HIGHER DIRECT IMAGES OF LOGARITHMIC IDEAL SHEAVES

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Contents

1. Introduction

A foundational problem in birational geometry, posed by Grothendieck in his 1958 ICM address [Gro60, Problem B], asked whether for every proper birational morphism of non-singular projective varieties $f\colon X \to Y$,

$$R^q f_* \mathcal{O}_X = 0 \text{ for } i > 0$$

or equivalently (via a Leray spectral sequence argument) whether the natural maps $H^i(Y, \mathcal{O}_Y) \to H^i(X, \mathcal{O}_X)$ are all isomorphisms. In characteristic 0 this was answered affirmatively by Hironaka as a corollary of resolution of singularities [Hir64, §7 Cor. 2]. It follows that the $H^i(X, \mathcal{O}_X)$ are *birational invariants* of nonsingular projective varieties over a fixed ground field k of characteristic 0; indeed, any birational morphism $\varphi\colon X \dashrightarrow Y$ may be factored as

$$\begin{tikzcd} & Z \arrow[dl, "r"'] \arrow[dr, "s"] & \\ X \arrow[rr, dashed, "\varphi"] & & Y \end{tikzcd} \tag{1.1}$$

where Z is another non-singular projective variety and r, s are projective morphisms, resulting in isomorphisms $H^i(X, \mathcal{O}_X) \xrightarrow{\simeq} H^i(Z, \mathcal{O}_Z) \xleftarrow{\simeq} H^i(Y, \mathcal{O}_Y)$.

In characteristic $p > 0$, where resolutions of singularities are not known to exist, answering Grothendieck's question proved much harder, remaining open until 2011 when Chatzistamatiou and Rülling proved the following theorem:

Theorem 1.2 ([CR11, Thm. 3.2.8])**.** *Let k be a perfect field and let S be a separated scheme of finite type over k. Suppose X and Y are two separated finite type S-schemes which are*

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(*i*) *smooth over k and*
(*ii*) ***properly birational*** *over S in the sense that there is a commutative diagram*

$$\begin{tikzcd} & Z \arrow[dl, "r"'] \arrow[dr, "s"] & \\ X \arrow[dr, "f"'] & \circlearrowleft & Y \arrow[dl, "g"] \\ & S & \end{tikzcd} \tag{1.3}$$

with r and s proper birational morphisms.
Set $n = \dim X = \dim Y = \dim Z$. *Then there are natural morphisms of sheaves*

$$\mathrm{cl}^j_Z : R^j f_* \Omega^i_X \to R^j g_* \Omega^i_Y \textit{ for all } i, \tag{1.4}$$

which are isomorphisms if $i = 0, n$.

In the special case char $k = 0$ this is a consequence of Hironaka's resolution of singularities [Hir64]. Analysis of the proof shows that the morphisms of 1.4 are obtained from morphisms of *complexes*

$$\mathrm{cl}_Z : Rf_* \Omega^i_X \to Rg_* \Omega^i_Y \text{ for all } i,$$

(for the cases $i = 0, n$ this is observed in [CR12; Kov20]).

One of the primary applications of Theorem 1.2 was to extend foundational results on rational singularities from characteristic zero to arbitrary characteristic.

Definition 1.5 ([Kol13, Def. 2.76])**.** Let S be a reduced, separated scheme of finite type over a field k. A **rational resolution** $f : X \to S$ is a proper birational morphism such that
(*i*) X is smooth over k,
(*ii*) $\mathcal{O}_S = Rf_* \mathcal{O}_X$ and
(*iii*) $R^i f_* \omega_X = 0$ for $i > 0$.
The scheme S is said to have **rational singularities** if and only if it has a rational resolution.

Corollary 1.6 ([CR11, Cor. 3.2.10])**.** *If S has a rational resolution, then every resolution of S is rational. In particular if S is smooth then it has rational singularities.*

This article concerns analogues of Theorem 1.2 for pairs.

Convention 1.7. In what follows a **pair** (X, Δ_X) will mean a reduced, equidimensional separated scheme X of finite type over k together with a reduced, effective divisor Δ_X on X. A pair (X, Δ_X) will be called a **simple normal crossing (snc) pair** if and only if X is smooth and Δ_X is a simple normal crossing divisor on X.

As observed in [Kol13, §2.5], to generalize Corollary 1.6 to pairs we must restrict attention to a special class of *thrifty resolutions* (Definition 3.5).

