ADAMS SPECTRAL SEQUENCES, ETC.

CHARLIE GODFREY

ABSTRACT. These notes will concern Adams spectral sequences and related topics like the homological algebra of comodules over Hopf algebroids. Most of the material will be coming from Adams's blue book and Ravenel's green and orange books. So, basically nothing here will be original.

1. ADAMS SPECTRAL SEQUENCES

1.1. **The "classical" Adams spectral sequence.** This section will be devoted to a proof of the following classical result:

Theorem 1.1. *(Adams) Let p be a prime number, and suppose X is a connective spectrum such that $H^*(X;\mathbb{F}_p)$ is of finite type. Recall that $H^*(X;\mathbb{F}_p)$ is a graded module over the mod p Steenrod algebra $\mathcal{A}^* := H\mathbb{F}_p^*(H\mathbb{F}_p)$. Then there's a spectral sequence $E_*^{**}(X)$ with differentials*

$$d_r : E_r^{s,t}(X) \to E_r^{s+r,t+r-1}(X)$$

so that

- *The second page can be identified as*
$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}^*}(H^*(X;\mathbb{F}_p),\mathbb{F}_p)$$
- *If X is a spectrum of finite type (say in the sense of Adams's blue book) then $E_\infty^{*,*}(X)$ can be identified as the associated bigraded group of a certain filtration of $\pi_*(X)\otimes\mathbb{Z}_p$, where $\mathbb{Z}_p$ denotes the p-adic integers.*

Remark 1.2. Here $\mathrm{Ext}^{*,*}_{\mathcal{A}^*}(H^*(X;\mathbb{F}_p),\mathbb{F}_p)$ is referring to Ext groups of graded modules over the Steenrod algebra $\mathcal{A}^*$. One grading is homological; the other comes from the fact that for any 2 graded $\mathcal{A}^*$-modules M^*, N^* $\mathrm{Hom}^{*,*}_{\mathcal{A}^*}(M^*,N^*)$ has a natural grading. More on this later.

Note that if $\pi_n(X)$ is a finitely generated abelian group for each n (for instance, this will be the case if X is a finite spectrum - will it be the case if X is a spectrum of finite type?) then $\pi_*(X)\otimes\mathbb{Z}_p$ coincides with the mod-p completion of $\pi_*(X)$. See Atiyah-Macdonald. Also, it's a good exercise to compute $A\otimes\mathbb{Z}_p$ for a finitely generated abelian group A.

Informally, the theorem asserts that if X is a connective spectrum of finite type there's a spectral sequence with E_2-page $\mathrm{Ext}^{*,*}_{\mathcal{A}^*}(H^*(X;\mathbb{F}_p),\mathbb{F}_p)$ converging to $\pi_*(X)\otimes\mathbb{Z}_p$.

1.1.1. *Preliminaries.* For the time being $H^*(X)$ will mean $H^*(X;\mathbb{F}_p)$.

We'll be making essential use of these facts:

- Suppose K is a locally finite wedge of suspensions of $H\mathbb{F}_p$, i.e. $K \simeq \bigvee_i \Sigma^{n_i} H\mathbb{F}$ and for each $n\in\mathbb{Z}$, $n_i = n$ for at most finitely many i (such a spectrum K will be called a generalized mod-p Eilenberg-Mac Lane spectrum). Then $\pi_*(K)$ is a graded $\mathbb{F}_p$-vector space of finite type with one generator for each wedge summand of $H\mathbb{F}_p$, and moreover the natural map
$$\pi_*(K) \to \mathrm{Hom}_{\mathcal{A}^*}(H^*(K),\mathbb{F}_p)$$
is an isomorphism. By "the natural map" I mean the following: if $f : \Sigma^n S \to K$ is a morphism of spectra, it induces an $\mathcal{A}^*$-module homomorphism $H^*(K) \to H^*(\Sigma^n S) \simeq H^{*-n}(S)$

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of mod-p homology. Of course, $H^*(S)$ can be identified with a copy of $\mathbb{F}_p$ concentrated in degree 0.

Proof. I think one must make some sort of appeal to the equivalence (in the stable homotopy category) of the locally finite wedge $\bigvee_i \Sigma^{n_i} H\mathbb{F}$ with the locally finite product $\prod_i \Sigma^{n_i} H\mathbb{F}$. From there it's clear that

$$\begin{aligned}\pi_*(K) &\simeq \pi_*(\bigvee_i \Sigma^{n_i} H\mathbb{F}) \simeq \pi_*(\prod_i \Sigma^{n_i} H\mathbb{F}) \\ &\simeq \prod_i \pi_*(\Sigma^{n_i} H\mathbb{F}) \simeq \prod_i \Sigma^{n_i} \mathbb{F}_p \simeq \bigoplus_i \Sigma^{n_i} \mathbb{F}_p\end{aligned}$$

where in the last step I've used the fact that the product is locally finite.

Notice that

$$H^*(K) \simeq H^*(\bigvee_i \Sigma^{n_i} H\mathbb{F}) \simeq \prod_i H^*(\Sigma^{n_i} H\mathbb{F}) \simeq \prod_i A^{*-n_i}$$

Since an $\mathcal{A}_p$-module homomorphism $\mathcal{A}_p \to \mathbb{F}_p$ must be a multiple of the augmentation, one sees that $\pi_*(K)$ and $\mathrm{Hom}_{\mathcal{A}^*}(H^*(K), \mathbb{F}_p)$ are abstractly isomorphic graded $\mathbb{F}_p$-vector spaces of finite type. So, to see that the above natural map is an isomorphism it'll suffice to show it's injective, which is more or less clear after definition-unravelling. Etc. □

- Let X be a connective spectrum such that $H^*(X)$ is a graded $\mathbb{F}_p$-vector space of finite type (for instance X might be a CW spectrum of finite type). Then a morphism of spectra $u : X \to K$ is equivalent to a locally finite collection of elements $u_i \in H^{-n_i}(X)$, one for each wedge summand of $H\mathbb{F}_p$.

  Proof. Again one must make some sort of appeal to the fact that K is equivalent to a locally finite product $\prod_i \Sigma^{n_i} H\mathbb{F}$. □

- Suppose the locally finite collection $u_i \in H^{-n_i}(X)$ generates $H^*(X)$ as a graded $\mathcal{A}_p$-module. Then the induced homomorphism $u^* : H^*(K) \to H^*(X)$ is surjective.

  Proof. Notice that u^* is a homomorphism of $\mathcal{A}_p$-modules that hits a set of generators. □

- $H\mathbb{F}_p \wedge X$ is (non-canonically) isomorphic to a wedge of suspensions of $H\mathbb{F}_p$, with one wedge summand for each element of an $\mathbb{F}_p$-vector space basis for $H^*(X)$. Moreover the natural homomorphism $\mathcal{A}^* \otimes H^*(X) \to H^*(H\mathbb{F}_p \wedge X)$ is an isomorphism and the map $X \simeq S \wedge X \xrightarrow{\eta \wedge \mathrm{id}} H\mathbb{F}_p \wedge X$ induces the $\mathcal{A}^*$-module structure

$$\mathcal{A}^* \otimes H^*(X) \simeq H^*(H\mathbb{F}_p \wedge X) \to H^*(X)$$

I think proofs of most of these facts appear in Adams's blue book.

1.1.2. *The construction.*

Definition 1.3. A **mod-p Adams tower for** X is a diagram of the form

$$\begin{tikzcd} X = X_0 \arrow[d, "f_0"'] & X_1 \arrow[l, "g_0"'] \arrow[d, "f_1"'] & X_2 \arrow[l, "g_1"'] \arrow[d, "f_2"'] & \dots \arrow[l, "g_2"'] \\ K_0 & K_1 & K_2 & \end{tikzcd} \tag{1.1}$$

where each K_i is a locally finite wedge of suspensions of $H\mathbb{F}_p$, the induced homomorphisms

$$f_i^* : H^*(K_i) \to H^*(X_i)$$

are all surjective, and each X_{i+1} is the fiber of the morphism $f_i : X_i \to K^i$.

Lemma 1.4. *Let X be a connective spectrum such that $H^*(X)$ is of finite type. Then there exists a mod-p Adams tower for X.*

Proof. In fact such a resolution can be constructed rather canonically (and I think even functorially). Set $K_0 := H\mathbb{F}_p \wedge X$ and let $f_0 : X \to K_0$ be the composition

$$X \simeq S \wedge X \xrightarrow{\eta \wedge \mathrm{id}} H\mathbb{F}_p \wedge X = K_0$$

where $\eta : S \to H\mathbb{F}_p$ is the structure map for the ring spectrum $H\mathbb{F}_p$. From the above bullet points, $K_0 = H\mathbb{F}_p \wedge X$ is a locally finite wedge sum of suspensions of $H\mathbb{F}_p$ and the induced map

$$f^* : H^*(K_0) \to H^*(X)$$

can be identified with the action map $\mathcal{A}^* \otimes H^*(X) \to H^*(X)$, so it's certainly surjective. Now let X_1 be the fiber of f_0, so we have a (co)fibration sequence

$$X_1 \xrightarrow{g_0} X_0 \xrightarrow{f_0} K_0$$

Analysis of the associated long exact sequence obtained by applying π_*, namely

$$\cdots \to \pi_{n+1}(K_0) \xrightarrow{\partial} \pi_n(X_1) \xrightarrow{g_*} \pi_n(X_0) \xrightarrow{f_*} \pi_n(K_0) \xrightarrow{\partial} \pi_{n-1}(X_1) \to \dots$$

shows that $\pi_n(X_1) = 0$ if $n < -1$, so X_1 isn't necessarily connective (but ΣX_1 is). Looking at the analogous long exact sequence for $H^*(-)$ shows that $H^*(X_1)$ of finite type. So, we may set $K_1 := H\mathbb{F}_p \wedge X$, define $f_1 : X_1 \to K_0$ as the composition $X_1 \simeq S \wedge X_1 \to H\mathbb{F}_p \wedge X_1 = K_1$, etc. Proceeding in this way yields the desired resolution.

□

Notice that as $X_{i+1} \xrightarrow{g_i} X_i \xrightarrow{f_i} K_i$ is a (co)fibration sequence, we can extract from it a morphism $K_i \to \Sigma X_{i+1}$. Together with the map $\Sigma f_{i+1} : \Sigma X_{i+1} \to \Sigma K_{i+1}$, this yields a morphism $h_i : K_i \to \Sigma K_{i+1}$, which can also be viewed as a degree -1 morphism of spectra $h_i : K_i \to K_{i+1}$ (that's what I'll do). In this way we obtain a sequence

$$X \xrightarrow{f_0} K_0 \xrightarrow{h_0} K_1 \xrightarrow{h_1} K_2 \xrightarrow{h_2} \dots$$

Lemma 1.5. *Applying $H^*(-)$ to the above sequence of morphisms gives a free resolution*

$$0 \leftarrow H^*(X) \xleftarrow{f_0^*} H^*(K_0) \xleftarrow{h_0^*} H^*(K_1) \xleftarrow{h_1^*} H^*(K_2) \xleftarrow{h_2^*} \dots$$

of the $\mathcal{A}^*$-module $H^*(X)$.

Proof. Certainly $H^*(K_i)$ is a free $\mathcal{A}^*$-module for all i. So, it'll suffice to show that the sequence is exact. For starters, f_0^* is surjective by the definition of an Adams tower.

Now consider the long exact sequence of mod-p cohomology arising from the (co)fibration sequence $X_{i+1} \xrightarrow{g_i} X_i \xrightarrow{f_i} K_i$: it looks like

$$\cdots \to H^{n-1}(X_{i+1}) \xrightarrow{\delta} H^n(K_i) \xrightarrow{f^*} H^n(X_i) \xrightarrow{g^*} H^n(X_{i+1}) \xrightarrow{\delta} H^{n+1}(K_i) \to \dots$$

Recall that all the homomorphisms f^* are *surjective*. Exactness then requires that the homomorphisms g^* are all 0, and the connecting homomorphisms δ are all *injective*. We may now check exactness at $H^n(K_i)$: observe that $h_i^* : H^{n-1}(K_{i+1}) \to H^n(K_i)$ can be factored as the composition

$$H^{n-1}(K_{i+1}) \xrightarrow{f^*} H^{n-1}(X_{i+1}) \xrightarrow{\delta} H^n(K_i)$$

, and similarly $h_{i-1}^* : H^n(K_i) \to H^{n+1}(K_{i-1})$ can be factored as

$$H^n(K_i) \xrightarrow{f^*} H^n(X_i) \xrightarrow{\delta} H^{n+1}(K_{i-1})$$

Since $f^* : H^{n-1}(K_{i+1}) \to H^{n-1}(X_{i+1})$ is surjective, the image of h_i^* is just the image of $\delta : H^{n-1}(X_{i+1}) \to H^n(K_i)$. By exactness, this is the kernel of $f^* : H^n(K_i) \to H^n(X_i)$, and since $\delta : H^n(X_i) \to H^{n+1}(K_{i-1})$ is injective, it's actually the kernel of h_{i-1}^*.

□

Applying $\pi_*(-)$ to an Adams tower yields an exact couple, and hence a spectral sequence, in the following way: first of all, we have a collection of long exact sequences

$$\dots \xrightarrow{f_*} \pi_{n+1}(K_i) \xrightarrow{\partial} \pi_n(X_{i+1}) \xrightarrow{g_*} \pi_n(X_i) \xrightarrow{f_*} \pi_n(K_i) \xrightarrow{\partial} \pi_{n-1}(X_{i+1}) \xrightarrow{g_*} \dots$$

To interweave them, one can define bigraded abelian groups

$$D_1^{s,t} = \pi_{t-s}(X_s) \text{ and } E_1^{s,t} = \pi_{t-s}(K_s)$$

(following a standard reindexing practice here - for one thing I think it ensures that $D_1^{s,t}0$ and $E_1^{s,t} = 0$ when $s < 0$ or $t < 0$ - so everything's in the first quadrant), and form the exact couple

$$\begin{tikzcd} D_1 \arrow[r, "g_*"] & D_1 \arrow[d, "f_*"] \\ E_1 \arrow[u, "\partial"] \arrow[r, equal] & E_1 \end{tikzcd} \tag{1.2}$$

Here we have homomorphisms

$$g_* : D_1^{s,t} = \pi_{t-s}(X_s) \to \pi_{t-1-(s-1)}(X_{s-1}) = D_1^{s-1,t-1}$$

(so the g_* have bidegree $(-1,-1)$),

$$f_* : D_1^{s,t} = \pi_{t-s}(X_s) \to \pi_{t-s}(K_s) = E_1^{s,t}$$

(so the f_* have bidegree $(0,0)$) and

$$\partial : E_1^{s,t} = \pi_{t-s}(K_s) \to \pi_{t-(s+1)}(X_{s+1}) = D_1^{s+1,t}$$

(so the ∂ have bidegree $(1,0)$).

From here one follows the usual recipe for obtaining a spectral sequence $E_*^{*,*}(X)$, called the **Adams spectral sequence**, from the above exact couple. The differential d_r on the page $E_r^{*,*}(X)$ has bidegree $(r, r-1)$, i.e. it consists of maps

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$$

Notice that this means the differential hitting $E_r^{s,t}$ is coming from $E_r^{s-r,t-r+1}$ - as mentioned above everything's in the first quadrant, so

$$E_r^{s-r,t-r+1} = 0 \text{ if } s < r \text{ or } t < r-1$$

Which is to say, no non-0 differentials hit $E_r^{s,t}$ if $s < r$ or $t < r-1$ - this is something to keep in mind.

Lemma 1.6. *There are natural isomorphisms $E_2^{s,t}(X) \simeq \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X), \mathbb{F}_p)$.*

Proof. As described above, $E_1^{s,t} = \pi_{t-s}(K_s)$. Since K_s is a locally finite wedge of suspensions of $H\mathbb{F}_p$, the natural map

$$\pi_{t-s}(K_s) \to \mathrm{Hom}_{\mathcal{A}}^{t-s}(H^*(K_s), \mathbb{F}_p)$$

is an isomorphism. One must now show that under this isomorphism the differentials $d_1 : E_1^{s,t} \to E_1^{s+1,t}$, defined to be the compositions

$$\pi_{t-s}(K_s) \xrightarrow{\partial} \pi_{t-s-1}(X_{s+1}) \xrightarrow{f_*} \pi_{t-s-1}(K_{s+1})$$

coincide with the homomorphisms $\mathrm{Hom}_{\mathcal{A}}^{t-s}(H^*(K_s), \mathbb{F}_p) \to \mathrm{Hom}_{\mathcal{A}}^{t-s-1}(H^*(K_{s+1}), \mathbb{F}_p)$ induced by the (degree -1) homomorphisms

$$h^* : H^*(K_{s+1}) \to H^*(K_s)$$

obtained by splicing long exact sequences (or alternatively building the maps $h : K_s \to \Sigma K_{s+1}$). This consists of a lot of "definition-unraveling" which I'm skipping out of sheer laziness.

Now recall that the sequence of maps

$$\ldots \xrightarrow{h^*} H^*(K_2) \xrightarrow{h^*} H^*(K_1) \xrightarrow{h^*} H^*(K_0) \xrightarrow{f^*} H^*(X) \to 0$$

is a free resolution of $H^*(X)$ over $\mathcal{A}$, and so by definition the homology of the sequence

$$\ldots \xrightarrow{d_1} \mathrm{Hom}_{\mathcal{A}}^{t-s+1}(H^*(K_{s-1}), \mathbb{F}_p) \xrightarrow{d_1} \mathrm{Hom}_{\mathcal{A}}^{t-s}(H^*(K_s), \mathbb{F}_p) \xrightarrow{d_1} \mathrm{Hom}_{\mathcal{A}}^{t-s-1}(H^*(K_{s+1}), \mathbb{F}_p) \xrightarrow{d_1} \ldots$$

at (s,t) is $\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X), \mathbb{F}_p)$. Here s is the (co)homological grading, whereas t comes from the grading of $\mathcal{A}$, $H^*(X)$ etc. The reason we go from homomorphisms of degree $t-s$ to Ext of degree t has to do with the fact that $H^*(K_s)$ is related to $H^*(X)$ by s homomorphisms of degree -1. □

Corollary 1.7. *If a morphism of spectra $f : X \to Y$ induces an isomorphism $f^* : H^*(Y) \to H^*(X)$ of mod-p cohomology, then it induces an isomorphism $f_* : E_*^{*,*}(X) \to E_*^{*,*}(Y)$ of Adams spectral sequences.*

Notice that from the Adams tower of X we obtain maps

$$g : X_i \to X$$

for each i, inducing homomorphisms $g_* : \pi_*(X_i) \to \pi_*(X)$. If $i < j$, we have a factorization

$$X_j \xrightarrow{X_i}\to X$$

of the map $g : X_j \to X$, and thus $\mathrm{im}\pi_*(X_j) \subset \mathrm{im}\pi_*(X_i)$ in $\pi_*(X)$. The upshot: the subgroups $\mathrm{im}\pi_*(X_i)$ form a *decreasing* filtration of $\pi_*(X)$. One can show that in certain cases the spectral sequence $E_*^{*,*}(X)$ converges to the filtered graded abelian group $\pi_*(X)$. More on this in a moment.

Definition 1.8. Let $X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \xleftarrow{f_4} \ldots$ be a sequence of spectra and morphisms. Then $\mathrm{ho}\lim\limits_{\leftarrow} X_i$ is the fiber of the morphism of spectra

$$\Delta : \prod_{i=0}^{\infty} X_i \to \prod_{i=0}^{\infty} X_i$$

whose ith coordinate is $\pi_i - f_{i+1} \circ \pi_{i+1}$.

So, by definition $\mathrm{ho}\lim\limits_{\leftarrow} X_i$ fits into a (co)fibration sequence

$$\mathrm{ho}\lim_{\leftarrow} X_i \to \prod_i X_i \xrightarrow{\Delta} \prod_i X_i$$

and so we have a long exact sequence of homotopy groups

$$\ldots \xrightarrow{\Delta_*} \prod_i \pi_{n+1}(X_i) \xrightarrow{\partial} \pi_n(\mathrm{ho}\lim_{\leftarrow} X_i) \to \prod_i \pi_n(X_i) \xrightarrow{\Delta_*} \prod_i \pi_n(X_i) \xrightarrow{\partial} \pi_{n-1}(\mathrm{ho}\lim_{\leftarrow} X_i) \to \ldots$$

For instance, this shows that if $\Delta_* : \prod_i \pi_*(X_i) \to \prod_i \pi_*(X_i)$ is surjective, or equivalently $\overset{1}{\lim\limits_{\leftarrow}} \pi_*(X_i) = 0$, then the natural map

$$\pi_*(\mathrm{ho}\lim_{\leftarrow} X_i) \to \lim_{\leftarrow} \pi_*(X_i)$$

is an isomorphism.

Lemma 1.9. *Suppose X has an Adams tower such that $ho \lim_{\leftarrow} X_i$ is contractible, i.e. $ho \lim_{\leftarrow} X_i \simeq \mathrm{pt}$ in the stable homotopy category. Then $\bigcap_s \mathrm{im}\pi_*(X_s) = 0 \subset \pi_*(X)$ and $E_\infty^{s,t}(X)$ can be naturally identified with the subquotient $\mathrm{im}\pi_{t-s}(X_s)/\mathrm{im}\pi_{t-s}(X_{s+1})$. Which is to say, $E_*^{*,*}(X)$ converges to the filtered graded abelian group $\pi_*(X)$.*

See Ravenel's proof. It's Lemma 2.1.12 in the green book.

Lemma 1.10. *Suppose that $\pi_n(X)$ is a finite p-group for all n. Then for any mod-p Adams tower of X, $\mathrm{ho}\lim_{\leftarrow} X_i$ is contractible.*

Notice that the π_* long exact sequence associated to the (co)fibration sequence $X_1 \to X \to K_0$, namely

$$\dots \xrightarrow{f_*} \pi_{n+1}(K_0) \xrightarrow{\partial} \pi_n(X_1) \xrightarrow{g_*} \pi_n(X) \xrightarrow{f_*} \pi_n(K_0) \xrightarrow{\partial} \pi_{n-1}(X_1) \xrightarrow{g_*} \dots$$

shows that $\pi_n(X_1)$ is a finite p-group for all n. Proceeding in this way one sees that all of the $\pi_n(X_s)$ are finite p-groups. It follows that $\lim_{\leftarrow}^1 \pi_*(X_i) = 0$, and thus the natural map

$$\pi_*(\mathrm{ho}\lim_{\leftarrow} X_i) \to \lim_{\leftarrow} \pi_*(X_i)$$

is an isomorphism. At this point it'll suffice to show that $\lim_{\leftarrow} \pi_*(X_i) = 0$, and at this point I'll defer again to Ravenel.

Definition 1.11. Recall (say from Adams's blue book) that for a spectrum Y and an abelian group G with Moore spectrum SG, one can define $YG := Y \wedge SG$. For a spectrum X as above, set $X^m := X\mathbb{Z}/(p^m)$ and $\hat{X} := X\mathbb{Z}_p$.

We'll need a few facts - they're basically all covered in the blue book:

- The natural map $X \to \hat{X}$ (obtained as the composition $X \simeq X \wedge S \xrightarrow{\mathrm{id}\wedge\eta} X \wedge S\mathbb{Z}_p$) induces an isomorphism mod-p (co)-homology, hence on mod-p Adams spectral sequences.
- There's a natural isomorphism $\pi_*(X) \otimes \mathbb{Z}_p \simeq \pi_*(\hat{X})$ - this can be viewed as a sort of universal coefficient theorem.
- If X is a CW spectrum of finite type, then there's a natural equivalence $\hat{X} \simeq \mathrm{ho}\lim_{\leftarrow} X^m$.

