NOTES ON HODGE/DE RHAM COHOMOLOGY OF STACKS

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1. Sheaves and complexes of differentials

This is mostly based off the intros of Behrend's "On the de Rham cohomology of differential and algebraic stacks."

Let $\mathcal{X}$ be a smooth algebraic stack over a base scheme S. The lisse-étale site (which I'm going to denote by $LE\mathcal{X}$) of $\mathcal{X}$ has objects the smooth $\mathcal{X}$-schemes $T \xrightarrow{x} \mathcal{X}$ and morphisms the morphisms $T \xrightarrow{\varphi} T'$ over $\mathcal{X}$. A collection of $\mathcal{X}$-morphisms $U_i \xrightarrow{\varphi_i} T$ in $LE\mathcal{X}$ is a cover if it's an etale cover of T (when we forget about the maps to $\mathcal{X}$).

The site $LE\mathcal{X}$ comes with a structure sheaf of rings $\mathcal{O}_{\mathcal{X}}$ whose restriction to each smooth $\mathcal{X}$-scheme $T \xrightarrow{x} \mathcal{X}$ is the structure sheaf $\mathcal{O}_T$. So, we can look at sheaves of $\mathcal{O}_{\mathcal{X}}$-modules on $LE\mathcal{X}$.

For each object $T \to \mathcal{X}$ in $LE\mathcal{X}$ there is a coherent sheaf $\Omega_{T|S}$ of relative differentials over the base, and for a morphism $\varphi : T \to T'$ in $LE\mathcal{X}$ there is an induced morphism of quasicoherent sheaves

$$d\varphi^\vee : \Omega_{T'|X} \to \varphi_*\Omega_{T|S} \text{ on } T'$$

If the map φ is etale, then the adjoint induced morphism of sheaves $\varphi^{-1}\Omega_{T'|S} \to \Omega_{T|S}$ on T is an isomorphism (this is basically what it *means* to be etale, right?). The criteria on p. 193 of Olsson's book can be used to verify that we've defined a sheaf Ω on $LE\mathcal{X}$ whose restriction to T is $\Omega_{T|S}$ and with restriction maps given by the morphisms $\Omega_{T'|S} \xrightarrow{d\varphi^\vee} \varphi_*\Omega_{T|S}$.

Since each $\Omega_{T|S}$ is a coherent sheaf of $\mathcal{O}_T$-modules and the morphisms $d\varphi^\vee : \Omega_{T'|X} \to \varphi_*\Omega_{T|S}$ are $\mathcal{O}_{T'}$ linear, it's clear that Ω is a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules. However it's *not* cartesian since the morphism $\varphi^*\Omega_{T'|S} \to \Omega_{T|S}$ adjoint to $d\varphi^\vee$ is not in general an isomorphism. So, Ω is not a quasi-coherent sheaf on $\mathcal{X}$ (in the sense of chapter 9 in Olsson's book). For this reason some people (Behrend in particular) call it a "big" sheaf on $LE\mathcal{X}$.

Remark 1.1. Since $\mathcal{X}$ is assumed to be smooth and the objects of $LE\mathcal{X}$ are *smooth* morphisms $T \xrightarrow{x} \mathcal{X}$, the schemes T are necessarily smooth over the base S, and so the sheaves $\Omega_{T|S}$ are locally free of finite rank.

So, if Ω *were* coherent, these sheaves $\Omega_{T|S}$ would all have the same rank - but that would imply that all the schemes T have the same relative dimension over S, which is absurd.

In fact the exterior powers $\Omega^p := \bigwedge^p \Omega$ fit into a complex Ω^* :

$$\mathcal{O}_{\mathcal{X}} \xrightarrow{\partial} \Omega \xrightarrow{\partial} \Omega^2 \xrightarrow{\partial} \dots$$

On any given smooth $\mathcal{X}$-scheme $T \to \mathcal{X}$ this restricts to the de Rham complex

$$\mathcal{O}_T \xrightarrow{\partial} \Omega_{T|S} \xrightarrow{\partial} \Omega^2_{T|S} \xrightarrow{\partial} \dots$$

where ∂ denotes the exterior derivative. Recall that ∂ is $\mathcal{O}_S$-linear but *not* $\mathcal{O}_T$-linear - it satisfies the Leibniz rule

$$\partial(f\alpha) = (\partial f) \wedge \alpha + f(\partial\alpha) \in \Omega^{p+1}_{T|S}(T) \text{ for } f \in \mathcal{O}_T(T),\ \alpha \in \Omega^p_{T|S}(T)$$

