

HIGHER DIRECT IMAGES OF LOGARITHMIC STRUCTURE SHEAVES

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CONTENTS

1. Introduction	1
2. Dual complexes	3
2.1. Morphisms of Dual Complexes	4
3. Thrifty morphisms of pairs	5
3.1. Thrifty proper birational equivalences	7
4. Structure sheaves of strata and their direct images	9
5. A morphism of restriction triangles	11
References	12

1. INTRODUCTION

In [CR11] Chatzistamatiou and Rülling prove the following theorem:

Theorem 1.1 ([CR11, Thm. 3.2.8]). *Let k be a perfect field and let S be a separated scheme of finite type over k . Suppose X and Y are two separated finite type S -schemes which are*

- (i) *smooth over k and*
- (ii) ***properly birational** over S in the sense that there is a commutative diagram*

$$\begin{array}{ccccc}
 & & Z & & \\
 & \swarrow r & & \searrow s & \\
 X & & \circlearrowleft & & Y \\
 & \searrow f & & \swarrow g & \\
 & & S & &
 \end{array} \tag{1.2}$$

with r and s proper birational morphisms.

Set $n = \dim X = \dim Y = \dim Z$. Then there are natural morphisms of sheaves

$$cl_Z^i : R^i f_* \Omega_X^i \rightarrow R^i g_* \Omega_Y^i \text{ for all } i, \tag{1.3}$$

which are isomorphisms if $i = 0, n$.

In the special case $\text{char } k = 0$ this is a consequence of Hironaka's resolution of singularities [Hir64]. Analysis of the proof shows that the morphisms of 1.3 are obtained from morphisms of complexes

$$cl_Z : Rf_* \Omega_X^i \rightarrow Rg_* \Omega_Y^i \text{ for all } i,$$

(for the cases $i = 0, n$ this is observed in [CR12; Kov19]).

One of the primary applications of [Theorem 1.1](#) was to extend foundational results on rational singularities from characteristic zero to arbitrary characteristic.

Definition 1.4 ([Kol13, Def. 2.76]). Let S be a reduced, separated scheme of finite type over a field k . A **rational resolution** $f : X \rightarrow S$ is a proper birational morphism such that

- (i) X is smooth over k ,
- (ii) $\mathcal{O}_S = Rf_*\mathcal{O}_X$ and
- (iii) $R^if_*\omega_X = 0$ for $i > 0$.

The scheme S is said to have **rational singularities** if and only if it has a rational resolution.

Corollary 1.5 ([CR11, Cor. 3.2.10]). *If S has a rational resolution, then every resolution of S is rational. In particular if S is smooth then it has rational singularities.*

This article concerns analogues of [Theorem 1.1](#) for pairs.

Convention 1.6. In what follows a **pair** (X, Δ_X) will mean a reduced, equidimensional separated scheme X of finite type over k together with a reduced, effective divisor Δ_X on X . A pair (X, Δ_X) will be called a **simple normal crossing (snc) pair** if and only if X is smooth and Δ_X is a simple normal crossing divisor on X .

As observed in [Kol13, §2.5], to generalize [Corollary 1.5](#) to pairs we must restrict attention to a special class of *thrifty resolutions* ([Definition 3.5](#)).

Theorem 1.7. *Let k be a perfect field and let S be a separated scheme of finite type over k . Let (X, Δ_X) and (Y, Δ_Y) be simple normal crossing pairs over S .*

Suppose $(X, \Delta_X), (Y, \Delta_Y)$ are properly birational over S in the sense that there is a commutative diagram

$$\begin{array}{ccccc}
 & (Z, \Delta_Z) & & & \\
 & \swarrow r & & \searrow s & \\
 (X, \Delta_X) & & \circlearrowright & & (Y, \Delta_Y) \\
 & \searrow f & & \swarrow g & \\
 & S & & &
 \end{array} \tag{1.8}$$

where r, s are proper and birational morphisms, and $\Delta_Z = r_*^{-1}\Delta_X = s_*^{-1}\Delta_Y$. Set $n = \dim X = \dim Y = \dim Z$. If r, s are thrifty then there are quasi-isomorphisms

$$Rf_*\mathcal{O}_X(-\Delta_X) \simeq Rg_*\mathcal{O}_Y(-\Delta_Y) \text{ and } Rf_*\omega_X(\Delta_X) \simeq Rg_*\omega_Y(\Delta_Y). \tag{1.9}$$

Definition 1.10 ([Kol13, Def. 2.78]). Let (S, Δ_S) be a pair as in [Convention 1.6](#), and suppose S is normal. A **rational resolution of (S, Δ_S)** is a proper birational morphism $f : X \rightarrow S$ such that if $\Delta_X = f_*^{-1}\Delta_S$ then

- (i) The pair (X, Δ_X) is snc,
- (ii) The natural map $\mathcal{O}_S(-\Delta_S) \rightarrow Rf_*\mathcal{O}_X(-\Delta_X)$ is a quasi-isomorphism, and
- (iii) $R^if_*\omega_X(\Delta_X) = 0$ for $i > 0$.

Remark 1.11 (description of the natural map in (ii)). Since Δ_X is the strict transform of Δ_S , so in particular $\Delta_X \subset f^{-1}(\Delta_S)$, there is a containment of ideal sheaves $\mathcal{I}_{f^{-1}(\Delta_S)} \subset \mathcal{I}_{\Delta_X} = \mathcal{O}_X(-\Delta_X)$ providing a morphism

$$f^*\mathcal{O}_S(-\Delta_S) = f^*\mathcal{I}_{\Delta_S} \rightarrow \mathcal{I}_{f^{-1}(\Delta_S)} \subset \mathcal{I}_{\Delta_X} = \mathcal{O}_X(-\Delta_X).$$

Taking the adjoint gives a morphism $\mathcal{O}_S(-\Delta_S) \rightarrow f_*\mathcal{O}_X(-\Delta_X)$, and composing with the natural map $f_*\mathcal{O}_X(-\Delta_X) \rightarrow Rf_*\mathcal{O}_X(-\Delta_X)$ gives (ii).

