` *and* `relyProp`*.*

Listing 3: Derived modalities equal concrete (DerivedModalities.lean)

```
theorem derived_diamond_eq_possiblyProp :
    derivedDiamond rhoSpan = possiblyProp := ...

theorem derived_box_eq_relyProp :
    derivedBox rhoSpan = relyProp := ...

theorem rho_galois_from_span :
    GaloisConnection (derivedDiamond rhoSpan)
                     (derivedBox rhoSpan) :=
  derived_galois rhoSpan
```

4 Constructor Category and Fibration

A key contribution of this formalization is the *constructor category*: a non-discrete category built from the sort-crossing constructors of any `LanguageDef`, replacing the discrete `Discrete R.Sorts` from the earlier categorical lift.

4.1 Sort Quiver and Free Category

Given a `LanguageDef`, we extract the *unary sort-crossing constructors*: grammar rules with exactly one `.simple` parameter whose base sort differs from the constructor's output sort. These become the arrows of a quiver on the language's sorts.

Listing 4: Constructor category (ConstructorCategory.lean)

```
-- Sort type: valid sort names
def LangSort (lang : LanguageDef) :=
  { s : String // s IN lang.types }

-- Sort-crossing arrows
structure SortArrow (lang : LanguageDef)
    (dom cod : LangSort lang) where
  label : String
  valid : (label, dom.val, cod.val) IN unaryCrossings lang

-- Free category: paths of sort-crossing arrows
inductive SortPath (lang : LanguageDef)
    : LangSort lang -> LangSort lang -> Type where
  | nil  : SortPath lang s s
  | cons : SortPath lang s t -> SortArrow lang t u
         -> SortPath lang s u
```

For the ϱ-calculus: 2 objects (Proc, Name), 2 arrows (`NQuote`: Proc $\to$ Name, `PDrop`: Name $\to$ Proc), and composites `PDrop` $\circ$ `NQuote` and `NQuote` $\circ$ `PDrop`.

Each arrow has a *semantic function* `arrowSem`: wrapping a pattern in the constructor's `.apply` node (e.g., $p \mapsto$ `NQuote`(p)). This extends to paths via `pathSem`, with a proven composition law `pathSem_comp`.

A *universal property* (free category lifting) is proven: any assignment of objects and arrows to a target category $\mathcal{C}$ lifts uniquely to a functor `liftFunctor`, with uniqueness proven in `lift_map_unique`.

4.2 SubobjectFibration and ChangeOfBase

Over the constructor category we build a fibration and change-of-base (`Framework/ConstructorFibration.lean`, 251 lines, 0 sorries):

Listing 5: Constructor fibration (ConstructorFibration.lean)

```
-- Each sort has fiber Pattern -> Prop (a Frame)
def constructorFibration (lang : LanguageDef) :
    SubobjectFibration (ConstructorObj lang) where
  Sub   := fun _ => Pattern -> Prop
  frame := fun _ => Pi.instFrame

-- Full change-of-base with proven adjunctions
def constructorChangeOfBase (lang : LanguageDef) :
    ChangeOfBase (constructorFibration lang) where
  pullback f       := pb (pathSem lang f)
  directImage f    := di (pathSem lang f)
  universalImage f := ui (pathSem lang f)
  direct_pullback_adj f := di_pb_adj (pathSem lang f)
  pullback_universal_adj f := pb_ui_adj (pathSem lang f)
  ...
```

The adjunctions $\exists_f \dashv f^* \dashv \forall_f$ are proven (not axiomatized), following from the generic `di_pb_adj` / `pb_ui_adj` in `DerivedModalities.lean`.

Key proven properties:

- **Pullback functoriality**: $(f \circ g)^* = g^* \circ f^*$ (from `pathSem_comp`), $id^*(\varphi) = \varphi$.
- **Frame morphism**: $f^*(\varphi \wedge \psi) = f^*(\varphi) \wedge f^*(\psi)$ and $f^*(\top) = \top$ (both by `rfl`).
- **Monotonicity** of all three operations (from adjunctions).

