Higher direct images of ideal sheaves and correspondences in log Hodge cohomology

Charles Godfrey

May 25, 2021

Abstract

This document consists of three mathematically independent and more or less thematically independent parts. Part I concerns invariance of the cohomology groups of divisorial ideal sheaves under (a restricted class of) birational morphisms of pairs in arbitrary characteristic, and as an application extends some foundational results in the theory of rational pairs that were previously known only in characteristic 0. Part II discusses correspondences in logarithmic Hodge theory related to an as-of-yet-unsuccessful alternative strategy for proving the main theorems of Part I.

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The author was partially supported by the University of Washington Department of Mathematics Graduate Research Fellowship, and by the NSF grant DMS-1440140, administered by the Mathematical Sciences Research Institute, while in residence at MSRI during the program Birational Geometry and Moduli Spaces.

Part I
Higher direct images of ideal sheaves

1 Introduction to Part I

A foundational problem in birational geometry, posed by Grothendieck in his 1958 ICM address [Gro60, Problem B], asked whether for every proper birational morphism of non-singular projective varieties $f : X \to Y$,

$$R^i f_* \mathcal{O}_X = 0 \text{ for } i > 0.$$

In characteristic 0 this was answered affirmatively by Hironaka as a corollary of resolution of singularities [Hir64, §7 Cor. 2]. In characteristic $p > 0$, where resolutions of singularities are not known to exist, answering Grothendieck's question proved much harder, remaining open until 2011 when Chatzistamatiou and Rülling proved the following theorem.

Theorem 1.1 ([CR11, Thm. 3.2.8], see also [CR15, Thm. 1.1] [Kov20, Thm. 1.6])**.** *Let k be a perfect field and let S be a separated scheme of finite type over k. Suppose X and Y are two separated finite type S-schemes which are*

(i) smooth over k and

(ii) ***properly birational*** *over S in the sense that there is a commutative diagram*

$$\begin{tikzcd} & Z \arrow[dl, "r"'] \arrow[dr, "s"] \arrow[dd, phantom, "\circlearrowleft"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd} \tag{1.2}$$

with r and s proper birational morphisms.

Set $n = \dim X = \dim Y = \dim Z$. Then there are isomorphisms of sheaves

$$R^i f_* \mathcal{O}_X \xrightarrow{\sim} R^i g_* \mathcal{O}_Y \text{ and } R^i f_* \omega_X \xrightarrow{\sim} R^i g_* \omega_Y \text{ for all } i, \tag{1.3}$$

One of the primary applications of Theorem 1.1 was to extend foundational results on rational singularities from characteristic 0 to arbitrary characteristic (for definitions of rational resolutions and rational singularities see Definition 6.2).

Corollary 1.4 ([CR11, Cor. 3.2.10],[Kov20, Thm. 1.4])**.** *If S has a rational resolution, then every resolution of S is rational.*

Part I concerns analogues of Theorem 1.1 for pairs.

Definition 1.5 (slightly more general version of [SingsMMP 2013, Def. 1.5])**.** In what follows a **pair** (X, Δ_X) will mean a reduced, equidimensional excellent scheme X admitting a dualizing complex together with a $\mathbb{Q}$-Weil divisor $\Delta_X = \sum_i a_i D_i$ on X such that no irreducible component D_i of Δ_X is contained in $\mathrm{Sing}(X)$.

Definition 1.6. A *simple normal crossing pair* is an equidimensional, regular excellent scheme X together with a reduced effective divisor $\Delta_X = \sum_i D_i$ such that for every subset $J \subseteq \{1, \dots, N\}$ the scheme-theoretic intersection

$$D_J := \cap_{j \in J} D_j \subseteq X$$

is regular of codimension $|J|$.

Remark 1.7. If X is regular as in Definition 1.6 then it admits a dualizing complex. By an amazing result of Kawasaki [Kaw02, Cor. 1.4], a noetherian ring admits a dualizing complex if and only if it is a homomorphic image of a finite-dimensional Gorenstein ring.

As observed in [SingsMMP 2013, §2.5], to generalize Corollary 1.4 to pairs we must restrict attention to a special class of *thrifty resolutions* (see Definition 3.5).

Theorem 1.8 (Corollary 5.21)**.** *Let S be an excellent noetherian scheme and let (X, Δ_X) and (Y, Δ_Y) be simple normal crossing pairs separated and of finite type over S. Suppose $(X, \Delta_X), (Y, \Delta_Y)$ are properly birational over S in the sense that there is a commutative diagram*

$$\begin{tikzcd} & (Z, \Delta_Z) \arrow[dl, "r"'] \arrow[dr, "s"] \arrow[d, phantom, "\circlearrowleft"] & \\ (X, \Delta_X) \arrow[dr, "f"'] & {} & (Y, \Delta_Y) \arrow[dl, "g"] \\ & S & \end{tikzcd} \tag{1.9}$$

where r, s are proper and birational morphisms, and assume $\Delta_Z = r_*^{-1}\Delta_X = s_*^{-1}\Delta_Y$. If r, s are thrifty *then there are quasi-isomorphisms*

$$Rf_*\mathcal{O}_X(-\Delta_X) \simeq Rg_*\mathcal{O}_Y(-\Delta_Y) \textit{ and } Rf_*\omega_X(\Delta_X) \simeq Rg_*\omega_Y(\Delta_Y). \tag{1.10}$$

Via the same methods as Theorem 1.8, we obtain an analogue of Corollary 1.4 (for definitions of rational resolutions and rational singularities for pairs see Definitions 6.2 and 6.5).

Theorem 1.11 ([SingsMMP 2013, Cor. 2.86] in characteristic 0, Lemma 6.6 in arbitrary characteristic)**.** *Let (S, Δ_S) be a pair, with Δ_S reduced and effective. If (S, Δ_S) has a thrifty rational resolution $f : (X, \Delta_X) \to (S, \Delta_S)$, then every thrifty resolution $g : (Y, \Delta_Y) \to (S, \Delta_S)$ is rational.*

Our methods rely on the machinery of semi-simplicial schemes; we feel obligated to provide some motivation for their use in the context of Theorem 1.8. To begin, we can translate the condition that a birational morphism $f : (X, \Delta_X) \to (S, \Delta_S)$ of simple normal crossing pairs[1] with $\Delta_X = f_*^{-1}\Delta_S$ is thrifty into the statement that the *dual complexes* $\mathcal{D}(\Delta_X)$ and $\mathcal{D}(\Delta_S)$ are isomorphic. The dual complex $\mathcal{D}(\Delta_X)$ is usually described as the Δ-complex (in the sense of [Hat02, §2.1]) with 0-cells the irreducible components D_i^X of $\Delta_X = \sum_i D_i^X$, 1-cells the components of intersections $D_i^X \cap D_j^X$ for $i < j$ with gluing maps corresponding to the inclusions $D_i^X \cap D_j^X \subseteq D_i^X$ and $D_i^X \cap D_j^X \subseteq D_j^X$, and so on (this is a generalization of the dual graph of a nodal curve). The topological properties of $\mathcal{D}(\Delta_X)$ have been extensively studied, for example in this non-exhaustive list of references: [ABW13; Dan75; FKX17; Ste06]. Upon inspection we see that a Δ-complex is precisely a semi-simplicial set, and that

[1] we can relax the condition that both pairs are snc, but it will make this motivational discussion simpler.

$\mathcal{D}(\Delta_X)$ is the semi-simplical set obtained by taking π_0 (connected components) of a semi-simplicial *scheme* $X_\bullet$, with

$$X_i = \coprod_{|J|=i+1} (\cap_{j\in J} D_j^X) \text{ for } i \geq 0$$

The thriftyness hypotheses of Theorem 1.8 ensure that (X, Δ_X) and (Y, Δ_Y) have the same dual complex, which provides enough rigidity to attempt to prove Theorem 1.8 by induction on $\dim X$ and the number of components of Δ_X, using Theorem 1.1 as a base case. For example, we have exact sequences

$$0 \to \mathcal{O}_X(-\Delta_X) \to \mathcal{O}_X(-\Delta_X + D_1^X) \to \mathcal{O}_{D_1^X}(-\Delta_X + D_1^X|_{D_1^X}) \to 0$$

and similarly on Y. We can even assume by induction the existence of already-defined quasi-isomorphisms in a diagram

$$\begin{tikzcd}
Rf_*\mathcal{O}_X(-\Delta_X) \arrow[r] \arrow[d, dashed, "\alpha"] & Rf_*\mathcal{O}_X(-\Delta_X + D_1^X) \arrow[r, "\rho^X"] \arrow[d, "\beta", "\simeq"'] \arrow[dr, phantom, "(*)"] & Rf_*\mathcal{O}_{D_1^X}(-\Delta_X + D_1^X|_{D_1^X}) \arrow[r] \arrow[d, "\gamma", "\simeq"'] & \cdots \\
Rg_*\mathcal{O}_Y(-\Delta_Y) \arrow[r] & Rg_*\mathcal{O}_Y(-\Delta_Y + D_1^Y) \arrow[r, "\rho^Y"] & Rg_*\mathcal{O}_{D_1^Y}(-\Delta_Y + D_1^Y|_{D_1^Y}) \arrow[r] & \cdots
\end{tikzcd} \tag{1.12}$$

If the square $(*)$ commutes, then using only the fact that $D^b_{\mathrm{coh}}(S)$ is a triangulated category we get a quasi-isomorphism α on the dashed arrow. However, in this approach β, γ are themselves defined by induction, and so to know $(*)$ commutes we must take one inductive step further, considering maps of distinguished triangles

$$\begin{tikzcd}
Rf_*\mathcal{O}_X(-\Delta_X + D_1^X) \arrow[r] \arrow[d] & Rf_*\mathcal{O}_X(-\Delta_X + D_1^X + D_2^X) \arrow[r] \arrow[d] & Rf_*\mathcal{O}_{D_2^X}(-\Delta_X + D_1^X|_{D_2^X}) \arrow[r] \arrow[d] & \cdots \\
Rg_*\mathcal{O}_Y(-\Delta_Y + D_1^Y) \arrow[r] & Rg_*\mathcal{O}_Y(-\Delta_Y + D_1^Y + D_2^Y) \arrow[r] & Rg_*\mathcal{O}_{D_2^Y}(-\Delta_Y + D_1^Y|_{D_2^Y}) \arrow[r] & \cdots
\end{tikzcd} \quad \text{and} \tag{1.13}$$

$$\begin{tikzcd}
Rf_*\mathcal{O}_{D_1^X}(-\Delta_X + D_1^X|_{D_1^X}) \arrow[r] \arrow[d] & Rf_*\mathcal{O}_{D_1^X}(-\Delta_X + D_1^X + D_2^X|_{D_1^X}) \arrow[r] \arrow[d] & Rf_*\mathcal{O}_{D_1^X \cap D_2^X}(-\Delta_X + D_1^X + D_2^X|_{D_1^X \cap D_2^X}) \arrow[r] \arrow[d] & \cdots \\
Rg_*\mathcal{O}_{D_1^Y}(-\Delta_Y + D_1^Y|_{D_1^Y}) \arrow[r] & Rg_*\mathcal{O}_{D_1^Y}(-\Delta_Y + D_1^Y + D_2^Y|_{D_1^Y}) \arrow[r] & Rg_*\mathcal{O}_{D_1^Y \cap D_2^Y}(-\Delta_Y + D_1^Y + D_2^Y|_{D_1^Y \cap D_2^Y}) \arrow[r] & \cdots
\end{tikzcd} \tag{1.14}$$

together with a map from (1.13) to (1.14) including the square $(*)$, and so on. It is certainly possible that the correct induction hypothesis (building in not only quasi-isomorphisms like β, γ in (1.12) but also commutativity hypotheses) and some careful analysis of diagrams in $D^b_{\mathrm{coh}}(S)$ could make this strategy work, but I had no such luck.

Another technical issue this approach encounters is that at some point in the base case, we must analyze how the isomorphisms of Theorem 1.1 behave with respect to restrictions, i.e. diagrams of schemes like

$$\begin{tikzcd}
D_1^X \arrow[d] & D_1^Z \arrow[l] \arrow[r] \arrow[d] & D_1^Y \arrow[d] \\
X & Z \arrow[l] \arrow[r] & Y
\end{tikzcd}$$

Delving into the methods of [CR11; CR15; Kov20], this analysis runs into subtle aspects of Grothendieck duality, *especially* since for this approach to work we do require morphisms in $D^b_{\mathrm{coh}}(S)$, not simply of cohomology sheaves as in Theorem 1.1.

What is clear is that this attempted induction takes place on the semi-simplicial schemes $X_\bullet$ and $Y_\bullet$ underlying the dual complexes $\mathcal{D}(\Delta_X)$ and $\mathcal{D}(\Delta_Y)$. Under necessary thriftiness hypotheses, in the situation of Theorem 1.8 we find that there is also an auxiliary semi-simplicial scheme $Z_\bullet$ together with morphisms $X_\bullet \xleftarrow{r_\bullet} Z_\bullet \xrightarrow{s_\bullet} Y_\bullet$ which are birational in each simplicial degree. Using refined forms of Chow's lemma and resolution of indeterminacies [Con07], together with the existence of Macaulayfications [es21; Kaw00], we can find such a $Z_\bullet$ where each scheme Z_i is Cohen-Macaulay and the morphisms $X_i \xleftarrow{r_i} Z_i \xrightarrow{s_i} Y_i$ are projective — this occupies Sections 3 and 5. We then make essential use of [Kov20, Thm. 1.4] to conclude that there are natural maps $\mathcal{O}_{X_i} \to Rr_{i*}\mathcal{O}_{Z_i}$ and $\mathcal{O}_{Y_i} \to Rs_{i*}\mathcal{O}_{Z_i}$ are quasi-isomorphisms. A more detailed overview of this construction is included at the beginning of Section 5.

The remainder of our proof is pure homological algebra: in Section 2 we show that when (X, Δ_X) is a simple normal crossing pair (more generally, when the components D_i^X of Δ_X form a *regular sequence*, see Definition 2.5) the ideal sheaf $\mathcal{O}_X(-\Delta_X)$ admits a Čech-type resolution of the form

$$\mathcal{O}_X(-\Delta_X) \to \mathcal{O}_X \to \mathcal{O}_{X_0} \to \mathcal{O}_{X_1} \to \cdots,$$

in other words we can recover $\mathcal{O}_X(-\Delta_X)$ from an augmentation morphism $X_\bullet \to X$. Moreover, we can recover the *cohomology* of $\mathcal{O}_X(-\Delta_X)$ from a descent-type spectral sequence Corollary 2.18 — the last major technical ingredient is a comparison of the resulting spectral sequences associated to X, Y and Z.

Section 6 deals with applications to rational pairs, in particular Corollary 1.4, and Section 4 includes some new examples illustrating the subtleties of thrifty and rational resolutions of pairs, for instance we affirmatively answer a question of Erickson and Prelli on whether there exists a non-thrifty rational resolution of a pair (S, Δ) — our (S, Δ) is even a rational pair, and the resolution is related to the famous Atiyah flop.

1.1 Acknowledgements

I would like to thank Jarod Alper, Chi-yu Cheng, Kristin DeVleming, Gabriel Dorfsman-Hopkins, Max Lieblich, Takumi Murayama, Karl Schwede and Tuomas Tajakka for helpful conversations, and my advisor Sándor Kovács for proposing the problem of extending results Theorem 1.1 to pairs and for many valuable suggestions.

2 Regular sequences of divisors and descent spectral sequences

2.1 Semi-simplicial schemes and their derived categories

To any simple normal crossing pair we can naturally associate a *semi-simplicial scheme*. A primary reference for the theory of semi-simplicial schemes is [SGA4II, Vbis]; since many elementary facts about *simplicial* schemes carry over to semi-semi-simplicial schemes, [Con03], [Ols16, §2.4] and [Stacks, Tag 0162] are also relevant. What follows is a condensed summary of the machinery we need.

Let Λ denote the category with objects the sets $[i] := \{0, 1, 2, \ldots, i\}$ for $i \in \mathbb{N}$ and with morphisms the *strictly increasing* functions $[j] \to [i]$; in particular $\mathrm{Hom}_\Lambda([j], [i]) = \emptyset$ if $j > i$.[2] A *semi-simplicial object* in a category $\mathcal{C}$ is a functor $\Lambda^{\mathrm{op}} \to \mathcal{C}$; semi-simplicial $\mathcal{C}$-objects naturally form a category, the functor category $\mathcal{C}^{\Lambda^{\mathrm{op}}}$. Any morphism $\varphi : [j] \to [i]$ can be written non-uniquely as a composition of the basic morphisms

$$\delta_k^i : [i-1] \longmapsto [i] \text{ defined by } \delta_k^i(x) = \begin{cases} x & \text{if } x < k \\ x+1 & \text{otherwise} \end{cases}$$

(so δ_k^i skips k) [Stacks, Tag 0164], and hence a semi-simplicial object $X : \Lambda^{\mathrm{op}} \to \mathcal{C}$ is equivalent to a sequence of objects $X_i := X([i])$ together with morphisms

$$d_k^i := X(\delta_k^i) : X_i \to X_{i-1} \text{ subject to the relations } d_k^{i-1} \circ d_l^i = d_{l-1}^{i-1} \circ d_k^i, \tag{2.1}$$

[2] In [SGA4II, Vbis] Λ is denoted by Δ^+ so this notation is non-standarad, but seemed necessary due to the number of divisors Δ and pairs (X, Δ) considered below. My apologies.

and all semi-simplicial objects below will be obtained from such an explicit description. In what follows semi-simplicial objects will be denoted with a $\bullet$, e.g. "the semi-simplicial scheme $X_\bullet$" (to distinguish them from plain schemes).

When $\mathcal{C}$ is a category of schemes, a *sheaf* on a semi-simplicial scheme $X_\bullet$ is the data of a sheaf $\mathcal{F}_i$ on each scheme X_i together with morphisms of sheaves $\delta^i_k : \mathcal{F}_{i-1} \to d^i_{k*}\mathcal{F}_i$ on X_{i-1} satisfying compatibilities coming from (2.1). These sheaves form a *topos* $\tilde{X}_\bullet$ such that morphisms of semi-simplicial schemes $f_\bullet : X_\bullet \to Y_\bullet$ induce functorial maps of topoi $\tilde{X}_\bullet \to \tilde{Y}_\bullet$ (see [SGA4II, Vbis, Prop. 1.2.15]) — the benefit of the topos-theoretic point of view is that it immediately implies the category of *abelian* sheaves $\mathbf{Ab}(X_\bullet)$ on $X_\bullet$ is an abelian category with enough injectives ([Stacks, Tag 01DL]), enables us to define pushforward functors $Rf_* : D^+(\mathbf{Ab}(X_\bullet)) \to D^+(\mathbf{Ab}(Y_\bullet))$ for morphisms of semi-simplicial schemes $f_\bullet : X_\bullet \to Y_\bullet$, and so on.

An *augmented* semi-simplicial scheme is a morphism of semi-simplicial schemes $\epsilon_\bullet : X_\bullet \to S_\bullet$ where $S_\bullet$ is a *constant* semi-simplicial scheme (that is, $S_i = S$ for all i for some fixed scheme S, and all $d^i_k = \mathrm{id}$). This is equivalent to the data of a semi-simplicial object of $\mathbf{Sch}_S$. For such a constant semi-simplicial scheme $S_\bullet$, $\mathbf{Ab}(S_\bullet)$ is equivalent to the category $\mathbf{Ab}(S)^\Lambda$ of co-semi-simplicial sheaves of abelian groups on S, that is, sequences of sheaves of abelian groups $\mathcal{G}_i$ on S together with morphisms $\delta^i_k : \mathcal{G}_{i-1} \to \mathcal{G}_i$ satisfying compatibilities forced by (2.1). As in the construction of the Čech complex setting $d^i = \sum_k (-1)^k : \delta^i_k : \mathcal{G}_{i-1} \to \mathcal{G}_i$ gives a complex of abelian sheaves on S and hence in particular an abelian sheaf $a(\mathcal{G}_\bullet) := \ker d^0$. Writing $\epsilon_* := a \circ \epsilon_{\bullet *}$, the composite derived functor

$$\begin{tikzcd} D^+(\mathbf{Ab}(X_\bullet)) \arrow[r, "R\epsilon_{\bullet *}"] \arrow[rr, bend left, "R\epsilon_*"] & D^+(\mathbf{Ab}(S_\bullet)) \arrow[r, "Ra"] & D^+(\mathbf{Ab}(S)) \end{tikzcd}$$

admits the following concrete description: given a sheaf $\mathcal{F}_\bullet$ on $X_\bullet$ one takes an injective resolution

$$\mathcal{F}_\bullet \to \mathcal{I}^0_\bullet \to \mathcal{I}^1_\bullet \to \mathcal{I}^2_\bullet \to \dots \text{ in } \mathbf{Ab}(X_\bullet)$$

Here the $\mathcal{I}^j_\bullet$ are in particular sheaves on $X_\bullet$ with each $\mathcal{I}^j_i$ an injective abelian sheaf on X_i — for further discussion of injective objects in $\mathbf{Ab}(X_\bullet)$ see [SGA4II, Vbis, Prop. 1.3.10] and [Con03, Lem. 6.4, comments on p. 42]. Then

$$\epsilon_{\bullet *}\mathcal{I}^0_\bullet \to \epsilon_{\bullet *}\mathcal{I}^1_\bullet \to \epsilon_{\bullet *}\mathcal{I}^2_\bullet \to \dots \text{ in } \mathbf{Ab}(S_\bullet)$$

is a complex of co-semi-simplicial abelian sheaves which via the Čech construction becomes a complex of complexes. Applying the sign trick gives a double complex whose Tot computes $R\epsilon_*\mathcal{F}_\bullet$. One of the spectral sequences of this double complex is displayed below. In our calculations it is *crucial* that this spectral sequence is (at least in a minimal sense) functorial.

Lemma 2.2 (Descent spectral sequence, [SGA4II, Vbis §2.3], [Con03, Thms. 6.11-6.12])**.** *If $\mathcal{F}_\bullet$ is an abelian sheaf on an augmented semi-simplicial scheme $\epsilon : X_\bullet \to S$ then there is a spectral sequence*

$$E_1^{pq} = R^q\epsilon_{p*}\mathcal{F}_p \to R^{p+q}\epsilon_*\mathcal{F}_\bullet$$

Moreover if $\mathcal{G}_\bullet$ is an abelian sheaf on another augmented semi-simplicial scheme $\epsilon' : Y_\bullet \to T$ and

$$\begin{tikzcd} Y_\bullet \arrow[r, "g_\bullet"] \arrow[d, "\epsilon'"] & X_\bullet \arrow[d, "\epsilon"] \\ Y \arrow[r, "g"] & S \end{tikzcd}$$

is a map of augmented semi-simplicial schemes together with a map of abelian sheaves $\varphi : \mathcal{F}_\bullet \to g_{\bullet *}\mathcal{G}_\bullet$ on $X_\bullet$, then φ induces a morphism of spectral sequences

$$E_1^{pq}(\mathcal{F}_\bullet) = R^q\epsilon_{p*}\mathcal{F}_p \to R^q(\epsilon \circ g_p)_*\mathcal{G}^p = E_1^{pq}(\mathcal{G}_\bullet)$$

converging to the morphism $R\epsilon_*(\varphi) : R\epsilon_*\mathcal{F}_\bullet \to R\epsilon_* Rg_{\bullet *}\mathcal{G}_\bullet = Rg_* R\epsilon'_*\mathcal{G}^\bullet$.

Proof of the "Moreover ...". We work with the abelian categories of sheaves of abelian groups on $Y_\bullet$, $X_\bullet$. Let $\mathcal{J}_\bullet^\bullet$ be an injective resolution of $\mathcal{G}_\bullet$ in $\mathbf{Ab}(Y_\bullet)$. Then $f_{\bullet *}\mathcal{J}_\bullet^\bullet$ is a complex of injectives (this uses the fact that $f_{\bullet *}$ has an *exact* left adjoint f^{-1}), $\mathcal{F}_\bullet \to \mathcal{I}_\bullet^\bullet$ is a quasi-isomorphism and we are given a map

$$\varphi : \mathcal{F}_\bullet \to f_{\bullet *}\mathcal{G}_\bullet \to f_{\bullet *}\mathcal{J}_\bullet^\bullet$$

By [Stacks, Tag 013P] (see also [Wei94, Thm. 2.2.6]) there is a map of complexes of abelian sheaves on $X_\bullet$ extending φ:

$$\tilde{\varphi} : \mathcal{I}_\bullet^\bullet \to f_{\bullet *}\mathcal{J}_\bullet^\bullet$$

Applying $\epsilon_{\bullet *}$ then gives a morphism of complexes of co-semi-simplicial abelian sheaves on S consisting of morphisms

$$\epsilon_{p*}\mathcal{I}_p^q \to \epsilon_{p*}g_{p*}\mathcal{J}_p^q$$

compatible with *both* the simplicial sheaf maps (in the p direction) *and* the injective resolution maps (in the q) direction, to which we may apply the Čech construction and sign trick to obtain a map of double complexes. This reduces us to the claim that a map of double complexes (or more generally a filtered map of filtered complexes) induces a map of spectral sequences, which we take as well known. □

Remark 2.3. The above proof is at least suggested in the last sentence of [Con03, Thm. 6.11]. An alternative method would be to use Deligne's trick of viewing φ as an abelian sheaf on the $\Lambda \times I$ scheme associated to $f_\bullet$ — for related discussion see [SGA4II, Vbis, §3.1].

Corollary 2.4. *In the situation of Lemma 2.2 suppose in addition that the morphisms $\varphi_p : \mathcal{F}_p \to Rf_{p*}\mathcal{G}_p$ are quasi-isomorphisms for all p. Then, the induced morphism*

$$R\epsilon_*(\varphi) : R\epsilon_*\mathcal{F}_\bullet \to R\epsilon_* Rg_{\bullet *}\mathcal{G}_\bullet = Rg_* R\epsilon'_*\mathcal{G}^\bullet$$

is a quasi-isomorphism.

2.2 Regular sequences of divisors

Definition 2.5. Let X be a locally noetherian scheme. A sequence of effective Cartier divisors $D_1, D_2, \dots, D_N \subseteq X$ is called *regular* if and only if for each point $x \in X$, letting $f_1, \cdots, f_N \in \mathcal{O}_{X,x}$ be local generators for the divisors D_i and letting $I(x) = \{i \mid x \in D_i\}$, the elements $(f_j \in \mathfrak{m}_x \mid j \in I(x))$ form a regular sequence.

This definition is designed to ensure that a permutation of a regular sequence of divisors is again a regular sequence (see [Mat80, §15, Thm. 27], [Stacks, Tag 00LJ]).

Let X be a locally noetherian scheme together with a regular sequence of effective Cartier divisors $D_1, D_2, \dots, D_N \subseteq X$. We define an augmented semi-simplicial scheme $X_\bullet$ as follows: $X_{-1} = X$, $X_0 = \coprod_i D_i$ and for $k > 0$,

$$X_k = \coprod_{I \subseteq \{1,\dots,N\} \mid |I| = k+1} D_I, \text{ where } D_I = \bigcap_{j \in I} D_j$$

The face maps are defined by the inclusions $d_k^j : D_I \hookrightarrow D_{I \setminus \{i_j\}}$ for $I = \{i_0, \dots, i_k\}$ and $0 \le j \le i$, as in a Čech complex, and for each k we have an augmentation map $\epsilon_p : X_k \to X$ obtained from the inclusions $D_I \subseteq X$. In this situation the descent spectral sequence of Lemma 2.2 degenerates: since the $\epsilon_p : X_p \to X$ are closed immersions and hence affine, $R^q\epsilon_{p*}\mathcal{O}_{X_p} = 0$ for $q > 0$. It follows that $R^i\epsilon_*\mathcal{O}_{X_\bullet}$ is the cohomology of the Čech type complex

$$\begin{aligned} &\epsilon_{0*}\mathcal{O}_{X_0} \xrightarrow{d^1} \epsilon_{1*}\mathcal{O}_{X_1} \xrightarrow{d^2} \cdots \xrightarrow{d^N} \epsilon_{N*}\mathcal{O}_{X_N} \\ &= \bigoplus_i \mathcal{O}_{D_i} \xrightarrow{d^1} \bigoplus_{i<j} \mathcal{O}_{D_i \cap D_j} \xrightarrow{d^2} \cdots \xrightarrow{d^N} \mathcal{O}_{\cap_i D_i} \end{aligned} \tag{2.6}$$

Lemma 2.7. *The complex (2.6) is exact in degrees $i > 0$, with* $\ker d^1 \simeq \mathcal{O}_{\cup_i D_i}$. *Equivalently, the extended complex*

$$0 \to \mathcal{O}_X(-\sum_i D_i) \to \mathcal{O}_X \xrightarrow{\gamma} \bigoplus_i \mathcal{O}_{D_i} \xrightarrow{d^1} \bigoplus_{i<j} \mathcal{O}_{D_i \cap D_j} \xrightarrow{d^2} \cdots \xrightarrow{d^N} \mathcal{O}_{\cap_i D_i} \to 0$$

where $\gamma : \mathcal{O}_X \to \bigoplus_i \mathcal{O}_{D_i}$ is restriction in each factor is exact, and hence there is a canonical quasi-isomorphism $\mathcal{O}_X(-\sum_i D_i) \simeq \mathrm{cone}(\mathcal{O}_X \to R\epsilon_* \mathcal{O}_{X_\bullet})[-1]$.

Proof. We proceed by induction on the number N of divisors. The base case $N = 0$ is vacuous ($X_\bullet$ is empty). If that seems too weird, the case $N = 1$ simply says that the sequence $0 \to \mathcal{O}_X(-D_1) \to \mathcal{O}_X \to \mathcal{O}_{D_1} \to 0$ is exact, which is indeed the case as D_1 is an effective Cartier divisor.