The natural equivalence can be described as follows: consideration of the $H^*(-;\mathbb{Z})$ long exact sequences associated to the (co)fibration sequence

$$\mathrm{ho}\lim_{\leftarrow} S\mathbb{Z}/(p^m) \to \prod_m S\mathbb{Z}/(p^m) \xrightarrow{\Delta} \prod_m S\mathbb{Z}/(p^m)$$

shows that $\mathrm{ho}\lim_{\leftarrow} S\mathbb{Z}/(p^m)$ is indeed a Moore spectrum for $\mathbb{Z}_p$, i.e. $S\mathbb{Z}_p \simeq \mathrm{ho}\lim_{\leftarrow} S\mathbb{Z}/(p^m)$. Thus we have a (co)fibration sequence $S\mathbb{Z}_p \to \prod_m S\mathbb{Z}/(p^m) \xrightarrow{\Delta} \prod_m S\mathbb{Z}/(p^m)$. The idea would be to smash with X and obtain a (co)fibration sequence

$$\hat{X} \to \prod_m X^m \xrightarrow{\Delta} \prod_m X^m$$

identifying $\hat{X}$ as $\mathrm{ho}\lim_{\leftarrow} X^m$ - for this to work one must show (as Adams does) that

$$X \wedge \prod_m S\mathbb{Z}/(p^m) \simeq \prod_m X \wedge S\mathbb{Z}/(p^m) = \prod_m X^m$$

Now suppose X is a connective CW spectrum of finite type. Then certainly it has an Adams filtration by CW spectra X_s of finite type. For each m smashing this with $S\mathbb{Z}/(p^m)$ yields an Adams filtration by spectra X_s^m for X^m. Now since $\pi_n(X^m)$ is a finite p-group for all n, it must be that $\operatorname{ho}\lim_s X_s^m$ is contractible. On the other hand, since each X_s is of finite type, we know that

$$\operatorname{ho}\lim_m X_s^m \simeq \hat{X}_s$$

Using lemma 2.1.11 from the green book, one concludes that

$$\begin{aligned}\operatorname{ho}\lim_s \hat{X}_s &\simeq \operatorname{ho}\lim_s \operatorname{ho}\lim_m X_s^m \\ &\simeq \operatorname{ho}\lim_m \operatorname{ho}\lim_s X_s^m \simeq \mathrm{pt}\end{aligned}$$

being a limit of contractible spectra. Since the spectra $\hat{X}_s$ form an Adams filtration for $\hat{X}$ obtained by smashing the one for X with $S\mathbb{Z}_p$, we see that the Adams spectral sequence $E_*^{*,*}(\hat{X})$ converges to the filtered graded abelian group $\pi_*(\hat{X}) \simeq \pi_*(X) \otimes \mathbb{Z}_p$. Since the natural map $X \to \hat{X}$ induces an isomorphism of Adams spectral sequences, we're done proving the above classical theorem.

Example 1.12. Take $X = H\mathbb{Z}$. According to Ravenel, a map $f : H\mathbb{Z} \to H\mathbb{F}_p$ corresponding to the usual quotient $\mathbb{Z} \to \mathbb{F}_p$ will induce a surjective homomorphism $\mathcal{A}^* = H^*(H\mathbb{F}_p) \xrightarrow{f^*} H^*(H\mathbb{Z})$ (**IS THIS OBVIOUS?**). Moreover such a map will fit into a (co)fibration sequence

$$H\mathbb{Z} \xrightarrow{p} H\mathbb{Z} \xrightarrow{f} H\mathbb{F}_p$$

(ah this shows why f^* is surjective - if you believe that multiplication by p map of spectra induces the 0 homomorphism on $H^*(-)$). The upshot is that $H\mathbb{Z}$ has an Adams tower in which each $X_s = H\mathbb{Z}$ and each $K_s = H\mathbb{F}_p$. It follows that the E_1-page looks like

$$E_1^{s,t} = \pi_{s-t}(H\mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \text{if } s = t \\ 0 & \text{otherwise} \end{cases}$$

and there can be no differentials for dimensional reasons. Thus the spectral sequence collapses immediately, and we see that

$$E_\infty^{s,t} = \begin{cases} \mathbb{F}_p & \text{if } s = t \\ 0 & \text{otherwise} \end{cases}$$

. Since one can build $H\mathbb{Z}$ as a connective CW spectrum of finite type, one knows that $E_*^{*,*}(H\mathbb{Z})$ converges to the filtered graded abelian group

$$\pi_*(H\mathbb{Z}) \otimes \mathbb{Z}_p = \mathbb{Z}_p, \text{ concentrated in degree } 0$$

and it seems clear that $E_\infty^{s,s} = \pi_0(H\mathbb{F}_p)$ corresponds to the subquotient $p^s\mathbb{Z}_p/p^{s+1}\mathbb{Z}_p$. The interesting upshot is that

$$\operatorname{Ext}_{\mathcal{A}}^{s,t}(H^*(H\mathbb{Z}), \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \text{if } s = t \\ 0 & \text{otherwise} \end{cases}$$

which at least to me was not at all obvious from the outset.

1.2. Generalizations.

1.2.1. *Generalized (co)homology operations.* Let E, F be CW spectra. Then there is a natural isomorphism of graded abelian groups

$$F^*(E) \simeq \text{ natural transformations } E^* \to F^*$$

where E^*, F^* are the "Yoneda functors" on the category of CW spectra taking $X \mapsto E^*(X) := [X, E]_{-*}$ and similarly for F.

Remark 1.13. If we consider the reduced cohomology theories $\tilde{E}^*, \tilde{F}^*$ on the category of pointed CW complexes given by $\tilde{E}^*(X) := [\Sigma^\infty X, E]_{-*}$ and similarly for F, the above natural transformations correspond to *stable* cohomology operations, in the classical sense. For instance, say A, B are abelian groups. An element $\theta \in HB^p(HA)$ corresponds to a morphism of spectra $\theta : HA \to \Sigma^p HB$, defining natural transformations

$$\tilde{H}^q(X; A) = [\Sigma^\infty X, HA]_{-q} \xrightarrow{\theta_*} [\Sigma^\infty X, HB]_{-p-q} = \tilde{H}^{p+q}(X; B)$$

by composition; these are clearly compatible with suspension, i.e. they form a stable cohomology operation.

Now suppose $F = E$, so we're considering $E^*(E)$, which can be viewed as the graded abelian group of natural transformations $E^* \to E^*$. In this situation we may also *compose* morphisms of spectra $\theta : E \to E$; if $\theta : E \to \Sigma^p E$ and $\psi : E \to \Sigma^q E$ are morphisms representing classes $\theta \in E^p(E), \psi \in E^q(E)$ then their composition gives a morphism

$$E \xrightarrow{\theta} \Sigma^p E \xrightarrow{\Sigma^p} \Sigma^{p+q} E$$

representing a class $\psi \circ \theta \in E^{p+q}(E)$. In this way $E^*(E)$ becomes a graded ring; it's just the graded ring of endomorphisms of the object E in the stable homotopy category of CW spectra, which is both graded and additive.

Per usual, $E^*(E)$ is an algebra over $E^0(E)$.

Example 1.14. Take $E = H\mathbb{F}_p$, representing singular (co)homology with $\mathbb{F}_p$-coefficients. In this case $H\mathbb{F}_p^*(H\mathbb{F}_p)$ is called the **mod-p Steenrod algebra** (by definition), and denoted by $\mathcal{A}_p$. It's a theorem (of Steenrod, Serre and Cartan say) that this is generated by the Bockstein β and the reduced pth powers P^i, subject to the Adem relations.

As mentioned above, $\mathcal{A}_p$ is an algebra over $\mathcal{A}_p^0 := H\mathbb{F}_p^0(H\mathbb{F}_p)$. This is the ring of degree 0 endomorphisms of the Eilenberg-MacLane spectrum $HH\mathbb{F}_p$, which by classic facts (Hatcher chapter 4, say) is the same as the ring of endomorphisms of the group $\mathbb{F}_p$, which is just $\mathbb{F}_p$ acting by (right) multiplication. Thus $\mathcal{A}_p$ is an $\mathbb{F}_p$-algebra.

Notice that for every CW spectrum X, composition of morphisms defines a natural operation

$$E^*(E) \otimes_{\mathbb{Z}} E^*(X) := [E, E]_{-*} \otimes_{\mathbb{Z}} [X, E]_{-*} \to [X, E]_{-*}$$

of the graded ring $E^*(E)$ on the graded abelian group $E^*(X)$. Thus $E^*(X)$ is a graded (left) module over the graded ring $E^*(E)$.

We can also consider the *E-homology* of E. This is

$$E_*(E) := \pi_*(E \wedge E)$$

In general it's just a graded abelian group, with a bilinear homomorphism

$$E_*(S) \otimes_{\mathbb{Z}} E_*(S) = \pi_*(E) \otimes_{\mathbb{Z}} \pi_*(E) \xrightarrow{\wedge} \pi_*(E \wedge E) = E_*(E)$$

Suppose now that E is an associative ring spectrum, with multiplication $\mu : E \wedge E \to E$ and structure map $\eta : S \to E$. Then using μ we may define a "Pontryagin product"

$$E_*(E) \otimes_{\mathbb{Z}} E_*(E) \xrightarrow{\times} E_*(E \wedge E) \xrightarrow{\mu_*} E_*(E)$$

which is associative (essentially since μ is), and graded commutative if μ is. The upshot is that $E_*(E)$ is a graded ring. Smashing η on the left and right with E gives 2 morphisms

$$E \simeq S \wedge E \xrightarrow{\eta \wedge \mathrm{id}} E \wedge E \text{ and}$$

$$E \simeq E \wedge S \xrightarrow{\mathrm{id} \wedge \eta} E \wedge E$$

which we might call η_L and η_R Applying π_* gives 2 homomorphisms

$$\eta_{L*}, \eta_{R*} : E_*(S) = \pi_*(E) \to \pi_*(E \wedge E) = E_*(E)$$

and one can check that these are both ring homomorphisms. Thus $E_*(E)$ is a bi-module over $E_*(S)$.

Observe that applying π_* to the multiplication $\mu : E \wedge E \to E$ yields a homomorphism

$$E_*(E) = \pi_*(E \wedge E) \to \pi_*(E) = E_*(S)$$

which can be viewed as a sort of augmentation.

Lemma 1.15. *Suppose $E_*(E)$ is flat as a right $E_*(S)$-module. Then for any CW spectrum X the natural map*

$$E_*(E) \otimes_{E_*(S)} E_*(X) \xrightarrow{\wedge} E_*(E \wedge X)$$

is an isomorphism.

It should be noted that $E_*(E)$ is flat as a right $E_*(S)$-module if and only if it's flat as a left $E_*(S)$-module (basically since η_L and η_R differ by a flip automorphism of $E \wedge E$).

Proof. By induction on the cells of X, so to speak.

In the case where $X = S^p$, this is trivial: we're essentially looking at the canonical isomorphism

$$E_*(E) \otimes_{E_*(S)} E_*(S) \simeq E_*(E)$$

Suppose now that

$$X_1 \to X_2 \to X_3 \to X_4 \to X_5$$

is a cofibration sequence. Applying E_* gives an exact sequence of the $E_*(X_i)$ and *because $E_*(E)$ is flat over $E_*(S)$* this stays exact after tensoring with $E_*(E)$.

On the other hand, smashing with E gives a cofibration sequence of the $E \wedge X_i$, and applying E_* gives an exact sequence of the $E_*(E \wedge X_i)$. We have a chain map of exact sequences $E_*(E) \otimes_{E_*(S)} E_*(X_i) \to E_*(E \wedge X_i)$, and by the 5-lemma if the natural map is an isomorphism for $i = 1, 2, 4, 5$, it's an isomorphism for $i = 3$.

This proves the lemma in the case X is a finite spectrum. Since both tensoring with $E_*(E)$ and smashing with E are compatible with filtered colimits, the general result follows. □

For the foreseeable future it'll be assumed that E is an associative CW ring spectrum and $E_*(E)$ is flat over $E_*(S)$.

The above lemma implies in particular that the natural map $E_*(E) \otimes_{E_*(S)} E_*(E) \to E_*(E \wedge E)$ is an isomorphism. Observe that one can define three obvious morphisms $E \wedge E \to E \wedge E \wedge E$, namely

$$E \wedge E \simeq S \wedge E \wedge E \xrightarrow{\eta \wedge \mathrm{id} \wedge \mathrm{id}} E \wedge E \wedge E,$$

$$E \wedge E \simeq E \wedge E \wedge S \xrightarrow{\mathrm{id} \wedge \mathrm{id} \wedge \eta} E \wedge E \wedge E \text{ and}$$

$$E \wedge E \simeq E \wedge S \wedge E \xrightarrow{\mathrm{id} \wedge \eta \wedge \mathrm{id}} E \wedge E \wedge E$$

Applying π_* one obtains three induced homomorphisms (of graded rings)

$$E_*(E) = \pi_*(E \wedge E) \to \pi_*(E \wedge E \wedge E) = E_*(E \wedge E) \simeq E_*(E) \otimes E_*(E)$$

According to Hopkins (see "Complex oriented cohomology theories and the language of stacks" a.k.a COCTALOS), the morphisms $\eta \wedge \mathrm{id} \wedge \mathrm{id}$ and $\mathrm{id} \wedge \mathrm{id} \wedge \eta$ induce the usual homomorphisms

$$E_*(E) \simeq E_*(S) \otimes_{E_*(S)} E_*(E) \xrightarrow{\eta_L \otimes \mathrm{id}} E_*(E) \otimes_{E_*(S)} E_*(E) \text{ and}$$

$$E_*(E) \simeq E_*(E) \otimes_{E_*(S)} E_*(S) \xrightarrow{\mathrm{id} \otimes \eta_R} E_*(E) \otimes_{E_*(S)} E_*(E)$$

but the morphism $\mathrm{id} \wedge \eta \wedge \mathrm{id}$ induces a more interesting homomorphism which we have to give a name, say

$$\Psi : E_*(E) \to E_*(E) \otimes_{E_*(S)} E_*(E)$$

More on this in a moment.

More generally if X is any CW spectrum then we can define a morphism of spectra

$$E \wedge X \simeq E \wedge S \wedge X \xrightarrow{\mathrm{id} \wedge \eta \wedge \mathrm{id}} E \wedge E \wedge X$$

and applying π_* gives a homomorphism

$$E_*(X) = \pi_*(E \wedge X) \to \pi_*(E \wedge E \wedge X) = E_*(E \wedge X) \simeq E_*(E) \otimes_{E_*(S)} E_*(X)$$

which should probably also be given a name.

Remark 1.16. In upcoming applications of this stuff E will be an associative, commutative CW ring spectrum (in fact with a complex orientation), but I have been leaving out some of these hypothesis to clarify the level of generality at which one can discuss cohomology operations. Moreover in what follows one really must pay attention to left and right actions, and using commutativity to identify those actions can lead to major confusion.

1.2.2. *Hopf algebroids.*

Definition 1.17. (Elegant but opaque version)

A **Hopf algebroid** over a commutative ring K is a cogroupoid object in the category of graded (or even bigraded) commutative K-algebras.

This requires some unraveling - to begin let's pin down the definition of a groupoid (or cogroupoid) object in a category.

Recall that a **groupoid** $\mathcal{G}$ is a small category in which every morphism is invertible. Consider the data that comes with $\mathcal{G}$:

- A set of objects, say $\mathrm{Ob}\mathcal{G}$.
- A set of morphisms $\mathrm{Mor}\mathcal{G}$ - here I mean all morphisms $x \xrightarrow{f} y$ between all pairs of objects x, y in $\mathcal{G}$. For the first two bullet points, recall that $\mathcal{G}$ is assumed to be small.
- Two functions $\sigma, \tau : \mathrm{Mor}\mathcal{G} \to \mathrm{Ob}\mathcal{G}$ which take a morphism $x \xrightarrow{f} y$ to it's *source* x and *target* y, respectively: i.e.
$$\sigma(f) = x \text{ and } \tau(f) = y$$
- Composition of morphisms. Given the source and target morphisms σ and τ, one can form the fiber product $\mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} \mathrm{Mor}\mathcal{G}$, which sits in the cartesian diagram of sets
$$\begin{array}{ccc} \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} \mathrm{Mor}\mathcal{G} & \longrightarrow & \mathrm{Mor}\mathcal{G} \\ \downarrow & & \downarrow{\scriptstyle \tau} \\ \mathrm{Mor}\mathcal{G} & \xrightarrow{\sigma} & \mathrm{Ob}\mathcal{G} \end{array} \tag{1.3}$$
Concretely, $\mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} \mathrm{Mor}\mathcal{G}$ is the collection of pairs of morphisms (f, g) so that $\sigma(f) = \tau(g)$, which is to say we're looking at morphisms
$$x \xrightarrow{g} y \text{ and } y \xrightarrow{f} z$$
so it makes sense to form the composition $x \xrightarrow{g} y \xrightarrow{f} z$. Thus composition can be considered as a function
$$\circ : \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} \mathrm{Mor}\mathcal{G} \to \mathrm{Mor}\mathcal{G} \text{ taking } (f, g) \mapsto f \circ g$$

Notice that we have some relations of the form $\sigma(f \circ g) = \sigma(g)$ and $\tau(f \circ g) = \tau(f)$, etc. Also, composition is associative, and this requires that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} \mathrm{Mor}\mathcal{G} & \xrightarrow{\mathrm{id}\times\circ} & \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} \mathrm{Mor}\mathcal{G} \\ {\scriptstyle \circ\times\mathrm{id}}\downarrow & & {\scriptstyle\circ}\downarrow \\ \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} \mathrm{Mor}\mathcal{G} & \xrightarrow{\circ} & \mathrm{Mor}\mathcal{G} \end{array} \tag{1.4}$$

- Identity morphisms. Each object $x \in \mathrm{Ob}\mathcal{G}$ is supposed to come with an identity morphism $\mathrm{id}_x \in \mathrm{Mor}\mathcal{G}$, satisfying $\mathrm{id}_x \circ f = f$ for every morphism $y \xrightarrow{f} x$ and $f \circ \mathrm{id}_x = f$ for every morphism $x \xrightarrow{f} y$. This means we have a function
$$\iota : \mathrm{Ob}\mathcal{G} \to \mathrm{Mor}\mathcal{G} \text{ taking } x \mapsto \mathrm{id}_x$$
satisfying
$$\sigma \circ \iota = \tau \circ \iota = \mathrm{id} : \mathrm{Ob}\mathcal{G} \to \mathrm{Ob}\mathcal{G}$$
and fitting into the commutative diagrams
$$\begin{array}{ccc} \mathrm{Ob}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} \mathrm{Mor}\mathcal{G} & \xrightarrow{\iota\times\mathrm{id}} & \mathrm{Mor}\mathcal{G} \otimes_{\mathrm{Ob}\mathcal{G}} \mathrm{Mor}\mathcal{G} \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\circ}\downarrow \\ \mathrm{Mor}\mathcal{G} & = & \mathrm{Mor}\mathcal{G} \end{array} \quad \text{and} \tag{1.5}$$
$$\begin{array}{ccc} \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} \mathrm{Ob}\mathcal{G} & \xrightarrow{\mathrm{id}\times\iota} & \mathrm{Mor}\mathcal{G} \otimes_{\mathrm{Ob}\mathcal{G}} \mathrm{Mor}\mathcal{G} \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\circ}\downarrow \\ \mathrm{Mor}\mathcal{G} & = & \mathrm{Mor}\mathcal{G} \end{array} \tag{1.6}$$
Here $\mathrm{Ob}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} \mathrm{Mor}\mathcal{G}$ is the fiber product occurring in the cartesian diagram
$$\begin{array}{ccc} \mathrm{Ob}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} \mathrm{Mor}\mathcal{G} & \longrightarrow & \mathrm{Mor}\mathcal{G} \\ \downarrow & & {\scriptstyle\tau}\downarrow \\ \mathrm{Ob}\mathcal{G} & = & \mathrm{Ob}\mathcal{G} \end{array} \tag{1.7}$$
Concretely it consists of pairs (x, f) of an object x and a morphism $y \xrightarrow{f} x$ with target x. The set $\mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} \mathrm{Ob}\mathcal{G}$ has an analogous description.
- Inverses. Each morphism $x \xrightarrow{f} y$ has a unique inverse $y \xrightarrow{f^{-1}} x$. This means we have an involution
$$\wedge^{-1} : \mathrm{Mor}\mathcal{G} \to \mathrm{Mor}\mathcal{G} \text{ taking } f \mapsto f^{-1}$$
(that is, $\wedge^{-1}$ is a bijection with $\wedge^{-1} \circ \wedge^{-1} = \mathrm{id}$) which must satisfy the relations
$$\sigma(f^{-1}) = \tau(f) \text{ and } \tau(f^{-1}) = \sigma(f)$$
and fit into the commutative diagrams
$$\begin{array}{ccc} \mathrm{Mor}\mathcal{G} & \xrightarrow{\mathrm{id}\times\wedge^{-1}} & \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} \mathrm{Mor}\mathcal{G} \\ {\scriptstyle\tau}\downarrow & & {\scriptstyle\circ}\downarrow \\ \mathrm{Ob}\mathcal{G} & \xrightarrow{\iota} & \mathrm{Mor}\mathcal{G} \end{array} \quad \text{and} \tag{1.8}$$

$$
\begin{array}{ccc}
\mathrm{Mor}\mathcal{G} & \xrightarrow{\wedge^{-1}\times \mathrm{id}} & \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} \mathrm{Mor}\mathcal{G} \\
\sigma\Big\downarrow & & \circ\Big\downarrow \\
\mathrm{Ob}\mathcal{G} & \xrightarrow{\iota} & \mathrm{Mor}\mathcal{G}
\end{array} \tag{1.9}
$$

At this point the following definition should seem reasonably well motivated:

Definition 1.18. A **groupoid object** in a category $\mathcal{C}$ consists of a pair of objects M, O in $\mathcal{C}$ (thought of as morphisms and objects) in $\mathcal{C}$ together with morphisms

$$\sigma, \tau : M \to O, \text{ thought of as "source and target"}$$

a morphism

$$\circ : M \times_O M \to M \text{ thought of as composition}$$

a morphism

$$\iota : O \to M \text{ thought of as the identities}$$

and a morphism

$$\wedge^{-1} : M \to M \text{ thought of as inversion}$$

subject to all the relations and satisfying all the commutative diagrams that appear in the above bullet points (I'm not going to copy them all down).

Notice that in this definition fiber products appear in an essential way. So one can only really talk about groupoid objects in categories with fiber products.

Similarly,

Definition 1.19. A **cogroupoid object** in a category $\mathcal{C}$ consists of a pair of objects O, M in $\mathcal{C}$ together with morphisms $\sigma, \tau : O \to M$, a morphism $\circ : M \to M \coprod_O M$, and a morphism $\iota : M \to O$, satisfying relations and fitting into commutative diagrams dual (or "co") to those of a groupoid object (that is, with arrows reversed and fiber products replaced with cofiber products).

Remark 1.20. I think it's more or less equivalent to say that a cogroupoid object in $\mathcal{C}$ consists of a pair of objects M, O in $\mathcal{C}$ together with a groupoid structure on the Yoneda functors $\mathrm{Hom}_{\mathcal{C}}(M, -), \mathrm{Hom}_{\mathcal{C}}(O, -)$.

To see that these are reasonable definitions, one can check that a groupoid object in the category of sets is, well, a groupoid.

Returning to the definition of a Hopf algebroid over a commutative ring K, one sees that it consists of a pair A, Γ of graded commutative K-algebras together with morphisms

$$\eta_L, \eta_R : A \to \Gamma \text{ thought of as left unit/target and right unit/source,}$$

$$\Psi : \Gamma \to \Gamma \otimes_A \Gamma \text{ thought of as composition,}$$

$$\epsilon : \Gamma \to A \text{ thought of as identities, and}$$

$$c : \Gamma \to \Gamma \text{ thought of as inversion.}$$

together with the relations and commutative diagrams discussed above (for a list, see the green book). Here I've changed notation somewhat to match up reasonably well with the green book and COCTALOS. Also, implicit in all of this is the fact that tensor product serves as a cofiber product in the category of graded commutative K-algebras.