As a straightforward corollary of [Theorem 1.7](#), one obtains:

Corollary 1.12. *Let (S, Δ_S) be a pair, with Δ_S reduced and effective. If (S, Δ_S) has a thrifty rational resolution $f : (X, \Delta_X) \rightarrow (S, \Delta_S)$, then every thrifty resolution $g : (Y, \Delta_Y) \rightarrow (S, \Delta_S)$ is rational. In particular, if (S, Δ_S) is snc then it is a rational pair.*

2. DUAL COMPLEXES

Definition 2.1 (cf. [\[dFKX14\]](#)). Let $Z = \bigcup_{i \in I} Z_i$ be a scheme with irreducible components Z_i . Say Z is an **expected-dimensional crossing scheme** if and only if

- (i) Z is pure dimensional and the components Z_i are normal, and
- (ii) For any $J \subset I$, set $Z_J := \bigcap_{j \in J} Z_j$. If $Z_J \neq \emptyset$ every connected component of Z_J is irreducible and of codimension $|J| - 1$ in Z .

A **stratum** of an expected-dimensional crossing scheme Z is an irreducible (or equivalently connected) component of $Z_J = \bigcap_{j \in J} Z_j$ for some $J \subset I$.

The main case of [Definition 2.1](#) considered here will be the case $\Delta = \Delta_X$ where (X, Δ_X) is a simple normal crossing pair, in which case all strata of Δ_X are smooth.

Let (X, Δ_X) be a simple normal crossing pair, and write $\Delta_X = \bigcup_{i \in I} D_i$ with D_i the irreducible components of Δ_X . For $J \subset I$, let $D_J = \bigcap_{j \in J} D_j$, and write $D_J = \bigcup_k D_J^k$ where the D_J^k are irreducible. Observe that $(\Delta_X - \sum_{i \in J} D_i)|_{D_J^k}$ is a (possibly empty) simple normal crossing divisor on each stratum D_J^k .

Definition 2.2 (strata as pairs).

$$\Delta_{D_J} := (\Delta_X - \sum_{i \in J} D_i)|_{D_J} \text{ and } \Delta_{D_J^k} := (\Delta_X - \sum_{i \in J} D_i)|_{D_J^k}$$

Definition 2.3. For an expected-dimensional crossing scheme $Z = \bigcup_{i \in I} Z_i$, the **dual complex** $\mathcal{D}(Z)$ is a Δ -complex [\[Hat02, §2.1\]](#) that can be described as follows: assume the index set I has been totally ordered. For each $d \in \mathbb{N}$, the d -simplices of $\mathcal{D}(Z)$ correspond to the irreducible components $Z_J^k \subset Z_J = \bigcap_{j \in J} Z_j$ where $J \subset I$ ranges over all subsets of size $|J| = d + 1$. Let σ_J^k be the d -simplex associated to Z_J^k .

If $j \in J$ write $\hat{J}(j) := J \setminus \{j\}$ – we have inclusions $Z_J \subset Z_{\hat{J}(j)}$, and the connected components of $Z_{\hat{J}(j)}$ are irreducible, for each component Z_J^k there is a *unique* component $Z_{\hat{J}(j)}^l \subset Z_{\hat{J}(j)}$ such that $Z_J^k \subset Z_{\hat{J}(j)}^l$. The face maps of $\mathcal{D}(Z)$ are obtained by setting

$$\partial_j \sigma_J^k = \sigma_{\hat{J}(j)}^l$$

Remark 2.4. In particular, $\mathcal{D}(Z)$ has

- 0-simplices σ_i corresponding to the irreducible components $Z_i \subset Z$,
- 1-simplices σ_{ij}^k corresponding to the components $Z_{ij}^k \subset Z_{ij} = Z_i \cap Z_j$ where $i < j$, with face maps ∂_0, ∂_1 corresponding to the inclusions $Z_{ij}^k \subset Z_i, Z_{ij}^k \subset Z_j$ respectively,

and so on.

Remark 2.5. From the description above one can see that $\mathcal{D}(Z)$ is a **regular** Δ -complex, meaning that if $\sigma \subseteq \mathcal{D}(Z)$ is a d -simplex, the corresponding map $\sigma : \Delta^d \rightarrow \mathcal{D}(Z)$ is injective. Indeed, if

$$\partial_j \sigma_J^k = \partial_{j'} \sigma_J^k$$

for $j \neq j'$, then $Z_{\hat{J}(j)} \cap Z_{\hat{J}(j')} = Z_J$ would contain a component of codimension $d - 1$, violating (ii) of [Definition 2.3](#).

Dual complexes have been extensively studied; to paraphrase Arapura, Bakhtary, and Włodarczyk, $\mathcal{D}(Z)$ governs the *combinatorial part* of the topology of Z [ABW13]. One underlying reason for this is

Lemma 2.6 (Special case of [Fri83, Prop. 1.5]¹). *If Δ is a simple normal crossing scheme and $n = \dim \mathcal{D}(\Delta)$, then there is a quasi-isomorphism*

$$\mathcal{O}_\Delta \simeq \left[\prod_{\sigma \in \mathcal{D}(\Delta)^0} \mathcal{O}_{D(\sigma)} \xrightarrow{d^1} \prod_{\sigma \in \mathcal{D}(\Delta)^1} \mathcal{O}_{D(\sigma)} \xrightarrow{d^2} \cdots \rightarrow \prod_{\sigma \in \mathcal{D}(\Delta)^n} \mathcal{O}_{D(\sigma)} \right] =: \check{C}(\Delta, \mathcal{O}) \text{ in } D^+(\Delta_X)$$

where the differential $d^i: \prod_{\sigma \in \mathcal{D}(\Delta)^{i-1}} \mathcal{O}_{D(\sigma)} \rightarrow \prod_{\sigma \in \mathcal{D}(\Delta)^i} \mathcal{O}_{D(\sigma)}$ has σ th coordinate

$$\prod_{\sigma \in \mathcal{D}(\Delta)^{i-1}} \mathcal{O}_{D(\sigma)} \rightarrow \prod_{j=0}^i \mathcal{O}_{D(\partial^j \sigma)} \xrightarrow{\sum_{j=0}^i (-1)^j \text{res}_j} \mathcal{O}_{D(\sigma)}$$

and where $\text{res}_j: \mathcal{O}_{D(\partial^j \sigma)} \rightarrow \mathcal{O}_{D(\sigma)}$ is the natural map restricting functions.

