

# HIGHER DIRECT IMAGES OF LOGARITHMIC STRUCTURE SHEAVES

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## CONTENTS

|  |    |
|--|----|
| 1. Introduction  | 1  |
| 1.1. Acknowledgements                                  | 3  |
| 2. Dual complexes                                      | 3  |
| 2.1. Morphisms of Dual Complexes                       | 5  |
| 3. Thrifty morphisms of pairs                          | 6  |
| 3.1. Thrifty proper birational equivalences            | 8  |
| 4. Structure sheaves of strata and their direct images | 9  |
| 5. A morphism of restriction triangles                 | 12 |
| References   | 12 |

## 1. INTRODUCTION

In [CR11] Chatzistamatiou and Rülling prove the following theorem:

**Theorem 1.1** ([CR11, Thm. 3.2.8]). *Let  $k$  be a perfect field and let  $S$  be a separated scheme of finite type over  $k$ . Suppose  $X$  and  $Y$  are two separated finite type  $S$ -schemes which are*

- (i) *smooth over  $k$  and*
- (ii) ***properly birational** over  $S$  in the sense that there is a commutative diagram*

$$\begin{array}{ccccc}
 & & Z & & \\
 & \swarrow r & & \searrow s & \\
 X & & & & Y \\
 & \searrow f & \circlearrowleft & \swarrow g & \\
 & & S & & 
 \end{array} \tag{1.2}$$

*with  $r$  and  $s$  proper birational morphisms.*

*Set  $n = \dim X = \dim Y = \dim Z$ . Then there are natural morphisms of sheaves*

$$cl_Z^i : R^j f_* \Omega_X^i \rightarrow R^j g_* \Omega_Y^i \text{ for all } i, \tag{1.3}$$

*which are isomorphisms if  $i = 0, n$ .*

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*Date:* October 5, 2020.

The author was partially supported by the University of Washington Department of Mathematics Graduate Research Fellowship, and by the NSF grant DMS-1440140, administered by the Mathematical Sciences Research Institute, while in residence at MSRI during the program Birational Geometry and Moduli Spaces.

In the special case  $\text{char } k = 0$  this is a consequence of Hironaka's resolution of singularities [Hir64]. Analysis of the proof shows that the morphisms of 1.3 are obtained from morphisms of complexes

$$\text{cl}_Z : Rf_*\Omega_X^i \rightarrow Rg_*\Omega_Y^i \text{ for all } i,$$

(for the cases  $i = 0, n$  this is observed in [CR12; Kov19]).

One of the primary applications of Theorem 1.1 was to extend foundational results on rational singularities from characteristic zero to arbitrary characteristic.

**Definition 1.4** ([Kol13, Def. 2.76]). Let  $S$  be a reduced, separated scheme of finite type over a field  $k$ . A **rational resolution**  $f : X \rightarrow S$  is a proper birational morphism such that

- (i)  $X$  is smooth over  $k$ ,
- (ii)  $\mathcal{O}_S = Rf_*\mathcal{O}_X$  and
- (iii)  $R^if_*\omega_X = 0$  for  $i > 0$ .

The scheme  $S$  is said to have **rational singularities** if and only if it has a rational resolution.

**Corollary 1.5** ([CR11, Cor. 3.2.10]). *If  $S$  has a rational resolution, then every resolution of  $S$  is rational. In particular if  $S$  is smooth then it has rational singularities.*

This article concerns analogues of Theorem 1.1 for pairs.

**Convention 1.6.** In what follows a **pair**  $(X, \Delta_X)$  will mean a reduced, equidimensional separated scheme  $X$  of finite type over  $k$  together with a reduced, effective divisor  $\Delta_X$  on  $X$ . A pair  $(X, \Delta_X)$  will be called a **simple normal crossing (snc) pair** if and only if  $X$  is smooth and  $\Delta_X$  is a simple normal crossing divisor on  $X$ .

As observed in [Kol13, §2.5], to generalize Corollary 1.5 to pairs we must restrict attention to a special class of *thrifty resolutions* (Definition 3.5).

**Theorem 1.7.** *Let  $k$  be a perfect field and let  $S$  be a separated scheme of finite type over  $k$ . Let  $(X, \Delta_X)$  and  $(Y, \Delta_Y)$  be simple normal crossing pairs over  $S$ .*

*Suppose  $(X, \Delta_X), (Y, \Delta_Y)$  are properly birational over  $S$  in the sense that there is a commutative diagram*

$$\begin{array}{ccc} & (Z, \Delta_Z) & \\ r \swarrow & & \searrow s \\ (X, \Delta_X) & \circlearrowleft & (Y, \Delta_Y) \\ f \searrow & & \swarrow g \\ & S & \end{array} \quad (1.8)$$

where  $r, s$  are proper and birational morphisms, and  $\Delta_Z = r_*^{-1}\Delta_X = s_*^{-1}\Delta_Y$ . Set  $n = \dim X = \dim Y = \dim Z$ . If  $r, s$  are thrifty then there are quasi-isomorphisms

$$Rf_*\mathcal{O}_X(-\Delta_X) \simeq Rg_*\mathcal{O}_Y(-\Delta_Y) \text{ and } Rf_*\omega_X(\Delta_X) \simeq Rg_*\omega_Y(\Delta_Y). \quad (1.9)$$

**Definition 1.10** ([Kol13, Def. 2.78]). Let  $(S, \Delta_S)$  be a pair as in Convention 1.6, and suppose  $S$  is normal. A **rational resolution of  $(S, \Delta_S)$**  is a proper birational morphism  $f : X \rightarrow S$  such that if  $\Delta_X = f_*^{-1}\Delta_S$  then

- (i) The pair  $(X, \Delta_X)$  is snc,
- (ii) The natural map  $\mathcal{O}_S(-\Delta_S) \rightarrow Rf_*\mathcal{O}_X(-\Delta_X)$  is a quasi-isomorphism, and
- (iii)  $R^if_*\omega_X(\Delta_X) = 0$  for  $i > 0$ .

*Remark 1.11* (description of the natural map in (ii)). Since  $\Delta_X$  is the strict transform of  $\Delta_S$ , so in particular  $\Delta_X \subset f^{-1}(\Delta_S)$ , there is a containment of ideal sheaves  $\mathcal{I}_{f^{-1}(\Delta_S)} \subset \mathcal{I}_{\Delta_X} = \mathcal{O}_X(-\Delta_X)$  providing a morphism

$$f^*\mathcal{O}_S(-\Delta_S) = f^*\mathcal{I}_{\Delta_S} \rightarrow \mathcal{I}_{f^{-1}(\Delta_S)} \subset \mathcal{I}_{\Delta_X} = \mathcal{O}_X(-\Delta_X).$$

Taking the adjoint gives a morphism  $\mathcal{O}_S(-\Delta_S) \rightarrow f_*\mathcal{O}_X(-\Delta_X)$ , and composing with the natural map  $f_*\mathcal{O}_X(-\Delta_X) \rightarrow Rf_*\mathcal{O}_X(-\Delta_X)$  gives (ii).

