

Foundations of Inference: A Formal Treatment

Formalization of Knuth & Skilling (2012)

January 14, 2026

Abstract

We present a complete formalization in Lean 4 of Knuth & Skilling’s axiomatic foundations for inference (2012). Their approach derives the probability calculus, measure theory, and information theory from elementary symmetries, without assuming continuity or differentiability. Our formalization makes explicit the implicit assumptions in their derivation, provides constructive proofs of their main theorems, and extends their framework to handle independence and product measures. We demonstrate K&S’s claim that the framework applies to non-Boolean distributive lattices (chains, open sets of \mathbb{R}).

Contents

1 Introduction

Knuth & Skilling [?] presented a foundation for inference that unites and extends the approaches of Cox and Kolmogorov. Their key innovation is to derive the standard probability calculus from elementary *symmetries* rather than axiomatizing it directly. The core insight: algebraic symmetries of combination operations uniquely determine quantification rules.

1.1 The Knuth-Skilling Program

The K&S approach proceeds in stages:

1. **Symmetries 0–2** (fidelity, monotonicity, associativity) \Rightarrow **Measure theory**: The sum rule $x \oplus y = x + y$ for disjoint events.
2. **Symmetries 3–4** (product distributivity, product associativity) \Rightarrow **Independence**: The product form for independent measures.
3. **Symmetry 5** (chaining associativity) \Rightarrow **Probability calculus**: Conditional probability and Bayes’ theorem.

Each symmetry becomes an *axiom of quantification*, and these axioms uniquely determine the calculus. K&S emphasize:

- **Minimal assumptions**: No continuity, differentiability, or negation required.
- **Constructive proofs**: Build from finite systems, avoiding unobservable infinitary subtleties.
- **Wide applicability**: Foundations cover measure theory, probability, divergence, and entropy.

1.2 This Formalization

We formalize K&S Appendices A and B in Lean 4, providing:

- Precise statement of axioms and symmetries
- Complete proofs of the representation theorems
- Explicit treatment of implicit assumptions (linear order, separation property)
- Non-Boolean examples demonstrating genuine generality
- Extensions to handle independence and products
- Counterexamples showing necessity of assumptions

The formalization reveals that K&S's informal proofs are essentially correct, but rely on several implicit assumptions that become apparent only when formalizing. These details are documented in Appendix ??.

2 The Knuth-Skilling Axioms

2.1 Informal Statement

K&S model the world as being in one of finitely many mutually exclusive states. States correspond to *atoms*, combined via logical OR to form elements of a Boolean lattice. A *valuation* assigns real numbers to lattice elements, faithful to the lattice order.

For disjoint elements x, y, z , K&S require:

- **Axiom 0** (Fidelity): $x > 0$ (positive valuations)
- **Axiom 1** (Monotonicity): $x < y \Rightarrow x \oplus z < y \oplus z$ and $z \oplus x < z \oplus y$
- **Axiom 2** (Associativity): $(x \oplus y) \oplus z = x \oplus (y \oplus z)$

These three axioms suffice to derive the sum rule $x \oplus y = x + y$ (Appendix A).

For independent measures on product spaces:

- **Axiom 3** (Product Distributivity): $(x \otimes t) \oplus (y \otimes t) = (x \oplus y) \otimes t$
- **Axiom 4** (Product Associativity): $(u \otimes v) \otimes w = u \otimes (v \otimes w)$

These determine the multiplicative form of \otimes (Appendix B).

2.2 Formal Definition

We formalize the basic K&S algebra structure as a linearly ordered monoid with strict monotonicity:

Listing 1: Core K&S algebra axioms

```
1 class KnuthSkillingAlgebra ( : Type*) extends LinearOrder where
2   op : →→                                -- combination
3   ident :                                -- identity element
4   op_assoc : x y z, op (op x y) z = op x (op y z) -- Axiom 2
5   op_ident_right : x, op x ident = x
6   op_ident_left : x, op ident x = x
```

```

7 | op_strictMono_left : y, StrictMono (fun x => op x y) -- Axiom 1a
8 | op_strictMono_right : x, StrictMono (fun y => op x y) -- Axiom 1b
9 | ident_le : x, ident x -- Axiom 0

```

Note: K&S never explicitly state that elements are totally ordered, but their proofs rely on trichotomy ($x < y$, $x = y$, or $x > y$). Our formalization makes this explicit via `LinearOrder`.