Corollary 2.7. *If (X, Δ_X) is a simple normal crossing pair let $n = \dim \mathcal{D}(\Delta_X)$, then there is a quasi-isomorphism*

$$\mathcal{O}_X(-\Delta_X) \simeq \left[\mathcal{O}_X \xrightarrow{d^0} \prod_{\sigma \in \mathcal{D}(\Delta)^0} \mathcal{O}_{D(\sigma)} \xrightarrow{d^1} \prod_{\sigma \in \mathcal{D}(\Delta)^1} \mathcal{O}_{D(\sigma)} \xrightarrow{d^2} \cdots \rightarrow \prod_{\sigma \in \mathcal{D}(\Delta)^n} \mathcal{O}_{D(\sigma)} \right] =: \check{C}(X, \Delta_X, \mathcal{O})$$

in $D^+(X)$.

Proof. We must show that the sequence

$$\mathcal{O}_X(-\Delta_X) \rightarrow \mathcal{O}_X \xrightarrow{d^0} \prod_{\sigma \in \mathcal{D}(\Delta)^0} \mathcal{O}_{D(\sigma)} \xrightarrow{d^1} \prod_{\sigma \in \mathcal{D}(\Delta)^1} \mathcal{O}_{D(\sigma)} \xrightarrow{d^2} \cdots \rightarrow \prod_{\sigma \in \mathcal{D}(\Delta)^n} \mathcal{O}_{D(\sigma)}$$

is exact – Lemma 2.6 already implies $\ker d^i = \text{im } d^{i-1}$ for $i > 1$ and $\ker d^1 = \mathcal{O}_\Delta$. Exactness of the sequence $0 \rightarrow \mathcal{O}_X(-\Delta_X) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{\Delta_X} \rightarrow 0$ tells us that $\mathcal{O}_X \rightarrow \mathcal{O}_{\Delta_X}$ is surjective with kernel $\mathcal{O}_X(-\Delta_X)$. Hence defining d^0 to be the composition

$$\begin{array}{ccccc} & & d^0 & & \\ & \searrow & & \swarrow & \\ \mathcal{O}_X & \longrightarrow & \mathcal{O}_{\Delta_X} & \longrightarrow & \prod_{\sigma \in \mathcal{D}(\Delta)^0} \mathcal{O}_{D(\sigma)} \end{array} \quad (2.8)$$

ensures that $\ker d^1 = \text{im } d^0$ and that $\ker d^0 = \mathcal{O}_X(-\Delta_X)$, as desired. \square

2.1. Morphisms of Dual Complexes. One can extract from the literature on dual complexes the following slogan:

Morphisms of pairs induce morphisms of dual complexes. Moreover, there is a “dictionary” relating properties of a morphism of pairs with corresponding properties of the induced morphism of dual complexes.

To precisify the slogan, we include a foundational result providing a weak sort of functoriality.

Lemma 2.9 (cf. [Wlo16, Def. 2.0.6]). *Let $Z = \cup_{i \in I} Z_i$ and $W = \cup_{j \in J} W_j$ be expected -dimensional crossing schemes and let $f: Z \dashrightarrow W$ be a rational morphism defined at the generic point of each stratum of Z . Then up to homotopy equivalence there is a unique induced morphism of Δ -complexes*

$$\mathcal{D}(f): \mathcal{D}(Z) \rightarrow \mathcal{D}(W)$$

such that if $\sigma \subset \mathcal{D}(Z)$ is a simplex and η_σ is the generic point of the corresponding stratum of Z , and if $\tau \subset \mathcal{D}(W)$ is the simplex corresponding to the unique minimal stratum $D(\tau) \subset W$ containing $f(\eta_\sigma)$, then $\mathcal{D}(f)(\sigma) \subset \tau$.

¹The cited proposition is stated over \mathbb{C} , but the proof works in arbitrary characteristic.

Proof in the case f is defined everywhere. Since $f(D(\sigma))$ is irreducible it is contained in some stratum of W (in particular, $f(D(\sigma)) \subset W_i$ for some i). Let

$$W_I := \cap \{W_j \subset W \mid f(D(\sigma)) \subset W_j\}$$

By (ii) of [Definition 2.1](#), the connected components of W_I are irreducible, and hence $f(D(\sigma))$ is contained in exactly one of them – let $\tau \subset \mathcal{D}(W)$ be the corresponding simplex. If $\dim \sigma = 0$ let $\mathcal{D}(f)(\sigma)$ be an interior point of τ .

One can now show by induction on $\dim \sigma$ that $\mathcal{D}(f)$ extends over all of $\mathcal{D}(Z)$ – so, assume $\dim \sigma > 1$. For each face $\sigma' \subset \sigma$ with corresponding stratum $D(\sigma') \subset Z$, let $D(\tau') \subset W$ be the smallest stratum containing $f(D(\sigma'))$. Now

$$f(D(\sigma)) \subset f(D(\sigma')) \text{ forces } D(\tau) \subset D(\tau')$$

and this gives an inclusion $\iota_{\tau'} : \tau' \rightarrow \tau$. By induction a map $\mathcal{D}(f)|_{\sigma'} : \sigma' \rightarrow \tau'$ has already been defined, so composing with ι one obtains

$$\sigma' \xrightarrow{\mathcal{D}(f)|_{\sigma'}} \tau' \xrightarrow{\iota} \tau \text{ for each face } \sigma' \subset \sigma$$

which together give a map $\partial\sigma \rightarrow \tau$, and as τ is contractible this map must extend over σ .

Uniqueness up to homotopy equivalence follows from [Lemma 2.10](#). \square

Lemma 2.10. *If $f, g: X \rightarrow Y$ are 2 maps of regular Δ -complexes such that for each simplex $\sigma \subseteq X$ there is a unique minimal simplex $\tau_\sigma \subseteq Y$ such that $f(\sigma), g(\sigma) \subseteq \tau_\sigma$ then there is a homotopy $h: X \times I \rightarrow Y$ from f to g such that $h(\sigma \times I) \subseteq \tau_\sigma$ for each simplex $\sigma \subseteq X$.*

Proof. We proceed by induction over the skeleta $X^d \subseteq X$. For the case $d = 0$ let $v \in X^0$ be a vertex. By hypothesis there's a unique minimal simplex $\tau_v \subseteq Y$ so that $f(v), g(v) \in \tau_v \subseteq Y$, so we may choose a path $\gamma_v: I \rightarrow \tau_v \subseteq Y$ with $\gamma_v(0) = f(v), \gamma_v(1) = g(v)$. Then the map

$$h^0: X^0 \times I \rightarrow Y \text{ defined by } h^0(v, t) = \gamma_v(t)$$

is a homotopy between $f|_{X^0}$ and $g|_{X^0}$ with $h^0(\{v\} \times I) \subseteq \tau_v$ for all v .

