HIGHER DIRECT IMAGES OF LOGARITHMIC STRUCTURE SHEAVES

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CONTENTS

1. INTRODUCTION

In [CR11] Chatzistamatiou and Rülling prove the following theorem:

Theorem 1.1 ([CR11, Thm. 3.2.8]). *Let k be a perfect field and let S be a separated scheme of finite type over k. Suppose X and Y are two separated finite type S-schemes which are*
(*i*) *smooth over k and*
(*ii*) ***properly birational*** *over S in the sense that there is a commutative diagram*

$$\begin{tikzcd} & Z \arrow[dl, "r"'] \arrow[dr, "s"] & \\ X \arrow[dr, "f"'] & \circlearrowleft & Y \arrow[dl, "g"] \\ & S & \end{tikzcd} \tag{1.2}$$

with r and s proper birational morphisms.
Set $n = \dim X = \dim Y = \dim Z$. Then there are natural morphisms of sheaves

$$\mathrm{cl}^j_Z : R^j f_* \Omega^i_X \to R^j g_* \Omega^i_Y \text{ for all } i, \tag{1.3}$$

which are isomorphisms if $i = 0, n$.

In the special case $\operatorname{char} k = 0$ this is a consequence of Hironaka's resolution of singularities [Hir64]. Analysis of the proof shows that the morphisms of 1.3 are obtained from morphisms of *complexes*

$$\mathrm{cl}_Z : R f_* \Omega^i_X \to R g_* \Omega^i_Y \text{ for all } i,$$

(for the cases $i = 0, n$ this is observed in [CR12; Kov19]).

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One of the primary applications of Theorem 1.1 was to extend foundational results on rational singularities from characteristic zero to arbitrary characteristic.

Definition 1.4 ([Kol13, Def. 2.76])**.** Let S be a reduced, separated scheme of finite type over a field k. A **rational resolution** $f : X \to S$ is a proper birational morphism such that

(i) X is smooth over k,
(ii) $\mathcal{O}_S = Rf_*\mathcal{O}_X$ and
(iii) $R^i f_* \omega_X = 0$ for $i > 0$.

The scheme S is said to have **rational singularities** if and only if it has a rational resolution.

Corollary 1.5 ([CR11, Cor. 3.2.10])**.** *If S has a rational resolution, then every resolution of S is rational. In particular if S is smooth then it has rational singularities.*

This article concerns analogues of Theorem 1.1 for pairs.

Convention 1.6. In what follows a **pair** (X, Δ_X) will mean a reduced, equidimensional separated scheme X of finite type over k together with a reduced, effective divisor Δ_X on X. A pair (X, Δ_X) will be called a **simple normal crossing (snc) pair** if and only if X is smooth and Δ_X is a simple normal crossing divisor on X.

As observed in [Kol13, §2.5], to generalize Corollary 1.5 to pairs we must restrict attention to a special class of *thrifty resolutions* (Definition 3.5).

Theorem 1.7. *Let k be a perfect field and let S be a separated scheme of finite type over k. Let (X, Δ_X) and (Y, Δ_Y) be simple normal crossing pairs over S.*

Suppose $(X, \Delta_X), (Y, \Delta_Y)$ are properly birational over S in the sense that there is a commutative diagram

$$\begin{tikzcd} & (Z, \Delta_Z) \arrow[dl, "r"'] \arrow[dr, "s"] & \\ (X, \Delta_X) \arrow[dr, "f"'] & \circlearrowleft & (Y, \Delta_Y) \arrow[dl, "g"] \\ & S & \end{tikzcd} \tag{1.8}$$

where r, s are proper and birational morphisms, and $\Delta_Z = r_*^{-1}\Delta_X = s_*^{-1}\Delta_Y$. Set $n = \dim X = \dim Y = \dim Z$. If r, s are thrifty *then there are quasi-isomorphisms*

$$Rf_*\mathcal{O}_X(-\Delta_X) \simeq Rg_*\mathcal{O}_Y(-\Delta_Y) \text{ and } Rf_*\omega_X(\Delta_X) \simeq Rg_*\omega_Y(\Delta_Y). \tag{1.9}$$

Definition 1.10 ([Kol13, Def. 2.78])**.** Let (S, Δ_S) be a pair as in Convention 1.6, and suppose S is normal. A **rational resolution of** (S, Δ_S) is a proper birational morphism $f\colon X \to S$ such that if $\Delta_X = f_*^{-1}\Delta_S$ then

(i) The pair (X, Δ_X) is snc,
(ii) The natural map $\mathcal{O}_S(-\Delta_S) \to Rf_*\mathcal{O}_X(-\Delta_X)$ is a quasi-isomorphism, and
(iii) $R^i f_* \omega_X(\Delta_X) = 0$ for $i > 0$.

Remark 1.11 (description of the natural map in (ii))*.* Since Δ_X is the strict transform of Δ_S, so in particular $\Delta_X \subset f^{-1}(\Delta_S)$, there is a containment of ideal sheaves $\mathcal{I}_{f^{-1}(\Delta_S)} \subset \mathcal{I}_{\Delta_X} = \mathcal{O}_X(-\Delta_X)$ providing a morphism

$$f^*\mathcal{O}_S(-\Delta_S) = f^*\mathcal{I}_{\Delta_S} \to \mathcal{I}_{f^{-1}(\Delta_S)} \subset \mathcal{I}_{\Delta_X} = \mathcal{O}_X(-\Delta_X).$$

Taking the adjoint gives a morphism $\mathcal{O}_S(-\Delta_S) \to f_*\mathcal{O}_X(-\Delta_X)$, and composing with the natural map $f_*\mathcal{O}_X(-\Delta_X) \to Rf_*\mathcal{O}_X(-\Delta_X)$ gives (ii).

As a straightforward corollary of Theorem 1.7, one obtains:

Corollary 1.12. *Let (S, Δ_S) be a pair, with Δ_S reduced and effective. If (S, Δ_S) has a thrifty rational resolution $f : (X, \Delta_X) \to (S, \Delta_S)$, then every thrifty resolution $g : (Y, \Delta_Y) \to (S, \Delta_S)$ is rational. In particular, if (S, Δ_S) is snc then it is a rational pair.*

2. Dual complexes

Definition 2.1 (cf. [dFKX14])**.** Let $Z = \bigcup_{i \in I} Z_i$ be a scheme with irreducible components Z_i. Say Z is an **expected-dimensional crossing scheme** if and only if

(i) Z is pure dimensional and the components Z_i are normal, and

(ii) For any $J \subset I$, set $Z_J := \bigcap_{j \in J} Z_j$. If $Z_J \neq \emptyset$ every connected component of Z_J is irreducible and of codimension $|J| - 1$ in Z.

A **stratum** of an expected-dimensional crossing scheme Z is an irreducible (or equivalently connected) component of $Z_J = \bigcap_{j \in J} Z_j$ for some $J \subset I$.

The main case of Definition 2.1 considered here will be the case $\Delta = \Delta_X$ where (X, Δ_X) is a simple normal crossing pair, in which case all strata of Δ_X are smooth.

Let (X, Δ_X) be a simple normal crossing pair, and write $\Delta_X = \bigcup_{i \in I} D_i$ with D_i the irreducible components of Δ_X. For $J \subset I$, let $D_J = \bigcap_{j \in J} D_j$, and write $D_J = \bigcup_k D_J^k$ where the D_J^k are irreducible. Observe that $(\Delta_X - \sum_{i \in J} D_i)|_{D_J^k}$ is a (possibly empty) simple normal crossing divisor on each stratum D_J^k.

