(LOGARITHMIC) CHOW-TO-HODGE CYCLE MAPS

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CONTENTS

1. PRELUDE

Let k be a *perfect* field.

1.1. **A classic Hartshorne problem.**

1.1.1. *Hartshorne exercise III.7.something.* Let X be a smooth projective variety over k and let $\iota : Z \hookrightarrow X$ be a *smooth* subvariety. Then the differential of ι gives a morphism of sheaves

$$d\iota^\vee : \Omega_X^{\dim Z}|_Z \to \Omega_Z^{\dim Z} = \omega_Z$$

and an induced map on cohomology

$$\mathrm{H}^{\dim Z}(X, \Omega_X^{\dim Z}) \xrightarrow{d\iota^\vee} \mathrm{H}^{\dim Z}(Z, \omega_Z) \underset{\simeq}{\xrightarrow{\mathrm{tr}}} k,$$

an element of $\mathrm{H}^{\dim Z}(X, \Omega_X^{\dim Z})^\vee$. Since we have a **perfect pairing**

$$\Omega_X^{\dim Z} \otimes \Omega_X^{\dim X - \dim Z} \xrightarrow{\wedge} \omega_X$$

$\Omega_X^{\dim Z} = \mathrm{Hom}(\Omega_X^{\dim X - \dim Z}, \omega_X)$ and so **Serre duality** gives an isomorphism

$$\mathrm{H}^{\dim Z}(X, \Omega_X^{\dim Z})^\vee \simeq \mathrm{H}^{\dim X - \dim Z}(X, \Omega_X^{\dim X - \dim Z})$$

In this way we get a **cycle class** $\mathrm{cl}_X(Z) \in \mathrm{H}^c(X, \Omega_X^c)$ with $c = \mathrm{codim}(Z, X)$.

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1.1.2. *Natural transformations out of Chow.* In fact, the above can be *upgraded* to show that Hodge cohomology $H^d(X) := \bigoplus_{p+q=d}(X, \Omega_X^p)$ is *almost* [1] an example of a *Weil cohomology theory*. This means among other things that as a functor on, say, smooth projective varieties it's

- contravariant for arbitrary morphisms,
- covariant for proper morphisms,
- satisfies a Künneth formula of the form

$$\mathrm{H}^d(X \times Y) = \bigoplus_{i+j=d} \mathrm{H}^i(X) \otimes \mathrm{H}^j(Y)$$

- comes with cycle classes $\mathrm{cl}_X(Z) \in H^c(X)$ for integral closed subschemes of codimension c, plus compatibilities for the above 3 bullet points. For example, for a dominant morphism $f : X \to \mathbb{P}^1$,

$$\mathrm{cl}_X([f^{-1}(0)]) = \mathrm{cl}([f^{-1}(\infty)]) \in \mathrm{H}(X)$$

See [dJ], [Mus]. As a consequence, the cycle class descends to a natural transformation $\mathrm{cl} : \mathrm{CH} \to \mathrm{H}$ compatible with pullbacks and pushforwards for proper morphisms.

Example 1.1. Set $d = 1$. Then we have a natural homomorphism

$$\mathrm{Pic}(X) \simeq \mathrm{CH}^1(X) \xrightarrow{\mathrm{cl}} \mathrm{H}^1(X, \Omega_X^1) \subset \mathrm{H}^1(X)$$

which can be viewed as a 1st Chern class in Hodge cohomology. When $k = \mathbb{C}$ we have a natural commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(X) \arrow[rr, "\mathrm{cl}"] \arrow[d, "c_1"] \arrow[drr, phantom, "\circlearrowleft"] & & \mathrm{H}^1(X, \Omega_X^1) \arrow[d, hook] \\
\mathrm{H}^2(X, \mathbb{Z}) \arrow[r] & \mathrm{H}^2(X, \mathbb{Z}) \otimes \mathbb{C} \simeq & \bigoplus_{p+q=2} \mathrm{H}^q(X, \Omega_X^p)
\end{tikzcd}$$

The **Lefschetz theorem on (1,1)-classes** states that the image of $\mathrm{Pic}(X)$ in $\mathrm{H}^2(X, \mathbb{Z})$ is the preimage of $\mathrm{H}^1(X, \Omega_X^1)$.

It's a remarkable fact that $\mathrm{H}^2(X, \mathbb{Z})$ classifies *topological* complex line bundles on X ("reason": $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$). Hence Lefschetz's theorem tells us when a topological complex line bundle on X is (topologically isomorphic to) an *algebraic* one.

