

THESIS

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ABSTRACT. This document consists of 2 mathematically independent and more or less thematically independent parts. The first concerns invariance of the cohomology groups $H^i(X, \mathcal{O}_X(-\Delta_X))$ under (a restricted class of) birational morphisms of pairs $f : (X, \Delta_X) \dashrightarrow (Y, \Delta_Y)$ in arbitrary characteristic, and as an application extends some foundational results in the theory of rational pairs that were previously known only in characteristic 0. The second part introduces and studies a condition on a proper scheme X over a field k of characteristic $p > 0$ defined in terms of the Frobenius action on $H^i(X, \mathcal{O}_X)$, which we call *globally F -full*.

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Date: May 9, 2021.

The author was partially supported by the University of Washington Department of Mathematics Graduate Research Fellowship, and by the NSF grant DMS-1440140, administered by the Mathematical Sciences Research Institute, while in residence at MSRI during the program Birational Geometry and Moduli Spaces.

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Part 1. Higher direct images of ideal sheaves

1. INTRODUCTION TO PART 1

A foundational problem in birational geometry, posed by Grothendieck in his 1958 ICM address [Gro60, Problem B], asked whether for every proper birational morphism of non-singular projective varieties $f: X \rightarrow Y$,

$$R^i f_* \mathcal{O}_X = 0 \text{ for } i > 0.$$

In characteristic 0 this was answered affirmatively by Hironaka as a corollary of resolution of singularities [Hir64, §7 Cor. 2]. In characteristic $p > 0$, where resolutions of singularities are not known to exist, answering Grothendieck's question proved much harder, remaining open until 2011 when Chatzistamatiou and Rülling proved the following theorem:

Theorem 1.1 ([CR11, Thm. 3.2.8], see also [CR15, Thm. 1.1] [Kov20, Thm. 1.4]). *Let k be a perfect field and let S be a separated scheme of finite type over k . Suppose X and Y are two separated finite type S -schemes which are*

- (i) *smooth over k and*
- (ii) ***properly birational** over S in the sense that there is a commutative diagram*

$$\begin{array}{ccc}
 & Z & \\
 r \swarrow & & \searrow s \\
 X & \hookrightarrow & Y \\
 f \searrow & & \swarrow g \\
 & S &
 \end{array} \tag{1.2}$$

with r and s proper birational morphisms.

Set $n = \dim X = \dim Y = \dim Z$. Then there are isomorphisms of sheaves

$$R^i f_* \mathcal{O}_X \xrightarrow{\sim} R^i g_* \mathcal{O}_Y \text{ and } R^i f_* \omega_X \xrightarrow{\sim} R^i g_* \omega_Y \text{ for all } i, \tag{1.3}$$

One of the primary applications of [Theorem 1.1](#) was to extend foundational results on rational singularities from characteristic 0 to arbitrary characteristic (for definitions of rational resolutions and rational singularities see [Definition 6.1](#)).

Corollary 1.4 ([CR11, Cor. 3.2.10]). *If S has a rational resolution, then every resolution of S is rational.*

[Part 1](#) concerns analogues of [Theorem 1.1](#) for pairs.

Convention 1.5 (slightly more general version of [Kol13, Def. 1.5]). In what follows a **pair** (X, Δ_X) will mean a reduced, equidimensional excellent scheme X together with a \mathbb{Q} -Weil divisor $\Delta_X = \sum_i a_i D_i$ on X such that no irreducible component D_i of Δ_X is contained in $\text{Sing}(X)$.

Convention 1.6. A pair (X, Δ_X) will be called a *simple normal crossing pair* (see [Definition 3.2](#)) if and only if X is regular and for every subset $J \subseteq \{1, \dots, N\}$ the scheme-theoretic intersection

$$D_J := \cap_{j \in J} D_j \subseteq X$$

is regular with codimension $|J|$.

As observed in [\[Kol13, §2.5\]](#), to generalize [Corollary 1.4](#) to pairs we must restrict attention to a special class of *thrifty resolutions* (see [Definition 3.6](#)).

Theorem 1.7 ([\[Kol13, Cor. 2.86\]](#) in characteristic 0, [Corollary 5.21](#) in arbitrary characteristic). *Let S be an excellent noetherian scheme admitting a dualizing complex and let (X, Δ_X) and (Y, Δ_Y) be simple normal crossing pairs separated and of finite type over S . Suppose $(X, \Delta_X), (Y, \Delta_Y)$ are properly birational over S in the sense that there is a commutative diagram*

$$\begin{array}{ccc} & (Z, \Delta_Z) & \\ r \swarrow & & \searrow s \\ (X, \Delta_X) & \circlearrowleft & (Y, \Delta_Y) \\ f \searrow & & \swarrow g \\ & S & \end{array} \quad (1.8)$$

where r, s are proper and birational morphisms, and assume $\Delta_Z = r_*^{-1}\Delta_X = s_*^{-1}\Delta_Y$. If r, s are thrifty then there are quasi-isomorphisms

$$Rf_*\mathcal{O}_X(-\Delta_X) \simeq Rg_*\mathcal{O}_Y(-\Delta_Y) \text{ and } Rf_*\omega_X(\Delta_X) \simeq Rg_*\omega_Y(\Delta_Y). \quad (1.9)$$

Via the same methods as [Theorem 1.7](#), we obtain an analogue of [Corollary 1.4](#) (for definitions of rational resolutions and rational singularities for pairs see [Definitions 6.1](#) and [6.3](#)).

Corollary 1.10. *Let (S, Δ_S) be a pair, with Δ_S reduced and effective. If (S, Δ_S) has a thrifty rational resolution $f : (X, \Delta_X) \rightarrow (S, \Delta_S)$, then every thrifty resolution $g : (Y, \Delta_Y) \rightarrow (S, \Delta_S)$ is rational.*

Our methods rely on the machinery of semi-simplicial schemes; we feel obligated to provide some motivation for their use in the context of [Theorem 1.7](#). To begin, one can translate the condition that a birational morphism $f : (X, \Delta_X) \rightarrow (S, \Delta_S)$ of simple normal crossing pairs¹ with $\Delta_X = f_*^{-1}\Delta_S$ is thrifty into the statement that the dual complexes $\mathcal{D}(\Delta_X)$ and $\mathcal{D}(\Delta_S)$ are isomorphic. The dual complex $\mathcal{D}(\Delta_X)$ is usually described as the Δ -complex (in the sense of [\[Hat02, §2.1\]](#)) with 0-cells the irreducible components D_i^X of $\Delta_X = \sum_i D_i^X$, 1-cells the components of intersections $D_i^X \cap D_j^X$ for $i < j$ with gluing maps corresponding to the inclusions $D_i^X \cap D_j^X \subseteq D_i^X$ and $D_i^X \cap D_j^X \subseteq D_j^X$, and so on (this is a generalization of the dual graph of a nodal curve). The topological properties of $\mathcal{D}(\Delta_X)$ have been extensively studied, for example in this non-exhaustive list of references: [\[ABW13; Dan75; FKX17; Ste06\]](#). Upon inspection one sees that a Δ -complex is precisely a semi-simplicial set, and that $\mathcal{D}(\Delta_X)$ is the semi-simplicial set obtained by taking π_0 (connected components) of a semi-simplicial scheme X_\bullet , with

$$X_i = \coprod_{|J|=i+1} (\cap_{j \in J} D_j^X) \text{ for } i \geq 0$$

The thriftness hypotheses of [Theorem 1.7](#) ensure that (X, Δ_X) and (Y, Δ_Y) have the same dual complex, which provides enough rigidity to attempt to prove [Theorem 1.7](#) by induction on $\dim X$

¹One can relax the condition that both pairs are snc, but it will make this motivational discussion simpler.

and the number of components of Δ_X , using [Theorem 1.1](#) as a base case. For example, we have exact sequences

$$0 \rightarrow \mathcal{O}_X(-\Delta_X) \rightarrow \mathcal{O}_X(-\Delta_X + D_1^X) \rightarrow \mathcal{O}_{D_1^X}(-\Delta_X + D_1^X|_{D_1^X}) \rightarrow 0$$

and similarly on Y , which push forward to distinguished triangles in $D_{\text{coh}}^b(S)$ of the form $Rf_*\mathcal{O}_X(-\Delta_X) \rightarrow Rf_*\mathcal{O}_X(-\Delta_X + D_1^X) \rightarrow Rf_*\mathcal{O}_{D_1^X}(-\Delta_X + D_1^X|_{D_1^X})$ and an a similar triangle coming from Y . One can even assume by induction the existence of already-defined quasi-isomorphisms in a diagram

$$\begin{array}{ccccccc} Rf_*\mathcal{O}_X(-\Delta_X) & \longrightarrow & Rf_*\mathcal{O}_X(-\Delta_X + D_1^X) & \xrightarrow{\rho^X} & Rf_*\mathcal{O}_{D_1^X}(-\Delta_X + D_1^X|_{D_1^X}) & \longrightarrow & \cdots \\ \downarrow \alpha & & \simeq \downarrow \beta & (*) & \simeq \downarrow \gamma & & \\ Rg_*\mathcal{O}_Y(-\Delta_Y) & \longrightarrow & Rg_*\mathcal{O}_Y(-\Delta_Y + D_1^Y) & \xrightarrow{\rho^Y} & Rg_*\mathcal{O}_{D_1^Y}(-\Delta_Y + D_1^Y|_{D_1^Y}) & \longrightarrow & \cdots \end{array} \quad (1.11)$$

If the square $(*)$ commutes, then using only the fact that $D_{\text{coh}}^b(S)$ is a triangulated category we get a quasi-isomorphism α on the dashed arrow. However, in this approach β, γ are themselves defined by induction, and so to know $(*)$ commutes we must take one inductive step further, considering maps of distinguished triangles

$$\begin{array}{ccccccc} Rf_*\mathcal{O}_X(-\Delta_X + D_1^X) & \longrightarrow & Rf_*\mathcal{O}_X(-\Delta_X + D_1^X + D_2^X) & \longrightarrow & Rf_*\mathcal{O}_{D_2^X}(-\Delta_X + D_1^X|_{D_2^X}) & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \text{and } (1.12) \\ Rg_*\mathcal{O}_Y(-\Delta_Y + D_1^Y) & \longrightarrow & Rg_*\mathcal{O}_Y(-\Delta_Y + D_1^Y + D_2^Y) & \longrightarrow & Rg_*\mathcal{O}_{D_2^Y}(-\Delta_Y + D_1^Y|_{D_2^Y}) & \longrightarrow & \cdots \\ \\ Rf_*\mathcal{O}_{D_1^X}(-\Delta_X + D_1^X|_{D_1^X}) & \rightarrow & Rf_*\mathcal{O}_{D_1^X}(-\Delta_X + D_1^X + D_2^X|_{D_1^X}) & \rightarrow & Rf_*\mathcal{O}_{D_1^X \cap D_2^X}(-\Delta_X + D_1^X + D_2^X|_{D_1^X \cap D_2^X}) & \rightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ Rg_*\mathcal{O}_{D_1^Y}(-\Delta_Y + D_1^Y|_{D_1^Y}) & \rightarrow & Rg_*\mathcal{O}_{D_1^Y}(-\Delta_Y + D_1^Y + D_2^Y|_{D_1^Y}) & \rightarrow & Rg_*\mathcal{O}_{D_1^Y \cap D_2^Y}(-\Delta_Y + D_1^Y + D_2^Y|_{D_1^Y \cap D_2^Y}) & \rightarrow & \cdots \end{array} \quad (1.13)$$

together with a map from (1.12) to (1.13) including the square $(*)$, and so on. It is certainly possible that the correct induction hypothesis (building in not only quasi-isomorphisms like β, γ in (1.11) but also commutativity hypotheses) and some careful analysis of diagrams in $D_{\text{coh}}^b(S)$ could make this strategy work, but I had no such luck.

Another technical issue this approach encounters is that at some point in the base case, we must analyze how the isomorphisms of [Theorem 1.1](#) behave with respect to restrictions, i.e. diagrams of schemes like

$$\begin{array}{ccccc} D_1^X & \longleftarrow & D_1^Z & \longrightarrow & D_1^Y \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & Z & \longrightarrow & Y \end{array}$$

This analysis runs into subtle aspects of Grothendieck duality, *especially* since for this approach to work we do require morphisms in $D_{\text{coh}}^b(S)$, not simply of cohomology sheaves as in [Theorem 1.1](#).

What is clear is that this attempted induction takes place on the semi-simplicial schemes X_\bullet and Y_\bullet underlying the dual complexes $\mathcal{D}(\Delta_X)$ and $\mathcal{D}(\Delta_Y)$. Under necessary thriftiness hypotheses, in the situation of [Theorem 1.7](#) we find that there is also an auxiliary semi-simplicial scheme Z_\bullet together with morphisms $X_\bullet \xleftarrow{r_\bullet} Z_\bullet \xrightarrow{s_\bullet} Y_\bullet$ which are birational in each simplicial degree. Using refined forms of Chow's lemma and resolution of indeterminacies [[Con07](#)], together with the existence of Macaulayfications [[Ces18](#); [Kaw00](#)], we can find such a Z_\bullet where each scheme Z_i is Cohen-Macaulay

and the morphisms $X_i \xleftarrow{r_i} Z_i \xrightarrow{s_i} Y_i$ are projective — this occupies [Sections 3 and 5](#). We then make essential use of [[Kov20](#), Thm. 1.4] to conclude that there are natural maps $\mathcal{O}_{X_i} \rightarrow Rr_{i*}\mathcal{O}_{Z_i}$ and $\mathcal{O}_{Y_i} \rightarrow Rs_{i*}\mathcal{O}_{Z_i}$ are quasi-isomorphisms.

The remainder of our proof is pure homological algebra: in [Section 2](#) we show that when (X, Δ_X) is a simple normal crossing pair (more generally, when the components D_i^X of Δ_X form a *regular sequence*, see [Definition 2.3](#)) the ideal sheaf $\mathcal{O}_X(-\Delta_X)$ admits a Čech-type resolution of the form

$$\mathcal{O}_X(-\Delta_X) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{X_0} \rightarrow \mathcal{O}_{X_1} \rightarrow \cdots,$$

in other words one can recover $\mathcal{O}_X(-\Delta_X)$ from an augmentation morphism $X_\bullet \rightarrow X$. Moreover, one can recover the *cohomology* of $\mathcal{O}_X(-\Delta_X)$ from a descent-type spectral sequence [Corollary 2.17](#) — the last major technical ingredient is a comparison of the resulting spectral sequences associated to X , Y and Z .

[Section 6](#) deals with applications to rational pairs, in particular [Corollary 1.4](#), and [Section 4](#) includes some new examples illustrating the subtleties of thrifty and rational resolutions of pairs, for instance we affirmatively answer a question of Erickson and Prelli on whether there exists a non-thrifty rational resolution of a pair (S, Δ) — our (S, Δ) is even a rational pair, and the resolution is related to the famous Atiyah flop. Lastly, [Section 7](#) includes mostly negative results coming from a failed attempt to replicate the original argument of [[CR11](#)], which was based on intersection theory and Hodge cohomology, using log Hodge cohomology (that is, cohomology of differential forms with log poles of the form $H^j(X, \Omega_X^i(\log \Delta_X))$).

1.1. Acknowledgements. The author would like to thank Jarod Alper, Daniel Bragg, Chi-yu Cheng, Kristin DeVleming, Gabriel Dorfsman-Hopkins, Giovanni Inchiostro, Max Lieblich, Takumi Murayama, Karl Schwede and Tuomas Tajakka for helpful conversations, and his advisor Sándor Kovács for years of guidance and encouragement as well as proposing the problem of extending results [Theorem 1.1](#) to pairs.

2. REGULAR SEQUENCES OF DIVISORS AND DESCENT SPECTRAL SEQUENCES

2.1. Semi-simplicial schemes and their derived categories. To any simple normal crossing pair we can naturally associate a *semi-simplicial scheme*. A primary reference for the theory of semi-simplicial schemes is [[SGA4II](#), Vbis]; since many elementary facts about *simplicial* schemes carry over to semi-semi-simplicial schemes, [[Con03](#); [Stacks](#)] are also relevant. What follows is a condensed summary of the machinery we need.

Let Λ denote the category with objects the sets $[i] := \{0, 1, 2, \dots, i\}$ for $i \in \mathbb{N}$ and with morphisms the *strictly increasing* functions $[j] \rightarrow [i]$; in particular $\text{Hom}_\Lambda([j], [i]) = \emptyset$ if $j > i$. A *semi-simplicial object* in a category \mathcal{C} is a functor $\Lambda^{\text{op}} \rightarrow \mathcal{C}$; semi-simplicial \mathcal{C} -objects naturally form a category, the functor category $\mathcal{C}^{\Lambda^{\text{op}}}$. Any morphism $\varphi : [j] \rightarrow [i]$ can be written non-uniquely as a composition of the basic morphisms

$$\delta_k^i : [i-1] \mapsto [i] \text{ defined by } \delta_k^i(x) = \begin{cases} x & \text{if } x < k \\ x+1 & \text{otherwise} \end{cases}$$

(so δ_k^i skips k) [[Stacks](#), [Tag 0164](#)], and hence a semi-simplicial object $X : \Lambda^{\text{op}} \rightarrow \mathcal{C}$ is equivalent to a sequence of objects $X_i := X([i])$ together with morphisms $d_k^i := X(\delta_k^i) : X_i \rightarrow X_{i-1}$ subject to the relations

$$d_k^{i-1} \circ d_l^i = d_{l-1}^{i-1} \circ d_k^i, \tag{2.1}$$

and all semi-simplicial objects below will be obtained from such an explicit description. In what follows semi-simplicial objects will be denoted with a \bullet , e.g. “the semi-simplicial scheme X_\bullet ” (to distinguish them from plain schemes).

When C is a category of schemes, a *sheaf* on a semi-simplicial scheme X_\bullet is the data of a sheaf \mathcal{F}_i on each scheme X_i together with morphisms of sheaves $\delta_k^i : \mathcal{F}_{i-1} \rightarrow d_{k*}^i \mathcal{F}_i$ on X_{i-1} satisfying compatibilities coming from (2.1). These sheaves form a *topos* \tilde{X}_\bullet such that morphisms of semi-simplicial schemes $f_\bullet : X_\bullet \rightarrow Y_\bullet$ induce functorial maps of topoi $\tilde{X}_\bullet \rightarrow \tilde{Y}_\bullet$ (see [SGA4II, Vbis, Prop. 1.2.15]) — the benefit of the topos-theoretic point of view is that it immediately implies the category of *abelian* sheaves $\mathbf{Ab}(X_\bullet)$ on X_\bullet is an abelian category with enough injectives ([Stacks, Tag 01DL]), enables us to define pushforward functors $Rf_* : D^+(\mathbf{Ab}(X_\bullet)) \rightarrow D^+(\mathbf{Ab}(Y_\bullet))$ for morphisms of semi-simplicial schemes $f_\bullet : X_\bullet \rightarrow Y_\bullet$, and so on.

An *augmented* semi-simplicial scheme is a morphism of semi-simplicial schemes $\epsilon_\bullet : X_\bullet \rightarrow S_\bullet$ where S_\bullet is a *constant* semi-simplicial scheme (that is, $S_i = S$ for all i for some fixed scheme S , and all $d_k^i = \text{id}$). This is equivalent to the data of a semi-simplicial object of \mathbf{Sch}_S . For such a constant semi-simplicial scheme S_\bullet , $\mathbf{Ab}(S_\bullet)$ is equivalent to the category $\mathbf{Ab}(S)^\Delta$ of co-semi-simplicial sheaves of abelian groups on S , that is, sequences of sheaves of abelian groups \mathcal{G}_i on S together with morphisms $\delta_k^i : \mathcal{G}_{i-1} \rightarrow \mathcal{G}_i$ satisfying compatibilities forced by (2.1). As in the construction of the Čech complex setting $d^i = \sum_k (-1)^k : \delta_k^i : \mathcal{G}_{i-1} \rightarrow \mathcal{G}_i$ gives a complex of abelian sheaves on S and hence in particular an abelian sheaf $a(\mathcal{G}_\bullet) := \ker d^0$. Writing $\epsilon_* := a \circ \epsilon_{**}$, the composite derived functor

$$\begin{array}{ccccc} & & R\epsilon_* & & \\ & \nearrow & & \searrow & \\ D^+(\mathbf{Ab}(X_\bullet)) & \xrightarrow{R\epsilon_{**}} & D^+(\mathbf{Ab}(S_\bullet)) & \xrightarrow{Ra} & D^+(\mathbf{Ab}(S)) \end{array}$$

admits the following concrete description: given a sheaf \mathcal{F}_\bullet on X_\bullet one finds a certain kind of injective resolution

$$\mathcal{F}_\bullet \rightarrow \mathcal{I}_\bullet^0 \rightarrow \mathcal{I}_\bullet^1 \rightarrow \mathcal{I}_\bullet^2 \rightarrow \dots \text{ in } \mathbf{Ab}(X_\bullet)$$

Here the \mathcal{I}_\bullet^j are in particular sheaves on X_\bullet with each \mathcal{I}_i^j an injective abelian sheaf on X_i — for further discussion of injective objects in $\mathbf{Ab}(X_\bullet)$ see [SGA4II, Vbis, Prop. 1.3.10] and [Con03, Lem. 6.4, comments on p. 42]. Then

$$f_* \mathcal{I}_\bullet^0 \rightarrow f_* \mathcal{I}_\bullet^1 \rightarrow f_* \mathcal{I}_\bullet^2 \rightarrow \dots \text{ in } \mathbf{Ab}(X_\bullet)$$

is a complex of co-semi-simplicial abelian sheaves which via the Čech construction becomes a complex of complexes. Applying the sign trick gives a double complex whose Tot computes $R\epsilon_* \mathcal{F}_\bullet$ (see [SGA4II, Vbis §2.3], [Con03, Thms. 6.11–6.12]). One of the spectral sequences of this double complex is:

Lemma 2.2 (Descent spectral sequence). *There is a spectral sequence*

$$E_1^{pq} = R^q \epsilon_{p*} \mathcal{F}_p \rightarrow R^{p+q} \epsilon_* \mathcal{F}_\bullet.$$

2.2. Regular sequences of divisors.

Definition 2.3. Let X be a locally noetherian scheme. A sequence of effective Cartier divisors $D_1, D_2, \dots, D_N \subseteq X$ is called *regular* if and only if for each point $x \in X$, letting $f_1, \dots, f_N \in \mathcal{O}_{X,x}$ be local generators for the divisors D_i and letting $I(x) = \{i \mid x \in D_i\}$, the elements $(f_j \in \mathfrak{m}_x \mid j \in I(x))$ form a regular sequence.

This definition is designed to ensure that a permutation of a regular sequence of divisors is again a regular sequence (see [Mat80, §15, Thm. 27]).

Let X be a locally noetherian scheme together with a regular sequence of effective Cartier divisors $D_1, D_2, \dots, D_N \subseteq X$. We define an augmented semi-simplicial scheme X_\bullet as follows: $X_{-1} = X$,

$X_0 = \coprod_i D_i$ and for $k > 0$,

$$X_k = \coprod_{I \subseteq \{1, \dots, N\} \mid |I|=k+1} D_I, \text{ where } D_I = \bigcap_{j \in I} D_j$$

The face maps are defined by the inclusions $d_k^j : D_I \hookrightarrow D_{I \setminus \{i_j\}}$ for $I = \{i_0, \dots, i_k\}$ and $0 \leq j \leq i$, as in a Čech complex, and for each k we have an augmentation map $\epsilon_p : X_k \rightarrow X$ obtained from the inclusions $D_I \subseteq X$. In this situation the descent spectral sequence of Lemma 2.2 degenerates: since the $\epsilon_p : X_p \rightarrow X$ are closed immersions and hence affine, $R^q \epsilon_{p*} \mathcal{O}_{X_p} = 0$ for $q > 0$. It follows that $R^i \epsilon_* \mathcal{O}_{X_\bullet}$ is the cohomology of the Čech type complex

$$\begin{aligned} \epsilon_{0*} \mathcal{O}_{X_0} &\xrightarrow{d^1} \epsilon_{1*} \mathcal{O}_{X_1} \xrightarrow{d^2} \dots \xrightarrow{d^N} \epsilon_{N*} \mathcal{O}_{X_N} \\ &= \bigoplus_i \mathcal{O}_{D_i} \xrightarrow{d^1} \bigoplus_{i < j} \mathcal{O}_{D_i \cap D_j} \xrightarrow{d^2} \dots \xrightarrow{d^N} \mathcal{O}_{\cap_i D_i} \end{aligned} \quad (2.4)$$

Lemma 2.5. *The complex (2.4) is exact in degrees $i > 0$, with $\ker d^1 \simeq \mathcal{O}_{\cup_i D_i}$. Equivalently, the extended complex*

$$0 \rightarrow \mathcal{O}_X(-\sum_i D_i) \rightarrow \mathcal{O}_X \xrightarrow{\gamma} \bigoplus_i \mathcal{O}_{D_i} \xrightarrow{d^1} \bigoplus_{i < j} \mathcal{O}_{D_i \cap D_j} \xrightarrow{d^2} \dots \xrightarrow{d^N} \mathcal{O}_{\cap_i D_i} \rightarrow 0$$

where $\gamma : \mathcal{O}_X \rightarrow \bigoplus_i \mathcal{O}_{D_i}$ is restriction in each factor is exact, and hence there is a canonical quasi-isomorphism $\mathcal{O}_X(-\sum_i D_i) \simeq \text{cone}(\mathcal{O}_X \rightarrow R\epsilon_* \mathcal{O}_{X_\bullet})[-1]$.

Proof. We proceed by induction on the number N of divisors. The base case $N = 0$ is vacuous (X_\bullet is empty). If that seems too weird, the case $N = 1$ simply says that the sequence $0 \rightarrow \mathcal{O}_X(-D_1) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{D_1} \rightarrow 0$ is exact, which is indeed true.

Suppose now that $N > 1$. Essentially by the definition of a regular sequence, $D_1 \cap D_2, D_1 \cap D_3, \dots, D_1 \cap D_N \subseteq D_1$ is a regular sequence of divisors, and by permutation invariance of regular sequences $D_2, \dots, D_N \subseteq X$ is a regular sequence. We form a short exact sequence of complexes (with cohomological degrees as indicated)

$$\begin{array}{ccccccc} C' : & 0 & \longrightarrow & \mathcal{O}_{D_1} & \xrightarrow{d'} & \bigoplus_{1 < j} \mathcal{O}_{D_1 \cap D_j} & \xrightarrow{d'} \bigoplus_{1 < j < k} \mathcal{O}_{D_1 \cap D_j \cap D_k} \xrightarrow{d'} \dots \\ \downarrow \alpha & \downarrow & & \downarrow \alpha & & \downarrow \alpha & \\ C : & \mathcal{O}_X & \xrightarrow{\gamma} & \bigoplus_i \mathcal{O}_{D_i} & \xrightarrow{d} & \bigoplus_{i < j} \mathcal{O}_{D_i \cap D_j} & \xrightarrow{d} \bigoplus_{i < j < k} \mathcal{O}_{D_i \cap D_j \cap D_k} \xrightarrow{d} \dots \\ \downarrow \beta & \parallel \beta & & \downarrow \beta & & \downarrow \beta & \\ C'' : & \mathcal{O}_X & \xrightarrow{\gamma''} & \bigoplus_{1 < i} \mathcal{O}_{D_i} & \xrightarrow{d''} & \bigoplus_{1 < i < j} \mathcal{O}_{D_i \cap D_j} & \xrightarrow{d''} \bigoplus_{1 < i < j < k} \mathcal{O}_{D_i \cap D_j \cap D_k} \xrightarrow{d''} \dots \end{array} \quad (2.6)$$

-1 0 1 2

(in fact the columns are *split* short exact sequences, by comparing indexed summands). By inductive hypotheses,

$$h^i(C') = \begin{cases} \mathcal{O}_{D_1}(-\sum_{1 < j} D_1 \cap D_j) & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases} \text{ and } h^i(C'') = \begin{cases} \mathcal{O}_X(-\sum_{1 < j} D_j) & \text{if } i = -1 \\ 0 & \text{otherwise} \end{cases}$$

showing that $h^i(C) = 0$ for $i > 0$, and that in low degrees there is an exact sequence

$$0 \rightarrow h^{-1}(C) \rightarrow \mathcal{O}_X(-\sum_{1 < j} D_j) = h^{-1}(C'') \xrightarrow{\delta} h^0(C') = \mathcal{O}_{D_1}(-\sum_{1 < j} D_1 \cap D_j) \rightarrow 0 \quad (2.7)$$

To complete the proof, we must verify that the connecting map δ is indeed restriction of sections, so that (2.7) coincides with the usual exact sequence

$$0 \rightarrow \mathcal{O}_X(-\sum_j D_j) \rightarrow \mathcal{O}_X(-\sum_{1 \leq j} D_j) \rightarrow \mathcal{O}_{D_1}(-\sum_{1 \leq j} D_1 \cap D_j) \rightarrow 0$$

and indeed, by the snake lemma construction of the connecting map δ we lift a local section $\sigma \in \ker \gamma'' \subseteq \mathcal{O}_X$ along β , apply $\gamma : \mathcal{O}_X \rightarrow \oplus_i \mathcal{O}_{D_i}$ to obtain a local section $(\sigma|_{D_i}) \in \ker \beta \subseteq \oplus_i \mathcal{O}_{D_i}$, and then lift along $\alpha : \mathcal{O}_{D_1} \rightarrow \oplus_i \mathcal{O}_{D_i}$ — the net result is $\sigma|_{D_1}$ as claimed. \square

Remark 2.8. Here we sketch a different proof of [Lemma 2.5](#), which could potentially shed more light on what happens if $D_1, \dots, D_N \subseteq X$ deviates from being a regular sequence. For each i let $\sigma_i : \mathcal{O}_X \rightarrow \mathcal{O}_X(D_i)$ be the canonical global section and let $\sigma_i^\vee : \mathcal{O}_X(-D_i) \rightarrow \mathcal{O}_X$ be its dual. For each subset $J \subseteq \{1, \dots, N\}$ let $\mathcal{E}_J := \oplus_{j \in J} \mathcal{O}_X(D_j)$. For each such J we have a section $\sigma_J = (\sigma_j|_{j \in J}) : \mathcal{O}_X \rightarrow \mathcal{E}_J$. There's a map of chain complexes

$$\begin{array}{ccccccc} 0 = \mathcal{E}_\emptyset & \longrightarrow & \bigoplus_{|J|=1} \mathcal{E}_J & \longrightarrow & \bigoplus_{|J|=2} \mathcal{E}_J & \longrightarrow & \bigoplus_{|J|=3} \mathcal{E}_J \longrightarrow \dots \\ \uparrow & & \uparrow \oplus \sigma_J & & \uparrow \oplus \sigma_J & & \uparrow \oplus \sigma_J \\ \mathcal{O}_X & \longrightarrow & \bigoplus_{|J|=1} \mathcal{O}_X & \longrightarrow & \bigoplus_{|J|=2} \mathcal{O}_X & \longrightarrow & \bigoplus_{|J|=3} \mathcal{O}_X \longrightarrow \dots \end{array}$$

where the horizontal differentials are alternating sums of summand inclusions (in effect, they come from the singular co-chain complex of the $N-1$ -simplex Δ^{N-1}) and the vertical maps are induced by the σ_i . Applying the Koszul construction to the individual maps $\sigma_J : \mathcal{O}_X \rightarrow \mathcal{E}_J$ (along with the usual sign trick) then results in a double complex $C^{\bullet\bullet}$ with $C^{pq} = \bigoplus_{|J|=p} \wedge^{-q} \mathcal{E}_J^\vee$.