As a straightforward corollary of [Theorem 1.7](#), one obtains:

**Corollary 1.12.** *Let  $(S, \Delta_S)$  be a pair, with  $\Delta_S$  reduced and effective. If  $(S, \Delta_S)$  has a thrifty rational resolution  $f : (X, \Delta_X) \rightarrow (S, \Delta_S)$ , then every thrifty resolution  $g : (Y, \Delta_Y) \rightarrow (S, \Delta_S)$  is rational. In particular, if  $(S, \Delta_S)$  is snc then it is a rational pair.*

**1.1. Acknowledgements.** The author would like to thank Jarod Alper, Daniel Bragg, Chi-yu Cheng, Kristin DeVleming, Gabriel Dorfsman-Hopkins, Max Lieblich, and Tuomas Tajakka for helpful conversations, and his advisor Sándor Kovács for constant support and encouragement, as well as proposing the problem of extending [Theorem 1.1](#) to pairs. Special thanks to Karl Schwede for suggesting the use of derived category techniques (rather than extending the Chow-to-Hodge cycle morphism used in [\[CR11\]](#)).

## 2. DUAL COMPLEXES

**Definition 2.1** (cf. [\[dFKX14\]](#)). Let  $Z = \bigcup_{i \in I} Z_i$  be a scheme with irreducible components  $Z_i$ . Say  $Z$  is an **expected-dimensional crossing scheme** if and only if

- (i)  $Z$  is pure dimensional and the components  $Z_i$  are normal, and
- (ii) For any  $J \subset I$ , set  $Z_J := \bigcap_{j \in J} Z_j$ . If  $Z_J \neq \emptyset$  every connected component of  $Z_J$  is irreducible and of codimension  $|J| - 1$  in  $Z$ .

A **stratum** of an expected-dimensional crossing scheme  $Z$  is an irreducible (or equivalently connected) component of  $Z_J = \bigcap_{j \in J} Z_j$  for some  $J \subset I$ .

The main case of [Definition 2.1](#) considered here will be the case  $\Delta = \Delta_X$  where  $(X, \Delta_X)$  is a simple normal crossing pair, in which case all strata of  $\Delta_X$  are smooth.

Let  $(X, \Delta_X)$  be a simple normal crossing pair, and write  $\Delta_X = \bigcup_{i \in I} D_i$  with  $D_i$  the irreducible components of  $\Delta_X$ . For  $J \subset I$ , let  $D_J = \bigcap_{j \in J} D_j$ , and write  $D_J = \bigcup_k D_J^k$  where the  $D_J^k$  are irreducible. Observe that  $(\Delta_X - \sum_{i \in J} D_i)|_{D_J^k}$  is a (possibly empty) simple normal crossing divisor on each stratum  $D_J^k$ .

**Definition 2.2** (strata as pairs).

$$\Delta_{D_J} := (\Delta_X - \sum_{i \in J} D_i)|_{D_J} \text{ and } \Delta_{D_J^k} := (\Delta_X - \sum_{i \in J} D_i)|_{D_J^k}$$

**Definition 2.3.** For an expected-dimensional crossing scheme  $Z = \bigcup_{i \in I} Z_i$ , the **dual complex**  $\mathcal{D}(Z)$  is a  $\Delta$ -complex [\[Hat02, §2.1\]](#) that can be described as follows: assume the index set  $I$  has been totally ordered. For each  $d \in \mathbb{N}$ , the  $d$ -simplices of  $\mathcal{D}(Z)$  correspond to the irreducible components  $Z_J^k \subset Z_J = \bigcap_{j \in J} Z_j$  where  $J \subset I$  ranges over all subsets of size  $|J| = d + 1$ . Let  $\sigma_J^k$  be the  $d$ -simplex associated to  $Z_J^k$ .

If  $j \in J$  write  $\hat{J}(j) := J \setminus \{j\}$  – we have inclusions  $Z_J \subset Z_{\hat{J}(j)}$ , and the connected components of  $Z_{\hat{J}(j)}$  are irreducible, for each component  $Z_J^k$  there is a *unique* component  $Z_{\hat{J}(j)}^l \subset Z_{\hat{J}(j)}$  such that  $Z_J^k \subset Z_{\hat{J}(j)}^l$ . The face maps of  $\mathcal{D}(Z)$  are obtained by setting

$$\partial_j \sigma_J^k = \sigma_{\hat{J}(j)}^l$$

*Remark 2.4.* In particular,  $\mathcal{D}(Z)$  has

- 0-simplices  $\sigma_i$  corresponding to the irreducible components  $Z_i \subset Z$ ,
- 1-simplices  $\sigma_{ij}^k$  corresponding to the components  $Z_{ij}^k \subset Z_{ij} = Z_i \cap Z_j$  where  $i < j$ , with face maps  $\partial_0, \partial_1$  corresponding to the inclusions  $Z_{ij}^k \subset Z_i, Z_{ij}^k \subset Z_j$  respectively,

and so on.

*Remark 2.5.* From the description above one can see that  $\mathcal{D}(Z)$  is a **regular**  $\Delta$ -complex, meaning that if  $\sigma \subseteq \mathcal{D}(Z)$  is a  $d$ -simplex, the corresponding map  $\sigma: \Delta^d \rightarrow \mathcal{D}(Z)$  is injective. Indeed, if

$$\partial_j \sigma_j^k = \partial_{j'} \sigma_j^k$$

for  $j \neq j'$ , then  $Z_{j(j)} \cap Z_{j'(j')} = Z_j$  would contain a component of codimension  $d - 1$ , violating (ii) of [Definition 2.3](#).

Dual complexes have been extensively studied; to paraphrase Arapura, Bakhtary, and Włodarczyk,  $\mathcal{D}(Z)$  governs the *combinatorial part* of the topology of  $Z$  [\[ABW13\]](#). One underlying reason for this is

**Lemma 2.6** (Special case of [\[Fri83, Prop. 1.5\]](#)<sup>1</sup>). *If  $\Delta$  is a simple normal crossing scheme and  $n = \dim \mathcal{D}(\Delta)$ , then there is a quasi-isomorphism*

$$\mathcal{O}_\Delta \simeq \left[ \prod_{\sigma \in \mathcal{D}(\Delta)^0} \mathcal{O}_{D(\sigma)} \xrightarrow{d^1} \prod_{\sigma \in \mathcal{D}(\Delta)^1} \mathcal{O}_{D(\sigma)} \xrightarrow{d^2} \cdots \rightarrow \prod_{\sigma \in \mathcal{D}(\Delta)^n} \mathcal{O}_{D(\sigma)} \right] =: \check{C}(\Delta, \mathcal{O}) \text{ in } D^+(\Delta_X)$$

where the differential  $d^i: \prod_{\sigma \in \mathcal{D}(\Delta)^{i-1}} \mathcal{O}_{D(\sigma)} \rightarrow \prod_{\sigma \in \mathcal{D}(\Delta)^i} \mathcal{O}_{D(\sigma)}$  has  $\sigma$ th coordinate

$$\prod_{\sigma \in \mathcal{D}(\Delta)^{i-1}} \mathcal{O}_{D(\sigma)} \rightarrow \prod_{j=0}^i \mathcal{O}_{D(\partial^j \sigma)} \xrightarrow{\sum_{j=0}^i (-1)^j \text{res}_j} \mathcal{O}_{D(\sigma)}$$

and where  $\text{res}_j: \mathcal{O}_{D(\partial^j \sigma)} \rightarrow \mathcal{O}_{D(\sigma)}$  is the natural map restricting functions.