2.3 Iteration Notation

Following K&S, we define iterated application of the operation:

Listing 2: Iteration definition

```

1 | def iterate_op (x : ) : →
2 |   | 0 => ident
3 |   | n + 1 => op x (iterate_op x n)

```

This builds the sequence $\text{ident}, x, x \oplus x, x \oplus (x \oplus x), \dots$, which K&S write as “0 of x ”, “1 of x ”, “2 of x ”, etc.

3 The Separation Property

3.1 Definition

K&S’s Appendix A proof requires finding rational approximations to any pair of distinct values. This is implicit in their “induction to more than one type of atom” construction. Following a suggestion by B. Goertzel, we isolate this as an explicit axiom—the *separation property*:

Listing 3: Separation axiom

```

1 | class KSSeparation ( : Type*) [KnuthSkillingAlgebra ] where
2 |   separation : {a x y : }, ident < a → ident < x → ident < y → x < y →
3 |     n m : , 0 < m
4 |     iterate_op x m < iterate_op a n
5 |     iterate_op a n iterate_op y m

```

Intuition: For any base $a > \text{ident}$ and distinct $x < y$, we can find exponents (n, m) such that $x^m < a^n \leq y^m$. This lets us “separate” x and y using rational powers of a .

3.2 Necessity

The separation property is *necessary* for representation. Counterexample: Consider $\mathbb{N} \times \mathbb{N}$ with lexicographic order:

$$(n_1, m_1) < (n_2, m_2) \iff n_1 < n_2 \text{ or } (n_1 = n_2 \text{ and } m_1 < m_2)$$

and operation:

$$(n_1, m_1) \oplus (n_2, m_2) = (n_1 + n_2, m_1 + m_2)$$

This satisfies Axioms 0–2 but *not* separation: $(1, 1) < (2, 0)$, but no powers of $(1, 0)$ can separate them. Consequently, no order embedding into \mathbb{R} exists.

3.3 Derivable Consequences

From separation, we derive:

- **Archimedean property:** For any $x > \text{ident}$ and y , there exists n such that $\text{iterate_op } x \ n > y$.
- **Commutativity:** $\text{op } x \ y = \text{op } y \ x$ for all x, y .

These are proven in `Separation/SandwichSeparation.lean` and `RepresentationTheorem/Core/Separation`.

4 The Representation Theorem (Appendix A)

4.1 Statement

Theorem 4.1 (K&S Appendix A). *Let α be a type with `KnuthSkillingAlgebra` and `KSSeparation`. Then there exists an order embedding $\Theta : \alpha \rightarrow \mathbb{R}$ such that:*

1. $\Theta(\text{ident}) = 0$
2. $\forall x, y : \alpha, \Theta(\text{op } x \ y) = \Theta(x) + \Theta(y)$

This is K&S's main result: Axioms 1–2 plus separation uniquely determine that \oplus corresponds to addition in \mathbb{R} (up to order-preserving regrade).

4.2 Proof Sketch

K&S's constructive proof proceeds by:

1. **One type:** For a single atom a , assign $\text{iterate_op } a \ n \mapsto n \cdot \Theta(a)$.
2. **Induction:** For a new atom d , use separation to find (u, v) such that $a^u < d^v \leq a^{u+1}$. Define $\Theta(d) = (u + \delta)/v$ for appropriate $\delta \in [0, 1]$.
3. **Globalization:** Show this extends to all combinations consistently.

Our formalization in `RepresentationTheorem/` provides complete Lean proofs of each step.

4.3 Formal Statement

Listing 4: Representation theorem in Lean

```

1 theorem associativity_representation
2   ( : Type*) [KnuthSkillingAlgebra ] [KSSeparation ]
3   [DenselyOrdered ] :  $\Theta$ 
4     :  $\rightarrow$  ,
5     ( a b : , a b  $\Theta$  a  $\Theta$  b)  $\Theta$ 
6     ident = 0
7     x y : ,  $\Theta$  (op x y) =  $\Theta$  x +  $\Theta$  y

```

5 The Product Theorem (Appendix B)

Lean status: COMPLETE (no sorry, no axiom).

After Appendix A regrades \oplus to real addition, K&S introduce a second scalar operation \otimes for the direct product of independent lattices. Appendix B shows that \otimes must be multiplication up to a single global scale constant.

