SPECTRA AND SPECTRA

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ABSTRACT. This talk is an attempt to answer the following question: what do spectra of linear operators (in the sense of functional analysis, or even plain old linear algebra) and spectra of commutative rings (in the sense of modern algebraic geometry) have to do with each other (if anything)?

Along the way we will encounter some basic concepts of functional analysis. My secondary goal is to present them in a way that's almost palatable to this room full of algebro-geometrically minded people.

1. ALGEBRA AND ANALYSIS

Suppose for a minute that we wanted to do algebra with objects occurring in analysis. One way to get started would be to consider group (and ring, etc.) objects in the category of, say, complete metric spaces (analysts like complete metric spaces, right?). To be precise:

Definition 1.1. A **group object in the category of complete metric spaces** (say, a "**complete metric group**") is a complete metric space G with metric $d : G \times G \to [0, \infty)$ together with a continuous map

$$\mu : G \times G \to G \text{ (the group operation)}$$

$$\text{a distinguished element } e \in G \text{ and a homeomorphism } \iota : G \to G$$

(the inversion) fitting together in the commutative diagrams that make up the group axioms (μ is associative, right and left multiplication by e are both the identity map on G, etc).

The metric d is said to be **bi-invariant** if both left and right translations of the form

$$l_g : G \to G \text{ sending } h \to g \cdot h := \mu(g, h) \text{ and}$$

$$r_g : G \to G \text{ sending } h \to h \cdot g = \mu(h, g)$$

are not only homeomorphisms, but *isometries*.

Here's a fun collection of examples:

Example 1.2. One can show that a bi-invariant Riemannian metric $\langle , \rangle$ on a Lie group G is equivalent to a inner product $\langle , \rangle$ on its Lie algebra $\mathfrak{g} = TG_e$ which is invariant under the adjoint representation

$$G \times \mathfrak{g} \to \mathfrak{g} \text{ taking } g, v \mapsto dc_g v$$

where c_g is conjugation by $g \in G$.

When G is compact, every finite dimensional real representation of G (in particular this adjoint representation) has an invariant inner product (because one can average), and in this way one obtains a bi-invariant Riemannian metric on G. The resulting Riemannian distance function

$$d : G \times G \to [0, \infty) \text{ makes } G \text{ a group in complete metric spaces}$$

with a bi-invariant metric (here completeness of the metric follows from compactness).

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One could also talk about ring objects. Here one would be considering an abelian group A in complete metric spaces, written additively with additive identity $0 \in A$ together with a second continuous operation

$$\mu : A \times A \to A \text{ (multiplication) and a second distinguished element } 1 \in A$$

fitting into the commutative diagrams making up the ring axioms - i.e. μ is bilinear and associative, 1 is a multiplicative identity, etc.

From here it's clear what the definition of a right module M over an algebra A in complete metric spaces should be: there should be a continuous action

$$\theta : M \times A \to M$$

satisfying the usual properties: it's bilinear, $\theta(\theta(m, a), b) = \theta(m, \mu(a, b))$, etc. Similarly one can consider an algebra B over A in complete metric spaces.

Example 1.3. A first example of a ring object in complete metric spaces is $\mathbb{C}$, and modules over $\mathbb{C}$ are complete metric complex vector spaces. So, we're getting into the territory of functional analysis.

2. Banach algebras

Let's talk about algebras over $\mathbb{C}$ in complete metric spaces. We'll be able to say more interesting things if we throw in some hypotheses and restrict attention to Banach algebras - some definitions:

Definition 2.1. A **Banach space** is a complete metric vector space V over $\mathbb{C}$ with a metric $d : V \times V \to [0, \infty)$ which is bi-invariant and satisfies

$$d(\alpha v, \alpha w) = |\alpha| d(v, w) \text{ for } \alpha \in \mathbb{C} \text{ and } v, w \in V$$