Suppose by inductive hypothesis that $d > 0$ and we have constructed a homotopy $h^{d-1}: X^{d-1} \times I \rightarrow Y$ from $f|_{X^{d-1}}$ to $g|_{X^{d-1}}$ with $h^{d-1}(\sigma \times I) \subseteq \tau_\sigma$ for all simplices $\sigma \subseteq X^{d-1}$. Let $\sigma \subset X$ be a d -simplex, and observe that if $\sigma' \subset \sigma$ is a face then $f(\sigma') \subseteq f(\sigma) \subseteq \tau_\sigma$, and similarly $g(\sigma') \subseteq \tau_\sigma$. By hypothesis this implies $\tau_{\sigma'} \subseteq \tau_\sigma$, so that the homotopy $h^{d-1}|_{\sigma'}: \sigma' \times I \rightarrow Y$ factors through τ_σ . We conclude that the map $\gamma|_\sigma: \sigma \times 0, 1 \cup \partial\sigma \rightarrow Y$ defined by

$$(x, t) \mapsto \begin{cases} f(x) & \text{if } t = 0, \\ g(x) & \text{if } t = 1, \text{ and} \\ h(x, t), & \text{otherwise} \end{cases}$$

factors through τ_σ ; since Y is regular τ_σ is contractible, and so $\gamma|_\sigma$ extends to a morphism $\gamma_\sigma: \sigma \times I \rightarrow Y$. As σ varies over the d -simplices of X , the γ_σ provide an extension of h^{d-1} to a homotopy

$$h^d: X^d \times I \rightarrow Y \text{ from } f|_{X^d} \text{ to } g|_{X^d}.$$

\square

3. THRIFTY MORPHISMS OF PAIRS

Let (S, Δ_S) be a pair (as in [Convention 1.6](#)).

Definition 3.1. The **snc locus** of (S, Δ_S) is the largest open $U \subset S$ so that $(U, \Delta_S|_U)$ is a simple normal crossing pair – it will be denoted $\text{snc}(S, \Delta_S)$. We also set

$$\text{non-snc}(S, \Delta_S) := S \setminus \text{snc}(S, \Delta_S) \tag{3.2}$$

Remark 3.3. When S is normal, $\text{non-snc}(S, \Delta_S)$ has codimension ≥ 2 in S .

In their work on dual complexes of Calabi-Yau pairs, introduced a natural generalization of thrifty resolutions to a class of *thrifty morphisms* where the domain is no longer required to be smooth.

Definition 3.4 ([KX16, Def. 9]). A crepant proper birational morphism of log canonical pairs $f: (X, \Delta_X) \dashrightarrow (S, \Delta_S)$ is **Kollár-Xu-thrifty** (KX-thrifty for short) if and only if there are closed subsets $Z_X \subset X$, $Z_S \subset S$ of codimension ≥ 1 so that

- Z_X contains no log canonical centers of (X, Δ_X) , and similarly for Z_S , and
- f restricts to an isomorphism $X \setminus Z_X \xrightarrow{f} S \setminus Z_S$.

Since rational pairs are not log canonical in general, for example since they are not necessarily \mathbb{Q} -Gorenstein², we adopt a slightly different definition of thrifty morphisms (see [Lemma 3.8](#) for a comparison).

Let (S, Δ_S) be a pair and let $f: X \rightarrow S$ be a proper birational morphism. Set $\Delta_X := f_*^{-1} \Delta_S$ (the strict transform).

Definition 3.5. The morphism f is **thrifty** if and only if

- (i) f is an isomorphism *over* the generic point of every stratum of $\text{snc}(S, \Delta_S)$ and
- (ii) f is an isomorphism *at* the generic point of every stratum of $\text{snc}(X, \Delta_X)$.

If in addition X is smooth and $f^{-1}(\Delta_S) \cup E$ is a simple normal crossing divisor (with E the exceptional locus) then f is called a **thrifty resolution**.

Remark 3.6. Equivalently, if $\text{Ex}(f) \subset X$ is the exceptional locus of f , then

- (i) $f(\text{Ex}(f))$ contains no stratum of $\text{snc}(S, \Delta_S)$ and
- (ii) $\text{Ex}(f)$ contains no stratum of $\text{snc}(X, \Delta_X)$.

Remark 3.7. Hence when X is smooth and $f^{-1}(\Delta_S) \cup E$ is a simple normal crossing divisor [Definition 3.5](#) reduces to [Kol13, Def. 2.79].

Lemma 3.8. Let $f: (X, \Delta_X) \rightarrow (S, \Delta_S)$ be a crepant proper birational morphism between dlt pairs. Then f is KX-thrifty ([Definition 3.4](#)) if and only if it is thrifty ([Definition 3.5](#)).

Proof. The map f is crepant, so $K_X + \Delta_X \sim_{\mathbb{Q}} f^*(K_S + \Delta_S)$ – equivalently,

$$\Delta_X \sim_{\mathbb{Q}} f_*^{-1}(\Delta_S) - \sum_i a_i E_i$$

where $a_i := a(E_i, S, \Delta_X)$ and the sum runs over all f -exceptional divisors $E_i \subset X$. Writing $\Delta_S = \sum_i c_i D_i$, we see that $\Delta_S^{-1} = \sum_{c_i=1} D_i$ and that $\Delta_X^{-1} = \sum_{c_i=1} f_*^{-1} D_i + \sum_{a_i=-1} E_i$. Both pairs are dlt, so the log canonical centers of (X, Δ_X) are the the strata of the expected-dimensional crossing scheme Δ_X^{-1} , and their generic points lie in $\text{snc}(X, \Delta_X)$ – similarly for (S, Δ_S) [Fuj07]. Moreover, if $a_i = -1$ then $f(E_i) \subset S$ is a log canonical center, so it must be a stratum of Δ_S^{-1} .

