(CO)BORDISM NOTES

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ABSTRACT. These are notes on (co)bordism, mostly from a geometric (rather than homotopy-theoretic) point of view in the spirit of Quillen's *Elementary proofs of some results of cobordism theory using Steenrod operations*. They begin with a lengthy discussion of oriented (co)bordism, this being in some sense the simplest structured (co)bordism theory. Then there's a discussion (not yet written) of (co)bordism theories in general (i.e. (co)bordism theories associated to fibrations over BO). The later sections deal with complex oriented cohomology theories (complex cobordism being the case of greatest interest) and their connection with formal group laws.

CAUTION: I recently realized that there are issues with the way I use real K-theory ($K0(-)$), reduced real K-theory ($\tilde{K}0(-)$) and groups of stable real vector bundles ($[-, BO]$) in sections 1 and 2. Basically I thought I could get away with using only the stable vector bundles $[-, BO]$, but now think that keeping track of dimensions by working with real K-theory $KO(-)$ is necessary in order to define things like the codimension of a map of manifolds, etc. So, these sections need serious revisions.

0.1. **Prologue: Stokes's theorem.** Recall the following piece of classic rock:

Let X be a smooth manifold and let ω be a n-form on X. Suppose M is a compact oriented smooth $n+1$-manifold with boundary and $f : M \to X$ is a smooth map. Then $f^*\omega$ is a n form on M, and

Theorem 0.1 (Stokes)**.**

$$\int_M d\, f^*\omega = \int_{\partial M} f^*\omega \in \mathbb{R}$$

In particular if ω is a *closed* n-form representing a de Rham cohomology class $[\omega] \in H^n_{dR}(M)$ and if ∂M can be written as a disjoint union of 2 closed oriented smooth n-manifolds (where a closed manifolds is just a compact manifold without boundary), say $\partial M = N_1 \coprod -N_2$ (here $-N_2$ denotes N_2 with its orientation reversed), then

$$\int_{N_1} f^*\omega = \int_{N_2} f^*\omega \in \mathbb{R}$$

Of course, in light of the de Rham theorem (which identifies $H^n_{dR}(M)$ with $H^n(M;\mathbb{R})$ and integration of forms over chains with the pairing

$$H^n(M;\mathbb{R}) \times H_n(M;\mathbb{Z}) \to \mathbb{R}$$

and so on) the equality of the above integrals is explained by the fact that if $f_* : H_*(M;\mathbb{Z}) \to H_*(X;\mathbb{Z})$ is the induced map on homology,

$$f_*[N_1] = f_*[N_2] \in H_n(X;\mathbb{Z})$$

where $[N_1]$, $[N_2]$ are the fundamental classes of N_1, N_2 respectively. Indeed, by the discussion at the end of chapter 3.3 in Hatcher's *Algebraic topology* there's a fundamental class $[M] \in H_{n+1}(M, \partial M;\mathbb{Z})$ and if

$$\partial : H_{n+1}(M, \partial M;\mathbb{Z}) \to H_n(\partial M;\mathbb{Z})$$

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is the boundary map in the long exact sequence of the pair $(M, \partial M)$ then $\partial[M] = [\partial M]$, the fundamental class of ∂M. Now if $\iota : \partial M \to M$ is the inclusion, we have $\iota_*[\partial M] = 0 \in H_n(M;\mathbb{Z})$, since $\iota_* \partial = 0$, and this shows

$$[N_1] - [N_2] = \iota_*[\partial M] = 0 \in H_n(M;\mathbb{Z})$$

In this situation one says $f : M \to X$ is a "bordism" between the smooth maps $f|_{N_i} : N_i \to X$, $i = 1, 2$. We've shown that if $g : N \to X$ is a smooth map from a closed oriented n-manifold N to X, then the homology class $g_*[N] \in H_n(X)$ depends only on the "bordism class" of g.

1. Oriented (co)bordism

1.1. **Stable normal bundles.** Let M be a compact smooth n-manifold, with tangent bundle τ_M. Suppose $\iota : M \to \mathbb{R}^N$ is a smooth embedding (by Whitney's embedding theorem we know one of these exists for sufficiently large N). Then viewing M as a submanifold of $\mathbb{R}^N$, we have a short exact sequence of vector bundles over M

$$0 \to \tau_M \to \tau_{\mathbb{R}^N}|_M \to \nu_{M|\mathbb{R}^N} \to 0$$

where $\tau_{\mathbb{R}^N}|_M$ is the tangent bundle of $\mathbb{R}^N$ restricted to M, and $\nu_{M|\mathbb{R}^N}$ is the normal bundle of M in $\mathbb{R}^N$. Of course, $\tau_{\mathbb{R}^N}|_M \simeq \epsilon^N$ is just a trivial bundle of rank N, and so we have a direct sum decomposition $\tau_M \oplus \nu_{M|\mathbb{R}^N} \simeq \epsilon^N$.

Definition 1.1. Two real vector bundles ξ, η over a finite CW complex X are said to be **stably isomorphic** if there are *trivial* real vector bundles ξ', η' over X so that

$$\xi \oplus \xi' \simeq \eta \oplus \eta'$$

Proposition 1.2. *The Whitney sum operation $\oplus$ on real vector bundles over X descends to an abelian group operation on the stable isomorphism classes.*

Proof. It's straightforward to check that $\oplus$ descends to an operation on stable classes - I won't prove it.

Associativity follows from associativity of $\oplus$ and the stable class of the 0 vector bundle serves as an identity. If ξ is a real vector bundle over X then there's an injection $\xi \to \epsilon^N$ for sufficiently large N (see Milnor and Stasheff's *Characteristic classes* or Atiyah and Anderson's *K-theory*). Let ξ' be the complement of ξ in ϵ^N, so $\xi \oplus \xi' \simeq \epsilon^N$. Then $\xi \oplus \xi' \simeq \epsilon^N$, and ϵ^N is stably isomorphic to 0. Hence ξ' serves as an inverse to ξ.

□

Proposition 1.3. *There's a natural abelian group isomorphism between the stable real vector bundles over X and the homotopy classes of maps $[X, BO]$.*

Proof. Recall that for any $k \in \mathbb{N}$, pulling back the tautological k-plane bundle γ_k on $BO(k)$ gives a natural bijection

$$[X, BO(k)] \simeq \mathrm{Vect}^k(X, \mathbb{R}) \text{ sending } [f] \mapsto f^*\gamma_k$$

where $\mathrm{Vect}^k(X, \mathbb{R})$ denotes the isomorphism classes of real k-plane bundles over X. Moreover these bijections piece together to give an isomorphism of directed systems; for each k we have a commutative diagram

$$\begin{array}{ccc} [X, BO(k)] & \xrightarrow{\simeq} & \mathrm{Vect}^k(X, \mathbb{R}) \\ {\scriptstyle B\iota_*}\big\downarrow & & \big\downarrow{\scriptstyle -\oplus\epsilon} \\ [X, BO(k+1)] & \xrightarrow{\simeq} & \mathrm{Vect}^{k+1}(X, \mathbb{R}) \end{array} \tag{1.1}$$

where the left vertical arrow is induced by the map $BO(k) \xrightarrow{B\iota} BO(k+1)$ classifying the usual inclusion $O(k) \xrightarrow{\iota} O(k+1)$ and the right vertical arrow corresponds to Whitney sum with a trivial line bundle. The result is an isomorphism of direct limits

$$\mathrm{colim}_k[X, BO(k)] \simeq \mathrm{colim}_k \mathrm{Vect}^k(X, \mathbb{R})$$

Now as X is a finite complex, the natural map $\mathrm{colim}_k[X, BO(k)] \to [X, BO]$ is an isomorphism, and by unraveling definitions one can identify $\mathrm{colim}_k \mathrm{Vect}^k(X, \mathbb{R})$ with the stable real vector bundles over X.

□

Remark 1.4. One can also identify the stable real vector bundles (equivalently, $[X, BO]$) with $KO^0(X)/H^0(X;\mathbb{Z})$, the quotient of the real K-theory of X with the copy of $H^0(X;\mathbb{Z})$ consisting of trivial vector bundles on the components of X.

Let's return to the matter at hand:

Definition 1.5. The **stable normal bundle** ν_M **of** M is the inverse of τ_M in the group of stable real vector bundles over M.

Thus the stable normal bundle is "$\nu_M = -\tau_M$".

Proposition 1.6. *An orientation of M is equivalent to an orientation of its stable normal bundle ν_M.*

Remark 1.7. Here an orientation of a stable vector bundle $[\xi]$ over M will mean an orientation of some representative bundle ξ. Equivalently, if $[f] \in [X, BO]$ corresponds to $[\xi]$ under the isomorphism of proposition 1.3, then an orientation will mean an equivalence class of lifts $\tilde{f} : X \to BSO$ of f fitting into

$$\begin{tikzcd} X \arrow[r, "\tilde{f}"] \arrow[d, "="'] & BSO \arrow[d, "B\iota"'] \\ X \arrow[r, "f"] & BO \end{tikzcd} \tag{1.2}$$

Proof. Let $0 \to V' \xrightarrow{i} V \xrightarrow{j} V'' \to 0$ be a short exact sequence of real vector spaces. Then specifying orientations for any 2 of the vector spaces V', V, V'' determines an orientation on the 3rd. This is a consequence of the following piece of linear algebra: if

$$\begin{tikzcd} 0 \arrow[r] & V' \arrow[r, "i"] \arrow[d, "\varphi'"'] & V \arrow[r, "j"] \arrow[d, "\varphi"'] & V'' \arrow[r] \arrow[d, "\varphi''"'] & 0 \\ 0 \arrow[r] & V' \arrow[r, "i"] & V \arrow[r, "j"] & V'' \arrow[r] & 0 \end{tikzcd} \tag{1.3}$$

is an endomorphism of the short exact sequence, then

$$\det \varphi = \det \varphi' \cdot \det \varphi'' \in \mathbb{R}$$

Similarly if $0 \to \xi' \xrightarrow{i} \xi \xrightarrow{j} \xi'' \to 0$ is a short exact sequence of real vector bundles on a CW complex X, specifying orientations on any 2 of ξ', ξ, ξ'' determines an orientation on the 3rd. Applying this in the case of a short exact sequence $0 \to \tau_M \to \epsilon^N \to \nu \to 0$, where ν is a representative of the stable normal bundle of M, and noting that the trivial bundle ϵ^N has a canonical orientation, we see that an orientation of τ_M determines an orientation of ν and vice versa.

□

1.2. **Geometric definition of oriented bordism groups.** Let X be a smooth manifold (without boundary). Define a **compact oriented smooth n-manifold in** X to be a smooth map $f : M \to X$ from a compact oriented smooth n-manifold M to X. Similarly, define a **closed oriented smooth n-manifold over** X to be a compact oriented smooth n-manifold $f : M \to X$ over X where M is closed, i.e. $\partial M = \emptyset$. Notice that if $f : M \to X$ is a compact oriented smooth n-manifold over X, then $f|_{\partial M} : \partial M \to X$ is a closed oriented smooth $n-1$-manifold over X.

The compact oriented smooth n-manifolds over X form a category, say $\mathcal{C}_n(X)$; we can define a morphism between $f_i : M_i \to X$, $i = 1, 2$ to consist of a boundary preserving oriented smooth map $g : M_1 \to M_2$ over X; that is, g is a smooth map $M_1 \to M_2$ such that $g(\partial M_1) \subset \partial M_2$ together with an isomorphism of principal C_2-bundles $\mathbb{O}_{M_1} \simeq g^*\mathbb{O}_{M_2}$ (where $\mathbb{O}_{M_i}$ is the orientation bundle of M_i), fitting into a commutative diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{g} & M_2 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow \\ X & \xrightarrow{=} & X \end{array} \tag{1.4}$$

The closed oriented smooth n-manifolds will form a full subcategory of $\mathcal{C}_n(X)$, say $\mathcal{Z}_n(X)$, and restricting to boundaries will define a functor $\partial : \mathcal{C}_n(X) \to \mathcal{Z}_{n-1}(X)$.

Define a **bordism** between 2 closed oriented smooth n-manifolds $f_i : M_i \to X$, $i = 0, 1$ over X to be a compact oriented smooth $n+1$-manifold $h : W \to X$ over X together with an isomorphism

$$\begin{array}{ccc} M_0 \coprod -M_1 & \xrightarrow{\simeq} & \partial W \\ {\scriptstyle f_0 \coprod -f_1}\downarrow & & {\scriptstyle h|_{\partial W}}\downarrow \\ X & \xrightarrow{=} & X \end{array} \tag{1.5}$$

of closed oriented n-manifolds over X (i.e. an isomorphism in the category $\mathcal{Z}_n(X)$). Here $-f_1 : -M_1 \to X$ is obtained from f_1 by reversing the orientation on M_1. In this situation say $f_i : M_i \to X$, $i = 0, 1$ are bordant, and write $f_0 \sim f_1$.

Proposition 1.8. *Bordism defines an equivalence relation on the (isomorphism classes of) the category $\mathcal{Z}_n(X)$.*

Sketch. Given a closed oriented smooth n-manifold $f : M \to X$, observe that

$$M \times I \xrightarrow{\text{proj}} M \xrightarrow{f} X$$

defines a bordism from f to itself; this shows bordism is reflexive. Bordism is symmetric basically because the disjoint union is. The most subtle point is showing bordism is transitive: suppose $h_0 : W_0 \to X$ is a bordism between $f_i : M_i \to X$, $i = 0, 1$ and $h_1 : W_1 \to X$ is a bordism between $f_i : M_i \to X$, $i = 1, 2$. Then one can show the map $h_0 \coprod h_1 : W_0 \coprod W_1 \to X$ factors like

$$W_0 \coprod W_1 \xrightarrow{\text{quot}} W \xrightarrow{h} X$$

where $W = W_0 \cup_{M_1} W_1$ is a compact oriented smooth $n+1$-manifold over X obtained by gluing W_0 to W_1 along M_1, and that $h : W \to X$ is a bordism between $f_i : M_i \to X$ for $i = 0, 2$. Of course this takes some work - see a good textbook on differential topology.

□

Proposition 1.9. *The disjoint union operation $\coprod$ on closed oriented smooth n-manifolds over X descends to an abelian group operation on bordism classes $\mathcal{Z}_n(X)/\sim$.*

Proof. It's straightforward to show the disjoint union descends to an associative operation on bordism classes. The class of the empty set $\emptyset \to X$ serves as an identity, and reversing orientation yields inverses.

□

Definition 1.10. The n**-th bordism group of** X is $\Omega_n(X) := \mathcal{Z}_n(X)/\sim$, under $\coprod$.

Now let $\varphi : X \to Y$ be a smooth map from X to another smooth manifold Y. Given any compact oriented smooth n-manifold $f : M \to X$ over X, the composition

$$M \xrightarrow{f} X \xrightarrow{\varphi} Y$$

is a compact oriented smooth n-manifold over Y. It's straightforward to check that this construction gives "pushforward functors"

$$\varphi_* : \mathcal{C}_n(X) \to \mathcal{C}_n(Y) \text{ for each } n \in \mathbb{N}$$

compatible with the boundary functors ∂, hence compatible with the bordism relation. Proceeding in this way we obtain homomorphisms of bordism groups

$$\varphi_* : \Omega_n(X) \to \Omega_n(Y) \text{ for } n \in \mathbb{N}$$

Proposition 1.11. *φ_* depends only on the homotopy class of φ_*. Thus Ω_n can be viewed as a covariant functor on the homotopy category of smooth manifolds, for $n \in \mathbb{N}$.*

Proof. Letting $\psi : X \times I \to Y$ be a homotopy between 2 smooth maps $\varphi_i : X \to Y$, $i = 0, 1$ and noting that $\varphi_i = \psi \circ \iota_i$, $i = 0, 1$, where ι_0, ι_1 are the usual inclusions

$$X \simeq X \times \{i\} \subset X \times I, \text{ for } i = 0, 1$$

we reduce to the case where $\psi = \mathrm{id} : X \times I \to X \times I$ is the identity giving a homotopy between $\iota_i : X \to X \times I$, $i = 0, 1$.

Now observe that if $f : M \to X$ is a closed oriented smooth n-manifold over X, then $f \times \mathrm{id} : M \times I \to X \times I$ defines a bordism between $\iota_i \circ f : M \to X \times I$, $i = 0, 1$.

□

Now suppose $f : M \to X$ is a closed oriented smooth n-manifold over X, and let $[M] \in H_n(M; \mathbb{Z})$ be the fundamental class. From this we obtain a homology class

$$f_*[M] \in H_n(X; \mathbb{Z}), \text{ depending only on the bordism class } [f] \in \Omega_n(X)$$

(as discussed in the prologue). In fact, the assignment $(M \xrightarrow{f} X) \mapsto f_*[M]$ defines an abelian group homorphism $\Omega_n(X) \mapsto H_n(X; \mathbb{Z})$. One can check that this construction defines a natural transformation $\Omega_n(-) \to H_n(-; \mathbb{Z})$ of abelian group valued functors on the homotopy category of smooth manifolds.

1.3. **Relative stable normal bundles.**

Let M be a (not necessarily compact) smooth manifold without boundary and let $f : M \to X$ be a smooth map. Observe that for sufficiently large $N \in \mathbb{N}$ we may factor f as

$$\begin{array}{ccc} M & \xrightarrow{\iota} & E(\xi) \\ f\big\downarrow & & \pi\big\downarrow \\ X & \xrightarrow{=} & X \end{array} \tag{1.6}$$

where ξ is an oriented real vector bundle of dimension N over X, and ι is a smooth embedding. For instance, take $g : M \to \mathbb{R}^N$ to be a smooth embedding, and set $\iota := f \times g : M \to X \times \mathbb{R}^N$.

Lemma 1.12. *The tangent bundle of the smooth manifold $E(\xi)$ can be identified as $\pi^*(\tau_X \oplus \xi)$.*

Sketch. Say $v \in E(\xi)$, and let $x = \pi(v) \in X$. There's a natural short exact sequence of tangent spaces

$$0 \to TE(\xi)_{x,v} \xrightarrow{d\iota} TE(\xi)_v \xrightarrow{d\pi} TX_x \to 0$$

where $\iota : E(\xi)_x \to E(\xi)$ is the inclusion of the fiber over $x \in X$. Now note that $E(\xi)_x$ is a real vector space, and hence there are canonical isomorphisms

$$TE(\xi)_{x,v} \xrightarrow{dl_v} TE(\xi)_{x,0} \xrightarrow{\exp} E(\xi)_x$$

Thus we can write the above short exact sequence as

$$0 \to E(\xi)_x \to TE(\xi)_v \xrightarrow{d\pi} TX_x \to 0$$

Now argue that as v varies over $E(\xi)$, these piece together to give a short exact sequence of real vector bundles $0 \to \pi^*\xi \to \tau_{E(\xi)} \to \pi^*\tau_X \to 0$ over $E(\xi)$.

□

In light of the lemma, we can rewrite the short exact sequence $0 \to \tau_M \xrightarrow{d\iota} \iota^*\tau_{E(\xi)} \to \nu_{M|E(\xi)} \to 0$ as

$$0 \to \tau_M \to f^*(\tau_X \oplus \xi) \to \nu_{M|E(\xi)} \to 0$$

and hence $\tau_M \oplus \nu_{M|E(\xi)} \simeq f^*(\tau_X \oplus \xi)$ (note that $\pi \circ \iota = f$).

The main case of interest is when $\xi = \epsilon^N$ is a trivial bundle over X, with its usual orientation. In that case we obtain the identity

$$\tau_M \oplus \nu_{M|E(\epsilon^N)} \simeq f^*(\tau_X \oplus \epsilon^N) \simeq f^*\tau_X \oplus \epsilon^N$$

. Notice that this means $\nu_{M|E(\epsilon^N)} = f^*\tau_X - \tau_M$ in the group of stable real vector bundles over M. With this motivation in hand:

Definition 1.13. The **stable normal bundle of** M **over** X is $f^*\tau_X - \tau_M$, viewed as an element of the group of stable real vector bundles over M.

Example 1.14. If $X = \text{pt}$ is a point and M is a closed smooth n-manifold, and if $f : M \to \text{pt}$ collapses M to a point, then the stable normal bundle of M over pt is just the stable normal bundle of M as defined in section 1.2.

Thus the previous definition provides a relative version of the stable normal bundle.

Definition 1.15. Let M be a (not necessarily compact) smooth manifold without boundary and let $f : M \to X$ be a smooth map - one could say "M is a smooth manifold over X." An **orientation of** M **over** X is an orientation of the stable normal bundle of M over X (for the definition of an orientation of a stable vector bundle see the remark after proposition 1.4).

Example 1.16. An orientation of M over X is equivalent to an isomorphism of principal C_2-bundles $\mathfrak{O}_M \simeq f^*\mathfrak{O}_X$, in the following way:

Given an embedding $\iota M \to X \times \mathbb{R}^N$ over X we'll obtain a short exact sequence of vector bundles

$$0 \to \tau_M \to f^*\tau_X \oplus \epsilon^N \to \nu_{M|E(\epsilon^N)} \to 0$$

where $\nu_{M|E(\epsilon^N)}$ is a representative for the stable normal bundle of M over X. This gives short exact sequences of vector spaces

$$0 \to TM_p \to TX_{f(p)} \oplus \mathbb{R}^N \to NM_p \to 0 \text{ for } p \in M$$

Note that there's a canonical bijection between orientations on $TX_{f(p)}$ and orientations on $TX_{f(p)} \oplus \mathbb{R}^N$. By the "3 for 2" property of orientations discussed in the proof of proposition 1.4, an orientation of NM_p is equivalent to a bijection between the orientations of TM_p and those of $TX_{f(p)}$, i.e. an isomorphism of principal C_2-bundles $\mathfrak{O}_M \simeq f^*\mathfrak{O}_X$.

In particular if X is oriented then an orientation of M over X is equivalent to an orientation of M.

1.4. **Geometric definition of oriented cobordism groups.** Let M be a (not necessarily compact) smooth manifold without boundary. Recall that a smooth map $f : M \to X$ is **proper** if for every compact subset $K \subset X$, the preimage $f^{-1}(X) \subset M$ is compact.

Definition 1.17. A **proper, oriented smooth manifold over** X is a proper smooth map $f : M \to X$ from a (not necessarily compact) smooth manifold M together with an orientation of M over X. The **codimension of** M **over** X is $\dim X - \dim M$.

Of course the proper oriented smooth manifolds over X with codimension n form a category, in which a morphism from $M_1 \xrightarrow{f_1} X$ to $M_2 \xrightarrow{f_2} X$ is just a smooth map $g : M_1 \to M_2$ over X; that is, g fits into a commutative diagram

$$\begin{tikzcd} M_1 \arrow[r, "g"] \arrow[d, "f_1"'] & M_2 \arrow[d, "f_2"] \\ X \arrow[r, "="] & X \end{tikzcd} \tag{1.7}$$

Call this category $\mathcal{Z}^n(X)$.

Recall that two smooth maps $f : Y \to X$ and $g : Z \to X$ from smooth manifolds Y, Z to X are said to be **transverse** if whenever $y \in Y, z \in Z$ and $f(y) = g(z) = x \in X$, the linear transformation

$$df \oplus dg : TY_y \oplus TZ_z \to TX_x \text{ is surjective}$$

We'll need the following fundamental

Theorem 1.18. *If $f : Y \to X$ and $g : Z \to X$ are transverse, then the fiber product*

$$Y \times_X Z = \{(y, z) \in Y \times Z \mid f(y) = g(z) \in X\} \subset Y \times Z$$

is a properly embedded submanifold of $Y \times Z$ with dimension $\dim Y + \dim Z - \dim X$; there's a cartesian square of smooth maps

$$\begin{tikzcd} Y \times_X Z \arrow[r, "f'"] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd} \tag{1.8}$$

See Theorem 6.30 in Lee's *Smooth manifolds*. Perhaps a more memorable way to phrase the statement about dimensions is: $\mathrm{codim} g' = \mathrm{codim} g$ and $\mathrm{codim} f' = \mathrm{codim} f$. Moreover:

Proposition 1.19. *Let $\iota : Z \to E(\xi)$ be a smooth embedding over X, where ξ is an oriented real vector bundle over X. Pulling this back over f gives a smooth embedding $\iota' : Y \times_X Z \to Y \times_X E(\xi) = E(f^*\xi)$ over Y, and we have a cartesian square of smooth maps*

$$\begin{tikzcd} Y \times_X Z \arrow[r, "f'"] \arrow[d, "\iota'"'] & Z \arrow[d, "\iota"] \\ E(f^*\xi) \arrow[r] & E(\xi) \end{tikzcd} \tag{1.9}$$

Furthermore, there's a canonical isomorphism $\nu_{Y \times_X Z | E(f^*\xi)} \simeq f'^* \nu_{Z|E(\xi)}$ of real vector bundles over $Y \times_X Z$.

The proof is relatively straightforward and I'm omitting it. Taking ξ to be a trivial vector bundle over X, we obtain the following result:

Proposition 1.20. *Let $\nu_{Z|X}$ be the stable normal bundle of Z over X, and let $\nu_{Y\times_X Z|Y}$ be the stable normal bundle of $Y \times_X Z$ over Y. Then $\nu_{Y\times_X Z|Y} = f'^*\nu_{Z|X}$. In particular an orientation on $\nu_{Z|X}$ determines an orientation on $\nu_{Y\times_X Z|Y}$.*

Note that if g is proper so is g': if $K \subset Y$ is a compact set then so is $f(K) \subset X$, and since g is proper $g^{-1}(f(K)) \subset Z$ is compact. Now observe that $g'^{-1}(K) \subset Y \times_X Z$ can be identified with a closed subset of the compact set $K \times g^{-1}(f(K)) \subset Y \times_X Z$. We've shown:

Proposition 1.21. *Let $g : M \to X$ be a proper, oriented smooth manifold over X, and suppose $f : Y \to X$ is a smooth map from a (not necessarily compact) smooth manifold without boundary Y. Then the fiber product $g' : Y \times_X M \to Y$ is a proper oriented smooth manifold over Y, and moreover $\nu_{Y\times_X M|Y} \simeq f'^*\nu_{M|X}$ as oriented stable real vector bundles over $Y \times_X M$, where g', f' fit into the cartesian square*

$$\begin{array}{ccc} Y\times_X M & \xrightarrow{f'} & M \\ {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow \\ Y & \xrightarrow{f} & X \end{array} \tag{1.10}$$

Definition 1.22. A **cobordism** between 2 proper, oriented smooth manifolds $f_i : M_i \to X$, $i = 0, 1$ of codimension n consists of a proper, oriented smooth manifold $h : W \to X \times \mathbb{R}$ of codimension n such that both inclusions $X \times \{i\} \subset X \times \mathbb{R}$, $i = 0, 1$ are transverse to h together with isomorphisms

$$\begin{array}{ccc} M_i & \xrightarrow{\simeq} & X\times_{\iota_i} W \\ {\scriptstyle f_i}\downarrow & & {\scriptstyle h'}\downarrow \\ X & \xrightarrow{\simeq} & X \end{array} \quad \text{for } i = 0, 1 \tag{1.11}$$

of proper, oriented smooth manifolds of codimension n over X. If there exists such a cobordism, say $M_i \xrightarrow{f_i} X$, $i = 0, 1$ are **cobordant**, and write $f_0 \sim f_1$.

Proposition 1.23. *Cobordism is an equivalence relation on the (isomorphism classes of the) category $\mathcal{Z}^n(X)$.*

The proof proceeds along the lines of the proof that bordism defines an equivalence relation on the isomorphism classes of $\mathcal{Z}_n(X)$. Again the only subtlety involves gluing manifolds along boundaries.

Proposition 1.24. *The disjoint union operation $\coprod$ on proper, oriented smooth manifolds over X with codimension n descends to an abelian group operation on the cobordism equivalence classes $\mathcal{Z}^n(X)/\sim$.*

Again the proof is similar to the proof that disjoint union induces an abelian group operation on bordism classes.

Definition 1.25. The n**-th cobordism group of** X is $\Omega^n(X) := \mathcal{Z}^n(X)/\sim$, under $\coprod$.

Remark 1.26. Evidently $\Omega^n(X) = 0$ when $n > \dim X$. However, we'll see later that when $X = \mathrm{pt}$, $\Omega^n(\mathrm{pt}) \neq 0$ for infinitely many negative values of n.

Now suppose Y is another smooth manifold without boundary, and let $\varphi : Y \to X$ be a smooth map. Let $f : M \to X$ be a proper oriented smooth manifold over X of codimension n, defining a cobordism class $[f] \in \Omega^n(X)$. By the homotopy transversality theorem (see theorem 6.36 in Lee's *Smooth manifolds*) φ is homotopic to a smooth map $\psi : Y \to X$ transverse to f, and pulling back $M \xrightarrow{f} X$ over ψ yields a proper oriented smooth manifold $\psi^* f : Y \times_\psi M \to Y$ over Y of codimension n, defining a cobordism class $[\psi^* f] \in \Omega^n(Y)$. We have a cartesian diagram

$$\begin{tikzcd} Y \times_\psi M \arrow[r, "\psi'"] \arrow[d, "\psi^* f"'] & M \arrow[d, "f"] \\ Y \arrow[r, "\psi"] & X \end{tikzcd} \tag{1.12}$$

Proposition 1.27. *The cobordism class $[\psi^* f] \in \Omega^n(Y)$ depends only on the cobordism class $[f] \in \Omega^n(X)$ and the homotopy class of $\varphi : Y \to X$.*

Proof. Suppose $h : W \to X \times \mathbb{R}$ is a cobordism between 2 proper, oriented smooth manifolds $f_i : M_i \to X$, $i = 0, 1$ of codimension n. Let $\psi_i : Y \to X$ be smooth maps homotopic to φ and transverse to f_i, $i = 0, 1$, and let $\psi : Y \times \mathbb{R} \to X$ be a smooth homotopy between ψ_0, ψ_1. Now consider the map

$$\psi \times \mathrm{id} : Y \times \mathbb{R} \to X \times \mathbb{R} \text{ sending } (y, t) \mapsto (\psi_t(y), t)$$

Observe that this is transverse to h over $X \times \{i\}$, $i = 0, 1$. By a relative version of the homotopy transversality theorem, $\psi \times \mathrm{id}$ is homotopic *rel* $Y \times \{i\}$ to a smooth map $\rho : Y \times \mathbb{R} \to X \times \mathbb{R}$ which is transverse to h. Pulling back $h : W \to X \times \mathbb{R}$ over ρ yields a cobordism $\rho^* h : Y \times \mathbb{R} \times_\rho W \to Y \times \mathbb{R}$ between $\psi_i^* f_i : Y \times_{\psi_i} M_i \to Y$, $i = 0, 1$.

□

Noting that disjoint unions are preserved under pullback, we obtain the following

Corollary 1.28. *The smooth map $\varphi : Y \to X$ induces abelian group homomorphisms*

$$\varphi^* : \Omega^n(X) \to \Omega^n(Y) \textit{ for } n \in \mathbb{N}$$

depending only on the homotopy class of φ. Thus Ω^n descends to an abelian group valued functor on the homotopy category of smooth manifolds.

Remark 1.29. There are natural transformations $\Omega^n(-) \to H^n(-; \mathbb{Z})$, analogous to the natural transformations $\Omega_n(-) \to H_n(-; \mathbb{Z})$ described in section 1.3. Suppose $f : M \to X$ is a proper, oriented smooth manifold over X of codimension n representing a cobordism class $[f] \in \Omega^n(X)$. Let $\iota : M \to E(\epsilon^N)$ be an embedding over X, with normal bundle $\nu_{M|E(\epsilon^N)}$.

By the tubular neighborhood theorem (see for instance theorem 6.24 of Lee's *Smooth manifolds*) there's a diffeomorphism of smooth manifolds with boundary $D(\nu_{M|E(\epsilon^N)}) \xrightarrow{\exp} D \subset E(\epsilon^N)$, where $D(\nu_{M|E(\epsilon^N)})$ is the disk bundle of the normal bundle $\nu_{M|E(\epsilon^N)}$ and D is a "regular domain" (i.e. a codimension-0 embedded submanifold with boundary) in $E(\epsilon^N)$ containing M, fitting into a commutative diagram

$$\begin{tikzcd} D(\xi) \arrow[r, "\exp"] \arrow[d] & D \arrow[d, "\subset"] \\ M \arrow[r, "\iota"] & E(\epsilon^N) \end{tikzcd} \tag{1.13}$$

Here int$D$ is the "tubular neighborhood" of M in $E(\epsilon^N)$ (I write "exp" for the map $D(\nu_{M|E(\epsilon^N)}) \xrightarrow{\exp} D$ because the standard way of constructing such a diffeomorphism is to put a Riemannian metric on $E(\xi)$ and use the exponential map).

Now define a continuous map

$$D \xrightarrow{\exp^{-1}} D(\nu_{M|E(\epsilon^N)}) \xrightarrow{\text{quot}} D(\nu_{M|E(\epsilon^N)})/S(\nu_{M|E(\epsilon^N)}) = \mathrm{Th}(\nu_{M|E(\epsilon^N)})$$

where $S(\nu_{M|E(\epsilon^N)})$ and $\mathrm{Th}(\nu_{M|E(\epsilon^N)})$ are the sphere bundle and Thom space of $\nu_{M|E(\epsilon^N)}$ respectively. Extend it to the Thom space $\Sigma^N X \simeq \mathrm{Th}(\epsilon^N)$ by sending $E(\epsilon^N) - \mathrm{int}D \to \{\infty\} \subset \mathrm{Th}(\nu_{M|E(\epsilon^N)})$; call this map $\varphi : \Sigma^N X \to \mathrm{Th}(\nu_{M|E(\epsilon^N)})$.

Note: There's a minor subtlety here: the reason sending $E(\epsilon^N) - \mathrm{int}D \to \{\infty\}$ yields an extension to the Thom space $\mathrm{Th}(\epsilon^N)$ is because f is proper, and so $M \subset E(\epsilon^N)$ lies in some disk bundle of ϵ^N.

Now observe that since $\nu_{M|E(\epsilon^N)}$ is an *oriented* real vector bundle over M of dimension $\dim X + N - \dim M = \mathrm{codim} f + N = n + N$, it comes with a Thom class

$$u(\nu_{M|E(\epsilon^N)}) \in H^{n+N}(\mathrm{Th}(\nu_{M|E(\epsilon^N)}); \mathbb{Z})$$

which we can pull back over the map φ to obtain a cohomology class $\varphi^* u(\nu_{M|E(\epsilon^N)}) \in H^{n+N}(\Sigma^N X; \mathbb{Z})$ Finally, using the suspension isomorphism $H^{n+N}(\Sigma^N X; \mathbb{Z}) \simeq H^n(X; \mathbb{Z})$ we may identify $\varphi^* u(\nu_{M|E(\epsilon^N)})$ with a class, say, $u([f]) \in H^n(X; \mathbb{Z})$.

Now one must argue that the assignment $[f] \mapsto u([f])$ gives a well-defined, natural homomorphism $\Omega^n(X) \to H^n(X; \mathbb{Z})$. In fact we'll obtain a cleaner construction of this natural homomorphism in the ensuing sections.

1.5. **External products and transfer maps.** Suppose X and Y are smooth manifolds (without boundary), let $f : M \to X$ be a closed oriented smooth m-manifold in X, and let $g : N \to Y$ be a closed oriented smooth n-manifold in Y. Let's make a few observations:

- The stable normal bundle of the closed smooth $m+n$-manifold $M \times N$ can be identified as $\nu_M \times \nu_N$, where ν_M, ν_N are the stable normal bundles of M, N respectively (this follows directly from the definitions). In particular given orientations of ν_M, ν_N we obtain an orientation of $\nu_M \times \nu_N$ (by the 3 for 2 property of orientations).
- It follows that the product $f \times g : M \times N \to X \times Y$ is a closed oriented smooth $m+n$-manifold in $X \times Y$. In fact, we have a functor

$$\mathcal{Z}_m(X) \times \mathcal{Z}_n(Y) \xrightarrow{\times} \mathcal{Z}_{m+n}(X \times Y) \text{ taking } (f, g) \mapsto f \times g$$

- Now suppose $h : W \to X$ is a bordism between 2 closed oriented smooth m-manifolds $f_i : M_i \to X$, $i = 0, 1$. Then the product $h \times g : W \times N \to X \times Y$ defines a bordism between $f_i \times g : M_i \times N \to X \times Y$, $i = 0, 1$ Here we're using the natural identification

$$M_0 \times N \coprod -M_1 \times N \simeq (M_0 \coprod -M_1) \times N \simeq \partial W \times N \simeq \partial(W \times N)$$

  Similarly if $k : V \to Y$ is a bordism between $g_i : N_i \to Y$, $i = 0, 1$ then $f \times k : M \times V \to X \times Y$ defines a bordism between $f \times g_i : M \times N_i \to X \times Y$, $i = 0, 1$ and hence the bordism class $[f \times g] \in \Omega_{m+n}(X \times Y)$ depends only on the classes $[f] \in \Omega_m(X)$, $[g] \in \Omega_n(Y)$.
- Say $f_i : M_i \to X$ $i = 0, 1$ are closed oriented smooth m-manifolds over X, and $g : N \to Y$ is a closed oriented smooth n-manifold over X. Then there's a natural identification

$$\begin{array}{ccc} (M_0 \coprod M_1) \times N & \xrightarrow{\simeq} & M_0 \times N \coprod M_1 \times N \\ {\scriptstyle (f_0 \coprod f_1) \times g} \downarrow & & {\scriptstyle f_0 \times g \coprod f_1 \times g} \downarrow \\ X \times Y & \xrightarrow{=} & X \times Y \end{array} \tag{1.14}$$

  of closed oriented smooth $m+n$-manifolds over $X \times Y$. There's an analogous statement about taking disjoint unions in the right factor.

