THE J-HOMOMORPHISM AND THE ADAMS CONJECTURE

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1. The groups $J(X)$

1.1. **Spherical fibrations and the classical J-homomorphism.** Let X be a finite CW complex. If $E \xrightarrow{\pi} X$ is an orthogonal real vector bundle, then one can consider the associated sphere bundle $S(E) \xrightarrow{\pi} X$ (it's a special sort of sphere bundle over X, as its structure group reduces to the orthogonal group). Our first step will be to show that the assignment $E \mapsto S(E)$ induces homomorphisms

$$KO^0(X) \xrightarrow{S} \mathrm{Sph}(X)$$

where $KO^0(X)$ is the Grothendeick group of stable isomorphism classes of orthogonal vector bundles under direct sum, and $\mathrm{Sph}(X)$ is the Grothendeick group of stable fiber homotopy equivalence classes of spherical fibrations under fiberwise join. I'll take the definition/construction of $KO^0(X)$ to be well known (see Atiyah's *K-theory*). But let's talk about $\mathrm{Sph}(X)$.

Spherical fibrations. Consider the category of spherical fibrations $E \xrightarrow{\pi} X$, i.e. fibrations over X in which each fiber $E_x = \pi^{-1}(x)$ has the homotopy type of a sphere, i.e. there's a homotopy equivalence $S^n \simeq E_x$ (note that the dimension n is constant on each path component of X, but we're not assuming it's constant on X). A morphism f from E to another spherical fibration $E' \xrightarrow{\pi'} X$ is a continuous map

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ X & \xrightarrow{=} & X \end{array} \tag{1.1}$$

over X. Define two such morphisms f_0, f_1 to be homotopic if there's a morphism

$$\begin{array}{ccc} E \times I & \xrightarrow{h} & E' \times I \\ \downarrow & & \downarrow \\ X \times I & \xrightarrow{=} & X \times I \end{array} \tag{1.2}$$

agreeing with f_0 over $X \times \{0\}$ and f_1 over $X \times \{1\}$ (under the usual identifications).

Given E, E' as above, we may form the fiberwise join $E * E' \to X$. Its fiber over a point $x \in X$ is the join $E_x * E'_x$, and it has the only reasonable topology. In detail, take the join of the projections π, π'

$$E * E' \xrightarrow{\pi * \pi'} X * X$$

and pull it back over the quotient $X \times I \times X \xrightarrow{q} X * X$ to obtain

$$\begin{array}{ccc} q^* E * E' & \longrightarrow & E * E' \\ \downarrow & & \downarrow \\ X \times I \times X & \xrightarrow{q} & X * X \end{array} \tag{1.3}$$

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Now consider the composition

$$q^*E * E' \to X \times I \times X \xrightarrow{\text{project}} X \times X$$

and unravel definitions to observe that its fiber over (x, y) is $E_x * E'_y$. To obtain the desired fibration $E * E' \to X$, just pull this back over the diagonal $X \xrightarrow{\Delta} X \times X$.

It's straightforward to check that the fiberwise join $*$ is associative and commutative, and that the "empty fibration" $\emptyset \to X$ serves as an identity in the sense that there's a canonical isomorphism $E * \emptyset \simeq E$ for any spherical fibration E.

In particular, given any spherical fibration $E \xrightarrow{\pi} X$ and any $n \in \mathbb{N}$, we can form the fibration $E * (X \times S^n) \to X$, which will be briefly denoted by $E * n$.

Definition 1.1. Two spherical fibrations $E, E' \xrightarrow{\pi,\pi'} X$ are **stably fiber homotopy equivalent** if for some $n \in \mathbb{N}$ there's a fiber homotopy equivalence $E * n \simeq E' * n$.

Now one can check that the fiberwise join $*$ descends to an operation on the set of stable fiber homotopy equivalence classes of spherical fibrations, which is associative, commutative and has identity the class of $\emptyset$ (or the class [n] of any trivial fibration $X \times S^n \to X$). Thus the stable fiber homotopy equivalence classes form an abelian monoid under $*$.

Definition 1.2. $\mathrm{Sph}(X)$ is the Grothendeick group of this abelian monoid.

Suppose $f : X \to Y$ is a continuous map from X to another finite CW complex Y. Then we may pull back a spherical fibration $E \xrightarrow{\pi} Y$ over Y to obtain a spherical fibration $f^*E \xrightarrow{f^*\pi} X$. Evidently f^* gives a functor from the category of spherical fibrations over Y to spherical fibrations over X, and one can check that it's compatible with fiberwise joins and stable fiber homotopy equivalences. Thus we obtain a homomorphism

$$f^* : \mathrm{Sph}(Y) \to \mathrm{Sph}(X)$$

and the "homotopy invariance of pullback" implies that this homomorphism depends only on the homotopy class of f. Hence $\mathrm{Sph}(-)$ defines a contravariant functor from the homotopy category of finite CW complexes to abelian groups.

Now suppose X is a finite *pointed* CW complex, with basepoint $x_0 \in X$. Then the inclusion $\iota : \{x_0\} \to X$ and projection $p : X \to \{x_0\}$ yield a canonically split surjection

$$\mathrm{Sph}(X) \xrightarrow{\iota^*} \mathrm{Sph}(\{x_0\}) \simeq \mathbb{Z}$$

(p^* gives the splitting; its image is the $\mathbb{Z}$ summand corresponding to the trivial spherical fibrations).

Definition 1.3. $\tilde{\mathrm{Sph}}(X) := \ker \iota^* \subset \mathrm{Sph}(X)$; equivalently, $\tilde{\mathrm{Sph}}(X) := \mathrm{coker} p^*$.

In words, $\tilde{\mathrm{Sph}}(X)$ consists of classes with "virtual dimension 0 on the basepoint component of X." In particular if X is connected, it consists of classes with virtual dimension 0. Equivalently, we may think of $\tilde{\mathrm{Sph}}(X)$ as the Grothendieck group of "loose" stable fiber homotopy equivalence classes of spherical fibrations over X under the fiberwise join $*$, where

Definition 1.4. Two spherical fibrations $E, E' \xrightarrow{\pi,\pi'} X$ are **loosely stably fiber homotopy equivalent** if for some $n, n' \in \mathbb{N}$ there's a fiber homotopy equivalence $E * n \simeq E' * n'$.

Note that this really is a "looser" notion of stable fiber homotopy equivalence; for instance, it doesn't even require that E, E' have the same fiber dimensions.

Suppose $f : (X, x_0) \to (Y, y_0)$ is a continuous map of finite pointed CW complexes. If $\iota' : y_0 \to Y$ is the inclusion of the basepoint of Y we'll have a commutative diagram of induced homomorphisms

$$\begin{array}{ccc} \mathrm{Sph}(Y) & \xrightarrow{f^*} & \mathrm{Sph}(X) \\ {\scriptstyle \iota'^*}\downarrow & & {\scriptstyle \iota^*}\downarrow \\ \mathrm{Sph}(\{y_0\}) & \longrightarrow & \mathrm{Sph}(\{x_0\}) \end{array} \tag{1.4}$$

and thus f^* restricts to a homomorphism $f^* : \tilde{\mathrm{Sph}}(Y) \to \tilde{\mathrm{Sph}}(X)$. So, $\tilde{Sph}(-)$ is a contravariant functor from the homotopy category of finite pointed CW complexes to abelian groups.

In the course of this discussion we've also shown:

Proposition 1.5. *Let X be a finite pointed CW complex. Then there's a natural direct sum decomposition*

$$\mathbb{Z} \oplus \tilde{\mathrm{Sph}}(X) \simeq \mathrm{Sph}(X)$$

The homomorphism of Grothendeick groups induced by "$E \mapsto S(E)$".

Proposition 1.6. *The assignment $E \mapsto S(E)$, viewed as a functor from the orthogonal vector bundles over X to the spherical fibrations over X, induces a natural homomorphism*

$$KO^0(X) \xrightarrow{S} \mathrm{Sph}(X)$$

Proof. Evidently the assignment $E \mapsto S(E)$ defines a functor from the category of orthogonal real vector bundles over X with isometries to the category of spherical fibrations over X with fiber homotopy equivalences (these are both (basically) groupoids).

If $E, E' \to X$ are two orthogonal vector bundles, there's a natural isomorphism of sphere bundles $S(E \oplus E') \simeq S(E) * S(E')$. Explicitly, this is given by the map

$$S(E) * S(E') \xrightarrow{\simeq} S(E \oplus E') \text{ sending}$$

$$[u, t, v] \mapsto \cos(\frac{\pi}{2}t)u + \sin(\frac{\pi}{2}t)v$$

Also, for any trivial vector bundle $X \times \mathbb{R}^n \to X$ we have $S(X \times \mathbb{R}^n) = X \times S^{n-1}$. It follows that the assignment $E \mapsto S(E)$ descends to a homomorphism of abelian monoids from the stable isomorphism classes of orthogonal real vector bundles to the stable fiber homotopy equivalence classes of spherical fibrations, and so it induces a homomorphism of Grothendeick groups $KO^0(X) \xrightarrow{S} \mathrm{Sph}(X)$.

To see that this homomorphism is natural with respect to pullback, just note that if $f : X \to Y$ is a continuous map of finite CW complexes and $E \xrightarrow{\pi} Y$ is an orthogonal real vector bundle over Y, then there's a canonical identification $S(f^*E) \simeq f^*S(E)$.

□

Note: In particular we have a natural homeomorphism

$$S^{m-1} * S^{n-1} \simeq S^{m+n-1} \text{ obtained from the map } S^{m-1} \times [0, \frac{\pi}{2}] \times S^{n-1} \to S^{m+n-1}$$

$$\text{sending } [u, t, v] \mapsto \cos(t)u + \sin(t)v$$

Note that the later map restricts to a diffeomorphism $\varphi : S^{m-1} \times (0, \frac{\pi}{2}) \times S^{n-1} \to S^{m+n-1}$ onto a dense open, with complement the evident copy of $S^{m-1} \cup S^{n-1} \subset S^{m+n-1}$. A direct computation shows that the differential $d\varphi : TS^{m-1} \times T(0, \frac{\pi}{2}) \times TS^{n-1} \simeq TS^{m-1} \times (0, \frac{\pi}{2}) \times S^{n-1} \to TS^{m+n-1}$ can be written as

$$''d\varphi = \cos(t)d\iota_m + (-\sin(t)u + \cos(t)v)dt + \sin(t)d\iota_n''$$

(really, this is the differential of the map $\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}m + n$ sending $(u, t, v) \mapsto \cos(t)u + \sin(t)v$, where $\iota_m : \mathbb{R}^m \to \mathbb{R}^{m+n}$ is the usual inclusion, and similarly for ι_n - so I'm implicitly viewing the spheres as embedded submanifolds). Then pulling back the standard (round) metric $\langle,\rangle_{m+n-1}$ on S^{m+n-1} along φ yields

$$\varphi^*\langle,\rangle_{m+n-1} = \cos(t)^2\langle,\rangle_{m-1} + dt^2 + \sin(t)^2\langle,\rangle_{n-1}$$

where $\langle,\rangle_{m-1}$ is the standard metric on S^{m-1}, and similarly for $\langle,\rangle_{n-1}$. It follows that pulling back the volume form dV_{m+n-1} on S^{m+n-1} along φ yields

$$\varphi^* dV_{m+n-1} = \cos(t)^{m-1} dV_{m-1} dt \sin(t)^{n-1} dV_{n-1}$$

So, we should have

$$\begin{aligned}\mathrm{vol}S^{m+n-1} = \int_{S^{m+n-1}} dV_{m+n-1} &= \int_{S^{m-1}\times(0,\frac{\pi}{2})\times S^{n-1}} \cos(t)^{m-1} dV_{m-1} dt \sin(t)^{n-1} dV_{n-1} \\ &= \int_{[0,\frac{\pi}{2}]} \cos(t)^{m-1}\sin(t)^{n-1}\,dt\ \mathrm{vol}S^{m-1}\mathrm{vol}S^{n-1}\end{aligned}$$

That is, $\mathrm{vol}S^{m+n-1} = \int_{[0,\frac{\pi}{2}]} \cos(t)^{m-1}\sin(t)^{n-1}\,dt\ \mathrm{vol}S^{m-1}\mathrm{vol}S^{n-1}$, a rather cool identity.

Definition 1.7. The image of the above homomorphism will be denoted by $J(X)$.

In words, $J(X)$ is the Grothendieck group of the stable fiber homotopy equivalence classes of *orthogonal* sphere *bundles* on X, under the fiberwise join $*$.

Since $KO^0(-)$ and $\mathrm{Sph}(-)$ are both contravariant functors from the homotopy category of finite CW complexes to abelian groups, and $KO^0(-) \xrightarrow{S} \mathrm{Sph}(-)$ is a natural transformation, it follows that $J(-)$ is also a contravariant functor from the homotopy category of finite CW complexes to abelian groups (again, the induced maps correspond to pullback).

Suppose X is a finite *pointed* CW complex, with basepoint $x_0 \in X$. Then we have a canonically split surjection

$$J(X) \xrightarrow{\iota^*} J(\{x_0\}) \simeq \mathbb{Z}$$

(the splitting is given by $p^* : \mathbb{Z} \simeq J(\{x_0\}) \to J(X)$, whose image is the $\mathbb{Z}$ summand corresponding to the trivial orthogonal sphere bundles).

Definition 1.8. $\tilde{J}(X) := \ker\iota^* \subset J(X)$. Equivalently, $\tilde{J}(X) := \mathrm{coker} p^*$.

So, $\tilde{J}(X)$ consists of classes with "virtual dimension 0 on the basepoint component of X." Alternatively we can think of it as the Grothendieck group of "loose" stable fiber homotopy equivalence classes of orthogonal sphere bundles over X under fiberwise join.

It's worth pointing out that these loose stable fiber homotopy equivalence classes of orthogonal sphere bundles form a group under $*$, even before passing to the Grothendeick group. This is because of the following standard fact (see Atiyah's *K-theory* or Milnor's *Characteristic classes*).

Proposition 1.9. *Let $E \to X$ be an orthogonal real vector bundle. Then E occurs as a direct summand of a trivial vector bundle; i.e. there's another orthogonal real vector bundle $F \to X$ and an isomorphism $E \oplus F \simeq X \times \mathbb{R}^n$.*

Hence we'll have $S(E) * S(F) \simeq S(E \oplus F) \simeq S(X \times \mathbb{R}^n)$ which is the identity.
In the course of this discussion we've also shown:

Proposition 1.10. *Let X be a finite pointed CW complex. Then there's a natural direct sum decomposition*

$$\mathbb{Z} \oplus \tilde{J}(X) \simeq J(X)$$

Notice that by *definition* $J(X)$ is a quotient of $KO^0(X)$. Observe that if $[E] - [E'] \in KO^0(X)$ and $[S(E)] - [S(E')] = 0 \in J(X)$, then there must be a fiber homotopy equivalence $S(E) * n \simeq S(E') * n$ over X for suitably large $n \in \mathbb{N}$. But this means that $E \oplus n$ and $E' \oplus n$ have fiber homotopy equivalent sphere bundles. It follows that the kernel of the surjection $KO^0(X) \to J(X)$ is the subgroup generated by elements of the form

$$[E] - [F] \in KO^0(X) \text{ where } S(E), S(E') \text{ are fiber homotopy equivalent}$$

Adams calls this subgroup $T(X)$.

Topological monoids and their classifying spaces. For any $n \in \mathbb{N}$ let $H(n)$ denote the space of homotopy equivalences $S^{n-1} \to S^{n-1}$, in the compact open topology. Equivalently, $H(n)$ is the subspace of $(S^{n-1})^{S^{n-1}}$ consisting of degree ± 1 maps.

Proposition 1.11. *$H(n)$ is a topological monoid (under composition).*

This will follow more or less immediately from

Proposition 1.12. *Let F be a locally compact Hausdorff space. Then the space F^F of continuous maps $F \to F$ with the compact open topology is a topological monoid (under composition).*

Proof. The only thing to check is that composition

$$F^F \times F^F \xrightarrow{\circ} F^F \text{ sending } (f, g) \mapsto f \circ g$$

is continuous (after all, we already know the underlying set of F^F is a monoid).

So, let $K \subset F$ be compact and let $U \subset F$ be open. Since F is locally compact Hausdorff, we can find an open set $V \subset U$ with compact closure $\bar{V} \subset U$. Now if $f, g \in F^F$ are continuous maps with $g(K) \subset V$ and $f(\bar{V}) \subset U$, we'll have $f \circ g(K) \subset U$.

□

Note: if F is a compact metric space, say with metric d then the compact open and sup norm topologies on F^F agree (see the appendix of Hatcher's *Algebraic topology*). In this situation we can argue as follows: let $f_1, g_1 \in F^F$ be continuous maps and let $\epsilon > 0$ be given. By compactness of F, f_1 is in fact uniformly continuous, so there's a $\delta > 0$ such that

$$d(f_1(x), f_1(y)) < \epsilon \text{ if } d(x, y) < \delta, \text{ for } x, y \in F$$

But then if $f_2, g_2 \in F^F$ are chosen so

$$\sup_{x \in F} d(f_1(x), f_2(x)) < \epsilon \text{ and } \sup_{x \in F} d(g_1(x), g_2(x)) < \delta$$

we'll have

$$\begin{aligned} \sup_{x \in F} d(f_1 \circ g_1(x), f_2 \circ g_2(x)) &\leq \sup_{x \in F} d(f_1 \circ g_1(x), f_1 \circ g_2(x)) + \sup_{x \in F} d(f_1 \circ g_2(x), f_2 \circ g_2(x)) \\ &\leq \epsilon + \epsilon = 2\epsilon \end{aligned}$$

We need a couple propositions, which can be viewed as partial extensions of the theory of classifying spaces of topological groups to the case of topological monoids like $H(n)$. Let F be a finite CW complex and let H denote the topological monoid of homotopy equivalences $F \to F$ (equivalently, H consists of the path components of F^F mapping to invertible elements of $[F, F]$).

Proposition 1.13. *Associated to the topological monoid H is a **classifying space** BH, characterized by the following property: there's a principal H-fibration $H \to EH \to BH$ (with EH contractible) so that for any CW complex X the natural map*

$$[X, BH] \to \{\textit{fiber homotopy equivalence classes of principal } H- \textit{ fibrations } H \to E \to X\}$$

$$\textit{taking } [f] \mapsto f^* EH$$

is a bijection.