Suppose f is KX-thrifty and let $Z_X \subset X$, $Z_S \subset S$ be closed sets as guaranteed in [Definition 3.4](#). Then f is an isomorphism over $S \setminus Z_S$ and Z_S contains no stratum of Δ_S^{-1} , giving condition (i) of [Definition 3.5](#). Also, we must have $a_i > -1$ for all i , and so $\Delta_X^{-1} = \sum_{c_i=1} f_*^{-1} D_i = f_*^{-1} \Delta_S^{-1}$. Since Z_X contains no stratum of Δ_X^{-1} , we obtain (ii) of [Definition 3.5](#). \square

In the next lemma we use a definition of a birational map general enough to encompass reducible schemes [Sta19, Tags 0A20, 0BX9]: a rational map $f: X \dashrightarrow Y$ between schemes with finitely many irreducible components is *birational* if and only if it is an isomorphism in the category with

²The cone over $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^{mn+m+n}$ embedded using the complete linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(m, n)|$ is rational for all $m, n > 0$, \mathbb{Q} -Gorenstein if and only if $m = n$.

- objects the schemes with finitely many irreducible components, and with
- morphisms the dominant rational maps between them.

When Y is locally of finite presentation over a field (as it will be in all cases considered here), the map f is birational if and only if it induces a bijection between the generic points of irreducible components of X and Y , and for each generic point of an irreducible component $\eta \in X$ the induced morphism $\mathcal{O}_{Y,f(\eta)} \rightarrow \mathcal{O}_{X,\eta}$ is an isomorphism.

Lemma 3.9. *Let $Z = \cup_{i=1}^N Z_i$ and $W = \cup_{j=1}^N W_j$ be expected-dimensional crossing schemes and let $f : Z \dashrightarrow W$ be a birational map defined at the generic point of each stratum of Z .*

- (i) *If f is an isomorphism at the generic point of every stratum $D(\sigma) \subset Z$, then $\mathcal{D}(f)$ can be realized as a subcomplex inclusion.*
- (ii) *If f is an isomorphism over the generic point of every stratum $D(\tau) \subset W$ then it is an isomorphism at the generic point of every stratum of Z , and $\mathcal{D}(f)$ can be realized as an isomorphism of Δ -complexes.*

Proof. In the case of (i), as f is birational it induces a bijection between the generic points of Z and W and hence a bijection on 0-skeleta

$$\mathcal{D}(f)_0 : \mathcal{D}(Z)_0 \xrightarrow{\sim} \mathcal{D}(W)_0$$

Without loss of generality we may assume f restricts to a birational maps $f_i : Z_i \dashrightarrow W_i$ for $i = 1, \dots, N$. Let $n = \dim Z = \dim W$.

Let $\sigma \subset \mathcal{D}(Z)$ be a simplex with corresponding stratum $D(\sigma) \subset Z$ – without loss of generality we may assume $D(\sigma) \subset Z_1$, and that $D(\sigma) \subseteq \cap_{j=1}^r Z_j$. Letting $\eta_\sigma \in D(\sigma)$ be the generic point, we see that $f(\eta_\sigma) \in \cap_{j=1}^r W_j$. Because f is an isomorphism at η_σ , it must be that $f(\eta_\sigma)$ is a generic point of a component $D(\tau) \subseteq \cap_{j=1}^r W_j$ corresponding to a simplex $\tau \subseteq \mathcal{D}(W)$. Let $\eta_\tau \in D(\tau)$ be the generic point; we have $\eta_\tau = f(\eta_\sigma)$.

At this point the only concern is that there could be another $r-1$ -simplex σ' such that $\mathcal{D}(f)(\sigma') = \tau$; any such σ' would correspond to another stratum $D(\sigma') \subseteq \cap_{j=1}^r Z_j$, hence another point $\eta_{\sigma'} \in Z_1$ of dimension $r-1$ with $f(\eta_{\sigma'}) = f(\eta_\tau)$. One can show this is impossible, using the normality of W_1 and Zariski's main theorem as follows.

The map f is an isomorphism at the generic point $n_\sigma \in D(\sigma)$, so its restriction $f|_{Z_1} : Z_1 \rightarrow W_1$ is also an isomorphism at n_σ . The scheme W_1 is normal and $f|_{Z_1}$ is birational by hypothesis, so by Zariski's main theorem [Sta19, Tag 05K0] $f|_{Z_1}$ is in fact an isomorphism over η_τ .

For (ii), observe that $f^{-1} : W \dashrightarrow Z$ satisfies the hypotheses of (i) and hence both $\mathcal{D}(f) : \mathcal{D}(Z) \rightarrow \mathcal{D}(W)$ and $\mathcal{D}(f^{-1}) : \mathcal{D}(W) \rightarrow \mathcal{D}(Z)$ may be realized as subcomplex inclusions; from the proof of (i), this can be done in such a way that $\mathcal{D}(f) \circ \mathcal{D}(f^{-1}) = \text{id}_{\mathcal{D}(W)}$. In particular this implies $\mathcal{D}(f)$ is a surjective subcomplex inclusion, hence an isomorphism. □

Corollary 3.10. *Let (S, Δ_S) be a pair and let $f : X \rightarrow S$ be a proper birational morphism and set $\Delta_X := f_*^{-1} \Delta_S$. Then f induces morphisms of Δ -complexes*

$$\mathcal{D}(\text{snc } \Delta_X) \xrightarrow{\mathcal{D}(f|_{\Delta})} \mathcal{D}(\text{snc } \Delta_S) \text{ and } \mathcal{D}(\text{snc}(X, \Delta_X)) \xrightarrow{\mathcal{D}(f)} \mathcal{D}(\text{snc}(S, \Delta_S))$$

which are isomorphisms if f is thrifty.

Proof. The induced morphisms come from Lemma 2.9; to see that they are isomorphisms when f is thrifty we may apply Definition 3.5 and Lemma 3.9. □

3.1. Thrifty proper birational equivalences. If S is a separated scheme of finite type over k and $f : X \rightarrow S, g : Y \rightarrow S$ are separated schemes of finite type over S , a **proper birational equivalence**

of X, Y over S is a commutative diagram

$$\begin{array}{ccc}
 & Z & \\
 r \swarrow & & \searrow s \\
 X & \circlearrowright & Y \\
 f \searrow & & \swarrow g \\
 & S &
 \end{array} \tag{3.11}$$

where r, s are proper birational morphisms.