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So, it seems like Ω^* should be viewed as a complex of $\mathcal{O}_S$-modules on $LE\mathfrak{X}$. Here $\mathcal{O}_S$ is the sheaf of rings on $LE\mathfrak{X}$ restricting to $\pi^{-1}\mathcal{O}_S$ on an object $T \to \mathfrak{X}$, where $\pi : T \to S$ is gives T the structure of an S-scheme.

2. The Hodge to de Rham spectral sequence

Definition 2.1. The **de Rham cohomology of** $\mathfrak{X}$ is $H^*_{dR}(\mathfrak{X}) := \mathbb{H}^*(\mathfrak{X}, \Omega^*)$.

Here the right hand side denotes the hyper-cohomology of the complex Ω^* of sheaves of $\mathcal{O}_S$-modules on $LE\mathfrak{X}$. Note that $H^*_{dR}(\mathfrak{X})$ has a natural graded commutative ring structure: the multiplication is given by the composition

$$\mathbb{H}^*(\mathfrak{X}, \Omega^*) \otimes \mathbb{H}^*(\mathfrak{X}, \Omega^*) \to \mathbb{H}^*(\mathfrak{X} \times_S \mathfrak{X}, \pi_1^*\Omega^* \otimes \pi_2^*\Omega^*)$$
$$\simeq \mathbb{H}^*(\mathfrak{X} \times_S \mathfrak{X}, \Omega^*) \xrightarrow{\Delta^*} \mathbb{H}^*(\mathfrak{X}, \Omega^*)$$

Here one way to obtain the first map is via the Eilenberg-Zilber theorem and the Kunneth formula (Totaro discusses this) and the second map is induced by the diagonal $\mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$. I think these should all be $\mathcal{O}_S(S)$-algebras, and in that case the tensors should be over $\mathcal{O}_S(S)$. At some point we'll set $S = \mathrm{Spec}k$ for k field and this will hopefully be more transparent.

The complex Ω^* comes with a descending filtration

$$\Omega^* = F^0\Omega^* \supset F^1\Omega^* \supset F^2\Omega^* \supset \dots$$

where $F^p\Omega^*$ is the complex obtained by truncating Ω^* before Ω^p: it looks like

$$0 \to \cdots \to 0 \to \Omega^p \xrightarrow{\partial} \Omega^{p+1} \xrightarrow{\partial} \dots$$

More precisely

$$(F^p\Omega^*)^i = \begin{cases} 0 & \text{if } i < p \\ \Omega^i & \text{otherwise} \end{cases}$$

From this we obtain a descending filtration

$$H^*_{dR}(\mathfrak{X}) = F^0H^*_{dR}(\mathfrak{X}) \supset F^1H^*_{dR}(\mathfrak{X}) \supset F^2H^*_{dR}(\mathfrak{X}) \supset \dots$$

Here

$$F^pH^*_{dR}(\mathfrak{X}) = \text{ the image of the homomorphism}$$
$$\mathbb{H}^*(\mathfrak{X}, F^p\Omega^*) \to \mathbb{H}^*(\mathfrak{X}, \Omega^*) = H^*_{dR}(\mathfrak{X})$$

There is a spectral sequence of the form

$$E_1^{pq} = H^q(\mathfrak{X}, \Omega^p) \implies \mathbb{H}^{p+q}(\mathfrak{X}, \Omega^*_{\mathfrak{X}|T}) = H^{p+q}_{dR}(\mathfrak{X})$$

This is the **Hodge-to-de-Rham spectral sequence** - it's just the hypercohomology spectral sequence of the complex Ω^*.