Suppose now that $N > 1$. Then by the definition of a regular sequence, $D_1 \cap D_2, D_1 \cap D_3, \dots, D_1 \cap D_N \subseteq D_1$ is a regular sequence of divisors, and by permutation invariance of regular sequences (for *noetherian local rings* [Mat80, §15, Thm. 27], [Stacks, Tag 00LJ] — this dictated Definition 2.5) $D_2, \dots, D_N \subseteq X$ is a regular sequence. We form a short exact sequence of complexes (with cohomological degrees as indicated)

$$\begin{array}{ccccccccccc}
C' : & & 0 & \longrightarrow & \mathcal{O}_{D_1} & \xrightarrow{d'} & \bigoplus_{1<j} \mathcal{O}_{D_1 \cap D_j} & \xrightarrow{d'} & \bigoplus_{1<j<k} \mathcal{O}_{D_1 \cap D_j \cap D_k} & \xrightarrow{d'} & \cdots \\
\downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \\
C : & & \mathcal{O}_X & \xrightarrow{\gamma} & \bigoplus_i \mathcal{O}_{D_i} & \xrightarrow{d} & \bigoplus_{i<j} \mathcal{O}_{D_i \cap D_j} & \xrightarrow{d} & \bigoplus_{i<j<k} \mathcal{O}_{D_i \cap D_j \cap D_k} & \xrightarrow{d} & \cdots \\
\downarrow{\scriptstyle\beta} & & \|{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & \\
C'' : & & \mathcal{O}_X & \xrightarrow{\gamma''} & \bigoplus_{1<i} \mathcal{O}_{D_i} & \xrightarrow{d''} & \bigoplus_{1<i<j} \mathcal{O}_{D_i \cap D_j} & \xrightarrow{d''} & \bigoplus_{1<i<j<k} \mathcal{O}_{D_i \cap D_j \cap D_k} & \xrightarrow{d''} & \cdots \\
& & & & & & & & & & \\
& & -1 & & 0 & & 1 & & 2 & &
\end{array} \tag{2.8}$$

(in fact by comparing ranges of indices we can see the columns are *split* short exact sequences). By inductive hypotheses,

$$h^i(C') = \begin{cases} \mathcal{O}_{D_1}(-\sum_{1<j} D_1 \cap D_j) & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases} \text{ and } h^i(C'') = \begin{cases} \mathcal{O}_X(-\sum_{1<j} D_j) & \text{if } i = -1 \\ 0 & \text{otherwise} \end{cases}$$

showing that $h^i(C) = 0$ for $i > 0$, and that in low degrees there is an exact sequence

$$0 \to h^{-1}(C) \to \mathcal{O}_X(-\sum_{1<j} D_j) = h^{-1}(C'') \xrightarrow{\delta} h^0(C') = \mathcal{O}_{D_1}(-\sum_{1<j} D_1 \cap D_j) \to 0 \tag{2.9}$$

To complete the proof, we must verify that the connecting map δ is indeed restriction of sections, so that (2.9) coincides with the usual exact sequence

$$0 \to \mathcal{O}_X(-\sum_j D_j) \to \mathcal{O}_X(-\sum_{1<j} D_j) \to \mathcal{O}_{D_1}(-\sum_{1<j} D_1 \cap D_j) \to 0$$

and indeed, by the snake lemma construction of the connecting map δ we lift a local section $\sigma \in \ker \gamma'' \subseteq \mathcal{O}_X$ along β, apply $\gamma : \mathcal{O}_X \to \oplus_i \mathcal{O}_{D_i}$ to obtain a local section $(\sigma|_{D_i}) \in \ker \beta \subseteq \oplus_i \mathcal{O}_{D_i}$, and then lift along $\alpha : \mathcal{O}_{D_1} \to \oplus_i \mathcal{O}_{D_i}$ — the net result is $\sigma|_{D_1}$ as claimed. □

Remark 2.10. Here we sketch a different proof of Lemma 2.7, which could potentially shed more light on what happens if $D_1, \dots, D_N \subseteq X$ deviates from being a regular sequence. For each i let $\sigma_i : \mathcal{O}_X \to \mathcal{O}_X(D_i)$ be the canonical global section and let $\sigma_i^\vee : \mathcal{O}_X(-D_i) \to \mathcal{O}_X$ be its dual. For each subset

$J \subseteq \{1, \dots, N\}$ let $\mathscr{E}_J := \oplus_{j\in J}\mathscr{O}_X(D_j)$. For each such J we have a section $\sigma_J = (\sigma_j | j \in J) : \mathscr{O}_X \to \mathscr{E}_J$. There's a map of chain complexes

$$\begin{tikzcd}
0 = \mathscr{E}_\emptyset \arrow[r] & \bigoplus_{|J|=1}\mathscr{E}_J \arrow[r] & \bigoplus_{|J|=2}\mathscr{E}_J \arrow[r] & \bigoplus_{|J|=3}\mathscr{E}_J \arrow[r] & \cdots \\
\mathscr{O}_X \arrow[r] \arrow[u] & \bigoplus_{|J|=1}\mathscr{O}_X \arrow[r] \arrow[u, "\oplus\sigma_J"] & \bigoplus_{|J|=2}\mathscr{O}_X \arrow[r] \arrow[u, "\oplus\sigma_J"] & \bigoplus_{|J|=3}\mathscr{O}_X \arrow[r] \arrow[u, "\oplus\sigma_J"] & \cdots
\end{tikzcd}$$

where the horizontal differentials are alternating sums of summand inclusions (in effect, they come from the singular co-chain complex of the $N-1$-simplex Δ^{N-1}) and the vertical maps are induced by the σ_i. Applying the Koszul construction to the individual maps $\sigma_J : \mathscr{O}_X \to \mathscr{E}_J$ (along with the usual sign trick) then results in a double complex $C^{\bullet\bullet}$ with $C^{pq} = \bigoplus_{|J|=p}\wedge^{-q}\mathscr{E}_J^\vee$.

I claim without proof[3] that the *horizontal* complexes

$$C^{\bullet q} : 0 \to \cdots \to 0 \to \bigoplus_{|J|=-q}\wedge^{-q}\mathscr{E}_J^\vee \to \bigoplus_{|J|=-q+1}\wedge^{-q}\mathscr{E}_J^\vee \to \cdots \to \bigoplus_{|J|=N}\wedge^{-q}\mathscr{E}_J^\vee = \wedge^{-q}(\bigoplus_{i=1}^N \mathscr{O}_X(-D_i))$$

are exact for $q > -N$, and hence $\mathrm{Tot}(C^{\bullet\bullet})$ is quasi-isomorphic to $\wedge^N(\oplus_{i=1}^N\mathscr{O}_X(-D_i)) = \mathscr{O}_X(-\sum_i D_i)$. On the other hand, the vertical complexes

$$C^{p\bullet} : 0 \to \cdots \to 0 \to \bigoplus_{|J|=p}\wedge^p\mathscr{E}_J^\vee \to \bigoplus_{|J|=p}\wedge^{p-1}\mathscr{E}_J^\vee \to \cdots \to \bigoplus_{|J|=p}\mathscr{E}_J^\vee \to \bigoplus_{|J|=p}\mathscr{O}_X$$

are direct sums of Koszul complexes by design, and so their cohomology is

$$h^q(C^{p\bullet}) = \bigoplus_{|J|=p}\mathscr{T}or_{-q}^{\mathscr{O}_X}(\mathscr{O}_{D_J}, \mathscr{O}_X),$$

which reduces to

$$h^q(C^{p\bullet}) = \begin{cases} \bigoplus_{|J|=p}\mathscr{O}_{D_J} & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

precisely when the sequence $D_1, \dots, D_N$ is regular [Mat80, §18 Thm. 43], [Ful98, Lem. A.5.3]. As a technical aside, this approach might show that Lemma 2.7 holds under slightly weaker hypotheses of *Koszul regularity* (see e.g. [Stacks, Tag 062D]).

2.3 Resolving sheaves of log-differentials

In the case where X is smooth over a perfect field and $\Delta_X := \cup_{i=1}^N D_i$ is snc in the strong sense that for each point $x \in X$ there are regular parameters $z_1 \cdots z_c$ so that $\Delta_X = V(z_1 \cdots z_r)$ on a Zariski neighborhood of x, we can say even more — however this additional information is not used in the sequel so the reader is welcome to proceed to Section 2.4.

Here the X_k are smooth, so in particular the sheaves of differential forms $\Omega^1_{X_k}$ are locally free, and for each p the standard Čech construction applied to the co-semi-simplicial sheaf $\Omega^p_{X_\bullet}$ gives a cochain complex

$$R\epsilon_*\Omega^p_{X_\bullet} : \epsilon_{0*}\Omega^p_{X_0} \to \epsilon_{1*}\Omega^p_{X_1} \to \epsilon_{2*}\Omega^p_{X_2} \to \cdots$$

on X, together with a morphism $\Omega^p_X \to R\epsilon_*\Omega^p_{X_\bullet}$ induced by the augmentation. The shifted cone $\underline{\Omega}^p_{X,\Delta_X} := \mathrm{cone}(\Omega^p_X \to R\epsilon_*\Omega^p_{X_\bullet})[-1]$ is then represented by the following complex, with derived

[3] It seems a proof by induction on N analogous to the argument in Lemma 2.7 works, although it is combinatorially more involved.

category degrees as indicated:[4]

$$\begin{array}{ccccccccc}
\Omega^p_X & \longrightarrow & \epsilon_{0*}\Omega^p_{X_0} & \longrightarrow & \epsilon_{1*}\Omega^p_{X_1} & \longrightarrow & \epsilon_{2*}\Omega^p_{X_2} & \longrightarrow & \cdots \\
= \Omega^p_X & \to & \prod_{\sigma\in\mathcal{D}((\Delta_X))^0}\Omega^p_{D(\sigma)} & \to & \prod_{\sigma\in\mathcal{D}((\Delta_X))^1}\Omega^p_{D(\sigma)} & \to & \prod_{\sigma\in\mathcal{D}((\Delta_X))^2}\Omega^p_{D(\sigma)} & \to & \cdots \\
0 & & 1 & & 2 & & 3 & &
\end{array} \tag{2.11}$$

Lemma 2.12 ([Fri83, Prop. 1.5], [DI87, Rem. 4.2.2]). *The complex*

$$0 \to \Omega^p_X(\log \Delta_X)(-\Delta_X) \to \Omega^p_X \to \prod_{\sigma\in\mathcal{D}((\Delta_X))^0}\Omega^p_{D(\sigma)} \to \prod_{\sigma\in\mathcal{D}((\Delta_X))^1}\Omega^p_{D(\sigma)} \to \cdots$$

is exact. Equivalently, the complex (2.11) *is a resolution of the sheaf* $\Omega^p_X(\log \Delta_X)(-\Delta_X)$*. In particular (for* $p = 0$*) the complex*

$$\mathcal{O}_X \to \prod_{\sigma\in\mathcal{D}(\Delta_X)^0}\mathcal{O}_{D(\sigma)} \to \prod_{\sigma\in\mathcal{D}(\Delta_X)^1}\mathcal{O}_{D(\sigma)} \to \cdots$$

is a resolution of $\mathcal{O}_X(-\Delta_X)$*.*

We include a proof merely to make clear that the lemma is valid in arbitrary characteristic — the argument given follows [Fri83, Prop. 1.5] very closely.

Proof. We can check exactness on Zariski stalks over a point $x \in X$. We may also check exactness after renumbering the divisors D_i, and so we may assume that $x \in D_1, \dots, D_k$ and $x \notin D_i$ for $i > k$. By hypothesis, there are local coordinates $z_1, \dots, z_c \in \mathcal{O}_{X,x}$ such that in a Zariski neighborhood of x, $\Delta_X = V(\prod_{i=1}^k z_i)$ and $D_i = V(z_i)$ for $i = 1, \dots, k$.

We now proceed by simultaneous induction on k and $\dim X$. Letting $\Delta_{D_1} = \sum_{i=2}^k D_i \cap D_1$, we have $\dim D_1 < \dim X$ and $k - 1 < k$, so denoting by $\epsilon' : D_{1\bullet} \to D_1$ the semi-simplicial scheme associated to (D_1, Δ_{D_1}), by inductive hypothesis the complex

$$0 \to \Omega^p_{D_1}(\log \Delta_{D_1})(-\Delta_{D_1}) \to \Omega^p_{D_1} \to \epsilon'_{0*}\Omega^p_{D_{1,0}} \to \epsilon'_{1*}\Omega_{D^*_{1,1}} \to \cdots \tag{2.13}$$

is exact. On the other hand, letting $\Delta^{>1} = \sum_{i=2}^r D_i$ we obtain a divisor with $k-1 < k$ components, so denoting $\epsilon'' : X^{>1}_\bullet \to X$ the semi-simplicial scheme associated to $(X, \Delta^{>1})$, by inductive hypothesis the complex

$$0 \to \Omega^p_X(\log \Delta^{>1})(-\Delta^{>1}) \to \Omega^p_X \to \epsilon''_{0*}\Omega^p_{X^{>1}_0} \to \epsilon''_{1*}\Omega^p_{X^{>1}_1} \to \cdots$$

is exact. Moreover, there is a sequence of complexes

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Omega^p_{D_1} \arrow[r, "d'"] & \epsilon'_{0*}\Omega^p_{D_{1,0}} \arrow[r, "d'"] & \epsilon'_{1*}\Omega^p_{D_{1,1}} \arrow[r, "d'"] & \cdots \\
\Omega^p_X \arrow[r, "\epsilon^\sharp"] \arrow[d, equal] & \epsilon_{0*}\Omega^p_{X_0} \arrow[r, "d"] \arrow[u, leftarrow, "\alpha"] \arrow[d, "\beta"] & \epsilon_{1*}\Omega^p_{X_1} \arrow[r, "d"] \arrow[u, leftarrow, "\alpha"] \arrow[d, "\beta"] & \epsilon_{2*}\Omega^p_{X_2} \arrow[r, "d"] \arrow[u, leftarrow, "\alpha"] \arrow[d, "\beta"] & \cdots \\
\Omega^p_X \arrow[r, "\epsilon''^\sharp"] & \epsilon''_{0*}\Omega^p_{X^{>1}_0} \arrow[r, "d''"] & \epsilon''_{1*}\Omega^p_{X^{>1}_1} \arrow[r, "d''"] & \epsilon''_{2*}\Omega^p_{X^{>1}_2} \arrow[r, "d''"] & \cdots \\
0 & 1 & 2 & 3 &
\end{tikzcd} \tag{2.14}$$

[4] This notation is chosen to align with the fact that over $\mathbb{C}$, the complex (2.11) represents the pth graded part of the Du Bois complex of the pair (X, Δ_X).

and since for each $k, X_k = X_k^{>1} \coprod D_{1,k-1}$ the columns are (split) exact. Using the long exact sequence of cohomology sheaves, the inductive hypotheses show that $h^i(\underline{\Omega}^p_{X,\Delta_X}) = 0$ for $i > 1$, and in low degrees we have an exact sequence

$$0 \to \Omega^p_X(\log \Delta_X)(-\Delta_X) \to \Omega^p_X(\log \Delta^{>1})(-\Delta^{>1}) \to \Omega^p_{D_1}(\log \Delta_{D_1})(-\Delta_{D_1}) \to h^1(\underline{\Omega}^p_{X,\Delta_X}) \to 0$$

It remains to show $h^1(\underline{\Omega}^p_{X,\Delta_X}) = 0$. For this consider a local section

$$(\varphi_i) = (\varphi_i | i = 1, \dots, k) \in \ker d \subseteq \epsilon_{0*}\Omega^p_{X_0} = \prod_{i=1}^k \Omega^p_{D_i}$$

As $d''\beta(\varphi_i) = \beta d(\varphi_i) = 0$, by inductive hypothesis there is a local section $\omega \in \Omega^p_X$ such that $\beta(\varphi_i) = \epsilon''^\sharp\omega$. Unravelling, $\beta(\varphi_i) = (\varphi_2, \dots, \varphi_k)$ and $\omega|_{D_i} = \varphi_i$ for $i = 2, \dots, k$. Since

$$0 = d(\varphi_i) = (\varphi_i|_{D_i \cap D_j} - \varphi_i|_{D_i \cap D_j} | 1 \le i < j \le N), \text{ so in particular for } i = 1$$
$$0 = \varphi_1|_{D_1 \cap D_j} - \varphi_j|_{D_1 \cap D_j} = \varphi_1|_{D_1 \cap D_j} - \omega|_{D_1 \cap D_j} \text{ for } j = 2, \dots, k$$

we find that $\varphi_1 - \omega|_{D_1}$ vanishes on Δ_{D_1}, and applying exactness of (2.13) once more we see $\varphi_1 - \omega|_{D_1} \in \Omega^p_{D_1}(\log \Delta_{D_1})(-\Delta_{D_1})$. At x, $\Omega^p_{D_1}(\log \Delta_{D_1})(-\Delta_{D_1})$ is generated by the forms

$$(\prod_{i=2}^k z_i) \cdot \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_l}}{z_{i_l}} \wedge dz_{i_{l+1}} \wedge \cdots \wedge dz_{i_p} \text{ where } 1 < i_1 < \cdots < i_l \le k < i_{l+1} < \cdots < i_p \le N$$

The key point is: each of these vanishes on D_i for $i > 1$ (since they each contain either a z_i or a dz_i for all $1 < i \le k$), and so we may find a local section $\xi \in \Omega^p_X$ with
(*i*) $\xi|_{D_1} = \varphi_1 - \omega|_{D_1}$;
(*ii*) $\xi|_{D_i} = 0$ for $i > 1$.
Rearranging shows $(\omega + \xi)|_{D_i} = \varphi_i$ for all i — in other words $(\varphi_i) = \epsilon^\sharp(\omega + \xi)$. □

Remark 2.15. As a byproduct we obtain an exact sequence

$$0 \to \Omega^p_X(\log \Delta_X)(-\Delta_X) \to \Omega^p_X(\log \Delta^{>1})(-\Delta^{>1}) \to \Omega^p_{D_1}(\log \Delta_{D_1})(-\Delta_{D_1}) \to 0,$$

and considering the snake-lemma definition of the connecting morphism shows this is, at least up to sign, restriction of log differential forms (see [EV92, §2])

2.4 Replacing the ideal sheaf with a filtered complex

Let X be a locally noetherian scheme and let $D_1, \dots, D_N \subseteq X$ be a regular sequence of effective Cartier divisors, with sum $\Delta_X := \sum_{i=1}^N D_i$. By Lemma 2.7 the ideal sheaf $\mathcal{O}_X(-\Delta_X)$ is quasi-isomorphic to $\operatorname{cone}(\mathcal{O}_X \to R\epsilon_*\mathcal{O}_{X_\bullet})[-1]$, which for convenience moving forward we give a name:[5]

Definition 2.16. $\underline{\Omega}^0_{X,\Delta_X} := \operatorname{cone}(\mathcal{O}_X \to R\epsilon_*\mathcal{O}_{X_\bullet})[-1]$.

By Lemma 2.7 and its proof this complex has the explicit representation

$$\begin{array}{ccccccccc} \mathcal{O}_X & \longrightarrow & \bigoplus_i \mathcal{O}_{D_i} & \longrightarrow & \bigoplus_{i<j} \mathcal{O}_{D_i \cap D_j} & \longrightarrow & \cdots & \longrightarrow & \mathcal{O}_{\cap_i D_i} \\ 0 & & 1 & & 2 & & \dots & & N \end{array}$$

[5]This notation is chosen to align with the fact that over $\mathbb{C}$ and when (X, Δ_X) is a simple normal crossing pair, the complex (2.11) represents the 0th graded part of the Du Bois complex of the pair (X, Δ_X).

We can give $\underline{\Omega}^0_{X,\Delta_X}$ a descending filtration by truncations

$$\underline{\Omega}^0_{X,\Delta_X} = \sigma_{\geq 0}\underline{\Omega}^0_{X,\Delta_X} \supset \sigma_{\geq 1}\underline{\Omega}^0_{X,\Delta_X} \supset \sigma_{\geq 2}\underline{\Omega}^0_{X,\Delta_X} \supset \cdots$$

where

$$(\sigma_{\geq i}\underline{\Omega}^0_{X,\Delta_X})^j = \begin{cases} 0 & \text{if } j < i \\ (\underline{\Omega}^0_{X,\Delta_X})^j = \epsilon_{j-1*}\mathcal{O}_{X_{j-1}} = \prod_{J\subseteq\{1,\ldots,N\}\,|\,|J|=j}\mathcal{O}_{D_J} & \text{otherwise} \end{cases} \tag{2.17}$$

Using this filtration we obtain a spectral sequence for higher direct images.

Corollary 2.18. *Let S be a locally noetherian scheme and let $f : X \to S$ be a finite type morphism. Let $D_1, \ldots, D_N \subseteq X$ be a regular sequence of effective Cartier divisors, with sum Δ_X. Then there is a filtered complex $(Rf_*\underline{\Omega}^0_{X,\Delta_X}, F)$ whose cohomology computes the higher direct images $R^{i+j}f_*\mathcal{O}_X(-\Delta_X)$. For each i there is a distinguished triangle*

$$F^{i+1}Rf_*\underline{\Omega}^0_{X,\Delta_X} \to F^iRf_*\underline{\Omega}^0_{X,\Delta_X} \to \prod_{J\subseteq\{1,\ldots,N\}\,|\,|J|=i} Rf_*\mathcal{O}_{D_J} \to \cdots$$

In particular, there is a spectral sequence

$$E_1^{ij} = \prod_{J\subseteq\{1,\ldots,N\}\,|\,|J|=i} R^jf_*\mathcal{O}_{D_J} \Rightarrow R^{i+j}f_*\mathcal{O}_X(-\Delta_X)$$

The filtration F is defined as $F = Rf_*\sigma$, and the resulting spectral sequence is just the usual hypercohomology spectral sequence.

Remark 2.19. Viewing $\epsilon : X_\bullet \to X$ as a sort of resolution of the pair (X, Δ_X), we can consider the spectral sequence of Corollary 2.18 as a sort of *descent* spectral sequence (see [SGA4II, Vbis], [Con03]).

3 Simple normal crossing divisors and thriftyness

3.1 Definitions and basic properties

Definition 3.1 ([EGAIV_2, §7.8])**.** A scheme X is *excellent* if and only if

- X is locally noetherian,
- for every point $x \in X$ the fibers of the natural map $\operatorname{Spec}\mathcal{O}^\wedge_{X,x} \to \operatorname{Spec}\mathcal{O}_{X,x}$ are regular,
- for every integral X-scheme Z that is finite over an affine open of X, there is a non-empty regular open subscheme $U \subseteq Z$, and
- every scheme X' locally of finite type over X is catenary (that is, if $x \in X'$ and $x \rightsquigarrow y$ is a specialization, then any 2 saturated chains of specializations $x = x_0 \rightsquigarrow x_1 \rightsquigarrow \cdots \rightsquigarrow x_n = y$ have the same length).

If X is excellent, then the locus

$$\operatorname{Reg}(X) = \{x \in X \mid \mathcal{O}_{X,x} \text{ is regular}\}$$

is open [EGAIV_2, Prop. 7.8.6]; we will make repeated use of this fact.

We first relate the notion of a simple normal crossing pair to the regular sequences of effective Cartier divisors considered in the previous section.

Lemma 3.2. *If $(X, \Delta_X = \sum_i D_i)$ is a simple normal crossing pair then (D_i) is a regular sequence of effective Cartier divisors.*

Proof. Let $x \in X$ be a point and as above let $I(x) = \{i \mid x \in D_i\}$. Let $f_j \in \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ be local generators for the D_j, for $j \in I(x)$. By hypothesis for any subset $J \subseteq I(x)$ the quotient $A/(f_j \mid j \in J)$ is regular, and so by induction we reduce to the commutative algebra statement that if A is a regular local ring, $f \in A$ and A/f is a regular local ring with dimension $\dim A - 1$ then f is a non-0-divisor (see for example [Stacks, Tag 0AGA]). $\square$

Lemma 3.3. *Let X be an integral excellent scheme with an effective Weil divisor $\Delta_X = \sum_i D_i$, and for each i let $\mathcal{I}_i \subseteq \mathcal{O}_X$ be the ideal sheaf of D_i. Then the locus*

$$\mathrm{snc}(X, \Delta_X) := \{x \in X \mid \sum_{i \in I(x)} \mathcal{I}_{i,x}^{\wedge} \subseteq \mathcal{O}_{X,x}^{\wedge} \textit{ is a simple normal crossing pair}\} \subseteq X$$

is open, and this is the largest open set $U \subseteq X$ such that $(U, \Delta_X|_U)$ is a simple normal crossing pair.

We could alternatively just declare $\mathrm{snc}(X, \Delta_X)$ to be the largest open set $U \subseteq X$ such that $(U, \Delta_X|_U)$ is a simple normal crossing pair; the content of the lemma is that in some sense the snc locus is "already open."

Proof. Suppose $J \subseteq \{1, \dots, N\}$, and write $\mathcal{I}_J = (f_j \in \mathcal{O}_{X,x} \mid j \in J) \subseteq \mathcal{O}_{X,x}$. Consider the co-cartesian diagram of noetherian local rings

$$\begin{tikzcd} \mathcal{O}_{X,x}^{\wedge} \arrow[r] & \mathcal{O}_{X,x}^{\wedge}/\mathcal{I}_J\mathcal{O}_{X,x}^{\wedge} \simeq (\mathcal{O}_{X,x}/\mathcal{I}_J)^{\wedge} \\ \mathcal{O}_{X,x} \arrow[u] \arrow[r] & \mathcal{O}_{X,x}/\mathcal{I}_J \arrow[u] \end{tikzcd}$$

The vertical homomorphisms are faithfully flat and by hypothesis $\mathcal{O}_{X,x}^{\wedge}/\mathcal{I}_J\mathcal{O}_{X,x}^{\wedge}$ is regular — since regularity satisfies faithfully flat descent, $\mathcal{O}_{X,x}/\mathcal{I}_J$ is also regular. Thus D_J is regular at the point $x \in D_J$, and as X is excellent by hypothesis the regular locus of D_J is open. Letting $x \in U_J \subseteq X$ be a neighborhood such that $D_J \cap U_J \subseteq D_J$ is regular and then letting $U = \cap_J D_J$ gives a neighborhood of x such that $(U, (D_i \cap U))$ is a simple normal crossing pair. $\square$

Note that for a simple normal crossing pair (X, Δ_X), since the intersections $D_J = \cap_{j \in J} D_j$ are regular their connected components and irreducible components coincide.

Definition 3.4. A *stratum* of a simple normal crossing pair $(X, \Delta_X = \sum_i D_i)$ is a connected (equivalently, irreducible) component of an intersection $D_J = \cap_{j \in J} D_j$.

Definition 3.5 (compare with [SingsMMP 2013, Def. 2.79-2.80], [KX16, §1, discussion before Def. 10]). Let $(S, \Delta_S = \sum_i D_i)$ be a pair in the sense of Definition 1.5, and assume Δ_S is reduced and effective. A separated, finite type birational morphism $f : X \to S$ is *thrifty with respect to* Δ_S if and only if

(*i*) f is an isomorphism over the generic point of every stratum of $\mathrm{snc}(S, \Delta_S)$ and
(*ii*) letting $\tilde{D}_i = f_*^{-1} D_i$ for $i = 1, \dots, N$ be the strict transforms of the divisors D_i, and setting $\Delta_X := \sum_i \tilde{D}_i$, the map f is an isomorphism at the generic point of every stratum of $\mathrm{snc}(X, \Delta_X)$.

The restriction that $D_i \cap \mathrm{Reg}(S) \neq \emptyset$ for all i ensures that if $\eta \in D_i$ is a generic point of a component, then $\eta \in \mathrm{Reg}(S)$. Since on a regular scheme every Weil divisor is Cartier, and as S is excellent and D_i is reduced by hypothesis, there is a neighborhood $\eta \in U \subseteq S$ such that $U, D_i \cap U$ is a simple normal crossing pair. In other words, $\eta \in \mathrm{snc}(S, \Delta_S)$ is the generic point of a stratum, so (i) implies $f^{-1}(\eta)$ is a single (non-closed) point. For our purposes the strict transform $\tilde{D}_i$ can be *defined* as

$$\tilde{D}_i := \bigcup_{\eta \in D_i \text{generic}} \overline{f^{-1}(\eta)} \subseteq X.$$

Since f is an isomorphism over η, we also see $f^{-1}(\eta) \subseteq \mathrm{snc}(X, \Delta_X)$.

Lemma 3.6. *Let S be an integral excellent noetherian scheme with a sequence of reduced effective Weil divisors $D_1, \cdots, D_N \subseteq S$ such that no component of $\cup_i D_i$ is contained in* $\mathrm{Sing}(X)$*, and let $f : X \to S$ be a separated, finite type birational morphism. Then, f is thrifty if and only if there is a diagram of separated finite type S-schemes*

$$S \hookleftarrow U \hookrightarrow X$$

with both morphisms (necessarily dense) open immersions, such that U contains all generic points of strata of $\mathrm{snc}(S, \Delta_S)$ *and* $\mathrm{snc}(X, \Delta_X)$*.*

Proof. Since the existence of a common dense open $S \hookleftarrow U \hookrightarrow X$ as in the statement of the lemma certainly guarantees (i) and (ii), we focus on the "only if," and in fact we show that one can take $U =$ the maximal domain of definition of $f^{-1} : S \dashrightarrow X$. By (i) of Definition 3.5 this U contains all generic points of strata of $\mathrm{snc}(S, \Delta_S)$.

Suppose $\xi \in \mathrm{snc}(X, \Delta_X)$ is a generic point of a stratum. By hypothesis there is a neighborhood $\xi \in V \subseteq X$ such that $f|_V : V \xrightarrow{\sim} S$ is an isomorphism onto its image. Then $W := f(V)$ is a Zariski neighborhood of $f(\xi)$ and the inverse of $f|_V$ gives a section of the birational map $X_W = X \times_S W \to W$.

$$\begin{tikzcd} V \arrow[r, hook] \arrow[dr, "f|_V"'] & X_W \arrow[d, "f_W"] \\ & W \end{tikzcd}$$

But then the inclusion $V \hookrightarrow X_W$ is a proper dense open immersion, hence an isomorphism. □

Remark 3.7. It seems that the above proof shows in addition that $f(\xi) \in S$ is the generic point of a stratum of $\mathrm{snc}(S, \Delta_S)$.

We will make repeated use of a few blowup lemmas from the construction of Nagata compactifications in Section 5 — here, they are used to show that thrifty morphisms can be dominated by certain admissible blowups.

Lemma 3.8 ([Con07, Lem. 2.4, Rmk. 2.5, Cor. 2.10])**.** *Let S be a quasi-compact, quasi-separated scheme.*

(*i*) *If X is a quasi-separated quasi-compact S-scheme and Y is a proper S-scheme, and if $f : U \to Y$ is an S-morphism defined on a dense open $U \subseteq X$, then there exists a U-admissible blowup $\tilde{X} \to X$ and an S-morphism $\tilde{f} : \tilde{X} \to Y$ extending f.*

(*ii*) *Let $j_i : U \to X_i$ be a finite collection of dense open immersions between finite type separated S-schemes. Then there exist U-admissible blowups $X_i' \to X_i$ and a separated finite type S-scheme X, together with open immersions $X_i' \hookrightarrow X$ over S, such that the X_i' cover X and the open immersions $U \hookrightarrow X_i' \hookrightarrow X$ are all the same.*

Corollary 3.9. *There exist U-admissible blowups*

$$\begin{tikzcd} \tilde{X} \arrow[r, hook, "\text{open imm.}"] \arrow[d] & \tilde{S} \arrow[d] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

In particular if f is proper then X and S have a common U-admissible blowup.