Remark 1.14. Let $F \to E \to X$ be an F-fibration over X. Then we can recover its underlying principal H-fibration: this is the fibration $Prin(E) \to X$ consisting of "local fiber homotopy equivalences" from the trivial fibration $X \times F \to X$ to $E \to X$. Its fiber over $x \in X$ is the space of homotopy equivalences $F \to E_x$. There's a natural action of H on $Prin(E)$ by precomposition, and the quotient by this action is just X.

On the other hand, given a principal H-fibration $E \to X$, a balanced product construction yields an associated F-fibration $E \times_H F \to X$. It seems clear that we have an equivalence between F-fibrations and principal H-fibrations.

The upshot is that BH also represents the functor

$$X \to \{\text{fiber homotopy equivalence classes of fibrations } F \to E \to X\}$$

Of course, we're interested in the case $F = S^{n-1}$.

Proposition 1.15. *Assigning to an orthogonal real n-plane bundle its associated $n-1$-sphere bundle corresponds to a map of universal fibrations*

$$\begin{tikzcd} O(n) \arrow[r, "="] \arrow[d] & H(n) \arrow[d] \\ EO(n) \arrow[r] \arrow[d] & EH(n) \arrow[d] \\ BO(n) \arrow[r] & BH(n) \end{tikzcd} \tag{1.5}$$

where $O(n) \to H(n)$ is the inclusion obtained from the usual action of $O(n)$ on $S^{n-1} \subset \mathbb{R}^n$.

More over, these are compatible with the maps of fibrations

$$\begin{tikzcd} O(n) \arrow[r] \arrow[d] & O(n+1) \arrow[d] & H(n) \arrow[r] \arrow[d] & H(n+1) \arrow[d] \\ EO(n) \arrow[r] \arrow[d] & EO(n+1) \arrow[d] & EH(n) \arrow[r] \arrow[d] & EH(n+1) \arrow[d] \\ BO(n) \arrow[r] & BO(n+1) & BH(n) \arrow[r] & BH(n+1) \end{tikzcd} \tag{1.6}$$

where $O(n) \subset O(n+1)$ corresponds to the inclusion $\mathbb{R}^n \subset \mathbb{R}^n \oplus \mathbb{R}$ and $H(n) \subset H(n+1)$ corresponds to the inclusion $S^{n-1} \subset S^{n-1} * S^0 \simeq S^n$.

See Dold and Lashof's *Principal quasifibrations and fiber homotopy equivalence of bundles* and Stasheff's *A classification theorem for fibre spaces*. One thing to keep in mind: for any finite complex F, the functor

$$X \to \{\text{fiber homotopy equivalence classes of fibrations } F \to E \to X\}$$

satisfies the hypotheses of Brown's representability theorem, so we know it's represented by a CW complex, say BH. For the rest of the above propositions (and much more information on $BH(n)$) see the above papers.

In the (co)limit as $n \to \infty$, we obtain a map of fibrations

$$\begin{array}{ccc} O & \xrightarrow{=} & H \\ \downarrow & & \downarrow \\ EO & \longrightarrow & EH \\ \downarrow & & \downarrow \\ BO & \longrightarrow & BH \end{array} \tag{1.7}$$

where $O = \text{colim} O(n)$ and $H = \text{colim} H(n)$. So, for any finite CW complex X, we obtain maps

$$[X, BO(n)] \to [X, BH(n)]$$

Taking the colimit and tacking on a copy of $\mathbb{Z}$ we obtain homomorphisms

$$KO^0(X) = [X, BO \times \mathbb{Z}] \to [X, BH \times \mathbb{Z}] = \text{Sph}(X)$$

where I'm taking the identification $KO^0(X) = [X, BO \times \mathbb{Z}]$ to be well known - the identification $[X, BH \times \mathbb{Z}] = \text{Sph}(X)$ is directly analogous.

Note: the copies of $\mathbb{Z}$ account for the possibility of bundles with fiber dimension that varies over the path components of X.

It's a fact (see the paper by Dold and Lashof) that the image of the map $[X, BO(n)] \to [X, BH(n)]$ can be identified with the fiber homotopy equivalence classes of *orthogonal* S^{n-1} bundles over X. Thus the image of $[X, BO \times \mathbb{Z}] \to [X, BH \times \mathbb{Z}]$ can be identified with the *stable* fiber homotopy equivalence classes of orthogonal S^{n-1} bundles over X, i.e. $J(X)$. This yields:

Proposition 1.16. *There's a natural identification of $J(X)$ with the image of the homomorphism*

$$[X, BO \times \mathbb{Z}] \to [X, BH \times \mathbb{Z}]$$

Similarly, if X is a finite pointed CW complex there's a natural identification of $\tilde{J}(X)$ with the image of the homomorphism

$$[X, BO \times \mathbb{Z}]_* \to [X, BH \times \mathbb{Z}]_*$$

where the $*$ denotes homotopy classes of pointed maps.

Homotopy groups of BH and the classical J-homomorphism. Observe that since $EH(n)$ is contractible, the π_* long exeact sequence of the universal bundle $H(n) \to EH(n) \to BH(n)$ gives isomorphisms

$$\pi_i BH(n) \simeq \pi_{i-1} H(n) \text{ for } i > 0$$

Moreover,

$$\pi_0 H(n) = \mathbb{Z}/2 \text{ and } \pi_i H(n) = \pi_i H(n)_0$$

where $H(n)_0 \subset H(n)$ is the identity component (consisting of self-homotopy equivalences S^{n-1} with degree 1).

Now observe that evaluation at the basepoint $s_0 \in S^{n-1}$ fits into a fibration

$$\Omega^{n-1} S_0^{n-1} \xrightarrow{ev_{s_0}} H(n)_0 \to S^{n-1}$$

where $ev_{s_0}(f) = f(s_0)$ and $\Omega^{n-1} S_0^{n-1}$ is the identity component of the *pointed* self homotopy equivalences of S^{n-1}. From its π_* long exact sequence and the fact that $\pi_i S^{n-1} = 0$ for $i < n-1$ we see that the induced map

$$\pi_i \Omega^{n-1} S_0^{n-1} \to \pi_i H(n)_0$$

is an isomorphism when $i < n-2$ (and a surjection when $i = n-2$). Finally, the $\Sigma - \Omega$ adjunction gives isomorphisms

$$\pi_i \Omega^{n-1} S_0^{n-1} \simeq \pi_{i+n-1} S^{n-1} \text{ for } i > 0$$

The conclusion is that there are canonical isomorphisms

$$\pi_i BH(n) \simeq \pi_{i+n-2} S^{n-1} \text{ for } 1 < i < n-1$$

From the above maps of universal bundles, the map of fibrations

$$\begin{array}{ccccc} \Omega^{n-1} S_0^{n-1} & \longrightarrow & H(n)_0 & \xrightarrow{ev_{s_0}} & S^{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^n S_0^n & \longrightarrow & H(n+1)_0 & \xrightarrow{ev_{s_0}} & S^n \end{array} \tag{1.8}$$

and a bit of unraveling of the details of the $\Sigma - \Omega$ adjunction, we see that these isomorphisms fit into commutative diagrams

$$\begin{array}{ccc} \pi_i BH(n) & \xrightarrow{\simeq} & \pi_{i+n-2} S^{n-1} \\ \downarrow & & \Sigma \downarrow \\ \pi_i BH(n+1) & \xrightarrow{\simeq} & \pi_{i+n-1} S^n \end{array} \quad \text{for } 1 < i < n-1 \text{ and } n \in \mathbb{N} \tag{1.9}$$

So, taking the (co)limit as $n \to \infty$ we obtain isomorphisms

$$\pi_i BH \simeq \operatorname{colim}_n \pi_{i+n-2} S^{n-1} = \pi_{i-1}^s S, \text{ for } i > 1$$

where $\pi_{i-1}^s S$ is the $i-1$st stable homotopy group of the sphere spectrum S. Of course we also have

$$\pi_0 BH = 0 \text{ and } \pi_1 BH = \pi_0 H = \mathbb{Z}/2$$

Now say $r \in \mathbb{N}$ and recall from the previous section that we may identify $\tilde{J}(S^r)$ with the image of the map

$$[S^r, BO \times \mathbb{Z}]_* \to [S^r, BH \times \mathbb{Z}]_* \text{ i.e. } \pi_r BO \to \pi_r BH$$

Since the map $BO \to BH$ fits into a map of universal bundles and and EO, EH are contractible, we have a commutative diagram

$$\begin{array}{ccc} \pi_{r-1} O & \longrightarrow & \pi_{r-1} H \\ \simeq \uparrow & & \simeq \uparrow \\ \pi_r BO & \longrightarrow & \pi_r BH \end{array} \tag{1.10}$$

From the above discussion, on the homotopy groups of BH, we have a canonical isomorphism $\pi_{r-1} H \simeq \pi_{r-1}^s S$. Thus we may identify $\tilde{J}(S^r)$ with the image of the homomorphism

$$\pi_{r-1} O \to \pi_{r-1} H \simeq \pi_{r-1}^s S, \text{ for } r > 1$$

This is the "J-homomorphism" introduced by Whitehead in his paper *On the homotopy groups of spheres and rotation groups*. It's worth mentioning that the homotopy groups $\pi_* O$ are computed in Bott's periodicity theorem. So, the question is what "survives" the J-homomorphism.

Of course, we have

$$\tilde{J}(S^0) = 0 \text{ and } \tilde{J}(S^1) = \mathbb{Z}/2$$

.

Since the stable homotopy groups $\pi_{r-1}^s S$, $r > 1$ are finite (this is a theorem in Serre's thesis *Homologie singuliere des espaces fibres*), we see that the groups $\tilde{J}(S^r)$ are all finite. More generally

Theorem 1.17. *Let X be a connected finite pointed CW complex. Then $\tilde{J}(X)$ is a finite abelian group.*

Proof. First note that as X is connected, we'll have $\tilde{J}(X) \simeq [X, BH]$ (as every pointed map $X \to BH \times \mathbb{Z}$ lands in $BH \times \{0\}$).

Now observe that if $n > \dim X + 1$, then every element $\alpha \in [X, BH]$ is represented by a homotopy class of maps $\alpha \in [X, BH(n)]$. To see this, observe that for every $n \in \mathbb{N}$ the above discussion shows that the map

$$BH(n) \to BH(n+1)$$

induces homomorphisms

$$\pi_i BH(n) \to \pi_i BH(n+1)$$

that are isomorphisms for $i < n-1$ and surjective if $i = n-1$, since it fits into the commutative diagram

$$\begin{array}{ccc} \pi_i BH(n) & \xrightarrow{\simeq} & \pi_{i+n-2} S^{n-1} \\ \downarrow & & \Sigma\downarrow \\ \pi_i BH(n+1) & \xrightarrow{\simeq} & \pi_{i+n-1} S^n \end{array} \quad \text{for } 1 < i < n-1 \text{ and } n \in \mathbb{N} \tag{1.11}$$

and the right vertical homomorphism is an isomorphism for $i < n-1$ and surjective for $i = n-1$ by the Fruedenthal suspension theorem. Now a standard argument shows that if $n > \dim X + 1$, the induced map $[X, BH(n)] \to [X, BH(n+1)]$ is surjective.

So, it'll suffice to show that $[X, BH(n)]$ is finite when $n > \dim X + 1$. Atiyah points out that in light of obstruction theory, it'll suffice to prove that that $\pi_i BH(n)$ is finite for $i < n-1$. In more detail, we're appealing to the fact that if $X_0 \subset X_1 \subset \cdots \subset X$ is the skeleton filtration of X, then the obstruction to extending a map $f : X_i \to BH(n)$ to X_{i+1} can be viewed as a class in $H^{i+1}(X; \pi_i BH(n))$, and if $f, g : X_{i+1} \to BH(n)$ are continuous maps and $h : X_i \times I \to BH(n)$ is a homotopy between $f|_{X_i}, g|_{X_i}$, the obstruction to extending it to a homotopy between f, g can be viewed as a class in $H^{i+2}(X; \pi_{i+1} BH(n))$. If X is a finite complex with $n > \dim X + 1$ and the groups $\pi_i BH(n)$ are finite for $i < n-1$, then all the cohomology groups $H^i(X; \pi_j BH(n)), i, j < n-1$ will be finite. By induction it'll follow that $[X, BH(n)]$ is finite.

Finally, the fact that $\pi_i BH(n)$ is finite for $i < n-1$ follows from Serre's thesis.

□

Remark 1.18. Here's an alternate proof, making use of the fact that BH is an H space, in fact an infinite loop space. Observe that if $A \xrightarrow{i} X$ is a cofibration of pointed CW complexes, then

$$[A, BH] \xleftarrow{i^*} [X, BH] \xleftarrow{j^*} [X/A, BH]$$

is an exact sequence of abelian groups. If X is a finite complex of dimension n and i is the inclusion of the $n-1$-skeleton $A = X^{n-1}$, we'll have $X/X^{n-1} \simeq \bigvee_\alpha S^n_\alpha$, a finite wedge of n-spheres, and so the above exact sequence reads

$$[X^{n-1}, BH] \xleftarrow{i^*} [X, BH] \xleftarrow{j^*} [\bigvee_\alpha S^n_\alpha, BH] \simeq \bigoplus_\alpha \pi_n(BH)$$

so by induction it'll suffice to show that $\pi_i(BH)$ is finite for all i, and we've already done that. Really, what we've proved is that $\widetilde{\mathrm{Sph}}(X)$ is a finite abelian group for every finite pointed CW-complex X.

1.2. **The stable homotopy type of Thom complexes.**

1.2.1. *Thom spaces of spherical fibrations and the Thom isomorphism.* If $F \xrightarrow{\pi} X$ is a spherical fibration over a CW complex X, we may define its **Thom space** $\mathrm{Th}F$ to be the cone of the projection: $\mathrm{Th}F := \mathrm{Cone}(\pi)$. Note that if $F = S(E)$ is the sphere bundle associated to a vector bundle E, then there's a natural identification of $\mathrm{Th}S(E)$ with the Thom space of E, defined to be the pointed space $(D(E)/S(E), S(E)/S(E))$ obtained from the disc bundle $D(E)$ by collapsing the sphere bundle $S(E)$ to a point (intuitively, $\mathrm{Th}E$ is obtained from E by adjoining a "common point at infinity" for all the vector spaces $E_x, x \in X$ - when X is finite and hence compact, $\mathrm{Th}E$ is the one point compactification of E). More precisely, identifying the disk bundle $D(E)$ with the mapping cylinder of the projection $S(E) \to X$ yields a homeomorphism $\mathrm{Th}S(E) \simeq \mathrm{Th}E$.

Given a spherical fibration $F \to X$ as above, say with fiber of dimension $n-1$, define its orientation bundle $\mathrm{Or}F \to X$ so that the fiber $\mathrm{Or}F_x$ over a point $x \in X$ consists of the components of the homotopy equivalences $S^{n-1} \to F_x$. Equivalently, $\mathrm{Or}F$ consists of the isomorphisms $S^{n-1} \to F_x$ in the *homotopy category* (say, of CW spaces). $\mathrm{Or}F$ is a principal C_2 bundle, with C_2 acting by precomposition with an orientation reversing self equivalence of S^{n-1}. One can check that if $F = S(E)$ is the sphere bundle of an orthogonal real vector bundle of rank n, then we'll have $\mathrm{Or}S(E) = S(\bigwedge^n E^\vee)$, the sphere bundle of the determinant line bundle of E.

Now let $\mathscr{A}$ to be the local coefficient system on X associated to the principal $\mathbb{Z}$ bundle $\mathrm{Or}F \times_{C_2} \mathbb{Z}$. This bundle can also be described as

$$\coprod_{x \in X} \tilde{H}_{n-1}(F_x; \mathbb{Z}) \to X, \text{ with the only reasonable topology.}$$

In the case where $F = S(E)$, the long exact sequences of the pairs $(D(E_x), S(E_x))$ give natural isomorphisms $\tilde{H}_{n-1}(S(E_x); \mathbb{Z}) \simeq H_n(D(E_x), S(E_x); \mathbb{Z})$, so this can be written as

$$\coprod_{x \in X} H_n(D(E_x), S(E_x); \mathbb{Z}) \to X, \text{ (again with the only reasonable topology)}$$

Note that for each $x \in X$, $\tilde{H}_n(D(E_x), S(E_x); \mathbb{Z}) \simeq \tilde{H}_n(\mathrm{Th}E_x; \mathbb{Z})$.

Theorem 1.19 (The Thom isomorphism)**.** *Let $F \xrightarrow{\pi} X$ be a $(n-1)$-sphere fibration over a CW-complex X. Then* $\mathrm{Th}F$ *is $n-1$-connected, and there are canonical isomorphisms*

$$H_k(X; \mathscr{A}) \simeq \tilde{H}_{k+n}(\mathrm{Th}F; \mathbb{Z}) \text{ and } H^k(X; \mathscr{A}) \simeq \tilde{H}^{k+n}(\mathrm{Th}F; \mathbb{Z})$$

Proof. First, from the π_* long exact sequence of the fibration $S^{n-1} \to F \to X$ and the fact that $\pi_i S^{n-1} = 0$ for $i < n-1$ we see that the induced map $\pi_i F \to \pi_i X$ is an isomorphism when $i < n-1$ and a surjection when $i = n-1$. Then if $\mathrm{Cyl}\pi$ is the mapping cylinder of the projection $F \xrightarrow{\pi} X$, the inclusion $F \subset \mathrm{Cyl}\pi$ induces isomorphisms $\pi_i F \to \pi_i \mathrm{Cyl}\pi$ for $i < n-1$ and a surjection when $i = n-1$, and together with the π_* long exact sequence of the inclusion this shows $\pi_i(\mathrm{Cyl}\pi, F) = 0$ for $i \leq n-1$, i.e. the *pair* $(\mathrm{Cyl}\pi, F)$ is $n-1$-connected. Now by Proposition 4.28 in Hatcher (a corollary of the "homotopy excision theorem" (which is a misnomer)) it follows that $\mathrm{Th}F = \mathrm{Cyl}\pi/F$ is $n-1$-connected.