3. The cotangent complex $\mathcal{L}_{\mathfrak{X}|S}$

Let $T \xrightarrow{x} \mathfrak{X}$ be an object of $LE\mathfrak{X}$. Letting $X \xrightarrow{q} \mathfrak{X}$ be a smooth presentation, we can form the cartesian diagram

$$\begin{array}{ccc} X \times_{\mathfrak{X}} T & \xrightarrow{q'} & T \\ {\scriptstyle x'}\downarrow & & \downarrow{\scriptstyle x} \\ X & \xrightarrow{q} & \mathfrak{X} \end{array} \tag{3.1}$$

Note that $q' : X \times_{\mathfrak{X}} T \to X$ is a smooth morphism of schemes, so $\Omega_{X \times_{\mathfrak{X}} T|X}$ is a locally free sheaf of finite rank on $X \times_{\mathfrak{X}} T$. It should be clear (by which I mean I haven't actually checked this, but

Behrend seems to be assuming this) that $\Omega_{X\times_{\mathfrak{X}}T|X}$ descends to a locally free sheaf of finite rank on T - call that sheaf $\Omega_{T|\mathfrak{X}}$ Note that if $T \xrightarrow{\varphi} T'$ is a morphism in $LE\mathfrak{X}$ we get a morphism of schemes $X \times_{\mathfrak{X}} T \xrightarrow{id\times\varphi} X \times_{\mathfrak{X}} T'$ over X and hence an induced morphism of sheaves $d(\mathrm{id}\times\varphi)^\vee : \Omega_{X\times_{\mathfrak{X}}T'|X} \to (\mathrm{id}\times\varphi)_*\Omega_{X\times_{\mathfrak{X}}T|X}$ which should descend to a morphism $d\varphi^\vee : \Omega_{T'|\mathfrak{X}} \to \varphi_*\Omega_{T|\mathfrak{X}}$ of coherent sheaves on T.

Proceeding in this way one obtains a sheaf $\Omega_{|\mathfrak{X}}$ of $\mathcal{O}_{\mathfrak{X}}$ modules on $LE\mathfrak{X}$ whose restriction to any $T \to \mathfrak{X}$ is $\Omega_{T|\mathfrak{X}}$. This is not a cartesian sheaf (for essentially the same reasons that Ω fails to be a cartesian sheaf).

NOTE: One should probably check that the sheaf $\Omega_{|\mathfrak{X}}$ so obtained doesn't depend on the choice of a presentation $X \to \mathfrak{X}$ (it'd be even better to give a definition of $\Omega_{|\mathfrak{X}}$ that makes no use of a presentation).

Observe that there's a natural morphism of sheaves $\Omega \to \Omega_{|\mathfrak{X}}$ - for a smooth $\mathfrak{X}$-scheme $T \xrightarrow{x} \mathfrak{X}$, form the Cartesian diagram

$$\begin{tikzcd} X\times_{\mathfrak{X}} T \arrow[r, "q'"] \arrow[d, "x'"'] & T \arrow[d, "x"] \\ X \arrow[r, "q"] & \mathfrak{X} \end{tikzcd} \tag{3.2}$$

and note that on $X \times_{\mathfrak{X}} T$ there's a short exact sequence

$$0 \to x'^*\Omega_{X|S} \xrightarrow{dx'^\vee} \Omega_{X\times_{\mathfrak{X}}T|S} \to \Omega_{X\times_{\mathfrak{X}}T|X} \to 0$$

(in general this sequence is only right exact, but here it's short exact since all schemes in sight are smooth). At this point one must show that the morphism $\Omega_{X\times_{\mathfrak{X}}T|S} \to \Omega_{X\times_{\mathfrak{X}}T|X}$ descends to a morphism $\Omega_{T|S} \to \Omega_{T|\mathfrak{X}}$ on T.

Claim (see p.3-4 of Behrend's paper): The cotangent complex of $\mathfrak{X}$ over S is the 2-term complex

$$\Omega \to \Omega_{|\mathfrak{X}}$$

I don't know how to verify this claim and haven't chased the references in his paper, but here's the heuristic reasoning (as I see it): for a given smooth $\mathfrak{X}$-scheme $T \xrightarrow{x} \mathfrak{X}$, we "*should*" have a short exact sequence of locally free coherent sheaves

$$0 \to \text{“}x^*\Omega_{\mathfrak{X}|S} \to \Omega_{T|S} \to \Omega_{T|\mathfrak{X}} \to 0$$

one T, which appears to identify "$x^*\Omega_{\mathfrak{X}|S}$" as the kernel of the morphism $\Omega_{T|S} \to \Omega_{T|\mathfrak{X}}$.

