CHERN CLASSES

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1. Chern classes as characteristic classes for vector bundles

Let X be a space that's at least something like a CW complex (really paracompact Hausdorffness is what we'll need), and let

$$E \xrightarrow{\pi} X$$

be a rank k complex vector bundle over X (clarify this if people ask). From this data we can concoct cohomology classes

$$c_i(E) \in H^{2i}(X; \mathbb{Z}) \text{ for } i \in \mathbb{N}$$

called **Chern classes** with the following (pleasant) properties:

(1) $c_0(E) = 1 \in H^0(X)$ and $c_i(E) = 0$ when $i > k$.
(2) The Chern classes are natural in the sense that if $f : Y \to X$ is a continuous map from another space Y and $f^*E \xrightarrow{\pi} Y$ is the pullback of E fitting into

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "\pi"'] & E \arrow[d, "\pi"'] \\ Y \arrow[r, "f"] & X \end{tikzcd} \tag{1.1}$$

then

$$c_i(f^*E) = f^*c_i(E) \in H^{2i}(X) \text{ for all } i$$

(3) Let $E, F \xrightarrow{\pi} X$ be 2 complex vector bundles on the same base space, and let $E \oplus F$ be their "Whitney" (fiberwise) sum. Then

$$c_k(E \oplus F) = \sum_{i+j=k} c_i(E) \smile c_j(F) \in H^{2k}(X) \text{ for all } k$$

More memorably, if we let $c(E) = \sum_i c_i(E) \in H^*(X)$ denote the "*total* Chern class" this reads $c(E \oplus F) = c(E) \smile c(F) \in H^*(X)$.
(4) Let $E(\gamma) \to \mathbb{C}P^n$ denote the "tautological" line bundle (corresponding to $\mathcal{O}(-1)$). Then $c_1(E(\gamma)) \in H^2(\mathbb{C}P^n)$ is *opposite* the canonical generator $\alpha \in H^2(\mathbb{C}P^n)$ (which is Poincare dual to a hyperplane $H \subset \mathbb{C}P^n$).

Come back to that last item, time permitting.

Remark 1.1. Another description of the "tautological bundle:" it's the blow-up of $\mathbb{C}^{n+1}$ at the origin.

Date: February 2016.

1.1. **The product formula in action and Chern classes as obstructions to "global generation".** Here's an illustration of the usefulness of the product formula: it's a classic fact that the tangent bundle $T\mathbb{P}^n_{\mathbb{C}}$ fits into a short exact sequence of bundles

$$0 \to \epsilon \to (\gamma^*)^{n+1} \to T\mathbb{P}^n_{\mathbb{C}} \to 0$$

where γ is the tautological bundle. This shows that

$$c(T\mathbb{P}^n_{\mathbb{C}}) = c(\gamma^*)^{n+1} = (1+\alpha)^{n+1} = \sum_{i=0}^{n} \binom{n+1}{i} \alpha^i$$

Comparing homogeneous terms, we see that $c_i(T\mathbb{P}^n_{\mathbb{C}}) = \binom{n+1}{i}\alpha^i$ for $i = 0, \ldots, n$.

Let $E \xrightarrow{\pi} X$ be a rank k complex vector bundle as above. Suppose $\sigma_1, \ldots, \sigma_l : X \to E$ are pointwise linearly independent global sections of E. Then they generate a trivial sub-bundle $\epsilon^l \subset E$ of rank l - we have $E \simeq \epsilon^l \oplus E/\epsilon^l$ and so

$$c(E) = c(\epsilon^l) \smile c(E/\epsilon^l) = c(E/\epsilon^l) \in H^*(X)$$

In particular, we have $c_i(E) = 0$ for $i > k - l$.

Remark 1.2. The sense in which Chern classes are obstructions to the existence of pointwise linearly independent global sections can be precisified within the framework of "obstruction theory."