- Taking products is associative in the obvious sense. It's *graded commutative* in the sense that we have a commutative diagram

$$\begin{tikzcd} M \times N \arrow[r, "T'"] \arrow[d, "f\times g"'] & N \times M \arrow[d, "g\times f"'] \\ X \times Y \arrow[r, "T"] & Y \times X \end{tikzcd} \tag{1.15}$$

  where T', T are the evident transpositions and the sign of the determinant of T' is $(-1)^{mn}$.

The upshot:

Proposition 1.30. *There are natural abelian group homomorphisms*

$$\Omega_m(X) \otimes \Omega_n(Y) \xrightarrow{\times} \Omega_{m+n}(X \times Y) \text{ for } m, n \in \mathbb{N}$$

These are associative in the obvious sense, and the following diagrams commute:

$$\begin{tikzcd} \Omega_m(X) \otimes \Omega_n(Y) \arrow[r, "\times"] \arrow[d, "S"'] & \Omega_{m+n}(X \times Y) \arrow[d, "T_*"'] \\ \Omega_n(Y) \otimes \Omega_m(X) \arrow[r, "\times"] & Omega_{m+n}(Y \times X) \end{tikzcd} \quad \text{for } m, n \in \mathbb{N} \tag{1.16}$$

where $S([f] \otimes [g]) = (-1)^{mn}[g] \otimes [f]$.

Sometimes it'll be useful to consider all bordism groups at once; in that case we'll look at the graded abelian group $\Omega_*(X) := \bigoplus_{n\in\mathbb{N}} \Omega_n(X)$. We could rephrase the above proposition as: there's a natural homomorphism of graded abelian groups $\Omega_*(X) \otimes \Omega_*(Y) \xrightarrow{\times} \Omega_*(X \times Y)$ where now $\otimes$ denotes the tensor product of graded abelian groups.

There's an analogous definition of products in cobordism. Another few observations:

- Suppose $f : M \to X$ is a proper oriented smooth manifold over X of codimension m, and $g : N \to Y$ is a proper oriented smooth manifold over Y with codimension n. Then the product $f \times g : M \times N \to X \times Y$ is a proper smooth manifold over $X \times Y$, with codimension $m + n$ (by direct calculation). Moreover there's a natural identification of its stable normal bundle as
$$\nu_{M\times N|X\times Y} = \nu_{M|X} \times \nu_{N|Y}$$
  In particular the orientations of $\nu_{M|X}, \nu_{N|Y}$ determine an orientation of $\nu_{M\times N|X\times Y}$, and so $f \times g : M \times N \to X \times Y$ is a proper oriented smooth manifold over $X \times Y$ of codimension $m + n$.
- Now suppose $h : W \to X \times \mathbb{R}$ is a cobordism between proper oriented smooth manifolds $f_i : M_i \to X$, $i = 0, 1$ over X of codimension m. One can check that $h \times g : W \times N \to X \times \mathbb{R} \times Y \simeq X \times Y \times \mathbb{R}$ defines a cobordism between $f_i \times g : M_i \times N \to X \times Y$, $i = 0, 1$, and there's an analogous statement about cobordisms in the second factor. Thus the cobordism class $[f \times g] \in \Omega^{m+n}(X \times Y)$ depends only on $[f] \in \Omega^m(X)$, $[g] \in \Omega^n(Y)$.
- As in the case of bordism, products are "bilinear" with respect to direct sums. They're associative in the evident sense, and we have commutative diagrams

$$\begin{tikzcd} N \times M \arrow[r, "T'"] \arrow[d, "g\times f"'] & M \times N \arrow[d, "f\times g"'] \\ Y \times X \arrow[r, "T"] & X \times Y \end{tikzcd} \tag{1.17}$$

  where the sign of the determinant of the resulting isomorphism of stable real vector bundles $\nu_{N\times M|Y\times X} \simeq T'^*\nu_{M\times N|X\times Y}$ is $(-1)^{mn}$.

We obtain the following

Proposition 1.31. *There are natural abelian group homomorphisms*

$$\Omega^m(X) \otimes \Omega^n(Y) \xrightarrow{\times} \Omega^{m+n}(X \times Y) \text{ for } m, n \in \mathbb{Z}$$

They are associative in the evident sense, and the following diagrams commute:

$$\begin{array}{ccc} \Omega^n(Y) \otimes \Omega^m(X) & \xrightarrow{\times} & \Omega^{m+n}(Y \times X) \\ {\scriptstyle S}\downarrow & & {\scriptstyle T^*}\downarrow \\ \Omega^m(X) \otimes \Omega^n(Y) & \xrightarrow{\times} & \Omega^{m+n}(X \times Y) \end{array} \quad \text{for } m, n \in \mathbb{Z} \tag{1.18}$$

where $S([g] \otimes [f]) = (-1)^{mn}[f] \otimes [g]$.

This can be phrased in terms of the graded cobordism group $\Omega^*(X)$ as: there is a natural homomorphism of graded abelian groups $\Omega^*(X) \otimes \Omega^*(Y) \simeq \Omega^*(X \times Y)$. It's associative and graded commutative as described above.

Now let $\Delta : X \to X \times X$ be the diagonal. The above proposition shows that the composition

$$\Omega^*(X) \otimes \Omega^*(X) \to \Omega^*(X \times X) \xrightarrow{\Delta^*} \Omega^*(X)$$

defines a multiplication giving $\Omega^*(X)$ the structure of a graded commutative ring.

Remark 1.32. The product on $\Omega^*(X)$ corresponds to taking *fiber products* over X. If $f : M \to X$ and $g : N \to X$ are proper oriented smooth manifolds over X, say of codimensions m, n respectively, and if the diagonal Δ is transverse to the product $f \times g$, then the proper oriented smooth manifold $\Delta^* f \times g = f \times_X g : M \times_X N \to X$ fits into the cartesian square

$$\begin{array}{ccc} M \times_X N & \longrightarrow & M \times N \\ {\scriptstyle f \times_X g}\downarrow & & {\scriptstyle f \times g}\downarrow \\ X & \xrightarrow{\Delta} & X \times X \end{array} \tag{1.19}$$

Important special cases to keep in mind: if $M, N \subset X$ are *submanifolds* intersecting transversely, then $M \times_X N = M \cap N \subset X$ can be identified with their intersection in X. If $X = \mathrm{pt}$, then $M \times_{\mathrm{pt}} N$ is just the regular old product of M and N.

Remark 1.33. We'll show later that the natural transformations $\Omega_* \to H_*$ and $\Omega^* \to H^*$ are compatible with products.

Now suppose X and Y are smooth manifolds without boundary and $\varphi : X \to Y$ is a proper oriented smooth map of codimension k. Let $f : M \to Y$ be a closed oriented smooth n-manifold in Y, and let $\varphi_0 : X \to Y$ be a proper oriented smooth map homotopic to φ transverse to f (to see that it's possible to choose such a φ_0, note that we may find maps φ_0 arbitrarily close to φ in the uniform topology on Y^X (in the C^∞ topology?) and this ensures they're proper. And stable normal bundles depend only on homotopy classes anyways).

Note that $f(M) \subset Y$ is compact (since M is) and $\varphi_0^{-1}(f(M)) \subset X$ is compact since φ_0 is proper. Since $X \times_{\varphi_0} M$ can be identified with a closed subset of the compact set $\varphi_0^{-1}(f(M)) \subset X \times M$, we see that $f' := \varphi_0^* f : X \times_{\varphi_0} M \to X$ is a closed smooth manifold in X of dimension $n + k$ (since $\dim X - \dim X \times_{\varphi_0} M = \dim Y - \dim M$). Now check it: we have a cartesian square

$$\begin{array}{ccc} X \times_{\varphi_0} M & \xrightarrow{\varphi_0'} & M \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ X & \xrightarrow{\varphi_0} & Y \end{array} \tag{1.20}$$

and from this we see that the stable normal bundle of $\varphi_0' : X \times_{\varphi_0} M \to M$ is identified as

$$\nu_{X\times_{\varphi_0}M|M} = f'^* \nu_{X|Y}$$

where $\nu_{X|Y}$ is the stable normal bundle of $\varphi_0 : X \to Y$ (which is just the stable normal bundle of φ). Hence the orientation on $\nu_{X|Y}$ induces an orientation on $\nu_{X\times_{\varphi_0}M|M}$. We'll need the following straightforward lemma:

Lemma 1.34. *Let $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ be oriented smooth maps of smooth manifolds without boundary. Then there's a natural short exact sequence of stable normal bundles on X*

$$0 \to \nu_{X|Y} \xrightarrow{d\psi} \nu_{X|Z} \to \varphi^* \nu_{Y|Z} \to 0$$

Equivalently $\nu_{X|Z} = \nu_{X|Y} \oplus \varphi^* \nu_{Y|Z}$. Applying this lemma to the maps

$$X \times_{\varphi_0} M \xrightarrow{\varphi_0'} M \to \mathrm{pt}$$

we see that the stable normal bundle of $X \times_{\varphi_0} M$ over a point is computed as $\nu_{X\times_{\varphi_0}M} = \nu_{X\times_{\varphi_0}M|M} \oplus \varphi_0'^* \nu_M$. The given orientations of the 2 summands determine an orientation of $\nu_{X\times_{\varphi_0}M}$. The upshot: $f' : X \times_{\varphi_0} M \to X$ is a closed oriented $n+k$ manifold in X. One can show that the bordism class $\varphi^*[f'] \in \Omega_{n+k}(X)$ depends only on the bordism class $[f] \in \Omega_n(Y)$ and the homotopy class of φ (here we should consider homotopies given by proper oriented smooth maps $X \times I \to Y$), and proceeding in this way we see that

Proposition 1.35. *There are natural homomorphisms*

$$\varphi^* : \Omega_n(Y) \to \Omega_{n+k}(X) \text{ for } n \in \mathbb{N}$$

These will be referred to as transfer maps.

Defining the analogous transfer maps for cobordism is comparatively simple. Suppose $f : M \to X$ is a proper oriented smooth manifold over X with codimension n. Since a composition of proper maps is proper, $M \xrightarrow{f} X \xrightarrow{\varphi} Y$ is proper and lemma 1.19 identifies the stable normal bundle of $\varphi \circ f$ as $\nu_{M|Y} = \nu_{M|X} \oplus f^* \nu_{X|Y}$ - the given orientations of the 2 summands define an orientation of $\nu_{M|Y}$. Thus $\varphi \circ f : M \to Y$ is a proper oriented smooth manifold over Y with codimension $n+k$. One can check that the cobordism class $\varphi_*[f] = [\varphi \circ f] \in \Omega^{n+k}(Y)$ depends only on $[f] \in \Omega^n(X)$ and the homotopy class of φ (again, referring to homotopies given by proper oriented smooth maps $X \times I \to Y$). From this we see that

Proposition 1.36. *There are natural homomorphisms*

$$\varphi_* : \Omega^n(X) \to \Omega^{n+k}(Y) \text{ for } n \in \mathbb{N}$$

These will also be called transfer maps.

1.6. **Poincare duality.** Suppose now that X is a closed oriented smooth n-manifold, and let $f : M \to X$ be a smooth map from a closed smooth k-manifold to X. Observe that an orientation of M is equivalent to an orientation of the map f - indeed, applying lemma 1.19 to the composition $M \xrightarrow{f} X \to \mathrm{pt}$ yields an identification $\nu_M = \nu_{M|X} \oplus f^* \nu_X$ of stable real vector bundles on M, and given the orientation of ν_X an orientation of ν_M is equivalent to an orientation of $\nu_{M|X}$. Note also that f is proper if and only if M is compact, and that We've basically just shown:

Proposition 1.37. *There's an equivalence of categories between the closed oriented smooth k-manifolds in X, i.e. $\mathcal{Z}_k(X)$, and the proper oriented smooth manifolds over X of codimension $n-k$, i.e. $\mathcal{Z}^{n-k}(X)$.*

Proposition 1.38. *Under the above equivalence, a bordism $h : W \to X$ between closed oriented smooth k-manifolds $f_i : M_i \to X$, $i = 0, 1$ is equivalent to a cobordism $k : V \to X \times \mathbb{R}$ between the f_i.*

Proof. Given a bordism h as above, let $U \subset W$ be a "collar neighborhood" of $\partial W \subset W$, so that we have a diffeomorphism of oriented smooth manifolds with boundary $\partial W \times [0,1) \simeq U \subset W$. Since $M_0 \coprod -M_1 \simeq \partial W$, we have a natural identification

$$M_0 \times [0,1) \coprod -M_1 \times [0,1) \simeq \partial W \times [0,1)$$

Now define a smooth map

$$\psi : M_0 \times [0, \frac{1}{2}] \coprod -M_1 \times [0, \frac{1}{2}] \to \mathbb{R}$$

by $\psi(p,t) = t$ if $p \in M_0$ and $\psi(p,t) = 1 - t$ if $p \in M_1$. Extend it to all of W by sending the rest of W to $\frac{1}{2}$, and argue that the resulting map $h \times \psi : W \to X \times \mathbb{R}$ defines a cobordism between the f_i.

On the other hand given a cobordism k as above, the preimage $k^{-1}(X \times I) \subset V$ is a compact oriented smooth $k+1$-manifold with boundary

$$\partial k^{-1}(X \times I) \simeq M_0 \coprod -M_1$$

(there are a few things to check here). Now argue that the map

$$k^{-1}(X \times I) \xrightarrow{k} X \times I \xrightarrow{\text{proj}} X$$

defines a bordism between the f_i.

□

Corollary 1.39. *There are canonical isomorphisms*

$$\Omega_k(X) \simeq \Omega^{n-k}(X) \textit{ for } k \in \mathbb{N}$$

Remark 1.40. The next question would be: in what sense are these duality isomorphisms natural? Here's one answer:

Suppose $\varphi : X \to Y$ is an oriented smooth map of closed oriented smooth manifolds, say with $\dim X = m$ and $\dim Y = n$. φ induces pushforward homomorphisms $\varphi_* : \Omega_k(X) \to \Omega_k(Y)$ along with transfer homomorphisms $\varphi^* : \Omega^{m-k}(X) \to \Omega^{n-k}(Y)$. The claim to make would be: the following diagrams commute

$$\begin{array}{ccc} \Omega_k(X) & \xrightarrow{\varphi_*} & \Omega_k(Y) \\ \simeq\downarrow & & \simeq\downarrow \\ \Omega^{m-k}(X) & \xrightarrow{\varphi_*} & \Omega^{n-k}(Y) \end{array} \quad \text{for } k \in \mathbb{N} \tag{1.21}$$

This seems clear; I won't prove it here.

2. The Pontryagin-Thom construction and the spectrum MSO

2.1. Thom spaces.

It'll be worthwhile to introduce a few pieces of notation.

Let ξ be an orthogonal real n-plane bundle over a CW complex X, with projection $\pi : E(\xi) \to X$, and let $\rho : X \to \mathbb{R}$ be a positive continuous function. Set

$$B(\xi,\rho) = \{v \in E(\xi) \mid |v| < \rho(\pi(v))\} \subset E(\xi),$$
$$D(\xi,\rho) = \{v \in E(\xi) \mid |v| \le \rho(\pi(v))\} \subset E(\xi),$$
$$\text{and } S(\xi,\rho) = \{v \in E(\xi) \mid |v| = \rho(\pi(v))\} \subset E(\xi)$$

These are the **ball, disk and sphere bundles of ξ with radius** ρ. Each comes with a natural projection to X obtained by restricting π - these projections define fiber bundles with fibers B^n, D^n and S^{n-1} (the unit ball, disk and sphere in $\mathbb{R}^n$) and structure group $O(n)$. One can show that different choices of ρ yields canonically isomorphic bundles. Also, define the Thom space of ξ with radius ρ to be the quotient space

$$\text{Th}(\xi,\rho) := D(\xi,\rho)/S(\xi,\rho)$$

Again, a different choice of ρ gives a canonically homeomorphic Thom space. The distinguished point $S(\xi,\rho)/S(\xi,\rho) \in \mathrm{Th}(\xi,\rho)$ will be denoted by ∞ **NOTE: expand on this.** In the case $\rho = 1 =$ const I'll just drop it from the notation altogether; so, $B(\xi) := B(\xi, 1)$, etc.

A few useful facts about Thom spaces:

- If $\xi : E(\xi) \to X$ and $\eta : E(\eta) \to Y$ are orthogonal real vector bundles over CW complexes and

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\tilde{f}} & E(\eta) \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} \tag{2.1}$$

  is a map of vector bundles such that each linear map $E(\xi)_x \to E(\eta)_{f(x)}$ is an isometric embedding, then we obtain a map of pairs

  $f : (D(\xi), S(\xi)) \to (D(\eta), S(\eta))$ inducing a continuous map $\mathrm{Th}f : \mathrm{Th}(\xi) \to \mathrm{Th}(\eta)$.

  This provides a sense in which forming Thom spaces is functorial.
- Let $\xi : E(\xi) \to X$ and $\eta : E(\eta) \to Y$ be orthogonal real vector bundles over CW complexes, and let $\xi \times \eta : E(\xi \times \eta) = E(\xi) \times E(\eta) \to X \times Y$ be their product. Then there's a natural homeomorphism
$$\mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta) \simeq \mathrm{Th}(\xi \times \eta)$$
  On the other hand, the Thom space of the disjoint union $E(\xi) \coprod E(\eta) \to X \coprod Y$ is computed as $\mathrm{Th}(\xi \coprod \eta) \simeq \mathrm{Th}(\xi) \vee \mathrm{Th}(\eta)$.
- Let $E(\epsilon^N) \to X$ be a trivial N-plane bundle over a CW complex X. Then there's a natural identification $\mathrm{Th}(\epsilon^N) \simeq \Sigma^N X_+$.

Let $\gamma_n : E(\gamma_n) \to BSO(n)$ be the tautological oriented real n-plane bundle, and define $MSO(n) := \mathrm{Th}(\gamma_n)$. The map of orthogonal real vector bundles

$$\begin{array}{ccc} E(\gamma_n \oplus \epsilon) & \longrightarrow & E(\gamma_{n+1}) \\ \downarrow & & \downarrow \\ BSO(n) & \xrightarrow{B\iota} & BSO(n+1) \end{array} \tag{2.2}$$

(where $B\iota$ classifies the oriented $n+1$-plane bundle $\gamma_n \oplus \epsilon$, or equivalently the usual inclusion $SO(n) \subset SO(n+1)$ induces a map of Thom spaces

$$MSO(n) \wedge S^1 \simeq \mathrm{Th}(\gamma_n \oplus \epsilon) \to Th(\gamma_n + 1) = MSO(n+1)$$

which we may as well call $M\iota$- if we begin with the usual models of the classifying spaces $BSO(n)$, i.e. the Grassmannians of oriented n-planes in $\mathbb{R}^\infty$, the Thom spaces $MSO(n)$ will be pointed CW complexes. If we choose reasonable models for the classifying maps $B\iota$, the maps $M\iota$ will be subcomplex inclusions. We've shown:

Proposition 2.1. *The Thom spaces $MSO(n)$, $n \in \mathbb{N}$ form a CW spectrum, which will be denoted by MSO.*

For any pointed CW complex X, let $\Sigma^\infty X$ denote the suspension spectrum of X, with n-th term $\Sigma^n X$ (take this to be a point if $n \leq 0$. From the spectrum MSO we obtain reduced generalized (co)homology theories $M\tilde{S}O_*$ and $M\tilde{S}O^*$ on the homotopy category of pointed CW complexes defined by

$$M\tilde{S}O_n(X) = [S, MSO \wedge \Sigma^\infty X]_n \text{ and } M\tilde{S}O^n(X) = [\Sigma^\infty X, MSO]_{-k}$$

There are corresponding non-reduced (co)homology theories on the homotopy category of CW complexes, defined by

$MSO_n(X) := \tilde{MSO}_n(X_+) = [S, MSO \wedge \Sigma^\infty X_+]_n$ and $MSO^n(X) := \tilde{MSO}^n(X_+) = [\Sigma^\infty X_+, MSO]_{-k}$

(see for instance chapter 4 in Hatcher's *Algebraic topology* or Adams's *Stable homotopy and generalised homology*).

Theorem 2.2 (Thom). *For a smooth manifold X, there are natural isomorphisms*

$$\Omega_n(X) \simeq MSO_n(X) \text{ and } \Omega^n(X) \simeq MSO^n(X)$$

In other words, the oriented cobordism functor Ω^* on the homotopy category of smooth manifolds is represented by the spectrum MSO, and a similar statement holds for bordism. As an upshot, we see:

Corollary 2.3. *Oriented (co)bordism extends to a generalized (co)homology theory on the homotopy category of CW complexes, with representing spectrum MSO.*

The rest of this section will be devoted to a sketched proof of this theorem. The key ingredients are what's come to be known as the Pontryagin-Thom construction and the homotopy transversality theorem.

2.2. **Pontryagin-Thom collapse maps.** Suppose $\iota : M \to X$ is a proper smooth embedding of a smooth m-manifold (without boundary) M into a Riemannian n-manifold X (again without boundary), and let $\nu_{M|X}$ be its normal bundle. Moving forward I'll just view M as a Riemannian closed submanifold of X (with the induced metric). Let $\exp : TX \to X$ be the exponential map of X. We may restrict this to the total space of $\nu_{M|X}$ to obtain a smooth map $\exp : NM \to X$.

Theorem 2.4 (Existence of tubular neighborhoods). *There's a positive smooth function $\rho : M \to \mathbb{R}$ so that the restriction*

$$\exp : B(\nu_{M|X}, \rho) \to X$$

maps $B(\nu_{M|X}, \rho)$ diffeomorphically onto a neighborhood $B \subset X$ of M.

The neighborhood $B \subset X$ of M is referred to as a "tubular neighborhood" of M. It follows that upon replacing ρ with a smaller positive smooth function (for instance $\frac{1}{2}\rho$) if necessary, the exponential defines a diffeomorphism of smooth manifolds with boundary $\exp : D(\nu_{M|X}, \rho) \to D \subset X$ where D is a "regular domain" in X, i.e. a codimension-0 closed submanifold with boundary.

Here's the fun part: extend the composition

$$D \xrightarrow{\log = \exp^{-1}} D(\nu_{M|X}, \rho) \xrightarrow{\text{quot}} \mathrm{Th}(\nu_{M|X}, \rho) \simeq \mathrm{Th}(\nu_{M|X})$$

to a continuous function on all of X by sending $X - \mathrm{int}D \to \{\infty\} \in \mathrm{Th}(\nu_{M|X})$. The result is a continuous map $\varphi : X \to \mathrm{Th}(\nu_{M|X})$, often referred to as a "collapse" map since it crushes the complement of a tubular neighborhood of M to a point.

Here's a useful generalization of the above construction. Suppose instead we begin with a proper smooth map $f : M \to X$ from a smooth manifold M to a Riemannian manifold X, together with a factorization

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[d, "="'] & E(\xi) \arrow[d, "\pi"'] \\ M \arrow[r, "f"] & X \end{tikzcd} \tag{2.3}$$

where ξ is an orthogonal real vector bundle over X and ι is an embedding. Note that there's a canonical Riemannian structure on $E(\xi)$ (consider the isomorphism $\tau_{E(\xi)} \simeq \pi^*(\tau_M \oplus \xi)$). Hence we can view M as a closed submanifold of the Riemannian manifold $E(\xi)$, with normal bundle $\nu_{M|E(\xi)}$, and the above construction yields a collapse map

$$\varphi : E(\xi) \to \mathrm{Th}(\nu_{M|E(\xi)})$$

Now recall that $\pi|_M$ is proper, so for any compact set $K \subset X$, $M \cap E(\xi)|_K = \pi^{-1}(K)$ is compact and hence the function $M \to \mathbb{R}$ taking $p \mapsto |p|$ is bounded on $M \cap E(\xi)|_K$. It follows that there's a positive smooth function $\rho : X \to \mathbb{R}$ so that $M \subset B(\xi, \rho)$. Evidently we may even arrange so that $B \subset B(\xi, \rho)$, where B is a tubular neighborhood of M in $E(\xi)$, and so we may extend φ to a continuous map

$$\varphi : \mathrm{Th}(\xi) \to \mathrm{Th}(\nu_{M|E(\xi)}) \text{ by sending } E(\xi) - B(\xi, \rho) \to \{\infty\}$$

In particular if $\xi = \epsilon^N$ is a trivial vector bundle, the result is a a continuous map $\varphi : \Sigma^N X_+ \to \mathrm{Th}(\nu_{M|E(\epsilon^N)})$.

2.3. **Sketch of the proof of Thom's theorem 2.2.** Suppose X is a smooth manifold, and let $f : M \to X$ be a closed oriented smooth n-manifold in X. Let $\iota : M \to \mathbb{R}^N$ be a smooth embedding with normal bundle $\nu_{M|\mathbb{R}^N}$ - since M is oriented over a point, this normal bundle comes with an orientation.

So, let $g : M \to BSO(N-n)$ be a map classifying $\nu_{M|\mathbb{R}^N}$ - it will fit into a commutative diagram

$$\begin{array}{ccc} NM & \xrightarrow{\tilde{g}} & E(\gamma_{N-n}) \\ \downarrow & & \downarrow \\ M & \xrightarrow{g} & BSO(N-n) \end{array} \tag{2.4}$$

Now take the product of g with f; the result is a map of vector bundles

$$\begin{array}{ccc} NM & \xrightarrow{\tilde{g}\times f} & E(\gamma_{N-n} \times 0) \\ \downarrow & & \downarrow \\ M & \xrightarrow{g\times f} & BSO(N-n) \times X \end{array} \tag{2.5}$$

giving isomorphisms of vector space fibers, and thus inducing a continuous map of Thom spaces

$$\mathrm{Th}(g \times f) : \mathrm{Th}(\nu_{M|\mathbb{R}^N}) \to \mathrm{Th}(\gamma_{N-n} \times 0) \simeq MSO(N-n) \wedge X_+$$

Here's the last ingredient: from the embedding $\iota : M \to \mathbb{R}^N$ we obtain a Pontryagin-Thom collapse map $\varphi : S^N \to \mathrm{Th}(\nu_{M|\mathbb{R}^N})$ and now the composition

$$S^N \xrightarrow{\varphi} \mathrm{Th}(\nu_{M|\mathbb{R}^N}) \xrightarrow{\mathrm{Th}(g\times f)} MSO(N-n) \wedge X_+$$

yields a continuous map $S^N \to MSO(N-n) \wedge X_+$, defining a stable homotopy class in $[S, MSO \wedge \Sigma^\infty X_+]_n$. One can show that it depends only on the bordism class $[f] \in \Omega_n(X)$. In this way we obtain a function $\Omega_n(X) \to MSO_n(X)$.

On the other hand, suppose we're given a stable homotopy class in $[S, MSO \wedge \Sigma^\infty X_+]_n$. Since the natural homomorphism

$$\mathrm{colim}_k \pi_{k+n}(MSO(k) \wedge X_+) \to [S, MSO \wedge \Sigma^\infty X_+]_n$$

is an isomorphism, we may represent this class by a pointed continuous map

$$f : S^{k+n} \to MSO(k) \wedge X_+ \simeq \mathrm{Th}(\gamma_k \times 0)$$

where 0 denotes the trivial 0-plane bundle on X. Since S^{k+n} is compact, for sufficiently large l this map f will actually factor through some continuous map

$$f : S^{k+n} \to \mathrm{Th}(\gamma_k^l \times 0)$$

where γ_k^l is the tautological k-plane bundle over $\tilde{G}_k\mathbb{R}^l$, the Grassmannian of oriented k-planes in $\mathbb{R}^l$. Now, except in the case $k = 1$ the Thom space $\mathrm{Th}(\gamma_k^l \times 0)$ is *not* a smooth manifold (one can see this via the Thom isomorphism and Poincare duality). But it *is* a smooth manifold away from the zero-section, and so using a combination of Whitney approximation and the homotopy transversality theorem one can obtain a map

$$\tilde{f} : S^{k+n} \to \mathrm{Th}(\gamma_k^l \times 0) \text{ homotopic to } f$$

with the properties that

- $\tilde{f}$ is smooth away from $\infty \in \mathrm{Th}(\gamma_k^l \times 0)$, i.e. it's smooth on the open submanifold $U := S^{k+n} - \tilde{f}^{-1}(\infty) \subset S^{k+n}$. Note that since $\tilde{f}$ sends the north pole $\infty \in S^{k+n}$ to $\infty \in \mathrm{Th}(\gamma_k^l \times 0)$, we may view U as an open set in $\mathbb{R}^{k+n} = S^{k+n} - \{\infty\}$.
- The smooth map $\tilde{f} : U \to E(\gamma_k^l \times 0)$ is transverse to the 0-section $\sigma : \tilde{G}_k\mathbb{R}^l \times X \to E(\gamma_k^l \times 0)$. Denote the image of σ by $Z \subset \mathrm{Th}(\gamma_k^l \times 0)$.

Now set $M := \tilde{f}^{-1}(Z) \subset S^{k+n}$. Since $\tilde{f}$ is smooth and transverse to the 0-section, M is a closed smooth submanifold of S^{k+n} of codimension k. Note that it fits into the cartesian square

$$\begin{array}{ccc} M & \xrightarrow{\sigma^{-1}\circ\tilde{f}} & \tilde{G}_k\mathbb{R}^l \times X \\ \iota\big\downarrow & & \sigma\big\downarrow \\ U & \xrightarrow{\tilde{f}} & E(\gamma_k^l \times 0) = E(\gamma_k^l) \times X \end{array} \tag{2.6}$$

Call the compositions

$$M \xrightarrow{\sigma^{-1}\circ\tilde{f}} \tilde{G}_k\mathbb{R}^l \times X \xrightarrow{\text{proj}} \tilde{G}_k\mathbb{R}^l \text{ and } M \xrightarrow{\sigma^{-1}\circ\tilde{f}} \tilde{G}_k\mathbb{R}^l \times X \xrightarrow{\text{proj}} X$$

φ and ψ respectively. Now observe that the normal bundle $\nu_{M|\mathbb{R}^{k+n}}$ of M in $U \subset \mathbb{R}^{k+n}$ is computed as

$$\nu_{M|\mathbb{R}^{k+n}} \simeq \nu_{\tilde{G}_k\mathbb{R}^l|E(\gamma_k^l)} \simeq \varphi^*\gamma_k^l$$

and in this way we obtain an orientation of the stable normal bundle of M. Thus the map $\psi : M \to X$ defines a closed oriented smooth n-manifold in X, representing a cobordism class $[\psi] \in \Omega_n(X)$. One can show it depends only on the stable homotopy class of f, and in this way we obtain a function $MSO_n(X) \to \Omega_n(X)$.

One now shows (as Thom did) that the functions between $\Omega_n(X)$ and $MSO_n(X)$ described above are mutual inverses. To see that they are in fact isomorphisms of abelian groups, suppose $f_i : M_i \to X$ are 2 closed oriented smooth n-manifolds in X, representing classes $[f_i] \in \Omega_n(X)$, $i = 0, 1$. Form their disjoint union over X, the map $f_0 \coprod f_1 : M_0 \coprod M_1 \to X$. Observe that we may construct an embedding $\iota : M_0 \coprod M_1 \to \mathbb{R}^N$ so that (identifying $M_0 \coprod M_1$ with its image)

$$x_1 < 0 \text{ when } (x_1, \ldots, x_N) \in M_0 \text{ and}$$
$$x_1 > 0 \text{ when } (x_1, \ldots, x_N) \in M_1$$

Note also that the normal bundle of this embedding will be

$$\nu_{M_0|\mathbb{R}^N} \coprod \nu_{M_1|\mathbb{R}^N} : NM_0 \coprod NM_1 \to M_0 \coprod M_1$$

Following the above construction we'll obtain a map

$$S^N \xrightarrow{\text{collapse}} \mathrm{Th}(\nu_{M_0|\mathbb{R}^N} \coprod \nu_{M_1|\mathbb{R}^N}) \xrightarrow{\mathrm{Th}(g_0 \coprod g_1)\times(f_0 \coprod f_1)} MSO(N-n) \wedge X_+$$

Given the conditions on ι together with the homeomorphism $\mathrm{Th}(\nu_{M_0|\mathbb{R}^N} \coprod \nu_{M_1|\mathbb{R}^N}) \simeq \mathrm{Th}(\nu_{M_0|\mathbb{R}^N}) \wedge \mathrm{Th}(\nu_{M_1|\mathbb{R}^N})$ we may write the collapse map as

$$S^N \xrightarrow{\text{cooperation}} S^N \vee S^N \xrightarrow{\varphi_0 \vee \varphi_1} \mathrm{Th}(\nu_{M_0|\mathbb{R}^N}) \wedge \mathrm{Th}(\nu_{M_1|\mathbb{R}^N})$$

where "cooperation" denotes the map $S^N \to S^N \vee S^N$ collapsing the $(x_i) \in S^N$ with $x_1 = 0$ (which makes S^N a cogroup object in the homotopy category of CW complexes), and φ_0, φ_1 denote the respective collapse maps associated to the embeddings of M_0, M_1 in $\mathbb{R}^N$. Proceeding in this way one sees that if $\psi_0, \psi_1, \psi : S^N \to MSO(N-n) \wedge X_+$ are the maps obtained from $f_0, f_1, f_0 \coprod f_1$ via the Pontryagin-Thom construction, then we have

$$\psi = \psi_0 + \psi_1 \in [S, MSO \wedge X_+]_n$$

In the case of cobordism the correspondence is a bit easier to describe. Given a proper oriented smooth manifold $f : M \to X$ of codimension n, let $\iota : M \to E(\epsilon^N)$ be a smooth embedding over X, with oriented normal bundle $\nu_{M|E(\epsilon^N)}$. From this we obtain a Pontryagin-Thom collapse map $\Sigma^N X_+ \simeq \mathrm{Th}(\epsilon^N) \xrightarrow{\varphi} \mathrm{Th}(\nu_{M|E(\epsilon^N)})$. If $\psi : M \to BSO(N+n)$ is a classifying map for $\nu_{M|E(\epsilon^N)}$ fitting into a commutative diagram

$$\begin{array}{ccc} NM & \xrightarrow{\tilde{\psi}} & E(\gamma_{N+n}) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\psi} & BSO(N+n) \end{array} \tag{2.7}$$

it induces a map of Thom spaces $\mathrm{Th}\tilde{\psi} : \mathrm{Th}(\nu_{M|E(\epsilon^N)}) \to MSO(N+n)$, and the composition

$$\Sigma^N X_+ \xrightarrow{\varphi} \mathrm{Th}(\nu_{M|E(\epsilon^N)}) \xrightarrow{\mathrm{Th}\tilde{\psi}} MSO(N-+n)$$

gives a map $\Sigma^N X_+ \to MSO(N+n)$, defining a class in $[\Sigma^\infty X_+, MSO]_{-n}$. One can show it depends only on the cobordism class of $f : M \to X$, and in this way we obtain a function $\Omega^n(X) \to MSO^n(X)$.

On the other hand given a stable homotopy class in $[\Sigma^\infty X_+, MSO]_{-n}$, arguing that the natural homomorphism

$$\mathrm{colim}_k[\Sigma^k X_+, MSO(k+n)] \to [\Sigma^\infty X_+, MSO]_{-n}$$

is an isomorphism so that our class is represented by a map $\Sigma^k X_+ \to MSO(k+n)$, and using the finite-dimensionality of $\Sigma^k X_+$ together with some form of cellular approximation to show this factors through a map

$$f : \mathrm{Th}(\epsilon^k) = \Sigma^k X_+ \to \mathrm{Th}(\gamma^l_{k+n})$$

Again carefully appealing to Whitney approximation and homotopy transversality, one obtains a map $\tilde{f} : \mathrm{Th}(\epsilon^k) \to \mathrm{Th}(\gamma^l_{k+n})$ homotopic to f so that

- $\tilde{f}$ is smooth away from $\infty \in \mathrm{Th}(\gamma^l_{k+n})$, i.e. it's smooth on the open set $U := \mathrm{Th}(\epsilon^k) - \tilde{f}^{-1}(\infty)$. Note that $\tilde{f}(\infty) = \infty$, so U may be viewed as an open submanifold of $X \times \mathbb{R}^k$.
- The smooth map $\tilde{f} : U \to E(\gamma^l_{k+n})$ is transverse to the 0-section $\sigma : \tilde{G}_{k+n}\mathbb{R}^l\mathbb{R}^l \to E(\gamma^l_{k+n})$. Denote the image of σ by $Z \subset E(\gamma^l_{k+n})$.