Definition 2.2 (strata as pairs)**.**

$$\Delta_{D_J} := (\Delta_X - \sum_{i \in J} D_i)|_{D_J} \text{ and } \Delta_{D_J^k} := (\Delta_X - \sum_{i \in J} D_i)|_{D_J^k}$$

Definition 2.3. For an expected-dimensional crossing scheme $Z = \bigcup_{i \in I} Z_i$, the **dual complex** $\mathcal{D}(Z)$ is a Δ-complex [Hat02, §2.1] that can be described as follows: assume the index set I has been totally ordered. For each $d \in \mathbb{N}$, the d-simplices of $\mathcal{D}(Z)$ correspond to the irreducible components $Z_J^k \subset Z_J = \cap_{j \in J} Z_j$ where $J \subset I$ ranges over all subsets of size $|J| = d + 1$. Let σ_J^k be the d-simplex associated to Z_J^k.

If $j \in J$ write $\hat{J}(j) := J \setminus \{j\}$ – we have inclusions $Z_J \subset Z_{\hat{J}(j)}$, and the connected components of $Z_{\hat{J}(j)}$ are irreducible, for each component Z_J^k there is a *unique* component $Z_{\hat{J}(j)}^l \subset Z_{\hat{J}(j)}$ such that $Z_J^k \subset Z_{\hat{J}(j)}^l$. The face maps of $\mathcal{D}(Z)$ are obtained by setting

$$\partial_j \sigma_J^k = \sigma_{\hat{J}(j)}^l$$

Remark 2.4. In particular, $\mathcal{D}(Z)$ has

- 0-simplices σ_i corresponding to the irreducible components $Z_i \subset Z$,
- 1-simplices σ_{ij}^k corresponding to the components $Z_{ij}^k \subset Z_{ij} = Z_i \cap Z_j$ where $i < j$, with face maps ∂_0, ∂_1 corresponding to the inclusions $Z_{ij}^k \subset Z_i$, $Z_{ij}^k \subset Z_j$ respectively,

and so on.

Remark 2.5. From the description above one can see that $\mathcal{D}(Z)$ is a **regular** Δ-complex, meaning that if $\sigma \subseteq \mathcal{D}(Z)$ is a d-simplex, the corresponding map $\sigma \colon \Delta^d \to \mathcal{D}(Z)$ is injective. Indeed, if

$$\partial_j \sigma_J^k = \partial_{j'} \sigma_J^k$$

for $j \neq j'$, then $Z_{\hat{J}(j)} \cap Z_{\hat{J}(j')} = Z_J$ would contain a component of codimension $d - 1$, violating (ii) of Definition 2.3.

Dual complexes have been extensively studied; to paraphrase Arapura, Bakhtary, and Włodarczyk, $\mathcal{D}(Z)$ governs the *combinatorial part* of the topology of Z [ABW13]. One underlying reason for this is

Lemma 2.6 (Special case of [Fri83, Prop. 1.5][1])**.** *If Δ is a simple normal crossing scheme and $n = \dim \mathcal{D}(\Delta)$, then there is a quasi-isomorphism*

$$\mathcal{O}_\Delta \simeq [\prod_{\sigma \in \mathcal{D}(\Delta)^0} \mathcal{O}_{D(\sigma)} \xrightarrow{d^1} \prod_{\sigma \in \mathcal{D}(\Delta)^1} \mathcal{O}_{D(\sigma)} \xrightarrow{d^2} \cdots \to \prod_{\sigma \in \mathcal{D}(\Delta)^n} \mathcal{O}_{D(\sigma)}] =: \check{C}(\Delta, \mathcal{O}) \text{ in } D^+(\Delta_X)$$

where the differential $d^i \colon \prod_{\sigma \in \mathcal{D}(\Delta)^{i-1}} \mathcal{O}_{D(\sigma)} \to \prod_{\sigma \in \mathcal{D}(\Delta)^i} \mathcal{O}_{D(\sigma)}$ has σth coordinate

$$\prod_{\sigma \in \mathcal{D}(\Delta)^{i-1}} \mathcal{O}_{D(\sigma)} \to \prod_{j=0}^{i} \mathcal{O}_{D(\partial^j \sigma)} \xrightarrow{\sum_{j=0}^{i} (-1)^j \mathrm{res}_j} \mathcal{O}_{D(\sigma)}$$

and where $\mathrm{res}_j : \mathcal{O}_{D(\partial^j \sigma)} \to \mathcal{O}_{D(\sigma)}$ is the natural map restricting functions.

Corollary 2.7. *If (X, Δ_X) is a simple normal crossing pair let $n = \dim \mathcal{D}(\Delta_X)$, then there is a quasi-isomorphism*

$$\mathcal{O}_X(-\Delta_X) \simeq [\mathcal{O}_X \xrightarrow{d^0} \prod_{\sigma \in \mathcal{D}(\Delta)^0} \mathcal{O}_{D(\sigma)} \xrightarrow{d^1} \prod_{\sigma \in \mathcal{D}(\Delta)^1} \mathcal{O}_{D(\sigma)} \xrightarrow{d^2} \cdots \to \prod_{\sigma \in \mathcal{D}(\Delta)^n} \mathcal{O}_{D(\sigma)}] =: \check{C}(X, \Delta_X, \mathcal{O})$$

in $D^+(X)$.

Proof. We must show that the sequence

$$\mathcal{O}_X(-\Delta_X) \to \mathcal{O}_X \xrightarrow{d^0} \prod_{\sigma \in \mathcal{D}(\Delta)^0} \mathcal{O}_{D(\sigma)} \xrightarrow{d^1} \prod_{\sigma \in \mathcal{D}(\Delta)^1} \mathcal{O}_{D(\sigma)} \xrightarrow{d^2} \cdots \to \prod_{\sigma \in \mathcal{D}(\Delta)^n} \mathcal{O}_{D(\sigma)}$$

is exact – Lemma 2.6 already implies $\ker d^i = \operatorname{im} d^{i-1}$ for $i > 1$ and $\ker d^1 = \mathcal{O}_\Delta$. Exactness of the sequence $0 \to \mathcal{O}_X(-\Delta_X) \to \mathcal{O}_X \to \mathcal{O}_{\Delta_X} \to 0$ tells us that $\mathcal{O}_X \to \mathcal{O}_{\Delta_X}$ is surjective with kernel $\mathcal{O}_X(-\Delta_X)$. Hence defining d^0 to be the composition

$$\begin{tikzcd} \mathcal{O}_X \arrow[r] \arrow[rr, bend left, "d^0"] & \mathcal{O}_{\Delta_X} \arrow[r] & \prod_{\sigma \in \mathcal{D}(\Delta)^0} \mathcal{O}_{D(\sigma)} \end{tikzcd} \tag{2.8}$$

ensures that $\ker d^1 = \operatorname{im} d^0$ and that $\ker d^0 = \mathcal{O}_X(-\Delta_X)$, as desired. □

2.1. **Morphisms of Dual Complexes.** One can extract from the literature on dual complexes the following slogan:

> *Morphisms of pairs induce morphisms of dual complexes. Moreover, there is a "dictionary" relating properties of a morphism of pairs with corresponding properties of the induced morphism of dual complexes.*

To precisify the slogan, we include a foundational result providing a weak sort of functoriality.