Behrend shows that the pullback of $\mathcal{L}_{\mathfrak{X}}$ along the presentation $X \xrightarrow{q} \mathfrak{X}$ has a nice description - $q^*\mathcal{L}_{\mathfrak{X}}$ looks like

$$\Omega_{X|S} \to \Omega_{X|\mathfrak{X}}$$

and

Proposition 3.1. *There is a canonical isomorphism* $\Omega_{X|\mathfrak{X}} \simeq \mathcal{N}^\vee_{X,X\times_{\mathfrak{X}}X}$.

Here $\Delta : X \to X \times_{\mathfrak{X}} X$ is the diagonal map, which can be viewed as the identity morphism for the groupoid presentation $(X \times_{\mathfrak{X}} X, X)$, and $\mathcal{N}^\vee_{X,X\times_{\mathfrak{X}}X}$ is the conormal bundle of the closed subscheme $X \xrightarrow{\Delta} X \times_{\mathfrak{X}} X$.

Proof. Consider again the cartesian diagram

$$\begin{array}{ccc} X \times_{\mathfrak{X}} X & \xrightarrow{\pi_2} & X \\ \pi_1 \downarrow & & \downarrow q \\ X & \xrightarrow{q} & \mathfrak{X} \end{array} \tag{3.3}$$

On $X \times_{\mathfrak{X}} X$ there's a commutative diagram of morphisms of sheaves

$$\begin{array}{ccc} \Omega_{X\times_{\mathfrak{X}} X|X} & \xleftarrow{d\pi_2^\vee} & \pi_2^*\Omega_{X|\mathfrak{X}} \\ \uparrow & & \uparrow \\ \Omega_{X\times_{\mathfrak{X}} X|S} & \xleftarrow{d\pi_2^\vee} & \pi_2^*\Omega_{X|S} \end{array} \tag{3.4}$$

where the top horizontal arrow is an isomorphism. Pulling this back along the diagonal map Δ and recalling that $\pi_2 \circ \Delta = \mathrm{id}$, we obtain a commutative diagram of the form

$$\begin{array}{ccc} \Delta^*\Omega_{X\times_{\mathfrak{X}} X|X} & \xleftarrow{\simeq} & \Omega_{X|\mathfrak{X}} \\ \uparrow & & \uparrow \\ \Delta^*\Omega_{X\times_{\mathfrak{X}} X|S} & \longleftarrow & \Omega_{X|S} \end{array} \tag{3.5}$$

The following lemma yields an isomorphism $\Delta^*\Omega_{X\times_{\mathfrak{X}} X|X} \simeq \mathcal{N}^\vee_{X,X\times_{\mathfrak{X}} X}$:

Lemma 3.2. *Let $f : E \to B$ be a smooth morphism of S-schemes and let $\sigma : B \to E$ be a section of f. Then σ is a regular embedding with conormal bundle $\mathcal{N}^\vee_{B,E} \simeq \sigma^*\Omega_{E|B}$*

Proof of (part of the) lemma. I'm not going to show here that σ is a regular embedding. Assuming that fact I'll derive the isomorphism $\mathcal{N}^\vee_{B,E} \simeq \sigma^*\Omega_{E|B}$.

Take the usual short exact sequence of sheaves on E

$$0 \to f^*\Omega_{B|S} \xrightarrow{df^\vee} \Omega_{E|S} \to \Omega_{E|B} \to 0$$

and pull it back over σ to obtain

$$0 \to \Omega_{B|S} \xrightarrow{df^\vee} \sigma^*\Omega_{E|S} \to \sigma^*\Omega_{E|B} \to 0$$

Note that the (dual of the) differential of σ gives a map $\sigma^*\Omega_{E|S} \to \Omega_{B|S}$ splitting $df^\vee$, yielding a direct sum decomposition $\sigma^*\Omega_{E|S} \simeq \Omega_{B|S} \oplus \sigma^*\Omega_{E|B}$.

On the other hand we have a canonical short exact sequence of sheaves on B

$$0 \to \mathcal{N}^\vee_{B,E} \to \sigma^*\Omega_{E|S} \xrightarrow{d\sigma^\vee} \Omega_{B|S} \to 0$$

and the (dual of the) differential of f gives a map $\Omega_{B|S} \to \sigma^*\Omega_{E|S}$ splitting $d\sigma^\vee$, giving another direct sum decomposition $\sigma^*\Omega_{E|S} \simeq \mathcal{N}^\vee_{B,E} \oplus \Omega_{B|S}$. Anyway, from here one can show that the composition

$$\mathcal{N}^\vee_{B,E} \to \sigma^*\Omega_{E|S} \to \sigma^*\Omega_{E|B}$$

is an isomorphism.