4.3 Modal Equivalence

The file `Framework/ModalEquivalence.lean` (311 lines) connects the constructor change-of-base to the OSLF modalities:

Theorem 4 (Modal = Change-of-Base, 0 sorries)**.** *The* OSLF *modalities are Set-level change-of-base along the reduction span:*

$$\Diamond_{lang} = \exists_{\mathrm{src}} \circ \mathrm{tgt}^* \qquad \textit{(definitional)}$$
$$\blacksquare_{lang} = \forall_{\mathrm{tgt}} \circ \mathrm{src}^* \qquad \textit{(definitional)}$$

For the ϱ-calculus, this gives the *typing actions*:

- `NQuote` (Proc → Name): $\varphi \mapsto \Diamond\varphi$ (“can reduce to φ”)
- `PDrop` (Name → Proc): $\alpha \mapsto \blacksquare\alpha$ (“all predecessors satisfy α”)

The composite `PDrop` $\circ$ `NQuote` gives $\blacksquare \circ \Diamond$ and `NQuote` $\circ$ `PDrop` gives $\Diamond \circ \blacksquare$. The typing action Galois connection $\Diamond \dashv \blacksquare$ is proven as an instance of the general language Galois connection.

4.4 Derived Typing Rules

The file `Framework/DerivedTyping.lean` (346 lines, 0 sorries) derives typing rules generically from the constructor category structure.

Each sort-crossing arrow is automatically classified:

Listing 6: Constructor classification (DerivedTyping.lean)

```
inductive ConstructorRole where
  | quoting     -- domain = procSort: introduces diamond
  | reflecting  -- codomain = procSort: introduces box
  | neutral     -- neither: identity

def classifyArrow (lang : LanguageDef) (procSort : String)
    (arr : SortArrow lang dom cod) : ConstructorRole :=
  if dom.val = procSort then .quoting
  else if cod.val = procSort then .reflecting
  else .neutral
```

Theorem 5 (Classification Correctness, 0 sorries)**.** *For the ϱ-calculus:*

- *`NQuote` is classified as* quoting*; its typing action equals $\Diamond$.*
- *`PDrop` is classified as* reflecting*; its typing action equals $\blacksquare$.*

The `DerivedHasType` judgment provides a generic typing rule for unary sort-crossing constructors: apply the constructor's typing action ($\Diamond$/$\blacksquare$/id) to the argument's predicate, then tag the result at the output sort.

4.5 Beck–Chevalley for Substitution

The file `Framework/BeckChevalleyOSLF.lean` (449 lines, 0 sorries) analyzes the COMM rule's substitution $p[@q/x]$ as a change-of-base map.

Definition 6 (COMM Substitution Map)**.** σ_q : *Pattern → Pattern defined by* $\sigma_q(\mathit{pBody}) = \mathit{openBVar}\ 0\ (\texttt{NQuote}(q))\ \mathit{pBody}$.

This induces the adjoint triple $\exists_{\sigma_q} \dashv \sigma_q^* \dashv \forall_{\sigma_q}$ via the same `pb`/`di`/`ui` infrastructure, and the modal+substitution Galois connections compose:

Theorem 7 (Composed Galois, 0 sorries)**.** $\Diamond \circ \exists_{\sigma_q} \dashv \sigma_q^* \circ \blacksquare$

The COMM rule's type preservation (`comm_preserves_type` from `Soundness.lean`) is re-expressed categorically as a pullback inequality:

Theorem 8 (Substitutability as Pullback, 0 sorries)**.** *For any typing context* Γ*, type* τ*, variable* x*, value* q*, and type* σ *with* $\Gamma \vdash q : \sigma$*:*

$$\mathit{typedAt}(\Gamma[x \mapsto \sigma], \tau) \ \leq \ \sigma_q^*(\mathit{typedAt}(\Gamma, \tau))$$

The COMM substitution map factors through the constructor semantics: $\sigma_q(p) = \mathit{openBVar}\ 0\ (\mathit{pathSem}\ \texttt{nquoteMor}\ q)\ p$.

Theorem 9 (Strong Beck–Chevalley Fails, 0 sorries)**.** *The GSLT's universal Beck–Chevalley condition* $f^* \circ \exists_g = \exists_{\pi_1} \circ \pi_2^*$ *does **not** hold for the constructor fibration.*

Concretely, for the commuting square $\mathit{PDrop} \circ \mathit{NQuote} = \mathit{PDrop} \circ \mathit{NQuote}$*:*

$$\mathit{PDrop}^*(\exists_{\mathit{PDrop}}(\top))(\mathit{fvar}\, x) = \top$$

but

$$\exists_{\mathit{NQuote}}(\mathit{NQuote}^*(\top))(\mathit{fvar}\, x) = \bot$$

because $\mathit{NQuote}(q) \neq \mathit{fvar}\, x$ *for all* q*.*

The counterexample is proven by exhibiting a concrete witness at `fvar` "x": the LHS is inhabited by $\langle \texttt{fvar}\ x, \mathit{rfl}, \top \rangle$ while the RHS requires some p with $\texttt{NQuote}(p) = \texttt{fvar}\ x$, which is impossible since $\texttt{NQuote}(p) = \texttt{.apply}$ "NQuote" $[p]$.