Theorem 1.8. *Let k be a perfect field and let S be a separated scheme of finite type over k. Let* (X, Δ_X) *and* (Y, Δ_Y) *be simple normal crossing pairs over S.*

Suppose $(X, \Delta_X), (Y, \Delta_Y)$ are properly birational over S in the sense that there is a commutative diagram

$$\begin{tikzcd} & (Z, \Delta_Z) \arrow[dl, "r"'] \arrow[dr, "s"] & \\ (X, \Delta_X) \arrow[dr, "f"'] \arrow[rr, phantom, "\circlearrowleft"] & & (Y, \Delta_Y) \arrow[dl, "g"] \\ & S & \end{tikzcd} \tag{1.9}$$

where r, s are proper and birational morphisms, and $\Delta_Z = r_*^{-1}\Delta_X = s_*^{-1}\Delta_Y$. Set $n = \dim X = \dim Y = \dim Z$. If r, s are thrifty *then there are quasi-isomorphisms*

$$Rf_*\mathcal{O}_X(-\Delta_X) \simeq Rg_*\mathcal{O}_Y(-\Delta_Y) \text{ and } Rf_*\omega_X(\Delta_X) \simeq Rg_*\omega_Y(\Delta_Y). \tag{1.10}$$

Definition 1.11 ([Kol13, Def. 2.78])**.** Let (S, Δ_S) be a pair as in Convention 1.7, and suppose S is normal. A **rational resolution of** (S, Δ_S) is a proper birational morphism $f\colon X \to S$ such that if $\Delta_X = f_*^{-1}\Delta_S$ then

(i) The pair (X, Δ_X) is snc,

(ii) The natural map $\mathcal{O}_S(-\Delta_S) \to Rf_*\mathcal{O}_X(-\Delta_X)$ is a quasi-isomorphism, and

(iii) $R^i f_*\omega_X(\Delta_X) = 0$ for $i > 0$.

Remark 1.12 (description of the natural map in (ii))*.* Since Δ_X is the strict transform of Δ_S, so in particular $\Delta_X \subset f^{-1}(\Delta_S)$, there is a containment of ideal sheaves $\mathcal{I}_{f^{-1}(\Delta_S)} \subset \mathcal{I}_{\Delta_X} = \mathcal{O}_X(-\Delta_X)$ providing a morphism

$$f^*\mathcal{O}_S(-\Delta_S) = f^*\mathcal{I}_{\Delta_S} \to \mathcal{I}_{f^{-1}(\Delta_S)} \subset \mathcal{I}_{\Delta_X} = \mathcal{O}_X(-\Delta_X).$$

Taking the adjoint gives a morphism $\mathcal{O}_S(-\Delta_S) \to f_*\mathcal{O}_X(-\Delta_X)$, and composing with the natural map $f_*\mathcal{O}_X(-\Delta_X) \to Rf_*\mathcal{O}_X(-\Delta_X)$ gives (ii).

As a straightforward corollary of Theorem 1.8, one obtains:

Corollary 1.13. *Let (S, Δ_S) be a pair, with Δ_S reduced and effective. If (S, Δ_S) has a thrifty rational resolution $f : (X, \Delta_X) \to (S, \Delta_S)$, then every thrifty resolution $g : (Y, \Delta_Y) \to (S, \Delta_S)$ is rational. In particular, if (S, Δ_S) is snc then it is a rational pair.*

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2. Dual complexes

Definition 2.1 (cf. [FKX17])**.** Let $Z = \bigcup_{i \in I} Z_i$ be a scheme with irreducible components Z_i. Say Z is an **expected-dimensional crossing scheme** if and only if

(i) Z is pure dimensional and the components Z_i are normal, and

(ii) For any $J \subset I$, set $Z_J := \bigcap_{j \in J} Z_j$. If $Z_J \neq \emptyset$ every connected component of Z_J is irreducible and of codimension $|J| - 1$ in Z.

A **stratum** of an expected-dimensional crossing scheme Z is an irreducible (or equivalently connected) component of $Z_J = \bigcap_{j \in J} Z_j$ for some $J \subset I$.

The main case of Definition 5.5 considered here will be the case $Z = \Delta_X$ where (X, Δ_X) is a simple normal crossing pair, in which case all strata of Δ_X are smooth. Let (X, Δ_X) be a simple normal crossing pair, and write $\Delta_X = \bigcup_{i\in I} D_i$ with D_i the irreducible components of Δ_X. For $J \subset I$, let $D_J = \bigcap_{j\in J} D_j$, and write $D_J = \bigcup_k D_J^k$ where the D_J^k are irreducible. Observe that $(\Delta_X - \sum_{i\in J} D_i)|_{D_J^k}$ is a (possibly empty) simple normal crossing divisor on each stratum D_J^k.