Remark 1.21. Notice that applying Spec to the above Hopf algebroid yields a groupoid object in the category of affine schemes over Spec$k$: indeed, Spec reverses all the arrows and turns the cofiber product $\Gamma \otimes_A \Gamma$ into the fiber product $\mathrm{Spec}\Gamma \times_{\mathrm{Spec}A} \mathrm{Spec}\Gamma$ (okay, I guess I'm treating A, Γ as commutative rings, say by restricting attention to the even degree stuff). Question: what extra structure does the grading of Γ and A put on these affine schemes?

In the previous section, we basically observed that

Proposition 1.22. *Let E be an associative, commutative CW ring spectrum such that $E_*(E)$ is flat over $E_*(S)$. Then the pair $E_*(S), E_*(E)$ together with the morphisms*

$$\eta_L, \eta_R : E_*(S) \to E_*(E),$$
$$\Psi : E_*(E) \to E_*(E) \otimes_{E_*(S)} E_*(E),$$
$$\epsilon : E_*(E) \to E_*(S), \text{ and}$$
$$c : E_*(E) \to E_*(E)$$

is a Hopf algebroid over $\mathbb{Z}$.

Remark 1.23. The last homomorphism c is induced by the flip morphism $E \wedge E \to E \wedge E$ interchanging the two factors.

A bit of terminology:

- A Hopf algebroid (A, Γ) as above is said to be connected if the natural maps $A \xrightarrow{\eta_L} A\Gamma_0$ and $A \xrightarrow{\eta_R} \Gamma_0 A$ are both isomorphisms.
  Recall that a groupoid $\mathcal{G}$ is connected if and only if for any two objects $x, y \in \mathrm{Ob}\mathcal{G}$ there is a morphism $x \xrightarrow{f} y$ in $\mathrm{Mor}\mathcal{G}$. Which is to say, the homomorphism $\mathrm{Mor}\mathcal{G} \xrightarrow{\sigma\times\tau} \mathrm{Ob}\mathcal{G} \times \mathrm{Ob}\mathcal{G}$ is surjective. It's not clear how this relates to the above definition (for instance, one might call a Hopf algebroid (A, Γ) as above connected if and only if the morphism of functors $\mathrm{Hom}_K(\Gamma, -) \xrightarrow{\eta_L^* \times \eta_R^*} \mathrm{Hom}_K(A, -) \times \mathrm{Hom}_K(A, -)$ is surjective, or something along those lines).
- Observe that if $\eta_L = \eta_R$, then Γ is a commutative (but not necessarily co-commutative) Hopf algebra over A, that is a cogroup object in the category of commutative A-algebras.
  Recall that a group is simply a groupoid with one object. Again, it's not clear how this relates to the above definition.

Definition 1.24. Let (A, Γ) be a Hopf algebroid over a commutative ring k. A **left Γ-comodule** is a (graded) left A-module M together with a homomorphism of (graded) left A-modules $\psi : M \to \Gamma \otimes_A M$ which is counitary in the sense that

$$M \xrightarrow{\psi} \Gamma \otimes_A M \xrightarrow{\epsilon\otimes\mathrm{id}} A \otimes_A M \simeq M$$

is the identity map of M and coassociative in the sense that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "\psi"'] & \Gamma \otimes_A M \arrow[d, "\mathrm{id}\otimes\psi"] \\ \Gamma \otimes_A M \arrow[r, "\Psi\otimes\mathrm{id}"] & \Gamma \otimes_A \Gamma \otimes_A M \end{tikzcd} \tag{1.10}$$

Similarly, a **left Γ-comodule algebra** is a left Γ-comodule M as above which is also a graded commutative A-algebra and where the structure homomorphism

$$\psi : M \to \Gamma \otimes_A M$$

is in fact a homomorphism of graded A-algebras.

Remark 1.25. Let's define an action of a groupoid object in a category (and dually, a coaction of a cogroupoid object in a category). One can take as a starting point the idea that an action of a groupoid $\mathcal{G}$ on a set should be equivalent to a functor $\rho : \mathcal{G} \to \mathbf{Set}$ (which is what I would call a representation of $\mathcal{G}$ in the category of sets). Each object $x \in \mathrm{Ob}\mathcal{G}$ will go to some set $\rho(x)$, and each morphism $x \xrightarrow{f} y$ will go to a bijection $\rho(x) \xrightarrow{\rho(f)} \rho(y)$. Now define $S := \coprod_{x\in\mathrm{Ob}\mathcal{G}} \rho(x)$. Observe that the maps $\rho(x) \to \{x\}$ induce a projection $\pi : S = \coprod_{x \in \mathrm{Ob}\mathcal{G}} \rho(x) \to \mathrm{Ob}\mathcal{G}$. Using this we can form the fiber product $\mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} S$ fitting into

$$\begin{tikzcd} \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} S \arrow[r] \arrow[d] & S \arrow[d, "\pi"'] \\ \mathrm{Mor}\mathcal{G} \arrow[r, "\sigma"] & \mathrm{Ob}\mathcal{G} \end{tikzcd} \tag{1.11}$$

which consists of pairs (f, s) so that $\sigma(f) = \pi(s)$ - now the functor ρ yields a function

$$\mu : \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} S \text{ taking } (f, s) \mapsto \rho(f)(s)$$

and the axioms for a functor ensure that

$$\begin{tikzcd} \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} S \arrow[r, "\mathrm{id}\times\mu"] \arrow[d, "\circ\times\mathrm{id}"'] & \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} S \arrow[d, "\mu"'] \\ \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} S \arrow[r, "\mu"] & S \end{tikzcd} \tag{1.12}$$

commutes and the composition

$$S \simeq \mathrm{Ob}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} S \xrightarrow{\iota\times\mathrm{id}} \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} S \xrightarrow{\mu} S$$

is the identity on S. Moreover μ is a function over $\mathrm{Ob}\mathcal{G}$ in the sense that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} S \arrow[r, "\mu"] \arrow[d] & S \arrow[d, "\pi"'] \\ \mathrm{Mor}\mathcal{G} \arrow[r, "\tau"] & \mathrm{Ob}\mathcal{G} \end{tikzcd} \tag{1.13}$$

This seems to be enough motivation to say: a **left action of a groupoid $\mathcal{G}$ on a set S** consists of a function $\pi : S \to \mathrm{Ob}\mathcal{G}$ together with a function

$$\mu : \mathrm{Mor}\mathcal{G} \times_{\mathrm{Ob}\mathcal{G}} S \to S$$

subject to the above associativity, identity, and source/target conditions. More generally

Definition 1.26. Let (M, O) be a groupoid object in a category $\mathcal{C}$. A **left action of (M, O) on another object X in $\mathcal{C}$** consists of a morphism $\pi : X \to O$ together with a morphism $\mu : M \times_O X \to X$ such that

$$\begin{tikzcd} M \times_O M \times_O \arrow[r, "\mathrm{id}\times\mu"] \arrow[d, "\circ\times\mathrm{id}"'] & M \times_O X \arrow[d, "\mu"'] \\ M \times_O X \arrow[r, "\mu"] & X \end{tikzcd} \tag{1.14}$$

commutes,

$$X \simeq O \times_O X \xrightarrow{\iota\times\mathrm{id}} M \times_O X \xrightarrow{\mu} X$$

is the identity on X, and

$$\begin{tikzcd} M \times_O X \arrow[r, "\mu"] \arrow[d] & X \arrow[d, "\pi"'] \\ M \arrow[r, "\tau"] & O \end{tikzcd} \tag{1.15}$$

commutes.

There's an analogous definition of a left coaction of a cogroupoid object. Unraveling definitions, one sees that a left coaction of a cogroupoid object in graded commutative k-algebras, i.e. a coaction of a Hopf algebroid (A,Γ) over k on a graded commutative k-algebra M, consists of a k-algebra homomorphism

$$\eta : A \to M \text{ making } M \text{ an } A\text{-algebra}$$

together with a homomorphism of A-algebras (this requires the source/target condition)

$$\psi : M \to \Gamma \otimes_A M$$

satisfying the evident coassociativity and coidentity conditions.

Thus a left coaction of (A,Γ) on a graded commutative k-algebra M is the same thing as a left comodule algebra M over the Hopf algebroid (A,Γ). Note also that applying Spec yields an action of the groupoid object $(\mathrm{Spec}\Gamma, \mathrm{Spec}A)$ on $\mathrm{Spec}M$ in the category of affine schemes over $\mathrm{Spec}k$.

One might ask for a similar description of a left Γ-comodule M - recall that M is a left A-module together with a homomorphism of left A-modules

$$\psi : M \to \Gamma \otimes_A M$$

satisfying coassociativity and coidentity conditions. This corresponds to a sheaf $\tilde{M}$ of $\mathcal{O}_{\mathrm{Spec}A}$-modules on $\mathrm{Spec}A$ together with a morphism of such sheaves $\psi : \tilde{M} \to \eta_{L*}\eta_R^*\tilde{M}$, or equivalently by adjunction $\psi : \eta_L^*\tilde{M} \to \eta_R^*\tilde{M}$. I'm not sure how to interpret coassociativity and coidentity - according to Dan Bragg, we should be looking at some sort of "descent data" describing a sheaf of modules on the quotient stack $\mathrm{Spec}A/\mathrm{Spec}\Gamma$. Note sure what that means.

In the previous section, we basically showed:

Proposition 1.27. *Let E be an associative, commutative CW ring spectrum such that $E_*(E)$ is flat over $E_*(S)$, and let X be another CW spectrum. Then $E_*(X)$ is a left $E_*(E)$-comodule with the usual left $E_*(S)$-modules structure and coaction*

$$\psi : E_*(X) \to E_*(E) \otimes_{E_*(S)} E_*(X)$$

obtained by applying π_* to the morphism of spectra

$$E \wedge X \simeq E \wedge S \wedge X \xrightarrow{\mathrm{id}\wedge\eta\wedge\mathrm{id}} E \wedge E \wedge X$$

If X is an associative, commutative CW ring spectrum then $E_*(X)$ is in fact a left $E_*(E)$-comodule algebra spectrum.

Question 1.28. Can we say more if X is a module spectrum over E?

Before moving on it's worth noting that a left Γ-comodule M is equivalent to the right Γ-comodule which consists of the right A-module M together with the structure homomorphism of right A-modules

$$M \xrightarrow{\psi} \Gamma \otimes_A M \xrightarrow{\mathrm{flip}} M \otimes_A \Gamma \xrightarrow{\mathrm{id}\otimes c} M \otimes_A \Gamma$$

This is analogous to using inversion to replace a left group action, and vice versa.

Definition 1.29. Let M, N be left Γ-comodules. Then their tensor product is the left Γ-comodule consisting of the left A-module $M \otimes_A N$ together with the coaction map

$$M \otimes_A N \xrightarrow{\psi_M \otimes \psi_N} \Gamma \otimes_A M \otimes_A \Gamma \otimes_A N$$
$$\xrightarrow{\mathrm{flip}} \Gamma \otimes_A \Gamma \otimes_A M \otimes_A N \xrightarrow{\mathrm{multiply}} \Gamma \otimes_A M \otimes_A N$$

Remark 1.30. Really, this definition only makes sense in light of the fact that there's an equivalence between left A-modules and right A-modules (since A is graded commutative). This equivalence implicitly enters into all of those tensor products over A.

In the examples of Hopf algebroids coming from stable homotopy theory, Γ is flat over A. It turns out that there's another good reason to include this flatness condition:

Proposition 1.31. *Let (A, Γ) be a Hopf algebroid over a commutative ring k. If Γ is flat over A then the category of left Γ-comodules is abelian.*

Remark 1.32. The homomorphism $\eta_L : A \to \Gamma$ is flat if and only if $\eta_R : A \to \Gamma$ is (the two are related by the automorphism $c : \Gamma \to \Gamma$). So, "Γ is flat over A" means that both of these homomorphisms are flat.

Sketch of a proof. Even if Γ is not necessarily flat over A, the category of left Γ-comodules is additive. In slightly more detail, the homomorphisms $\mathrm{Hom}_\Gamma(M, N)$ between left Γ-comodules M, N form the k-submodule (but *not* necessarily A-submodule!) of $\mathrm{Hom}_A(M, N)$ of maps $\varphi : M \to N$ making

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle\psi_M}\downarrow & & {\scriptstyle\psi_N}\downarrow \\ \Gamma \otimes_A M & \xrightarrow{\mathrm{id}\otimes\varphi} & \Gamma \otimes_A N \end{array} \tag{1.16}$$

commute. For an arbitrary collection $\{M_i \,|\, i \in I\}$ of left Γ-comodules, there's an evident left Γ-comodule structure on the direct sum of A-modules $\bigoplus_i M_i$.

Flatness ensures that kernels and co-kernels exist. See Ravenel.

□

In light of this proposition and the fact that we'd like to consider the homological algebra of Γ-comodules, it will be assumed from here on out that Γ is flat over A.

Definition 1.33. Let M be a right Γ-comodule and let N be a left Γ-comodule. Then the **cotensor product of** M **and** N is the k-module $M\square_\Gamma N$ appearing in the exact sequence

$$0 \to M\square_\Gamma N \to M \otimes_A N \xrightarrow{\psi_M\otimes\mathrm{id}-\mathrm{id}\otimes_A\psi_N} M \otimes_A \Gamma \otimes_A N$$

Remark 1.34. The reason that $M\square_\Gamma N$ is only a k-module is that the homomorphism $\psi_M \otimes \mathrm{id} - \mathrm{id} \otimes_A \psi_N$ is not even A-linear, since ψ_M is A-linear with respect to η_R while ψ_N is A-linear with respect to η_L.

Proposition 1.35. *There's a canonical isomorphism $N\square_\Gamma M \simeq M\square_\Gamma N$.*

Here in order to form $N\square_\Gamma M$ one must first convert N to a right Γ-comodule and M to a left Γ-comodule. I'm omitting the proof.

Proposition 1.36. *Let M be a left Γ-comodule which is finitely generated and projective over A, and let N be any left Γ-comodule. Then there's a canonical right Γ-comodule structure on $\mathrm{Hom}_A(M, A)$ and a natural isomorphism $\mathrm{Hom}_\Gamma(M, N) \simeq \mathrm{Hom}_A(M, A)\square_\Gamma N$.*

Sketch of proof. First observe that if $\psi_M : M \to M \otimes_A \Gamma$ is the coaction of Γ on M, we can turn a homomorphism $\varphi : M \to A$ into a homomorphism

$$\psi_M : M \to M \otimes_A \Gamma \xrightarrow{\varphi\otimes\mathrm{id}} A \otimes_A \simeq \Gamma \simeq \Gamma$$

which we may as well call $\psi_M^*\varphi$ - this yields a homomorphism $\psi_M^* : \mathrm{Hom}_A(M, A) \to \mathrm{Hom}_A(M, \Gamma)$. Since M is a finitely generated projective A-module, the natural map

$$\mathrm{Hom}_A(M, A) \otimes_A \Gamma \to \mathrm{Hom}_A(M, \Gamma)$$

is an isomorphism, and we can view ψ_M^* as a homomorphism of right A-modules

$$\mathrm{Hom}_A(M, A) \to \mathrm{Hom}_A(M, A) \otimes_A \Gamma$$

One must now check that this serves as a coaction map for a right Γ-module structure on $\mathrm{Hom}_A(M, A)$.

Now for the fun part: let's define a homomorphism of k-modules

$$\mathrm{Hom}_\Gamma(M, N) \to \mathrm{Hom}_A(M, A)\Box_\Gamma N$$

as follows. Since a homomorphism of left Γ-comodules $\varphi : M \to N$ is in particular a homomorphism of left A-modules, it can be viewed as an element

$$\varphi \in \mathrm{Hom}_A(M, N) \simeq \mathrm{Hom}_A(M, A) \otimes_A N$$

where I'm again using the fact that M is finitely generated and projective over A. Now the fact that

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \psi_M \big\downarrow & & \psi_N \big\downarrow \\ \Gamma \otimes_A M & \xrightarrow{\mathrm{id}\otimes\varphi} & \Gamma \otimes_A N \end{array} \tag{1.17}$$

commutes says precisely that $\varphi \in \mathrm{Hom}_A(M, A) \otimes_A N$ lies in the kernel of

$$\mathrm{Hom}_A(M, A) \otimes_A N \xrightarrow{\psi_M^*\otimes\mathrm{id}-\mathrm{id}\otimes\psi_N} \mathrm{Hom}_A(M, A) \otimes_A \Gamma \otimes_A N$$

In this way we obtain the desired homomorphism $\mathrm{Hom}_\Gamma(M, N) \to \mathrm{Hom}_A(M, A)\Box_\Gamma N$. It's basically definition-unraveling to show that this is an isomorphism: indeed we have

$$\mathrm{Hom}_\Gamma(M, N) \subset \mathrm{Hom}_A(M, N) \text{ and } \mathrm{Hom}_A(M, A)\Box_\Gamma N \subset \mathrm{Hom}_A(M, A) \otimes_A N$$

and the natural isomorphism $\mathrm{Hom}_A(M, A) \otimes_A N \to \mathrm{Hom}_A(M, A)$ identifies the two subgroups. $\square$

Definition 1.37. A morphism $f : (A, \Gamma) \to (B, \Sigma)$ of Hopf algebroids over a (graded) commutative ring k is a morphism of cogroupoid objects in the category of graded commutative k-algebras - which is to say, it's a pair of homomorphisms of k-algebras

$$f : A \to B \text{ and } \tilde{f} : \Gamma \to \Sigma$$

so that

$$\tilde{f}\eta_L = \eta_L f, \quad \tilde{f}\eta_R = \eta_R f, \quad f\epsilon = \epsilon\tilde{f},$$
$$\Delta\tilde{f} = (\tilde{f} \otimes \tilde{f})\Delta \text{ and } \tilde{f}c = c\tilde{f}$$

All of these identities are dual to the identities characterizing a functor between two groupoids (and that's the way to remember them!).

1.2.3. *Homological algebra of comodules over Hopf algebroids.* Let (A, Γ) be a Hopf algebroid over a commutative ring k - let's continue to assume Γ is flat over A, to ensure that the category of left Γ-comodules is abelian. In that case it makes sense to talk about the homological algebra of those comodules.

There is an evident exact forgetful functor res : Γ-**comod** $\to A$-**mod**. On the other hand, if N is a left A-module, then $\Gamma \otimes_A N$ is a left A-module together with a coaction homomorphism

$$\Gamma \otimes_A N \xrightarrow{\Psi\otimes\mathrm{id}} \Gamma \otimes_A \Gamma \otimes_A N$$

i.e. a left Γ-comodule. This provides us with an additive functor ind : A-**mod** $\to \Gamma$-**comod**. In fact

Proposition 1.38. *Let M be a left Γ-comodule and let N be a left A-module. Then there is a natural isomorphism*

$$\mathrm{Hom}_A(\mathrm{res}M, N) \simeq \mathrm{Hom}_\Gamma(M, \mathrm{ind}N)$$

Which is to say, ind *is the* right *adjoint to* res.

Sketch of proof. Let $\varphi : \mathrm{res}M \to N$ be a homomorphism of left A-modules. Then we may use it to define a homomorphism

$$M \xrightarrow{\psi} \Gamma \otimes_A M \xrightarrow{\mathrm{id}\otimes\varphi} \Gamma \otimes_A N = \mathrm{ind}N$$

and it's not hard to see that this is a map of left Γ-comodules. On the other hand, given a homomorphism of left Γ-comodules $\varphi : M \to \mathrm{ind}N$, one can form the homomorphism

$$M \xrightarrow{\varphi} \mathrm{ind}N = \Gamma \otimes_A N \xrightarrow{\epsilon\otimes\mathrm{id}} A \otimes_A N = N$$

At this point one must check that the maps $\mathrm{Hom}_A(\mathrm{res}M, N) \to \mathrm{Hom}_\Gamma(M, \mathrm{ind}N)$ and $\mathrm{Hom}_\Gamma(M, \mathrm{ind}N) \to \mathrm{Hom}_A(\mathrm{res}M, N)$ defined above are mutual inverses.

□

Corollary 1.39. *The category of left* Γ*-comodules has enough injectives.*

Proof. From the above proposition, ind : A-**mod** $\to$ Γ-**comod** is right adjoint the the exact restriction functor, and so it preserves injectives. Which is to say, if N is an injective A-module, then ind$N$ is an injective Γ-comodule. Now suppose M is a left Γ-comodule. Since the category of left A-modules has enough injectives, we can find an injection $\iota : \mathrm{res}M \to I$, where I is injective. By res- ind adjunction, this corresponds to a homomorphism of left Γ-comodules

$$\mathrm{M} \xrightarrow{\psi} \Gamma \otimes_A M \xrightarrow{\mathrm{id}\otimes\iota} \Gamma \otimes_A I = \mathrm{ind}I$$

which is in fact injective - indeed, the coidentity condition shows $M \xrightarrow{\psi} \Gamma \otimes_A M$ is injective and flatness of Γ over A guarantees that $\Gamma \otimes_A -$ is exact, so that $\Gamma \otimes_A M \xrightarrow{\mathrm{id}\otimes\iota} \Gamma \otimes_A I$ is injective. Thus we've obtained an embedding $M \to \mathrm{ind}I$ of M in an injective left Γ-comodule.

□

Since ■-**comod** has enough injectives, we can safely conclude that any left exact covariant functor on ■-**comod** has right derived functors. In particular, observe that for a fixed left Γ-comodule M,

$$\mathrm{Hom}_\Gamma(M, -) : \Gamma\text{-}\mathbf{comod} \to k\text{-}\mathbf{mod}$$

is left exact (after all, we're talking about Hom in an abelian category), and so we can define it's derived functors

$$\mathrm{Ext}^i_\Gamma(M, -) := R^i\mathrm{Hom}_\Gamma(M, -)$$

Remark 1.40. In our applications (Adams spectral sequences) we talk about bigraded Ext groups. They're actually present above - indeed, Γ-comodules are in particular *graded* A-modules, and Γ-comodule homomorphisms are in particular homomorphisms of graded A-modules, so $\mathrm{Hom}_\Gamma(M, -)$ car really be viewed as a left exact functor to *graded* k-modules. Here a degree d homomorphism of Γ-comodules $\varphi : M \to N$, i.e. an element $\varphi \in \mathrm{Hom}^d_\Gamma(M, N)$ consists of homomorphisms $M_{i+d} \to N_i$ (*decreasing* degree by d - trying to stick with the grading conventions from Adams's blue book). In that case the derived functors $\mathrm{Ext}^i_\Gamma(M, -)$ can also be considered as functors to graded k-modules. By convention $\mathrm{Ext}^{s,t}_\Gamma(M, N)$ denotes the tth part of the graded k-module $\mathrm{Ext}^s_\Gamma(M, N)$. So, s is the (co)homological index, t is the grading index.

Similarly, suppose M is a right Γ-comodule which is flat as a right A-module - then the cotensor product with M defines a left exact functor

$$M\Box_\Gamma - : \Gamma\text{-}\mathbf{comod} \to k\text{-}\mathbf{mod}$$

(again, we can consider it as a functor to graded k-modules). To see this, let $0 \to N' \xrightarrow{i} N \xrightarrow{j} N'' \to 0$ be a short exact sequence of left Γ-comodules. Since both Γ and M are assumed to be flat over A, we obtain a chain map of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M \otimes_A N' & \longrightarrow & M \otimes_A N & \longrightarrow & M \otimes_A N'' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M \otimes_A \Gamma \otimes_A N' & \longrightarrow & M \otimes_A \Gamma \otimes_A N & \longrightarrow & M \otimes_A \Gamma \otimes_A N'' & \longrightarrow & 0
\end{array} \tag{1.18}
$$

and now the snake lemma provides a left exact sequence

$$0 \to M \square_\Gamma N' \xrightarrow{\mathrm{id} \square_\Gamma i} M \square_\Gamma N \xrightarrow{\mathrm{id} \square_\Gamma j} M \square_\Gamma N'' \tag{1.19}$$

So, we may define the right derived functors

$$\mathrm{Cotor}_\Gamma^i(M, -) := R^i M \square_\Gamma -$$

Again they can be viewed as functors to graded k-modules.