5 Executable Rewrite Engine

A formalization that only *specifies* reduction is incomplete for verification of actual implementations. We provide `reduceStep`, a computable function that enumerates all one-step reducts, proven sound w.r.t. the propositional `Reduces`.

5.1 Engine Design

Listing 7: The executable engine (Engine.lean)

```
def reduceStep (p : Pattern) (fuel : Nat := 100)
    : List Pattern :=
  match fuel with
  | 0 => []
  | fuel + 1 =>
    match p with
    | .collection .hashBag elems none =>
      let commReducts := findAllComm elems
```

```
    let parReducts :=
      reduceElemsAux (reduceStep . fuel) elems
      |>.map fun (i, elem') =>
        .collection .hashBag (elems.set i elem') none
    commReducts ++ parReducts
  | .apply "PDrop" [.apply "NQuote" [inner]] =>
    [inner]
  | _ => []
```

The engine handles COMM (all output-input pairs on matching channels), DROP ($*(@P) \rightsquigarrow P$), and PAR (recursive reduction under parallel composition). Non-deterministic races produce multiple reducts in the output list.

5.2 Soundness

Theorem 10 (Engine Soundness, 0 sorries)**.** *Every reduct computed by* `reduceStep` *corresponds to a valid* `Reduces`*:*

$$q \in \texttt{reduceStep}(p, \mathit{fuel}) \implies \exists d : p \rightsquigarrow q$$

The proof proceeds by case analysis:

- **COMM**: Each output-input pair is located by `findAllComm`, whose specification is proven to identify valid COMM redex positions. The soundness chain is: list permutation (`perm_extract_two`) $\rightarrow$ structural congruence (`par_perm`) $\rightarrow$ `Reduces.comm` $\rightarrow$ `Reduces.equiv`.
- **DROP**: Direct pattern match yields `Reduces.drop`.
- **PAR**: The `reduceElemsAux_spec` lemma extracts the sub-element index and recursive reduct. List decomposition via `List.set_eq_take_append_cons_drop` connects `List.set` to the $\mathit{before} \mathbin{+\!\!+} [p] \mathbin{+\!\!+} \mathit{after}$ form required by `Reduces.par_any`.

6 Generic MeTTaIL Rewrite Framework

The specialized ϱ-calculus engine is lifted to a language-parametric framework. Given any `LanguageDef` (a list of sorts, constructors, equations, and rewrite rules), the generic engine automatically:

1. matches concrete terms against rule LHS patterns (multiset matching with rest variables),
2. produces variable bindings via alpha-renaming for binders,
3. applies bindings to rule RHS patterns (including substitution evaluation), and
4. iterates under congruence (subterm rewriting inside parallel compositions).

Listing 8: Premise-aware generic rewrite step (Engine.lean)

```
structure RelationEnv where
  tuples : String -> List Pattern -> List (List Pattern)

def applyRuleWithPremisesUsing
    (relEnv : RelationEnv) (lang : LanguageDef)
    (rule : RewriteRule) (term : Pattern) : List Pattern :=
  (matchPattern rule.left term).flatMap fun bs =>
    (applyPremisesWithEnv relEnv lang rule.premises bs).map fun bs'
         =>
      applyBindings bs' rule.right

def rewriteStepWithPremisesUsing
    (relEnv : RelationEnv) (lang : LanguageDef) (term : Pattern) :
        List Pattern :=
  lang.rewrites.flatMap fun rule => applyRuleWithPremisesUsing
      relEnv lang rule term
```

Multiset matching. The key algorithmic contribution is `matchBag`: for a collection pattern with n elements and an optional rest variable, it enumerates all ways to match the n pattern elements against term elements (backtracking search over permutations), binding unmatched elements to the rest variable.

Declarative reduction. The files `MeTTaIL/DeclReduces.lean` and `MeTTaIL/DeclReducesWithPremises.lean` provide declarative inductive reduction relations independent of the executable engine. The generic premise-aware engine is proven both *sound* and *complete* with respect to the premise-aware specification (0 sorries).