Proof. By Lemma 3.8 there are a separated, finite type S-scheme Y, U-admissible blowups $\tilde{S} \to S$ and $\tilde{X} \to X$ and dense open immersions $\tilde{S} \hookrightarrow Y \hookleftarrow \tilde{S}$ over S such that the diagram

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, hook] & \tilde{X} \arrow[d, hook] \\ \tilde{S} \arrow[r, hook] & Y \end{tikzcd}$$

commutes. Since $\tilde{S}$ is proper over S, the bottom arrow is necessarily an isomorphism, in other words $Y = \tilde{S}$. If f is proper then $\tilde{X}$ is proper over S, so $Y = \tilde{X}$ as well. □

Remark 3.10. If (S, Δ_S) is a simple normal crossing pair and $U \subseteq S$ is an open containing all strata, a U-admissible blowup $f : X \to S$ need not be thrifty, see Example 4.12.

3.2 The "regular to regular" case

Using Corollary 2.18 we can obtain a restricted form of Theorem 1.8, the case of a thrifty proper birational morphism of simple normal crossing pairs.

Theorem 3.11. *Let (Y, Δ_Y) be a simple normal crossing pair and let $f : X \to Y$ be a thrifty proper birational morphism. Assume (X, Δ_X) is also a simple normal crossing pair. Then the natural map*

$$\mathcal{O}_Y(-\Delta_Y) \to Rf_*\mathcal{O}_X(-\Delta_X) \text{ is a quasi-isomorphism.}$$

Proof. Let $X_\bullet$ (resp. $Y_\bullet$) be the semi-simplicial scheme associated to (X, Δ_X) (resp. (Y, Δ_Y)). For any $J \subseteq \{1, \ldots, N\}$ f restricts to a morphism $\cap_{j\in J}\tilde{D}_j \to \cap_{j\in J} D_j$, and in this way we obtain a morphism of semi-simplicial schemes

$$\begin{tikzcd} \cdots \arrow[r] & X_2 \arrow[r] \arrow[d, "f_2"] & X_1 \arrow[r] \arrow[d, "f_1"] & X_0 \arrow[r] \arrow[d, "f_2"] & X \arrow[d, "f"] \\ \cdots \arrow[r] & Y_2 \arrow[r] & Y_1 \arrow[r] & Y_0 \arrow[r] & Y \end{tikzcd} \tag{3.12}$$

The hypothesis that both pairs have simple normal crossings and f is thrifty implies that for each i, $f_i : X_i \to Y_i$ is a proper birational morphism of (possibly disconnected) regular schemes over k. By [CR11, Cor. 3.2.10] (or [CR15, Thm. 1.1], [Kov20, Thm. 1.4])

$$\mathcal{O}_{Y_i} \xrightarrow{\sim} Rf_*\mathcal{O}_{X_i} \text{ is a quasi-isomorphism for all } i \tag{3.13}$$

The diagram (3.12) induces a morphism of *filtered* complexes $f^\sharp : \underline{\Omega}^0_{Y,\Delta_Y} \to Rf_*\underline{\Omega}^0_{X,\Delta_X}$, and by Lemma 2.7 and Corollary 2.18 it will suffice to show that the resulting map of descent spectral sequences

$$E_1^{ij}(Y) = \begin{cases} \prod_{\sigma\in\mathcal{D}(\Delta_Y)^{i-1}} \mathcal{O}_{D(\sigma)} & j = 0 \\ 0 & \text{otherwise} \end{cases} \to \prod_{\sigma\in\mathcal{D}(\Delta_X)^{i-1}} R^j f_*\mathcal{O}_{D(\sigma)} = E_1^{ij}(X)$$

is an isomorphism, and this last step is a consequence of (3.13). □

4 (Non-)examples of thrift

In this section we work over a field k. Our first example is not new, and likely served as the original motivation for considering thrify morphisms.

Example 4.1. Let $S = \mathbb{A}^2_{xy}$ and $\Delta = V(xy)$. Then $f : X = \mathrm{Bl}_0 S \to S$ is neither thrifty nor rational. Indeed, letting $D_1 = V(x), D_2 = V(y)$ we see that Δ is the union of the 2 lines D_1, D_2 meeting at the origin. Let $\tilde{D}_i = f_*^{-1}D_i$ be the strict transforms, $E = f^{-1}(0)$ the exceptional divisor, and $\tilde{\Delta} = \tilde{D}_1 + \tilde{D}_2$ (see Figure 1). The map $f : X \to S$ fails to be thrifty since it is not an isomorphism over the stratum $0 = D_1 \cap D_2$ of (S, Δ). We will calculate cohomology to show f isn't rational either.

Since $S = \mathbb{A}^2_{xy}$ is affine, we can identify the sheaves $R^i f_*\mathcal{O}_X(-\tilde{\Delta})$ as the sheaves associated to the $k[x,y]$-modules $H^i(X, \mathcal{O}_X(-\tilde{\Delta}))$. Observe that X can be identified with the geometric line bundle $\mathrm{Spec}_{\mathbb{P}^1} \mathrm{Sym}\, \mathcal{O}_{\mathbb{P}^1}(1)$ associated to $\mathcal{O}_{\mathbb{P}^1}(-1)$. Under this identification, the projection π :

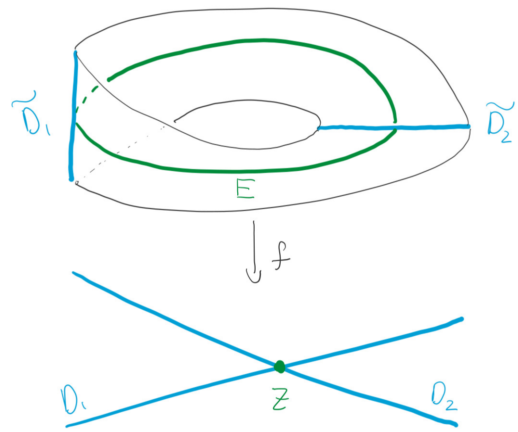

Figure 1: The blowup appearing in Example 4.1

$\operatorname{Spec}_{\mathbb{P}^1} \operatorname{Sym} \mathcal{O}_{\mathbb{P}^1}(1) \to \mathbb{P}^1$ corresponds to the composition $\mathrm{Bl}_0 S \subseteq \mathbb{A}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$, and the blowup map $f : \operatorname{Spec}_{\mathbb{P}^1} \operatorname{Sym} \mathcal{O}_{\mathbb{P}^1}(1) \to \mathbb{A}^2$ corresponds to the natural map

$$\operatorname{Spec}_{\mathbb{P}^1} \operatorname{Sym} \mathcal{O}_{\mathbb{P}^1}(1) \to \operatorname{Spec}_k H^0(\mathbb{P}^1, \operatorname{Sym} \mathcal{O}_{\mathbb{P}^1}(-1)) = \operatorname{Spec}_k k[x,y] = \mathbb{A}^2$$

Hence $\tilde{\Delta} = \pi^*(0+\infty)$. Now since π is affine its Leray spectral sequence degenerates to give

$$H^i(X, \mathcal{O}_X(-\tilde{\Delta})) = H^i(\mathbb{P}^1, \pi_*\mathcal{O}_X(-\tilde{\Delta})) \text{ and via projection formula}$$
$$\pi_*\mathcal{O}_X(-\tilde{\Delta}) = \pi_*\mathcal{O}_X(-\pi^*(0+\infty)) = (\pi_*\mathcal{O}_X)(-0-\infty)$$

By the correspondence between affine schemes and sheaves of algebras,

$$\pi_*\mathcal{O}_X = \operatorname{Sym} \mathcal{O}_{\mathbb{P}^1}(1) = \bigoplus_{d \geq 0} \mathcal{O}_{\mathbb{P}^1}(d)$$

Hence $H^i(X, \mathcal{O}_X(-\tilde{\Delta})) = \oplus_{d\geq 0} H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-2))$. In particular, when $i = 1$ and $d = 0$, we see $H^1(X, \mathcal{O}_X(-\tilde{\Delta})) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \simeq k$ by [Har77, Thm. III.5.1].

An elaboration of Example 4.1 shows in general that if (S, Δ) is a simple normal crossing pair and $Z \subseteq S$ is a stratum, then $f : X = \mathrm{Bl}_Z S \to S$ fails to be thrifty. Localizing at the generic point $\eta \in Z$ we can reduce to the case where Z is replaced by a closed point $\eta \in S$ and $\Delta = V(x_1 \cdot x_2 \cdots x_n)$ where $x_1, \cdots, x_n \in \mathfrak{m}_\eta$ is a regular system of parameters. Then the long exact sequence obtained by pushing forward $\mathcal{O}_X(-\tilde{\Delta} - E) \to \mathcal{O}_X(-\tilde{\Delta}) \to \mathcal{O}_E(-\tilde{\Delta}|_E)$ ends in

$$R^{n-1}f_*\mathcal{O}_X(-\tilde{\Delta}-E) \to R^{n-1}f_*\mathcal{O}_X(-\tilde{\Delta}) \to R^{n-1}f_*\mathcal{O}_E(-\tilde{\Delta}|_E) \to R^n f_*\mathcal{O}_X(-\tilde{\Delta}-E) = 0$$

where the vanishing on the right holds since the maximal fiber dimension of f is $n-1$ [Har77, Cor. III.11.2]. Thus $R^{n-1}f_*\mathcal{O}_X(-\tilde{\Delta}) \to R^{n-1}f_*\mathcal{O}_E(-\tilde{\Delta}|_E) = H^{n-1}(E, \mathcal{O}_E(-\tilde{\Delta}|_E))$ is surjective, and identifying E with the projectivized Zariski tangent space $\mathbb{P}(TS_\eta)$ with homogeneous coordinates $x_1, \dots, x_n$ and $\tilde{\Delta}|_E$ with $V(\prod_i x_i)$ shows $H^{n-1}(E, \mathcal{O}_E(-\tilde{\Delta}|_E)) \simeq H^{n-1}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(-n)) \simeq k$. For related discussion see [SingsMMP 2013, p. 86].

The next example answers (in the affirmative!) a question of Erickson [Eri14a, p.2] and Prelli [Pre17, p.3] about whether there exists a resolution which is thrifty but not rational. In fact, we give such an example where the underlying pair (S, Δ) is rational.

Example 4.2. Let $S = V(xy - zw) \subseteq \mathbb{A}^4_{xyzw}$, $D_0 = V(x,w)$ and $D_\infty = V(y,z)$; finally let $C_\infty = V(w,y)$. We can identify $S = C(\mathbb{P}^1 \times \mathbb{P}^1)$ as the affine cone over the Segre embedding $\mathbb{P}^1_s \times \mathbb{P}^1_t \hookrightarrow \mathbb{P}^3_{xyzw}$ given by

$$\begin{bmatrix} x & w \\ z & y \end{bmatrix} = \begin{bmatrix} s_0 \\ s_1 \end{bmatrix} \begin{bmatrix} t_0 & t_1 \end{bmatrix} = \begin{bmatrix} s_0t_0 & s_0t_1 \\ s_1t_0 & s_1t_1 \end{bmatrix} \tag{4.3}$$

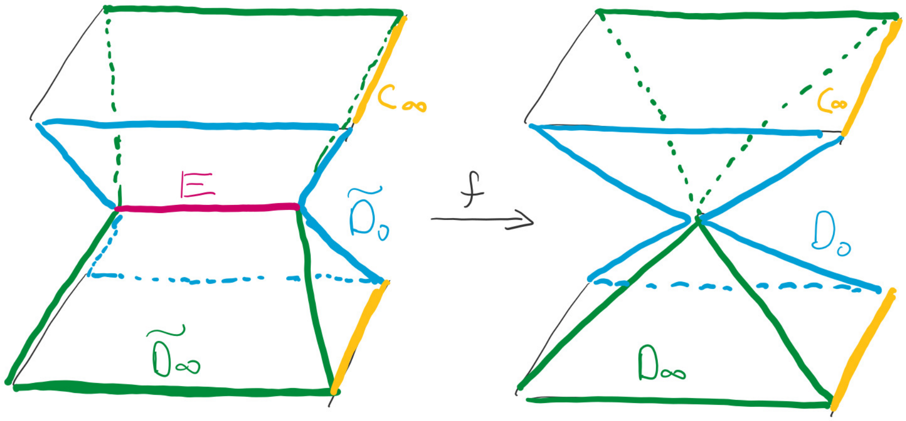

Figure 2: The small resolution of Example 4.2

Hence $D_0 = C(\{0\}\times\mathbb{P}^1)$, $D_\infty = C(\{\infty\}\times\mathbb{P}^1)$ and $C_\infty = C(\mathbb{P}^1\times\{\infty\})$.

Let $\Delta = D_0 + D_\infty + C_\infty$. Note that Δ is *not* Cartier, as it is not linearly equivalent to any multiple of $C(\{0\}\times\mathbb{P}^1) + C(\mathbb{P}^1\times\{0\})$ (here $\{0\}\times\mathbb{P}^1 + \mathbb{P}^1\times\{0\}$ is a hyperplane section of the Segre embedding) — see e.g. [Har77, Ex. II.6.3],[SingsMMP 2013, Prop. 3.14]. Since K_S *is* $\mathbb{Q}$-Cartier, it follows that the pair $(S, \Delta = D_0 + D_\infty)$ is not $\mathbb{Q}$-Gorenstein — in particular it isn't dlt, so we are not at risk of violating [SingsMMP 2013, Thm. 2.87] which implies that a resolution of a dlt pair is thrifty *if and only if* it is rational.

Now let $f : X = \mathrm{Bl}_{D_0} S \to S$ be the blowup at D_0, let $\tilde{D}_i = f_*^{-1}D_i$ for $i = 0, \infty$ and $\tilde{C}_\infty = f_*^{-1}C_\infty$, and let $\tilde{\Delta} = \tilde{D}_0 + \tilde{D}_\infty + \tilde{C}_\infty$. The map f is a small resolution of S (as mentioned in [KM98, Ex. 2.7]). This means we are not at risk of violating [Eri14a, Prop. 1.6] which states that if a log resolution of a pair is rational then it is thrifty. Indeed, the ambient blowup is described as

$$\mathrm{Bl}_{D_0}\mathbb{A}^4 \subseteq \{(x,y,z,w),[u,v] \mid (x,w)\propto(u,v)\} \subseteq \mathbb{A}^4\times\mathbb{P}^1_{uv}$$

so on the $D(u)$ patch $(x,w) = \lambda(1,v)$ and

$$xy - zw = \lambda y - z\lambda v = \lambda(y - zv)$$

Since $V(\lambda)$ is the exceptional divisor we see the strict transform $X \subseteq \mathrm{Bl}_{D_0}\mathbb{A}^4$ of S is $V(y - zv)$ on the $u = 1$ patch — this is smooth as it's a graph. By symmetry in x, w, we conclude X is smooth.

Even better, this allows us to parametrize $X \cap D(u)$ with coordinates z, λ, v:

$$\begin{aligned}&\mathbb{A}^3_{z\lambda v} \simeq \mathrm{Bl}_{D_0} S \cap D(u) \subseteq D(u) \simeq \mathbb{A}^5_{xyzwv}\\ &\text{sending } (z,\lambda,v) \longmapsto (\lambda, zv, z, \lambda v, v) = (x,y,z,w,v)\end{aligned} \tag{4.4}$$

So in particular the restriction of f looks like $(z,\lambda,v)\longmapsto(\lambda, zv, z, \lambda v)$ and we see that the exceptional locus is the v-axis. In this coordinate patch the strict transforms $\tilde{D}_0$ and $\tilde{D}_\infty$ are $V(\lambda)$ and $V(z)$ respectively, which *intersect along the v-axis* $V(\lambda, z)$! Thus $\tilde{\Delta}$ has a stratum in $\mathrm{Ex}(f)$ and f isn't thrifty. We also see that on this patch $\tilde{C}_\infty = V(v)$. As a philosophical aside, the blowup coordinates $[u,v]$ correspond to $[x,w] = [s_0t_0, s_0t_1] = [t_0,t_1]$ as long as $s_0 \neq 0$, so $\mathrm{Ex}\, f$ can be viewed as a copy of the $\mathbb{P}^1_t$ appearing in $D_0 = C(\{0\}\times\mathbb{P}^1_t)$ — see Figure 2.

To show that f *is* in fact a rational resolution we will use an alternative description of X. Starting with the ample invertible sheaf $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$ we have natural morphisms of relative spectra

$$\mathrm{Spec}_{\mathbb{P}^1_s\times\mathbb{P}^1_t}\mathrm{Sym}\,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1) \to \mathrm{Spec}_{\mathbb{P}^1_t}\mathrm{Sym}\,\mathrm{pr}_{t*}\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1) \xrightarrow{f'} \mathrm{Spec}_k H^0(\mathrm{Sym}\,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)) \tag{4.5}$$

where $\mathrm{pr}_t : \mathbb{P}^1_s\times\mathbb{P}^1_t \to \mathbb{P}^1_t$ is the projection. It is well known that the scheme on the left can be identified with the blowup $\mathrm{Bl}_0 S$, and the scheme on the right is S.

Claim 4.6. There is an isomorphism of $S \times \mathbb{P}^1$-schemes

$$X = \mathrm{Bl}_{D_0} S \simeq \mathrm{Spec}_{\mathbb{P}^1_t} \mathrm{Sym}\, \mathrm{pr}_{t*} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$$

This can be proved via the universal property. On the other hand, at least when k is algebraically closed, a quick, dirty and more illuminating proof is possible: we *have* a morphism (f', π) : $\mathrm{Spec}_{\mathbb{P}^1_t} \mathrm{Sym}\, \mathrm{pr}_{t*} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \to S \times \mathbb{P}^1_t$: the first factor is the second map of (4.5), the second is the canonical projection

$$\pi : \mathrm{Spec}_{\mathbb{P}^1_t} \mathrm{Sym}\, \mathrm{pr}_{t*} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \to \mathbb{P}^1_t$$

from the relative Spec construction. $X \subseteq S \times \mathbb{P}^1_t$ by construction, and we can check φ maps the k-points of $\mathrm{Spec}_{\mathbb{P}^1_t} \mathrm{Sym}\, \mathrm{pr}_{t*} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$ bijectively onto those of X. Indeed, the fiber of $\mathrm{Spec}_{\mathbb{P}^1_t} \mathrm{Sym}\, \mathrm{pr}_{t*} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$ over $t \in \mathbb{P}^1_t$ can be described as follows: Note by projection formula

$$\begin{aligned} \mathrm{pr}_{t*} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) &\simeq H^0(\mathbb{P}^1_s, \mathcal{O}_{\mathbb{P}^1_s}(1)) \otimes_k \mathcal{O}_{\mathbb{P}^1_t}(1), \\ \text{so } \mathrm{Sym}\, \mathrm{pr}_{t*} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) &\simeq \mathrm{Sym}\, H^0(\mathbb{P}^1_s, \mathcal{O}_{\mathbb{P}^1_s}(1)) \otimes_k \mathcal{O}_{\mathbb{P}^1_t}(1) \end{aligned} \tag{4.7}$$

Explicitly $\mathrm{Sym}\, H^0(\mathbb{P}^1_s, \mathcal{O}_{\mathbb{P}^1_s}(1)) \otimes_k \mathcal{O}_{\mathbb{P}^1_t}(1) = \oplus_d k[s_0, s_1]_d \otimes \mathcal{O}_{\mathbb{P}^1_t}(d) = k[s_0, s_1] \times \mathrm{Sym}\, \mathcal{O}_{\mathbb{P}^1_t}(1)$ where $\times$ denotes the product of graded rings of [Har77, Ex. II.5.11] and hence for a k-point t,

$$\pi^{-1}(t) \simeq \mathrm{Spec}\, k[s_0, s_1], \text{ so that } f'|_{\pi^{-1}(t)} : \pi^{-1}(t) \to S$$

is a map $\mathbb{A}^2_{s_0 s_1} \to S \subseteq \mathbb{A}^4_{xyzw}$. Writing down the map of algebras corresponding to f' shows that it is none other than the linear transformation of (4.3). Finally, referencing (4.4) we see that the fibers of $X \to \mathbb{P}^1_t$ have the same description.[6]

Using the claim, we proceed as in Example 4.1 using degeneration of the Leray spectral sequence for the affine map $\pi : X \to \mathbb{P}^1_t$ to calculate

$$H^i(X, \mathcal{O}_X(-\tilde{\Delta})) = H^i(\mathbb{P}^1_t, \pi_* \mathcal{O}_X(-\tilde{\Delta}))$$

On $\mathbb{P}^1_t$, noting that $\tilde{C}_\infty = \pi^*(\infty)$, the projection formula gives

$$\pi_* \mathcal{O}_X(-\tilde{\Delta}) = (\pi_* \mathcal{O}_X(-\tilde{D}_0 - \tilde{D}_\infty))(-\infty) \tag{4.8}$$

and $\pi_* \mathcal{O}_X(-\tilde{D}_0 - \tilde{D}_\infty) \subseteq \pi_* \mathcal{O}_X$ is the sheaf of ideals $(s_0 \cdot s_1) \subseteq k[s_0, s_1] \times \mathrm{Sym}\, \mathcal{O}_{\mathbb{P}^1_t}(1)$. Letting $(s_0 \cdot s_1)_d \subseteq k[s_0, s_1]$ denote the d-th graded part, we see

$$\pi_* \mathcal{O}_X(-\tilde{\Delta}) = \bigoplus_{d \geq 0} (s_0 \cdot s_1)_d \otimes_k \mathcal{O}_{\mathbb{P}^1_t}(d-1)$$

where the "-1" comes from the twist "$(-\infty)$" in (4.8). This yields:

$$\begin{aligned} H^i(\mathbb{P}^1_t, \pi_* \mathcal{O}_X(-\tilde{\Delta})) &= \bigoplus_{d \geq 0} (s_0 \cdot s_1)_d \otimes_k H^i(\mathbb{P}^1_t, \mathcal{O}_{\mathbb{P}^1_t}(d-1)) \\ &= \begin{cases} \bigoplus_{d \geq 0} (s_0 \cdot s_1)_d \otimes_k (t_1)_d \subseteq k[s_0, s_1] \times k[t_0, t_1] = H^0(S, \mathcal{O}_S) & \text{if } i = 0 \\ 0 & \text{if } i = 1 \end{cases} \end{aligned} \tag{4.9}$$

the key point being that $H^1(\mathbb{P}^1_t, \mathcal{O}_{\mathbb{P}^1_t}(d-1)) = 0$ for $d \geq 0$. This calculation shows $f_* \mathcal{O}_X(-\tilde{\Delta}) = \mathcal{O}_S(-\Delta)$ (this holds for more general reasons, namely S is normal [Pre17, Lem. 2.1]) and $R^1 f_* \mathcal{O}_X(-\tilde{\Delta}) = 0$.

Finally, (S, Δ) *is* a rational pair, as a consequence of the theorem below — this was the main reason for including the additional divisor C_∞. If we had left it out, the above calculations would still show that $f : X \to (S, D_0 + D_\infty)$ is a non-thrifty rational resolution, however the pair $(S, D_0 + D_\infty)$ isn't rational (also by the theorem below).

[6] In slogan form: $X = \mathrm{Bl}_{D_0} S$ is a pencil of 2-planes on S corresponding to the pencil of rulings $\mathbb{P}^1_s \times \{t\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 4.10 ([Pre17, Thm. 3.2]). *Let (Y, B) be a pair such that Y is a normal variety over k and B is a reduced effective Weil divisor on Y (for example a simple normal crossing pair) and let $\mathscr{L}$ be an ample invertible sheaf on Y. Let (CY, CB) be the abstract affine cone over (Y, B) with respect to $\mathscr{L}$: $CY = \operatorname{Spec}_k H^0(Y, \operatorname{Sym}\mathscr{L})$ and CB is the image of $\operatorname{Spec}_k H^0(B, \operatorname{Sym}\mathscr{L}|_B) \to \operatorname{Spec}_k H^0(Y, \operatorname{Sym}\mathscr{L}) = CY$ with its reduced subscheme structure. Then (CY, CB) is a rational pair if and only if (Y, B) is a rational pair and*

$$H^i(Y, \mathscr{L}^d(-B)) = 0 \text{ for } i > 0, d \geq 0$$

Applying the theorem to $Y = \mathbb{P}^1 \times \mathbb{P}^1$ with the divisor $B = \{0, \infty\} \times \mathbb{P}^1 + \mathbb{P}^1 \times \{\infty\}$ which has associated invertible sheaf $\mathcal{O}_Y(B) \simeq \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,1)$, together with the ample invertible sheaf $\mathscr{L} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$ we calculate (using Künneth)

$$H^i(Y, \mathscr{L}^d(-B)) = H^i(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d-2, d-1)) = \bigoplus_{j+k=i} H^j(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-2)) \otimes_k H^k(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-1)) \tag{4.11}$$

Noting that $H^k(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-1)) = 0$ for $k > 0$ and $d \geq 0$, we see that $H^2(Y, \mathscr{L}^d(-B)) = 0$ for $d \geq 0$, and

$$H^1(Y, \mathscr{L}^d(-B)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-2)) \otimes_k H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-1))$$

Now $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-2)) = 0$ for $d \neq 0$, but $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$, so the tensor product is always 0.

The last example of this section shows that even when (S, Δ) is a simple normal crossing pair and $f : X \to S$ is a U-admissible blowup for some $U \subseteq S$ containing all strata, and $\tilde{\Delta} = f_*^{-1}\Delta$ is snc, f may still fail to be thrifty. Unfortunately our presentation only makes sense in characteristic 0, but I would be shocked and appalled if this example doesn't work in any characteristic $p > 2$.

Example 4.12. Let $S = \mathbb{A}^3_{xyz}$, let $\Delta = V((z-x)(z+x))$ and let $Z = V(x, y)$; let $U = S \setminus Z$. Then there is a U-admissible blowup $f : X \to S$ such that $f_*^{-1}\Delta$ is a simple normal crossing divisor but f is not thrifty.

We first blow up Z to obtain $g : \operatorname{Bl}_Z S \to S$, and claim that the strict transform of Δ is no longer snc. Letting $D_\pm = V(z \pm x)$ we can work in blowup coordinates described like

$$\operatorname{Bl}_Z S = \{((x, y, z), [u, v]) \in \mathbb{A}^3 \mid (x, y) \propto (u, v)\}$$

so that on the $D(u)$ patch $(x, y) = \lambda(1, v)$ and

$$z \pm x = z \pm \lambda, \text{ so in } (z, \lambda, v) \text{ coordinates } \tilde{D}_\pm \cap D(u) = V(z \pm \lambda)$$

in other words $\tilde{\Delta}$ *is* snc on the $D(u)$ patch (as is expected since on $D(x) \subseteq \mathbb{A}^3$, Δ is smooth). But on the $D(v)$ patch where $(x, y) = \lambda(u, 1)$,

$$z \pm x = z \pm \lambda u, \text{ so in } (z, \lambda, u) \text{ coordinates } \tilde{D}_\pm \cap D(v) = V(z \pm \lambda u) \tag{4.13}$$

and here we see the strict transforms intersect along $V(\lambda u)$ and hence fail to be snc (Figure 3).

A global description of the situation: $\operatorname{Bl}_Z S$ is isomorphic to $\mathbb{A}^1_z \times \operatorname{Bl}_0 \mathbb{A}^2_{xy}$, and $\tilde{D}_\pm$ are 2 copies of $\operatorname{Bl}_0 \mathbb{A}^2_{xy}$ embedded via the maps

$$(\pm x, \mathrm{id}) : \operatorname{Bl}_0 \mathbb{A}^2_{xy} \to \mathbb{A}^1_z \times \operatorname{Bl}_0 \mathbb{A}^2_{xy}$$

where the map $\pm x : \operatorname{Bl}_0 \mathbb{A}^2_{xy} \to \mathbb{A}^1_z$ really means the composition $\operatorname{Bl}_0 \mathbb{A}^2_{xy} \to \mathbb{A}^2_{xy} \xrightarrow{\pm x} \mathbb{A}^1_z$. From this perspective $\tilde{D}_+ \cap \tilde{D}_-$ is the preimage of $V(x)$ under the blowup map $\operatorname{Bl}_0 \mathbb{A}^2_{xy} \to \mathbb{A}^2_{xy}$, the union $\mathbb{P}^1_{xy} \cup \mathbb{A}^1_y$ glued along the points $[0, 1] \in \mathbb{P}^1_{xy}$ and $0 \in \mathbb{A}^1_y$. Let p denote the point in $\mathbb{P}^1_{xy} \cap \mathbb{A}^1_y$. Equivalently $\operatorname{Sing}(\tilde{D}_+ \cap \tilde{D}_-)$ consists of a single closed point which we call p.

This discussion shows that the snc locus of $(\operatorname{Bl}_Z S, \tilde{\Delta})$ is

$$\operatorname{snc}(\operatorname{Bl}_Z S, \tilde{\Delta}) = \operatorname{Bl}_Z S \setminus \{p\}$$

By work of Szabó and Bierstone-Milman [BM97; Sza94] (this is where we use the characteristic 0 hypothesis) there exists a further blowup $h : X \to \operatorname{Bl}_Z S$ such that $h_*^{-1}\tilde{\Delta} + \operatorname{Ex} h$ is a simple

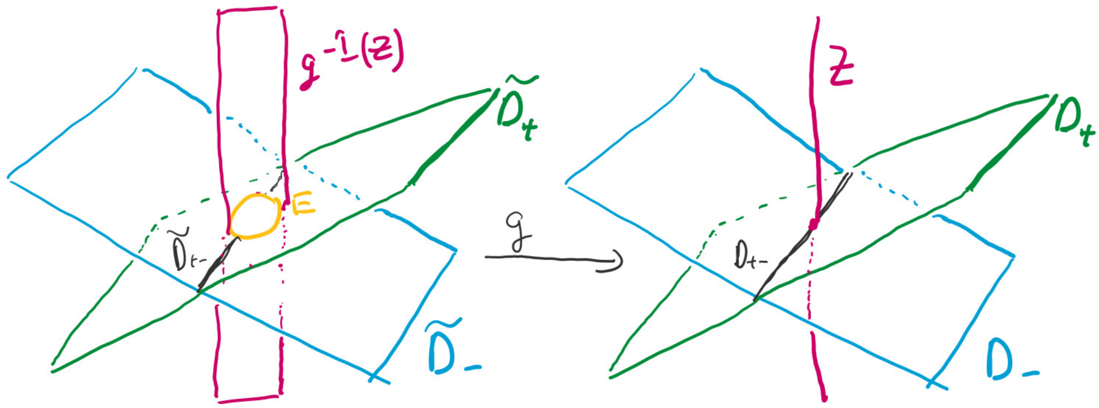

Figure 3: The blowup of Example 4.12

normal crossing divisor *and* h is an isomorphism over $\mathrm{snc}(\mathrm{Bl}_Z S, \tilde{\Delta})$, that is, h must be a $\mathrm{snc}(\mathrm{Bl}_Z S, \tilde{\Delta})$-admissible blowup. Now by [Har77, Thm. II.7.17] we know that $f := g \circ h : X \to S$ is a blowup at *some* closed subscheme $W \subseteq S$ and since $g(p) \in Z$ (equivalently) $g^{-1}(U) \subseteq \mathrm{snc}(\mathrm{Bl}_Z S, \tilde{\Delta})$, it must be that $W \subseteq Z$ *as closed sets* (see also [RG71, Lem. 5.1.4]), hence $f : X \to S$ is a U-admissible blowup.