Now set $M := \tilde{f}^{-1}(Z) \subset U$ - it's a closed submanifold of U with codimension $K+n$, and so its codimension over X is n. By a more detailed analysis one can arrange so that M lies in the unit ball bundel of ϵ^k, so that it's proper over X. Also note that M it fits into a cartesian diagram

$$\begin{array}{ccc} M & \xrightarrow{\sigma^{-1}\circ \tilde{f}} & \tilde{G}_{k+n}\mathbb{R}^l \\ {\scriptstyle \iota}\big\downarrow & & {\scriptstyle \sigma}\big\downarrow \\ U & \longrightarrow & E(\gamma^l_{k+n}) \end{array} \tag{2.8}$$

and this can be used to identify the stable normal bundle of M over X with $(\sigma^{-1}\circ \tilde{f})^*\gamma^l_{k+n}$. This provides an orientation of M over X. Hence $M \to X$ is a proper oriented smooth manifold over X of codimension n, defining a cobordism class in $\Omega^n(X)$, which one can show depends only on the stable homotopy class of f in $[\Sigma^\infty X_+, MSO]_{-n}$, so we get a function $MSO^n(X) \to \Omega^n(X)$. One can show it's mutually inverse to $\Omega^n(X) \to MSO^n(X)$. In fact these functions are homomorphisms - this follows by an argument similar to the one given in the case of bordism.

2.4. **A stable-homotopy-theoretic description of products, transfers and Poincare duality.** Let γ_m, γ_n be the tautological oriented real m,n-plane bundles over $BSO(m), BSO(n)$ respectively, and let $\gamma_m \times \gamma_n$ be their product- it's an oriented real $m+n$-plane bundle over $BSO(m)\times BSO(n)$, and so its classified by a map of vector bundles

$$\begin{array}{ccc} E(\gamma_m)\times E(\gamma_n) & \longrightarrow & E(\gamma_{m+n}) \\ \big\downarrow & & \big\downarrow \\ BSO(m)\times BSO(n) & \xrightarrow{B\iota} & BSO(m+n) \end{array} \tag{2.9}$$

giving isomorphisms on fibers. The lower horizontal map $B\iota$ classifies the usual inclusion $SO(m)\times SO(n) \subset SO(m+n)$ as block diagonal matrices. From this we obtain a map of Thom spaces

$$\mu_{mn} : MSO(m) \wedge MSO(n) \to MSO(m+n)$$

One can check that these are associative in the sense that the following diagrams homotopy commute

$$\begin{array}{ccc} MSO(l)\wedge MSO(m)\wedge MSO(n) & \xrightarrow{\mu_{lm}\wedge \mathrm{id}} & MSO(l+m)\wedge MSO(n) \\ {\scriptstyle \mathrm{id}\wedge\mu_{mn}}\big\downarrow & & {\scriptstyle \mu_{(l+m)n}}\big\downarrow \\ MSO(l)\wedge MSO(m+n) & \xrightarrow{\mu_{l(m+n)}} & MSO(l+m+n) \end{array} \quad \text{for } l,m,n\in\mathbb{N} \tag{2.10}$$

and commutative in the sense that the following diagrams homotopy commute

$$\begin{array}{ccc} MSO(m)\wedge MSO(n) & \xrightarrow{\tau} & MSO(n)\wedge MSO(m) \\ {\scriptstyle \mu_{mn}}\big\downarrow & & {\scriptstyle \mu_{nm}}\big\downarrow \\ MSO(m+n) & \xrightarrow{=} & MSO(m+n) \end{array} \quad \text{for } m,n\in\mathbb{N} \tag{2.11}$$

(in both cases one begins with an evidently commutative diagram of classifying maps of vector bundles, and then passes to the induced maps of Thom spaces). Now consider $\mathbb{R}^n$ as a trivial oriented real vector bundle $\epsilon^n : E(\epsilon^n) \to \mathrm{pt}$ over a point. It's classified by a map of vector bundles

$$\begin{array}{ccc} E(\epsilon^n) & \longrightarrow & E(\gamma_n) \\ \big\downarrow & & \big\downarrow \\ \mathrm{pt} & \longrightarrow & BSO(n) \end{array} \tag{2.12}$$

which induces a map of Thom spaces $\eta_n : S^n = \mathrm{Th}(\epsilon^n) \to MSO(n)$. Notice that the classifying map of the product $\gamma_m \times \epsilon^n$ induces a map of Thom spaces

$$\Sigma^n MSO(m) = \mathrm{Th}(\gamma_m \times \epsilon^n) \to MSO(m+n)$$

which can be identified as the composition

$$MSO(m) \wedge S^n \xrightarrow{\mathrm{id}\wedge\eta_n} MSO(m) \wedge MSO(n) \xrightarrow{\mu_{mn}} MSO(m+n)$$

and it's just a composition of structure maps in the spectrum MSO. We've essentially shown:

Proposition 2.5. *The maps μ_{mn}, $m, n \in \mathbb{N}$ and $\eta_n : S^n \to MSO(n)$ induce an operation*

$$\mu : MSO \wedge MSO \to MSO \text{ and structure map } \eta : S \to MSO$$

making MSO an associative, commutative ring CW spectrum.

Let's show that the (external) products in oriented (co)bordism defined in section 1 coincide with those obtained from the ring structure on MSO under the isomorphisms of theorem 2.2. So, let X, Y be smooth manifolds and let $f : M \to X, g : N \to Y$ be closed oriented smooth m, n manifolds in X, Y respectively. If $\iota_M : M \to \mathbb{R}^k, \iota_N : N \to \mathbb{R}^l$ are smooth embeddings with normal bundles ν_M, ν_N respectively then $\iota_{M\times N} := \iota_M \times \iota_N : M \times N \to \mathbb{R}^{k+l}$ is a smooth embedding with normal bundle $\nu_{M\times N} \simeq \nu_{M|\mathbb{R}^k} \times \nu_{N|\mathbb{R}^l}$. Proceeding in this way one may rewrite the map

$$S^{k+l} \xrightarrow{\text{collapse}} \mathrm{Th}(\nu_{M\times N}) \to MSO(k+l-m-n) \wedge (X \times Y)_+$$

as

$$S^k \wedge S^l \xrightarrow{\text{collapse}\wedge\text{collapse}} \mathrm{Th}(\nu_M) \wedge \mathrm{Th}(\nu_N) \to MSO(k-m) \wedge X_+ \wedge MSO(l-n) \wedge Y_+$$

$$\to MSO(k-m) \wedge MSO(l-n) \wedge X_+ \wedge Y_+ \xrightarrow{\mu_{(k-m)(l-n)}} MSO(k+l-m-n) \wedge (X \times Y)_+$$

(obviously there are details to check). Similarly, suppose $f : M \to X, g : N \to Y$ are proper oriented smooth manifolds over X, Y with codimensions m, n - then $f \times g : M \times N \to X \times Y$ is a proper oriented smooth manifold over $X \times Y$ of codimension $m+n$. If $\iota_M : M \to X \times \mathbb{R}^k, \iota_N : N \to Y \times \mathbb{R}^l$ are embeddings over X, Y with normal bundles ν_M, ν_N then the product

$$M \times N \xrightarrow{\iota_M\times\iota_N} X \times \mathbb{R}^k \times Y \times \mathbb{R}^l \simeq X \times Y \times \mathbb{R}^{k+l}$$

is an embedding over $X \times Y$. From here one shows that the map

$$\Sigma^{k+l}(X \times Y)_+ \xrightarrow{\text{collapse}} \mathrm{Th}(\nu_{M\times N}) \xrightarrow{\text{classify}} MSO(k+l+m+n)$$

obtained via the Pontryagin-Thom construction can be identified with the map

$$\Sigma^{k+l}(X \times Y)_+ \simeq \Sigma^k X_+ \wedge \Sigma^l Y_+ \xrightarrow{\text{collapse}\wedge\text{collapse}} \mathrm{Th}(\nu_M) \wedge \mathrm{Th}(\nu_N)$$

$$\xrightarrow{\text{classify}\wedge\text{classify}} MSO(k+m) \wedge MSO(l+n) \xrightarrow{\mu_{(k+m)(l+n)}} MSO(k+l+m+n)$$

(again, there are plenty of details to check).

Among other things, this shows:

Proposition 2.6. *Let X be a smooth manifold. Then the isomorphisms $\Omega^n(X) \simeq MSO^n(X)$ fit together to give a natural isomorphism of graded rings*

$$\Omega^*(X) \simeq MSO^*(X)$$

To carry on with a homotopy-theoretic discussion of transfer maps and Poincare duality, we'll need Thom isomorphisms in (co)bordism

Theorem 2.7. *Let X be a smooth manifold and let $\xi : E(\xi) \to X$ be an oriented real n-plane bundle over X. Then the cobordism class $u(\xi) \in \tilde{\Omega}^n(\mathrm{Th}\xi)$ represented by the "0-section" $X \xrightarrow{0} E(\xi) \xrightarrow{quot} \mathrm{Th}\xi$ generates $\tilde{\Omega}^*(\mathrm{Th}\xi)$ as a free $\Omega^*(X)$-module of rank 1. In particular, there are natural isomorphisms $\Omega^m(X) \simeq \tilde{\Omega}^{m+n}(\mathrm{Th}\xi)$ for all $m \in \mathbb{Z}$. Similarly, taking slant products with $u(\xi)$ defines natural isomorphisms $\tilde{\Omega}_{m+n}(\mathrm{Th}\xi) \simeq \Omega_m(X)$ for all $m \in \mathbb{Z}$.*

Note that under the isomorphism $\tilde{\Omega}^n(\mathrm{Th}\xi) \simeq \tilde{MSO}^n(\mathrm{Th}\xi)$, $u(\xi)$ corresponds to the stable homotopy class of the map $\mathrm{Th}\xi \to MSO(n)$ on Thom spaces induced by a classifying map

$$\begin{array}{ccc} E(\xi) & \longrightarrow & E(\gamma_n) \\ \downarrow & & \downarrow \\ X & \longrightarrow & BSO(n) \end{array} \tag{2.13}$$

NOTE: provide a proof (or at least a sketch) of this theorem; also describe external and slant products from a homotopy-theoretic perspective.

Now suppose $f : X \to Y$ is a proper oriented smooth map of codimension n, and factor it through an embedding $\iota : X \to E(\epsilon^N)$ over Y, where ϵ^N denotes the trivial real N-plane bundle over Y. Let $\nu_{X|E(\epsilon^N)}$ be the oriented normal bundle of X in $E(\epsilon^N)$. We can construct a collapse map

$$\varphi : \Sigma^N Y_+ \simeq \mathrm{Th}(\epsilon^N) \to \mathrm{Th}(\nu_{X|E(\epsilon^N)})$$

and this induces homomorphisms

$$\tilde{\Omega}_*(\Sigma^N Y_+) \xrightarrow{\varphi_*} \tilde{\Omega}_*(\mathrm{Th}\nu_{X|E(\epsilon^N)}) \text{ and}$$

$$\tilde{\Omega}^*(\mathrm{Th}\nu_{X|E(\epsilon^N)}) \xrightarrow{\varphi^*} \tilde{\Omega}^*(\Sigma^N Y_+)$$

Using the suspension and Thom isomorphisms, we may identify these with the transfer homomorphisms of section 1.

NOTE: explain how Atiyah's identification of the SW dual of a closed smooth manifold M as $\mathrm{Th}\nu_M$ where ν_M is the stable normal bundle of M gives a stable-homotopy-theoretic description of Poincare duality. See notes on *The J-homomorphism and the Adams conjecture*.

Partial computations of the cobordism ring $\Omega^*(pt)$.

3. Generalized (co)bordism theories

3.1. Proper maps, relative stable normal bundles and transversality.

3.2. Fibrations over BO and structures on stable normal bundles.

3.3. B (co)bordism theory.

3.4. Thom spectra and the stable-homotopy-theoretic point of view.

3.5. B oriented cohomology theories.

4. Complex oriented cohomology theories and formal group laws

Let $\mathcal{Z}$ denote the category of smooth manifolds (without boundary), with morphisms the smooth maps; for each $n \in \mathbb{N}$ let $\mathcal{Z}_n$ denote the full subcategory of smooth n-manifolds.

Note that within $\mathcal{Z}$ there's a distinguished class of morphisms, the proper complex oriented maps; note that a composition of proper complex oriented maps is again proper, with a natural complex orientation, and of course identity maps are proper and complex oriented.

Thus restricting the morphisms of $\mathcal{Z}$ to the proper complex oriented ones, we obtain a category, say $\mathcal{Z}_{PCO}$, with a faithful embedding $\mathcal{Z}_{PCO} \to \mathcal{Z}$.

Definition 4.1. A **complex oriented set-valued functor** h **on the category of smooth manifolds** is a contravariant functor $h : \mathcal{Z} \to \text{Set}$ which is also covariant when restricted to $\mathcal{Z}_{PCO}$ (Quillen describes h as a covariant functor together with "Gysin maps" $f_* : h(X) \to h(Y)$ for proper complex oriented maps $f : X \to Y$), subject to the following additional restrictions:

- If $f_0, f_1 : X \to Y$ are homotopic smooth maps, then $f_0^* = f_1^* : h(Y) \to h(X)$.
- If

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{g'} & Z \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & X \end{array} \tag{4.1}$$

  is a cartesian square of manifolds where f is proper and complex oriented, g is transverse to f and f' is given the pullback of the complex orientation of f, then

$$f'_* g'^* = g^* f_* : h(Z) \to h(Y)$$

A natural transformation from h to another such functor k will be a natural transformation θ between the contravariant functors $h, k : \mathcal{Z} \to \text{Set}$ which is also a natural transformation of covariant functors $h, k : \mathcal{Z}_{PCO} \to \text{Set}$.

Of course, one could also consider complex oriented functors on the category of smooth manifolds taking values in more interesting categories. The main case of interest will be the following: suppose E is an associative, commutative ring CW spectrum representing a cohomology theory E^*; then E^* defines a contravariant functor on the category of smooth manifolds taking values in graded commutative rings (here I mean graded commutative in the topologist's sense), which is homotopy invariant. Suppose in addition that for every proper complex oriented map $f : X \to Y$ of codimension n there's an induced homomorphism of graded abelian groups

$$f_* : E^*(X) \to E^*(Y) \text{ of degree } -n, \text{ i.e. } f_* \text{ sends } E^i(X) \to E^{i+n}(Y)$$

and the contravariant/covariant induced maps are compatible with cartesian squares in the sense described above. Then we'll say E^* is a **complex oriented cohomology theory**. A natural transformation from E^* to another such cohomology theory F^* will consist of a morphism $\theta : E \to F$ of associative, commutative ring CW spectra so that the resulting natural transformation of cohomology theories $\theta : E^* \to F^*$ is also a natural transformation of covariant functors on $\mathcal{Z}_{PCO}$ (briefly, θ must be compatible with "Gysin homomorphisms").

Theorem 4.2. *Let h be a complex oriented set-valued functor on the category of smooth manifolds and suppose $a \in h(\text{pt})$. Then there exists a unique natural transformation of complex oriented functors*

$$\theta : MU^* \to h \text{ so } \theta(1) = a \in h(\text{pt})$$

where $1 \in MU^*(\text{pt})$ is the cobordism class of the identity map $\text{pt} \to \text{pt}$.

Moreover E^* is a complex oriented cohomology theory then there is a unique natural transformation of complex oriented cohomology theories $\theta : MU^* \to E^*$.

Thus MU^* is the universal complex oriented cohomology theory, and in fact the universal complex oriented set-valued functor on the category of smooth manifolds.

Proof. Let $f : M \to X$ be a proper complex oriented map representing a class $[f] \in MU^*(X)$. Certainly $[f] = f_*[\mathrm{id}_M]$, with id_M the identity of M; also, note that $[\mathrm{id}_M] = \pi_M^* 1 \in MU^*(M)$, where $\pi_M : M \to \mathrm{pt}$ is the usual projection. Thus

$$[f] = f_*\pi_M^* 1, \text{ and so } \theta[f] = f_*\pi_M^*\theta(1) \in h(X)$$

and so there can be at most 1 natural transformation θ with $\theta(1) = a$. We'll now show that one can *define* a natural transformation θ with $\theta(1) = a$ by setting $\theta[f] := f_*\pi_M^* a \in h(X)$.

One must prove that the element $f_*\pi_M^* a$ depends only on the cobordism class of f. So, let $g : W \to X \times \mathbb{R}$ be a cobordism between proper complex oriented smooth maps $f_i : M_i \to X$ representing classes $[f_i] \in MU^*(X)$, with $i = 0, 1$. Then W is complex oriented, the 2 inclusions $\iota_i : X \simeq X \times \{i\} \subset X \times \mathbb{R}$, $i = 0, 1$ are transverse to W, and we're given natural isomorphisms $M_i \simeq X \times_{\iota_i} W$ over X for $i = 0, 1$; thus we have cartesian squares

$$\begin{array}{ccc} M_i \simeq X \times_{\iota_i} W & \xrightarrow{\iota_i'} & W \\ {\scriptstyle f_i}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ X & \xrightarrow{\iota_i} & X \times \mathbb{R} \end{array} \tag{4.2}$$

and so

$$f_{i*}\iota_i'^* = \iota_i^* g_* : h(W) \to h(X) \text{ for } i = 0, 1$$

The maps ι_i are obviously homotopic, and hence $\iota_0^* = \iota_1^* : h(X \times \mathbb{R}) \to h(X)$; this implies that $f_{0*}\iota_0'^* = f_{1*}\iota_1'^* : h(W) \to h(X)$. Now observe that the projection $\pi_{M_i} M_i \to \mathrm{pt}$ may be factored as $M_i \xrightarrow{\iota_i'} W \xrightarrow{\pi_W} \mathrm{pt}$ and hence

$$f_{i*}\pi_{M_i}^* a = f_{i*}\iota_i'^*\pi_W^* a \in h(X) \text{ for } i = 0, 1$$

Thus

$$f_{0*}\pi_{M_0}^* a = f_{0*}\iota_0'^*\pi_W^* a = f_{1*}\iota_1'^*\pi_W^* a = f_{1*}\pi_{M_i}^* a \in h(X)$$

as desired.

Now suppose E^* is a complex oriented cohomology theory. By the first part of this theorem, we know there's a unique natural transformation of complex oriented *set-valued* functors on the category of smooth manifolds $\theta : MU^* \to E^*$. Our first goal is to show that this is indeed a natural transformation of contravariant functors taking values in graded commutative rings.

So, suppose $f_i : M_i \to X$, $i = 0, 1$ are 2 proper complex oriented smooth maps representing classes $[f_0], [f_1] \in MU^*(X)$. Then $f_0 \coprod f_1 : M_0 \coprod M_1 \to X$ represents $[f_0] + [f_1] \in MU^*(X)$. One must show that

$$(f_0 \coprod f_1)_* \pi_{M_0 \coprod M_1}^* 1 = f_{0*}\pi_{M_0}^* 1 + f_{1*}\pi_{M_1}^* 1 \in E^*(X)$$

The key fact is that E^* is a generalized cohomology theory, so in particular if X_α, $\alpha \in J$ is an indexed collection of spaces (CW complexes if you want) then the inclusions $\iota_\alpha : X_\alpha \to \coprod_{\alpha \in J} X_\alpha$ induce a natural isomorphism of graded commutative rings

$$E^*(\coprod_\alpha X_\alpha) \xrightarrow{\prod_\alpha \iota_\alpha^*} \prod_\alpha E^*(X_\alpha)$$

Now observe that the projection

$$\pi_{M_0 \coprod M_1} : M_0 \coprod M_1 \to \mathrm{pt} \text{ factors as}$$

$$M_0 \coprod M_1 \xrightarrow{\pi_{M_0} \coprod \pi_{M_1}} \mathrm{pt}_0 \coprod \mathrm{pt}_1 \xrightarrow{\pi} \mathrm{pt}$$

Thus $\pi^*_{M_0 \coprod M_1} 1 = \pi^*_{M_0} \times \pi^*_{M_1}(1,1)$ so to speak, with $(1,1)$ the identity in $E^*(\mathrm{pt}_0 \coprod \mathrm{pt}_1) \simeq E^*(\mathrm{pt}) \times E^*(\mathrm{pt})$. I'm going to assume that the product decomposition described above is natural with respect to Gysin homomorphisms in the sense that if $\coprod_i f_i : \coprod M_i \to X$ is a finite disjoint union of proper complex oriented manifolds over X (hence a proper complex oriented manifold over X in its own right), then there's a natural isomorphism of homomorphims of abelian groups

$$\begin{array}{ccc} E^*(\coprod_i M_i) & \xrightarrow{\simeq} & \bigoplus_i E^*(M_i) \\ {\scriptstyle \coprod_i f_{i*}} \big\downarrow & & \big\downarrow {\scriptstyle \oplus_i f_{i*}} \\ E^*(X) & \xrightarrow{=} & E^*(X) \end{array} \tag{4.3}$$

(here I'm using the fact that finite products are the same as finite direct sums for abelian groups). It will follow that $(f_0 \coprod f_1)_* \pi^*_{M_0 \coprod M_1} 1 = f_{0*} \times f_{1*} \pi^*_{M_0} \times \pi^*_{M_1}(1,1) \in E^*(X)$, and writing $(1,1) = (1,0) + (0,1)$ yields the desired result. Thus $\theta : MU^*(X) \to E^*(X)$ is a homomorphism of abelian groups, and the proof that it's a homomorphism of rings is similar.

To see that it respects gradings, just observe that if $f : M \to X$ is a proper complex oriented map of codimension n representing a class $[f] \in MU^n(X)$, then since $\pi^*_M 1 \in E^0(M)$ and the Gysin homomorphism $f_* : E^*(M) \to E^*(X)$ has degree $-n$, we'll have

$$\theta[f] = f_* \pi^*_M 1 \in E^n(X)$$

What remains to be shown is that θ is induced by a morphism of associative commutative ring CW spectra $\theta : MU \to E$. The idea is that such a morphism θ will be constructed from morphisms $\Sigma^\infty MU(n) \to E$ of degree $-2n$ representing the "universal E-Thom classes" $u(\gamma_n) \in E^{2n}(MU(n))$ of the tautological complex n-plane bundles $\gamma_n : E(\gamma_n) \to BU(n)$. I'll pursue this idea in later sections.

□

4.1. **Orientations in generalized cohomology.**

Let E be an associative, commutative ring CW spectrum representing a cohomology theory E^* on the category of CW complexes. Let $\xi : E(\xi) \to X$ be a real n-plane bundle over a CW complex X

Definition 4.3. An ***E*-orientation of** ξ consists of a class $u(\xi) \in \tilde{E}^*(\mathrm{Th}\xi)$ so that for each $x \in X$, the restriction $u(\xi_x) \in \tilde{E}^*(\mathrm{Th}(\xi_x))$ generates $\tilde{E}^*(\mathrm{Th}(\xi_x))$ as a free $E^*(\mathrm{pt})$-module of rank 1.

Here $\xi_x : E(\xi)_x \to \{x\}$ is the fiber of ξ over x and $\iota : \mathrm{Th}(\xi_x) \to \mathrm{Th}(\xi)$ is the inclusion of Thom spaces induced by the inclusion $E(\xi)_x \to E(\xi)$ (of course $\mathrm{Th}(\xi_x) \simeq S^n$, since $E(\xi)_x \simeq \mathbb{R}^n$).

Remark 4.4. A choice of isomorphism $\mathbb{R}^n \simeq E(\xi)_x$ provides an isomorphism $\tilde{E}^*(\mathrm{Th}(\xi_x)) \simeq \tilde{E}^*(S^n)$ and the suspension isomorphism $E^*(\mathrm{pt}) \simeq \tilde{E}^*(S^0) \simeq \tilde{E}^*(S^n)$ then gives one choice of a generator for $\tilde{E}^*(\mathrm{Th}(\xi_x))$ as a free $E^*(\mathrm{pt})$-module of rank 1. The above definition does *NOT* require that $u(\xi_x)$ is this generator. It doesn't even require that $u(\xi_x)$ is homogeneous of degree n.

However, suppose ξ is an *oriented* real n-plane bundle. Then the local coefficient system corresponding to the bundle of groups

$$\coprod_{x \in X} \tilde{E}^*(\mathrm{Th}\xi_x) \to X$$

is trivial. One way to see this: as ξ is oriented, the representation of the fundamental groupoid $\Pi(X)$ in the homotopy category of CW complexes assigning $x \mapsto \mathrm{Th}\xi_x$ and assigning to a homotopy class of paths $\gamma : I \to X$ from $x_0 \to x_1$ the homotopy class of the "monodromy action" $\mathrm{Th}\xi_{x_0} \xrightarrow{\gamma_*} \mathrm{Th}\xi_{x_1}$ consists entirely of acts exclusively via "degree 1" homotopy equivalences. Of course this requires some precisification. Thus for each $x \in X$ we may choose an *oriented* isomorphism $\mathbb{R}^n \simeq E(\xi)_x$, yielding an isomorphism $\tilde{E}^*(\mathrm{Th}(\xi_x)) \simeq \tilde{E}^*(S^n) \simeq E^*(\mathrm{pt})$, and for any path

$\gamma : I \to X$ from $x_0 \to x_1$ the following diagram will commute:

$$\begin{tikzcd} \tilde{E}^*(\mathrm{Th}(\xi_{x_1})) \arrow[r, "\simeq"] \arrow[d, "\gamma^*"'] & \tilde{E}^*(S^n) \arrow[d, "\mathrm{id}"] \\ \tilde{E}^*(\mathrm{Th}(\xi_{x_0})) \arrow[r, "\simeq"] & \tilde{E}^*(S^n) \end{tikzcd} \tag{4.4}$$

As a consequence, we could ask that for each $x \in X$, $u(\xi_x)$ is "the" canonical generator for $\tilde{E}^*(\mathrm{Th}(\xi_x))$, and in particular that it's homogeneous of degree n. In the following we'll often require this.

Observe that $\tilde{E}^*(\mathrm{Th}(\xi))$ has a natural graded $E^*(X)$-module structure - in fact the action

$$E^*(X) \otimes \tilde{E}^*(\mathrm{Th}(\xi)) \to \tilde{E}^*(\mathrm{Th}(\xi))$$

is induced by the "Thom diagonal"

$$\mathrm{Th}(\xi) \to X_+ \wedge \mathrm{Th}(\xi)(\simeq \mathrm{Th}(0 \times \xi))$$

where 0 denotes the 0-vector bundle over X, and $0 \times \xi : E(0 \times \xi)(\simeq X \times E(\xi)) \to X \times X$ is the product of ξ with 0. The above map of Thom spaces is induced by the evident map of vector bundles

$$\begin{tikzcd} E(\xi) \arrow[r] \arrow[d] & E(0 \times \xi)(= X \times E(\xi)) \arrow[d] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd} \tag{4.5}$$

Proposition 4.5. *Let $u(\xi) \in \tilde{E}^*(\mathrm{Th}\xi)$ be an E-orientation of ξ. Then $u(\xi)$ generates $\tilde{E}^*(\mathrm{Th}\xi)$ as a free $E^*(X)$-module of rank* 1.

Proof. **NOTE: Use Atiyah-Hirzebruch-Serre spectral sequences? Still seems hard.** □

Thus an E-orientation of ξ is precisely what one needs to construct a Thom isomorphism in E-cohomology.

Definition 4.6. Let E be an associative, commutative ring CW-spectrum representing a cohomology theory E^* on the category of CW complexes. An **E-orientation for complex vector bundles** consists of an E-orientation $u(\xi) \in \tilde{E}^*(\mathrm{Th}(\xi))$ for each complex vector bundle $\xi : E(\xi) \to X$ over a CW complex X, which is

- natural in the sense that if

$$\begin{tikzcd} E(\xi) \arrow[r, "\tilde{f}"] \arrow[d] & E(\eta) \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{4.6}$$

  is a morphism of complex vector bundles (giving isomorphisms on fibers) and $\mathrm{Th}f : \mathrm{Th}\xi \to \mathrm{Th}\eta$ is the induced map of Thom spaces, then $\mathrm{Th}f^* u(\eta) = u(\xi) \in \tilde{E}^*(\mathrm{Th}\xi)$, and
- multiplicative in the sense that if $\xi : E(\xi) \to X$ and $\eta : E(\eta) \to Y$ are 2 complex vector bundles over CW complexes X and Y and $\xi \times \eta : E(\xi \times \eta) = E(\xi) \times E(\eta) \to X \times Y$ is their product, then

$$u(\xi \times \eta) = u(\xi) \wedge u(\eta) \in \tilde{E}^*(\mathrm{Th}(\xi \times \eta))$$

  Here I'm using the canonical identification $\mathrm{Th}\xi \wedge \mathrm{Th}\eta \simeq \mathrm{Th}(\xi \times \eta)$ and the resulting external product $\tilde{E}^*(\mathrm{Th}\xi) \otimes \tilde{E}^*(\mathrm{Th}\eta) \to \tilde{E}^*(\mathrm{Th}(\xi \times \eta))$.

- for each $x \in X$, the restriction $u(\xi_x) \in \tilde{E}^*(\mathrm{Th}\xi_x)$ is "the" canonical generator (here we're using the fact that the underlying real bundle of ξ has a canonical orientation).

Let's now give an alternative, more homotopy-theoretic definition of a complex oriented cohomology theory:

Definition 4.7. A **complex oriented cohomology theory** is an associative, commutative ring CW spectrum E together with an E-orientation for complex vector bundles.

Proposition 4.8. *The two competing definitions of a complex oriented cohomology theory E are equivalent.*

Proof. Suppose we're given an associative, commutative ring CW spectrum E together with an E-orientation for complex vector bundles. We must show that there are natural Gysin homomorphisms $f_* : E^*(X) \to E^*(Y)$ of degree $-2n$ associated to every proper complex oriented smooth map of codimension n making E^* a covariant functor on the category of smooth manifolds with proper complex oriented maps, and that these Gysin homomorphisms are compatible with cartesian squares in the appropriate sense.

Let $\iota : X \to E(\epsilon^N) = Y \times \mathbb{R}^N$ be an embedding over Y; then the normal bundle $\nu_{X|E(\epsilon^N)}$ represents the stable normal bundle of f. Suppose N is large enough so that $\nu_{X|E(\epsilon^N)}$ itself has a complex structure. Then the Pontryagin-Thom construction gives a collapse map

$$\varphi : \Sigma^N Y_+ = \mathrm{Th}(\epsilon^N) \to \mathrm{Th}(\nu_{X|E(\epsilon^N)})$$

inducing a homomorphism

$$\varphi^* : \tilde{E}^*(\mathrm{Th}\nu_{X|E(\epsilon^N)}) \to \tilde{E}^*(\Sigma^N Y_+)$$

Since E is complex oriented and $\nu_{X|E(\epsilon^N}$ is a complex vector bundle of rank $n+N$, we have a Thom isomorphism $E^*(X) \simeq \tilde{E}^*(\mathrm{Th}\nu_{X|E(\epsilon^N)})$ of degree $-2n-N$ and of course we have a suspension isomorphism $E^*(Y) \simeq \tilde{E}^*(\Sigma^N Y_+)$ of degree $-N$. Combining these homomorphisms in the evident way yields a homomorphism

$$f_* : E^*(X) \to E^*(Y) \text{ of degree} - 2n$$

I'm not going to get into functoriality and cartesian squares.

Suppose on the other hand we're given an associative, commutative ring CW spectrum E whose associated cohomology theoryE^* comes with Gysin homomorphisms for proper complex oriented smooth maps as described in the previous section. One must use these Gysin homomorphisms to define an E-orientation for complex vector bundles.

Let $\xi : E(\xi) \to X$ be a complex n-plane bundle over a smooth manifold X. Observe that the 0-section $\sigma : X \to E(\xi)$ is a proper complex oriented map of codimension $2n$, and so it induces a Gysin homomorphism $\sigma_* : E^*(X) \to E^*(E(\xi))$ of degree $-2n$. The claim to make is that $\sigma_* 1 \in E^{2n}(E(\xi))$ can in fact be identified with a unique class $u(\xi) \in \tilde{E}^{2n}(\mathrm{Th}\xi)$ providing an E-orientation for ξ (or more generally that σ_* can be viewed as a homomorphism $E^*(X) \to \tilde{E}^*(\mathrm{Th}\xi)$ of degree $-2n$). Not sure of the best way to go about this. In any case I won't get in to naturality/multiplicativity.

NOTE: IN LIGHT OF THE NEXT SECTION, IT WOULD SUFFICE TO PROVIDE A THOM CLASS FOR THE DUAL OF THE TAUTOLOGICAL LINE BUNDLE ON PROJECTIVE SPACE - AND THIS IS TOTALLY DOABLE.

□

So, let E be a complex oriented cohomology theory. Suppose $\xi : E(\xi) \to X$ is a complex n-plane bundle over a CW complex X, and let $f : X \to BU(n)$ be a map classifying ξ, fitting into a commutative diagram

$$
\begin{array}{ccc}
E(\xi) & \xrightarrow{\tilde{f}} & E(\gamma_n) \\
\Big\downarrow & & \Big\downarrow \\
X & \xrightarrow{f} & BU(n)
\end{array}
\tag{4.7}
$$

By naturality, we know that $\mathrm{Th}f^*u(\gamma_n) = u(\xi) \in \tilde{E}^*(\mathrm{Th}\xi)$. Note also that if $\xi \times \eta : E(\xi) \times E(\eta) \to X \times Y$ is a product of a complex m plane bundle ξ with a complex n-plane bundle η, then we can construct a classifying map of the form

$$X \times Y \xrightarrow{f \times g} BU(m) \times BU(n) \xrightarrow{B\iota} BU(m+n)$$

Proceeding in this way one sees that the induced map of Thom spaces $\mathrm{Th}(\xi \times \eta) \to MU(m+n)$ has the form

$$\mathrm{Th}\xi \wedge \mathrm{Th}\eta \xrightarrow{\mathrm{Th}f \wedge \mathrm{Th}g} MU(m) \wedge MU(n) \xrightarrow{M\iota} MU(m+n)$$

the upshot is that requiring the orienation classes u to be multiplicative is equivalent to requiring that

$$M\iota^* u(\gamma_m + n) = u(\gamma_m) \wedge u(\gamma_n) \in \tilde{E}^*(MU(m) \wedge MU(n))$$

Thus the given E-orientation for complex vector bundles is equivalent to E-orientations $u(\gamma_n) \in \tilde{E}^*(MU(n))$, $n \in \mathbb{N}$ with the property that $M\iota^* u(\gamma_m + n) = u(\gamma_m) \wedge u(\gamma_n) \in \tilde{E}^*(MU(m) \wedge MU(n))$ for $m, n \in \mathbb{N}$.

Remark 4.9. We've essentially shown that a complex oriented cohomology theory E is equivalent to an associative, commutative MU-algebra; the classes $u(\gamma_n) \in \tilde{E}^{2n}(MU(n)) = [MU(n), E]_{-2n}$ piece together to define a morphism of CW spectra $u : MU \to E$, the condition that $M\iota^* u(\gamma_m + n) = u(\gamma_m) \wedge u(\gamma_n) \in \tilde{E}^*(MU(m) \wedge MU(n))$ is precisely what's needed to ensure that this morphism commutes with the respective multiplications

$$
\begin{array}{ccc}
MU \wedge MU & \xrightarrow{\mu} & MU \\
{\scriptstyle u \wedge u}\Big\downarrow & & {\scriptstyle u}\Big\downarrow \\
E \wedge E & \xrightarrow{\mu} & E
\end{array}
\tag{4.8}
$$

and the condition that E-orientations restrict to "the canonical generators" on fibers ensures that u commutes with structure maps from S

$$
\begin{array}{ccc}
MU & \xrightarrow{u} & E \\
\Big\uparrow & & \Big\uparrow \\
S & \xrightarrow{=} & S
\end{array}
\tag{4.9}
$$

4.2. Generalized Chern classes.

The major result of this section will be the following

Theorem 4.10. *A complex oriented cohomology theory is equivalent to an associative, commutative ring CW spectrum together with a class $x \in \tilde{E}^2(\mathbb{C}P^\infty)$ such that $\iota^* x \in \tilde{E}^2(\mathbb{C}P^1)$ is the usual generator, where $\iota : \mathbb{C}P^1 \to \mathbb{C}P^\infty$ is a linear inclusion.*

To be specific, one could require that $\iota[a_0, a_1] = [a_0, a_1, 0, \dots] \in \mathbb{C}P^\infty$.

To begin suppose E is an associative, commutative ring CW spectrum together with a class $x \in \tilde{E}^2(\mathbb{C}P^\infty)$ as described in the statement of the theorem.

Remark 4.11. The inclusion $\iota : \{x_0\} \to \mathbb{C}P^\infty$ and projection $\pi : \mathbb{C}P^\infty \to \{x_0\}$ give a split short exact sequence

$$0 \to \tilde{E}^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty) \xrightarrow{\iota^*} E^*(\mathrm{pt}) \to 0$$

For this reason x can be viewed as an element $x \in E^2(\mathbb{C}P^\infty)$.