Definition 2.2 (strata as pairs)**.**

$$\Delta_{D_J} := (\Delta_X - \sum_{i\in J} D_i)|_{D_J} \text{ and } \Delta_{D_J^k} := (\Delta_X - \sum_{i\in J} D_i)|_{D_J^k}$$

Definition 2.3. For an expected-dimensional crossing scheme $Z = \bigcup_{i\in I} Z_i$, the **dual complex** $\mathcal{D}(Z)$ is a Δ-complex [Hat02, §2.1] that can be described as follows: assume the index set I has been totally ordered. For each $d \in \mathbb{N}$, the d-simplices of $\mathcal{D}(Z)$ correspond to the irreducible components $Z_J^k \subset Z_J = \cap_{j\in J} Z_j$ where $J \subset I$ ranges over all subsets of size $|J| = d + 1$. Let σ_J^k be the d-simplex associated to Z_J^k.

If $j \in J$ write $\hat{J}(j) := J \setminus \{j\}$ – we have inclusions $Z_J \subset Z_{\hat{J}(j)}$, and the connected components of $Z_{\hat{J}(j)}$ are irreducible, for each component Z_J^k there is a *unique* component $Z^l_{\hat{J}(j)} \subset Z_{\hat{J}(j)}$ such that $Z_J^k \subset Z^l_{\hat{J}(j)}$. The face maps of $\mathcal{D}(Z)$ are obtained by setting

$$d_j \sigma_J^k = \sigma^l_{\hat{J}(j)}$$

Remark 2.4. In particular, $\mathcal{D}(Z)$ has

- 0-simplices σ_i corresponding to the irreducible components $Z_i \subset Z$,
- 1-simplices σ_{ij}^k corresponding to the components $Z_{ij}^k \subset Z_{ij} = Z_i \cap Z_j$ where $i < j$, with face maps d_0, d_1 corresponding to the inclusions $Z_{ij}^k \subset Z_i$, $Z_{ij}^k \subset Z_j$ respectively,

and so on. In the case where $\dim Z = 1$, this is definition agrees with the usual dual graph of a curve.

Remark 2.5. From the description above one can see that $\mathcal{D}(Z)$ is a **regular** Δ-complex, meaning that if $\sigma \subseteq \mathcal{D}(Z)$ is a d-simplex, the corresponding map $\sigma\colon \Delta^d \to \mathcal{D}(Z)$ is injective. Indeed, if

$$d_j \sigma_J^k = d_{j'} \sigma_J^k$$

for $j \neq j'$, then $Z_{\hat{J}(j)} \cap Z_{\hat{J}(j')} = Z_J$ would contain a component of codimension $d - 1$, violating (ii) of Definition 2.3.

Dual complexes have been extensively studied; to paraphrase Arapura, Bakhtary, and Włodarczyk, $\mathcal{D}(Z)$ governs the *combinatorial part* of the topology of Z [ABW13]. For a precise statement see Lemma 4.2. One can extract from the literature on dual complexes the following slogan:

> *Morphisms of pairs induce morphisms of dual complexes. Moreover, there is a "dictionary" relating properties of a morphism of pairs with corresponding properties of the induced morphism of dual complexes.*

To precisify the slogan, we include a foundational result providing a weak sort of functoriality.

Lemma 2.6 (cf. [Wlo16, Def. 2.0.6])**.** *Let $Z = \cup_{i\in I} Z_i$ and $W = \cup_{j\in J} W_j$ be expected -dimensional crossing schemes and let $f : Z \dashrightarrow W$ be a rational morphism defined at the generic point of each stratum of Z. Then up to homotopy equivalence there is a unique induced morphism of Δ-complexes*

$$\mathcal{D}(f) : \mathcal{D}(Z) \to \mathcal{D}(W)$$

such that if $\sigma \subset \mathcal{D}(Z)$ is a simplex and η_σ is the generic piont of the corresponding stratum of Z, and if $\tau \subset \mathcal{D}(W)$ is the simplex corresponding to the unique minimal stratum $D(\tau) \subset W$ containing $f(\eta_\sigma)$, then $\mathcal{D}(f)(\sigma) \subset \tau$.