I claim without proof² that the *horizontal* complexes

$$C^{\bullet q} : 0 \rightarrow \dots \rightarrow 0 \rightarrow \bigoplus_{|J|=-q} \wedge^{-q} \mathcal{E}_J^\vee \rightarrow \bigoplus_{|J|=-q+1} \wedge^{-q} \mathcal{E}_J^\vee \rightarrow \dots \rightarrow \bigoplus_{|J|=N} \wedge^{-q} \mathcal{E}_J^\vee = \wedge^{-q} \left(\bigoplus_{i=1}^N \mathcal{O}_X(-D_i) \right)$$

are exact for $q > -N$, and hence $\text{Tot}(C^{\bullet\bullet})$ is quasi-isomorphic to $\wedge^N (\oplus_{i=1}^N \mathcal{O}_X(-D_i)) = \mathcal{O}_X(-\sum_i D_i)$. On the other hand, the vertical complexes

$$C^{p\bullet} : 0 \rightarrow \dots \rightarrow 0 \rightarrow \bigoplus_{|J|=p} \wedge^p \mathcal{E}_J^\vee \rightarrow \bigoplus_{|J|=p} \wedge^{p-1} \mathcal{E}_J^\vee \rightarrow \dots \rightarrow \bigoplus_{|J|=p} \mathcal{E}_J^\vee \rightarrow \bigoplus_{|J|=p} \mathcal{O}_X$$

are direct sums of Koszul complexes by design, and so their cohomology is $h^q(C^{p\bullet}) = \bigoplus_{|J|=p} \mathcal{T}or_{-q}^{\mathcal{O}_X}(\mathcal{O}_{D_J}, \mathcal{O}_X)$, which reduces to

$$h^q(C^{p\bullet}) = \begin{cases} \bigoplus_{|J|=p} \mathcal{O}_{D_J} & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

precisely when the sequence D_1, \dots, D_N is regular [[Mat80](#), §18 Thm. 43], [[Ful98](#), Lem. A.5.3].

We close this section by noting that the locus where a collection of divisors forms a regular sequence is open in a strong sense. This is not used directly below (although see [Definition 3.2](#) and [Lemma 3.3](#)).

Lemma 2.9. *If X is an integral excellent scheme with a sequence of effective Weil divisors $D_1, \dots, D_N \subseteq X$ with ideal sheaves $\mathcal{I}_1, \dots, \mathcal{I}_N \subseteq \mathcal{O}_X$ then the locus*

$$\text{regseq}(X, (D_i)) := \{x \in X \mid \sum_{i \in I(x)} \mathcal{I}_i \subseteq \mathcal{O}_{X,x}^\wedge \text{ is a regular sequence of effective Cartier divisors}\} \subseteq X$$

²It seems a proof by induction on N analogous to the argument in [Lemma 2.5](#) works, although it is combinatorially more involved.

(where as in [Definition 2.3](#) $I(x) = \{i \mid x \in D_i\}$) is open, and this is the largest open set $U \subseteq X$ such that $(U, (D_i \cap U))$ is a simple normal crossing pair.

Proof. If $\mathcal{I}_{i,x}^\wedge \subseteq \mathcal{O}_{X,x}^\wedge$ is a principal ideal, say generated by $f_i \in \mathcal{O}_{X,x}^\wedge$, then letting $k(x)$ denote the residue field at x we see

$$\mathcal{I}_i \otimes_{\mathcal{O}_X} k(x) = \mathcal{I}_{i,x}^\wedge \otimes_{\mathcal{O}_{X,x}^\wedge} k(x) = (f_i) \otimes_{\mathcal{O}_{X,x}^\wedge} k(x)$$

is rank 1, generated by the reduction of f_i . It follows that

$$\{x \in X \mid \mathcal{I}_{i,x}^\wedge \subseteq \mathcal{O}_{X,x}^\wedge \text{ is principal}\} = \{x \in X \mid \mathcal{I}_i \otimes_{\mathcal{O}_X} k(x) \text{ is 1-dimensional}\} =: U_i$$

Since \mathcal{I}_i is a coherent sheaf with generic rank 1, semi-continuity of the fiber dimensions $\mathcal{I}_i \otimes_{\mathcal{O}_X} k(x)$ shows that the right hand side is open. Hence we may reduce to the case where the divisors $D_1, \dots, D_N \subseteq X$ are Cartier.

We will show that given any $x \in \text{regseq}(X, (D_i))$, there is a neighborhood $x \in U \subseteq X$ such that $(U, (D_i \cap U))$ is a regular sequence of effective Cartier divisors. Replacing X with $X \setminus \bigcup_{i \mid x \notin D_i} D_i$ we may assume $x \in \cap_i D_i$. Let \mathcal{K}^\bullet be the Koszul complex of the section $(\sigma_i) : \mathcal{O}_X \rightarrow \oplus_i \mathcal{O}_X(D_i)$, \mathcal{K}_x^\bullet be its localization at x and $\hat{\mathcal{K}}_x^\bullet$ the completion thereof. Since completion is faithfully flat, the natural maps $h^i(\mathcal{K}_x^\bullet) \rightarrow h^i(\hat{\mathcal{K}}_x^\bullet)$ are injective, and by hypothesis $(f_i) \subseteq \mathcal{O}_{X,x}^\wedge$ is a regular sequence so $h^i(\hat{\mathcal{K}}_x^\bullet) = 0$ for $i > 0$. Thus $h^i(\mathcal{K}_x^\bullet) = 0$ for $i > 0$, in other words $x \notin \bigcup_{i>0} \text{supp } h^i(\mathcal{K}^\bullet)$. Since the cohomology sheaves $h^i(\mathcal{K}^\bullet)$ are coherent, the supports $\text{supp } h^i(\mathcal{K}^\bullet)$ are closed, so that $X \setminus \bigcup_{i>0} \text{supp } h^i(\mathcal{K}^\bullet)$ is a neighborhood of x where the D_i form a regular sequence. \square

2.3. Resolving sheaves of log-differentials. In the case where X is smooth over a perfect field and $\Delta_X := \bigcup_{i=1}^N D_i$ is a simple normal crossing divisor, we can say even more — however this additional information is not used in the sequel so the reader is welcome to proceed to [section 2.4](#).

Here the X_k are smooth, so in particular the sheaves of differential forms $\Omega_{X_k}^1$ are locally free, and for each p the standard Čech construction applied to the co-semi-simplicial sheaf $\Omega_{X_\bullet}^p$ gives a cochain complex

$$R\epsilon_* \Omega_{X_\bullet}^p : \epsilon_{0*} \Omega_{X_0}^p \rightarrow \epsilon_{1*} \Omega_{X_1}^p \rightarrow \epsilon_{2*} \Omega_{X_2}^p \rightarrow \dots$$

on X , together with a morphism $\Omega_X^p \rightarrow R\epsilon_* \Omega_{X_\bullet}^p$ induced by the augmentation. The shifted cone $\underline{\Omega}_{X, \Delta_X}^p := \text{cone}(\Omega_X^p \rightarrow R\epsilon_* \Omega_{X_\bullet}^p)[-1]$ is then represented by the following complex, with derived category degrees as indicated:³

$$\begin{array}{ccccccc} \Omega_X^p & \longrightarrow & \epsilon_{0*} \Omega_{X_0}^p & \longrightarrow & \epsilon_{1*} \Omega_{X_1}^p & \longrightarrow & \epsilon_{2*} \Omega_{X_2}^p \longrightarrow \dots \\ & & & & & & \\ = \Omega_X^p & \rightarrow & \prod_{\sigma \in \mathcal{D}((\Delta_X))^0} \Omega_{D(\sigma)}^p & \rightarrow & \prod_{\sigma \in \mathcal{D}((\Delta_X))^1} \Omega_{D(\sigma)}^p & \rightarrow & \prod_{\sigma \in \mathcal{D}((\Delta_X))^2} \Omega_{D(\sigma)}^p \rightarrow \dots \end{array} \quad (2.10)$$

0
1
2
3

Lemma 2.11 (Cf. [\[Fri83, Prop. 1.5\]](#), [\[DI87, Rem. 4.2.2\]](#)). *The complex*

$$0 \rightarrow \Omega_X^p(\log \Delta_X)(-\Delta_X) \rightarrow \Omega_X^p \rightarrow \prod_{\sigma \in \mathcal{D}((\Delta_X))^0} \Omega_{D(\sigma)}^p \rightarrow \prod_{\sigma \in \mathcal{D}((\Delta_X))^1} \Omega_{D(\sigma)}^p \rightarrow \dots$$

³This notation is chosen to align with the fact that over \mathbb{C} , the complex (2.10) represents the p th graded part of the Du Bois complex of the pair (X, Δ_X) .

is exact. Equivalently, the complex (2.10) is a resolution of the sheaf $\Omega_X^p(\log \Delta_X)(-\Delta_X)$. In particular (for $p = 0$) the complex

$$\mathcal{O}_X \rightarrow \prod_{\sigma \in \mathcal{D}(\Delta_X)^0} \mathcal{O}_{D(\sigma)} \rightarrow \prod_{\sigma \in \mathcal{D}(\Delta_X)^1} \mathcal{O}_{D(\sigma)} \rightarrow \cdots$$

is a resolution of $\mathcal{O}_X(-\Delta_X)$.

We include a proof merely to make clear that the lemma is valid in arbitrary characteristic — the argument given follows [Fri83, Prop. 1.5] very closely.

Proof. We can check exactness on Zariski stalks over a point $x \in X$. We may also check exactness after renumbering the divisors D_i , and so we may assume that $x \in D_1, \dots, D_k$ and $x \notin D_i$ for $i > k$. By hypothesis, there are local coordinates $z_1, \dots, z_c \in \mathcal{O}_{X,x}$ such that in a Zariski neighborhood of x , $\Delta_X = V(\prod_{i=1}^k z_i)$ and $D_i = V(z_i)$ for $i = 1, \dots, k$.

We now proceed by simultaneous induction on k and $\dim X$. Letting $\Delta_{D_1} = \sum_{i=2}^k D_i \cap D_1$, we have $\dim D_1 < \dim X$ and $k-1 < k$, so denoting by $\epsilon' : D_{1\bullet} \rightarrow D_1$ the semi-simplicial scheme associated to (D_1, Δ_{D_1}) , by inductive hypothesis the complex

$$0 \rightarrow \Omega_{D_1}^p(\log \Delta_{D_1})(-\Delta_{D_1}) \rightarrow \Omega_{D_1}^p \rightarrow \epsilon'_{0*} \Omega_{D_{1,0}}^p \rightarrow \epsilon'_{1*} \Omega_{D_{1,1}}^p \rightarrow \cdots \quad (2.12)$$

is exact. On the other hand, letting $\Delta^{>1} = \sum_{i=2}^r D_i$ we obtain a divisor with $k-1 < k$ components, so denoting $\epsilon'' : X_{\bullet}^{>1} \rightarrow X$ the semi-simplicial scheme associated to $(X, \Delta^{>1})$, by inductive hypothesis the complex

$$0 \rightarrow \Omega_X^p(\log \Delta^{>1})(-\Delta^{>1}) \rightarrow \Omega_X^p \rightarrow \epsilon''_{0*} \Omega_{X_0^{>1}}^p \rightarrow \epsilon''_{1*} \Omega_{X_1^{>1}}^p \rightarrow \cdots$$

is exact. Moreover, there is a sequence of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{D_1}^p & \xrightarrow{d'} & \epsilon'_{0*} \Omega_{D_{1,0}}^p & \xrightarrow{d'} & \epsilon'_{1*} \Omega_{D_{1,1}}^p \xrightarrow{d'} \cdots \\ \downarrow & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha \\ \Omega_X^p & \xrightarrow{\epsilon^\#} & \epsilon_{0*} \Omega_{X_0}^p & \xrightarrow{d} & \epsilon_{1*} \Omega_{X_1}^p & \xrightarrow{d} & \epsilon_{2*} \Omega_{X_2}^p \xrightarrow{d} \cdots \\ \parallel & & \downarrow \beta & & \downarrow \beta & & \downarrow \beta \\ \Omega_X^p & \xrightarrow{\epsilon''^\#} & \epsilon''_{0*} \Omega_{X_0^{>1}}^p & \xrightarrow{d''} & \epsilon''_{1*} \Omega_{X_1^{>1}}^p & \xrightarrow{d''} & \epsilon''_{2*} \Omega_{X_2^{>1}}^p \xrightarrow{d''} \cdots \end{array} \quad (2.13)$$

0 1 2 3

and since for each k , $X_k = X_k^{>1} \amalg D_{1,k-1}$ the columns are (split) exact. Using the long exact sequence of cohomology sheaves, the inductive hypotheses show that $h^i(\underline{\Omega}_{X,\Delta_X}^p) = 0$ for $i > 1$, and in low degrees we have an exact sequence

$$0 \rightarrow \Omega_X^p(\log \Delta_X)(-\Delta_X) \rightarrow \Omega_X^p(\log \Delta^{>1})(-\Delta^{>1}) \rightarrow \Omega_{D_1}^p(\log \Delta_{D_1})(-\Delta_{D_1}) \rightarrow h^1(\underline{\Omega}_{X,\Delta_X}^p) \rightarrow 0$$

It remains to show $h^1(\underline{\Omega}_{X,\Delta_X}^p) = 0$. For this consider a local section

$$(\varphi_i) = (\varphi_i | i = 1, \dots, k) \in \ker d \subseteq \epsilon_{0*} \Omega_{X_0}^p = \prod_{i=1}^k \Omega_{D_i}^p$$

As $d''\beta(\varphi_i) = \beta d(\varphi_i) = 0$, by inductive hypothesis there is a local section $\omega \in \Omega_X^p$ such that $\beta(\varphi_i) = \epsilon''^\# \omega$. Unravelling, $\beta(\varphi_i) = (\varphi_2, \dots, \varphi_k)$ and $\omega|_{D_i} = \varphi_i$ for $i = 2, \dots, k$. Since

$$0 = d(\varphi_i) = (\varphi_i|_{D_i \cap D_j} - \varphi_i|_{D_i \cap D_j}|1 \leq i < j \leq N), \text{ so in particular for } i = 1$$

$$0 = \varphi_1|_{D_1 \cap D_j} - \varphi_j|_{D_1 \cap D_j} = \varphi_1|_{D_1 \cap D_j} - \omega|_{D_1 \cap D_j} \text{ for } j = 2, \dots, k$$

we find that $\varphi_1 - \omega|_{D_1}$ vanishes on Δ_{D_1} , and applying exactness of (2.12) once more we see $\varphi_1 - \omega|_{D_1} \in \Omega_{D_1}^p(\log \Delta_{D_1})(-\Delta_{D_1})$. At x , $\Omega_{D_1}^p(\log \Delta_{D_1})(-\Delta_{D_1})$ is generated by the forms

$$\left(\prod_{i=2}^k z_i \right) \cdot \frac{dz_{i_1}}{z_{i_1}} \wedge \dots \wedge \frac{dz_{i_l}}{z_{i_l}} \wedge dz_{i_{l+1}} \wedge \dots \wedge dz_{i_p} \text{ where } 1 < i_1 < \dots < i_l \leq k < i_{l+1} < \dots < i_p \leq N$$

The key point is: each of these vanishes on D_i for $i > 1$ (since they each contain either a z_i or a dz_i for all $1 < i \leq k$), and so we may find a local section $\xi \in \Omega_X^p$ with

$$(i) \quad \xi|_{D_1} = \varphi_1 - \omega|_{D_1};$$

$$(ii) \quad \xi|_{D_i} = 0 \text{ for } i > 1.$$

Rearranging shows $(\omega + \xi)|_{D_i} = \varphi_i$ for all i — in other words $(\varphi_i) = \epsilon^\#(\omega + \xi)$. \square

Remark 2.14. As a byproduct we obtain an exact sequence

$$0 \rightarrow \Omega_X^p(\log \Delta_X)(-\Delta_X) \rightarrow \Omega_X^p(\log \Delta^{>1})(-\Delta^{>1}) \rightarrow \Omega_{D_1}^p(\log \Delta_{D_1})(-\Delta_{D_1}) \rightarrow 0,$$

and considering the snake-lemma definition of the connecting morphism shows this is, at least up to sign, restriction of log differential forms (see [EV92, §2])

2.4. Replacing $\mathcal{O}(-\Delta)$ with a filtered complex. Let X be a locally noetherian scheme and let $D_1, \dots, D_N \subseteq X$ be a regular sequence of effective Cartier divisors, with sum $\Delta_X := \sum_{i=1}^N D_i$. By Lemma 2.5 the ideal sheaf $\mathcal{O}_X(-\Delta_X)$ is quasi-isomorphic to $\text{cone}(\mathcal{O}_X \rightarrow R\epsilon_*\mathcal{O}_{X_\bullet})[-1]$, which for convenience moving forward we give a name:

Definition 2.15. $\underline{\Omega}_{X, \Delta_X}^0 := \text{cone}(\mathcal{O}_X \rightarrow R\epsilon_*\mathcal{O}_{X_\bullet})[-1]$.

By Lemma 2.5 and its proof this complex has the explicit representation

$$\begin{array}{ccccccc} \mathcal{O}_X & \longrightarrow & \bigoplus_i \mathcal{O}_{D_i} & \longrightarrow & \bigoplus_{i < j} \mathcal{O}_{D_i \cap D_j} & \longrightarrow & \dots \longrightarrow \mathcal{O}_{\cap_i D_i} \\ 0 & & 1 & & 2 & & \dots & N \end{array}$$

We can give $\underline{\Omega}_{X, \Delta_X}^0$ a descending filtration by truncations

$$\underline{\Omega}_{X, \Delta_X}^0 = \sigma_{\geq 0} \underline{\Omega}_{X, \Delta_X}^0 \supset \sigma_{\geq 1} \underline{\Omega}_{X, \Delta_X}^0 \supset \sigma_{\geq 2} \underline{\Omega}_{X, \Delta_X}^0 \supset \dots$$

where

$$(\sigma_{\geq i} \underline{\Omega}_{X, \Delta_X}^0)^j = \begin{cases} 0 & \text{if } j < i \\ (\underline{\Omega}_{X, \Delta_X}^0)^j = \epsilon_{j-1*} \mathcal{O}_{X_{j-1}} = \prod_{J \subseteq \{1, \dots, N\} \mid |J|=j} \mathcal{O}_{D_J} & \text{otherwise} \end{cases} \quad (2.16)$$

Using this filtration we obtain a spectral sequence for higher direct images.

Corollary 2.17. *Let S be a locally noetherian scheme and let $f : X \rightarrow S$ be a finite type morphism. Let $D_1, \dots, D_N \subseteq X$ be a regular sequence of effective Cartier divisors, with sum Δ_X . Then there is a filtered complex $(Rf_* \underline{\Omega}_{X, \Delta_X}^0, F)$ whose cohomology computes the higher direct images $R^{i+j} f_* \mathcal{O}_X(-\Delta_X)$. For each i there is a distinguished triangle*

$$F^{i+1} Rf_* \underline{\Omega}_{X, \Delta_X}^0 \rightarrow F^i Rf_* \underline{\Omega}_{X, \Delta_X}^0 \rightarrow \prod_{J \subseteq \{1, \dots, N\} \mid |J|=i} Rf_* \mathcal{O}_{D_J} \rightarrow \dots$$

In particular, there is a spectral sequence

$$E_1^{ij} = \prod_{J \subseteq \{1, \dots, N\} \mid |J|=i} R^j f_* \mathcal{O}_{D_J} \implies R^{i+j} f_* \mathcal{O}_X(-\Delta_X)$$

The filtration F is defined as $F = Rf_*\sigma$, and the resulting spectral sequence is just the usual hypercohomology spectral sequence.

Remark 2.18. Viewing $\epsilon : X_\bullet \rightarrow X$ as a sort of resolution of the pair (X, Δ_X) , we can consider the spectral sequence of [Corollary 2.17](#) as a sort of *descent* spectral sequence (see [\[SGA4II, Vbis\]](#), [\[Con03\]](#)).

3. SIMPLE NORMAL CROSSING DIVISORS AND THRIFTYNESS

3.1. Definitions and basic properties.

Definition 3.1 ([\[EGAIV₂, §7.8\]](#)). A scheme X is *excellent* if and only if

- X is locally noetherian,
- for every point $x \in X$ the fibers of the natural map $\mathrm{Spec} \mathcal{O}_{X,x}^\wedge \rightarrow \mathrm{Spec} \mathcal{O}_{X,x}$ are regular,
- for every integral X -scheme Z that is finite over an affine open of X , there is a non-empty regular open subscheme $U \subseteq Z$, and
- every scheme X' locally of finite type over X is catenary (that is, if $x \in X'$ and $x \rightsquigarrow y$ is a specialization, then any 2 saturated chains of specializations $x = x_0 \rightsquigarrow x_1 \rightsquigarrow \dots \rightsquigarrow x_n = y$ have the same length).

If X is excellent, then the locus

$$\mathrm{Reg}(X) = \{x \in X \mid \mathcal{O}_{X,x} \text{ is regular}\}$$

is open [\[EGAIV₂, Prop. 7.8.6\]](#); we will make repeated use of this fact.

Definition 3.2. A *simple normal crossing pair* is an equidimensional, regular excellent scheme X together with a reduced effective divisor $\Delta_X = \sum_i D_i$ such that for every subset $J \subseteq \{1, \dots, N\}$ the scheme-theoretic intersection

$$D_J := \cap_{j \in J} D_j \subseteq X$$

is regular of codimension $|J|$.

We first relate the notion of a simple normal crossing pair to the regular sequences of effective Cartier divisors considered in the previous section.

Lemma 3.3. *If $(X, \Delta_X = \sum_i D_i)$ is a simple normal crossing pair then (D_i) is a regular sequence of effective Cartier divisors.*

Proof. This boils down to the commutative algebra statement that if A is a regular local ring, $f \in A$ and A/f is a regular local ring with dimension $\dim A - 1$ then f is a non-0-divisor (see for example [\[Stacks, Tag 0AGA\]](#)). \square

Lemma 3.4. *Let X be an integral excellent scheme with an effective Weil divisor $\Delta_X = \sum_i D_i$, and for each i let $\mathcal{I}_i \subseteq \mathcal{O}_X$ be the ideal sheaf of D_i . Then the locus*

$$\mathrm{snc}(X, \Delta_X) := \{x \in X \mid \sum_{i \in I(x)} \mathcal{I}_{i,x}^\wedge \subseteq \mathcal{O}_{X,x}^\wedge \text{ is a simple normal crossing pair}\} \subseteq X$$

is open, and this is the largest open set $U \subseteq X$ such that $(U, (D_i \cap U))$ is a simple normal crossing pair.

In particular, simple normal crossing pairs are simple normal crossing pairs.

Proof. Suppose $J \subseteq \{1, \dots, N\}$, and write $\mathcal{J}_J = (f_j \in \mathcal{O}_{X,x} \mid j \in J) \subseteq \mathcal{O}_{X,x}$. Consider the co-cartesian diagram of noetherian local rings

$$\begin{array}{ccc} \mathcal{O}_{X,x}^\wedge & \longrightarrow & \mathcal{O}_{X,x}^\wedge / \mathcal{J}_J \mathcal{O}_{X,x}^\wedge \simeq (\mathcal{O}_{X,x} / \mathcal{J}_J)^\wedge \\ \uparrow & & \uparrow \\ \mathcal{O}_{X,x} & \longrightarrow & \mathcal{O}_{X,x} / \mathcal{J}_J \end{array}$$

The vertical homomorphisms are faithfully flat and by hypothesis $\mathcal{O}_{X,x}^\wedge / \mathcal{J}_J \mathcal{O}_{X,x}^\wedge$ is regular — since regularity satisfies faithfully flat descent, $\mathcal{O}_{X,x} / \mathcal{J}_J$ is also regular. Thus D_J is regular at the point $x \in D_J$, and as X is excellent by hypothesis the regular locus of D_J is open. Letting $x \in U_J \subseteq X$ be a neighborhood such that $D_J \cap U_J \subseteq D_J$ is regular and then letting $U = \cap_J D_J$ gives a neighborhood of x such that $(U, (D_i \cap U))$ is a simple normal crossing pair. \square

Note that for a simple normal crossing pair (X, Δ_X) , since the intersections $D_J = \cap_{j \in J} D_j$ are regular their connected components and irreducible components coincide.

Definition 3.5. A *stratum* of a simple normal crossing pair $X, (D_i)$ is a connected (equivalently, irreducible) component of an intersection $D_J = \cap_{j \in J} D_j$.

Definition 3.6 (cf. [Kol13, Def. 2.79-2.80]). Let $(S, \Delta_S = \sum_i D_i)$ be a pair in the sense of [Convention 1.5](#), and assume Δ_S is reduced and effective. A separated, finite type birational morphism $f : X \rightarrow S$ is *thrifty with respect to Δ_S* if and only if

- (i) f is an isomorphism over the generic point of every stratum of $\text{snc}(S, \Delta_S)$ and
- (ii) letting $\tilde{D}_i = f_*^{-1} D_i$ for $i = 1, \dots, N$ be the strict transforms of the divisors D_i , and setting $\Delta_X := \sum_i \tilde{D}_i$ f is an isomorphism at the generic point of every stratum of $\text{snc}(X, \Delta_X)$.

The restriction that $D_i \cap \text{Reg}(S) \neq \emptyset$ for all i ensures that if $\eta \in D_i$ is a generic point of a component, then $\eta \in \text{Reg}(S)$. Since on a regular scheme every Weil divisor is Cartier, and as S is excellent and D_i is reduced by hypothesis, there is a neighborhood $\eta \in U \subseteq S$ such that $U, D_i \cap U$ is a simple normal crossing pair. In other words, $\eta \in \text{snc}(S, \Delta_S)$ is the generic point of a stratum, so (i) implies $f^{-1}(\eta)$ is a single (non-closed) point. For our purposes the strict transform \tilde{D}_i is *defined* to be

$$\tilde{D}_i := \bigcup_{\eta \in D_i \text{ generic}} \overline{f^{-1}(\eta)} \subseteq X.$$

Since f is an isomorphism over η , we also see $f^{-1}(\eta) \subseteq \text{snc}(X, \Delta_X)$.

Lemma 3.7. Let S be an integral excellent noetherian scheme with a sequence of reduced effective Weil divisors $D_1, \dots, D_N \subseteq S$ such that no component of $\cup_i D_i$ is contained in $\text{Sing}(X)$, and let $f : X \rightarrow S$ be a separated, finite type birational morphism. Then, f is thrifty if and only if there is a diagram of separated finite type S -schemes

$$S \leftarrow U \hookrightarrow X$$

with both morphisms (necessarily dense) open immersions, such that U contains all generic points of strata of $\text{snc}(S, \Delta_S)$ and $\text{snc}(X, \Delta_X)$.

Proof. Since the existence of a common dense open $S \leftarrow U \hookrightarrow X$ as in the statement of the lemma certainly guarantees (i) and (ii), we focus on the “only if,” and in fact we show that one can take $U =$ the maximal domain of definition of $f^{-1} : S \dashrightarrow X$.

Suppose $\xi \in \text{snc}(X, \Delta_X)$ is a generic point of a stratum. By hypothesis there is a neighborhood $\xi \in V \subseteq X$ such that $f|_V : V \xrightarrow{\sim} S$ is an isomorphism onto its image. Then $W := f(V)$ is a Zariski

neighborhood of $f(\xi)$ and the inverse of $f|_V$ gives a section of the birational map $X_W = X \times_S W \rightarrow W$.

$$\begin{array}{ccc} V & \hookrightarrow & X_W \\ & \searrow f|_V & \downarrow f_W \\ & & W \end{array}$$

But then the inclusion $V \hookrightarrow X_W$ is a proper dense open immersion, hence an isomorphism. \square

Remark 3.8. It seems that the above proof shows in addition that $f(\xi) \in S$ is the generic point of a stratum of $\text{sn}(S, \Delta_S)$.

We will make repeated use of a few blowup lemmas from the construction of Nagata compactifications in [section 5](#) — here, they are used to show that thrifty morphisms can be dominated by certain admissible blowups.

Lemma 3.9 ([Con07, Lem. 2.4, Rmk. 2.5, Cor. 2.10]). *Let S be a quasi-compact, quasi-separated scheme. If X is a quasi-separated quasi-compact S -scheme and Y is a proper S -scheme, and if $f : U \rightarrow Y$ is an S -morphism defined on a dense open $U \subseteq X$, then there exists a U -admissible blowup $\tilde{X} \rightarrow X$ and an S -morphism $\tilde{f} : \tilde{X} \rightarrow Y$ extending f .*

Let $j_i : U \rightarrow X_i$ be a finite collection of dense open immersions between finite type separated S -schemes. Then there exist U -admissible blowups $X'_i \rightarrow X_i$ and a separated finite type S -scheme X , together with open immersions $X'_i \hookrightarrow X$ over S , such that the X'_i cover X and the open immersions $U \hookrightarrow X'_i \hookrightarrow X$ are all the same.

Corollary 3.10. *There exist U -admissible blowups*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\text{open imm.}} & \tilde{S} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S \end{array}$$

In particular if f is proper then X and S have a common U -admissible blowup.

Proof. By [Lemma 3.9](#) there are a separated, finite type S -scheme Y , U -admissible blowups $\tilde{S} \rightarrow S$ and $\tilde{X} \rightarrow X$ and dense open immersions $\tilde{S} \hookrightarrow Y \hookrightarrow \tilde{X}$ over S such that the diagram

$$\begin{array}{ccc} U & \hookrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ \tilde{S} & \hookrightarrow & Y \end{array}$$

commutes. Since \tilde{S} is proper over S , the bottom arrow is necessarily an isomorphism, in other words $Y = \tilde{S}$. If f is proper then \tilde{X} is proper over S , so $Y = \tilde{X}$ as well. \square

Remark 3.11. If (S, Δ_S) is a simple normal crossing pair and $U \subseteq S$ is an open containing all strata, a U -admissible blowup $f : X \rightarrow S$ need not be thrifty, see ??.

3.2. The “regular to regular” case. Using [Corollary 2.17](#) we can obtain a restricted form of [Theorem 1.7](#), the case of a thrifty proper birational morphism of simple normal crossing pairs.

Theorem 3.12. *Let $(Y, (D_i))$ be a simple normal crossing pair and let $f : X \rightarrow Y$ be a thrifty proper birational morphism. Assume $(X, (\tilde{D}_i))$ is also a simple normal crossing pair. Then the natural map*

$$\mathcal{O}_Y(-\Delta_Y) \rightarrow Rf_*\mathcal{O}_X(-\Delta_X) \text{ is a quasi-isomorphism.}$$

Proof. Let X_\bullet (resp. Y_\bullet) be the semi-simplicial scheme associated to $\tilde{D}_1, \dots, \tilde{D}_N \subseteq X$ (resp. $D_1, \dots, D_N \subseteq Y$). For any $J \subseteq \{1, \dots, N\}$ f restricts to a morphism $\cap_{j \in J} \tilde{D}_j \rightarrow \cap_{j \in J} D_j$, and in this way we obtain a morphism of semi-simplicial schemes

$$\begin{array}{ccccccc} \cdots & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & X \\ & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f \\ \cdots & \longrightarrow & Y_2 & \longrightarrow & Y_1 & \longrightarrow & Y_0 & \longrightarrow & Y \end{array} \quad (3.13)$$

The hypothesis that both pairs have simple normal crossings and f is thrifty implies that for each i , $f_i : X_i \rightarrow Y_i$ is a proper birational morphism of (possibly disconnected) regular schemes over k . By [CR11, Cor. 3.2.10] (or [CR15, Thm. 1.1])

$$\mathcal{O}_{Y_i} \xrightarrow{\sim} Rf_{*}\mathcal{O}_{X_i} \text{ is a quasi-isomorphism for all } i \quad (3.14)$$

The diagram (3.13) induces a morphism of *filtered* complexes $f^\# : \underline{\Omega}_{Y, \Delta_Y}^0 \rightarrow Rf_*\underline{\Omega}_{X, \Delta_X}^0$, and by Lemma 2.5 and Corollary 2.17 it will suffice to show that the resulting map of descent spectral sequences

$$E_1^{ij}(Y) = \begin{cases} \prod_{\sigma \in \mathcal{D}(\Delta_Y)^{i-1}} \mathcal{O}_{D(\sigma)} & j = 0 \\ 0 & \text{otherwise} \end{cases} \rightarrow \prod_{\sigma \in \mathcal{D}(\Delta_X)^{i-1}} R^j f_* \mathcal{O}_{D(\sigma)} = E_1^{ij}(X)$$

is an isomorphism, and this last step is a consequence of (3.14). \square

4. (NON-)EXAMPLES OF THRIFT

In this section we work over a field k . Our first example is not new, and likely served as the original motivation for considering thrifty morphisms.