**Corollary 2.7.** *If  $(X, \Delta_X)$  is a simple normal crossing pair let  $n = \dim \mathcal{D}(\Delta_X)$ , then there is a quasi-isomorphism*

$$\mathcal{O}_X(-\Delta_X) \simeq \left[ \mathcal{O}_X \xrightarrow{d^0} \prod_{\sigma \in \mathcal{D}(\Delta)^0} \mathcal{O}_{D(\sigma)} \xrightarrow{d^1} \prod_{\sigma \in \mathcal{D}(\Delta)^1} \mathcal{O}_{D(\sigma)} \xrightarrow{d^2} \cdots \rightarrow \prod_{\sigma \in \mathcal{D}(\Delta)^n} \mathcal{O}_{D(\sigma)} \right] =: \check{C}(X, \Delta_X, \mathcal{O})$$

in  $D^+(X)$ .

*Proof.* We must show that the sequence

$$\mathcal{O}_X(-\Delta_X) \rightarrow \mathcal{O}_X \xrightarrow{d^0} \prod_{\sigma \in \mathcal{D}(\Delta)^0} \mathcal{O}_{D(\sigma)} \xrightarrow{d^1} \prod_{\sigma \in \mathcal{D}(\Delta)^1} \mathcal{O}_{D(\sigma)} \xrightarrow{d^2} \cdots \rightarrow \prod_{\sigma \in \mathcal{D}(\Delta)^n} \mathcal{O}_{D(\sigma)}$$

is exact – [Lemma 2.6](#) already implies  $\ker d^i = \text{im } d^{i-1}$  for  $i > 1$  and  $\ker d^1 = \mathcal{O}_\Delta$ . Exactness of the sequence  $0 \rightarrow \mathcal{O}_X(-\Delta_X) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{\Delta_X} \rightarrow 0$  tells us that  $\mathcal{O}_X \rightarrow \mathcal{O}_{\Delta_X}$  is surjective with kernel  $\mathcal{O}_X(-\Delta_X)$ . Hence defining  $d^0$  to be the composition

$$\begin{array}{c} \mathcal{O}_X \xrightarrow{\quad d^0 \quad} \mathcal{O}_{\Delta_X} \longrightarrow \prod_{\sigma \in \mathcal{D}(\Delta)^0} \mathcal{O}_{D(\sigma)} \end{array} \quad (2.8)$$

ensures that  $\ker d^1 = \text{im } d^0$  and that  $\ker d^0 = \mathcal{O}_X(-\Delta_X)$ , as desired.  $\square$

<sup>1</sup>The cited proposition is stated over  $\mathbb{C}$ , but the proof works in arbitrary characteristic.

**2.1. Morphisms of Dual Complexes.** One can extract from the literature on dual complexes the following slogan:

*Morphisms of pairs induce morphisms of dual complexes. Moreover, there is a “dictionary” relating properties of a morphism of pairs with corresponding properties of the induced morphism of dual complexes.*

To precisify the slogan, we include a foundational result providing a weak sort of functoriality.

**Lemma 2.9** (cf. [Wlo16, Def. 2.0.6]). *Let  $Z = \cup_{i \in I} Z_i$  and  $W = \cup_{j \in J} W_j$  be expected -dimensional crossing schemes and let  $f : Z \dashrightarrow W$  be a rational morphism defined at the generic point of each stratum of  $Z$ . Then up to homotopy equivalence there is a unique induced morphism of  $\Delta$ -complexes*

$$\mathcal{D}(f) : \mathcal{D}(Z) \rightarrow \mathcal{D}(W)$$

*such that if  $\sigma \subset \mathcal{D}(Z)$  is a simplex and  $\eta_\sigma$  is the generic point of the corresponding stratum of  $Z$ , and if  $\tau \subset \mathcal{D}(W)$  is the simplex corresponding to the unique minimal stratum  $D(\tau) \subset W$  containing  $f(\eta_\sigma)$ , then  $\mathcal{D}(f)(\sigma) \subset \tau$ .*

*Proof in the case  $f$  is defined everywhere.* Since  $f(D(\sigma))$  is irreducible it is contained in some stratum of  $W$  (in particular,  $f(D(\sigma)) \subset W_i$  for some  $i$ ). Let

$$W_J := \cap \{W_j \subset W \mid f(D(\sigma)) \subset W_j\}$$

By (ii) of Definition 2.1, the connected components of  $W_J$  are irreducible, and hence  $f(D(\sigma))$  is contained in exactly one of them – let  $\tau \subset \mathcal{D}(W)$  be the corresponding simplex. If  $\dim \sigma = 0$  let  $\mathcal{D}(f)(\sigma)$  be an interior point of  $\tau$ .

One can now show by induction on  $\dim \sigma$  that  $\mathcal{D}(f)$  extends over all of  $\mathcal{D}(Z)$  – so, assume  $\dim \sigma > 1$ . For each face  $\sigma' \subset \sigma$  with corresponding stratum  $D(\sigma') \subset Z$ , let  $D(\tau') \subset W$  be the smallest stratum containing  $f(D(\sigma'))$ . Now

$$f(D(\sigma)) \subset f(D(\sigma')) \text{ forces } D(\tau) \subset D(\tau')$$

and this gives an inclusion  $\iota_{\tau'} : \tau' \rightarrow \tau$ . By induction a map  $\mathcal{D}(f)|_{\sigma'} : \sigma' \rightarrow \tau'$  has already been defined, so composing with  $\iota$  one obtains

$$\sigma' \xrightarrow{\mathcal{D}(f)|_{\sigma'}} \tau' \xrightarrow{\iota} \tau \text{ for each face } \sigma' \subset \sigma$$

which together give a map  $\partial \sigma \rightarrow \tau$ , and as  $\tau$  is contractible this map must extend over  $\sigma$ .

Uniqueness up to homotopy equivalence follows from Lemma 2.10.  $\square$

**Lemma 2.10.** *If  $f, g : X \rightarrow Y$  are 2 maps of regular  $\Delta$ -complexes such that for each simplex  $\sigma \subseteq X$  there is a unique minimal simplex  $\tau_\sigma \subseteq Y$  such that  $f(\sigma), g(\sigma) \subseteq \tau_\sigma$  then there is a homotopy  $h : X \times I \rightarrow Y$  from  $f$  to  $g$  such that  $h(\sigma \times I) \subseteq \tau_\sigma$  for each simplex  $\sigma \subset X$ .*

*Proof.* We proceed by induction over the skeleta  $X^d \subseteq X$ . For the case  $d = 0$  let  $v \in X^0$  be a vertex. By hypothesis there's a unique minimal simplex  $\tau_v \subseteq Y$  so that  $f(v), g(v) \in \tau_v \subseteq Y$ , so we may choose a path  $\gamma_v : I \rightarrow \tau_v \subseteq Y$  with  $\gamma_v(0) = f(v), \gamma_v(1) = g(v)$ . Then the map

$$h^0 : X^0 \times I \rightarrow Y \text{ defined by } h^0(v, t) = \gamma_v(t)$$

is a homotopy between  $f|_{X^0}$  and  $g|_{X^0}$  with  $h^0(\{v\} \times I) \subseteq \tau_v$  for all  $v$ .