It seems clear that the canonical projection $\mathrm{Cyl}\pi \to X$ is also a fibration, with fiber D^n. Moreover F includes in $\mathrm{Cyl}\pi$ as a "sub-fibration" - we have a map of fibrations

$$\begin{array}{ccc} F & \longrightarrow & \mathrm{Cyl}\pi \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X & \xrightarrow{=} & X \end{array} \tag{1.12}$$

This comes with relative Serre spectral sequences; the homology version begins with $E^2_{pq} = H_p(X; \mathscr{H}_q(D^n, \partial D^n; \mathbb{Z})$ and converges to $H_{p+q}(\mathrm{Cyl}\pi, F; \mathbb{Z})$. But we have

$$\mathscr{H}_q(D^n, \partial D^n; \mathbb{Z})) = 0 \text{ unless } q = n$$

and it's not hard to show using the isomorphisms $H_n(D^n, \partial D^n; \mathbb{Z}) \simeq \tilde{H}_{n-1}(\partial D^n; \mathbb{Z})$ that there's an isomorphism of local coefficient systems $\mathscr{H}_n(D^n, \partial D^n; \mathbb{Z})) \simeq \mathscr{A}$ Thus the spectral sequence E^*_{**} collapses on page 2 (with no extension issues) leaving

$$H_{k+n}(\mathrm{Cyl}\pi, F; \mathbb{Z}) \simeq H_k(X; \mathscr{A}) \text{ for all } k$$

Finally, since $\mathrm{Th}F = \mathrm{Cyl}\pi / F$ we have excision isomorphisms

$$H_{k+n}(\mathrm{Cyl}\pi, F; \mathbb{Z}) \simeq \tilde{H}_{k+n}(\mathrm{Th}F; \mathbb{Z}) \text{ for all } k$$

The cohomology version is similar.

□

It follows from the definitions that $F \to X$ is orientable if and only if $\mathscr{A}$ is trivial (in fact we could just take this as a definition of orientability for a spherical fibration).

In the case where $F = S(E)$ is the sphere bundle of an orthogonal real vector bundle $E \xrightarrow{\pi} X$ of rank n which is trivial over every cell $e \subset X$, one can give a more illuminating description of the Thom isomorphism.

Proposition 1.20. *Suppose that for every cell $e \subset X$, the restriction $E|_{\bar{e}} \to \bar{e}$ of E to the closure of e is trivial. Then* Th*E has a canonical CW complex structure with a basepoint ∞ (corresponding to $S(E)/S(E)$) and one $k+n$-cell for each k-cell $e^k_\alpha \subset X$.*

Remark 1.21. The hypotheses are satisfied if X is a simplicial complex. Since every finite CW complex is homotopy equivalent to a finite simplicial complex, the hypotheses are always satisfied "up to homotopy."

Sketch. As $E|_{\bar{e}^k_\alpha} \to \bar{e}^k_\alpha$ is trivial, we'll have maps of vector bundles (giving isomorphisms on fibers)

$$\begin{array}{ccccc} \bar{e}^k_\alpha \times \mathbb{R}^n & \xrightarrow{\simeq} & E|_{\bar{e}^k_\alpha} & \longrightarrow & E \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \bar{e}^k_\alpha & \xrightarrow{=} & \bar{e}^k_\alpha & \longrightarrow & X \end{array} \tag{1.13}$$

inducing maps of Thom spaces

$$\mathrm{Th}(\bar{e}^k_\alpha \times \mathbb{R}^n) \simeq \mathrm{Th}E|_{\bar{e}^k_\alpha} \to \mathrm{Th}E$$

Note that $\mathrm{Th}(\bar{e}^k_\alpha \times \mathbb{R}^n) = \bar{e}^k_\alpha \times D^n / \bar{e}^k_\alpha \times \partial D^n$; by the way, this is the same as $\Sigma^n(\bar{e}^k_\alpha)+$. So, if $\Phi : D^k_\alpha \to X$ is the characteristic map of e^k_α, then we may define a map

$$D^{k+n}_\alpha \simeq D^k_\alpha \times D^n \xrightarrow{\Phi \times id} \bar{e}^k_\alpha \times D^n \to \mathrm{Th}(\bar{e}^k_\alpha \times \mathbb{R}^n) \to \mathrm{Th}E$$

and the claim to make is that this is the characteristic map of a cell $\mathrm{Th}(E|_{e^k_\alpha}) \subset \mathrm{Th}E$.

□

Theorem 1.22. Th*E is $n-1$-connected, and there are natural isomorphisms*

$$H_k(X; \mathscr{A}) \xrightarrow{\simeq} \tilde{H}_{k+n}(\mathrm{ThE}; \mathbb{Z}) \text{ and } H^k(X; \mathscr{A}) \xrightarrow{\simeq} \tilde{H}^{k+n}(\mathrm{Th}E; \mathbb{Z})$$

for all k.

Proof. The previous proposition shows that ThE is a CW-complex obtained from a 0-cell by attaching cells of dimension $\geq n$. So, it's $n-1$-connected by cellular approximation.

To obtain the above isomorphisms it'll suffice to write down an explicit isomorphism of cellular chain complexes $C_*(X; \mathscr{A}) \xrightarrow{[+n]} C_*(\mathrm{Th}E, \infty; \mathbb{Z})$, shifting degree by n. The natural candidate for such an isomorphism is given by the isomorphisms of chain groups

$$C_k(X; \mathscr{A}) \xrightarrow{[+n]} C_{k+n}(\mathrm{Th}E, \infty; \mathbb{Z}) \text{ sending}$$

$$e_\alpha^k \mapsto \mathrm{Th}(E|_{e_\alpha^k}) \text{ and "extending linearly"}$$

The only thing to check is that these are compatible with the respective boundary maps. To see this, note that if $\varphi_\alpha : \partial D_\alpha^k \to X^{k-1}$ is the attaching map for e_α^k, and for each $k-1$-cell $e_\beta^{k-1} \subset X$ the composition $\partial D_\alpha^k \xrightarrow{\varphi_\alpha} X^{k-1} \to X^{k-1}/X^{k-2} \simeq \bigvee_\beta D_\beta^{k-1}/\partial D_\beta^{k-1} \to D_\beta^{k-1}/\partial D_\beta^{k-1}$ is denoted by $\varphi_{\alpha\beta}$, the boundary

$$C_k(X;\mathscr{A}) \xrightarrow{\partial} C_{k-1}(X;\mathscr{A}) \text{ takes}$$
$$e_\alpha^k \mapsto \sum_\beta (\operatorname{sgn}\det\tau_{\alpha\beta})(\deg\varphi_{\alpha\beta})e_\beta^{k-1}$$

where $\operatorname{sgn}\det\tau_{\alpha\beta} \in \pm 1$ is defined as follows: let $\Phi_\alpha : D_\alpha^k \to X$ and $\Phi_\beta : D_\beta^{k-1} \to X$ be the characteristic maps of e_α^k and e_β^{k-1} respectively. We have pullback diagrams of vector bundles

$$\begin{array}{ccc} D_\alpha^k \times \mathbb{R}^n \simeq \Phi_\alpha^* E & \longrightarrow & E \\ \downarrow & & \downarrow \\ D_\alpha^k & \xrightarrow{\Phi_\alpha} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} D_\beta^{k-1} \times \mathbb{R}^n \simeq \Phi_\beta^* E & \longrightarrow & E \\ \downarrow & & \downarrow \\ D_\beta^{k-1} & \xrightarrow{\Phi_\beta} & X \end{array} \tag{1.14}$$

If $\deg\varphi_{\alpha\beta} \neq 0$, then $\Phi_\alpha(D_\alpha^k) \cap \Phi_\beta(D_\beta^{k-1}) \neq \emptyset$, so there's a point $x \in X$ for which we have isometries

$$\mathbb{R}^n \xrightarrow{\simeq} E_x \xleftarrow{\simeq} \mathbb{R}^n$$

where the 2 isomorphisms come from the above 2 pullback diagrams. Let $\tau_{\alpha\beta}$ be this isometry.

On the other hand, if $\mathrm{Th}\varphi_\alpha : \partial D_\alpha^{k+n} \to \mathrm{Th}E$ is the attaching map for $\mathrm{Th}(E|_{e_\alpha^k})$, which from the above discussion factors through $\Sigma^n(\partial D_\alpha^{k+n})_+$, and if the composition

$$\Sigma^n(\partial D_\alpha^{k+n})_+ \xrightarrow{\mathrm{Th}\varphi_\alpha} \mathrm{Th}E^{k+n-1} \xrightarrow{\text{quot}} \mathrm{Th}E^{k+n-1}/\mathrm{Th}E^{k+n-2} \simeq \bigvee_\beta \Sigma^n(D_\beta^{k+n-1}/\partial D_\beta^{k+n-1})_+$$
$$\to \Sigma^n(D_\beta^{k+n-1}/\partial D_\beta^{k+n-1})_+$$

is denoted by $\mathrm{Th}\varphi_{\alpha\beta}$, then the boundary

$$C_{k+n}(\mathrm{Th}E,\infty;\mathbb{Z}) \xrightarrow{\partial} C_{k+n-1}(\mathrm{Th}E,\infty;\mathbb{Z}) \text{ takes}$$
$$\mathrm{Th}(E|_{e_\alpha^k} \mapsto \sum_\beta (\deg\mathrm{Th}\varphi_{\alpha\beta})\mathrm{Th}(E|_{e_\beta^{k-1}}$$

It'll suffice to observe that $\deg\mathrm{Th}\varphi_{\alpha\beta} = (\operatorname{sgn}\det\tau_{\alpha\beta})(\deg\varphi_{\alpha\beta})$, and this follows from the fact that $\mathrm{Th}\varphi_{\alpha\beta}$ can be identified with the smash product of $\varphi_{\alpha\beta}$ with the one point compactification $\mathbb{R}^n \cup \infty \xrightarrow{\tau_{\alpha\beta}} \mathbb{R}^n$ of the isometry $\tau_{\alpha\beta}$, a degree $\operatorname{sgn}\det\tau_{\alpha\beta}$ map.

□

Proposition 1.23. *With all the above assumptions, the following are equivalent: (i) $E \to X$ is orientable; (ii) $\mathscr{A}$ is trivial; (iii) $w_1(E) = 0$, where $w_1(E)$ is the first Stiefel-Whitney class of E. Moreover, $\mathscr{A} \otimes \mathbb{Z}/2$ is always trivial.*

See Atiyah's paper.

Let $E \to X$ and $E' \to X'$ be orthogonal real vector bundles on connected, finite, pointed CW-complexes X and X'. Then we can form the product vector bundle $E \times E' \to X \times X'$.

Proposition 1.24. *There's a natural homeomorphism* $\mathrm{Th}E \wedge \mathrm{Th}E' \xrightarrow{\simeq} \mathrm{Th}(E \times E')$.

Proof. This is obtained from the map $E \times E' \to E \times E'$ taking $(u,v) \in E_x \times E'_y \mapsto u + v \in E \times E'_{(x,y)} := E_x \oplus E_y$ by declaring that $u + \infty = \infty, \infty + v = \infty$ for all $u \in E, v \in E'$. □

In particular, note that when $E' \to Y$ is $\mathbb{R}^n \to \mathrm{pt}$, we obtain a natural homeomorphism

$$\Sigma^n \mathrm{Th} E \simeq \mathrm{Th}(E \oplus n)$$

One would like an analogous result for spherical fibrations. So, let $F \xrightarrow{\pi} X$ and $F' \xrightarrow{\pi'} X'$ be spherical fibrations over connected, finite, pointed CW complexes X, X'. Then we can form the fiberwise join $F * F' \xrightarrow{\pi * \pi'} X \times X$.

Proposition 1.25. *There's a natural homeomorphism* $\mathrm{Th}F \wedge \mathrm{Th}F' \simeq \mathrm{Th}(F * F')$.

Proof. It will suffice to define a map of pairs

$$(\mathrm{Cyl}\pi, F) \times (\mathrm{Cyl}\pi', F') \to (\mathrm{Cyl}\pi * \pi', F * F')$$

giving a homeomorphism $\mathrm{Cyl}\pi \times \mathrm{Cyl}\pi' - \mathrm{Cyl}\pi \times F' \cup F \times \mathrm{Cyl}\pi' \simeq \mathrm{Cyl}\pi * \pi' - F * F'$. To do this, identify the unit interval with $[0, \infty]$, view $\mathrm{Cyl}\pi$ as the adjunction space obtained by gluing $F \times [0, \infty]$ to X along $F \times \{0\} \simeq F \xrightarrow{\pi} X$, and similarly for $\pi, \pi * \pi'$. Write elements of $\mathrm{Cyl}\pi'$ like $[u, t, \pi(u)]$ and similarly for $\pi, \pi * \pi'$ Now send

$$[u, s, \pi(u)], [v, t, \pi(v)] \mapsto [[u, \arccos(\frac{s}{\sqrt{s^2 + t^2}}), v], \sqrt{s^2 + t^2}, (\pi(u), \pi'(v))]$$

Check that this does the trick. $\square$

In particular, if $F' \to Y$ is the fibration $S^{m-1} \to \mathrm{pt}$ we obtain a natural isomorphism

$$\Sigma^m \mathrm{Th}F \simeq \mathrm{Th}(F * (m-1))$$

where $m - 1$ denotes the trivial S^{m-1} fibration over X.

Corollary 1.26. *Suppose* $E, E' \xrightarrow{\pi, \pi'} X$ *are orthogonal real vector bundles over a finite pointed CW-complex and* $[E] = [E'] \in KO^0(X)$. *Then there's a degree 0 stable homotopy equivalence* $\Sigma^* \mathrm{Th}E \simeq \Sigma^* \mathrm{Th}E'$. *Similarly* $[E] = [E'] \in \tilde{KO}^0(X)$ *then there's a stable homotopy equivalence* $\Sigma^* \mathrm{Th}E \simeq \Sigma^* \mathrm{Th}E'$, *possibly shifting degrees.*

Proof. After unraveling definitions, one sees that $[E] = [E'] \in KO^0(X)$ if and only if for some $m \in \mathbb{N}$ we have an isomorphism of vector bundles $E \oplus X \times \mathbb{R}^m \simeq E' \oplus X \times \mathbb{R}^m$ over X. But then there's a homeomorphism of Thom spaces

$$\Sigma^m \mathrm{Th}E \simeq \Sigma^m \mathrm{Th}E'$$

The statement about reduced K-theory is proved similarly.

$\square$

The above discussion motivates the following: suppose we have an element $\alpha \in KO^0(X)$; we know $\alpha = [E] - n$ for some orthogonal real vector bundle $E \to X$ and some $n \in \mathbb{N}$ (See Atiyah's *K-theory*). Define the suspension spectrum $\Sigma^* \mathrm{Th}\alpha$ of the Thom space of α by

$$\Sigma^m \mathrm{Th}\alpha = \Sigma^{m-n} \mathrm{Th}E$$

.

Proposition 1.27. *Let* $F, F' \xrightarrow{\pi, \pi'} X$ *be spherical fibrations over a connected, finite, pointed CW complex* X *and suppose there's a fiber homotopy equivalence* $F \xrightarrow{f} F'$ *over* X. *Then* f *induces a homotopy equivalence* $\mathrm{Th}F \xrightarrow{\mathrm{Th}f} \mathrm{Th}F'$.

Proof. This follows from the more general fact that if $Y, Y' \xrightarrow{\pi, \pi'} X$ are, say, CW spaces over X, then a homotopy equivalence $f : Y \to Y'$ over X induces a homotopy equivalence $\mathrm{Cone}f : \mathrm{Cone}\pi \to \mathrm{Cone}\pi'$. I won't prove that fact here.

$\square$

Corollary 1.28. *Let $F, F' \xrightarrow{\pi,\pi'} X$ be spherical fibrations over a connected, finite, pointed CW complex X. Then a stable fiber homotopy equivalence $F \sim F'$ induces a stable homotopy equivalence $\Sigma^*\mathrm{Th}F \simeq \Sigma^*\mathrm{Th}F'$, possibly shifting degrees.*

This follows from the previous propositions.

Question 1.29. Can any element of $\tilde{\mathrm{Sph}}(X)$ be written in the form $[F] - n$ for some spherical fibration $F \to X$? If so, we could conclude from the above proposition that if $F, F' \xrightarrow{\pi,\pi'} X$ are spherical fibrations over X and $[F] = [F'] \in \tilde{\mathrm{Sph}}(X)$, then there's a stable homotopy equivalence $\Sigma^*\mathrm{Th}F \simeq \Sigma^*\mathrm{Th}F'$, possibly shifting degrees.

Note that this question can be rephrased as follows: if $F \xrightarrow{\pi} X$ is a spherical fibration over a finite CW complex X, is there another spherical fibration $F' \xrightarrow{\pi'} X$ over X so that $F * F'$ is fiber homotopically trivial? Certainly this is the case if $F = S(E)$ is the sphere bundle of an orthogonal real vector bundle E over X, and according to Adams it's the case if F is a sphere *bundle*.

In any case, we have the following:

Proposition 1.30. *Let $E, E' \xrightarrow{\pi,\pi'} X$ be orthogonal real vector bundles over a connected, finite, pointed CW complex X. If $[S(E)] = [S(E')] \in \tilde{J}(X)$ then there's a stable homotopy equivalence $\Sigma^*\mathrm{Th}E \simeq \Sigma^*\mathrm{Th}E'$, possibly shifting degrees.*

Proof. Since we can view $\tilde{J}(X)$ as a subgroup of the abelian monoid of stable fiber homotopy equivalence classes of spherical fibrations over X, $[S(E)] = [S(E')] \in \tilde{J}(X)$ implies that there's a stable fiber homotopy equivalence $S(E) \sim S(E')$. We can then appeal to the above corollary. □

Here's a fun application of these ideas:

First let's recall the definition of the Stiefel-Whitney classes of an orthogonal real vector bundle $E \xrightarrow{\pi} X$ of rank n over a CW complex X given in Milnor and Stasheff's *Characteristic classes*.

I'm going to state without proof a mod 2 analogue of the above Thom isomorphisms - the point is that there are no orientation issues mod 2.