Recall that when M is a left Γ-comodule which is finitely generated and projective as a left A-module, then for any left Γ-comodule N there's a natural isomorphism

$$\mathrm{Hom}_\Gamma(M, N) \simeq \mathrm{Hom}_A(M, A) \square_\Gamma N$$

Note that in this situation $\mathrm{Hom}_A(M, A)$ is flat over A, and so the derived functors $\mathrm{Cotor}_\Gamma^i(\mathrm{Hom}_A(M, A), -)$ are defined. Moreover we'll obtain a natural isomorphism of derived functors

$$\mathrm{Ext}_\Gamma^i(M, -) \simeq \mathrm{Cotor}_\Gamma^i(\mathrm{Hom}_A(M, A), -)$$

For this reason the Ext_Γ and Cotor_Γ are more or less interchangeable when the left argument (M, in the above) is finitely generated and projective over A.

Remark 1.41. I'm being careful to not make any appeal to something like "finitely generated projective $\iff$ finitely generated flat $\iff$ finitely generated locally free" over A, which holds only when A is noetherian, because many of the rings A which arise in applications fail to be noetherian. For example, A might be the dual Steenrod algebra $\mathcal{A}^*$ or the Lazard ring L.

Recall that for left Γ-comodules the groups $\mathrm{Ext}_\Gamma^i(M, N)$ can be computed using a resolution

$$0 \to N \to J^0 \to J^1 \to J^2 \to \cdots$$

by left Γ-comodules J^i which are merely $\mathrm{Hom}_\Gamma(M, -)$-acyclic, i.e. satisfy

$$\mathrm{Ext}_\Gamma^i(M, J^i) = 0 \text{ for } i > 0$$

(that is, the J^i don't need to be injective). Similarly the groups $\mathrm{Cotor}_\Gamma^i(M, N)$ can be computed using a $M \square_\Gamma -$ acyclic resolution of N. We're going to define an important class of left Γ-comodules which are $\mathrm{Hom}_\Gamma(M, -)$ and $M \square_\Gamma -$ when M is (finitely generated?) projective **(FIGURE THIS OUT)**.

Definition 1.42. An **extended left Γ-comodule** is a left Γ-comodule M isomorphic to an induced comodule of the form $\mathrm{ind} N = \Gamma \otimes_A N$ for some left A-module N. A **relatively injective left Γ-comodule** is a left Γ-comodule occurring as a direct summand of an extended comodule.

Proposition 1.43. *A left Γ-comodule S is relatively injective if and only if whenever $\iota : M \to N$ is an injective homomorphism of left Γ-comodules which is split as a homomorphism of left A-modules, and $\varphi : M \to S$ is any homomorphism of left Γ-comodules, there is an extension of φ to a homomorphism of left Γ-comodules $\tilde{\varphi} : N \to S$. Moreover if M is a left Γ-comodule which is (**finitely generated?**) projective over A, then any relatively injective left Γ-comodule S is $\mathrm{Hom}_\Gamma(M, -)$ and $M \square_\Gamma -$ acyclic.*

Sketch of proof. Let $\pi : N \to M$ be a homomorphism of left A-modules splitting ι. Then we can define a homomorphism of left A-modules $N \xrightarrow{\pi} M \xrightarrow{\varphi} S$ which under the adjunction $\mathrm{Hom}_A(\mathrm{res}N, \mathrm{res}S) \simeq \mathrm{Hom}_\Gamma(N, \mathrm{indres}S)$ is equivalent to a homomorphism of left Γ-comodules $\mathrm{ind}(\varphi \circ \pi) : N \to \mathrm{indres}S = \Gamma \otimes_A S$. The claim to make is that the composition

$$N \xrightarrow{\mathrm{ind}(\varphi\circ\pi)} \Gamma \otimes_A S \xrightarrow{\epsilon\otimes\mathrm{id}} A \otimes_A S = S$$

is an extension $\tilde{\varphi}$ of φ.

On the other hand suppose S has the property that whenever $\varphi : M \to N$ is an injection of left Γ-comodules split over A, every homomorphism of left Γ-comodules $\varphi : M \to S$ extends to a homomorphism $\tilde{\varphi} : N \to S$. I claim that S is a summand of the extended comodule $\mathrm{indres}S = \Gamma \otimes_A S$, and hence a relative injective comodule. Indeed, we have a homomorphism of left Γ-comodules

$$S \xrightarrow{\psi} \Gamma \otimes_A S = \mathrm{indres}S$$

which is injective by the coidentity condition - in fact it's a *split* injection of left A-modules. By hypothesis, the identity map $S \xrightarrow{\mathrm{id}} S$ extends to a homomorphism of left Γ-comodules $\Gamma \otimes_A S \to S$ splitting ψ.

Now suppose M is a left Γ-comodule which is projective as a left A-module, and let N be any left A-module. If

$$0 \to N \to I^0 \to I^1 \to I^2 \to \cdots$$

is an injective resolution of N, then tensoring with Γ yields an injective resolution

$$0 \to \Gamma \otimes_A N \to \Gamma \otimes_A I^0 \to \Gamma \otimes_A I^1 \to \Gamma \otimes_A I^2 \to \cdots$$

of $\mathrm{ind}N = \Gamma \otimes_A N$ (since ind preserves injectives, and Γ is flat over A so the sequence remains exact). Then applying $\mathrm{Hom}_\Gamma(M, -)$ yields a complex

$$\mathrm{Hom}_\Gamma(M, \Gamma \otimes_A I^0) \to \mathrm{Hom}_\Gamma(M, \Gamma \otimes_A I^1) \to \mathrm{Hom}_\Gamma(M, \Gamma \otimes_A I^2) \to \cdots$$

whose cohomology is $\mathrm{Ext}^*_\Gamma(M, \Gamma \otimes_A N)$. On the other hand, res-ind adjunction gives natural isomorphisms

$$\mathrm{Hom}_\Gamma(M, \Gamma \otimes_A I^i) \simeq \mathrm{Hom}_A(\mathrm{res}M, I^i) \text{ for all } i$$

and so the cohomology of this complex is also $\mathrm{Ext}^*_A(\mathrm{res}M, N)$, which vanishes in positive degrees since M is projective. Thus

$$\mathrm{Ext}^i_\Gamma(M, \Gamma \otimes_A N) = 0 \text{ for } i > 0$$

and we've shown that every extended comodule is $\mathrm{Hom}_\Gamma(M, -)$ acyclic. From here it's not hard to show that every relatively injective left Γ-comodule is $\mathrm{Hom}_\Gamma(M, -)$ acyclic, since it occurs as a summand of an extended comodule.

The proof of the analogous statement for $\mathrm{Cotor}_\Gamma(M, -)$ should be similar, but I'm a little confused about whether or not it requires M to be finitely generated and projective over A. Ravenel seems to be implicitly assuming finite generation at various times.

□

Definition 1.44. A **resolution by relative injectives of a left Γ-comodule** M is an exact sequence of left Γ-comodules

$$0 \to M \to R^0 \to R^1 \to R^2 \to \cdots$$

that is split exact over A in which each R^i is relatively injective.

Proposition 1.45. *Let M be a left Γ-comodule. Then there exists a resolution of M by relative injectives.*

In fact the proof will show that such a resolution can be constructed *functorially*. The construction is obviously motivated by the bar resolution of group cohomology. **NOTE: EVEN AFTER A LOT OF SCRUTINY I THINK THERE ARE STILL INDEXING ISSUES BELOW. AGH! APOLOGIES IN ADVANCE.**

Proof plus lots of digressions. Let $\mathcal{G}$ be a groupoid, or even a groupoid object in some category $\mathcal{C}$ with morphisms M and objects O. Once can define a simplicial object $B\mathcal{G}$ (hopefully this isn't too far from the notation people actually use) in $\mathcal{C}$ as follows: the idea is that $B\mathcal{G}([n])$ consists of ordered sequences

$$x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} x_2 \xrightarrow{g_3} \cdots \xrightarrow{g_n} x_n$$

of n composable morphisms in $\mathcal{G}$. To make this precise, define

$$B\mathcal{G}([n]) := M \times_O M \times_O \cdots \times_O M \text{ (n factors of } M)$$

(when $n = 0$ we'll have $B\mathcal{G}([0]) = O$). If $\delta^i : [n-1] \to [n]$ is the inclusion skipping $i \in [n]$, define

$$d_i := B\mathcal{G}(\delta^i) : B\mathcal{G}([n]) \to B\mathcal{G}([n-1])$$

to be the map

$$M \times_O M \times_O \cdots \times_O M \xrightarrow{\prod_{j=1}^{i-2} \mathrm{id} \times \circ \times \prod_{j=i+1}^{n} \mathrm{id}} M \times_O M \times_O \cdots \times_O M$$

If $\mathcal{G}$ is an actual groupoid, this will take a sequence of morphisms

$$x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} x_n$$

$$\text{to } x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{i-1}} x_{i-1} \xrightarrow{g_{i+1} \circ g_i} x_{i+1} \xrightarrow{g_{i+2}} \cdots \xrightarrow{g_n} x_n$$

If $\sigma^i : [n+1] \to [n]$ is the surjection sending $i, i+1$ to i, define

$$s_i := B\mathcal{G}(\sigma^i) : B\mathcal{G}([n]) \to B\mathcal{G}([n+1])$$

to be the map

$$M \times_O M \times_O \cdots \times_O M \xrightarrow{\prod_{j=1}^{i} \mathrm{id} \times \iota \times \prod_{j=i+1}^{n} \mathrm{id}} M \times_O M \times_O \cdots \times_O M$$

If $\mathcal{G}$ is a groupoid, this will take the sequence

$$x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} x_n$$

$$\text{to } x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} \cdots \xrightarrow{g_i} x_i \xrightarrow{\mathrm{id}} x_i \xrightarrow{g_{i+1}} \cdots \xrightarrow{g_n} x_n$$

Using the defining axioms of a groupoid object (really just the axioms that say $\mathcal{G}$ is a category) one checks that $B\mathcal{G}$ is in fact a simplicial object in $\mathcal{C}$. Of course, when $\mathcal{G}$ is a groupoid, $B\mathcal{G}$ is just the simplicial set usually referred to as the nerve of the category $\mathcal{G}$ - its geometric realization is often referred to as the classifying space of the category $\mathcal{G}$.

It should be noted that each $B\mathcal{G}([n])$ comes with a $\mathcal{G}$ action, given by the morphism

$$M \times_O B\mathcal{G}([n]) = M \times_O (M \times_O M \times_O \cdots \times_O M)$$

$$= (M \times_O M) \times_O M \times_O \cdots \times_O M \xrightarrow{\circ \times \mathrm{id} \times \cdots \times \mathrm{id}} M \times_O \cdots \times_O M = B\mathcal{G}([n])$$

and the morphisms d_i, s_i are equivariant. Thus $B\mathcal{G}$ is a simplicial object in the category of "$\mathcal{G}$-objects in $\mathcal{C}$."

Similarly, beginning with a cogroupoid object $\mathcal{G}$ in a category $\mathcal{C}$ and reversing arrows/replacing fiber products with cofiber products in the above construction, one obtains a *cosimplicial* object in the category of "$\mathcal{G}$-objects in $\mathcal{C}$." In the case of a Hopf algebroid (A, Γ) over a commutative ring

k, we obtain a cosimplicial object $B\Gamma$ (if you'll permit this suggestive notation) in the category of Γ-comodule algebras, with

$$B\Gamma([n]) = \Gamma \otimes_A \Gamma \otimes_A \cdots \otimes_A \Gamma \text{ (n factors of } \Gamma)$$

(here $B\Gamma([0]) = A$) with face maps $\delta^i : B\Gamma([n]) \to B\Gamma([n+1])$ taking

$$\gamma_1 \otimes \cdots \otimes \gamma_n \mapsto \gamma_1 \otimes \cdots \gamma_{i-1} \otimes \Psi(\gamma_i) \otimes \gamma_{i+1} \otimes \cdots \otimes \gamma_n$$

and degeneracy maps $\sigma^i : B\Gamma([n]) \to B\Gamma([n-1])$ taking

$$\gamma_1 \otimes \cdots \otimes \gamma_n \mapsto \epsilon(\gamma_i)\gamma_1 \otimes \cdots \otimes \gamma_{i-1} \otimes \gamma_{i+1} \otimes \cdots \otimes \gamma_n$$

The coaction map $\psi : B\Gamma([n]) \to \Gamma \otimes_A B\Gamma([n])$ sends

$$\gamma_1 \otimes \cdots \otimes \gamma_n \mapsto \Psi(\gamma_1) \otimes \gamma_2 \otimes \cdots \otimes \gamma_n$$

Which is to say, as left Γ-comodules,

$$B\Gamma([n]) \simeq \text{indres} B\Gamma([n-1]) = \Gamma \otimes_A B\Gamma([n-1])$$

and so each $B\Gamma([n])$ is an extended comodule.

From this point the usual Dold-Kan correspondence can be used to convert the cosimplicial left Γ-comodule $B\Gamma$ into a cochain complex of left Γ-comodules, say $B\Gamma^*$, of the form

$$0 \to A \to \Gamma \to \Gamma \otimes_A \Gamma \to \Gamma \otimes_A \Gamma \otimes_A \Gamma \to \cdots$$

My first claim is that this provides a resolution of A by relative injectives. We saw above that each $B\Gamma([n])$ is an extended comodule, hence a relative injective comodule, so it will suffice to show that this cochain complex is split exact over A. In fact one can use the degeneracy maps $\sigma_0 : B\Gamma([n]) \to B\Gamma([n-1])$ to define a nullhomotopy of the complex $B\Gamma^*$. See Ravenel.

Now let M be any left Γ-comodule. Then one can simply tensor the complex $B\Gamma^*$ with M to obtain a resolution

$$0 \to M \to \Gamma \otimes_A M \to \Gamma \otimes_A \Gamma \otimes_A M \to \cdots$$

of M by relative injectives (clearly this is still a complex of relative injectives, and it remains exact since $B\Gamma^*$ is *split* exact over A) (I'll probably call this resolution $B\Gamma^* \otimes_A M$ or something). If $\varphi : M \to N$ is a homomorphism of left Γ-comodules, it's clear that tensoring $B\Gamma^*$ with φ yields a chain map $\varphi_* : B\Gamma^* \otimes_A M \to B\Gamma^* \otimes_A N$. Thus the resolution $B\Gamma^* \otimes_A M$ is functorial in M.

□

Remark 1.46. Often people like to replace a simplicial set with its non-degenerate simplices, a simplicial abelian group with its non-degenerate elements, etc. In the case of the simplicial left Γ-comodule algebra $B\Gamma$, I think this amounts to replacing $B\Gamma([n]) = \Gamma \otimes_A \cdots \otimes_A \Gamma$ with $\bar{B}\Gamma([n]) := \Gamma \otimes_A \bar{\Gamma} \otimes_A \cdots \otimes_A \bar{\Gamma}$, where $\bar{\Gamma}$ is defined to be the kernel of the augmentation $\epsilon : \Gamma \to A$ (it's a homogeneous ideal in Γ). People seem to refer to $B\Gamma^*$ as a bar resolution and $\bar{B}\Gamma^*$ as a normalized bar resolution.

The following is a relative version of the "fundamental theorem of homological algebra."

Proposition 1.47. *Let M, N be left Γ-comodules and let*

$$0 \to M \to R^0 \to R^1 \to R^2 \to \cdots \text{ and}$$

$$0 \to N \to S^0 \to S^1 \to S^2 \to \cdots$$

be resolutions by relative injectives. Given a homomorphism of left Γ-comodules $\varphi : M \to N$ there exists a chain map $\tilde{\varphi} : R^* \to S^*$ extending φ, and moreover $\tilde{\varphi}$ is unique up to chain homotopy.

See Ravenel for a proof.

1.2.4. *Generalized Adams spectral sequences.* Let E be a commutative CW ring spectrum, and let X be another CW spectrum.

Definition 1.48. An E**-Adams tower for** X is a diagram of the form

$$\begin{tikzcd} X = X_0 \arrow[d, "f_0"'] & X_1 \arrow[l, "g_0"'] \arrow[d, "f_1"'] & X_2 \arrow[l, "g_1"'] \arrow[d, "f_2"'] & \cdots \arrow[l, "g_2"'] \\ K_0 & K_1 & K_2 & \end{tikzcd} \tag{1.20}$$

where

- Each X_{s+1} is the fiber of $f : X_s \to K_s$,
- Each $E \wedge X_s$ is a retract of $E \wedge K_s$ - more precisely, there's a left inverse $\sigma : E \wedge K_s \to E \wedge X_s$ of the morphism $\mathrm{id} \wedge f : E \wedge X_s \to E \wedge K_s$. In particular this implies that the induced homomorphism $f_{s*} : E_*(X_s) \to E_*(K_s)$ is a split injection.
- Each K_s is a retract of $E \wedge K_s$ - more precisely there's a left inverse of the usual morphism $K_s \simeq S \wedge K_s \to E \wedge K_s$. In particular this implies that $E_*(K_s) \to E_*(E \wedge K_s)$ is a split injection.

Proposition 1.49. *There exists an E-Adams tower for X.*

In fact the proof will show that such a resolution can be constructed functorialy. Compare this with the bar resolution of a comodule over a Hopf algebroid described above.

Proof. First one may construct an E-Adams tower (X_s) for the sphere spectrum S as follows: set $X_0 := S$, $K_0 := E$ and

$$f_0 := \eta : S \to E \text{ the structure map of } E$$

Then let X_1 be the fiber of η - people usually write $\bar{E}$ for the cofiber of η, fitting into the cofibration sequence $S\eta E \to \bar{E}$, so we'll have $X_1 := \Sigma^{-1}\bar{E}$. Set $K_1 := E \wedge X_1 = E \wedge \Sigma^{-1}\bar{E}$ and define

$$f_1 := \eta \wedge \mathrm{id} : X_1 = \Sigma^{-1}\bar{E} \simeq S \wedge \Sigma^{-1}\bar{E} \to E \wedge \Sigma^{-1}\bar{E} = K_1$$

To identify the fiber of this morphism, recall that the smash product $- \wedge \Sigma^{-1}\bar{E}$ with $\Sigma^{-1}\bar{E}$ preserves (co)fibration sequences. Since $\Sigma^{-1}\bar{E} \xrightarrow{g_0} S \xrightarrow{\eta} E$ is a fibration sequence (basically by the definition of $\Sigma^{-1}\bar{E}$ so is

$$\Sigma^{-1}\bar{E} \wedge \Sigma^{-1}\bar{E} \xrightarrow{g_0 \wedge \mathrm{id}} \Sigma^{-1}\bar{E} \xrightarrow{\eta \wedge \mathrm{id}} E \wedge \Sigma^{-1}\bar{E} \text{ that is,}$$

$$\Sigma^{-1}\bar{E} \wedge \Sigma^{-1}\bar{E} \to X_1 \xrightarrow{f_1} K_1$$

So, $X_2 := \Sigma^{-1}\bar{E} \wedge \Sigma^{-1}\bar{E}$ is the fiber of f_1. Proceeding in this way, one obtains a diagram of the form

$$\begin{tikzcd} S \arrow[d, "f_0"'] & \Sigma^{-1}\bar{E} \arrow[l, "g_0"'] \arrow[d, "f_1"'] & \Sigma^{-1}\bar{E} \wedge \Sigma^{-1}\bar{E} \arrow[l, "g_1"'] \arrow[d, "f_2"'] & \Sigma^{-1}\bar{E} \wedge \Sigma^{-1}\bar{E} \wedge \Sigma^{-1}\bar{E} \arrow[l, "g_2"'] \arrow[d, "f_3"'] & \cdots \arrow[l, "g_3"'] \\ E & E \wedge \Sigma^{-1}\bar{E} & E \wedge \Sigma^{-1}\bar{E} \wedge \Sigma^{-1}\bar{E} & E \wedge \Sigma^{-1}\bar{E} \wedge \Sigma^{-1}\bar{E} \wedge \Sigma^{-1}\bar{E} & \end{tikzcd} \tag{1.21}$$

By construction each $X_{s+1} = \Sigma^{-s-1}\bar{E}^{\wedge s+1}$ is the fiber of $f_s : X_s = \Sigma^{-s}\bar{E}^{\wedge s} \to E \wedge \Sigma^{-s}\bar{E}^{\wedge s} = K_s$. Recall that the composition

$$E \simeq E \wedge S \xrightarrow{\mathrm{id} \wedge \eta} E \wedge E \xrightarrow{\mu} E$$

is the identity on E, and so

$$E \wedge \Sigma^{-s}\bar{E}^{\wedge s} \simeq E \wedge S \wedge \Sigma^{-s}\bar{E}^{\wedge s} \xrightarrow{\mathrm{id} \wedge \eta \wedge \mathrm{id}} E \wedge E \wedge \Sigma^{-s}\bar{E}^{\wedge s} \xrightarrow{\mu \wedge \mathrm{id}} E \wedge \Sigma^{-s}\bar{E}^{\wedge s}$$

is the identity on $E \wedge X_s = E \wedge \Sigma^{-s}\bar{E}^{\wedge s}$ - moreover the above sequence of maps can be rewritten as

$$E \wedge X_s \xrightarrow{\mathrm{id} \wedge f_s} E \wedge K_s \xrightarrow{r_s} E \wedge X_s$$

where $r_s := \mu \wedge \mathrm{id}$. This shows that $E \wedge X_s$ is a retract of K_s. Actually, since $K_s = E \wedge X_s$, it also shows that K_s is a retract of $E \wedge K_s$! Thus we've constructed an E-Adams tower for S.

At this point it's straightforward to show that if X is any other CW spectrum, smashing the above tower with X yields an E-Adams tower for X.

$\square$

Remark 1.50. Let's take another look at the above towers. To begin with say $X = S$ - the sequence of maps

$$K_0 \to \Sigma K_1 \to \Sigma^2 K_2 \to \cdots$$

obtained by splicing together the cofibration sequences

$$X_{s+1} \xrightarrow{g_s} X_s \xrightarrow{f_s} K_s \xrightarrow{\partial} \Sigma X_{s+1} \xrightarrow{\Sigma g_s} \Sigma X_s \xrightarrow{\Sigma f_s} \Sigma K_s \to \cdots$$

can be written as

$$E \to E \wedge \bar{E} \to E \wedge \bar{E} \wedge \bar{E} \to \cdots$$

which is starting to look a lot like the bar resolutions of Hopf algebroids constructed above - more precisely it would correspond to a normalized bar resolution. In fact a slightly different E-Adams tower for S would result in the "bar resolution" described below. See Hopkins.

The first observation is that $(E, E \wedge E)$ is a cogroupoid object in the category of commutative CW ring spectra. Indeed, we have morphisms

$$E \simeq S \wedge E \xrightarrow{\eta \wedge \mathrm{id}} E \wedge E \text{ and}$$

$$E \simeq E \wedge S \xrightarrow{\mathrm{id} \wedge \eta} E \wedge E$$

say η_L and η_R respectively, serving as the co-source and co-target morphisms, along with a morphism

$$E \wedge E \simeq E \wedge S \wedge E \xrightarrow{\mathrm{id} \wedge \eta \wedge \mathrm{id}} E \wedge E \wedge E$$

serving as the "co-composition" (it's not totally obvious that $E \wedge E \wedge E$ is the cofiber product $(E \wedge E) \wedge_E (E \wedge E)$ - **THINK ABOUT THIS!**). The multiplication $\mu : E \wedge E \to E$ serves as a "co-inclusion-of-identities" and the flip morphism $E \wedge E \xrightarrow{\text{flip}} E \wedge E$ serves as a "co-inversion." One must now check that these morphisms satisfy all the requisite commutativity conditions.