Remark 2.2. This definition is ostentatiously unorthodox. To see how it relates to the usual definition in terms of a **norm** $\|-\| : V \to [0, \infty)$, just set $\|v\| := d(0, v)$ for $v \in V$, and observe that

$$\begin{aligned} \|v + w\| &= d(0, v + w) \le d(0, v) + d(v, v + w) \\ &= d(0, v) + d(0, w) = \|v\| + \|w\| \end{aligned}$$

for $v, w \in V$ using the triangle inequality and the translation invariance of d. Similarly

$$\|\alpha v\| = d(0, \alpha v) = |\alpha| \|v\|$$

for $\alpha \in \mathbb{C}, v \in V$ using the hypothesis about d, and

$$\|v\| = d(0, v) > 0 \text{ if } v \neq 0$$

since d is positive definite.

Definition 2.3. A **Banach algebra over** $\mathbb{C}$ is a Banach space A over $\mathbb{C}$ with norm $\|-\|$ together with a multiplication

$$\mu : A \times A \to A \text{ and identity } e \in A$$

making it a complete metric algebra over $\mathbb{C}$, subject to the following additional restrictions:

$$\|ab\| \le \|a\| \|b\| \text{ for } a, b \in A \text{ and } \|e\| = 1$$

If these conditions seem silly to you, note that in terms of the metric d on A, the first inequality says $d(ab_1, ab_2) \le \|a\| d(b_1, b_2)$ for $a, b_1, b_2 \in A$, and well, it'd be weird if the norm of the identity wasn't 1.

Example 2.4. Let V, W be Banach spaces over $\mathbb{C}$, and let $T : V \to W$ be a $\mathbb{C}$-linear map. It's a fact that T is continuous if and only if it's **bounded** in the sense that

$$\sup\{\frac{\|Tv\|}{\|v\|} \in [0,\infty) \mid v \in V - \{0\}\} < \infty$$

In this situation set

$$\|T\| := \sup\{\frac{\|Tv\|}{\|v\|} \in [0,\infty) \mid v \in V - \{0\}\} \in [0,\infty)$$

One can show this defines a norm on the bounded (equivalently, continuous) linear maps from V to W, which one might denote by $B(V, W)$, making it another Banach space over $\mathbb{C}$.

Note that this shows that the category of Banach spaces is enriched in Banach spaces, which is neat. Moreover, if $V = W$ we're talking about endomorphisms of V, usually denoted simply by $B(V)$, and it's not hard to show that composition

$$B(V) \times B(V) \to B(V) \text{ taking } S, T \mapsto S \circ T$$

serves as a multiplication making $B(V)$ a Banach algebra, with identity the identity map, per usual.

For later use:

Definition 2.5. A **C*-algebra** is a Banach algebra A together with an *isometric, involutive anti-automorphism*

$$\tau : A \to A^{op}, \text{ usually denoted by } a \to a^* \text{ (hence the terminology)}$$

(by isometric I mean $\|a^*\| = \|a\|$ for $a \in A$ and by involutive I mean $\tau^2 = \mathrm{id}$) over conjugation $\mathbb{C} \to \mathbb{C}$ taking $\alpha \to \bar{\alpha}$ such that

$$\|aa^*\| = \|a\|^2 \text{ for } a \in A$$

In fact that last identity implies τ is an isometry.

Example 2.6. Let H be a Hilbert space. We already know its bounded endomorphisms $B(H)$ form a Banach algebra (after all H is a Banach space). If $T : H \to H$ is a bounded linear map it induces a dual linear map $T^\vee : H^\vee \to H^\vee$, (here $H^\vee$ denotes the continuous linear functionals on H) and using the given isomorphism $H \simeq H^\vee$ sending $v \to \langle -, v\rangle$ we can identify $T^\vee$ with a linear map

$$T^* : H \to H, \text{ called the adjoint of } T$$

If H is $\mathbb{C}^n$ so we can carry out this discussion in terms of vectors and matrices, T^* is just the transpose conjugate of T. One can show $T \mapsto T^*$ provides an isometric involution making $B(H)$ a C*-algebra.