Proposition 1.31. *Let $F \xrightarrow{\pi} X$ be a $(n-1)$-sphere fibration over a CW-complex X. Then $\mathrm{Th}F$ is $n-1$-connected, and there are canonical isomorphisms*

$$H_k(X;\mathbb{F}_2) \simeq \tilde{H}_{k+n}(\mathrm{Th}F;\mathbb{F}_2) \text{ and } H^k(X;\mathbb{F}_2) \simeq \tilde{H}^{k+n}(\mathrm{Th}F;\mathbb{F}_2)$$

In the case where $F = S(E)$ is the sphere bundle of an orthogonal real n-plane bundle $E \xrightarrow{\pi} X$, we obtain canonical isomorphisms

$$H_k(X;\mathbb{F}_2) \simeq \tilde{H}_{k+n}(\mathrm{Th}E;\mathbb{F}_2) \text{ and } H^k(X;\mathbb{F}_2) \simeq \tilde{H}^{k+n}(\mathrm{Th}E;\mathbb{F}_2)$$

Let $u \in \tilde{H}^n(\mathrm{Th}E;\mathbb{F}_2)$ be the **Thom class** of E, i.e. the class corresponding to $1 \in H^0(X;\mathbb{F}_2)$ under the Thom isomorphism $\varphi : H^*(X;\mathbb{F}_2) \simeq \tilde{H}^*(\mathrm{Th}E;\mathbb{F}_2)[+n]$ Then we can consider its Steenrod squares

$$\mathrm{Sq}^k u \in \tilde{H}^{k+n}(\mathrm{Th}E;\mathbb{F}_2) \text{ for } k = 0,\ldots,n$$

(note that $\mathrm{Sq}^0 u = u$ and $\mathrm{Sq}^n u = u \smile u$). From these we obtain classes

$$\varphi^{-1}\mathrm{Sq}^k u \in H^k(X;\mathbb{F}_2) \text{ for } k = 0,\ldots,n$$

Definition 1.32. The kth Stiefel-Whitney class of $E \xrightarrow{\pi} X$ is $w_k(E) := \varphi^{-1}\mathrm{Sq}^k u$.

Remark 1.33. It's not at all obvious that this definition of the Stiefel-Whitney classes (apparently due to Thom in his paper *Espaces fibres en spheres et carres de Steenrod*) agrees with the original obstruction-theoretic definition. See *Characteristic classes*.

The thing to notice is that we can now define the Stiefel-Whitney classes of a $n-1$-sphere fibration $F \xrightarrow{\pi} X$ over a CW-complex X; if $\varphi : \varphi : H^*(X;\mathbb{F}_2) \simeq \tilde{H}^*(\mathrm{Th}F;\mathbb{F}_2)[+n]$ is the Thom isomorphism and $u \in \tilde{H}^n(\mathrm{Th}F;\mathbb{F}_2)$ is the Thom class, then

$$w_k(F) := \varphi^{-1}\mathrm{Sq}^k(u) \in H^k(X;\mathbb{F}_2) \text{ for } k = 0, \dots, n$$

When $F = S(E)$ for an orthogonal real n-plane bundle $E \xrightarrow{\pi} X$ we'll have $w_k(S(E)) = w_k(E)$ for all k.

Proposition 1.34. *Let $E, E' \xrightarrow{\pi,\pi'} X$ be orthogonal real vector bundles over a connected, finite, pointed CW complex X. If $[S(E)] = [S(E')] \in \tilde{J}(X)$ then $w_i(E) = w_i(E')$ for all i; that is, E and E' have the same Stiefel-Whitney classes.*

Proof. As discussed above, the hypotheses imply that for some $m, m' \in \mathbb{N}$ there's a fiber homotopy equivalence of sphere bundles $f : S(E \oplus m) \simeq S(E' \oplus m')$ over X, which will yield a homotopy equivalence of Thom spaces $\mathrm{Th}f : \mathrm{Th}S(E \oplus m) \simeq \mathrm{Th}S(E' \oplus m')$.

Then by the evident naturality of the Thom isomorphism we'll have a commutative diagram

$$\begin{tikzcd} \tilde{H}^*(\mathrm{Th}S(E \oplus m);\mathbb{F}_2)[+n] & \tilde{H}^*(\mathrm{Th}S(E' \oplus m');\mathbb{F}_2)[+n] \arrow[l, "\mathrm{Th}f^*"'] \\ H^*(X;\mathbb{F}_2) \arrow[u, "\varphi"] & H^*(X;\mathbb{F}_2) \arrow[l, "="'] \arrow[u, "\varphi'"'] \end{tikzcd} \tag{1.15}$$

where φ, φ' are the Thom isomorphisms. From this we see that if $u \in \tilde{H}^*(\mathrm{Th}S(E \oplus m);\mathbb{F}_2), u' \in \tilde{H}^*(\mathrm{Th}S(E' \oplus m');\mathbb{F}_2)$ are the Thom classes, then $u = \mathrm{Th}f^*u'$, and by the naturality of the Steenrod squares we see that $\mathrm{Sq}^k u = \mathrm{Th}f^*\mathrm{Sq}^k u'$ for all k, and so

$$\begin{aligned} w_k(E \oplus m) &= \varphi^{-1}\mathrm{Sq}^k u = \varphi^{-1}\mathrm{Th}f^*\mathrm{Sq}^k u' \\ &= \varphi'^{-1}\mathrm{Sq}^k u' = w_k(E' \oplus m') \text{ for all } k \end{aligned}$$

Finally the Whitney product formula gives $w_k(E \oplus m) = w_k(E)$ and $w_k(E' \oplus m') = w_k(E')$ for all k.

□

Let Y be a pointed CW complex. Then Atiyah says Y is n-reducible if there's a pointed map $S^n \xrightarrow{f} Y$ inducing isomorphisms $\tilde{H}_i(S^n;\mathbb{Z}) \xrightarrow{f_*} \tilde{H}_i(Y;\mathbb{Z})$ for $i \geq n$. Dually, Y is said to be n-coreducible if there's a pointed map $Y \xrightarrow{g} S^n$ inducing isomorphisms $\tilde{H}^i(S^n;\mathbb{Z}) \xrightarrow{g^*} \tilde{H}^i(Y;\mathbb{Z})$ for $i \leq n$. Similarly, Y is said to be stably n-reducible if there's a stable map $S \xrightarrow{f} \Sigma^*X$ of degree n inducing isomorphisms $\tilde{H}_i(S;\mathbb{Z}) \xrightarrow{f_*} \tilde{H}_{i+n}(\Sigma^*Y;\mathbb{Z})$ for $i \geq 0$, and stably n-coreducible if there's a stable map $\Sigma^*Y \xrightarrow{g} S$ of degree $-n$ inducing isomorphisms $\tilde{H}^i(S;\mathbb{Z}) \xrightarrow{g^*} \tilde{H}^{i+n}(\Sigma^*Y;\mathbb{Z})$ for $i \leq 0$ We can make some trivial observations:

Proposition 1.35. *Let Y, Y' be pointed CW complexes and let $Y \xrightarrow{f} Y'$ be a pointed homotopy equivalence. Then Y is n-(co)reducible if and only if Y' is. Similarly, if $\sigma^*Y \xrightarrow{f} \Sigma^*Y'$ is stable homotopy equivalence then Y is stably n-(co)reducible if and only if Y' is.*

Let Y be a finite pointed CW complex with suspension spectrum Σ^*Y and let S^{Σ^*Y} be the Spanier Whitehead dual of Σ^*Y. Then a stable map $\Sigma^*Y \xrightarrow{f} S$ of degree $-n$ is equivalent to a stable map $S \xrightarrow{f^\vee} S^{\Sigma^*Y}$ of degree n, and by naturality f induces isomorphisms

$$\tilde{H}^i(S;\mathbb{Z}) \xrightarrow{f^*} \tilde{H}^{i+n}(\Sigma^*Y;\mathbb{Z})$$

for $i \geq 0$ if and only if $f^\vee$ induces isomorphisms $\tilde{H}_i(S;\mathbb{Z}) \xrightarrow{f^{\vee *}} \tilde{H}_{i+n}(\Sigma^* Y;\mathbb{Z})$ for $i \leq 0$ (see Adams's *Stable homotopy and generalised homology*). We can say "Y is stably n-coreducible if and only if it's Spanier-Whitehead dual is stably n-reducible." A similar statement holds with coreducible and reducible interchanged.

Proposition 1.36. *Let X be a connected, finite, pointed CW complex and let $S^{n-1} \to F \xrightarrow{\pi} X$ be a spherical fibration. If* ThF *is stably n-coreducible, then F is orientable.*

Proof. If ThF is stably n-coreducible we'll have a stable map $\Sigma^*\mathrm{Th}F \xrightarrow{f} S$ of degree $-n$ inducing isomorphisms $\tilde{H}^i(S;\mathbb{Z}) \xrightarrow{f^*} \tilde{H}^{i+n}(\Sigma^*\mathrm{Th}F;\mathbb{Z})$ for $i \geq 0$. Since X is finite, this is equivalent to a map $\Sigma^m\mathrm{Th}F \xrightarrow{f} S^{m+n}$ inducing isomorphisms $\tilde{H}^i(S^{m+n};\mathbb{Z}) \xrightarrow{f^*} \tilde{H}^i(\Sigma^m\mathrm{Th}F;\mathbb{Z})$ for $i \leq m+n$. In particular, it must be that

$$\tilde{H}^{m+n}(\mathrm{ThF} \oplus (\mathrm{m}-1);\mathbb{Z})\tilde{H}^{m+n}(\Sigma^m\mathrm{Th}F;\mathbb{Z}) \simeq \mathbb{Z}$$

From the above discussion of the Thom isomorphism, it follows that $F \oplus (m-1)$ is orientable. □

Also,

Proposition 1.37. *Let X be a connected, finite, pointed CW complex and let $E \xrightarrow{\pi} X$ be an orthogonal real vector bundle of rank n. Then* ThE *is stably n-coreducible if and only if $[S(E)] = 0 \in \tilde{J}(X)$.*

Proof. If $[S(E)] = 0 \in \tilde{J}(X)$ then there's a fiber homotopy equivalence $S(E) * (m-1) \simeq X \times S^{m+n-1}$ over X, resulting in a homotopy equivalence of Thom spaces $\Sigma^m\mathrm{Th}E \simeq \Sigma^{m+n}X_+$. Now just observe that the "projection" $\Sigma^{m+n}X_+ \xrightarrow{f} S^{m+n}$ (really, this is obtained from the projection $X \times S^{m+n} \to S^{m+n}$, which sends $X \times \{\infty\} \to \{\infty\}$) induces isomorphisms

$$\tilde{H}^i(S^{m+n};\mathbb{Z}) \xrightarrow{f^*} \tilde{H}^i(\Sigma^{m+n}X_+;\mathbb{Z})$$

for $i \leq m+n$ (this can be seen from the Kunneth formulas).

For the converse see Atiyah's paper. □

Corollary 1.38. *Let $E \xrightarrow{\pi} X$ be an orthogonal real vector bundle of rank n on a connected, finite, pointed CW complex X. Then there's a stable homotopy equivalence $\Sigma^*X \simeq \Sigma^*\mathrm{Th}E$ of degree n if and only if $[S(E)] = 0 \in \tilde{J}(X)$.*

Corollary 1.39. *Let $E \xrightarrow{\pi} X$ be an orthogonal real vector bundle of rank n on a connected, finite, pointed CW complex X. Then there's a $d \in \mathbb{N}$ so that there's a stable homotopy equivalence $\Sigma^*X \simeq \Sigma^*\mathrm{Th}mE$ of degree degree mn if and only if $d|m$. Namely, d is the order of $[S(E)] \in \tilde{J}(X)$.*

1.2.2. *Spanier-Whitehead duality for compact smooth manifolds.* See Adams's *Stable homotopy and generalised homology* for a crash course on Spanier-Whitehead duality.

Proposition 1.40. *Let M be a compact smooth n-manifold with boundary ∂M. If $N > 2n+1$, then there's a smooth embedding $\iota : M \to \mathbb{R}^N$ so that (i)* int$M$ *lies in the open unit cube $\{(x_i) \in \mathbb{R}^N | 0 < x_i < 1 \text{ for all } i\}$, (ii) ∂M lies in the open face $\{(x_i) \in \mathbb{R}^N | 0 < x_i < 1 \text{ for } i < N \text{ and } x_N = 0\}$ and (iii) M intersects the hyperplane $x_N = 0$ transversely.*

Proof. Atiyah gives a slick proof (which only requires $N > 2n$) that I can't understand. Here's a different approach:

By Whitney's embedding theorem we can find smooth embedding $M \xrightarrow{f} \mathbb{R}^{N-1}$. Now, recall that the normal bundle $N\partial M \to \partial M$ is trivial, as we can build an "inward pointing" unit vector field $V : \partial M \to TM$. So, for sufficiently small $\epsilon > 0$, the map

$$\partial M \times I \to \bar{B}(0,\epsilon) \subset N\partial M \xrightarrow{\exp} M$$
$$\text{sending } (x,t) \mapsto \epsilon t V_x \mapsto \exp(\epsilon t V_x)$$

will be a homeomorphism onto a "collar neighborhood" $C \subset M$ of ∂M. Thus we can define a smooth function $g : C \to \mathbb{R}$ by the composition $C \simeq \partial M \times I \xrightarrow{\text{project}} I \subset \mathbb{R}$. The function g will send the components of ∂C corresponding to $\partial M \times \{0\}, \partial M \times \{1\}$ to $0, 1$ respectively. In fact, we can extend g to all of M by declaring it to be 1 on $M - C$. The claim is that the resulting smooth map

$$\iota := f \times g : M \to \mathbb{R}^N \text{ sending } x \mapsto (f(x), g(x))$$

has all the desired properties after rescaling (note that M is compact, so its image is closed and bounded). I won't check this.

□

Theorem 1.41. *Let M be a compact smooth n-manifold with boundary ∂M, and let $TM \xrightarrow{\pi} M$ be its tangent bundle. Then $\Sigma^*\mathrm{Th}(-TM)$ is the Spanier-Whitehead dual of $\Sigma^* M/\partial M$.*

Proof. Let $\iota : M \to I^N$ be a smooth embedding as described in the previous proposition. I'm going to systematically identify M with its image $\iota(M)$. Include $I^N \subset I^N \times I = I^{N+1}$ in the usual way, let e_{N+1} be the last standard basis vector, and form the join $I^N * e_{N+1}$; this is just a cone on I^N; it's boundary is a topological N-sphere, which I'll just call S^N. Moreover, conditions (i-ii) of the previous proposition guarantee that if $\partial M * e_{N+1} \subset I^N * e_{N+1}$ is the join of ∂M with e_{N+1}, then we have a canonical homeomorphism $M \cup C\partial M \simeq M \cup \partial M * e_{N+1}$. Since the canonical map $M \cup C\partial M \to M/\partial M$ collapsing the cone is a homotopy equivalence, it follows that $\Sigma^*(S^N - M \cup C\partial M)[-N+1]$ will be Spanier-Whitehead dual to $\Sigma^* M/\partial M$.

Observe that projection from e_{N+1} gives a deformation retraction $S^N - M \cup C\partial M \xrightarrow{\text{proj}} I^N - M$. Also, if $C \subset I^N$ is a (closed) tubular neighborhood of M diffeomorphic (via the exponential) to the disc bundle $D(NM)$ of the normal bundle $NM \xrightarrow{\pi} M$, then the deformation retraction $D(NM) - M \to S(NM)$ (sending $v \mapsto \frac{v}{|v|}$) extends to a deformation retraction $I^N - M \to I^N - U$ where $U := \operatorname{int} C$ is the open tubular neighborhood of M corresponding to $\operatorname{int} D(NM)$. We'll need the following lemma:

Lemma 1.42. *Let (Y, X) be a CW pair, with Y contractible. Then there's a natural homotopy equivalence $Y/X \simeq \Sigma X$.*

Proof of lemma. Consider the cofibration sequence

$$X \xrightarrow{i} Y \xrightarrow{j} Y/X \to \Sigma X \xrightarrow{\Sigma i} \Sigma Y \xrightarrow{\Sigma j} \dots$$

Since Y and ΣY are contractible, it must be that the natural map $Y/X \to \Sigma X$ is a homotopy equivalence.