Now as discussed in the previous section, we obtain a co-simplicial object B^* in the category of commutative CW ring spectra with $E \wedge E$ co-actions. It's n-th term is the spectrum

$$B([n]) = (E \wedge E) \wedge_E \cdots \wedge_E (E \wedge E) \text{ (n factors of } E \wedge E)$$

$$\simeq E \wedge E \wedge \cdots \wedge E \text{ (n+1 factors of } E)$$

In particular $B([0]) = E$. As the category of CW spectra is at least additive, we can apply the "alternating sum construction" to obtain a sequence of morphisms

$$B([0]) \xrightarrow{\delta} B([1]) \xrightarrow{\delta} B([2]) \xrightarrow{\delta} \cdots \text{ i.e.}$$

$$E \xrightarrow{\delta} E \wedge E \xrightarrow{\delta} E \wedge E \wedge E \xrightarrow{\delta} \cdots$$

We have yet another analogue of the fundamental theorem of homological algebra:

Proposition 1.51. *Let X, X' be CW spectra with E-Adams towers $(X_s), (X'_s)$ respectively, and let $\varphi : X \to X'$ be a morphism of CW spectra. Then φ lifts to a morphism $(\varphi_i) : (X_s) \to (X'_s)$ of E-Adams towers which is unique up to chain homotopy of such morphisms in a suitable sense of the word.*

See Ravenel and Hopkins (for two very different approaches).

Sketch of proof. Recall that $E \wedge X_0$ is a retract of $E \wedge K_0$, say via $r_0 : E \wedge K_s \to E \wedge X_s$, and K'_0 is a retract of $E \wedge K'_0$, say via $s'_0 : E \wedge K'_0 \to K'_0$. Using these two morphisms, we may define $\psi_0 : K_0 \to K'_0$ to be the composition

$$K_0 \xrightarrow{\eta\wedge\mathrm{id}} E \wedge K_0 \xrightarrow{r_0} E \wedge X_0 \xrightarrow{\mathrm{id}\wedge\varphi_0} E \wedge X'_0 \xrightarrow{f'_0} E \wedge K'_0 \xrightarrow{s'_0} K'_0$$

One must now argue that the diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & K_0 \\ {\scriptstyle\varphi_0}\downarrow & & {\scriptstyle\psi_0}\downarrow \\ X'_0 & \xrightarrow{f'_0} & K'_0 \end{array} \tag{1.22}$$

commutes (in the stable homotopy category, of course). In that case it presumably induces a morphism of fibration sequences

$$\begin{array}{ccccc} X_1 & \xrightarrow{g_0} & X_0 & \xrightarrow{f_0} & K_0 \\ {\scriptstyle\varphi_1}\downarrow & & {\scriptstyle\varphi_0}\downarrow & & {\scriptstyle\psi_0}\downarrow \\ X'_1 & \xrightarrow{g'_0} & X'_0 & \xrightarrow{f'_0} & K'_0 \end{array} \tag{1.23}$$

Proceed. I won't get into the statement about chain homotopies of maps of towers (since I don't know how to go about it).

□

From an E-Adams tower for X one obtains bigraded groups

$$D_1^{s,t} = \pi_{t-s}(X_s) \text{ and } E_1^{s,t} = \pi_{t-s}(K_s)$$

and interweaving the long exact sequences

$$\cdots \xrightarrow{f_*} \pi_{n+1}(K_s) \xrightarrow{\partial} \pi_n(X_{s+1}) \xrightarrow{g_*} \pi_n(X_s) \xrightarrow{f_*} \pi_n(K_s) \xrightarrow{\partial} \pi_{n-1}(X_{s+1}) \xrightarrow{g_*} \cdots$$

associated to the (co)fibration sequences $X_{s+1} \xrightarrow{g} X_s \xrightarrow{f} K_s$ one obtains an exact triangle

$$\begin{array}{ccc} D_1 & \xrightarrow{g_*} & D_1 \\ {\scriptstyle\partial}\uparrow & & {\scriptstyle f_*}\downarrow \\ E_1 & = & E_1 \end{array} \tag{1.24}$$

The above exact couple yields a spectral sequence $E_*^* * (X)$ with $E_1^{s,t} = \pi_{t-s}(K_s)$, called the E-Adams spectral sequence of X. The previous proposition is exactly what's needed to show that this spectral sequence is natural (from the E_2 page onward).

We'll show that in certain cases there's a natural identification

$$E_2^{s,t} \simeq \mathrm{Ext}^{s,t}_{E_*(E)}(E_*(S), E_*(X))$$

where the right hand side consists of Ext-groups of comodules over the Hopf algebroid $(E_*(S), E_*(E))$, and the spectral sequence converges to the associated bigraded group of $\pi_*(\hat{X})$ together with a certain filtration.

In fact for the homological identification of the E_2-term of the spectral sequence the only additional assumption is flatness of $E_*(E)$ over $E_*(S)$, which we've been assuming for a while for various reasons. As the spectral sequence $E_*^{**}(X)$ is independent of the particular E-Adams tower (X_s) from the E_2 page onward, we may as well work with the tower in which the resolution

$$K_0 \to \Sigma K_1 \to \Sigma^2 K_2 \to \cdots \text{ is the bar resolution}$$

$$E \wedge X \xrightarrow{\delta} E \wedge E \wedge X \xrightarrow{\delta} E \wedge E \wedge E \wedge X$$

In this case the E_2 page is the cohomology of the complex

$$\pi_*(E \wedge X) \xrightarrow{\delta_*} \pi_*(E \wedge E \wedge X) \xrightarrow{\delta_*} \pi_*(E \wedge E \wedge E \wedge X) \xrightarrow{\delta_*} \cdots$$

Since $E_*(E)$ is flat over E_*, we have isomorphisms

$$E_*(E)^{\otimes n} \otimes_{E_*(S)} E_*(X) \simeq E_*(E^{\wedge n} \wedge X) = \pi_*(E^{\wedge n+1} X)$$

and so this complex can be written as

$$E_*(X) \xrightarrow{\delta_*} E_*(E) \otimes_{E_*(S)} E_*(X) \xrightarrow{\delta_*} E_*(E) \otimes_{E_*(S)} E_*(E) \otimes_{E_*(S)} E_*(X) \xrightarrow{\delta_*} \cdots$$

One must argue (I'm skipping this) that the differentials are precisely those of the bar resolution for $E_*(X)$ as a comodule over the Hopf algebroid $(E_*(S), E_*(E))$, in which case it's immediate that the E_2-page is identified as

$$E_2^{s,t}(X) \simeq \mathrm{Ext}^{s,t}_{E_*(E)}(E_*(S), E_*(X))$$

As a simple corollary, we see that

Proposition 1.52. *Let $\varphi : X \to Y$ be a morphism of spectra which induces an isomorphism $\varphi_* : E_*(X) \simeq E_*(Y)$ on E-homology. Then φ induces an isomorphism of E-Adams spectral sequences $\varphi_* : E_*^{**}(X) \simeq E_*^{**}(Y)$ from the E_2-pages onward.*

Per usual convergence questions are far more subtle.

Definition 1.53. An E**-completion of** X is spectrum $\hat{X}$ with an E-Adams resolution $\hat{X}_s$ so that $\mathrm{ho}\varprojlim \hat{X}_s$ is contractible together with a morphism $X \to \hat{X}$ inducing an isomorphism $E_*(X) \simeq E_*(\hat{X})$.

In particular, if $X \to \hat{X}$ is an E-completion of X we have an isomorphism of E-Adams spectral sequences $E_*^{**}(X) \simeq E_*^{**}(\hat{X})$. Moreover by analogy with the classical (mod-p) case, one can show (**NOT SURE IF WE NEED (**E **and or** X**) TO BE CONNECTIVE HERE - THERE MIGHT EVEN BE ADDITIONAL NECESSARY HYPOTHESES**)

Proposition 1.54. *If X is a CW spectrum with an Adams tower (X_s) so that $\mathrm{ho}\varprojlim X_s$ is contractible, then the E_∞ page $E_\infty^{**}(X)$ of the E-Adams spectral sequence for X is the associated bigraded group of $\pi_*(X)$ together with the filtration*

$$\pi_*(X) = \mathrm{im}\pi_*(X_0) \supset \mathrm{im}\pi_*(X_1) \supset \mathrm{im}\pi_*(X_2) \supset \cdots$$

The upshot is that if X has an E-completion $X \to \hat{X}$ (and if I've included all the requisite hypotheses on E, X and $\hat{X}$, etc.) then the E-Adams spectral sequence $E_*^{**}(X)$ for X converges to $\pi_*(\hat{X})$ with the filtration given above. In order for this statement to have content, we must know something about the existence of such completions. So suppose in addition that

- E is *connective* ($\pi_n(E) = 0$ when $n < 0$),
- The multiplication map $E_*(S) \otimes_{\mathbb{Z}} E_*(S) \to E_*(S)$ gives an isomorphism $\pi_0(E) \otimes_{\mathbb{Z}} \pi_0(E) \to \pi_0(E)$. For instance, this will be the case if $\pi_0(E)$ is a subring of the rationals, or if $\pi_0(E) = \mathbb{Z}/(n)$ for some integer n.
- Let $\theta : \mathbb{Z} \to \pi_0(E)$ be the unique ring homomorphism, and let $R \subset \mathbb{Q}$ be the largest subring to which θ extends. Then $H_n(E; R)$ is finitely generated over R for all R.

Man, I'm getting rather confused about what is actually proved in Ravenel - going to take a break from this section.

1.3. **Some applications.** One major application of the classical (mod-p) Adams spectral sequence is the computation of the complex cobordism ring $\pi_*(MU)$ (due to Milnor). To begin, recall (say from Steenrod and Epstein or Mosher and Tangora) that for an odd prime p the dual Steenrod algebra $A_* := H\mathbb{F}_{p*}H\mathbb{F}_p$ is a commutative but not co-commutative Hopf *algebra* over $\mathbb{F}_p$ of the form $\mathbb{F}_p[\xi_n] \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}[\tau_n]$ (where $\Lambda_{\mathbb{F}_p}$ denotes the "exterior algebra over $\mathbb{F}_p$") where $\deg \xi_n = 2(p^n - 1)$ and $\deg \tau_n = 2p^n - 1$. Its identity and augmentation maps are given by the usual homomorphisms $\mathbb{F}_p \to \mathbb{F}_p[\xi_n] \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}[\tau_n]$ including the coefficients and $\mathbb{F}_p[\xi_n] \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}[\tau_n] \to \mathbb{F}_p$ sending $\xi_n \mapsto 0$, $\tau_n \mapsto 0$. The coproduct $\Delta : A_* \to A_* \otimes_{\mathbb{F}_p} A_*$ is described by

$$\Delta(\xi_n) = \sum_{i+j=n} \xi_i^{p^j} \otimes \xi_j \text{ and } \Delta(\tau_n) = \tau_n \otimes 1 + \sum_{i+j=n} \xi_i^{p^j} \otimes \tau_j \text{ for all } n$$

The conjugation antiautomorphism $c : A_* \to A_*$ is described recursively by

$$\sum_{i+j=n} \xi_i^{p^j} c(\xi_j) = 0 \text{ and } \tau_n + \sum_{i+j=n} \xi_i^{p^j} c(\tau_j) = 0 \text{ for all } n$$

At the prime $p = 2$, we have $A_* = \mathbb{F}_p[\xi_n]$ where $\deg \xi_n = 2^n - 1$ and the coproduct Δ is described by $\Delta(\xi_n) = \sum_{i+j=n} \xi_i^{p^j} \otimes \xi_j$ for all n and the conjugation is described recusively by $\sum_{i+j=n} \xi_i^{p^j} c(\xi_j) = 0$ for all n. The identity and augmentation maps are still the usual ones.

To get started we'll need to know the structure of $H_*(MU; \mathbb{F}_p)$ as an A_*-comodule algebra. Recall (say from Adams's blue book) that for any complex oriented cohomology theory E there is a canonical isomorphism of E_*-algebras $E_*[b_i] \simeq E_*(MU)$ where $\deg b_i = 2i$, so in particular

$$\mathbb{Z}[b_i] \simeq H_*(MU; \mathbb{Z}) \text{ and hence } \mathbb{F}_p[b_i] \simeq H_*(MU; \mathbb{F}_p)$$

Proposition 1.55. *Let M_* be a left A_*-comodule which is concentrated in (non-negative) even dimensions. Then M_* is a comodule over the sub-Hopf algebra $P_* \subset A_*$ defined to be $P_* = \mathbb{F}_p[\xi_n]$ when $p > 2$ and $P_* = \mathbb{F}_p[\xi_n^2]$ when $p = 2$.*

Proof. Let $\psi : M_* \to A_* \otimes_{\mathbb{F}_p} M_*$ be the coaction of A_* on M_*. We must show that it factors as

$$M_* \xrightarrow{\psi} P_* \otimes_{\mathbb{F}_p} M_* \xrightarrow{\iota \otimes \mathrm{id}} A_* \otimes_{\mathbb{F}_p} M_*$$

If $m \in M_*$ is a homogeneous element, say of degree $2n$ for some natural number n, write

$$\psi(m) = 1 \otimes m + m \otimes 1 + \sum_{0<i<n} a_i \otimes n_i \in A_* \otimes_{\mathbb{F}_p} M_*$$

where $\deg n_i = 2i$, so $\deg a_i = 2(n-i)$. Note that $1 \otimes m, m \otimes 1 \in P_* \otimes M_*$ trivially, so we just need to show $\sum_{0<i<n} a_i \otimes n_i \in P_* \otimes M_*$. By induction on the degree of M we may assume that $\psi(m) \in P_* \otimes M_*$ for all i; by coassociativity of the action ψ, we see that

$$\sum_{0<i<n} \Delta(a_i) \otimes n_i = \sum_{0<i<n} a_i \otimes \psi(n_i) \in A_* \otimes P_* \otimes M_*$$

Thus each a_i has even degree and a coproduct expansion $\Delta(a_i) \in A_* \otimes P_*$. From here it shouldn't be hard to show that $\Delta(a_i) \in P_*$.

□

Remark 1.56. It seems like P_* can be alternatively described as the sub-Hopf algebra of A_* consisting of elements $a \in A_*$ whose coproduct expansions $\Delta(a)$ consist entirely of even degree terms.

In particular we see that $H^*(MU; \mathbb{F}_p)$ is a comodule algebra over the sub-Hopf algebra $P_* \subset A_*$.

Proposition 1.57. *As a left A_*-comodule, $H_*(MU; \mathbb{F}_p) \simeq P_* \otimes_{\mathbb{F}_p} C$ where $C \simeq \mathbb{F}_p[u_i]$ and $\deg u_i = 2i$ and i is any positive integer not of the form $p^n - 1$.*

AGH... this is going to be quite involved. Might come back to it later.

1.4. **Adams-Novikov spectral sequences and BP-theory.**

1.4.1. *Formal group laws.* Let R be a $\mathbb{Z}$-graded commutative ring.

Definition 1.58. A **1-dimensional abelian formal group law over** R consists of homomorphisms of complete graded R-algebras

$$\mu : R[[x]] \to R[[x,y]]$$

(which of course is equivalent to a formal power series $\mu(x,y) = \sum_{i,j} a_{ij}x^i y^j$, the image of x under μ) and

$$\iota : R[[x]] \to R[[x]]$$

(which of course is equivalent to a formal power series $\iota(x) = \sum_i b_i x^i$, the image of x under ι) so that the following diagram commutes,

$$\begin{tikzcd} R[[x]] \arrow[r, "\mu"] \arrow[d, "\mu"'] & R[[x,y]] \arrow[d, "\mu\otimes\mathrm{id}"] \\ R[[x,y]] \arrow[r, "\mathrm{id}\otimes\mu"] & R[[x,y,z]] \end{tikzcd} \tag{1.25}$$

both compositions

$$R[[x]] \xrightarrow{\mu} R[[x,y]] \xrightarrow{\epsilon\otimes\mathrm{id}} R \otimes_R R[[x]] \simeq R[[x]] \text{ and}$$

$$R[[x]] \xrightarrow{\mu} R[[x,y]] \xrightarrow{\mathrm{id}\otimes\epsilon} R[[x]] \otimes_R R \simeq R[[x]]$$

are the identity and both compositions

$$R[[x]] \xrightarrow{\mu} R[[x,y]] \xrightarrow{\mathrm{id}\otimes\iota} R[[x,y]] \xrightarrow{x,y\mapsto x} R[[x]] \text{ and}$$

$$R[[x]] \xrightarrow{\mu} R[[x,y]] \xrightarrow{\iota\otimes\mathrm{id}} R[[x,y]] \xrightarrow{x,y\mapsto x} R[[x]] \text{ and}$$

coincide with

$$R[[x]] \xrightarrow{\epsilon} R \to R[[x]]$$

Here $\epsilon : R[[x]] \to R$ is the map taking $x \to 0$. Also, the following diagram commutes:

$$\begin{tikzcd} R[[x]] \arrow[r, "\mu"] \arrow[d, "\mu"'] & R[[x,y]] \arrow[d, "x\mapsto y, y\mapsto x"] \\ R[[x,y]] \arrow[r, equal] & R[[x,y]] \end{tikzcd} \tag{1.26}$$

I'm going to follow the convention in which $\deg x = -2$.

In terms of formal power series, the coassociativity condition states that

$$\mu(\mu(x,y),z) = \mu(x,\mu(y,z))$$

Expanding both sides using the formal power series $\mu(x,y) = \sum_{i,j} a_{ij}x^i y^j$ yields relations

$$p_{ijk}(a) = q_{ijk}(a) \text{ for all } i,j,k$$

where the p_{ijk}, q_{ijk} are polynomials in the coefficients a_{ijk}. The coidentity condition requires $\mu(x,0) = x$ and $\mu(0,y) = y$ which translates to $a_{i0} = 0$ for $i \neq 1$ and $a_{0j} = 0$ for $j \neq 1$, or more concretely

$$\mu(x,y) = x + y + \sum_{i,j>1} a_{ij}x^i y^j$$

The coinversion condition states that

$$\mu(x,\iota(x)) = \mu(\iota(x),x) = 0$$

and since $\mu(x, \iota(x)) = x + \iota(x) + \sum_{i,j>1} a_{ij} x^i \iota(x)^j$ this requires that $\iota(x) = -x + \sum_{i>1} b_i x^i$. In fact given only μ one can use the above formula to recursively compute the b_i, so the power series $\iota(x)$ turns out to be a redundant piece of information. Finally the commutativity condition translates to $\mu(x, y) = \mu(y, x)$, so that $a_{ij} = a_{ji}$ for all i, j.

Remark 1.59. The adjectives "1-dimensional" and "abelian" will be dropped from now on, as these are the only sorts of formal group laws that will be discussed.

Definition 1.60. Let R be a $\mathbb{Z}$-graded commutative ring and let μ, μ' be two formal group laws over R. A **homomorphism** $\psi : \mu \to \mu'$ is a homomorphism of complete graded R-algebras

$$\psi : R[[x]] \to R[[x]]$$

(which of course is equivalent to a formal power series of the form $\psi(x) = \sum_i r_i x^{i+1}$ - the constant term must be 0) so that the following diagram commutes:

$$\begin{array}{ccc} R[[x]] & \xrightarrow{\psi} & R[[x]] \\ {\scriptstyle \mu'}\downarrow & & {\scriptstyle \mu}\downarrow \\ R[[x,y]] & \xrightarrow{\psi\otimes\psi} & R[[x,y]] \end{array} \tag{1.27}$$

(which is to say, $\psi(\mu(x, y)) = \mu'(\psi(x), \psi(y))$). One can show that such a homomorphism is invertible (i.e. an isomorphism) if and only if the coefficient r_0 on x (or alternatively, $\psi'(0)$) is a *unit* in R. By definition a **strict isomorphism from** μ **to** μ' is a homomorphism as above given by a power series of the form $\psi(x) = x + \sum_{i>0} r_i x^{i+1}$ (that is, the coefficient on x is 1 - equivalently $\psi'(0) = 1$).

Observe that the "formal chain rule" shows that a composition of strict isomorphisms is a strict isomorphism, and this ensures that the formal group laws over R together with strict isomorphisms form a *groupoid*. Let $FGL(R)$ denote the set of formal group laws over R, and let $SI(R)$ denote the set of strict isomorphisms between those formal group laws. We can look at $(SI(R), FGL(R))$ as a groupoid object in **Set**.

Moreover if $\varphi : R \to R'$ is a homomorphism of graded commutative rings and μ is a formal group law over R, then its pushforward $\mu_* \mu$, obtained by tensoring the map

$$\mu : R[[x]] \to R[[x, y]]$$

with R' along φ, is a formal group law over R'. Concretely if $\mu(x, y) = \sum_{ij} a_{ij} x^i y^j$ then $\varphi_* \mu(x, y) = \sum_{ij} \varphi(a_{ij}) x^i y^j$. Similarly one can push forward a morphism of formal group laws $\psi : \mu \to \mu'$ over R to obtain a morphism $\varphi_* \psi : \varphi_* \mu \to \varphi_* \mu'$ - if $\psi(x) = \sum_i r_i x^{i+1}$ then we'll have $\varphi_* \psi(x) = \sum_i \varphi(r_i) x^{i+1}$, so in particular if ψ is a strict isomorphism so is $\varphi_* \psi$. In this way one shows that $(SI(-), FGL(-))$ can be viewed as a covariant groupoid valued functor on the category of graded commutative rings.

Proposition 1.61. *$(SI(-), FGL(-))$ is corepresented by a cogroupoid object in the category of graded commutative rings, i.e. a Hopf algebroid. That is, there is a Hopf algebroid (L, LB) together with a natural isomorphism of covariant groupoid valued functors*

$$(\mathrm{Hom}(LB, -), \mathrm{Hom}(L, -)) \simeq (SI(-), FGL(-))$$

Proof. To begin, observe that a formal group law μ over a graded commutative ring R is determined by the formal power series $\mu(x, y) = \sum_{i,j} a_{ij} x^i y^j$, which in turn is determined by the coefficients a_{ij} - these define a homomorphism

$$\mathbb{Z}[a_{ij}] \to R$$

Notice that if x, y have degree -2, then the coefficients a_{ij} have degree $2(i+j-1)$. Now recall that the a_{ij} must satisfy the associativity relations $p_{ijk}(a) = q_{ijk}(a)$, the identity relations

$$a_{i0} = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \text{ and } a_{0j} = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

and the commutativity relations $a_{ij} = a_{ji}$. Let $I \subset \mathbb{Z}[a_{ij}]$ be the ideal corresponding to all these relations, and set $L = \mathbb{Z}[a_{ij}]/I$ - as I is certainly a homogeneous ideal, L is a graded commutative ring. Moreover it follows that our map $\mathbb{Z}[a_{ij}] \to R$ factors through L:

$$\mathbb{Z}[a_{ij}] \to L \to R$$

At this point it's clear that L co-represents $FGL(-)$.

Now suppose $\psi : \mu \to \mu'$ is a strict isomorphism of formal group laws over R. Since $\mu'(x,y) = \psi(\mu(\psi^{-1}(x), \psi^{-1}(y)))$, this strict isomorphism is completely determined by the formal power series $\mu(x,y) = \sum_{i,j} a_{ij} x^i y^j$ and $\psi(x) = \sum_i b_i x^{i+1}$ (where $b_0 = 1$). We've already seen that $\mu(x,y)$ determines a homomorphism $L \to R$, and the coefficients b_i for $i > 0$ of $\psi(x)$ give a homomorphism $\mathbb{Z}[b_i] \to R$ - notice that the coefficients b_i have degree $2i$. In this way we obtain a homomorphism

$$L[b_i] = L \otimes_{\mathbb{Z}} \mathbb{Z}[b_i] \to R$$

It's important to note that

Lemma 1.62. *Let $\psi(x) = \sum_i b_i x^{i+1}$, where $b_0 = 1$. Then $\psi(x)$ defines a strict isomorphism from $\mu(x,y)$ to the formal group law $\mu'(x,y) := \psi(\mu(\psi^{-1}(x), \psi^{-1}(y)))$.*

Using this lemma one can concluded that LB corepresents $SI(-)$. It follows from Yoneda nonsense that (L, LB) is a Hopf algebroid. I'll give some description of this structure once we have a better idea what L looks like.

□

Definition 1.63. Let R be a graded commutative ring. The **additive formal group law over** R is the one defined by

$$x + y \in R[[x,y]]$$

Definition 1.64. Let μ be a formal group law over a graded commutative ring R. A **logarithm for** μ is a strict isomorphism $\log_\mu$ from μ to the additive formal group law. That is,

$$\log_\mu(\mu(x,y)) = \log_\mu(x) + \log \mu(y)$$

Proposition 1.65. *Suppose R is a $\mathbb{Q}$-algebra. Then every formal group law μ over R has a logarithm.*

Note that this means that every object of $(SI(R), FGL(R))$ is uniquely isomorphic to the additive formal group law, so this groupoid is contractible - right?