In Lean we formalize this conclusion in three complementary ways (all sorry-free):

1. **Product-equation route** (closest to K&S’s presentation): assume an additive order-isomorphism representation of \otimes on $\mathbb{R}_{>0}$, i.e. $\Theta(x \otimes y) = \Theta x + \Theta y$, together with distributivity over $+$ (Axiom 3). This yields the Appendix B product equation for $\Psi := \Theta^{-1}$, which we solve to obtain $\Psi(x) = Ce^{Ax}$ and hence $(x \otimes y) = (xy)/C$. (Lean: `ProductTheorem/Main.lean`.)
2. **Lean-friendly “avoid Appendix A again” route**: do *not* assume a representation theorem for \otimes . Instead, package the regularity properties as `TensorRegularity` (associativity + injectivity of $t \mapsto 1 \otimes t$) and combine with distributivity (Axiom 3) to derive $(x \otimes y) = (xy)/C$ directly. (Lean: `ProductTheorem/AczelTheorem.lean`.)
3. **Event-level bridge** (closes the gap from events to scalars): For $\otimes = \cdot$ (ordinary multiplication), the required properties (distributivity, associativity, commutativity) are **field laws**—Mathlib theorems, not K&S axioms. Therefore `multTensor_is_scaled_mul` requires **no K&S hypotheses at all**. (Lean: `ProductTheorem/EventBridge.lean`.)

Key point: the Appendix A axiom `ident_le` (identity is the bottom element) is appropriate for \oplus but structurally incompatible with \otimes on $(0, \infty)$, where the identity is 1 and is *not* a minimum element. Our Appendix B development avoids this mismatch by never trying to model \otimes as a “bottomed” additive monoid.

Axiom audit: We checked all `ProductTheorem/*.lean` files for `axiom`, `sorry`, or `admit`—none found. The injectivity requirement in `TensorRegularity` is *derived* from distributivity + commutativity via `TensorRegularity.of_assoc_of_distrib_of_comm`, not assumed.

5.1 K&S’s Approach: Fibonacci Recurrence

For independent random variables, K&S derive that the measure function Ψ must satisfy:

$$\Psi(\tau + \xi) + \Psi(\tau + \eta) = \Psi(\tau + \zeta(\xi, \eta))$$

where τ, ξ, η are independent real variables.

K&S prove the unique solution $\Psi(x) = C \cdot e^{Ax}$ via:

1. **2-term recurrence** (special case $\xi = \eta$): Shows $\Psi(\theta + na) = 2^n \Psi(\theta)$
2. **3-term recurrence** (Fibonacci-like): Produces golden ratio behavior
3. **Density argument**: Since b/a is irrational, offsets $mb - na$ approximate any real

This clever proof avoids continuity assumptions and works purely combinatorially.

5.2 Formalization Notes

- **K&S’s Fibonacci proof:** not formalized (golden-ratio density argument is Lean-hostile).
- **Regularity assumptions:** made explicit (product-equation route: continuity/strict monotonicity of Ψ ; direct route: injectivity of $t \mapsto 1 \otimes t$ via `TensorRegularity`); these are derivable from an order-isomorphism representation when one is available.
- **Mathlib support:** we use `Real.log_mul` and related properties of \exp/\log for the log-coordinate arguments.

6 Probability and Conditional Inference

K&S’s Axiom 5 (chaining associativity) extends the framework to conditional probability. This formalization is future work, building on the measure-theoretic foundations established here.

7 Conclusion

The K&S program successfully derives probability calculus from symmetries. Our formalization confirms their approach is sound, while making explicit:

- Linear order (implicit in their trichotomy arguments)
- Separation property (implicit in rational approximation)
- Archimedean property (derivable from separation)
- Commutativity (derivable from separation + order)

The formalized representation theorem provides a constructive, computationally meaningful foundation for inference.

A Corrections to the K&S Axioms

Formalization revealed several implicit assumptions and derivable properties in K&S. We document these findings with a generous reading of their intent.

A.1 The Implicit Linear Order

K&S never explicitly state that elements are totally ordered, yet their proofs rely on trichotomy: for any x, y , exactly one of $x < y$, $x = y$, or $x > y$ holds.

Evidence: K&S line 1339:

“By interchanging x and y in axiom 1, the same relationship holds when ‘ $<$ ’ is replaced throughout by ‘ $>$ ’, and replacement by ‘ $=$ ’ holds trivially. So, in effect, the axiom makes a three-fold assertion”

This implicitly assumes trichotomy, which we make explicit via `LinearOrder`.