Suppose by inductive hypothesis that  $d > 0$  and we have constructed a homotopy  $h^{d-1} : X^{d-1} \times I \rightarrow Y$  from  $f|_{X^{d-1}}$  to  $g|_{X^{d-1}}$  with  $h^{d-1}(\sigma \times I) \subseteq \tau_\sigma$  for all simplices  $\sigma \subseteq X^{d-1}$ . Let  $\sigma \subset X$  be a  $d$ -simplex, and observe that if  $\sigma' \subset \sigma$  is a face then  $f(\sigma') \subseteq f(\sigma) \subseteq \tau_\sigma$ , and similarly  $g(\sigma') \subseteq \tau_\sigma$ . By hypothesis this implies  $\tau_{\sigma'} \subseteq \tau_\sigma$ , so that the homotopy  $h^{d-1}|_{\sigma'} : \sigma' \times I \rightarrow Y$  factors through  $\tau_\sigma$ . We

conclude that the map  $\gamma|_{\sigma}: \sigma \times 0, 1 \cup \partial\sigma \rightarrow Y$  defined by

$$(x, t) \mapsto \begin{cases} f(x) & \text{if } t = 0, \\ g(x) & \text{if } t = 1, \text{ and} \\ h(x, t), & \text{otherwise} \end{cases}$$

factors through  $\tau_{\sigma}$ ; since  $Y$  is regular  $\tau_{\sigma}$  is contractible, and so  $\gamma|_{\sigma}$  extends to a morphism  $\gamma_{\sigma}: \sigma \times I \rightarrow Y$ . As  $\sigma$  varies over the  $d$ -simplices of  $X$ , the  $\gamma_{\sigma}$  provide an extension of  $h^{d-1}$  to a homotopy

$$h^d: X^d \times I \rightarrow Y \text{ from } f|_{X^d} \text{ to } g|_{X^d}.$$

□

### 3. THRIFTY MORPHISMS OF PAIRS

Let  $(S, \Delta_S)$  be a pair (as in [Convention 1.6](#)).

**Definition 3.1.** The **snc locus** of  $(S, \Delta_S)$  is the largest open  $U \subset S$  so that  $(U, \Delta_S|_U)$  is a simple normal crossing pair – it will be denoted  $\text{snc}(S, \Delta_S)$ . We also set

$$\text{non-snc}(S, \Delta_S) := S \setminus \text{snc}(S, \Delta_S) \quad (3.2)$$

*Remark 3.3.* When  $S$  is normal,  $\text{non-snc}(S, \Delta_S)$  has codimension  $\geq 2$  in  $S$ .

In their work on dual complexes of Calabi-Yau pairs, introduced a natural generalization of thrifty resolutions to a class of *thrifty morphisms* where the domain is no longer required to be smooth.

**Definition 3.4** ([[KX16](#), Def. 9]). A crepant proper birational morphism of log canonical pairs  $f: (X, \Delta_X) \dashrightarrow (S, \Delta_S)$  is **Kollár-Xu-thrifty** (KX-thrifty for short) if and only if there are closed subsets  $Z_X \subset X, Z_S \subset S$  of codimension  $\geq 1$  so that

- $Z_X$  contains no log canonical centers of  $(X, \Delta_X)$ , and similarly for  $Z_S$ , and
- $f$  restricts to an isomorphism  $X \setminus Z_X \xrightarrow{f} S \setminus Z_S$ .

Since rational pairs are not log canonical in general, for example since they are not necessarily  $\mathbb{Q}$ -Gorenstein<sup>2</sup>, we adopt a slightly different definition of thrifty morphisms (see [Lemma 3.8](#) for a comparison).

Let  $(S, \Delta_S)$  be a pair and let  $f: X \rightarrow S$  be a proper birational morphism. Set  $\Delta_X := f_*^{-1}\Delta_S$  (the strict transform).

**Definition 3.5.** The morphism  $f$  is **thrifty** if and only if

- (i)  $f$  is an isomorphism *over* the generic point of every stratum of  $\text{snc}(S, \Delta_S)$  and
- (ii)  $f$  is an isomorphism *at* the generic point of every stratum of  $\text{snc}(X, \Delta_X)$ .

If in addition  $X$  is smooth and  $f^{-1}(\Delta_S) \cup E$  is a simple normal crossing divisor (with  $E$  the exceptional locus) then  $f$  is called a **thrifty resolution**.

*Remark 3.6.* Equivalently, if  $\text{Ex}(f) \subset X$  is the exceptional locus of  $f$ , then

- (i)  $f(\text{Ex}(f))$  contains no stratum of  $\text{snc}(S, \Delta_S)$  and
- (ii)  $\text{Ex}(f)$  contains no stratum of  $\text{snc}(X, \Delta_X)$ .

*Remark 3.7.* Hence when  $X$  is smooth and  $f^{-1}(\Delta_S) \cup E$  is a simple normal crossing divisor [Definition 3.5](#) reduces to [[Kol13](#), Def. 2.79].

**Lemma 3.8.** Let  $f: (X, \Delta_X) \rightarrow (S, \Delta_S)$  be a crepant proper birational morphism between dlt pairs. Then  $f$  is KX-thrifty ([Definition 3.4](#)) if and only if it is thrifty ([Definition 3.5](#)).

<sup>2</sup>The cone over  $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^{mn+m+n}$  embedded using the complete linear system  $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(m, n)|$  is rational for all  $m, n > 0$ ,  $\mathbb{Q}$ -Gorenstein if and only if  $m = n$ .



*Proof.* The map  $f$  is crepant, so  $K_X + \Delta_X \sim_{\mathbb{Q}} f^*(K_S + \Delta_S)$  – equivalently,

$$\Delta_X \sim_{\mathbb{Q}} f_*^{-1}(\Delta_S) - \sum_i a_i E_i$$

where  $a_i := a(E_i, S, \Delta_S)$  and the sum runs over all  $f$ -exceptional divisors  $E_i \subset X$ . Writing  $\Delta_S = \sum_i c_i D_i$ , we see that  $\Delta_S^{\leq 1} = \sum_{c_i=1} D_i$  and that  $\Delta_X^{\leq 1} = \sum_{c_i=1} f_*^{-1} D_i + \sum_{a_i=-1} E_i$ . Both pairs are dlt, so the log canonical centers of  $(X, \Delta_X)$  are the strata of the expected-dimensional crossing scheme  $\Delta_X^{\leq 1}$ , and their generic points lie in  $\text{snc}(X, \Delta_X)$  – similarly for  $(S, \Delta_S)$  [Fuj07]. Moreover, if  $a_i = -1$  then  $f(E_i) \subset S$  is a log canonical center, so it must be a stratum of  $\Delta_S^{\leq 1}$ .