Proof. In general (that is, even when R is not necessarily a $\mathbb{Q}$-algebra) *if* μ has a logarithm $\log_\mu$, then differentiating the equation

$$\log_\mu(\mu(x,y)) = \log_\mu(x) + \log \mu(y)$$

with respect to y yields

$$\log'_\mu(\mu(x,y)) \frac{\partial \mu}{\partial y}(x,y) = \log'_\mu(y)$$

Now setting $y = 0$ and recalling that $\mu(x,0) = x$ and $\log'_\mu(0) = 1$, we see that

$$\log'_\mu(x) \frac{\partial \mu}{\partial y}(x,0) = 1$$

Which is to say,

$$\log_\mu'(x) = (\frac{\partial \mu}{\partial y}(x,0))^{-1}$$

When R is a $\mathbb{Q}$-algebra, every formal power series $f(x) = \sum_n c_n x^n$ has a unique anti-derivative $F(x)$ with $F(0) = 0$, namely $F(x) = \sum_n \frac{c_n}{n+1} x^{n+1}$ - one might write $F(x) = \int_0^x f(t)dt$. In this situation,

$$\log_\mu(x) = \int_0^x (\frac{\partial \mu}{\partial y}(t,0))^{-1} dt$$

Now just observe that when R is a $\mathbb{Q}$-algebra the right hand side of the above equation can be used to *define* $\log_\mu(x)$.

□

Remark 1.66. Let R be a graded commutative ring, and observe that formal differentiation can be viewed as an R-module homomorphism

$$d : R[[x]] \to R[[x]]dx \text{ sending } f(x) = \sum_n c_n x^n \mapsto \sum_n n c_n x^{n-1} dx = f'(x)dx$$

We can then consider the "de Rham complex" $R[[x]] \xrightarrow{d} R[[x]]dx$, and compute it's cohomology. Notice that $\sum_n n c_n x^{n-1} dx = f'(x)dx = 0$ if and only if $nc_n = 0$ for all n, and if $g(x) = \sum_n d_n x^n$ and for some $f(x) = \sum_n c_n x^n$

$$\sum_n d_n x^n = g(x) = f'(x) = \sum_n n c_n x^{n-1}$$

we'll have $d_n = (n+1)c_{n+1}$ for all n. Clearly, when R is a $\mathbb{Q}$ algebra, then the kernel of d is just the usual copy of $R \subset R[[x]]$ (the constant power series) and the cokernel of d is 0. However this de Rham complex could obviously have more interesting cohomology if R is not a $\mathbb{Q}$-algebra.

Corollary 1.67. *There is a canonical isomorphism of graded* $\mathbb{Q}$*-algebras* $L \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}[m_i | i > 0]$.

Proof. Notice that for a graded commutative ring R, a homomorphism $L \otimes_{\mathbb{Z}} \mathbb{Q} \to R$ consists of a pair of homomorphisms $L \to R$ and $\mathbb{Q} \to R$; the map $\mathbb{Q} \to L$ just says R is a $\mathbb{Q}$ algebra, and the homomorphism $L \to R$ corresponds to a formal group law over R (okay, I might be assuming that R is not the 0 ring ... but if R is the 0 ring there's not much to say!). So, $L \otimes_{\mathbb{Z}} \mathbb{Q}$ corepresents the functor $FGL(-)$ on the category of graded $\mathbb{Q}$-algebras.

On the other hand, if R is a $\mathbb{Q}$-algebra and μ is a formal group law over R, then from the above discussion there's a logarithm $\log_\mu(x) = \sum_i m_i x^{i+1}$ (with $m_0 = 1$) for μ, so that

$$\log_\mu(\mu(x,y)) = \log_\mu(x) + \log_\mu(y) \text{ or alternatively}$$

$$\mu(x,y) = \exp_\mu(\log_\mu(x) + \log_\mu(y))$$

where $\exp_\mu = \log_\mu^{-1}$. This means that μ is totally determined by its logarithm $\log_\mu(x) = \sum_i m_i x^i$, and the coefficients m_i for $i > 0$ (recall $m_0 = 1$) of this logarithm correspond to a homomorphism of graded $\mathbb{Q}$-algebras $\mathbb{Q}[m_i] \to R$ Notice that $\deg m_i = 2i$. Evidently $\mathbb{Q}[m_i]$ also corepresents $FGL(-)$ on the category of graded $\mathbb{Q}$-algebras, so there must be a (unique) isomorphism $L \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}[m_i]$.

In fact it's not hard to be specific about what this isomorphism is: *define* the formal power series

$$\log(x) := \sum_i m_i x^{i+1} \in \mathbb{Q}[m_i][[x]]$$

and *define* the formal group law

$$\mu(x,y) := \exp(\log(x) + \log(y)) \in \mathbb{Q}[m_i][[x,y]]$$

This corresponds to a homomorphism of $\mathbb{Q}$-algebras

$$L \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}[m_i]$$

From the above discussion it's more or less clear that this is an isomorphism.

□

Notice that in fact the image of L under

$$L \to L \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}[m_i]$$

lies in the subring $\mathbb{Z}[m_i] \subset \mathbb{Q}[m_i]$, since it corresponds to the formal group law

$$\mu(x,y) = \exp(\log(x) + \log(y)), \text{ where } \log(x) = \sum_i m_i x^{i+1}$$

Since $\log(x)$ has coefficients in $\mathbb{Z}[m_i]$, so does its compositional inverse $\exp(x)$, and from there one can see that $\mu(x,y)$ has coefficients in $\mathbb{Z}[m_i]$. So, the homomorphism $L \to \mathbb{Q}[m_i]$ takes the formal group law coefficients a_{ij} into $\mathbb{Z}[m_i]$, and we know the a_{ij} generate L as a ring... etc.

Recall that for an augmented algebra A over a commutative ring R, with augmentation $\epsilon : A \to R$ and augmentation ideal $IA = \ker \epsilon$, the R-module of indecomposables QA fits into the exact sequence

$$IA \otimes_R IA \xrightarrow{\text{multiply}} IA \to QA \to 0$$

which is to say, "$QA = IA/IA^2$." Here's a geometric idea of what QA is: say $V \subset \mathbb{P}^n_k$ is a projective variety over a field k, defined by a collection of homogeneous polynomials $f_1, \ldots, f_r \in k[x_0, \ldots, x_n]$ and let $S(V) = k[x_i]/(f_i)$ be its homogeneous coordinate ring - $S(V)$ will also be the *affine* coordinate ring of the cone $C(V) \subset \mathbb{A}^{n+1}_k$ over V. Note that there's an evident augmentation $\epsilon : S(V) \to k$ (induced by sending $x_i \to 0$ for all i) which corresponds to evaluation at the vertex $0 \in C(V)$. So, the augmentation ideal $IS(V) = \ker \epsilon$ is the ideal of the vertex, and so $QS(V) = IS(V)/IS(V)^2$ corresponds to the *conormal sheaf* of the vertex $0 \in C(V)$ - alternatively, this is the cotangent space to $C(V)$ at the vertex 0. This seems like a reasonable thing to care about, as presumably some of the geometry of V will be captured by the (co)-tangent space at the vertex of its cone.

The following is a beautiful theorem of Lazard:

Theorem 1.68. *There exist generators $x_i \in L$, with $\deg x_i = 2i$, so that the map*

$$\mathbb{Z}[x_i] \to L$$

is an isomorphism. Moreover these generators can be chosen so that the image of x_i in

$$QL \otimes_{\mathbb{Z}} \mathbb{Q} \simeq Q\mathbb{Q}[m_i] \simeq \bigoplus_i \mathbb{Q} m_i$$

is

$$\begin{cases} pm_i & \text{if } i = p^k \text{ for some prime } p \text{ and some positive integer } k \\ 0 & \text{otherwise} \end{cases}$$

Finally, the homomorphism

$$L \to \mathbb{Z}[m_i] \subset \mathbb{Q}[m_i] \simeq L \otimes_{\mathbb{Z}} \mathbb{Q}$$

discussed above is injective.

For a proof see Adams or Ravenel.

Let's describe the structure maps of the Hopf algebroid (L, LB). To begin, recall that if $\psi : \mu \to \mu'$ is a strict isomorphism of formal group laws over a graded commutative ring R, then it's completely described by the formal power series $\mu(x,y) \in R[[x,y]]$ and $\psi(x) \in R[[x]]$ - the source $\mu(x,y)$ determines a homomorphism $L \to R$ and then $\psi(x)$ determines a homomorphism $LB \to R$. From this it's clear that the cosource $\eta_L : L \to LB$ is the usual inclusion $L \subset LB = L[b_i]$.

The target of the above strict isomorphism is the formal group law $\mu'(x,y) = \psi(\mu(\psi^{-1}(x), \psi^{-1}(y)))$, which determines another homomorphism $L \to R$. Note that the coefficients of $\mu'(x,y)$ will be polynomials in the coefficients of $\mu(x,y)$ and $\psi(x)$, so at least in principle the above formula describes $\eta_R : L \to LB$. However we can get a more explicit description of η_R after base-changing to $\mathbb{Q}$: recall that there are canonical isomorphisms $L \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}[m_i]$ and $LB \otimes_{\mathbb{Z}} \mathbb{Q} \simeq Q[m_i][b_i]$ and so $\eta_R \otimes \mathrm{id} : L \otimes_{\mathbb{Z}} \mathbb{Q} \to LB \otimes_{\mathbb{Z}} \mathbb{Q}$ can be identified with a homomorphism

$$\mathbb{Q}[m_i] \to \mathbb{Q}[m_i][b_i]$$

Recall that if R is a $\mathbb{Q}$-algebra, then μ and μ' have unique logarithms, whose coefficients determine maps $\mathbb{Q}[m_i] \to R$. Observe that since ψ^{-1} is a strict isomorphism from $\mu' \to \mu$ and $\log_\mu$ is a strict isomorphism from μ to $x+y$, it must be that

$$\log_{\mu'}(x) = \log_\mu(\psi^{-1}(x)) \in R[[x]]$$

Restricting attention to the universal case where $R = L \otimes_{\mathbb{Z}} \mathbb{Q}$, μ is the universal formal group law and $\psi(x) = \sum_i b_i x^{i+1}$ this reads

$$\log_{\mu'}(x) = \sum_i m_i (\sum_j c(b_j) x^{j+1})^{i+1}$$

where $\sum_i c(b_i) x^{i+1}$ is the compositional inverse of $\sum_i b_i x^{i+1}$. This means that the homomorphism $\eta_R \otimes \mathrm{id} : L \otimes_{\mathbb{Z}} \mathbb{Q} \to LB \otimes_{\mathbb{Z}} \mathbb{Q}$ must satisfy

$$\sum_i \eta_R \otimes \mathrm{id}(m_i) x^{i+1} = \sum_i m_i (\sum_j c(b_j) x^{j+1})^{i+1}$$

and comparing coefficients gives a calculation of the $\eta_R \otimes \mathrm{id}(m_i)$ as polynomials in the m_i, b_i.

Notice that the identity homomorphism from μ to itself is given by the formal power series x. From this one sees that the augmentation $\epsilon : LB \to L$ is the usual map $L[b_i] \to L$ sending $b_i \mapsto 0$ for all i.

Let $\psi : \mu \to \mu'$ and $\psi' : \mu' \to \mu''$ be strict isomorphisms given by power series

$$\psi(x) = \sum_i b_i x^{i+1} \text{ and } \psi'(x) = \sum_i b'_i x^{i+1}$$

Then their composition is the strict isomorphism $\psi' \circ \psi : \mu \to \mu''$ given by the formal power series

$$\psi'(\psi(x)) = \sum_i b'_i (\sum_j b_j x^{j+1})^{i+1}$$

From this it's clear that the cocomposition

$$\Delta : LB \to LB \otimes LB = L[b_i] \otimes L[b'_i]$$

must satisfy

$$\sum_i \Delta(b_i) x^{i+1} = \sum_i b'_i (\sum_j b_j x^{j+1})^{i+1}$$

Finally, note that the inverse of $\psi : \mu \to \mu'$ is $\psi^{-1} : \mu' \to \mu$, which is described by its source $\mu'(x,y)$ together with $\psi^{-1}(x)$. This shows that the coinversion $c : LB \to LB$ takes L to LB via η_R and sends $b_i \mapsto c(b_i)$.

Let R be a graded commutative ring and let μ be a formal group law over R. Observe that for any R-algebra A, say with structure homomorphism $\varphi : R \to A$, we obtain a formal group law $\varphi_* \mu$ over A, and it doesn't seem too silly to abbreviate $\varphi_* \mu$ to μ, as long as we're thinking of A as a R-algebra.

Definition 1.69. If $x, y \in A$ and it makes sense to evaluate $\mu(x,y) \in A$, then write

$$x +_\mu y := \mu(x,y) \in A$$

Similarly, write

$$x +_\mu y +_\mu z = \mu(x, \mu(y,z)) \in A$$

provided it makes sense to evaluate $\mu(x,\mu(y,z)) \in A$. (recall μ is associative and commutative, so it doesn't really matter what order we apply μ in). One can also define

$$x -_\mu y := \mu(x, \iota(y)) \in A$$

(again, provided it makes sense to evaluate these power series in A). One can recursively define

$$[n]_\mu(x) = \mu(x, [n-1]_\mu(x)) \in A$$

for a non-negative integer n, with the condition that $[0]_\mu(x) = 0$. In fact one may then define $[n]_\mu(x)$ for all integers n by setting

$$[n]_\mu(x) = \iota([-n]_\mu(x)) \text{ if } n < 0$$

As a trivial observation, notice that if μ is the additive formal group law, then $x +_\mu y = \mu(x,y) = x + y \in A$, so in that case $+_\mu$ is just the usual addition in A.

Proposition 1.70. *Let K be a subring of $\mathbb{Q}$, let R be a graded commutative K-algebra and let μ be a formal group law over R. Then for each $r \in K$ there is a unique formal power series $[r]_\mu(x) \in R[[x]]$ so that*

- *if $r \in \mathbb{N} \subset K$ is a non-negative integer then $[r]_\mu(x)$ is the formal power series defined above,*
- *$[r_1 + r_2]_\mu(x) = [r_1]_\mu(x) +_\mu [r_2]_\mu(x)$, and*
- *$[r_1 r_2]_\mu(x) = [r_1]_\mu([r_2]_\mu(x))$.*

Proof. The above definition yields $[n]_\mu(x) \in R[[x]]$ for any integer $n \in \mathbb{Z}$ (notice that if $f, g \in (x) \subset R[[x]]$ then it makes sense to evaluate $\mu(f,g) \in R[[x]]$). Observe that if $n \in \mathbb{Z}$ is invertible in K, the third bullet point would require that

$$x = [1]_\mu(x) = [n^{-1}n]_\mu(x) = [n^{-1}]_\mu([n]_\mu(x))$$

which would require that $[n^{-1}]_\mu(x) = [n]_\mu^{-1}(x)$. Since $\mu(x,y) \equiv x + y \mod (x,y)^2$, it's not hard to show by induction that $[n]_\mu(x) \equiv nx \mod (x)^2$, and since n is a unit in K by hypothesis this series does indeed have a compositional inverse. Now for any $r \in K$, we may write $r = \frac{m}{n}$ for integers $m, n \in \mathbb{Z}$, and define

$$[r]_\mu(x) := [m]_\mu([n]_\mu(x)) \in R[[x]]$$

From here it's not hard to verify the conditions in the three bullet points are satisfied. □

Notice that $(x) \subset R[[x]]$ is an abelian group under $+_\mu$. This proposition yields an action $K \times (x) \to (x)$ of K on the abelian group $(x) \subset R[[x]]$, taking

$$(r, f) \mapsto [r]_\mu(f) \in R[[x]]$$

This can be viewed as a K-module structure on (x) - but it's not necessarily the same as the K-module structure on (x) given by the inclusion $K \subset R$ (this will only be the case if μ is the additive formal group law). Suppose μ' is another formal group law over R and let $\psi : \mu \to \mu'$ be a strict isomorphism, so $\psi(\mu(x,y)) = \mu'(\psi(x), \psi(y)) \in R[[x,y]]$ which is to say

$$\psi(x +_\mu y) = \psi(x) +_{\mu'} \psi(y) \in R[[x,y]]$$

From here it seems clear that we'll have $\psi([r]_\mu(x)) = [r]_{\mu'}(x) \in (x) \subset R[[x]]$ for any $r \in K$, so that $x \to \psi(x)$ defines an isomorphism of the K-module (x) (replacing the structure corresponding to μ with that of μ').

Definition 1.71. If $q \in \mathbb{N}$ is a natural number which is invertible in K, then

$$f_q(x) := [\frac{1}{q}]_\mu(\sum_{i=1}^{q} \xi^i x)_\mu \in R[[x]]$$

where ξ is a primitive q-th root of unity.

A priori $f_q(x) \in R[\xi][[x]]$, but since it's invariant under Galois permutations of the ξ^i it must actually lie in $R[[x]]$.

As discussed above, we have a K-module structure on the abelian group (x) given by

$$r \cdot g = [r]_\mu(g) \in (x) \subset R[[x]]$$

. $q \in \mathbb{N}$ is a unit in K and we adjoin a primitive q-th root of unity ξ we can extend this to a $K[\xi]$-module structure on $xR[\xi][[x]] \subset R[\xi][[x]]$. We then have an action of the cyclic group C_q on this module where the generator acts as ξ. In that case $f_q(x)$ is the image of x under the "averaging operator"

$$xR[\xi][[x]] \to xR[[x]] \text{ taking } g \mapsto [\frac{1}{q}]_\mu(\sum_{i=1}^{q} \xi^i g)_\mu$$

Alternatively, $f_q(x)$ *is* the averaging operator, since $f_q(g) = [\frac{1}{q}]_\mu(\sum_{i=1}^{q} \xi^i g)_\mu$.

Definition 1.72. A formal group law μ over a $\mathbb{Z}_{(p)}$-algebra R is **p-typical** if and only if $f_q(x) = 0$ for all primes $q \neq p$.

Remark 1.73. Let R be a graded commutative $\mathbb{Z}_{(p)}$ algebra. Then the (p)-typical formal group laws over R with strict isomorphisms form a full sub-groupoid of $(SI(R), FGL(R))$, say $(SI(R), FGL_{ptyp}(R))$.

Proposition 1.74. *Let R be a graded commutative $\mathbb{Z}_{(p)}$-algebra and let μ be a p-typical formal group law over R. Let $\psi(x) = \sum_i r_i x^{i+1} \in$ be a strict isomorphism (so $r_0 = 1$) from μ to the formal group law*

$$\mu'(x,y) := \psi(\mu(\psi^{-1}(x), \psi^{-1}(y))) \in R[[x,y]]$$

Then μ' is p-typical if and only if $\psi^{-1}(x)$ is of the form

$$\psi^{-1}(x) = (\sum_i t_i x^{p^i})_\mu \in R[[x]]$$

Proof. Recall that μ' will be p-typical if and only if

$$f_{q,\mu'}(x) := [\frac{1}{q}]_{\mu'}(\sum_{i=1}^{q} \xi^i x)_{\mu'} = 0$$

for all primes $q \neq p$. Certainly $f_{q,\mu'}(x) = 0$ if and only if $\psi^{-1}(f_{q,\mu'}(x)) = 0$, and we have

$$\psi^{-1}(f_{q,\mu'}(x)) = \psi^{-1}([\frac{1}{q}]_{\mu'}(\sum_{i=1}^{q} \xi^i x)_{\mu'}) = [\frac{1}{q}]_\mu(\sum_{i=1}^{q} \psi^{-1}(\xi^i x))_\mu$$

Now according to Ravenel, there exist unique constants $s_i \in R$ so that

$$\psi^{-1}(x) = (\sum_i s_i x^{i+1})_\mu \in R[[x]]$$

(just use the above formula to solve for the s_i inductively I guess). Assuming this, one obtains

$$[\frac{1}{q}]_\mu(\sum_{i=1}^{q} \psi^{-1}(\xi^i x))_\mu = [\frac{1}{q}]_\mu(\sum_{i=1}^{q} \sum_j s_j(\xi^i x)^{j+1})_\mu$$

$$= (\sum_j([\frac{1}{q}]_\mu(\sum_{i=1}^{q} \xi^{i(j+1)} s_j x^{j+1})_\mu)_\mu$$

Now observe that if $q \nmid j+1$, then ξ^{j+1} is a primitive q-th root of 1 and so

$$[\frac{1}{q}]_\mu(\sum_{i=1}^{q} \xi^{i(j+1)} s_j x^{j+1})_\mu = f_{q,\mu}(s_j x^{j+1}) = 0$$

(since μ is p-typical by hypothesis). On the other hand if $q|j+1$ then $\xi^{j+1} = 1$ and so

$$\begin{aligned}[\frac{1}{q}]_\mu(\sum_{i=1}^q \xi^{i(j+1)} s_j x^{j+1})_\mu &= [\frac{1}{q}]_\mu(\sum_{i=1}^q s_j x^{j+1})_\mu \\ &= [\frac{1}{q}]_\mu([q]_\mu s_j x^{j+1}) = s_j x^{j+1}\end{aligned}$$

In this way one sees that

$$\begin{aligned}(\sum_j([\frac{1}{q}]_\mu(\sum_{i=1}^q \xi^{i(j+1)} s_j x^{j+1})_\mu)_\mu &= (\sum_{q\nmid j+1}([\frac{1}{q}]_\mu(\sum_{i=1}^q \xi^{i(j+1)} s_j x^{j+1})_\mu)_\mu +_\mu (\sum_{q|j+1}([\frac{1}{q}]_\mu(\sum_{i=1}^q \xi^{i(j+1)} s_j x^{j+1})_\mu)_\mu \\ &= 0 +_\mu (\sum_{q|j+1} s_j x^{j+1})_\mu = (\sum_{q|j+1} s_j x^{j+1})_\mu\end{aligned}$$

Presumably (have to be a little careful since this is a formal sum with respect to μ) this will vanish if and only if $s_j = 0$ whenever $q|j+1$. Considering this for all primes $q \neq p$, we see that μ' will be p-typical if and only if $s_i = 0$ whenever $j+1$ is not a power of p, which is to say $\psi^{-1}(x) = (\sum_i s_{p^i-1} x^{p^i})_\mu$, or setting $t_i := s_{p^i-1}$, $\psi^{-1}(x) = (\sum_i t_i x^{p^i})_\mu$ as claimed.

□

The following special case will be particularly useful:

Corollary 1.75. *Let μ be a formal group law over a* torsion-free *$\mathbb{Z}_{(p)}$-algebra R (this means that the natural map $R \to R \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ is injective). Then μ is p-typical if and only if the logarithm for μ over $R \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ is of the form*

$$\log_\mu(x) = \sum_i t_i x^{p^i} \in R \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}[[x]]$$

Proof. Since because $\exp_\mu$ gives an isomorphism from the additive formal group law to μ over $R \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$, with inverse $\log_\mu$, it will suffice to observe that the additive formal group law over $R \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ is p-typical. For the additive formal group we have

$$f_q(x) = \frac{1}{q}\sum_{i=1}^q \xi^i x \text{ where } \xi \text{ is a primitive } q\text{-th root}$$

and we may appeal to the following

Lemma 1.76.

$$\sum_{i=1}^q \xi^i = 0$$

Proof of lemma. Letting the generator act by ξ defines a faithful representation of C_q on $\mathbb{C}$, and $\frac{1}{q}\sum_{i=1}^q \xi^{i(j+1)}$ is the average operator giving a projection from $\mathbb{C}$ to the isotypical summand corresponding to the trivial representation of C_q lying in $\mathbb{C}$, which is obviously 0.