7 MeTTaIL and MeTTa Core

7.1 MeTTaIL: The Meta-Language

MeTTaIL ("MeTTa Intermediate Language") defines the AST shared by all process calculi in the formalization, using locally nameless representation. The core type is `Pattern` with 7 constructors:

Listing 9: Pattern AST — locally nameless (Syntax.lean)

```
inductive Pattern where
  | bvar       : Nat -> Pattern         -- bound variable (de
      Bruijn)
  | fvar       : String -> Pattern      -- free variable /
      metavariable
  | apply      : String -> List Pattern -> Pattern
  | lambda     : Pattern -> Pattern     -- binder (no name)
  | multiLambda: Nat -> Pattern -> Pattern
  | subst      : Pattern -> Pattern -> Pattern
  | collection : CollType -> List Pattern
                  -> Option String -> Pattern
```

Key proven properties of substitution (29 theorems, 0 sorries):

- `subst_empty`: empty substitution is the identity;
- `subst_fresh`: substitution on a fresh variable is the identity;
- `commSubst`: the COMM-rule substitution $p[@q/x]$ respects freshness.

7.2 MeTTa Core Interpreter

The `MeTTaCore/` directory provides a complete interpreter specification for Hyperon Experimental MeTTa (2,946 lines, 0 sorries), including:

- `Atom`: the universal term type with `DecidableEq`;
- `Bindings`: variable resolution with merge and transitive lookup;
- `MeTTaState`: the 4-register $\langle i, k, w, o \rangle$ machine;
- `PatternMatch`: bidirectional unification;
- `MinimalOps`: grounded operations ($+, -, *, /, <$, etc.);
- `RewriteRules`: equation-driven rewriting;
- `Atomspace`: multiset-based knowledge base with query operations;
- `Properties`: progress, confluence, and barbed bisimulation.

8 Type Soundness

Theorem 11 (Substitutability, 0 sorries)**.** *If P and Q are bisimilar processes, then they have the same native types:*

$$P \sim Q \implies \forall \tau.\, P : \tau \iff Q : \tau$$

Theorem 12 (Type Preservation, 0 sorries)**.** *The COMM and DROP rules preserve types:*

$$\Gamma \vdash P : \tau \ \wedge\ P \rightsquigarrow Q \implies \Gamma \vdash Q : \tau$$

Both theorems are proven in `RhoCalculus/Soundness.lean` with a 10-constructor typing judgment `HasType` and explicit typing contexts.

9 ϱ-Calculus Instance

The ϱ-calculus is formalized with full reduction semantics (COMM, DROP), structural congruence (11 rules), and multi-step reduction. The OSLF algorithm derives its spatial-behavioral type system, with the Galois connection $\Diamond \dashv \blacksquare$ proven in `RhoCalculus/Soundness.lean` (0 sorries).

10 Type Synthesis: `LanguageDef` $\to$ `OSLFTypeSystem`

The culmination of the formalization is the *type synthesis pipeline*: given any `LanguageDef`, mechanically produce a full `OSLFTypeSystem` with a **proven Galois connection**.

10.1 The Pipeline

The pipeline proceeds in five steps, all implemented in `Framework/TypeSynthesis.lean`:

1. `langReduces lang p q` wraps the executable engine: $q \in$ `rewriteWithContext` *lang* p.
2. `langRewriteSystem lang` assembles a `RewriteSystem`.
3. `langSpan lang` builds the reduction span (edges = reductions).
4. `langDiamond`/`langBox` derive modal operators via `derivedDiamond`/`derivedBox` from the adjoint triple.
5. `langOSLF lang` packages everything into an `OSLFTypeSystem`, with the Galois connection proven by `derived_galois`.

Theorem 13 (Automatic Galois Connection, 0 sorries)**.** *For any `LanguageDef` `lang`:*

$$\Diamond_{lang} \dashv \blacksquare_{lang}$$

where $\Diamond_{lang} = \exists_{\text{src}} \circ \text{tgt}^*$ *and* $\blacksquare_{lang} = \forall_{\text{tgt}} \circ \text{src}^*$ *are derived from the reduction span. No manual proof is needed per language.*

11 Language Instances

The pipeline is validated by four language instances of increasing complexity:

11.1 Lambda Calculus (1 sort, 0 crossings)

Untyped lambda calculus with β-reduction (`Framework/LambdaInstance.lean`, 218 lines). One sort (`Term`), two constructors (`App`, `Lam`), one reduction rule. The constructor category is discrete (no sort-crossing constructors). Six executable demos verify β-reduction, multi-step normalization, and formula checking.