□

Applying this lemma to the inclusion $I^N - U \subset I^N$, we see that there's a natural homotopy equivalence $I^N/(I^N - U) \simeq \Sigma(I^N - U)$, yielding a stable homotopy equivalence $\Sigma^*(I^N - U)[+1] \simeq \Sigma^* I^N/(I^N - U)$ Now observe that condition (iii) of the previous proposition guarantees that if $D(NM) \xrightarrow{\pi} M$ is the disk bundle associated to the normal bundle of M in I^N, then there's an "excision homeomorphism"

$$D(NM)/S(NM) \simeq I^N/(I^N - U)$$

And by definition $D(NM)/S(NM)$ is ThNM, so we have an equivalence $\Sigma^* I^N/(I^N \bar{-} M) \simeq \Sigma^*ThNM$ The upshot is that we have stable homotopy equivalences

$$\Sigma^*(S^N - M \cup C\partial M)[-N+1] \simeq \Sigma^* I^N/(I^N \bar{-} M)[-N] \simeq \Sigma^*\mathrm{Th}NM[-N]$$

Since there's an isomorphism of vector bundles $TM \oplus NM \simeq M \times \mathbb{R}^N$ over M, we have $-[TM] = [NM] - N \in KO^0(M)$, and hence by definition $\Sigma^*\mathrm{Th}(-TM)$ is the spectrum $\Sigma^*\mathrm{Th}NM[-N]$. □

Remark 1.43. Note that in the case where $\partial M = \emptyset$, we see that the Spanier-Whitehead dual of $\Sigma^* M_+$ is $\Sigma^*\mathrm{Th}(-TM)$ (here $M_+ = M \coprod \mathrm{pt}$ is the disjoint union of M with a point). In this (important) special case we can give a more straightforward proof of the above theorem:

If $N > 2n$ then by Whitney's theorem there's a smooth embedding $\iota M \to \mathbb{R}^N$. I'll systematically identify M with its image $\iota(M)$. Extend this to an embedding $M_+ \to \mathbb{R}^N \cup \{\infty\} = \mathbb{S}^N$ sending the additional point to ∞. Now observe that $S^N - M_+ = \mathbb{R}^N - M$. Let $U \subset \mathbb{R}^N$ be a tubular neighborhood of M diffeomorphic to the (open) disk bundle of the normal bundle $NM \xrightarrow{\pi} M$; then we have a deformation retraction $U \to M$, and this yields a homotopy equivalence $\mathbb{R}^N - M \simeq \mathbb{R}^N - U$. Again as $\mathbb{R}^N$ is contractible we can show that $\mathbb{R}^N/(\mathbb{R}^N - U) \simeq \Sigma(\mathbb{R}^N - U)$. Now, we have an "excision homeomorphism"

$$\mathbb{R}^N/(\mathbb{R}^N - U) \simeq D(NM)/S(NM) = \mathrm{Th}NM$$

The result of all this is an identification of the Spanier Whitehead dual of $\Sigma^* M_+$ as

$$\Sigma^*(\mathbb{R}^N - M)[-N+1] \simeq \Sigma^*\mathbb{R}^N/(\mathbb{R}^N - U)[-N] \simeq \Sigma^*\mathrm{Th}NM[-N] = \Sigma^*\mathrm{Th}(-TM)$$

Corollary 1.44 (Poincare Duality)**.** *Let M be a compact oriented smooth n-manifold with boundary ∂M. Then there are canonical isomorphisms*

$$H^i(M, \partial M; \mathbb{Z}) \simeq H_{n-i}(M; \mathbb{Z}) \text{ and}$$

$$H_i(M, \partial M; \mathbb{Z}) \simeq H^{n-i}(M; \mathbb{Z}) \text{ for all } i$$

Proof. I'll suppress the coefficients. We have

$$H^i(M, \partial M) \simeq \tilde{H}^i(M/\partial M) \simeq \tilde{H}_{-i}(\Sigma^*\mathrm{Th}(-TM))$$

by the previous result and Spanier-Whitehead duality. By the proof of the previous result if $\iota : M \to \mathbb{R}^N$ is a smooth embedding with normal bundle $NM \to M$, then we'll have $\Sigma^*\mathrm{Th}(-TM) = \Sigma^*\mathrm{Th}(NM)[-N]$, and hence

$$\tilde{H}_{-i}(\Sigma^*\mathrm{Th}(-TM)) \simeq \tilde{H}_{N-i}(\Sigma^*\mathrm{Th}(NM)) \simeq H_{n-i}(M)$$

where the second isomorphism is the Thom isomorphism for the oriented vector bundle $NM \to M$ (NM is oriented because TM is, and we have $N = n + \dim NM$). The proof of the second statement is similar (Use the fact that "$M/\partial M^{\vee\vee} \simeq M/\partial M$"). □

More importantly, this proposition opens up the possibility of proving Poincare-duality theorems for generalized (co)homology theories.

Proposition 1.45. *Let M be a compact smooth n-manifold (without boundary) and let $E \xrightarrow{\pi} M$ be a smooth orthogonal real vector bundle of rank m. Then the Spanier-Whitehead dual of* $\Sigma^*\mathrm{Th}E$ *is* $\Sigma^*\mathrm{Th}(-E - TM)$.

Remark 1.46. There's not much loss of generality in assuming that $E \xrightarrow{\pi} M$ is smooth. More precisely: if $E \xrightarrow{\pi} M$ is any orthogonal real vector bundle, there's a smooth orthogonal real vector bundle $E' \xrightarrow{\pi} M$ and an isometry $E \simeq E'$ over M. Moreover if $E, E' \xrightarrow{\pi} M$ are smooth orthogonal real vector bundles and there's a continuous isometry $E \simeq E'$ over M, then there's a smooth isometry $E \simeq E'$ over M.

This can be seen as follows. By compactness of M, any continuous classifying map $M \xrightarrow{f} G_m\mathbb{R}^\infty$ will factor through a finite-dimensional Grassmannian $G_m\mathbb{R}^N$, for sufficiently large N. By Whitney approximation, f will be homotopy to a smooth map $M \xrightarrow{g} G_m\mathbb{R}^N$. Similarly, if $M \times I \xrightarrow{h} G_m\mathbb{R}^\infty$ is a continuous homotopy between smooth classifying maps $M \xrightarrow{f,g} G_m\mathbb{R}^N \subset G_m\mathbb{R}^\infty$, then h will factor through some finite-dimensional Grassmannian $G_m\mathbb{R}^{N'}$, and now Whitney approximation shows that there's a smooth homotopy k from f to g.

Basically the same argument shows that if G is any compact Lie group, then every continuous principal G-bundle $P \xrightarrow{\pi} M$ is isomorphic to a smooth principal G-bundle, and if two smooth principal G-bundles $P, P' \xrightarrow{\pi,\pi'} M$ are continuously isomorphic then they're smoothly isomorphic.

Proof. By definition $\mathrm{Th}E = D(E)/S(E)$. Since $E \xrightarrow{\pi} M$ is a smooth orthogonal vector bundle, $D(E)$ is a compact smooth manifold with boundary $S(E)$, and so the Spanier-Whitehead dual of $\Sigma^* D(E)/S(E)$ can be identified as $\Sigma^*\mathrm{Th}(-TD(E))$.

Now, $D(E_x) \xrightarrow{\iota} D(E) \xrightarrow{\pi} M$ is a smooth fiber bundle, and for each $v \in D(E)$ over a point $x \in M$ the differentials $d\iota, d\pi$ give a natural short exact sequence

$$0 \to TD(E_x)_v \xrightarrow{d\iota} TD(E)_v \xrightarrow{d\pi} TM_x \to 0$$

Now note that E_x has the structure of a real vector space, and so the left invariant vector fields on E_x give a canonical trivialization $E_x \times E_x \simeq TE_x$; by restriction we obtain a trivialization $D(E_x) \times E_x \simeq TD(E_x)$. It follows that the above short exact sequence is equivalent to

$$0 \to E_x \to TD(E)_v \to TM_x \to 0$$

From this we see that there's a "bundle map" (i.e. a map of vector bundles giving isomorphisms on fibers)

$$\begin{array}{ccc} TD(E) & \longrightarrow & E \oplus TM \\ \downarrow & & \downarrow \\ D(E) & \xrightarrow{\pi} & M \end{array} \tag{1.16}$$

So, $TD(E) \simeq \pi^*E \oplus \pi^*TM$, and $-[TD(E)] = -\pi^*[E] - \pi^*[TM]$. From this and the fact that $D(E) \to M$ is a deformation retraction it's not hard to obtain a stable homotopy equivalence $\Sigma^*\mathrm{Th}(-TD(E)) \simeq \Sigma^*\mathrm{Th}(-E - TM)$. More precisely, let $F, NM \to M$ be orthogonal vector bundles so $E \oplus F$ and $TM \oplus NM$ are trivial. Then taking the direct sum of the above exact sequence with the exact sequence $0 \to \pi^*F \to \pi^*(F \oplus NM) \to \pi^*NM \to 0$ shows that $\pi^*(F \oplus NM)$ is an orthogonal vector bundle so $TD(E) \oplus \pi^*(F \oplus NM)$ is trivial. We have a bundle map

$$\begin{array}{ccc} \pi^*(F \oplus NM) & \longrightarrow & F \oplus NM \\ \downarrow & & \downarrow \\ D(E) & \xrightarrow{\pi} & M \end{array} \tag{1.17}$$

giving a stable homotopy equivalence

$$\Sigma^*\mathrm{Th}(-TD(E)) \simeq \Sigma^*\mathrm{Th}\pi^*(F \oplus NM)[-N] \simeq \Sigma^*\mathrm{Th}(F \oplus NM)[-N] \simeq \Sigma^*\mathrm{Th}(-E - TM)$$

where $N = \dim E \oplus TM \oplus F \oplus NM$.

□

Corollary 1.47. *Let M be a connected compact smooth n-manifold (without boundary) and let $E \xrightarrow{\pi} M$ be an orthogonal real vector bundle of rank m. Then* ThE *is stably m-reducible if and only if $[S(E)] = -[S(TM)] \in \tilde{J}(M)$.*

Proof. Note that ThE is stably m-reducible if and only if the Spanier Whitehead dual of $\Sigma^*\mathrm{Th}E$, which by the previous result is $\Sigma^*\mathrm{Th}(-E - TM)$, is stably coreducible. From one of the above propositions this occurs if and only if

$$-[S(E)] - [S(TM)] = 0, \text{ i.e. } [S(E)] = -[S(TM)] \text{ in } \tilde{J}(M)$$

□

Definition 1.48. An **oriented map** $M \xrightarrow{f} N$ between compact smooth manifolds M and N is a smooth map together with an isomorphism of local coefficient systems $\mathscr{A} \simeq f^*\mathscr{B}$, where $\mathscr{A}, \mathscr{B}$ are the orientation local coefficient systems of M, N respectively.

A smooth map $M \xrightarrow{f} N$ is said to be *orientable* if there exists an isomorphism $\mathscr{A} \simeq f^*\mathscr{B}$. Atiyah shows in his paper that this occurs if and only if $w_1(TM) \simeq f^*w_1(TN)$.

Note that an oriented map $M \xrightarrow{f} N$ as above induces homomorphisms

$$H_*(M;\mathscr{A}) \xrightarrow{f_*} H_*(N;\mathscr{B}) \text{ and } H^*(M;\mathscr{A}) \xleftarrow{f^*} H^*(N;\mathscr{B})$$

Proposition 1.49. *Let $M \xrightarrow{f} N$ be an oriented map of connected compact smooth manifolds inducing an isomorphism*

$$H_*(M;\mathscr{A}) \xrightarrow{f_*} H_*(N;\mathscr{B})$$

Then $[S(TM)] = [S(f^*TN)] \in \tilde{J}(M)$.

Atiyah points out that a homotopy equivalence $M \xrightarrow{f} N$ is necessarily orientable and induces an isomorphism on homology with local coefficients. This yields:

Corollary 1.50. *Suppose $M \xrightarrow{f} N$ is a homotopy equivalence of connected compact smooth manifolds. Then $[S(TM)] = [S(f^*TN)] \in \tilde{J}(M)$*

See Atiyah's paper for proofs.

2. The Adams conjecture

2.1. The "k-local" version of Dold's theorem.

Theorem 2.1. *Let X be a finite CW complex and let $F, F' \xrightarrow{\pi} X$ be sphere bundles over X (with the same fiber dimension). Let $k \in \mathbb{N}$ be a positive integer, and suppose there's a map of sphere bundles*

$$\begin{array}{ccc} F & \xrightarrow{f} & F' \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'} \\ X & \xrightarrow{=} & X \end{array} \tag{2.1}$$

of degree $\pm k$ on fibers (i.e. for each $x \in X$, the map of spheres $F_x \xrightarrow{f} F'_x$ has degree $\pm k$). Then there is a non-negative integer $e \in \mathbb{N}$ so that k^eF and k^eF' are fiber homotopy equivalent.

The case where $k = 1$ is a theorem of Dold from his paper *Uber fasemweise Homotopieaquivalenz von Faserraumen*. This "k-local" generalization of Dold's theorem is due to Adams.

Remark 2.2. Just to fix ideas: let $F, F' \xrightarrow{\pi,\pi'} X$ be $n-1$-sphere bundles over a finite CW-complex X, and let $F \xrightarrow{f} F'$ be a map over X. For each $x \in X$ we can look at the map of $n-1$-sphere fibers $F_x \xrightarrow{f_x} F'_x$.

Important note: f_x does not have a well-defined degree, because F_x and F'_x don't come with orientations (at least not under the current hypotheses). What *is* well defined is $\pm \deg f_x \in \mathbb{Z}/\{\pm 1\}$. To compute this, note that the sphere-bundle data that comes with F, F' gives homeomorphisms $F_x, F'_x \simeq S^{n-1}$, allowing us to represent f_x by a map $S^{n-1} \to S^{n-1}$, which has a degree; now mod out by the fact that we don't know orientations.

The fiberwise degree $\pm \deg f_x \in \mathbb{Z}/\{\pm 1\}$ will be locally constant as a function of $x \in X$ - i.e. it'll be constant on the path components of X. To see this, just note that we can cover X by (path) connected neighborhoods $U \subset X$ with local trivializations $U \times S^{n-1} \simeq F|_U, F'|_U$ in which case over U, f is represented by a map $U \times S^{n-1} \to U \times S^{n-1}$ of the form $(x, v) \mapsto (x, g(x, v))$; where g is a map $U \times S^{n-1} \to S^{n-1}$. For each $x \in U$ the map $F_x \xrightarrow{f_x} F'_x$ is represented by the map $S^{n-1} \to S^{n-1}$ taking $v \mapsto g(x, v)$, and (since U is assumed to be connected) all these representing maps are homotopic.

I'm not going to include a proof of Theorem 2.1 - the original proof appearing in Adams's paper *On the groups $J(X)$ - I* is fantastic. I did take some notes on the tools he develops along the way, which seem interesting in their own right.

2.1.1. *The topological monoid $G(n)$.* For any $n \in \mathbb{N}$, let $G(n) = (S^{n-1})^{S^{n-1}}$ denote the topological monoid of continuous maps from S^{n-1} to itself . Recall that two maps $f, g : S^{n-1} \to S^{n-1}$ are homotopic if and only if they have the same degree (this theorem dates back to Brouwer and Hopf); so, degree defines a bijection $\deg : \pi_0 G(n) \to \mathbb{Z}$. Furthermore, if $f, g : S^{n-1} \to S^{n-1}$ are continuous maps then $\deg f \circ g = \deg f \cdot \deg g$, i.e. degree is multiplicative on compositions. It follows that deg defines an isomorphism of monoids $\pi_0 G(n) \simeq \mathbb{Z}$, where $\mathbb{Z}$ is viewed as a monoid under multiplication. For any $d \in \mathbb{Z}$, let $G(n, d) = \deg^{-1}(d)$ denote the maps of degree d. We have $H(n) = G(n, 1) \cup G(n, -1)$.

As one might guess (in light of the discussion of the topological monoid $H(n)$) for any $g \in G(n)$ we have canonical isomorphisms

$$\pi_i(G(n), g) \simeq \pi_i^s S \text{ for } i < n-2$$

These will be described presently - the new wrinkle is that they'll *depend on the degree of g*.

Observe that evaluating self maps $f \in G(n)$ at a base point $s_0 \in S^{n-1}$ results in a fibration

$$F(n) \to G(n) \xrightarrow{\mathrm{ev}_{s_0}} S^{n-1}$$

where $\mathrm{ev}_{s_0} f = f(s_0)$ and by definition $F(n) \subset G(n)$ is the topological submonoid of *pointed* self maps of S^{n-1}; that is, $f \in F(n) \subset G(n)$ if and only if $f(s_0) = s_0 \in S^{n-1}$.

Remark 2.3. Loosely speaking $G(n)$ acts transitively on S^{n-1} and the isotropy subgroup of s_0 is the closed submonoid $F(n)$.

Given a basepoint $g \in F(n) \subset G(n)$, we'll have a π_* long exact sequence

$$\ldots \xrightarrow{\partial} \pi_i(F(n), g) \xrightarrow{\iota_*} \pi_i(G(n), g) \xrightarrow{\mathrm{ev}_{s_0 *}} \pi_i(S^{n-1}, s_0) \xrightarrow{\partial} \ldots$$

Since $\pi_i(S^{n-1}, s_0) = 0$ when $i < n-1$, we see that the homomorphism $\pi_i(F(n), g) \xrightarrow{\iota_*} \pi_i(G(n), g)$ is an isomorphism when $i < n-2$ and a surjection when $i = n-2$.

As observed above, $F(n)$ is a topological monoid under the composition operation

$$F(n) \times F(n) \mapsto F(n) \text{ taking } (f, g) \mapsto f \circ g$$

Degree defines an isomorphism of (discrete) monoids

$$\deg : \pi_0 F(n) \mapsto \mathbb{Z}, \text{ where } \mathbb{Z} \text{ is viewed as a monoid under multiplication}$$

The cool thing is that $F(n)$ comes with *another* H-space structure, coming from the "co-H-space" structure of S^{n-1}, i.e. the co-operation

$$S^{n-1} \xrightarrow{\nu} S^{n-1} \vee S^{n-1}$$

obtained by collapsing an equatorial copy of S^{n-2} containing the basepoint. To be explicit it's useful to view S^{n-1} as the quotient $I^{n-1}/\partial I^{n-1}$, with basepoint $\partial I^{n-1}/\partial I^{n-1}$. In that case we may view ν as the map obtained by stretching I^{n-1} with

$$I^{n-1} \to [0,2] \times I^{n-2} = \text{ sending } (x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1})$$

and then collapsing $\partial[0,1] \times I^{n-2} \cup \partial[1,2] \times I^{n-2} \subset [0,2] \times I^{n-2}$ to a point. The usual arguments used to set up the machinery of higher homotopy groups (see the beginning of chapter 4.1 in Hatcher) show that ν is homotopy co-associative, with co-unit the constant map $S^{n-1} \mapsto \{s_0\}$, co-antipode $j : S^{n-1} \to S^{n-1}$ obtained by reversing the x_1-coordinate; that is, j is obtained from the map

$$\tilde{j} : I^{n-1} \to I^{n-1} \text{ taking } (x_1, \ldots, x_{n-1}) \mapsto (1 - x_1, \ldots, x_{n-1})$$

Now the co-H-space structure on S^{n-1} induces an H-space structure on $F(n)$ - the operation

$$F(n) \times F(n) \xrightarrow{\mu} F(n)$$

can be described as follows: if $f, g \in F(n)$ are pointed self maps of S^{n-1}, then $\mu(f,g)$ is the map

$$S^{n-1} \xrightarrow{\nu} S^{n-1} \vee S^{n-1} \xrightarrow{f \vee g} S^{n-1}$$

Since ν is homotopy co-associative, μ is homotopy associative. The homotopy co-unit $S^{n-1} \to \{s_0\}$ corresponds to a homotopy unit $s_0 = \text{const} \in F(n)$, and precomposition with the homotopy co-antipode reflecting over $x_0 = 0$ gives a homotopy antipode $i : F(n) \to F(n)$ sending $f \mapsto f \circ j$. Finally if $n > 2$, μ is homotopy-commutative.