Suppose  $f$  is KX-thrifty and let  $Z_X \subset X$ ,  $Z_S \subset S$  be closed sets as guaranteed in Definition 3.4. Then  $f$  is an isomorphism over  $S \setminus Z_S$  and  $Z_S$  contains no stratum of  $\Delta_S^{\leq 1}$ , giving condition (i) of Definition 3.5. Also, we must have  $a_i > -1$  for all  $i$ , and so  $\Delta_X^{\leq 1} = \sum_{c_i=1} f_*^{-1} D_i = f_*^{-1} \Delta_S^{\leq 1}$ . Since  $Z_X$  contains no stratum of  $\Delta_X^{\leq 1}$ , we obtain (ii) of Definition 3.5.  $\square$

In the next lemma we use a definition of a birational map general enough to encompass reducible schemes [Sta19, Tags 0A20, 0BX9]: a rational map  $f: X \dashrightarrow Y$  between schemes with finitely many irreducible components is *birational* if and only if it is an isomorphism in the category with

- objects the schemes with finitely many irreducible components, and with
- morphisms the dominant rational maps between them.

When  $Y$  is locally of finite presentation over a field (as it will be in all cases considered here), the map  $f$  is birational if and only if it induces a bijection between the generic points of irreducible components of  $X$  and  $Y$ , and for each generic point of an irreducible component  $\eta \in X$  the induced morphism  $\mathcal{O}_{Y, f(\eta)} \rightarrow \mathcal{O}_{X, \eta}$  is an isomorphism.

**Lemma 3.9.** *Let  $Z = \cup_{i=1}^N Z_i$  and  $W = \cup_{j=1}^N W_j$  be expected-dimensional crossing schemes and let  $f: Z \dashrightarrow W$  be a birational map defined at the generic point of each stratum of  $Z$ .*

- If  $f$  is an isomorphism at the generic point of every stratum  $D(\sigma) \subset Z$ , then  $\mathcal{D}(f)$  can be realized as a subcomplex inclusion.*
- If  $f$  is an isomorphism over the generic point of every stratum  $D(\tau) \subset W$  then it is an isomorphism at the generic point of every stratum of  $Z$ , and  $\mathcal{D}(f)$  can be realized as an isomorphism of  $\Delta$ -complexes.*

*Proof.* In the case of (i), as  $f$  is birational it induces a bijection between the generic points of  $Z$  and  $W$  and hence a bijection on 0-skeleta

$$\mathcal{D}(f)_0: \mathcal{D}(Z)_0 \xrightarrow{\sim} \mathcal{D}(W)_0$$

Without loss of generality we may assume  $f$  restricts to a birational maps  $f_i: Z_i \dashrightarrow W_i$  for  $i = 1, \dots, N$ . Let  $n = \dim Z = \dim W$ .

Let  $\sigma \in \mathcal{D}(Z)$  be a simplex with corresponding stratum  $D(\sigma) \subset Z$  – without loss of generality we may assume  $D(\sigma) \subset Z_1$ , and that  $D(\sigma) \subseteq \cap_{j=1}^r Z_j$ . Letting  $\eta_\sigma \in D(\sigma)$  be the generic point, we see that  $f(\eta_\sigma) \in \cap_{j=1}^r W_j$ . Because  $f$  is an isomorphism at  $\eta_\sigma$ , it must be that  $f(\eta_\sigma)$  is a generic point of a component  $D(\tau) \subseteq \cap_{j=1}^r W_j$  corresponding to a simplex  $\tau \subseteq \mathcal{D}(W)$ . Let  $\eta_\tau \in D(\tau)$  be the generic point; we have  $\eta_\tau = f(\eta_\sigma)$ .

At this point the only concern is that there could be another  $r-1$ -simplex  $\sigma'$  such that  $\mathcal{D}(f)(\sigma') = \tau$ ; any such  $\sigma'$  would correspond to another stratum  $D(\sigma') \subseteq \cap_{j=1}^r Z_j$ , hence another point  $\eta_{\sigma'} \in Z_1$  of dimension  $r-1$  with  $f(\eta_{\sigma'}) = f(\eta_\tau)$ . One can show this is impossible, using the normality of  $W_1$  and Zariski's main theorem as follows.

The map  $f$  is an isomorphism at the generic point  $n_\sigma \in D(\sigma)$ , so its restriction  $f|_{Z_1}: Z_1 \rightarrow W_1$  is also an isomorphism at  $n_\sigma$ . The scheme  $W_1$  is normal and  $f|_{Z_1}$  is birational by hypothesis, so by Zariski's main theorem [Sta19, Tag 05K0]  $f|_{Z_1}$  is in fact an isomorphism over  $\eta_\tau$ .

For (ii), observe that  $f^{-1}: W \dashrightarrow Z$  satisfies the hypotheses of (i) and hence both  $\mathcal{D}(f): \mathcal{D}(Z) \rightarrow \mathcal{D}(W)$  and  $\mathcal{D}(f^{-1}): \mathcal{D}(W) \rightarrow \mathcal{D}(Z)$  may be realized as subcomplex inclusions; from the proof of

(i), this can be done in such a way that  $\mathcal{D}(f) \circ \mathcal{D}(f^{-1}) = \text{id}_{\mathcal{D}(W)}$ . In particular this implies  $\mathcal{D}(f)$  is a surjective subcomplex inclusion, hence an isomorphism.  $\square$

**Corollary 3.10.** *Let  $(S, \Delta_S)$  be a pair and let  $f : X \rightarrow S$  be a proper birational morphism and set  $\Delta_X := f_*^{-1} \Delta_S$ . Then  $f$  induces morphisms of  $\Delta$ -complexes*

$$\mathcal{D}(\text{snc } \Delta_X) \xrightarrow{\mathcal{D}(f|_{\Delta})} \mathcal{D}(\text{snc } \Delta_S) \text{ and } \mathcal{D}(\text{snc}(X, \Delta_X)) \xrightarrow{\mathcal{D}(f)} \mathcal{D}(\text{snc}(S, \Delta_S))$$

which are isomorphisms if  $f$  is thrifty.

*Proof.* The induced morphisms come from [Lemma 2.9](#); to see that they are isomorphisms when  $f$  is thrifty we may apply [Definition 3.5](#) and [Lemma 3.9](#).  $\square$

**3.1. Thrifty proper birational equivalences.** If  $S$  is a separated scheme of finite type over  $k$  and  $f : X \rightarrow S, g : Y \rightarrow S$  are separated schemes of finite type over  $S$ , a **proper birational equivalence of  $X, Y$  over  $S$**  is a commutative diagram

$$\begin{array}{ccc} & Z & \\ r \swarrow & & \searrow s \\ X & \circlearrowright & Y \\ f \searrow & & \swarrow g \\ & S & \end{array} \quad (3.11)$$

where  $r, s$  are proper birational morphisms.