□

□

Theorem 1.77 (Cartier)**.** *Let R be a graded commutative $\mathbb{Z}_{(p)}$-algebra. Then every formal group law μ over R is canonically strictly isomorphic to a p-typical formal group law over R.*

In fact, the proof will yield a natural retraction of groupoids $(SI(R), FGL(R)) \to (SI(R), FGL_{\text{ptyp}}(R))$. (Actually (and correct me if I'm wrong) shouldn't the theorem also show that the "inclusion" $(SI(R), FGL_{\text{ptyp}}(R)) \to (SI(R), FGL(R))$ is an equivalence of groupoids (being fully faithful and essentially surjective)?)

Proof. First observe that μ is equivalent to a homomorphism $L \to R$, and since R is a $\mathbb{Z}_{(p)}$-algebra this is the same as a homomorphism of $\mathbb{Z}_{(p)}$-algebras $L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to R$. Evidently $L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ represents the set valued functor $FGL(-)$ on the category of $\mathbb{Z}_{(p)}$-algebras.

So, it will suffice to construct a canonical strict isomorphism $\psi(x)$ from the universal formal group law $\mu(x,y) = \sum_{i,j} a_{ij}x^iy^j \in L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}[[x,y]]$ to a p-typical one of the form $\mu'(x,y) := \psi(\mu(\psi^{-1}(x), \psi^{-1}(y)))$. Note that since we know $L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is torsion-free, a formal group law μ' over $L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ will be p-typical if and only if its logarithm $\log_{\mu'}(x) \in L \otimes_{\mathbb{Z}} \mathbb{Q}[[x]]$ has the form

$$\log_{\mu'}(x) = \sum_i t_i x^{p^i}$$

Since ψ^{-1} will be a strict isomorphism from μ' to μ and $\log_\mu$ is an isomorphism from μ to the additive formal group law, $\log_\mu \circ \psi^{-1}(x)$ will be a strict isomorphism from μ' to the additive formal group law - which is to say,

$$\log_{\mu'}(x) = \log_\mu(\psi^{-1}(x)) \in R \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}[[x]]$$

The claim is that we can even find a strict isomorphism $\psi(x)$ so that

$$\log_{\mu'}(x) = \log_\mu(\psi^{-1}(x)) = \sum_i m_{p^i-1} x^{p^i}$$

In fact (and I have no idea how Cartier came up with this) setting

$$\psi^{-1}(x) := (\sum_{p \nmid q} [\nu(q)]_\mu f_{q,\mu}(x))_\mu$$

does the trick. Here ν is the "Mobius function" on $\mathbb{N}$ defined by

$$\nu(q) = \begin{cases} 0 & \text{if } q \text{ is divisible by a square} \\ (-1)^r & \text{if } q \text{ is a product of } r \text{ distinct primes} \end{cases}$$

Remark 1.78. Typically the function I'm calling ν is denoted by μ. I can't use μ right now because I've (perhaps foolishly) been calling my formal group laws μ. My sincere apologies to the number-theoretic community.

Remark 1.79. Observe that if $\nu(x,y)$ is a p-typical formal group law over a graded commutative $\mathbb{Z}_{(p)}$-algebra R and $\varphi : L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to R$ is the homomorphism with $\varphi_*\mu = \nu$, then the pushforward of $\psi(x)$ along φ is the identity (since $\varphi_* f_{q,\mu}(x) = f_{q,\nu}(x) = 0$ for all primes $q \neq p$).

To see that the above formal sum actually converges, note that

Lemma 1.80. *$f_{q,\mu}(x) \equiv 0 \mod (x^q)$ for all q.*

Proof. Notice that

$$f_{q,\mu}(\xi x) = [\frac{1}{q}]_\mu (\sum_{i=1}^{q} \xi^i \xi x)_\mu = f_{q,\mu}(x)$$

since $\xi^i\xi$ ranges over all the q-th roots of 1 as i goes from 1 to q. But then if $f_{q,\mu}(x) = \sum_i c_i x^i$ we have

$$\sum_i c_i \xi^i x^i = \sum_i c_i x^i, \text{ so } c_i \xi^i = c_i \text{ for all } i$$

In fact this requires that $c_i = 0$ unless $p|i$, which means $f_{q,\mu} = \sum_i c_{pi} x^{pi}$ (notice that this is a formal power series in x^p).

∎

Now observe that

$$\log_\mu(\psi^{-1}(x)) = \log_\mu(\sum_{p \not| q}[\nu(q)]_\mu f_{q,\mu}(x))_\mu$$

$$= \sum_{p \not| q} \nu(q) \frac{1}{q} \sum_{i=1}^{q} \log_\mu(\xi^i x) = \sum_{p \not| q} \nu(q) \frac{1}{q} \sum_{i=1}^{q} \sum_j m_j (\xi^i x)^{j+1}$$

$$= \sum_{p \not| q} \nu(q) \sum_j \frac{1}{q} \sum_{i=1}^{q} \xi^{i(j+1)} m_j x^{j+1}$$

Again we have

$$\frac{1}{q} \sum_{i=1}^{q} \xi^{i(j+1)} = \begin{cases} 1 & \text{if } q | j+1 \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\sum_{p \not| q} \nu(q) \sum_j \frac{1}{q} \sum_{i=1}^{q} \xi^{i(j+1)} m_j x^{j+1} = \sum_{p \not| q} \nu(q) \sum_j m_{qj-1} x^{qj}$$

$$\sum_n \sum_{q|n, p \not| q} \nu(q) m_{n-1} x^n$$

At this point we need to know

Lemma 1.81. *For any (positive) natural number n,*

$$\sum_{q|n, p \not| q} \nu(q) = \begin{cases} 1 & \textit{if } n = p^k \textit{ for some } k \\ 0 & \textit{otherwise} \end{cases}$$

I'm not going to prove this here because there's probably a slick proof that's "obvious" (at least to the numerotheoretically inclined). But I don't know it off the top of my head.

□

Proposition 1.82. *The groupoid valued functor on the category of graded commutative $\mathbb{Z}_{(p)}$-algebras assigning*

$$R \mapsto (SI(R), FGL_{ptyp}(R))$$

is co-representable, say by a graded commutative $\mathbb{Z}_{(p)}$-algebra V.

Proof. Notice that a p-typical formal group law $\mu(x, y) = \sum_{i,j} a_{ij} x^i y^j$ over a graded commutative $\mathbb{Z}_{(p)}$-algebra R defines a homomorphism

$$\varphi : L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to R$$

taking the $a_{ij} \in L$ to the coefficients $a_{ij} \in R$ of $\mu(x, y)$ (I realize this is kind of sloppy). Now observe that the p-typicality condition

$$f_{q,\mu}(x) = 0 \in R[[x]] \text{ for all primes } q \not| p$$

provides a whole host of additional relations on the $a_{ij} \in R$ (since the coefficients of $f_{q,\mu}$ are polynomials in these $a_i j \in R$, I think with $\mathbb{Z}_{(p)}$-coefficients). Let $J \subset L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ be the ideal generated by those relations on the $a_{ij} \in L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, and set $V = L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}/J$. Then the homomorphism φ factors as

$$L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}/J \to R$$

From here it's not hard to show V corepresents $FGL_{\text{ptyp}}(-)$.

Recall from proposition 1.29 that if R is a graded commutative $\mathbb{Z}_{(p)}$-algebra, $\mu(x,y)$ is a p-typical formal group law over R and $\psi(x)$ is a strict isomorphism from μ to $\mu'(x,y) := \psi(\mu(\psi^{-1}(x), \psi^{-1}(y)))$, then μ' is p-typical if and only if $\psi^{-1}(x)$ is of the form

$$\psi^{-1}(x) = (\sum_i t_i x^{p^i})_\mu \in R[[x]]$$

Now consider a strict isomorphism of p-typical formal group laws $\psi : \mu \to \mu'$ over R. Note that this strict isomorphism is completely determined by $\mu(x,y)$ and $\psi^{-1}(x)$ - $\mu(x,y)$ corresponds to a homomorphism $V \to R$, and the coefficients t_i of $\psi^{-1}(x) = (\sum_i t_i x^{p^i})_\mu \in R[[x]]$ then define a homomorphism $V[t_i] \to R$. Thus $\psi : \mu \to \mu'$ is equivalent to a homomorphism

$$VT := V[t_i | i \in \mathbb{N}] \to R$$

Here $\deg t_i = 2(p^i - 1)$ (and $t_0 = 1$ - sorry if this convention annoys you).

□

In fact one can say quite a bit more about the structure of V:

Through the lens of standard Yoneda nonsense one sees that the inclusion of groupoid-valued functors (on the category of graded commutative $\mathbb{Z}_{(p)}$-algebras)

$$(SI(-), FGL_{\text{ptyp}}(-)) \to (SI(-), FGL(-))$$

corresponds to a homomorphism $\pi : L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to V$ (which in turn corresponds to the universal p-typical formal group law). Evidently this is just the quotient map described in the previous proposition. On the other hand, Cartier's theorem provides a retraction of groupoid-valued functors

$$(SI(-), FGL(-)) \to (SI(-), FGL_{\text{ptyp}}(-))$$

which must correspond to a homomorphism $\iota : V \to L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. It's not too hard to describe this explicitly: as described above, Cartier's theorem provides a strict isomorphism $\psi(x)$ from the universal formal group law $\mu(x,y)$ over $L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ to a p-typical formal group law $\mu'(x,y) = \psi(\mu(\psi^{-1}(x), \psi^{-1}(y)))$. This p-typical formal group law μ' corresponds to the desired homomorphism $\iota : V \to L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

Since the composition of groupoid-valued functors

$$(SI(-), FGL_{\text{ptyp}}(-)) \to (SI(-), FGL(-)) \to (SI(-), FGL_{\text{ptyp}}(-))$$

is the identity, the composition

$$V \xrightarrow{\iota} L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \xrightarrow{\varphi} V$$

is also the identity. Thus φ is surjective (which we already kind of knew, by construction) and ι is injective. So, we can figure out the structure of V by identifying its image $\iota(V) \subset L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Note that this is also the image of the homomorphism $g := \iota \circ \varphi : L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. It's easier to analyze this map after base-changing to $\mathbb{Q}$: we'll have a commutative diagram

$$\begin{array}{ccc} L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} & \xrightarrow{g} & L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow \\ L \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{g \otimes \mathrm{id}} & L \otimes_{\mathbb{Z}} \mathbb{Q} \end{array} \tag{1.28}$$

Recall that we have a canonical isomorphism $L \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}[m_i]$ where the m_i correspond to the coefficients of the logarithm of the universal formal group law. With respect to this isomorphism, the map $g \otimes \mathrm{id}$ can be identified with the homomorphism

$$\mathbb{Q}[m_i] \to \mathbb{Q}[m_i] \text{ taking } m_i \mapsto \begin{cases} m_i & \text{if } i = p^k - 1 \\ 0 & \text{otherwise} \end{cases}$$

(since $\log_{\mu'}(x) = \sum_i m_{p^k-1} x^{p^k} \in L \otimes_{\mathbb{Z}} \mathbb{Q}$) - notice that this vividly illustrates the fact that g is an idempotent homomorphism. Proceeding in this way one obtains a canonical isomorphism

$$\mathrm{im} g \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}[m_{p^k-1}]$$

and hence $V \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}[m_{p^k-1}]$. According to Ravenel, an argument along the lines of the proof of Lazard's theorem shows that $V \simeq \mathbb{Z}_{(p)}[v_i | i \in \mathbb{N}]$ where $\deg v_i = 2(p^i - 1)$ (and where $v_0 = 1$). In fact one can describe a choice of generators v_i quite explicitly (more on this later). Note that this is in stark contrast to the case of the Lazard ring L - while Lazard's theorem shows that $L \simeq \mathbb{Z}[x_i]$ is a polynomial ring on generators x_i of degree $2i$, according to all of the complex cobordists I've spoken with it's difficult to pin down a choice of these x_i.

Since (V, VT) corepresents the groupoid-valued functor $(SI(-), FGL_{\text{ptyp}}())$ on the category of graded commutative $\mathbb{Z}_{(p)}$-algebras, we know it's a Hopf algebroid over $\mathbb{Z}_{(p)}$. Here's a description of its structure maps:

Say $\psi : \mu \to \mu'$ is a strict isomorphism of p-typical formal group laws over a $\mathbb{Z}_{(p)}$-algebra R. Recall that it's completely described by the source $\mu(x, y)$, which determines a homomorphism $V \to R$, and $\psi^{-1}(x)$, which determines a homomorphism $VT \to R$ as described above. From this it's already clear that the cosource $\eta_L : V \to VT$ is the usual inclusion $V \subset V[t_i] = VT$.

Note that the target of ψ is $\mu'(x, y) = \psi(\mu(\psi^{-1}(x), \psi^{-1}(y)))$, and this determines a homomorphism $V \to R$. At least in principle the coefficients of $\mu'(x, y)$ can be computed as polynomials in the coefficients of $\mu(x, y)$ and $\psi^{-1}(x)$, and this will give a description of η_R, but again it's easier to work with

$$\eta_R \otimes \mathrm{id} : V \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \to VT \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$$

Let $\lambda_i \in V \otimes \mathbb{Q}$ be the image of $m_{p^i-1} \in L \otimes \mathbb{Q}$ under the map $L \otimes \mathbb{Q} \to V \otimes \mathbb{Q}$. As discussed above we have a canonical isomorphism $V \otimes \mathbb{Q} \simeq \mathbb{Q}[\lambda_i]$. If R is a $\mathbb{Q}$ algebra then every p-typical formal group law μ over R has a unique logarithm which must be of the form $\log_\mu(x) = \sum_i r_i x^{p^i}$ and its coefficients determine a homomorphism $\mathbb{Q}[\lambda_i] \to R$. As discussed above, the logarithm of μ' is related to that of μ by

$$\log_{\mu'}(x) = \log_\mu(\psi^{-1}(x))$$

and restricting attention to the universal case, where $R = VT \otimes \mathbb{Q}$, μ is the universal p-typical formal group law (with logarithm $\sum_i \lambda_i x^{p^i}$) and $\psi^{-1}(x) = (\sum_i t_i x^{p^i})_\mu$, this reads

$$\log_{\mu'}(x) = \log_\mu(\sum_i t_i x^{p^i})_\mu = \sum_i \log_\mu(t_i x^{p^i})$$

$$= \sum_i \sum_j \lambda_j (t_i x^{p^i})^{p^j} = \sum_{i,j} \lambda_j t_i^{p^j} x^{p^{i+j}}$$

It must be that $\eta_R \otimes \mathrm{id}$ satisfies

$$\sum_i \eta_R \otimes \mathrm{id}(\lambda_i) x^{p^i} = \sum_{i,j} \lambda_j t_i^{p^j} x^{p^{i+j}} = \sum_i (\sum_{j+k=i} \lambda_k t_j^{p^k}) x^{p^i}$$

so that $\eta_R \otimes \mathrm{id}(\lambda_i) = \sum_{j+k=i} \lambda_k t_j^{p^k}$.

Again the coidentity $\epsilon : VT \to V$ is just the usual map $VT = V[t_i] \to V$ taking $t_i \mapsto 0$ for all i.

Suppose $\psi : \mu \to \mu'$ and $\psi' : \mu' \to \mu''$ are strict isomorphisms of p-typical formal group laws over R. Then their composition is the strict isomorphism $\psi' \circ \psi : \mu \to \mu''$, which will be determined by its source $\mu(x,y)$ and $(\psi' \circ \psi)^{-1}(x) = \psi^{-1}(\psi'^{-1}(x))$. Restricting to the case where $R = VT \otimes VT = V[t_i] \otimes V[t_i']$, μ is the universal p-typical formal group law and $\psi^{-1}(x) = (\sum_i t_i x^{p^i})_\mu$, $\psi'^{-1}(x) = (\sum_i t_i' x^{p^i})_{\mu'}$, this becomes

$$(\psi' \circ \psi)^{-1}(x) = \psi^{-1}(\sum_i t_i' x^{p^i})_{\mu'} = (\sum_i \psi^{-1}(t_i' x^{p^i}))_\mu$$

$$= (\sum_i (\sum_j t_j (t_i' x^{p^i})^{p^j})_\mu)_\mu = (\sum_{i,j} t_j t_i'^{p^j} x^{p^{i+j}})_\mu$$

It must be that the cocomposition $\Delta : VT \to VT \otimes VT$ satisfies

$$(\sum_i \Delta(t_i) x^{p^i})_\mu = (\sum_{i,j} t_j t_i'^{p^j} x^{p^{i+j}})_\mu = (\sum_i \sum_{j+k=i} t_k t_j'^{p^k} x^{p^i})_\mu$$

One can also take the logarithm $\log_\mu$ of this identity to obtain

$$\sum_i \log_\mu(\Delta(t_i) x^{p^i}) = \sum_{i,j} \log_\mu(t_j t_i'^{p^j} x^{p^{i+j}}) \text{ which is to say}$$

$$\sum_{i,j} \lambda_j (\Delta(t_i) x^{p^i})^{p^j} = \sum_{i,j,k} \lambda_k (t_j t_i'^{p^j} x^{p^{i+j}})^{p^k} \text{ or}$$

$$\sum_i \sum_{j+k=i} \lambda_k \Delta(t_j)^{p^k} x^{p^i} = \sum_i \sum_{j+k+l=i} \lambda_l t_k^{p^l} t_j'^{p^{k+l}} x^{p^i}$$

and comparing coefficients shows that

$$\sum_{j+k=i} \lambda_k \Delta(t_j)^{p^k} = \sum_{j+k+l=i} \lambda_l t_k^{p^l} t_j'^{p^{k+l}} \text{ for all } i$$

Notice that the inverse of a strict isomorphism $\psi : \mu \to \mu'$ is $\psi^{-1} : \mu' \to \mu$, which is determined by its source μ' and $\psi(x)$ - we have $\psi^{-1}(\psi(x)) = x$. Let's again restrict attention to the universal case, where $R = VT$, μ is the universal p-typical formal group law and $\psi^{-1}(x) = (\sum_i t_i x^{p^i})_\mu$ - in this case $\psi(x) = (\sum_i c(t_i) x^{p^i})_{\mu'}$. Observe that

$$x = \psi^{-1}(\psi(x)) = \psi^{-1}(\sum_i c(t_i) x^{p^i})_{\mu'}$$

$$= (\sum_i \psi^{-1}(c(t_i) x^{p^i}))_\mu = (\sum_{i,j} t_j c(t_i)^{p^j} x^{p^{i+j}})_\mu$$

One can also take the logarithm of this identity to obtain

$$\sum_i \lambda_i x^{p^i} = \sum_{i,j} \log_\mu(t_j c(t_i)^{p^j} x^{p^{i+j}})$$

$$= \sum_{i,j,k} \lambda_k t_j^{p^k} c(t_i)^{p^{j+k}} x^{p^{i+j+k}} = \sum_i \sum_{j+k+l=i} \lambda_l t_k^{p^l} c(t_j)^{p^{k+l}} x^i$$

so that $\lambda_i = \sum_{j+k+l=i} \lambda_l t_k^{p^l} c(t_j)^{p^{k+l}}$ for all i.

I'll now describe an explicit choice of elements $v_i \in V$ of degree $2(p^i - 1)$ generating V as a polynomial algebra over $\mathbb{Z}_{(p)}$, i.e. so that $V = \mathbb{Z}_{(p)}[v_i]$. The **Araki generators** $v_i \in V$ are described implicitly by the formula

$$[p]_\mu(x) = (\sum_i v_i x^{p^i})_\mu \in V[[x]]$$

Of course this only makes sense once one argues that $[p]_\mu(x)$ is power series of this form. To see this just recall that $[p]_\mu(x) = \exp_\mu(p\log_\mu(x))$ - since both μ and the additive formal group law are p-typical, we know $\log_\mu(x) = \sum_i \lambda_i x^{p^i}$ and $\exp_\mu(x) = (\sum_i c(\lambda_i)x^{p^i})_\mu$. In that case

$$[p]_\mu(x) = \exp_\mu(p\log_\mu(x)) = \exp_\mu(\sum_i p\lambda_i x^{p^i})$$

$$= (\sum_i \exp_\mu(p\lambda_i x^{p^i}))_\mu = (\sum_{i,j} c(\lambda_j)(p\lambda_i x^{p^i})^{p^j})_\mu$$

from which it's clear that $[p]_\mu(x)$ can contain only p^ith powers of x.

Remark 1.83. There should be a more direct verification of this fact, beginning with the identities

$$f_{q,\mu}(x) = [\frac{1}{q}]_\mu(\sum_{i=0}^{q-1} \zeta_q^i x)_\mu = 0 \text{ for all primes } q \neq p$$

For instance, the above identities are the same as

$$(\sum_{i=0}^{q-1} \zeta_q^i x)_\mu = 0 \text{ for all primes } q \neq p$$

and from there we should have

$$0 = [p]_\mu(\sum_{i=0}^{q-1} \zeta_q^i x)_\mu = (\sum_{i=0}^{q-1} [p]_\mu(\zeta_q^i x))_\mu$$

Suppose for the sake of argument that $[p]_\mu(x) = (\sum_i c_i x^i)_\mu$ - then this would become

$$(\sum_{i=0}^{q-1} [p]_\mu(\zeta_q^i x))_\mu = (\sum_j \sum_{i=0}^{q-1} c_j \zeta_q^{ij} x^j)_\mu$$

The idea is that the "μ-coefficients $\sum_i c_j \zeta_q^{ij}$" are all 0, and since $\sum_i \zeta_q^{ij} = q$ if $j|q$, we must have $c_j = 0$ unless j is a power of p. Obviously it would take a bit more work to make this precise.

Applying $\log_\mu$ to the identity $[p]_\mu(x) = (\sum_i v_i x^{p^i})_\mu$, one sees that

$$p\log_\mu(x) = \sum_i \log_\mu(v_i x^{p^i})$$

and expanding this further one obtains

$$\sum_i p\lambda_i x^{p^i} = \sum_{i,j} \lambda_j v_i^{p^j} x^{p^{i+j}} = \sum_i \sum_{j+k=i} \lambda_k v_j^{p^k} x^{p^i}$$

so that $p\lambda_i = \sum_{j+k=i} \lambda_k v_j^{p^k}$ for all i. I'm not going to actually prove that the v_i serve as polynomial generators - see Ravenel.

Proposition 1.84. *In terms of the isomorphisms $\mathbb{Z}_{(p)}[v_i] \simeq V$ and $\mathbb{Z}_{(p)}[v_i][t_i] \simeq VT$ the structure map $\eta_R : V \to VT$ can be described as*

$$(\sum_{i,j} t_i \eta_R(v_j)^{p^i})_\mu = (\sum_{i,j} v_i t_j^{p^i})_\mu$$

Proof. Let $\psi : \mu \to \mu'$ be the universal strict isomorphism of p-typical formal group laws, with $\psi^{-1}(x) = (\sum_i t_i x^{p^i})_\mu$. It seems clear enough that

$$[p]_{\mu'}(x) = (\sum_i \eta_R(v_i) x^{p^i})_{\mu'}$$

Applying ψ^{-1} to both sides then yields

$$[p]_\mu(\psi^{-1}(x)) = (\sum_i \psi^{-1}(\eta_R(v_i) x^{p^i}))_\mu$$

and expanding this further one obtains

$$(\sum_{i,j} v_j t_i^{p^j} x^{p^{i+j}})_\mu = (\sum_{i,j} t_j \eta_R(v_i)^{p^j} x^{p^{i+j}})_\mu$$

□

Evidently by using this proposition one can provide a more intrinsic description of the Hopf algebroid structure of (V, VT) (making less use of the logarithm).

There are a lot of interesting things to be said about formal group laws over graded commutative $\mathbb{F}_p$-algebras.

Just to lay some groundwork, recall that we have two distinct module structures on $(x) \subset R[[x]]$ - the usual R-module structure as well as a $\mathbb{Z}_{(p)}$-module structure given by μ. The map of formal group laws ψ determines an endomorphism of the *latter* module.