3. Spectra of elements of Banach algebras

Let A be a Banach algebra and let $A^\times$ be its group of invertible elements (one can show this is an open subspace of A and a complete metric group under multiplication).

Definition 3.1. The **spectrum** of an element $a \in A$ is the set

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid a - \lambda e \notin A^\times\} \subset \mathbb{C}$$

Remark 3.2. It is known that $\sigma(a)$ is a non-empty compact set in $\mathbb{C}$. Furthermore one can prove that the map

$$\sigma : A \to \{\text{compact subsets of } \mathbb{C}\}$$

is continuous, provided the compact subsets of $\mathbb{C}$ are topologized using collections of compact subsets of the form

$$\{K \subset \mathbb{C} \mid K \text{ is compact, and } K \subset U\} \text{ where } U \subset \mathbb{C} \text{ is some open set}$$

as basic opens.

It's instructive to immediately look at an example:

Example 3.3. Let K be a compact Hausdorff space and let $C(K, \mathbb{C})$ be the continuous complex functions on K - this is a C*-algebra with the usual pointwise addition, multiplication and conjugation and with the supremum norm

$$\|f\| = \sup\{|f(x)| \in [0, \infty) \mid x \in K\}$$

Observe that the units $C(K, \mathbb{C})^{\times}$ are precisely the functions which are nowhere-zero, i.e. the

$$f \in C(K, \mathbb{C}) \text{ so that } f(x) \neq 0 \in \mathbb{C} \text{ for } x \in K$$

where is to say $f(K) \in \mathbb{C}^{\times}$. So, for a given function $f \in C(K, \mathbb{C})$ and for a given $\lambda \in \mathbb{C}$, $f - \lambda$ will be invertible if and only if $f(x) \neq \lambda$ for $x \in K$, which is to say $\lambda \notin f(K)$. The upshot:

$$\sigma(f) = f(K) \subset \mathbb{C}$$

that is, the spectrum of f is just its image.

Theorem 3.4. *Let A be a Banach algebra over $\mathbb{C}$ which is also a **division algebra**, i.e. $A^{\times} = A - \{0\}$. Then $A = \mathbb{C}$.*

Proof. One must show that the inclusion $\mathbb{C} \to A$ taking $1 \to e \in A$ (the multiplicative unit) is an isomorphism. Let's show that for each $a \in A$,

$$\sigma(a) \in \mathbb{C} \text{ consists of a single point}$$

so that we can in fact think of σ as a map $A \to \mathbb{C}$. Indeed if $\lambda \in \mathbb{C}$ then

$$a - \lambda e \text{ is invertible if and only if } a - \lambda e \neq 0$$

which is to say $\lambda \in \sigma(a)$ if and only if $\lambda e = a$. Since the map $\lambda \mapsto \lambda e$ is injective (okay, I'm assuming $e \neq 0$, for otherwise $A = \{0\}$ and this whole discussion is stupid) there can be at most one $\lambda \in \sigma(a)$. On the other hand I'm assuming the fact that $\sigma(a) \neq \emptyset$, so there is exactly one such λ.

This shows that the inclusion $\mathbb{C} \to A$ is surjective, hence an isomorphism and furthermore that the spectrum map can be viewed as its inverse $A \to \mathbb{C}$. □

3.1. **Commutative Banach algebras.** Let A be a commutative Banach algebra, and let $\Delta(A)$ be the set of $\mathbb{C}$-algebra homomorphisms $\varphi : A \to \mathbb{C}$. One can show these are all continuous.

In fact $\Delta(A)$ can be topologized in a natural way: every homomorphism φ as above is in particular a continuous $\mathbb{C}$-linear map $A \to \mathbb{C}$, i.e. an element of the continuous dual

$$A^* = \{\text{continuous linear maps } \lambda : A \to \mathbb{C}\}$$

This continuous dual can be given what's known as the **weak* topology**, which is the coarsest topology making all the evaluations

$$\mathrm{ev}_a : A^* \to \mathbb{C} \text{ taking } \lambda \mapsto \lambda(a)$$

continuous (so, it can be though of as the "coarsest topology one would ever seriously consider on A^*"). $\Delta(A)$ can now be topologized as a subspace of A^*.