1.2. **Idea: look at analogues for pairs/log schemes.**

2. Pairs

Definition 2.1.

(1) A **simple normal crossing pair** (X, Δ_X) is a smooth scheme over k together with a *reduced*, *effective* simple normal crossing divisor $\Delta_X \subset X$. The **interior** $U_X \subset X$ of a simple normal crossing pair is $U_X := X \setminus \Delta_X$.
(2) A **pulling morphism** $f : (X, \Delta_X) \to (Y, \Delta_Y)$ of simple normal crossing pairs is a map of schemes $f : X \to Y$ such that $f(U_X) \subset U_Y$.
(3) A **pushing morphism** $f : (X, \Delta_X) \to (Y, \Delta_Y)$ of simple normal crossing pairs is a *proper* map of schemes $f : X \to Y$ such that $f(U_X) \subset U_Y$ and $f^*\Delta_Y - \Delta_X$ is effective..

2.1. **Log differentials.**

[1] if $\operatorname{char} k > 0$ then the "coefficient field" will have positive charactersitic.

2.1.1. *Classical case: differentials with log poles.* A log smooth pair (X, Δ_X) comes with a sheaf of **differentials with log poles** $\Omega^1_X(\log \Delta_X)$. This naturally exists as the sheaf of differentials in the world of log geometry, but there's also a nice local description:

Proposition 2.2. *Let $x \in X$ be a closed point and let $z_1, \dots, z_n$ be local coordinates at x such that in a neighborhood of x*

$$\Delta_X = V(z_1 \cdots z_r)$$

Then near x the sheaf $\Omega_X(\log \Delta_X)$ is freely generated by

$$d \log z_1, \dots, d \log z_r, dz_{r+1}, \dots, dz_n$$

Definition 2.3. The **log Hodge cohomology of a simple normal crossing pair** (X, Δ_X) is the graded abelian group $\mathrm{H}^\bullet(X, \Delta_X)$

$$\mathrm{H}^d(X, \Delta_X) := \bigoplus_{p+q=d} \mathrm{H}^q(X, \Omega_X(\log \Delta_X))$$

Example 2.4. When X is a smooth projective curve of genus g, there are only 2 sheaves of log differential forms to consider:

$$\Omega^0_X(\log \Delta_X) = \mathcal{O}_X \text{ and } \Omega^1_X(\log \Delta_X) = \omega_X(\Delta_X)$$

$h^0(\mathcal{O}_X) = 1$ and $h^1(\mathcal{O}_X) = g$ per usual. Assume $\Delta_X \neq 0$ – then Δ_X is ample and since Kodaira vanishing always holds for curves, $h^1(\omega_X(\Delta_X)) = 0$. So, $h^0(\omega_X(\Delta_X))$ can be calculated with Riemann-Roch:

$$h^0(\omega_X(\Delta_X)) = \chi(\omega_X(\Delta_X)) = g - 1 + \deg \Delta_X$$

2.1.2. *Log Hartshorne II.8.*

3. Chow-of-the-complement

Chow for log schemes is a very active area of research. Here we use the most naïve possible version. For more interesting approaches, see e.g. [Bar], [BS17], [RS18]. There is also a growing body of work on algebraic K-theory of log schemes; see [Niz08], [Hag03].

3.1. **Complements and their Chow.**

Definition 3.1. The **Chow groups of a simple normal crossing pair** (X, Δ_X) are

$$\mathrm{CH}(X, \Delta_X) := \mathrm{CH}(U_X)$$

If $f : (X, \Delta_X) \to (Y, \Delta_Y)$ is a pulling morphism, since $f(U_X) \subset U_Y$ there's an induced morphism $f^* : \mathrm{CH}(Y, \Delta_Y) \to \mathrm{CH}(X, \Delta_X)$. If f is a pushing map, then the conditions $f(U_X) \subset U_Y$ and $f^*\Delta_Y - \Delta_X$ together require that $U_X = f^{-1}(U_Y)$, and hence $f|_{U_X}$ is *proper*. So there's a pushforward $f_* : \mathrm{CH}(X, \Delta_X) \to \mathrm{CH}(Y, \Delta_Y)$.