On the other hand, by a proposition of Erickson [Eri14a, Prop. 1.4], since $h_*^{-1}\tilde{\Delta} + \mathrm{Ex}\, h$ is snc the map h is thrifty and so the strata of $f_*^{-1}\Delta = h_*^{-1}\tilde{\Delta}$ are in 1-1 birational correspondence with those of $\tilde{\Delta}$, in particular $f_*^{-1}\Delta$ has a stratum in $\mathrm{Ex}\, f$.

While the application of [BM97; Sza94] is heavy-handed for this toy example, we point out that h is not simply the blowup at p as one might initially guess: starting from (4.13), blowing up the origin $0 \in \mathbb{A}^3_{z\lambda u}$ and introducing blowup coordinates

$$\mathrm{Bl}_0 \mathbb{A}^3_{z\lambda u} = \{((z, \lambda, u), [r, s, t]) \in \mathbb{A}^3_{z\lambda u} \times \mathbb{P}^2_{rst} \mid (z, \lambda, u) \propto (r, s, t)\}$$

we note that since $V((z - \lambda u) \cdot (z + \lambda u))$ is smooth on $D(z)$ we can check that the strict transform remains smooth on the $D(r)$ patch. We will investigate the $D(s)$ patch — by symmetry of λ, u in the equation $(z - \lambda u) \cdot (z + \lambda u)$ the situation is similar on the $D(t)$ patch. On $D(s)$ we have $(z, \lambda, u) = \mu(r, 1, t)$ and so

$$z \pm \lambda u = \mu r \pm \mu^2 t = \mu(r \pm \mu t)$$

Here $V(\mu)$ is a copy of the exceptional divisor of $\mathrm{Bl}_0 \mathbb{A}^3_{z\lambda u} \to \mathbb{A}^3_{z\lambda u}$ but we are still left with strict transforms $(r \pm \mu t)$ of exactly the same form as $z \pm \lambda u$; in other words, blowing up $0 \in \mathbb{A}^3_{z\lambda u}$ does not help! This is quite similar to the classical fact that blowing up the origin of the pinch point $V(z^2 - \lambda u^2) \subseteq \mathbb{A}^3_{z\lambda u}$ gives another pinch point singularity. In fact, since $(z - \lambda u) \cdot (z + \lambda u) = z^2 - \lambda^2 u^2$ our example is a double cover of the pinch point (that is, it is the preimage of the pinch point with respect to $(z, \lambda, u) \longmapsto (z, \lambda^2, u)$).

5 Constructing semi-simplicial projective Macaulayfications

5.1 Setup

In the situation of Theorem 1.8, if Z is smooth and Δ_Z is snc, then Theorem 3.11 applied to both r and s shows

$$Rf_*\mathcal{O}_X(-\Delta_X) \simeq Rf_*Rr_*\mathcal{O}_Z(-\Delta_Z) = Rg_*Rs_*\mathcal{O}_Z(-\Delta_Z) \simeq Rg_*\mathcal{O}_Y(-\Delta_Y).$$

Of course, Z need not be smooth and in the absence of resolution of singularities away from characteristic 0,[7] we cannot replace it by a resolution. In characteristic $p > 0$ we could replace Z with an

[7] At least at the time of this writing ...

alteration, but only at the cost of allowing r, s to be generically finite but not necessarily birational, and as such using alterations seems incompatible with the strategy of Theorem 3.11. Moreover, to the best of my knowledge at the level of generality Theorem 1.8 is stated, even alterations are unavailable.[8]

Instead, we replace Z with a mildly singular (specifically Cohen-Macaulay) semi-simplicial scheme $Z_\bullet$ together with morphisms $X_\bullet \xleftarrow{r_\bullet} Z_\bullet \xrightarrow{s_\bullet} Y_\bullet$ over S which are term-by-term proper birational equivalences over S. This construction is made possible by the existence of Macaulayfications.

Theorem 5.1 ([es21, Thm. 1.6], see also [Kaw00, Thm. 1.1]). *For every a CM-quasi-excellent noetherian scheme X there exists a projective birational morphism $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is Cohen-Macaulay and π is an isomorphism over the Cohen-Macaulay locus* $\mathrm{CM}(X) \subset X$.

The notion of CM-quasi-excellence is a weakening of excellence introduced by Česnavičius, so in particular the theorem applies to excellent noetherian schemes. The usefulness of Macaulayfications for the problem at hand stems from an extension of the results of Chatzistamatiou-Rülling due to Kovács.

Theorem 5.2 ([Kov20, Thm. 1.4]). *Let $f : X \to Y$ be a locally projective birational morphism of excellent Cohen-Macaulay schemes. If Y has pseudo-rational singularities then*

$$\mathscr{O}_Y = Rf_*\mathscr{O}_X \text{ and } Rf_*\omega_X = \omega_Y.$$

By a result of Lipman-Teissier, if Y is regular (so in particular if it is smooth over k) then Y is pseudo-rational [LT81, §4], hence Theorem 5.2 applies when Y is regular. Without further ado, here is the statement of the construction:

Lemma 5.3. *Let S be an excellent noetherian scheme and let (X, Δ_X) and (Y, Δ_Y) be simple normal crossing pairs separated and of finite type over S, and let $X \xleftarrow{r} Z \xrightarrow{s} Y$ be a thrifty proper birational equivalence over S. Then their exists a semi-simplicial separated finite type S-scheme $Z_\bullet$ and morphisms of semi-simplicial S-schemes $X_\bullet \xleftarrow{r_\bullet} Z_\bullet \xrightarrow{s_\bullet} Y_\bullet$ such that for all i,*

(*i*) *Z_i is Cohen-Macaulay and*

(*ii*) *$X_i \xleftarrow{r_i} Z_i \xrightarrow{s_i} Y_i$ is a projective birational equivalence over S.*

In fact, we obtain this lemma as a consequence of the more general Lemma 5.13, which is flexible enough to apply to the situations of both Theorem 1.8 and Theorem 1.11.

5.2 Gluing on simplices

To prove Lemma 5.3 we need a few preliminaries. The first describes an inductive method for constructing a sequence of truncated semi-simplicial schemes converging to $Z_\bullet$. Here for any $i \in \mathbb{N}$ an *i-truncated* semi-simplicial object in a category $\mathcal{C}$ is a functor $\Lambda^{\mathrm{op}}_{\leq i} \to \mathcal{C}$, where $\Lambda^{\mathrm{op}}_{\leq i}$ is the full subcategory of Λ^{op} generated by the objects $[j]$ with $j \leq i$. Given an $i-1$-truncated semi-simplicial object $X_\bullet$ of $\mathcal{C}$, let

$$[i]^2_< := \{j, k \in [i] \mid j < k\}$$

and define two morphisms

$$\delta_+, \delta_- : X^{[i]}_{i-1} \to X^{[i]^2_<}_{i-2}$$

by $\delta_+(x_0, \dots, x_i) = (d^{i-1}_j(x_k) \mid j < k)$ and $\delta_-(x_0, \dots, x_i) = (d^{i-1}_{k-1}(x_j) \mid j < k)$. Assuming $\mathcal{C}$ has finite limits we may form the equalizer

$$E(X_\bullet) := \mathrm{Eq}(\delta_+, \delta_-) \longrightarrow X^{[i]}_{i-1} \underset{\delta_-}{\overset{\delta_+}{\rightrightarrows}} X^{[i]^2_<}_{i-2} \tag{5.4}$$

[8] Ditto.

one can check that this construction is *functorial* in $X_\bullet$: indeed if $Y_\bullet$ is another $i-1$-truncated semi-simplicial object then given a morphism $X_\bullet \to Y_\bullet$ we can form a commutative diagram

$$\begin{tikzcd} E(X_\bullet) := \mathrm{Eq}(\delta_+, \delta_-) \arrow[r] \arrow[d, dashed] & X_{i-1}^{[i]} \arrow[r, shift left, "\delta_+"] \arrow[r, shift right, "\delta_-"'] \arrow[d] & X_{i-2}^{[i]^2_<} \arrow[d] \\ E(Y_\bullet) := \mathrm{Eq}(\delta_+, \delta_-) \arrow[r] & Y_{i-1}^{[i]} \arrow[r, shift left, "\delta_+"] \arrow[r, shift right, "\delta_-"'] & Y_{i-2}^{[i]^2_<} \end{tikzcd} \tag{5.5}$$

and obtain a unique morphism on the dashed arrow by functoriality of equalizers. Finally, let I denote the category $0 \to 1$ (thought of as the "unit interval"). An object of $\mathcal{C}^I$ is a morphism $f : X \to Y$ in $\mathcal{C}$ and there are 2 functors $s : \mathcal{C}^I \to \mathcal{C}$ defined by $s(f) = X, t(f) = Y$ (source and target).

Lemma 5.6 (compare with [SGA4II, Vbis, Prop. 5.1.3], [Stacks, Tag 0AMA]). *Let $\mathcal{C}$ be a category containing finite limits. The functor*

$$\Phi_i : \mathcal{C}^{\Lambda^{\mathrm{op}}_{\leq i}} \to \mathcal{C}^{\Lambda^{\mathrm{op}}_{\leq i-1}} \times_{\mathcal{C}} \mathcal{C}^I$$

to the 2-fiber product with respect to the functors $E : \mathcal{C}^{\Lambda^{\mathrm{op}}_{\leq i-1}} \to \mathcal{C}$ and $t : \mathcal{C}^I \to \mathcal{C}$ that sends an i-truncated semi-simplicial object $X_\bullet$ to the pair $(\mathrm{sk}_{i-1}X_\bullet, X_i \to E(\mathrm{sk}_{i-1}X))$ is an equivalence of categories.

Proof. We first check that Φ_i is fully faithful. For faithfulness, note that for any 2 i-truncated semi-simplicial objects $X_\bullet, Y_\bullet$ there is an *injection*

$$\mathrm{Hom}_{\mathcal{C}^{\Lambda^{\mathrm{op}}_{\leq i}}}(X_\bullet, Y_\bullet) \hookrightarrow \prod_{j=0}^{i} \mathrm{Hom}_{\mathcal{C}}(X_j, Y_j) \tag{5.7}$$

since a morphism $\alpha : X_\bullet \to Y_\bullet$ is equivalent to a sequence of morphisms $\alpha_i : X_i \to Y_i$ commuting with differentials. By the definition of the 2-fiber product, the morphism $\Phi_i(\alpha) : \Phi_i(X_\bullet) \to \Phi_i(Y_\bullet)$ induced by α consists of the morphism $\mathrm{sk}_{i-1}\alpha : \mathrm{sk}_{i-1}X_\bullet \to \mathrm{sk}_{i-1}Y_\bullet$, and the commutative diagram

$$\begin{tikzcd} X_i \arrow[r] \arrow[d, "\alpha_i"] & E(\mathrm{sk}_{i-1}X) \arrow[d, "E(\alpha)"] \\ Y_i \arrow[r] & E(\mathrm{sk}_{i-1}Y) \end{tikzcd}$$

This shows that (5.7) *factors* as

$$\mathrm{Hom}_{\mathcal{C}^{\Lambda^{\mathrm{op}}_{\leq i}}}(X_\bullet, Y_\bullet) \xrightarrow{\Phi_i} \mathrm{Hom}_{\mathcal{C}^{\Lambda^{\mathrm{op}}_{\leq i-1}} \times_{\mathcal{C}} \mathcal{C}^I}\big(\Phi_i(X_\bullet), \Phi_i(Y_\bullet)\big) \to \prod_{j=0}^{i} \mathrm{Hom}_{\mathcal{C}}(X_j, Y_j) \tag{5.8}$$

hence the first map is injective, or in other words Φ_i is faithful. On the other hand given an arbitrary morphism $\Phi_i(X_\bullet) \to \Phi_i(X_\bullet)$ consisting of a map $\beta : \mathrm{sk}_{i-1}X_\bullet \to \mathrm{sk}_{i-1}Y_\bullet$, a map $\gamma : X_i \to Y_i$ and a commutative diagram

$$\begin{tikzcd} X_i \arrow[r] \arrow[d, "\gamma"] & E(\mathrm{sk}_{i-1}X) \arrow[d, "E(\beta)"] \\ Y_i \arrow[r] & E(\mathrm{sk}_{i-1}Y) \end{tikzcd} \tag{5.9}$$

we may verify commutativity of

$$\begin{tikzcd} X_i \arrow[r] \arrow[d, "\gamma"] \arrow[rr, bend left, "d^i_k"] \arrow[dr, phantom, "(1)"] & E(\mathrm{sk}_{i-1}X) \arrow[r, "\mathrm{pr}_k"] \arrow[d, "E(\beta)"] \arrow[dr, phantom, "(2)"] & X_{i-1} \arrow[d, "\beta_{i-1}"] \\ Y_i \arrow[r] \arrow[rr, bend right, "d^i_k"'] & E(\mathrm{sk}_{i-1}Y) \arrow[r, "\mathrm{pr}_k"] & Y_{i-1} \end{tikzcd}$$

as follows: commutativity of (1) is exactly (5.9), and commutativity of (2) can be deduced from that of the left square of (5.5). Hence β and γ define a map $X_\bullet \to Y_\bullet$ and so Φ_i is full.

Next we show Φ_i is essentially surjective — this argument is inspired by and closely follows the proof of [Stacks, Tag 0186]. For this we consider an object of the 2-fiber product $\mathcal{C}^{\Lambda^{\mathrm{op}}_{\le i-1}} \times_{\mathcal{C}} \mathcal{C}^I$ consisting of an $i-1$-truncated semi-simplicial object $X_\bullet$, and object Y and a morphism $f : Y \to E(X_\bullet)$, and we must prove that there exists an i-truncated semi-simplicial object $Z_\bullet$ and an isomorphism $\Phi_i(Z_\bullet) \simeq (X_\bullet, f)$. We first let $Z_j = X_j$ for $j < i$ and let $Z(\varphi) = X(\varphi)$ for any $\varphi : [j'] \to [j]$ with $j' < j < i$. Then we set $Z_i = Y$, and we must define morphisms $Z(\varphi) : Z_i = Y \to X_j = Z_j$ for increasing maps $[j] \to [i]$ which are functorial in φ, in the sense that for any increasing $\psi : [j'] \to [j]$ the diagram

$$\begin{tikzcd} Y \arrow[r, "Z(\varphi)"] \arrow[rr, bend right, "Z(\varphi\circ\psi)"'] & X_j \arrow[r, "X(\psi)"] & X_{j'} \end{tikzcd} \tag{5.10}$$

commutes (note that the data of $X(\psi)$ is already included in $X_\bullet$). We may assume $j < i$ (otherwise $\varphi = \mathrm{id}$ and we must set $Z(\varphi) = \mathrm{id}$), and so φ must factor as

$$[j] \xrightarrow{\psi} [i-1] \xrightarrow{\delta^i_k} [i]$$

for some k and some ψ. We define $Z(\varphi)$ to be the composition

$$Y \xrightarrow{f} E(X_\bullet) \to X^{[i]}_{i-1} \xrightarrow{\mathrm{pr}_k} X_{i-1} \xrightarrow{X(\psi)} X_j$$

(so in particular we define $Z(\delta^i_k) = \mathrm{pr}_k \circ f =: f_k$). To verify this definition is independent of ψ, suppose that there is another factorization

$$[j] \xrightarrow{\psi'} [i-1] \xrightarrow{\delta^i_l} [i]$$

Note that if $j = i-1$ then $\psi = \psi' = \mathrm{id}$ and $k = l$ for trivial reasons, so we may assume $j < i-1$ and in that case φ misses *both* k and l, so we may factor through $[i-2]$ as follows:

$$\begin{tikzcd} & & {[i-1]} \arrow[dr, "\delta^i_k"] & \\ {[j]} \arrow[r, "\rho"] \arrow[urr, bend left, "\psi"] \arrow[drr, bend right, "\psi'"'] & {[i-2]} \arrow[ur, "\delta^{i-1}_{l-1}"] \arrow[dr, "\delta^{i-1}_k"'] & & {[i]} \\ & & {[i-1]} \arrow[ur, "\delta^i_l"'] & \end{tikzcd} \tag{5.11}$$

By the defining property of the equalizer $E(X_\bullet)$, we know $X(\delta^{i-1}_{j-1}) \circ f_k = X(\delta^{i-1}_k) \circ f_l$, and

$$X(\rho) \circ X(\delta^{i-1}_{j-1}) = X(\psi) \text{ and } X(\rho) \circ X(\delta^{i-1}_k) = X(\psi')$$

because $X_\bullet$ is an $i-1$-truncated semi-simplicial object. It follows that $X(\psi)\circ f_k = X(\psi')\circ f_l$ as desired.

We now prove to prove the commutativity statement in (5.10). Again we may assume $j < i$, since otherwise $\varphi = \mathrm{id}$ and $\psi = \varphi\circ\psi$ so commutativity is implied by the above proof that the $Z(\varphi)$ are well defined. When $j < k$ the map φ, and hence also $\varphi\circ\psi$ must factor through some $\delta^i_k : [i-1] \to [i]$ and we obtain the following situation:

$$\begin{tikzcd}
{[j]} \arrow[r, "\psi"] \arrow[rrr, bend right, "\varphi\circ\psi"'] & {[j]} \arrow[r, "\rho"] \arrow[rr, bend left, "\varphi"] & {[i-1]} \arrow[r, "\delta^i_k"] & {[i]}
\end{tikzcd}$$

Now *by definition* $Z(\varphi) = X(\rho)\circ f_k$ and $Z(\varphi\circ\psi) = X(\rho\circ\psi)\circ f_k$, and since $X_\bullet$ is an $i-1$-truncated semi-simplicial object $X(\rho\circ\psi) = X(\psi)\circ X(\rho)$, so that

$$X(\psi)\circ Z(\varphi) = X(\psi)\circ X(\rho)\circ f_k = X(\rho\circ\psi)\circ f_k = Z(\varphi\circ\psi)$$

as claimed.

□

5.3 Common admissible blowups

Using Lemma 5.6 to build the semi-simplicial scheme $Z_\bullet$ inductively, at each step we encounter the situation of the lemma below.

Lemma 5.12. *Suppose*

$$\begin{tikzcd}
X \arrow[d, "\varphi"] & U \arrow[l, "\iota"'] \arrow[r, "\jmath"] \arrow[d, "\rho^0"] & Y \arrow[d, "\psi"] \\
F & E \arrow[l, "f"] \arrow[r, "g"] & G
\end{tikzcd}$$

is a commutative diagram of schemes of finite type over a quasi-compact quasi-separated base scheme S, and assume that f, g, φ and ψ are proper and ι and $\jmath$ are dense open immersions. Then, there is a commutative diagram

$$\begin{tikzcd}
X \arrow[d, "\varphi"] & Z \arrow[l, "r"'] \arrow[r, "s"] \arrow[d, "\rho"] & Y \arrow[d, "\psi"] \\
F & E \arrow[l, "f"] \arrow[r, "g"] & G
\end{tikzcd}$$

where r and s are U-admissible blowups (hence in particular projective*).*

If in addition S is a CM-quasi-excellent noetherian scheme and U is Cohen-Macaulay, we may ensure that Z is also Cohen-Macaulay.

Proof. First, X and E are proper over the scheme F, which is quasi-compact and quasi-separated since it is of finite type over S. By the first part of Lemma 3.8 applied to the map of F-schemes $\rho^0 : U \to E$ defined on the dense open $U \subseteq X$, there is a U-admissible blowup $V_X \to X$ and an F-morphism $V_X \to E$ extending ρ^0. A similar argument produces a U-admissible blowup $V_Y \to Y$ and a G-morphism $V_Y \to E$ extending ρ^0. The current situation is summarized below:

$$\begin{tikzcd}
U \arrow[r] \arrow[drr] \arrow[ddr, "\iota"] \arrow[dddrr, bend right, "\jmath"'] & V_Y \arrow[rr] \arrow[dd] & & E \arrow[dr, equal] \arrow[dd, "g"] & \\
& & V_X \arrow[rr] \arrow[dd] & & E \arrow[dd, "f"] \\
& Y \arrow[rr, "\psi"] & & G & \\
& & X \arrow[rr, "\varphi"] & & F
\end{tikzcd}$$

Since V_X, V_Y are U-admissible blowups of X, Y respectively, they still contain U as a *dense* open ([Con07, comments before Lem. 1.1]). Note that since $V_X \to X$ is a blowup, φ is proper and f is proper the morphism $V_X \to E$ is also proper; similarly V_Y is proper over E. Now applying the second part of Lemma 3.8 to V_X and V_Y *over* E we obtain a separated finite type morphism $\rho : Z \to E$, U admissible blowups $\tilde{V}_X \to V_X$ and $\tilde{V}_Y \to V_Y$ and open immersions $\tilde{V}_X \hookrightarrow Z \hookleftarrow \tilde{V}_Y$ over E such that the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \tilde{V}_Y \arrow[d] \\ \tilde{V}_X \arrow[r] & Z \end{tikzcd}$$

commutes and $E = \tilde{V}_X \cup \tilde{V}_Y$. Since U is dense in both $\tilde{V}_X$ and $\tilde{V}_Y$, we see that $\tilde{V}_X$ and $\tilde{V}_Y$ are both dense in Z. Then as $\tilde{V}_X \to Z$ is a dense open immersion of separated finite type E-schemes where $\tilde{V}_X$ is *proper* over E, it must be that $\tilde{V}_X = Z$; similarly, $\tilde{V}_Y = Z$ (see also the comments following [Con07, Cor. 2.10]). Finally, we define r and s to be the compositions

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[rrr, bend left, "r"] & \tilde{V}_X \arrow[r] & V_X \arrow[r] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Z \arrow[r, equal] \arrow[rrr, bend left, "s"] & \tilde{V}_Y \arrow[r] & V_Y \arrow[r] & Y \end{tikzcd}$$

Finally if S is CM-quasi-excellent, then since Z is of finite type over S it is also CM-quasi-excellent by [es21, Rmk.1.5]. By hypothesis $U \subseteq \mathrm{CM}(Z)$, and by Theorem 5.1 there is a $\mathrm{CM}(X)$-admissible (hence also U-admissible) blowup $\tilde{Z} \to Z$ such that $\tilde{Z}$ is Cohen-Macaulay. In this case we replace Z with $\tilde{Z}$. □

Lemma 5.13. *Let S be a quasi-compact quasi-separated base scheme and let*

$$\begin{tikzcd} X_\bullet \arrow[d] & U_\bullet \arrow[l, "\iota_\bullet"'] \arrow[r, "j_\bullet"] \arrow[d] & Y_\bullet \arrow[d] \\ X_{-1} & U_{-1} \arrow[l, "\iota_{-1}"'] \arrow[r, "j_{-1}"] & Y_{-1} \end{tikzcd} \tag{5.14}$$

be morphisms of augmented semi-simplicial schemes of finite type over S. Assume that all differentials and augmentations of $X_\bullet$ and $Y_\bullet$ are proper,[9] and that the morphisms $X_i \xleftarrow{\iota_i} U_i \xrightarrow{j_i} Y_i$ are dense open immersions for all i (including $i = -1$). If there exists a finite-type S-scheme Z_{-1} and U_{-1}-admissible blowups $X_{-1} \xleftarrow{r_{-1}} Z_{-1} \xrightarrow{s_{-1}} Y_{-1}$, then there exists an augmented semi-simplicial S-scheme $Z_\bullet \to Z_{-1}$ together with morphisms

$$\begin{tikzcd} X_\bullet \arrow[d] & Z_\bullet \arrow[l, "r_\bullet"'] \arrow[r, "s_\bullet"] \arrow[d] & Y_\bullet \arrow[d] \\ X_{-1} & Z_{-1} \arrow[l, "r_{-1}"'] \arrow[r, "s_{-1}"] & Y_{-1} \end{tikzcd} \tag{5.15}$$

such that for all i the morphisms $X_i \xleftarrow{r_i} Z_i \xrightarrow{s_i} Y_i$ are U_i-admissible blowups (hence in particular projective and birational).

Moreover if S is a CM-quasi-excellent noetherian scheme, and each U_i is Cohen-Macaulay, we may ensure that the Z_i are also Cohen-Macaulay.

Proof. We construct a sequence of i-truncated semi-simplicial S-schemes $\tilde{Z}_{i\bullet}$ converging to $Z_\bullet$, with the additional requirement that the morphisms $\mathrm{sk}_{i-1}(U_\bullet) \to \mathrm{sk}_{i-1}(X_\bullet)$ and $\mathrm{sk}_{i-1}(U_\bullet) \to \mathrm{sk}_{i-1}(Y_\bullet)$ factor through $\tilde{Z}_{i\bullet}$.[10] The $i = -1$ case is included in the hypotheses. At the inductive step we may

[9] This is equivalent to requiring that $X_\bullet$ is a semi-semi-simplicial object in the category of proper X_{-1}-schemes (and similarly for $Y_\bullet$).

[10] I *think* that this isn't actually an additional restriction, but including it makes the inductive step easier.

assume that there is an $i-1$-truncated semi-simplicial S-scheme $\tilde{Z}_{i-1\bullet}$ together with a commutative diagram

$$\begin{tikzcd}
& \mathrm{sk}_{i-1}(U_\bullet), \arrow[dl, "\mathrm{sk}_{i-1}(\iota_\bullet)"'] \arrow[d, "k_\bullet"] \arrow[dr, "\mathrm{sk}_{i-1}(\jmath_\bullet)"] & \\
\mathrm{sk}_{i-1}(X_\bullet) \arrow[d] & \tilde{Z}_{i-1\bullet} \arrow[l, "\tilde{r}_{i-1\bullet}"] \arrow[r, "\tilde{s}_{i-1\bullet}"] \arrow[d] & \mathrm{sk}_{i-1}(Y_\bullet) \arrow[d] \\
X_{-1} & Z_{-1} \arrow[l, "r_{-1}"] \arrow[r, "s_{-1}"] & Y_{-1}
\end{tikzcd} \tag{5.16}$$

such that for all $j < i$ the morphisms $X_j \xleftarrow{\tilde{r}_{i-1,j}} \tilde{Z}_{i-1,j} \xrightarrow{\tilde{s}_{i-1,j}} Y_j$ are U_j-admissible blowups. Letting E denote the equalizer functor of Lemma 5.6, we obtain a commutative diagram of the form

$$\begin{tikzcd}
X_i \arrow[dd, "(X(\delta^i_k))"] & U_i \arrow[l, "\iota_i"] \arrow[r, "\jmath_i"] \arrow[d, "(U(\delta^i_k))"] & Y_i \arrow[dd, "(Y(\delta^i_k))"] \\
& E(\mathrm{sk}_{i-1}(U_\bullet)) \arrow[d, "E(k_\bullet)"] & \\
E(\mathrm{sk}_{i-1}(X_\bullet)) & E(\tilde{Z}_{i-1\bullet}) \arrow[l, "E(\tilde{r}_{i-1\bullet})"] \arrow[r, "E(\tilde{s}_{i-1\bullet})"] & E(\mathrm{sk}_{i-1}(Y_\bullet))
\end{tikzcd} \tag{5.17}$$

Next, we verify that (5.17) satisfies the hypotheses of Lemma 5.12, making repeated reference to the constructions in (5.4) and (5.5). Note that the bottom horizontal arrows are proper, since they are obtained as limits of the blowup maps $\tilde{r}_{i-1,j} : \tilde{Z}_{i-1,j} \to X_j$ and $\tilde{s}_{i-1,j} : \tilde{Z}_{i-1,j} \to Y_j$ for $j = i-1, i-2$. The vertical maps on the outside edges are proper since the differentials $X(\delta^i_k) : X_i \to X_{i-1}$ and $Y(\delta^i_k) : Y_i \to Y_{i-1}$ are proper by hypothesis. Hence applying Lemma 5.12 we obtain a commutative diagram

$$\begin{tikzcd}
U_i \arrow[r, "\iota_i"] \arrow[rr, bend left] \arrow[rrr, bend left, "\jmath_i"] \arrow[d, "(U(\delta^i_k))"] & X_i \arrow[d, "(X(\delta^i_k))"] & Z_i \arrow[l, "\tilde{r}_{i-1,i}"] \arrow[r, "\tilde{s}_{i-1,i}"] \arrow[d, "\rho"] & Y_i \arrow[d, "(Y(\delta^i_k))"] \\
E(\mathrm{sk}_{i-1}(U_\bullet)) \arrow[r] \arrow[rr, bend right, "E(k_\bullet)"] \arrow[rrr, bend right] & E(\mathrm{sk}_{i-1}(X_\bullet)) & E(\tilde{Z}_{i-1\bullet}) \arrow[l, "E(\tilde{r}_{i-1\bullet})"] \arrow[r, "E(\tilde{s}_{i-1\bullet})"] & E(\mathrm{sk}_{i-1}(Y_\bullet))
\end{tikzcd} \tag{5.18}$$

in which the maps $\tilde{r}_{i-1,i} : Z_i \to X_i$ and $\tilde{s}_{i-1,i} : Z_i \to Y_i$ are U_i-admissible blowups. In the case where S is CM-quasi-excellent we apply Lemma 5.12 to ensure that Z_i is Cohen-Macaulay.