Remark 4.12. Say $n \in \mathbb{N}$ and let $j : \mathbb{C}P^n \to \mathbb{C}P^\infty$ be a linear injection. Then evidently $j^*x \in \tilde{E}^2(\mathbb{C}P^n)$ is a class with the property that if $\iota : \mathbb{C}P^1 \to \mathbb{C}P^n$ is a linear embedding then $\iota^*j^*x \in \tilde{E}^*(\mathbb{C}P^1)$ is the canonical generator. Abusing notation to avoid excessive notation I'll often write x when I mean j^*x. Again we can view this as a class in $E^2(\mathbb{C}P^n)$.

The class $x \in E^2(\mathbb{C}P^n)$ defines a homomorphism of graded commutative rings $E^*(\mathrm{pt})[x] \to E^*(\mathbb{C}P^n)$. Moreover, we have $x^{n+1} = 0 \in E^*(\mathbb{C}P^n)$ since $\mathbb{C}P^n$ can be covered by $n+1$ contractible open subspaces $U_0, \dots, U_n \subset \mathbb{C}P^n$ (namely, the usual affine opens) and x lies in the kernel of the restriction homomorphisms $E^*(\mathbb{C}P^n) \to E^*(U_i)$. Here I'm appealing to something along the lines of the following trivial but useful fact:

Lemma 4.13. *Let F be an associative, commutative ring CW spectrum representing a generalized cohomology theory F^* on the category of CW complexes, and suppose X is a CW complex that admits a covering $X = \coprod_{i=0}^n A_i$ by $n+1$ contractible subcomplexes. Then for any $n+1$ classes $x_i \in h^*(X)$, $i = 0, \dots, n$ with x_i in the kernel of the restiction $h^*(X) \to h^*(A_i)$, the product $\prod_{i=0}^n x_i \in h^*(X)$ vanishes.*

The proof uses only the formal properties of relative products.

Thus our homomorphism $E^*(\mathrm{pt})[x] \to E^*(\mathbb{C}P^n)$ factors through the quotient $E^*(\mathrm{pt})[x]/(x^{n+1})$.

Proposition 4.14. *The resulting homomorphism of graded commutative rings*

$$E^*(\mathrm{pt})[x]/(x^{n+1}) \to E^*(\mathbb{C}P^n)$$

is an isomorphism. Moreover these isomorphisms are compatible with the evident restriction homomorphisms

$$E^*(\mathrm{pt})[x]/(x^{n+1}) \to E^*(\mathrm{pt})[x]/(x^n) \text{ and } E^*(\mathbb{C}P^n) \to E^*(\mathbb{C}P^{n-1})$$

and it follows that

$$E^*(\mathrm{pt})[[x]] = \lim_n E^*(\mathrm{pt})[x]/(x^{n+1}) \simeq \lim_n E^*(\mathbb{C}P^n) \simeq E^*(\mathbb{C}P^\infty)$$

where the last isomorphism comes from the vanishing if the $\lim^1$ *term in the Milnor short exact sequence for the E-cohomology of $\mathbb{C}P^\infty$.*

Proof. The proof will make use of the Atiyah-Hirzebruch spectral sequence for the E-cohomology of the CW complex $\mathbb{C}P^n$ (with its usual cell structure). In fact we'll want to open the black box that is the construction of this spectral sequence just a little bit: the skeleton filtration of $\mathbb{C}P^n$ looks like

$$\mathrm{pt} = X_0 \subset X_1 \subset \cdots \subset X_{2n} = \mathbb{C}P^n \text{ where for } i \in \{0, \dots, n\}$$

$$X_{2i} = X_{2i+1} = \mathbb{C}P^i \subset \mathbb{C}P^n$$

where we can view $\mathbb{C}P^i$ as the linear subvariety consisting of points like $[a_0, \dots, a_i, 0, \dots, 0]$ (last $n-i$ coordinates 0). This filtration of $\mathbb{C}P^n$ induces a *descending* filtration $F^*E^*(\mathbb{C}P^n)$ of its cohomology where

$$F^iE^*(\mathbb{C}P^n) := \ker(E^*(\mathbb{C}P^n) \to E^*(X_{i-1})) \text{ for } i \in \mathbb{N}$$

Taking the product of the long exact sequences of the pairs (X_i, X_{i-1}) yields an exact couple

$$\begin{array}{ccc} \prod_i E^*(X_i) & \xrightarrow{i^*} & \prod_i E^*(X_i) \\ {\scriptstyle j^*}\uparrow & & \downarrow{\scriptstyle \delta} \\ \prod_i \tilde{E}^*(X_i/X_{i-1}) & \xrightarrow{=} & \prod_i \tilde{E}^*(X_i/X_{i-1}) \end{array} \tag{4.10}$$

From this we cook up an spectral sequence E_*^{**} with E_1-page given by $E_1^{pq} = \tilde{E}^{p+q}(X_p/X_{p-1})$ converging to $E^*(\mathbb{C}P^n)$ with the filtration $F^*E^*(\mathbb{C}P^n)$ in the sense that we'll have isomorphisms $E_\infty^{pq} \simeq F^pE^{p+q}(\mathbb{C}P^n)/F^{p-1}E^{p+q}(\mathbb{C}P^n)$.

Remark 4.15. In general Atiyah-Hirzebruch spectral sequences present convergence issues. However for a finite CW complex X we are dealing with a bounded (in the sense that $E_1^{pq} = 0$ for $p > \dim X$) right half plane spectral sequence (which converges "in finite time" since it has only finitely many non-0 differentials).

Noting that

$$\tilde{E}^{p+q}(X_p/X_{p-1}) \simeq \prod_{e_\alpha \subset \mathbb{C}P^n \text{ a p-cell}} \tilde{E}^{p+q}(D_\alpha^p/\partial D_\alpha^p) \simeq \prod_{e_\alpha \subset \mathbb{C}P^n \text{ a p-cell}} E^q(\mathrm{pt}) = C^p(\mathbb{C}P^n; E^q(\mathrm{pt}))$$

where the right hand side denotes cellular cochains with coefficients in $E^q(\mathrm{pt})$, and showing that the differentials on the E_1-page are just the differentials coming from the cellular cochain complex, we may pass to the E_2 page:

$$E_2^{pq} = H^p(\mathbb{C}P^n; E^q(\mathrm{pt}))$$

It's worth noting that in this spectral sequence d_1 is trivial, so one can meaningfully say $E_1^{**} = E_2^{**}$.

Now observe that whenever $m \leq n$ the linear inclusion $\iota : \mathbb{C}P^m \to \mathbb{C}P^n$ as the points of the form $[a_0, \ldots, a_m, 0, \ldots, 0]$ induces a morphism of Atiyah-Hirzebruch spectral sequences, say, $E_*^{**}(\mathbb{C}P^n) \xrightarrow{\iota^*} E_*^{**}(\mathbb{C}P^m)$ converging to the filtered homomorphism of graded commutative rings $\iota^* : E^*(\mathbb{C}P^n) \to E^*(\mathbb{C}P^m)$. We may describe the map of E_1 and E_2 pages rather explicitly: it gives isomorphisms

$$E_1^{pq}(\mathbb{C}P^n) \simeq \tilde{E}^{p+q}(X_p/X_{p-1}) \simeq E_1^{pq}(\mathbb{C}P^m) \text{ for } p \leq 2m$$

and kills $E_1^{pq}(\mathbb{C}P^n)$ when $p > 2m$. Of course the map of E_2 pages is just the map on cellular cohomology induced by the inclusion ι.

In particular taking $m = 1$ we obtain a morphism of spectral sequences

$$\iota^* E_*^{**}(\mathbb{C}P^n) \to E_*^{**}(\mathbb{C}P^1) \text{ converging to the filtred homomorphism } \iota^* E^*(\mathbb{C}P^n) \to E^*(\mathbb{C}P^1)$$

The spectral sequence $E_*^{**}(\mathbb{C}P^1)$ is rather uncomplicated: its E_1-page has just 2 non-0 columns $E_1^{0*}(\mathbb{C}P^1) \simeq E^*(\mathrm{pt})$ and $E_1^{2*} \simeq \tilde{E}^*(\mathbb{C}P^1)$. Moreover we've observed that d_1 is 0, and d_2 is 0 by the well known fact that the differentials in the cellular cochain complex of $\mathbb{C}P^1$ are 0. Thus the spectral sequence collapses on page 1, and the "answer" is given by the usual split short exact sequence

$$0 \to \tilde{E}^*(\mathbb{C}P^1) \to E^*(\mathbb{C}P^1) \to E^*(\mathrm{pt}) \to 0$$

By hypothesis, $x \in \tilde{E}^2(\mathbb{C}P^n)$ restricts to the usual generator $\iota^* x \in \tilde{E}^2(\mathbb{C}P^1)$. Hence it must be that there's a permanent cycle $\tilde{x} \in \prod_{p+q=2} E_1^{pq}(\mathbb{C}P^n) = \prod_{p+q=2} \tilde{E}^{p+q}(X_p/X_{p-1})$ surviving to x with components $0 \in E_1^{02}(\mathbb{C}P^n)$ and $\iota^* x \in E_1^{20}(\mathbb{C}P^n) \simeq \tilde{E}^2(\mathbb{C}P^1)$.

Remark 4.16. It seems like everyone assumes that the components for higher values of p are 0, but it's not obvious to me why that has to be the case.

In any case, noting that under the isomorphism

$$E_1^{20}(\mathbb{C}P^n) = \tilde{E}^2(\mathbb{C}P^1) \simeq H^2(\mathbb{C}P^n; E^0(\mathrm{pt})) = E_2^{20}(\mathbb{C}P^n)$$

$\iota^* x$ goes to the element "$y \otimes 1 \in H^2(\mathbb{C}P^n; E^0(\mathrm{pt}))$" by which I mean the class represented by the cellular cochain assigning $1 \in E^0(\mathrm{pt})$ to the 2-cell of $\mathbb{C}P^n$. Here y denotes the canonical generator of $H^2(\mathbb{C}P^n; \mathbb{Z})$. It follows that our permanent cycle $\tilde{x} \in \prod_{p+q=2} E_1^{pq}(\mathbb{C}P^n)$ can be viewed as an element of the form "$y \otimes 1 + y^2 \otimes a_2 + \cdots + y^n \otimes a_n \in \prod_{p+q=2} H^p(\mathbb{C}P^n; E^q(\mathrm{pt}))$" where $a_i \in E^{-2i+2}(\mathrm{pt})$ for all i. Thus the powers $1, \tilde{x}, \tilde{x}^2, \ldots, \tilde{x}^n$ of $\tilde{x}$ are permanent cycles surviving to

$1, x, x^2, \dots, x^n \in E^*(\mathbb{C}P^n)$ forming a basis for $E_2^{**}(\mathbb{C}P^n)$ as a free $E^*(\mathrm{pt})$-module (using some multiplicative properties of the spectral sequence here). Since the differentials of the spectral sequence are $E^*(\mathrm{pt})$-linear. $E_*^{**}(\mathbb{C}P^n)$ collapses *on the first page*.

Remark 4.17. One might say the existence of this class x with the property that $\iota^* x$ is the usual generator *triggers* the collapse of the Atiyah-Hirzebruch spectral sequence for the E-cohomology of $\mathbb{C}P^n$.

Now, define a (descending) filtration $F^* E^*(\mathrm{pt})[x]/(x^{n+1})$ of $E^*(\mathrm{pt})[x]/(x^{n+1})$ by setting

$$F^{2i} E^*(\mathrm{pt})[x]/(x^{n+1}) = F^{2i+1} E^*(\mathrm{pt})[x]/(x^{n+1}) = (x^i)$$

and observe that our homomorphism $E^*(\mathrm{pt})[x]/(x^{n+1}) \to E^*(\mathbb{C}P^n)$ is now a *filtered* homomorphism (say, since x^i vanishes on $\mathbb{C}P^{i-1}$, which can be covered by i contractible affine opens). Using the collapse of the Atiyah-Hirzebruch spectral sequence, one sees that the induced homomorphism of associated gradeds looks like

$$(x^i)/(x^{i+1}) \to E_2^{2i*}(\mathbb{C}P^n) \simeq H^{2i}(\mathbb{C}P^n; E^*(\mathrm{pt}))$$

taking $x^i \mapsto y^i \otimes 1$ - in particular it's an isomorphism of free $E^*(\mathrm{pt})$-modules of rank 1. Thus the induced map of associated gradeds is an isomorphism, and we may conclude that $E^*(\mathrm{pt})[x]/(x^{n+1}) \to E^*(\mathbb{C}P^n)$ is an isomorphism. It's straightforward to show that the evident diagrams

$$\begin{array}{ccc} E^*(\mathrm{pt})[x]/(x^{n+1}) & \xrightarrow{\simeq} & E^*(\mathbb{C}P^n) \\ \downarrow & & \downarrow \\ E^*(\mathrm{pt})[x]/(x^n) & \xrightarrow{\simeq} & E^*(\mathbb{C}P^{n-1}) \end{array} \tag{4.11}$$

commute, and this shows by direct calculation that all maps in the inverse system

$$\iota^* : E^*(\mathbb{C}P^n) \to E^*(\mathbb{C}P^{n-1}) \text{ for } n \in \mathbb{N}$$

are *surjective*. But then $\lim_n^1 E^*(\mathbb{C}P^n) = 0$ and hence the Milnor short exact sequence

$$0 \to \lim_n^1 E^*(\mathbb{C}P^n)[-1] \to E^*(\mathbb{C}P^\infty) \to \lim_n E^*(\mathbb{C}P^n) \to 0$$

gives the desired isomorphism

$$E^*(\mathbb{C}P^\infty) \simeq \lim_n E^*(\mathbb{C}P^n) \simeq \lim_n E^*(\mathrm{pt})[x]/(x^{n+1}) \simeq E^*(\mathrm{pt})[[x]]$$

□

Remark 4.18. Really $E^*(\mathrm{pt})[[x]]$ should be interpreted as short-hand for the completion of $E^*(\mathrm{pt})[x]$ at the ideal (x). It's still a graded commutative ring, and any element $u \in E^*(\mathrm{pt})[[x]]$ is a finite sum of homogeneous terms. However, for each i we'll have $E^*(\mathrm{pt})[[x]]_i = \lim_n (E^*(\mathrm{pt})[x]/(x^{n+1}))_i$ and since $E^*(\mathrm{pt})$ may contain elements of arbitrarily large negative degree, the ith summand $E^*(\mathrm{pt})[[x]]_i$ may contain legitimate power series of the form

$$\sum_{j \in \mathbb{N}} a_j x^j \text{ where } \deg a_j = i - 2j \text{ for all } j$$

Say $m, n \in \mathbb{N}$ and let $\pi_1 : \mathbb{C}P^m \times \mathbb{C}P^n \to \mathbb{C}P^m, \pi_2 : \mathbb{C}P^m \times \mathbb{C}P^n \to \mathbb{C}P^n$ be the projections onto the first and second factors. Let $x_i := \pi_i^* x \in \tilde{E}^2(\mathbb{C}P^m \times \mathbb{C}P^n)$ (abusing notation and identifying a lot of "xs" here but you get the idea). Then the external product in E-cohomology yields a homomorphism of graded rings

$$E^*(\mathbb{C}P^m) \otimes E^*(\mathbb{C}P^n) \xrightarrow{\times} E^*(\mathbb{C}P^m \times \mathbb{C}P^n)$$

and using proposition 4.6 we may identify this as a homomorphism $E^*(\mathrm{pt})[x_1, x_2]/(x_1^{m+1}, x_2^{n+1}) \to E^*(\mathbb{C}P^m \times \mathbb{C}P^n)$. A similar Atiyah-Hirzebruch spectral sequence based argument shows that

Proposition 4.19. *The homomorphism of graded commutative rings*

$$E^*(\mathrm{pt})[x_1, x_2]/(x_1^{m+1}, x_2^{n+1}) \to E^*(\mathbb{C}P^m \times \mathbb{C}P^n)$$

is an isomorphism. Moreover these isomorphisms are compatible with the evident restriction homomorphisms (not going to write them out here) and it follows that

$$E^*(\mathrm{pt})[[x_1, x_2]] = \lim_{m,n} E^*(\mathrm{pt})[x_1, x_2]/(x_1^{m+1}, x_2^{n+1}) \simeq \lim_{m,n} E^*(\mathbb{C}P^m \times \mathbb{C}P^n) \simeq E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

where the last isomorphism comes from the vanishing of the $\lim^1$ *term in the Milnor exact sequence for the E-cohomology of* $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$.

Our next goal will be to compute the *E-homology* of complex projective spaces. The result is

Proposition 4.20. *There exist unique classes* $\beta_i \in E_{2i}(\mathbb{C}P^n)$ *for* $i = 0, \ldots, n$ *dual to the classes* $x^i \in E^*(\mathbb{C}P^n)$ *in the sense that*

$$\langle x^i, \beta_j \rangle = \delta_{ij} \in E^*(\mathrm{pt}) \text{ for } i, j \in \{0, \ldots, n\}$$

and the β_i *generate* $E_*(\mathbb{C}P^n)$ *as a free module over* $E_*(\mathrm{pt})$*. The Kronecker product*

$$\langle , \rangle : E^*(\mathbb{C}P^n) \times E_*(\mathbb{C}P^n) \to E^*(\mathrm{pt})$$

is perfect, and thus $E^*(\mathbb{C}P^n)$ *and* $E_*(\mathbb{C}P^n)$ *are dual free* $E_*(\mathrm{pt})$*-modules of rank* $n+1$*. It follows that there are unique classes* $\beta_i \in E_{2i}(\mathbb{C}P^\infty) = \mathrm{colim}_n E_{2i}(\mathbb{C}P^n)$ *dual to the* $x^i \in E^2 i(\mathbb{C}P^\infty)$ *which generate* $E_*(\mathbb{C}P^n)$ *as a free module over* $E_*(\mathrm{pt})$*.*

Proof. Let's look at the Atiyah-Hirzebruch spectral sequence for the *E-homology* of $\mathbb{C}P^n$. Again it's worth opening the black box of the construction of this spectral sequence. Again letting X_i denote the i-skeleton of $\mathbb{C}P^n$ we get an increasing filtration $F_* E_*(\mathbb{C}P^n)$ of $E_*(\mathbb{C}P^n)$ defined by

$$F_p E_*(\mathbb{C}P^n) = \mathrm{im}(E_*(X_p) \to E^*(\mathbb{C}P^n)) \text{ for } p \in \mathbb{N}$$

Taking the direct sum of the long exact sequences of the pairs (X_p, X_{p-1}) yields an exact couple

$$\begin{tikzcd} \bigoplus_p E_*(X_p) \arrow[r, "\iota_*"] & \bigoplus_p E_*(X_p) \arrow[d, "j_*"] \\ \bigoplus_p \tilde{E}_*(X_p/X_{p-1}) \arrow[u, "\partial"] \arrow[r, "="] & \bigoplus_p \tilde{E}_*(X_p/X_{p-1}) \end{tikzcd} \tag{4.12}$$

and from this we obtain a homological spectral sequence E^*_{**} with E^1-page $E^1_{pq} = \tilde{E}_{p+q}(X_p/X_{p-1})$ converging to $E_*(\mathbb{C}P^n)$ with the given filtration, in the sense that we'll have isomorphisms $E^\infty_{pq} \simeq F_p E_*(\mathbb{C}P^n)/F_{p-1}E_*(\mathbb{C}P^n)$.

Remark 4.21. *Homological* Atiyah-Hirzebruch spectral sequences are better behaved when it comes to convergence questions. Because they live in the $p \geq 0$ half plane, for any given index pq the differentials leaving the E^*_{pq} terms are eventually 0 (there's probably a name for this phenomena - see Weibel's *Homological algebra*). Of course in the case at hand we're dealing with a bounded right half plane spectral sequence, which must converge in finite time.

Observing that

$$\tilde{E}_{p+q}(X_p/X_{p-1}) \simeq \bigoplus_{e_\alpha \subset \mathbb{C}P^n \text{ a p-cell}} \tilde{E}_{p+q}(D^p_\alpha/\partial D^p_\alpha) \simeq \bigoplus_{e_\alpha \subset \mathbb{C}P^n \text{ a p-cell}} E_q(\mathrm{pt}) = C_p(\mathbb{C}P^n; E_q(\mathrm{pt}))$$

where the right hand side denotes cellular p-chains in $\mathbb{C}P^n$ with coefficients in $E_q(\mathrm{pt})$, and showing that the differentials d^1 on the E^1-page are those coming from the cellular chain complex, we obtain the E^2-page:

$$E^2_{pq} = H_p(\mathbb{C}P^n; E_q(\mathrm{pt}))$$

Note that d^1 is 0, and so $E^1_{**} = E^2_{**}$.

One can make use of the fact that

Lemma 4.22. *There exists a Kronnecker product pairing*

$$\langle , \rangle : E^{**}_* \times E^*_{**} \to E^*(\mathrm{pt})$$

of Atiyah-Hirzebruch spectral sequences for the E-(co)homology of $\mathbb{C}P^n$ *converging to the pairing* $\langle , \rangle : E^*(\mathbb{C}P^n) \times E_*(\mathbb{C}P^n) \to E^*(\mathrm{pt})$ *(I won't get into exactly what convergence means just yet). The pairing of spectral sequences respects differentials in the sense that if "*$\langle d\alpha, \beta\rangle = \langle \alpha, d\beta\rangle$*." Moreover the pairing of 2nd pages is just the usual pairing of singular cohomology and homology (combined with the pairing of* $E^*(\mathrm{pt})$ *and* $E_*(\mathrm{pt})$*.*

The usual calculations of the singular cohomology and homology of $\mathbb{C}P^n$, along with the fact that the pairing $E^*(\mathrm{pt}) \times E_*(\mathrm{pt}) \to E^*(\mathrm{pt})$ is perfect (it's basically multiplication in $E^*(\mathrm{pt})$, show that the pairing of 2nd pages is perfect. Here we're using in a serious way the fact that $H_*(\mathbb{C}P^n; E_*(\mathrm{pt}))$ is a free $E_*(\mathrm{pt})$module and similarly $H^*(\mathbb{C}P^n; E^*(\mathrm{pt}))$ is a free $E^*(\mathrm{pt})$-module, and appealing to the universal coefficient theorem for singular cohomology. Thus the 2nd pages are dual free $E_*(\mathrm{pt})$-modules of rank $n+1$.

Now let's show that the homological spectral sequence E^*_{**} collapses. Recall that the cohomological spectral sequence $E^*_*{*}$ collapses on page 1. Now for any element $\beta \in E^2_{**}$, observe that

$$\langle \alpha, d\beta\rangle = \langle d\alpha, d\beta\rangle = 0 \in E^*(\mathrm{pt})$$

for each $\alpha \in E^{**}_2$. Since the pairing of 2nd pages is perfect, it must be that $d\beta = 0$. Proceeding in this way on shows (perhaps by induction) that all differentials in E^*_{**} are 0, and thus it collapses on page 1. At this point we've shown that $E_*(\mathbb{C}P^n)$ is a free $E_*(\mathrm{pt})$-module of rank $n+1$.

Finally, observe that the Kronecker product $E^*(\mathbb{C}P^n) \times E_*(\mathbb{C}P^n) \to E^*(\mathrm{pt})$ is filtered in the following sense; if $\iota : X_p \to \mathbb{C}P^n$ is the inclusion of the p-skeleton, then we'll have

$$\langle \iota^*\alpha, \beta\rangle = \langle \alpha, \iota_*\beta\rangle \in E^*(\mathrm{pt})$$

for $\alpha \in E^*(\mathbb{C}P^n), \beta \in E_*(X_p)$ and thus the following diagram commutes:

$$\begin{tikzcd}
E^*(\mathbb{C}P^n) \arrow[r, "\iota^*"] \arrow[d] & E^*(X_p) \arrow[d] \\
\mathrm{Hom}_{E^*(\mathrm{pt})}(E_*(\mathbb{C}P^n), E^*(\mathrm{pt})) \arrow[r, "\iota_*^*"] & \mathrm{Hom}_{E^*(\mathrm{pt})}(E_*(X_p), E^*(\mathrm{pt}))
\end{tikzcd} \tag{4.13}$$

Note that the increasing filtration $F_*E_*(\mathbb{C}P^n)$ of $E_*(\mathbb{C}P^n)$ induces a decreasing filtration $F^*\mathrm{Hom}_{E^*(\mathrm{pt})}(E_*(\mathbb{C}P^n), E^*($ of $\mathrm{Hom}_{E^*(\mathrm{pt})}(E_*(\mathbb{C}P^n), E^*(\mathrm{pt}))$ where

$$F^p\mathrm{Hom}_{E^*(\mathrm{pt})}(E_*(\mathbb{C}P^n), E^*(\mathrm{pt})) := \ker(\mathrm{Hom}_{E^*(\mathrm{pt})}(E_*(\mathbb{C}P^n), E^*(\mathrm{pt})) \xrightarrow{\iota_*^*} \mathrm{Hom}_{E^*(\mathrm{pt})}(E_*(X_{p-1}), E^*(\mathrm{pt})))$$

From the above commutative diagram one sees that the homomorphism

$$E^*(\mathbb{C}P^n) \to \mathrm{Hom}_{E^*(\mathrm{pt})}(E_*(\mathbb{C}P^n), E^*(\mathrm{pt})) \text{ induced by } \langle , \rangle$$

respects the filtrations $F^*E^*(\mathbb{C}P^n)$ and $F^*\mathrm{Hom}_{E^*(\mathrm{pt})}(E_*(\mathbb{C}P^n), E^*(\mathrm{pt}))$. Since the pairing of Atiyah-Hirzebruch spectral sequences converges to the Kronecker product in E-(co)homology and both spectral sequences collapse, one sees that the induced homomorphism of associated gradeds looks like

$$H^p(\mathbb{C}P^n; E^*(\mathrm{pt})) = F^pE^*(\mathbb{C}P^n)/F^{p+1}E^*(\mathbb{C}P^n)$$

$$\to \frac{F^p\mathrm{Hom}_{E^*(\mathrm{pt})}(E_*(\mathbb{C}P^n), E^*(\mathrm{pt}))}{F^{p+1}\mathrm{Hom}_{E^*(\mathrm{pt})}(E_*(\mathbb{C}P^n), E^*(\mathrm{pt}))} \simeq \mathrm{Hom}_{E^*(\mathrm{pt})}(H_p(\mathbb{C}P^n; E_*(\mathrm{pt})), E^*(\mathrm{pt}))$$

where the last isomorphism should follow from extensive diagram chasing. Now argue that this composition is precisely the map induced by the pairing of 2nd pages, which we know is perfect, so it's an isomorphism. It will follow that the Kronecker product in the E-(co)homology of $\mathbb{C}P^n$ is perfect.

□

There's a homology analogue of proposition 4.7:

Proposition 4.23. *For $m, n \in \mathbb{N}$ the external product in E-homology*

$$E_*(\mathbb{C}P^m) \otimes_{E_*(\mathrm{pt})} E_*(\mathbb{C}P^n) \xrightarrow{\times} E_*(\mathbb{C}P^m \times \mathbb{C}P^n)$$

is an isomorphism, and so $E_*(\mathbb{C}P^m \times \mathbb{C}P^n)$ is a free $E_*(\mathrm{pt})$-module of rank $(m+1)(n+1)$ on a basis consisting of the external products $\beta_i \times \beta'_j \in E_{2(i+j)}(\mathbb{C}^m \times \mathbb{C}P^n)$, where $\beta_i \in E_{2i}(\mathbb{C}P^m), \beta'_j \in E_{2j}(\mathbb{C}P^n)$ are the classes described in proposition 4.8. It follows that the external product

$$E_*(\mathbb{C}P^\infty) \otimes_{E_*(\mathrm{pt})} E_*(\mathbb{C}P^\infty) \xrightarrow{\times} E_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

is an isomorphism, so $E_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ is a free $E_*(\mathrm{pt})$-module on the external products $\beta_i \times \beta'_j$.

We now turn to the E-(co)homology of the classifying spaces $BU(n)$ and their colimit BU, this time starting with homology. The idea is that $\mathbb{C}P^\infty$ should serve as a "generating complex" for the E-homology of BU, as is the case when $E = H\mathbb{Z}$.

So, say $n \in \mathbb{N}$ and let

$$j : \prod_{i=1}^n \mathbb{C}P^\infty \to BU(n)$$

be a map classifying the n-fold product $\gamma_1^* \times \cdots \times \gamma_1^*$ of the dual of the tautological line bundle on $\mathbb{C}P^\infty$ (i.e. the line bundle corresponding to $\mathcal{O}(1)$). It's not hard to show that up to homotopy this map is invariant under the right action of Σ_n on $\prod_{i=1}^n \mathbb{C}P^\infty$ permuting coordinates. From j we obtain an induced homomorphism

$$j_* : E_*(\prod_{i=1}^n \mathbb{C}P^\infty) \to E_*(BU(n))$$

which is invariant under the evident right Σ_n action on $E_*(\prod_{i=1}^n \mathbb{C}P^\infty)$. From proposition 4.10 we know that the external product gives a natural isomorphism

$$\bigotimes_{i=1}^n E_*(\mathbb{C}P^\infty) \xrightarrow{\times} E_*(\prod_{i=1}^n \mathbb{C}P^\infty)$$

(where the tensor product is taken over $E_*(\mathrm{pt})$) and with respect to this isomorphism the Σ_n action looks like

$$\alpha_1 \otimes \cdots \otimes \alpha_n \cdot \sigma ='' \mathrm{sgn}(\sigma, \alpha)'' \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(n)} \in \bigotimes_{i=1}^n E_*(\mathbb{C}P^\infty)$$

for $\alpha := \alpha_1 \otimes \cdots \otimes \alpha_n \in \bigotimes_{i=1}^n E_*(\mathbb{C}P^\infty), \sigma \in \Sigma_n$. Here "$\mathrm{sgn}(\sigma, \alpha)$" $= \prod_{i<j, \sigma(i)>\sigma(j)} (-1)^{\deg \alpha_i \deg \alpha_j}$ is the generalized sign of the permutation σ acting on α. Thus we may view j_* as a homomorphism

$$\bigotimes_{i=1}^n E_*(\mathbb{C}P^\infty) \to E_*(BU(n)) \text{ invariant under the } \Sigma_n \text{ action described above}$$

which will factor through the coinvariants $(\bigotimes_{i=1}^n E_*(\mathbb{C}P^\infty))_{\Sigma_n} = S_n E_*(\mathbb{C}P^\infty)$, the nth symmetric power of the graded $E_*(\mathrm{pt})$-module $E_*(\mathbb{C}P^\infty)$. Note that the natural direct sum decomposition $E_*(\mathrm{pt}) \oplus \tilde{E}_*(\mathbb{C}P^\infty) \simeq E_*(\mathbb{C}P^\infty)$ together with the natural isomorphism

$$S_n(E_*(\mathrm{pt}) \oplus \tilde{E}_*(\mathbb{C}P^\infty)) \simeq \bigoplus_{i+j=n} S_i(E_*(\mathrm{pt})) \otimes_{E_*(\mathrm{pt})} S_j(\tilde{E}_*(\mathbb{C}P^\infty)) = \bigoplus_{m \le n} S_m(\tilde{E}_*(\mathbb{C}P^\infty))$$

show that there's a natural identification $S_n E_*(\mathbb{C}P^\infty) \simeq \bigoplus_{m \le n} S_m(\tilde{E}_*(\mathbb{C}P^\infty))$.

Recall that the classifying maps $BU(m) \times BU(n) \to BU(m+n)$ of the "direct sum" homomorphisms $U(m) \times U(n) \to U(m+n)$ colimit to an H-space operation

$$\mu : BU \times BU \to BU$$

making BU an abelian group object in the homotopy category of CW complexes (of course, by Bott periodicity BU is an infinite loopspace and one can show this direct sum operation agrees with the infinite loopspace operation).

Remark 4.24. If one is willing to branch out of the CW world a bit and consider the model for BU presented at the end of Atiyah's *K-theory*, namely the Calkin algebra obtained as the quotient of the bounded operators $\mathrm{End}(H)$ on a separable Hilbert space H over $\mathbb{C}$ by the 2-sided ideal $I \subset \mathrm{End}(H)$ consisting of compact operators, then BU will be a topological abelian group under the addition in the Calkin algebra. In fact it will be a topological $\mathbb{C}$ algebra, on the nose!

The H-space operation μ will induce a Pontryagin product

$$E_*(BU) \otimes_{E_*(\mathrm{pt})} E_*(BU) \xrightarrow{\times} E_*(BU \times BU) \xrightarrow{\mu_*} E_*(BU)$$

making $E_*(BU)$ an associative, graded commutative $E_*(\mathrm{pt})$-algebra. Considering the commutative diagrams

$$\begin{array}{ccc} \prod_{i=1}^m \mathbb{C}P^\infty \times \prod_{i=1}^n \mathbb{C}P^\infty & \xrightarrow{=} & \prod_{i=1}^{m+n} \mathbb{C}P^\infty \\ {\scriptstyle j_m \times j_n}\downarrow & & {\scriptstyle j_{m+n}}\downarrow \\ BU(m) \times BU(n) & \longrightarrow & BU(m+n) \end{array} \tag{4.14}$$

and the resulting homomorphisms of E-homology, and keeping track of the symmetric group actions in play, one sees that the homomorphism

$$j_* : S_*(\tilde{E}_*(\mathbb{C}P^\infty)) = \bigoplus_{n=0}^{\infty} S_n(\tilde{E}_*(\mathbb{C}P^\infty)) \to E_*(BU)$$

obtained as the colimit of the homomorphisms $j_* : \bigoplus_{m \le n} S_m(\tilde{E}_*(\mathbb{C}P^\infty)) \to E_*(BU(n))$ is in fact a homomorphism of graded commutative $E_*(\mathrm{pt})$ *algebras*.

Proposition 4.25. *The homomorphism of graded* $E_*(\mathrm{pt})$*-modules*

$$j_* : \bigoplus_{m \le n} S_m(\tilde{E}_*(\mathbb{C}P^\infty)) \to E_*(BU(n))$$

is an isomorphism, and

$$j_* : S_*(\tilde{E}_*(\mathbb{C}P^\infty)) \to E_*(BU)$$

is an isomorphism of graded commutative $E_*(\mathrm{pt})$*-algebras.*

Of course by proposition 4.8 $\tilde{E}_*(\mathbb{C}P^\infty)$ is a free $E_*(\mathrm{pt})$-module on the generators β_i, where $i \in \mathbb{N}, i > 0$ and so $\bigoplus_{m\le n} S_m(\tilde{E}_*(\mathbb{C}P^\infty))$ is a free $E_*(\mathrm{pt})$-module on the monomials $\beta_{i_1} \cdot \beta_{i_r}$, where $i_j \in \mathbb{N}, i_j > 0$ and $r \le n$, i.e. the monomials in the β_i with *length* at most n. Note that we really can view $\bigoplus_{m\le n} S_m(\tilde{E}_*(\mathbb{C}P^\infty))$ as a bigraded $E_*(\mathrm{pt})$-module, where the 2 gradings are topological degree and length. Similarly $S_*(\tilde{E}_*(\mathbb{C}P^\infty))$ is a bigraded $E_*(\mathrm{pt})$-algebra, and in fact it can be identified as the polynomial algebra $E_*[\beta_i \mid i \in \mathbb{N}, i > 0]$.

Proof. Consider the morphism of Atiyah-Hirzebruch spectral sequences for E-homology induced by the map $j : \prod_{i=1}^n \mathbb{C}P^\infty \to BU(n)$; denote it by

$$j_* : E^*_{**}(\prod_{i=1}^n \mathbb{C}P^\infty) \to E^*_{**}(BU(n))$$

From the usual calculation of the singular homology of $BU(n)$ we know that the morphism of 2nd pages $j_* : H_*(\prod_{i=1}^n \mathbb{C}P^\infty; E_*(\mathrm{pt})) \to H_*(BU(n); E_*(\mathrm{pt}))$ is surjective, factoring through an isomorphism

$$\bigoplus_{m\le n} S_m \tilde{H}_*(\mathbb{C}P^\infty; E_*(\mathrm{pt})) \simeq H_*(\prod_{i=1}^n \mathbb{C}P^\infty; E_*(\mathrm{pt}))_{\Sigma_n} \simeq H_*(BU(n); E_*(\mathrm{pt}))$$

arguing as in the proof of proposition 4.6 we see that there are permanent cycles $\tilde{\beta}_i \in H_*(\mathbb{C}P^\infty; E_*(\mathrm{pt}))$ freely generating the 2nd page of the Atiyah-Hirzebruch spectral sequence for $E_*(\mathbb{C}P^\infty)$ as a module over $E_*(\mathrm{pt})$ corresponding to the elements $\beta_i \in E_{2i}(\mathbb{C}P^\infty)$. Moreover it should be that the $\tilde{\beta}_i$ look like

$$\tilde{\beta}_i ='' b_i \otimes 1 + b_i + 1 \otimes a_{i+1} + \dots ,''$$

where $b_j \otimes a_j$ is the class in $H_{2j}(\mathbb{C}P^\infty; E_{2(i-j)}(\mathrm{pt}))$ of the cochain assigning $a_j \in E_{2(i-j)}(\mathrm{pt})$ to the $2j$-cell of $\mathbb{C}P^\infty$. Again appealing to the usual calculation of the singular cohomology of $BU(n)$ one sees that j_* sends the monomials $\tilde{\beta}_{i_0} \cdot \dots \cdot \tilde{\beta}_{i_r} \in \bigoplus_{m\le n} S_m \tilde{H}_*(\mathbb{C}P^\infty; E_*(\mathrm{pt}))$ to permanent cycles freely generating $H_*(BU(n); E_*(\mathrm{pt}))$ as a module over $E_*(\mathrm{pt})$. It follows that the spectral sequence $E^*_{**}(BU(n))$ collapses on page 2 (and in fact on page 1).