11.2 Petri Net (1 sort, 0 crossings)

A simple Petri net with four places and two transitions (`Framework/PetriNetInstance.lean`, 233 lines). Validates that multiset (bag) matching works correctly without any abstraction or substitution machinery. Key properties proven by `native_decide`:

- $\{D\}$ is a dead marking (0 reducts);
- $\{A, B\}$ has exactly 1 reduct via transition T_1.

11.3 ϱ-Calculus (2 sorts, 2 crossings)

The primary instance with sorts Proc and Name, constructors `NQuote`: Proc $\to$ Name and `PDrop`: Name $\to$ Proc. This is the most extensively developed instance, with the full type soundness proof, specialized engine, and Beck–Chevalley analysis (Sections 4–4.5).

11.4 TinyML (2 sorts, 2 crossings, 6 rules)

New: A call-by-value λ-calculus with booleans, pairs, and thunks (`Framework/TinyMLInstance.lean`, 528 lines, 0 sorries).

Listing 10: TinyML language definition (TinyMLInstance.lean)

```
-- Sorts: Expr (process sort), Val (data sort)
-- Constructors:
--   Expr: App, If, Fst, Snd, Inject(v:Val)
--   Val:  BoolT, BoolF, Lam(^body), PairV, Thunk(e:Expr)
-- Reductions:
--   Beta:     App(Inject(Lam(^body)), Inject(v)) ~> body[v/x]
--   Force:    Inject(Thunk(e)) ~> e
--   IfTrue:   If(Inject(BoolT), t, e) ~> t
--   IfFalse:  If(Inject(BoolF), t, e) ~> e
--   FstPair:  Fst(Inject(PairV(a, b))) ~> Inject(a)
--   SndPair:  Snd(Inject(PairV(a, b))) ~> Inject(b)
```

TinyML mirrors the ϱ-calculus sort structure:

TinyML	**ϱ-Calculus**	**Role**
Expr	Proc	Process sort (carries reduction)
Val	Name	Data sort
`Thunk`: Expr $\to$ Val	`NQuote`: Proc $\to$ Name	Quoting ($\Diamond$)
`Inject`: Val $\to$ Expr	`PDrop`: Name $\to$ Proc	Reflecting ($\blacksquare$)
`Beta`	`COMM`	Main computation rule
`Force`	`DROP`	Quoting/reflecting cancel

The CBV strategy is encoded syntactically: β-reduction requires both the function and argument to be wrapped in `Inject(-)`, ensuring arguments are evaluated to values before substitution.

Key proven results:

- `tinyML_crossings`: exactly 2 sort-crossing constructors (`Inject`, `Thunk`) – `native_decide`.
- `thunk_is_quoting`: `Thunk` classified as quoting; typing action = $\Diamond$.
- `inject_is_reflecting`: `Inject` classified as reflecting; typing action = $\blacksquare$.
- `tinyML_typing_action_galois`: $\Diamond \dashv \blacksquare$ for TinyML typing actions.
- 11 executable demos including a 3-step reduction chain (`force` $\to$ `ifTrue` $\to$ `fstPair`).

12 Verified Formula Checker

The file `Formula.lean` provides a verified bounded model checker for OSLF formulas (582 lines, 0 sorries). The formula type `OSLFFormula` supports $\Diamond$, $\blacksquare$, $\wedge$, $\vee$, $\implies$, $\top$, $\bot$, and atomic predicates.

Theorem 14 (Checker Soundness, 0 sorries)**.** *If the checker returns* `.sat` *for formula* φ *at term* p*, then* $p \models \varphi$ *in the denotational semantics.*

The checker supports:

- `checkWithPred`: checking with external predecessor functions, enabling bounded $\blacksquare$ verification;
- `aggregateBox`: universal checking of $\blacksquare\varphi$ over a predecessor list, with proven soundness (`aggregateBox_sat`).

13 Categorical Lift

The file `Framework/CategoryBridge.lean` lifts the Set-level OSLF construction to Mathlib's categorical infrastructure (247 lines, 0 sorries).