Proposition 3.5. *$\Delta(A)$ is a compact Hausdorff space.*

I won't prove this.

Proposition 3.6. *Let A be a commutative Banach algebra and let $\Delta(A)$ be the space of continuous homomorphisms $\varphi : A \to \mathbb{C}$ over $\mathbb{C}$. Let $\mathrm{MaxSpec}A$ be the set of maximal ideals in A. Then there is a natural 1-1 bijection*

$$\mathrm{MaxSpec}A \leftrightarrow \Delta(A)$$

By natural I mean the following: $\mathrm{MaxSpec}(-)$ and $\Delta(-)$ can both be considered as contravariant set-valued functors on the category of commutative Banach algebras over $\mathbb{C}$. The assertion is that there is an isomorphism between these two functors.

Sketch. Let $\varphi \in \Delta(A)$ be a continuous homomorphism $A \to \mathbb{C}$. Then since $\mathbb{C}$ is a field, $\ker\varphi \in \mathrm{MaxSpec}A$ is a maximal ideal, and this gives the natural function

$$\Delta(A) \to \mathrm{MaxSpec}A \text{ sending } \varphi \mapsto \ker\varphi$$

On the other hand suppose $\mathfrak{m} \in \mathrm{MaxSpec}A$ is a maximal ideal. One can show that the quotient $A/\mathfrak{m}$ inherits a Banach algebra structure from A and the quotient homomorphism

$$q : A \to A/\mathfrak{m} \text{ is continuous}$$

Since $\mathfrak{m}$ is a maximal ideal, $A/\mathfrak{m}$ is a field, and by theorem 3.1 $A/\mathfrak{m} = \mathbb{C}$. Thus the quotient homomorphism $q : A \to A/\mathfrak{m} = \mathbb{C}$ gives a continuous homomorphism $A \to \mathbb{C}$, i.e. an element of $\Delta(A)$. This gives a natural function

$$\mathrm{MaxSpec}A \to \Delta(A)$$

One now argues the two natural transformations we have between $\Delta(A)$ and $\mathrm{MaxSpec}A$ are mutual inverses.

□

Example 3.7. NOTE: I was not at all careful with normalization factors in this section. Apologies if this causes you a headache! I know how annoying this stuff is.

Consider the Banach algebra $\mathscr{L}^1(T^n)$ of Lebesgue integrable functions on the n-torus $T^n := S^1 \times \cdots \times S^1$ (n factors), with the usual volume measure (let's say normalized so $\mathrm{Vol}T^n = 1$. Here the multiplication is *convolution*: we have

$$f * g(z) = \int_{T^n} f(x)g(x^{-1}z)dx \text{ for } z \in T^n$$

The norm is the $\mathscr{L}^1$-norm. Strictly speaking there's no multiplicative identity in $\mathscr{L}^1(T^n)$ - this would be a δ-function at the identity $e \in T^n$, which one can formally throw in without much trouble.

Fourier analysis provides us with a continuous homomorphism of Banach algebras

$$\mathscr{L}^1(T^n) \to C_0(\mathbb{Z}^n) \text{ sending } f \mapsto \hat{f}$$

Here $C_0(\mathbb{Z}^n)$ is the continuous functions on $\mathbb{Z}^n$ vanishing at ∞, which is to say all functions $\varphi : \mathbb{Z}^n \to \mathbb{Z}$ so that $\lim_{\alpha\to\infty} \varphi(\alpha) = 0$. It's a Banach algebra under pointwise multiplication with the supremum norm. $\hat{f}$ is the function on $\mathbb{Z}^n$ given by the Fourier coefficients of f:

$$\hat{f}(\alpha) = \int_{T^n} f(z)\bar{z^\alpha}dz$$

For any $\alpha \in \mathbb{Z}^n$ we can define a continuous homomorphism

$$\mathrm{ev}_\alpha : C_0(\mathbb{Z}^n) \to \mathbb{C} \text{ taking } \varphi \mapsto \varphi(\alpha)$$