Example 4.1. Let $S = \mathbb{A}_{xy}^2$ and $\Delta = V(xy)$. Then $f : X = \text{Bl}_0 S \rightarrow S$ is neither thrifty nor rational. Indeed, letting $D_1 = V(x)$, $D_2 = V(y)$ we see that Δ is the union of the 2 lines D_1, D_2 meeting at the origin. Let $\tilde{D}_i = f_*^{-1}D_i$ be the strict transforms, $E = f^{-1}(0)$ the exceptional divisor, and $\tilde{\Delta} = \tilde{D}_1 + \tilde{D}_2$ (see Figure 1). The map $f : X \rightarrow S$ fails to be thrifty since it is not an isomorphism over the stratum

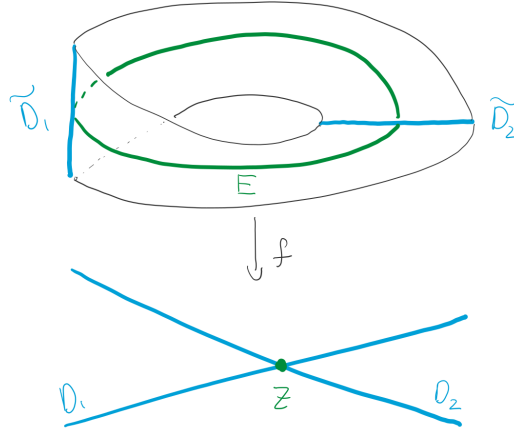


FIGURE 1. The blowup appearing in Example 4.1

$0 = D_1 \cap D_2$ of (S, Δ) . We will calculate cohomology to show f isn't rational either.

Since $S = \mathbb{A}_{xy}^2$ is affine, we can identify the sheaves $R^i f_* \mathcal{O}_X(\tilde{\Delta})$ as the sheaves associated to the $k[x, y]$ -modules $H^i(X, \mathcal{O}_X(-\tilde{\Delta}))$. Observe that X can be identified with the geometric

line bundle $\mathrm{Spec}_{\mathbb{P}^1} \mathrm{Sym} \mathcal{O}_{\mathbb{P}^1}(1)$ associated to $\mathcal{O}_{\mathbb{P}^1}(-1)$. Under this identification, the projection $\pi : \mathrm{Spec}_{\mathbb{P}^1} \mathrm{Sym} \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow \mathbb{P}^1$ corresponds to the composition $\mathrm{Bl}_0 S \subseteq \mathbb{A}^2 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$, and the blowup map $f : \mathrm{Spec}_{\mathbb{P}^1} \mathrm{Sym} \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow \mathbb{A}^2$ corresponds to the natural map

$$\mathrm{Spec}_{\mathbb{P}^1} \mathrm{Sym} \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow \mathrm{Spec}_k H^0(\mathbb{P}^1, \mathrm{Sym} \mathcal{O}_{\mathbb{P}^1}(-1)) = \mathrm{Spec}_k k[x, y] = \mathbb{A}^2$$

Hence $\tilde{\Delta} = \pi^*(0 + \infty)$. Now since π is affine its Leray spectral sequence degenerates to give

$$H^i(X, \mathcal{O}_X(-\tilde{\Delta})) = H^i(\mathbb{P}^1, \pi_* \mathcal{O}_X(-\tilde{\Delta})) \text{ and via projection formula} \\ \pi_* \mathcal{O}_X(-\tilde{\Delta}) = \pi_* \mathcal{O}_X(-\pi^*(0 + \infty)) = (\pi_* \mathcal{O}_X)(-0 - \infty)$$

By the correspondence between affine schemes and sheaves of algebras,

$$\pi_* \mathcal{O}_X = \mathrm{Sym} \mathcal{O}_{\mathbb{P}^1}(1) = \bigoplus_{d \geq 0} \mathcal{O}_{\mathbb{P}^1}(d)$$

Hence $H^i(X, \mathcal{O}_X(-\tilde{\Delta})) = \bigoplus_{d \geq 0} H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d - 2))$. In particular, when $i = 1$ and $d = 0$, we see $H^1(X, \mathcal{O}_X(-\tilde{\Delta})) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \simeq k$ by [Har77, Thm. III.5.1].

An elaboration of [Example 4.1](#) shows in general that if (S, Δ) is a simple normal crossing pair and $Z \subseteq S$ is a stratum, then $f : X = \mathrm{Bl}_Z S \rightarrow S$ fails to be thrifty. Localizing at the generic point $\eta \in Z$ we can reduce to the case where Z is replaced by a closed point $\eta \in S$ and $\Delta = V(x_1 \cdot x_2 \cdots x_n)$ where $x_1, \dots, x_n \in \mathfrak{m}_\eta$ is a regular system of parameters. Then the long exact sequence obtained by pushing forward $\mathcal{O}_X(-\tilde{\Delta} - E) \rightarrow \mathcal{O}_X(-\tilde{\Delta}) \rightarrow \mathcal{O}_E(-\tilde{\Delta}|_E)$ ends in

$$R^{n-1} f_* \mathcal{O}_X(-\tilde{\Delta} - E) \rightarrow R^{n-1} f_* \mathcal{O}_X(-\tilde{\Delta}) \rightarrow R^{n-1} f_* \mathcal{O}_E(-\tilde{\Delta}|_E) \rightarrow R^n f_* \mathcal{O}_X(-\tilde{\Delta} - E) = 0$$

where the vanishing on the right holds since the maximal fiber dimension of f is $n - 1$ [Har77, Cor. III.11.2]. Thus $R^{n-1} f_* \mathcal{O}_X(-\tilde{\Delta}) \rightarrow R^{n-1} f_* \mathcal{O}_E(-\tilde{\Delta}|_E) = H^{n-1}(E, \mathcal{O}_E(-\tilde{\Delta}|_E))$ is surjective, and identifying E with the projectivized Zariski tangent space $\mathbb{P}(TS_\eta)$ with homogeneous coordinates x_1, \dots, x_n and $\tilde{\Delta}|_E$ with $V(\prod_i x_i)$ shows $H^{n-1}(E, \mathcal{O}_E(-\tilde{\Delta}|_E)) \simeq H^{n-1}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(-n)) \simeq k$. For related discussion see [Kol13, p. 86, Thm. 2.87].

The next example answers (in the affirmative!) a question of Erickson and Prelli about whether there exists a resolution which is thrifty but not rational. In fact, we give such an example where the underlying pair (S, Δ) is rational.

Example 4.2 (cf. [Eri14, p.2] [Pre17, p.3]). Let $S = V(xy - zw) \subseteq \mathbb{A}_{xyzw}^4$, $D_0 = V(x, w)$ and $D_\infty = V(y, z)$; finally let $C_\infty = V(w, y)$. We can identify $S = C(\mathbb{P}^1 \times \mathbb{P}^1)$ as the affine cone over the Segre embedding $\mathbb{P}_s^1 \times \mathbb{P}_t^1 \hookrightarrow \mathbb{P}_{xyzw}^3$ given by

$$\begin{bmatrix} x & w \\ z & y \end{bmatrix} = \begin{bmatrix} s_0 \\ s_1 \end{bmatrix} \begin{bmatrix} t_0 & t_1 \end{bmatrix} = \begin{bmatrix} s_0 t_0 & s_0 t_1 \\ s_1 t_0 & s_1 t_1 \end{bmatrix} \quad (4.3)$$

Hence $D_0 = C(\{0\} \times \mathbb{P}^1)$, $D_\infty = C(\{\infty\} \times \mathbb{P}^1)$ and $C_\infty = C(\mathbb{P}^1 \times \{\infty\})$.

Let $\Delta = D_0 + D_\infty + C_\infty$. Note that Δ is *not* Cartier, as it is not linearly equivalent to any multiple of $C(\{0\} \times \mathbb{P}^1) + C(\mathbb{P}^1 \times \{0\})$ (here $\{0\} \times \mathbb{P}^1 + \mathbb{P}^1 \times \{0\}$ is a hyperplane section of the Segre embedding) — see e.g. [Har77, Ex. II.6.3], [Kol13, Prop. 3.14]. Since K_S is \mathbb{Q} -Cartier, it follows that the pair $(S, \Delta = D_0 + D_\infty)$ is not \mathbb{Q} -Gorenstein — in particular it isn't dlt.

Now let $f : X = \mathrm{Bl}_{D_0} S \rightarrow S$ be the blowup at D_0 , let $\tilde{D}_i = f_*^{-1} D_i$ for $i = 0, \infty$ and $\tilde{C}_\infty = f_*^{-1} C_\infty$, and let $\tilde{\Delta} = \tilde{D}_0 + \tilde{D}_\infty + \tilde{C}_\infty$. The map f is a small resolution of S (as mentioned in [KM98, Ex. 2.7]). Indeed, the ambient blowup is described as

$$\mathrm{Bl}_{D_0} \mathbb{A}^4 \subseteq \{(x, y, z, w), [u, v] \mid (x, w) \propto (u, v)\} \subseteq \mathbb{A}^4 \times \mathbb{P}_{uv}^1$$

so on the $D(u)$ patch $(x, w) = \lambda(1, v)$ and

$$xy - zw = \lambda y - z\lambda v = \lambda(y - zv)$$

Since $V(\lambda)$ is the exceptional divisor we see the strict transform $X \subseteq \text{Bl}_{D_0} \mathbb{A}^4$ of S is $V(y - zv)$ on the $u = 1$ patch — this is smooth as it's a graph. By symmetry in x, w , we conclude X is smooth.

Even better, this allows us to parametrize $X \cap D(u)$ with coordinates z, λ, v :

$$\mathbb{A}_{z\lambda v}^3 \simeq \text{Bl}_{D_0} S \cap D(u) \subseteq D(u) \simeq \mathbb{A}_{xyzwv}^5 \text{ sending } (z, \lambda, v) \mapsto (\lambda, zv, z, \lambda v, v) = (x, y, z, w, v) \quad (4.4)$$

So in particular the restriction of f looks like $(z, \lambda, v) \mapsto (\lambda, zv, z, \lambda v)$,

and we see that the exceptional locus is the v -axis. In this coordinate patch the strict transforms \tilde{D}_0 and \tilde{D}_∞ are $V(\lambda)$ and $V(z)$ respectively, which *intersect along the v -axis* $V(\lambda, z)$! Thus $\tilde{\Delta}$ has a stratum in $\text{Ex}(f)$ and f isn't thrifty. We also see that on this patch $\tilde{C}_\infty = V(v)$. As a philosophical aside, the blowup coordinates $[u, v]$ correspond to $[x, w] = [s_0 t_0, s_0 t_1] = [t_0, t_1]$ as long as $s_0 \neq 0$, so $\text{Ex } f$ can be viewed as a copy of the \mathbb{P}_t^1 appearing in $D_0 = C(\{0\} \times \mathbb{P}_t^1)$ — see Figure 2.

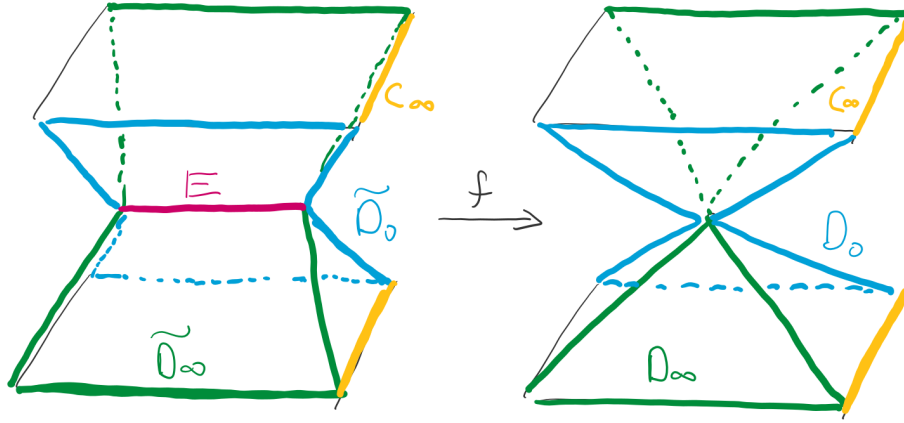


FIGURE 2. The small resolution of Example 4.2

To show that f is in fact a rational resolution we will use an alternative description of X . Starting with the ample invertible sheaf $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ we have natural morphisms of relative spectra

$$\text{Spec}_{\mathbb{P}_s^1 \times \mathbb{P}_t^1} \text{Sym } \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1) \rightarrow \text{Spec}_{\mathbb{P}_t^1} \text{Sym } \text{pr}_{t*} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1) \xrightarrow{f'} \text{Spec}_k H^0(\text{Sym } \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)) \quad (4.5)$$

where $\text{pr}_t : \mathbb{P}_s^1 \times \mathbb{P}_t^1 \rightarrow \mathbb{P}_t^1$ is the projection. It is well known that the scheme on the left can be identified with the blowup $\text{Bl}_0 S$, and the scheme on the right is S .

Claim 4.6. There is an isomorphism of $S \times \mathbb{P}^1$ -schemes $X = \text{Bl}_{D_0} S \simeq \text{Spec}_{\mathbb{P}_t^1} \text{Sym } \text{pr}_{t*} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$

This can be proved via the universal property. On the other hand, at least when k is algebraically closed, a quick, dirty and more illuminating proof is possible: we *have* a morphism $(f', \pi) : \text{Spec}_{\mathbb{P}_t^1} \text{Sym } \text{pr}_{t*} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1) \rightarrow S \times \mathbb{P}_t^1$: the first factor is the second map of (4.5), the second is the canonical projection

$$\pi : \text{Spec}_{\mathbb{P}_t^1} \text{Sym } \text{pr}_{t*} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1) \rightarrow \mathbb{P}_t^1$$

from the relative Spec construction. $X \subseteq S \times \mathbb{P}_t^1$ by construction, and we can check φ maps the k -points of $\text{Spec}_{\mathbb{P}_t^1} \text{Sym } \text{pr}_{t*} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ bijectively onto those of X . Indeed, the fiber of $\text{Spec}_{\mathbb{P}_t^1} \text{Sym } \text{pr}_{t*} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ over $t \in \mathbb{P}_t^1$ can be described as follows: Note by projection formula

$$\text{pr}_{t*} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1) \simeq H^0(\mathbb{P}_s^1, \mathcal{O}_{\mathbb{P}_s^1}(1)) \otimes_k \mathcal{O}_{\mathbb{P}_t^1}(1), \text{ so } \text{Sym } \text{pr}_{t*} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1) \simeq \text{Sym } H^0(\mathbb{P}_s^1, \mathcal{O}_{\mathbb{P}_s^1}(1)) \otimes_k \mathcal{O}_{\mathbb{P}_t^1}(1)$$

Explicitly $\mathrm{Sym} H^0(\mathbb{P}_s^1, \mathcal{O}_{\mathbb{P}_s^1}(1)) \otimes_k \mathcal{O}_{\mathbb{P}_t^1}(1) = \oplus_d k[s_0, s_1]_d \otimes \mathcal{O}_{\mathbb{P}_t^1}(d) = k[s_0, s_1] \times \mathrm{Sym} \mathcal{O}_{\mathbb{P}_t^1}(1)$ where \times denotes the product of graded rings of [Har77, Ex. II.5.11] and hence for a k -point t ,

$$\pi^{-1}(t) \simeq \mathrm{Spec} k[s_0, s_1], \text{ so that } f'|_{\pi^{-1}(t)} : \pi^{-1}(t) \rightarrow S$$

is a map $\mathbb{A}_{s_0 s_1}^2 \rightarrow S \subseteq \mathbb{A}_{xyzw}^4$. Writing down the map of algebras corresponding to f' shows that it is none other than the linear transformation of (4.3). Finally, referencing (4.4) we see that the fibers of $X \rightarrow \mathbb{P}_t^1$ have the same description.⁴

Using the claim, we proceed as in Example 4.1 using degeneration of the Leray spectral sequence for the affine map $\pi : X \rightarrow \mathbb{P}_t^1$ to calculate

$$H^i(X, \mathcal{O}_X(-\tilde{\Delta})) = H^i(\mathbb{P}_t^1, \pi_* \mathcal{O}_X(-\tilde{\Delta}))$$

On \mathbb{P}_t^1 , noting that $\tilde{C}_\infty = \pi^*(\infty)$, the projection formula gives

$$\pi_* \mathcal{O}_X(-\tilde{\Delta}) = (\pi_* \mathcal{O}_X(-\tilde{D}_0 - \tilde{D}_\infty))(-\infty) \quad (4.7)$$

and $\pi_* \mathcal{O}_X(-\tilde{D}_0 - \tilde{D}_\infty) \subseteq \pi_* \mathcal{O}_X$ is the sheaf of ideals $(s_0 \cdot s_1) \subseteq k[s_0, s_1] \times \mathrm{Sym} \mathcal{O}_{\mathbb{P}_t^1}(1)$. Letting $(s_0 \cdot s_1)_d \subseteq k[s_0, s_1]$ denote the d -th graded part, we see

$$\pi_* \mathcal{O}_X(-\tilde{\Delta}) = \bigoplus_{d \geq 0} (s_0 \cdot s_1)_d \otimes_k \mathcal{O}_{\mathbb{P}_t^1}(d-1)$$

where the “-1” comes from the twist “ $(-\infty)$ ” in (4.7). This yields:

$$\begin{aligned} H^i(\mathbb{P}_t^1, \pi_* \mathcal{O}_X(-\tilde{\Delta})) &= \bigoplus_{d \geq 0} (s_0 \cdot s_1)_d \otimes_k H^i(\mathbb{P}_t^1, \mathcal{O}_{\mathbb{P}_t^1}(d-1)) \\ &= \begin{cases} \bigoplus_{d \geq 0} (s_0 \cdot s_1)_d \otimes_k (t_1)_d \subseteq k[s_0, s_1] \times k[t_0, t_1] = H^0(S, \mathcal{O}_S) & \text{if } i = 0 \\ 0 & \text{if } i = 1 \end{cases} \end{aligned} \quad (4.8)$$

the key point being that $H^1(\mathbb{P}_t^1, \mathcal{O}_{\mathbb{P}_t^1}(d-1)) = 0$ for $d \geq 0$. This calculation shows $f_* \mathcal{O}_X(-\tilde{\Delta}) = \mathcal{O}_S(-\Delta)$ (this holds for more general reasons, namely S is normal [Pre17, Lem. 2.1]) and $R^1 f_* \mathcal{O}_X(-\tilde{\Delta}) = 0$.

Finally, (S, Δ) is a rational pair, as a consequence of the theorem below — this was the main reason for including the additional divisor C_∞ . If we had left it out, the above calculations would still show that $f : X \rightarrow (S, D_0 + D_\infty)$ is a non-thrifty rational resolution, however the pair $(S, D_0 + D_\infty)$ isn't rational.

Theorem 4.9 ([Pre17, Thm. 3.2]). *Let (Y, B) be a pair such that Y is a normal variety over k and B is a reduced effective Weil divisor on Y (for example a simple normal crossing pair) and let \mathcal{L} be an ample invertible sheaf on Y . Let (CY, CB) be the abstract affine cone over (Y, B) with respect to \mathcal{L} : $CY = \mathrm{Spec}_k H^0(Y, \mathrm{Sym} \mathcal{L})$ and CB is the image of $\mathrm{Spec}_k H^0(B, \mathrm{Sym} \mathcal{L}|_B) \rightarrow \mathrm{Spec}_k H^0(Y, \mathrm{Sym} \mathcal{L}) = CY$ with its reduced subscheme structure. Then (CY, CB) is a rational pair if and only if (Y, B) is a rational pair and*

$$H^i(Y, \mathcal{L}^d(-B)) = 0 \text{ for } i > 0, d \geq 0$$

Applying the theorem to $Y = \mathbb{P}^1 \times \mathbb{P}^1$ with the divisor $B = \{0, \infty\} \times \mathbb{P}^1 + \mathbb{P}^1 \times \{\infty\}$ which has associated invertible sheaf $\mathcal{O}_Y(B) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 1)$, together with the ample invertible sheaf $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ we calculate (using Künneth)

$$H^i(Y, \mathcal{L}^d(-B)) = H^i(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d-2, d-1)) = \bigoplus_{j+k=i} H^j(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-2)) \otimes_k H^k(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-1)) \quad (4.10)$$

⁴In slogan form: $X = \mathrm{Bl}_{D_0} S$ is a pencil of 2-planes on S corresponding to the pencil of rulings $\mathbb{P}_s^1 \times \{t\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$.

Noting that $H^k(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-1)) = 0$ for $k > 0$ and $d \geq 0$, we see that $H^2(Y, \mathcal{L}^d(-B)) = 0$ for $d \geq 0$, and

$$H^1(Y, \mathcal{L}^d(-B)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-2)) \otimes_k H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-1))$$

Now $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-2)) = 0$ for $d \neq 0$, but $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$.

The last example of this section shows that even when (S, Δ) is a simple normal crossing pair and $f : X \rightarrow S$ is a U -admissible blowup for some $U \subseteq S$ containing all strata, and $\tilde{\Delta} = f_*^{-1}\Delta$ is snc, f may still fail to be thrifty. Unfortunately our presentation only makes sense in characteristic 0, but the author would be shocked and appalled if this example doesn't work in any characteristic $p > 2$.

Example 4.11. Let $S = \mathbb{A}_{xyz}^3$, let $\Delta = V((z-x)(z+x))$ and let $Z = V(x, y)$; let $U = S \setminus Z$. Then there is a U -admissible blowup $f : X \rightarrow S$ such that $f_*^{-1}\Delta$ is a simple normal crossing divisor but f is not thrifty.

We first blow up Z to obtain $g : \text{Bl}_Z S \rightarrow S$, and claim that the strict transform of Δ is no longer snc. Letting $D_{\pm} = V(z \pm x)$ we can work in blowup coordinates described like

$$\text{Bl}_Z S = \{((x, y, z), [u, v]) \in \mathbb{A}^3 \mid (x, y) \propto (u, v)\}$$

so that on the $D(u)$ patch $(x, y) = \lambda(1, v)$ and

$$z \pm x = z \pm \lambda, \text{ so in } (z, \lambda, v) \text{ coordinates } \tilde{D}_{\pm} \cap D(u) = V(z \pm \lambda)$$

in other words $\tilde{\Delta}$ is snc on the $D(u)$ patch (as is expected since on $D(x) \subseteq \mathbb{A}^3$, Δ is smooth). But on the $D(v)$ patch where $(x, y) = \lambda(u, 1)$,

$$z \pm x = z \pm \lambda u, \text{ so in } (z, \lambda, u) \text{ coordinates } \tilde{D}_{\pm} \cap D(v) = V(z \pm \lambda u) \quad (4.12)$$

and here we see the strict transforms intersect along $V(\lambda u)$ and hence fail to be snc (Figure 3).

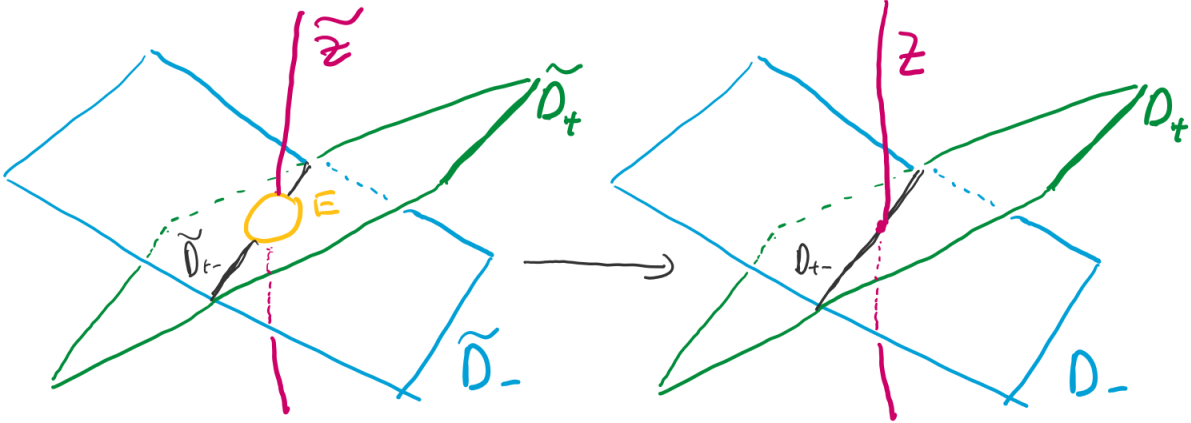


FIGURE 3. The blowup of Example 4.11

A global description of the situation: $\text{Bl}_Z S$ is isomorphic to $\mathbb{A}_z^1 \times \text{Bl}_0 \mathbb{A}_{xy}^2$, and \tilde{D}_{\pm} are 2 copies of $\text{Bl}_0 \mathbb{A}_{xy}^2$ embedded via the maps

$$(\pm x, \text{id}) : \text{Bl}_0 \mathbb{A}_{xy}^2 \rightarrow \mathbb{A}_z^1 \times \text{Bl}_0 \mathbb{A}_{xy}^2$$

where the map $\pm x : \text{Bl}_0 \mathbb{A}_{xy}^2 \rightarrow \mathbb{A}_z^1$ really means the composition $\text{Bl}_0 \mathbb{A}_{xy}^2 \rightarrow \mathbb{A}_{xy}^2 \xrightarrow{\pm x} \mathbb{A}_z^1$. From this perspective $\tilde{D}_+ \cap \tilde{D}_-$ is the preimage of $V(x)$ under the blowup map $\text{Bl}_0 \mathbb{A}_{xy}^2 \rightarrow \mathbb{A}_{xy}^2$, the union $\mathbb{P}_{xy}^1 \cup \mathbb{A}_y^1$ glued along the points $[0, 1] \in \mathbb{P}_{xy}^1$ and $0 \in \mathbb{A}_y^1$. Let p denote the point in $\mathbb{P}_{xy}^1 \cap \mathbb{A}_y^1$. Equivalently $\text{Sing}(\tilde{D}_+ \cap \tilde{D}_-)$ consists of a single closed point which we call p .

This discussion shows that the snc locus of $(\mathrm{Bl}_Z S, \tilde{\Delta})$ is

$$\mathrm{snc}(\mathrm{Bl}_Z S, \tilde{\Delta}) = \mathrm{Bl}_Z S \setminus \{p\}$$

By work of Szabó and Bierstone-Milman ([BM97; Sza94], this is where we use the characteristic 0 hypothesis) there exists a further blowup $h : X \rightarrow \mathrm{Bl}_Z S$ such that $h_*^{-1}\tilde{\Delta} + \mathrm{Ex} h$ is a simple normal crossing divisor *and* h is an isomorphism over $\mathrm{snc}(\mathrm{Bl}_Z S, \tilde{\Delta})$, that is, h must be a $\mathrm{snc}(\mathrm{Bl}_Z S, \tilde{\Delta})$ -admissible blowup. Now by [Har77, Thm. II.7.17] we know that $f := g \circ h : X \rightarrow S$ is a blowup at *some* closed subscheme $W \subseteq S$ and since $g(p) \in Z$ (equivalently) $g^{-1}(U) \subseteq \mathrm{snc}(\mathrm{Bl}_Z S, \tilde{\Delta})$, it must be that $W \subseteq Z$ as closed sets (see also [RG71, Lem. 5.1.4]), hence $f : X \rightarrow S$ is a U -admissible blowup.

On the other hand, by a proposition of Erickson [Eri14, Prop. 1.4], since $h_*^{-1}\tilde{\Delta} + \mathrm{Ex} h$ is snc the map h is thrifty and so the strata of $f_*^{-1}\Delta = h_*^{-1}\tilde{\Delta}$ are in 1-1 birational correspondence with those of $\tilde{\Delta}$, in particular $f_*^{-1}\Delta$ has a stratum in $\mathrm{Ex} f$.

While the application of [BM97; Sza94] here is heavy-handed for this toy example, we point out that h is not simply the blowup at p as one might initially guess: starting from (4.12), blowing up the origin $0 \in \mathbb{A}_{z\lambda u}^3$ and introducing blowup coordinates

$$\mathrm{Bl}_0 \mathbb{A}_{z\lambda u}^3 = \{((z, \lambda, u), [r, s, t]) \in \mathbb{A}_{z\lambda u}^3 \times \mathbb{P}_{rst}^2 \mid (z, \lambda, u) \propto (r, s, t)\}$$

we note that since $V((z - \lambda u) \cdot (z + \lambda u))$ is smooth on $D(z)$ one can check that the strict transform remains smooth on the $D(r)$ patch. We will investigate the $D(s)$ patch — by symmetry of λ, u in the equation $(z - \lambda u) \cdot (z + \lambda u)$ the situation is similar on the $D(t)$ patch. On $D(s)$ we have $(z, \lambda, u) = \mu(r, 1, t)$ and so

$$z \pm \lambda u = \mu r \pm \mu^2 t = \mu(r \pm \mu t)$$

Here $V(\mu)$ is a copy of the exceptional divisor of $\mathrm{Bl}_0 \mathbb{A}_{z\lambda u}^3 \rightarrow \mathbb{A}_{z\lambda u}^3$ but we are still left with strict transforms $(r \pm \mu t)$ of exactly the same form as $z \pm \lambda u$; in other words, blowing up $0 \in \mathbb{A}_{z\lambda u}^3$ does not help! This is quite similar to the classical fact that blowing up the origin of the pinch point $V(z^2 - \lambda u^2) \subseteq \mathbb{A}_{z\lambda u}^3$ gives another pinch point singularity. In fact, since $(z - \lambda u) \cdot (z + \lambda u) = z^2 - \lambda^2 u^2$ our example is a double cover of the pinch point (that is, it is the preimage of the pinch point with respect to $(z, \lambda, u) \mapsto (z, \lambda^2, u)$).

5. CONSTRUCTING SEMI-SIMPLICIAL PROJECTIVE MACAULAYIFICATIONS

5.1. Setup. For the purposes of preliminary discussion, suppose k is a perfect field and $(X, \Delta_X), (Y, \Delta_Y)$ are snc pairs over a finite-type k -scheme S with structure morphisms $X \xrightarrow{f} S \xleftarrow{g} Y$, related by a thrifty proper birational equivalence $X \xleftarrow{r} Z \xrightarrow{s} Y$ over S : that is, r and s are thrifty proper birational morphisms and $r_*^{-1}\Delta_X = s_*^{-1}\Delta_Y =: \Delta_Z$. If Z is smooth and Δ_Z is snc, then Theorem 3.12 applied to both r and s shows

$$Rf_*\mathcal{O}_X(-\Delta_X) \simeq Rf_*Rr_*\mathcal{O}_Z(-\Delta_Z) = Rg_*Rs_*\mathcal{O}_Z(-\Delta_Z) \simeq Rg_*\mathcal{O}_Y(-\Delta_Y)$$

Of course, Z need not be smooth and in the absence of resolution of singularities away from characteristic 0,⁵ we cannot replace it by a resolution. In characteristic $p > 0$ one could replace Z with an alteration, but only at the cost of allowing r, s to be generically finite but not necessarily birational, and as such using alterations seems incompatible with the strategy of Theorem 3.12. Moreover, in more general mixed characteristic situations to be considered below, even alterations are unavailable.⁶

⁵At least at the time of this writing ...

⁶Ditto.

Instead, we replace Z with a mildly singular (specifically Cohen-Macaulay) semi-simplicial scheme Z_\bullet together with morphisms $X_\bullet \xleftarrow{r_\bullet} Z_\bullet \xrightarrow{s_\bullet} Y_\bullet$ over S which are term-by-term proper birational equivalences over S . This construction is made possible by the existence of Macaulayfications.

Theorem 5.1 ([Ces18, Thm. 1.6], cf. also [Kaw00, Thm. 1.1]). *For every a CM-quasi-excellent noetherian scheme X there exists a projective birational morphism $\pi : \tilde{X} \rightarrow X$ such that \tilde{X} is Cohen-Macaulay and π is an isomorphism over the Cohen-Macaulay locus $\text{CM}(X) \subset X$.*

The usefulness of Macaulayfications for the problem at hand stems from an extension of the results of Chatzistamatiou-Rülling due to Kovács.