A.2 Commutativity: Derivable, But Only With Separation

K&S state (Section 3.1):

“We also confirm that commutativity is not a necessary assumption. Rather, commutativity of measure is imposed by the associativity and order required of a scalar representation.”

Our finding: Commutativity is *not* derivable from Axioms 0–2 alone. It requires the *separation property*.

Proof: Theorem in `RepresentationTheorem/Core/SeparationImpliesCommutative.lean`:

```
1 theorem op_comm_of_associativity
2   ( : Type*) [KnuthSkillingAlgebra ] [KSSeparation ] :
3   x y : , op x y = op y x
```

Generous reading: K&S likely intended “associativity and order” to include the implicit separation property that enables their rational approximation construction.

A.3 Archimedean: Derivable from Separation

K&S’s proofs implicitly use the Archimedean property: for any $a > \text{ident}$ and x , there exists n such that $a^n > x$.

Our finding: The Archimedean property is *derivable* from separation, not an independent axiom.

Proof: Theorem in `Separation/SandwichSeparation.lean`:

```
1 theorem op_archimedean_of_separation
2   [KSSeparation ] (a x : ) (ha : ident < a) :
3   n : , x < iterate_op a n
```

This is formalization-revealed structure, not a correction to K&S.

A.4 The Separation Property Itself

K&S’s Appendix A construction implicitly assumes we can “separate” any two distinct values using rational powers of a base. We make this explicit as `KSSeparation`.

Evidence: K&S’s “induction to more than one type of atom” (Appendix A, starting line 1437) constructs rationals u/v to approximate new atom values. This requires separation.

A.5 Summary of Necessary Assumptions

For the representation theorem, we require:

1. **LinearOrder** (implicit in K&S)
2. **Associativity, Identity, Strict Monotonicity** (K&S Axioms 1–2)
3. **Separation** (implicit in K&S’s rational approximation)

From these, we *derive*:

- Archimedean property
- Commutativity
- Additive representation

B Philosophical Context: Cox, Kolmogorov, and K&S

We compare three foundational approaches to probability, all formalized in Mettapedia.

B.1 Kolmogorov (1933): Measure-Theoretic

Kolmogorov’s axioms define probability as a normalized measure on a σ -algebra. Formalized in Mathlib as `IsProbabilityMeasure`:

Listing 5: Kolmogorov probability in Mathlib

```
1 class IsProbabilityMeasure ( : Measure  $\Omega$ ) : Prop where  
2   measure_univ : Set.univ = 1
```

Key properties (all in Mathlib):

- σ -additivity: $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$ for disjoint A_i
- Monotonicity: $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$
- Normalization: $\mu(\Omega) = 1, \mu(\emptyset) = 0$

Comparison with K&S: Kolmogorov *axiomatizes* additivity; K&S *derives* it from associativity symmetry.

B.2 Cox (1946, 1961): Logical Plausibility

Formalization: See `Cox/Basic.lean`

B.2.1 Philosophical Setting

Cox takes an *epistemological* starting point: How should a rational agent assign degrees of belief to propositions? Unlike Kolmogorov (who starts with sets and measures), Cox starts with propositions and asks what constraints logic imposes on plausibility assignments.

The key philosophical move: rather than axiomatizing what probability *is*, Cox axiomatizes how plausibility *behaves* and proves probability emerges uniquely.

B.2.2 The Cox Axioms

1. **Real-valued plausibility:** Each proposition A given background B has a plausibility $p(A|B) \in [0, 1]$.
2. **Order preservation:** If A is more plausible than A' given B , then $p(A|B) > p(A'|B)$.
3. **Product rule** (the key axiom): There exists *some* function F such that:

$$p(A \wedge B|C) = F(p(A|C), p(B|A \wedge C))$$

4. **Negation rule:** There exists *some* function G such that:

$$p(\neg A|B) = G(p(A|B))$$

The crucial point: Cox does *not* assume $F(x, y) = xy$ or $G(x) = 1 - x$. He proves these are the *only* possibilities (up to reparametrization).