Now Lemma 5.6 implies that there is an i-truncated semi-simplicial S-scheme $\tilde{Z}_{i\bullet}$ such that $\mathrm{sk}_{i-1}(\tilde{Z}_{i\bullet}) = \tilde{Z}_{i-1\bullet}$ and $\tilde{Z}_{i,i} = Z_i$, together with a commutative diagram

$$\begin{tikzcd}
& \mathrm{sk}_{i}(U_\bullet), \arrow[dl, "\mathrm{sk}_{i}(\iota_\bullet)"'] \arrow[d, "k_\bullet"] \arrow[dr, "\mathrm{sk}_{i}(\jmath_\bullet)"] & \\
\mathrm{sk}_{i}(X_\bullet) \arrow[d] & \tilde{Z}_{i\bullet} \arrow[l, "\tilde{r}_{i\bullet}"] \arrow[r, "\tilde{s}_{i\bullet}"] \arrow[d] & \mathrm{sk}_{i}(Y_\bullet) \arrow[d] \\
X_{-1} & Z_{-1} \arrow[l, "r_{-1}"] \arrow[r, "s_{-1}"] & Y_{-1}
\end{tikzcd} \tag{5.19}$$

such that for all $j \leq i$ the morphisms $X_j \xleftarrow{\tilde{r}_{i-1,j}} \tilde{Z}_{i-1,j} \xrightarrow{\tilde{s}_{i-1,j}} Y_j$ are U_j-admissible blowups. □

5.4 Constructions and corollaries

Proof of Lemma 5.3. Set $\Delta_Z = r_*^{-1}\Delta_X = s_*^{-1}\Delta_Y$. By Lemma 3.6 there is a dense open set $X \hookleftarrow U_X \hookrightarrow Z$ (resp $Z \hookleftarrow U_Y \hookrightarrow Y$) containing all generic points of strata of $\mathrm{snc}(X, \Delta_X)$ and $\mathrm{snc}(Z, \Delta_Z)$ (resp. $\mathrm{snc}(Y, \Delta_Y)$ and $\mathrm{snc}(Z, \Delta_Z)$). Then $U := U_X \cap U_Z$ is a dense open containing all generic points of strata of $\mathrm{snc}(X, \Delta_X)$, $\mathrm{snc}(Y, \Delta_Y)$ and $\mathrm{snc}(Z, \Delta_Z)$. Set $\Delta_U := \Delta_Z|_U$, so that (U, Δ_U) is simple normal crossing pair together with thrifty birational (but not necessarily projective) morphisms $(X, \Delta_X) \xleftarrow{r|_U} (U, \Delta_U) \xrightarrow{s|_U} (Y, \Delta_Y)$. We now let $X_\bullet$, $Y_\bullet$ and $U_\bullet$ be the augmented semi-simplicial schemes associated to $(X, \Delta_X), (Y, \Delta_Y)$ and (U, Δ_U) as in the discussion at the beginning of Section 2, and consider the resulting morphisms

$$\begin{tikzcd} X_\bullet \arrow[d] & U_\bullet \arrow[l] \arrow[r] \arrow[d] & Y_\bullet \arrow[d] \\ X_{i-1} = X & U_{i-1} = U \arrow[l] \arrow[r] & Y_{i-1} = Y \end{tikzcd} \tag{5.20}$$

Since U contains the generic points of all strata of $\mathrm{snc}(Z, \Delta_Z)$, the morphisms $X_i \leftarrow U_i \to Y_i$ are dense open immersions for all i, and the differentials and augmentations of $X_\bullet$ and $Y_\bullet$ are closed immersions, hence proper. Finally applying Lemma 3.8 to the collection of open immersions $U \subseteq X, Z$ over X, we obtain U-admissible blowups $\tilde{X}, \tilde{Y}$ of X, Y respectively, as well as a separated finite type X-scheme W with open immersions $\tilde{X}, \tilde{Z} \subseteq W$ covering W. Again properness of $\tilde{X}, \tilde{Y}$ over X forces $\tilde{X} = \tilde{Z} = W$, hence replacing Z with $\tilde{Z}$ we can ensure $r : Z \to X$ is a U-admissible blowup. Repeating this construction with Y, Z in place of X, Z, we may ensure $s : Z \to Y$ is also a U-admissible blowup. Thus the hypotheses of Lemma 5.13 are satisfied. □

Corollary 5.21. *With the same hypotheses as Lemma 5.3, there exists a complex $\mathcal{K}$ on Z together with quasi-isomorphisms $\mathcal{O}_X(-\Delta_X) \simeq Rr_*\mathcal{K}$ and $\mathcal{O}_Y(-\Delta_Y) \simeq Rs_*\mathcal{K}$. In particular there are quasi-isomorphisms $Rf_*\mathcal{O}_X(-\Delta_X) \simeq Rf_*Rr_*\mathcal{K} = Rg_*Rs_*\mathcal{K} \simeq Rg_*\mathcal{O}_Y(-\Delta_Y)$.*

Proof. By Lemma 5.3 there is a commutative diagram of augmented semi-simplicial S-schemes

$$\begin{tikzcd} X_\bullet \arrow[d, "\epsilon^X"] & Z_\bullet \arrow[l, "r_\bullet"'] \arrow[r, "s_\bullet"] \arrow[d, "\epsilon^Z"] & Y_\bullet \arrow[d, "\epsilon^Y"] \\ X & Z \arrow[l, "r"'] \arrow[r, "s"] & Y \end{tikzcd} \tag{5.22}$$

such that for each i the scheme Z_i is Cohen-Macaulay and the maps $X_i \xleftarrow{r_i} Z_i \xrightarrow{s_i} Y_i$ define a projective birational equivalence over S. Defining $\mathcal{K} = \mathrm{cone}(\mathcal{O}_Z \to R\epsilon_*^Z\mathcal{O}_{Z_\bullet})[-1]$, from (5.22) we obtain a map of complexes $\underline{\Omega}^0_{X,\Delta_X} \to Rr_*\mathcal{K}$ appearing in a map of distinguished triangles

$$\begin{tikzcd} \underline{\Omega}^0_{X,\Delta_X} \arrow[r] \arrow[d, "\alpha"] & \mathcal{O}_X \arrow[r] \arrow[d, "\beta"] & R\epsilon_*^X\mathcal{O}_{X_\bullet} \arrow[r, "+1"] \arrow[d, "\gamma"] & \cdots \\ Rr_*\mathcal{K} \arrow[r] & Rr_*\mathcal{O}_Z \arrow[r] & Rr_*R\epsilon_*^Z\mathcal{O}_{Z_\bullet} \arrow[r, "+1"] & \cdots \end{tikzcd}$$

By [Kov20, Thm. 1.4] β is a quasi-isomorphism. Using commutativity of (5.22) we may identify γ with the morphism

$$R\epsilon_*^X\mathcal{O}_{X_\bullet} \to R\epsilon_*^X Rr_{\bullet*}\mathcal{O}_{Z_\bullet}. \tag{5.23}$$

The morphisms on cohomology induced by (5.23) are the abutment of a map of descent spectral sequences (see Lemma 2.2); the map of E_1 pages reads

$$E_1^{ij}(X) = \begin{cases} \epsilon_*^X\mathcal{O}_{X_i} & \text{if } j = 0 \\ 0 & \text{else} \end{cases} \to R^j\epsilon_{i*}^X Rr_{i*}\mathcal{O}_{Z_i} = E_1^{ij}(Z) \tag{5.24}$$

By [Kov20, Thm. 1.4] again, for each i the natural map $\mathcal{O}_{X_i} \to Rr_{i*}\mathcal{O}_{Z_i}$ is a quasi-isomorphism . We conclude via Corollary 2.4 that (5.24) an isomorphism, and so γ is a quasi-isomorphism.

By the 5-lemma, we conclude α is a quasi-isomorphism. Applying Rf_* and using Lemma 2.7 then gives a quasi-isomorphism

$$Rf_*\mathcal{O}_X(-\Delta_X) \simeq Rf_*\underline{\Omega}^0_{X,\Delta_X} \simeq Rf_*Rr_*\mathcal{K}.$$

A symmetric argument applied on the Y side gives the desired quasi-isomorphism $Rg_*\mathcal{O}_Y(-\Delta_Y) \simeq Rg_*Rs_*\mathcal{K}$. □

6 Applications to rational pairs

In this section we make use of Grothendieck duality, as formulated in [Con00; R&D].[11]

Theorem 6.1 (Grothendieck duality, [R&D, Cor. VII.3.4], [Con00, Thm. 3.4.4]). *Let $f : X \to Y$ be a proper morphism of finite-dimensional noetherian schemes and assume Y admits a dualizing complex (for example X and Y could be schemes of finite type over k). Then for any pair of objects $\mathcal{F}^\bullet \in D^-_{qc}(X)$ and $\mathcal{G}^\bullet \in D^+_c(Y)$ there is a natural isomorphism*

$$Rf_*R\underline{Hom}_X(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet) \simeq R\underline{Hom}_Y(Rf_*\mathcal{F}^\bullet, \mathcal{G}^\bullet) \text{ in } D^b_c(Y)$$

If $\omega^\bullet_Y$ is a dualizing complex on Y then $\omega^\bullet_X := f^!\omega^\bullet_Y$ is a dualizing complex on X [R&D, §V.10, Cor. VI.3.5], and so in the case $\mathcal{G} = \omega^\bullet_Y$ we obtain a natural isomorphism

$$Rf_*R\underline{Hom}_X(\mathcal{F}^\bullet, \omega^\bullet_X) \simeq R\underline{Hom}_Y(Rf_*\mathcal{F}^\bullet, \omega^\bullet_Y) \text{ in } D^b_c(Y)$$

In the case where f is *smooth* of relative dimension n, there is a quasi-isomorphism $f^!\mathcal{O}_Y \simeq \omega_{X/Y}[n]$.

Now, a definition:

Definition 6.2 (compare with [SingsMMP 2013, Def. 2.78]). Let (S, Δ_S) be a pair as in Definition 1.5 and assume Δ_S is reduced and effective. A proper birational morphism $f : X \to S$ is a *rational resolution* if and only if

(*i*) X is regular and the strict transform $\Delta_X := f^{-1}_*\Delta_S$ has simple normal crossings,

(*ii*) the natural morphism $\mathcal{O}_S(-\Delta_S) \to Rf_*\mathcal{O}_X(-\Delta_X)$ is a quasi-isomorphism, and

letting $\omega_X = h^{-\dim X}\omega^\bullet_X$ where we use $\omega^\bullet_X = f^!\omega^\bullet_S$ as a normalized dualizing complex on X,

(*iii*) $R^if_*\omega_X(\Delta_X) = 0$ for $i > 0$.

In the situation of Definition 6.2, the map $\mathcal{O}_S(-\Delta_S) \to Rf_*\mathcal{O}_X(-\Delta_X)$ appearing in condition (ii) is Grothendieck dual to a morphism

$$\begin{aligned} &Rf_*\omega^\bullet_X(\Delta_X) \xrightarrow[(1)]{=} Rf_*R\mathcal{H}om_X(\mathcal{O}_X(-\Delta_X), \omega^\bullet_X) \\ &\xrightarrow[(2)]{\simeq} R\mathcal{H}om_S(Rf_*\mathcal{O}_X(-\Delta_X), \omega^\bullet_S) \xrightarrow[(3)]{} R\mathcal{H}om_S(\mathcal{O}_S(-\Delta_S), \omega^\bullet_S) \end{aligned} \tag{6.3}$$

where the equality (1) comes from the fact that Δ_X is a Cartier divisor ((X, Δ_X) is snc by hypothesis), the isomorphism (2) comes from Grothendieck duality and the map (3) is obtained from the morphism of (ii) by applying the derived functor $R\mathcal{H}om_S(-, \omega^\bullet_S)$. As X is regular and the dualizing complex $\omega^\bullet_X$ is normalized $h^i\omega^\bullet_X = 0$ for $i \neq -\dim X$; in other words, $\omega^\bullet_X \simeq \omega_X[\dim X]$. Twisting this equation with the Cartier divisor Δ_X gives $\omega^\bullet_X(\Delta_X) \simeq \omega_X(\Delta_X)[\dim X]$. If $\mathcal{O}_S(-\Delta_S) \to Rf_*\mathcal{O}_X(-\Delta_X)$ is a quasi-isomorphism, so is

$$Rf_*\omega_X(\Delta_X)[\dim X] \simeq Rf_*\omega^\bullet_X(\Delta_X) \to R\mathcal{H}om_S(\mathcal{O}_S(-\Delta_S), \omega^\bullet_S)$$

[11] Which is to say we make *explicit* use of Grothendieck duality — that is, it has already been used implicitly via dependence on references quite a few times!

and taking cohomology sheaves we see that $R^{i+\dim X} f_* \omega_X(\Delta_X) \simeq h^i R\mathcal{H}om_S(\mathcal{O}_S(-\Delta_S), \omega_S^\bullet)$ for all i.

Thus *given* conditions (i) and (ii) of Definition 6.2, condition (iii) is equivalent to Cohen-Macaulayness of the *sheaf* $\mathcal{O}_S(-\Delta_S)$. We record these observations as a lemma.

Lemma 6.4 (compare with [SingsMMP 2013, Cor. 2.73, Props. 2.82-2.23], [Kov20, Def. 1.3])**.** *With notation and setup as in Definition 6.2, the morphism $f : X \to S$ is a rational resolution if and only if*

(i) X is regular and the strict transform $\Delta_X := f_*^{-1}\Delta_S$ has simple normal crossings,

(ii) the natural morphism $\mathcal{O}_S(-\Delta_S) \to Rf_*\mathcal{O}_X(-\Delta_X)$ is a quasi-isomorphism, and

(iii) the sheaf $\mathcal{O}_S(-\Delta_S)$ is Cohen-Macaulay.

As illustrated in the examples of Section 4, even simple normal crossing pairs (S, Δ_S) may have non-rational resolutions in the absence of additional thriftiness restrictions, hence the following definition of rational singularities for pairs.

Definition 6.5. Let (S, Δ_S) be a pair such that Δ_S is a reduced effective Weil divisor. Then, (S, Δ_S) is *resolution-rational* if and only if it has a thrifty rational resolution.

6.1 All for one

In the case where S is a normal variety over a field of characteristic 0, it is known that if (S, Δ_S) has a thrifty rational resolution then *every* thrifty resolution is rational [SingsMMP 2013, Cor. 2.86]. The proof of this fact shows more generally that if $f : X \to S$ and $g : Y \to S$ are thrifty resolutions, then there are isomorphisms $R^i f_*\mathcal{O}_X(-\Delta_X) \simeq R^i g_*\mathcal{O}_Y(-\Delta_Y)$ for all i. This remains true in arbitrary characteristic.

Lemma 6.6 ([SingsMMP 2013, Cor. 2.86] in characteristic 0)**.** *Let (S, Δ_S) be a pair such that Δ_S is a reduced effective Weil divisor, and let $f : X \to S$ and $g : Y \to S$ be thrifty resolutions. Then there is a quasi-isomorphism $Rf_*\mathcal{O}_X(-\Delta_X) \simeq Rg_*\mathcal{O}_Y(-\Delta_Y)$. In particular, f is a rational resolution if and only if g is.*

Note that this includes Theorem 3.11 as a special case: indeed, if (S, Δ_S) is a simple normal crossing pair then given any thrifty resolution $f : X \to S$ we may choose g to be the identity.

Proof. By Lemma 3.6, there are dense open immersions $S \hookleftarrow U_X \hookrightarrow X$ and $S \hookleftarrow U_Y \hookrightarrow Y$ such that U_X (resp. U_Y) contains all strata of $\operatorname{snc}(S, \Delta_S)$ and (X, Δ_X) (resp. $\operatorname{snc}(S, \Delta_S)$ and (Y, Δ_Y)). Then $U := U_X \cap U_X$ also contains all strata of $\operatorname{snc}(S, \Delta_S)$ — moreover since f and g are thrifty, the strata of (X, Δ_X) and (Y, Δ_Y) are in one-to-one birational correspondence with those of (S, Δ_S), so it remains true that U contains all strata of (X, Δ_X) and (Y, Δ_Y). Replacing U with $U \cap \operatorname{snc}(X, \Delta_X)$, we may assume $(U, \Delta_U := \Delta_S \cap U)$ is an snc pair. We now have morphisms $\imath : U \hookrightarrow X$, $\jmath : U \hookrightarrow Y$ which are thrifty and birational, but not necessarily proper.

Now let $X_\bullet \to X_{-1} =: X$, $Y_\bullet \to Y_{-1} =: Y$ and $U_\bullet \to U_{-1} =: U$ be the augmented semi-simplicial schemes associated to these simple normal crossing pairs. The inclusions $\imath$ and $\jmath$ induce a diagram as in (5.14); we proceed to verify that the hypotheses of Lemma 5.13 are satisfied. All schemes in sight are defined over the noetherian and hence quasi-compact quasi-separated S. The differentials and augmentations are all closed immersions and hence proper, and thriftiness of $\imath$ and $\jmath$ implies that the morphisms $X_i \xleftarrow{\imath_i} U_i \xrightarrow{\jmath_i} Y_i$ are dense open immersions for all i. Applying Lemma 3.8 to the collection of S-schemes S, X and Y with the common dense open U gives a common U-admissible blowup $X \leftarrow Z \to Y$. Finally (for the moreover part of the lemma) S is excellent by hypothesis and the U_i are regular, hence Cohen-Macaulay.

The output of Lemma 5.13 is an augmented semi-simplicial scheme $Z_\bullet \to Z_{-1} =: Z$ such that each scheme Z_i is Cohen-Macaulay, together with morphisms $X_\bullet \xleftarrow{r_\bullet} Z_\bullet \xrightarrow{s_\bullet} Y_\bullet$ such that for each i the $X_i \xleftarrow{r_i} Z_i \xrightarrow{s_i} Y_i$ are U_i-admissible blowups. For the remainder of the proof we argue exactly as in Corollary 5.21. $\square$

Part II

Correspondences in log Hodge cohomology

7 Introduction to Part II

The original proof of [CR11, Thm. 3.2.8] makes use of a cycle morphism $\mathrm{cl} : CH^*(X) \to H^*(X, \Omega_X^*)$ from Chow cohomology to Hodge cohomology, which is ultimately applied to a cycle $Z \subset X \times Y$ obtained from a proper birational equivalence. That cycle morphism satisfies 2 essential properties: the first is that it is compatible with *correspondences*: here Chow correspondences are homomorphisms

$$CH^*(X) \to CH^*(Y) \text{ of the form } \alpha \longmapsto \mathrm{pr}_{Y*}(\mathrm{pr}_X^* \alpha \smile \gamma) \text{ for some } \gamma \in CH^*(X \times Y)$$

where $\smile$ is the cup product induced by intersecting cycles; Hodge correspondences are defined in a similar way. The second key property is a compatibility with the filtrations

$$CH^n(X \times Y) = F^0 CH^n(X \times Y) \supseteq F^1 CH^n(X \times Y) \supseteq \cdots \supseteq F^{\dim Y} CH^n(X \times Y) \supseteq 0$$

where $F^c CH^n(X \times Y)$ is the subgroup generated by cycles $Z \subseteq X \times Y$ such that $\mathrm{codim}(\mathrm{pr}_Y Z \subseteq Y) \geq c$, and

$$H^n(X \times Y, \Omega^m_{X \times Y}) = F^0 H^n(X \times Y, \Omega^m_{X \times Y}) \supseteq F^1 CH^*(X \times Y) \supseteq \cdots \supseteq F^{\dim Y} H^n(X \times Y, \Omega^m_{X \times Y}) \supseteq 0$$

where $F^c H^n(X \times Y, \Omega^m_{X \times Y})$ is the image of the map $H^n(X \times Y, \oplus_{j=c}^m \Omega_X^{m-j} \boxtimes \Omega_Y^j) \to H^n(X \times Y, \Omega^m_{X \times Y})$ coming from the Künneth decomposition.

It is natural to ask if a similar method can be applied to prove Theorem 1.8, by replacing the ordinary sheaves of differentials Ω_X appearing in Hodge cohomology with sheaves of differentials with log poles $\Omega_X(\log \Delta_X)$. Many of results on Hodge cohomology in [CR11, §2] have been extended to include log poles in [BPØ20, §9], and Section 8 is a rapid expository summary of those results. While [BPØ20, §9] does also construct correspondences, only *finite* correspondences are considered, with additional strictness (in the sense of logarithmic geometry) conditions. To the best of my understanding, if the projections

$$Z \subseteq X \times Y \to X \text{ and } Z \subseteq X \times Y \to Y$$

are allowed to be proper birational (but not necessarily finite) maps, even with necessary thriftiness hypotheses we encounter cases where finite correspondences are no longer applicable. We describe a different type of correspondences in Section 10, obtained from certain Hodge classes with both log poles *and* log zeroes. In order to do so we prove a base change formula on the interaction of pushforward and pullback operations in cartesian squares in Section 9.

Ultimately even the correspondences of Section 10 seem to be insufficient to deal with thrifty proper birational equivalences, as we illustrate in Section 11. The problem we encounter is elementary: looking at the recipe for the Hodge class $\mathrm{cl}(Z)$ of a subvariety $Z \subseteq X$, where Z and X are smooth an projective (outlined in [Har77, Ex. III.7.4]), we see that $\mathrm{cl}(Z)$ ultimately comes from the trace linear functional $\mathrm{tr} : H^{\dim Z}(Z, \omega_Z) \to k$, or Serre-dually the element $1 \in H^0(Z, \mathcal{O}_Z)$. Due to the introduction of log poles and zeroes in Section 10, trying to follow that recipe we pass through cohomology groups of the form $H^{\dim Z}(Z, \omega_Z(D))$, or dually $H^0(Z, \mathcal{O}_Z(-D))$ where D is an (often non-0 in cases of interest) effective Cartier divisor on Z, and so there simply is no "1" to be had.

7.1 Acknowledgements

Thanks to Daniel Bragg, Giovanni Inchiostro and Yun Hao for helpful conversations, to Lawrence Jack Barrott for illuminating email correspondence regarding logarithmic aspects of Chow and Hodge, and to my advisor Sándor Kovács for many insightful discussions. Thanks also to the participants of the Spring 2019 MSRI graduate student seminar, in particular organizer Fatemeh Rezaee, for feedback on a talk related to early work on this problem.

8 Functoriality properties of log Hodge cohomology with supports

8.1 Supports

In order to obtain results that apply to proper birational equivalences $X \dashrightarrow Y$ where neither X nor Y is proper, it is necessary to work with cohomology with *supports*, also known as local cohomology. A primary source for the material of this subsection is [R&D, §IV]. Let X be a noetherian scheme.

Definition 8.1 ([R&D, §IV], [CR11, §1.1])**.** A **family of supports** Φ **on** X is a non-empty collection Φ of closed subsets of X such that

- If $C \in \Phi$ and $D \subset C$ is a closed subset, then $D \in \Phi$.
- If $C, D \in \Phi$ then $C \cup D \in \Phi$.

Example 8.2. $\Phi = \{$ all closed subsets of X $\}$ is a family of supports. More generally if $\mathcal{C}$ is any collection of closed subsets $C \subset X$, there's a *smallest* family of supports $\Phi(\mathcal{C})$ containing $\mathcal{C}$ (explicitly, $\Phi(\mathcal{C})$ consists of finite unions $\bigcup_i Z_i$ of closed subsets $Z_i \subset C_i$ of elements $C_i \in \mathcal{C}$). Taking $\Phi = \Phi(\{X\})$ recovers the previous example. A more interesting example is the case where for some fixed $p \in \mathbb{N}$, $\Phi = \{\text{closed sets } Z \subseteq X \mid \dim Z \leq p\}$.

There is a close relationship between families of supports on X and certain collections of specialization-closed subsets of points on X, and we can also consider sheaves of families of supports — for further details we refer to [R&D, §IV.1].

If $f : X \to Y$ is a morphism of noetherian schemes and Ψ is a family of supports on Y, then $\{f^{-1}(Z) \mid Z \in \Psi\}$ is a family of closed subsets of X, and is closed under unions, but is *not* in general closed under taking closed subsets.

Definition 8.3. $f^{-1}(\Psi)$ is the smallest family of supports on X containing $\{f^{-1}(Z) \mid Z \in \Psi\}$.

Let Φ be a family of supports on X. The notation/terminology $f|_\Phi$ **is proper** will mean $f|_C$ is proper for every $C \in \Phi$. If $f|_\Phi$ is proper then $f(C) \subset Y$ is closed for every $C \in \Phi$ and in fact

$$f(\Phi) = \{f(C) \subset Y \mid C \in \Phi\} \tag{8.4}$$

is a family of supports on Y. The key point here is that if $D \subset f(C)$ is closed, then $f^{-1}(D) \cap C \in \Phi$ and $D = f(f^{-1}(D) \cap C)$.

Definition 8.5. A **scheme with supports** (X, Φ_X) is a scheme X together with a family of supports Φ_X on X.

Definition 8.6. A **pushing morphism** $f : (X, \Phi_X) \to (Y, \Phi_Y)$ of schemes with supports is a morphism $f : X \to Y$ of underlying schemes such that $f|_{\Phi_X}$ is proper and $f(\Phi_X) \subset \Phi_Y$. A **pulling morphism** $f : X \to Y$ is a morphism $f : X \to Y$ such that $f^{-1}(\Phi_Y) \subset \Phi_X$.

These morphisms provide 2 different categories with underlying set of objects schemes with supports (X, Φ_X), and pushing/pulling morphisms respectively (the verification is elementary; for instance a composition of pushing morphisms is again a pushing morphism since compositions of proper morphisms are proper). Schemes with supports provide a natural setting for describing functoriality properties of local cohomology. Let $\mathcal{F}$ be a sheaf of abelian groups on a scheme with supports (X, Φ_X) (more precisely $\mathcal{F}$ is just a sheaf of abelian groups on X).

Definition 8.7. The **sheaf of sections with supports** of $\mathcal{F}$, denoted $\underline{\Gamma}_\Phi(\mathcal{F})$, is obtained by setting

$$\underline{\Gamma}_\Phi(\mathcal{F})(U) = \{\sigma \in \mathcal{F}(U) \mid \operatorname{supp} \sigma \in \Phi_X|_U\} \tag{8.8}$$

for each open $U \subset X$ (here $\Phi_X|_U$ is short for $\iota^{-1}\Phi_X$ where $\iota : U \to X$ is the inclusion). More explicitly: for a local section $\sigma \in \mathcal{F}(U)$, $\sigma \in \underline{\Gamma}_\Phi(\mathcal{F})(U)$ means $\operatorname{supp} \sigma = C \cap U$ for a closed set $C \subset \Phi_X$.

The functor $\underline{\Gamma}_\Phi$ is right adjoint to an exact functor, for instance the inclusion of the subcategory $\mathbf{Ab}_\Phi(X) \subset \mathbf{Ab}(X)$ of abelian sheaves on X with supports in Φ; so, $\underline{\Gamma}_\Phi$ is left exact and preserves injectives. In the case $\Phi = \Phi(Z)$ for some closed $Z \subset X$, this is proved in [Stacks, Tag 0A39, Tag 0G6Y, Tag 0G7F] — the general case can then be obtained by writing $\underline{\Gamma}_\Phi$ as a filtered colimit:

$$\underline{\Gamma}_\Phi = \operatorname{colim}_{Z\in\Phi} \underline{\Gamma}_Z.$$

The right derived functor of $\underline{\Gamma}_\Phi$ will be denoted $R\underline{\Gamma}_\Phi$. Taking global sections on X gives the **sections with supports** of $\mathcal{F}$: $\Gamma_\Phi(\mathcal{F}) := \Gamma_X(\underline{\Gamma}_\Phi(\mathcal{F}))$ This is also left exact, and (the cohomologies of) its derived functor give the **cohomology with supports in** Φ: $H^i_\Phi(X, \mathcal{F}) := R^i\Gamma_\Phi(\mathcal{F})$.

Proposition 8.9. *Cohomology with supports enjoys the following functoriality properties:*

(*i*) *If $f : (X, \Phi_X) \to (Y, \Phi_Y)$ is a* pulling *morphism of schemes with supports, $\mathcal{F}, \mathcal{G}$ are sheaves of abelian groups on X, Y respectively, and if*

$$\varphi : \mathcal{G} \to f_*\mathcal{F} \text{ is a morphism of sheaves,} \tag{8.10}$$

then there is a natural morphism $R\underline{\Gamma}_\Phi\mathcal{G} \to Rf_*R\underline{\Gamma}_\Phi\mathcal{F}$. Similarly if $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent then there are natural morphisms $R\underline{\Gamma}_\Phi\mathcal{G} \to Rf_*R\underline{\Gamma}_\Phi\mathcal{F}$.

(*ii*) *If $f : (X, \Phi_X) \to (Y, \Phi_Y)$ is a* pushing *morphism, $\mathcal{F}, \mathcal{G}$ are sheaves of abelian groups on X, Y respectively, and*

$$\psi : Rf_*\mathcal{F} \to \mathcal{G} \text{ is a morphism in the derived category of } X, \tag{8.11}$$

then there is a natural morphism $Rf_*R\underline{\Gamma}_\Phi(\mathcal{F}) \to R\underline{\Gamma}_\Phi\mathcal{G}$.

Both parts of the proposition follow from [Stacks, Tag 0G78]; (i) is discussed in detail in [CR11, §2.1] and (ii) can be extracted from [CR11, §2.2] (although it doesn't appear to be stated explicitly). See also [BPØ20, Constructions 9.4.2, 9.5.3]

8.2 Differential forms with log poles

Let k be a perfect field.

Definition 8.12. A **snc pair with supports** (X, Δ_X, Φ_X) over k is a smooth scheme X separated and of finite type over k with a family of supports Φ_X together with a reduced, effective divisor Δ_X on X such that $\operatorname{supp}\Delta_X$ has simple normal crossings, in the sense that for any point $x \in X$ there are regular parameters $z_1, \dots, z_c \in \mathcal{O}_{X,x}$ such that $\operatorname{supp}\Delta_X = V(z_1 \cdot z_2 \cdots z_r)$ on a Zariski neighborhood of x.[12] The **interior** U_X of a snc pair with supports (X, Δ_X, Φ_X) is

$$U_X := X \setminus \operatorname{supp}\Delta_X \tag{8.13}$$

The inclusion of U_X in X is denoted by $\iota_X : U_X \to X$.

Here $\operatorname{supp}\Delta_X$ denotes the **support** of Δ_X (if $\Delta_X = \sum_i a_i D_i$ where the D_i are prime divisors, then $\operatorname{supp}\Delta_X = \cup_i D_i$). Similarly let $j_X : \operatorname{supp}\Delta_X \to X$ denote the evident inclusion.

Definition 8.14 (compare with [CR11, Def. 1.1.4]). A **pulling morphism** $f : (X, \Delta_X, \Phi_X) \to (Y, \Delta_Y, \Phi_Y)$ **of snc pairs with supports** is a pulling morphism $f : X \to Y$ of underlying schemes with support such that $f^{-1}(\operatorname{supp}\Delta_Y) \subset \operatorname{supp}\Delta_X$; equivalently, f restricts to a morphism $f|_{U_X} : U_X \to U_Y$. A **pushing morphism** $f : (X, \Delta_X, \Phi_X) \to (Y, \Delta_Y, \Phi_Y)$ **of snc pairs with supports** is a pushing morphism of underlying schemes with support such that $f^*\Delta_Y = \Delta_X$.