Proof in the case f is defined everywhere. Since $f(D(\sigma))$ is irreducible it is contained in some stratum of W (in particular, $f(D(\sigma)) \subset W_i$ for some i). Let

$$W_J := \cap\{W_j \subset W \mid f(D(\sigma)) \subset W_j\}$$

By (ii) of Definition 5.5, the connected components of W_J are irreducible, and hence $f(D(\sigma))$ is contained in exactly one of them – let $\tau \subset \mathcal{D}(W)$ be the corresponding simplex. If $\dim \sigma = 0$ let $\mathcal{D}(f)(\sigma)$ be an interior point of τ.

One can now show by induction on $\dim \sigma$ that $\mathcal{D}(f)$ extends over all of $\mathcal{D}(Z)$ – so, assume $\dim \sigma > 1$. For each face $\sigma' \subset \sigma$ with corresponding stratum $D(\sigma') \subset Z$, let $D(\tau') \subset W$ be the smallest stratum containing $f(D(\sigma'))$. Now

$$f(D(\sigma)) \subset f(D(\sigma')) \text{ forces } D(\tau) \subset D(\tau')$$

and this gives an inclusion $\iota_{\tau'} : \tau' \to \tau$. By induction a map $\mathcal{D}(f)|_{\sigma'} : \sigma' \to \tau'$ has already been defined, so composing with ι one obtains

$$\sigma' \xrightarrow{\mathcal{D}(f)|_{\sigma'}} \tau' \xrightarrow{\iota} \tau \text{ for each face } \sigma' \subset \sigma$$

which together give a map $d\sigma \to \tau$, and as τ is contractible this map must extend over σ.

Uniqueness up to homotopy equivalence follows from Lemma 2.7. □

Lemma 2.7. *If $f, g\colon X \to Y$ are 2 maps of regular Δ-complexes such that for each simplex $\sigma \subseteq X$ there is a unique minimal simplex $\tau_\sigma \subseteq Y$ such that $f(\sigma), g(\sigma) \subseteq \tau_\sigma$ then there is a homotopy $h\colon X \times I \to Y$ from f to g such that $h(\sigma \times I) \subseteq \tau_\sigma$ for each simplex $\sigma \subset X$.*

Proof. We proceed by induction over the skeleta $X^d \subseteq X$. For the case $d = 0$ let $v \in X^0$ be a vertex. By hypothesis there's a unique minimal simplex $\tau_v \subseteq Y$ so that $f(v), g(v) \in \tau_v \subseteq Y$, so we may choose a path $\gamma_v\colon I \to \tau_v \subseteq Y$ with $\gamma_v(0) = f(v), \gamma_v(1) = g(v)$. Then the map

$$h^0\colon X^0 \times I \to Y \text{ defined by } h^0(v,t) = \gamma_v(t)$$

is a homotopy between $f|_{X^0}$ and $g|_{X^0}$ with $h^0(\{v\} \times I) \subseteq \tau_v$ for all v.

Suppose by inductive hypothesis that $d > 0$ and we have constructed a homotopy $h^{d-1}\colon X^{d-1} \times I \to Y$ from $f|_{X^{d-1}}$ to $g|_{X^{d-1}}$ with $h^{d-1}(\sigma \times I) \subseteq \tau_\sigma$ for all simplices $\sigma \subseteq X^{d-1}$. Let $\sigma \subset X$ be a d-simplex, and observe that if $\sigma' \subset \sigma$ is a face then $f(\sigma') \subseteq f(\sigma) \subseteq \tau_\sigma$, and similarly $g(\sigma') \subseteq \tau_\sigma$. By hypothesis this implies $\tau_{\sigma'} \subseteq \tau_\sigma$, so that the homotopy $h^{d-1}|_{\sigma'}\colon \sigma' \times I \to Y$ factors through τ_σ. We conclude that the map $\tilde{\gamma|}_\sigma\colon \sigma \times 0,1 \cup d\sigma \to Y$ defined by

$$(x,t) \mapsto \begin{cases} f(x) & \text{if } t = 0, \\ g(x) & \text{if } t = 1, and \\ h(x,t), & \text{otherwise} \end{cases}$$

factors through τ_σ; since Y is regular τ_σ is contractible, and so $\tilde{\gamma|}_\sigma$ extends to a morphism $\gamma_\sigma\colon \sigma \times I \to Y$. As σ varies over the d-simplices of X, the γ_σ provide an extension of h^{d-1} to a homotopy

$$h^d\colon X^d \times I \to Y \text{ from } f|_{X^d} \text{ to } g|_{X^d}.$$

□

3. Thrifty morphisms of pairs

Let (S, Δ_S) be a pair (as in Convention 1.7).