□

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Corollary 3.3. *If G is a smooth group scheme over S and X is a smooth G-scheme over S, then the cotangent complex $\mathcal{L}_{[X/G]|S}$ corresponds to the 2-term complex of G-equivariant locally free sheaves $d\mu^\vee : \Omega^1_X \to \mathcal{O}_X \otimes \mathfrak{g}^\vee$ on X obtained by differentiating the action map $\mu : X \times G \to X$ at the identity section of G. In this relative context $\mathfrak{g}^\vee$ should be the locally free coherent sheaf $e^*\Omega^1_{G|S}$ on S, where $e : S \to G$ is the identity section.*

Proof. We already have a description of the pullback of $\mathcal{L}_{[X/G]|S}$ to X as the 2 term complex $\Omega_X \to \Omega_{X|[X/G]}$ and a description of $\Omega_{X|[X/G]}$ as $\mathcal{N}^\vee_{X,X\times_{[X/G]}X}$. Moreover there's a canonical isomorphism $X \times_{[X/G]} X \simeq X \times G$, fitting into the commutative diagram

$$\begin{tikzcd} X \times_{[X/G]} X \arrow[r, "\simeq"] \arrow[d, "\pi"'] & X \times G \arrow[d, "\mu"] \\ X \arrow[r, "\mathrm{id}"] & X \end{tikzcd} \tag{3.6}$$

So, the diagonal of $X \times_{[X/G]} X$ corresponds to $X \times \{e\}$ and the projection corresponds to μ, and this identifies $\mathcal{N}^\vee_{X,X\times_{[X/G]}X}$ as $\mathcal{N}^\vee_{X\times\{e\},X\times G} \simeq \mathcal{O}_X \otimes \mathfrak{g}^\vee$, and shows that the map $\Omega^1_X \to \mathcal{O}_X \otimes \mathfrak{g}^\vee$ is $d\mu^\vee$.

□

Claim (on p.4 of Behrend's paper): the natural map of complexes $\Omega \to \mathcal{L}_{\mathfrak{X}|S}$ induces isomorphisms

$$H^*(\mathfrak{X}, \Omega^p) \simeq \mathbb{H}^*(\mathfrak{X}, \bigwedge^p \mathcal{L}_{\mathfrak{X}|S})$$

for all p.

Notes: Behrend attributes this to Teleman. I find it sort of surprising since it seems to be saying something to the effect of"$\Omega_{|\mathfrak{X}}$ is acyclic." To be honest I am having a really hard time finding a proof of this theorem in the generality it's stated. Apparently it's proved in characteristic 0 by Teleman and Simpson in "de Rham cohomology of ∞-stacks" (notes on the Berkeley math department website), and it seems to be proved in the case where $\mathfrak{X} = [X/G]$ for X a smooth projective variety (in characteristic 0) and G a linear algebraic group acting linearly on X.

4. Totaro's work

Let R be a commutative ring and let X be a smooth affine scheme over Spec$R$. Let G be a smooth affine group scheme over Spec$R$ acting on X.

Theorem 4.1 (Totaro)**.** *The natural maps*

$$H^i([X/G], \Omega^j) \xrightarrow{\simeq} H^i([X/G], \bigwedge^j \mathcal{L}_{[X/G]|R})$$

are isomorphisms for all i, j. Moreover $\bigwedge^j \mathcal{L}_{[X/G]|R}$ has the following explicit description as a complex of G-equivariant locally free sheaves on X, in degrees 0 to j:

$$0 \to \Omega^j_X \to \Omega^{j-1}_X \otimes \mathfrak{g}^\vee \to \cdots \to S^j(\mathfrak{g}^\vee) \to 0$$

Here all the maps are obtained from the differential in $\mathcal{L}_{[X/G]|R}$:

$$\Omega^1_X \to \mathcal{O}_X \otimes \mathfrak{g}^\vee$$

which in turn has a pleasant geometric descritpion: let $X \times G \xrightarrow{\mu} X$ be the action map. Differentiating μ we obtain a morphism of locally free sheaves on $X \times G$:

$$d\mu^\vee : \mu^*\Omega^1_X \to \Omega^1_{X\times G} \simeq \pi_X^*\Omega^1_X \oplus \pi_G^*\Omega^1_G$$

We may restrict this to $X \times e$ and project onto the Ω^1_G factor to obtain a morphism

$$\Omega^1_X \to \mathcal{O}_X \otimes (\Omega^1_G \otimes k(e)) \simeq \mathcal{O}_X \otimes \mathfrak{g}^\vee$$

(making some simplifications on the fly).