Note that if $f : (X, \Delta_X, \Phi_X) \to (Y, \Delta_Y, \Phi_Y)$ is a pushing morphism then $U_X = f^{-1}(U_Y)$, so for example if $f : X \to Y$ is proper then so is the induced map $U_X \to U_Y$.

[12] This is equivalent to the more general definition [BPØ20, Def. 7.2.1] in the case where the base scheme is Spec k, which is all we need. Note that this is *not equivalent in general to* Definition 1.6.

Convention 8.15 (compare with [CR11, p. 1.1.5]). A morphism of snc pairs with supports $f : (X, \Delta_X, \Phi_X) \to (Y, \Delta_Y, \Phi_Y)$ is flat, proper, an immersion, etc. if and only if the same is true of the underlying morphism of schemes $f : X \to Y$. A diagram of snc pairs with supports

$$\begin{tikzcd} (X', \Delta_{X'}, \Phi_{X'}) \arrow[r, "g'"] \arrow[d, "f'"'] & (X, \Delta_X, \Phi_X) \arrow[d, "f"] \\ (Y', \Delta_{Y'}, \Phi_{Y'}) \arrow[r, "g"'] & (Y, \Delta_Y, \Phi_Y) \end{tikzcd} \tag{8.16}$$

is **cartesian** if and only if the induced diagram of underlying schemes

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd} \tag{8.17}$$

is cartesian.[13]

The terminology is meant to suggest that pushing (resp. pulling) morphisms induce pushforward (resp. pullback) maps on log Hodge cohomology, as we now describe.

If (X, Δ_X) is an snc pair, or more generally a normal separated scheme of finite type X over k together with a sequence of effective Cartier divisors $D_1, \dots, D_N \subseteq X$ with sum $\Delta_X = \sum_i D_i$, then it comes with a sheaf of *differential forms with log poles* $\Omega_X(\log \Delta_X)$. In the case where (X, Δ_X, Φ_X) is snc, this sheaf and its properties are described in [EV92, §2]. For a definition and treatment of $\Omega_X(\log \Delta_X)$ in the much greater generality of logarithmic schemes we refer to [Ogu18, §IV]. However, we emphasize that we only require the case of effective Cartier divisors on a normal scheme mentioned above (in the terminology of log geometry this is the case of a Deligne-Faltings log structure).

In some of the calculations below the following concrete local description will be very useful. Let $z_1, z_2, \dots, z_n$ be local coordinates at a point $x \in X$ such that $\operatorname{supp} \Delta_X = V(z_1 z_2 \cdots z_r)$ in a neighborhood of x. Recall that as X is smooth the differentials $d\,z_1, d\,z_2, \dots, d\,z_n$ freely generate Ω_X on a neighborhood of x.

Lemma 8.18 (see e.g. [EV92, §2]). *The sections* $\frac{d\,z_1}{z_1}, \dots, \frac{d\,z_r}{z_r}, d\,z_{r+1}, \dots, d\,z_n$ *freely generate* $\Omega_X(\log \Delta_X)$ *on a neighborhood of* x.

Given $\Omega_X(\log \Delta_X)$, we can form the exterior powers

$$\Omega_X^p(\log \Delta_X) := \bigwedge^p \Omega_X(\log \Delta_X), \tag{8.19}$$

and combining Lemma 8.18 with (8.19) gives concrete local descriptions of the $\Omega_X^p(\log \Delta_X)$; in particular, we see that $\Omega_X^{\dim X}(\log \Delta_X) = \omega_X(\Delta_X)$.

Definition 8.20. The **log-Hodge cohomology with supports** of a log-smooth pair with supports (X, Δ_X, Φ_X) is defined by

$$H^d(X, \Delta_X, \Phi_X) = \bigoplus_{p+q=d} H^q_\Phi(X, \Omega_X^p(\log \Delta_X)) \tag{8.21}$$

Here H^q_Φ denotes local cohomology with respect to the family of supports Φ_X. For connected X, we *define* $H_d(X, \Delta_X, \Phi_X) := H^{2\dim X - d}(X, \Delta_X, \Phi_X)$, and in general we set $H_d(X, \Delta_X, \Phi_X) = \bigoplus_i H_d(X_i, \Delta_{X_i}, \Phi_{X_i})$ where X_i are the connected components of X.

[13] If we take the red pill of logarithmic geometry, it starts to seem almost more reasonable to only require flatness, properness, cartesianness and so on of the induced maps of *interiors* $U_X \to U_Y$. However we do use the stronger restrictions of the given definition in some of the proofs below.

Let $f : (X, \Delta_X, \Phi_X) \to (Y, \Delta_Y, \Phi_Y)$ be pulling morphism of snc pairs with supports.

Lemma 8.22 ([Ogu18, Prop. 2.3.1] + (8.19))**.** *The map f induces a morphism of sheaves*

$$\begin{aligned} f^*\Omega^p_Y(\log \Delta_Y) &\xrightarrow{df^\vee} \Omega^p_X(\log \Delta_X) \text{ adjoint to a morphism} \\ f^*\Omega^p_Y(\log \Delta_Y) &\xrightarrow{df^\vee} \Omega^p_X(\log \Delta_X) \text{ for all } p. \end{aligned} \tag{8.23}$$

The essential content of this lemma is that when we pull back a log differential form σ on (Y, Δ_Y), it doesn't *develop* poles of order ≥ 1 along Δ_X. To see why, I find it illuminating to look at the following 2 examples:

Example 8.24. Consider the morphism of pairs $f : (\mathbb{A}^1_z, 0) \to (\mathbb{A}^1_z, 0)$ defined by $f(z) = z^n$, where $n \in \mathbb{Z}, n \neq 0$. When we pull back $\frac{dz}{z}$, we get

$$\frac{d(f(z))}{f(z)} = \frac{d(z^n)}{z^n} = n \cdot \frac{dz}{z} \tag{8.25}$$

Of course, if char $k|n$ this is 0, but regardless it has a pole of order ≤ 1 at $0 \in \mathbb{A}^1$.

Example 8.26. Take the pair $(\mathbb{A}^2_x, L_1 + L_2)$, where $L_i = V(x_i)$ for $i = 1, 2$ and blow up the origin to obtain $\mathrm{Bl}_0(\mathbb{A}^2)$; let $\pi : \mathrm{Bl}_0(\mathbb{A}^2) \to \mathbb{A}^2$ be the projection, let $E \subset \mathrm{Bl}_0(\mathbb{A}^2)$ be the exceptional divisor and let $\tilde{L}_1, \tilde{L}_2 \subset \mathrm{Bl}_0(\mathbb{A}^2)$ be the strict transforms of L_1, L_2 respectively. We obtain a morphism of pairs

$$\pi : (\mathrm{Bl}_0(\mathbb{A}^2), \tilde{L}_1 + \tilde{L}_2 + E) \to (\mathbb{A}^2, L_1 + L_2) \tag{8.27}$$

Note that with $\tilde{U} := \mathrm{Bl}_0(\mathbb{A}^2) \setminus (\tilde{L}_1 + \tilde{L}_2 + E)$ and $U := \mathbb{A}^2 \setminus (L_1 + L_2)$, we have $\pi(\tilde{U}) \subset U$ (this would *not* hold if we didn't include E in the divisor on $\mathrm{Bl}_0(\mathbb{A}^2)$).

Now let's pull back $\frac{dx_1}{x_1}$: recall that$\mathrm{Bl}_0(\mathbb{A}^2) = V(x_1y_2 - x_2y_1) \subset \mathbb{A}^2_x \times \mathbb{P}^1_y$ On the $D(y_1) \subset \mathrm{Bl}_0(\mathbb{A}^2)$ affine neighborhood, π looks like

$$\mathbb{A}^2_{x_1,y_2} \simeq D(y_1) \xrightarrow{\pi} \mathbb{A}^2_{x_1,x_2} \text{ sending } (x_1, y_2) \longmapsto (x_1, x_1y_2) \tag{8.28}$$

(note that the exceptional divisor corresponds to $V(x_1) \subset \mathbb{A}^2_{x_1,y_2}$, i.e. the y_2-axis). So, the pullback of $\frac{dx_1}{x_1}$ is still $\frac{dx_1}{x_1}$, but the pullback of $\frac{dx_2}{x_2}$ is

$$\frac{d(x_1y_2)}{x_1y_2} = \frac{dx_1}{x_1} + \frac{dy_2}{y_2}$$

We see that $d\pi^\vee(\frac{dx_2}{x_2})$ has a pole of order 1 along E.

Combining the previous lemma with proposition 8.9 gives:

Proposition 8.29 ([BPØ20, §9.1-2], see also [CR11, §2.1])**.** *For every pulling morphism $f : (X, \Delta_X, \Phi_X) \to (Y, \Delta_Y, \Phi_Y)$ there are functorial morphisms*

$$R\underline{\Gamma}_\Phi \Omega^p_Y(\log \Delta_Y) \to Rf_* R\underline{\Gamma}_\Phi \Omega^p_Y(\log \Delta_Y) \text{ for all } p \tag{8.30}$$

In particular, for each p, q there are functorial homomorphisms

$$f^* : H^q_\Phi(Y, \Omega^p_Y(\log \Delta_Y)) \to H^q_\Phi(X, \Omega^p_X(\log \Delta_X)) \tag{8.31}$$

and hence (summing over $p + q = d$) functorial homomorphisms

$$f^* : H^d(X, \Delta_X, \Phi_X) \to H^d(Y, \Delta_Y, \Phi_Y) \tag{8.32}$$

The maps $f_* : H_d(X, \Delta_X, \Phi_X) \to H_d(Y, \Delta_Y, \Phi_Y)$ induced by a pushing morphism $f : (X, \Delta_X, \Phi_X) \to (Y, \Delta_Y, \Phi_Y)$ can be obtained from a combination of Nagata compactification and Grothendieck duality.

Lemma 8.33 ([BPØ20, §9.5], see also [CR11, §2.3]). *Let $f : (X, \Delta_X, \Phi_X) \to (Y, \Delta_Y, \Phi_Y)$ be a pushing morphism of equidimensional log-smooth pairs with support such that. Then letting $c = \dim Y - \dim X$, for each p there are functorial morphisms of complexes of coherent sheaves*

$$Rf_* R\underline{\Gamma}_{\Phi_X}(\Omega^p_X(\log \Delta_X)) \to R\underline{\Gamma}_{\Phi_Y}\Omega^{p+c}_Y(\log \Delta_Y)[c] \tag{8.34}$$

inducing maps on cohomology

$$f_* : H^q_{\Phi_X}(X, \Omega^p_X(\log \Delta_X)) \to H^{q+c}_{\Phi_Y}(Y, \Omega^{p+c}_Y(\log \Delta_Y)) \tag{8.35}$$

for all q.

Since they enter into the calculations below we give a description of these pushforward morphisms. Before beginning, a word on duality in our current setup: since we are working exclusively over $\operatorname{Spec} k$, we can make use of compatible normalized dualizing complexes — namely, if $\pi : Z \to \operatorname{Spec} k$ is a separated finite type k-scheme then $\omega^\bullet_Z$ is a dualizing complex [Stacks, Tag 0E2S, Tag 0FVU]. We will make repeated use of the behavior of dualizing with respect to differentials: as a consequence of Lemma 8.18, wedge product gives a perfect pairing

$$\Omega^p_X(\log \Delta_X)(-\Delta_X) \otimes \Omega^{\dim X - p}_X(\log \Delta_X) \to \omega_X \tag{8.36}$$

(see also [Har77, Cor. III.7.13]) and so $\Omega^{\dim X-p}_X(\log \Delta_X) \simeq R\mathcal{H}om_X(\Omega^p_X(\log \Delta_X)(-\Delta_X), \omega_X)$. Here the derived sheaf Hom $R\mathcal{H}om_X$ agrees with the regular sheaf Hom as $\Omega^p_X(\log \Delta_X)(-\Delta_X)$ is locally free. On the other hand, the *dualizing functor* of X is $R\mathcal{H}om_X(\Omega^p_X(\log \Delta_X)(-\Delta_X), \omega_X[\dim X])$ where $\omega_X = \Omega^{\dim X}_X$. An upshot is that Grothendieck duality calculations involving the sheaves of differential forms become more symmetric and predictable if we work with the shifted versions $\Omega^p_X(\log \Delta_X)(-\Delta_X)[p]$; for example then we have the identity

$$\Omega^{\dim X-p}_X(\log \Delta_X)[\dim X - p] \simeq R\mathcal{H}om_X(\Omega^p_X(\log \Delta_X)(-\Delta_X)[p], \omega_X[\dim X])$$

Now, we need to compactify $f : X \to Y$.

Theorem 8.37 ([Nag63, §4 Thm. 2], [Con07, Thm. 4.1]). *Let S be a quasi-compact quasi-separated scheme and let $X \to S$ be a separated morphism of finite type. Then there is a dense open immersion of S-schemes $X \hookrightarrow \overline{X}$ such that $\overline{X}$ is proper.*

Using Theorem 8.37 we obtain a dense open immersion of Y-schemes $\iota : X \to \bar{X}$ where $\bar{f} : \bar{X} \to Y$ is *proper*; we let $\Delta_{\bar{X}} = \overline{\Delta_X}$ be the closure. Replacing $\bar{X}$ with its normalization and blowing up the components of $\Delta_{\bar{X}}$ if necessary we may assume $\bar{X}$ is normal and $\Delta_{\bar{X}}$ is Cartier. Note that it remains true that $\bar{f}^*\Delta_Y = \Delta_{\bar{X}}$ since this can be checked at generic points of components of $\bar{f}^*\Delta_Y, \Delta_{\bar{X}}$ all of which lie in X. The pushforward morphisms of Lemma 8.33 are defined using the sheaves $\Omega^p_{\overline{X}}(\log \Delta_{\overline{X}})$ of log differential p-forms over k as described in [Ogu18, §IV.1]. The essential properties that we need are:

- $\Omega^p_{\overline{X}}(\log \Delta_{\overline{X}})$ is a coherent sheaf on $\overline{X}$ together with a functorial morphism

$$\Omega^p_Y(\log \Delta_Y) \to \overline{f}_* \Omega^p_{\overline{X}}(\log \Delta_{\overline{X}});$$

- there is a natural isomorphism $\Omega^p_{\overline{X}}(\log \Delta_{\overline{X}})|_X \simeq \Omega^p_X(\Delta_X)$.

Hence in particular $\Omega^p_{\overline{X}}(\log \Delta_{\overline{X}})$ is a *functorial coherent extension* of $\Omega^p_X(\Delta_X)$ to the possibly non-snc $(\overline{X}, \Delta_{\overline{X}})$. Starting with the log differential

$$d\,\mathrm{pr}^\vee_Y : \Omega^p_Y(\log \Delta_Y)[p] \to R\overline{f}_* \Omega^p_{\overline{X}}(\log \Delta_{\overline{X}})[p],$$

twisting by $-\Delta_Y$ and using the projection formula gives a morphism (*note*: this is where we use the assumption that $f^*\Delta_Y = \Delta_X$)

$$\Omega^p_Y(\log \Delta_Y)(-\Delta_Y)[p] \to R\overline{f}_* \Omega^p_{\overline{X}}(\log \Delta_{\overline{X}})(-\Delta_{\overline{X}})[p] \tag{8.38}$$

to which we apply Grothendieck duality (Theorem 6.1) — this gives a morphism

$$\begin{array}{c} R\overline{f}_*R\mathcal{H}om_{\overline{X}}(\Omega^p_{\overline{X}}(\log\Delta_{\overline{X}})(-\Delta_{\overline{X}})[p],\omega^\bullet_{\overline{X}}) = R\mathcal{H}om_Y(R\overline{f}_*\Omega^p_{\overline{X}}(\log\Delta_{\overline{X}})(-\Delta_{\overline{X}})[p],\omega_Y[\dim Y]) \\ \downarrow \\ R\mathcal{H}om_Y(\Omega^p_Y(\log\Delta_Y)(-\Delta_Y)[p],\omega_Y[\dim Y]) \end{array} \tag{8.39}$$

Where the equality is Theorem 6.1 and the vertical map is induced by (8.38). Adding supports gives a morphism

$$\begin{array}{c} Rf_*R\Gamma_{\underline{\Phi}_X}R\mathcal{H}om_X(\Omega^p_X(\log\Delta_X)(-\Delta_X)[p],\omega_X[\dim X]) = R\overline{f}_*R\Gamma_{\underline{\Phi}_X}R\mathcal{H}om_{\overline{X}}(\Omega^p_{\overline{X}}(\log\Delta_{\overline{X}})(-\Delta_{\overline{X}})[p],\omega^\bullet_{\overline{X}}) \\ \downarrow \\ R\Gamma_{\underline{\Phi}_Y}R\mathcal{H}om_Y(\Omega^p_Y(\log\Delta_Y)(-\Delta_Y)[p],\omega_Y[\dim Y]) \end{array} \tag{8.40}$$

where the equality is obtained from the *excision* property of local cohomology, compatibility of the dualizing functor with restriction *and* the natural isomorphism $\Omega^p_{\overline{X}}(\log\Delta_{\overline{X}})|_X \simeq \Omega^p_X(\Delta_X)$. Using (8.36) we obtain

$$\Omega_X^{\dim X-p}(\log\Delta_X) \simeq \mathcal{H}om_X(\Omega^p_X(\log\Delta_X)(-\Delta_X),\omega_X) = R\mathcal{H}om_X(\Omega^p_X(\log\Delta_X)(-\Delta_X),\omega_X)$$

where the last equality uses the fact that $\Omega^p_X(\log\Delta_X)(-\Delta_X)$ is locally free. A similar calculation on Y transforms (8.40) into:

$$Rf_*R\Gamma_{\underline{\Phi}_X}\Omega_X^{\dim X-p}(\log\Delta_X)[\dim X-p] \to R\Gamma_{\underline{\Phi}_Y}\Omega_Y^{\dim Y-p}(\log\Delta_Y)[\dim Y-p]$$

and reindexing like $p \leftrightarrow \dim X - p$ recovers Lemma 8.33.

9 A base change formula

Lemma 9.1 (compare with [CR11, Prop. 2.3.7]). *Let*

$$\begin{tikzcd} (X',\Delta_{X'},\Phi_{X'}) \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & (X,\Delta_X,\Phi_X) \arrow[d, "f"] \\ (Y',\Delta_{Y'},\Phi_{Y'}) \arrow[r, "g"] & (Y,\Delta_Y,\Phi_Y) \end{tikzcd} \tag{9.2}$$

be a cartesian diagram of equidimensional snc pairs with supports, where f, f' (resp. g, g') are pushing (resp. pulling) morphisms and g is either flat or a closed immersion transverse to f. Then

$$g^*f_* = f'_*g'^* : H^*(X,\Delta_X,\Phi_X) \to H^*(Y',\Delta_{Y'},\Phi_{Y'}).$$

We will prove this following Chatzistamatiou and Rülling's argument [CR11, Prop. 2.3.7] quite closely, at various points reducing to statements proved therein. In the proofs we will make use of a slight variant of Definition 8.3.

Definition 9.3. If $f : X \to Y$ is a morphism of noetherian schemes and let Φ_Y is a family of supports on Y, then

$$f_*^{-1}(\Phi_Y) := \{Z \subseteq X \mid f|_Z \text{ is proper and } f(Z) \in \Phi_Y\}$$

Lemma 9.4. *It suffices to prove Lemma 9.1 in the cases where f is either*

(*i*) *a projection morphism of the form $\mathrm{pr}_Y : (X\times Y, \mathrm{pr}_Y^*\Delta_Y, \mathrm{pr}_{Y*}^{-1}(\Phi_Y)) \to (Y,\Delta_Y,\Phi_Y)$, or* (*ii*) *a closed immersion.*

Remark 9.5. This lemma makes essential use of the *functoriality* part of Lemma 8.33.

Proof. We can decompose (9.2) as a concatenation of cartisian diagrams

$$\begin{tikzcd}
(X', \Delta_{X'}, \Phi_{X'}) \arrow[r, "g'"] \arrow[d, "h'"] \arrow[dr, phantom, "(2)"] & (X, \Delta_X, \Phi_X) \arrow[d, "h"] \\
(X \times Y', \mathrm{pr}_{Y'}^*\Delta_Y, \mathrm{pr}_{Y'*}^{-1}(\Phi'_Y)) \arrow[r, "\mathrm{id}\times g"] \arrow[d, "\mathrm{pr}_{Y'}"] \arrow[dr, phantom, "(1)"] & (X \times Y, \mathrm{pr}_Y^*\Delta_Y, \mathrm{pr}_{Y*}^{-1}(\Phi_Y)) \arrow[d, "\mathrm{pr}_Y"] \\
(Y', \Delta_{Y'}, \Phi_{Y'}) \arrow[r, "g"] & (Y, \Delta_Y, \Phi_Y)
\end{tikzcd} \tag{9.6}$$

where $h = \mathrm{id} \times f$ is the graph morphism of f and $h' = g' \times f'$. If g is flat or a closed immersion transverse to f then $\mathrm{id} \times g$ is flat or a closed immersion transverse to h (by base change).

Here the only new feature not covered in [CR11, Prop. 2.3.7] is the presence of divisors, and we simply note that $\Delta_X = f^*\Delta_X = h^*\mathrm{pr}_Y^*\Delta_Y$ and similarly for $\Delta_{X'}$, so that both pr_Y and h are pushing morphisms in the sense of Definition 8.14, and similarly for the left vertical maps. In other words, the supports and divisors in the middle row have been chosen precisely so that the vertical morphisms are all "pushing." ☐

We proceed to consider case (i), and wish to point out that for this case g can be arbitrary (we will need the flatness/transversality restrictions in case (ii)). In what follows we set $d_X = \dim X$, $d_Y = \dim Y$ and similarly for X', Y'. Using Theorem 8.37 we obtain a compactification $\iota : X \hookrightarrow \overline{X}$ over k of the smooth, separated and finite type k-scheme X in the upper right corner of (9.2) and (9.6). This results in a compactification of the square (1) in (9.6) which we write as

$$\begin{tikzcd}
(X \times Y', \mathrm{pr}_{Y'}^*\Delta_Y, \mathrm{pr}_{Y'*}^{-1}(\Phi'_Y)) \arrow[r, "\mathrm{id}\times g"] \arrow[d, "\iota\times\mathrm{id}"] & (X \times Y, \mathrm{pr}_Y^*\Delta_Y, \mathrm{pr}_{Y*}^{-1}(\Phi_Y)) \arrow[d, "\iota\times\mathrm{id}"] \\
(\overline{X} \times Y', \overline{\mathrm{pr}}_{Y'}^*\Delta_Y, \overline{\mathrm{pr}}_{Y'*}^{-1}(\Phi'_Y)) \arrow[r, "\mathrm{id}\times g"] \arrow[d, "\overline{\mathrm{pr}}_{Y'}"] & (\overline{X} \times Y, \overline{\mathrm{pr}}_Y^*\Delta_Y, \overline{\mathrm{pr}}_{Y*}^{-1}(\Phi_Y)) \arrow[d, "\overline{\mathrm{pr}}_Y"] \\
(Y', \Delta_{Y'}, \Phi_{Y'}) \arrow[r, "g"] & (Y, \Delta_Y, \Phi_Y)
\end{tikzcd} \tag{9.7}$$

By the description following Lemma 8.33, we know that

$$\mathrm{pr}_{Y*} : H^*(X \times Y, \mathrm{pr}_Y^*\Delta_Y, \mathrm{pr}_{Y*}^{-1}(\Phi_Y)) \to H^*(Y, \Delta_Y, \Phi_Y)$$

stems from a morphism

$$R\overline{\mathrm{pr}}_{Y*} R\mathcal{H}om_{\overline{X}\times Y}(\Omega^p_{\overline{X}\times Y}(\log \mathrm{pr}_Y^*\Delta_Y)(-\mathrm{pr}_Y^*\Delta_Y)[p], \omega^\bullet_{\overline{X}\times Y}) \to \Omega_Y^{d_Y - p}(\log \Delta_Y)[d_Y - p] \tag{9.8}$$

obtained as the Grothendieck dual of a log differential of $\overline{\mathrm{pr}}_Y$ (here and throughout what follows, a similar statement holds for $\overline{\mathrm{pr}}_{Y'}$). By an observation of Chatzistamatiou-Rülling , this map factors as

$$\begin{aligned}
&R\overline{\mathrm{pr}}_{Y*} R\mathcal{H}om_{\overline{X}\times Y}(\Omega^p_{\overline{X}\times Y}(\log \overline{\mathrm{pr}}_Y^*\Delta_Y)(-\overline{\mathrm{pr}}_Y^*\Delta_Y)[p], \omega^\bullet_{\overline{X}\times Y}) \\
&\to R\overline{\mathrm{pr}}_{Y*} R\mathcal{H}om_{\overline{X}\times Y}(L\overline{\mathrm{pr}}_Y^*\Omega^p_Y(\log \Delta_Y)(-\Delta_Y)[p], \omega^\bullet_{\overline{X}\times Y}) \\
&\xrightarrow[\text{adjunction}]{\simeq} R\mathcal{H}om_Y(\Omega^p_Y(\log \Delta_Y)(-\Delta_Y)[p], R\overline{\mathrm{pr}}_{Y*}\omega^\bullet_{\overline{X}\times Y}) \\
&\xrightarrow[\text{trace}]{} R\mathcal{H}om_Y(\Omega^p_Y(\log \Delta_Y)(-\Delta_Y)[p], \omega_Y[d_Y]) \\
&\xrightarrow{\simeq} \Omega_Y^{d_Y-p}(\log \Delta_Y)[d_Y - p]
\end{aligned} \tag{9.9}$$

where the adjunction isomorphism is [R&D, Prop. II.5.10], and the map labeled trace is induced by the Grothendieck trace $R\overline{\mathrm{pr}}_{Y*}\omega^{\bullet}_{\overline{X}\times Y} \to \omega_Y[d_Y]$. *If* it were the case that $\overline{X}$ were smooth, then the usual "box product" decomposition

$$\omega^{\bullet}_{\overline{X}\times Y} \simeq \omega_{\overline{X}}[d_X] \boxtimes \omega_Y[d_Y] := \mathrm{pr}^*_{\overline{X}}\omega_{\overline{X}}[d_X] \otimes \overline{\mathrm{pr}}_{Y*}\omega_Y[d_Y]$$

together with the perect pairings (8.36) and the local freeness of $\Omega^p_Y(\log \Delta_Y)(-\Delta_Y)[p]$ would give an identification

$$R\mathcal{H}om_{\overline{X}\times Y}(L\overline{\mathrm{pr}}^*_Y\Omega^p_Y(\log \Delta_Y)(-\Delta_Y)[p], \omega^{\bullet}_{\overline{X}\times Y}) \simeq \mathrm{pr}^*_{\overline{X}}\omega_{\overline{X}}[d_X] \otimes \overline{\mathrm{pr}}^*_Y\Omega^{d_Y-p}_Y(\log \Delta_Y)[d_Y - p] \quad (9.10)$$

In fact a more careful version of this argument, carrying out the above calculation on the smooth locus $X \times Y$ and using excision, shows that $H^*(X \times Y, \mathrm{pr}^*_Y\Delta_Y, \mathrm{pr}^{-1}_{Y*}(\Phi_Y)) \to H^*(Y, \Delta_Y, \Phi_Y)$ *always* factors through the summand $H^*_{\Phi_X}(X \times Y, \mathrm{pr}^*_{\overline{X}}\omega_{\overline{X}} \otimes \overline{\mathrm{pr}}^*_Y\Omega^{d_Y-p}_Y(\log \Delta_Y))$.

Our next lemma implies that even when $\overline{X}$ is not known to be smooth, (9.8) still factors through something like $R\overline{\mathrm{pr}}_{Y*}(\mathrm{pr}^*_{\overline{X}}\omega_{\overline{X}}[d_X] \otimes \overline{\mathrm{pr}}^*_Y\Omega^{d_Y-p}_Y(\log \Delta_Y)[d_Y - p])$, provided we replace $\mathrm{pr}^*_{\overline{X}}\omega_{\overline{X}}[d_X]$ with $\overline{\mathrm{pr}}^!_Y\mathcal{O}_Y$.

Lemma 9.11 (compare with [CR11, Lem. 2.2.16]). *For each p there is a natural map*

$$\gamma : \overline{\mathrm{pr}}^!_Y\mathcal{O}_Y \otimes \overline{\mathrm{pr}}^*_Y\Omega^{d_Y-p}_Y(\log \Delta_Y)(-\Delta_Y)[d_Y - p] \to R\mathcal{H}om_{\overline{X}\times Y}(\overline{\mathrm{pr}}^*_Y\Omega^p_Y(\log \Delta_Y)(-\Delta_Y)[p], \omega^{\bullet}_{\overline{X}\times Y})$$

such that the restriction of γ to $X \times Y$ agrees with the isomorphism

$$\mathrm{pr}^*_X\,\omega_X[d_X] \otimes \mathrm{pr}^*_Y\,\Omega^{d_Y-p}_Y(\log \Delta_Y)(-\Delta_Y)[d_Y - p] \xrightarrow{\simeq} R\mathcal{H}om_{X\times Y}(L\,\mathrm{pr}^*_Y\,\Omega^p_Y(\log \Delta_Y)(-\Delta_Y)[p], \omega^{\bullet}_{X\times Y})$$

and such that the composition

$$\begin{aligned}
&R\overline{\mathrm{pr}}_{Y*}(\mathrm{pr}^*_X\,\omega_X[d_X] \otimes \mathrm{pr}^*_Y\,\Omega^{d_Y-p}_Y(\log \Delta_Y)(-\Delta_Y)[d_Y - p]) \\
&\xrightarrow{R\overline{\mathrm{pr}}_{Y*}(\gamma)} R\overline{\mathrm{pr}}_{Y*}R\mathcal{H}om_{X\times Y}(\mathrm{pr}^*_Y\,\Omega^p_Y(\log \Delta_Y)(-\Delta_Y)[p], \omega^{\bullet}_{X\times Y}) \\
&\xrightarrow[\text{adjunction}]{\simeq} R\mathcal{H}om_{X\times Y}(\Omega^p_Y(\log \Delta_Y)(-\Delta_Y)[p], R\overline{\mathrm{pr}}_{Y*}\omega^{\bullet}_{X\times Y}) \\
&\xrightarrow{\text{trace}} R\mathcal{H}om_{X\times Y}(\Omega^p_Y(\log \Delta_Y)(-\Delta_Y)[p], \omega_Y[d_Y]) \simeq \Omega^{d_Y-p}_Y(\log \Delta_Y)(-\Delta_Y)[d_Y - p]
\end{aligned} \quad (9.12)$$

coincides with the composition

$$\begin{aligned}
&R\overline{\mathrm{pr}}_{Y*}(\overline{\mathrm{pr}}^!_Y\mathcal{O}_Y \otimes \overline{\mathrm{pr}}^*_Y\Omega^{d_Y-p}_Y(\log \Delta_Y)(-\Delta_Y)[d_Y - p]) \\
&\xrightarrow[\text{form.}]{\text{proj.}} R\overline{\mathrm{pr}}_{Y*}(\overline{\mathrm{pr}}^!_Y\mathcal{O}_Y) \otimes \overline{\mathrm{pr}}^*_Y\Omega^{d_Y-p}_Y(\log \Delta_Y)(-\Delta_Y)[d_Y - p] \\
&\xrightarrow{\mathrm{tr}\otimes\mathrm{id}} \Omega^{d_Y-p}_Y(\log \Delta_Y)(-\Delta_Y)[d_Y - p]
\end{aligned} \quad (9.13)$$

By base change for dualizing complexes ([Stacks, Tag 0BZX, Tag 0E2S]) applied to the cartesian diagram

$$\begin{array}{ccc}
\overline{X}\times Y & \longrightarrow & \overline{X} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & \operatorname{Spec} k
\end{array}$$

(note that this is a very mild situation: $\overline{X} \to \operatorname{Spec} k$ is flat and proper and $Y \to \operatorname{Spec} k$ is smooth) we see that $\overline{\mathrm{pr}}^!_Y\mathcal{O}_Y \simeq \mathrm{pr}^*_{\overline{X}}\omega^{\bullet}_{\overline{X}}$. This makes the map γ look even more like (9.10).