Definition 3.1. The **snc locus of** (S, Δ_S) is the largest open $U \subset S$ so that $(U, \Delta_S|_U)$ is a simple normal crossing pair – it will be denoted $\mathrm{snc}(S, \Delta_S)$. We also set

$$\text{non-snc}(S, \Delta_S) := S \setminus \mathrm{snc}(S, \Delta_S) \tag{3.2}$$

Remark 3.3. When S is normal, non-snc(S, Δ_S) has codimension ≥ 2 in S.

In their work on dual complexes of Calabi-Yau pairs, introduced a natural generalization of thrifty resolutions to a class of *thrifty morphisms* where the domain is no longer required to be smooth.

Definition 3.4 ([KX16, Def. 9]). A crepant proper birational morphism of log canonical pairs $f\colon (X, \Delta_X) \dashrightarrow (S, \Delta_S)$ is **Kollár-Xu-thrifty** (KX-thrifty for short) if and only if there are closed subsets $Z_X \subset X$, $Z_S \subset S$ of codimension ≥ 1 so that

- Z_X contains no log canonical centers of (X, Δ_X), and similarly for Z_S, and
- f restricts to an isomorphism $X \setminus Z_X \underset{\simeq}{\xrightarrow{f}} S \setminus Z_S$.

Since rational pairs are not log canonical in general, for example since they are not necessarily $\mathbb{Q}$-Gorenstein[1], we adopt a slightly different definition of thrifty morphisms (see Lemma 3.8 for a comparison).

Let (S, Δ_S) be a pair and let $f : X \to S$ be a proper birational morphism. Set $\Delta_X := f_*^{-1}\Delta_S$ (the strict transform).

Definition 3.5. The morphism f is **thrifty** if and only if

(i) f is an isomorphism *over* the generic point of every stratum of snc(S, Δ_S) and
(ii) f is an isomorphism *at* the generic point of every stratum of snc(X, Δ_X).

If in addition X is smooth and $f^{-1}(\Delta_S) \cup E$ is a simple normal crossing divisor (with E the exceptional locus) then f is called a **thrifty resolution**.

Remark 3.6. Equivalently, if $\mathrm{Ex}(f) \subset X$ is the exceptional locus of f, then

(i) $f(\mathrm{Ex}(f))$ contains no stratum of snc(S, Δ_S) and
(ii) $\mathrm{Ex}(f)$ contains no stratum of snc(X, Δ_X).

Remark 3.7. Hence when X is smooth and $f^{-1}(\Delta_S) \cup E$ is a simple normal crossing divisor Definition 3.5 reduces to [Kol13, Def. 2.79].

Lemma 3.8. *Let $f\colon (X, \Delta_X) \to (S, \Delta_S)$ be a crepant proper birational morphism between* dlt *pairs. Then f is KX-thrifty (Definition 3.4) if and only if it is thrifty (Definition 3.5).*

Proof. The map f is crepant, so $K_X + \Delta_X \sim_{\mathbb{Q}} f^*(K_S + \Delta_S)$ – equivalently,

$$\Delta_X \sim_{\mathbb{Q}} f_*^{-1}(\Delta_S) - \sum_i a_i E_i$$

where $a_i := a(E_i, S, \Delta_X)$ and the sum runs over all f-exceptional divisors $E_i \subset X$. Writing $\Delta_S = \sum_i c_i D_i$, we see that $\Delta_S^{=1} = \sum_{c_i=1} D_i$ and that $\Delta_X^{=1} = \sum_{c_i=1} f_*^{-1} D_i + \sum_{a_i=-1} E_i$. Both pairs are dlt, so the log canonical centers of (X, Δ_X) are the the strata of the expected-dimensional crossing scheme $\Delta_X^{=1}$, and their generic points lie in snc(X, Δ_X) – similarly for (S, Δ_S) [Fuj07]. Moreover, if $a_i = -1$ then $f(E_i) \subset S$ is a log canonical center, so it must be a stratum of $\Delta_S^{=1}$.