Remark 2.4. Obviously lots of these statements generalize - for instance if (X, x_0) is a pointed co-H-space and (Y, y_0) is a pointed space, it should be that the space of pointed maps $(Y, y_0)^{(X, x_0)}$ in the compact-open topology is an H-space ... at least under reasonable point-set hypotheses on X, Y. Etc.

For simplicity I'll start writing this H-space operation μ additively, i.e. $\mu(f,g) = f + g \in F(n)$ for $f, g \in F(n)$.

Question 2.5. How does this H-space operation relate the the H-space operation $\circ$ given by composition? More specifically, do the operations $+$ and $\circ$ make $F(n)$ into a "H-ring" (more precisely, this would mean a "ring object in the homotopy category of pointed (CW) spaces"). Of course this question only makes sense with $n > 2$.

The only thing to check would be that $\circ$ is bilinear with respect to $+$, at least up to homotopy; on the one hand if $f, g_1, g_2 \in F(n)$ we have $f \circ (g_1 + g_2) = f \circ g_1 + f \circ g_2$ "on the nose." However, it's not at all clear that if $f_1, f_2, g \in F(n)$ then "$(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$" (this *is* true stably; see the end if Chapter 4.2 in Hatcher).

Provided that we view the identification $S^{m-1} * S^{n-1} \simeq S^{m+n-1}$ as a homeomorphism of pointed spaces (Adams points out that we just need to agree on a basepoint somewhere along the line segment in $S^{m-1} * S^{n-1}$ between the basepoints for S^{m-1}, S^{n-1}- for instance, we could just declare the basepoint for any sphere $S^{m-1} \subset \mathbb{R}^m$ to be the first standard basis vector e_1), the join $G(m) \times G(n) \xrightarrow{*} G(m+n)$ restricts to a homomorphism of topological monoids

$$F(m) \times F(n) \xrightarrow{*} F(m+n) \text{ taking } (f,g) \mapsto f * g$$

If $g \in F(n,t)$, then adding g (on the right, say) defines homotopy equivalences

$$r_g : F(n,s) \to F(n,s+t) \text{ sending } f \mapsto f + g$$

with homotopy inverse the map

$$r_{-g} : F(n,s+t) \to F(n,s) \text{ sending } f \mapsto f - g$$

for all s. Thus it defines a homotopy equivalence $F(n) \xrightarrow{r_g} F(n)$ inducing translation by t on $\pi_0 F(n) \simeq \mathbb{Z}$ (this requires some direct verification, which I'm going to omit). In fact we should think of r_g as a homotopy equivalence of *pointed* spaces $(F(n), s_0) \simeq (F(n), g)$ (where $s_0 \in F(n,0)$ is the constant map to the basepoint), inducing isomorphisms

$$r_{g*}\pi_i(F(n), s_0) \simeq \pi_i(F(n), g) \text{ for all } i$$

This is because by definition $(F(n,0), s_0) = (\Omega^{n-1}S^{n-1}, s_0)$, and so by the $\Sigma - \Omega$ adjunction we have

$$\pi_i(F(n), s_0) = \pi_i(\Omega^{n-1}S^{n-1}, s_0) = \pi_{i+n-1}S^{n-1} \text{ for all } i$$

and by the Freudenthal suspension theorem

$$\pi_{i+n-1}S^{n-1} = \pi_i^s S \text{ for } i < n-2$$

Conclusion: if $g \in F(n)$ and $i < n-2$ then we have isomorphisms

$$\pi_i(G(n), g) \simeq \pi_i(F(n), g) \simeq \pi_i(F(n), s_0)$$
$$= \pi_i(\Omega^{n-1}S^{n-1}, s_0) = \pi_{i+n-1}S^{n-1} = \pi_i^s S$$

These certainly depend on g - to see that they only depend on the path component of g, or equivalently the degree $\deg g$, one must check that a path $\gamma : I \to F(n)$ between $g_0, g_1 \in F(n)$ defines a homotopy between the maps

$$(F(n), s_0) \xrightarrow{r_{g_i}} F(n, g_i) \text{ where } i = 1, 2$$

(I won't check this here).

Now consider the (left) action of $G(n)$ on itself by post composition; so, an element $f \in G(n)$ acts by the map $l_f : G(n) \mapsto G(n)$ taking $g \mapsto f \circ g$. If $\deg f = t$ then l_f maps $G(n,s) \to G(n,st)$ for $s \in \mathbb{Z}$.

Proposition 2.6. *In the situation described above, suppose* $[g] \in \pi_i G(n,s)$ *so that* $l_{f*}[g] \in \pi_i G(n,st)$. *Assume* $i < n-2$ *so we have canonical isomorphisms* $\pi_i G(n,s) \simeq \pi_i^s S, \pi_i G(n,st) \simeq \pi_i^s S$. *Then under all these identifications,*

$$l_{f*}[g] = t[g] \in \pi_i^s S$$

Recall that the identification $S^{m-1} * S^{n-1} \simeq S^{m+n-1}$ yields a homomorphism of topological monoids

$$G(m) \times G(n) \xrightarrow{*} G(m+n) \text{ taking } (f,g) \mapsto f * g$$

(nothing too deep going on here; everything follows from the fact that "join" is a continuous functor, say on the category of finite CW complexes).

Proposition 2.7. *The homomorphism* $G(m) \times G(n) \xrightarrow{*} G(m+n)$ *sends* $G(m,s) \times G(n,t) \to G(m+n,st)$. *That is,* $\deg f * g = \deg f \cdot \deg g$ *for* $f \in G(m), g \in G(n)$.

Proof. Milnor proves in his paper *Construction of universal bundles* that if X, Y are pointed CW complexes and G is any abelian group, then there are natural short exact sequences

$$0 \to \tilde{H}_{i+1}(X * Y; G) \to \tilde{H}_i(X \times Y; G) \to \tilde{H}_i(X; G) \oplus \tilde{H}_i(Y; G) \to 0$$

In fact there are similar natural short exact sequences for any generalized homology theory - see Proposition 2.12 below. Applying this to the case where $X = S^{m-1}, Y = S^{n-1}$ and $G = \mathbb{Z}$ and we're looking at the map $f * g : S^{m-1} * S^{n-1} \to S^{m-1} * S^{n-1}$, (and disregarding the case where $m = 1$ or $n = 1$, in which case we're just talking about suspensions) we obtain a commutative diagram

$$\begin{array}{ccc} \tilde{H}_{m+n-1}(S^{m-1} * S^{n-1}; \mathbb{Z}) & \xrightarrow{\simeq} & \tilde{H}_{m+n-2}(S^{m-1} \times S^{n-1}; \mathbb{Z}) \\ {\scriptstyle f*g_*}\downarrow & & {\scriptstyle f\times g_*}\downarrow \\ \tilde{H}_{m+n-1}(S^{m-1} * S^{n-1}; \mathbb{Z}) & \xrightarrow{\simeq} & \tilde{H}_{m+n-2}(S^{m-1} \times S^{n-1}; \mathbb{Z}) \end{array} \tag{2.2}$$

showing that the degree of $f * g$ is the same as the degree of $f \times g : S^{m-1} \times S^{n-1} \to S^{m-1} \times S^{n-1}$. We can now appeal to the following more general fact:

Lemma 2.8. *Let $f : M \to M'$ and $g : N \to N'$ be continuous maps of connected compact oriented manifolds, where $\dim M = \dim M' = m$ and $\dim N = \dim N' = n$. Then the degree of the resulting map $f \times g : M \times N \to M' \times N'$ of connected compact oriented $m + n$-manifolds is computed as $\deg f \times g = \deg f \cdot \deg g$.*

Proof of the lemma. Let's work with singular homology with coefficients in $\mathbb{Z}$ - I'm going to systematically suppress the $\mathbb{Z}$-coefficients to avoid notational clutter.

First observe that the cross product $H_*(M) \otimes H_*(N) \xrightarrow{\times} H_*(M \times N)$ gives natural isomorphism

$$H_m(M) \otimes H_n(N) \simeq H_{m+n}(M \times N) \text{ taking } [M] \otimes [N] \mapsto [M \otimes N]$$

where $[M], [N]$ and $[M \times N]$ are the fundamental classes. To see this, note that the Kunneth formula gives a natural short exact sequence

$$0 \to \bigoplus_{i+j=m+n} H_i(M) \otimes H_j(N) \xrightarrow{\times} H_{m+n}(M \times N) \to \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(M), H_j(N)) \to 0$$

Since

$$H_i(M) \simeq \begin{cases} \mathbb{Z}, \text{ generated by } [M] & \text{if } i = m \\ 0 & \text{if } i > m \end{cases} \text{ and } H_i(N) = \begin{cases} \mathbb{Z}, \text{ generated by } [N] & \text{if } i = n \\ 0 & \text{if } i > n \end{cases}$$

we see that

$$\bigoplus_{i+j=m+n} H_i(M) = H_m(M) \otimes H_n(N) \text{ and } \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(M), H_j(N)) = 0$$

Similarly, the cross product gives a natural isomorphism

$$H_m(M') \otimes H_n(N') \simeq H_{m+n}(M' \times N') \text{ taking } [M'] \otimes [N'] \mapsto [M' \otimes N']$$

Now we have a commutative diagram

$$\begin{array}{ccc} H_m(M) \otimes H_n(N) & \xrightarrow{\simeq} & H_{m+n}(M \times N) \\ {\scriptstyle f_*\otimes g_*}\downarrow & & {\scriptstyle f\times g_*}\downarrow \\ H_m(M') \otimes H_n(N') & \xrightarrow{\simeq} & H_{m+n}(M' \times N') \end{array} \tag{2.3}$$

from which we see that

$$f \times g_*[M \times N] = f_*[M] \times g_*[N] = \deg f[M'] \times \deg g[N'] = \deg f \deg g[M' \times N']$$

Note that if all the manifolds and map are assumed to be smooth, we can give a dirtier proof:

Let $p \in M', q \in N'$ be regular values of f, g respectively. Then because we have decompositions of tangent bundles

$$T(M \times N) = TM \times TN \text{ and } T(M' \times N') = TM' \times TN'$$

and from this perspective the differential of $f \times g$ looks like

$$df \times dg : TM \times TN \to TM' \times TN'$$

we see that $(p, q) \in M' \times N'$ is a regular value of $f \times g$. Also,

$$f \times g^{-1}(p, q) = \{(x, y) \in M \times N \,|\, f(x) = p, g(y) = q\}$$

and for each $(x, y) \in f \times g^{-1}(p, q)$ we have $df \times g_{x,y} = df_x \times dg_y$ so that

$$\operatorname{sgn}\det df \times g_{x,y} = \operatorname{sgn}\det df_x \cdot \operatorname{sgn}\det dg_y$$

It follows that

$$\deg f \times g = \sum_{(x,y)\in f\times g^{-1}(p,q)} \operatorname{sgn}\det df \times g_{x,y} = \sum_{x\in f^{-1}(p), y\in g^{-1}(q)} \operatorname{sgn}\det df_x \cdot \operatorname{sgn}\det dg_y$$

$$(\sum_{x\in f^{-1}(p)} \operatorname{sgn}\det df_x)(\sum_{y\in g^{-1}(q)} \operatorname{sgn}\det dg_y) = \deg f \cdot \deg g$$

□

□

From the continuous maps $G(m, s) \times G(n, t) \xrightarrow{*} G(m + n, st)$ we obtain homomorphisms of homotopy groups

$$\pi_i G(m, s) \times \pi_i G(n, t) \simeq \pi_i G(m, s) \times G(n, t) \xrightarrow{*} \pi_i G(m + n, st)$$

As long as $0 < i < \min\{m - 2, n - 2\}$ we have canonical identifications of each of the groups $\pi_i G(m, s), \pi_i G(n, t)$ and $\pi_i G(m + n, st)$ with $\pi_i^s S$.

Proposition 2.9. *Suppose $[f] \in \pi_i G(m, s), [g] \in \pi_i G(n, t)$ so that $[f * g] \in \pi_i G(m + n, st)$. Assume $0 < i < \min\{m - 2, n - 2\}$ and identify all these homotopy groups with $\pi_i^s S$ as described above. Then*

$$[f * g] = t[f] + s[g] \in \pi_i^s S$$

Let's derive a few corollaries. For any $m_1, \ldots, m_N \in \mathbb{N}$ the identification $*_{i=1}^n S^{m_i - 1} \simeq S^{\sum_{i=1}^N -1}$ induces a homomorphism of topological monoids $\prod_{i=1}^N G(m_i) \xrightarrow{*} G(\sum_{i=1}^N m_i)$, and since we can factor it like

$$\prod_{i=1}^N G(m_i) \xrightarrow{*\times\mathrm{id}} G(m_1 + m_2) \times \prod_{i=3}^N G(m_i) \xrightarrow{*\times\mathrm{id}} \cdots \xrightarrow{*\times\mathrm{id}} G(\sum_{i=1}^{N-1} m_i) \times G(m_N) \xrightarrow{*} G(\sum_{i=1}^N m_i)$$

we can apply the previous results to show that for any $s_1, \ldots, s_N \in \mathbb{Z}$ the join homomorphism sends $\prod_{i=1}^N G(m_i, s_i) \to G(\sum_{i=1}^N m_i, \prod_{i=1}^N s_i)$, i.e. the induced homomorphism of (discrete) monoids $\prod_{i=1}^N \pi_0 G(m_i) \to \pi_0 G(\sum_{i=1}^N m_i)$ is just the homomorphism $\prod_{i=1}^N \mathbb{Z}_i \to \mathbb{Z}$ taking $(s_i) \mapsto \prod_{i=1}^N s_i$. Then we can show

Corollary 2.10. *Suppose $[f_j] \in \pi_i G(m_i, s_i)$ for $j = 1, \ldots, N$ so that $[*_{i=1}^N f_j] \in \pi_i G(\sum_{i=1}^N m_i, \prod_{i=1}^N s_i)$, and $0 < i < \min\{m_i - 2\}$ so that we can identify all relevant homotopy groups with $\pi_i^s S$. Then*

$$[*_{i=1}^N f_j] = \sum_{j=1}^N (\prod_{k\neq j} s_k)[f_j] \in \pi_i^s S$$

Proof. By induction on N - the base cases ($N = 1$ or 2) have already been dealt with. So, assume $N > 2$ and write

$$[*_{i=1}^{N} f_j] = [*_{i=1}^{N-1} f_j * f_N] = s_N[*_{i=1}^{N-1} f_j] + \prod_{k=1}^{N-1} s_k[f_N]$$

where we've applied the previous lemma. By inductive hypothesis, this is

$$s_N \sum_{j=1}^{N-1} (\prod_{k \neq j} s_k)[f_j] + \prod_{k=1}^{N-1} s_k[f_N]$$

and rearranging a little bit gives the desired result. □

In the special case where $m_i = m \in \mathbb{N}$, $s_i = s \in \mathbb{Z}$ and $[f_j] = [f] \in \pi_i G(m, s)$ for all i, we obtain

Corollary 2.11. *Suppose $[f] \in \pi_i G(m, s)$, so that $[*^N f] \in \pi_i G(Nm, s^N)$, and assume $0 < i < m - 2$ so we can identify all relevant homotopy groups with $\pi_i^s S$. Then*

$$[*^N f] = N s^{N-1}[f] \in \pi_i^s S$$

Based on our discussion of the homotopy groups of $G(n)$, it will suffice to prove the variants of Propositions 2.4 and 2.7 in which the topological monoids $G(n)$ are replaced with the submonoids $F(n)$. Here are precise statements of these variants:

Proposition 2.12. *Suppose $f \in F(n, t)$ so that composition with f defines a map $l_f : F(n, s) \to F(n, st)$ sending $g \mapsto f \circ g$; suppose $[g] \in \pi_i F(n, s)$ so that $l_{f*}[g] \in \pi_i F(n, st)$. Assume $i < n - 2$ so we have canonical isomorphisms $\pi_i F(n, s) \simeq \pi_i^s S, \pi_i F(n, st) \simeq \pi_i^s S$. Then under all these identifications,*

$$l_{f*}[g] = t[g] \in \pi_i^s S$$

Proposition 2.13. *Suppose $[f] \in \pi_i F(m, s), [g] \in \pi_i F(n, t)$ so that $[f * g] \in \pi_i F(m + n, st)$. Assume $0 < i < \min\{m - 2, n - 2\}$ and identify all these homotopy groups with $\pi_i^s S$ as described above. Then*

$$[f * g] = t[f] + s[g] \in \pi_i^s S$$

Proof of Lemma 2.10. Suppose first that $s = 0$ and recall that our identification $\pi_i F(n, 0) \simeq \pi_{i+n-1} S^{n-1} \simeq \pi_i^s S$ takes $g : S^i \to F(n, 0)$ to the adjoint map $S^i \wedge S^{n-1} \mapsto S^{n-1}$ taking $x \wedge y \mapsto g(x)(y)$.

Similarly $l_f g : S^i \to F(n, 0)$ will correspond to the map $S^i \wedge S^{n-1} \mapsto S^{n-1}$ taking $x \wedge y \mapsto l_f g(x)(y) = f \circ g(x)(y)$ - but this is just the composition

$$S^i \wedge S^{n-1} \mapsto S^{n-1} \xrightarrow{f} S^{n-1}$$

$$\text{sending } x \wedge y \mapsto g(x)(y) \mapsto f(g(x)(y))$$

Thus in $\pi_i^s S$, $l_{f*}[g]$ corresponds to the composition product of a map $S^{i+n-1} \to S^{n-1}$ representing $[g]$ with $S^{n-1} \xrightarrow{f} S^{n-1}$. Since $\deg f = t$ we'll have $l_{f*}[g] = t[g]$. See the end of chapter 4.2 in Hatcher.

To obtain the general case (where s might not be 0), note that our identification $\pi_i F(n, s) \simeq \pi_i^s S$ takes $g : S^i \to F(n, s)$ to the map $S^i \wedge S^{n-1} \to S^{n-1}$ sending $x \wedge y \mapsto (g(x) - h)(y)$ where $h \in F(n, s)$ is some fixed degree s map, so adding h gives a homotopy equivalence $F(n, 0) \simeq F(n, s)$.