Proof. Following [CR11, Lem. 2.2.16] we begin with the morphism

$$e : \overline{\mathrm{pr}}_Y^! \mathcal{O}_Y \otimes^L L\overline{\mathrm{pr}}_Y^* \omega_Y^\bullet \to \overline{\mathrm{pr}}_Y^! \omega_Y^\bullet =: \omega_{\overline{X}\times Y}^\bullet$$

of [Con00, p. 4.3.12], which as explained therein agrees with

$$\mathrm{pr}_X^* \,\omega_X[d_X] \otimes \mathrm{pr}_Y^* \,\omega_Y[d_Y] \xrightarrow{\simeq} \omega_{X\times Y}[d_X + d_Y]$$

on locus $X \times Y$,[14] and has the property that

$$\begin{array}{ccc} R\overline{pr}_{Y*}(\overline{\mathrm{pr}}_Y^! \mathcal{O}_Y \otimes^L L\overline{\mathrm{pr}}_Y^* \omega_Y^\bullet) & \xrightarrow{R\overline{pr}_{Y*}e} & R\overline{pr}_{Y*}\omega_{\overline{X}\times Y}^\bullet \\ \downarrow{\scriptstyle \text{proj. form}} & & \downarrow{\scriptstyle \mathrm{tr}} \\ R\overline{pr}_{Y*}\overline{\mathrm{pr}}_Y^! \mathcal{O}_Y \otimes^L \omega_Y^\bullet & \xrightarrow{\mathrm{tr}\otimes \mathrm{id}} & \omega_Y^\bullet \end{array}$$

commutes [Con00, Thm. 4.4.1]. We then define our version of γ as the composition

$$\begin{aligned} &\overline{\mathrm{pr}}_Y^! \mathcal{O}_Y \otimes^L L\overline{\mathrm{pr}}_Y^* \Omega_Y^{d_Y - p}(\log \Delta_Y)(-\Delta_Y)[d_Y - p] \\ &\xrightarrow{\mathrm{id}\otimes^L (8.36)} \overline{\mathrm{pr}}_Y^! \mathcal{O}_Y \otimes^L L\overline{\mathrm{pr}}_Y^* R\mathcal{H}om_Y(\Omega_Y^p(\log \Delta_Y)[p], \omega_Y^\bullet) \\ &\xrightarrow[\text{of } L\overline{\mathrm{pr}}_Y^*, \otimes^L]{\text{functoriality}} R\mathcal{H}om_{\overline{X}\times Y}(L\overline{\mathrm{pr}}_Y^* \Omega_Y^p(\log \Delta_Y)[p], \overline{\mathrm{pr}}_Y^! \mathcal{O}_Y \otimes^L \omega_Y^\bullet) \\ &\xrightarrow[e]{\text{induced by}} R\mathcal{H}om_{\overline{X}\times Y}(L\overline{\mathrm{pr}}_Y^* \Omega_Y^p(\log \Delta_Y)[p], \omega_{\overline{X}\times Y}^\bullet) \end{aligned} \tag{9.14}$$

Note that we may drop the "L"s as $\Omega_Y^{d_Y-p}(\log \Delta_Y)(-\Delta_Y)$ and $\Omega_Y^p(\log \Delta_Y)$ are locally free. Verification of the stated compatibilities is as in [CR11, Lem. 2.2.16]. □

Remark 9.15. It seems like we could have also used the more general version of [Con00, p. 4.3.12]

$$e' : \overline{\mathrm{pr}}_Y^! \mathcal{O}_Y \otimes^L L\overline{\mathrm{pr}}_Y^* \Omega_Y^{d_Y-p}(\log \Delta_Y)(-\Delta_Y)[d_Y - p] \to \overline{\mathrm{pr}}_Y^! \Omega_Y^{d_Y-p}(\log \Delta_Y)(-\Delta_Y)[d_Y - p]$$

together with the description

$$\overline{\mathrm{pr}}_Y^! \Omega_Y^{d_Y-p}(\log \Delta_Y)(-\Delta_Y)[d_Y - p] = D_{\overline{X}\times Y}(L\overline{\mathrm{pr}}_Y^* D_Y(\Omega_Y^{d_Y-p}(\log \Delta_Y)(-\Delta_Y)[d_Y - p]))$$

where $D_Y(-) = R\mathcal{H}om(-, \omega_Y^\bullet)$ and similarly for $D_{\overline{X}\times Y}$.

Using this modified γ, we obtain a modified version of the diagram [CR11, p. 732 during Lem. 2.3.4], namely (9.16) in Figure 4). To make this diagram legible, we use a few abbreviations: all functors are derived, we use the dualizing functors of the form $D_Y(-) = R\mathcal{H}om_Y(-, \omega_Y^\bullet)$ and we let $d = d_X + d_Y$. Lemma 9.11 shows that triangles involving γ commute, and (9.9) gives commutativity of the rest of the diagram. The usefulness of this diagram is that by *definition* beginning in the top left corner and following the path $\to\downarrow$ we obtain the pushforward on Hodge cohomology

$$\mathrm{pr}_{Y*}\,\underline{\Gamma}_{\mathrm{pr}_{Y*}^{-1}\Phi_Y}\,\Omega_{X\times Y}^{d-p}(\log \mathrm{pr}_Y^*\Delta_Y)[d-p] \to \underline{\Gamma}_{\Phi_Y}\,\Omega_{\times Y}^{d_Y-p}(\log \Delta_Y)(-\Delta_Y)[d_Y - p]$$

but following $\downarrow\to$ gives a composition whose behavior with respect to (9.7) is easier to analyze. Namely, we have a diagram like (9.16) on Y', and in fact a map from (9.16) to g_* of the analogous diagram on Y', and hence from the preceding discussion it will suffice to prove commutativity of (9.17) of Figure 4.

[14]See Conrad's comment "It is easy to check that e_f coincides with (3.3.21) in the smooth case and is compatible with composites in f (using (4.3.6)."

$$
\begin{tikzcd}
\mathrm{pr}_{Y*}\underline{\Gamma}_{\mathrm{pr}_{Y*}^{-1}\Phi_Y}\Omega^{d-p}_{X\times Y}(\log \mathrm{pr}_Y^*\Delta_Y)[d-p] \arrow[rr, "\text{excision}"] \arrow[dd, "\text{excision+}\textit{Lemma}\ 9.11"] \arrow[dr] & & \overline{\mathrm{pr}}_{Y*}\underline{\Gamma}_{\overline{\mathrm{pr}}_{Y*}^{-1}\Phi_Y} D_{\overline{X}\times Y}(\Omega^p_{\overline{X}\times Y}(\log \mathrm{pr}_Y^*\Delta_Y)(-\mathrm{pr}_Y^*\Delta_Y)[p]) \arrow[dl, "d\overline{\mathrm{pr}}_Y^\vee"'] \arrow[dd, "d\overline{\mathrm{pr}}_Y^\vee"] \\
& \overline{\mathrm{pr}}_{Y*}\underline{\Gamma}_{\overline{\mathrm{pr}}_{Y*}^{-1}\Phi_Y} D_{\overline{X}\times Y}(\overline{\mathrm{pr}}_Y^*\Omega^p_{\times Y}(\log \Delta_Y)(-\Delta_Y)[p]) \arrow[dr] & \\
\overline{\mathrm{pr}}_{Y*}\underline{\Gamma}_{\overline{\mathrm{pr}}_{Y*}^{-1}\Phi_Y}(\overline{\mathrm{pr}}_Y^!\mathscr{O}_Y \otimes \overline{\mathrm{pr}}_Y^*\Omega_Y^{d_Y-p}(\log \Delta_Y)(-\Delta_Y)[d_Y-p]) \arrow[ur, "\overline{\mathrm{pr}}_{Y*}(\gamma)"] \arrow[rr, "(9.13)"] & & \underline{\Gamma}_{\Phi_Y} D_Y(\Omega_Y^p(\log \Delta_Y)(-\Delta_Y)[p]) = \underline{\Gamma}_{\Phi_Y}\Omega_Y^{d_Y-p}(\log \Delta_Y)(-\Delta_Y)[d_Y-p]
\end{tikzcd}
\tag{9.16}
$$

$$
\begin{tikzcd}
\mathrm{pr}_{Y*}\underline{\Gamma}_{\overline{\mathrm{pr}}_{Y*}^{-1}\Phi_Y}\Omega^{d-p}_{X\times Y}(\log \mathrm{pr}_Y^*\Delta_Y)[d-p] \arrow[r] \arrow[d] & \overline{\mathrm{pr}}_{Y*}\underline{\Gamma}_{\overline{\mathrm{pr}}_{Y*}^{-1}\Phi_Y}(\overline{\mathrm{pr}}_Y^!\mathscr{O}_Y \otimes \overline{\mathrm{pr}}_Y^*\Omega_Y^{d_Y-p}(\log \Delta_Y)(-\Delta_Y)[d_Y-p]) \arrow[r] \arrow[d] & \underline{\Gamma}_{\Phi_Y}\Omega_Y^{d_Y-p}(\log \Delta_Y)(-\Delta_Y)[d_Y-p] \arrow[d] \\
g_*\mathrm{pr}_{Y'*}\underline{\Gamma}_{\overline{\mathrm{pr}}_{Y'*}^{-1}\Phi_{Y'}}\Omega^{d-p}_{X\times Y'}(\log \mathrm{pr}_{Y'}^*\Delta_{Y'})[d-p] \arrow[r] & g_*\overline{\mathrm{pr}}_{Y'*}\underline{\Gamma}_{\overline{\mathrm{pr}}_{Y'*}^{-1}\Phi_{Y'}}(\overline{\mathrm{pr}}_{Y'}^!\mathscr{O}_{Y'} \otimes \overline{\mathrm{pr}}_{Y'}^*\Omega_{Y'}^{d_Y-p}(\log \Delta_{Y'})(-\Delta_Y)[d_Y-p]) \arrow[r] & g_*\underline{\Gamma}_{\Phi_{Y'}}\Omega_{Y'}^{d_Y-p}(\log \Delta_{Y'})(-\Delta_{Y'})[d_Y-p]
\end{tikzcd}
\tag{9.17}
$$

Figure 4: Modified versions of diagrams appearing in the proof of [CR11, Lem. 2.3.4] (all functors derived)

Applying excision together with Lemma 9.11 we may rewrite the top row of (9.17) as

$$\begin{aligned}
&R\,\mathrm{pr}_{Y*}\,R\underline{\Gamma}_{\mathrm{pr}_{Y*}^{-1}\Phi_Y}\,\Omega^{d-p}_{X\times Y}(\log \mathrm{pr}_Y^*\Delta_Y)[d-p]\\
&\xrightarrow{\text{project}} R\,\mathrm{pr}_{Y*}\,R\underline{\Gamma}_{\mathrm{pr}_{Y*}^{-1}\Phi_Y}(\mathrm{pr}_X^*\,\omega_X[d_X]\otimes \mathrm{pr}_Y^*\,\Omega_Y^{d_Y-p}(\log\Delta_Y)(-\Delta_Y)[d_Y-p])\\
&\xrightarrow[\text{form.}]{\text{proj.}} R\,\mathrm{pr}_{Y*}\,R\underline{\Gamma}_{\mathrm{pr}_{Y*}^{-1}\Phi_Y}(\mathrm{pr}_X^*\,\omega_X[d_X])\otimes\Omega_Y^{d_Y-p}(\log\Delta_Y)(-\Delta_Y)[d_Y-p]\\
&\xrightarrow{\mathrm{tr}\otimes\mathrm{id}} R\underline{\Gamma}_{\Phi_Y}\,\Omega_Y^{d_Y-p}(\log\Delta_Y)(-\Delta_Y)[d_Y-p]
\end{aligned}\tag{9.18}$$

where the first map is induced by a projection

$$\Omega^{d-p}_{X\times Y}(\log \mathrm{pr}_Y^*\Delta_Y)[d-p]\to \mathrm{pr}_X^*\,\omega_X[d_X]\otimes \mathrm{pr}_Y^*\,\Omega_Y^{d_Y-p}(\log\Delta_Y)(-\Delta_Y)[d_Y-p]$$

coming from a Künneth-type decomposition of $\Omega^{d-p}_{X\times Y}(\log \mathrm{pr}_Y^*\Delta_Y)$, the second is the projection formula, and the last map is induced by a trace map with supports defined as the composition

$$\begin{aligned}
&R\,\mathrm{pr}_{Y*}\,R\underline{\Gamma}_{\mathrm{pr}_{Y*}^{-1}\Phi_Y}(\mathrm{pr}_X^*\,\omega_X[d_X])\xrightarrow{\text{excision}} R\overline{\mathrm{pr}}_{Y*}R\underline{\Gamma}_{\overline{\mathrm{pr}}_{Y*}^{-1}\Phi_Y}(\overline{\mathrm{pr}}_Y^!\mathcal{O}_Y)\\
&\xrightarrow{\text{Proposition 8.9}} R\underline{\Gamma}_{\Phi_Y}R\overline{\mathrm{pr}}_{Y*}(\overline{\mathrm{pr}}_Y^!\mathcal{O}_Y)\xrightarrow{\mathrm{tr}} R\underline{\Gamma}_{\Phi_Y}\mathcal{O}_Y
\end{aligned}\tag{9.19}$$

Here the second map comes from the functoriality properties of Proposition 8.9, since there is an inclusion $\mathrm{pr}_{Y*}^{-1}\Phi_Y\subseteq \mathrm{pr}_Y^{-1}\Phi_Y$. The decomposition (9.18) maps to a similar decomposition of the bottom row of (9.17), and the only commutativity not guaranteed by standard functoriality properties (e.g. functoriality of the projection formula appearing in the second map of (9.18)) is that of

$$\begin{array}{ccc}
R\,\mathrm{pr}_{Y*}\,R\underline{\Gamma}_{\mathrm{pr}_{Y*}^{-1}\Phi_Y}(\mathrm{pr}_X^*\,\omega_X[d_X])\otimes\Omega_Y^{d_Y-p}(\log\Delta_Y)(-\Delta_Y)[d_Y-p] & \xrightarrow{\mathrm{tr}\otimes\mathrm{id}} & R\underline{\Gamma}_{\Phi_Y}\,\Omega_Y^{d_Y-p}(\log\Delta_Y)(-\Delta_Y)[d_Y-p]\\
\downarrow & & \downarrow\\
Rg_*(R\,\mathrm{pr}_{Y'*}\,R\underline{\Gamma}_{\mathrm{pr}_{Y'*}^{-1}\Phi_{Y'}}(\mathrm{pr}_X^*\,\omega_X[d_X])\otimes\Omega_{Y'}^{d_Y-p}(\log\Delta_{Y'})(-\Delta_{Y'})[d_Y-p]) & \xrightarrow{\mathrm{tr}'\otimes\mathrm{id}} & Rg_*(R\underline{\Gamma}_{\Phi_{Y'}}\,\Omega_{Y'}^{d_Y-p}(\log\Delta_{Y'})(-\Delta_{Y'})[d_Y-p])
\end{array}\tag{9.20}$$

But applying one more projection formula to the bottom row of (9.20), we see (9.20) is obtained by tensoring the differential

$$\Omega_Y^{d_Y-p}(\log\Delta_Y)(-\Delta_Y)[d_Y-p]\to Rg_*\Omega_{Y'}^{d_Y-p}(\log\Delta_{Y'})(-\Delta_{Y'})[d_Y-p]$$

with

$$\begin{array}{ccc}
R\,\mathrm{pr}_{Y*}\,R\underline{\Gamma}_{\mathrm{pr}_{Y*}^{-1}\Phi_Y}(\mathrm{pr}_X^*\,\omega_X[d_X]) & \xrightarrow{\mathrm{tr}\otimes\mathrm{id}} & R\underline{\Gamma}_{\Phi_Y}\mathcal{O}_Y\\
\downarrow & & \downarrow\\
Rg_*(R\,\mathrm{pr}_{Y'*}\,R\underline{\Gamma}_{\mathrm{pr}_{Y'*}^{-1}\Phi_{Y'}}(\mathrm{pr}_X^*\,\omega_X[d_X])) & \xrightarrow{\mathrm{tr}'\otimes\mathrm{id}} & Rg_*(R\underline{\Gamma}_{\Phi_{Y'}}\mathcal{O}_{Y'})
\end{array}\tag{9.21}$$

and the commutativity of (9.21) is proved in [CR11, Lem. 2.3.4]. So far we have proved:

Lemma 9.22. *Lemma 9.1 holds in case (i) of Lemma 9.4.*

It remains to deal with case (ii) of Lemma 9.4, and for this we use the following lemma.

Lemma 9.23 (compare with [CR11, Cor. 2.2.22])**.** *Consider a diagram of pure-dimensional snc pairs*

$$\begin{array}{ccc}
(X',\Delta_{X'}) & \xrightarrow{g'} & (X,\Delta_X)\\
\downarrow{\scriptstyle\iota'} & & \downarrow{\scriptstyle\iota}\\
(Y',\Delta_{Y'}) & \xrightarrow{g} & (Y,\Delta_Y)
\end{array}\tag{9.24}$$

where ι, ι' are pushing closed immersions and $\dim Y - \dim X = \dim Y' - \dim X' =: c$. *Then, for all q the diagram*

$$\begin{tikzcd} \iota_* \Omega^q_X(\log \Delta_X)[q] \arrow[r, "dg'^\vee"] \arrow[d] & Rg_* \iota'_* \Omega^q_{X'}(\log \Delta_{X'}) \arrow[d] \\ \Omega^{q+c}_Y(\log \Delta_Y)[q+c] \arrow[r, "dg^\vee"] & Rg_* \Omega^{q+c}_{Y'}(\log \Delta_{Y'})[q+c] \end{tikzcd} \tag{9.25}$$

commutes, where the horizontal maps are induced by log differentials and the left vertical map is the composition

$$\begin{aligned} &\iota_* \Omega^q_X(\log \Delta_X)[q] \xrightarrow{\simeq} \iota_* R\mathcal{H}om(\Omega^{d_X - q}_X(\log \Delta_X)(-\Delta_X)[d_X - q], \omega^\bullet_X) \\ &\xrightarrow{\text{duality}} R\mathcal{H}om(\iota_* \Omega^{d_X - q}_X(\log \Delta_X)(-\Delta_X)[d_X - q], \omega^\bullet_Y) \xrightarrow{d\iota^\vee} R\mathcal{H}om(\Omega^{d_X - q}_Y(\log \Delta_Y)(-\Delta_Y)[d_X - q], \omega^\bullet_Y) \\ &\xrightarrow{\simeq} \Omega^{q+c}_Y(\log \Delta_Y)[q+c] \end{aligned} \tag{9.26}$$

and the right vertical arrow is Rg_* of a similar composition on Y'.

Note that the codimension hypotheses hold if g is flat or a closed immersion transverse to ι.

Proof. While it seems a proof following [CR11, Cor. 2.2.22] step-by-step is possible, we instead *reduce* to the case proved there as follows: first, observe that there is an evident map from the cartesian diagram

$$\begin{tikzcd} U_{X'} \arrow[r] \arrow[d] & U_X \arrow[d] \\ U_{Y'} \arrow[r] & U_Y \end{tikzcd} \tag{9.27}$$

of interiors to (9.24). Noting that (9.25) will map to a similar diagram obtained from (9.27), that the compositions (9.26) are at least compatible with Zariski localization, *and* that the situation of (9.27) is covered by [CR11, Cor. 2.2.22], it will suffice to show that the natural map

$$h^0 R\mathcal{H}om_Y(\iota_* \Omega^q_X(\log \Delta_X)[q], Rg_* \Omega^{q+c}_{Y'}(\log \Delta_{Y'})[q+c]) \to h^0 R\mathcal{H}om_{U_Y}(\iota_* \Omega^q_{U_X}[q], Rg_* \Omega^{q+c}_{U_{Y'}}[q+c]) \tag{9.28}$$

is *injective*. This can be checked Zariski-locally at a point $x \in X \subseteq Y$, so we may assume $X \subseteq Y$ is a global complete intersection, say of $t_1, \ldots, t_c \in \mathcal{O}_Y$. In that case the t_i define a Koszul resolution $\mathcal{K}^\bullet(t_i) \to \mathcal{O}_X$, and *because* $X' = Y' \times_Y X = V(t_1 \circ g, \cdots t_c \circ g)$ is smooth of codimension c by hypotheses, it must be that the $t_i \circ g$ are also a regular sequence, hence

$$L^i g^* \mathcal{O}_X = h^{-i} g^* \mathcal{K}^\bullet(t_i) = \begin{cases} \mathcal{O}_{X'}, & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

in other words $Lg^* \mathcal{O}_X = \mathcal{O}_{X'}$. Now using the fact that $\Omega^q_X(\log \Delta_X)$ is locally free on X' we conclude

$$Lg^* \iota_* \Omega^q_X(\log \Delta_X)[q] = g^* \iota_* \Omega^q_X(\log \Delta_X)[q] = \iota'_* g'^* \Omega^q_X(\log \Delta_X)[q]$$

Next, applying derived adjunction to both sides of (9.28) gives a commutative diagram

$$\begin{tikzcd}
R\mathcal{H}om_Y(\iota_*\Omega^q_X(\log\Delta_X)[q], Rg_*\Omega^{q+c}_{Y'}(\log\Delta_{Y'})[q+c]) \arrow[r] \arrow[d, equal] & R\mathcal{H}om_{U_Y}(\iota_*\Omega^q_{U_X}[q], Rg_*\Omega^{q+c}_{U_{Y'}}[q+c]) \arrow[d, equal] \\
Rg_*R\mathcal{H}om_{Y'}(Lg^*\iota_*\Omega^q_X(\log\Delta_X)[q], \Omega^{q+c}_{Y'}(\log\Delta_{Y'})[q+c]) \arrow[r] \arrow[d, equal] & Rg_*R\mathcal{H}om_{U_{Y'}}(Lg^*\iota_*\Omega^q_{U_X}[q], \Omega^{q+c}_{U_{Y'}}[q+c]) \arrow[d, equal] \\
Rg_*R\mathcal{H}om_{Y'}(\iota'_*g'^*\Omega^q_X(\log\Delta_X)[q], \Omega^{q+c}_{Y'}(\log\Delta_{Y'})[q+c]) \arrow[r] & Rg_*R\mathcal{H}om_{U_{Y'}}(\iota'_*g'^*\Omega^q_{U_X}[q], \Omega^{q+c}_{U_{Y'}}[q+c])
\end{tikzcd} \tag{9.29}$$

Getting even more Zariski-local we may assume $\Omega^q_X(\log\Delta_X)$ is *free*, say generated by $dx_1, \dots, dx_n$ and in that case

$$R\mathcal{H}om_{Y'}(\iota'_*g'^*\Omega^q_X(\log\Delta_X)[q], \Omega^{q+c}_{Y'}(\log\Delta_{Y'})[q+c]) = (\prod_i R\mathcal{H}om_{Y'}(\mathcal{O}_{X'}dx_i[q], \mathcal{O}_{Y'}[q+c]))\otimes\Omega^{q+c}_{Y'}(\log\Delta_{Y'}) \tag{9.30}$$

and by Grothendieck's fundamental local isomorphism [Con00, §2.5]

$$R\mathcal{H}om_{Y'}(\mathcal{O}_{X'}[q], \mathcal{O}_{Y'}[q+c])) \simeq \mathcal{E}xt^c_{Y'}(\mathcal{O}_{X'}, \mathcal{O}_{Y'}) \simeq \det(\mathcal{I}_{X'}/\mathcal{I}_{X'})^\vee \tag{9.31}$$

(the last 2 as sheaves supported in degree 0). In particular, this is an *invertible sheaf on X'*, and it follows that the left hand side of (9.30) is a locally free sheaf (supported in degree 0) on X'. Recalling X' is smooth and so in particular reduced, and since $U_{Y'} \cap X'$ is a dense open (this is part of the hypothesis that $X' \to Y'$ is a pulling map) the natural map

$$\begin{aligned}
&h^0R\mathcal{H}om_{Y'}(\iota'_*g'^*\Omega^q_X(\log\Delta_X)[q], \Omega^{q+c}_{Y'}(\log\Delta_{Y'})[q+c]) \\
&\to h^0R\mathcal{H}om_{Y'}(\iota'_*g'^*\Omega^q_X(\log\Delta_X)[q], \Omega^{q+c}_{Y'}(\log\Delta_{Y'})[q+c])|_{U_{Y'}} \\
&\simeq h^0R\mathcal{H}om_{U_{Y'}}(\iota'_*g'^*\Omega^q_X(\log\Delta_X)|_{U_{Y'}}[q], \Omega^{q+c}_{Y'}(\log\Delta_{Y'})|_{U_{Y'}}[q+c])
\end{aligned} \tag{9.32}$$

is injective, where on the third line we have applied localization for $\mathcal{E}xt$. Now left-exactness of g_* gives an injection

$$\begin{aligned}
&h^0Rg_*R\mathcal{H}om_{Y'}(\iota'_*g'^*\Omega^q_X(\log\Delta_X)[q], \Omega^{q+c}_{Y'}(\log\Delta_{Y'})[q+c]) \\
&\to h^0Rg_*R\mathcal{H}om_{U_{Y'}}(\iota'_*g'^*\Omega^q_X(\log\Delta_X)|_{U_{Y'}}[q], \Omega^{q+c}_{Y'}(\log\Delta_{Y'})|_{U_{Y'}}[q+c])
\end{aligned} \tag{9.33}$$

To complete the proof, we use (9.29) to identify the *map* (9.33) with (9.28). □

Corollary 9.34. *Lemma 9.1 holds in case (ii) of Lemma 9.4.*

Proof. This follows by applying cohomology with supports to (9.25). □

This completes our proof of Lemma 9.1.

Corollary 9.35 (projection formula, compare with [CR11, Prop. 1.1.16])**.** *Let $f : X \to Y$ be a map of smooth schemes admitting two different enhancements to maps of smooth schemes with supports,*

$$(X, \Delta_X, \Phi_X) \to (Y, \Delta_Y, f(\Phi_X)) \text{ pushing and } (X, f^*(\Delta'_Y), f^{-1}(\Phi_Y)) \to (Y, \Delta'_Y, \Phi_Y) \text{ pulling}$$

*Assume in addition that $\Delta_X + f^*(\Delta'_Y)$ and $\Delta_Y + \Delta'_Y$ are (*reduced*) snc divisors. Then*

$$(X, \Delta_X + f^*(\Delta'_Y), \Phi_X \cap f^{-1}(\Phi_Y)) \to (Y, \Delta_Y + \Delta'_Y, f(\Phi_X) \cap \Phi_Y)$$

is also a pushing map, and

$$f_*(\beta \smile f^*\alpha) = f_*\beta \smile \alpha \in H^*(Y, \Delta_Y + \Delta'_Y, f(\Phi_X) \cap \Phi_Y)$$

for any $\alpha \in H^*(Y, \Delta'_Y, \Phi_Y)$ and $\beta \in (X, \Delta_X, \Phi_X)$, where $\smile$ is the cup product on log Hodge cohomology defined along the lines of [CR11, §1.1.4, 2.4]

Proof. This is a formal consequence of Lemma 9.1 and can be derived following the proof of [CR11, Prop. 1.1.16]. Again we use a factorization through the graph like

$$\begin{tikzcd} (X, \Delta_X + f^*(\Delta'_Y), \Phi_X \cap f^{-1}(\Phi_Y)) \arrow[r, "f"] \arrow[d, "\mathrm{id}_X \times \mathrm{id}_X"] & (Y, \Delta_Y + \Delta'_Y, f(\Phi_X) \cap \Phi_Y) \arrow[dd, "\mathrm{id}_Y \times \mathrm{id}_Y"] \\ (X \times X, \mathrm{pr}_1^* \Delta_X + \mathrm{pr}_2^* f^*(\Delta'_Y), \Phi_X \times f^{-1}(\Phi_Y)) \arrow[d, "\mathrm{id}_X \times f"] & \\ (X \times Y, \mathrm{pr}_1^* \Delta_X + \mathrm{pr}_2^* \Delta'_Y, \Phi_X \times \Phi_Y) \arrow[r, "f \times \mathrm{id}_Y"] & (Y \times Y, \mathrm{pr}_1^* \Delta_Y + \mathrm{pr}_2^* \Delta'_Y, f(\Phi_X) \times \Phi_Y) \end{tikzcd} \tag{9.36}$$

Here $f \times \mathrm{id}_Y$ on the bottom is a pushing morphism (since $f|_{\Phi_X}$ is proper and $f^*\Delta_Y = \Delta_X$) and the right vertical map $\mathrm{id}_Y \times \mathrm{id}_Y$ is a closed immersion transverse to $f \times \mathrm{id}_Y$ since the outer rectangle is cartesian and X is smooth of the correct codimension. This means we are in a situation to apply Lemma 9.1, and that lemma plus the definition of cup products in terms of pullbacks along diagonals gives the desired identity. □

Following the approach of [CR11], the next step would be to construct a cycle class $\mathrm{cl}(Z) \in H^*_{\Phi_X}(X, \Omega^*_X(\log \Delta_X))$ for a subvariety $Z \subset X$ with $Z \in \Phi_X$. This is possible, and is carried out in [BPØ20, §9], however it seems that for compatibility with correspondences in the absence of additional finiteness/strictness conditions, a more refined cycle class would be needed. For this reason we turn now to log Hodge correspondences and then return to the issue of cycle classes.