B.2.3 The Derivation

The associativity of conjunction— $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$ —forces:

$$F(F(x, y), z) = F(x, F(y, z))$$

This is the same functional equation as K&S Axiom 2! Combined with **continuity** of F , it forces F to be multiplication after a monotone reparametrization Θ :

$$F(x, y) = \Theta^{-1}(\Theta(x) \cdot \Theta(y))$$

The standard choice $\Theta = \text{id}$ gives the probability product rule $p(A \wedge B | C) = p(A | C) \cdot p(B | A \wedge C)$.

B.2.4 Continuity vs. Separation: A Key Contrast

Both Cox and K&S face the same challenge: the associativity equation $F(F(x, y), z) = F(x, F(y, z))$ has pathological solutions (Cauchy’s discontinuous additive functions). They rule these out differently:

	Cox	K&S
Setting	$F : \mathbb{R} \rightarrow \mathbb{R} \rightarrow \mathbb{R}$	Abstract algebra α
Regularity	F is continuous	Separation property
Domain	Already on \mathbb{R}	Must embed into \mathbb{R}

What they share: Both conditions rule out pathological non-measurable solutions.

How they differ:

- **Continuity** (Cox): A topological condition on \mathbb{R} . Directly invokes the topology of the reals.
- **Separation** (K&S): An algebraic condition on abstract α . For any $x < y$, there exist n, m with $x^m < a^n \leq y^m$. This is *density of rational powers*—no topology needed!

Key insight: Separation and continuity are *similar conditions in different settings*.

The precise relationship:

1. **From α to \mathbb{R} :** Separation + base axioms \Rightarrow embedding $\Theta : \alpha \rightarrow \mathbb{R}$ exists. Once embedded, the operation becomes addition on $\text{Im}(\Theta) \subseteq \mathbb{R}$, which is automatically continuous.
2. **On \mathbb{R} :** Any continuous, monotone, associative $F : \mathbb{R} \rightarrow \mathbb{R} \rightarrow \mathbb{R}$ satisfies separation. Why? Separation on $(\mathbb{R}, +)$ says: for $x < y$, $\exists n, m$ with $mx < na \leq my$. This is just $x < (n/m) \cdot a \leq y$ for some rational—density of $\mathbb{Q} \cdot a$ in \mathbb{R} .
3. **Failure mode:** What if separation *fails*? Then no embedding into \mathbb{R} exists at all.

Counterexample: `NatProdLex` ($\mathbb{N} \times \mathbb{N}$ with lexicographic order and componentwise addition) satisfies the base K&S axioms but *fails* separation. The gap between $(0, k)$ and $(1, 0)$ cannot be bridged by rational powers. Consequently, no order-preserving additive embedding into \mathbb{R} exists.

See: `RepresentationTheorem/Counterexamples/`

`ProductFailsSeparation.lean`

Connection theorem: `cox_comm_of_additiveOrderIsoRep` in

`Literature/CoxTheorem.lean`

B.3 K&S (2012): Lattice Symmetries

K&S unify both approaches via lattice symmetries:

- Like Kolmogorov: works with lattices (Boolean for sets, more general for logic)
- Like Cox: derives rules from functional equations, not axiomatizes them
- Beyond both: minimal assumptions (no continuity, no differentiability)

Summary of relationships:

	Kolmogorov	Cox	K&S
Additivity	Axiom	Derived	Derived
Commutativity	Assumed	Derived	Derived
Continuity	Needed	Assumed	Not needed
Domain	σ -algebra	Propositions	Lattice

C The Direct Cuts Approach

K&S prove the representation theorem via **inductive extension**: given a representation for n atoms, extend it to $n + 1$ atoms by finding the unique real value for the new atom.

Our formalization provides **both** proofs:

1. **K&S's approach** (grid/induction): `RepresentationTheorem/Main.lean`
Uses `RepresentationGlobalization` typeclass machinery.
2. **Direct cuts** (our alternative): `RepresentationTheorem/Alternative/Main.lean`
Constructs Θ via Dedekind cuts explicitly. Smaller and more direct.

Both prove the same theorem; the cuts version is more concise.

C.1 Connection to Hölder and Alimov

The direct cuts construction is modeled on classical embedding theorems for ordered algebraic structures:

Theorem	Structure	Result
Hölder (1901)	Archimedean ordered abelian groups	Embed in $(\mathbb{R}, +)$
Alimov (1950)	Cancellative ordered semigroups	Embed in $(\mathbb{R}, +)$
K&S + Separation	Ordered monoids with separation	Embed in $(\mathbb{R}, +)$

Our cuts construction adapts Hölder's technique to the K&S monoid setting, bypassing the need for group completion. The separation axiom (Section ??) provides the density condition that makes this construction work.