**Definition 3.12.** Suppose  $(X, \Delta_X), (Y, \Delta_Y)$  are pairs over  $S$ , with  $X$  and  $Y$  normal and  $\Delta_X, \Delta_Y$  reduced and effective. A **thrifty proper birational equivalence of  $(X, \Delta_X)$  and  $(Y, \Delta_Y)$  over  $S$**  is a proper birational equivalence as in diagram 3.11, where  $r$  and  $s$  are thrifty.

*Remark 3.13.* By [Corollary 3.10](#), a thrifty proper birational equivalence  $X \xleftarrow{r} Z \xrightarrow{s} Y$  between  $(X, \Delta_X)$  and  $(Y, \Delta_Y)$  induces an isomorphism  $\mathcal{D}(\Delta_X) \simeq \mathcal{D}(\Delta_Y)$ .

**Proposition 3.14.** *Let  $(S, \Delta_S)$  be a pair with  $\Delta_S$  reduced and effective, and let  $f : X \rightarrow S, g : Y \rightarrow S$  be 2 thrifty resolutions of  $(S, \Delta_S)$ . Then there is a thrifty proper birational equivalence of  $X$  and  $Y$  over  $S$ .*

*Proof.* Let  $U \subset S$  be an open set such that both  $f$  and  $g$  are isomorphisms over  $U$ ; then we have an isomorphism

$$g^{-1} \circ f : f^{-1}(U) \rightarrow g^{-1}(U)$$

Set

$$Z := \overline{\Gamma_{g^{-1} \circ f}} \subset X \times_S Y$$

and let  $p : Z \rightarrow X, s : Z \rightarrow Y$  be the projections. The claim is that  $X \xleftarrow{r} Z \xrightarrow{s} Y$  is a thrifty proper birational equivalence over  $S$ . It is birational by design, and proper since  $X, Y$  and hence  $X \times_Y Z$  are proper over  $S$  and  $Z$  is closed in  $X \times_S Y$ . It remains to show that  $r, s$  are thrifty.

**Lemma 3.15.** *Let  $\text{Ex}(r), \text{Ex}(s) \subset Z$  be the exceptional loci of  $r, s$  respectively; let  $\text{Ex}(f) \subset X, \text{Ex}(g) \subset Y$  be the exceptional loci of  $f$  and  $g$ . Then*

$$r(\text{Ex}(r)) \subset f^{-1}(g(\text{Ex}(g))) \text{ and } s(\text{Ex}(s)) \subset g^{-1}(f(\text{Ex}(f)))$$

*Proof of Lemma 3.15.* Let  $U \subset S$  and  $V \subset Y$  be a maximal pair of open sets such that  $g|_V : V \xrightarrow{\sim} U$  is an isomorphism; note that since  $g$  is an honest morphism  $\text{Ex}(g) = Y \setminus V$  and  $g(\text{Ex}(g)) = S \setminus U$ . Then  $W := f^{-1}(U) \subset X$  is an open set such that  $g^{-1} \circ f : X \dashrightarrow Y$  is defined on  $W$ . This implies



the projection  $\Gamma_{g^{-1} \circ f} \xrightarrow{r} X$  is an isomorphism over  $W$ , but what we need to know is that the same is true for  $Z = \bar{\Gamma}_{g^{-1} \circ f} \xrightarrow{r} X$ . For this, note that

$$\bar{\Gamma}_{g^{-1} \circ f} \cap r^{-1}(W) = \overline{\Gamma_{g^{-1} \circ f} \cap r^{-1}(W)} = \overline{\Gamma_{g^{-1} \circ f}|_W} \subset W \times_S Y$$

Since  $W$  and  $Y$  are both separated over  $S$ , the graph  $\Gamma_{g^{-1} \circ f|_W}$  is already closed, so we conclude  $\bar{\Gamma}_{g^{-1} \circ f} \cap r^{-1}(W) = \Gamma_{g^{-1} \circ f|_W}$ .  $\square$

Now suppose  $W \subset X$  is a stratum of  $(X, \Delta_X)$  – we must show  $r$  is an isomorphism over the generic point  $\eta \in W$ . First,  $f$  is an isomorphism at  $\eta$  by hypothesis, and so by the proof of [Lemma 3.9](#),  $f(\eta)$  is the generic point of a stratum of  $\text{sn}c(S, \Delta_S)$ . Then  $g$  is an isomorphism over  $f(\eta)$  by hypothesis, so in particular  $f(\eta) \notin g(\text{Ex}(g))$ . By [Lemma 3.15](#) we conclude that  $\eta \notin r(\text{Ex}(r))$ , as desired.

Finally we show that  $s$  is an isomorphism at the generic point of every stratum of  $\Delta_Z := r_*^{-1} f_*^{-1} \Delta_S$ , using a more general lemma:

**Lemma 3.16.** *Let  $r: (Z, \Delta_Z) \rightarrow (X, \Delta_X)$  be a proper birational morphism. If  $(X, \Delta_X)$  is a simple normal crossing pair, then  $r$  is thrifty if and only if it satisfies condition (i) of [Definition 3.5](#). Explicitly,  $r$  is thrifty if and only if it is an isomorphism over every stratum of  $\Delta_X$ .*

*Proof of Lemma 3.16.* In this situation there is an honest morphism  $\text{sn}c(\Delta_Z) \rightarrow \Delta_X$ , so the hypotheses of [Lemma 3.9](#) are satisfied. We then apply [Lemma 3.9 \(ii\)](#).  $\square$

$\square$

*Remark 3.17.* In the case where the morphism  $r: Z \rightarrow X$  of [Lemma 3.16](#) is projective, [[Har77](#), Thm. 7.17] implies that  $r$  is the blowup of some sheaf of ideals  $\mathcal{I} \subseteq \mathcal{O}_X$  such that  $V(\mathcal{I}) \subset X$  contains no stratum of  $\Delta_X$ . If in addition  $V(\mathcal{I})$  has *simple normal crossings* with  $\Delta_X$  [[Kol07](#), Def. 3.24], [Lemma 3.16](#) can be obtained from known results on the effect of blowing up on dual complexes [[Ste06](#), §2], [[dFKX14](#), §9], [[Wlo16](#), Prop. 2.1.6].

#### 4. STRUCTURE SHEAVES OF STRATA AND THEIR DIRECT IMAGES

In this section we prove weak functoriality statements about the quasi-isomorphisms in [Theorem 1.1](#), or alternatively those of [[Kov19](#)].