Lemma 2.9 (cf. [Wlo16, Def. 2.0.6])**.** *Let $Z = \cup_{i \in I} Z_i$ and $W = \cup_{j \in J} W_j$ be expected -dimensional crossing schemes and let $f : Z \dashrightarrow W$ be a rational morphism defined at the generic point of each stratum of Z. Then up to homotopy equivalence there is a unique induced morphism of Δ-complexes*

$$\mathcal{D}(f) : \mathcal{D}(Z) \to \mathcal{D}(W)$$

such that if $\sigma \subset \mathcal{D}(Z)$ is a simplex and η_σ is the generic piont of the corresponding stratum of Z, and if $\tau \subset \mathcal{D}(W)$ is the simplex corresponding to the unique minimal stratum $D(\tau) \subset W$ containing $f(\eta_\sigma)$, then $\mathcal{D}(f)(\sigma) \subset \tau$.

[1] The cited proposition is stated over $\mathbb{C}$, but the proof works in arbitrary characteristic.

Proof in the case f is defined everywhere. Since $f(D(\sigma))$ is irreducible it is contained in some stratum of W (in particular, $f(D(\sigma)) \subset W_i$ for some i). Let

$$W_J := \cap\{W_j \subset W \mid f(D(\sigma)) \subset W_j\}$$

By (ii) of Definition 2.1, the connected components of W_J are irreducible, and hence $f(D(\sigma))$ is contained in exactly one of them – let $\tau \subset \mathcal{D}(W)$ be the corresponding simplex. If $\dim \sigma = 0$ let $\mathcal{D}(f)(\sigma)$ be an interior point of τ.

One can now show by induction on $\dim \sigma$ that $\mathcal{D}(f)$ extends over all of $\mathcal{D}(Z)$ – so, assume $\dim \sigma > 1$. For each face $\sigma' \subset \sigma$ with corresponding stratum $D(\sigma') \subset Z$, let $D(\tau') \subset W$ be the smallest stratum containing $f(D(\sigma'))$. Now

$$f(D(\sigma)) \subset f(D(\sigma')) \text{ forces } D(\tau) \subset D(\tau')$$

and this gives an inclusion $\iota_{\tau'} : \tau' \to \tau$. By induction a map $\mathcal{D}(f)|_{\sigma'} : \sigma' \to \tau'$ has already been defined, so composing with ι one obtains

$$\sigma' \xrightarrow{\mathcal{D}(f)|_{\sigma'}} \tau' \xrightarrow{\iota} \tau \text{ for each face } \sigma' \subset \sigma$$

which together give a map $\partial\sigma \to \tau$, and as τ is contractible this map must extend over σ.

Uniqueness up to homotopy equivalence follows from Lemma 2.10. □

Lemma 2.10. *If $f, g\colon X \to Y$ are 2 maps of regular Δ-complexes such that for each simplex $\sigma \subseteq X$ there is a unique minimal simplex $\tau_\sigma \subseteq Y$ such that $f(\sigma), g(\sigma) \subseteq \tau_\sigma$ then there is a homotopy $h\colon X \times I \to Y$ from f to g such that $h(\sigma \times I) \subseteq \tau_\sigma$ for each simplex $\sigma \subset X$.*

Proof. We proceed by induction over the skeleta $X^d \subseteq X$. For the case $d = 0$ let $v \in X^0$ be a vertex. By hypothesis there's a unique minimal simplex $\tau_v \subseteq Y$ so that $f(v), g(v) \in \tau_v \subseteq Y$, so we may choose a path $\gamma_v\colon I \to \tau_v \subseteq Y$ with $\gamma_v(0) = f(v), \gamma_v(1) = g(v)$. Then the map

$$h^0\colon X^0 \times I \to Y \text{ defined by } h^0(v,t) = \gamma_v(t)$$

is a homotopy between $f|_{X^0}$ and $g|_{X^0}$ with $h^0(\{v\} \times I) \subseteq \tau_v$ for all v.

Suppose by inductive hypothesis that $d > 0$ and we have constructed a homotopy $h^{d-1}\colon X^{d-1} \times I \to Y$ from $f|_{X^{d-1}}$ to $g|_{X^{d-1}}$ with $h^{d-1}(\sigma \times I) \subseteq \tau_\sigma$ for all simplices $\sigma \subseteq X^{d-1}$. Let $\sigma \subset X$ be a d-simplex, and observe that if $\sigma' \subset \sigma$ is a face then $f(\sigma') \subseteq f(\sigma) \subseteq \tau_\sigma$, and similarly $g(\sigma') \subseteq \tau_\sigma$. By hypothesis this implies $\tau_{\sigma'} \subseteq \tau_\sigma$, so that the homotopy $h^{d-1}|_{\sigma'}\colon \sigma' \times I \to Y$ factors through τ_σ. We conclude that the map $\tilde{\gamma|}_\sigma\colon \sigma \times 0,1 \cup \partial\sigma \to Y$ defined by

$$(x,t) \mapsto \begin{cases} f(x) & \text{if } t = 0, \\ g(x) & \text{if } t = 1, \textit{and} \\ h(x,t), & \text{otherwise} \end{cases}$$

factors through τ_σ; since Y is regular τ_σ is contractible, and so $\tilde{\gamma|}_\sigma$ extends to a morphism $\gamma_\sigma\colon \sigma \times I \to Y$. As σ varies over the d-simplices of X, the γ_σ provide an extension of h^{d-1} to a homotopy

$$h^d\colon X^d \times I \to Y \text{ from } f|_{X^d} \text{ to } g|_{X^d}.$$

□

3. Thrifty morphisms of pairs

Let (S, Δ_S) be a pair (as in Convention 1.6).

Definition 3.1. The **snc locus of** (S, Δ_S) is the largest open $U \subset S$ so that $(U, \Delta_S|_U)$ is a simple normal crossing pair – it will be denoted $\mathrm{snc}(S, \Delta_S)$. We also set

$$\text{non-snc}(S, \Delta_S) := S \setminus \mathrm{snc}(S, \Delta_S) \tag{3.2}$$

Remark 3.3. When S is normal, non-snc(S, Δ_S) has codimension ≥ 2 in S.

In their work on dual complexes of Calabi-Yau pairs, introduced a natural generalization of thrifty resolutions to a class of *thrifty morphisms* where the domain is no longer required to be smooth.

Definition 3.4 ([KX16, Def. 9])**.** A crepant proper birational morphism of log canonical pairs $f\colon (X, \Delta_X) \dashrightarrow (S, \Delta_S)$ is **Kollár-Xu-thrifty** (KX-thrifty for short) if and only if there are closed subsets $Z_X \subset X$, $Z_S \subset S$ of codimension ≥ 1 so that

- Z_X contains no log canonical centers of (X, Δ_X), and similarly for Z_S, and
- f restricts to an isomorphism $X \setminus Z_X \xrightarrow[\simeq]{f} S \setminus Z_S$.

Since rational pairs are not log canonical in general, for example since they are not necessarily $\mathbb{Q}$-Gorenstein[2], we adopt a slightly different definition of thrifty morphisms (see Lemma 3.8 for a comparison).

Let (S, Δ_S) be a pair and let $f : X \to S$ be a proper birational morphism. Set $\Delta_X := f_*^{-1}\Delta_S$ (the strict transform).

Definition 3.5. The morphism f is **thrifty** if and only if

(i) f is an isomorphism *over* the generic point of every stratum of snc(S, Δ_S) and
(ii) f is an isomorphism *at* the generic point of every stratum of snc(X, Δ_X).