Formalization: See `Literature/SemigroupRepresentations.lean` for the connection to Hölder's theorem via Mathlib's `Archimedean.exists_orderAddMonoidHom_real_injective`.

C.2 The A/B/C Partition

K&S partition grid points when extending the representation to a new atom d :

- **Set A:** Points below some power of d : $\mu(F, r) < d^u$
- **Set B:** Points equal to some power of d : $\mu(F, r) = d^u$
- **Set C:** Points above some power of d : $\mu(F, r) > d^u$

C.3 Dedekind Cut Property (Our Reformulation)

When $B = \emptyset$ (the generic case), we show that $\delta := \Theta(d)$ is determined as a Dedekind cut:

Listing 6: Dedekind cut tightness lemma

```

1 lemma delta_cut_tight (hB_empty : r u, r extensionSetB F d u) :
2   > 0, rA rC uA uC,
3   rA extensionSetA F d uA
4   rC extensionSetC F d uC
5   |stat(rC, uC) - | <
6   | - stat(rA, uA)| <

```

Meaning: $\delta = \sup(\text{A-statistics}) = \inf(\text{C-statistics})$, a tight Dedekind cut.

C.4 Why This Works

The separation property (**KSSeparation**) guarantees that for any $x < y$, we can find exponents that “separate” them. This provides the density needed for the Dedekind cut to uniquely determine δ .

Key insight: The K&S proof is essentially constructing Dedekind cuts to embed the abstract algebra into \mathbb{R} , one atom at a time.

D The Probability Hypercube

We formalize a multi-axis classification of probability theories, placing K&S among its neighbors.

D.1 Hypercube Axes

Each axis represents a binary or multi-valued property of probability theories.

Formalization: See `Hypercube/Basic.lean`

- **Commutativity:** $P(A \cap B) = P(B \cap A)$? (Classical: yes; Quantum: no)
- **Precision:** Point-valued or interval-valued plausibilities?
- **Additivity:** $P(A \cup B) = P(A) + P(B)$ for disjoint A, B ?
- **Lattice structure:** Boolean, orthomodular, distributive?
- **Separation:** Can distinct values be separated by rationals?

D.2 Named Vertices

Key probability theories correspond to vertices of this hypercube:

Framework	Result	Comm.	Precise	Add.	Lattice	Method
Kolmogorov	Classical prob.	Yes	Yes	Axiom	Boolean	Axiom
Cox	Classical prob.	Yes	Yes	Derived	Boolean	Continuity
K&S (Boolean)	Classical prob.	Yes	Yes	Derived	Boolean	Separation
K&S (general)	Lattice plaus.	Yes	Yes	Derived	Distrib.	Separation
de Finetti	Subjective prob.	Yes	Yes	Finite	Boolean	Coherence
Dempster-Shafer	Belief functions	Yes	No	Sub	Boolean	Combination
Walley	Imprecise prob.	Yes	No	Sub	Boolean	Coherence
von Neumann	Quantum prob.	No	Yes	Axiom	Orthomod.	Axiom

Key insight: Kolmogorov, Cox, and K&S are *different vertices* (different derivation methods) that *converge* to the same theory on Boolean lattices. They differ in what they assume vs. derive:

- **Kolmogorov:** Axiomatizes additivity directly (no derivation)
- **Cox:** Derives additivity from continuity of plausibility functions
- **K&S:** Derives additivity from separation (no continuity needed)

D.3 Distributive vs. Boolean: What’s Different?

K&S on general distributive lattices differs from classical probability in one key way: **no negation required**.

- **Boolean lattice:** Every element a has a complement \bar{a} with $a \vee \bar{a} = \top$ and $a \wedge \bar{a} = \perp$. This gives the “negation” rule $P(\bar{A}) = 1 - P(A)$.
- **Distributive lattice:** No complements required. K&S axioms never mention negation—this is the “first fruit of our minimalist approach” (K&S).

When does this matter? Problems where not all combinations of states are allowed. Rather than “padding” a distributive lattice to Boolean, K&S works directly on the natural structure.

Literature: Valuations on distributive lattices appear in Klain & Rota’s *Introduction to Geometric Probability* (cited by K&S) and the theory of lattice valuations in combinatorics.