**Lemma 4.1.** *Let  $S$  be scheme over a field  $k$  and let  $f: X \rightarrow S, g: Y \rightarrow S$  are  $S$ -schemes that are smooth over  $k$ . Suppose  $X \xleftarrow{r} Z \xrightarrow{s} Y$  is a proper birational equivalence over  $S$  such that both  $r$  and  $s$  are projective. Let  $\mathcal{C}(Z)$  denote the category with objects the pairs  $(E \subseteq X, F \subseteq Y)$  of smooth closed subschemes of  $X$  and  $Y$  such that*

- (i)  *$r$  and  $s$  are isomorphisms over the generic points of  $E$  and  $F$  respectively, and*
- (ii) *the birational map  $s \circ r^{-1}: X \dashrightarrow Y$  sends the generic point of  $E$  to the generic point of  $F$ , and with morphisms  $(E_1, F_1) \rightarrow (E_2, F_2)$  given by inclusions  $E_1 \subseteq E_2, F_1 \subseteq F_2$ . If  $\mathcal{K} \subset \mathcal{C}(Z)$  is a finite subcategory, then there are proper birational equivalences  $E \xleftarrow{r'} W \xrightarrow{s'} F$  compatible with  $Z$  in the sense that*

$$\begin{array}{ccccc} E & \xleftarrow{r'} & W & \xrightarrow{s'} & F \\ \downarrow & \circlearrowleft & \downarrow k & \circlearrowleft & \downarrow \\ X & \xleftarrow{r} & Z & \xrightarrow{s} & Y \end{array} \quad (4.2)$$

commutes, and commutative diagrams

$$\begin{array}{ccc}
 Rf_*\mathcal{O}_X & \xrightarrow[\simeq]{\gamma_{X,Y}} & Rf_*\mathcal{O}_Y \\
 \downarrow & \circlearrowleft & \downarrow \\
 Rf_*\mathcal{O}_E & \xrightarrow[\gamma_{E,F}]{\simeq} & Rf_*\mathcal{O}_F
 \end{array} \quad \text{in } D^+(S). \tag{4.3}$$

defining a natural transformation of functors  $\mathcal{K}^{\text{op}} \rightarrow D^+(S)$ .

*Proof.* We proceed by descending induction over the poset  $\mathcal{K}$ .

For each object  $(E, F) \in \text{Ob}(\mathcal{K})$ , since  $X$  is smooth and  $r$  is an isomorphism over the generic point  $\xi \in E$  we see that if  $\tilde{E} \subseteq Z$  is the strict transform of  $E$  then  $\tilde{E} \not\subseteq \text{Sing}(Z)$ , so in particular if  $\text{non-CM}(Z) \subseteq Z$  is the non-Cohen-Macaulay locus then  $\tilde{E} \not\subseteq \text{non-CM}(Z)$  – similarly for  $F$ . By a theorem of Česnavičius, there exists *Macaulayfication*  $\pi: \tilde{Z} \rightarrow Z$  such that  $\pi$  is an isomorphism over  $Z \setminus \text{non-CM}(Z)$  – explicitly,  $\tilde{Z}$  is Cohen-Macaulay and  $\pi$  is a projective birational morphism [Ces18, Thm. 1.6] (see also [Kaw00, Thm. 5.1]). It follows that  $r \circ \pi$  and  $s \circ \pi$  are projective and isomorphisms over the generic points of  $E$  and  $F$  respectively, for all  $(E, F) \in \text{Ob}(\mathcal{K})$ . From now on we may assume  $Z$  is Cohen-Macaulay.

Now suppose  $(E, F) \in \text{Ob}(\mathcal{K})$  is *maximal* (categorically final), and let  $W \subseteq Z \times_{X \times Y} E \times F$  be the component dominating  $E$  and  $F$ , and form the commutative diagram of  $S$ -schemes

$$\begin{array}{ccccc}
 & W & \xrightarrow{k} & Z & \\
 & \searrow s' & & \searrow s & \\
 E & & F & \xrightarrow{j} & Y \\
 & \nearrow r' & & \nearrow r & \\
 & W & \xrightarrow{k} & Z & \\
 E & \xrightarrow{\iota} & X & & 
 \end{array} \tag{4.4}$$

Replacing  $W$  with a Macaulayfication  $\pi': \tilde{W} \rightarrow W$  if necessary, we may assume  $W$  is Cohen-Macaulay. Now by functoriality we have commutative diagrams

$$\begin{array}{ccc}
 \mathcal{O}_X & \xrightarrow{\iota^\#} & R\iota_*\mathcal{O}_E \\
 r^\# \downarrow \simeq & \circlearrowleft & \simeq \downarrow R\iota_*r'^\# \\
 Rr_*\mathcal{O}_Z & \xrightarrow{Rr_*k^\#} & R(r \circ k)_*\mathcal{O}_W = R(\iota \circ r')_*\mathcal{O}_W
 \end{array} \quad \text{and} \quad \begin{array}{ccc}
 \mathcal{O}_Y & \xrightarrow{j^\#} & Rj_*\mathcal{O}_F \\
 s^\# \downarrow \simeq & \circlearrowleft & \simeq \downarrow Rj_*s'^\# \\
 Rs_*\mathcal{O}_Z & \xrightarrow{Rs_*k^\#} & R(s \circ k)_*\mathcal{O}_W = R(j \circ s')_*\mathcal{O}_W
 \end{array} \tag{4.5}$$

in  $D(X)$  and  $D(Y)$  respectively. The vertical arrows are isomorphisms since  $X, Y, E$  and  $F$  are all smooth, so in particular they have rational singularities, and  $W$  and  $Z$  are Cohen-Macaulay, so we

may apply [Kov19, Thm. 8.6]. Finally, pushing forward along  $f$  and  $g$  we obtain

$$\begin{array}{ccccc}
 Rf_*\mathcal{O}_X & \xrightarrow{Rf_*t^\sharp} & R(f \circ t)_*\mathcal{O}_E & & \\
 Rf_*r^\sharp \downarrow \simeq & \circlearrowleft & \simeq \downarrow R(f \circ t)_*r'^\sharp & & \\
 R(f \circ r)_*\mathcal{O}_Z & \xrightarrow{R(f \circ r)_*k^\sharp} & R(f \circ r \circ k)_*\mathcal{O}_W = R(f \circ t \circ r')_*\mathcal{O}_W & & \\
 \parallel & & \parallel & & \\
 R(g \circ s)_*\mathcal{O}_Z & \xrightarrow{R(g \circ s)_*k^\sharp} & R(g \circ s \circ k)_*\mathcal{O}_W = R(g \circ j \circ s')_*\mathcal{O}_W & & \\
 \simeq \uparrow Rg_*s^\sharp & \circlearrowleft & R(g \circ j)_*s'^\sharp \uparrow \simeq & & \\
 Rg_*\mathcal{O}_Y & \xrightarrow{Rg_*j^\sharp} & R(g \circ j)_*\mathcal{O}_F & & 
 \end{array} \tag{4.6}$$