Theorem 5.2 ([Kov20, Thm. 1.4]). *Let $f : X \rightarrow Y$ be a locally projective birational morphism of excellent Cohen-Macaulay schemes. If Y has pseudo-rational singularities then*

$$\mathcal{O}_Y = Rf_*\mathcal{O}_X \text{ and } Rf_*\omega_X = \omega_Y.$$

By a result of Lipman-Teissier, if Y is regular (so in particular if it is smooth over k) then Y is pseudo-rational [LT81, §4], hence Theorem 5.2 applies when Y is regular. Without further ado, here is the statement of the construction:

Lemma 5.3. *Let S be an excellent noetherian scheme and let (X, Δ_X) and (Y, Δ_Y) be simple normal crossing pairs separated and of finite type over S , and let $X \xleftarrow{r} Z \xrightarrow{s} Y$ be a thrifty proper birational equivalence over S . Then there exists a semi-simplicial separated finite type S -scheme Z_\bullet and morphisms of semi-simplicial S -schemes $X_\bullet \xleftarrow{r_\bullet} Z_\bullet \xrightarrow{s_\bullet} Y_\bullet$ such that for all i ,*

(i) Z_i is Cohen-Macaulay and

(ii) $X_i \xleftarrow{r_i} Z_i \xrightarrow{s_i} Y_i$ is a projective birational equivalence over S .

5.2. Gluing on simplices. To prove Lemma 5.3 we need a few preliminaries. The first describes an inductive method for constructing a sequence of truncated semi-simplicial schemes converging to Z_\bullet . Here for any $i \in \mathbb{N}$ an i -truncated semi-simplicial object in a category \mathcal{C} is a functor $\Lambda_{\leq i}^{\text{op}} \rightarrow \mathcal{C}$, where $\Lambda_{\leq i}^{\text{op}}$ is the full subcategory of Λ^{op} generated by the objects $[j]$ with $j \leq i$. Given an $i-1$ -truncated semi-simplicial object X_\bullet of \mathcal{C} , let

$$[i]_{<}^2 := \{j, k \in [i] \mid j < k\}$$

and define two morphisms

$$\delta_+, \delta_- : X_{i-1}^{[i]} \rightarrow X_{i-2}^{[i]^2}$$

by $\delta_+(x_0, \dots, x_i) = (d_j^{i-1}(x_k) \mid j < k)$ and $\delta_-(x_0, \dots, x_i) = (d_{k-1}^{i-1}(x_j) \mid j < k)$. Assuming \mathcal{C} has finite limits we may form the equalizer

$$E(X) := \text{Eq}(\delta_+, \delta_-) \longrightarrow X_{i-1}^{[i]} \xrightleftharpoons[\delta_-]{\delta_+} X_{i-2}^{[i]^2} \quad (5.4)$$

one can check that this construction is *functorial* in X_\bullet : indeed if Y_\bullet is another $i-1$ -truncated semi-simplicial object then given a morphism $X_\bullet \rightarrow Y_\bullet$ we can form a commutative diagram

$$\begin{array}{ccccc} E(X) := \text{Eq}(\delta_+, \delta_-) & \longrightarrow & X_{i-1}^{[i]} & \xrightleftharpoons[\delta_-]{\delta_+} & X_{i-2}^{[i]^2} \\ \downarrow & & \downarrow & & \downarrow \\ E(Y) := \text{Eq}(\delta_+, \delta_-) & \longrightarrow & Y_{i-1}^{[i]} & \xrightleftharpoons[\delta_-]{\delta_+} & Y_{i-2}^{[i]^2} \end{array} \quad (5.5)$$

and obtain a unique morphism on the dashed arrow by functoriality of equalizers. Finally, let I denote the category $0 \rightarrow 1$ (thought of as the “unit interval”). An object of \mathcal{C}^I is a morphism

$f : X \rightarrow Y$ in C and there are 2 functors $s : C^I \rightarrow C$ defined by $s(f) = X, t(f) = Y$ (source and target).

Lemma 5.6 (cf. [SGA4II, Vbis, Prop. 5.1.3], [Stacks, Tag 0AMA]). *Let C be a category containing finite limits. The functor*

$$\Phi_i : C^{\Lambda_{\leq i}^{\text{op}}} \rightarrow C^{\Lambda_{\leq i-1}^{\text{op}}} \times_C C^I$$

to the 2-fiber product with respect to the functors $E : C^{\Lambda_{\leq i-1}^{\text{op}}} \rightarrow C$ and $t : C^I \rightarrow C$ that sends an i -truncated semi-simplicial object X_\bullet to the pair $(\text{sk}_{i-1}X_\bullet, X_i \rightarrow E(\text{sk}_{i-1}X))$ is an equivalence of categories.

Proof. We first check that Φ_i is fully faithful. For faithfulness, note that for any 2 i -truncated semi-simplicial objects X_\bullet, Y_\bullet there is an *injection*

$$\text{Hom}_{C^{\Lambda_{\leq i}^{\text{op}}}}(X_\bullet, Y_\bullet) \hookrightarrow \prod_{j=0}^i \text{Hom}_C(X_j, Y_j) \quad (5.7)$$

since a morphism $\alpha : X_\bullet \rightarrow Y_\bullet$ is equivalent to a sequence of morphisms $\alpha_i : X_i \rightarrow Y_i$ commuting with differentials. Unpacking the definition of the 2-fiber product, the morphism $\Phi_i(\alpha) : \Phi_i(X_\bullet) \rightarrow \Phi_i(Y_\bullet)$ induced by α consists of the morphism $\text{sk}_{i-1}\alpha : \text{sk}_{i-1}X_\bullet \rightarrow \text{sk}_{i-1}Y_\bullet$, and the commutative diagram

$$\begin{array}{ccc} X_i & \longrightarrow & E(\text{sk}_{i-1}X) \\ \downarrow \alpha_i & & \downarrow E(\alpha) \\ Y_i & \longrightarrow & E(\text{sk}_{i-1}Y) \end{array}$$

This shows that (5.7) factors as

$$\text{Hom}_{C^{\Lambda_{\leq i}^{\text{op}}}}(X_\bullet, Y_\bullet) \xrightarrow{\Phi_i} \text{Hom}_{C^{\Lambda_{\leq i-1}^{\text{op}}} \times_C C^I}(\Phi_i(X_\bullet), \Phi_i(Y_\bullet)) \rightarrow \prod_{j=0}^i \text{Hom}_C(X_j, Y_j) \quad (5.8)$$

hence the first map is injective, or in other words Φ_i is faithful. On the other hand given an arbitrary morphism $\Phi_i(X_\bullet) \rightarrow \Phi_i(Y_\bullet)$ consisting of a map $\beta : \text{sk}_{i-1}X_\bullet \rightarrow \text{sk}_{i-1}Y_\bullet$, a map $\gamma : X_i \rightarrow Y_i$ and a commutative diagram

$$\begin{array}{ccc} X_i & \longrightarrow & E(\text{sk}_{i-1}X) \\ \downarrow \gamma & & \downarrow E(\beta) \\ Y_i & \longrightarrow & E(\text{sk}_{i-1}Y) \end{array} \quad (5.9)$$

we may verify commutativity of

$$\begin{array}{ccccc} & & d_k^i & & \\ & \searrow & & \nearrow & \\ X_i & \longrightarrow & E(\text{sk}_{i-1}X) & \xrightarrow{\text{pr}_k} & X_{i-1} \\ \downarrow \gamma & (1) & \downarrow E(\beta) & (2) & \downarrow \beta_{i-1} \\ Y_i & \longrightarrow & E(\text{sk}_{i-1}Y) & \xrightarrow{\text{pr}_k} & Y_{i-1} \\ & \nearrow & & \searrow & \\ & & d_k^i & & \end{array}$$

as follows: commutativity of (1) is exactly (5.9), and commutativity of (2) can be deduced from that of the left square of (5.5). Hence β and γ define a map $X_\bullet \rightarrow Y_\bullet$ and so Φ_i is full.

Next we show Φ_i is essentially surjective. For this we consider an object of the 2-fiber product $C^{\Lambda_{\leq i-1}^{\text{op}}} \times_C C^I$ consisting of an $i-1$ -truncated semi-simplicial object X_\bullet , and object Y and a morphism

$f : Y \rightarrow E(X)$. We will prove that there exists an i -truncated semi-simplicial object Z_\bullet and an isomorphism $\Phi_i(Z_\bullet) \simeq (X_\bullet, f)$. We first let $Z_j = X_j$ for $j < i$ and let $Z(\varphi) = X(\varphi)$ for any $\varphi : [j'] \rightarrow [j]$ with $j' < j < i$. Then we set $Z_i = Y$, and we must define morphisms $Z(\varphi) : Z_i = Y \rightarrow X_j = Z_j$ for increasing maps $[j] \rightarrow [i]$ which are functorial in φ , in the sense that for any increasing $\psi : [j'] \rightarrow [j]$ the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{Z(\varphi)} & X_j & \xrightarrow{X(\psi)} & X_{j'} \\ & \searrow & & \nearrow & \\ & & Z(\varphi \circ \psi) & & \end{array} \quad (5.10)$$

commutes (note that the data of $X(\psi)$ is already included in X_\bullet). We may assume $j < i$ (otherwise $\varphi = \text{id}$ and we must set $Z(\varphi) = \text{id}$), and so φ must factor as

$$[j] \xrightarrow{\psi} [i-1] \xrightarrow{\delta_k^i} [i]$$

for some k and some ψ . We define $Z(\varphi)$ to be the composition

$$Y \xrightarrow{f} E(X) \rightarrow X_{i-1}^{[i]} \xrightarrow{\text{pr}_k} X_{i-1} \xrightarrow{X(\psi)} X_j$$

(so in particular we define $Z(\delta_k^i) = \text{pr}_k \circ f =: f_k$). To verify this definition is independent of ψ , suppose that there is another factorization

$$[j] \xrightarrow{\psi'} [i-1] \xrightarrow{\delta_l^i} [i]$$

Note that if $j = i-1$ then $\psi = \psi' = \text{id}$ and $k = l$ for trivial reasons, so we may assume $j < i-1$ and in that case φ misses *both* k and l , so we may factor through $[i-2]$ as follows:

$$\begin{array}{ccccc} & & \psi & & \\ & & \nearrow & & \\ [j] & \xrightarrow{\rho} & [i-2] & \xrightarrow{\delta_{l-1}^{i-1}} & [i-1] \\ & \searrow & & \searrow & \\ & & \delta_k^{i-1} & & \\ & & \searrow & & \\ & & [i-1] & \xrightarrow{\delta_l^i} & [i] \end{array} \quad (5.11)$$

By the defining property of the equalizer $E(X)$, we know $X(\delta_{j-1}^{i-1}) \circ f_k = X(\delta_k^{i-1}) \circ f_l$, and

$$X(\rho) \circ X(\delta_{j-1}^{i-1}) = X(\psi) \text{ and } X(\rho) \circ X(\delta_k^{i-1}) = X(\psi')$$

because X_\bullet is an $i-1$ -truncated semi-simplicial object. It follows that $X(\psi) \circ f_k = X(\psi') \circ f_l$ as desired.

We now prove to prove the commutativity statement in (5.10). Again we may assume $j < i$, since otherwise $\varphi = \text{id}$ and $\psi = \varphi \circ \psi$ so commutativity is implied by the above proof that the $Z(\varphi)$ are well defined. When $j < k$ the map φ , and hence also $\varphi \circ \psi$ must factor through some $\delta_k^i : [i-1] \rightarrow [i]$ and we obtain the following situation:

$$\begin{array}{ccccc} & & \varphi & & \\ & & \nearrow & & \\ [j] & \xrightarrow{\psi} & [j] & \xrightarrow{\rho} & [i-1] \xrightarrow{\delta_k^i} [i] \\ & \searrow & & \searrow & \\ & & \varphi \circ \psi & & \end{array}$$

Now *by definition* $Z(\varphi) = X(\rho) \circ f_k$ and $Z(\varphi \circ \psi) = X(\rho \circ \psi) \circ f_k$, and since X_\bullet is an $i - 1$ -truncated semi-simplicial object $X(\rho \circ \psi) = X(\psi) \circ X(\rho)$, so that

$$X(\psi) \circ Z(\varphi) = X(\psi) \circ X(\rho) \circ f_k = X(\rho \circ \psi) \circ f_k = Z(\varphi \circ \psi)$$

as claimed. □

5.3. Common admissible blowups.

Lemma 5.12. *Suppose*

$$\begin{array}{ccccc} X & \xleftarrow{\iota} & U & \xrightarrow{j} & Y \\ \downarrow \varphi & & \downarrow \rho^0 & & \downarrow \psi \\ F & \xleftarrow{f} & E & \xrightarrow{g} & G \end{array}$$

is a commutative diagram of schemes of finite type over a quasi-compact quasi-separated base scheme S , and assume that f, g, φ and ψ are proper and ι and j are dense open immersions. Then, there is a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{r} & Z & \xrightarrow{s} & Y \\ \downarrow \varphi & & \downarrow \rho & & \downarrow \psi \\ F & \xleftarrow{f} & E & \xrightarrow{g} & G \end{array}$$

where r and s are U -admissible blowups (hence in particular projective).

If in addition S is a CM-quasi-excellent noetherian scheme and U is Cohen-Macaulay, we may ensure that Z is also Cohen-Macaulay.

Proof. First, X and E are proper over the scheme F , which is quasi-compact and quasi-separated since it is of finite type over S . By the first part of [Lemma 3.9](#) applied to the map of F -schemes $\rho^0 : U \rightarrow E$ defined on the dense open $U \subseteq X$, there is a U -admissible blowup $V_X \rightarrow X$ and an F -morphism $V_X \rightarrow E$ extending ρ^0 . A similar argument produces a U -admissible blowup $V_Y \rightarrow Y$ and a G -morphism $V_Y \rightarrow E$ extending ρ^0 . The current situation is summarized below:

$$\begin{array}{ccccccc} U & \xrightarrow{\quad} & V_Y & \xrightarrow{\quad} & E & & \\ & \searrow & \downarrow & & \downarrow & \nearrow & \\ & & V_X & \xrightarrow{\quad} & E & & \\ & \searrow & \downarrow & & \downarrow g & & \\ & & Y & \xrightarrow{\quad} & G & & \\ & \searrow & \downarrow \psi & & \downarrow & & \\ & & X & \xrightarrow{\quad} & F & & \\ & \searrow & \downarrow \varphi & & \downarrow f & & \end{array}$$

Since V_X, V_Y are U admissible blowups of X, Y respectively, they still contain U as a *dense* open. Note that since $V_X \rightarrow X$ is a blowup, φ is proper and f is proper the morphism $V_X \rightarrow E$ is also proper; similarly V_Y is proper over E . Now applying the second part of [Lemma 3.9](#) to V_X and V_Y over E we obtain a separated finite type morphism $\rho : Z \rightarrow E$, U admissible blowups $\tilde{V}_X \rightarrow V_X$ and

$\tilde{V}_Y \rightarrow V_Y$ and open immersions $\tilde{V}_X \hookrightarrow Z \hookleftarrow \tilde{V}_Y$ over E such that the diagram

$$\begin{array}{ccc} U & \longrightarrow & \tilde{V}_Y \\ \downarrow & & \downarrow \\ \tilde{V}_X & \longrightarrow & Z \end{array}$$

commutes and $E = \tilde{V}_X \cup \tilde{V}_Y$. Since U is dense in both \tilde{V}_X and \tilde{V}_Y , we see that \tilde{V}_X and \tilde{V}_Y are both dense in Z . Then as $\tilde{V}_X \rightarrow Z$ is a dense open immersion of separated finite type E -schemes where \tilde{V}_X is *proper* over E , it must be that $\tilde{V}_X = Z$; similarly, $\tilde{V}_Y = Z$ (see also the comments following [Con07, Cor. 2.10]). Finally, we define r and s to be the compositions

$$Z \xlongequal{\quad} \tilde{V}_X \longrightarrow V_X \xrightarrow{\quad} X \quad \text{and} \quad Z \xlongequal{\quad} \tilde{V}_Y \longrightarrow V_Y \xrightarrow{\quad} Y$$

Finally if S is CM-quasi-excellent, then since Z is of finite type over S it is also CM-quasi-excellent by [Ces18, Rmk.1.5]. By hypothesis $U \subseteq \text{CM}(Z)$, and by Theorem 5.1 there is a $\text{CM}(X)$ -admissible (hence also U -admissible) blowup $\tilde{Z} \rightarrow Z$ such that \tilde{Z} is Cohen-Macaulay. In this case we replace Z with \tilde{Z} . \square

Lemma 5.13. *Let S be a quasi-compact quasi-separated base scheme and let*

$$\begin{array}{ccccc} X_\bullet & \xleftarrow{i_\bullet} & U_\bullet & \xrightarrow{j_\bullet} & Y_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ X_{-1} & \xleftarrow{i_{-1}} & U_{-1} & \xrightarrow{j_{-1}} & Y_{-1} \end{array} \quad (5.14)$$

be morphisms of augmented semi-simplicial schemes of finite type over S . Assume that all differentials and augmentations of X_\bullet and Y_\bullet are proper,⁷ and that the morphisms $X_i \xleftarrow{i_i} U_i \xrightarrow{j_i} Y_i$ are dense open immersions for all i (including $i = -1$). If there exists a finite-type S -scheme Z_{-1} and U_{-1} -admissible blowups $X_{-1} \xleftarrow{r_{-1}} Z_{-1} \xrightarrow{s_{-1}} Y_{-1}$, then there exists an augmented semi-simplicial S -scheme $Z_\bullet \rightarrow Z_{-1}$ together with morphisms

$$\begin{array}{ccccc} X_\bullet & \xleftarrow{r_\bullet} & Z_\bullet & \xrightarrow{s_\bullet} & Y_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ X_{-1} & \xleftarrow{r_{-1}} & Z_{-1} & \xrightarrow{s_{-1}} & Y_{-1} \end{array} \quad (5.15)$$

such that for all i the morphisms $X_i \xleftarrow{r_i} Z_i \xrightarrow{s_i} Y_i$ are U_i -admissible blowups (hence in particular projective and birational).

Moreover if S is a CM-quasi-excellent noetherian scheme, and each U_i is Cohen-Macaulay, we may ensure that the Z_i are also Cohen-Macaulay.

Proof. We construct a sequence of i -truncated semi-simplicial S -schemes $\tilde{Z}_{i\bullet}$ converging to Z_\bullet , with the additional requirement that the morphisms $\text{sk}_{i-1}(U_\bullet) \rightarrow \text{sk}_{i-1}(X_\bullet)$ and $\text{sk}_{i-1}(U_\bullet) \rightarrow \text{sk}_{i-1}(Y_\bullet)$ factor through $\tilde{Z}_{i\bullet}$.⁸ The $i = -1$ case is included in the hypotheses. At the inductive step we may

⁷This is equivalent to requiring that X_\bullet is a semi-semi-simplicial object in the category of proper X_{-1} -schemes (and similarly for Y_\bullet).

⁸I think that this isn't actually an additional restriction, but including it makes the inductive step easier.

assume that there is an $i - 1$ -truncated semi-simplicial S -scheme $\tilde{Z}_{i-1\bullet}$ together with a commutative diagram

$$\begin{array}{ccccc}
 & & \text{sk}_{i-1}(U_{\bullet}) & & \\
 & \swarrow \text{sk}_{i-1}(i_{\bullet}) & \downarrow k_{\bullet} & \searrow \text{sk}_{i-1}(j_{\bullet}) & \\
 \text{sk}_{i-1}(X_{\bullet}) & \xleftarrow{\tilde{r}_{i-1\bullet}} & \tilde{Z}_{i-1\bullet} & \xrightarrow{\tilde{s}_{i-1\bullet}} & \text{sk}_{i-1}(Y_{\bullet}) \\
 \downarrow & & \downarrow & & \downarrow \\
 X_{-1} & \xleftarrow{r_{-1}} & Z_{-1} & \xrightarrow{s_{-1}} & Y_{-1}
 \end{array} \tag{5.16}$$

such that for all $j < i$ the morphisms $X_j \xleftarrow{\tilde{r}_{i-1,j}} \tilde{Z}_{i-1,j} \xrightarrow{\tilde{s}_{i-1,j}} Y_j$ are U_j -admissible blowups. Letting E denote the equalizer functor of [Lemma 5.6](#), we obtain a commutative diagram of the form

$$\begin{array}{ccccc}
 X_i & \xleftarrow{i_i} & U_i & \xrightarrow{j_i} & Y_i \\
 \downarrow (X(\delta_k^i)) & & \downarrow (U(\delta_k^i)) & & \downarrow (Y(\delta_k^i)) \\
 & & E(\text{sk}_{i-1}(U_{\bullet})) & & \\
 & & \downarrow E(k_{\bullet}) & & \\
 E(\text{sk}_{i-1}(X_{\bullet})) & \xleftarrow{E(\tilde{r}_{i-1\bullet})} & E(\tilde{Z}_{i-1\bullet}) & \xrightarrow{E(\tilde{s}_{i-1\bullet})} & E(\text{sk}_{i-1}(Y_{\bullet}))
 \end{array} \tag{5.17}$$

Next, we verify that (5.17) satisfies the hypotheses of [Lemma 5.12](#), making repeated reference to the constructions in (5.4) and (5.5). Note that the bottom horizontal arrows are proper, since they are obtained as limits of the blowup maps $\tilde{r}_{i-1,j} : \tilde{Z}_{i-1,j} \rightarrow X_j$ and $\tilde{s}_{i-1,j} : \tilde{Z}_{i-1,j} \rightarrow Y_j$ for $j = i - 1, i - 2$. The vertical maps on the outside edges are proper since the differentials $X(\delta_k^i) : X_i \rightarrow X_{i-1}$ and $Y(\delta_k^i) : Y_i \rightarrow Y_{i-1}$ are proper by hypothesis. Hence applying [Lemma 5.12](#) we obtain a commutative diagram

$$\begin{array}{ccccccc}
 & & & & j_i & & \\
 & & & & \curvearrowright & & \\
 U_i & \xrightarrow{i_i} & X_i & \xleftarrow{\tilde{r}_{i-1,i}} & Z_i & \xrightarrow{\tilde{s}_{i-1,i}} & Y_i \\
 \downarrow (U(\delta_k^i)) & & \downarrow (X(\delta_k^i)) & & \downarrow \rho & & \downarrow (Y(\delta_k^i)) \\
 E(\text{sk}_{i-1}(U_{\bullet})) & \longrightarrow & E(\text{sk}_{i-1}(X_{\bullet})) & \xleftarrow{E(\tilde{r}_{i-1\bullet})} & E(\tilde{Z}_{i-1\bullet}) & \xrightarrow{E(\tilde{s}_{i-1\bullet})} & E(\text{sk}_{i-1}(Y_{\bullet})) \\
 & & & & \curvearrowleft E(k_{\bullet}) & &
 \end{array} \tag{5.18}$$

in which the maps $\tilde{r}_{i-1,i} : Z_i \rightarrow X_i$ and $\tilde{s}_{i-1,i} : Z_i \rightarrow Y_i$ are U_i -admissible blowups. In the case where S is CM-quasi-excellent we apply [Lemma 5.12](#) to ensure that Z_i is Cohen-Macaulay.

Now [Lemma 5.6](#) implies that there is an i -truncated semi-simplicial S -scheme $\tilde{Z}_{i\bullet}$ such that $\mathrm{sk}_{i-1}(\tilde{Z}_{i\bullet}) = \tilde{Z}_{i-1\bullet}$ and $\tilde{Z}_{i,i} = Z_i$, together with a commutative diagram

$$\begin{array}{ccccc}
 & & \mathrm{sk}_i(U_\bullet), & & \\
 & \swarrow \mathrm{sk}_i(i_\bullet) & \downarrow k_\bullet & \searrow \mathrm{sk}_i(j_\bullet) & \\
 \mathrm{sk}_i(X_\bullet) & \xleftarrow{\tilde{r}_{i\bullet}} & \tilde{Z}_{i\bullet} & \xrightarrow{\tilde{s}_{i\bullet}} & \mathrm{sk}_i(Y_\bullet) \\
 \downarrow & & \downarrow & & \downarrow \\
 X_{-1} & \xleftarrow{r_{-1}} & Z_{-1} & \xrightarrow{s_{-1}} & Y_{-1}
 \end{array} \tag{5.19}$$

such that for all $j \leq i$ the morphisms $X_j \xleftarrow{\tilde{r}_{i-1,j}} \tilde{Z}_{i-1,j} \xrightarrow{\tilde{s}_{i-1,j}} Y_j$ are U_j -admissible blowups. \square

5.4. Construction and corollaries.

Proof of [Lemma 5.3](#). Set $\Delta_Z = r_*^{-1}\Delta_X = s_*^{-1}\Delta_Y$. By [Lemma 3.7](#) there is a dense open set $X \hookleftarrow U_X \hookrightarrow Z$ (resp $Z \hookleftarrow U_Y \hookrightarrow Y$) containing all generic points of strata of $\mathrm{snc}(X, \Delta_X)$ and $\mathrm{snc}(Z, \Delta_Z)$ (resp. $\mathrm{snc}(Y, \Delta_Y)$ and $\mathrm{snc}(Z, \Delta_Z)$). Then $U := U_X \cap U_Z$ is a dense open containing all generic points of strata of $\mathrm{snc}(X, \Delta_X)$, $\mathrm{snc}(Y, \Delta_Y)$ and $\mathrm{snc}(Z, \Delta_Z)$. Set $\Delta_U := \Delta_Z|_U$, so that (U, Δ_U) is simple normal crossing pair together with thrifty birational (but not necessarily projective) morphisms $(X, \Delta_X) \xleftarrow{r|_U} (U, \Delta_U) \xrightarrow{s|_U} (Y, \Delta_Y)$. We now let X_\bullet , Y_\bullet and U_\bullet be the augmented semi-simplicial schemes associated to (X, Δ_X) , (Y, Δ_Y) and (U, Δ_U) as in the discussion at the beginning of [section 2](#), and consider the resulting morphisms

$$\begin{array}{ccccc}
 X_\bullet & \xleftarrow{\quad} & U_\bullet & \xrightarrow{\quad} & Y_\bullet \\
 \downarrow & & \downarrow & & \downarrow \\
 X_{i-1} = X & \xleftarrow{\quad} & U_{i-1} = U & \xrightarrow{\quad} & Y_{i-1} = Y
 \end{array} \tag{5.20}$$

Since U contains the generic points of all strata of $\mathrm{snc}(Z, \Delta_Z)$, the morphisms $X_i \leftarrow U_i \rightarrow Y_i$ are dense open immersions for all i , and the differentials and augmentations of X_\bullet and Y_\bullet are closed immersions, hence proper. Finally applying [Lemma 3.9](#) to the collection of open immersions $U \subseteq X, Z$ over X , we obtain U -admissible blowups \tilde{X}, \tilde{Y} of X, Y respectively, as well as a separated finite type X -scheme W with open immersions $\tilde{X}, \tilde{Z} \subseteq W$ covering W . Again properness of \tilde{X}, \tilde{Y} over X forces $\tilde{X} = \tilde{Z} = W$, hence replacing Z with \tilde{Z} we can ensure $r : Z \rightarrow X$ is a U -admissible blowup. Repeating this construction with Y, Z in place of X, Z , we may ensure $s : Z \rightarrow Y$ is also a U -admissible blowup. Thus the hypotheses of [Lemma 5.13](#) are satisfied. \square

Corollary 5.21. *With the same hypotheses as [Lemma 5.3](#), there exists a complex \mathcal{K} on Z together with quasi-isomorphisms $\mathcal{O}_X(-\Delta_X) \simeq Rr_*\mathcal{K}$ and $\mathcal{O}_Y(-\Delta_Y) \simeq Rs_*\mathcal{K}$. In particular there are quasi-isomorphisms $Rf_*\mathcal{O}_X(-\Delta_X) \simeq Rf_*Rr_*\mathcal{K} = Rg_*Rs_*\mathcal{K} \simeq Rg_*\mathcal{O}_Y(-\Delta_Y)$.*

Proof. By [Lemma 5.3](#) there is a commutative diagram of augmented semi-simplicial S -schemes

$$\begin{array}{ccccc}
 X_\bullet & \xleftarrow{r_\bullet} & Z_\bullet & \xrightarrow{s_\bullet} & Y_\bullet \\
 \downarrow \epsilon^X & & \downarrow \epsilon^Z & & \downarrow \epsilon^Y \\
 X & \xleftarrow{r} & Z & \xrightarrow{s} & Y
 \end{array} \tag{5.22}$$

such that for each i the scheme Z_i is Cohen-Macaulay and the maps $X_i \xleftarrow{r_i} Z_i \xrightarrow{s_i} Y_i$ define a projective birational equivalence over S . Defining $\mathcal{K} = \mathrm{cone}(\mathcal{O}_Z \rightarrow R\epsilon_*^Z \mathcal{O}_{Z_\bullet})[-1]$, from [\(5.22\)](#) we

obtain a map of complexes $\underline{\Omega}_{X,\Delta_X}^0 \rightarrow Rr_*\mathcal{K}$ appearing in a map of distinguished triangles

$$\begin{array}{ccccccc} \underline{\Omega}_{X,\Delta_X}^0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & R\epsilon_*^X \mathcal{O}_X & \xrightarrow{+1} & \cdots \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ Rr_*\mathcal{K} & \longrightarrow & Rr_*\mathcal{O}_Z & \longrightarrow & Rr_*R\epsilon_*^Z \mathcal{O}_Z & \xrightarrow{+1} & \cdots \end{array}$$

By [Kov20, Thm. 1.4] β is a quasi-isomorphism. Using commutativity of (5.22) we may identify γ with the morphism

$$R\epsilon_*^X \mathcal{O}_X \rightarrow R\epsilon_*^X Rr_*\mathcal{O}_Z. \quad (5.23)$$

The morphisms on cohomology induced by (5.23) are the abutment of a map of descent spectral sequences (see Lemma 2.2); the map of E_1 pages reads

$$E_1^{ij}(X) = \begin{cases} \epsilon_*^X \mathcal{O}_{X_i} & \text{if } j = 0 \\ 0 & \text{else} \end{cases} \rightarrow R^j \epsilon_{i*}^X Rr_{i*} \mathcal{O}_{Z_i} = E_1^{ij}(Z) \quad (5.24)$$

By [Kov20, Thm. 1.4] again, for each i the natural map $\mathcal{O}_{X_i} \rightarrow Rr_{i*} \mathcal{O}_{Z_i}$ is a quasi-isomorphism. We conclude that (5.24) is an isomorphism, and so γ is a quasi-isomorphism.

By the 5-lemma, we conclude α is a quasi-isomorphism. Now applying Rf_* and using Lemma 2.5 then gives a quasi-isomorphism

$$Rf_* \mathcal{O}_X(-\Delta_X) \simeq Rf_* \underline{\Omega}_{X,\Delta_X}^0 \simeq Rf_* Rr_* \mathcal{K}.$$

A symmetric argument applied on the Y side gives the desired quasi-isomorphism $Rf_* \mathcal{O}_X(-\Delta_X) \simeq Rg_* Rr_* \mathcal{K}$. \square

6. APPLICATIONS TO RATIONAL PAIRS

In this section we make use of Grothendieck duality, as formulated in [Con00; R&D].⁹ To begin, a definition:

Definition 6.1 (cf. [Kol13, Def. 2.78]). Let (S, Δ_S) be a pair as in Convention 1.5, with the additional hypothesis that S admits a dualizing complex, and the divisor Δ_S is reduced and effective. A proper birational morphism $f : X \rightarrow S$ is a *rational resolution* if and only if

- (i) X is regular and the strict transform $\Delta_X := f_*^{-1} \Delta_S$ has simple normal crossings,
- (ii) the natural morphism $\mathcal{O}_S(-\Delta_S) \rightarrow Rf_* \mathcal{O}_X(-\Delta_X)$ is a quasi-isomorphism, and letting $\omega_X = h^{-\dim X} \omega_X^\bullet$ where we use $\omega_X^\bullet = f^! \omega_S^\bullet$ as a normalized dualizing complex on X ,
- (iii) $R^i f_* \omega_X(\Delta_X) = 0$ for $i > 0$.