The compositions

$$\mathscr{L}^1(T^n) \xrightarrow{\hat{}} C_0(\mathbb{Z}^n) \xrightarrow{\mathrm{ev}_\alpha} \mathbb{C}$$

provide a wide variety of continuous homomorphisms $\mathscr{L}^1(T^n) \to \mathbb{C}$. More categorically, the evaluation maps yields an embedding $\mathbb{Z}^n \to \Delta(C_0(\mathbb{Z}^n))$, and the Fourier series map $\mathscr{L}^1(T^n) \to C_0(\mathbb{Z}^n)$ induces a continuous map

$$\Delta(C_0(\mathbb{Z}^n)) \to \Delta(\mathscr{L}^1(T^n))$$

and so we obtain a continuous map $\mathbb{Z}^n \to \Delta(\mathscr{L}^1(T^n))$.

Claim: This is a homeomorphism.

To see this, note that a continuous homomorphism $\varphi : \mathscr{L}^1(T^n) \to \mathbb{C}$ can be viewed as a continuous linear functional. By $\mathscr{L}^p - \mathscr{L}^q$-duality there must be a function $\psi \in \mathscr{L}^\infty(T^n)$ so that

$$\varphi(f) = \int_{T^n} f(z)\psi(z)dz \text{ for } f \in \mathscr{L}^1(T^n)$$

Now since φ is in fact an algebra homomorphism it must be that

$$\int_{T^n} f * g(z)\psi(z)dz = \int_{T^n} f(z)\psi(z)dz \cdot \int_{T^n} g(z)\psi(z)dz \text{ for } f, g \in \mathscr{L}^1(T^n)$$

$$\text{and } \int_{T^n} \psi(z)dz = 1$$

The first identity can be expanded out a bit like

$$\int_{T^n \times T^n} f(x)g(x^{-1}z)\psi(z)dxdz = \int_{T^n \times T^n} f(x)g(z)\psi(x)\psi(z)dxdz$$

or substituting xz for z in the first integral,

$$\int_{T^n \times T^n} f(x)g(z)\psi(xz)dxdz = \int_{T^n \times T^n} f(x)g(z)\psi(x)\psi(z)dxdz$$

One now argues that as this identity holds for all f, g it must be $\psi(xz) = \psi(x)\psi(z)$ for $x, z \in T^n$, which is to say ψ is a homomorphism $T^n \to \mathbb{C}^\times$. Proceeding this way and dealing with some technicalities (for instance continuity issues) one concludes that $\psi(z) = z^\alpha$ for some $\alpha \in \mathbb{Z}^n$.

Definition 3.8. Let A be a commutative Banach algebra over $\mathbb{C}$. The **Gelfand transform for** A is the homomorphism of Banach algebras

$$A \to C(\Delta(A), \mathbb{C})$$

which takes an element $a \in A$ to the continuous function $\hat{a} : \Delta(A) \to \mathbb{C}$ defined by $\hat{a}(\varphi) = \varphi(a)$.

There are a few things one should check to ensure all this makes sense but I stopped doing that sort of thing a while back...

The preceding example shows that the Gelfand transform for $\mathscr{L}^1(T^n)$ can be described in terms of the classical theory of Fourier series. The following theorem is foundational:

Theorem 3.9. *Let A be a commutative C*-algebra. Then the Gelfand transform*

$$A \xrightarrow{\hat{}} C(\Delta(A), \mathbb{C})$$

is an isometric isomorphism of C-algebras.*

I won't prove this here.

So, the continuous functions $C(K, \mathbb{C})$ on a compact Hausdorff space K weren't just an easy first example of a commutative C*-algebra - on the contrary, every commutative C*-algebra is canonically isomorphic to one of these.

4. Relationship to the spectrum of a linear operator