1.2. **Sketch of construction.** We'll need to know:

Theorem 1.3. *Let $E \xrightarrow{\pi} X$ be an oriented real vector bundle of rank k (this includes complex vector bundles). Then there exists a unique class $\tau \in H^k(E, E - X; \mathbb{Z})$ restricting to the preferred generator on every fiber $(E_x, E_x - x) \simeq (\mathbb{C}^k, \mathbb{C}^k - 0)$ (moreover the external product with τ induces isomorphisms $H^i(X) \xrightarrow{\times\tau} H^{i+k}(E, E - X)$).*

Explain the idea behind it (what it looks like in the de Rham setting)

Definition 1.4. The **Euler class** of $E \xrightarrow{\pi} X$ is the pullback $e(E) = \sigma^*\tau \in H^k(X)$ of τ along the 0-section $\sigma : X \to E$.

Explain the name.

Now we're ready to build the Chern classes by induction on rank. Let $E \xrightarrow{\pi} X$ be a complex vector bundle of rank k. *Define* $c_k(E) = e(E) \in H^{2k}(X)$. Here's the fun part: Pull E back along the projection $E - X \xrightarrow{\pi} X$ to obtain a pullback diagram

$$\begin{array}{ccc} \pi^*E & \longrightarrow & E \\ \downarrow & & \downarrow \\ E - X & \xrightarrow{\pi} & X \end{array} \tag{1.2}$$

Since π^*E comes with a nowhere-0 section we get to split off a line bundle, call it L, and we get a decomposition $\pi^*E = L \oplus \pi^*E/L$ over $E - X$. By inductive hypothesis the classes

$$c_i(\pi^*/L) \in H^{2i}(E - X) \text{ with } i = 0, \ldots, k-1$$

are already defined, and now we're going to say:

$$\text{For } i < k, c_i(E) \in H^{2i}(X) \text{ is the unique class pulling back to } c_i(\pi^*E/L)$$

In order for this to make any sense, we need to know that the induced maps on cohomology $H^i(X) \to H^i(E - X)$ are isomorphisms when $i < 2k$ - this is indeed the case.

1.3. **The splitting principle.** This argument is based on what's come to be known as the "splitting principle," the idea that we can find an appropriate base change to split up a vector bundle into a sum of line bundles without destroying cohomological information. Here's a precise statement:

Theorem 1.5. *Let $E \xrightarrow{\pi} X$ be a rank k complex vector bundle on a CW complex X. Let $F(E) \xrightarrow{\pi} X$ be the complete flag bundle associated to E. Then the pullback $\pi^*E \to F(E)$ splits as a sum of line bundles, and the induced map on cohomology $H^*(X) \xrightarrow{\pi^*} H^*(F(E))$ is injective.*

It's immediate that the pullback splits as a sum of line bundles: a point in $F(E)$ is a complete flag $V_1 \subset V_2 \subset \cdots \subset V_k = E_x$ in a fiber $E_x \subset E$ over a point $x \in X$. The fiber of π^*E over this flag is E_x, and we can decompose it as the sum of the composition factors V_{i+1}/V_i in the flag. The fact that the induced map on cohomology is injective takes some work.

Say $\pi^*E \simeq \bigoplus_{i=1}^k L_i$ is the decomposition of π^*E as a sum of line bundles. Then in $H^*(F(E))$ we have

$$\pi^*c(E) = c(\bigoplus_i L_i) = \prod_i c(L_i), \text{ which we can expand as}$$

$$\sum_i \pi^* c_i(E) = \prod_i (1 + c_1(L_i)) = \sum_i \sigma_i(c_1(L_j))$$

where σ_i is the ith elementary symmetric function.