If in addition X is smooth and $f^{-1}(\Delta_S) \cup E$ is a simple normal crossing divisor (with E the exceptional locus) then f is called a **thrifty resolution**.

Remark 3.6. Equivalently, if Ex$(f) \subset X$ is the exceptional locus of f, then

(i) $f(\mathrm{Ex}(f))$ contains no stratum of snc(S, Δ_S) and
(ii) Ex(f) contains no stratum of snc(X, Δ_X).

Remark 3.7. Hence when X is smooth and $f^{-1}(\Delta_S) \cup E$ is a simple normal crossing divisor Definition 3.5 reduces to [Kol13, Def. 2.79].

Lemma 3.8. *Let $f\colon (X, \Delta_X) \to (S, \Delta_S)$ be a crepant proper birational morphism between* dlt *pairs. Then f is KX-thrifty (Definition 3.4) if and only if it is thrifty (Definition 3.5).*

Proof. The map f is crepant, so $K_X + \Delta_X \sim_{\mathbb{Q}} f^*(K_S + \Delta_S)$ – equivalently,

$$\Delta_X \sim_{\mathbb{Q}} f_*^{-1}(\Delta_S) - \sum_i a_i E_i$$

where $a_i := a(E_i, S, \Delta_X)$ and the sum runs over all f-exceptional divisors $E_i \subset X$. Writing $\Delta_S = \sum_i c_i D_i$, we see that $\Delta_S^{=1} = \sum_{c_i=1} D_i$ and that $\Delta_X^{=1} = \sum_{c_i=1} f_*^{-1} D_i + \sum_{a_i=-1} E_i$. Both pairs are dlt, so the log canonical centers of (X, Δ_X) are the the strata of the expected-dimensional crossing scheme $\Delta_X^{=1}$, and their generic points lie in snc(X, Δ_X) – similarly for (S, Δ_S) [Fuj07]. Moreover, if $a_i = -1$ then $f(E_i) \subset S$ is a log canonical center, so it must be a stratum of $\Delta_S^{=1}$.

Suppose f is KX-thrifty and let $Z_X \subset X$, $Z_S \subset S$ be closed sets as guaranteed in Definition 3.4. Then f is an isomorphism over $S \setminus Z_S$ and Z_S contains no stratum of $\Delta_S^{=1}$, giving condition (i) of Definition 3.5. Also, we must have $a_i > -1$ for all i, and so $\Delta_X^{=1} = \sum_{c_i=1} f_*^{-1} D_i = f_*^{-1}\Delta_S^{=1}$. Since Z_X contains no stratum of $\Delta_X^{=1}$, we obtain (ii) of Definition 3.5. ◻

In the next lemma we use a definition of a birational map general enough to encompass reducible schemes [Sta19, Tags 0A20, 0BX9]: a rational map $f\colon X \dashrightarrow Y$ between schemes with finitely many irreducible components is *birational* if and only if it is an isomorphism in the category with

[2]The cone over $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^{mn+m+n}$ embedded using the complete linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(m, n)|$ is rational for all $m, n > 0$, $\mathbb{Q}$-Gorenstein if and only if $m = n$.

- objects the schemes with finitely many irreducible components, and with
- morphisms the dominant rational maps between them.

When Y is locally of finite presentation over a field (as it will be in all cases considered here), the map f is birational if and only if it induces a bijection between the generic points of irreducible components of X and Y, and for each generic point of an irreducible component $\eta \in X$ the induced morphism $\mathcal{O}_{Y,f(\eta)} \to \mathcal{O}_{X,\eta}$ is an isomorphism.

Lemma 3.9. *Let $Z = \cup_{i=1}^N Z$ and $W = \cup_{j=1}^N W_j$ be expected-dimensional crossing schemes and let $f : Z \dashrightarrow W$ be a birational map defined at the generic point of each stratum of Z.*

(i) If f is an isomorphism at the generic point of every stratum $D(\sigma) \subset Z$, then $\mathcal{D}(f)$ can be realized as a subcomplex inclusion.

(ii) If f is an isomorphism over the generic point of every stratum $D(\tau) \subset W$ then it is an isomorphism at the generic point of every stratum of Z, and $\mathcal{D}(f)$ can be realized as an isomorphism of Δ-complexes.

Proof. In the case of (i), as f is birational it induces a bijection between the generic points of Z and W and hence a bijection on 0-skeleta

$$\mathcal{D}(f)_0 : \mathcal{D}(Z)_0 \xrightarrow{\simeq} \mathcal{D}(W)_0$$

Without loss of generality we may assume f restricts to a birational maps $f_i : Z_i \dashrightarrow W_i$ for $i = 1, \dots, N$. Let $n = \dim Z = \dim W$.

Let $\sigma \subset \mathcal{D}(Z)$ be a simplex with corresponding stratum $D(\sigma) \subset Z$ – without loss of generality we may assume $D(\sigma) \subset Z_1$, and that $D(\sigma) \subseteq \cap_{j=1}^r Z_j$. Letting $\eta_\sigma \in D(\sigma)$ be the generic point, we see that $f(\eta_\sigma) \subset \cap_{j=1}^r W_j$. Because f is an isomorphism at η_σ, it must be that $f(\eta_\sigma)$ is a generic point of a component $D(\tau) \subseteq \cap_{j=1}^r W_j$ corresponding to a simplex $\tau \subseteq \mathcal{D}(W)$. Let $\eta_\tau \in D(\tau)$ be the generic point; we have $\eta_\tau = f(\eta_\sigma)$.

At this point the only concern is that there could be another $r-1$-simplex σ' such that $\mathcal{D}(f)(\sigma') = \tau$; any such σ' would correspond to another stratum $D(\sigma') \subseteq \cap_{j=1}^r Z_j$, hence another point $\eta_{\sigma'} \in Z_1$ of dimension $r-1$ with $f(\eta'_\sigma) = f(\eta_\tau)$. One can show this is impossible, using the normality of W_1 and Zariski's main theorem as follows.

The map f is an isomorphism at the generic point $n_\sigma \in D(\sigma)$, so its restriction $f|_{Z_1} \colon Z_1 \to W_1$ is also an isomorphism at n_σ. The scheme W_1 is normal and $f|_{Z_1}$ is birational by hypothesis, so by Zariski's main theorem [Sta19, Tag 05K0] $f|_{Z_1}$ is in fact an isomorphism *over* η_τ.