Definition 3.12. Suppose $(X, \Delta_X), (Y, \Delta_Y)$ are pairs over S , with X and Y normal and Δ_X, Δ_Y reduced and effective. A **thrifty proper birational equivalence of (X, Δ_X) and (Y, Δ_Y) over S** is a proper birational equivalence as in diagram 3.11, where r and s are thrifty.

Remark 3.13. By [Corollary 3.10](#), a thrifty proper birational equivalence $X \xleftarrow{r} Z \xrightarrow{s} Y$ between (X, Δ_X) and (Y, Δ_Y) induces an isomorphism $\mathcal{D}(\Delta_X) \simeq \mathcal{D}(\Delta_Y)$.

Proposition 3.14. Let (S, Δ_S) be a pair with Δ_S reduced and effective, and let $f : X \rightarrow S, g : Y \rightarrow S$ be 2 thrifty resolutions of (S, Δ_S) . Then there is a thrifty proper birational equivalence of X and Y over S .

Proof. Let $U \subset S$ be an open set such that both f and g are isomorphisms over U ; then we have an isomorphism

$$g^{-1} \circ f : f^{-1}(U) \rightarrow g^{-1}(U)$$

Set

$$Z := \overline{\Gamma_{g^{-1} \circ f}} \subset X \times_S Y$$

and let $p : Z \rightarrow X, s : Z \rightarrow Y$ be the projections. The claim is that $X \xleftarrow{r} Z \xrightarrow{s} Y$ is a thrifty proper birational equivalence over S . It is birational by design, and proper since X, Y and hence $X \times_Y Z$ are proper over S and Z is closed in $X \times_S Y$. It remains to show that r, s are thrifty.

Lemma 3.15. Let $\text{Ex}(r), \text{Ex}(s) \subset Z$ be the exceptional loci of r, s respectively; let $\text{Ex}(f) \subset X, \text{Ex}(g) \subset Y$ be the exceptional loci of f and g . Then

$$r(\text{Ex}(r)) \subset f^{-1}(g(\text{Ex}(g))) \text{ and } s(\text{Ex}(s)) \subset g^{-1}(f(\text{Ex}(f)))$$

Proof of Lemma 3.15. Let $U \subset S$ and $V \subset Y$ be a maximal pair of open sets such that $g|_V : V \xrightarrow{\sim} U$ is an isomorphism; note that since g is an honest morphism $\text{Ex}(g) = Y \setminus V$ and $g(\text{Ex}(g)) = S \setminus U$. Then $W := f^{-1}(U) \subset X$ is an open set such that $g^{-1} \circ f : X \dashrightarrow Y$ is defined on W . This implies the projection $\Gamma_{g^{-1} \circ f} \xrightarrow{r} X$ is an isomorphism over W , but what we need to know is that the same is true for $Z = \overline{\Gamma_{g^{-1} \circ f}} \xrightarrow{r} X$. For this, note that

$$\overline{\Gamma_{g^{-1} \circ f}} \cap r^{-1}(W) = \overline{\Gamma_{g^{-1} \circ f} \cap r^{-1}(W)} = \overline{\Gamma_{g^{-1} \circ f|_W}} \subset W \times_S Y$$

Since W and Y are both separated over S , the graph $\Gamma_{g^{-1} \circ f|_W}$ is already closed, so we conclude $\overline{\Gamma_{g^{-1} \circ f}} \cap r^{-1}(W) = \Gamma_{g^{-1} \circ f|_W}$. \square

Now suppose $W \subset X$ is a stratum of (X, Δ_X) – we must show r is an isomorphism over the generic point $\eta \in W$. First, f is an isomorphism at η by hypothesis, and so by the proof of [Lemma 3.9](#), $f(\eta)$ is the generic point of a stratum of $\text{snc}(S, \Delta_S)$. Then g is an isomorphism over $f(\eta)$ by hypothesis, so in particular $f(\eta) \notin g(\text{Ex}(g))$. By [Lemma 3.15](#) we conclude that $\eta \notin r(\text{Ex}(r))$, as desired.

Finally we show that s is an isomorphism at the generic point of every stratum of $\Delta_Z := r_*^{-1} f_*^{-1} \Delta_S$, using a more general lemma:

Lemma 3.16. *Let $r: (Z, \Delta_Z) \rightarrow (X, \Delta_X)$ be a proper birational morphism. If (X, Δ_X) is a simple normal crossing pair, then r is thrifty if and only if it satisfies condition (i) of Definition 3.5. Explicitly, r is thrifty if and only if it is an isomorphism over every stratum of Δ_X .*

Proof of Lemma 3.16. In this situation there is an honest morphism $\text{snc}(\Delta_Z) \rightarrow \Delta_X$, so the hypotheses of Lemma 3.9 are satisfied. We then apply Lemma 3.9 (ii). \square

\square

Remark 3.17. In the case where the morphism $r: Z \rightarrow X$ of Lemma 3.16 is projective, [Har77, Thm. 7.17] implies that r is the blowup of some sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$ such that $V(\mathcal{I}) \subset X$ contains no stratum of Δ_X . If in addition $V(\mathcal{I})$ has simple normal crossings with Δ_X [Kol07, Def. 3.24], Lemma 3.16 can be obtained from known results on the effect of blowing up on dual complexes [Ste06, §2], [dFKX14, §9], [Wlo16, Prop. 2.1.6].

4. STRUCTURE SHEAVES OF STRATA AND THEIR DIRECT IMAGES

In this section we prove weak functoriality statements about the quasi-isomorphisms in Theorem 1.1, or alternatively those of [Kov19].