Now argue that the homomorphism of filtered graded $E_*(\mathrm{pt})$-modules

$$j_* : \bigoplus_{m\le n} S_m \tilde{E}_*(\mathbb{C}P^\infty) \to E_*(BU(n))$$

(here the filtration of $\bigoplus_{m\le n} S_m \tilde{E}_*(\mathbb{C}P^\infty)$ is induced in the evident way by that on $E_*(\prod_{i=1}^n \mathbb{C}P^\infty)$ - in fact $\bigoplus_{m\le n} S_m \tilde{E}_*(\mathbb{C}P^\infty)$ and $E_*(BU(n))$ are simply being filtered by topological degree (i.e. the pth term F_p in their filtrations consists of classes of topological degree $\le p$)) induces a map of associated gradeds which is precisely the isomorphism

$$\bigoplus_{m\le n} S_m \tilde{H}_*(\mathbb{C}P^\infty; E_*(\mathrm{pt})) \simeq H_*(BU(n); E_*(\mathrm{pt}))$$

This will yield the first part of the proposition. The second follows by taking colimits as $n \to \infty$. $\square$

Remark 4.26. One could probably construct a spectral sequence of co-invariants $E^*_{**}(\prod_{i=1}^n \mathbb{C}P^\infty)_{\Sigma_n}$ converging to the co-invariants $E_*(\prod_{i=1}^n \mathbb{C}P^\infty)_{\Sigma_n}$ with the evident filtration. In this case one could phrase the above argument as follows: the map of spectral sequences

$$j_* : E^*_{**}(\prod_{i=1}^n \mathbb{C}P^\infty)_{\Sigma_n} \to E^*_{**}(BU(n)) \text{ gives an isomorphism of 2nd pages}$$

and so it must converge to an isomorphism $E_*(\prod_{i=1}^n \mathbb{C}P^\infty)_{\Sigma_n} \simeq E_*(BU(n))$.

Remark 4.27. It seems that in the above proof we made use of the trivial but useful fact: if $\varphi : E^*_{**} \to F^*_{**}$ is a morphism of spectral sequences then

- if φ is surjective and E^*_{**} collapses on page r, then so does F^*_{**}.
- if φ is injective and F^*_{**} collapses on page r, then so does E^*_{**}

Basically, the proof of proposition 4.11 needs to be rewritten to incorporate these remarks.

We'll also want to know that

Proposition 4.28. *The external products*

$$E_*(BU(m)) \otimes_{E_*(\mathrm{pt})} E_*(BU(n)) \xrightarrow{\times} E_*(BU(m) \times BU(n))$$

and

$$E_*(BU) \otimes_{E_*(\mathrm{pt})} E_*(BU) \xrightarrow{\times} E_*(BU \times BU)$$

are isomorphisms, and the coproducts

$$\Delta_* : E_*(BU(n)) \to E_*(BU(n) \times BU(n)) \simeq E_*(BU(n)) \otimes_{E_*(\mathrm{pt})} E_*(BU(n)) \text{ and}$$
$$\Delta_* : E_*(BU) \to E_*(BU \times BU) \simeq E_*(BU) \otimes_{E_*(\mathrm{pt})} E_*(BU)$$

are described by $\Delta_*\beta_n = \sum_{i+j=n} \beta_i \otimes \beta_j$.

Proof. The proof that the external products are isomorphisms can be carried out using the Atiyah-Hirzebruch spectral sequences - as we've done enough of this already, I'll skip it.

To figure out the coproducts Δ_*, consider a map

$$j : \mathbb{C}P^\infty \to BU(n) \text{ classifying the } n-\text{plane bundle } \gamma_1^* \oplus \epsilon^{n-1} \text{ over } \mathbb{C}P^\infty$$

since the diagram

$$\begin{tikzcd} \mathbb{C}P^\infty \arrow[r, "j"] \arrow[d, "\Delta"'] & BU(n) \arrow[d, "\Delta"'] \\ \mathbb{C}P^\infty \arrow[r, "j\times j"] & BU(n) \times BU(n) \end{tikzcd} \tag{4.15}$$

obviously commutes we have a commutative diagram of coproducts

$$\begin{tikzcd} E_*(\mathbb{C}P^\infty) \arrow[r, "j_*"] \arrow[d, "\Delta_*"'] & E_*(BU(n)) \arrow[d, "\Delta_*"'] \\ E_*(\mathbb{C}P^\infty) \otimes_{E_*(\mathrm{pt})} E_*(\mathbb{C}P^\infty) \arrow[r, "j_*\otimes j_*"] & E_*(BU(n)) \otimes_{E_*(\mathrm{pt})} E_*(BU(n)) \end{tikzcd} \tag{4.16}$$

and so $\Delta_* j_* \beta_n = j_* \otimes j_* \Delta_* \beta_n$. It will suffice then to show that for $\beta_n \in E_*(\mathbb{C}P^\infty)$, $\Delta_*\beta_n = \sum_{i+j=n} \beta_i \otimes \beta_j \in E_*(\mathbb{C}P^\infty) \otimes_{E_*(\mathrm{pt})} E_*(\mathbb{C}P^\infty)$. Note that it will suffice to show that

$$\langle \Delta_*\beta_n, x^k \otimes y^l \rangle = \langle \sum_{i+j=n} \beta_i \otimes \beta_j, x^k \otimes y^l \rangle \in E^*(\mathrm{pt}) \text{ for } k+l=n$$

where $x, y \in E^*(\mathrm{pt})[[x,y]] \simeq E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ are the usual generators. This is indeed the case, since

$$\langle \Delta_*\beta_n, x^k \otimes y^l \rangle = \langle \beta_n, \Delta^* x^k \otimes y^l \rangle = \langle \beta_n, x^n \rangle = 1 \text{ and}$$
$$\langle \sum_{i+j=n} \beta_i \otimes \beta_j, x^k \otimes y^l \rangle = \sum_{i+j=n} \langle \beta_i, x^k \rangle \langle \beta_j, y^l \rangle = \sum_{i+j=n} \delta_{ik}\delta_{jl} = 1$$

□

To obtain the E-cohomology of $BU(n)$ and BU by "dualizing" this result, we'll need the following

Lemma 4.29. *Let X be a CW complex (or more generally a connective CW spectrum). Suppose that $H_*(X; E_*(\mathrm{pt}))$ is a free $E_*(\mathrm{pt})$-module and the Atiyah-Hirzebruch spectral sequence E^*_{**} for the E-homology of X (which begins with $E^2_{pq} = H_p(X; E_q(\mathrm{pt}))$ and converges to $E_*(X)$) collapses on page 2. Let F be a module spectrum over the ring spectrum E. Then the Atiyah-Hirzebruch spectral sequences for the F homology and cohomology of X (which begin with $E^2_{pq} = H_p(X; F_q(\mathrm{pt}))$ and $E_2^{pq} = H^p(X; F^q(\mathrm{pt}))$ and converge to $F_*(X)$ and $F^*(X)$ respectively) also collapse on page 2, and the natural maps*

$$E_*(X) \otimes_{E_*(\mathrm{pt})} F_*(\mathrm{pt}) \to F_*(X) \textit{ and } F^*(X) \to \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(X), F_*(\mathrm{pt}))$$

are isomorphisms.

By ”the natural maps” I mean the following: suppose $f : S \to E \wedge X$ and $g : S \to F$ are morphisms of spectra with degrees p and q respectively representing classes $[f] \in E_p(X), [g] \in F_q(\mathrm{pt})$. Smashing them together yields a morphism $g \wedge f : S \wedge S \to F \wedge E \wedge X$, and using the identification $S \simeq S \wedge S$ along with the action $\mu : F \wedge E \to F$ one obtains a map

$$S \simeq S \wedge S \to F \wedge E \wedge X \to F \wedge X$$

of degree $p + q$ representing a class, say $g \wedge f \in F_*(X)$. Proceeding in this way one obtains the first natural map. On the other hand, given a morphism of spectra $f : X \to F$ of degree $-q$ representing a class $[f] \in F^q(X)$, applying E-homology yields a homomorphism of graded abelian groups $f_* : E_*(X) \to E_*(F)$ with degree $-q$ and the action $\mu : F \wedge E \to F$ induces a map of stable homotopy groups $\mu_* : E_*(F) \to F_*(\mathrm{pt})$. Putting these together we get a homomorphism

$$E_*(X) \xrightarrow{f_*} E_*(F) \xrightarrow{\mu_*} F_*(\mathrm{pt})$$

Proceeding in this way one obtains the second natural map.

I won’t prove the lemma here.

Applying the lemma to the matter at hand, we see that the natural maps

$$E^*(BU(n)) \to \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(BU(n)), E_*(\mathrm{pt})) \text{ and } E^*(BU) \to \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(BU), E_*(\mathrm{pt}))$$

are isomorphisms; so are the natural maps

$$E^*(BU(n) \times BU(n)) \to \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(BU(n)) \otimes_{E_*(\mathrm{pt})} E_*(BU(n)), E_*(\mathrm{pt})) \text{ and}$$

$$E^*(BU \times BU) \to \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(BU) \otimes_{E_*(\mathrm{pt})} E_*(BU), E_*(\mathrm{pt}))$$

We see that there exist unique classes $c_i \in E^{2i}(BU(n))$ dual to $\beta_1^i \in E_{2i}(BU(n))$ for $i = 1, \ldots, n$ in the sense that for a monomial $m = \beta_{i_1} \cdot \beta_{i_r} \in E_*(BU(n))$ in the β_i,

$$\langle c_i, m \rangle = \begin{cases} 1 & \text{if } m = \beta_1^i \\ 0 & \text{otherwise} \end{cases} \in E^*(\mathrm{pt})$$

Similarly there are unique classes $c_i \in E^{2i}(BU)$ dual to the β_1^i for $i \in \mathbb{N}, i > 0$. The fact that the β_i serve as polynomial generators for $E_*(BU)$ implies that

$$\mu^* c_n = \sum_{i+j=n} c_i \times c_j \in E^*(BU \times BU)$$

Indeed from the above remarks it’ll suffice to show that

$$\langle \mu^* c_n, m_1 \otimes m_2 \rangle = \langle \sum_{i+j=n} c_i \times c_j, m_1 \otimes m_2 \rangle \in E^*(\mathrm{pt})$$

for any monomials $m_1, m_2 \in E_*(BU)$ in the β_i, and this is indeed the case, as

$$\langle \mu^* c_n, m_1 \otimes m_2 \rangle = \langle c_n, \mu_* m_1 \otimes m_2 \rangle = \begin{cases} 1 & \text{if } m_1 = \beta_1^k, m_2 = \beta_1^l \text{ and } k + l = n \\ 0 & \text{otherwise} \end{cases}$$

and this is precisely

$$\langle \sum_{i+j=n} c_i \times c_j, m_1 \otimes m_2 \rangle = \sum_{i+j=n} \langle c_i, m_1 \rangle \langle c_j, m_2 \rangle \in E^*(\mathrm{pt})$$

since each term in the sum is 1 if $m_1 = \beta_1^i$ and $m_2 = \beta_1^j$ and 0 otherwise. We've proved most of:

Proposition 4.30. *The maps*

$$E^*(\mathrm{pt})[[c_1, \ldots, c_n]] \to E^*(BU(n)) \text{ and } E^*(\mathrm{pt})[[c_i \,|\, i \in \mathbb{N}, i > 0]] \to E^*(BU)$$

are isomorphisms, and the "coproduct" for $E^*(BU)$ is described by $\mu^* c_n = \sum_{i+j=n} c_i \times c_j$.

Proof. Since the natural map $E^*(BU(n)) \to \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(BU(n)), E_*(\mathrm{pt}))$ is an isomorphism, it'll suffice to show that the composition

$$E^*(\mathrm{pt})[[c_1, \ldots, c_n]] \to E^*(BU(n)) \to \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(BU(n)), E_*(\mathrm{pt}))$$

is an isomorphism of graded $E^*(\mathrm{pt})$-modules. Similarly it'll suffice to show that the composition

$$E^*(\mathrm{pt})[[c_i \,|\, i \in \mathbb{N}, i > 0]] \to E^*(BU) \to \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(BU), E_*(\mathrm{pt}))$$

is an isomorphism. At this point we may appeal to the usual (purely algebraic) fact that for a graded commutative ring R, $R[[c_1, \ldots, c_n]]$ is the R-algebra dual to the R-coalgebra $\bigoplus_{m \le n} S_m(\bigoplus_{i=1}^{\infty} R\beta_i)$ and $R[[c_i \,|\, i \in \mathbb{N}, i > 0]]$ is the R-Hopf algebra dual to the R-Hopf algebra $R[\beta_i \,|\, i \in \mathbb{N}, i > 0]$ where the duality between the c_i and the β_i and the respective products/coproducts are described as above.

□

Remark 4.31. Let $\prod_{j=1}^r c_{i_j} \in E^*(\mathrm{pt})[[c_1, \ldots, c_n]]$ be a monomial in the c_i and let $m \in E_*(BU(n))$ be a monomial in the β_i. Observe that

$$\langle \prod_{j=1}^r c_{i_j}, m \rangle = \langle c_{i_1} \times \cdots \times c_{i_r}, \sum_{\alpha} m_{\alpha_1} \times \cdots \times m_{\alpha_r} \rangle \in E^*(\mathrm{pt})$$

where $\sum_\alpha m_{\alpha_1} \times \cdots \times m_{\alpha_r}$ is the image of m under the homomorphism

$$E_*(BU(n)) \to E_*(\prod_{i=1}^r BU(n))$$

induced by the "r-fold" diagonal. We have

$$\langle c_{i_1} \times \cdots \times c_{i_r}, \sum_{\alpha} m_{\alpha_1} \times \cdots \times m_{\alpha_r} \rangle = \sum_{\alpha} \prod_{j=1}^r \langle c_{i_j}, m_{\alpha_j} \rangle \in E^*(\mathrm{pt})$$

and at this point we observe that $\langle c_{i_j}, m_{\alpha_j} \rangle$ is 1 if $m_{\alpha_j} = \beta_1^{i_j}$ and 0 otherwise; the upshot is that $\langle \prod_{j=1}^r c_{i_j}, m \rangle \in \mathbb{N}$ is a *non-negative integer* depending only on the combinatorics of the monomials $\prod_{j=1}^r c_{i_j}$ and m (it's *indpendent* of the particular spectrum E).

Proposition 4.32. *Let $j : \prod_{i=1}^n \mathbb{C}P^\infty \to BU(n)$ be a map classifying the n-fold product $\prod_{i=1}^n \gamma_1^*$ of the dual of the tautological line bundle on $\mathbb{C}P^\infty$. Then the induced homomorphism of graded $E^*(\mathrm{pt})$-algebras*

$$j^* : E^*(BU(n)) \to E^*(\prod_{i=1}^n \mathbb{C}P^\infty)$$

is injective with image the symmetric sub-algebra $E^*(\prod_{i=1}^n \mathbb{C}P^\infty)^{\Sigma_n}$. Moreover

$$j^*(c_i) = \sigma_i(x_1, \ldots, x_n) \in E^*(\mathrm{pt})[[x_1, \ldots, x_n]] \simeq E^*(\prod_{i=1}^n \mathbb{C}P^\infty)$$

where $\sigma_i(x_1, \ldots, x_n)$ is the ith elementary symmetric function.

Proof. In essence one must "dualize" proposition 4.11 and its proof. There it was shown that the homomorphism $j_* : E_*(\prod_{i=1}^n \mathbb{C}P^\infty) \to E_*(BU(n))$ is a split surjection factoring through an isomorphism $E_*(\prod_{i=1}^n \mathbb{C}P^\infty)_{\Sigma_n} \simeq E_*(BU(n))$. Appealing to the fact that everything in sight is free (and finite type) over $E_*(\mathrm{pt})$, it follows that the homomorphism $j_*^* : \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(BU(n)), E_*(\mathrm{pt})) \to \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(\prod_{i=1}^n \mathbb{C}P^\infty), E_*(\mathrm{pt}))$ is a split injection factoring through an isomorphism

$$\mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(BU(n)), E_*(\mathrm{pt})) \simeq \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(\prod_{i=1}^n \mathbb{C}P^\infty)_{\Sigma_n}, E_*(\mathrm{pt}))$$

$$\simeq \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(\prod_{i=1}^n \mathbb{C}P^\infty), E_*(\mathrm{pt}))^{\Sigma_n}$$

(**NOTE**: I think we need at least freeness over $E_*(\mathrm{pt})$ (and perhaps also finite type (i.e. finite generation in each degree) over $E_*(\mathrm{pt})$, to obtain that last isomorphism). Now by a mild generalization of lemma 4.13 (incorporating the Σ_n-action) there is a commutative diagram

$$\begin{array}{ccc} E^*(BU(n)) & \xrightarrow{j^*} & E^*(\prod_{i=1}^n \mathbb{C}P^\infty)^{\Sigma_n} \\ \simeq\downarrow & & \simeq\downarrow \\ \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(BU(n)), E_*(\mathrm{pt})) & \xrightarrow{\simeq} & \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(\prod_{i=1}^n \mathbb{C}P^\infty), E_*(\mathrm{pt}))^{\Sigma_n} \end{array} \tag{4.17}$$

Thus j^* gives an isomorphism $E^*(BU(n)) \simeq E^*(\prod_{i=1}^n \mathbb{C}P^\infty)^{\Sigma_n}$. With respect to the isomorphism $E^*(\mathrm{pt})[[x_1, \dots, x_n]] \simeq E^*(\prod_{i=1}^n \mathbb{C}P^\infty)$ described in proposition 4.7, the Σ_n action just permutes the x_i, and so we obtain an identification $E^*(\mathrm{pt})[[x_1, \dots, x_n]]^{\Sigma_n} \simeq E^*(\prod_{i=1}^n \mathbb{C}P^\infty)^{\Sigma_n}$ where the left hand side is the usual symmetric subalgebra. It remains to show that under this identification $j^*(c_i) = \sigma_i(x_1, \dots, x_n)$ for $i = 1, \dots, n$. Notice that

$$\langle j^* c_i, \otimes_{k=1}^r \beta_{j_k} \rangle = \langle c_i, j_* \otimes_{k=1}^r \beta_{j_k} \rangle$$

$$= \langle c_i, \prod_{k=1}^r \beta_{j_k} \rangle \in E^*(\mathrm{pt})$$

for any basic tensor product $\otimes_{k=1}^r \beta_{j_k} \in \bigoplus_{m \le n} S_m(\tilde{E}_*(\mathbb{C}P^\infty)) \simeq E_*(\prod_{i=1}^n \mathbb{C}P^\infty)$. So, it will suffice to verify the identity

$$\langle \sigma_i(x_1, \dots, x_n), \otimes_{k=1}^r \beta_{j_k} \rangle = \langle c_i, \prod_{k=1}^r \beta_{j_k} \rangle \in E^*(\mathrm{pt})$$

And indeed, writing $\sigma_i(x_1, \dots, x_n) = \sum \sigma \prod_{j \le i} x_j$ (where $\sigma \prod_{j \le i} x_j = \prod_{j \le i} x_{\sigma(j)}$, and the sum runs over the orbit $\Sigma_n \prod_{j \le i} x_j = \{\sigma \prod_{j \le i} x_j \mid \sigma \in \Sigma_n\}$) as the sum over the Σ_n-orbit of $\prod_{j \le i} x_j$, we see that

$$\langle \sigma_i(x_1, \dots, x_n), \otimes_{k=1}^r \beta_{j_k} \rangle = \sum \langle \sigma \prod_{j \le i} x_j, \otimes_{k=1}^r \beta_{j_k} \rangle$$

Now observe that each term $\langle \sigma \prod_{j \le i} x_j, \otimes_{k=1}^r \beta_{j_k} \rangle$ in the sum is 1 if the tensor product $\otimes_{k=1}^r \beta_{j_k}$ consists of β_1s in the positions $\sigma(1), \dots, \sigma(i)$, and 0 otherwise. It follows that $\langle \sigma_i(x_1, \dots, x_n), \otimes_{k=1}^r \beta_{j_k} \rangle$ is 1 if $\otimes_{k=1}^r \beta_{j_k}$ consists of i β_1s and 0 otherwise. And by *definition* $\langle c_i, \prod_{k=1}^r \beta_{j_k} \rangle$ is 1 if $\prod_{k=1}^r \beta_{j_k} = \beta_1^r$ and 0 otherwise.

□

Let MU be the Thom spectrum representing complex (co)bordism - it's an associative commutative ring CW spectrum. This makes the E-homology $E_*(MU)$ a graded commutative $E_*(\mathrm{pt})$-algebra in the following way: the multiplication $\mu : MU \wedge MU \to MU$ induces a homomorphism $\mu_* : E_*(MU \wedge MU) \to E_*(MU)$, and combining this with the exterior product in E-homology $E_*(MU) \otimes_{E_*(\mathrm{pt})} E_*(MU) \xrightarrow{\times} E_*(MU \wedge MU)$ yields an operation $E_*(MU) \otimes_{E_*(\mathrm{pt})} E_*(MU) \to E_*(MU)$, which is associative and (graded) commutative because μ is. The Hurewicz morphism $\eta : S \to MU$ induces a homomorphism $\eta_* : E_*(\mathrm{pt}) \to E_*(MU)$ and the fact that both compositions

$$MU \simeq S \wedge MU \xrightarrow{\eta\wedge\mathrm{id}} MU \wedge MU \xrightarrow{\mu} MU \text{ and}$$

$$MU \simeq MU \wedge S \xrightarrow{\mathrm{id}\wedge\eta} MU \wedge MU \xrightarrow{\mu} MU$$

are the identity implies that η_* is a homomorphism of graded commutative algebras.

Now recall that the 2nd term of MU is $MU(1)$, and so there's a canonical morphism of spectra

$$\iota : \Sigma^{-2}MU(1) \to MU$$

Applying E-homology gives a homomorphism of graded $E_*(\mathrm{pt})$-modules

$$\iota_* : \tilde{E}_*(\Sigma^{-2}MU(1)) \to E_*(MU)$$

(i.e. we have homomorphisms $\tilde{E}_p(MU(1)) \to E_{p-2}(MU)$ for $p \in \mathbb{Z}$).

We'll make essential use of the fact that there's a canonical homeomorphism $\mathbb{C}P^\infty \simeq MU(1)$. Let $\gamma_1 : E(\gamma_1) \to \mathbb{C}P^{n-1}$ be the tautological line bundle, and let γ_1^* be its dual.

Proposition 4.33. *There is a canonical homeomorphism* $\mathrm{Th}(\gamma_1^*) \simeq \mathbb{CP}^n$*, and under this homeomorphism the 0-section* $\mathbb{C}P^{n-1} \subset \mathrm{Th}(\gamma_1^*)$ *corresponds to a hyperplane section. In the colimit as* $n \to \infty$ *we obtain a canonical homeomorphism* $MU(1) = \mathrm{Th}(\gamma_1^*) \simeq \mathbb{C}P^\infty$*; again the 0-section corresponds to a hyperplane section.*

Sketch. Unraveling definitions, we see that an element of $E(\gamma_1^*)$ consists of a line $L \subset \mathbb{C}^n$ together with a linear functional $\lambda : L \to \mathbb{C}$. Consider the linear map

$$\iota \times \lambda : L \to \mathbb{C}^n \times \mathbb{C} = \mathbb{C}^{n+1}$$

where $\iota : L \to \mathbb{C}^n$ is the inclusion. Since ι is injective, $\iota \times \lambda$ is too and so its image $\iota \times \lambda(L) \subset \mathbb{C}^{n+1}$ is a line. Thus we may define a continuous (in fact *smooth*) map

$$f : E(\gamma_1^*) \to \mathbb{C}P^n \text{ by } f(L) = \acute{} \times \lambda(L)$$

Now argue that this is a homeomorphism (in fact *diffeomorphism*) onto $\mathbb{C}P^n - \{[0,\dots,0,1]\}$ and declaring $f(\infty) = [0,\dots,0,1]$ extends it to the desired homeomorphism, etc. etc.

□

The above homeomorphism gives an isomorphism $\tilde{E}_*(\mathbb{C}P^\infty) \simeq \tilde{E}_*(MU(1))$, and in this way we obtain elements $\beta_i \in \tilde{E}_{2i}(MU(1))$, for $i \in \mathbb{N}, i > 0$ (I'm identifying the $\beta_i \in \tilde{E}_*(\mathbb{C}P^\infty)$ with their images in $\tilde{E}_*(MU(1))$. Set

$$\alpha_i := \iota_*\beta_{i+1} \in E_{2i}(MU) \text{ for } i \in \mathbb{N}$$

It's clear enough that $\alpha_0 = 1$.

Now, the homomorphism of pointed graded $E_*(\mathrm{pt})$-modules

$$\iota_* : \tilde{E}_*(\Sigma^{-2}MU(1)) \to E_*(MU) \text{ sending } \beta_1 \mapsto 1$$

induces (by adjunction) a homomorphism of graded $E_*(\mathrm{pt})$-algebras

$$S_\infty\iota_* : S_\infty(\tilde{E}_*(\Sigma^{-2}MU(1))) \to E_*(MU)$$

where the left hand side denotes the infinite symmetric product of $\tilde{E}_*(\Sigma^{-2}MU(1))$ with β_1 acting as 1, or equivalently the quotient of $S_*(\tilde{E}_*(\Sigma^{-2}MU(1))) = \bigoplus_{n\in\mathbb{N}} S_n(\tilde{E}_*(\Sigma^{-2}MU(1)))$ by the ideal $(\beta_1 - 1)$. In terms of the α_i we may identify $S_\infty\iota_*$ as the evident homomorphism

$$E_*(\mathrm{pt})[\alpha_i \mid i \in \mathbb{N}, i > 0] \to E_*(MU)$$

Proposition 4.34. *Again let $j : \prod_{i=1}^n \mathbb{C}P^\infty \to BU(n)$ be the map classifying the product $\prod_{i=1}^n \gamma_1^*$ and let $\mathrm{Th}j : \bigwedge_{i=1}^n MU(1) \to MU(n)$ be the induced map of Thom spaces. Then the induced homomorphism $\mathrm{Th}j_* : \tilde{E}_*(\bigwedge_{i=1}^n MU(1)) \to \tilde{E}_*(MU(n))$ is surjective and factors through an isomorphism of graded $E_*(\mathrm{pt})$-modules $S_n(\tilde{E}_*(MU(1))) \simeq \tilde{E}_*(MU(n))$. Taking the colimit of the resulting homomorphisms*

$$S_n(\tilde{E}_*(\Sigma^{-2}MU(1))) \simeq \tilde{E}_*(\Sigma^{-2n}MU(n)) \to E_*(MU)$$

one obtains an isomorphism of graded $E_*(\mathrm{pt})$-algebras $S_\infty(\tilde{E}_*(\Sigma^{-2}MU(1))) \to E_*(MU)$. Moreover the external product gives isomorphisms

$$E_*(MU(m)) \otimes_{E_*(\mathrm{pt})} E_*(MU(n)) \xrightarrow{\times} E_*(MU(m) \wedge MU(n)) \text{ and}$$

$$E_*(MU) \otimes_{E_*(\mathrm{pt})} E_*(MU) \xrightarrow{\times} E_*(MU \wedge MU)$$

The argument is similar to the one used to compute $E_*(BU(n))$ and $E_*(BU)$ in proposition 4.11, so I won't go into much detail. To get started one considers the morphism of Atiyah-Hirzebruch spectral sequences $E_* *^* (\bigwedge_{i=1}^n MU(1)) \to E_* *^* (MU(n))$ induced by $\mathrm{Th}j$; on the 2nd page it looks like

$$\tilde{H}_*(\bigwedge_{i=1}^n MU(1); E_*(\mathrm{pt})) \to \tilde{H}_*(MU(n); E_*(\mathrm{pt}))$$

and one knows this is surjective, factoring through an isomorphism $S_n(\tilde{H}_*(MU(1); E_*(\mathrm{pt}))) \to \tilde{H}_*(MU(n); E_*(\mathrm{pt}))$ from the classical computations of this singular homology of $MU(n)$.

Using Adams's lemma 4.13 to dualize, one sees that the natural homomorphisms

$$E^*(MU(n)) \to \mathrm{Hom}_{E_*(\mathrm{pt})}(\tilde{E}_*(MU(n)), E_*(\mathrm{pt})) \text{ and } E^*(MU) \to \mathrm{Hom}_{E_*(\mathrm{pt})}(\tilde{E}_*(MU), E_*(\mathrm{pt}))$$

are isomorphisms. We have commutative diagrams

$$\begin{array}{ccc} \tilde{E}^*(MU(n)) & \xrightarrow{\mathrm{Th}j^*} & \tilde{E}^*(\bigwedge_{i=1}^n MU(1)) \\ \simeq\downarrow & & \simeq\downarrow \\ \mathrm{Hom}_{E\_*(\mathrm{pt})}(\tilde{E}_*(MU(n)), E_*(\mathrm{pt})) & \xrightarrow{\mathrm{Th}j_*^*} & \mathrm{Hom}_{E_*(\mathrm{pt})}(\tilde{E}_*(\bigwedge_{i=1}^n MU(1)), E_*(\mathrm{pt})) \end{array} \tag{4.18}$$

where proposition 4.16 implies the bottom horizontal arrow is a split injection. Thus the top horizontal arrow is a split injection.

Proposition 4.35. *Let $u_1 \in \tilde{E}^2(MU(1))$ be the image of $x \in \tilde{E}^2(\mathbb{C}P^\infty)$ under the canonical isomorphism $\tilde{E}^2(\mathbb{C}P^\infty) \simeq \tilde{E}^2(MU(1))$, and for $n \in \mathbb{N}, n > 0$ let $u_n \in \tilde{E}^{2n}(MU(n))$ be the unique class such that $\mathrm{Th}j^* u_n = \wedge_{i=1}^n u_1 \in \tilde{E}^{2n}(\bigwedge_{i=1}^n MU(1))$. Then u_n generates $\tilde{E}^*(MU(n))$ as a free $E^*(BU(n))$-module of rank 1. Furthermore, if $\mu_{mn} : MU(m) \wedge MU(n) \to MU(m+n)$ is the map of Thom spaces induced by a map classifying the bundle $\gamma_m \times \gamma_n$ over $BU(m) \times BU(n)$, then $\mu_{mn}^* u_{m+n} = u_m \wedge u_n \in \tilde{E}^{2(m+n)}(MU(m) \wedge MU(n))$ and if $\iota : S^{2n} \to MU(n)$ is the inclusion of the Thom space of a fiber of γ_n, then $\iota^* u_n \in \tilde{E}^{2n}(S^{2n})$ is the usual generator.*

Proof. It seems one could resort to another (perhaps lengthy) Atiyah-Hirzebruch spectral sequence based argument. Alternatively, observe that $\tilde{E}^*(MU(n)) \xrightarrow{\mathrm{Th}j^*} \tilde{E}^*(\bigwedge_{i=1}^n MU(1))$ is an injective homomorphism of modules over the ring homomorphism $j^* : E^*(BU(n)) \to E^*(\prod_{i=1}^n \mathbb{C}P^\infty)$. Using proposition 4.15 we may identify j^* with the homomorphism $E^*(\mathrm{pt})[[c_1, \ldots, c_n]] \to E^*(\mathrm{pt})[[x_1, \ldots, x_n]]$ taking $c_i \mapsto \sigma_i(x_1, \ldots, x_n)$. It's clear that u_1 generates $\tilde{E}^*(MU(1))$ as a free $E^*(\mathbb{C}P^\infty)$-module of rank 1; indeed using the identification $\tilde{E}^*(MU(1)) \simeq \tilde{E}^*(\mathbb{C}P^\infty)$ this reduces to the observation that $\tilde{E}^*(\mathbb{C}P^\infty) \simeq x \cdot E^*(\mathbb{C}P^\infty) \subset E^*(\mathbb{C}P^\infty)$. But then certainly $\wedge_{i=1}^n$ generates $\tilde{E}^*(\bigwedge_{i=1}^n MU(1))$ as a free $E^*(\prod_{i=1}^n \mathbb{C}P^\infty)$-module of rank 1, and moreover using the identification $\tilde{E}^*(\bigwedge_{i=1}^n MU(1)) \simeq \tilde{E}^*(\bigwedge_{i=1}^n \mathbb{C}P^\infty) \subset E^*(\prod_{i=1}^n \mathbb{C}P^\infty)$ we can identify $\tilde{E}^*(\bigwedge_{i=1}^n MU(1))$ with the ideal $\sigma_n(x_1, \ldots, x_n) \cdot$

$E^*(\prod_{i=1}^n \mathbb{C}P^\infty) \subset E^*(\prod_{i=1}^n \mathbb{C}P^\infty)$. Noting that under all the identifications in play $\mathrm{Th} j^*$ gives an isomorphism $\tilde{E}^*(MU(n)) \simeq \sigma_n(x_1, \ldots, x_n) \cdot E^*(\prod_{i=1}^n \mathbb{C}P^\infty) \cap E^*(\prod_{i=1}^n \mathbb{C}P^\infty)^{\Sigma_n}$, it will suffice to prove the purely algebraic fact that

$$\sigma_n(x_1, \ldots, x_n) \cdot E^*(\prod_{i=1}^n \mathbb{C}P^\infty) \cap E^*(\prod_{i=1}^n \mathbb{C}P^\infty)^{\Sigma_n} = \sigma_n(x_1, \ldots, x_n) \cdot E^*(\prod_{i=1}^n \mathbb{C}P^\infty)^{\Sigma_n}$$

I won't prove it here.

□

Theorem 4.36. *There is a unique (up to homotopy of course) homomorphism of ring CW spectra $\varphi : MU \to E$ so that $\varphi_* x_{MU} = x_E \in \tilde{E}^2(\mathbb{C}P^\infty)$ where φ_* denotes the induced homomorphism $\varphi_* : MU^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty)$, which is to say the following diagram of CW spectra commutes:*

$$\begin{array}{ccc} E & \xrightarrow{=} & E \\ {\scriptstyle x_E}\uparrow & & {\scriptstyle \varphi}\uparrow \\ \Sigma^{-2}\mathbb{C}P^\infty = \Sigma^{-2}MU(1) & \xrightarrow{x_{MU}} & MU \end{array} \tag{4.19}$$

This can be viewed as the proof of theorem 4.4.

Proof. By definition a homomorphism of ring CW spectra $\varphi : MU \to E$ is a morphism of spectra such that the following diagrams commute:

$$\begin{array}{ccc} MU \wedge MU & \xrightarrow{\varphi\wedge\varphi} & E \wedge E \\ {\scriptstyle \mu_{MU}}\downarrow & & {\scriptstyle \mu_E}\downarrow \\ MU & \xrightarrow{\varphi} & E \end{array} \quad \text{and} \quad \begin{array}{ccc} MU & \xrightarrow{\varphi} & E \\ {\scriptstyle \eta_{MU}}\uparrow & & {\scriptstyle \eta_E}\uparrow \\ S & \xrightarrow{=} & S \end{array} \tag{4.20}$$

This is equivalent to a cohomology class $\theta \in E^0(MU)$ so that $\mu_{MU}^* \varphi = \varphi \wedge \varphi \in E^0(MU \wedge MU)$ and $\eta_{MU}^* \varphi = \eta_E = 1 \in E^*(\mathrm{pt})$. Using the fact that the canonical maps

$$E^*(MU) \to \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(MU), E_*(\mathrm{pt})) \text{ and}$$

$$E^*(MU \wedge MU) \to \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(MU \wedge MU), E_*(\mathrm{pt})) \simeq \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(MU) \otimes_{E_*(\mathrm{pt})} E_*(MU), E_*(\mathrm{pt}))$$

are isomorphisms, we see that such a class φ is equivalent to a homomorphisms of graded $E_*(\mathrm{pt})$-modules $\varphi : E_*(MU) \to E_*(\mathrm{pt})$ so that $\varphi\mu_{MU*} = \varphi \otimes \varphi$ as homomorphisms $E_*(MU) \otimes_{E_*(\mathrm{pt})} E_*(MU) \to E_*(\mathrm{pt})$ and $\varphi 1 = 1 \in E_*(\mathrm{pt})$, which is to say, a homomorphism of graded $E_*(\mathrm{pt})$-algebras $\varphi : E_*(MU) \to E_*(\mathrm{pt})$.