Since $f \circ h \in F(n, st)$ is a degree st map, $l_f g : S^i \to F(n, st)$ will correspond to the map $S^i \wedge S^{n-1} \to S^{n-1}$ sending $x \wedge y \mapsto (l_f g(x) - f \circ h)(y) = f \circ (g(x) - h)(y)$, which is just the composition

$$S^i \wedge S^{n-1} \to S^{n-1} \xrightarrow{f} S^{n-1}$$

$$\text{sending } x \wedge y \mapsto (g(x) - h)(y) \mapsto f \circ (g(x) - h)(y)$$

Again as $\deg f = t$, we conclude that $l_{f*}[g] = t[g]$.

□

Before jumping into a proof of Lemma 2.11, let's introduce a reduced version of the join. Let (X, x_0) and (Y, y_0) be pointed CW-complexes, and observe that we have canonical subcomplex inclusions

$$X * \{y_0\}, \{x_0\} * Y \to X * Y$$

(I'll identify $X * \{y_0\}, \{x_0\} * Y$ with their images). There are evident homeomorphisms $CX \simeq X * \{y_0\}, CY \simeq \{x_0\} * Y$ and we have

$$X * \{y_0\}, \{x_0\} * Y = \{x_0\} * \{y_0\} \subset X * Y$$

and from this it follows that $X * \{y_0\} \cup \{x_0\} * Y \subset X * Y$ is homeomorphic to the adjunction space obtained by gluing CX to CY along the identification $\{x_0\} \times I \simeq \{y_0\} \times I$. In particular, $X * \{y_0\} \cup \{x_0\} * Y$ is a contractible subcomplex.

Define the **reduced join** of X and Y to be the pointed CW complex $X * Y_{\text{red}} := \frac{X*Y}{X*\{y_0\}\cup\{x_0\}*Y}$ (with basepoint $\frac{X*\{y_0\}\cup\{x_0\}*Y}{X*\{y_0\}\cup\{x_0\}*Y}$). The above discussion shows that the quotient map $X * Y \xrightarrow{\text{quot}} X * Y_{\text{red}}$ is a homotopy equivalence. From now on (or at least for the rest of this section), the join of *pointed* CW complexes $(X, x_0), (Y, y_0)$ is to be understood as the reduced join.

Observation: there's a natural homeomorphism

$$X \wedge S^1 \wedge Y \simeq X * Y_{\text{red}}$$

Essentially, these spaces are obtained as quotients of $X \times I \times Y$ by the same equivalence relation. Just observe that $X * Y_{\text{red}}$ is obtained from $X \times I \times Y$ by first modding out by the equivalence relation $\sim$ where

$$(x, 0, y) \sim (x, 0, y_0) \text{ and } (x, 1, y) \sim (x_0, 1, y) \text{ for all } x \in X, y \in Y$$

to obtain $X * Y$ and then collapsing $X * \{y_0\} \cup \{x_0\} * Y$ to a point; the net effect of all this is to collapse $\{x_0\} \times I \times Y \cup X \times \partial I \times Y \cup X \times I \times \{y_0\}$ to a point. Among other things, this shows

Proposition 2.14. *Let X and Y be pointed CW complexes and let $\tilde{h}_*$ be a generalized homology theory. Then there are canonical isomorphisms*

$$\tilde{h}_{i+1}(X * Y_{red}) \simeq \tilde{h}_i(X \wedge Y)$$

Of course we can replace the reduced join with $X * Y$, since the quotient $X * Y \to X * Y_{\text{red}}$ is a homotopy equivalence. To relate this to Milnor's computation of the homology of joins, note that the inclusion-retractions $X \times \{y_0\}, \{x_0\} \times Y \leftrightarrow X \times Y$ give natural split short exact sequences

$$0 \to \tilde{h}_i(X) \oplus \tilde{h}_i(Y) \to \tilde{h}(X \times Y) \to \tilde{h}_i(X \wedge Y) \to 0$$

Returning to the matter at hand, where $X = S^{m-1}, Y = S(n-1)$, we have an identification of pointed spaces

$$S^{m-1} * S^{n-1}_{\text{red}} \simeq S^{m-1} \wedge S^1 \wedge S^{n-1} \simeq S^{m+n-1}$$

and so we can view the resulting homomorphism of topological monoids

$$F(m) \times F(n) \xrightarrow{*} F(m+n) \text{ as the map}$$
$$(f, g) \mapsto f \wedge \text{id} \wedge g$$

Sketch of proof of Lemma 2.11. Adams points out that we're talking about a *homomorphism* $\pi_i^s S \times \pi_i^s S \to \pi_i^s$, which is to say

$$[f * g] = [f * 0] + [0astg] \in \pi_i^s S$$

Thus it'll suffice to prove that $[f * 0] = t[f], [0 * g] = s[g] \in \pi_i^s S$.

I'll show $[f * 0] = t[f]$; the proof that $[0 * g] = s[g]$ is entirely similar. Observe that if $h \in F(m,s), k \in F(n,t)$ are fixed degree s,t maps, which we can view as the basepoints of $F(m,s), F(n,t)$, then $[f * 0]$ is represented by the map $f * k : S^i \to F(m+n, st)$ sending $x \mapsto f(x) * h$. Since $h * k \in F(m+n, st)$ is a degree st map, the stable map of spheres corresponding to $f * k$ is given by

$$S^i \wedge S^{m-1} \wedge S^1 \wedge S^{n-1} \to S^{m-1} \wedge S^1 \wedge S^{n-1}$$

$$\text{taking } w \wedge x \wedge t \wedge y \mapsto (f(w) \wedge \mathrm{id} \wedge k - h \wedge \mathrm{id} \wedge k)(x \wedge t \wedge y)$$

At least stably speaking, this is obtained from the stable map of spheres $S^i \wedge S^{m-1} \to S^{m-1}$ sending $w \wedge x \mapsto (f(w) - h)(x)$ associated to f by smashing with the degree t map k. Hence (again appealing to chapter 4.2 of Hatcher) $[f * 0] = t[f] \in \pi_i^s S$.

□

2.2. **The conjecture and it's proof in a special case.** In his paper *On the groups* $J(X)$*-I* Adams puts forth the following

Conjecture 2.15. *Let k be an integer. Suppose X is a finite CW complex and $\alpha \in KO^0(X)$ is a virtual orthogonal real vector bundle on X. Then there is a non-negative integer $e \in \mathbb{N}$ (depending on k and α) so that $k^e(\psi^k - 1)\alpha \in KO^0(X)$ maps to $0 \in J(X)$.*

Here ψ^k is the k-th Adams operation on KO^0. One way to rephrase the conjecture is as follows. If k is an integer and X is a finite CW complex, then the following diagram of homomorphisms of abelian groups commutes when localized at k:

$$\begin{array}{ccc} KO^0(X) & \xrightarrow{\psi^k} & KO^0(X) \\ \downarrow & & \downarrow \\ J(X) & \xrightarrow{\mathrm{id}} & J(X) \end{array} \tag{2.4}$$

Here and in what follows it will be useful to consider the "dependence" of this conjecture on its various "inputs," i.e. the integer k, the complex X and the virtual orthogonal bundle $\alpha \in KO^0(X)$. Statements like "the Adams conjecture is true for the integer $k \in \mathbb{Z}$ and "the Adams conjecture is true for the virtual orthogonal bundle $\alpha \in KO^0(X)$ are to be interpreted in the only reasonable way.

Using Theorem 2.1, he establishes the following special case of the conjecture:

Theorem 2.16. *Let k be an integer. Let X be a finite CW complex and suppose $\alpha \in KO^0(X)$ is a virtual vector bundle that can be written as a linear combination of classes $[E] \in KO^0(X)$ where $E \xrightarrow{\pi} X$ is an orthogonal real vector bundle with $\dim E \leq 2$. Then there's a non-negative integer $e \in \mathbb{N}$ so that $k^e(\psi^k - 1)\alpha$ maps to 0 in $J(X)$ - i.e. $k^e(\psi^k - 1)\alpha \in T(X) \subset KO^0(X)$.*

Briefly, the Adams conjecture is true for linear combinations of line bundles and 2-plane bundles.

NOTE: the notation in this proof degenerates into a complete mess. Suggestion: if ρ is an orthogonal real representation of a compact Lie group G acting on a vector space V, let $\chi_\rho \in RO(G)$ denote its character, let $E(\rho) := EG \times_G V \to BG$ be the associated orthogonal real vector bundle over BG and let $[E(\rho)] \in KO^0(BG)$ be the associated class in K-theory. Identifying all these things in the notation leads to a kerfuffle.

Proof. First, observe that for any $e \in \mathbb{N}$, $k^e(\psi^k - 1)$ is an *endomorphism* of the abelian group $KO^0(X)$. For this reason it will suffice to prove: if $E \xrightarrow{\pi} X$ is an orthogonal real vector bundle over X with $\dim E \leq 2$, then there's an $e \in \mathbb{N}$ so $k^e(\psi^k - 1)[E] \in T(X)$. The general case will follow because we may write $\alpha = \sum_i n_i[E_i]$, as a $\mathbb{Z}$-linear combination of classes of 1 and 2-plane bundles $[E_i]$, choose $e_i \in \mathbb{N}$ sufficiently large so that

$$k^{e_i}(\psi^k - 1)[E_i] \in T(X) \text{ for each } i$$

and then choose $e \in \mathbb{N}$ so that $e \geq \max\{e_i\}$. Then we'll have

$$k^e(\psi^k - 1)\alpha = \sum_i k^e(\psi^k - 1)[E_i] = \sum_i k^{e-e_i}k^{e_i}(\psi^k - 1)[E_i] = 0$$

Suppose $\dim E = 1$. In this situation E is obtained from the universal line bundle $E(\gamma_1) \to \mathbb{R}P^\infty (= BO(1))$ by pulling back along some continuous map $f : X \to \mathbb{R}P^\infty$. Note that by cellular approximation and homotopy invariance of pullback we may assume the image of f lies in $\mathbb{R}P^n \subset \mathbb{R}P^\infty$, where $n = \dim X$. Thus E is pulled back from the "n-universal" line bundle $E(\gamma_1^n) \to \mathbb{R}P^n$.

Similarly if $\dim E = 2$, E is obtained from the universal 2-plane bundle $E(\gamma_2) \to G_2\mathbb{R}^\infty (= BO(2))$ by pulling back along a continuous map $f : X \to G_2\mathbb{R}^\infty$, and again by cellular approximation and homotopy invariance of pullback we may assume the image of f lies in $G_2\mathbb{R}^m \subset G_2\mathbb{R}^\infty$, where $\dim X = \dim G_2\mathbb{R}^m = 2(m-2)$. So, E is pulled back from the "n-universal" 2-plane bundle $E(\gamma_2^m) \to G_2\mathbb{R}^m$.

Now, some general observations: first, the operation $k^e(\psi^k - 1)$ is natural with respect to pullback (because ψ^k is). Thus if $f : X \to Y$ is a continuous map of finite CW complexes and $E' \xrightarrow{\pi} Y$ is an orthogonal real vector bundle over Y, with pullback $f^*E \to X$ over X, we'll have

$$k^e(\psi^k - 1)f^*[E] = f^*k^e(\psi^k - 1)[E] \in KO^0(X)$$

Also, the homomorphism $S : KO^0(X) \to J(X)$ is natural, so we'll have a chain map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & T(Y) \arrow[r] \arrow[d, "f^*"'] & KO^0(Y) \arrow[r, "S"] \arrow[d, "f^*"'] & J(Y) \arrow[r] \arrow[d, "f^*"'] & 0 \\
0 \arrow[r] & T(X) \arrow[r] & KO^0(X) \arrow[r, "S"] & J(X) \arrow[r] & 0
\end{tikzcd} \tag{2.5}$$

Hence if $k^e(\psi^k - 1)[E] \in T(Y)$, then $k^e(\psi^k - 1)f^*[E] \in T(X)$. Briefly, if the Adams conjecture is true for E, it's true for f^*E. Even more briefly (and loosely), "the Adams conjecture is compatible with pullback."

Based on these general observations, it will be enough to show that for sufficiently large $e, e' \in \mathbb{N}$,

$$k^e(\psi^k - 1)[\gamma_1^n] \in T(\mathbb{R}P^n) \text{ and } k^{e'}(\psi^k - 1)[\gamma_2^m] \in T(G_2\mathbb{R}^m)$$

Actually, one can show that for large enough e, $k^e(\psi^k - 1)[\gamma_1^n] = 0$.

At this point we need at least a partial computation of the ring $KO^0(X)$, the subgroup $T(X) \subset KO^0(X)$ and the action of ψ^k on $KO^0(X)$ in the cases $X = \mathbb{R}P^n, G_2\mathbb{R}^m$. A brief digression:

Let G be a compact Lie group. For any $n \in \mathbb{N}$, let $EG_n := *^n G$ denote the n-fold join of G. It comes with a free right G-action given by $(g_1 * g_2 * \cdots * g_n)g = g_1g * g_2g * \cdots * g_ng$, and we have a principal G-bundle $EG_n \to EG_n/G := BG_n$ - taking the colimit of these yields a universal principal G-bundle $EG \to BG$. See Milnor's *Construction of universal bundles II*.

Given a k-dimensional orthogonal real representation V of G, one can construct an orthogonal real k-plane bundle $EG_n \times_G V \to BG_n$ over a classifying space BG. The functor $\mathrm{Rep}(G, \mathbb{R}) \xrightarrow{EG_n \times_G -} \mathrm{Vect}(BG_n, \mathbb{R})$ is compatible with direct sums and tensor products, and so it induces a ring homomorphism $RO(G) \xrightarrow{\varphi_n} KO^0(BG_n)$ compatible with Adams operations. Let $I(G) \subset RO(G)$ be the augmentation ideal, i.e. the kernel of the homomorphism $RO(G) \to RO(\{e\}) \simeq \mathbb{Z}$ induced by the inclusion of the identity $e \in G$. One can show that the n-th power $I(G)^n$ lies in the kernel of the homomorphism φ_n. It's a special case of the Atiyah-Segal completion theorem that:

Theorem 2.17. *The homomorphisms $\varphi_n : RO(G)/I(G)^n \to KO^0(BG_n)$ induce and isomorphism of rings $\hat{RO}(G) \to KO^0(BG)$ compatible with Adams operations, where $\hat{RO}(G)$ denotes the $I(G)$-adic completion of $RO(G)$.*

See Atiyah and Segal's *Equivariant K-theory and completion* or Adams et. al.'s *A generalization of the Atiyah-Segal completion theorem*. Benson points out (in his *Representations and cohomology II*) that one can divide the above theorem into two parts: first, that the homomorphisms φ_n induce an isomorphism $RO(G) \simeq \lim KO^0(BG_n)$, and second, that in the Milnor exact sequence

$$0 \to \lim^1 KO^{-1}(BG_n) \to KO^0(BG) \to \lim KO^0(BG_n) \to 0$$

the $\lim^1$ term vanishes.

The upshot is that we may carry out the necessary computations in the representation rings $RO(O(1))$ and $RO(O(2))$.

By abuse of notation, let γ_1 denote the standard representation of $O(1)$. Then

$$\psi^k \gamma_1 = (\gamma_1)^k = \begin{cases} \gamma_1 & \text{if } k \text{ is odd} \\ 1 \text{ (the trivial 1-dimensional representation)} & \text{if } k \text{ is even} \end{cases}$$

and so if k is odd, $(\psi^k - 1)\gamma_1 = \gamma_1 - \gamma_1 = 0$. If k is even, then $k^e(\psi^k - 1)\gamma_1 = k^e(1 - \gamma_1)$, and if $k = 2l$ this is $l^e 2^e (1 - \gamma_1)$. It'll suffice to show that for large enough e,

$$2^e(1 - \gamma_1^n) = 0 \in KO^0(\mathbb{R}P^n)$$

In fact this is immediate from Adams's explicit calculation of $KO^0(\mathbb{R}P^n)$ in his paper *Vector fields on spheres*. To see it a little bit more heuristically from the Atiyah-Segal perspective, note that

$$RO(O(1)) \simeq \mathbb{Z}[\gamma_1]/((\gamma_1)^2 - 1) \simeq \mathbb{Z}[\lambda]/(\lambda^2 - 2\lambda)$$

where $\lambda = 1 - \gamma_1$. Under this isomorphism, we have $I(O(1)) \simeq (\lambda)$, and so for any $j \in \mathbb{N}$

$$RO(O(1))/I(O(1))^j \simeq \mathbb{Z}[\lambda]/(\lambda^j, \lambda^2 - 2\lambda)$$

Since $\lambda^2 = 2\lambda \in \mathbb{Z}[\lambda]/(\lambda^2 - 2\lambda)$, we'll have $\lambda^j = 2^{j-1}\lambda$, and so

$$\mathbb{Z}[\lambda]/(\lambda^j, \lambda^2 - 2\lambda) \simeq \mathbb{Z}[\lambda]/(2^{j-1}\lambda, \lambda^2 - 2\lambda)$$

So, we have an isomorphism of abelian groups

$$\mathbb{Z} \oplus \mathbb{Z}/(2^{j-1}) \simeq RO(O(1))/I(O(1))^j$$

where the $\mathbb{Z}$-summand is generated by 1 and the $\mathbb{Z}/(2^{j-1})$ summand is generated by λ. In particular, if $e \geq j$ we have $2^e(1 - \gamma_1) = 0 \in RO(O(1))/I(O(1))^j$.

Again abusing notation, let γ_2 denote the standard representation of $O(2)$ acting by rotations of $\mathbb{R}^2$ - we must figure out the effect of the Adams operation ψ^k on γ_2. Let $r_\theta \in SO(2) \subset O(2)$ denote counterclockwise rotation by $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, and let τ denote reflection across the x-axis; then elements of $O(2)$ may be written like $\tau^i r_\theta$, for $i \in \{0, 1\}$. Note that we have the conjugation relation $\tau r_\theta \tau = r_{-\theta}$. From this perspective $O(2)$ looks like a "continuous dihedral group."