For (ii), observe that $f^{-1} \colon W \dashrightarrow Z$ satisfies the hypotheses of (i) and hence both $\mathcal{D}(f) \colon \mathcal{D}(Z) \to \mathcal{D}(W)$ and $\mathcal{D}(f^{-1}) \colon \mathcal{D}(W) \to \mathcal{D}(W)$ may be realized as subcomplex inclusions; from the proof of (i), this can be done in such a way that $\mathcal{D}(f) \circ \mathcal{D}(f^{-1}) = \mathrm{id}_{\mathcal{D}(W)}$. In particular this implies $\mathcal{D}(f)$ is a surjective subcomplex inclusion, hence an isomorphism. □

Corollary 3.10. *Let (S, Δ_S) be a pair and let $f : X \to S$ be a proper birational morphism and set $\Delta_X := f_*^{-1}\Delta_S$. Then f induces morphisms of Δ-complexes*

$$\mathcal{D}(\operatorname{snc} \Delta_X) \xrightarrow{\mathcal{D}(f|_\Delta)} \mathcal{D}(\operatorname{snc} \Delta_S) \text{ and } \mathcal{D}(\operatorname{snc}(X, \Delta_X)) \xrightarrow{\mathcal{D}(f)} \mathcal{D}(\operatorname{snc}(S, \Delta_S))$$

which are isomorphisms *if f is thrifty.*

Proof. The induced morphisms come from Lemma 2.9; to see that they are isomorphisms when f is thrifty we may apply Definition 3.5 and Lemma 3.9. □

3.1. **Thrifty proper birational equivalences.** If S is a separated scheme of finite type over k and $f : X \to S$, $g : Y \to S$ are separated schemes of finite type over S, a **proper birational equivalence**

of X, Y **over** S is a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "r"'] \arrow[dr, "s"] & \\ X \arrow[dr, "f"'] & \circlearrowleft & Y \arrow[dl, "g"] \\ & S & \end{tikzcd} \tag{3.11}$$

where r, s are proper birational morphisms.

Definition 3.12. Suppose (X, Δ_X), (Y, Δ_Y) are pairs over S, with X and Y normal and Δ_X, Δ_Y reduced and effective. A **thrifty proper birational equivalence of** (X, Δ_X) **and** (Y, Δ_Y) **over** S is a proper birational equivalence as in diagram 3.11, where r and s are thrifty.

Remark 3.13. By Corollary 3.10, a thrifty proper birational equivalence $X \xleftarrow{r} Z \xrightarrow{s} Y$ between (X, Δ_X) and (Y, Δ_Y) induces an isomorphism $\mathcal{D}(\Delta_X) \simeq \mathcal{D}(\Delta_Y)$.

Proposition 3.14. *Let (S, Δ_S) be a pair with Δ_S reduced and effective, and let $f : X \to S$, $g : Y \to S$ be 2 thrifty resolutions of (S, Δ_S). Then there is a thrifty proper birational equivalence of X and Y over S.*

Proof. Let $U \subset S$ be an open set such that both f and g are isomorphisms over U; then we have an isomorphism

$$g^{-1} \circ f : f^{-1}(U) \to g^{-1}(U)$$

Set

$$Z := \overline{\Gamma_{g^{-1} \circ f}} \subset X \times_S Y$$

and let $p : Z \to X$, $s : Z \to Y$ be the projections. The claim is that $X \xleftarrow{r} Z \xrightarrow{s} Y$ is a thrifty proper birational equivalence over S. It is birational by design, and proper since X, Y and hence $X \times_Y Z$ are proper over S and Z is closed in $X \times_S Y$. It remains to show that r, s are thrifty.

Lemma 3.15. *Let* $\mathrm{Ex}(r), \mathrm{Ex}(s) \subset Z$ *be the exceptional loci of r, s respectively; let* $\mathrm{Ex}(f) \subset X$, $\mathrm{Ex}(g) \subset Y$ *be the exceptional loci of f and g. Then*

$$r(\mathrm{Ex}(r)) \subset f^{-1}(g(\mathrm{Ex}(g))) \text{ and } s(\mathrm{Ex}(s)) \subset g^{-1}(f(\mathrm{Ex}(f)))$$

Proof of Lemma 3.15. Let $U \subset S$ and $V \subset Y$ be a maximal pair of open sets such that $g|_V : V \xrightarrow{\simeq} U$ is an isomorphism; note that since g is an honest morphism $\mathrm{Ex}(g) = Y \setminus V$ and $g(\mathrm{Ex}(g)) = S \setminus U$. Then $W := f^{-1}(U) \subset X$ is an open set such that $g^{-1} \circ f : X \dashrightarrow Y$ is defined on W. This implies the projection $\Gamma_{g^{-1} \circ f} \xrightarrow{r} X$ is an isomorphism over W, but what we need to know is that the same is true for $Z = \bar{\Gamma}_{g^{-1} \circ f} \xrightarrow{r} X$. For this, note that

$$\bar{\Gamma}_{g^{-1} \circ f} \cap r^{-1}(W) = \overline{\Gamma_{g^{-1} \circ f} \cap r^{-1}(W)} = \overline{\Gamma_{g^{-1} \circ f|_W}} \subset W \times_S Y$$

Since W and Y are both separated over S, the graph $\Gamma_{g^{-1} \circ f|_W}$ is already closed, so we conclude $\bar{\Gamma}_{g^{-1} \circ f} \cap r^{-1}(W) = \Gamma_{g^{-1} \circ f|_W}$. □

Now suppose $W \subset X$ is a stratum of (X, Δ_X) – we must show r is an isomorphism over the generic point $\eta \in W$. First, f is an isomorphism at η by hypothesis, and so by the proof of Lemma 3.9, $f(\eta)$ is the generic point of a stratum of $\mathrm{snc}(S, \Delta_S)$. Then g is an isomorphism over $f(\eta)$ by hypothesis, so in particular $f(\eta) \notin g(\mathrm{Ex}(g))$. By Lemma 3.15 we conclude that $\eta \notin r(\mathrm{Ex}(r))$, as desired.

Finally we show that s is an isomorphism at the generic point of every stratum of $\Delta_Z := r_*^{-1} f_*^{-1} \Delta_S$, using a more general lemma:

Lemma 3.16. *Let $r\colon (Z,\Delta_Z) \to (X,\Delta_X)$ be a proper birational morphism. If (X,Δ_X) is a simple normal crossing pair, then r is thrifty if and only if it satisfies condition (i) of Definition 3.5. Explicitly, r is thrifty if and only if it is an isomorphism over every stratum of Δ_X.*

Proof of Lemma 3.16. In this situation there is an honest morphism $\mathrm{snc}(\Delta_Z) \to \Delta_X$, so the hypotheses of Lemma 3.9 are satisfied. We then apply Lemma 3.9 (ii). □

□

Remark 3.17. In the case where the morphism $r\colon Z \to X$ of Lemma 3.16 is projective, [Har77, Thm. 7.17] implies that r is the blowup of some sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$ such that $V(\mathcal{I}) \subset X$ contains no stratum of Δ_X. If in addition $V(\mathcal{I})$ has *simple normal crossings* with Δ_X [Kol07, Def. 3.24], Lemma 3.16 can be obtained from known results on the effect of blowing up on dual complexes [Ste06, §2], [dFKX14, §9], [Wlo16, Prop. 2.1.6].

4. Structure sheaves of strata and their direct images

In this section we prove weak functoriality statements about the quasi-isomorphisms in Theorem 1.1, or alternatively those of [Kov19].