The characterization of $\mathcal{L}_{[X/G]|R}$ in terms of $d\mu^\vee$ comes from the previous section, and from there the description of $\bigwedge^j \mathcal{L}_{[X/G]|R}$ comes from a standard recipe for the exterior powers of a cochain complex. So I'll focus on the proof that the maps $H^i([X/G], \Omega^j) \xrightarrow{\simeq} H^i([X/G], \bigwedge^j \mathcal{L}_{[X/G]|R})$ are isomorphisms. But first an important corollary, obtained by taking $X = \operatorname{Spec} R$ with the trivial G-action (in that case $[X/G] = BG$:

Corollary 4.2. *The natural maps $H^i(BG, \Omega^j) \xrightarrow{\simeq} H^i(BG, \bigwedge^j \mathcal{L}_{BG|R})$ are all isomorphisms. Moreover the cotangent complex $\mathcal{L}_{BG|R})$ is just $\mathfrak{g}^\vee$ sitting in degree 1, and hence $\bigwedge^j \mathcal{L}_{BG|R})$ is just $S^j(\mathfrak{g}^\vee)$ sitting in degree j, so we have isomorphisms*

$$H^i(BG, \Omega^j) \xrightarrow{\simeq} H^{i-j}(G, S^j(\mathfrak{g}^\vee))$$

where the cohomology on the right hand side is the group cohomology of the G-representation $S^j(\mathfrak{g}^\vee)$.

of the theorem. Totaro's bright idea is to introduce a sheaf $\mathcal{F}$ on the lisse-etale site of $[X/G]$, defined as follows: given a smooth $[X/G]$-scheme $f : T \to [X/G]$, consider the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{f'} & X \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ T & \xrightarrow{f} & [X/G] \end{array} \tag{4.1}$$

From this we get a short exact sequence of locally free sheaves on P:

$$0 \to \pi'^{*}\Omega_{T|R} \to \Omega_{P|R} \to \Omega_{P|T} \to 0$$

Note that since P is a principal G-bundle, we have a natural isomorphism $\Omega_{P|T} \simeq \mathcal{O}_P \otimes \mathfrak{g}^\vee$. In fact this is a short exact sequence of G-locally free sheaves, equivalent to a short exact sequence of locally free sheaves on the quotient $T = P/G$, which I will denote by

$$0 \to \Omega_{T|R} \to \bar{\Omega}_{P|R} \to \mathcal{O}_P \bar{\otimes} \mathfrak{g}^\vee \to 0$$

where the vector bundles $\bar{\Omega}_{P|R}$ and $\mathcal{O}_P \bar{\otimes} \mathfrak{g}^\vee$ on T are obtained by descent (geometrically, $\mathcal{O}_P \bar{\otimes} \mathfrak{g}^\vee$ is the balanced product $P \times_G \mathfrak{g}^\vee$ where we have the given G-action on P and the adjoint representation gives the G-action on $\mathfrak{g}^\vee$). So, finally the definition of $\mathcal{F}$ is:

$$\mathcal{F}|_T = \bar{\Omega}_{P|R}$$

Note: the locally free sheaf $\mathcal{O}_P \bar{\otimes} \mathfrak{g}^\vee$ can also be described as $\bar{\mathfrak{g}^\vee}|_T$, the pullback of the vector bundle $\bar{\mathfrak{g}^\vee}$ on BG corresponding to the G-representation $\mathfrak{g}^\vee$ over the map $T \xrightarrow{f} [X/G] \to BG$.

Note: the sheaf $\mathcal{F}$ is *not* cartesian: we'll have $\operatorname{rk}\mathcal{F}|_T = \operatorname{rk}\Omega_{T|R} + \dim G$ and since $\operatorname{rk}\Omega_{T|R}$ isn't constant, $\operatorname{rk}\mathcal{F}|_T$ won't be constant either.