By proposition 4.17 the homomorphism $\tilde{E}_*(\Sigma^{-2}MU(1)) \to E_*(MU)$ induces an isomorphism $S_\infty \tilde{E}_*(\Sigma^{-2}MU(1)) \simeq E_*(MU)$, so a homomorphism of algebras φ as above is equivalent to a homomorphism of graded $E_*(\mathrm{pt})$-modules

$$\tilde{E}_*(\Sigma^{-2}MU(1)) \to E_*(\mathrm{pt}) \text{ so that the map } \tilde{E}_2(MU(1)) \to E_0(\mathrm{pt})$$

takes $\beta_1 \mapsto 1$. But given the isomorphism $\tilde{E}^2(\mathbb{C}P^\infty) \simeq \mathrm{Hom}_{E_*(\mathrm{pt})}(\tilde{E}_*(\Sigma^{-2}MU(1)), E_*(\mathrm{pt}))$, this is the same as a class $x_E \in \tilde{E}^2(\mathbb{C}P^\infty)$ so that $\langle x_E, \beta_1 \rangle = 1$; it's not hard to show the last condition means $\iota^* x_E \in \tilde{E}^2(\mathbb{C}P^1)$ is the usual generator.

□

4.3. **Some overdue examples.** At this point it is comparatively easy to exhibit some complex oriented cohomology theories. There are three important examples to keep in mind:

Example 4.37. $H\mathbb{Z}$, the Eilenberg-Maclane spectrum representing singular cohomology with coefficients in $\mathbb{Z}$ - I'll often drop the $\mathbb{Z}$ from the notation. In this case if $\iota : \mathbb{C}P^1 \to \mathbb{C}P^\infty$ is the inclusion of a line the restriction $\iota^* : \tilde{H}^2(\mathbb{C}P^\infty) \to \tilde{H}^2(\mathbb{C}P^1)$ is an *isomorphism*, so there's a unique class $x_H \in \tilde{H}^2(\mathbb{C}P^\infty)$ restricting to the usual generator of $\tilde{H}^2(\mathbb{C}P^1)$. This class x_H makes H a complex oriented cohomology theory.

Example 4.38. K, the spectrum representing complex K-theory (consisting of $BU \times \mathbb{Z}$ in even dimensions, U in odd dimensions). In this case one can show that if γ_1^* is the dual of the tautological line bundle on $\mathbb{C}P^\infty$, then $x_K = \beta^{-1}([\gamma_1^*] - 1) \in \tilde{K}^2(\mathbb{C}P^\infty)$ is a class restricting to the usual generator of $\tilde{K}^2(\mathbb{C}P^1)$; here $\beta : K \to \Omega^2 K$ is the Bott periodicity isomorphism, and β^{-1} is its inverse.

Example 4.39. MU, the Thom spectrum representing complex cobordism. Consider a hyperplane section $H \subset \mathbb{C}P^n$; to be specific one can take $H = Z(x_0)$, consisting of points of the form $[0, a_1, \dots, a_n]$. Let $\iota : H \to \mathbb{C}P^n$ be the inclusion, and define $x_n \in MU^2(\mathbb{C}P^n)$ to be the cobordism class of ι (since H is disjoint from the basepoint $[1, 0, \dots, 0]$, this can be viewed as a reduced class). Notice that if $j : \mathbb{C}P^1 \to \mathbb{C}P^n$ is the usual inclusion of $\mathbb{C}P^1$ as the points like $[a_0, a_1, 0, \dots, 0]$, then $j^* x_n \in \tilde{MU}^2(\mathbb{C}P^1)$ can be identified with $\mathbb{C}P^1 \cap H$, which is just the point $\{[0,1]\} \subset \mathbb{C}P^1$, and it's not hard to show this is the usual generator.

Passing to the limit as $n \to \infty$ (carefully, by proving most of proposition 4.6 in the case $E = MU$) one can obtain a class $x_{MU} \in \tilde{MU}^2(\mathbb{C}P^\infty)$ restricting to the usual generator in $\tilde{MU}^2(\mathbb{C}P^1)$. Alternatively, recall that we have a canonical homeomorphism $\mathbb{C}P^\infty \simeq MU(1)$, and together with the canonical morphism $\Sigma^{-2} MU(1) \to MU$ this gives a morphism of spectra

$$\Sigma^{-2} \mathbb{C}P^\infty \simeq \Sigma^{-2} MU(1) \to MU$$

representing the class x_{MU}.

Thus we may summarize some of the main results of the last few sections as:

Theorem 4.40. *MU is the universal complex oriented cohomology theory.*

Example 4.41. There is a homomorphism of ring CW spectra $\varphi : MU \to H$ so that $\varphi_* x_{MU} = x_H \in \tilde{H}^2(\mathbb{C}P^\infty)$. Here's a geometric description: suppose X is a smooth manifold, and let $f : M \to X$ be a proper complex oriented smooth map of codimension n; let $\iota : M \to E(\epsilon^N)$ be a smooth embedding over X, with complex normal bundle ν. Then we obtain a Pontryagin Thom collapse map $\Sigma^N X \to \mathrm{Th}\nu$, and pulling back the Thom class $u(\nu) \in \tilde{H}^{n+N}(\mathrm{Th}\nu)$ yields a class in $\tilde{H}^{n+N}(\Sigma^N X) \simeq \tilde{H}^n(X)$; call this $\varphi[f]$. In the case where M is an embedded submanifold with a complex normal bundle, this is the classical Poincare dual of M.

Alternatively, note that the universal Thom classes $u(\gamma_n) \in \tilde{H}^{2n}(MU(n))$ define morphisms of spectra $\Sigma^{-2n} MU(n) \to H$ which piece together (certainly something to prove here) to yield a homomorphism of ring CW spectra $\varphi : MU \to H$.

Question 4.42. Is there a pleasant description of the analogous homomorphism $MU \to K$? In particular, isn't there a nice description of Thom classes in K-theory?

4.4. **Formal group laws and their dual Hopf algebras.** Let E be a complex oriented cohomology theory.

The computations of section 4.2 provide Chern classes c_i for complex vector bundles in E-cohomology. Given a complex n-plane bundle $\xi : E(\xi) \to X$, let $f : X \to BU$ be a map classifying the stable bundle ξ. Then we may set

$$c_i(\xi) := f^* c_i \in E^{2i}(X) \text{ for } i \in \mathbb{N}$$

where the classes $c_i \in E^*(BU)$ are those described in proposition 4.14 (I'm introducing the convention $c_0 = 1$). Evidently we've defined a natural transformation $\mathrm{Vect}^n(X,\mathbb{C}) \xrightarrow{c_i} E^{2n}(X)$, i.e. a characteristic class in the usual sense of the word; also note that these classes are stable by definition; the above map f also classifies the bundles $\xi \oplus \epsilon^n$ for $n \in \mathbb{N}$. Furthermore

- $c_0(\xi) = 1$ (by definition) and $c_i(\xi) = 0$ for $i > n$, because f factors through a classifying map $X \to BU(n)$ and the restriction homomorphism $E^*(BU) \to E^*(BU(n))$ sends $c_i \mapsto 0$ for $i > n$.
- Let $\eta : E(\eta) \to X$ be a complex m-plane bundle on X and consider the Whitney sum $\xi \oplus \eta$. If $g : X \to BU$ classifies the stable bundle η, then the composition
$$X \xrightarrow{\Delta} X \times X \xrightarrow{f\times g} BU \times BU \xrightarrow{\mu} BU$$
classifies $\xi \oplus \eta$, where μ is the H-space operation on BU (corresponding to direct sum). By proposition 4.14 the coproduct for $E^*(BU)$ is described by $\mu^*(c_n) = \sum_{i+j=n} c_i \times c_j \in E^*(BU \times BU)$, so it's immediate that
$$c_n(\xi \oplus \eta) = \sum_{i+j=n} c_i(\xi) \smile c_j(\eta) \in E^{2n}(X)$$
Thus the c_i satisfy a Whitney product formula.
- If γ_1^* is the dual of the tautological line bundle on $\mathbb{C}P^\infty$, then $c_1(\gamma_1^*) = x_E \in E^2(\mathbb{C}P^\infty)$. In particular $c_1(\gamma_1^*)$ restricts to the usual generator of $\tilde{E}^2(\mathbb{C}P^1)$. This can be seen by tracing through the definitions in section 4.2.

Of course, when $E = H$ these are the usual Chern classes (and the above list of properties *proves* that they are).

The behavior of these E-cohomology Chern classes with respect to tensor products of vector bundles is more subtle; more specifically, it depends on the complex oriented cohomology theory E. Note that in light of the "splitting principle" described in proposition 4.15, it would be enough to figure out the behavior of the c_i with respect to tensor products of *line* bundles.

Example 4.43. Taking $E = H$ (so we're talking about the usual Chern classes) it's a classic theorem that if λ_1 and λ_2 are 2 complex line bundles on a CW complex X, then
$$c_1(\lambda_1 \otimes \lambda_2) = c_1(\lambda_1) + c_1(\lambda_2) \in H^2(X)$$
and moreover c_1 defines an isomorphism between the topological Picard group of (isomorphism classes of) complex line bundles over X and $H^2(X)$.

Example 4.44. When $E = K$, the 1st K-theory Chern class of a complex line bundle λ_1 on a CW complex X can be identified as $\beta^{-1}([\lambda_1] - 1) \in K^2(X)$. If λ_2 is another complex line bundle on X, observe that
$$\beta^{-1}([\lambda_1 \otimes \lambda_2] - 1) = \beta^{-1}([\lambda_1] - 1)([\lambda_2] - 1) + \beta^{-1}([\lambda_1] - 1) + \beta^{-1}([\lambda_2] - 1)$$
Thus in K-theory $c_1(\lambda_1 \otimes \lambda_2) = c_1(\lambda_1) + c_1(\lambda_2) + \beta c_1(\lambda_1)c_1(\lambda_2)$.

Let γ_1 be the tautological line bundle on $\mathbb{C}P^\infty$; let $\pi_1, \pi_2 : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ be the projections onto the 2 factors, and consider the universal tensor product of 2 line bundles $\pi_1^*\gamma_1 \otimes \pi_2^*\gamma_1$ over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$. It's classified by the H-space operation $\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. Thus $\mu^*\gamma^1 = \pi_1^*\gamma_1 \otimes \pi_2^*\gamma_1$; of course $\mu^*\gamma_1^* = \pi_1^*\gamma_1^* \otimes \pi_2^*\gamma_1^*$.

Remark 4.45. One way to describe μ is as the map $BS^1 \times BS^1 \to BS^1$ classifying the multiplication homomorphism $S^1 \times S^1 \to S^1$. Here's another description:

Consider effective divisors on $\mathbb{C}P^1$, i.e. (finite) linear combinations $D = \sum_i n_i p_i$ of points $p_i \in \mathbb{C}P^1$ with coefficients $n_i \in \mathbb{N}$ - denote these by $\mathrm{Div}_+(\mathbb{C}P^1)$. Recall that the degree of such an effective divisor is $\deg D = \sum_i n_i \in \mathbb{N}$. For each $n \in \mathbb{N}$ the set of effective divisors of degree n, say $\mathrm{Div}_+(\mathbb{C}P^1)_n$ can be identified with the projectivization of

$$\mathcal{O}(n)(\mathbb{C}P^1) \simeq \mathbb{C}[x,y]_n$$

the homogeneous polynomials of degree n. Thus it's copy of $\mathbb{C}P^n$. Moreover we have canonical inclusions $\mathrm{Div}_+(\mathbb{C}P^1)_n \xrightarrow{\infty} \mathrm{Div}_+(\mathbb{C}P^1)_{n+1}$; just add $\infty = [1,0]$. This corresponds to the injection

$$\mathcal{O}(n)(\mathbb{C}P^1) \xrightarrow{y} \mathcal{O}(n+1)(\mathbb{C}P^1) \text{ multiplying by } y$$

which we can view as a linear embedding $\mathbb{C}P^n \to \mathbb{C}P^{n+1}$. The upshot: we've exhibited a natural bijection between the effective divisors $\mathrm{Div}_+(\mathbb{C}P^1)$ and $\mathbb{C}P^\infty$- evidently we can identify $\mathrm{Div}_+(\mathbb{C}P^1)$ with the infinite symmetric product $SP^\infty\mathbb{C}P^1$, in which case we've obtained the classical homeomorphism $SP^\infty\mathbb{C}P^1 \simeq \mathbb{C}P^\infty$.

Now addition of effective divisors gives $\mathbb{C}P^\infty$ the structure of a topological abelian monoid (this is clear at least at the level of sets - I'm going to assume without proof that addition is continuous).

Implicit in the above discussion was the fact that for a given effective divisor D of degree n the global sections $f \in \mathcal{O}(n)(\mathbb{C}P^1)$ with zeroes exactly D span a 1-dimensional subspace (in essence this is saying a polynomial in one variable is determined by its roots (and their multiplicities)). In this way we obtain a canonical complex line bundle over $\mathrm{Div}_+(\mathbb{C}P^1)$, and one can show it corresponds to the tautological line bundle over $\mathbb{C}P^\infty$.

Now suppose $f_1 \in \mathcal{O}(m)(\mathbb{C}P^1)$ and $f_2 \in \mathcal{O}(n)(\mathbb{C}P^1)$ are non-0 global sections with 0-set the effective divisors $D_1, D_2 \in \mathrm{Div}_+(\mathbb{C}P^1)$, with degrees m, n respectively. Then $f \cdot g \in \mathcal{O}(m+n)(\mathbb{C}P^1)$ is a non-0 global section with 0-set the sum $D_1 + D_2 \in \mathrm{Div}_+(\mathbb{C}P^1)$. This construction gives a map $E(\gamma_1) \times E(\gamma_1) \to E(\gamma_1)$ over the addition operation $\mu : \mathrm{Div}_+(\mathbb{C}P^1) \times \mathrm{Div}_+(\mathbb{C}P^1) \to \mathrm{Div}_+(\mathbb{C}P^1)$ which is $\mathbb{C}$-bilinear on fibers, and induces an isomorphism $\pi_1^*\gamma_1 \otimes \pi_2^*\gamma_1 \simeq \mu^*\gamma_1$ (recall that multiplication defines isomorphisms $\mathcal{O}(m)(\mathbb{C}P^1) \otimes \mathcal{O}(n)(\mathbb{C}P^1) \simeq \mathcal{O}(m+n)(\mathbb{C}P^1)$).

Observe that the 1st E-cohomology Chern class of $\pi_1^*\gamma_1^* \otimes \pi_2^*\gamma_1^*$ satisfies

$$c_1(\pi_1^*\gamma_1^* \otimes \pi_2^*\gamma_1^*) = \mu^* c_1(\gamma_1^*) = \mu^* x \in \tilde{E}^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

(I'm going to abbreviate; $x = x_E$). From proposition 4.7 we know there's an isomorphism $E^*(\mathrm{pt})[[x_1, x_2]] \simeq E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ where $x_i = \pi_i^* x$; thus we may expand

$$\mu^* x = \mu(x_1, x_2) = \sum_{i,j \in \mathbb{N}} a_{ij} x_1^i x_2^j \in \tilde{E}^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

as a formal power series in x_1, x_2. Note that the x_i have degree 2, so it must be that $a_{ij} \in E^{2(1-i-j)}(\mathrm{pt})$. Since μ is an abelian monoid operation, it's immediate that:

- $\mathrm{id} \times \mu^* \mu^* x = \mu \times \mathrm{id}^* \mu^* x \in \tilde{E}^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty \times \mathbb{C}P^\infty)$, i.e. $\mu(x_1, \mu(x_2, x_3)) = \mu(\mu(x_1, x_2), x_3)$.
- $\mathrm{id} \times \epsilon^* \mu^* x = \epsilon \times \mathrm{id}^* \mu^* x = x \in \tilde{E}^2(\mathbb{C}P^\infty)$, i.e. $\mu(x, 0) = \mu(0, x) = x$. Here $\epsilon : \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ is the map sending all of $\mathbb{C}P^\infty$ to the basepoint $e = [1, 0, 0, \dots] \in \mathbb{C}P^\infty$, so $\mathrm{id} \times \epsilon$ is the map $\mathbb{C}P^\infty \simeq \mathbb{C}P^\infty \times \{e\} \subset \mathbb{C}P^\infty \times \mathbb{C}P^\infty$. Similarly for $\epsilon \times \mathrm{id}$.
- $\tau^* \mu^* x = \mu^* x \in \tilde{E}^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$, i.e. $\mu(x_2, x_1) = \mu(x_1, x_2)$.

Recall that there's an inversion map $\iota : \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ making $\mathbb{C}P^\infty$ an abelian group object in the homotopy category of CW complexes - it classifies the inversion of S^1 and can be realized as complex conjugation sending $[a_i] \mapsto [\bar{a}_i]$. I'm not sure how to describe this in terms of divisors and such. In any case, it will induce an involution $\iota^* : E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty)$ which given the isomorphism $E^*(\mathrm{pt})[[x]] \simeq E^*(\mathbb{C}P^\infty)$ is equivalent to a power series $\iota^* x = \iota(x) \in E^*(\mathbb{C}P^\infty)$ so that $\iota^* \iota^* x = x$, i.e. $\iota(\iota(x)) = x$. We'll also have

- $\mathrm{id} \times \iota^* \mu^* x = \iota \times \mathrm{id}^* x = 0$, i.e. $\mu(x, \iota(x)) = \mu(\iota(x), x) = 0$.

Definition 4.46. Let R be a $\mathbb{Z}$-graded commutative ring. An **abelian formal group law of dimension 1 over** R is a homomorphism of graded commutative R-algebras $\mu^* : R[[x]] \to R[[x_1, x_2]]$ together with an involution $\iota^* : R[[x]] \to R[[x]]$ (where the indeterminates have degree 2) so that

$$\mathrm{id} \times \mu^* \mu^* = \mu \times \mathrm{id}^* \mu^* : R[[x]] \to R[[x_1, x_2, x_3]], \quad \mathrm{id} \times \epsilon^* \mu^* = \epsilon \times \mathrm{id}^* \mu^* = \mathrm{id} : R[[x]] \to R[[x]]$$

$$\tau^* \mu^* = \mu^* : R[[x]] \to R[[x_1, x_2]] \text{ and } \mathrm{id} \times \iota^* \mu^* = \iota \times \mathrm{id}^* = \epsilon^* : R[[x]] \to R[[x]]$$

where the homomorphisms are defined by analogy with those above; in particular ϵ^* is the composition

$$R[[x]] \xrightarrow{x \mapsto 0} R \xrightarrow{\mathrm{inc}} R[[x]]$$

This data is equivalent to formal power series $\mu(x_1, x_2) = \mu^* x \in R[[x_1, x_2]]$ and $\iota(x) = \iota^* x \in R[[x]]$ of degree 2 so that

$$\mu(x_1, \mu(x_2, x_3)) = \mu(\mu(x_1, x_2), x_3), \quad \mu(x, 0) = \mu(0, x) = x \quad \mu(x_2, x_1) = \mu(x_1, x_2)$$

$$\iota(\iota(x)) = x \text{ and } \mu(x, \iota(x)) = \mu(\iota(x), x) = 0$$

There's an evident general definition of a **formal group law over** R, but it won't be necessary. I'll often drop the adjectives "abelian" and "dimension 1," since this is the only sort of formal group law we'll encounter.

Unpacking some of the definitions, write $\mu(x_1, x_2) = \sum_{i,j} a_{ij} x_1^i x_2^j \in R[[x_1, x_2]]$, where $a_{ij} \in R$ and $\deg a_{ij} = 2(1 - i - j)$. The requirement $\mu(x_2, x_1) = \mu(x_1, x_2)$ simply says that $a_{ji} = a_{ij}$, so the coefficients a_{ij} are symmetric in i and j, and conditions $\mu(x_1, 0) = x_1$ and $\mu(0, x_2) = x_2$ just say that $a_{i0} = \delta_{i1}$ and $a_{0j} = \delta_{1j}$, i.e.

$$\mu(x_1, x_2) = \sum_{i,j} a_{ij} x_1^i x_2^j = x_1 + x_2 + \sum_{i>0, j>0} a_{ij} x_1^i x_2^j$$

Expanding both sides of the equation $\mu(x_1, \mu(x_2, x_3)) = \mu(\mu(x_1, x_2), x_3)$ as formal power series in the x_i (note that this only makes sense because the constant term of $\mu(x_1, x_2)$ is 0), say

$$\mu(x_1, \mu(x_2, x_3)) = \sum_{i,j,k} p_{ijk}(a) x_1^i x_2^j x_3^k \text{ and } \mu(\mu(x_1, x_2), x_3) = \sum_{i,j,k} q_{ijk}(a) x_1^i x_2^j x_3^k$$

where $p_{ijk}(a)$ and $q_{ijk}(a)$ are homogeneous polynomials of degree $2(1 - i - j - k)$ in the a_{ij} and equating the coefficients of $x_1^i x_2^j x_3^k$ will yield a collection of complicated homogeneous relations $p_{ijk}(a) = q_{ijk}(a)$ on the a_{ij}- more on this later.

Write $\iota(x) = \sum_i b_i x^i$. Then the equation $\iota(\iota(x)) = x$ expands as $\sum_i b_i (\sum_j b_j x^j)^i = x$ - in particular this implies $b_0 = 0$ and $b_1 = \pm 1$. The condition $\mu(x, \iota(x)) = 0$ reads

$$0 = x + \iota(x) + \sum_{i>0, j>0} a_{ij} x^i (\iota(x))^j = x + \sum_i b_i x^i + \sum_{i>0, j>0} a_{ij} x^i (\sum_k b_k x^k)^j$$

From this we see that $b_1 = -1$, i.e. $\iota(x) = -x + \sum_{i>1} b_i x^i$. In fact, we may continue expanding power series like

$$(\sum_k b_k x^k)^j = \sum_k (\sum_{\lambda \vDash k, l(\lambda) = j} b_\lambda) x^k$$

where the sum $\sum_{\lambda \vDash k, l(\lambda) = j} b_\lambda$ is to be taken over all "weak compositions of k with length j", i.e. all $\lambda = (\lambda_1, \ldots, \lambda_j) \in \mathbb{N}^j$ so that $\sum_i \lambda_i = k$, and $b_\lambda := \prod_{i=1}^j b_{\lambda_i}$. In that case

$$\sum_{i>0, j>0} a_{ij} x^i (\sum_k b_k x^k)^j = \sum_{i>0, j>0} a_{ij} x^i \sum_k (\sum_{\lambda \vDash k, l(\lambda) = j} b_\lambda) x^k$$

$$= \sum_{n>0} (\sum_{i+k=n, i>0} \sum_{j>0} a_{ij} (\sum_{\lambda \vDash k, l(\lambda) = j} b_\lambda)) x^n$$

Note that for each $n \in \mathbb{N}$ the sum $\sum_{i+k=n,i>0} \sum_{j>0} a_{ij}(\sum_{\lambda \vDash k, l(\lambda)=j} b_\lambda)$ is indeed finite: since $b_0 = 0$, we have $\sum_{\lambda \vDash k, l(\lambda)=j} b_\lambda = 0$ for $j > k$. The upshot: comparing coefficients in the equation

$$x + \sum_i b_i x^i + \sum_{i>0,j>0} a_{ij} x^i (\sum_k b_k x^k)^j = 0$$

we see that

$$b_n = -\sum_{i+k=n,i>0} \sum_{j>0} a_{ij}(\sum_{\lambda \vDash k, l(\lambda)=j} b_\lambda) \text{ for } n \in \mathbb{N}, n > 1$$

Unraveling all the indexing, one sees that the right hand side only involves the coefficients a_{ij} for $0 < i, j < n$ and b_i for $i < n$. Thus b_n can be computed inductively as a polynomial in the a_{ij}. We've just proved most of:

Proposition 4.47. *Let R be a $\mathbb{Z}$-graded commutative ring and suppose $\mu^* : R[[x]] \to R[[x_1, x_2]]$ is a homomorphism of graded R-algebras determined by a formal power series $\mu(x_1, x_2) = \mu^* x$ so that*

$$\mu(x_1, \mu(x_2, x_3)) = \mu(\mu(x_1, x_2), x_3), \quad \mu(x, 0) = \mu(0, x) = x \quad \mu(x_2, x_1) = \mu(x_1, x_2)$$

Here the indeterminates have degree 2. Then there exists a unique involution $\iota^* : R[[x]] \to R[[x]]$ given by a formal power series $\iota(x)$ such that

$$\iota(\iota(x)) = x \text{ and } \mu(x, \iota(x)) = \mu(\iota(x), x) = 0$$

Indeed, from the above discussion the equation

$$\mu(x, \iota(x)) = 0$$

admits a unique solution $\iota(x) \in R[[x]]$ - and it is *automatic* that $\iota(\iota(x)) = x$, since $\mu(x, \iota(x)) = 0$ and $\mu(\iota(x), \iota(\iota(x)) = 0$, so

$$\begin{aligned} \iota(\iota(x)) &= \mu(0, \iota(\iota(x))) = \mu(\mu(x, \iota(x)), \iota(\iota(x))) \\ &= \mu(x, \mu(\iota(x), \iota(\iota(x)))) = \mu(x, 0) = x \end{aligned}$$

Due to proposition 4.21 most people omit from the definition of a formal group law the existence of an involution $\iota^* : R[[x]] \to R[[x]]$ corresponding to inversion.

Let's rephrase some of our earlier discussion in the language of formal group laws:

Proposition 4.48. *The homomorphism $\mu^* : E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$, together with the identifications $E^*(\mathrm{pt})[[x]] \simeq E^*(\mathbb{C}P^\infty)$ and $E^*(\mathrm{pt})[[x_1, x_2]] \simeq E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$, defines a formal group law over $E^*(\mathrm{pt})$. Equivalently, expanding $c_1(\pi_1^*\gamma_1^* \otimes \pi_2^*\gamma_1^*)$ as a formal power series on $\pi_1^* c_1(\gamma_1^*), \pi_2^* c_1(\gamma_1^*)$ yields a formal group law over $E^*(\mathrm{pt})$.*

Remark 4.49. Here's another way one runs into formal group laws: let k be an algebraically closed field and let G be an abelian group variety of dimension 1 over k. For instance, G might be an elliptic curve with a distinguished point.

Let $\mu : G \times G \to G$ be the multiplication morphism, and localize and complete it at the identities: the result is a homomorphism of complete local k-algebras

$$\mu^* : \hat{\mathcal{O}}_{G,e} \to \hat{\mathcal{O}}_{G \times G,(e,e)}$$

Let x be a local parameter at $e \in G$, and let $x_i = \pi_i^* x$, $i = 1, 2$ be the resulting local parameters at $(e, e) \in G \times G$. Then it's classic rock (the Cohen structure theorem) that the natural maps $k[[x]] \to \hat{\mathcal{O}}_{G,e}$ and $k[[x_1, x_2]] \to \hat{\mathcal{O}}_{G \times G,(e,e)}$ are isomorphisms, and in this way we may identify μ^* with a homomorphism $k[[x]] \to k[[x_1, x_2]]$, equivalent to a formal power series

$$\mu^* x = \mu(x_1, x_2) \in k[[x_1, x_2]]$$

The fact μ is an abelian group operation ensures that $\mu(x_1, \mu(x_2, x_3)) = \mu(\mu(x_1, x_2), x_3), \mu(x, 0) = \mu(0, x) = x$ and $\mu(x_2, x_1) = \mu(x_1, x_2)$. Similarly localizing and completing the inversion $\iota : G \to G$

at the identity yields and involution $\iota^* : \hat{\mathcal{O}}_{G,e} \to \hat{\mathcal{O}}_{G,e}$, which we may identify as an involution of $k[[x]]$ determined by a formal power series $\iota(x)$. Hence we've obtained a formal group law over k.

NOTE: there's no grading involved in this context. So one might want to modify the definitions a bit.

Apparently Grothendeick has papers comparing formal group laws to group objects in the category of formal schemes.

Examples 7 shows that the formal group law associated to H is given by $\mu(x_1, x_2 = x_1 + x_2)$. People call this the *additive* formal group law, because it's what one obtains by carring out the construction of the previous remark starting with the group variety $G = \mathbb{A}^1_k$, the affine line under addition.

On the other hand example 8 shows that the formal group law associated to K is given by $\mu(x_1, x_2) = x_1 + x_2 + \beta x_1 x_2$ People call this the *multiplicative* formal group law since it's what one obtains from the group variety $G = \mathbb{A}^1_k - \{0\}$, the affine line minus the origin under multiplication.

Recall from proposition 4.8 that $E_*(\mathbb{C}P^\infty)$ is a free graded $E_*(\mathrm{pt})$-module on classes $\beta_i \in E_{2i}(\mathbb{C}P^\infty)$ dual to the x^i. Moreover the external product $E_*(\mathbb{C}P^\infty) \otimes_{E_*(\mathrm{pt})} E_*(\mathbb{C}P^\infty) \xrightarrow{\times} E_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ is an isomorphism, and we have a coproduct

$$\Delta_* : E_*(\mathbb{C}P^\infty) \to E_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq E_*(\mathbb{C}P^\infty) \otimes_{E_*(\mathrm{pt})} E_*(\mathbb{C}P^\infty)$$

described by $\Delta_* \beta_n = \sum_{i+j=n} \beta_i \otimes \beta_j$ making $E_*(\mathbb{C}P^\infty)$ a graded co-algebra over $E_*(\mathrm{pt})$ (which is co-associative, co-commutative, etc.). The H-space operation $\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ induces a Pontryagin product

$$E_*(\mathbb{C}P^\infty) \otimes_{E_*(\mathrm{pt})} E_*(\mathbb{C}P^\infty) \xrightarrow{\times} E_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \xrightarrow{\mu_*} E_*(\mathbb{C}P^\infty)$$

(which is associative, commutative etc. because μ is) making $E_*(\mathbb{C}P^\infty)$ a Hopf algebra over $E_*(\mathrm{pt})$.

Given that the natural map $E^*(\mathbb{C}P^\infty) \to \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(\mathbb{C}P^\infty), E_*(\mathrm{pt}))$ is an isomorphism, one would expect that the Hopf algebra structure on $E_*(\mathbb{C}P^\infty)$ provided by μ_* will be just as interesting as the formal group law $\mu^* : E^*(\mathbb{C}P^\infty) \to E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$- they carry the same information.

Note that μ_* will be completely determined by the products $\mu_* \beta_i \otimes \beta_j \in E_*(\mathbb{C}P^\infty)$; expanding over the basis β_k, we'll have

$$\mu_* \beta_i \otimes \beta_j = \sum_k \langle x^k, \mu_* \beta_i \otimes \beta_j \rangle \beta_k$$

and the Kronecker products can be computed as

$$\begin{aligned} \langle x^k, \mu_* \beta_i \otimes \beta_j \rangle &= \langle \mu^*(x^k), \beta_i \otimes \beta_j \rangle \\ &= \langle \mu(x_1, x_2)^k, \beta_i \otimes \beta_j \rangle \end{aligned}$$

This is the coefficient on $x_1^i x_2^j$ in the formal power series $\mu(x_1, x_2)^k = (\sum_{m,n} a_{mn} x_1^m x_2^n)^k$ which of course can be computed by expanding things out - i.e. the coefficient on $x_1^i x_2^j$ will be a homogeneous polynomial of degree $2(i + j - k)$ in the a_{mn} (note that we're taking coefficients in the homology ring $E_*(\mathrm{pt})$ now, so the degrees change sign). For instance, this shows that

$$\mu_* \beta_i \otimes \beta_j = \delta_{i0} \delta_{j0} \beta_0 + a_{ij} \beta_1 + \sum_{k>1} \langle x^k, \mu_* \beta_i \otimes \beta_j \rangle \beta_k$$

(where I'll be following the convention that $\beta_0 = 1$).

Similarly the involution $\iota_* : E_*(\mathbb{C}P^\infty) \to E_*(\mathbb{C}P^\infty)$ will be completely determined by the elements $\iota_* \beta_i$, and these can be expanded over the basis β_j like

$$\iota_* \beta_i = \sum_j \langle x^j, \iota_* \beta_i \rangle \beta_j$$

We'll have

$$\langle x^j, \iota_*\beta_i\rangle = \langle \iota^* x^j, \beta_i\rangle = \langle \iota(x)^j, \beta_i\rangle$$

the coefficient on x^i in the formal power series $\iota(x)^j = (\sum_k b_k x^k)^j$ (in the notation used above). Of course, this will be a homogeneous polynomial of degree $2(i-j)$ in the b_k.

More generally, let R be a $\mathbb{Z}$-graded commutative ring. Let $R\{\beta_i \,|\, i \in \mathbb{N}\}$ be the free graded R-module on generators β_i of degree $2i$, and make it a graded R-coalgebra with coproduct

$$\Delta_* : R\{\beta_i\} \to R\{\beta_i\} \otimes_R R\{\beta_i\} \simeq R\{\beta_i \otimes \beta_j \,|\, i, j \in \mathbb{N}\}$$

described by $\Delta_*\beta_n = \sum_{i+j=n} \beta_i \otimes \beta_j$ -clearly this is co-associative, co-commutative, etc.

Recall that there is an isomorphism $R^{op} \to \mathrm{End}_R(R, R)$ Proceeding by analogy with the case of spectra one sees that $\mathrm{End}_R(R, R)$ should be graded so that right multiplication by $a \in R$ has degree $-\deg a$. Thus as a graded abelian group R^{op} is just R with the signs of degrees flipped.

Let $x \in \mathrm{Hom}_R(R\{\beta_i\}, R)$ (note: this should be interpreted as the *graded* dual of $R\{\beta_i\}$) be the homomorphism sending $\beta_1 \to 1 \in R$ and all other β_i to 0. Again defining the grading on $\mathrm{Hom}_R(R\{\beta_i\}, R)$ by analogy with spectra, this has degree 2.

Notice that the co-algebra structure on $R\{\beta_i\}$ induces a graded commutative R^{op}-algebra structure on $\mathrm{Hom}_R(R\{\beta_i\}, R)$ - indeed it induces a map of graded R^{op}-modules

$$\mathrm{Hom}_R(R\{\beta_i\}, R) \otimes_{R^{op}} \mathrm{Hom}_R(R\{\beta_i\}, R) \to \mathrm{Hom}_R(R\{\beta_i\} \otimes_R R\{\beta_i\}, R) \to \mathrm{Hom}_R(R\{\beta_i\}, R)$$

which is associative, commutative etc. because Δ_* is. Observe that x^n is dual to β_n, in the sense that it's the homomorphism $R\{\beta_i\} \to R$ taking β_n to 1 and all other β_m to 0. Indeed, we have

$$\begin{aligned}\langle x^n, \beta_m\rangle &= \langle x^{n-1} \otimes x, \Delta_*\beta m\rangle \\ &= \sum_{i+j=m} \langle x^{n-1}, \beta_i\rangle\langle x, \beta_j\rangle\end{aligned}$$

and this facilitates an inductive argument that $\langle x^n, \beta_m\rangle = \delta_{mn}$. Now observe that a degree n homomorphism $\varphi_i : R\{\beta_i\} \to R$ is totally determined by the $\varphi(\beta_i)$, which must be degree $2i - n$ elements of R - it must be that

$$\varphi = \sum_i \varphi(\beta_i)x^i \in \mathrm{Hom}_R(R\{\beta_i\}, R)$$

We've essentially shown:

Proposition 4.50. *The natural map $R^{op}[[x]] \to \mathrm{Hom}_R(R\{\beta_i\}, R)$ is an isomorphism. Similarly, the natural map $R^{op}[[x_1, x_2]] \to \mathrm{Hom}_R(R\{\beta_i\} \otimes_R R\{\beta_i\}, R)$ is an isomorphism.*

Now suppose we're given a product

$$\mu_* : R\{\beta_i\} \otimes_R R\{\beta_i\} \to R\{\beta_i\}$$

and an involution $\iota_* : R\{\beta_i\} \to R\{\beta_i\}$ making $R\{\beta_i\}$ an associative, commutative etc. Hopf algebra over R. Then it's not hard to show that the induced map

$$\mu^*\mathrm{Hom}_R(R\{\beta_i\}, R) \to \mathrm{Hom}_R(R\{\beta_i\} \otimes_R R\{\beta_i\}, R)$$

together with the identifications of proposition 4.23 yield a formal group law μ^* over R^{op}. In fact

Proposition 4.51. *The above constructions define a natural 1-1 correspondence between associative, commutative etc. Hopf algebra structures on the coalgebra $R\{\beta_i\}$ and formal group laws over R^{op}.*

I'm going to skip a detailed proof (we've basically done all of the work).

Remark 4.52. Note that the relationship between $E^*(\mathrm{pt})$ and $E_*(\mathrm{pt})$ is best expressed by saying the natural map

$$E^*(\mathrm{pt}) \to \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(\mathrm{pt}), E_*(\mathrm{pt}))$$

identifies $E^*(\mathrm{pt})$ with $E_*(\mathrm{pt})^{op}$. Thus all this funny business with opposite rings.