Lemma 4.1. *Let S be scheme over a field k and let $f\colon X \to S$, $g\colon Y \to S$ are S-schemes that are smooth over k. Suppose $X \xleftarrow{r} Z \xrightarrow{s} Y$ is a proper birational equivalence over S such that both r and s are projective. Let $\mathcal{C}(Z)$ denote the category with objects the pairs $(E \subseteq X, F \subseteq Y)$ of smooth closed subschemes of X and Y such that*

(i) r and s are isomorphisms over the generic points of E and F respectively, and

(ii) the birational map $s \circ r^{-1}\colon X \dashrightarrow Y$ sends the generic point of E to the generic point of F,

and with morphisms $(E_1,F_1) \to (E_2,F_2)$ given by inclusions $E_1 \subseteq E_2$, $F_1 \subseteq F_2$. If $\mathcal{K} \subset \mathcal{C}(Z)$ is a finite subcategory, then there are proper birational equivalences $E \xleftarrow{r'} W \xrightarrow{s'} F$ compatible with Z in the sense that

$$\begin{tikzcd}
E \arrow[d] & W \arrow[l, "r'"'] \arrow[r, "s'"] \arrow[d, "k"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & F \arrow[d] \\
X & Z \arrow[l, "r"] \arrow[r, "s"] & Y
\end{tikzcd} \tag{4.2}$$

commutes, and commutative diagrams

$$\begin{tikzcd}
Rf_*\mathcal{O}_X \arrow[r, "\gamma_{X,Y}", "\simeq"'] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Rf_*\mathcal{O}_Y \arrow[d] \\
Rf_*\mathcal{O}_E \arrow[r, "\simeq", "\gamma_{E,F}"'] & Rf_*\mathcal{O}_F
\end{tikzcd} \quad \text{in } D^+(S). \tag{4.3}$$

defining a natural transformation of functors $\mathcal{K}^{\mathrm{op}} \to D^+(S)$.

Proof. We proceed by descending induction over the poset $\mathcal{K}$.

For each object $(E,F) \in \mathrm{Ob}(\mathcal{K})$, since X is smooth and r is an isomorphism over the generic point $\xi \in E$ we see that if $\tilde{E} \subseteq Z$ is the strict transform of E then $\tilde{E} \not\subset \mathrm{Sing}(Z)$, so in particular if non-CM$(Z) \subseteq Z$ is the non-Cohen-Macaulay locus then $\tilde{E} \not\subset$ non-CM(Z) – similarly for F. By a theorem of Česnavičius, there exists *Macaulayfication* $\pi\colon \tilde{Z} \to Z$ such that π is an isomorphism over $Z \setminus$ non-CM(Z) – explicitly, $\tilde{Z}$ is Cohen-Macaulay and π is a projective birational morphism [Ces18, Thm. 1.6] (see also [Kaw00, Thm. 5.1]). It follows that $r \circ \pi$ and $s \circ \pi$ are projective and isomorphisms over the generic points of E and F respectively, for all $(E,F) \in \mathrm{Ob}(\mathcal{K})$. From now on we may assume Z is Cohen-Macaulay.

Now suppose $(E, F) \in \mathrm{Ob}(\mathcal{K})$ is *maximal* (categorically final), and let $W \subseteq Z \times_{X\times Y} E \times F$ be the component dominating E and F, and form the commutative diagram of S-schemes

$$\begin{tikzcd} & W \arrow[rr, "k"] \arrow[dr, "s'"] \arrow[ddl, "r'"'] & & Z \arrow[dr, "s"] \arrow[ddl, "r"] & \\ & & F \arrow[rr, "\jmath"] & & Y \\ E \arrow[rr, "\imath"] & & X & & \end{tikzcd} \tag{4.4}$$

Replacing W with a Macaulayfication $\pi' \colon \tilde{W} \to W$ if necessary, we may assume W is Cohen-Macaulay. Now by functoriality we have commutative diagrams

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "\imath^\sharp"] \arrow[d, "r^\sharp"', "\simeq"] \arrow[dr, phantom, "\circlearrowleft"] & R\imath_*\mathcal{O}_E \arrow[d, "\simeq"', "R\imath_* r'^\sharp"] \\ Rr_*\mathcal{O}_Z \arrow[r, "Rr_* k^\sharp"] & R(r\circ k)_*\mathcal{O}_W = R(\imath \circ r')_*\mathcal{O}_W \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{O}_Y \arrow[r, "\jmath^\sharp"] \arrow[d, "s^\sharp"', "\simeq"] \arrow[dr, phantom, "\circlearrowleft"] & R\jmath_*\mathcal{O}_F \arrow[d, "\simeq"', "R\jmath_* s'^\sharp"] \\ Rs_*\mathcal{O}_Z \arrow[r, "Rs_* k^\sharp"] & R(s\circ k)_*\mathcal{O}_W = R(\jmath \circ s')_*\mathcal{O}_W \end{tikzcd} \tag{4.5}$$

in $D(X)$ and $D(Y)$ respectively. The vertical arrows are isomorphisms since X, Y, E and F are all smooth, so in particular they have rational singularities, and W and Z are Cohen-Macaulay, so we may apply [Kov19, Thm. 8.6]. Finally, pushing forward along f and g we obtain

$$\begin{tikzcd} Rf_*\mathcal{O}_X \arrow[r, "Rf_*\imath^\sharp"] \arrow[d, "Rf_* r^\sharp"', "\simeq"] \arrow[dr, phantom, "\circlearrowleft"] & R(f\circ\imath)_*\mathcal{O}_E \arrow[d, "\simeq"', "R(f\circ\imath)_* r'^\sharp"] \\ R(f\circ r)_*\mathcal{O}_Z \arrow[r, "R(f\circ r)_* k^\sharp"] \arrow[d, equal] & R(f\circ r\circ k)_*\mathcal{O}_W = R(f\circ\imath\circ r')_*\mathcal{O}_W \arrow[d, equal] \\ R(g\circ s)_*\mathcal{O}_Z \arrow[r, "R(g\circ s)_* k^\sharp"] \arrow[dr, phantom, "\circlearrowleft"] & R(g\circ s\circ k)_*\mathcal{O}_W = R(g\circ \jmath\circ s')_*\mathcal{O}_W \\ Rg_*\mathcal{O}_Y \arrow[r, "Rg_*\jmath^\sharp"] \arrow[u, "\simeq", "Rg_* s^\sharp"'] & R(g\circ\jmath)_*\mathcal{O}_F \arrow[u, "R(g\circ\jmath)_* s'^\sharp", "\simeq"'] \end{tikzcd} \tag{4.6}$$

For the inductive step, suppose $(E, F) \in \mathrm{Ob}(\mathcal{K})$ and let $\alpha_i \colon (E, F) \to (E_i, F_i)$, $i = 1, \ldots, r$ be the morphisms in $\mathcal{K}$ with source (E, F). By inductive hypothesis, for each i we have a Cohen-Macaulay S-scheme W_i and a projective birational equivalence $E_i \xleftarrow{r_i} W_i \xrightarrow{s_i} F_i$ inducing a morphism $\gamma_{E_i,F_i} \colon Rf_*\mathcal{O}_{E_i} \to Rg_*\mathcal{O}_{F_i}$ – using the above construction, we can ensure that for any $\mathcal{K}$-morphism $(E', F') \to (E_i, F_i)$ the map r_i is an isomorphism over E', and similarly for F_i. Consider the cartesian diagram

$$\begin{tikzcd} W \arrow[r, hook] \arrow[dr, "r'\times_S s'"'] & (E\times_S F) \times_{\prod_{S,i=1}^r (E_i\times_S F_i)} \prod_{S,i=1}^r W_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_{S,i=1}^r W_i \arrow[d, "\prod_{S,i}(r_i\times_S s_i)"] \\ & E\times_S F \arrow[r] & \prod_{S,i=1}^r (E_i \times_S F_i) \end{tikzcd} \tag{4.7}$$

and let $W \subseteq (E \times_S F) \times_{\prod_{S,i=1}^r (E_i\times F_i)} \prod_{S,i=1}^r W_i$ be the component dominating E and F. Note that r', s' are projective since $\prod_{S,i}(r_i \times_S s_i)$ is projective by hypothesis. As above, we may replace W by a projective Macaulayfication while retaining the property that r', s' are isomorphisms over the generic points of $E' \subset E$, $F' \subset F$ for every $\mathcal{K}$-morphism $(E', F') \to (E, F)$. Now by design for each i there is a commutative diagram

$$\begin{tikzcd}
& W \arrow[rr, "k"] \arrow[dl, "r'"'] \arrow[dr, "s'"] & & W_i \arrow[dr, "s"] \arrow[dl, "r"] & \\
E \arrow[rr, "\iota"'] & & E_i & F \arrow[r, "\jmath"] & F_i
\end{tikzcd}
\tag{4.8}$$

and arguing as in the base case we obtain from (4.8) a commutative diagram in $D^+(S)$ of the form

$$\begin{tikzcd}
Rf_*\mathcal{O}_{E_i} \arrow[r, "\gamma_{E_i,F_i}", "\simeq"'] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Rf_*\mathcal{O}_{F_i} \arrow[d] \\
Rf_*\mathcal{O}_{E} \arrow[r, "\simeq", "\gamma_{E,F}"'] & Rf_*\mathcal{O}_{F}
\end{tikzcd}
\tag{4.9}$$