In the situation of Definition 6.1, the map $\mathcal{O}_S(-\Delta_S) \rightarrow Rf_* \mathcal{O}_X(-\Delta_X)$ appearing in condition (ii) is Grothendieck dual to a morphism

$$Rf_* \omega_X^\bullet(\Delta_X) = Rf_* R\mathcal{H}om_X(\mathcal{O}_X(-\Delta_X), \omega_X^\bullet) \xrightarrow[(2)]{\simeq} R\mathcal{H}om_S(Rf_* \mathcal{O}_X(-\Delta_X), \omega_S^\bullet) \xrightarrow[(3)]{\rightarrow} R\mathcal{H}om_S(\mathcal{O}_S(-\Delta_S), \omega_S^\bullet)$$

where the equality (1) comes from the fact that Δ_X is a Cartier divisor ((X, Δ_X) is snc by hypothesis), the isomorphism (2) comes from Grothendieck duality and the map (3) is obtained from the morphism of (ii) by applying the derived functor $R\mathcal{H}om_S(-, \omega_S^\bullet)$. As X is regular and the dualizing complex ω_X^\bullet is normalized $h^i \omega_X^\bullet = 0$ for $i \neq -\dim X$; in other words, $\omega_X^\bullet \simeq \omega_X[\dim X]$. Twisting this equation with the Cartier divisor Δ_X gives $\omega_X^\bullet(\Delta_X) \simeq \omega_X(\Delta_X)[\dim X]$. If $\mathcal{O}_S \rightarrow Rf_* \mathcal{O}_X$ is a quasi-isomorphism, so is

$$Rf_* \omega_X(\Delta_X)[\dim X] \simeq Rf_* \omega_X^\bullet(\Delta_X) \rightarrow R\mathcal{H}om_S(\mathcal{O}_S(-\Delta_S), \omega_S^\bullet)$$

⁹Which is to say we make *explicit* use of Grothendieck duality — that is, it has already been used implicitly via dependence on references quite a few times!

and taking cohomology sheaves we see that $R^{i+\dim X} f_* \omega_X(\Delta_X) \simeq h^i R\mathcal{H}om_S(\mathcal{O}_S(-\Delta_S), \omega_S^\bullet)$ for all i .

Thus given conditions (i) and (ii) of Definition 6.1, condition (iii) is equivalent to Cohen-Macaulayness of the sheaf $\mathcal{O}_S(-\Delta_S)$. We record these observations as a lemma.

Lemma 6.2 (cf. [Kol13, Cor. 2.73, Props. 2.82-2.23], [Kov20, Def. 1.3]). *With notation and setup as in Definition 6.1, the morphism $f : X \rightarrow S$ is a rational resolution if and only if*

- (i) *X is regular and the strict transform $\Delta_X := f_*^{-1} \Delta_S$ has simple normal crossings,*
- (ii) *the natural morphism $\mathcal{O}_S(-\Delta_S) \rightarrow Rf_* \mathcal{O}_X(-\Delta_X)$ is a quasi-isomorphism, and*
- (iii) *the sheaf $\mathcal{O}_S(-\Delta_S)$ is Cohen-Macaulay.*

As illustrated in the examples of section 4, even simple normal crossing pairs (S, Δ_S) may have non-rational resolutions in the absence of additional thriftiness restrictions, hence the following definition of rational singularities for pairs.

Definition 6.3. Let (S, Δ_S) be a pair such that S admits a dualizing complex and Δ_S is a reduced effective Weil divisor. Then, (S, Δ_S) is *resolution-rational* if and only if it has a thrifty rational resolution.

6.1. All for one. In the case where S is a normal variety over a field of characteristic 0, it is known that if (S, Δ_S) has a thrifty rational resolution then *every* thrifty resolution is rational [Kol13, Cor. 2.86]. The proof of this fact shows more generally that if $f : X \rightarrow S$ and $g : Y \rightarrow S$ are thrifty resolutions, then there are isomorphisms $R^i f_* \mathcal{O}_X(-\Delta_X) \simeq R^i g_* \mathcal{O}_Y(-\Delta_Y)$ for all i . This remains true in arbitrary characteristic.

Lemma 6.4. *Let (S, Δ_S) be a pair such that S admits a dualizing complex and Δ_S is a reduced effective Weil divisor, and let $f : X \rightarrow S$ and $g : Y \rightarrow S$ be thrifty resolutions. Then there is a quasi-isomorphism $Rf_* \mathcal{O}_X(-\Delta_X) \simeq Rg_* \mathcal{O}_Y(-\Delta_Y)$. In particular, f is a rational resolution if and only if g is.*

Note that this includes Theorem 3.12 as a special case: indeed, if (S, Δ_S) is a simple normal crossing pair (note that in this case S is regular and hence admits a dualizing complex), then given any thrifty resolution $f : X \rightarrow S$ we may choose g to be the identity.

Proof. By Lemma 3.7, there are dense open immersions $S \hookleftarrow U_X \hookrightarrow X$ and $S \hookleftarrow U_Y \hookrightarrow Y$ such that U_X (resp. U_Y) contains all strata of $\text{snc}(S, \Delta_S)$ and (X, Δ_X) (resp. $\text{snc}(S, \Delta_S)$ and (Y, Δ_Y)). Then $U := U_X \cap U_Y$ also contains all strata of $\text{snc}(S, \Delta_S)$ — moreover since f and g are thrifty, the strata of (X, Δ_X) and (Y, Δ_Y) are in one-to-one birational correspondence with those of (S, Δ_S) , so it remains true that U contains all strata of (X, Δ_X) and (Y, Δ_Y) . Replacing U with $U \cap \text{snc}(X, \Delta_X)$, we may assume $(U, \Delta_U := \Delta_S \cap U)$ is an snc pair. We now have morphisms $\iota : U \hookrightarrow X$, $j : U \hookrightarrow Y$ which are thrifty and birational, but not necessarily proper.

Now let $X_\bullet \rightarrow X_{-1} =: X$, $Y_\bullet \rightarrow Y_{-1} =: Y$ and $U_\bullet \rightarrow U_{-1} =: U$ be the augmented semi-simplicial schemes associated to these simple normal crossing pairs. The inclusions ι and j induce a diagram as in (5.14); we proceed to verify that the hypotheses of Lemma 5.13 are satisfied. All schemes in sight are defined over the noetherian and hence quasi-compact quasi-separated S . The differentials and augmentations are all closed immersions and hence proper, and thriftiness of ι and j implies that the morphisms $X_i \xleftarrow{\iota_i} U_i \xrightarrow{j_i} Y_i$ are dense open immersions for all i . Applying Lemma 3.9 to the collection of S -schemes S, X and Y with the common dense open U gives a common U -admissible blowup $X \leftarrow Z \rightarrow Y$. Finally (for the moreover part of the lemma) S is excellent by hypothesis and the U_i are regular, hence Cohen-Macaulay.

The output of Lemma 5.13 is an augmented semi-simplicial scheme $Z_\bullet \rightarrow Z_{-1} =: Z$ such that each scheme Z_i is Cohen-Macaulay, together with morphisms $X_\bullet \xleftarrow{r_\bullet} Z_\bullet \xrightarrow{s_\bullet} Y_\bullet$ such that for each i the $X_i \xleftarrow{r_i} Z_i \xrightarrow{s_i} Y_i$ are U_i -admissible blowups. For the remainder of the proof we argue exactly as in Corollary 5.21. \square

7. CYCLE MORPHISMS TO LOG HODGE COHOMOLOGY

The original proof of [CR11, Thm. 3.2.8] makes use of a cycle morphism $\text{cl} : CH^*(X) \rightarrow H^*(X, \Omega_X^*)$ from Chow cohomology to Hodge cohomology, which is ultimately applied to a cycle $Z \subset X \times Y$ obtained from a proper birational equivalence. That cycle morphism satisfies 2 key properties: the first is that it is compatible with *correspondences*: here Chow correspondences are homomorphisms

$$CH^*(X) \rightarrow CH^*(Y) \text{ of the form } \alpha \mapsto \text{pr}_{Y*}(\text{pr}_X^* \alpha \smile \gamma) \text{ for some } \gamma \in CH^*(X \times Y)$$

where \smile is the cup product induced by intersecting cycles; Hodge correspondences are defined in a similar way. The second key property is a compatibility with the filtrations

$$CH^n(X \times Y) = F^0 CH^n(X \times Y) \supseteq F^1 CH^n(X \times Y) \supseteq \dots \supseteq F^{\dim Y} CH^n(X \times Y) \supseteq 0$$

where $F^c CH^n(X \times Y)$ is the subgroup generated by cycles $Z \subseteq X \times Y$ such that $\text{codim}(\text{pr}_Y Z \subseteq Y) \geq c$, and

$$H^n(X \times Y, \Omega_{X \times Y}^m) = F^0 H^n(X \times Y, \Omega_{X \times Y}^m) \supseteq F^1 H^n(X \times Y, \Omega_{X \times Y}^m) \supseteq \dots \supseteq F^{\dim Y} H^n(X \times Y, \Omega_{X \times Y}^m) \supseteq 0$$

where $F^c H^n(X \times Y, \Omega_{X \times Y}^m)$ is the image of the map $H^n(X \times Y, \bigoplus_{j=c}^m \Omega_X^{m-j} \boxtimes \Omega_Y^j) \rightarrow H^n(X \times Y, \Omega_{X \times Y}^m)$ coming from the Künneth decomposition.

It is natural to ask if a similar method can be applied to prove [Theorem 1.7](#), by replacing the ordinary sheaves of differentials Ω_X appearing in Hodge cohomology with sheaves of differentials with log poles $\Omega_X(\log \Delta_X)$. Many of the preliminary results on Hodge cohomology in [CR11, §2] carry over without difficulty, however log poles add complications when one begins to deal with correspondences $H^*(X, \Omega_X(\log \Delta_X)) \rightarrow H^*(Y, \Omega_Y(\log \Delta_Y))$ associated to certain Hodge classes with log poles on $X \times Y$.

[Section 7.1](#) has substantial overlap with [BPØ20, §9]. However in that article only *finite* correspondences are considered, with additional strictness (in the sense of logarithmic geometry) conditions. Such correspondences seem to be insufficient to deal with proper birational equivalences, as we illustrate in [sections 7.2](#) and [7.3](#).

7.1. Functoriality properties of log Hodge cohomology with supports. Let X be a noetherian scheme.

Definition 7.1 ([R&D], [CR11]). A **family of supports** Φ on X is a non-empty collection Φ of closed subsets of X such that

- If $C \in \Phi$ and $D \subset C$ is a closed subset, then $D \in \Phi$.
- If $C, D \in \Phi$ then $C \cup D \in \Phi$.

Example 7.2. $\Phi = \{ \text{all closed subsets of } X \}$ is a family of supports. More generally if \mathcal{C} is any collection of closed subsets $C \subset X$, there's a *smallest* family of supports $\Phi(\mathcal{C})$ containing \mathcal{C} (explicitly, $\Phi(\mathcal{C})$ consists of finite unions $\bigcup_i Z_i$ of closed subsets $Z_i \subset C_i$ of elements $C_i \in \mathcal{C}$). Taking $\Phi = \Phi(\{X\})$ recovers the previous example. For a closed subset $Z \subset X$ we will use the abbreviation $\Phi(Z) := \Phi(\{Z\})$.

There is a close relationship between families of supports on X and certain collections of specialization-closed subsets of points on X . One can also consider sheaves of families of supports. See [R&D].

If $f : X \rightarrow Y$ is a morphism of noetherian schemes and Ψ is a family of supports on Y , then $\{f^{-1}(Z) \mid Z \in \Psi\}$ is a family of closed subsets of X , and is closed under unions, but is *not* in general closed under taking closed subsets.

Definition 7.3. $f^{-1}(\Psi)$ be the smallest family of supports on X containing $\{f^{-1}(Z) \mid Z \in \Psi\}$.

Let Φ be a family of supports on X . The notation/terminology $f|_\Phi$ is **proper** will mean $f|_C$ is proper for every $C \in \Phi$. If $f|_\Phi$ is proper then $f(C) \subset Y$ is closed for every $C \in \Phi$ and in fact

$$f(\Phi) = \{f(C) \subset Y \mid C \in \Phi\} \quad (7.4)$$

is a family of supports on Y . The key point here is that if $D \subset f(C)$ is closed, then $f^{-1}(D) \cap C \in \Phi$ and $D = f(f^{-1}(D) \cap C)$.

Definition 7.5. A **scheme with supports** (X, Φ_X) is a scheme X together with a family of supports Φ_X on X .

When no confusion is likely to arise we will abbreviate (X, Φ_X) by simply X .

Definition 7.6. A **pushing morphism** $f : (X, \Phi_X) \rightarrow (Y, \Phi_Y)$ of schemes with supports is a morphism $f : X \rightarrow Y$ of underlying schemes such that $f|_{\Phi_X}$ is proper and $f(\Phi_X) \subset \Phi_Y$. A **pulling morphism** $f : X \rightarrow Y$ is a morphism $f : X \rightarrow Y$ such that $f^{-1}(\Phi_Y) \subset \Phi_X$.

These morphisms provide 2 different categories with underlying set of objects schemes with supports (X, Φ_X) , and pushing/pulling morphisms respectively (the verification is elementary; for instance a composition of pushing morphisms is again a pushing morphism since compositions of proper morphisms are proper).

Schemes with supports provide a natural setting for local cohomology [R&D]. Let \mathcal{F} be a sheaf of abelian groups on a scheme with supports (X, Φ_X) (more precisely \mathcal{F} is just a sheaf of abelian groups on X).

Definition 7.7. The **sheaf of sections with supports** of \mathcal{F} , denoted $\underline{\Gamma}_\Phi(\mathcal{F})$, is obtained by setting

$$\underline{\Gamma}_\Phi(\mathcal{F})(U) = \{\sigma \in \mathcal{F}(U) \mid \text{supp } \sigma \in \Phi_X|_U\} \quad (7.8)$$

for each open $U \subset X$ (here $\Phi_X|_U$ is short for $\iota^{-1}\Phi_X$ where $\iota : U \rightarrow X$ is the inclusion). More explicitly: for a local section $\sigma \in \mathcal{F}(U)$, $\sigma \in \underline{\Gamma}_\Phi(\mathcal{F})(U)$ means $\text{supp } \sigma = C \cap U$ for a closed set $C \subset \Phi_X$.

The functor $\underline{\Gamma}_\Phi$ is right adjoint to an exact functor, for instance the inclusion of the subcategory $\mathbf{Ab}_\Phi(X) \subset \mathbf{Ab}(X)$ of abelian sheaves on X with supports in Φ ; so, $\underline{\Gamma}_\Phi$ is left exact and preserves injectives. In the case $\Phi = \Phi(Z)$ for some closed $Z \subset X$, this is proved in [Stacks, Tag 0A39, Tag 0G6Y, Tag 0G7F] — the general case can then be obtained by writing $\underline{\Gamma}_\Phi$ as a filtered colimit:

$$\underline{\Gamma}_\Phi = \text{colim}_{Z \in \Phi} \underline{\Gamma}_Z.$$

The right derived functor of $\underline{\Gamma}_\Phi$ will be denoted $R\underline{\Gamma}_\Phi$. Taking global sections on X gives the **sections with supports** of \mathcal{F} :

$$\Gamma_\Phi(\mathcal{F}) := \Gamma_X(\underline{\Gamma}_\Phi(\mathcal{F})) \quad (7.9)$$

This is also left exact, and (the cohomologies of) its derived functor give the **cohomology with supports in Φ** :

$$H_\Phi^i(X, \mathcal{F}) := R^i \Gamma_\Phi(\mathcal{F}) \quad (7.10)$$

Proposition 7.11. *Cohomology with supports enjoys the following functoriality properties:*

- (i) *If $f : (X, \Phi_X) \rightarrow (Y, \Phi_Y)$ is a pulling morphism of schemes with supports, \mathcal{F}, \mathcal{G} are sheaves of abelian groups on X, Y respectively, and if*

$$\varphi : \mathcal{G} \rightarrow f_* \mathcal{F} \text{ is a morphism of sheaves,} \quad (7.12)$$

then there is a natural morphism $R\underline{\Gamma}_\Phi \mathcal{G} \rightarrow Rf_ R\underline{\Gamma}_\Phi \mathcal{F}$. Similarly if \mathcal{F} and \mathcal{G} are quasicoherent then there are natural morphisms $R\underline{\Gamma}_\Phi \mathcal{G} \rightarrow Rf_* R\underline{\Gamma}_\Phi \mathcal{F}$.*

(ii) If $f : (X, \Phi_X) \rightarrow (Y, \Phi_Y)$ is a pushing morphism, \mathcal{F}, \mathcal{G} are sheaves of abelian groups on X, Y respectively, and

$$\psi : Rf_*\mathcal{F} \rightarrow \mathcal{G} \text{ is a morphism in the derived category of } X, \quad (7.13)$$

then there is a natural morphism $Rf_*R\Gamma_{\Phi}(\mathcal{F}) \rightarrow R\Gamma_{\Phi}\mathcal{G}$.

Let k be a perfect field.

Definition 7.14. A **snc pair with supports** (X, Δ_X, Φ_X) over k is a smooth scheme X over k with a family of supports Φ_X together with a \mathbb{Q} -divisor Δ_X on X such that $\text{supp } \Delta_X$ has simple normal crossings. The **interior** U_X of a snc pair with supports (X, Δ_X, Φ_X) is

$$U_X := X \setminus \text{supp } \Delta_X \quad (7.15)$$

The inclusion of U_X in X is denoted by $\iota_X : U_X \rightarrow X$.

When no confusion is likely to arise we may abbreviate (X, Δ_X, Φ_X) to simply X , and drop subscripts. Here $\text{supp } \Delta_X$ denotes the **support** of Δ_X (if $\Delta_X = \sum_i a_i D_i$ where the D_i are prime divisors, then $\text{supp } \Delta_X = \cup_i D_i$). Similarly let $j_X : \text{supp } \Delta_X \rightarrow X$ denote the evident inclusion.

Observation 7.16. U_X inherits a family of supports from X , namely

$$\Phi_{U_X} := \iota_X^{-1}(\Phi_X) \quad (7.17)$$

Moreover $\iota_X : (U_X, \Phi_{U_X}) \rightarrow (X, \Phi_X)$ is a *pulling* morphism (but generally not a pushing morphism) From now on we will promote the interior of X to the scheme with supports (U_X, Φ_{U_X}) .

Definition 7.18. A **pulling morphism** $f : (X, \Delta_X, \Phi_X) \rightarrow (Y, \Delta_Y, \Phi_Y)$ of **snc pairs with supports** is a pulling morphism $f : X \rightarrow Y$ of underlying schemes with support such that¹⁰ $f^{-1}(\text{supp } \Delta_Y) \subset \text{supp } \Delta_X$. A **pushing morphism** $f : (X, \Delta_X, \Phi_X) \rightarrow (Y, \Delta_Y, \Phi_Y)$ of **snc pairs with supports** is a pushing morphism of underlying schemes with support such that $f^*\Delta_Y = \Delta_X$.

Definition 7.19 (conventions). A morphism of snc pairs with supports $f : (X, \Delta_X, \Phi_X) \rightarrow (Y, \Delta_Y, \Phi_Y)$ is flat, proper, an immersion, etc. if and only if the same is true of the induced morphism $f|_{U_X} : U_X \rightarrow U_Y$. A diagram of snc pairs with supports

$$\begin{array}{ccc} (X', \Delta_{X'}, \Phi_{X'}) & \xrightarrow{g'} & (X, \Delta_X, \Phi_X) \\ f' \downarrow & & \downarrow f \\ (Y', \Delta_{Y'}, \Phi_{Y'}) & \xrightarrow{g} & (Y, \Delta_Y, \Phi_Y) \end{array} \quad (7.20)$$

is **cartesian** if and only if the induced diagram of interiors

$$\begin{array}{ccc} U_{X'} & \xrightarrow{g'} & U_X \\ f' \downarrow & \square & \downarrow f \\ U_{Y'} & \xrightarrow{g} & U_Y \end{array} \quad (7.21)$$

is cartesian.

The terminology is meant to suggest that pushing (resp. pulling) morphisms induce pushforward (resp. pullback) maps on log Hodge cohomology, as we now describe.

Let (X, Δ_X) be a log-smooth pair, let $U_X = X \setminus \Delta_X$ and let $\iota_X : U_X \rightarrow X$ be the inclusion. Let Ω_X^\bullet be the de Rham complex of X and recall that while each term Ω_X^p is a locally free coherent sheaf,

¹⁰In slogan form: “ f maps the interior to the interior.”

Ω_X^\bullet is only a complex of sheaves of k -vector spaces (the differential d is k -linear and satisfies the Leibniz rule

$$d(f\sigma) = df \wedge \sigma + f d\sigma$$

where f and σ are local sections of \mathcal{O}_X and Ω_X^p respectively). The same remarks apply to the de Rham complex $\Omega_{U_X}^\bullet$. Since $\Omega_{U_X}^\bullet$ is a complex on U_X , by functoriality $\iota_{X*}\Omega_{U_X}^\bullet$ is a complex on X and adjunction gives a natural morphism of complexes $d\iota^\vee : \Omega_X^\bullet \rightarrow \iota_{X*}\Omega_{U_X}^\bullet$

Proposition 7.22. *Let \mathcal{F} be a sheaf on a noetherian normal scheme and let D be an effective Cartier divisor on X ; let $U := X \setminus D$. Then there is a natural isomorphism*

$$\operatorname{colim}_{r \rightarrow \infty} \mathcal{F}(rD) \xrightarrow{\cong} \iota_{X*}(\mathcal{F}|_U) \quad (7.23)$$

Proposition 7.22 gives isomorphisms $\iota_{X*}\Omega_{U_X}^p \simeq \operatorname{colim} \Omega_X^p(r \operatorname{supp} \Delta_X)$ and so in particular there are natural morphisms of sheaves $\Omega_X^p(r \operatorname{supp} \Delta_X) \rightarrow \iota_{X*}\Omega_{U_X}^p$, for all p and all $r \geq 0$. At least in the context at hand, where X is smooth and Δ_X has simple normal crossings, these natural maps are injective.

Definition 7.24 (cf. [Del71]). The complex $\Omega_X^\bullet(\log \Delta_X)$ of **differential forms on X with log poles along Δ_X** is the *largest* subcomplex of $\iota_{X*}\Omega_{U_X}^\bullet$ such that

$$\Omega_X^p(\log \Delta_X) \subset \Omega_X^p(\operatorname{supp} \Delta_X) \text{ for all } p$$

More explicitly, on a neighborhood $W \subset X$ a local section $\sigma \in \Omega_X^p(\log \Delta)(W)$ is a section $\sigma \in \iota_{X*}\Omega_{U_X}^p(W)$ such that $\sigma \in \Omega_X^p(\operatorname{supp} \Delta_X)(W)$ and $d\sigma \in \Omega_X^{p+1}(\operatorname{supp} \Delta_X)(W)$ so that less formally but more memorably,

$$\Omega_X^p(\log \Delta) = \{\sigma \in \iota_{X*}\Omega_{U_X}^p \mid \sigma \in \Omega_X^p(\operatorname{supp} \Delta_X) \text{ and } d\sigma \in \Omega_X^{p+1}(\operatorname{supp} \Delta_X)\} \quad (7.25)$$

Let z_1, z_2, \dots, z_n be local coordinates at a point $x \in X$ such that

$$\operatorname{supp} \Delta_X = V(z_1 z_2 \cdots z_r)$$

in a neighborhood of x (the existence of such local coordinates is essentially the *definition* of the simple normal crossing condition given in [Kol13]). Recall that as X is smooth the differentials dz_1, dz_2, \dots, dz_n freely generate Ω_X on a neighborhood of x . In this situation we have the following useful description of $\Omega_X(\log \Delta_X)$:

Lemma 7.26 (see e.g. [EV92]). *The sections $\frac{dz_1}{z_1}, \dots, \frac{dz_r}{z_r}, dz_{r+1}, \dots, dz_n$ freely generate $\Omega_X(\log \Delta_X)$ on a neighborhood of x . For every p the natural map*

$$\wedge^p \Omega_X(\log \Delta_X) \rightarrow \Omega_X^p(\log \Delta_X)$$

is an isomorphism.

Definition 7.27. The **log-Hodge cohomology with supports** of a log-smooth pair with supports (X, Δ_X, Φ_X) is defined by

$$H^d(X, \Delta_X, \Phi_X) = \bigoplus_{p+q=d} H_\Phi^q(X, \Omega_X^p(\log \Delta_X)) \quad (7.28)$$

Here H_Φ^q denotes local cohomology with respect to the family of supports Φ_X . For connected X , we define $H_d(X, \Delta_X, \Phi_X) := H^{2 \dim X - d}(X, \Delta_X, \Phi_X)$, and in general we set $H_d(X, \Delta_X, \Phi_X) = \bigoplus_i H_d(X_i, \Delta_{X_i}, \Phi_{X_i})$ where X_i are the connected components of X .

Let $f : (X, \Delta_X, \Phi_X) \rightarrow (Y, \Delta_Y, \Phi_Y)$ be a pulling morphism of snc pairs with supports.

Lemma 7.29 (cf. [Ogu18, Prop. 2.3.1]). *The map f induces a morphism of complexes of sheaves of k -vector spaces*

$$\begin{aligned} f^*\Omega_Y^\bullet(\log \Delta_Y) &\xrightarrow{df^\vee} \Omega_X^\bullet(\log \Delta_X) \text{ adjoint to a morphism} \\ f^*\Omega_Y^\bullet(\log \Delta_Y) &\xrightarrow{df^\vee} \Omega_X^\bullet(\log \Delta_X) \end{aligned} \quad (7.30)$$

fitting into the following commutative diagram:

$$\begin{array}{ccccc} f_*\iota_{X*}\Omega_{U_X}^\bullet & \longleftarrow & f_*\Omega_X^\bullet(\log \Delta_X) & \longleftarrow & f_*\Omega_X^\bullet \\ df|_U^\vee \uparrow & & \cup & & df^\vee \uparrow & & \cup & & \uparrow df^\vee \\ \iota_{Y*}\Omega_{U_Y}^\bullet & \longleftarrow & \Omega_Y^\bullet(\log \Delta_Y) & \longleftarrow & \Omega_Y^\bullet \end{array} \quad (7.31)$$

of complexes of k -vector spaces on Y .

The essential content of this lemma is that when we pull back a log differential form σ on (Y, Δ_Y) , it doesn't *develop* poles of order ≥ 1 along Δ_X . To see why, it's illuminating to look at the following 2 examples:

Example 7.32. Consider the morphism of pairs $f : (\mathbb{A}_z^1, 0) \rightarrow (\mathbb{A}_z^1, 0)$ defined by $f(z) = z^n$, where $n \in \mathbb{Z}, n \neq 0$. When we pull back $\frac{dz}{z}$, we get

$$\frac{d(f(z))}{f(z)} = \frac{d(z^n)}{z^n} = n \cdot \frac{dz}{z} \quad (7.33)$$

Of course, if $\text{char } k | n$ this is 0, but regardless it has a pole of order ≤ 1 at $0 \in \mathbb{A}^1$.

Example 7.34. Take the pair $(\mathbb{A}_x^2, L_1 + L_2)$, where $L_i = V(x_i)$ for $i = 1, 2$ and blow up the origin to obtain $\text{Bl}_0(\mathbb{A}^2)$; let $\pi : \text{Bl}_0(\mathbb{A}^2) \rightarrow \mathbb{A}^2$ be the projection, let $E \subset \text{Bl}_0(\mathbb{A}^2)$ be the exceptional divisor and let $\tilde{L}_1, \tilde{L}_2 \subset \text{Bl}_0(\mathbb{A}^2)$ be the strict transforms of L_1, L_2 respectively. We obtain a morphism of pairs

$$\pi : (\text{Bl}_0(\mathbb{A}^2), \tilde{L}_1 + \tilde{L}_2 + E) \rightarrow (\mathbb{A}^2, L_1 + L_2) \quad (7.35)$$

Note that with $\tilde{U} := \text{Bl}_0(\mathbb{A}^2) \setminus (\tilde{L}_1 + \tilde{L}_2 + E)$ and $U := \mathbb{A}^2 \setminus (L_1 + L_2)$, we have $\pi(\tilde{U}) \subset U$ (this would *not* hold if we didn't include E in the divisor on $\text{Bl}_0(\mathbb{A}^2)$).

Now let's pull back $\frac{dx_1}{x_1}$: recall that

$$\text{Bl}_0(\mathbb{A}^2) = V(x_1y_2 - x_2y_1) \subset \mathbb{A}_x^2 \times \mathbb{P}_y^1$$

On the $D(y_1) \subset \text{Bl}_0(\mathbb{A}^2)$ affine neighborhood, π looks like

$$\begin{aligned} \mathbb{A}_{x_1, y_2}^2 &\simeq D(y_1) \xrightarrow{\pi} \mathbb{A}_{x_1, x_2}^2 \text{ sending} \\ (x_1, y_2) &\mapsto (x_1, x_1y_2) \end{aligned} \quad (7.36)$$

(note that the exceptional divisor corresponds to $V(x_1) \subset \mathbb{A}_{x_1, y_2}^2$, i.e. the y_2 -axis). So, the pullback of $\frac{dx_1}{x_1}$ is still $\frac{dx_1}{x_1}$, but the pullback of $\frac{dx_2}{x_2}$ is

$$\frac{d(x_1y_2)}{x_1y_2} = \frac{dx_1}{x_1} + \frac{dy_2}{y_2}$$

We see that $d\pi^\vee(\frac{dx_2}{x_2})$ has a pole of order 1 along E .

Proof. Note that since $f(U_X) \subset U_Y$, $U_X \subset f^{-1}(U_Y)$.

Case 1 ($U_X = f^{-1}(U_Y)$): in this case we have a *cartesian* diagram

$$\begin{array}{ccc} U_X & \hookrightarrow & X \\ f|_U \downarrow & \circlearrowleft & \downarrow f \\ U_Y & \hookrightarrow & Y \end{array} \quad (7.37)$$

First, functoriality of the de Rham complex yields morphisms

$$df|_{U_X}^\vee : \Omega_{U_Y}^\bullet \rightarrow f|_{U_X*} \Omega_{U_X}^\bullet \text{ and } df^\vee : \Omega_Y^\bullet \rightarrow f_* \Omega_X^\bullet \quad (7.38)$$

where $df|_{U_X}^\vee$ is the restriction of df^\vee in the sense that applying ι_Y^* to df^\vee and using the isomorphism

$$\iota_Y^* f_* \Omega_X^\bullet \simeq f_{U_X*} \iota_X^* \Omega_X^\bullet = f|_{U_X*} \Omega_{U_X}^\bullet$$

obtained from flat base change¹¹ yields $df|_{U_X}^\vee$. From this we obtain a commutative diagram

$$\begin{array}{ccc} f_* \iota_{X*} \Omega_{U_X}^\bullet & \longleftarrow & f_* \Omega_X^\bullet \\ df|_{U_X}^\vee \uparrow & \circlearrowleft & \uparrow df^\vee \\ \iota_{Y*} \Omega_{U_Y}^\bullet & \longleftarrow & \Omega_Y^\bullet \end{array} \quad (7.39)$$

Finally commutativity of diagram 7.37 provides an isomorphism

$$f_* \iota_{X*} \Omega_{U_X}^\bullet \simeq \iota_{Y*} f|_{U_X*} \Omega_{U_X}^\bullet \quad (7.40)$$

Case 2 ($U_X \subset f^{-1}(U_Y)$): Since $U_X \subset f^{-1}(U_Y)$ we have a natural restriction

$$\iota_{X*} \Omega_{f^{-1}(U_Y)}^\bullet \rightarrow \iota_{X*} \Omega_{U_X}^\bullet$$

In either case, we obtain a commutative diagram of complexes of k -vector spaces as in equation 7.39. Finally we must check that the composition

$$\Omega_Y^p(\log \Delta_Y) \rightarrow \iota_{Y*} \Omega_{U_Y}^p \xrightarrow{df|_{U_X}^\vee} f_* \iota_{X*} \Omega_{U_X}^p \quad (7.41)$$

(where the second map $df|_{U_X}^\vee$ is taken from diagram 7.39) factors through $f_* \Omega_X^p(\log \Delta_X) \subset f_* \iota_{X*} \Omega_{U_X}^p$. This is a local calculation: say $x \in X$ is a closed point and let $y = f(x) \in Y$. From lemma 7.29, if z_1, \dots, z_n are local coordinates at y so that $\Delta_Y = V(z_1 \cdot z_2 \cdots z_r)$ in a neighborhood of y , then the local sections

$$\frac{dz_1}{z_1}, \dots, \frac{dz_r}{z_r}, dz_{r+1}, \dots, dz_n \text{ freely generate } \Omega_Y^1(\log \Delta_Y) \text{ at } y.$$

From the same lemma, we know the natural maps

$$\bigwedge^p \Omega_Y^1(\log \Delta_Y) \xrightarrow{\sim} \Omega_Y^p(\log \Delta_Y) \text{ and } \bigwedge^p \Omega_X^1(\log \Delta_X) \xrightarrow{\sim} \Omega_X^p(\log \Delta_X) \quad (7.42)$$

are isomorphisms, and in this way we reduce to showing:

$$\text{For } i = 1, \dots, r, \text{ the local section } df|_U^\vee \left(\frac{dz_i}{z_i} \right) \text{ factors through } \Omega_X^1(\log \Delta_X) \quad (7.43)$$

Getting even more explicit, say $\tilde{z}_1, \dots, \tilde{z}_m$ are local coordinates at x such that $\Delta_X = V(\tilde{z}_1 \cdot \tilde{z}_2 \cdots \tilde{z}_q)$ in a neighborhood of x .