Now let μ_k denote the 2-dimensional representation obtained by pulling back γ_2 over the homomorphism

$$O(2) \to O(2) \text{ taking } \tau^i r_\theta \mapsto \tau^i r_{k\theta}$$

Let $\bigwedge^2 \gamma_2^\vee$ denote the determinant 1-dimensional representation of $O(2)$ ($\tau^i r_\theta$ acts on $\bigwedge^2 \gamma_2^\vee$ as multiplication by $(-1)^i$). Claim:

$$\psi^k \gamma_2 = \begin{cases} \mu_k & \text{if } k \text{ is odd} \\ \mu_k - \bigwedge^2 \gamma_2^\vee + 1 & \text{if } k \text{ is even} \end{cases}$$

We may verify this claim on the level of characters, viewed as functions $O(2) \to \mathbb{R}$. Note that the action of $\tau^i r_\theta$ on the representation γ_2 is given by the matrix

$$\begin{pmatrix} (-1)^i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

so by direct computation the character of γ_2 looks like

$$\gamma_2(\tau^i r_\theta) = \begin{cases} 2\cos\theta & \text{if } i = 0 \\ 0 & \text{if } i = 1 \end{cases}$$

Similarly, the action of $\tau^i r_\theta$ on the representation μ_k is given by the matrix

$$\begin{pmatrix} (-1)^i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}$$

and so the character of μ_k looks like

$$\mu_k(\tau^i r_\theta) = \begin{cases} 2\cos k\theta & \text{if } i = 0 \\ 0 & \text{if } i = 1 \end{cases}$$

It's easy to check that the character of $\bigwedge^2 \lambda_2^\vee$ looks like $\bigwedge^2 \lambda_2^\vee(\tau^i r_\theta) = (-1)^i$, and of course the character of 1 is, well, 1.

Now a direct calculation shows that $(r_\theta)^k = r_{k\theta}$, and

$$(\tau r_\theta)^k = \begin{cases} \tau r_\theta & \text{if } k \text{ is odd} \\ \mathrm{id} & \text{if } k \text{ is even} \end{cases}$$

It follows that the character $\psi^k\gamma_2$ looks like

$$\psi^k\gamma_2(\tau^i r_\theta) = \gamma_2((\tau^i r_\theta)^k)$$

and the right hand side is computed as $\gamma_2((r_\theta)^k) = \gamma_2(r_{k\theta}) = 2\cos k\theta$, and

$$\gamma_2((\tau r_\theta)^k) = \begin{cases} \gamma_2(\tau r_\theta) = 0 & \text{if } k \text{ is odd} \\ \gamma_2(\mathrm{id}) = 2 & \text{if } k \text{ is even} \end{cases}$$

The claim now follows by an explicit comparison of characters.

It follows that

$$(\psi^k - 1)\gamma_2 = \begin{cases} \mu_k - \gamma_2 & \text{if } k \text{ is odd} \\ \mu_k - \gamma_2 + (1 - \bigwedge^2 \gamma_2^\vee) & \text{if } k \text{ is even} \end{cases}$$

Note that $\bigwedge^2 \gamma_2^\vee$ is obtained from the standard representation of $O(1)$ by pulling back over the determinant homomorphism $O(2) \xrightarrow{\det} O(1)$. Arguing along the lines of our previous observation that "the Adams conjecture is compatible with pullback" one can show that for sufficiently large e, $2^e(1 - \bigwedge^2 \gamma_2^{m\vee}) = 0 \in KO^0(G_2\mathbb{R}^m)$. So, if k is even for large enough e $k^e(1 - \bigwedge^2 \gamma_2^{m\vee}) = 0 \in KO^0(G_2\mathbb{R}^m)$.

At this point the only thing left to prove is that for sufficiently large e

$$k^e(\mu_k - \gamma_2) \in T(G_2\mathbb{R}^m)$$

For this it'll suffice to prove that there's a fiber homotopy equivalence of sphere bundles $k^e S(\gamma_2) \simeq k^e S(\mu_k)$ over $G_2\mathbb{R}^m$, and by the "k-local" version of Dold's theorem, we'll obtain such a fiber homotopy equivalence if we can construct a map of sphere bundles $S(\gamma_2) \xrightarrow{f} S(\mu_k)$ over $G_2\mathbb{R}^m$ with degree $\pm k$ on fibers. Suppose we can define an $O(2)$-equivariant map $g : S^1_{\gamma_2} \to S^1_{\mu_k}$ with degree k, where here I'm referring to the unit spheres in the *representations* γ_2 and μ_k. Then we'll obtain the desired map of sphere bundles f as the balanced product

$$V_2\mathbb{R}^m \times_{O(2)} S^1_{\gamma_2} \xrightarrow{\mathrm{id}\times_{O(2)} g} V_2\mathbb{R}^m \times_{O(2)} S^1_{\mu_k}$$

Finally, to define g, view S^1 as the unit circle in $\mathbb{C}$, with $O(2)$ acting on $S^1_{\gamma_2}$ like

$$\tau^i r_\theta \cdot z = \begin{cases} e^{\imath\theta} z & \text{if } i = 0 \\ e^{-\imath\theta}\bar{z} & \text{if } i = 1 \end{cases}$$

and on $S^1_{\mu_k}$ like

$$\tau^i r_\theta \cdot z = \begin{cases} e^{\imath k\theta} z & \text{if } i = 0 \\ e^{-\imath k\theta}\bar{z} & \text{if } i = 1 \end{cases}$$

Set $g(z) = z^k$; it's straightforward to verify that g is $O(2)$-equivariant.

□

3. Quillen's proof of the Adams conjecture

This section will consist of notes on Quillen's elementary proof of the Adams conjecture (appearing in his paper *The Adams conjecture*). Here "elementary" is to be understood in the sense that there were earlier proofs, but they relied on etale homotopy theory.

In Quillen's own words: "Put briefly, one first shows that the conjecture is true for vector bundles with finite structural group and then using modular character theory one produces enough examples of virtual representations of finite groups to deduce the general case of the conjecture from this special case." In slightly more detail:

3.1. **Outline of the proof.** Let X be a finite CW complex. Let G be a finite group, and suppose $P \to X$ is a principal G-bundle over X. Given an orthogonal real representation V of G, we can construct an orthogonal real vector bundle $P \times_G V \to X$ over X - this defines a functor $\mathrm{Rep}(G, \mathbb{R}) \to \mathrm{Vect}(X, \mathbb{R})$ compatible with direct sums and tensor products, inducing a ring homomorphism $RO(G) \xrightarrow{\varphi_P} KO^0(X)$. If $\alpha \in KO^0(X)$ is a virtual orthogonal vector bundle lying in the image of φ_P, we'll say the structure group of α reduces to G.

Theorem 3.1. *The Adams conjecture is true for any virtual orthogonal vector bundle whose structure group reduces to a finite group.*

This will be proved in later sections.

Now, let's make a straightforward observation:

Proposition 3.2. *If the Adams conjecture holds for every prime natural number $p \in \mathbb{N}$, then it holds for any integer $k \in \mathbb{Z}$.*

Proof. First, observe that the Adams conjecture holds trivially when $k = 0$. It also holds trivially when $k = \pm 1$ because ψ^1, ψ^{-1} are both the identity on $KO^0(X)$ (see, for instance, Adams's *Vector fields on spheres*).

Since any integer $k \in \mathbb{Z}$ has a unique factorization $k = (\pm 1)\prod_i p_i^{m_i}$ as a product of primes, by induction (say on $|k|$) it'll suffice to prove:

Lemma 3.3. *Suppose the Adams conjecture holds for $k, l \in \mathbb{Z}$. Then it also holds for $kl \in \mathbb{Z}$.*

Proof of the lemma. The Adams operations satisfy the relation $\psi^k\psi^l = \psi^{kl}$ (again see Adams's *Vector fields on spheres*). Hence

$$\psi^{kl} - 1 = \psi^{kl} - \psi^l + \psi^l - 1 = \psi^l(\psi^k - 1) + (\psi^l - 1)$$

Now if $\alpha \in KO^0(X)$, choose $e, e' \in \mathbb{N}$ large enough so $k^e(\psi^k - 1)\alpha = 0$ and $l^{e'}(\psi^l - 1)\alpha = 0$, and observe that provided $e'' \geq \max\{e, e'\}$,

$$(kl)^{e''}(\psi^{kl} - 1)\alpha = l^{e''}\psi^l k^{e''}(\psi^k - 1)\alpha + k^{e''}l^{e''}(\psi^l - 1)\alpha = 0$$

□

□

For the remainder of this section, $p \in \mathbb{N}$ will denote a fixed prime natural number. Let k be an algebraic closure of the field $\mathbb{F}_p$ with p elements. We may embed the group of units $k^\times$ in $\mathbb{C}^\times$ as the roots of unity with order prime to p. Fix (once and for all) such an embedding, and denote it by $\phi : k^\times \to \mathbb{C}^\times$. As Quillen remarks in his paper on the higher algebraic K-theory of finite fields, all of our future constructions will depend on φ, but "in a sense that is well understood in the theory of etale cohomology."

Definition 3.4. Let G be a finite group and let V be a finite-dimensional representation of G over k. The **modular character of** V **with respect to** ϕ is the complex function $\chi_V : G \to \mathbb{C}$ given by

$$\chi_V(g) := \sum_i \phi(\alpha_i) \text{ where}$$

$$\alpha_i \in k^\times \text{ are the eigenvalues of the automorphism } g : V \to V, \text{ counted by multiplicity.}$$

Remark 3.5. Here's one way to think about χ: taking the eigenvalues of the automorphisms $g : V \to V$ defines a function

$$G \to SP^n k^\times \text{ taking } g \mapsto [\alpha_1, \ldots, \alpha_n]$$

where the α_i are the eigenvalues of g, counted by multiplicity. Here $n = \dim_k V$, and $SP^n k^\times$ is the n-fold symmetric product of $k^\times$. Now the embedding $\phi : k^\times \to \mathbb{C}^\times$ induces an embedding $SP^n\phi : SP^n k^\times \to SP^n\mathbb{C}^\times$, and of course addition defines a function $SP^n\mathbb{C}^\times \to \mathbb{C}$ taking $[c_1, \ldots, c_n] \mapsto \sum_i c_i$.

It is a theorem of Green (see his paper *The characters of the finite general linear groups*) that the function χ_V is in fact the character of a virtual complex representation of G, i.e. $\chi_V \in R(G, \mathbb{C})$. Suppose V, W are two finite-dimensional representations of G over k, say of dimensions m, n. If $g \in G$ and $[\alpha_i] \in SP^m k^\times, [\beta_i] \in SP^n k^\times$ are the eigenvalues of the automorphisms $V \xrightarrow{g} V, W \xrightarrow{g} W$ respectively, then it's not hard to show that the eigenvalues of $V \oplus W \xrightarrow{g \oplus g} V \oplus W$ and $V \otimes W \xrightarrow{g \otimes g} V \otimes W$ are given by $[\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n] \in SP^{m+n}k^\times$ and $[\alpha_1\beta_1, \ldots, \alpha_m\beta_n] \in SP^{mn}k^\times$. It follows that

$$\chi_{V \oplus W}(g) = \sum_i \phi(\alpha_i) + \sum_i \phi(\beta_i) = \chi_V(g) + \chi_W(g) \text{ and}$$

$$\chi_{V \otimes W}(g) = \sum_{i,j} \phi(\alpha_i\beta_j) = (\sum_i \phi(\alpha_i))(\sum_j \phi(\beta_j)) = \chi_V(g)\chi_W(g)$$

So in fact the above construction of modular characters defines a homomorphism of representation rings

$$R(G, k) \to R(G, \mathbb{C}) \text{ sending } [V] \mapsto \chi_V$$

This is generally referred to as the "Brauer lift."

Note: For the above discussion to make sense, one must define $R(G, k)$ to be the Grothendieck ring of the commutative semi-ring $\mathrm{Rep}(G, \bar{k})$ of finite-dimensional representations of G over $\bar{k}$, under direct sum and tensor product. If $\mathrm{char} k \not| |G|$ (so that Maschke's theorem ensures every representation is projective) then this is just $K(\bar{k}G)$- but generally (in the modular case) the two will be different.

Proposition 3.6. *Assume p is odd. If V is a finite-dimensional representation of G over k with a G-invariant non-degenerate symmetric k-bilinear form $\beta : V \times V \to k$, then χ_V is in fact the character of a virtual* real *representation of G, i.e. $\chi_V \in R(G, \mathbb{R})$.*

This will be proved in later sections.

Now let q be a power of p and let $\mathbb{F}_q \subset k$ be the subfield with q elements. Then the standard representation of $GL(n, \mathbb{F}_q)$ acting on $\mathbb{F}_q^n$ yields, by extension of scalars, a representation of $GL(n, \mathbb{F}_q)$ acting on k^n. The above construction gives a modular character $\chi_{k^n} \in R(GL(n, \mathbb{F}_q), \mathbb{C})$ and from this we obtain a map $BGL(n, \mathbb{F}_q) \to BU$, unique up to homotopy. Similarly, assuming in addition that p is odd, the usual representation of $O(n, \mathbb{F}_q)$ acting on $\mathbb{F}_q^n$ extends to a representation of $O(n, \mathbb{F}_q)$ acting on k^n together with an invariant non-degenerate symmetric bilinear form $\beta : k^n \times k^n \to k$, defined by $\beta(v, w) = \sum_i v_i w_i$. From this we obtain a modular character $\chi_k^n \in R(O(n, \mathbb{F}_q), \mathbb{R})$, which in turn defines a map $BO(n, \mathbb{F}_q) \to BO$, unique up to homotopy. To see how we obtain these classifying maps, look at the above discussion of the Atiyah-Segal completion theorem.

One can check that the maps $BGL(n, \mathbb{F}_q) \to BU$ and $BO(n, \mathbb{F}_q) \to BO$ are compatible with the usual directed systems obtained as we let n and q go to infinity. Hence they define elements of $\lim_{n,q}[BGL(n, \mathbb{F}_q), BU]$ and $\lim_{n,q}[BO(n, \mathbb{F}_q), BO]$. Now using the Milnor exact sequence and the fact that for any compact Lie group G (so in particular any finite group), $K^{-1}(BG) = 0$ and $KO^{-1}(BG) = 0$, we obtain maps

$$BGL(k) \to BU \text{ and } BO(k) \to BO$$

unique up to homotopy - here $GL(k) = \mathrm{colim}_{n,q} GL(n, \mathbb{F}_q)$ and $O(k) = \mathrm{colim}_{n,q} O(n, \mathbb{F}_q)$. See Quillen's paper *On the cohomology and K-theory of the general linear groups over a finite field*.

Theorem 3.7. *These maps induce isomorphisms on cohomology with coefficients in $\mathbb{Z}/(d)$ where $d \in \mathbb{Z}$ is any integer prime to p.*

This will also be proved later. Assuming all the above facts, here's a proof of the Adams conjecture for an odd prime p:

Recall that for any CW complex X there's a natural isomorphism of abelian groups

$$\mathrm{Sph}(X) \simeq [X, BH \times \mathbb{Z}]$$

where $H = \mathrm{colim} H(n)$, and $H(n)$ is the monoid of homotopy equivalences $S^{n-1} \to S^{n-1}$. Let $Y = (BH \times \mathbb{Z})[p^{-1}]$ be the localization of the *space* $BH \times \mathbb{Z}$ at the prime p. There's two ways to describe Y:

First, observe that $\tilde{Sph}(-)[p^-1]$ is a representable functor from the homotopy category of pointed CW complexes to abelian groups - to see this just check the hypotheses of the Brown representability theorem. Then take Y to be a CW complex representing $\tilde{\mathrm{Sph}}(-)[p^-1]$. Note that Y will come with a pointed map $BH \times \mathbb{Z} \to Y$ representing localization at p (and this map will be unique up to unique isomorphism in the homotopy category of pointed CW complexes). Also note that from this perspective it's clear that Z is an H-space (in fact an abelian group object in the homotopy category of pointed CW complexes).

Alternatively, form the directed system

$$BH(p) \times \mathbb{Z} \to BH(p^2) \times \mathbb{Z} \to BH(p^n) \times \mathbb{Z} \to \cdots$$

where the maps correspond to the p-fold join of spherical fibrations. One can check that the homotopy colimit of this system is a localization of $BH \times \mathbb{Z}$ at p.

Let Z denote the identity component of Y.

Lemma 3.8. *Let X be a connected pointed CW complex that can be expressed as a union of an ascending sequence of connected finite pointed subcomplexes*

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$$

Then the natural map $[X, Z] \to \lim_n [X_n, Z]$ is an isomorphism.

Proof. First, observe that as Z is a connected H-space, there's no difference between homotopy classes of maps and homotopy classes of pointed maps. So, it'll suffice to show that the natural map

$$[X, Z]_* \to \lim_n [X_n, Z]_*$$

is an isomorphism. For this we may appeal to the Milnor exact sequence

$$0 \to \overset{1}{\lim}[\Sigma X_n, Z]_* \to [X, Z]_* \to \lim[X_n, Z] \to 0$$

The groups $[\Sigma X_n, Z]$ may be identified as $\tilde{\mathrm{Sph}}(\Sigma X_n)$ - in particular they are finite, and it's a general fact that $\lim^1$ of an inverse system of finite abelian groups is 0 (the Mittag-Lefler condition holds for trivial reasons).

□

Now consider the map

$$BO \times \mathbb{Z} \xrightarrow{\psi^p - 1} BO \times \mathbb{Z} \xrightarrow{S} BH \times \mathbb{Z} \to (BH \times \mathbb{Z})[p^{-1}] = Y$$

representing the natural transformation $KO^0(-) \to \mathrm{Sph}(-)[p^{-1}]$ taking a virtual orthogonal bundle $\alpha \in KO^0(X)$ to $S((\psi^p - 1)\alpha) \in \mathrm{Sph}(X)[p^{-1}]$. Call this μ; note that it's an H-map, and sends the identity component $BO \subset BO \times \mathbb{Z}$ to the identity component $Z \subset Y$. Now look at the composition

$$BO(k) \xrightarrow{\alpha} BO \xrightarrow{\mu} Z$$

where α the map obtained from the Brauer lift.

Proposition 3.9. *$\mu \circ \alpha$ is null-homotopic. Hence $\mu : BO \to Z$ factors through the mapping cone $C(\alpha)$ of α:*

(3.1)
$$\begin{array}{ccc} BO & \longrightarrow & C(\alpha) \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \bar{\mu}} \\ Z & \xrightarrow{=} & Z \end{array}$$

Proof. □