□

Corollary 4.10. *Let S be a scheme over a field k and let (X, Δ_X) and (Y, Δ_Y) be simple normal crossing pairs over k with morphisms $f\colon X \to S$ and $g\colon Y \to S$. Suppose $X \xleftarrow{r} Z \xrightarrow{s} Y$ is a thrifty proper birational equivalence over S such that both r and s are projective. Let $\mathcal{D}$ be the common dual complex of Δ_X and Δ_Y (see Remark 3.13) and for a simplex $\sigma \subseteq \mathcal{D}$ let $D_X(\sigma) \subseteq X$, $D_Y(\sigma) \subseteq Y$ be the corresponding strata. In this situation there is a natural transformation of functors* $\mathrm{Face}(\mathcal{D}) \to D^+(S)$ *from $Rf_*\mathcal{O}_{D_X(\sigma)}$ to $Rg_*\mathcal{O}_{D_Y(\sigma)}$, compatible with restrictions from $Rf_*\mathcal{O}_X$ and $Rg_*\mathcal{O}_Y$, and hence a commutative diagram in $D^+(S)$ of the form*

$$\begin{tikzcd}
Rf_*\mathcal{O}_X \arrow[r] \arrow[d] & \prod_{\sigma\in\mathcal{D}^0} Rf_*\mathcal{O}_{D_X(\sigma)} \arrow[r] \arrow[d, "\gamma^0"] & \prod_{\sigma\in\mathcal{D}^1} Rf_*\mathcal{O}_{D_X(\sigma)} \arrow[r] \arrow[d, "\gamma^1"] & \prod_{\sigma\in\mathcal{D}^2} Rf_*\mathcal{O}_{D_X(\sigma)} \arrow[r] \arrow[d, "\gamma^2"] & \cdots \\
Rg_*\mathcal{O}_Y \arrow[r] & \prod_{\sigma\in\mathcal{D}^0} Rg_*\mathcal{O}_{D_Y(\sigma)} \arrow[r] & \prod_{\sigma\in\mathcal{D}^1} Rg_*\mathcal{O}_{D_Y(\sigma)} \arrow[r] & \prod_{\sigma\in\mathcal{D}^2} Rg_*\mathcal{O}_{D_Y(\sigma)} \arrow[r] & \cdots
\end{tikzcd}
\tag{4.11}$$

Proof. We apply Lemma 4.1 to the finite subcategory $\mathcal{K} \subset \mathcal{C}(Z)$ with objects $(D_X(\sigma), D_Y(\sigma))$ for $\sigma \subseteq \mathcal{D}$. Evidently, this $\mathcal{K}$ is equivalent to $\mathrm{Face}(\mathcal{D})$. □

5. A morphism of restriction triangles

The main result of this section is

Lemma 5.1. *Let S be a base scheme over a field k and let and let (X, Δ_X) and (Y, Δ_Y) be simple normal crossing schemes over k with morphisms $f\colon X \to S$, $f\colon Y \to S$. If $X \xleftarrow{r} Z \xrightarrow{s} Y$ is a thrifty proper birational equivalence over S then there is an isomorphism of distinguished triangles*

$$\begin{tikzcd}
Rf_*\mathcal{O}_X(-\Delta_X) \arrow[r] \arrow[d, "\gamma'"] & Rf_*\mathcal{O}_X \arrow[r] \arrow[d, "\gamma"] & Rf_*\mathcal{O}_{\Delta_X} \arrow[r, "+1"] \arrow[d, "\gamma''"] & \cdots \\
Rg_*\mathcal{O}_Y(-\Delta_Y) \arrow[r] & Rg_*\mathcal{O}_Y \arrow[r] & Rg_*\mathcal{O}_{\Delta_Y} \arrow[r, "+1"] & \cdots
\end{tikzcd}
\quad \textit{in } D^+(S).
\tag{5.2}$$

For the most part, this consists of combining Corollaries 2.7 and 4.10 to obtain the isomorphisms γ and γ'' – after that, the existence of γ' is guaranteed since $D^+(S)$ is triangulated, and the fact that γ' is an isomorphism follows from the 5-lemma.

Proof. Let $n = \dim \mathcal{D}(\Delta_X) = \dim \mathcal{D}(\Delta_Y)$. By Corollary 2.7 there are quasi-isomorphisms $\mathcal{O}_X(-\Delta_X) \simeq \check{C}(X, \Delta_X, \mathcal{O})$ in $D^+(X)$ and $\mathcal{O}_Y(-\Delta_Y) \simeq \check{C}(Y, \Delta_Y, \mathcal{O})$ in $D^+(Y)$. For each $d = 0, \dots, n$, we have a truncated complex

$$\tau_{\leq d}\check{C}(X, \Delta_X, \mathcal{O}) := [\mathcal{O}_X \xrightarrow{d_X^0} \prod_{\sigma \in \mathcal{D}(\Delta_X)^0} \mathcal{O}_{D(\sigma)} \xrightarrow{d_X^1} \prod_{\sigma \in \mathcal{D}(\Delta_X)^1} \mathcal{O}_{D(\sigma)} \xrightarrow{d_X^2} \cdots \to \prod_{\sigma \in \mathcal{D}(\Delta_X)^{d-1}} \mathcal{O}_{D(\sigma)}]$$

and these truncations are related via distinguished triangles

$$\prod_{\sigma \in \mathcal{D}(\Delta_X)^{d-1}} \mathcal{O}_{D(\sigma)}[-d] \to \tau_{\leq d}\check{C}(\Delta_X, \mathcal{O}) \xrightarrow{\rho_d} \tau_{\leq d-1}\check{C}(\Delta_X, \mathcal{O}) \xrightarrow[\delta]{[+1]} \cdots \text{ for } d > 0;$$

similarly for (Y, Δ_Y). Shifting by $[d]$, rotating and pushing forward along f and g, we obtain distinguished triangles in $D^+(S)$ on the horizontal rows of

$$\begin{tikzcd}
Rf_*\tau_{\leq d-1}\check{C}(\Delta_X, \mathcal{O})[d-1] \arrow[r, "\delta_X"] \arrow[d, dashed, "\beta_{d-1}[d-1]"] \arrow[dr, phantom, "(1)"] & \prod_{\sigma \in \mathcal{D}(\Delta_X)^{d-1}} Rf_*\mathcal{O}_{D(\sigma)} \arrow[r, "\iota_X"] \arrow[d, "\gamma_d"] & Rf_*\tau_{\leq d}\check{C}(\Delta_X, \mathcal{O})[d] \arrow[r, "{[+1]}", "\rho_d"'] \arrow[d, dashed, "\beta_d[d]"] & \cdots \\
Rg_*\tau_{\leq d-1}\check{C}(\Delta_Y, \mathcal{O})[d-1] \arrow[r, "\delta_Y"] & \prod_{\sigma \in \mathcal{D}(\Delta_Y)^{d-1}} Rg_*\mathcal{O}_{D(\sigma)} \arrow[r, "\iota_Y"] & Rg_*\tau_{\leq d}\check{C}(\Delta_Y, \mathcal{O})[d] \arrow[r, "{[+1]}", "\rho_d"'] & \cdots
\end{tikzcd} \tag{5.3}$$