¹¹Here is where we use the fact that diagram 7.37 is cartesian and ι_X is flat (it's an open immersion)

Claim 7.44.

$$f^*(z_i) (= z_i \circ f) = u \tilde{z}_1^{a_1} \cdot \tilde{z}_2^{a_2} \cdots \tilde{z}_q^{a_q} \quad (7.45)$$

where u is nowhere-vanishing on a neighborhood of x and the a_i are non-negative integers to be described below. *Given* claim 7.44, we obtain the following calculation:

$$df|_U^V \frac{dz_i}{z_i} = \frac{df^* z_i}{f^* z_i} = \frac{d(u \tilde{z}_1^{v_1} \cdots \tilde{z}_q^{v_q})}{(u \tilde{z}_1^{v_1} \cdots \tilde{z}_q^{v_q})} = \frac{du}{u} + \sum_{i=1}^q v_i \frac{d\tilde{z}_i}{\tilde{z}_i} \quad (7.46)$$

Since u is nowhere-vanishing at x , the first term $\frac{du}{u}$ has no poles near x , and appealing once more to lemma 7.26 we have verified equation 7.43.

Proof of (7.45). By hypothesis,

$$\text{supp } f^{-1}(\Delta_Y) \subset \text{supp } \Delta_X, \text{ so locally } \text{supp } f^{-1}(V(\prod_{i=1}^r z_i)) \subset \text{supp } V(\prod_{i=1}^q \tilde{z}_i)$$

Since $V(z_i) \subset \Delta_Y$, it must be that

$$\text{supp } V(z_i \circ f) = \text{supp } f^{-1}(V(z_i)) \subset \text{supp } f^{-1}(V(\prod_{i=1}^r z_i)) \subset \text{supp } V(\prod_{i=1}^q \tilde{z}_i) \quad (7.47)$$

So, $V(z_i \circ f)$ is a divisor with support contained in $\text{supp } V(\prod_{i=1}^q \tilde{z}_i)$. For each j , let $\eta_j \in X$ be the generic point of $V(z_j)$, and recall \mathcal{O}_{X, η_j} is a discrete valuation ring; let v_j be its discrete valuation. Now set

$$a_j = v_j(z_i \circ f) \text{ for } ij = 1, 2, \dots, q \quad (7.48)$$

Then by construction, $z_i \circ f$ and $\prod_j^q \tilde{z}_j^{a_j}$ are 2 local sections of \mathcal{O}_X at x with the same associated divisor, so they must differ by a unit, say $u \in \mathcal{O}_{X, x}^\times$. \square

\square

Combining the previous lemma with proposition 7.11 we find:

Proposition 7.49. *For every pulling morphism $f : (X, \Delta_X, \Phi_X) \rightarrow (Y, \Delta_Y, \Phi_Y)$ in PS^* there are natural morphisms*

$$R\Gamma_{\Phi_Y} \Omega_Y^p(\log \Delta_Y) \rightarrow Rf_* R\Gamma_{\Phi_X} \Omega_X^p(\log \Delta_X) \text{ for all } p \quad (7.50)$$

Proof. Combining the morphism $\Omega_Y^p(\log \Delta_Y) \rightarrow f_* \Omega_X^p(\log \Delta_X)$ of (7.30) with the natural map in the derived category $f_* \Omega_X^p(\log \Delta_X) \rightarrow Rf_* \Omega_X^p(\log \Delta_X)$ (coming from the fact that $f_* \Omega_X^p(\log \Delta_X)$ is the bottom non-0 cohomology sheaf of $Rf_* \Omega_X^p(\log \Delta_X)$) gives a functorial morphism $\Omega_Y^p(\log \Delta_Y) \rightarrow Rf_* \Omega_X^p(\log \Delta_X)$. Taking sections with support along Φ_Y we obtain

$$R\Gamma_{\Phi_Y} \Omega_Y^p(\log \Delta_Y) \rightarrow R\Gamma_{\Phi_Y} Rf_* \Omega_X^p(\log \Delta_X)$$

Composing with the natural morphism

$$R\Gamma_{\Phi_Y} Rf_* \Omega_X^p(\log \Delta_X) \rightarrow Rf_* R\Gamma_{\Phi_X} \Omega_X^p(\log \Delta_X)$$

obtained from the inclusion $f^{-1}(\Phi_Y) \subset \Phi_X$ completes the proof. \square

Corollary 7.51. *For each p there are functorial homomorphisms*

$$f^* : H_{\Phi}^q(Y, \Omega_Y^p(\log \Delta_Y)) \rightarrow H_{\Phi}^q(X, \Omega_X^p(\log \Delta_X)) \quad (7.52)$$

and hence (summing over $p + q = d$) functorial homomorphisms

$$f^* : H^d(X, \Delta_X, \Phi_X) \rightarrow H^d(Y, \Delta_Y, \Phi_Y) \quad (7.53)$$

The maps $f_* : H_d(X, \Delta_X, \Phi_X) \rightarrow H_d(Y, \Delta_Y, \Phi_Y)$ induced by a pushing morphism $f : (X, \Delta_X, \Phi_X) \rightarrow (Y, \Delta_Y, \Phi_Y)$ can be obtained from a combination of Nagata compactification and Grothendieck duality, and this is carried out in [BPØ20, §9.5]. Here we merely explain the construction in the case where f itself is proper.

Theorem 7.54 (Grothendieck duality, [R&D], [Con00]). *Let $f : X \rightarrow Y$ be a proper morphism of finite-dimensional noetherian schemes admitting dualizing complexes ω_X^\bullet and ω_Y^\bullet respectively (for example X and Y could be schemes of finite type over k). Then for any object \mathcal{F}^* in the bounded derived category $D_c^b(X)$ of X there is a natural isomorphism*

$$Rf_* R\mathcal{H}om_X(\mathcal{F}^*, \omega_X^\bullet) \simeq R\mathcal{H}om_Y(Rf_* \mathcal{F}^*, \omega_Y^\bullet) \text{ in } D_c^b(Y)$$

Lemma 7.55. *Let $f : (X, \Delta_X) \rightarrow (Y, \Delta_Y)$ be a morphism of equidimensional log-smooth pairs such that the map $X \xrightarrow{f} Y$ of underlying schemes is proper. Then for each p there are natural morphisms of complexes of coherent sheaves*

$$Rf_*(\Omega_X^{\dim X - p}(\log \Delta_X)(f^* \Delta_Y - \Delta_X)) \rightarrow \Omega_Y^{\dim Y - p}(\log \Delta_Y)[\text{codim } f] \quad (7.56)$$

where $\text{codim } f := \dim Y - \dim X$, inducing maps on cohomology

$$f_* : H^q(X, \Omega_X^{\dim X - p}(\log \Delta_X)(-\Delta_X + f^* \Delta_Y)) \rightarrow H^{q+\text{codim } f}(Y, \Omega_Y^{\dim Y - p}(\log \Delta_Y)) \quad (7.57)$$

for all q . Alternatively, reindexing like $p \leftarrow \dim X - p$, we can write these as

$$\begin{aligned} Rf_*(\Omega_X^p(\log \Delta_X)(f^* \Delta_Y - \Delta_X)) &\rightarrow \Omega_Y^{p+\text{codim } f}(\log \Delta_Y)[\text{codim } f] \text{ and} \\ H^q(X, \Omega_X^p(\log \Delta_X)(-\Delta_X + f^* \Delta_Y)) &\rightarrow H^{q+\text{codim } f}(Y, \Omega_Y^{p+\text{codim } f}(\log \Delta_Y)) \end{aligned} \quad (7.58)$$

In the proof, it will be convenient to work with objects of the form $\Omega_X^p(\log \Delta_X)[p]$ in $D(X)$ — this is not at all essential but it makes the indexing as symmetric as possible.

Proof. Since X and Y are smooth, we have

$$\omega_X^\bullet \simeq \omega_X[\dim X] \text{ and } \omega_Y^\bullet \simeq \omega_Y[\dim Y] \quad (7.59)$$

Grothendieck duality for the object $\Omega_X^p(\log \Delta_X)[p]$ in $D(X)$ says that

$$Rf_* R\mathcal{H}om_X(\Omega_X^p(\log \Delta_X)[p], \omega_X[\dim X]) \simeq R\mathcal{H}om_Y(Rf_* \Omega_X^p(\log \Delta_X)[p], \omega_Y[\dim Y]) \quad (7.60)$$

We now make a couple observations. Focusing first on the left hand side of equation 7.60 note that by lemma 7.26

- $\Omega_X^{\dim X}(\log \Delta_X) \simeq \omega_X(\Delta_X)$ and
- The pairing $\Omega_X^p(\log \Delta_X) \otimes \Omega_X^{\dim X - p}(\log \Delta_X) \rightarrow \omega_X(\Delta_X)$ is perfect. Equivalently (twisting by $-\Delta_X$) $\Omega_X^p(\log \Delta_X) \otimes \Omega_X^{\dim X - p}(\log \Delta_X)(-\Delta_X) \rightarrow \omega_X$ is perfect.

In this way we obtain an isomorphism

$$R\mathcal{H}om_X(\Omega_X^p(\log \Delta_X), \omega_X) \xrightarrow{\simeq} \Omega_X^{\dim X - p}(\log \Delta_X)(-\Delta_X) \quad (7.61)$$

and hence introducing shifts on both sides an isomorphism

$$R\mathcal{H}om_X(\Omega_X^p(\log \Delta_X)[p], \omega_X[\dim X]) \xrightarrow{\simeq} \Omega_X^{\dim X - p}(\log \Delta_X)(-\Delta_X)[\dim X - p] \quad (7.62)$$

Turning to the right hand side, note that the differential $f^* \Omega_Y^p(\log \Delta_Y) \rightarrow \Omega_X^p(\log \Delta_X)$ from lemma 7.29 is adjoint to a morphism $\Omega_Y^p(\log \Delta_Y) \rightarrow Rf_* \Omega_X^p(\log \Delta_X)$. Shifting by $[p]$ and applying $R\mathcal{H}om_Y(-, \omega_Y[\dim Y])$ yields a morphism

$$\begin{aligned} R\mathcal{H}om_Y(Rf_* \Omega_X^p(\log \Delta_X)[p], \omega_Y[\dim Y]) &\rightarrow R\mathcal{H}om_Y(\Omega_Y^p(\log \Delta_Y)[p], \omega_Y[\dim Y]) \\ &\simeq \Omega_Y^{\dim Y - p}(\log \Delta_Y)(-\Delta_Y)[\dim Y - p] \end{aligned} \quad (7.63)$$

Putting everything together, we obtain a natural morphism

$$Rf_*(\Omega_X^{\dim X - p}(\log \Delta_X)(-\Delta_X)[\dim X - p]) \rightarrow \Omega_Y^{\dim Y - p}(\log \Delta_Y)(-\Delta_Y)[\dim Y - p] \quad (7.64)$$

Twisting by Δ_Y , applying the projection formula and shifting by $p - \dim X$ gives

$$Rf_*(\Omega_X^{\dim X - p}(\log \Delta_X)(f^*\Delta_Y - \Delta_X)) \rightarrow \Omega_Y^{\dim Y - p}(\log \Delta_Y)[\dim Y - \dim X] = \Omega_Y^{\dim Y - p}(\log \Delta_Y)[\text{codim } f] \quad (7.65)$$

which is (7.56); the remaining statements of the lemma follow from taking global sections and reindexing. \square

Lemma 7.66. *Suppose in addition that $f^*\Delta_Y - \Delta_X$ is effective. Then there is a natural morphism of complexes*

$$Rf_*(\Omega_X^{\dim X - p}(\log \Delta_X)) \rightarrow \Omega_Y^{\dim Y - p}(\log \Delta_Y)[\text{codim } f] \quad (7.67)$$

inducing maps on cohomology

$$f_* : H^q(X, \Omega_X^p(\log \Delta_X)) \rightarrow H^{q+\text{codim } f}(Y, \Omega_Y^{p+\text{codim } f}(\log \Delta_Y)) \quad (7.68)$$

Proof. When $f^*(\Delta_Y) - \Delta_X$ is effective, there's an inclusion

$$\Omega_X^{\dim X - p}(\log \Delta_X)(-\Delta_X + f^*\Delta_Y) \subseteq \Omega_X^{\dim X - p}(\log \Delta_X)$$

\square

The pushforward/pullback morphisms f_*/f^* satisfy a *projection formula*.

Lemma 7.69 (cf. [BPØ20, Lem 9.4.6-7]). *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & \square & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of snc pairs with supports, where f, f' (resp. g, g') are pushing (resp. pulling) morphisms and g is either flat or a closed immersion transverse to f . Then

$$g_* f_* = f'_* g'^* : H^*(X, \Delta_X, \Phi_X) \rightarrow H^*(Y', \Delta_{Y'}, \Phi_{Y'}).$$

Proof. **Under construction.** (follows along the lines of [CR11, Prop. 2.3.7]) \square

Following the approach of [CR11], the next step would be to construct a cycle class $\text{cl}(Z) \in H_{\Phi_X}^*(X, \Omega_X^*(\log \Delta_X))$ for a subvariety $Z \subset X$ with $Z \in \Phi_X$. This is possible, and is carried out in [BPØ20, §9], however it seems that for compatibility with correspondences in the absence of additional finiteness/strictness conditions, a more refined cycle class would be needed. For this reason we turn now to log Hodge correspondences and then return to the issue of cycle classes.

7.2. Correspondences. Given snc pairs with families of supports (X, Δ_X, Φ_X) and (Y, Δ_Y, Φ_Y) with dimensions d_X and d_Y , as in [CR11, §1.3] we may define a family of supports $P(\Phi_X, \Phi_Y)$ on $X \times Y$ by

$$P(\Phi_X, \Phi_Y) := \{\text{closed subsets } Z \subseteq X \times Y \mid \text{pr}_Y|_Z \text{ is proper and for all } W \in \Phi_X, \\ \text{pr}_Y(\text{pr}_X^{-1}(W) \cap Z) \in \Phi_Y\}$$

(the conditions of Definition 7.1 are straightforward to verify). For convenience we will let $\Delta_{X \times Y} := \text{pr}_X^* \Delta_X + \text{pr}_Y^* \Delta_Y$.

Lemma 7.70. *A class $\gamma \in H_{P(\Phi_X, \Phi_Y)}^j(X \times Y, \Omega_{X \times Y}^i(\log \Delta_{X \times Y})(-\text{pr}_X^* \Delta_X))$ defines homomorphisms*

$$\text{cor}(\gamma) : H_{\Phi_X}^q(X, \Omega_X^p(\log \Delta_X)) \rightarrow H_{\Phi_Y}^{q+j-d_X}(Y, \Omega_Y^{p+i-d_X}(\log \Delta_Y))$$

by the formula $\text{cor}(\gamma)(\alpha) := \text{pr}_{Y}(\text{pr}_X^*(\alpha) \smile \gamma)$. Moreover if (Z, Δ_Z, Φ_Z) is another snc pair with supports and $\delta \in H_{P(\Phi_Y, \Phi_Z)}^{j'}(Y \times Z, \Omega_{Y \times Z}^{i'}(\log \Delta_{Y \times Z})(-\text{pr}_Y^* \Delta_Y))$, then*

$$\text{pr}_{X \times Z*}(\text{pr}_{X \times Y}^*(\gamma) \smile \text{pr}_{Y \times Z}^*(\delta)) \in H_{P(\Phi_X, \Phi_Z)}^{j+j'-d_Y}(X \times Z, \Omega_{X \times Z}^{i+i'-d_Y}(\log \Delta_{X \times Z})(-\text{pr}_X^* \Delta_X)) \text{ and}$$

$$\text{cor}(\text{pr}_{X \times Z*}(\text{pr}_{X \times Y}^*(\gamma) \smile \text{pr}_{Y \times Z}^*(\delta))) = \text{cor}(\delta) \circ \text{cor}(\gamma)$$

as homomorphisms $H_{\Phi_X}^q(X, \Omega_X^p(\log \Delta_X)) \rightarrow H_{\Phi_Z}^{q+j+j'-d_X-d_Y}(Z, \Omega_Z^{p+i+i'-d_X-d_Y}(\log \Delta_Z))$.

Such correspondences involving both log poles and “log zeroes” appear to have been considered before at least in the setting of crystalline cohomology, for example in work of Mieda [Mie09a; Mie09b]. However, I was unable to find any published proof of this fact in the literature (Mieda sketches the construction, but not in the level of detail presented below). If you or anyone you know is aware of previous work related [Lemma 7.70](#), please email me!

Proof. We make two observations: first, there are natural wedge product pairings¹²

$$\Omega_{X \times Y}^p(\log \Delta_{X \times Y}) \otimes \Omega_{X \times Y}^i(\log \Delta_{X \times Y})(-\text{pr}_X^* \Delta_X) \xrightarrow{\wedge} \Omega_{X \times Y}^{p+i}(\log \Delta_Y)$$

Second, essentially by the definition of $P(\Phi_X, \Phi_Y)$ the Künneth morphism on cohomology for the tensor product $\Omega_{X \times Y}^p(\log \Delta_{X \times Y}) \otimes \Omega_{X \times Y}^i(\log \Delta_{X \times Y})(-\text{pr}_X^* \Delta_X)$ can be enhanced with supports as

$$\begin{aligned} H_{\text{pr}_X^{-1}(\Phi_X)}^q(X \times Y, \Omega_{X \times Y}^p(\log \Delta_{X \times Y})) \otimes H_{P(\Phi_X, \Phi_Y)}^j(X \times Y, \Omega_{X \times Y}^i(\log \Delta_{X \times Y})(-\text{pr}_X^* \Delta_X)) \\ \rightarrow H_{\Psi}^{p+j}(X \times Y, \Omega_{X \times Y}^p(\log \Delta_{X \times Y}) \otimes \Omega_{X \times Y}^i(\log \Delta_{X \times Y})(-\text{pr}_X^* \Delta_X)) \end{aligned}$$

where $\Psi := \{\text{closed subsets } Z \in X \times Y \mid \text{pr}_Y|_Z \text{ is proper and } \text{pr}_Y(Z) \in \Phi_Z\}$. Combining these 2 observations gives a pairing

$$\begin{aligned} H_{\text{pr}_X^{-1}(\Phi_X)}^q(X \times Y, \Omega_{X \times Y}^p(\log \Delta_{X \times Y})) \otimes H_{P(\Phi_X, \Phi_Y)}^j(X \times Y, \Omega_{X \times Y}^i(\log \Delta_{X \times Y})(-\text{pr}_X^* \Delta_X)) \\ \xrightarrow{\sim} H_{\Psi}^{p+j}(X \times Y, \Omega_{X \times Y}^{p+i}(\log \Delta_Y)) \end{aligned}$$

Now note that $\text{pr}_X : (X \times Y, \Delta_{X \times Y}, \text{pr}_X^{-1}(\Phi_X)) \rightarrow (X, \Delta_X, \Phi_X)$ is a pulling morphism, so by [Corollary 7.51](#) there is an induced map $\text{pr}_X^* : H_{\Phi_X}^q(X, \Omega_X^p(\log \Delta_X)) \rightarrow H_{\text{pr}_X^{-1}(\Phi_X)}^q(X \times Y, \Omega_{X \times Y}^p(\log \Delta_{X \times Y}))$. On the other hand since $\text{pr}_Y : (X \times Y, \Delta_Y, \Psi) \rightarrow (Y, \Delta_Y, \Phi_Y)$ is a pushing morphism, [Lemma 7.66](#) provides a morphism $\text{pr}_{Y*} : H_{\Psi}^{p+j}(X \times Y, \Omega_{X \times Y}^{p+i}(\log \Delta_Y)) \rightarrow H_{\Phi_Y}^{q+j-d_X}(Y, \Omega_Y^{p+i-d_X}(\log \Delta_Y))$. Composing, we obtain the desired homomorphism

$$\begin{aligned} H_{\Phi_X}^q(X, \Omega_X^p(\log \Delta_X)) &\xrightarrow{\text{pr}_X^*} H_{\text{pr}_X^{-1}(\Phi_X)}^q(X \times Y, \Omega_{X \times Y}^p(\log \Delta_{X \times Y})) \\ &\xrightarrow{\sim} H_{\Psi}^{p+j}(X \times Y, \Omega_{X \times Y}^{p+i}(\log \Delta_Y)) \\ &\xrightarrow{\text{pr}_{Y*}} H_{\Phi_Y}^{q+j-d_X}(Y, \Omega_Y^{p+i-d_X}(\log \Delta_Y)) \end{aligned}$$

¹²This is perhaps easiest to see by a verification in local coordinates.

For the “moreover” half of the lemma, we again begin with a certain wedge product pairing, this time on $X \times Y \times Z$:

$$\begin{aligned} & \Omega_{X \times Y \times Z}^i(\log \text{pr}_{X \times Y}^* \Delta_{X \times Y})(-\text{pr}_X^* \Delta_X) \otimes \Omega_{X \times Y \times Z}^{i'}(\log \text{pr}_{Y \times Z}^* \Delta_{Y \times Z})(-\text{pr}_Y^* \Delta_Y) \\ & \xrightarrow{\wedge} \Omega_{X \times Y \times Z}^{i+i'}(\log \text{pr}_{X \times Z}^* \Delta_{X \times Z})(-\text{pr}_X^* \Delta_X) \end{aligned} \quad (7.71)$$

If $V \in P(\Phi_X, \Phi_Y)$, $W \in P(\Phi_Y, \Phi_Z)$ then unravelling definitions we find:

- $\text{pr}_{X \times Z}|_{\text{pr}_{X \times Y}^{-1}(V) \cap \text{pr}_{Y \times Z}^{-1}(W)}$ is proper and
- $\text{pr}_{X \times Z}(\text{pr}_{X \times Y}^{-1}(V) \cap \text{pr}_{Y \times Z}^{-1}(W)) \in P(\Phi_X, \Phi_Z)$

so that the Künneth morphism on cohomology associated to the middle term of (7.71) can be enhanced with supports like

$$\begin{aligned} & H_{\text{pr}_{X \times Y}^{-1}(P(\Phi_X, \Phi_Y))}^j(X \times Y \times Z, \Omega_{X \times Y \times Z}^i(\log \text{pr}_{X \times Y}^* \Delta_{X \times Y})(-\text{pr}_X^* \Delta_X)) \\ & \otimes H_{\text{pr}_{Y \times Z}^{-1}(P(\Phi_Y, \Phi_Z))}^{j'}(X \times Y \times Z, \Omega_{X \times Y \times Z}^{i'}(\log \text{pr}_{Y \times Z}^* \Delta_{Y \times Z})(-\text{pr}_Y^* \Delta_Y)) \\ & \rightarrow H_{\Sigma}^{j+j'}(X \times Y \times Z, \Omega_{X \times Y \times Z}^i(\log \text{pr}_{X \times Y}^* \Delta_{X \times Y})(-\text{pr}_X^* \Delta_X) \otimes \Omega_{X \times Y \times Z}^{i'}(\log \text{pr}_{Y \times Z}^* \Delta_{Y \times Z})(-\text{pr}_Y^* \Delta_Y)) \end{aligned}$$

where $\Sigma := \{\text{closed sets } W \subseteq X \times Y \times Z \mid \text{pr}_{X \times Z}|_W \text{ is proper and } \text{pr}_{X \times Z}(W) \in P(\Phi_X, \Phi_Z)\}$.

Since $\text{pr}_{X \times Y} : (X \times Y \times Z, \text{pr}_{X \times Y}^* \Delta_{X \times Y}, \text{pr}_{X \times Y}^{-1}(P(\Phi_X, \Phi_Y))) \rightarrow (X \times Y, \Delta_{X \times Y}, P(\Phi_X, \Phi_Y))$ is a pulling morphism, [Corollary 7.51](#) gives an induced morphism $\Omega_{X \times Y}^i(\log \Delta_{X \times Y}) \rightarrow Rf_* \Omega_{X \times Y \times Z}^i(\log \text{pr}_{X \times Y}^* \Delta_{X \times Y})$; twisting by $-\Delta_{X \times Y}$ and applying the projection formula gives a morphism

$$\Omega_{X \times Y}^i(\log \Delta_{X \times Y})(-\Delta_{X \times Y}) \rightarrow Rf_*(\Omega_{X \times Y \times Z}^i(\log \text{pr}_{X \times Y}^* \Delta_{X \times Y})(-\text{pr}_{X \times Y}^* \Delta_{X \times Y}))$$

and then taking cohomology with supports along $P(\Phi_X, \Phi_Y)$ and using [Proposition 7.11](#) gives a modified pullback map

$$H_{P(\Phi_X, \Phi_Y)}^j(X \times Y, \Omega_{X \times Y}^i(\log \Delta_{X \times Y})(-\Delta_{X \times Y})) \rightarrow H_{\text{pr}_{X \times Y}^{-1}(P(\Phi_X, \Phi_Y))}^j(X \times Y \times Z, \Omega_{X \times Y \times Z}^i(\log \text{pr}_{X \times Y}^* \Delta_{X \times Y})(-\text{pr}_X^* \Delta_X))$$

and a similar argument gives a modified pullback

$$H_{P(\Phi_Y, \Phi_Z)}^{j'}(Y \times Z, \Omega_{Y \times Z}^{i'}(\log \Delta_{Y \times Z})(-\Delta_{Y \times Z})) \rightarrow H_{\text{pr}_{Y \times Z}^{-1}(P(\Phi_Y, \Phi_Z))}^{j'}(X \times Y \times Z, \Omega_{X \times Y \times Z}^{i'}(\log \text{pr}_{Y \times Z}^* \Delta_{Y \times Z})(-\text{pr}_Y^* \Delta_Y))$$

On the other hand, $\text{pr}_{X \times Z} : (X \times Y \times Z, \text{pr}_{X \times Z}^* \Delta_{X \times Z}, \Sigma) \rightarrow (X \times Z, \Delta_{X \times Z}, P(\Phi_X, \Phi_Z))$ is a pushing morphism and hence by [Lemma 7.66](#) induces morphisms

$$R\text{pr}_{X \times Z*} R\Gamma_{-\Sigma}(\Omega_{X \times Y \times Z}^{\dim X \times Y \times Z - k}(\log \text{pr}_{X \times Z}^* \Delta_{X \times Z})) \rightarrow R\Gamma_{-P(\Phi_X, \Phi_Z)}^{\dim X \times Z - k}(\log \Delta_{X \times Z})[-\dim Z]$$

for all k ; twisting by $-\text{pr}_X^* \Delta_X$ and applying the projection formula this becomes

$$R\text{pr}_{X \times Z*} R\Gamma_{-\Sigma}(\Omega_{X \times Y \times Z}^{\dim X \times Y \times Z - k}(\log \text{pr}_{X \times Z}^* \Delta_{X \times Z})(-\text{pr}_X^* \Delta_X)) \rightarrow R\Gamma_{-P(\Phi_X, \Phi_Z)}^{\dim X \times Z - k}(\log \Delta_{X \times Z})(-\text{pr}_X^* \Delta_X)[- \dim Z]$$

Now letting $k = \dim X \times Y \times Z - i - i'$, the induced morphisms of cohomology with supports are

$$H_{\Sigma}^{j+j'}(X \times Y \times Z, \Omega_{X \times Y \times Z}^{i+i'}(\log \text{pr}_{X \times Z}^* \Delta_{X \times Z})(-\text{pr}_X^* \Delta_X)) \rightarrow H_{P(\Phi_X, \Phi_Z)}^{j+j' - \dim Z}(X \times Z, \Omega_{X \times Z}^{i+i' - \dim Z}(\log \Delta_{X \times Z})(-\text{pr}_X^* \Delta_X))$$

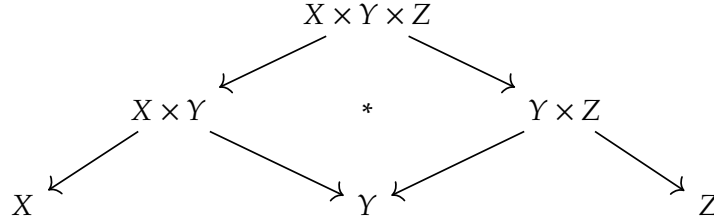
Combining the above ingredients, we obtain a bilinear pairing

$$\begin{aligned} & H_{P(\Phi_X, \Phi_Y)}^j(X \times Y, \Omega_{X \times Y}^i(\log \Delta_{X \times Y})(-\Delta_{X \times Y})) \otimes H_{P(\Phi_Y, \Phi_Z)}^{j'}(Y \times Z, \Omega_{Y \times Z}^{i'}(\log \Delta_{Y \times Z})(-\Delta_{Y \times Z})) \\ & \rightarrow H_{P(\Phi_X, \Phi_Z)}^{j+j' - \dim Z}(X \times Z, \Omega_{X \times Z}^{i+i' - \dim Z}(\log \Delta_{X \times Z})(-\text{pr}_X^* \Delta_X)) \end{aligned}$$

sending $\gamma \otimes \delta \mapsto \text{pr}_{X \times Z*}(\text{pr}_{X \times Y}^*(\gamma) \smile \text{pr}_{Y \times Z}^*(\delta))$. It remains to be seen that

$$\text{cor}(\text{pr}_{X \times Z*}(\text{pr}_{X \times Y}^*(\gamma) \smile \text{pr}_{Y \times Z}^*(\delta))) = \text{cor}(\delta) \circ \text{cor}(\gamma)$$

and for this we will make repeated use of [Lemma 7.69](#). Consider the diagram of smooth schemes



where all morphisms are projections. There are various ways to enhance this to include supports; here we add the family of supports Ψ on $X \times Y$ defined above. Then in the cartesian diagram $(*)$, $\text{pr}_Y : (X \times Y, \Psi) \rightarrow (Y, \Phi_Y)$ and $\text{pr}_{Y \times Z} : (X \times Y \times Z, \text{pr}_{X \times Y}^{-1} \Psi) \rightarrow (Y \times Z, \text{pr}_Y^{-1} \Phi_Y)$ are pushing morphisms, whereas $\text{pr}_{X \times Y}$ and pr_Y are pulling morphisms. At the same time, we have a pulling morphism $\text{pr}_{X \times Z} : (X \times Y \times Z, \text{pr}_{X \times Z}^{-1}(P(\Phi_Y, \Phi_Z))) \rightarrow (Y \times Z, P(\Phi_Y, \Phi_Z))$. To be precise in what follows, whenever ambiguity is possible we will use notation like $\text{pr}_X^{X \times Y}$ to denote the projection $X \times Y \rightarrow X$, $\text{pr}_X^{X \times Y \times Z}$ to denote the projection $X \times Y \times Z \rightarrow X$ and so on.