By construction we have a short exact sequence of sheaves of $\mathcal{O}_{[X/G]}$-modules on $[X/G]$ of the form

$$0 \to \Omega \to \mathcal{F} \to \bar{\mathfrak{g}^\vee} \to 0$$

Hence in the derived category of $[X/G]$ (should this be the derived category of bounded complexes of $\mathcal{O}_{[X/G]}$-modules with coherent cohomology?) there's a quasi-iso

$$\Omega \xrightarrow{\text{q-iso}} (\mathcal{F} \to \bar{\mathfrak{g}^\vee})$$

By functoriality we obtain quasi-isomorphisms

$$\Omega^j \simeq \bigwedge^j(\mathcal{F} \to \bar{\mathfrak{g}}^\vee)$$

for all j and hence isomorphisms of hypercohomology

$$\mathbb{H}^*([X/G], \Omega^j) \simeq \mathbb{H}^*([X/G], \bigwedge^j(\mathcal{F} \to \bar{\mathfrak{g}}^\vee))$$

Returning to the above commutative diagram, observe that it also induces morphisms of sheaves fitting into

$$\begin{tikzcd} f^*\Omega_X \arrow[r] \arrow[d] & f^*\Omega_{X|[X/G]} \arrow[d] \\ \Omega_P \arrow[r] & \Omega_{P|T} \end{tikzcd} \tag{4.2}$$

which can be rewritten as

$$\begin{tikzcd} f^*\Omega_X \arrow[r] \arrow[d] & \mathfrak{g}^\vee \otimes \mathcal{O}_P \arrow[d, "="'] \\ \Omega_P \arrow[r] & \mathfrak{g}^\vee \otimes \mathcal{O}_P \end{tikzcd} \tag{4.3}$$

In fact this is a commutative diagram of G-equivariant locally free sheaves on P, descending to a commutative diagram of locally free sheaves on T:

$$\begin{tikzcd} \bar{\Omega}_X|_T = f^*\bar{\Omega}_X \arrow[r] \arrow[d] & \bar{\mathfrak{g}}^\vee \arrow[d, "="'] \\ \mathcal{F}|_T \arrow[r] & \bar{\mathfrak{g}}^\vee \end{tikzcd} \tag{4.4}$$

Here $\bar{\Omega}_X$ is the locally free sheaf on $[X/G]$ corresponding to the G-equivariant locally free sheaf Ω_X on X. Notice that the top row of this diagram is none other than $\mathcal{L}_{[X/G]}|_T$, the cotangent complex of $[X/G]$. Thus we've constructed a morphism of complexes

$$\mathcal{L}_{[X/G]} \to (\mathcal{F} \to \bar{\mathfrak{g}}^\vee)$$

The rest of the proof will be devoted to showing that the resulting morphisms of complexes

$$\bigwedge^p \mathcal{L}_{[X/G]} \to \bigwedge^p(\mathcal{F} \to \bar{\mathfrak{g}}^\vee)$$

induce isomorphisms on hypercohomology

$$\mathbb{H}^*([X/G], \bigwedge^p \mathcal{L}_{[X/G]}) \simeq \mathbb{H}^*([X/G], \bigwedge^p(\mathcal{F} \to \bar{\mathfrak{g}}^\vee))$$

for all p.

First, observe that the morphism of complexes $\bigwedge^p \mathcal{L}_{[X/G]} \to \bigwedge^p(\mathcal{F} \to \bar{\mathfrak{g}}^\vee)$ will induce a morphism of hypercohomology spectral sequences with morhpism of E_2-pages consisting of the homomorphisms (**agh maybe of E_1-pages? can never remember**)

$$H^i([X/G], \bar{\Omega}_X^{p-j} \otimes S^j(\mathfrak{g}^\vee)) \to H^i([X/G], \bigwedge^{p-j} \mathcal{F} \otimes S^j(\mathfrak{g}^\vee))$$

and so it'll suffice to show these are all isomorphisms.