We will prove by induction on d that there are morphisms $\beta_d \colon Rf_*\tau_{\leq d}\check{C}(\Delta_X, \mathcal{O}) \to Rg_*\tau_{\leq d}\check{C}(\Delta_Y, \mathcal{O})$, $d = 0, \dots, n$ as indicated in the left and right vertical arrows of (5.3), such that (5.3) is a morphism of distinguished triangles. In the base case $d = 0$, we have $\tau_{\leq 0}\check{C}(\Delta_X, \mathcal{O}) = \mathcal{O}_X$ and $\tau_{\leq 0}\check{C}(\Delta_Y, \mathcal{O}) = \mathcal{O}_Y$ and so a morphism $\beta_0 \colon Rf_*\mathcal{O}_X \to Rg_*\mathcal{O}_Y$ is constructed in Theorem 1.1.

Now suppose $d > 0$ and assume by inductive hypothesis that for $e < d$ there are morphisms $\beta_e \colon Rf_*\tau_{\leq e}\check{C}(\Delta_X, \mathcal{O}) \to Rg_*\tau_{\leq e}\check{C}(\Delta_Y, \mathcal{O})$ appearing in morphisms of distinguished triangles of the form (5.3). To obtain a morphism $\beta_d \colon Rf_*\tau_{\leq d}\check{C}(\Delta_X, \mathcal{O}) \to Rg_*\tau_{\leq d}\check{C}(\Delta_Y, \mathcal{O})$, we show that the square (1) of (5.3) necessarily commutes, which is to say

$$\gamma_d \circ \delta_X - \delta_Y \circ \beta_{d-1}[d-1] = 0 \in \operatorname{Hom}_{D^+(S)}(Rf_*\tau_{\leq d-1}\check{C}(\Delta_X, \mathcal{O})[d-1], \prod_{\sigma \in \mathcal{D}(\Delta_Y)^{d-1}} Rg_*\mathcal{O}_{D(\sigma)}).$$

Consider now

$$\begin{tikzcd}
\prod_{\sigma \in \mathcal{D}(\Delta_X)^{d-2}} Rf_*\mathcal{O}_{D(\sigma)} \arrow[r, "\iota_X"] \arrow[d, "\gamma_{d-1}"] \arrow[dr, phantom, "(2)"] \arrow[rr, bend left, "d_X^{d-1}"] & Rf_*\tau_{\leq d-1}\check{C}(\Delta_X, \mathcal{O})[d-1] \arrow[r, "\delta_X"] \arrow[d, "\beta_{d-1}[d-1]"] \arrow[dr, phantom, "(1)"] & \prod_{\sigma \in \mathcal{D}(\Delta_X)^{d-1}} Rf_*\mathcal{O}_{D(\sigma)} \arrow[d, "\gamma_d"] \\
\prod_{\sigma \in \mathcal{D}(\Delta_X)^{d-2}} Rf_*\mathcal{O}_{D(\sigma)} \arrow[r, "\iota_Y"] \arrow[rr, bend right, "d_Y^{d-1}"'] & Rg_*\tau_{\leq d-1}\check{C}(\Delta_Y, \mathcal{O})[d-1] \arrow[r, "\delta_Y"] & \prod_{\sigma \in \mathcal{D}(\Delta_Y)^{d-1}} Rg_*\mathcal{O}_{D(\sigma)}
\end{tikzcd} \tag{5.4}$$

By inductive hypothesis, the left square (2) commute. Suppose for a moment that we can verify $\delta_X \circ \imath_X = d_X^{d-1}$, $\delta_Y \circ \imath_Y = d_Y^{d-1}$ as illustrated in (5.4), in which case Corollary 4.10 shows that the outer rectangle commutes.

Notation 5.5. For any 2 objects $\mathcal{F}, \mathcal{G}$ in $D^+(S)$, $[\mathcal{F}, \mathcal{G}] := \mathrm{Hom}_{D^+(S)}(\mathcal{F}, \mathcal{G})$.

Applying the exact functor $[-, \prod_{\sigma \in \mathcal{D}(\Delta_Y)^{d-1}} Rg_*\mathcal{O}_{D(\sigma)}]$ to the distinguished triangle

$$\prod_{\sigma \in \mathcal{D}(\Delta_X)^{d-2}} Rf_*\mathcal{O}_{D(\sigma)} \xrightarrow{\imath_X} Rf_*\tau_{\leq d-1}\check{C}(\Delta_X, \mathcal{O})[d-1] \xrightarrow{\rho_d[d-1]} Rf_*\tau_{\leq d-2}\check{C}(\Delta_X, \mathcal{O})[d-1]$$

we obtain an exact sequence

$$\begin{aligned}&[Rf_*\tau_{\leq d-2}\check{C}(\Delta_X, \mathcal{O})[d-1], \prod_{\sigma \in \mathcal{D}(\Delta_Y)^{d-1}} Rg_*\mathcal{O}_{D(\sigma)}] \xrightarrow{\rho_d[d-1]^*} [Rf_*\tau_{\leq d-1}\check{C}(\Delta_X, \mathcal{O})[d-1], \prod_{\sigma \in \mathcal{D}(\Delta_Y)^{d-1}} Rg_*\mathcal{O}_{D(\sigma)}] \\ &\xrightarrow{\imath_X^*} [\prod_{\sigma \in \mathcal{D}(\Delta_X)^{d-2}} Rf_*\mathcal{O}_{D(\sigma)}, \prod_{\sigma \in \mathcal{D}(\Delta_Y)^{d-1}} Rg_*\mathcal{O}_{D(\sigma)}]\end{aligned} \tag{5.6}$$

Our temporary hypothesis that $\delta_X \circ \imath_X = d_X^{d-1}$, $\delta_Y \circ \imath_Y = d_Y^{d-1}$ implies that

$$\begin{aligned}\imath_X^*(\gamma_d \circ \delta_X - \delta_Y \circ \beta_{d-1}[d-1]) &= \gamma_d \circ \delta_X \circ \imath_X - \delta_Y \circ \beta_{d-1}[d-1] \circ \imath_X \\ &= \gamma_d \circ d_X^{d-1} - \delta_Y \circ \imath_Y \circ \gamma_{d-1} \\ &= \gamma_d \circ d_X^{d-1} - d_Y^{d-1} \circ \gamma_{d-1} \\ &= 0 \text{ by Corollary 4.10}\end{aligned} \tag{5.7}$$

in which case $\gamma_d \circ \delta_X - \delta_Y \circ \beta_{d-1}[d-1] \in \mathrm{im}\, \rho_d[d-1]^*$. □

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