Applying the projection formula first to $\text{pr}_{X \times Z}$ we see that

$$\text{pr}_{Y \times Z*}(\text{pr}_{X \times Y}^*(\text{pr}_X^{X \times Y*} \alpha - \gamma) - \text{pr}_{Y \times Z}^* \delta) = \text{pr}_{Y \times Z*}(\text{pr}_{X \times Y}^*(\text{pr}_X^{X \times Y*} \alpha - \gamma)) - \delta$$

and then applying the projection formula to $(*)$ shows

$$\text{pr}_{Y \times Z*}(\text{pr}_{X \times Y}^*(\text{pr}_X^{X \times Y*} \alpha - \gamma)) = \text{pr}_Y^{Y \times Z*}(\text{pr}_{Y*}^{X \times Y}(\text{pr}_X^{X \times Y*} \alpha - \gamma)) = \text{pr}_Y^{Y \times Z*} \text{cor}(\gamma)(\alpha)$$

so that

$$\text{pr}_{Y \times Z*}(\text{pr}_{X \times Y}^*(\text{pr}_X^{X \times Y*} \alpha - \gamma) - \text{pr}_{Y \times Z}^* \delta) = \text{pr}_Y^{Y \times Z*} \text{cor}(\gamma)(\alpha) - \delta$$

Applying $\text{pr}_{Z*}^{Y \times Z}$ we conclude that

$$(\text{cor } \delta \circ \text{cor } \gamma)(\alpha) = \text{pr}_{Z*}^{X \times Y \times Z}(\text{pr}_X^{X \times Y \times Z*} \alpha - \text{pr}_{X \times Y}^* \gamma - \text{pr}_{Y \times Z}^* \delta) \quad (7.72)$$

Finally, we rewrite the right hand side as

$$\text{pr}_{Z*}^{X \times Z} \text{pr}_{X \times Z*}(\text{pr}_{X \times Z}^* \text{pr}_X^{X \times Z*} \alpha - \text{pr}_{X \times Y}^* \gamma - \text{pr}_{Y \times Z}^* \delta)$$

and apply the projection formula to $\text{pr}_{X \times Z}$ (with the pushing morphism $(X \times Y \times Z, \Sigma) \rightarrow (X \times Z, P(\Phi_X, \Phi_Z))$ and pulling morphism $(X \times Y \times Z, \text{pr}_X^{X \times Y \times Z-1}(\Phi_X)) \rightarrow (X \times Z, \text{pr}_X^{X \times Z-1}(\Phi_X))$) to arrive at

$$\text{pr}_{X \times Z*}(\text{pr}_{X \times Z}^* \text{pr}_X^{X \times Z*} \alpha - \text{pr}_{X \times Y}^* \gamma - \text{pr}_{Y \times Z}^* \delta) = \text{pr}_X^{X \times Z*} \alpha - \text{pr}_{X \times Z*}(\text{pr}_{X \times Y}^* \gamma - \text{pr}_{Y \times Z}^* \delta)$$

Applying $\text{pr}_{Z*}^{X \times Z}$ on both sides shows that the right hand side of (7.72) is $\text{cor}(\text{pr}_{X \times Z*}(\text{pr}_{X \times Y}^* \gamma - \text{pr}_{Y \times Z}^* \delta))(\alpha)$, as desired. \square

Remark 7.73. There is a Grothendieck-Serre dual approach to such correspondences, where classes $\gamma \in H_{P(\Phi_X, \Phi_Y)}^j(X \times Y, \Omega_{X \times Y}^i(\log \Delta_{X \times Y})(-\text{pr}_Y^* \Delta_Y))$ define homomorphisms

$$H^q(X, \Omega_X^p(\log \Delta_X)(-\Delta_X)) \rightarrow H^{q+j-d_X}(Y, \Omega_Y^{p+i-d_X}(\log \Delta_Y)(-\Delta_Y)).$$

The construction is formally similar.

7.3. Attempts to construct a fundamental class of a thrifty birational equivalence. Let $(X, \Delta_X), (Y, \Delta_Y)$ be simple normal crossing pairs, and assume in addition that X, Y are connected and proper. Let $Z \subseteq X \times Y$ be a smooth closed subvariety with codimension c . In this situation the fundamental class of $\text{cl}(Z) \in H^c(X \times Y, \Omega_{X \times Y}^c)$ (no log poles yet) can be described using only Serre duality, as follows: the composition

$$H^{\dim Z}(X \times Y, \Omega_{X \times Y}^{\dim Z}) \rightarrow H^{\dim Z}(Z, \Omega_Z^{\dim Z}) \xrightarrow{\text{tr}} k \quad (7.74)$$

(where tr is the trace map of Serre duality) is an element of

$$H^{\dim Z}(X \times Y, \Omega_{X \times Y}^{\dim Z})^\vee \simeq H^c(X \times Y, \Omega_{X \times Y}^c) \quad (7.75)$$

which we may *define* to be $\text{cl}(Z)$.¹³ In light of [Lemma 7.70](#) one might hope to modify [eqs. \(7.74\)](#) and [\(7.75\)](#) to obtain a class in $H^c(X \times Y, \Omega_{X \times Y}^c(\log \Delta_{X \times Y})(-\text{pr}_X^* \Delta_X))$. Let us focus on the case where

- $\text{pr}_X|_Z : Z \rightarrow X, \text{pr}_Y|_Z : Z \rightarrow Y$ are both thrifty and birational, so in particular $c = \dim X = \dim Y =: d$ and
- $(\text{pr}_X|_Z)_*^{-1} \Delta_X = (\text{pr}_Y|_Z)_*^{-1} \Delta_Y =: \Delta_Z$

To keep the notation under control, set $\pi_X := \text{pr}_X|_Z$ and $\pi_Y := \text{pr}_Y|_Z$.

In this situation letting $\iota : Z \rightarrow X \times Y$ be the inclusion there is a natural map

$$d\iota^\vee : \Omega_{X \times Y}^d(\log \Delta_{X \times Y}) \rightarrow \iota_* \Omega_Z^d(\log \Delta_{X \times Y}|_Z) \text{ and twisting by } -\text{pr}_Y^* \Delta_Y \text{ gives a map}$$

$$\Omega_{X \times Y}^d(\log \Delta_{X \times Y})(-\text{pr}_Y^* \Delta_Y) \rightarrow \iota_* \Omega_Z^d(\log \Delta_{X \times Y}|_Z)(-\text{pr}_Y^* \Delta_Y|_Z) = \iota_* \Omega_Z^d(\log \Delta_{X \times Y}|_Z)(-\pi_Y^* \Delta_Y)$$

To identify $\Omega_Z^d(\log \Delta_{X \times Y}|_Z)(-\pi_X^* \Delta_X|_Z)$, write

$$(\pi_X)^* \Delta_X = (\pi_X)_*^{-1} \Delta_X + E_X = \Delta_Z + E_X \text{ and}$$

$$(\pi_Y)^* \Delta_Y = (\pi_Y)_*^{-1} \Delta_Y + E_Y = \Delta_Z + E_Y$$

so that $\Delta_{X \times Y}|_Z = (\pi_X)^* \Delta_X + (\pi_Y)^* \Delta_Y = 2\Delta_Z + E_X + E_Y$. While the hypotheses guarantee Δ_Z is reduced it may be that E_X, E_Y are non-reduced — however something can be said about their multiplicities. If $E_X = \sum_i a_X^i E_X^i, E_Y = \sum_i a_Y^i E_Y^i$ where the E_X^i, E_Y^i are irreducible, then by a generalization of [\[Har77, Prop. 3.6\]](#),

$$a_X^i = \text{mlt}(\pi_X(E_X^i) \subseteq \Delta_X)$$

and since Δ_X is a reduced effective simple normal crossing divisor, if in addition we write $\Delta_X = \sum_i D_X^i$ $\text{mlt}(\pi_X(E_X^i) \subseteq \Delta_X) = |\{i \mid \pi_X(E_X^i) \subseteq D_X^i\}|$ (see [\[Kol13, §2.10\]](#)). The thriftiness hypothesis that $\pi_X(E_X^i)$ is not a stratum then implies $a_X^i = \text{mlt}(\pi_X(E_X^i) \subseteq \Delta_X) < \text{codim}(\pi_X(E_X^i) \subset X)$. Since differentials with log poles are insensitive to multiplicities, we have

$$\Omega_Z^d(\log \Delta_{X \times Y}|_Z) = \omega_Z(\Delta_Z + E_X^{\text{red}} + E_Y^{\text{red}})$$

where $-^{\text{red}}$ denotes the associated reduced effective divisor. Then

$$\Omega_Z^d(\log \Delta_{X \times Y}|_Z)(-\pi_Y^* \Delta_Y) = \omega_Z(\Delta_Z + E_X^{\text{red}} + E_Y^{\text{red}} - \Delta_Z - E_Y)$$

$$\omega_Z(E_X^{\text{red}} + (E_Y^{\text{red}} - E_Y)) = \omega_Z\left(\sum_i E_X^i + \sum_i (1 - a_Y^i) E_Y^i\right)$$

The upshot is that we have an induced map

$$H^d(X \times Y, \Omega_{X \times Y}^d(\log \Delta_{X \times Y})(-\text{pr}_Y^* \Delta_Y)) \rightarrow H^d(Z, \omega_Z(E_X^{\text{red}} + (E_Y^{\text{red}} - E_Y))) \quad (7.76)$$

¹³It may then be non-trivial to verify this agrees with other definitions, especially if one cares about signs, but we will not need that level of detail for what follows.

Here the left hand side is Serre dual to $H^d(X \times Y, \Omega_{X \times Y}^d(\log \Delta_{X \times Y})(-\text{pr}_X^* \Delta_X))$, so the k -linear dual of (7.76) is a morphism

$$H^d(Z, \omega_Z(E_X^{\text{red}} + (E_Y^{\text{red}} - E_Y)))^\vee \rightarrow H^d(X \times Y, \Omega_{X \times Y}^d(\log \Delta_{X \times Y})(-\text{pr}_X^* \Delta_X))$$

Unfortunately¹⁴ $H^d(Z, \omega_Z(E_X^{\text{red}} + (E_Y^{\text{red}} - E_Y)))$ is often 0. If E_X and E_Y are both reduced (an explicit example where this holds will be given below), then $H^d(Z, \omega_Z(E_X^{\text{red}} + (E_Y^{\text{red}} - E_Y))) = H^d(Z, \omega_Z(E_X))$. If in addition $E_X \neq 0$, we obtain $H^d(Z, \omega_Z(E_X)) = 0$ by an extremely weak (but characteristic independent) sort of Kodaira vanishing:

Lemma 7.77. *Let Z be a proper variety over a field k with dimension d , and assume Z is normal and Cohen-Macaulay. If $D \subset Z$ is a non-0 effective Cartier divisor on Z then $H^d(Z, \omega_Z(D)) = 0$.*

Proof. By Serre duality $H^d(Z, \omega_Z(D)) = H^0(Z, \mathcal{O}_Z(-D))$, which vanishes by the classic fact that “a nontrivial line bundle and its inverse can’t both have non-0 global sections.” Since I am not aware of a reference, here is a proof:

Suppose towards contradiction that there is a non-0 global section $\sigma \in H^0(Z, \mathcal{O}_Z(-D))$ — then the composition

$$\begin{array}{ccccc} \mathcal{O}_Z & \xrightarrow{\sigma} & \mathcal{O}_Z(-D) & \hookrightarrow & \mathcal{O}_Z \\ & & \searrow \tau & & \nearrow \end{array}$$

is non-0. By [Stacks, Tag 0358] $H^0(Z, \mathcal{O}_Z)$ is a (normal) domain, and since it’s also a finite dimensional k -vector space it must be an extension field of k . But then $\tau \in H^0(Z, \mathcal{O}_Z)$ is invertible hence surjective, so $\mathcal{O}_Z(-D) \hookrightarrow \mathcal{O}_Z$ is surjective, which is a contradiction since by hypothesis the cokernel $\mathcal{O}_D \neq 0$. \square

Example 7.78. Let $X = \mathbb{P}^2$ and let $\Delta_X \subset X$ be a line. Let $p \in L$ be a k -point, let $Y = \text{Bl}_p X$ and let $\Delta_Y = \tilde{L}$ = the strict transform of L . Finally let $f : Y \rightarrow X$ be the blowup map and let $Z = (f \times \text{id})(Y) \subset X \times Y$. In this case (with all notation as above) $\pi_X \circ (f \times \text{id}) = f$ and $\pi_Y \circ (f \times \text{id}) = \text{id}_Y$, so under the isomorphism $f \times \text{id} : Y \simeq Z$, E_X is the exceptional divisor of f (with multiplicity 1). On the other hand $E_Y = 0$. In particular E_X and E_Y are reduced and $E_X \neq 0$ so from the above discussion $H^2(Z, \omega_Z(E_X)) = 0$.

Part 2. Globally F -full varieties

8. INTRODUCTION TO PART 2

We begin by considering a theorem of Du Bois-Jarraud.

Theorem 8.1 ([Du 81, Thm. 4.6], see also [DJ74]). *If $f : X \rightarrow B$ is a flat proper morphism of schemes of finite type over \mathbb{C} , and if the geometric fibers of f are reduced with at worst Du Bois singularities, then the higher direct images of the structure sheaf $R^i f_* \mathcal{O}_X$ are locally free and compatible with arbitrary base change.*

All known characterizations of Du Bois singularities are somewhat technical, so rather than giving definitions we summarize a few of their properties. For applications of Theorem 8.1, the important facts are that both normal crossing and semi-log canonical schemes of finite type over \mathbb{C} have Du Bois singularities (for proofs as well as the necessary definitions see [Kol13, §6] – the log canonical case is [KK10, Thm. 1.4]). On the other hand, the proof hinges on the fact that if X is a proper \mathbb{C} -scheme with Du Bois singularities then the natural maps

$$H^i(X, \mathbb{C}) \rightarrow H^i(X, \mathcal{O}_X)$$

are surjective for all i (when X is smooth this is an immediate consequence of degeneration of the Hodge-to-de-Rham spectral sequence). Theorem 8.1 has found various striking applications: for example, in [KK10, Thm. 1.8] it is used to show that for a family as above, the cohomology

¹⁴at least for the purposes of constructing log Hodge cohomology classes of subvarieties ...

sheaves $h^i(\omega_f^\bullet)$ (including the relative dualizing sheaf $\omega_{X/B}$) are flat over B and compatible with base change. In a different direction, it was noticed by Kollár that [Theorem 8.1](#) combined with a hypothetical strong form of semi-stable reduction would recover one of his theorems on higher direct images of dualizing sheaves [[Kol86](#), Thm. 2.6 Rmk. 2.7].

As mentioned above, the proof of [Theorem 8.1](#) makes essential use of Hodge theory, and moreover it is currently unknown how to even define Du Bois singularities away from characteristic 0. Below we present a variant, [Proposition 9.6](#), which differs in several aspects: first, it applies exclusively to flat proper families in characteristic $p > 0$, and second, it replaces local singularity conditions on the closed fibers with global arithmetic restrictions which we call *global F -fullness* ([Definition 9.2](#)). Finally, while in characteristic 0 semi-stable reduction shows that if $f_0 : X_0 \rightarrow C \setminus \{0\}$ is a smooth morphism from a variety to a punctured smooth curve then after a finite base change $\pi : C' \rightarrow C$ such that $\pi^{-1}(0) = 0'$ is a single closed point, the family $g_0 : X'_0 \rightarrow C' \setminus \{0'\}$ obtained from f_0 by base change can be completed to a family $g : X' \rightarrow C'$ such that the fiber $g^{-1}(0)$ has at worst normal crossing singularities, hence in particular Du Bois singularities. Based on a few examples (e.g. [Example 10.24](#)) it seems unlikely that a similar semi-stable reduction statement holds for global F -fullness.¹⁵

[Section 9](#) begins with a recollection of relative Frobenii to set up the definition of globally F -full ([Definition 9.2](#)) and state our main result, [Proposition 9.6](#). This is proved in [Section 10](#), where after preliminary reductions using cohomology and base change and the theorem of formal functions, the lemma on which our entire argument rests is [Lemma 10.12](#). [Section 11](#) begins to investigate the question of when a generic cyclic cover of a globally F -full variety is again globally F -full.

8.1. Acknowledgements. The author would like to thank Jarod Alper, Johan de Jong, Gabriel Dorfsman-Hopkins, Takumi Murayama and the participants of Sándor Kovács’s “What are you working on?” seminar for helpful conversations. Special thanks to Tuomas Tajakka for reading and providing feedback on parts of this paper, and to Karl Schwede for suggesting that a global condition on Frobenius might give a positive characteristic analogue of [Theorem 8.1](#).

9. GLOBAL F -FULLNESS

Let B be a scheme of characteristic $p > 0$ and let $f : X \rightarrow B$ be a B -scheme. The *absolute* Frobenius $F_X : X \rightarrow X$ of X is the identity on the underlying topological space and the p -th power endomorphism $\mathcal{O}_X \rightarrow \mathcal{O}_X$ sending $f \rightarrow f^p$ on the structure sheaf. Iterating F_X e times gives the e -th Frobenius $F_X^e : X \rightarrow X$ given by p^e -th powers $f \rightarrow f^{p^e}$ on the structure sheaf. The e -th *relative* (or k -linear) Frobenius $F_{X/k}^e : X \rightarrow X^{(e)} := X \times_{B, F_B^e} B$ is then defined by the commutative diagram ([\[Stacks, Tag 0CC6\]](#).)

$$\begin{array}{ccccc}
 & & F_X^e & & \\
 & \nearrow & & \searrow & \\
 X & \xrightarrow{F_f^e} & X^{(e)} & \xrightarrow{\quad} & X \\
 & \searrow f & \downarrow f^{(e)} & \square & \downarrow f \\
 & & B & \xrightarrow{F_B^e} & B
 \end{array} \tag{9.1}$$

For a ring R of characteristic $p > 0$ we will denote by R^{p^e} the image of the e -th Frobenius $F_R^e : R \rightarrow R$ sending $f \mapsto f^{p^e}$. When R is reduced this is a subring of R — in particular for a field k of characteristic $p > 0$ we obtain a descending chain of subfields

$$k \supseteq k^p \supseteq k^{p^2} \supseteq \dots$$

¹⁵So maybe the connection is tenuous at best, but at least my initial motivations have been made clear!

If X is a proper k -scheme considering (9.1) (with $B = \operatorname{Spec} k$) we get a morphism of k -schemes $F_{X/k}^e : X \rightarrow X^{(e)}$ which induces k -linear maps of cohomology of the structure sheaf

$$F_{X/k}^{e*} : H^i(X^{(e)}, \mathcal{O}_{X^{(e)}}) \rightarrow H^i(X, \mathcal{O}_X) \text{ for } i \geq 0$$

Since k is and so $F_k^e : k \rightarrow k$ is flat, we may apply flat base change to the cartesian part of (9.1) to obtain isomorphisms $H^i(X^{(e)}, \mathcal{O}_{X^{(e)}}) \simeq H^i(X, \mathcal{O}_X) \otimes_{k^{p^e}} k$.

Definition 9.2. Let k be a field of characteristic $p > 0$. A proper k -scheme X is *globally F -full* if and only if the natural morphisms induced by the relative or k -linear Frobenius of X/k

$$H^i(X, \mathcal{O}_X) \otimes_{k^{p^e}} k \twoheadrightarrow H^i(X, \mathcal{O}_X) \text{ are surjective for all } e, i \in \mathbb{N}. \quad (9.3)$$

Remark 9.4. In general, (9.3) is a map of k -vector spaces of the same finite dimension, so it is surjective if and only if it is an isomorphism. In the case k is *perfect*, (9.3) is equivalent to the condition that the adjoint morphisms

$$F_X^* : H^i(X, \mathcal{O}_X) \rightarrow H^i(X, \mathcal{O}_X)$$

induced by the absolute Frobenius are isomorphisms (or equivalently injective or surjective) for all $e, i \in \mathbb{N}$. Indeed, we may factor (9.3) as

$$H^i(X, \mathcal{O}_X) \rightarrow H^i(X, \mathcal{O}_X) \otimes_{k^{p^e}} k \xrightarrow{F_{X/k}^*} H^i(X, \mathcal{O}_X)$$

where the first map is a unit of adjunction for restriction/extension of scalars along k , hence an isomorphism when k is perfect. An alternative factorization of (9.3) is

$$H^i(X, \mathcal{O}_X) \otimes_{k^{p^e}} k \xrightarrow{F_X^* \otimes \operatorname{id}} H^i(X, \mathcal{O}_X) \otimes_{k^{p^e}} k \rightarrow H^i(X, \mathcal{O}_X)$$

where the second map is the other unit of adjunction (also an isomorphism if k is perfect). Moreover, (9.3) can be checked on geometric (or perfect) fibers — see [Remark 10.23](#).

This shows that [Definition 9.2](#) is a strengthening of the weak ordinarity condition of [\[MS11\]](#), which would only require an injection $H^{\dim X}(X, \mathcal{O}_X) \hookrightarrow F_*^e H^{\dim X}(X, \mathcal{O}_X)$.¹⁶

Remark 9.5. The terminology “globally F -full” is chosen to mirror the notion of F -full defined in [\[MQ18, Def. 2.3\]](#), which requires a surjectivity similar to the one appearing in (9.3) but for local cohomology modules.

With terminology and definitions in hand, we now state the proposition whose proof will occupy the next section.

Proposition 9.6. *Let B be a locally noetherian scheme of characteristic $p > 0$ and let $f : X \rightarrow B$ be a flat proper morphism. Let $b \in B$ be a closed point and assume that the fiber X_b is globally F -full over $k(b)$. Then $R^i f_* \mathcal{O}_X$ is locally free and compatible with arbitrary base change for all $i \in \mathbb{N}$ on Zariski neighborhood $b \in U \subseteq B$.*

10. COHOMOLOGY OF THE STRUCTURE SHEAF

10.1. Restriction maps from thickened fibers. Following the approach in [\[DJ74\]](#), we immediately apply [\[EGAIII₂, Prop. 7.7.10\]](#) which shows:

Proposition 10.1. *Let B be a locally noetherian scheme, let $f : X \rightarrow B$ be a proper morphism and let \mathcal{F} be a coherent sheaf on X flat over B . Let $b \in B$ be a closed point with corresponding maximal ideal sheaf $\mathfrak{m}_b \subseteq \mathcal{O}_B$. Write $X_{b,n} := f^{-1}(V(\mathfrak{m}_b^{n+1})) \subseteq X$ and $\mathcal{F}_{b,n} := \mathcal{F}|_{X_{b,n}}$ (with the exception that we write $X_b = X_{b,0}$ and*

¹⁶Hence globally F -full could have been called not-so-weakly ordinary.

$\mathcal{F}_b = \mathcal{F}_{b,0}$). Then, the sheaves $R^i f_* \mathcal{F}$ are locally free and compatible with arbitrary base change on a Zariski neighborhood $U \subseteq B$ containing b for all $i \in \mathbb{N}$ if and only if the restriction morphisms

$$H^i(X_{b,n}, \mathcal{F}_{b,n}) \twoheadrightarrow H^i(X_b, \mathcal{F}_b) \text{ are surjective for all } n, i \in \mathbb{N}. \quad (10.2)$$

We briefly explain how to derive [Proposition 10.1](#) from cohomology and base change and the theorem of formal functions as discussed in e.g. [\[Har77, §III.11-12\]](#), [\[Mum08, §II.5\]](#).

Proof. By [\[Har77, Thm. III.12.11\]](#) it will suffice to prove that the natural maps $R^i f_* \mathcal{F} \otimes k(b) \rightarrow H^i(X_b, \mathcal{F}_b)$ are surjective for all i . Since completion with respect to \mathfrak{m}_b is faithfully flat and since $(R^i f_* \mathcal{F})_b^\wedge \otimes k(b) = R^i f_* \mathcal{F} \otimes k(b)$, we reduce to showing surjectivity of the maps

$$(R^i f_* \mathcal{F})_b^\wedge \otimes k(b) \rightarrow H^i(X_b, \mathcal{F}_b)$$

Now by [\[Har77, Thm. III.11.1\]](#) the natural map $(R^i f_* \mathcal{F})_b^\wedge \rightarrow \lim H^i(X_{b,n}, \mathcal{F}_{b,n})$ is an isomorphism, hence we reduce to showing that the surjectivity condition (10.2) implies that the natural map $\lim H^i(X_{b,n}, \mathcal{F}_{b,n}) \rightarrow H^i(X_b, \mathcal{F}_b)$ is surjective for all i . As n varies, the short exact sequences

$$0 \rightarrow \mathfrak{m}_b \mathcal{F}_{b,n} \rightarrow \mathcal{F}_{b,n} \rightarrow \mathcal{F}_b \rightarrow 0$$

form an inverse system and taking cohomology we obtain a short exact sequence of inverse systems

$$0 \rightarrow (H^i(X_{b,n}, \mathfrak{m}_b \mathcal{F}_{b,n})) \rightarrow (H^i(X_{b,n}, \mathcal{F}_{b,n})) \rightarrow (H^i(X_b, \mathcal{F}_b)) \rightarrow 0;$$

on the right we have a constant inverse system, and surjectivity on the right is *exactly* (10.2). At this point we only need to know that the sequence of inverse limits remains an exact, and for that it suffices to know $(H^i(X_{b,n}, \mathfrak{m}_b \mathcal{F}_{b,n}))$ satisfies the Mittag-Leffler, which is automatic since each $H^i(X_{b,n}, \mathfrak{m}_b \mathcal{F}_{b,n})$ is a finitely generated module over the artinian local ring $\mathcal{O}_{B,b}/\mathfrak{m}_b^{n+1}$. \square

It will be useful to consider not only the inclusion of a fiber X_b into its n -th thickening $X_{b,n}$, but the entire sequence of inclusions $X_{b,n-1} \subseteq X_{b,n}$. This not only decomposes the maps (10.2) but also yields useful long exact sequences.

Lemma 10.3. *In the situation of [Proposition 10.1](#) there are long exact sequences*

$$\cdots \longrightarrow H^i(X_b, \mathcal{F}_b) \otimes_{k(b)} (\mathfrak{m}_b^n / \mathfrak{m}_b^{n+1}) \longrightarrow H^i(X_{b,n}, \mathcal{F}_{b,n}) \longrightarrow H^i(X_{b,n-1}, \mathcal{F}_{b,n-1}) \longrightarrow \cdots \quad (10.4)$$

which are natural in the sense that if $g : Y \rightarrow B$ is another proper morphism and \mathcal{G} is a coherent sheaf on Y flat over B , and if we are given a B -morphism $h : X \rightarrow Y$ together with a map of sheaves $\varphi : \mathcal{G} \rightarrow h_* \mathcal{F}$, there is a functorial morphism of long exact sequences (of modules over the local ring $\mathcal{O}_{B,b}$)

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^i(Y_b, \mathcal{G}_b) \otimes_{k(b)} (\mathfrak{m}_b^n / \mathfrak{m}_b^{n+1}) & \longrightarrow & H^i(Y_{b,n}, \mathcal{G}_{b,n}) & \longrightarrow & H^i(Y_{b,n-1}, \mathcal{G}_{b,n-1}) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & H^i(X_b, \mathcal{F}_b) \otimes_{k(b)} (\mathfrak{m}_b^n / \mathfrak{m}_b^{n+1}) & \longrightarrow & H^i(X_{b,n}, \mathcal{F}_{b,n}) & \longrightarrow & H^i(X_{b,n-1}, \mathcal{F}_{b,n-1}) \longrightarrow \cdots \end{array} \quad (10.5)$$

We give 2 proofs of different flavors, the first in terms of the derived projection formula and the second at the level of complexes. In fact as the proof(s) will show it's not necessary for f, g to be proper, although that is the only case we will use in what follows; we also omit proof of full functoriality (which isn't used in what follows either).

Proof 1. We derive (10.5) as it includes (10.4) as a special case (e.g. with $\varphi = \text{id}$). By functoriality of derived pushforwards, we have a morphism $Rg_* \mathcal{G} \rightarrow Rf_* \mathcal{F}$ in $D_{\text{coh}}^b(B)$. Taking the derived tensor

product of this with the distinguished triangle $\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1} \rightarrow \mathcal{O}_B/\mathfrak{m}_b^{n+1} \rightarrow \mathcal{O}_B/\mathfrak{m}_b^n$ and applying the derived projection formula [Stacks, Tag 08ET] yields a morphism of distinguished triangles

$$\begin{array}{ccccccc} Rg_*(\mathcal{G} \otimes^L Lg^*(\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1})) & \rightarrow & Rg_*(\mathcal{G} \otimes^L Lg^*(\mathcal{O}_B/\mathfrak{m}_b^{n+1})) & \rightarrow & Rg_*(\mathcal{G} \otimes^L Lg^*(\mathcal{O}_B/\mathfrak{m}_b^n)) & \rightarrow & \cdots \\ \downarrow & & \downarrow & & * & & \downarrow \\ Rf_*(\mathcal{F} \otimes^L Lf^*(\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1})) & \rightarrow & Rf_*(\mathcal{F} \otimes^L Lf^*(\mathcal{O}_B/\mathfrak{m}_b^{n+1})) & \rightarrow & Rf_*(\mathcal{F} \otimes^L Lf^*(\mathcal{O}_B/\mathfrak{m}_b^n)) & \rightarrow & \cdots \end{array} \quad (10.6)$$

Since \mathcal{F}, \mathcal{G} are flat over B the derived pullbacks/tensor products simplify; we have

$$\mathcal{F} \otimes^L Lf^*(\mathcal{O}_B/\mathfrak{m}_b^{n+1}) \simeq \mathcal{F} \otimes_{f^{-1}\mathcal{O}_B}^L f^{-1}(\mathcal{O}_B/\mathfrak{m}_b^{n+1}) = \mathcal{F}_{b,n}$$

and similarly for the other terms on the corners of $(*)$ in (10.6). Moreover since $\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}$ is a $k(b)$ -vector space a similar tensor product manipulation gives

$$\mathcal{F} \otimes^L Lf^*(\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) \simeq \mathcal{F} \otimes_{f^{-1}\mathcal{O}_B}^L f^{-1}(\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) \simeq \mathcal{F} \otimes_{f^{-1}\mathcal{O}_B}^L f^{-1}k(b) \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) = \mathcal{F}_b \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1})$$

Applying Künneth gives a natural isomorphism $Rf_*(\mathcal{F}_b \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1})) \simeq Rf_*\mathcal{F}_b \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1})$. Similarly for the top right corner of (10.6).

Hence the map of distinguished triangles (10.6) is isomorphic to

$$\begin{array}{ccccccc} Rg_*\mathcal{G}_b \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) & \rightarrow & Rg_*(\mathcal{G}_{b,n}) & \rightarrow & Rg_*(\mathcal{G}_{b,n-1}) & \rightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ Rf_*\mathcal{F}_b \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) & \rightarrow & Rf_*(\mathcal{F}_{b,n}) & \rightarrow & Rf_*(\mathcal{F}_{b,n-1}) & \rightarrow & \cdots \end{array} \quad (10.7)$$

and taking cohomology yields (10.5). \square

Proof 2. Replacing B with $\text{Spec } \mathcal{O}_{B,b}$ and X, Y, \mathcal{F} and \mathcal{G} with their corresponding localizations we may assume $B = \text{Spec } A$ for some noetherian local ring A with maximal ideal \mathfrak{m} . Let $\mathfrak{U} = \{U_i \subseteq X\}$ be an affine open cover of X and let $\mathfrak{V} = \{V_i \subseteq Y\}$ be an affine open cover refining $\{h^{-1}(U_i)\}$. Then as discussed in [Mum70, §II.5] the Čech complexes $\check{C}(\mathfrak{U}, \mathcal{F})$ and $\check{C}(\mathfrak{V}, \mathcal{G})$ are bounded complexes of flat A -modules together with functorial isomorphisms

$$H^i(\check{C}(\mathfrak{U}, \mathcal{F}) \otimes A') \simeq H^i(X_{A'}, \mathcal{F}_{A'}) \text{ and } H^i(\check{C}(\mathfrak{V}, \mathcal{G}) \otimes A') \simeq H^i(Y_{A'}, \mathcal{G}_{A'}) \quad (10.8)$$

on the category of A -algebras A' , where $X_{A'} := X \times_{\text{Spec } A} \text{Spec } A'$ and $\mathcal{F}_{A'}$ is the pullback of \mathcal{F} along the projection $X_{A'} \rightarrow X$, etc.