Legend:

- **Comm.:** Is the conjunction operation commutative? ($P(A \cap B) = P(B \cap A)$)
- **Precise:** Single-valued (Yes) vs. interval-valued credal sets (No)
- **Additive:** Full σ -additivity (Yes), finite additivity (Finite), or subadditivity (Sub)
- **Lattice:** Boolean (σ -algebra), distributive (K&S), or orthomodular (quantum)

D.4 Novel Theories

The hypercube suggests unexplored combinations:

- **Imprecise K&S:** K&S with interval-valued plausibilities (credal sets)
- **Quantum D-S:** Dempster-Shafer on orthomodular lattices
- **Non-commutative classical:** Classical probability without $P(A \cap B) = P(B \cap A)$

Formalization: See `Hypercube/NovelTheories.lean` and `Hypercube/NeighborTheories.lean`

E To-Do: Open Gaps in Formalization

E.1 Event Lattice to Plausibility Scale

Status: resolved.

The bridge is implemented in `Model.lean` as:

```
1 structure KSMModel (E S : Type*)  
2   [PlausibilitySpace E] [KnuthSkillingAlgebraBase S] where  
3   v : E → S  
4   mono : Monotone v  
5   v_bot : v = ident  
6   v_sup_of_disjoint : {a b}, Disjoint a b →  
7     v (a ∪ b) = op (v a) (v b)
```

The scale S is equipped with its K&S algebra structure from the start. The representation theorem tells us this algebra embeds into $(\mathbb{R}, +)$, so using $\mathbb{R}_{\geq 0}$ with addition as our scale is canonical.

`KSMModelWithRepresentation` extends this by composing with the representation $\Theta : S \rightarrow \mathbb{R}$, yielding the full pipeline:

$$E \xrightarrow{v} S \xrightarrow{\Theta} \mathbb{R}$$

Concrete example: `Examples/CoinFlip.lean` demonstrates a coin flip (Bernoulli trial) with 4-element Boolean algebra mapping through $\mathbb{R}_{\geq 0}$ to classical probability in \mathbb{R} .

E.2 Appendix B: Product Theorem

Status: COMPLETE (no sorry, no axiom smuggling).

Lean proves that any $\otimes : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ satisfying distributivity over $+$ (Axiom 3) and mild regularity (associativity, plus injectivity of $t \mapsto 1 \otimes t$) is multiplication up to a global scale constant. This is formalized as `TensorRegularity` in `ProductTheorem/AczelTheorem.lean`.

Key: Event-Level Bridge. The file `ProductTheorem/EventBridge.lean` closes the event-to-scalar gap. For $\otimes = \cdot$ (real multiplication), the required properties are:

- Distributivity: $(x + y) \cdot t = x \cdot t + y \cdot t$ — field law (Mathlib theorem)
- Associativity: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ — field law (Mathlib theorem)
- Commutativity: $x \cdot y = y \cdot x$ — field law (Mathlib theorem)

All three are **Mathlib theorems**, not assumed K&S axioms! Therefore `mulTensor_is_scaled_mul` is a *pure field-theory result* requiring **no K&S axioms whatsoever**.

The Product Theorem confirms: for multiplication induced by independence $v(A \times B) = v(A) \cdot v(B)$, the product rule emerges from field arithmetic alone.

E.3 Cauchy’s Functional Equation

Status: proved and used.

We prove the needed Cauchy lemma as `continuous_additive_eq_mul` in `Mettapedia/ProbabilityTheory/Kn` using Mathlib’s continuous additive $\Rightarrow \mathbb{R}$ -linear infrastructure.

E.4 Concrete Examples

Status: demonstrated.

See `Mettapedia/ProbabilityTheory/KnuthSkilling/Examples/` for:

- `CoinFlip.lean`: Full K&S pipeline on a coin flip (Bernoulli trial, Durrett Example 1.6.12). Demonstrates: 4-element Boolean algebra $\rightarrow \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$. Verifies: $P(\emptyset) = 0$, $P(\Omega) = 1$, additivity $P(H \cup T) = P(H) + P(T)$.
- `ThreeElementChain.lean`: Minimal non-Boolean distributive lattice (3-element chain).
- `OpenSets.lean`: Open sets of \mathbb{R} (distributive, typically not complemented).

References

- [1] Kevin H. Knuth and John Skilling. *Foundations of Inference*. Axioms, 1(1):38–73, 2012.