For this Totaro resorts to a "Cech cohomology" calculation using the canonical simplicial resolution of $[X/G]$:

$$\cdots \to X \times G^2 \to X \times G \to X \to [X/G]$$

Here we get into spectral sequences on spectral sequences: there's a spectral sequence

$$H^l(X \times G^k, \Omega_X^{p-j} \otimes S^j(\mathfrak{g}^\vee)|_{X\times G^k}) \implies H^{k+l}([X/G], \Omega_X^{p-j} \otimes S^j(\mathfrak{g}^\vee))$$

and similarly for $\bigwedge^{p-j} \mathcal{F} \otimes S^j(\mathfrak{g}^\vee)$, and in fact we'll have a morphism of E_2-pages of the form

$$H^l(X \times G^k, \Omega_X^{p-j} \otimes S^j(\mathfrak{g}^\vee)|_{X\times G^k}) \to H^l(X \times G^k, \bigwedge^{p-j} \mathcal{F} \otimes S^j(\mathfrak{g}^\vee)|_{X\times G^k})$$

Here's where the affine-ness hypotheses come into play: *because G and X are affine* we have

$$H^l(X \times G^k, \Omega_X^{p-j} \otimes S^j(\mathfrak{g}^\vee)|_{X\times G^k}) \simeq 0 \text{ for } l > 0$$

and similarly for $\bigwedge^{p-j} \mathcal{F} \otimes S^j(\mathfrak{g}^\vee)|_{X\times G^k}$. Thus this morphism of pages of the spectral sequence is essentially a morphism of *complexes*

$$H^0(X \times G^k, \Omega_X^{p-j} \otimes S^j(\mathfrak{g}^\vee)|_{X\times G^k}) \to H^0(X \times G^k, \bigwedge^{p-j} \mathcal{F} \otimes S^j(\mathfrak{g}^\vee)|_{X\times G^k})$$

(recall that p and j are fixed here - the index for the complexes is k). The proof will be complete if we can show these morphisms of complexes are quasi-isomorphisms.

At this point it will be helpful to calculate $\bigwedge^{p-j} \mathcal{F} \otimes S^j(\mathfrak{g}^\vee)|_{X\times G^k}$. First observe that by definition $\mathcal{F}|_{X\times G^k}$ fits into a short exact sequence $0 \to \Omega_{X\times G^k} \to \mathcal{F}|_{X\times G^k} \to \bar{\mathfrak{g}}^\vee|_{X\times G^k} \to 0$. In the following steps one should pay more attention to the face/degeneracy maps of the simplicial scheme $(X \times G^k | k \in \mathbb{N})$ than I will. Recalling the construction of $\mathcal{F}$ we identify the first map in this short exact sequence as

$$d\bar{\mu}^\vee : \Omega_{X\times G^k} \to \Omega_{X\bar{\times} G^{k+1}}$$

induced by the morphism $\mu^*\Omega_{X\times G^k} \to \Omega_{X\times G^{k+1}}$ of sheaves on $X \times G^{k+1}$. In fact we should have an isomorphism

$$\mathcal{F}|_{X\times G^k} \simeq \Omega_{X\bar{\times} G^{k+1}} \simeq e^*\Omega_{X\times G^{k+1}}$$

, the pullback of $\Omega_{X\times G^{k+1}}$ over the section $e : X \times G^k \to X \times G^{k+1}$ ... **but why?** And should it actually be a pushforward along π or μ instead? Totaro just writes $\Omega_{X\times G^{k+1}}$...

Presumably these isomorphisms can be chosen compatibly, to give an isomorphism of simplicial sheaves. From this we'll obtain an isomorphism

$$\Omega^{p-j}_{X\times G^{k+1}} \otimes S^j(\mathfrak{g}^\vee) \simeq \bigwedge^{p-j} \mathcal{F} \otimes S^j(\mathfrak{g}^\vee)|_{X\times G^k}$$

and it should be that the "obvious" morphisms

$$\Omega_X^{p-j} \otimes S^j(\mathfrak{g}^\vee)|_{X\times G^k} \to \Omega^{p-j}_{X\times G^{k+1}} \otimes S^j(\mathfrak{g}^\vee)$$

induce the map of complexes

$$H^0(X \times G^k, \Omega_X^{p-j} \otimes S^j(\mathfrak{g}^\vee)|_{X\times G^k}) \to H^0(X \times G^k, \Omega^{p-j}_{X\times G^{k+1}} \otimes S^j(\mathfrak{g}^\vee))$$

under consideration. Note that there's a natural isomorphism

$\square$