Moreover φ induces a map of complexes

$$\check{C}(\varphi) : \check{C}(\mathfrak{U}, \mathcal{F}) \rightarrow \check{C}(\mathfrak{V}, \mathcal{G})$$

and since the terms of $\check{C}(\mathfrak{U}, \mathcal{F}), \check{C}(\mathfrak{V}, \mathcal{G})$ are flat, tensoring $\check{C}(\varphi)$ with the short exact sequence of A -modules $\mathfrak{m}^n/\mathfrak{m}^{n+1} \hookrightarrow A/\mathfrak{m}^{n+1} \twoheadrightarrow A/\mathfrak{m}^n$ results in a short exact sequence of chain complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \check{C}(\mathfrak{U}, \mathcal{F}) \otimes \mathfrak{m}^n/\mathfrak{m}^{n+1} & \longrightarrow & \check{C}(\mathfrak{U}, \mathcal{F}) \otimes A/\mathfrak{m}^{n+1} & \longrightarrow & \check{C}(\mathfrak{U}, \mathcal{F}) \otimes A/\mathfrak{m}^n \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \check{C}(\mathfrak{V}, \mathcal{G}) \otimes \mathfrak{m}^n/\mathfrak{m}^{n+1} & \longrightarrow & \check{C}(\mathfrak{V}, \mathcal{G}) \otimes A/\mathfrak{m}^{n+1} & \longrightarrow & \check{C}(\mathfrak{V}, \mathcal{G}) \otimes A/\mathfrak{m}^n \longrightarrow 0 \end{array} \quad (10.9)$$

whose induced map of long exact sequences of cohomology is (10.5) \square

10.2. Thickened fibers of Frobenius twists. Returning to [Proposition 9.6](#): we have a flat proper morphism $f : X \rightarrow B$ where B is locally noetherian of characteristic $p > 0$ and a closed point $b \in B$ such that X_b is globally F -full over $k(b)$, and we wish to show the sheaves $R^i f_* \mathcal{O}_X$ are locally free and compatible with base change on a neighborhood $U \subseteq B$ containing b . Since we aim to prove this using [Proposition 10.1](#), localizing with respect to $\text{Spec } \mathcal{O}_{B,b} \rightarrow B$ we reduce to proving the following lemma.

Lemma 10.10. *Let A be a noetherian local ring of characteristic p with maximal ideal \mathfrak{m} and residue field $k = A/\mathfrak{m}$, and let $f : X \rightarrow \text{Spec } A = B$ be a flat proper morphism. For each $n \in \mathbb{N}$ let $X_n = f^{-1}(V(\mathfrak{m}^{n+1}))$. If X_0 is globally F -full over k then the restriction maps*

$$H^i(X_n, \mathcal{O}_{X_n}) \rightarrow H^i(X_0, \mathcal{O}_{X_0}) \text{ are surjective for all } n, i \in \mathbb{N}.$$

Applying [Lemma 10.3](#) to the relative Frobenius F_f^e of [\(9.1\)](#) (which automatically comes with a map of sheaves $\mathcal{O}_{X^{(e)}} \rightarrow F_{f*}^e \mathcal{O}_X$) gives us a map of long exact sequences

$$\begin{array}{ccccccc} \cdots & \rightarrow & H^i(X_0^{(e)}, \mathcal{O}_{X_0}) \otimes_k (\mathfrak{m}^n / \mathfrak{m}^{n+1}) & \rightarrow & H^i(X_n^{(e)}, \mathcal{O}_{X_n}) & \rightarrow & H^i(X_{n-1}^{(e)}, \mathcal{O}_{X_{n-1}}) \rightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \rightarrow & H^i(X_0, \mathcal{O}_{X_0}) \otimes_k (\mathfrak{m}^n / \mathfrak{m}^{n+1}) & \rightarrow & H^i(X_n, \mathcal{O}_{X_n}) & \rightarrow & H^i(X_{n-1}, \mathcal{O}_{X_{n-1}}) \rightarrow \cdots \end{array} \quad (10.11)$$

For large e , the top row simplifies considerably.

Lemma 10.12. *For fixed n and $e \gg 0$, the composite $A \xrightarrow{F_A^e} A \rightarrow A/\mathfrak{m}^n$ factors through k . Equivalently, for e in this range A/\mathfrak{m}^n is a k^{p^e} -algebra.*

Proof. We must show that the kernel I of $A \xrightarrow{F_A^e} A \rightarrow A/\mathfrak{m}^n$ is \mathfrak{m} . Explicitly this kernel is

$$I = \{x \in A \mid x^{p^e} \in \mathfrak{m}^n\}$$

from which we see $I = \mathfrak{m}$ for $p^e \geq n$. □

Remark 10.13. [Lemma 10.12](#) is equivalent to the trivial inclusion $\mathfrak{m}^{[p^e]} \subseteq \mathfrak{m}^n$ for $p^e \geq n$.

Corollary 10.14. *For fixed n and $e \gg 0$, there is a natural isomorphism of finite-type A/\mathfrak{m}^n -schemes $X_{n-1}^{(e)} \simeq X_0 \otimes_{k^{p^e}} A/\mathfrak{m}^n$. Explicitly, $X_0 \otimes_{k^{p^e}} A/\mathfrak{m}^n$ is the fiber product in*

$$\begin{array}{ccc} X_0 \otimes_{k^{p^e}} A/\mathfrak{m}^n & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ \text{Spec } A/\mathfrak{m}^n & \longrightarrow & \text{Spec } k \end{array}$$

where the bottom horizontal arrow is given by [Lemma 10.12](#).

We now apply [Corollary 10.14](#) to rewrite the top row of [\(10.11\)](#). For $e \geq \log_p(n+1)$ there are isomorphisms

$$H^i(X_{n-1}^{(e)}, \mathcal{O}_{X_{n-1}}) \simeq H^i(X_0, \mathcal{O}_{X_0}) \otimes_{k^{p^e}} A/\mathfrak{m}^n$$

and similarly $H^i(X_n^{(e)}, \mathcal{O}_{X_n}) \simeq H^i(X_0, \mathcal{O}_{X_0}) \otimes_{k^{p^e}} A/\mathfrak{m}^{n+1}$. In particular for $n = 0$ we have $H^i(X_0^{(e)}, \mathcal{O}_X) \simeq H^i(X_0, \mathcal{O}_{X_0}) \otimes_{k^{p^e}} k$.¹⁷ Using these identifications, (10.11) becomes

$$\begin{array}{ccccccc} \cdots & \rightarrow & H^i(X_0, \mathcal{O}_{X_0}) \otimes_{k^{p^e}} (\mathfrak{m}^n / \mathfrak{m}^{n+1}) & \rightarrow & H^i(X_0, \mathcal{O}_{X_0}) \otimes_{k^{p^e}} A/\mathfrak{m}^{n+1} & \xrightarrow{\rho_n^{(e),i}} & H^i(X_0, \mathcal{O}_{X_0}) \otimes_{k^{p^e}} A/\mathfrak{m}^n \rightarrow \cdots \\ & & \downarrow \psi_n^{(e),i} & & \downarrow \varphi_n^{(e),i} & & \downarrow \varphi_{n-1}^{(e),i} \\ \cdots & \rightarrow & H^i(X_0, \mathcal{O}_{X_0}) \otimes_k (\mathfrak{m}^n / \mathfrak{m}^{n+1}) & \longrightarrow & H^i(X_n, \mathcal{O}_{X_n}) & \xrightarrow{\rho_n^i} & H^i(X_{n-1}, \mathcal{O}_{X_{n-1}}) \longrightarrow \cdots \end{array} \quad (10.15)$$

10.3. Surjectivity of relative Frobenii.

Proposition 10.16. *If X_0 is globally F -full then for fixed n and $e \gg 0$, the homomorphisms $\rho_n^{(e),i}$ and $\varphi_{n-1}^{(e),i}$ (and hence also ρ_n^i) are surjective for all $i \in \mathbb{N}$.*

Proof. Fixing n , choose $e \geq \log_p(n+1)$ (so $p^e \geq n+1$). Then the homomorphisms $\rho_n^{(e),i}$ are all surjective, since the reductions $A/\mathfrak{m}^{n+1} \twoheadrightarrow A/\mathfrak{m}^n$ are surjective, and because tensoring over k^{p^e} is exact. Moreover global F -fullness of X_0 guarantees the vertical maps $\psi_n^{(e),i}$ are all surjective: after choosing a basis for $\mathfrak{m}^n / \mathfrak{m}^{n+1}$, the map $\psi_n^{(e),i}$ can be written as a direct sum $\dim \mathfrak{m}^n / \mathfrak{m}^{n+1}$ copies of the map appearing in (9.3)).

We now show by induction on $m \leq n$ (with a subsidiary induction on i) that the $\varphi_m^{(e),i}$ and ρ_m^i are all surjective — surjectivity of $\varphi_m^{(e),i}$ in the base case $m = 0$ is exactly global F -fullness of X_0 , and surjectivity of ρ_m^0 is covered in the $m = 1$ -st inductive step below. Now suppose $0 < m \leq n$ and consider

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(X_0, \mathcal{O}_{X_0}) \otimes_{k^{p^e}} (\mathfrak{m}^m / \mathfrak{m}^{m+1}) & \rightarrow & H^0(X_0, \mathcal{O}_{X_0}) \otimes_{k^{p^e}} (A/\mathfrak{m}^{m+1}) & \xrightarrow{\rho_m^{(e),0}} & H^0(X_0, \mathcal{O}_{X_0}) \otimes_{k^{p^e}} (A/\mathfrak{m}^m) \rightarrow 0 \\ & & \downarrow \psi_m^{(e),0} & & \downarrow \varphi_m^{(e),0} & & \downarrow \varphi_{m-1}^{(e),0} \\ 0 & \rightarrow & H^0(X_0, \mathcal{O}_{X_0}) \otimes_k (\mathfrak{m}^m / \mathfrak{m}^{m+1}) & \longrightarrow & H^0(X_m, \mathcal{O}_{X_m}) & \xrightarrow{\rho_m^0} & H^0(X_{m-1}, \mathcal{O}_{X_{m-1}}) \xrightarrow{\delta_m^1} \cdots \end{array} \quad (10.17)$$

where in the top row we have applied the surjectivity of $\rho_m^{(e),0}$ mentioned above to obtain a short exact sequence, and in the left vertical map we have applied the surjectivity of $\psi_m^{(e),0}$. By inductive hypothesis we may assume the right vertical arrow $\varphi_{m-1}^{(e),0}$ is surjective. Now the snake lemma [Stacks, Tag 07JV] gives us an exact sequence

$$0 = \operatorname{coker} \psi_m^{(e),0} \rightarrow \operatorname{coker} \varphi_m^{(e),0} \rightarrow \varphi_{m-1}^{(e),0} = 0$$

and hence $\operatorname{coker} \varphi_m^{(e),0} = 0$.

We also conclude from surjectivity of $\rho_m^{(e),0}$ and $\varphi_{m-1}^{(e),0}$ that ρ_m^0 is surjective, and so the connecting map $\delta_m^1 = 0$. This means that for $i > 0$, we obtain a diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & H^i(X_0, \mathcal{O}_{X_0}) \otimes_{k^{p^e}} (\mathfrak{m}^m / \mathfrak{m}^{m+1}) & \rightarrow & H^i(X_0, \mathcal{O}_{X_0}) \otimes_{k^{p^e}} (A/\mathfrak{m}^{m+1}) & \xrightarrow{\rho_m^{(e),i}} & H^i(X_0, \mathcal{O}_{X_0}) \otimes_{k^{p^e}} (A/\mathfrak{m}^m) \rightarrow 0 \\ & & \downarrow \psi_m^{(e),i} & & \downarrow \varphi_m^{(e),i} & & \downarrow \varphi_{m-1}^{(e),i} \\ 0 & \rightarrow & H^i(X_0, \mathcal{O}_{X_0}) \otimes_k (\mathfrak{m}^m / \mathfrak{m}^{m+1}) & \longrightarrow & H^i(X_m, \mathcal{O}_{X_m}) & \xrightarrow{\rho_m^i} & H^i(X_{m-1}, \mathcal{O}_{X_{m-1}}) \xrightarrow{\delta_m^{i+1}} \cdots \end{array} \quad (10.18)$$

where now exactness on the left is obtained the inductive hypothesis that $\rho_m^{(e),i-1}$ and ρ_m^{i-1} are surjective. Again we may assume by inductive hypothesis that the vertical map $\varphi_{m-1}^{(e),i}$ on the right

¹⁷this last isomorphism of course doesn't need restrictions on e .

is surjective, and then the snake lemma shows $\varphi_m^{(e),i}$ is surjective. Since $\rho_m^{(e),i}$ and $\varphi_{m-1}^{(e),i}$ are both surjective we conclude ρ_m^i is surjective, completing the inductive step. \square

Proof of Lemma 10.10. Proposition 10.16 shows that the restriction maps

$$\rho_n^i : H^i(X_n, \mathcal{O}_{X_n}) \rightarrow H^i(X_{n-1}, \mathcal{O}_{X_{n-1}})$$

are surjective for all $n, i \in \mathbb{N}$, and so the composite

$$H^i(X_n, \mathcal{O}_{X_n}) \xrightarrow{\rho_n^i} H^i(X_{n-1}, \mathcal{O}_{X_{n-1}}) \rightarrow \cdots \rightarrow H^i(X_{n-1}, \mathcal{O}_{X_1}) \xrightarrow{\rho_1^i} H^i(X_0, \mathcal{O}_{X_0})$$

is surjective. This is precisely the restriction morphism (10.2). \square

Proof of Proposition 9.6. Combine Proposition 10.1 and Lemma 10.10. \square

Corollary 10.19. *The set of points $b \in B$ such that X_b is globally F -full is open.*

Proof. If X_b is globally F -full then by Proposition 9.6 there is a neighborhood $U \subseteq B$ such that the sheaves $R^i f_* \mathcal{O}_X|_U$ are locally free and compatible with base change — replacing B with U we can assume that the $R^i f_* \mathcal{O}_X$ themselves are locally free and compatible with base change.

In particular applying compatibility to base change to the e -th relative Frobenius (9.1) gives an isomorphism

$$LF_B^{e*} Rf_* \mathcal{O}_X = Rf_*^{(e)} \mathcal{O}_{X^{(e)}} \text{ in } D_{\text{coh}}^b(B) \quad (10.20)$$

so that the cohomology sheaves of $Rf_*^{(e)} \mathcal{O}_{X^{(e)}}$ are also locally free and compatible with base change (alternatively, this could be deduced from the fact that $f^{(e)}$ and f have the same fibers over closed points, i.e. applying Proposition 9.6 to $f^{(e)}$ as well). We now consider the map $\varphi^{(e)} : Rf_*^{(e)} \mathcal{O}_{X^{(e)}} \rightarrow Rf_* \mathcal{O}_X$ induced by F_f^e , and claim $\varphi^{(e)}$ is a quasi-isomorphism on a neighborhood of b . For each i the induced morphism

$$R^i f_*^{(e)} \mathcal{O}_{X^{(e)}} \xrightarrow{\varphi^{(e)}} R^i f_* \mathcal{O}_X \quad (10.21)$$

is a map of locally free sheaves whose reduction mod \mathfrak{m}_b is

$$H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)^{p^e}} k(b) \simeq H^i(X_b^{(e)}, \mathcal{O}_{X_b^{(e)}}) \rightarrow H^i(X_b, \mathcal{O}_{X_b}),$$

by hypothesis an isomorphism. By Nakayama's lemma (10.21) is an isomorphism on a neighborhood $U \subseteq B$ of b . Choosing U small enough so that (10.21) is an isomorphism for all i , for any $b' \in U$ tensoring with $k(b')$ gives

$$\begin{array}{ccc} R^i f_*^{(e)} \mathcal{O}_{X^{(e)}} \otimes k(b') & \xrightarrow[\simeq]{\varphi^{(e)} \otimes \text{id}} & R^i f_* \mathcal{O}_X \otimes k(b') \\ \downarrow \simeq & & \downarrow \simeq \\ H^i(X_b', \mathcal{O}_{X_b'}) \otimes_{k(b')^{p^e}} k(b') & \longrightarrow & H^i(X_b', \mathcal{O}_{X_b'}) \end{array} \quad (10.22)$$

\square

Remark 10.23. Note that the relative Frobenius defined in (9.1) is compatible with base change in the following sense:¹⁸ indeed, if $t : B' \rightarrow B$ is a morphism of locally noetherian schemes of characteristic

¹⁸Slogan: it's the p th power on coordinate functions, not coefficients, hence unaffected by changing coefficients.

$p > 0$ then since absolute Frobenii are functorial, setting $X' := X \times_B B' \xrightarrow{f'} B'$ the diagram

$$\begin{array}{ccccc}
 & & X & \xrightarrow{F_X^e} & X \\
 & \nearrow t' & \downarrow f & \nearrow t' & \downarrow f \\
 X' & \xrightarrow{F_{X'}^e} & X' & & \\
 \downarrow f' & & \downarrow f' & & \\
 B & \xrightarrow{F_B^e} & B & & \\
 \nearrow t & & \nearrow t & & \\
 B' & \xrightarrow{F_{B'}^e} & B' & &
 \end{array}$$

commutes, where the squares on the left and right sides are cartesian. This implies not only that $X'^{(e)} = X^{(e)} \times_B B'$ but also that the map

$$X' = X \times_B B' \xrightarrow{F_f^{(e)} \times \text{id}} X^{(e)} \times_B B' = X'^{(e)}$$

agrees with the relative Frobenius of f' (for example, they are both adjoint to the map of B schemes $F_X^e \circ t' = t' \circ F_{X'}^e$, over the morphism $F_B^e \circ t = t \circ F_{B'}^e$). Combining this with flat base change, we see that if $\bar{b} = \text{Spec } \overline{k(b)}$ then there is a commutative diagram

$$\begin{array}{ccc}
 H^i(X_b^{(e)}, \mathcal{O}_{X_b^{(e)}} \otimes_{k(b)} \overline{k(b)}) & \longrightarrow & H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} \overline{k(b)} \\
 \downarrow \simeq & & \downarrow \simeq \\
 H^i(X_{\bar{b}}^{(e)}, \mathcal{O}_{X_{\bar{b}}^{(e)}}) & \longrightarrow & H^i(X_{\bar{b}}, \mathcal{O}_{X_{\bar{b}}})
 \end{array}$$

where the top arrow is the relative Frobenius on X_b tensored with $\overline{k(b)}$ and the bottom is the relative Frobenius on $X_{\bar{b}}$. Since the top horizontal arrow is surjective if and only if $H^i(X_b^{(e)}, \mathcal{O}_{X_b^{(e)}}) \otimes_{k(b)} \overline{k(b)} \rightarrow H^i(X_b, \mathcal{O}_{X_b})$ is, we see that X_b is globally F -full if and only if $X_{\bar{b}}$ is. The same discussion applies with the perfection $k(b)^{\text{perf}}$ in place of the algebraic closure $\overline{k(b)}$.

10.4. Examples.

Example 10.24 (Suggested by Johan de Jong; shows (9.3) is sufficient but not necessary). Let k be an algebraically closed field of characteristic $p > 2$ ¹⁹, let $B = \mathbb{A}_\lambda^1$ and let $X = V(y^2z - x(x-z)(x-\lambda z)) \subseteq \mathbb{A}_\lambda^1 \times \mathbb{P}_{xyz}^2$. Let $f : X \rightarrow B$ be the projection.

By [Har77, Cor. 4.22] the locus of closed points $b \in \mathbb{A}_\lambda^1$ where (9.3) holds is the *non-vanishing* $D(h_p)$ of the polynomial

$$h_p(\lambda) = \sum_{i=0}^{p-1} \binom{\frac{p-1}{2}}{i} \lambda^i$$

¹⁹I think this works with k perfect, but it references [Har77, Ch. IV] which begins with a blanket assumption that the ground field is algebraically closed ...

so in particular it is a *proper* open subset. However in this case the higher direct images $R^i f_* \mathcal{O}_X$ are still locally free: identifying them with the $k[\lambda]$ -modules $H^i(X, \mathcal{O}_X)$ and using the exact sequence

$$\cdots \longrightarrow H^i(\mathbb{A}_\lambda^1 \times \mathbb{P}_{xyz}^2, \mathcal{O}(-3)) \longrightarrow H^i(\mathbb{A}_\lambda^1 \times \mathbb{P}_{xyz}^2, \mathcal{O}) \longrightarrow H^i(X, \mathcal{O}_X) \longrightarrow \cdots \quad (10.25)$$

induced by the section $y^2 z - x(x - z)(x - \lambda z) \in H^0(\mathbb{A}_\lambda^1 \times \mathbb{P}_{xyz}^2, \mathcal{O}(3))$ we get isomorphisms

$$H^0(X, \mathcal{O}_X) \simeq H^0(\mathbb{A}_\lambda^1 \times \mathbb{P}_{xyz}^2, \mathcal{O}) \text{ and } H^1(X, \mathcal{O}_X) \simeq H^2(\mathbb{A}_\lambda^1 \times \mathbb{P}_{xyz}^2, \mathcal{O}(-3))$$

and the latter 2 modules are free of rank 1 by [Har77, Thm. III.5.1].

11. GENERIC CYCLIC COVERS OF GLOBALLY F -FULL VARIETIES

In [KK10], Theorem 8.1 is used to derive flatness properties of relative dualizing sheaves. A key ingredient in that argument is the following fact: if X is a projective variety with Du Bois singularities and $Y \rightarrow X$ is a cyclic cover branched along a general hyperplane section of X , then Y also has Du Bois singularities. Here we begin to consider the question of when cyclic covers of globally F -full varieties are again globally F -full.

11.1. Frobenius actions on cyclic covers. Let X be a proper scheme over a field k of characteristic $p > 0$, which in this section we assume to be perfect, and let \mathcal{L} be an invertible sheaf on X together with a global section $\sigma \in H^0(X, \mathcal{L}^d)$. Given this data, one can form an associated *cyclic cover* $\pi : Y \rightarrow X$. While this construction is standard (a good reference is [EV92, §3.5]) we will need many of its details, and so we give a complete description.

There is a natural inclusion of \mathcal{O}_X algebras

$$\iota : \text{Sym } \mathcal{L}^{-d} = \bigoplus_{j \in \mathbb{N}} \mathcal{L}^{-dj} \subseteq \bigoplus_{j \in \mathbb{N}} \mathcal{L}^{-j} = \text{Sym } \mathcal{L}^{-1}$$

and the map $\sigma^\vee \mathcal{L}^{-d} \rightarrow \mathcal{O}_X$ dual to σ induces a morphism of \mathcal{O}_X -algebras $\text{Sym}(\sigma^\vee) : \text{Sym } \mathcal{L}^{-d} \rightarrow \mathcal{O}_X$. Hence both $\text{Sym } \mathcal{L}^{-1}$ and \mathcal{O}_X are $\text{Sym } \mathcal{L}^{-d}$ -algebras, and we define Y to be the affine X -scheme associated to the tensor product of \mathcal{O}_X -algebras

$$\text{Sym } \mathcal{L}^{-1} \otimes_{\text{Sym } \mathcal{L}^{-d}} \mathcal{O}_X$$

Since $\text{Sym}(\sigma^\vee)$ is surjective, there is also a natural isomorphism

$$\pi_* \mathcal{O}_Y = \text{Sym } \mathcal{L}^{-1} \otimes_{\text{Sym } \mathcal{L}^{-d}} \mathcal{O}_X \simeq \text{Sym } \mathcal{L}^{-1} / \iota(\ker \text{Sym}(\sigma^\vee)) \quad (11.1)$$

Geometrically: if $L = \text{Spec } \text{Sym } \mathcal{L}^{-1}$ and $L^d = \text{Spec } \text{Sym } \mathcal{L}^{-d}$ are the associated line bundles, then ι corresponds to a d -th power map $L \rightarrow L^d$, and Y is the fiber product

$$\begin{array}{ccc} Y & \longrightarrow & L \\ \downarrow & & \downarrow (-)^d \\ X & \xrightarrow{\sigma} & L^d \end{array}$$

We claim that the composition $\bigoplus_{0 \leq j < d} \mathcal{L}^{-j} \subseteq \text{Sym } \mathcal{L}^{-1} \twoheadrightarrow \pi_* \mathcal{O}_Y$ is an isomorphism of \mathcal{O}_X -modules. This can be checked on an affine open $U = \text{Spec } A$ of X where there is a nowhere-0 global section $\tau : \mathcal{O}_U \rightarrow \mathcal{L}^{-1}|_U$, hence also a nowhere-0 global section $\tau^d : \mathcal{O}_U \rightarrow \mathcal{L}^{-d}$, which together give identifications (here $\tilde{}$ denotes “sheaf associated to”)

$$\text{Sym } \mathcal{L}^{-1} = A[\tilde{t}] \text{ and } \text{Sym } \mathcal{L}^{-d} = A[\tilde{t}^d]$$

Moreover the composition

$$\mathcal{O}_U \xrightarrow{\tau^d} \mathcal{L}^{-d} \xrightarrow{\sigma^\vee} \mathcal{O}_U$$

gives us an element $f \in A$ such that the map $\text{Sym } \mathcal{L}^{-d} \rightarrow \mathcal{O}_U$ can be identified with the map $A[t^d] \rightarrow A$ taking $t^d \mapsto f$. Using (11.1) we finally obtain an identification $Y|_U \simeq \text{Spec } A[t]/(t^d - f)$ which is a free A -module on the basis $1, t, \dots, t^{d-1}$. Hence $\pi_* \mathcal{O}_Y \simeq \bigoplus_{0 \leq j < d} \mathcal{L}^{-j}$ with algebra structure induced by the map of sheaves $\sigma^\vee : \mathcal{L}^{-d} \rightarrow \mathcal{O}_X$ dual to σ in the sense that for any $0 \leq j, k < d$, writing $j + k = qd + r$ with $0 \leq r < d$ we have a multiplication operation

$$\mathcal{L}^{-j} \otimes \mathcal{L}^{-k} \simeq \mathcal{L}^{-j-k} \xrightarrow{(\sigma^\vee)^q} \mathcal{L}^{-r}$$

Applying this discussion to the p th power map $\pi_*(F_Y)$ on $\pi_* \mathcal{O}_Y$ we see that its restriction to \mathcal{L}^{-j} is

$$\mathcal{L}^{-j} \rightarrow F_{X*} \mathcal{L}^{-pj} \xrightarrow{F_{X*}(\sigma^{\vee qj})} F_{X*} \mathcal{L}^{-rj} \text{ where } pj = qj + r_j \text{ and } 0 \leq r_j < d. \quad (11.2)$$

Taking cohomology yields the following lemma.

Lemma 11.3. *Assume X is proper over a field k of characteristic $p > 0$. Then the action of Frobenius on $H^i(Y, \mathcal{O}_Y)$ is compatible with the direct sum decomposition*

$$H^i(Y, \mathcal{O}_Y) = \bigoplus_{0 \leq j < d} H^i(X, \mathcal{L}^{-j})$$

in the following sense: for each j , writing $pj = qj + r_j$ where $0 \leq r_j < d$,²⁰ Frobenius induces a map of k -vector spaces

$$H^i(X, \mathcal{L}^{-j}) \rightarrow F_* H^i(X, \mathcal{L}^{-pj}) \xrightarrow{F_*(\sigma^{qj})} F_* H^i(X, \mathcal{L}^{-rj}) \quad (11.4)$$

and the Frobenius morphism $H^i(Y, \mathcal{O}_Y) \rightarrow F_* H^i(Y, \mathcal{O}_Y)$ is the sum of the (11.4) as j varies.

In particular if Y is globally F -full if and only if X is globally F -full and the maps (11.4) are isomorphisms for $0 < j < d$.

One immediate consequence of Lemma 11.3 is that Y is unlikely to be globally F -full when $p \not\equiv 1 \pmod{d}$, since then there will be values of j for which $r_j \neq j$ and the left and right hand sides of (11.4) may have different dimensions.

It will be useful to have a Grothendieck-Serre dual formulation of Lemma 11.3. Let $f : X \rightarrow \text{Spec } k$ be the structure map and let $\omega_X^\bullet := f^! k$ be the resulting normalized dualizing complex on X . Applying $R\mathcal{H}om_X(-, \omega_X^\bullet)$ to (11.2) and using Grothendieck duality for the absolute Frobenius F_X (which is finite since k is perfect) we obtain dual morphisms

$$F_{X*}(\omega_X^\bullet \otimes \mathcal{L}^{rj}) \xrightarrow{F_{X*}(\text{id} \otimes \sigma^{qj})} F_{X*}(\omega_X^\bullet \otimes \mathcal{L}^{pj}) \xrightarrow{\text{tr}_{F_X} \otimes \text{id}} \omega_X^\bullet \otimes \mathcal{L}^j$$

where the first map is obtained from $\sigma^{qj} : \mathcal{L}^{rj} \rightarrow \mathcal{L}^{pj}$ by tensoring with ω_X^\bullet and applying F_{X*} , and the second is obtained from the trace map of Grothendieck duality $\text{tr}_{F_X} : F_{X*}(\omega_X^\bullet) \rightarrow \omega_X^\bullet$ by tensoring with \mathcal{L}^j and using the projection formula. Taking hypercohomology results in the following lemma.

Lemma 11.5. *The k -linear dual of the map (11.4) is*

$$F_* \mathbb{H}^{-i}(X, \omega_X^\bullet \otimes \mathcal{L}^{rj}) \xrightarrow{F_*(\text{id} \otimes \sigma^{qj})} F_* \mathbb{H}^{-i}(X, \omega_X^\bullet \otimes \mathcal{L}^{pj}) \xrightarrow{\text{tr}_{F_X} \otimes \text{id}} \mathbb{H}^{-i}(X, \omega_X^\bullet \otimes \mathcal{L}^j) \quad (11.6)$$

In particular Y is globally F -full if and only if X is globally F -full and (11.6) is an isomorphism for $0 < j < d$.

Remark 11.7. If X is Cohen-Macaulay with dualizing sheaf $\omega_X = h^{-\dim X} \omega_X^\bullet$, then since $\omega_X^\bullet \simeq \omega_X[\dim X]$, the terms of (11.6) simplify to:

$$F_* H^{\dim X - i}(X, \omega_X \otimes \mathcal{L}^{rj}) \xrightarrow{F_*(\text{id} \otimes \sigma^{qj})} F_* H^{\dim X - i}(X, \omega_X \otimes \mathcal{L}^{pj}) \xrightarrow{\text{tr}_{F_X} \otimes \text{id}} H^{\dim X - i}(X, \omega_X \otimes \mathcal{L}^j)$$

²⁰Equivalently, $qj = \lfloor \frac{pj}{d} \rfloor$ and $r_j = pj - \lfloor \frac{pj}{d} \rfloor d$.

Definition 11.8. An invertible sheaf \mathcal{L} on X is called *extra ample* if and only if \mathcal{L} is ample and

$$H^j(X, h^i(\omega_X^\bullet) \otimes \mathcal{L}^k) = 0 \text{ for all } j, k > 0 \text{ and for all } i.$$

If X is smooth, then \mathcal{L} is extra ample if it is ample and satisfies the vanishing that would be guaranteed by Kodaira vanishing if we were in characteristic 0. By Serre vanishing for any ample invertible sheaf \mathcal{L} there is a $d_0 \geq 0$ such that \mathcal{L}^d is extra ample for all $d \geq d_0$. We record a trivial consequence of extra-ameness.

Lemma 11.9. *If \mathcal{L} is extra ample then there are natural isomorphisms*

$$\mathbb{H}^{-i}(X, \omega_X^\bullet \otimes \mathcal{L}^k) \simeq H^0(X, h^{-i}(\omega_X^\bullet) \otimes \mathcal{L}^k)$$

Proof. In general there is a hypercohomology spectral sequence of the form

$$H^j(X, h^i(\omega_X^\bullet) \otimes \mathcal{L}^k) \implies \mathbb{H}^{i+j}(X, \omega_X^\bullet \otimes \mathcal{L}^k)$$

When \mathcal{L} is extra ample this collapses, with an edge map giving the desired isomorphism. \square

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