Lemma 4.1. *Let S be scheme over a field k and let $f: X \rightarrow S, g: Y \rightarrow S$ are S -schemes that are smooth over k . Suppose $X \xleftarrow{r} Z \xrightarrow{s} Y$ is a proper birational equivalence over S such that both r and s are projective. Let $\mathcal{C}(Z)$ denote the category with objects the pairs $(E \subseteq X, F \subseteq Y)$ of smooth closed subschemes of X and Y such that*

- (i) *r and s are isomorphisms over the generic points of E and F respectively, and*
- (ii) *the birational map $s \circ r^{-1}: X \dashrightarrow Y$ sends the generic point of E to the generic point of F , and with morphisms $(E_1, F_1) \rightarrow (E_2, F_2)$ given by inclusions $E_1 \subseteq E_2, F_1 \subseteq F_2$. If $\mathcal{K} \subset \mathcal{C}(Z)$ is a finite subcategory, then there are proper birational equivalences $E \xleftarrow{r'} W \xrightarrow{s'} F$ compatible with Z in the sense that*

$$\begin{array}{ccccc} E & \xleftarrow{\quad} & W & \xrightarrow{s'} & F \\ \downarrow & \circlearrowleft & \downarrow k & \circlearrowleft & \downarrow \\ X & \xleftarrow{\quad r \quad} & Z & \xrightarrow{\quad s \quad} & Y \end{array} \quad (4.2)$$

commutes, and commutative diagrams

$$\begin{array}{ccc} Rf_* \mathcal{O}_X & \xrightarrow[\simeq]{\gamma_{X,Y}} & Rf_* \mathcal{O}_Y \\ \downarrow & \circlearrowleft & \downarrow \\ Rf_* \mathcal{O}_E & \xrightarrow[\gamma_{E,F}]{\simeq} & Rf_* \mathcal{O}_F \end{array} \quad \text{in } D^+(S). \quad (4.3)$$

defining a natural transformation of functors $\mathcal{K}^{\text{op}} \rightarrow D^+(S)$.

Proof. We proceed by descending induction over the poset \mathcal{K} .

For each object $(E, F) \in \text{Ob}(\mathcal{K})$, since X is smooth and r is an isomorphism over the generic point $\xi \in E$ we see that if $\tilde{E} \subseteq Z$ is the strict transform of E then $\tilde{E} \not\subseteq \text{Sing}(Z)$, so in particular if $\text{non-CM}(Z) \subseteq Z$ is the non-Cohen-Macaulay locus then $\tilde{E} \not\subseteq \text{non-CM}(Z)$ – similarly for F . By a theorem of Česnavičius, there exists *Macaulayfication* $\pi: \tilde{Z} \rightarrow Z$ such that π is an isomorphism over $Z \setminus \text{non-CM}(Z)$ – explicitly, \tilde{Z} is Cohen-Macaulay and π is a projective birational morphism [Ces18, Thm. 1.6] (see also [Kaw00, Thm. 5.1]). It follows that $r \circ \pi$ and $s \circ \pi$ are projective and isomorphisms over the generic points of E and F respectively, for all $(E, F) \in \text{Ob}(\mathcal{K})$. From now on we may assume Z is Cohen-Macaulay.

Now suppose $(E, F) \in \text{Ob}(\mathcal{K})$ is *maximal* (categorically final), and let $W \subseteq Z \times_{X \times Y} E \times F$ be the component dominating E and F , and form the commutative diagram of S -schemes

$$\begin{array}{ccccc}
 & W & \xrightarrow{k} & Z & \\
 & \searrow s' & & \searrow s & \\
 & F & \xrightarrow{j} & Y & \\
 \swarrow r' & & \swarrow r & & \\
 E & \xrightarrow{i} & X & &
 \end{array} \tag{4.4}$$

Replacing W with a Macaulayfication $\pi': \tilde{W} \rightarrow W$ if necessary, we may assume W is Cohen-Macaulay. Now by functoriality we have commutative diagrams

$$\begin{array}{ccc}
 \mathcal{O}_X \xrightarrow{i^\#} R\iota_* \mathcal{O}_E & & \mathcal{O}_Y \xrightarrow{j^\#} Rj_* \mathcal{O}_F \\
 r^\# \downarrow \simeq & \circlearrowleft & s^\# \downarrow \simeq \\
 Rr_* \mathcal{O}_Z \xrightarrow{Rr_* k^\#} R(r \circ k)_* \mathcal{O}_W = R(\iota \circ r')_* \mathcal{O}_W & \text{and} & Rs_* \mathcal{O}_Z \xrightarrow{Rs_* k^\#} R(s \circ k)_* \mathcal{O}_W = R(j \circ s')_* \mathcal{O}_W
 \end{array} \tag{4.5}$$

in $D(X)$ and $D(Y)$ respectively. The vertical arrows are isomorphisms since X, Y, E and F are all smooth, so in particular they have rational singularities, and W and Z are Cohen-Macaulay, so we may apply [Kov19, Thm. 8.6]. Finally, pushing forward along f and g we obtain

$$\begin{array}{ccc}
 Rf_* \mathcal{O}_X \xrightarrow{Rf_* i^\#} R(f \circ \iota)_* \mathcal{O}_E & & \\
 Rf_* r^\# \downarrow \simeq & \circlearrowleft & \simeq \downarrow R(f \circ \iota)_* r'^\# \\
 R(f \circ r)_* \mathcal{O}_Z \xrightarrow{R(f \circ r)_* k^\#} R(f \circ r \circ k)_* \mathcal{O}_W = R(f \circ \iota \circ r')_* \mathcal{O}_W & & \\
 \parallel & & \parallel \\
 R(g \circ s)_* \mathcal{O}_Z \xrightarrow{R(g \circ s)_* k^\#} R(g \circ s \circ k)_* \mathcal{O}_W = R(g \circ j \circ s')_* \mathcal{O}_W & & \\
 \simeq \uparrow Rg_* s^\# & \circlearrowleft & R(g \circ j)_* s'^\# \uparrow \simeq \\
 Rg_* \mathcal{O}_Y \xrightarrow{Rg_* j^\#} R(g \circ j)_* \mathcal{O}_F & &
 \end{array} \tag{4.6}$$