Note also we have a Frobenius endomorphism

$$\wedge^p : R[[x]] \to R[[x]] \text{ taking } f(x) \mapsto f(x)^p$$

over the Frobenius endomorphism $R \to R$ taking $r \mapsto r^p$. This provides a homomorphism from the $\mathbb{Z}_{(p)}$-module (x) with the operation μ to the $\mathbb{Z}_{(p)}$-module (x) with the operation $\mu^{(1)}$ defined implicitly by the equation

$$\mu(x,y)^p = \mu^{(1)}(p^*(x), p^*(y)) = \mu^{(1)}(x^p, y^p)$$

To unravel this note that if $\mu(x,y) = \sum_{i,j} a_{ij} x^i y^j$ then

$$\mu(x,y)^p = \sum_{i,j} a_{ij}^p x^{pi} y^{pj} = \sum_{i,j} a_{ij}^p (x^p)^i (y^p)^j$$

so the coefficients of $\mu^{(1)}(x,y)$ are a_{ij}^p. Which is to say, $\mu^{(1)}$ is obtained by pushing μ forward along the Frobenius $\wedge^p : R \to R$. In a similar fashion one can define iterated Frobenius endomorphisms

$$\wedge^{p^i} : R[[x]] \to R[[x]] \text{ taking } f(x) \mapsto f(x)^{p^i}$$

and these will give homomorphisms from μ to the formal group law $\mu^{(i)}$ defined implicitly by

$$\mu(x,y)^{p^i} = \mu^{(i)}(x^{p^i}, y^{p^i})$$

and one sees that the coefficients of $\mu^{(i)}(x,y)$ are $a_{jk}^{p^i}$.

Proposition 1.85. *Let μ be a formal group law over a graded commutative $\mathbb{F}_p$-algebra R and let $\psi : \mu \to \mu$ be a non-zero endomorphism of μ. Then for some (unique) $n \in \mathbb{N}$, there's a homomorphism of formal group laws $\rho : \mu^{(i)} \to \mu$ with $\rho'(0) \neq 0$ so that $\psi(x) = \rho(x^{p^n})$.*

Proof. It will suffice to show that if $\psi'(0) = 0$, then there's a non-zero homomorphism $\rho : \mu^{(1)} \to \mu$ so that $\psi(x) = \rho(x^p)$ - the desired result then follows by induction.

As ψ is an endomorphism of μ, we have

$$\psi(\mu(x,y)) = \mu(\psi(x), \psi(y))$$

Differentiating with respect to y shows that

$$\psi'(\mu(x,y))\frac{\partial\mu}{\partial y}(x,y) = \frac{\partial\mu}{\partial y}(\psi(x),\psi(y))\psi'(y)$$

Setting $y = 0$ and recalling that $\mu(x,0) = x$, $\frac{\partial\mu}{\partial y}(x,0) \in 1 + (x) \subset R[[x,y]]$ and similarly (since $\psi(0) = 0$) $\frac{\partial\mu}{\partial y}(\psi(x),\psi(0)) \in 1 + (x) \subset R[[x,y]]$, we obtain an equation of the form

$$\psi'(x)\frac{\partial\mu}{\partial y}(x,0) = \frac{\partial\mu}{\partial y}(\psi(x),0)\psi'(0) = 0$$

and conclude that $\psi'(x) = 0$. But then certainly $\psi(x) = \rho(x^p)$ for some non-zero formal power series $\rho(x)$ (we're in characteristic p - here's where that comes into play!) , and we have

$$\rho(\mu^{(1)}(x^p,y^p)) = \rho(\mu(x,y)^p)$$

$$= f(\mu(x,y)) = \mu(f(x),f(y)) = \mu(\rho(x^p),\rho(y^p))$$

so it must be that $\rho(\mu^{(1)}(x,y)) = \mu(\rho(x),\rho(y))$, i.e. ρ gives a homomorphism from $\mu^{(1)}$ to μ.

□

Remark 1.86. It's clear from the proof of this lemma that if ψ is an endomorphism of formal group laws over a torsion-free ring R (I just mean R is torsion free as an abelian group), and if $\psi'(0) = 0$, then $\psi'(x) = 0$ and hence $\psi(x) = 0$.

Remark 1.87. By the way, I should point out that the endomorphisms of a formal group law μ over an arbitrary graded commutative ring R always form a ring: given endomorphisms $\psi(x), \rho(x)$ of μ we can form their sum like

$$(\psi + \rho)(x) = \psi(x) +_\mu \rho(x) = \mu(\psi(x),\rho(x))$$

and their composition like $(\rho \circ \psi)(x) = \rho(\psi(x))$. From this perspective the above remark shows that if R is torsion-free, the map

$$\psi(x) \mapsto \psi'(0) \text{ provides an embedding } \mathrm{End}(\mu) \to R$$

In particular $\mathrm{End}(\mu)$ is a graded commutative ring (which at least to me is a few steps removed from obvious). To see a vivid illustration of this fact, note that if R is a $\mathbb{Q}$-algebra, then μ is strictly isomorphic to the additive formal group law $x + y$, and such a strict isomorphism will give an induced isomorphism $\mathrm{End}(\mu) \simeq \mathrm{End}(x + y)$. The endomorphism ring of $x + y$ is obviously just R, and so we see $\mathrm{End}(\mu) \simeq R$. With a little more care one can show the derivative map $\mathrm{End}(\mu) \to R$ is an isomorphism.

The above proposition shows that when R is an $\mathbb{F}_p$-algebra, the kernel of the derivative homomorphism

$$\mathrm{End}(\mu) \to R \text{ taking } \psi(x) \mapsto \psi'(0)$$

is precisely the image of the map $\mathrm{Hom}(\mu^{(1)},\mu) \to \mathrm{End}(\mu)$ obtained by precomposing with the Frobenius $\wedge^p : \mu \to \mu^{(1)}$.

For a formal group law μ over a graded commutative $\mathbb{F}_p$ algebra R one particularly interesting endomorphism is formal multiplication by p, given by $[p]_\mu(x)$. It's entirely possible that $[p]_\mu(x) = 0$ - this is the case when μ is the additive formal group law. On the other hand if $[p]_\mu(x) \neq 0$, then from the above proposition we know that for some unique $n \in \mathbb{N}$ there's a homomorphism $\rho : \mu^{(n)} \to \mu$ with $\rho'(0) \neq 0$ so that $[p]_\mu(x) = \rho(x^{p^n})$, and since the leading term of $[p]_\mu(x)$ is $px = 0$, it's clear that $n > 0$.

Definition 1.88. Let μ be a formal group law over a graded commutative $\mathbb{F}_p$-algebra R. If $[p]_\mu(x) \neq 0$, the **height of** μ is defined to be the smallest positive integer n so that $[p]_\mu(x) = \rho(x^{p^n})$ where $\rho : \mu^{(n)} \to \mu$ is a homomorphism with $\rho'(0) \neq 0$. If $[p]_\mu(x) = 0$ then the height of μ is defined to be ∞.

Perhaps more concretely, the height of μ is the integer n so that $[p]_\mu(x)$ has leading term ax^{p^n} for some non-zero $a \in R$ (or ∞ if $[p]_\mu(x) = 0$).

Remark 1.89. In the case where μ is p-typical we know that $[p]_\mu(x)$ is of the form $[p]_\mu(x) = (\sum_i r_i x^{p^i})_\mu$ and from this it's pretty clear that $[p]_\mu(x)$ must be a formal power series in x^{p^n} for some n.

Proposition 1.90. *Height is an isomorphism invariant (hence in particular a strict isomorphism invariant) of formal group laws over a graded commutative $\mathbb{F}_p$-algebra R.*

Proof. Let $\psi : \mu \to \mu'$ be an isomorphism of formal group laws over R. Then

$$\psi([p]_\mu(x)) = [p]_{\mu'}(\psi(x))$$

and the result follows by comparing leading terms. □

The following observation seems to be useful:

Proposition 1.91. *Let μ be a formal group law over a commutative $\mathbb{F}_p$-algebra R, and let $\psi : \mu \to \mu'$ be the isomorphism from μ to a p-typical formal group law μ' constructed in the proof of Cartier's theorem. Let*

$$\varphi : V \to R$$

be the homomorphism corresponding to μ'. Then the height of μ is the first n such that $\varphi(v_n) \neq 0 \in R$ (or ∞ if no such n exists).

Proof. Since height is an isomorphism invariant, heightμ = heightμ', and (basically by the definition of the Araki generators)

$$[p]_{\mu'}(x) = (\sum_i \varphi(v_i) x^{p^i})_{\mu'}$$

From this we see that $[p]_{\mu'}(x) = 0$ if and only if $\varphi(v_n) = 0$ for all n. On the other hand, if $[p]_{\mu'}(x) \neq 0$, then clearly its leading term will be $\varphi(v_n)x^{p^n}$ where n is the smallest positive integer so $\varphi(v_n) \neq 0$. Thus heightμ' is the smallest n so that $\varphi(v_n) \neq 0$ or ∞ if no such n exists. □

In a similar flavor:

Proposition 1.92. *Let R be a torsion-free graded commutative ring and let μ be a formal group law over R. Let $\sum_i m_i x^{i+1}$ be the logarithm of μ over $R \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the height of the mod-p reduction of μ over $R \otimes_{\mathbb{Z}} \mathbb{F}_p$ is the first positive integer n so that $\lambda_n := m_{p^n-1} \notin R \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.*

Proof. Recall that if $\psi : \mu \to \mu'$ is the isomorphism over $R \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ described in Cartier's theorem, we have $\log_{\mu'}(x) = \sum_i m_{p^i-1} x^{p^i}$ - letting $\lambda_i = m_{p^n-1}$, this reads $\log_{\mu'}(x) = \sum_i \lambda_i x^{p^i}$.

Now recall that if $[p]_{\mu'}(x) = \sum_i v_i x^{p^i}$, then we'll have

$$p\lambda_n = \sum_{j+k=n} \lambda_k v_j^{p^k} \in R \otimes_{\mathbb{Z}} \mathbb{Q} \text{ for all } n$$

From the previous proposition we know that heightμ is the first n so that $v_n \neq 0 \in R \otimes_{\mathbb{Z}} \mathbb{F}_p$ (or ∞ if no such n exists). Now one must stare at the above formula and deduce that this is the first n so that $\lambda_n \notin R \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

Here's a good way to see it: notice that if $v_m = 0 \in R \otimes_{\mathbb{Z}} \mathbb{F}_p$ for $m \leq n$, then it must be that $p | v_m$ in $R \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ for $m \leq n$. If $\lambda_i \in R \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ for $i < n$, then the equation

$$p\lambda_n = \sum_{j+k=n} \lambda_k v_j^{p^k}$$

can be divided by p to show that

$$\lambda_n = \sum_{j+k=n} \lambda_k \frac{v_j^{p^k}}{p}, \text{ an element of } R \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$$

On the other hand if $\lambda_i \in R \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ for $i \leq n$ and $v_i = 0 \in R \otimes_{\mathbb{Z}} \mathbb{F}_p$ for $i < n$, the equation

$$p\lambda_n = \sum_{j+k=n} \lambda_k v_j^{p^k} \text{ can be rearranged as}$$

$$v_n = p\lambda_n - \sum_{j+k=n, j<n} \lambda_k v_j^{p^k}$$

shows that $v_n = 0 \in R \otimes_{\mathbb{Z}} \mathbb{F}_p$.

□

Remark 1.93. At this point it's worth observing that we have groupoid-valued functors

$$R \mapsto (SI(R), FGL^{\geq n}_{p\text{typ}}(R))$$

on the category of graded commutative $\mathbb{F}_p$-algebras, where $FGL^{n}_{p\text{typ}}(R)$ denotes the p-typical formal group laws of height at least n over R, and n is either a positive integer or ∞ (we could of course omit the p-typicality condition, but it should be clear from the preceding propositions that it's frequently convenient to use Cartier's theorem to replace a formal group law by an isomorphic p-typical one). I would like to show that these functors are corepresentable by Hopf algebroids over $\mathbb{F}_p$.

To begin, notice that p-typical formal group laws with strict isomorphisms over $\mathbb{F}_p$-algebras are corepresented by $(V \otimes \mathbb{F}_p, VT \otimes \mathbb{F}_p)$. I'd like to describe a Hopf algebroid corepresenting those of height at least n as a quotient of $(V \otimes \mathbb{F}_p, VT \otimes \mathbb{F}_p)$.

So, let μ be a p-typical formal group law of height at least n over a graded commutative $\mathbb{F}_p$-algebra R. We know that it's classified by a homomorphism $\varphi : \mathbb{F}_p[v_i] = V \otimes \mathbb{F}_p \to R$, and according to proposition 1.39

$$\varphi(v_i) = 0 \text{ for } i < n$$

Thus φ factors through the quotient $\mathbb{F}_p[v_n, v_{n+1}, \dots] = V \otimes \mathbb{F}_p / (v_1, v_2, \dots, v_{n-1})$. It seems clear that $V \otimes \mathbb{F}_p / (v_1, v_2, \dots, v_{n-1})$ corepresents the set-valued functor $FGL^{\geq n}_{p\text{typ}}(-)$.

Now let $\psi : \mu \to \mu'$ be a strict isomorphism between p-typical formal group laws of height at least n over R. Recall from proposition 1.36 that the formal p-series of μ and μ' are related by

$$\psi^{-1}([p]'_\mu(x)) = [p]_\mu(\psi^{-1}(x))$$

and (yes, I realize this is getting sloppy) this leads to a relation of the form

$$(\sum_{i,j} v_j t_i^{p^j} x^{p^{i+j}})_\mu = (\sum_{i,j} t_j \eta_R(v_i)^{p^j} x^{p^{i+j}})_\mu$$

Restricting to the universal case and taking logarithms on both sides shows

$$\sum_{i,j} \log_\mu(v_j t_i^{p^j} x^{p^{i+j}}) = \sum_{i,j} \log_\mu(t_j \eta_R(v_i)^{p^j} x^{p^{i+j}}) \text{ or expanding further}$$

$$\sum_{i,j,k} \lambda_k v_j^{p^k} t_i^{p^{j+k}} x^{p^{i+j+k}} = \sum_{i,j,k} \lambda_k t_j^{p^k} \eta_R(v_i)^{p^{j+k}} x^{p^{i+j+k}} \text{ and comparing coefficients,}$$

$$\sum_{i+j+k=n} \lambda_k v_j^{p^k} t_i^{p^{j+k}} = \sum_{i+j+k=n} \lambda_k t_j^{p^k} \eta_R(v_i)^{p^{j+k}}$$

I was hoping that this would lead to some conditions on the t_i and give a description of the quotient of $VT \otimes \mathbb{F}_p$ corresponding to strict isomorphisms of p-typical formal group laws of height at least n - but I can't see it.

This remark shows that the notion of "height at least n" is in some senses better behaved than that of "height n" defined above. In fact Lurie's definitions are as follows: let μ be a formal group law over a graded commutative $\mathbb{F}_p$-algebra. We know that $[p]_\mu(x)$ is and endomorphism of μ, so for some (unique) m there's a homomorphism $\rho : \mu^{(m)} \to \mu$ with $\rho'(0) \neq 0$ so that $[p]_\mu(x) = \rho(x^{p^m})$. The formal group law μ is said to have height at least n if $n \leq m$ (equivalently, $[p]_\mu(x)$ vanishes to order x^{p^m}). It is said to have height exactly n if $m = n$ and $\rho'(0) \in R^\times$ is a unit (equivalently, the leading term of $[p]_\mu(x)$ is ax^{p^n}, where $a \in R^\times$) - note that in this situation ρ is an *isomorphism* of formal group laws. There are lots of ways in which Lurie's definitions are better. See his notes on chromatic homotopy theory.

Definition 1.94. Let R be a graded commutative $\mathbb{F}_p$-algebra. For each positive integer n define the p-typical formal group law μ_n of height n over R to be the one obtained from the homomorphism $\varphi_n : V \otimes \mathbb{F}_p \to R$ taking $v_n \mapsto 1$ and $v_i \mapsto 0$ for $i \neq n$.

It should be pointed out that the formal group laws μ_n may as well be defined over $\mathbb{F}_p$ - they can then be pushed forward along structure maps $\mathbb{F}_p \to R$.

Theorem 1.95 (Lazard)**.** *Suppose K is a separably closed field of characteristic p, and let μ be a formal group law of height n over K. Then there exists a (non-canonical) isomorphism $\psi : \mu_n \to \mu$.*

Sketch. By Cartier's theorem μ is (canonically) isomorphic to a p-typical formal group law of height n, and so we may safely assume from the get-go that it is p-typical.

Suppose $\psi : \mu_n \to \mu$ were an isomorphism of formal group laws, say with $\psi^{-1}(x) = (\sum_i t_i x^{p^i})_{\mu_n}$ - then by (a mild generalization of) proposition 1.36 we'd obtain an equation of the form

$$(\sum_{i,j} \delta_{jn} t_i^{p^j} x^{p^{i+j}})_{\mu_n} = (\sum_{i,j} t_j v_i^{p^j} x^{p^{i+j}})_{\mu_n}$$

where δ_{ij} is the Kronecker delta and I'm letting $v_i \in K$ be the coefficients of $[p]_\mu(x)$. This equation simplifies to

$$(\sum_i t_i^{p^n} x^{p^{i+n}})_{\mu_n} = (\sum_{i,j} t_j v_i^{p^j} x^{p^{i+j}})_{\mu_n}$$

The claim is that this identity can solved (inductively) for the coefficients t_i of $\psi^{-1}(x)$. To begin, compare leading terms to see that

$$t_0^{p^n} x^{p^n} = t_0 v_n x^{p^n}, \text{ so that } t_0^{p^n} = t_0 v_n$$

(since $v_i = 0$ when $i < n$ by hypothesis). We are trying to find a non-zero root t_0 of the polynomial

$$t_0^{p^n} - t_0 v_n = t_0(t_0^{p^n-1} - v_n)$$

Since the factor $t_0^{p^n-1} - v_n$ is separable (it's relatively prime to it's formal derivative $(p^n-1)t_0^{p^n-2}$ - this is where it's important that $v_n \neq 0$!) and by hypothesis K is separably possible, we can find such a root in K. So, we've solved for t_0.

Remark 1.96. Unless $v_n = 1$, we won't be able to choose $t_0 = 1$, and so we won't end up with a *strict isomorphism* ψ.

I'm not actually going to write out the inductive argument. Hopefully the above gives an idea of how one would go about it.

□

One can give a rather complete description of the endomorphism rings of these "standard" p-typical formal group laws of height n that I'm calling μ_n. To get started:

Proposition 1.97. *Let K be a field of characteristic p and let μ be a formal group law over K. Let $E = \mathrm{End}(\mu)$ denote the endomorphisms of μ. Then*

- *E is a (not necessarily commutative) ring under formal addition with respect to μ and composition.*
- *E is a domain.*
- *E is a torsion-free $\mathbb{Z}_p$-algebra if μ has finite height and a $\mathbb{F}_p$-algebra μ has infinite height.*

Sketch. The first bullet point has already been discussed above. The second bullet point is basically a statement about composition of non-zero power series in $(x) \subset K[[x]]$, so I'll skip that too.

Notice that we've already defined a $\mathbb{Z}_{(p)}$-algebra structure on E - for any $q \in \mathbb{Z}_{(p)}$, the power series $[q]_\mu(x)$ defined in proposition 1.28 gives an endomorphism of μ and in this way we obtain a homomorphism $\mathbb{Z}_{(p)} \to E$. If height$\mu = \infty$, then $[p]_\mu(x) = 0$ so the homomorphism $\mathbb{Z}_{(p)} \to E$ factors through $\mathbb{F}_p$ - in this case E is an $\mathbb{F}_p$-algebra.

This is a good time to point out that E is a complete topological ring - its topology comes from the descending filtration

$$E \supset E \cap (x) \supset E \cap (x)^2 \supset \cdots$$

(there are probably words for all these things but I forget them). When height$\mu = n < \infty$ the structure homomorphism $\mathbb{Z}_{(p)} \to E$ sends $p \mapsto [p]_\mu(x) \subset E \cap (x)^{p^n}$ and so it is a map of topological rings. Then by the universal property of $\mathbb{Z}_p$ we obtain a commutative diagram of ring homomorphisms

$$\begin{tikzcd} \mathbb{Z}_{(p)} \arrow[r] \arrow[d] & E \arrow[d, "="'] \\ \mathbb{Z}_p \arrow[r] & E \end{tikzcd} \tag{1.29}$$

so E is a $\mathbb{Z}_p$-algebra. Finally since E is a domain, to conclude it's a torsion-free $\mathbb{Z}_p$-algebra we just need to know that the structure homomorphism $\mathbb{Z}_p \to E$ is injective, and this isn't hard to see (say by looking at the leading term of $[q]_\mu(x)$ for $q \in \mathbb{Z}_p$).

□

See Ravenel for further calculations.

1.4.2. *MU and The Brown-Peterson spectrum BP.*

Definition 1.98. Let E be a commutative ring spectrum. A **complex orientation for** $E$is a class $x_E \in \tilde{E}^2(\mathbb{C}P^\infty)$ whose restriction to $\tilde{E}^2(\mathbb{C}P^1) = \tilde{E}^2(S^2) \simeq \pi_0(E)$ is 1.

Let MU be the Thom spectrum for complex bordism, which can be described as follows: for each $n \in \mathbb{N}$ there is a "tautological" complex vector bundle $\gamma_n : E(\gamma_n) \to BU(n)$ over the classifying space $BU(n)$. Let $MU(n) = \mathrm{Th}\gamma(n)$ be its Thom space. For each n the inclusion $\iota : U(n) \to U(n+1)$ gives rise to a map of complex vector bundles

$$\begin{tikzcd} E(\gamma_n \oplus 1) \arrow[r] \arrow[d] & E(\gamma_{n+1}) \arrow[d] \\ BU(n) \arrow[r, "B\iota"] & BU(n+1) \end{tikzcd} \tag{1.30}$$

which is injective on fibers, and this yields an induced map of Thom spaces

$$\Sigma^2 MU(n) = \mathrm{Th}(\gamma_n \oplus 1) \to \mathrm{Th}(\gamma_{n+1}) = MU(n+1)$$

Declaring $MU_{2n} = MU(n)$ and $MU_{2n+1} = \Sigma MU(n)$ and using the structure maps

$$\Sigma MU_{2n} = \Sigma MU(n) = MU_{2n+1} \text{ and } \Sigma MU_{2n+1} = \Sigma^2 MU(n) \to MU(n+1) = MU_{2n+2}$$

one obtains a spectrum in the sense of Adams (or a prespectrum in the sense of May, which would need to be "spectrified"). The inclusions $U(m) \otimes U(n) \subset U(m+n)$ give rise to maps of vector bundles

$$\begin{array}{ccc} E(\gamma_m \otimes \gamma_n) & \longrightarrow & E(\gamma_{m+n}) \\ \big\downarrow & & \big\downarrow \\ BU(m) \times BU(n) & \longrightarrow & BU(m+n) \end{array} \tag{1.31}$$

These induce maps of Thom spaces

$$MU(m) \wedge MU(n) = \mathrm{Th}(\gamma_m \times \gamma_n) \to \mathrm{Th}(\gamma_{m+n}) = MU(m+n)$$

and one can show these assemble to give a morphism of spectra

$$\mu : MU \wedge MU \to MU$$

Also, the inclusions of vector space fibers $\mathbb{C}^n \subset E(\gamma_n)$ induce maps of Thom spaces $S^{2n} \to MU(n)$ which assemble to give a morphism of spectra $\eta : S \to MU$. It's known that η and μ serve as structure maps making MU a commutative ring spectrum.

There is a natural identification $MU_2 = MU(1) = \mathrm{Th}(\gamma_1) \simeq \mathbb{C}P^\infty$, and from this one obtains a morphism of spectra

$$\Sigma^{-2}\mathbb{C}P^\infty \to MU$$

where on the left I mean the suspension spectrum of $\mathbb{C}P^\infty$, desuspended by 2. One can show that its restriction to $\Sigma^{-2}\mathbb{C}P^1 = S$ represents $1 \in \pi_* MU$. Thus the class in $\tilde{MU}^2(\mathbb{C}P^\infty)$ represented by this morphism $\Sigma^{-2}\mathbb{C}P^\infty \to MU$ serves as a complex orientation for MU. It is a theorem that MU is the universal complex oriented cohomology theory in the following sense:

Theorem 1.99. *Let E be a commutative ring spectrum. Then there is a natural one-to-one correspondence between*