2. Chern classes of the "universal bundles" and fun with Yoneda arguments

Theorem 2.1. *The assignment $X \to \mathrm{Vect}^k(X, \mathbb{C})$, the isomorphism classes of rank k complex vector bundles on X, is a contravariant functor on the homotopy category of CW complexes. Furthermore, Chern classes define natural transformations $\mathrm{Vect}^k(-, \mathbb{C}) \to H^*(-; \mathbb{Z})$.*

It turns out that Vect^k is representable, and in fact it's represented by the Grassmannian $G_k\mathbb{C}^\infty$ of k-planes in $\mathbb{C}^\infty$ - more precisely, this is the colimit of the $G_k\mathbb{C}^n$ with respect to the maps

$$G_k\mathbb{C}^n \to G_k\mathbb{C}^{n+1} \text{ for } n > k$$

coming from the usual inclusions $\mathbb{C}^n \subset \mathbb{C}^{n+1}$. The universal vector bundle is the "tautological" k-plane bundle $E(\gamma_k) \xrightarrow{\pi} G_k\mathbb{C}^\infty$, where

$$E(\gamma_k) = \{(V, w) \in G_k\mathbb{C}^\infty \times \mathbb{C}^\infty | w \in V\}$$

All of which is to say:

Theorem 2.2. *For any CW complex X, there's a natural bijection*

$$[X, G_k\mathbb{C}^\infty] \to \mathrm{Vect}^k(X, \mathbb{C}) \textit{ taking } f \mapsto f^*E(\gamma_k)$$

An upshot: if $E \xrightarrow{\pi} X$ is a complex k-plane bundle and $f : X \to G_k\mathbb{C}^\infty$ is a map "classifying" E, then $c(E) = f^*c(E(\gamma_k))$. Thus the $c_i(E(\gamma_k)) \in H^*(G_k\mathbb{C}^\infty)$ are in this sense the *universal* Chern classes. In fact, it's now clear that a characteristic class for complex k-plane bundles is just a class in the cohomology of $G_k\mathbb{C}^\infty$

Theorem 2.3. *The inclusion of the universal Chern classes $c_i := c_i(E(\gamma_k))$ induces an isomorphism of rings*

$$\mathbb{Z}[c_1, \ldots, c_k] \xrightarrow{\simeq} H^*(G_k\mathbb{C}^\infty)$$

The proof takes some work.

2.1. The "universal splitting". Let $E(\gamma_1) \to \mathbb{C}P^\infty$ be the universal line bundle, and let $\prod_1^k E(\gamma_1) \to \prod_1^k \mathbb{C}P^\infty$ be its k-fold product. This is a complex k-plane bundle, classified by a map

$$f : \prod_1^k \mathbb{C}P^\infty \to G_k\mathbb{C}^\infty$$

By the Kunneth theorem, we have a canonical isomorphism $\mathbb{Z}[\pi_1^* c_1, \ldots, \pi_k^* c_1] \simeq H^*(\prod_1^k \mathbb{C}P^\infty)$. With respect to this isomorphism,

Theorem 2.4. *The induced map on cohomology $f^* : H^*(G_k\mathbb{C}^\infty) \to H^*(\prod_1^k \mathbb{C}P^\infty)$ is injective with image precisely the symmetric subalgebra, and sends $c_i \mapsto \sigma_i(\pi_j^* c_1)$, where σ_i is the ith elementary symmetric function.*

3. Chern numbers of compact complex manifolds

Let M be a compact complex manifold of complex dimension n, and let $TM \xrightarrow{\pi} M$ be its tangent bundle - this is a complex n-plane bundle. So, we can look at its Chern classes $c_i(TM) \in H^{2i}(M;\mathbb{Z})$ for $i = 0, \ldots, n$. For each *partition* $I = i_1, \ldots, i_l$ of n,

$$c_I(TM) := \prod_j c_{i_j}(TM) \in H^{2n}(M)$$

is a top dimensional class, which can be integrated over M (more precisely, paired with the fundamental class $[M] \in H_{2n}(M;\mathbb{Z})$) to yield an *integer*

$$c_I[M] = \langle c_I(TM), [M] \rangle \in \mathbb{Z}$$

called the Ith **Chern number** of M.