For the inductive step, suppose $(E, F) \in \text{Ob}(\mathcal{K})$ and let $\alpha_i: (E, F) \rightarrow (E_i, F_i)$, $i = 1, \dots, r$ be the morphisms in \mathcal{K} with source (E, F) . By inductive hypothesis, for each i we have a Cohen-Macaulay S -scheme W_i and a projective birational equivalence $E_i \xleftarrow{r_i} W_i \xrightarrow{s_i} F_i$ inducing a morphism $\gamma_{E_i, F_i}: Rf_* \mathcal{O}_{E_i} \rightarrow Rg_* \mathcal{O}_{F_i}$ – using the above construction, we can ensure that for any \mathcal{K} -morphism $(E', F') \rightarrow (E_i, F_i)$ the map r_i is an isomorphism over E' , and similarly for F_i . Consider the cartesian diagram

$$\begin{array}{ccccc}
 W & \hookrightarrow & (E \times_S F) \times_{\prod_{i=1}^r (E_i \times_S F_i)} \prod_{i=1}^r W_i & \longrightarrow & \prod_{i=1}^r W_i \\
 & \searrow r' \times_S s' & \downarrow & \square & \downarrow \prod_{i=1}^r (r_i \times_S s_i) \\
 & & E \times_S F & \longrightarrow & \prod_{i=1}^r (E_i \times_S F_i)
 \end{array} \tag{4.7}$$

and let $W \subseteq (E \times_S F) \times_{\prod_{i=1}^r (E_i \times_S F_i)} \prod_{i=1}^r W_i$ be the component dominating E and F . Note that r', s' are projective since $\prod_{i=1}^r (r_i \times_S s_i)$ is projective by hypothesis. As above, we may replace W by a projective Macaulayfication while retaining the property that r', s' are isomorphisms over the

generic points of $E' \subset E$, $F' \subset F$ for every \mathcal{K} -morphism $(E', F') \rightarrow (E, F)$. Now by design for each i there is a commutative diagram

$$\begin{array}{ccccc}
 W & \xrightarrow{k} & W_i & & \\
 \searrow s' & & \searrow s & & \\
 & F & \xrightarrow{j} & F_i & \\
 \swarrow r' & & \swarrow r & & \\
 E & \xrightarrow{i} & E_i & &
 \end{array} \tag{4.8}$$

and arguing as in the base case we obtain from (4.8) a commutative diagram in $D^+(S)$ of the form

$$\begin{array}{ccc}
 Rf_*\mathcal{O}_{E_i} & \xrightarrow[\simeq]{\gamma_{E_i, F_i}} & Rf_*\mathcal{O}_{F_i} \\
 \downarrow & \circlearrowleft & \downarrow \\
 Rf_*\mathcal{O}_E & \xrightarrow[\gamma_{E, F}]{\simeq} & Rf_*\mathcal{O}_F
 \end{array} \tag{4.9}$$

□

Corollary 4.10. *Let S be a scheme over a field k and let (X, Δ_X) and (Y, Δ_Y) be simple normal crossing pairs over k with morphisms $f: X \rightarrow S$ and $g: Y \rightarrow S$. Suppose $X \xleftarrow{r} Z \xrightarrow{s} Y$ is a thrifty proper birational equivalence over S such that both r and s are projective. Let \mathcal{D} be the common dual complex of Δ_X and Δ_Y (see Remark 3.13) and for a simplex $\sigma \subseteq \mathcal{D}$ let $D_X(\sigma) \subseteq X$, $D_Y(\sigma) \subseteq Y$ be the corresponding strata. In this situation there is a natural transformation of functors $\text{Face}(\mathcal{D}) \rightarrow D^+(S)$ from $Rf_*\mathcal{O}_{D_X(\sigma)}$ to $Rg_*\mathcal{O}_{D_Y(\sigma)}$, compatible with restrictions from $Rf_*\mathcal{O}_X$ and $Rg_*\mathcal{O}_Y$, and hence a commutative diagram in $D^+(S)$ of the form*

$$\begin{array}{ccccccc}
 Rf_*\mathcal{O}_X & \rightarrow & \prod_{\sigma \in \mathcal{D}^0} Rf_*\mathcal{O}_{D_X(\sigma)} & \rightarrow & \prod_{\sigma \in \mathcal{D}^1} Rf_*\mathcal{O}_{D_X(\sigma)} & \rightarrow & \prod_{\sigma \in \mathcal{D}^2} Rf_*\mathcal{O}_{D_X(\sigma)} \rightarrow \dots \\
 \downarrow & & \downarrow \gamma^0 & & \downarrow \gamma^1 & & \downarrow \gamma^2 \\
 Rg_*\mathcal{O}_Y & \rightarrow & \prod_{\sigma \in \mathcal{D}^0} Rg_*\mathcal{O}_{D_Y(\sigma)} & \rightarrow & \prod_{\sigma \in \mathcal{D}^1} Rg_*\mathcal{O}_{D_Y(\sigma)} & \rightarrow & \prod_{\sigma \in \mathcal{D}^2} Rg_*\mathcal{O}_{D_Y(\sigma)} \rightarrow \dots
 \end{array} \tag{4.11}$$

Proof. We apply Lemma 4.1 to the finite subcategory $\mathcal{K} \subset \mathcal{C}(Z)$ with objects $(D_X(\sigma), D_Y(\sigma))$ for $\sigma \subseteq \mathcal{D}$. Evidently, this \mathcal{K} is equivalent to $\text{Face}(\mathcal{D})$. □

5. A MORPHISM OF RESTRICTION TRIANGLES

The main result of this section is

Lemma 5.1. *Let S be a base scheme over a field k and let (X, Δ_X) and (Y, Δ_Y) be simple normal crossing schemes over k with morphisms $f: X \rightarrow S$, $g: Y \rightarrow S$. If $X \xleftarrow{r} Z \xrightarrow{s} Y$ is a thrifty proper birational equivalence over S then there is an isomorphism of distinguished triangles*

$$\begin{array}{ccccccc}
 Rf_*\mathcal{O}_X(-\Delta_X) & \longrightarrow & Rf_*\mathcal{O}_X & \longrightarrow & Rf_*\mathcal{O}_{\Delta_X} & \xrightarrow{+1} & \dots \\
 \downarrow \gamma' & & \downarrow \gamma & & \downarrow \gamma'' & & \\
 Rg_*\mathcal{O}_Y(-\Delta_Y) & \longrightarrow & Rg_*\mathcal{O}_Y & \longrightarrow & Rg_*\mathcal{O}_{\Delta_Y} & \xrightarrow{+1} & \dots
 \end{array} \text{ in } D^+(S). \tag{5.2}$$

For the most part, this consists of combining [Corollaries 2.7](#) and [4.10](#) to obtain the isomorphisms γ and γ'' – after that, the existence of γ' is guaranteed since $D^+(S)$ is triangulated, and the fact that γ' is an isomorphism follows from the 5-lemma.

Proof. Under construction! □

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