10 Correspondences

Given snc pairs with familes of supports (X, Δ_X, Φ_X) and (Y, Δ_Y, Φ_Y) with dimensions d_X and d_Y, as in [CR11, §1.3] we may define a family of supports $P(\Phi_X, \Phi_Y)$ on $X \times Y$ by

$$\begin{aligned} P(\Phi_X, \Phi_Y) := \{&\text{closed subsets } Z \subseteq X \times Y \mid \mathrm{pr}_Y|_Z \text{ is proper and for all } W \in \Phi_X, \\ &\mathrm{pr}_Y(\mathrm{pr}_X^{-1}(W) \cap Z) \in \Phi_Y\} \end{aligned}$$

(the conditions of Definition 8.1 are straightforward to verify). For convenience we will let $\Delta_{X\times Y} := \mathrm{pr}_X^*\Delta_X + \mathrm{pr}_Y^*\Delta_Y$.

Lemma 10.1. *A class $\gamma \in H^j_{P(\Phi_X,\Phi_Y)}(X \times Y, \Omega^i_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X))$ defines homomorphisms*

$$\mathrm{cor}(\gamma) : H^q_{\Phi_X}(X, \Omega^p_X(\log \Delta_X)) \to H^{q+j-d_X}_{\Phi_Y}(Y, \Omega^{p+i-d_X}_Y(\log \Delta_Y))$$

by the formula $\mathrm{cor}(\gamma)(\alpha) := \mathrm{pr}_{Y*}(\mathrm{pr}_X^*(\alpha) \smile \gamma)$. Moreover if (Z, Δ_Z, Φ_Z) is another snc pair with supports and $\delta \in H^{j'}_{P(\Phi_Y,\Phi_Z)}(Y \times Z, \Omega^{i'}_{Y\times Z}(\log \Delta_{Y\times Z})(-\mathrm{pr}_Y^*\Delta_Y))$, then

$$\mathrm{pr}_{X\times Z*}(\mathrm{pr}_{X\times Y}^*(\gamma) \smile \mathrm{pr}_{Y\times Z}^*(\delta)) \in H^{j+j'-d_Y}_{P(\Phi_X,\Phi_Z)}(X \times Z, \Omega^{i+i'-d_Y}_{X\times Z}(\log \Delta_{X\times Z})(-\mathrm{pr}_X^*\Delta_X)) \text{ and}$$

$$\mathrm{cor}(\mathrm{pr}_{X\times Z*}(\mathrm{pr}_{X\times Y}^*(\gamma) \smile \mathrm{pr}_{Y\times Z}^*(\delta))) = \mathrm{cor}(\delta) \circ \mathrm{cor}(\gamma)$$

as homomorphisms $H^q_{\Phi_X}(X, \Omega^p_X(\log \Delta_X)) \to H^{q+j+j'-d_X-d_Y}_{\Phi_Z}(Z, \Omega^{p+i+i'-d_X-d_Y}_Z(\log \Delta_Z))$.

Such correspondences involving both log poles and "log zeroes" appear to have been considered before at least in crystalline cohomology, for example in work of Mieda [Mie09a; Mie09b]. However, I was unable to find any published proof of Lemma 10.1 in the literature. If you or anyone you know is aware of previous work related Lemma 10.1, please email me!

Proof. We make two observations: first, using Lemma 8.18 there are natural wedge product pairings

$$\Omega^p_{X\times Y}(\log \Delta_{X\times Y}) \otimes \Omega^i_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X) \xrightarrow{\wedge} \Omega^{p+i}_{X\times Y}(\log \Delta_Y)$$

Second, essentially by the definition of $P(\Phi_X, \Phi_Y)$ the Künneth morphism on cohomology for the tensor product $\Omega^p_{X\times Y}(\log \Delta_{X\times Y}) \otimes \Omega^i_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X)$ can be enhanced with supports as

$$\begin{aligned} &H^q_{\mathrm{pr}_X^{-1}(\Phi_X)}(X\times Y, \Omega^p_{X\times Y}(\log \Delta_{X\times Y})) \otimes H^j_{P(\Phi_X,\Phi_Y)}(X\times Y, \Omega^i_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X)) \\ &\qquad \to H^{p+j}_{\Psi}(X\times Y, \Omega^p_{X\times Y}(\log \Delta_{X\times Y}) \otimes \Omega^i_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X)) \end{aligned}$$

where $\Psi := \mathrm{pr}_{Y*}^{-1}(\Phi_Z)$ (see [CR11, §1.3.7, Prop. 1.3.10]). Combining these 2 observations gives a pairing

$$\begin{aligned} &H^q_{\mathrm{pr}_X^{-1}(\Phi_X)}(X\times Y, \Omega^p_{X\times Y}(\log \Delta_{X\times Y})) \otimes H^j_{P(\Phi_X,\Phi_Y)}(X\times Y, \Omega^i_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X)) \\ &\qquad \xrightarrow{\smile} H^{p+j}_{\Psi}(X\times Y, \Omega^{p+i}_{X\times Y}(\log \Delta_Y)) \end{aligned}$$

Now note that $\mathrm{pr}_X : (X\times Y, \Delta_{X\times Y}, \mathrm{pr}_X^{-1}(\Phi_X)) \to (X, \Delta_X, \Phi_X)$ is a pulling morphism, so by Proposition 8.29 there is an induced map $\mathrm{pr}_X^* : H^q_{\Phi_X}(X, \Omega^p_X(\log \Delta_X)) \to H^q_{\mathrm{pr}_X^{-1}(\Phi_X)}(X\times Y, \Omega^p_{X\times Y}(\log \Delta_{X\times Y}))$. On the other hand since $\mathrm{pr}_Y : (X\times Y, \Delta_Y, \Psi) \to (Y, \Delta_Y, \Phi_Y)$ is a pushing morphism, Lemma 8.33 provides a morphism $\mathrm{pr}_{Y*} : H^{p+j}_\Psi(X\times Y, \Omega^{p+i}_{X\times Y}(\log \Delta_Y)) \to H^{q+j-d_X}_{\Phi_Y}(Y, \Omega^{p+i-d_X}_Y(\log \Delta_Y))$. Composing, we obtain the desired homomorphism

$$\begin{aligned} H^q_{\Phi_X}(X, \Omega^p_X(\log \Delta_X)) &\xrightarrow{\mathrm{pr}_X^*} H^q_{\mathrm{pr}_X^{-1}(\Phi_X)}(X\times Y, \Omega^p_{X\times Y}(\log \Delta_{X\times Y})) \\ &\xrightarrow{\smile\gamma} H^{p+j}_\Psi(X\times Y, \Omega^{p+i}_{X\times Y}(\log \Delta_Y)) \\ &\xrightarrow{\mathrm{pr}_{Y*}} H^{q+j-d_X}_{\Phi_Y}(Y, \Omega^{p+i-d_X}_Y(\log \Delta_Y)) \end{aligned}$$

For the "moreover" half of the lemma, we again begin with a certain wedge product pairing, this time on $X\times Y\times Z$:

$$\begin{aligned} &\Omega^i_{X\times Y\times Z}(\log \mathrm{pr}_{X\times Y}^*\Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X) \otimes \Omega^{i'}_{X\times Y\times Z}(\log \mathrm{pr}_{Y\times Z}^*\Delta_{Y\times Z})(-\mathrm{pr}_Y^*\Delta_Y) \\ &\xrightarrow{\wedge} \Omega^{i+i'}_{X\times Y\times Z}(\log \mathrm{pr}_{X\times Z}^*\Delta_{X\times Z})(-\mathrm{pr}_X^*\Delta_X) \end{aligned} \tag{10.2}$$

If $V \in P(\Phi_X, \Phi_Y)$, $W \in P(\Phi_Y, \Phi_Z)$ then unravelling definitions (again we refer to [CR11, §1.3.7, Prop. 1.3.10] for a similar claim) we find:

- $\mathrm{pr}_{X\times Z}|_{\mathrm{pr}_{X\times Y}^{-1}(V)\cap \mathrm{pr}_{Y\times Z}^{-1}(W)}$ is proper and
- $\mathrm{pr}_{X\times Z}(\mathrm{pr}_{X\times Y}^{-1}(V) \cap \mathrm{pr}_{Y\times Z}^{-1}(W)) \in P(\Phi_X, \Phi_Z)$

so that the Künneth morphism on cohomology associated to the left hand side of (10.2) can be enhanced with supports like

$$\begin{aligned} &H^j_{\mathrm{pr}_{X\times Y}^{-1}(P(\Phi_X,\Phi_Y))}(X\times Y\times Z, \Omega^i_{X\times Y\times Z}(\log \mathrm{pr}_{X\times Y}^*\Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X)) \\ &\otimes H^{j'}_{\mathrm{pr}_{Y\times Z}^{-1}(P(\Phi_Y,\Phi_Z))}(X\times Y\times Z, \Omega^{i'}_{X\times Y\times Z}(\log \mathrm{pr}_{Y\times Z}^*\Delta_{Y\times Z})(-\mathrm{pr}_Y^*\Delta_Y)) \\ &\to H^{j+j'}_\Sigma(X\times Y\times Z, \Omega^i_{X\times Y\times Z}(\log \mathrm{pr}_{X\times Y}^*\Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X) \otimes \Omega^{i'}_{X\times Y\times Z}(\log \mathrm{pr}_{Y\times Z}^*\Delta_{Y\times Z})(-\mathrm{pr}_Y^*\Delta_Y)) \end{aligned}$$

where $\Sigma := \mathrm{pr}_{X\times Z*}^{-1}(P(\Phi_X, \Phi_Z))$.

Since $\mathrm{pr}_{X\times Y} : (X\times Y\times Z, \mathrm{pr}_{X\times Y}^*\Delta_{X\times Y}, \mathrm{pr}_{X\times Y}^{-1}(P(\Phi_X,\Phi_Y))) \to (X\times Y, \Delta_{X\times Y}, P(\Phi_X, \Phi_Y))$ is a pulling morphism, Proposition 8.29 gives an induced morphism

$$\Omega^i_{X\times Y}(\log \Delta_{X\times Y}) \to Rf_*\Omega^i_{X\times Y\times Z}(\log \mathrm{pr}_{X\times Y}^*\Delta_{X\times Y});$$

twisting by $-\Delta_{X\times Y}$ and applying the projection formula gives a morphism

$$\Omega^i_{X\times Y}(\log \Delta_{X\times Y})(-\Delta_{X\times Y}) \to Rf_*\big(\Omega^i_{X\times Y\times Z}(\log \mathrm{pr}^*_{X\times Y}\Delta_{X\times Y})(-\mathrm{pr}^*_{X\times Y}\Delta_{X\times Y})\big)$$

and then taking cohomology with supports along $P(\Phi_X,\Phi_Y)$ and using Proposition 8.9 gives a modified pullback map

$$\begin{aligned} &H^j_{P(\Phi_X,\Phi_Y)}(X\times Y,\Omega^i_{X\times Y}(\log \Delta_{X\times Y})(-\Delta_{X\times Y})) \\ &\to H^j_{\mathrm{pr}^{-1}_{X\times Y}(P(\Phi_X,\Phi_Y))}(X\times Y\times Z,\Omega^i_{X\times Y\times Z}(\log \mathrm{pr}^*_{X\times Y}\Delta_{X\times Y})(-\mathrm{pr}^*_X\Delta_X)) \end{aligned} \tag{10.3}$$

and a similar argument gives a modified pullback

$$\begin{aligned} &H^{j'}_{P(\Phi_Y,\Phi_Z)}(Y\times Z,\Omega^{i'}_{Y\times Z}(\log \Delta_{Y\times Z})(-\Delta_{Y\times Z})) \\ &\to H^{j'}_{\mathrm{pr}^{-1}_{Y\times Z}(P(\Phi_Y,\Phi_Z))}(X\times Y\times Z,\Omega^{i'}_{X\times Y\times Z}(\log \mathrm{pr}^*_{Y\times Z}\Delta_{Y\times Z})(-\mathrm{pr}^*_X\Delta_Y)) \end{aligned} \tag{10.4}$$

On the other hand, $\mathrm{pr}_{X\times Z} : (X\times Y\times Z, \mathrm{pr}^*_{X\times Z}\Delta_{X\times Y},\Sigma) \to (X\times Z,\Delta_{X\times Z},P(\Phi_X,\Phi_Z))$ is a pushing morphism and hence by Lemma 8.33 induces morphisms

$$R\mathrm{pr}_{X\times Z*}R\underline{\Gamma}_\Sigma(\Omega^{\dim X\times Y\times Z-k}_{X\times Y\times Z}(\log \mathrm{pr}^*_{X\times Z}\Delta_{X\times Y})) \to R\underline{\Gamma}_{P(\Phi_X,\Phi_Z)}\Omega^{\dim X\times Z-k}_{X\times Z}(\log \Delta_{X\times Z})[-\dim Z]$$

for all k; twisting by $-\mathrm{pr}^*_X\Delta_X$ and applying the projection formula this becomes

$$\begin{aligned} &R\mathrm{pr}_{X\times Z*}R\underline{\Gamma}_\Sigma(\Omega^{\dim X\times Y\times Z-k}_{X\times Y\times Z}(\log \mathrm{pr}^*_{X\times Z}\Delta_{X\times Y})(-\mathrm{pr}^*_X\Delta_X)) \\ &\to R\underline{\Gamma}_{P(\Phi_X,\Phi_Z)}\Omega^{\dim X\times Z-k}_{X\times Z}(\log \Delta_{X\times Z})(-\mathrm{pr}^*_X\Delta_X)[-\dim Z] \end{aligned} \tag{10.5}$$

Now letting $k = \dim X\times Y\times Z - i - i'$, the induced morphisms of cohomology with supports are

$$H^{j+j'}_\Sigma(X\times Y\times Z,\Omega^{i+i'}_{X\times Y\times Z}(\log \mathrm{pr}^*_{X\times Z}\Delta_{X\times Y})(-\mathrm{pr}^*_X\Delta_X)) \to H^{j+j'-\dim Z}_{P(\Phi_X,\Phi_Z)}(X\times Z,\Omega^{i+i'-\dim Z}_{X\times Z}(\log \Delta_{X\times Z})(-\mathrm{pr}^*_X\Delta_X)) \tag{10.6}$$

Combining the above ingredients, we obtain a bilinear pairing

$$\begin{aligned} &H^j_{P(\Phi_X,\Phi_Y)}(X\times Y,\Omega^i_{X\times Y}(\log \Delta_{X\times Y})(-\Delta_{X\times Y}))\otimes H^{j'}_{P(\Phi_Y,\Phi_Z)}(Y\times Z,\Omega^{i'}_{Y\times Z}(\log \Delta_{Y\times Z})(-\Delta_{Y\times Z})) \\ &\to H^{j+j'-\dim Z}_{P(\Phi_X,\Phi_Z)}(X\times Z,\Omega^{i+i'-\dim Z}_{X\times Z}(\log \Delta_{X\times Z})(-\mathrm{pr}^*_X\Delta_X)) \end{aligned}$$

sending $\gamma\otimes\delta \longmapsto \mathrm{pr}_{X\times Z*}(\mathrm{pr}^*_{X\times Y}(\gamma)\smile \mathrm{pr}^*_{Y\times Z}(\delta))$. It remains to be seen that

$$\mathrm{cor}(\mathrm{pr}_{X\times Z*}(\mathrm{pr}^*_{X\times Y}(\gamma)\smile \mathrm{pr}^*_{Y\times Z}(\delta))) = \mathrm{cor}(\delta)\circ\mathrm{cor}(\gamma)$$

and for this we will make repeated use of Lemma 9.1. Consider the diagram of smooth schemes

$$\begin{tikzcd} & & X\times Y\times Z \arrow[dl] \arrow[dr] & & \\ & X\times Y \arrow[dl] \arrow[dr] & * & Y\times Z \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

where all morphisms are projections. There are various ways to enhance this to include supports; here we add the family of supports Ψ on $X\times Y$ defined above. Then in the cartesian diagram $(*)$, $\mathrm{pr}_Y : (X\times Y,\Psi)\to(Y,\Phi_Y)$ and $\mathrm{pr}_{Y\times Z} : (X\times Y\times Z,\mathrm{pr}^{-1}_{X\times Y}\Psi)\to(Y\times Z,\mathrm{pr}^{-1}_Y\Phi_Y)$ are pushing morphisms, whereas $\mathrm{pr}_{X\times Y}$ and pr_Y are pulling morphisms. At the same time, we have a pulling morphism $\mathrm{pr}_{X\times Z} : (X\times Y\times Z,\mathrm{pr}^{-1}_{X\times Z}(P(\Phi_Y,\Phi_Z)))\to(Y\times Z,P(\Phi_Y,\Phi_Z))$. To be precise in what follows, whenever ambiguity is possible we will use notation like $\mathrm{pr}_X^{X\times Y}$ to denote the projection $X\times Y\to X$, $\mathrm{pr}_X^{X\times Y\times Z}$ to denote the projection $X\times Y\times Z\to X$ and so on.

Applying Corollary 9.35 first to $\mathrm{pr}_{X\times Z}$ we see that

$$\mathrm{pr}_{Y\times Z*}(\mathrm{pr}_{X\times Y}^*(\mathrm{pr}_X^{X\times Y*}\alpha\smile\gamma)\smile\mathrm{pr}_{Y\times Z}^*\delta)=\mathrm{pr}_{Y\times Z*}(\mathrm{pr}_{X\times Y}^*(\mathrm{pr}_X^{X\times Y*}\alpha\smile\gamma))\smile\delta$$

and then applying Lemma 9.1 to $(*)$ shows

$$\mathrm{pr}_{Y\times Z*}(\mathrm{pr}_{X\times Y}^*(\mathrm{pr}_X^{X\times Y*}\alpha\smile\gamma))=\mathrm{pr}_Y^{Y\times Z*}(\mathrm{pr}_{Y*}^{X\times Y}(\mathrm{pr}_X^{X\times Y*}\alpha\smile\gamma))=\mathrm{pr}_Y^{Y\times Z*}\operatorname{cor}(\gamma)(\alpha)$$

so that

$$\mathrm{pr}_{Y\times Z*}(\mathrm{pr}_{X\times Y}^*(\mathrm{pr}_X^{X\times Y*}\alpha\smile\gamma)\smile\mathrm{pr}_{Y\times Z}^*\delta)=\mathrm{pr}_Y^{Y\times Z*}\operatorname{cor}(\gamma)(\alpha)\smile\delta$$

Applying $\mathrm{pr}_{Z*}^{Y\times Z}$ we conclude that

$$\operatorname{cor}\delta(\operatorname{cor}\gamma)(\alpha))=\mathrm{pr}_{Z*}^{X\times Y\times Z}(\mathrm{pr}_X^{X\times Y\times Z*}\alpha\smile\mathrm{pr}_{X\times Y}^*\gamma\smile\mathrm{pr}_{Y\times Z}^*\delta)\tag{10.7}$$

Finally, we rewrite the right hand side as

$$\mathrm{pr}_{Z*}^{X\times Z}\mathrm{pr}_{X\times Z*}(\mathrm{pr}_{X\times Z}^*\mathrm{pr}_X^{X\times Z*}\alpha\smile\mathrm{pr}_{X\times Y}^*\gamma\smile\mathrm{pr}_{Y\times Z}^*\delta)$$

and apply Corollary 9.35 to $\mathrm{pr}_{X\times Z}$ (with the pushing morphism $(X\times Y\times Z,\Sigma)\to(X\times Z,P(\Phi_X,\Phi_Z))$ and pulling morphism $(X\times Y\times Z,\mathrm{pr}_X^{X\times Y\times Z-1}(\Phi_X))\to(X\times Z,\mathrm{pr}_X^{X\times Z-1}(\Phi_X)))$ to arrive at

$$\mathrm{pr}_{X\times Z*}(\mathrm{pr}_{X\times Z}^*\mathrm{pr}_X^{X\times Z*}\alpha\smile\mathrm{pr}_{X\times Y}^*\gamma\smile\mathrm{pr}_{Y\times Z}^*\delta)=\mathrm{pr}_X^{X\times Z*}\alpha\smile\mathrm{pr}_{X\times Z*}(\mathrm{pr}_{X\times Y}^*\gamma\smile\mathrm{pr}_{Y\times Z}^*\delta)$$

Applying $\mathrm{pr}_{Z*}^{X\times Z}$ on both sides shows that the right hand side of (10.7) is $\operatorname{cor}(\mathrm{pr}_{X\times Z*}(\mathrm{pr}_{X\times Y}^*\gamma\smile\mathrm{pr}_{Y\times Z}^*\delta)(\alpha)$, as desired. □

Remark 10.8. There is a Grothendieck-Serre dual approach to such correspondences, where classes $\gamma\in H^j_{P(\Phi_X,\Phi_Y)}(X\times Y,\Omega^i_{X\times Y}(\log\Delta_{X\times Y})(-\mathrm{pr}_Y^*\Delta_Y))$ define homomorphisms

$$H^q(X,\Omega_X^p(\log\Delta_X)(-\Delta_X))\to H^{q+j-d_X}(Y,\Omega_Y^{p+i-d_X}(\log\Delta_Y)(-\Delta_Y)).$$

The construction is formally similar.

11 Attempts to construct a fundamental class of a thrifty birational equivalence

Let $(X,\Delta_X),(Y,\Delta_Y)$ be simple normal crossing pairs, and assume in addition that X,Y are connected and proper. Let $Z\subseteq X\times Y$ be a smooth closed subvariety with codimension c. In this situation the fundamental class of $\operatorname{cl}(Z)\in H^c(X\times Y,\Omega^c_{X\times Y})$ (no log poles yet) can be described using only Serre duality, as follows (we refer to [Har77, Ex. III.7.4]). the composition

$$H^{\dim Z}(X\times Y,\Omega^{\dim Z}_{X\times Y})\to H^{\dim Z}(Z,\Omega_Z^{\dim Z})\xrightarrow{\mathrm{tr}}k\tag{11.1}$$

(where tr is the trace map of Serre duality) is an element of

$$H^{\dim Z}(X\times Y,\Omega^{\dim Z}_{X\times Y})^\vee\simeq H^c(X\times Y,\Omega^c_{X\times Y})\tag{11.2}$$

which we may *define* to be $\operatorname{cl}(Z)$.[15] In light of Lemma 10.1 we might hope to modify eqs. (11.1) and (11.2) to obtain a class in $H^c(X\times Y,\Omega^c_{X\times Y}(\log\Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X))$. Let us focus on the case where

[15] It may then be non-trivial to verify this agrees with other definitions, especially if we worry about signs, but we will not need that level of detail for what follows.

- $\mathrm{pr}_X|_Z : Z \to X$, $\mathrm{pr}_Y|_Z : Z \to Y$ are both thrifty and birational, so in particular $c = \dim X = \dim Y =: d$ and
- $(\mathrm{pr}_X|_Z)_*^{-1}\Delta_X = (\mathrm{pr}_Y|_Z)_*^{-1}\Delta_Y =: \Delta_Z$

To keep the notation under control, set $\pi_X := \mathrm{pr}_X|_Z$ and $\pi_Y := \mathrm{pr}_Y|_Z$.

In this situation letting $\iota : Z \to X \times Y$ be the inclusion there is a natural map

$d\iota^\vee : \Omega^d_{X\times Y}(\log \Delta_{X\times Y}) \to \iota_*\Omega^d_Z(\log \Delta_{X\times Y}|_Z)$ and twisting by $-\mathrm{pr}_Y^*\Delta_Y$ gives a map

$$\Omega^d_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_Y^*\Delta_Y) \to \iota_*\Omega^d_Z(\log \Delta_{X\times Y}|_Z)(-\mathrm{pr}_Y^*\Delta_Y|_Z) = \iota_*\Omega^d_Z(\log \Delta_{X\times Y}|_Z)(-\pi_Y^*\Delta_Y)$$

To identify $\Omega^d_Z(\log \Delta_{X\times Y}|_Z)(-\mathrm{pr}_X^*\Delta_X|_Z)$, write

$$(\pi_X)^*\Delta_X = (\pi_X)_*^{-1}\Delta_X + E_X = \Delta_Z + E_X \text{ and}$$
$$(\pi_Y)^*\Delta_Y = (\pi_Y)_*^{-1}\Delta_Y + E_Y = \Delta_Z + E_Y$$

so that $\Delta_{X\times Y}|_Z = (\pi_X)^*\Delta_X + (\pi_Y)^*\Delta_Y = 2\Delta_Z + E_X + E_Y$. While the hypotheses guarantee Δ_Z is reduced it may be that E_X, E_Y are non-reduced — however something can be said about their multiplicities. If $E_X = \sum_i a^i_X E^i_X$, $E_Y = \sum_i a^i_Y E^i_Y$ where the E^i_X, E^i_Y are irreducible, then by a generalization of [Har77, Prop. 3.6] (see also [SingsMMP 2013, §2.10]),

$$a^i_X = \mathrm{mlt}(\pi_X(E^i_X) \subseteq \Delta_X)$$

and since Δ_X is a reduced effective simple normal crossing divisor, if in addition we write $\Delta_X = \sum_i D^i_X$, then $\mathrm{mlt}(\pi_X(E^i_X) \subseteq \Delta_X) = |\{i \mid \pi_X(E^i_X) \subseteq D^i_X\}|$. The thriftiness hypothesis that $\pi_X(E^i_X)$ is not a stratum then implies $a^i_X = \mathrm{mlt}(\pi_X(E^i_X) \subseteq \Delta_X) < \mathrm{codim}(\pi_X(E^i_X) \subset X)$. Since differentials with log poles are insensitive to multiplicities, we have

$$\Omega^d_Z(\log \Delta_{X\times Y}|_Z) = \omega_Z(\Delta_Z + E_X^{\mathrm{red}} + E_Y^{\mathrm{red}})$$

where $-^{\mathrm{red}}$ denotes the associated reduced effective divisor. Then

$$\Omega^d_Z(\log \Delta_{X\times Y}|_Z)(-\pi_Y^*\Delta_Y) = \omega_Z(\Delta_Z + E_X^{\mathrm{red}} + E_Y^{\mathrm{red}} - \Delta_Z - E_Y)$$
$$\omega_Z(E_X^{\mathrm{red}} + (E_Y^{\mathrm{red}} - E_Y)) = \omega_Z(\sum_i E^i_X + \sum_i (1 - a^i_Y)E^i_Y)$$

The upshot is that we have an induced map

$$H^d(X \times Y, \Omega^d_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_Y^*\Delta_Y)) \to H^d(Z, \omega_Z(E_X^{\mathrm{red}} + (E_Y^{\mathrm{red}} - E_Y))) \tag{11.3}$$

Here the left hand side is Serre dual to $H^d(X \times Y, \Omega^d_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X))$, so the k-linear dual of (11.3) is a morphism

$$H^d(Z, \omega_Z(E_X^{\mathrm{red}} + (E_Y^{\mathrm{red}} - E_Y)))^\vee \to H^d(X \times Y, \Omega^d_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X))$$

Unfortunately[16] $H^d(Z, \omega_Z(E_X^{\mathrm{red}} + (E_Y^{\mathrm{red}} - E_Y)))$ is often 0. If E_X and E_Y are both reduced (an explicit example where this holds will be given below), then $H^d(Z, \omega_Z(E_X^{\mathrm{red}} + (E_Y^{\mathrm{red}} - E_Y))) = H^d(Z, \omega_Z(E_X))$. If in addition $E_X \neq 0$, we obtain $H^d(Z, \omega_Z(E_X)) = 0$ by an extremely weak (but characteristic independent) sort of Kodaira vanishing:

Lemma 11.4. *Let Z be a proper variety over a field k with dimension d, and assume Z is normal and Cohen-Macaulay. If $D \subset Z$ is a non-0 effective Cartier divisor on Z then $H^d(Z, \omega_Z(D)) = 0$.*

Proof. By Serre duality $H^d(Z, \omega_Z(D)) = H^0(Z, \mathcal{O}_Z(-D))$, which vanishes by the classic fact that "a nontrivial line bundle and its inverse can't both have non-0 global sections." Since I am not aware of a specific reference, here is a proof:

[16] at least for the purposes of constructing log Hodge cohomology classes of subvarieties ...

Suppose towards contraditction that there is a non-0 global section $\sigma \in H^0(Z, \mathcal{O}_Z(-D))$ — then the composition

$$\begin{tikzcd} \mathcal{O}_Z \arrow[r, "\sigma"] \arrow[rr, bend right, "\tau"'] & \mathcal{O}_Z(-D) \arrow[r, hook] & \mathcal{O}_Z \end{tikzcd}$$

is non-0. By [Stacks, Tag 0358] $H^0(Z, \mathcal{O}_Z)$ is a (normal) domain, and since it's also a finite dimensional k-vector space it must be an extension field of k. But then $\tau \in H^0(Z, \mathcal{O}_Z)$ is invertible hence surjective, so $\mathcal{O}_Z(-D) \hookrightarrow \mathcal{O}_Z$ is surjective, which is a contradiction since by hypothesis the cokernel $\mathcal{O}_D \neq 0$. $\square$

Example 11.5. Let $X = \mathbb{P}^2$ and let $\Delta_X \subset X$ be a line. Let $p \in L$ be a k-point, let $Y = \mathrm{Bl}_p X$ and let $\Delta_Y = \tilde{L}$ = the strict transform of L. Finally let $f : Y \to X$ be the blowup map and let $Z = (f \times \mathrm{id})(Y) \subset X \times Y$. In this case (with all notation as above) $\pi_X \circ (f \times \mathrm{id}) = f$ and $\pi_Y \circ (f \times \mathrm{id}) = \mathrm{id}_Y$, so under the isomorphism $f \times \mathrm{id} : Y \simeq Z$, E_X is the exceptional divisor of f (with multiplicity 1). On the other hand $E_Y = 0$. In particular E_X and E_Y are reduced and $E_X \neq 0$ so from the above discussion $H^2(Z, \omega_Z(E_X)) = 0$.

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Department of Mathematics
University of Washington
Seattle, WA 98105
cgodfrey@uw.edu
https://math.washington.edu/~cgodfrey