Remark 4.53. Let G be an abelian group variety of dimension 1 over a field k of characteristic 0 - in fact we may as well take $k = \mathbb{C}$, since this is just for kicks. Let x be a local parameter at the identity $e \in G$, and let $\frac{\partial}{\partial x}$ be the associated derivation at e. If f, g are functions on a neighborhood of e, then

$$\frac{1}{n!}\frac{\partial^n}{\partial x^n}(fg)|_e = \sum_{l+m=n} \frac{1}{l!}\frac{\partial^l f}{\partial x^l}|_e \frac{1}{m!}\frac{\partial^m g}{\partial x^m}|_e$$

In this way we see that the vector space of differential operators $\mathbb{C}\{\frac{1}{n!}\frac{\partial^n}{\partial x^n}\}$ has a canonical co-algebra structure given by the Leibniz rule

$$\frac{1}{n!}\frac{\partial^n}{\partial x^n} \mapsto \sum_{l+m=n} \frac{1}{l!}\frac{\partial^l}{\partial x^l} \otimes \frac{1}{m!}\frac{\partial^m}{\partial x^m}$$

Now let f be a function on a neighborhood of $e \in G$; then $f \circ \mu$ is a function on a neighborhood of $(e, e) \in G \times G$, so we can consider

$$\mu_*(\frac{1}{l!}\frac{\partial^l}{\partial x^l} \otimes \frac{1}{m!}\frac{\partial^m}{\partial x^m})f|_e := \frac{1}{l!}\frac{1}{m!}\frac{\partial^{l+m} f \circ \mu}{\partial x^l \partial y^m}|_{e,e}$$

In this way we can associate to $\frac{1}{l!}\frac{\partial^l}{\partial x^l} \otimes \frac{1}{m!}\frac{\partial^m}{\partial x^m}$ a differential operator at e; to expand it over the basis $\frac{1}{n!}\frac{\partial^n}{\partial x^n}$, just evaluate on the various powers x^n; we'll have

$$\mu_*(\frac{1}{l!}\frac{\partial^l}{\partial x^l} \otimes \frac{1}{m!}\frac{\partial^m}{\partial x^m}) = \sum_n \mu_*(\frac{1}{l!}\frac{\partial^l}{\partial x^l} \otimes \frac{1}{m!}\frac{\partial^m}{\partial x^m})x^n|_e \frac{1}{n!}\frac{\partial^n}{\partial x^n}$$

where evidently

$$\mu_*(\frac{1}{l!}\frac{\partial^l}{\partial x^l} \otimes \frac{1}{m!}\frac{\partial^m}{\partial x^m})x^n|_e = \frac{1}{l!}\frac{1}{m!}\frac{\partial^{l+m}\mu(x,y)^n}{\partial x^l \partial y^m}|_{e,e}$$

the coefficient on $x^l y^m$ in the power series $\mu(x, y)^n$.

Thus the duality between the $\beta_n \in E_*(\mathbb{C}P^\infty)$ and the $x^n \in E^*(\mathbb{C}P^\infty)$ is analogous to the duality between differential operators $\frac{1}{n!}\frac{\partial^n}{\partial x^n}$ and the powers of x.

While we're on the subject:

Definition 4.54. Let R, R' be $\mathbb{Z}$-graded commutative rings and let $\mu^*, \mu^{'*}$ be formal group laws over R, R' respectively. A **homomorphism of formal group laws** $\varphi : \mu^* \to \mu^{'*}$ consists of a homomorphism of graded rings $\varphi : R \to R'$ together with a homomorphism of graded complete rings $\psi : R'[[x]] \to R'[[x']]$ making the following diagram commute:

$$\begin{array}{ccc} R'[[x]] & \xrightarrow{\psi} & R'[[x']] \\ {\scriptstyle \varphi_*\mu^*}\downarrow & & {\scriptstyle \mu^{'*}}\downarrow \\ R'[[x]] \otimes_{R'} R'[[x]] & \xrightarrow{\psi\otimes\psi} & R'[[x']] \otimes_{R'} R'[[x']] \end{array} \tag{4.21}$$

Here $\varphi_*\mu^*$ is obtained from μ^* by extending scalars to R'. The conditions on ψ should imply that it takes the ideal generated by x to the ideal generated by x'. This should amount to saying that ψ is determined by a degree 2 formal power series $\psi(x) \in R'[[x']]$ of the form $\psi(x) = \sum_{i>0} r'_i x'^i$.

More concretely, if $\mu(x, y) = \sum_{i,j} a_{ij} x^i y^j \in R[[x, y]]$ and $\mu'(x, y) = \sum_{i,j} a'_{ij} x^i y^j \in R'[[x, y]]$ are the formal power series describing $\mu^*, \mu'*$ then $\varphi_*\mu(x, y) = \sum_{i,j} \varphi(a_{ij}) x^i y^j \in R'[[x, y]]$ describes $\varphi_*\mu^*$ and we must have

$$\psi(\mu'(x, y)) = \varphi_*\mu(\psi(x), \psi(y)) \in R'[[x, y]]$$

There should be a dual definition of a homomorphism of Hopf algebra structures on $R\{\beta_i\}, R'\{\beta'_i\}$.

Lemma 4.55. *Let R be a graded commutative ring and let $\psi(x) = \sum_{i \in \mathbb{N}} c_i x^i \in R[[x]]$ be a formal power series of degree (2) with $c_0 = 0, c_1 = 1$ (so, $\psi(x) = x + \sum_{i>1} c_i x^i$). Let $\mu(x,y) \in R[[x,y]]$ be a formal group law over R. Then $\psi(x)$ defines an isomorphism of formal group laws over R from μ to the formal group law given by*

$$\mu'(x,y) := \psi^{-1}(\mu(\psi(x), \psi(y)))$$

An automorphism $\psi^* : R[[x]] \to R[[x]]$ described by a formal power series $\psi(x) \in R[[x]]$ as in the lemma is often referred to as a "strict automorphism" (here "strict" just means the coefficient on x is 1, not just any unit $u \in R^\times$).

With these definitions in hand, we can expand on proposition 4.22 and say that there is a *functor* taking a complex oriented cohomology theory E to the formal group law μ^* over $E^*(\mathrm{pt})$. Moreover if $\varphi : E \to E'$ is a homomorphism of complex oriented cohomology theories the induced homomorphism $\varphi_* : \mu^* \to \mu'^*$ takes $x_E \mapsto x_{E'}$, i.e., $\mu'^* = \varphi_* \mu^*$ is obtained from μ^* by extension of scalars along the coefficient homomorphism $\varphi_* : E^*(\mathrm{pt}) \to E'^*(\mathrm{pt})$.

Taking $E = MU$, we see that there is a formal group law over $MU^*(\mathrm{pt})$ described by a formal power series

$$\mu(x_1, x_2) = \sum_{i,j} a_{ij} x_1^i x_2^j \in MU^*(\mathrm{pt})[[x_1, x_2]] \simeq MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

where $a_{ij} \in MU^{2(1-i-j)}(\mathrm{pt}) = \pi_{2(i+j-1)} MU$ is a degree $2(i+j-1)$ element of the complex cobordism ring. There is a dual Hopf algebra structure on $MU_*(\mathrm{pt})\{\beta_i\} \simeq MU_*(\mathbb{C}P^\infty)$.

Here's an interesting

Question 4.56. Can one describe a closed $2(i+j-1)$-manifold M_{ij} with a complex structure on its stable normal bundle representing the cobordism class $a_{ij} \in \pi_{2(i+j-1} MU$?

Let M be a closed n-manifold with a complex structure on its stable normal bundle. Let $\iota : M \to S^{n+2k}$ be a stable embedding with complex normal bundle $\nu : E(\nu) \to M$, and let

$$S^{n+2k} \xrightarrow{\varphi} \mathrm{Th}\nu \xrightarrow{\mathrm{Th}\tilde{f}} MU(k)$$

be the map $S^{n+2k} \to MU(k)$ obtained via the Pontryagin-Thom construction; here $\mathrm{Th}\tilde{f} : \mathrm{Th}\nu \to MU(k)$ is the induced map of Thom spaces corresponding to a classifying map $f : M \to BU(k)$ for ν. Evidently the following diagram commutes:

$$\begin{array}{ccc} S^{n+2k} & \xrightarrow{\mathrm{Th}\tilde{f}} & MU(k) \\ \iota\uparrow & & \sigma\uparrow \\ M & \xrightarrow{f} & BU(k) \end{array} \tag{4.22}$$

Unravelling definitions, one sees that if

$$u(\gamma_k) \cap - : \tilde{H}_{n+2k}(MU(k)) \to H_n(BU(k))$$

is the Thom isomorphism obtained as the cap product with the Thom class $u(\gamma_k) \in \tilde{H}^{2k}(MU(k))$ then we'll have

$$u(\gamma_k) \cap \mathrm{Th}\tilde{f} \circ \varphi_*[S^{n+2k}] = f_*[M] \in H_n(BU(k))$$

From this one concludes that we have a commutative diagram (at least of graded abelian groups, but I think of graded commutative *rings*)

$$\begin{array}{ccc} \pi_*(MU) & \xrightarrow{\eta\wedge \mathrm{id}_*} & H_*(MU) \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ \Omega^U_*(\mathrm{pt}) & \xrightarrow{\nu} & H_*(BU) \end{array} \tag{4.23}$$

Here the left vertical arrow is given by the Pontryagin-Thom construction. The top horizontal arrow is the Hurewicz homomorphism, and the right vertical arrow is the stable Thom isomorphism. The bottom horizontal arrow ν takes the bordism class $[M] \in \Omega^U_*(\mathrm{pt})$ to $f_*[M] \in H_n(BU)$ - one can check that it's a ring homomorphism.

Since $H_*(BU)$ and $H^*(BU)$ are dual Hopf algebras, $f_*[M]$ is completely determined by the **characteristic numbers** of its stable normal bundle ν, i.e. the integers

$$\langle \alpha, f_*[M]\rangle = \langle \alpha(\nu), [M]\rangle \in \mathbb{Z}, \text{ for } \alpha \in H^n(BU)$$

where $\alpha(\nu) = f^*\alpha \in H^n(M)$. In fact it's not too hard to expand $f_*[M]$ as a homogeneous polynomial of degree n in the polynomial generators $\beta_i \in H_{2i}(BU)$.

Let $\lambda \vdash n$ be a partition of an integer $n \in \mathbb{N}$; say $\lambda = (\lambda_1, \ldots, \lambda_l)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$ and $\sum_{i=1}^l \lambda_i = n$ - here $l = l(\lambda)$ is the *length* of λ. Define

$$m_\lambda(x_1, x_2, \ldots, x_l) := \sum_{\sigma x^\lambda \in \Sigma_l x^\lambda} \sigma x^\lambda \in \mathbb{Z}[x_1, \ldots, x_l] \simeq H^*(\prod_{i=1}^l \mathbb{C}P^\infty)$$

Here $x^\lambda = \prod_{i=1}^l x_i^{\lambda_i}$ and $\sigma x^\lambda = \prod_{i=1}^l x_{\sigma(i)}^{\lambda_i}$ - the sum is to be taken over the Σ_l-orbit of x^λ. $m_\lambda(x_i, \ldots, x_l)$ is referred to as the *monomial* symmetric polynomial associated to the partition λ - note that it has degree n. From our computation of $H^*(BU)$ and discussion of the splitting principle, we know that if $j : \prod_{i=1}^l \mathbb{C}P^\infty \to BU$ classifies $\prod_{i=1}^l \gamma_1^*$ then there's a symmetric polynomial $s_\lambda(c_1, \ldots, c_l) \in H^{2n}(BU)$ so that $j^* s_\lambda(c_1, \ldots, c_l) = s_\lambda(\sigma_1(x_j), \ldots, \sigma_l(x_j)) = m_\lambda(x_1, \ldots, x_l) \in H^*(\prod_{i=1}^l \mathbb{C}P^\infty)$ and if $l \geq n$ the polynomial s_λ is *unique*.

For any partition λ as above, set $\beta_\lambda := \prod_{i=1}^l \beta_{\lambda_i} \in H_{2n}(BU)$.

Proposition 4.57. *Let λ and μ be partitions. Then*

$$\langle s_\lambda, \beta_\mu\rangle = \delta_{\lambda\mu} \in \mathbb{Z}$$

Thus $\{s_\lambda \mid \lambda \vdash n, n \in \mathbb{N}\}$ is the basis for $H^*(BU)$ dual to the basis $\{\beta_\lambda \mid \lambda \vdash n, n \in \mathbb{N}\}$ for $H_*(BU)$.

Among other things it follows that

$$f_*[M] = \sum_{\lambda \vdash n} \langle s_\lambda(\nu), [M]\rangle \beta_\lambda \in H_*(BU)$$

so that $f_*[M]$ is characterized by the "s-characteristic numbers" of the stable normal bundle ν.

Proof. We can make the simplifying assumption that $\lambda, \mu \vdash n$ are partitions of the same integer n, and that they have the same length l. We may also assume $l \geq n$.

Now look at the map $j : \prod_{i=1}^l \mathbb{C}P^\infty \to BU$ and its induced homomorphisms on (co)homology - we'll have

$$\begin{aligned} \langle s_\lambda, \beta_\mu\rangle &= \langle j^* s_\lambda, \otimes_{i=1}^l \beta_{\mu_i}\rangle = \langle m_\lambda(x_1, \ldots, x_l), \otimes_{i=1}^l \beta_{\mu_i}\rangle \\ &= \sum_{\sigma x^\lambda \in \Sigma_l x^\lambda} \prod_{i=1}^l \langle x_i^{\lambda_{\sigma(i)}}, \beta_{\mu_i}\rangle = \sum_{\sigma\lambda \in \Sigma_l \lambda} \prod_{i=1}^l \delta_{\lambda_{\sigma(i)}\mu_i} \end{aligned}$$

where the last sum runs over the Σ_l-orbit of $\lambda \in \mathbb{N}^l$. Now argue that the far right hand side is 1 if $\lambda = \mu$ and 0 otherwise.

□

Compare with the later sections of Milnor's *Characteristic classes*. Let's continue studying the Hurewicz homomorphism $\pi_*(MU) \to H_*(MU)$.

Let X and Y be CW spectra, and suppose E is an associative, commutative ring CW spectrum. Then the unit homomorphism $\eta : S \to E$ can be used to define a morphism of spectra

$$Y \simeq S \wedge Y \xrightarrow{\eta \wedge \mathrm{id}} E \wedge Y$$

inducing a homomorphism of graded abelian groups $[X, Y]_* \xrightarrow{\eta \wedge \mathrm{id}_*} [X, E \wedge Y]_*$, called a **Boardman homomorphism**.

Notice that $E \wedge Y$ is a module spectrum over the ring spectrum E; the action map is given by

$$E \wedge E \wedge Y \xrightarrow{\mu \wedge \mathrm{id}} E \wedge Y, \text{ where } \mu \text{ is multiplication in } E$$

In fact we could probably think of $E \wedge Y$ as a sort of "free E-module on Y" - given a morphism $\varphi : Y \to Z$, we can form the morphism of E-module spectra

$$E \wedge Y \xrightarrow{\mathrm{id} \wedge \varphi} E \wedge Z \xrightarrow{\theta} Z$$

where θ gives the action of E on Z. On the other hand, given a morphism of E-module spectra $\psi : E \wedge Y \to Z$ we obtain a morphism of spectra

$$Y \simeq S \wedge Y \xrightarrow{\eta \wedge \mathrm{id}} E \wedge Y \xrightarrow{\psi} Z$$

where η is the unit map of E. This should show that a morphism of spectra from Y to a module spectrum Z over E is equivalent to a morphism of E-modules spectra $E \wedge Y \to Z$.

In the case where $X = S$, $E = H$, the Boardman homomorphism is just the Hurewicz homomorphism. We have the following result when $E = H$:

Theorem 4.58. *$H \wedge Y$ is (non-canonically) stably homotopy equivalent to a product of Eilenberg-MacLane spectra with homotopy groups the homology groups $H_n(Y)$; that is, there is a non-canonical stable homotopy equivalence*

$$H \wedge Y \simeq \prod_{n \in \mathbb{Z}} \Sigma^n H(H_n(Y))$$

See Adams's *Stable homotopy and generalized homology*. Compare with the Dold-Thom theorem, which says that if X is a pointed CW complex then the infinite symmetric product $SP^\infty X$ is non-canonically homotopy equivalent to a product of Eilenberg-MacLane spaces of type $K(\tilde{H}_n(X), n)$, i.e. there's a non-canonical homotopy equivalence (which I think can be taken to be a homomorphism of H-spaces) $SP^\infty X \simeq \prod_{n \in \mathbb{N}} K(\tilde{H}_n(Y), n)$.

It follows that for any CW spectrum X an equivalence as in the above proposition induces isomorphisms

$$[X, H \wedge Y]_r \simeq \prod_{n \in \mathbb{Z}} H^{-r}(X; \Sigma^n H_n(Y)) \simeq \prod_{n \in \mathbb{Z}} H^{n-r}(X; H_n(Y))$$

The point is that in the case $E = H$ the Boardman homomorphism is relatively tractable.

There is an evident natural homomorphism of graded abelian groups

$$\alpha : [X, Y]_* \to \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(X), E_*(Y)) \text{ taking } [f] \mapsto f_*$$

In fact there's also a natural homomorphism

$$\alpha' : [X, E \wedge Y]_* \to \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(X), E_*(Y))$$

described as follows: a morphism of spectra $f : X \to E \wedge Y$ induces a morphism

$$E \wedge X \xrightarrow{\mathrm{id} \wedge f} E \wedge E \wedge Y \xrightarrow{\mu \wedge \mathrm{id}} E \wedge Y$$

and the induced map of stable homotopy groups is our desired homomorphism $E_*(X) \to E_*(Y)$. Unraveling definitions one shows:

Lemma 4.59. *The following diagram commutes:*

$$\begin{tikzcd} {[X,Y]_*} \arrow[r, "b"] \arrow[d, "\alpha"] & {[X, E\wedge Y]_*} \arrow[d, "\alpha'"] \\ \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(X), E_*(Y)) \arrow[r, "="] & \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(X), E_*(Y)) \end{tikzcd} \tag{4.24}$$

where the top horizontal map is the Boardman homomorphism, and the vertical maps are those described above.

Adams points out that the lemma is mostly useful when the right vertical map α' is an isomorphism. Since $E \wedge Y$ is a module spectrum over the ring spectrum E, lemma 4.13 shows that α' will be an isomorphism when

- the spectrum X is connective,
- $H_*(X; E_*(\mathrm{pt}))$ is a free $E_*(\mathrm{pt})$-module and
- the Atiyah-Hirzebruch spectral sequence for the E-homology of X collapses on page 2 (so in particular $E_*(X)$ is a free $E_*(\mathrm{pt})$-module).

From the calculations of section 4.2, these hypotheses are satisfied when $X = \mathbb{C}P^\infty, BU$ or MU and E is complex oriented.

Let E be a complex oriented cohomology theory, and consider the 2 morphisms of ring spectra

$$E \simeq E \wedge S \xrightarrow{\mathrm{id}\wedge\eta_E} E \wedge MU \text{ and } MU \simeq S \wedge MU \xrightarrow{\eta_{MU}\wedge\mathrm{id}} E \wedge MU$$

where η_E, η_{MU} are the unit morphisms of E, MU respectively. We obtain Boardman ring homomorphisms

$$E^*(\mathbb{C}P^\infty) \to (E \wedge MU)^*(\mathbb{C}P^\infty) \text{ and } MU^*(\mathbb{C}P^\infty) \to (E \wedge MU)^*(\mathbb{C}P^\infty)$$

over the homomorphisms of coefficient rings $E^*(\mathrm{pt}) \to (E \wedge MU)^*(\mathrm{pt})$ and $MU^*(\mathrm{pt}) \to (E \wedge MU)^*(\mathrm{pt})$ - notice that

$$(E \wedge MU)^*(\mathrm{pt}) \simeq (E \wedge MU)_*(\mathrm{pt})^{op} = E_*(MU)^{op}$$

(which can pretty safely be thought of as $E_*(MU)$ with signs of degrees flipped). Moreover we've seen that the natural map

$$(E \wedge MU)^*(\mathbb{C}P^\infty) \to \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(\mathbb{C}P^\infty), E_*(MU))$$

is an isomorphism. Now, let $x_E \in \tilde{E}^2(\mathbb{C}P^\infty), x_{MU} \in \tilde{MU}^2(\mathbb{C}P^\infty)$ be the respective generators; we can consider their images $x_E, x_{MU} \in (E \wedge \tilde{} MU)^2(\mathbb{C}P^\infty)$ (not going to give these new names). Observe that they both restrict to the usual generator of $(E \wedge \tilde{} MU)^2(\mathbb{C}P^1)$, since we have commutative diagrams

$$\begin{tikzcd} \tilde{E}^2(\mathbb{C}P^\infty) \arrow[r] \arrow[d, "\iota^*"] & (E \wedge \tilde{} MU)^2(\mathbb{C}P^\infty) \arrow[d, "\iota^*"] \\ \tilde{E}^2(\mathbb{C}P^2) \arrow[r] & (E \wedge \tilde{} MU)^2(\mathbb{C}P^2) \end{tikzcd} \text{ and } \begin{tikzcd} \tilde{MU}^2(\mathbb{C}P^\infty) \arrow[r] \arrow[d, "\iota^*"] & (E \wedge \tilde{} MU)^2(\mathbb{C}P^\infty) \arrow[d, "\iota^*"] \\ \tilde{MU}^2(\mathbb{C}P^2) \arrow[r] & (E \wedge \tilde{} MU)^2(\mathbb{C}P^2) \end{tikzcd} \tag{4.25}$$

where the bottom maps are just suspensions of the homomorphisms $E^0(\mathrm{pt}) \to (E \wedge MU)^0(\mathrm{pt})$ and $MU^0(\mathrm{pt}) \to (E \wedge MU)^0(\mathrm{pt})$, which obviously send $1 \mapsto 1$. Thus both of the classes x_E, x_{MU} make $E \wedge MU$ a complex oriented cohomology theory (they correspond to the 2 MU-algebra structures given by the homomorphisms $MU \to E \to E \wedge MU$ and $MU \simeq MU \to E \wedge MU$, respectively). Thus there are isomorphisms

$$(E \wedge MU)^*(\mathrm{pt})[[x_E]] \to (E \wedge MU)^*(\mathbb{C}P^\infty) \text{ and } (E \wedge MU)^*(\mathrm{pt})[[x_{MU}]] \to (E \wedge MU)^*(\mathbb{C}P^\infty)$$

It follows that we may expand x_{MU} as a formal power series in x_E, with coefficients in $(E \wedge MU)^*(\mathrm{pt})$ (and vice versa); that is, there's a formal power series $g(x) \in (E \wedge MU)^*(\mathrm{pt})[[x]]$ so that

$$x_{MU} = g(x_E) \in (E \wedge MU)^*(\mathbb{C}P^\infty)$$

Of course we'll have $x_E = g^{-1}(x_{MU}) \in (E \wedge MU)^*(\mathbb{C}P^\infty)$ where $g^{-1}(x)$ is the formal power series inverse to $g(x)$.

Lemma 4.60.

$$x_{MU} = \sum_{i \in \mathbb{N}} \alpha_i x_E^{i+1} \in (E \wedge MU)^*(\mathbb{C}P^\infty)$$

That is, $g(x) = \sum_{i \in \mathbb{N}} \alpha_i x^{i+1} \in (E \wedge MU)^*(\mathrm{pt})[[x]]$.

Here the coefficients are the classes $\alpha_i \in E_{2i}(MU)$ (using the identification $(E \wedge MU)^*(\mathrm{pt}) \simeq E_*(MU)^{op}$) described in propositions 4.16 and 4.17. Note that all the degrees add up: viewed as classes in $(E \wedge MU)^*(\mathrm{pt})$ the α_i have degree $-2i$, and x_E has degree 2, so each term in the power series has degree 2, as it must.

Proof. Under the isomorphism

$$(E \wedge MU)^*(\mathbb{C}P^\infty) \simeq \mathrm{Hom}_{E_*(\mathrm{pt})}(E_*(\mathbb{C}P^\infty), E_*(MU))$$

x_{MU} corresponds the homomorphism induced by the usual "inclusion"

$$\Sigma^{-2}\mathbb{C}P^\infty \simeq \Sigma^{-2}MU(1) \to MU$$

and (essentially by *definition* of the α_i) this takes $\beta_{i+1} \mapsto \alpha_i$. Similarly, since x_E^{i+1} is dual to β_{i+1} the element $\sum_{i \in \mathbb{N}} \alpha_i x_E^{i+1}$ corresponds to the homomorphism taking $\beta_{i+1} \mapsto \alpha_i$. $\square$

Let μ_E^*, μ_{MU}^* be the formal group laws over $(E \wedge MU)^*(\mathrm{pt})$ obtained from the formal group laws over $E^*(\mathrm{pt})$, $MU^*(\mathrm{pt})$ by extending scalars along the homomorphisms $E^*(\mathrm{pt}), MU^*(\mathrm{pt}) \to (E \wedge MU)^*(\mathrm{pt})$; more geometrically, μ_E^*, μ_{MU}^* are obtained by rewriting the homomorphism

$$\mu^* : (E \wedge MU)^*(\mathbb{C}P^\infty) \to (E \wedge MU)^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

in terms of x_E, x_{MU} respectively.

Lemma 4.61. *The formal power series $g(x)$ defines an isomorphism of formal group laws $\mu_{MU}^* \to \mu_E^*$ over $(E \wedge MU)^*(\mathrm{pt})$.*

Proof. We know that the homomorphism $\mu^* : (E \wedge MU)^*(\mathbb{C}P^\infty) \to (E \wedge MU)^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ satisfies

$$\mu^*(x_E) = \mu_E^*(x_E, y_E) \text{ and } \mu^*(x_{MU}) = \mu_{MU}^*(x_{MU}, y_{MU})$$

Recalling that $x_{MU} = g(x_E)$, we see that

$$\mu^*(g(x_E)) = \mu_{MU}^*(g(x_E), g(y_E))$$

Now arguing that μ^* is a homomorphism of complete graded rings one shows that

$$\mu^*(g(x_E)) = g(\mu^*(x_E)) = g(\mu_E^*(x_E, y_E))$$

Thus $g(\mu_E^*(x_E, y_E)) = \mu_{MU}^*(g(x_E), g(y_E))$ and we see that the following diagram commutes:

$$\begin{array}{ccc} (E \wedge MU)^*(\mathrm{pt})[[x_{MU}]] & \xrightarrow{x_{MU} \mapsto g(x_E)} & (E \wedge MU)^*(\mathrm{pt})[[x_E]] \\ {\scriptstyle \mu_{MU}^*}\big\downarrow & & {\scriptstyle \mu_E^*}\big\downarrow \\ (E \wedge MU)^*(\mathrm{pt})[[x_{MU}, y_{MU}]] & \xrightarrow{x_{MU} \mapsto g(x_E), y_{MU} \mapsto g(y_E)} & (E \wedge MU)^*(\mathrm{pt})[[x_E, y_E]] \end{array} \tag{4.26}$$

□

Taking $E = H$, we obtain a homomorphism of formal group laws $\mu_H \to \mu_{MU}$ over $(H \wedge MU)^*(\mathrm{pt}) \simeq H_*(MU)^{op}$. Recall that μ_H is just the *additive* formal group law, and so we obtain an equation

$$g(x_H + y_H) = \mu_{MU}^*(g(x_H), g(y_H)) \in (H \wedge MU)^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

This looks more suggestive if we write $g(x) = \exp(x)$ (which isn't too crazy, since $g(x) = \sum_i \alpha_i x^{i+1}$, and the $\alpha_i \in H_*(MU)$ come from the $\beta_{i+1} \in H_*(MU(1))$, which we've seen behave like the differential operators $\frac{1}{(i+1!)} \frac{\partial^{i+1}}{\partial x^{i+1}}$ ... and so on):

$$\exp(x_H + y_H) = \mu_{MU}^*(\exp(x_H), \exp(y_H)) \in (H \wedge MU)^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

Replacing x_H with $g^{-1}(x_{MU})$ yields an equation

$$\mu_{MU}^*(x_{MU}, y_{MU}) = g(g^{-1}(x_{MU}) + g^{-1}(y_{MU})) \in (H \wedge MU)^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

If we're writing $g(x) = \exp(x)$ we should write $g^{-1}(x) = \log(x)$, so this becomes

$$\mu_{MU}^*(x_{MU}, y_{MU}) = \exp(\log(x_{MU}) + \log(y_{MU})) \in (H \wedge MU)^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

In particular, we see that the image of $a_{ij} \in MU^*(\mathrm{pt})$ under the Hurewicz homomorphism $MU^*(\mathrm{pt}) \to (H \wedge MU)^*(\mathrm{pt})$ is just the coefficient on $x_{MU}^i y_{MU}^j$ in the formal power series $\exp(\log(x_{MU}) + \log(y_{MU}))$. This provides a way to compute the Hurewicz images of the a_{ij} as polynomials in the α_i. One can carry out an analogous discussion in the case $E = K$.

Consider now the case $E = MU$ itself. In this case we must distinguish between the "left and right" copies of MU; denote by η_L and η_R the homomorphisms

$$MU \simeq MU \wedge S \xrightarrow{\mathrm{id} \wedge \eta} MU \wedge MU \text{ and } MU \simeq S \wedge MU \xrightarrow{\eta \wedge \mathrm{id}} MU \wedge MU$$

We obtain generators $x_L := \eta_{L*} x_{MU}$ and $x_R := \eta_{R*} x_{MU}$ for $(MU \tilde{\wedge} MU)^2(\mathbb{C}P^\infty)$ related by a formula

$$x_R = g(x_L) = \sum_i \alpha_i x_L^{i+1} \in (MU \tilde{\wedge} MU)^2(\mathbb{C}P^\infty)$$

We also obtain formal group laws $\mu_L^* := \eta_{L*} \mu_{MU}^*$ and $\mu_R^* := \eta_{R*} \mu_{MU}^*$ over $(MU \wedge MU)^*(\mathrm{pt}) \simeq MU_*(MU)^{\mathrm{op}}$, and the formal power series $g(x)$ defines a homomorphism of formal group laws $\mu_R^* \to \mu_L^*$; we obtain an equation

$$g(\mu_L^*(x_L, y_L)) = \mu_R^*(g(x_L), g(y_L)) \in (MU \wedge MU)^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

Equivalently, replacing x_L, y_L with $g^{-1}(x_R), g^{-1}(y_R)$ gives an equation

$$\mu_R^*(x_R, y_R) = g(\mu_L^*(g^{-1}(x_R), g^{-1}(y_R)) \in (MU \wedge MU)^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

This provides a way to compute the images of the a_{ij} under the generalized Hurewicz homomorphism $\pi_*(MU) \to MU_*(MU)$ as polynomials in the $\alpha_i \in MU_*(MU)$.

4.5. **The Lazard ring.** Given a graded commutative ring R, let $\mathrm{FGL}(R)$ denote the formal group laws over R. Note that FGL is a category: if $\mu^*, \mu^{'*} : R[[x]] \to R[[x,y]]$ are formal group laws over R, then a homomorphism $\mu \to \mu'$ will be a homomorphism of complete graded rings $\psi : R[[x]] \to R[[x]]$ so that the following diagram commutes:

$$\begin{array}{ccc} R[[x]] & \xrightarrow{\psi} & R[[x]] \\ {\scriptstyle \mu^*}\downarrow & & {\scriptstyle \mu^{'*}}\downarrow \\ R[[x,y]] & \xrightarrow{\psi\otimes\psi} & R[[x,y]] \end{array} \tag{4.27}$$

or in terms of power series, $\psi(\mu^{'}(x,y)) = \mu(\psi(x),\psi(y))$. Moreover, if $\varphi : R \to R'$ is a homomorphism of graded rings, than to each formal group law $\mu^* : R[[x]] \to R[[x,y]]$ wa can assign the formal group law $\varphi_*\mu^* : R'[[x]] \to R'[[x,y]]$ obtained from μ^* by extending scalars along φ. In terms of power series, if $\mu(x,y) = \sum_{i,j} a_{ij}x^iy^j$ then $\varphi_*\mu(x,y) = \sum_{i,j}\varphi(a_{ij})x^iy^j$. In this way we obtain a functor

$$\varphi_* : \mathrm{FGL}(R) \to \mathrm{FGL}(R') \text{ assigning } \mu^* \mapsto \varphi_*\mu^*$$

So, FGL can be viewed as a (covariant) category-valued functor on graded commutative rings. Of course, we may also view it as a set-valued functor on graded commutative rings. That's what I'll be doing from here on out (otherwise we'd need to talk about stacks).

Proposition 4.62. *The set-valued functor* FGL *on the category of graded commutative rings is co-representable. That is, there's a graded commutative ring L together with a formal group law λ^* over L so that for every graded commutative ring R the natural function*

$$\mathrm{Hom}(L,R) \to \mathrm{FGL}(R) \textit{ taking } \varphi \mapsto \varphi_*\lambda^*$$

is a bijection.

L is called the **Lazard ring** (after Michel Lazard).

Proof. As we've observed many times now, a formal group law $\mu^* : R[[x]] \to R[[x_1,x_2]]$ over a graded commutative ring R is equivalent to a formal power series

$$\mu(x,y) = \sum_{i,j} a_{ij}x_1^ix_2^j \in R[[x_1,x_2]]$$

where the coefficients $a_{ij} \in R$ have degrees $2(1-i-j)$ so that

$$\mu(\mu(x_1,x_2),x_3) = \mu(x_1,\mu(x_2,x_3)), \quad \mu(x,0) = \mu(0,x) = 0 \text{ and } \mu(x,y) = \mu(y,x)$$

The condition $\mu(x,y) = \mu(y,x)$ is equivalent to $a_{ji} = a_{ij}$ for all i,j and the condition $\mu(x,0) = \mu(0,x) = 0$ is equivalent to $a_{i0} = \delta_{i1}$ and $a_{0j} = \delta_{j1}$ for all i,j. The associativity condition is equivalent to $p_{ijk}(a) = q_{ijk}(a)$ for all i,j,k, where $p_{ijk}(a)$ and $q_{ijk}(a)$ are homogeneous polynomials in the a_{ij} of degree $2(1-i-j-k)$ defined implicitly as the coefficients in the power series expansions

$$\mu(\mu(x_1,x_2),x_3) = \sum_{i,j,k} p_{ijk}(a)x_1^ix_2^jx_3^k \text{ and } \mu(x_1,\mu(x_2,x_3)) = \sum_{i,j,k} q_{ijk}(a)x_1^ix_2^jx_3^k$$

So, define L to be the quotient ring $L := \mathbb{Z}[a_{ij} \mid i,j \in \mathbb{N}]/I$ where $I \subset \mathbb{Z}[a_{ij} \mid i,j \in \mathbb{N}]$ is the ideal generated by the relations

$$a_{ji} = a_{ij}, \quad a_{i0} = a_{0i} = \delta_{i1} \text{ and } p_{ijk}(a) = q_{ijk}(a)$$

and define the formal group law λ^* over L by the formal power series

$$\lambda(x_1,x_2) := \sum_{i,j} a_{ij}x_1^ix_2^j \in L[[x_1,x_2]]$$

The rest of the proof is straightforward.

□

Especially in light of the above proposition, the functor FGL is ”obviously co-representable” (quoting lots of people here). What’s not at all obvious is that

Theorem 4.63 (Lazard)**.** *L is a polynomial ring over $\mathbb{Z}$ on generators of degree $-2i$, for $i \in \mathbb{N}$.*

Here’s the idea: one knows how to construct homomorphisms *out of* L; a homomorphism $\varphi : L \to R$ is the same as a formal group law μ^* over R. Thus to determine the structure of L, one might attempt to construct such a formal group law μ^* so that the resulting homomorphism φ is an embedding, with an identifiable image. This will be the strategy used to prove theorem 4.31 (following Adams here). Much of this is motivated by lemmas 4.28 and 4.29.

Form the graded commutative ring $R := \mathbb{Z}[\alpha_i \mid i \in \mathbb{N}]/(\alpha_0 = 1)$, where the α_i are generators of degree $-2i$, and $\alpha_0 = 1$ (okay, clearly this is just $\mathbb{Z}[\alpha_i \mid i > 0]$, but the convention $\alpha_0 = 1$ always simplifies life). Define a degree 2 formal power series

$$\exp(x) := \sum_{i \in \mathbb{N}} \alpha_i x^{i+1} \in R[[x]]$$

and let $\log(x)$ be its inverse, so that $\log(\exp(x)) = \exp(\log(x)) = x \in R[[x]]$. Our first lemma identifies the series $\log(x)$ rather explicitly:

Lemma 4.64. *Write $\log(x) := \sum_{i \in \mathbb{N}} m_i x^{i+1} \in R[[x]]$. Then $(n+1)m_n$ is the coefficient of x^n in the formal power series expansion of $(\frac{\exp(x)}{x})^{-n-1}$.*

I’ll sketch 2 ways to discover this formula.

Let $f : U \to \mathbb{C}$ be a holomorphic function on a neighborhood of 0 so that $f(0) = 0$ and $f'(0) = 1$. By the (holomorphic) inverse function theorem, we can find an inverse function $f^{-1} : V \to U$ on a neighborhood V of 0. In fact we might as well assume f defines a holomorphic isomorphism $U \simeq V$.