13.1 Modal Adjunction

The predicate type `Pattern → Prop` carries conflicting category instances (`Preorder.smallCategory` vs. `CategoryTheory.Pi`). We introduce a type wrapper `PredLattice` to disambiguate, then lift:

Listing 11: Modal adjunction (CategoryBridge.lean)

```
noncomputable def langModalAdjunction (lang : LanguageDef) :
    (langGaloisL lang).monotone_l.functor |-
    (langGaloisL lang).monotone_u.functor :=
  (langGaloisL lang).adjunction
```

This provides a categorical `Adjunction` between the $\Diamond$ and $\blacksquare$ endofunctors on the predicate preorder category, for any `LanguageDef`.

14 Status and Roadmap

14.1 Milestones

✓ **Milestone 1**: Executable ϱ-calculus engine. `reduceStep` computes all one-step reducts; `reduceStep_sound` proven with 0 sorries; 6 executable tests pass.

✓ **Milestone 2**: Generic MeTTaIL framework. Language-parametric pattern matching (`Match.lean`) and rewriting (`MeTTaIL/Engine.lean`) for any `LanguageDef`. 8-test agreement suite confirms equivalence with specialized engine.

- ✓ **Milestone 3**: OSLF type synthesis. `langOSLF` mechanically generates an `OSLFTypeSystem` from any `LanguageDef` with auto-proven Galois connection. `GenHasType` provides generated typing rules. Three-layer bridge established.

- ✓ **Milestone 4**: Correctness infrastructure. Totalized `Match.lean` (no `partial def` in core matching). Declarative `DeclReduces` with engine soundness and completeness. Enhanced formula checker with predecessor-based $\blacksquare$ checking.

- ✓ **Milestone 5**: Language instances. Four languages—lambda calculus, Petri nets, ϱ-calculus, TinyML—each with auto-generated `OSLFTypeSystem` and Galois connection.

- ✓ **Milestone 6**: Presheaf-primary categorical lift. Interface-selected bases (`SortCategoryInterface`) and presheaf-primary predicate fibrations are wired via `CategoryBridge`, including representable-fiber characteristic equivalences and checker-to-fiber soundness.

- ✓ **Milestone 7**: Graph-object BC path and end-to-end clients. `ConstructorCategory`: free category on sort-crossing constructors. `ConstructorFibration`: `SubobjectFibration` + `ChangeOfBase` with proven adjunctions. `ModalEquivalence` and `DerivedTyping`: modal/typing synthesis from constructor structure. `ToposReduction/BeckChevalleyOSLF`: reduction graph objects, explicit substitution squares, and graph-level $\Diamond/\blacksquare$ compatibility consumed by TinyML, MeTTaMinimal, and MeTTaFull checker-to-semantics theorems. All 0 sorries in this core track.

15 Conclusion

We have presented the first machine-checked formalization of the OSLF algorithm, spanning 22,300+ lines of Lean 4 across 58 files with 0 sorries in the core pipeline. The formalization demonstrates that the entire pipeline from rewrite rules to spatial-behavioral types can be made rigorous in a modern proof assistant.

The Galois connection at the heart of OSLF is proven at three levels: (1) directly for the ϱ-calculus, (2) categorically for any reduction span via adjoint composition, and (3) lifted to a Mathlib `Adjunction` between endofunctors on the predicate preorder category. All three levels are shown to agree.

The constructor category (Section 4) provides a genuine categorical backbone: sort-crossing constructors become morphisms, the fibered change-of-base gives the adjoint triple $\exists_f \dashv f^* \dashv \forall_f$ for each constructor, and the modal operators $\Diamond/\blacksquare$ are shown to be the typing actions of quoting/reflecting constructors. The derived typing rules (Section 4.4) make explicit that the assignment `NQuote` $\mapsto \Diamond$, `PDrop` $\mapsto \blacksquare$ is not ad hoc but follows from the constructor's position in the category. The Beck–Chevalley analysis (Section 4.5) shows that while substitution commutes with change-of-base in the COMM-specific case (proven via `comm_preserves_type`), the universal condition fails—a mathematically interesting negative result proven by explicit counterexample.

The full type synthesis pipeline is validated by four language instances (ϱ-calculus, lambda calculus, Petri nets, and TinyML), each with auto-proven Galois connections. TinyML demonstrates the framework's ability to handle multi-sort CBV languages with binders, conditionals, and sort-crossing constructors, mirroring the ϱ-calculus's NQuote/PDrop structure with Thunk/Inject.

A declarative reduction relation provides an engine-independent specification, and the executable engines are proven both sound and complete with respect to it. All seven milestones are complete with 0 sorries in the core pipeline.

References