3.1.1. *Example: curves.* Suppose X is a smooth projective curve. Then Δ_X is just a bunch of points on X – say $\Delta_X = \{p_0, \dots, p_N\}$. For $d = 0, 1$ we have right exact sequences

$$\mathrm{CH}_d(\Delta_X) \xrightarrow{j_*} \mathrm{CH}_d(X) \xrightarrow{i^*} \mathrm{CH}_d(U_X) \to 0$$

when $d = 1$ this shows $\mathrm{CH}_1(U_X) \simeq \mathrm{CH}_1(X) \simeq \mathbb{Z}$. When $d = 0$, $\mathrm{CH}_0(\Delta_X) = \bigoplus_{i=0}^N \mathbb{Z}[p_i]$, and we have the identifications $\mathrm{CH}_0(X) = \mathrm{Cl}(X)$ and $\mathrm{CH}_0(U_X) = \mathrm{Cl}(U_X)$. Choose p_0 as a basepoint for

$\mathrm{Cl}(X)$ to get a splitting of the degree map $\mathrm{Cl}(X) \xrightarrow{\deg} \mathbb{Z}$, hence a decomposition

$$\mathrm{Cl}(X) \simeq \mathbb{Z}[p_0] \times \mathrm{Cl}^0(X)$$

We now get a diagram

$$\begin{tikzcd}
\mathbb{Z}p_0 \arrow[r, "\simeq"] \arrow[d] & \mathbb{Z}p_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
\bigoplus_{i=0}^N \mathbb{Z}p_i \arrow[r, "j_*"] \arrow[d] & \mathbb{Z}p_0 \times \mathrm{Cl}^0(X) \arrow[r] \arrow[d] & \mathrm{CH}_0(U) \arrow[r] \arrow[d, "\simeq"] & 0 \\
\bigoplus_{i=1}^N \mathbb{Z}p_i \arrow[r] & \mathrm{Cl}^0(X) \arrow[r] & \mathrm{CH}_0(U) \arrow[r] & 0
\end{tikzcd}$$

identifying $\mathrm{CH}_0(U)$ with the cokernel of the homomorphism

$$\bigoplus_{i=1}^{N} \mathbb{Z}p_i \to \mathrm{Cl}^0(X) \text{ sending } [p_i] \mapsto [p_i] - [p_0]$$

4. Construction of a cycle class

Even in the absolute case $\Delta_X = 0$, the construction of a cycle class $\mathrm{cl}_X(Z)$ for a subvariety $Z \subset X$ is non-trivial (since Z may be arbitrarily singular). It was first carried out by El Zein in [EZ78], and the key ideas remain the same in the logarithmic setting.

4.1. **Setup.** Let (X, Δ_X) be a simple normal crossing pair of dimension n and suppose $Z \subset X$ is a closed subvariety (possibly singular) of co-dimension c, with $Z \cap U_X \neq \emptyset$. This means if $\varphi^* : Z \to X$ is the inclusion then $\varphi^*\Delta_X$ is a Cartier divisor on Z.

The construction that follows appears in [BS17]:

4.2. **Case 1 (Z is normal).** In this case the smooth locus of Z contains the generic points of all components of $\varphi^*\Delta_X$. Since k is perfect $\operatorname{supp} \Delta_X$ is generically smooth. Moreover the *non-simple normal crossing locus* of (Z, Δ_Z) has codimension > 1 in Z and hence $> c+1$ in X.

So, after removing a closed subset $W \subset X$ with codimension $> c+1$ we may assume: Z is smooth and $\varphi^*\Delta_X$ is a simple normal crossing divisor.

The local cohomology exact sequence for the sheaf $\Omega^c_X(\log \Delta_X)$ at W reads

$$\begin{aligned}
\cdots \to &\mathrm{H}^c_W(X, \Omega^c_X(\log \Delta_X)) \to \mathrm{H}^c(X, \Omega^c_X(\log \Delta_X)) \\
\cdots \to &\mathrm{H}^c(X \setminus W, \Omega^c_{X\setminus W}(\log \Delta_{X\setminus W})) \to \mathrm{H}^{c+1}_W(X, \Omega^c_X(\log \Delta_X)) \to \cdots
\end{aligned}$$

We will make use of a lemma:

Lemma 4.1. *For a closed subset $W \subset X$ of codimension r,*

$$\mathrm{H}^i_W(X, \Omega^c_X(\log \Delta_X)) = 0 \textit{ for } i < r$$

Hence $\mathrm{H}^c(X, \Omega^c_X(\log \Delta_X)) = \mathrm{H}^c(X \setminus W, \Omega^c_{X\setminus W}(\log \Delta_{X\setminus W}))$.