On might attempt to calculate f^{-1} using Cauchy’s integral formula: say $w \in V$, and write

$$f^{-1}(w) = \frac{1}{2\pi\imath} \oint \frac{f^{-1}(\zeta)\, d\zeta}{\zeta - w}$$

where the integral is to be taken over a small loop around w. If w is close enough to 0, we may find such a loop of the form $f \circ \gamma : S^1 \to V$, where $\gamma : S^1 \to U$ is a loop around 0. In that case the integral can be rewritten as

$$\frac{1}{2\pi\imath} \oint \frac{f^{-1}(\zeta)\, d\zeta}{\zeta - w} = \frac{1}{2\pi\imath} \oint_\gamma \frac{z\, dz}{f(z) - w}$$

The next step is to expand this integral as a series in w: we have

$$\frac{1}{f(z) - w} = \frac{1}{f(z)} \frac{1}{1 - \frac{w}{f(z)}} = \frac{1}{f(z)} \sum_{n=0}^{\infty} (\frac{w}{f(z)})^n = \sum_{n=0}^{\infty} \frac{w^n}{f(z)^{n+1}}$$

$$\text{and so } \frac{1}{2\pi\imath} \oint_\gamma \frac{z\, dz}{f(z) - w} = \sum_{n=0}^{\infty} \frac{1}{2\pi\imath} \oint_\gamma \frac{z\, dz}{f(z)^{n+1}} \cdot w^n$$

Since $(\frac{z}{f(z)^n})' = \frac{1}{f(z)^n} - n\frac{z}{f(z)^{n+1}}$, an integration by parts gives

$$\frac{1}{2\pi\imath} \oint_\gamma \frac{z\, dz}{f(z)^{n+1}} = \frac{1}{2\pi\imath} \oint_\gamma \frac{dz}{nf(z)^n} \text{ for } n \in \mathbb{N}$$

Now, write $f(z) = \sum_{n=0}^{\infty} \alpha_n z^{n+1}$, where $\alpha_0 = 1$, and $f^{-1}(w) = \sum_{n=0}^{\infty} m_n w^{n+1}$, where $m_0 = 1$ - here I'm making use of the fact that $f(0) = 0, f'(0) = 1$ and similarly for f^{-1}. From the above discussion it must be that

$$m_n = \frac{1}{2\pi\imath} \oint_{\gamma} \frac{dz}{(n+1)f(z)^{n+1}} \text{ for } n \in \mathbb{N}$$

As $f(z)$ is divisible by z, this can be written as

$$\frac{1}{2\pi\imath} \oint_{\gamma} \frac{dz}{(n+1)f(z)^{n+1}} = \frac{1}{2\pi\imath} \oint_{\gamma} \frac{dz}{(n+1)z^{n+1}(\frac{f(z)}{z})^{n+1}}$$

This is $\frac{1}{n+1}$ times the coefficient on z^n in the power series expansion of $(\frac{f(z)}{z})^{-n-1}$, and we obtain the desired result.

Remark 4.65. How does one proceed from here? Some thoughts:
NOTE: There are lots of factors of (-1) missing in this remark.
Observe that

$$\frac{f(z)}{z} = \sum_{n=0}^{\infty} \alpha_n z^n = 1 + \sum_{n=1}^{\infty} \alpha_n z^n, \text{ and so}$$

$$(\frac{f(z)}{z})^{-1} = \frac{1}{1 + \sum_{n=1}^{\infty} \alpha_n z^n} = \sum_{n=0}^{\infty} (\sum_{m=1}^{\infty} \alpha_m z^m)^n$$

Recall that

$$(\sum_{m=1}^{\infty} \alpha_m z^m)^n = \sum_{m=n}^{\infty} (\sum_{\lambda \vDash m, l(\lambda)=n} \alpha_\lambda) z^m \text{ and so}$$

$$\sum_{n=0}^{\infty} (\sum_{m=1}^{\infty} \alpha_m z^m)^n = \sum_{n=0}^{\infty} (\sum_{\lambda \vDash n} \alpha_\lambda) z^n$$

Here $\lambda \vDash n$ means λ is a composition of n, i.e. $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{N}^l$ and $\sum_i \lambda_i = n$, and $\alpha_\lambda := \prod_i \alpha_{\lambda_i}$. It follows that

$$(\frac{f(z)}{z})^{-n-1} = (\sum_{m=0}^{\infty} (\sum_{\lambda \vDash n} \alpha_\lambda) z^m)^{n+1} = \sum_{m=0}^{\infty} (\sum_{\lambda \vDash m, l(\lambda)=n+1} \prod_{i=1}^{n+1} (\sum_{\mu \vDash \lambda_i} \alpha_\mu)) z^m$$

Thus the coefficient on z^n in $(\frac{f(z)}{z})^{-n-1}$ is $\sum_{\lambda \vDash n, l(\lambda)=n+1} \prod_{i=1}^{n+1} (\sum_{\mu \vDash \lambda_i} \alpha_\mu)$ and so

$$m_n = \frac{1}{n+1} \sum_{\lambda \vDash n, l(\lambda)=n+1} \prod_{i=1}^{n+1} (\sum_{\mu \vDash \lambda_i} \alpha_\mu)$$

Alternatively, one could write

$$(\frac{f(z)}{z})^{-n-1} = (\frac{1}{1 + \sum_{m=1}^{\infty} \alpha_m z^n})^{n+1} = \frac{1}{\sum_{k=0}^{n+1} \binom{n+1}{k} (\sum_{m=1}^{\infty} \alpha_m z^m)^k}$$

and since $(\sum_{m=1}^{\infty} \alpha_m z^m)^k = \sum_{m=k}^{\infty} (\sum_{\lambda \vDash m, l(\lambda)=k} \alpha_\lambda) z^m$ the denominator rearranges as

$$\sum_{k=0}^{n+1} \binom{n+1}{k} (\sum_{m=1}^{\infty} \alpha_m z^m)^k = \sum_{k=0}^{n+1} \binom{n+1}{k} \sum_{m=k}^{\infty} (\sum_{\lambda \vDash m, l(\lambda)=k} \alpha_\lambda) z^m$$

$$= \sum_{m=0}^{\infty} (\sum_{k=0}^{n+1} \binom{n+1}{k} (\sum_{\lambda \vDash m, l(\lambda)=k} \alpha_\lambda)) z^m = 1 + \sum_{m=1}^{\infty} (\sum_{k=0}^{n+1} \binom{n+1}{k} (\sum_{\lambda \vDash m, l(\lambda)=k} \alpha_\lambda)) z^m$$

and proceeding in this way one sees that the coefficient on z^n in $(\frac{f(z)}{z})^{-n-1}$ is given by

$$\sum_{\lambda \vDash n} \prod_i (\sum_{k=0}^{n+1} \binom{n+1}{k} (\sum_{\mu \vDash \lambda_i, l(\mu)=k} \alpha_\mu)), \text{ so that}$$

$$m_n = \frac{1}{n+1} \sum_{\lambda \vDash n} \prod_i (\sum_{k=0}^{n+1} \binom{n+1}{k} (\sum_{\mu \vDash \lambda_i, l(\mu)=k} \alpha_\mu))$$

Here's another way to proceed: observe that if $f(z) = \sum_i \alpha_i x^{i+1}$ and $f^{-1}(x) = \sum_i m_i x^{i+1}$, then

$$x = f^{-1}(f(x)) = \sum_i m_i f(x)^{i+1} \text{ and since}$$

$$f(x)^{i+1} = \sum_{j=i+1}^{\infty} (\sum_{\lambda \vDash j, l(\lambda)=i+1} \alpha_{\lambda-1}) x^j$$

where $\alpha_{\lambda-1}) := \prod_i \alpha_{\lambda_i - 1}$ we see that

$$x = \sum_{j=0}^{\infty} (\sum_{i=1}^{\infty} m_i \sum_{\lambda \vDash j, l(\lambda)=i+1} \alpha_{\lambda-1}) x^j$$

from this we obtain a sequence of equations

$$\sum_{i=1}^{\infty} m_i \sum_{\lambda \vDash j+1, l(\lambda)=i+1} a_{\lambda_i - 1} = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases}$$

Notice that the sums are finite, because $\sum_{\lambda \vDash j+1, l(\lambda)=i+1} a_{\lambda_i-1} = 0$ when $i+1 > j+1$. When $i = j$, we have $\sum_{\lambda \vDash j+1, l(\lambda)=j+1} a_{\lambda_i-1} = 1$, and so we can rewrite the above equation as

$$m_j + \sum_{i=1}^{j-1} m_i \sum_{\lambda \vDash j+1, l(\lambda)=i+1} a_{\lambda_i - 1} = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases}$$

Perhaps this facilitates an inductive proof of the lemma.

By lemma 4.25, we know that $\log(x)$ defines an isomorphism between the additive formal group law over R defined by $x + y \in R[[x, y]]$ and the formal group law over R defined by

$$\mu(x, y) := \exp(\log(x) + \log(y)) \in R[[x, y]]$$

By proposition 4.31, there's a unique homomorphism of graded rings

$$\varphi : L \to R \text{ so that } \varphi_* \lambda = \mu$$

where λ is the universal formal group law over L.

Lemma 4.66. *The homomorphism φ is injective.*

Recall that a graded commutative ring $S = \bigoplus_{i \in \mathbb{Z}} S_i$ over $\mathbb{Z}$ (so we have a structure homomorphism of graded rings $\eta : \mathbb{Z} \to S$, where $\mathbb{Z}$ is concentrated in degree 0) is said to be **connected** if it comes with an augmentation homomorphism of graded rings $\epsilon : S \to \mathbb{Z}$ so that $\epsilon \circ \eta = \mathrm{id} : \mathbb{Z} \to \mathbb{Z}$, and if the resulting maps

$$\mathbb{Z} \xrightarrow{\eta} S_0 \text{ and } S_0 \simeq S / \bigoplus_{i \neq 0} S_i \xrightarrow{\epsilon} \mathbb{Z}$$

are mutual inverses (so in particular $S_0 \simeq \mathbb{Z}$).

Let $I = \ker \epsilon \subset S$ be the augmentation ideal. The **group of indecomposables in** S is defined to be $Q(A) = I/I^2$ (note that it doesn't consist of elements of S- it's a graded $\mathbb{Z} = S/I$-module). Algebro-geometrically speaking this corresponds to the global sections of the co-normal bundle over the closed subscheme $V(I) \subset \mathrm{Spec} S$. The group of indecomposables defines a functor Q from connected graded commutative rings over $\mathbb{Z}$ to abelian groups, with the evident induced homomorphisms.

Notice that both L and R are *connected* graded commutative rings over $\mathbb{Z}$, and the map $\varphi : L \to R$ is a homomorphism of connected graded rings over $\mathbb{Z}$. This can be seen from the constructions provided above. The augmentation homomorphism $L \to \mathbb{Z}$ corresponds to the additive formal group law over $\mathbb{Z}$ given by $x + y \in \mathbb{Z}[[x, y]]$; the augmentation $R \to \mathbb{Z}$ sends $\alpha_i \to 0$ for $i > 0$. So, we obtain an induced homomorphism of graded abelian groups of indecomposables

$$\varphi_* : Q(L) \to Q(R)$$

Also from the constructions, L and R are concentrated in degrees $-2i$ for $i \in \mathbb{N}$, hence the same is true of the indecomposables $Q(L), Q(R)$. It's clear enough that

$$Q_{-2i}(R) = \mathbb{Z}\bar{\alpha}_i$$

where $\bar{\alpha}_i$ is the image of $\alpha_i \in I$ under the quotient map $I \to I/I^2 = Q(R)$.

Recall that we defined L as a quotient of the polynomial ring $\mathbb{Z}[a_{ij}]$ (where $\deg a_{ij} = 2(1 - i - j)$). Notice that the a_{ij} generate $Q(\mathbb{Z}[a_{ij}])$ - indeed it should be that $Q_{-2n}(\mathbb{Z}[a_{ij}]) = \bigoplus_{i+j-1=n} \mathbb{Z}\bar{a}_{ij}$. Let $\pi : \mathbb{Z}[a_{ij}] \to L$ be the quotient map. The claim is that $\pi_* : Q(\mathbb{Z}[a_{ij}]) \to Q(L)$ is surjective, so the coefficients a_{ij} generate the group of indecomposables $Q(L)$. More generally:

Proposition 4.67. *Let $\varphi : S \to S'$ be a surjective homomorphism of connected graded commutative rings over $\mathbb{Z}$. Then the induced homomorphism of graded abelian groups $\varphi_* : Q(S) \to Q(S')$ is also surjective.*

The proof is trivial.

Lemma 4.68.
- $m_i \equiv -\alpha_i \mod I^2$
- $\varphi(a_{ij}) \equiv \binom{i+j}{i}\alpha_{i+j-1} \mod I^2$ *for $i, j > 0$.*
- *The image of the induced map $\varphi_* : Q_{-2n}(L) \to Q_{-2n}(R)$ consists of the multiples of $d_n\bar{\alpha}_n$, where*

$$d_n = \begin{cases} p & \text{if } n+1 = p^e \text{ where } p \text{ is a prime, } e > 0 \\ 1 & \text{otherwise} \end{cases}$$

Proof. Recall that $(n+1)m_i$ is the coefficient on x^n in the formal power series expansion of $(\frac{\exp(x)}{x})^{-n-1}$. Writing $\exp(x) = \sum_{i\in\mathbb{N}} \alpha_i x^{i+1}$, so

$$\frac{\exp(x)}{x} = \sum_{i\in\mathbb{N}} \alpha_i x^i \text{ and } (\frac{\exp(x)}{x})^{-n-1} = (\frac{1}{1 + \sum_{i>0} \alpha_i x^i})^{n+1}$$

Now observe that if we reduce the coefficients mod I^2, this can be computed as

$$1 - (n+1)\sum_{i>0} \alpha_i x^i$$

Thus $(n+1)m_i \equiv -(n+1)\alpha_n \mod I^2$, as desired.

Recall that $\varphi(a_{ij})$ is the ij-th coefficient of the formal group law

$$\mu(x, y) := \exp(\log(x) + \log(y)) \text{ over } R$$

Writing $\log(x) = \sum_{i\in\mathbb{N}} m_i x^{i+1}$, we have

$$\begin{aligned}\exp(\log(x) + \log(y)) &= \sum_{i\in\mathbb{N}} \alpha_i (\sum_{j\in\mathbb{N}} m_j x^{j+1} + \sum_{j\in\mathbb{N}} m_j y^{j+1})^{i+1} \\ &= \sum_{i\in\mathbb{N}} \alpha_i (\sum_{j\in\mathbb{N}} m_j (x^{j+1} + y^{j+1}))^{i+1}\end{aligned}$$

At this point observe that if we reduce all coefficients mod I^2, we obtain

$$(\sum_{j\in\mathbb{N}} m_j(x^{j+1}+y^{j+1}))^{i+1} \equiv (x+y)^{i+1} + (i+1)(x+y)^i \sum_{j>0} m_j(x^{j+1}+y^{j+1})$$

and so

$$\sum_{i\in\mathbb{N}} \alpha_i(\sum_{j\in\mathbb{N}} m_j(x^{j+1}+y^{j+1}))^{i+1} \equiv \sum_{i\in\mathbb{N}} \alpha_i((x+y)^{i+1} + (i+1)(x+y)^i \sum_{j>0} m_j(x^{j+1}+y^{j+1}))$$

$$\equiv \sum_{i\in\mathbb{N}} \alpha_i(x+y)^{i+1} + \sum_{i>0} m_i(x^{i+1}+y^{i+1})$$

Recalling that $m_i \equiv -\alpha_i \mod I^2$, this becomes

$$\sum_{i\in\mathbb{N}} \alpha_i((x+y)^{i+1} - x^{i+1} - y^{i+1}) = \sum_{i\in\mathbb{N}} \alpha_i(\sum_{j+k=i+1, j>0, k>0} \binom{j+k}{j} x^j y^k)$$

This shows that modulo I^2, $\varphi(a_{jk}) = \binom{j+k}{j}\alpha_{j+k-1}$, the desired result.

By proposition 4.35, the coefficients a_{ij} generate the indecomposables $Q(L)$, and so their images $\varphi(a_{ij})$ will generate the image of $\varphi_* : Q(L) \to Q(R)$. So, for each $n \in \mathbb{N}$ we must figure out the span of

$$\varphi(a_{ij}) = \binom{i+j}{i}\alpha_{i+j-1} \in Q_{-2n}(R) \text{ for } i+j-1 = n$$

i.e. $i+j = n+1$. That is, we must figure out the span of

$$\binom{n+1}{i}\alpha_n \in Q_{-2n}(R) \text{ for } 0 < i < n+1$$

The following purely number-theoretic computation of the g.c.d. of the binomial coefficients $\binom{n+1}{i}$ for $0 < i < n+1$ wraps up the proof.

□

Proposition 4.69. *Let d_n be the g.c.d. of the binomial coefficients $\binom{n+1}{i}$ for $0 < i < n+1$. Then*

$$d_n = \begin{cases} p & \text{if } n+1 = p^e \text{ where } p \text{ is a prime, } e > 0 \\ 1 & \text{otherwise} \end{cases}$$

Proof. Consider the polynomial

$$(1+x)^{n+1} = \sum_{i=0}^{n} \binom{n+1}{i} x^i \in \mathbb{Z}[x]$$

and notice that a prime p divides d_n if and only if

$$(1+x)^{n+1} \equiv 1 + x^{n+1} \mod p$$

Among other things this requires that $p | n+1 = \binom{n+1}{1}$.

Suppose $n+1$ is not a prime power - then for any prime $p|n+1$ we'll have a factorization $n+1 = p^e m$ where $e > 0$ and m is a positive integer prime to p. But then

$$(1+x)^{n+1} = ((1+x)^{p^e})^m = (1+x^{p^e})^m = \sum_{i=0}^{m} \binom{m}{i} x^{p^e i}$$

and this can't be $1 + x^{n+1} \mod p$, for instance because the coefficient on x^{p^e} is $m \neq 0 \mod p$. Thus no prime p divides d_n, and it must be that $d_n = 1$.

On the other hand if $n+1 = p^e$ for some prime p and some $e > 0$ (in which case d_n must be a power of p), then certainly $(1+x)^{n+1} \equiv 1 + x^{n+1} \mod p$ because the "freshman's dream" is true in $\mathbb{F}_p[x]$ - this means that $p|d_n$. On the other hand, notice that

$$(1+x)^{n+1} = ((1+x)^{p^{e-1}})^p \equiv (1+x^{p^{e-1}})^p \mod p^2$$
$$\equiv \sum_{i=0}^{p} \binom{p}{i} x^{p^{e-1}i} \mod p^2$$

so that

$$\binom{p^e}{p^{e-1}i} \equiv \binom{p}{i} \mod p^2$$

and this is *not* zero mod p^2. The conclusion is that $p^2 \not| d_n$, so that $d_n = p$.

□

Let A be any abelian group and let n be an integer. We can make $\mathbb{Z} \oplus A$ into a connected, graded commutative ring over $\mathbb{Z}$ with

$$(\mathbb{Z} \oplus A)_0 = \mathbb{Z}, (\mathbb{Z} \oplus A)_{-2n} = A$$

by defining multiplication as

$$(a_1, b_1)(a_2, c_2) = (a_1 a_2, a_1 b_2 + a_2 b_1) \in \mathbb{Z} \oplus A \text{ for } (a_1, b_1), (a_2, c_2) \in \mathbb{Z} \oplus A$$

Call this ring $R_{-2n}(A)$. Evidently

$$Q(R_{-2n}(A)) = A, \text{ concentrated in degree } -2n$$

Now observe that for any connected graded commutative ring S over $\mathbb{Z}$, a homomorphism of connected graded rings $S \to R_{-2n}(A)$ is equivalent to a homomorphism of abelian groups $Q_{-2n}(S) \to A$. Indeed, given a homomorphism $\psi : S \to R_{-2n}(A)$ we get an induced map of indecomposables $\varphi_* : Q_{-2n}(S) \to Q_{-2n}(R_{-2n}(A)) = A$, and given a homomorphism of abelian groups $\rho : Q_{-2n}(S) \to A$ we may *define* a homomorphism of connected graded rings

$$\tilde{\rho} : S \to R_{-2n}(A) \text{ by } \tilde{\rho}(a) = (\epsilon(a), \rho(\bar{a}))$$

where $\bar{a}$ is the image of a in $Q(S)$. In other words, the functor R_{-2n} is *right adjoint* to the functor Q_{-2n}.

In particular, we see that a homomorphism $\psi : L \to R_{-2n}(A)$ (which is the same as a formal group law over $R_{-2n}(A)$) is equivalent to a homomorphism $\psi_* : Q_{-2n}(L) \to A$. From a Yoneda viewpoint, we can describe $Q_{-2n}(L)$ by considering formal groups over rings of the form $R_{-2n}(A)$.

Let $T_{-2n} \subset Q_{-2n}(R)$ be the image of the homomorphism $\varphi_* : Q_{-2n}(L) \to Q_{-2n}(R)$.

Lemma 4.70 (Lazard, Frohlich). *For any abelian group A and any formal group law over $R_{-2n}(A)$, the classifying map $L \to A$ factors through $\varphi(L) \subset R$. Equivalently, the classifying map $Q_{-2n}(L) \to A$ factors through T_{-2n}.*

Outline of a proof. First, notice that $Q_{-2n}(A)$ is finitely generated (say by the coefficients a_{ij} with $i + j - 1 = n$), so its image under a classifying map $Q_{-2n}(L) \to A$ will be finitely generated. In this way we reduce to the case where A is finitely generated.

Now by the structure theorem A has a direct sum decomposition of the form $A \simeq \mathbb{Z}^r \oplus \bigoplus_{i=1}^{s} \mathbb{Z}/(p_i^{m_i})$ where the p_i are primes and the m_i are positive integers. Note that a homomorphism $\rho : Q_{-2n}(L) \to A$ will be equivalent to a collection of homomorphisms $\rho_0 : Q_{-2n}(L) \to \mathbb{Z}^r, \rho_i : Q_{-2n}(L) \to \mathbb{Z}/(p_i^{m_i})$; in this way we reduce to the case where $A = \mathbb{Z}/(p^m)$ for some prime p and some $m > 0$.

At this point things get rather technical. See Adams's blue book.

□

Corollary 4.71. *The induced homomorphism of indecomposables $\varphi_* : Q(L) \to Q(R)$ is injective.*

Proof. By lemma 4.38, for each n the homomorphism

$$Q_{-2n}(L) \to T_{-2n} \subset Q_{-2n}(R)$$

induces an isomorphism of functors $\mathrm{Hom}(T_{-2n}, -) \to \mathrm{Hom}(Q_{-2n}(L), -)$. By the Yoneda lemma the homomorphism $Q_{-2n}(L) \to T_{-2n}$ must be an isomorphism, so φ_* maps $Q_{-2n}(L)$ isomorphically onto its image. □

Proof of theorem 4.32. The idea is supposed to be: for each $n \in \mathbb{N}$, $n > 0$, choose an element $t_n \in L_{-2n}$ whose image under the map

$$L_{-2n} \xrightarrow{\text{quotient}} Q_{-2n}(L) \xrightarrow{\varphi_*} Q_{-2n}(R)$$

generates $T_{-2n} = \varphi_*(Q_{-2n}(L)) \subset Q_{-2n}(R)$. We obtain a homomorphism of connected graded commutative rings over $\mathbb{Z}$

$$\psi : \mathbb{Z}[t_i \,|\, i \in \mathbb{N}, i > 0] \to L$$

and by design, the induced homomorphisms of indecomposables

$$\mathbb{Z}\bar{t}_n = Q_{-2n}(\mathbb{Z}[t_i]) \xrightarrow{\psi_*} Q_{-2n}(L)$$

can be viewed as the inverses of the maps $Q_{-2n}(L) \to T_{-2n}$- so, they are isomorphisms. According to Adams, the fact that $\psi_* : Q(\mathbb{Z}[t_i]) \to Q(L)$ is an isomorphism implies that ψ is surjective (**NOTE**: see Luries notes, lecture 2). Also, the composition

$$\mathbb{Z}[t_i] \xrightarrow{\psi} L \xrightarrow{\varphi} R$$

is injective, basically because it sends t_n to $d_n\alpha_n +$ decomposables (again see Lurie's notes), and this implies that ψ is injective, hence an isomorphism. It follows that φ is injective. □

From Lazard's theorem we obtain various immediate (and interesting) corollaries:

Corollary 4.72. *Let $\psi : R \to R'$ be a surjective homomorphism of graded commutative rings, and let μ' be a formal group law over R'. Then there exists a formal group law μ over R so that $\psi * \mu = \mu'$.*

Indeed, this is equivalent to the statement that given a surjective homomorphism $\psi : R \to R'$ and a homomorphism $\rho : L \to R'$, there exists a lift $\tilde{\rho} : L \to R$ so that $\psi \circ \tilde{\rho} = \rho$:

$$\begin{array}{ccc} L & \xrightarrow{\tilde{\rho}} & R \\ {\scriptstyle =}\downarrow & & {\scriptstyle \psi}\downarrow \\ L & \xrightarrow{\rho} & R' \end{array} \tag{4.28}$$

The existence of such a $\tilde{\rho}$ is obvious as L is a polynomial ring.

Corollary 4.73. *Let R be a graded commutative ring over $\mathbb{Q}$. Then every formal group law μ over R is strictly isomorphic to the additive formal group law.*

Proof. Observe that since R is a $\mathbb{Q}$-algebra, the formal group law μ is equivalent to a homomorphism of graded rings $\rho : L \to R$ which in turn is equivalent to the homomorphism $\rho : L \otimes_{\mathbb{Z}} \mathbb{Q} \to R$. At this point it will suffice to show that there's an isomorphism of formal groups over $L \otimes \mathbb{Q}$, say g so that

$$\lambda(x, y) = g(g^{-1}(x) + g^{-1}(y)) \in L \otimes \mathbb{Q}[[x, y]]$$

(where to be uber-precise λ means the formal group law over $L \otimes \mathbb{Q}$ obtained from the universal formal group law over L).

To see this, note that the homomorphism $\varphi : L \to R$ induces an isomorphism $L \otimes \mathbb{Q} \simeq R \otimes \mathbb{Q}$. Also, $\varphi_* \lambda(x, y) = \exp(\log(x) + \log(y)) \in R \otimes \mathbb{Q}[[x, y]]$. Thus we may take $g(x) = \varphi^{-1}(\exp(x)) \in L \otimes \mathbb{Q}[[x]]$.

□

5. The theorems of Milnor and Quillen on $MU^*(\mathrm{pt})$

5.1. **Generalized (co)homology operations.** Let E, F be CW spectra. Then there is a natural isomorphism of graded abelian groups

$$F^*(E) \simeq \text{ natural transformations } E^* \to F^*$$

where E^*, F^* are the "Yoneda functors" on the category of CW spectra taking $X \mapsto E^*(X) := [X, E]_{-*}$ and similarly for F.

Remark 5.1. If we consider the reduced cohomology theories $\tilde{E}^*, \tilde{F}^*$ on the category of pointed CW complexes given by $\tilde{E}^*(X) := [\Sigma^\infty X, E]_{-*}$ and similarly for F, the above natural transformations correspond to *stable* cohomology operations, in the classical sense. For instance, say A, B are abelian groups. An element $\theta \in HB^p(HA)$ corresponds to a morphism of spectra $\theta : HA \to \Sigma^p HB$, defining natural transformations

$$\tilde{H}^q(X; A) = [\Sigma^\infty X, HA]_{-q} \xrightarrow{\theta_*} [\Sigma^\infty X, HB]_{-p-q} = \tilde{H}^{p+q}(X; B)$$

by composition; these are clearly compatible with suspension, i.e. they form a stable cohomology operation.

Now suppose $F = E$, so we're considering $E^*(E)$, which can be viewed as the graded abelian group of natural transformations $E^* \to E^*$. In this situation we may also *compose* morphisms of spectra $\theta : E \to E$; if $\theta : E \to \Sigma^p E$ and $\psi : E \to \Sigma^q E$ are morphisms representing classes $\theta \in E^p(E), \psi \in E^q(E)$ then their composition gives a morphism

$$E \xrightarrow{\theta} \Sigma^p E \xrightarrow{\Sigma^p} \Sigma^{p+q} E$$

representing a class $\psi \circ \theta \in E^{p+q}(E)$. In this way $E^*(E)$ becomes a graded ring; it's just the graded ring of endomorphisms of the object E in the stable homotopy category of CW spectra, which is both graded and additive.

Per usual, $E^*(E)$ is an algebra over $E^0(E)$.

Example 5.2. Take $E = H\mathbb{F}_p$, representing singular (co)homology with $\mathbb{F}_p$-coefficients. In this case $H\mathbb{F}_p^*(H\mathbb{F}_p)$ is called the **mod-p Steenrod algebra** (by definition), and denoted by $\mathcal{A}_p$. It's a theorem (of Steenrod, Serre and Cartan say) that this is generated by the Bockstein β and the reduced pth powers P^i, subject to the Adem relations.

As mentioned above, $\mathcal{A}_p$ is an algebra over $\mathcal{A}_p^0 := H\mathbb{F}_p^0(H\mathbb{F}_p)$. This is the ring of degree 0 endomorphisms of the Eilenberg-MacLane spectrum $HH\mathbb{F}_p$, which by classic facts (Hatcher chapter 4, say) is the same as the ring of endomorphisms of the group $\mathbb{F}_p$, which is just $\mathbb{F}_p$ acting by (right) multiplication. Thus $\mathcal{A}_p$ is an $\mathbb{F}_p$-algebra.

Notice that for every CW spectrum X, composition of morphisms defines a natural operation

$$E^*(E) \otimes_{\mathbb{Z}} E^*(X) := [E, E]_{-*} \otimes_{\mathbb{Z}} [X, E]_{-*} \to [X, E]_{-*}$$

of the graded ring $E^*(E)$ on the graded abelian group $E^*(X)$. Thus $E^*(X)$ is a graded (left) module over the graded ring $E^*(E)$.

We can also consider the *E-homology* of E. This is

$$E_*(E) := \pi_*(E \wedge E)$$

In general it's just a graded abelian group, with a bilinear homomorphism

$$E_*(S) \otimes_{\mathbb{Z}} E_*(S) = \pi_*(E) \otimes_{\mathbb{Z}} \pi_*(E) \xrightarrow{\wedge} \pi_*(E \wedge E) = E_*(E)$$

Suppose now that E is an associative ring spectrum, with multiplication $\mu : E \wedge E \to E$ and structure map $\eta : S \to E$. Then using μ we may define a "Pontryagin product"

$$E_*(E) \otimes_{\mathbb{Z}} E_*(E) \xrightarrow{\times} E_*(E \wedge E) \xrightarrow{\mu_*} E_*(E)$$

which is associative and graded commutative (essentially since μ is). On the other hand, smashing η on the left and right with E gives 2 morphisms

$$E \simeq S \wedge E \xrightarrow{\eta \wedge \mathrm{id}} E \wedge E \text{ and}$$

$$E \simeq E \wedge S \xrightarrow{\mathrm{id} \wedge \eta} E \wedge E$$

which we might call η_L and η_R Applying π_* gives 2 homomorphisms

$$\eta_{L*}, \eta_{R*} : E_*(S) = \pi_*(E) \to \pi_*(E \wedge E) = E_*(E)$$

and one can check that these are both ring homomorphisms. Thus $E_*(E)$ is a bi-module over $E_*(S)$.

Observe that applying π_* to the multiplication $\mu : E \wedge E \to E$ yields a homomorphism

$$E_*(E) = \pi_*(E \wedge E) \to \pi_*(E) = E_*(S)$$

and applying E_* to η_L, η_R yields 2 homomorphisms

$$\eta_{L*}, \eta_{R*} : E_*(E) \to E_*(E \wedge E)$$

Lemma 5.3. *Suppose $E_*(E)$ is flat as a right $E_*(S)$-module. Then for any CW spectrum X the natural map*

$$E_*(E) \otimes_{E_*(S)} E_*(X) \xrightarrow{\wedge} E_*(E \wedge X)$$

is an isomorphism.

Proof. By induction on the cells of X, so to speak.

In the case where $X = S^p$, this is trivial: we're essentially looking at the canonical isomorphism

$$E_*(E) \otimes_{E_*(S)} E_*(S) \simeq E_*(E)$$

Suppose now that

$$X_1 \to X_2 \to X_3 \to X_4 \to X_5$$

is a cofibration sequence. Applying E_* gives an exact sequence of the $E_*(X_i)$ and *because $E_*(E)$ is flat over $E_*(S)$* this stays exact after tensoring with $E_*(E)$.

On the other hand, smashing with E gives a cofibration sequence of the $E \wedge X_i$, and applying E_* gives an exact sequence of the $E_*(E \wedge X_i)$. We have a chain map of exact sequences $E_*(E) \otimes_{E_*(S)} E_*(X_i) \to E_*(E \wedge X_i)$, and by the 5-lemma if the natural map is an isomorphism for $i = 1, 2, 4, 5$, it's an isomorphism for $i = 3$.

This proves the lemma in the case X is a finite spectrum. Since both tensoring with $E_*(E)$ and smashing with E are compatible with filtered colimits, the general result follows. □

In particular the natural map $E_*(E) \otimes_{E_*(S)} E_*(E) \to E_*(E \wedge E)$ is an isomorphism, so both η_L, η_R provide homomorphisms

$$\eta_{L*}, \eta_{R*} : E_*(E) \to E_*(E) \otimes_{E_*(S)} E_*(E)$$

More generally, for any CW spectrum X the morphism

$$X \simeq S \wedge X \xrightarrow{\eta \wedge id} E \wedge X$$

which we might call θ_L induces a homomorphism

$$E_*(X) \xrightarrow{\theta_{L*}} E_*(E \wedge X) \simeq E_*(E) \otimes_{E_*(S)} E_*(X)$$

Of course, if $E_*(E)$ is also flat as a *left* $E_*(S)$ module (for instance, if E is commutative then $E_*(E)$ is flat as a left $E_*(S)$-module if and only if it's flat as a right module) then the morphism

$$X \simeq X \wedge S \xrightarrow{\mathrm{id} \wedge \eta} X \wedge E$$

which we might call θ_R induces a homomorphism

$$E_*(X) \xrightarrow{\theta_{R*}} E_*(X \wedge E) \simeq E_*(X) \otimes_{E_*(S)} E_*(E)$$

Suppose now that E is an (associative) CW ring spectrum, with multiplication $\mu : E \wedge E \to E$ and structure map $\eta : S \to E$. Note that by applying E-homology to these maps we obtain graded abelian group homomorphisms

$$\mu^* E^*(E) \to E^*(E \wedge E) \text{ and } \eta^* : E^*(S) \to E^*(E)$$

Note however that it's rare that η^* will be a ring homomorphism (the ring structure on $E^*(S)$ comes from μ, *not* the composition of operations), and it's rare that we can decompose $E^*(E \wedge E)$ as $E^*(E) \otimes_{E^*} E^*(E)$ to obtain something like a co-multiplication.

On the other hand, we can define a homomorphism of graded abelian groups

$$E^*(E) \to \mathrm{Hom}_{E_*(S)}(E_*(E), E_*(S))$$

as follows: given an element $\theta \in E^p(E)$, represented by a morphism $\theta : E \to \Sigma^p E$, applying E_* gives a homomorphism of graded $E_*(S)$-modules

$$\theta_* : E_*(E) \to E_*(E)$$

In this way we obtain a graded ring homomorphism

$$E^*(E) \to \mathrm{End}_{E_*(S)}(E_*(E), E_*(E))$$

(for the basic reason that the covariant additive functor E_* takes endomorphisms of E to endomorphisms of $E_*(E)$). Now observe that the multiplication operation μ of E induces a homomorphism

$$E^*(E) = \pi_*(E \wedge E) \xrightarrow{\mu_*} \pi_*(E) = E_*(S)$$

and composition with μ_* provides a homomorphism of graded abelian groups

$$\mathrm{End}_{E_*(S)}(E_*(E), E_*(E)) \to \mathrm{Hom}_{E_*(S)}(E_*(E), E_*(S))$$

If θ is as above, then precomposition with θ_* induces an endomorphism of $\mathrm{Hom}_{E_*(S)}(E_*(E), E_*(S))$ taking $\lambda_* \mapsto \lambda_* \circ \theta_*$ and in this way we obtain a (right) action of $E^*(E)^{op}$ on $\mathrm{Hom}_{E_*(S)}(E_*(E), E_*(S))$ (for the basic reason that the contravariant additive functor $\mathrm{Hom}_{E_*(S)}(E_*(-), E_*(S))$ takes endomorphisms of E to endomorphisms of $\mathrm{Hom}_{E_*(S)}(E_*(E), E_*(S))$, reversing the order of composition).

Remark 5.4. In upcoming applications of this stuff E will be an associative, commutative CW ring spectrum (in fact with a complex orientation), but I am leaving out these hypotheses to clarify the level of generality at which one can discuss cohomology operations. Moreover in what follows one really must pay attention to left and right actions, and using commutativity to identify those actions can lead to major confusion.