For the inductive step, suppose  $(E, F) \in \text{Ob}(\mathcal{K})$  and let  $\alpha_i: (E, F) \rightarrow (E_i, F_i)$ ,  $i = 1, \dots, r$  be the morphisms in  $\mathcal{K}$  with source  $(E, F)$ . By inductive hypothesis, for each  $i$  we have a Cohen-Macaulay  $S$ -scheme  $W_i$  and a projective birational equivalence  $E_i \xleftarrow{r_i} W_i \xrightarrow{s_i} F_i$  inducing a morphism  $\gamma_{E_i, F_i}: Rf_*\mathcal{O}_{E_i} \rightarrow Rg_*\mathcal{O}_{F_i}$  – using the above construction, we can ensure that for any  $\mathcal{K}$ -morphism  $(E', F') \rightarrow (E_i, F_i)$  the map  $r_i$  is an isomorphism over  $E'$ , and similarly for  $F_i$ . Consider the cartesian diagram

$$\begin{array}{ccccc}
 W & \hookrightarrow & (E \times_S F) \times_{\prod_{S, i=1}^r (E_i \times_S F_i)} \prod_{S, i=1}^r W_i & \longrightarrow & \prod_{S, i=1}^r W_i \\
 & \searrow r' \times_S s' & \downarrow & \square & \downarrow \prod_{S, i=1}^r (r_i \times_S s_i) \\
 & & E \times_S F & \longrightarrow & \prod_{S, i=1}^r (E_i \times_S F_i)
 \end{array} \tag{4.7}$$

and let  $W \subseteq (E \times_S F) \times_{\prod_{S, i=1}^r (E_i \times_S F_i)} \prod_{S, i=1}^r W_i$  be the component dominating  $E$  and  $F$ . Note that  $r', s'$  are projective since  $\prod_{S, i=1}^r (r_i \times_S s_i)$  is projective by hypothesis. As above, we may replace  $W$  by a projective Macaulayfication while retaining the property that  $r', s'$  are isomorphisms over the generic points of  $E' \subset E$ ,  $F' \subset F$  for every  $\mathcal{K}$ -morphism  $(E', F') \rightarrow (E, F)$ . Now by design for each  $i$  there is a commutative diagram

$$\begin{array}{ccccc}
 W & \xrightarrow{k} & W_i & & \\
 \downarrow r' & \searrow s' & \downarrow s & & \\
 & & F & \xrightarrow{j} & F_i \\
 \downarrow & & \downarrow r & & \\
 E & \xrightarrow{t} & E_i & & 
 \end{array} \tag{4.8}$$

and arguing as in the base case we obtain from (4.8) a commutative diagram in  $D^+(S)$  of the form

$$\begin{array}{ccc}
 Rf_*\mathcal{O}_{E_i} & \xrightarrow{\gamma_{E_i, F_i}} & Rf_*\mathcal{O}_{F_i} \\
 \downarrow & \circlearrowleft & \downarrow \\
 Rf_*\mathcal{O}_E & \xrightarrow{\gamma_{E, F}} & Rf_*\mathcal{O}_F
 \end{array} \tag{4.9}$$

□

**Corollary 4.10.** *Let  $S$  be a scheme over a field  $k$  and let  $(X, \Delta_X)$  and  $(Y, \Delta_Y)$  be simple normal crossing pairs over  $k$  with morphisms  $f: X \rightarrow S$  and  $g: Y \rightarrow S$ . Suppose  $X \xleftarrow{r} Z \xrightarrow{s} Y$  is a thrifty proper birational*

equivalence over  $S$  such that both  $r$  and  $s$  are projective. Let  $\mathcal{D}$  be the common dual complex of  $\Delta_X$  and  $\Delta_Y$  (see [Remark 3.13](#)) and for a simplex  $\sigma \subseteq \mathcal{D}$  let  $D_X(\sigma) \subseteq X$ ,  $D_Y(\sigma) \subseteq Y$  be the corresponding strata. In this situation there is a natural transformation of functors  $\text{Face}(\mathcal{D}) \rightarrow D^+(S)$  from  $Rf_*\mathcal{O}_{D_X(\sigma)}$  to  $Rg_*\mathcal{O}_{D_Y(\sigma)}$ , compatible with restrictions from  $Rf_*\mathcal{O}_X$  and  $Rg_*\mathcal{O}_Y$ , and hence a commutative diagram in  $D^+(S)$  of the form

$$\begin{array}{ccccccc} Rf_*\mathcal{O}_X & \rightarrow & \prod_{\sigma \in \mathcal{D}^0} Rf_*\mathcal{O}_{D_X(\sigma)} & \rightarrow & \prod_{\sigma \in \mathcal{D}^1} Rf_*\mathcal{O}_{D_X(\sigma)} & \rightarrow & \prod_{\sigma \in \mathcal{D}^2} Rf_*\mathcal{O}_{D_X(\sigma)} \rightarrow \cdots \\ \downarrow & & \downarrow \gamma^0 & & \downarrow \gamma^1 & & \downarrow \gamma^2 \\ Rg_*\mathcal{O}_Y & \rightarrow & \prod_{\sigma \in \mathcal{D}^0} Rg_*\mathcal{O}_{D_Y(\sigma)} & \rightarrow & \prod_{\sigma \in \mathcal{D}^1} Rg_*\mathcal{O}_{D_Y(\sigma)} & \rightarrow & \prod_{\sigma \in \mathcal{D}^2} Rg_*\mathcal{O}_{D_Y(\sigma)} \rightarrow \cdots \end{array} \quad (4.11)$$

*Proof.* We apply [Lemma 4.1](#) to the finite subcategory  $\mathcal{K} \subset \mathcal{C}(Z)$  with objects  $(D_X(\sigma), D_Y(\sigma))$  for  $\sigma \subseteq \mathcal{D}$ . Evidently, this  $\mathcal{K}$  is equivalent to  $\text{Face}(\mathcal{D})$ .  $\square$

## 5. A MORPHISM OF RESTRICTION TRIANGLES

The main result of this section is

**Lemma 5.1.** *Let  $S$  be a base scheme over a field  $k$  and let  $(X, \Delta_X)$  and  $(Y, \Delta_Y)$  be simple normal crossing schemes over  $k$  with morphisms  $f: X \rightarrow S$ ,  $g: Y \rightarrow S$ . If  $X \xleftarrow{r} Z \xrightarrow{s} Y$  is a thrifty proper birational equivalence over  $S$  then there is an isomorphism of distinguished triangles*

$$\begin{array}{ccccccc} Rf_*\mathcal{O}_X(-\Delta_X) & \longrightarrow & Rf_*\mathcal{O}_X & \longrightarrow & Rf_*\mathcal{O}_{\Delta_X} & \xrightarrow{+1} & \cdots \\ \downarrow \gamma' & & \downarrow \gamma & & \downarrow \gamma'' & & \\ Rg_*\mathcal{O}_Y(-\Delta_Y) & \longrightarrow & Rg_*\mathcal{O}_Y & \longrightarrow & Rg_*\mathcal{O}_{\Delta_Y} & \xrightarrow{+1} & \cdots \end{array} \quad \text{in } D^+(S). \quad (5.2)$$

For the most part, this consists of combining [Corollaries 2.7](#) and [4.10](#) to obtain the isomorphisms  $\gamma$  and  $\gamma''$  – after that, the existence of  $\gamma'$  is guaranteed since  $D^+(S)$  is triangulated, and the fact that  $\gamma'$  is an isomorphism follows from the 5-lemma.

*Proof.* Under construction! :)  $\square$

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