In the case where (Z, Δ_Z) is smooth with simple normal crossings, apply Grothendieck Duality to the inclusion $\varphi : Z \hookrightarrow X$ and the coherent sheaf $\omega_Z(\Delta_Z)[\dim Z]$ to get a morphism

$$\begin{aligned}
&\varphi_* \mathcal{RH}om_Z(\omega_Z(\Delta_Z)[\dim Z], \omega_Z[\dim Z]) \simeq \mathcal{RH}om_X(\varphi_*\omega_Z(\Delta_Z)[\dim Z], \omega_X[\dim X]) \\
&\xrightarrow{D(d\varphi^\vee)} \mathcal{RH}om_X(\Omega^{\dim Z}_X(\log \Delta_X)[\dim Z], \omega_X[\dim X])
\end{aligned}$$

Using the perfect pairing

$$\Omega^p_X(\log \Delta_X) \otimes \Omega^{\dim X - p}_X(\log \Delta_X) \xrightarrow{\wedge} \omega_X(\Delta_X)$$

we have $\mathcal{RH}om_X(\Omega^p_X(\log \Delta_X), \omega_X) \simeq \Omega^{\dim X - p}_X(\log \Delta_X)(-\Delta_X)$ and similarly for Z, so that the morphism of Theorem 4.2 can be rewritten as

$$\varphi_* \mathcal{O}_Z(-\Delta_Z) \xrightarrow{D(df^\vee)} \Omega^c_X(\log \Delta_X)(-\Delta_X)[c]$$

or using the projection formula,

$$\varphi_* \mathcal{O}_Z = \varphi_* \mathcal{O}_Z(\varphi^* \Delta_X - \Delta_Z) \xrightarrow{D(d\varphi^\vee)} \Omega^c_X(\log \Delta_X)[c]$$

Now take global sections and let $\mathrm{cl}_{(X,\Delta_X)}(Z)$ be the image of $1_Z \in H^0(Z, \mathcal{O}_Z)$

4.3. **Case 2 (reduction to the normal case).** Since Z is a variety, its *normalization* $\pi : \tilde{Z} \to Z$ is finite, and hence projective in the sense that there's a locally free sheaf $\mathcal{F}$ on Z and a closed immersion $\psi : \tilde{Z} \hookrightarrow \mathbb{P}(\mathcal{F})$ over Z. Since X is smooth we can find a $\mathcal{F}$ of the form $\mathcal{F} = \mathcal{E}|_Z$ where $\mathcal{E}$ is locally free on X, and in this way we get a commutative diagram

$$\begin{tikzcd}
\tilde{Z} \arrow[r, hook, "\psi"] \arrow[dr, "\pi"'] & \mathbb{P}(\mathcal{E}|_Z) \arrow[r, "\varphi'"] \arrow[d, "\rho'"] \arrow[dr, phantom, "\square"] & \mathbb{P}(\mathcal{E}) \arrow[d, "\rho"] \\
 & Z \arrow[r, "\varphi"'] & X
\end{tikzcd}$$

Here $\tilde{Z} \subset \mathbb{P}(\mathcal{E})$ is normal and $\mathbb{P}(\mathcal{E})$ is smooth. Setting $\Delta_{\mathbb{P}(\mathcal{E})} = \rho^* \Delta_X$, we obtain a class $\mathrm{cl}_{\mathbb{P}(\mathcal{E})}(\tilde{Z}) \in \mathrm{H}^\bullet(\mathbb{P}(\mathcal{E}), \Omega^\bullet_{\mathbb{P}(\mathcal{E})}(\log \Delta_{\mathbb{P}(\mathcal{E})}))$.

The trick now is to set $\mathrm{cl}_X(Z) = \rho_* \mathrm{cl}_{\mathbb{P}(\mathcal{E})}(\tilde{Z})$.

References

[Bar] Lawrence J. Barrott, *Logarithmic chow theory*.

[BS17] F. Binda and S. Saito, *Relative cycles with moduli and regulator maps*, Journal of the Institute of Mathematics of Jussieu (2017), 1–61.

[dJ] Aise Johan de Jong, *Weil cohomology theories*.

[EZ78] Fouad El Zein, *Complexe dualisant et applications a la classe fondamentale d'un cycle*, Memoire (Societe mathematique de France) ; no 58, Societe mathematique de France, Paris, 1978 (fre).

[Hag03] K Hagihara, *Structure theorem of kummer etale k-group*, K-Theory **29** (2003), no. 2, 75–99 (English).

[Mus] Mircea Mustata, *Weil cohomology theories and the weil conjectures*.

[Niz08] W Niziol, *K-theory of log-schemes i*, Documenta Mathematica **13** (2008), 505–551 (English).

[RS18] Kay Rlling and Shuji Saito, *Higher chow groups with modulus and relative milnor K-theory*, Transactions of the American Mathematical Society **370** (2018), no. 2, 987–1043 (eng).