As an example, we saw that for $i = 0, \ldots, n$, $c_i(T\mathbb{P}^n_{\mathbb{C}}) = \binom{n+1}{i}\alpha^i$. Hence for any partition $I = i_1, \ldots, i_l$ of n,

$$c_I(T\mathbb{P}^n_{\mathbb{C}}) = \prod_j \binom{n+1}{i_j}\alpha^{i_j} = \prod_j \binom{n+1}{i_j}\alpha^n \text{ and so}$$

$$c_I[\mathbb{P}^n_{\mathbb{C}}] = \prod_j \binom{n+1}{i_j} \langle \alpha^n, [\mathbb{P}^n_{\mathbb{C}}] \rangle = \prod_j \binom{n+1}{i_j}$$

Definition 3.1. For any partition $I = i_1, \ldots, i_l$ of some $m \in \mathbb{N}$ and any $n \in \mathbb{N}$ with $n \geq l$, let $\sum_{f \in S_n \prod_j x_j^{i_j}} f \in \mathbb{Z}[x_1, \ldots, x_n]$ denote the sum over the orbit of the monomial $\prod_j x_j^{i_j}$ under the usual action of the symmetric group S_n on $\mathbb{Z}[x_1, \ldots, x_n]$ permuting the x_i. This sum is (by construction) S_n-invariant, so (by a classic theorem) it can be written as a polynomial in the elementary symmetric functions $\sigma_0(x_i), \ldots, \sigma_n(x_i) \in \mathbb{Z}[x_1, \ldots, x_n]$. Call this polynomial $s_I(\sigma_i)$. So, s_I is defined implicitly by

$$s_I(\sigma_i(x_j)) = \sum_{f \in S_n \prod_j x_j^{i_j}} f$$

It can be shown that s_I depends only on I, and not on n, provided $n \geq l$ of course.

Definition 3.2. For any complex vector bundle $E \xrightarrow{\pi} X$ of rank n over a CW complex X, and any partition $I = i_1, \ldots, i_l$ of an integer $m \in \mathbb{N}$ with $m \leq n$, define the cohomology class $s_I(E) \in H^{2m}(X;\mathbb{Z})$ by

$$s_I(E) = s_I(c_1(E), \ldots, c_n(E))$$

If M is a compact complex n-manifold and $TM \xrightarrow{\pi} M$ its tangent bundle, then for any partition I of n define the characteristic number $s_I[M]$ by

$$s_I[M] = \langle s_I(TM), [M] \rangle \in \mathbb{Z}$$

Lemma 3.3 (Thom)**.** *The characteristic class $s_I(E \oplus F)$ of a Whitney sum is computed as*

$$s_I(E \oplus F) = \sum_{JK=I} s_J(E) \smile s_K(F)$$

where the summation runs over all pairs of partitions J, K that juxtapose to I.

Corollary 3.4. *Let M and N be compact complex manifolds of dimension m and n respectively, and let I be a partition of $m + n$. Then the characteristic number $s_I[M \times N]$ is computed as*

$$s_I[M \times N] = \sum_{JK=I} s_J[M] s_K[N]$$

where the summation runs over all pairs of partitions J of m and K of n with juxtaposition $JK = I$.

Theorem 3.5 (Thom)**.** *Let $M_1, \ldots, M_n$ be complex manifolds of dimensions $1, 2, \ldots, n$ respectively with $s_i[M_i] \neq 0$ for each i. For each partition $I = i_1, \ldots, i_l$ of n, let $M^I = \prod_j M_{i_j}$ (where we should really be specifying an ordering on I, for instance, writing it in non-decreasing order). Then the $p(n) \times p(n)$ matrix of integers*

$$(c_I[M^J]), \text{ where } I \text{ and } J \text{ range over partitions of } n$$

is non-singular. Thus there are no linear relations between the Chern numbers.

In fact, since the Chern numbers and s-numbers are related by a change of basis, one can prove this result by showing that the analogous matrix $(s_I[M^J])$ is non-singular, which is easier because we have the above product formula.