Suppose f is KX-thrifty and let $Z_X \subset X$, $Z_S \subset S$ be closed sets as guaranteed in Definition 3.4. Then f is an isomorphism over $S \setminus Z_S$ and Z_S contains no stratum of $\Delta_S^{=1}$, giving condition (i) of Definition 3.5. Also, we must have $a_i > -1$ for all i, and so $\Delta_X^{=1} = \sum_{c_i=1} f_*^{-1} D_i = f_*^{-1}\Delta_S^{=1}$. Since Z_X contains no stratum of $\Delta_X^{=1}$, we obtain (ii) of Definition 3.5. □

In the next lemma we use a definition of a birational map general enough to encompass reducible schemes [Stacks, Tags 0A20, 0BX9]: a rational map $f\colon X \dashrightarrow Y$ between schemes with finitely many irreducible components is *birational* if and only if it is an isomorphism in the category with

- objects the schemes with finitely many irreducible components, and with

[1]The cone over $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^{mn+m+n}$ embedded using the complete linear system $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(m,n)|$ is rational for all $m, n > 0$, $\mathbb{Q}$-Gorenstein if and only if $m = n$.

- morphisms the dominant rational maps between them.

When Y is locally of finite presentation over a field (as it will be in all cases considered here), the map f is birational if and only if it induces a bijection between the generic points of irreducible components of X and Y, and for each generic point of an irreducible component $\eta \in X$ the induced morphism $\mathcal{O}_{Y,f(\eta)} \to \mathcal{O}_{X,\eta}$ is an isomorphism.

Lemma 3.9. *Let $Z = \cup_{i=1}^N Z$ and $W = \cup_{j=1}^N W_j$ be expected-dimensional crossing schemes and let $f : Z \dashrightarrow W$ be a birational map defined at the generic point of each stratum of Z.*

(*i*) *If f is an isomorphism at the generic point of every stratum $D(\sigma) \subset Z$, then $\mathcal{D}(f)$ can be realized as a subcomplex inclusion.*

(*ii*) *If f is an isomorphism over the generic point of every stratum $D(\tau) \subset W$ then it is an isomorphism at the generic point of every stratum of Z, and $\mathcal{D}(f)$ can be realized as an isomorphism of Δ-complexes.*

Proof. In the case of (i), as f is birational it induces a bijection between the generic points of Z and W and hence a bijection on 0-skeleta

$$\mathcal{D}(f)_0 : \mathcal{D}(Z)_0 \xrightarrow{\simeq} \mathcal{D}(W)_0$$

Without loss of generality we may assume f restricts to a birational maps $f_i : Z_i \dashrightarrow W_i$ for $i = 1, \ldots, N$. Let $n = \dim Z = \dim W$.

Let $\sigma \subset \mathcal{D}(Z)$ be a simplex with corresponding stratum $D(\sigma) \subset Z$ – without loss of generality we may assume $D(\sigma) \subset Z_1$, and that $D(\sigma) \subseteq \cap_{j=1}^r Z_j$. Letting $\eta_\sigma \in D(\sigma)$ be the generic point, we see that $f(\eta_\sigma) \subset \cap_{j=1}^r W_j$. Because f is an isomorphism at η_σ, it must be that $f(\eta_\sigma)$ is a generic point of a component $D(\tau) \subseteq \cap_{j=1}^r W_j$ corresponding to a simplex $\tau \subseteq \mathcal{D}(W)$. Let $\eta_\tau \in D(\tau)$ be the generic point; we have $\eta_\tau = f(\eta_\sigma)$.

At this point the only concern is that there could be another $r-1$-simplex σ' such that $\mathcal{D}(f)(\sigma') = \tau$; any such σ' would correspond to another stratum $D(\sigma') \subseteq \cap_{j=1}^r Z_j$, hence another point $\eta_{\sigma'} \in Z_1$ of dimension $r-1$ with $f(\eta'_\sigma) = f(\eta_\tau)$. One can show this is impossible, using the normality of W_1 and Zariski's main theorem as follows.

The map f is an isomorphism at the generic point $n_\sigma \in D(\sigma)$, so its restriction $f|_{Z_1} : Z_1 \to W_1$ is also an isomorphism at n_σ. The scheme W_1 is normal and $f|_{Z_1}$ is birational by hypothesis, so by Zariski's main theorem [Stacks, Tag 05K0] $f|_{Z_1}$ is in fact an isomorphism *over* η_τ.

For (ii), observe that $f^{-1} : W \dashrightarrow Z$ satisfies the hypotheses of (i) and hence both $\mathcal{D}(f) : \mathcal{D}(Z) \to \mathcal{D}(W)$ and $\mathcal{D}(f^{-1}) : \mathcal{D}(W) \to \mathcal{D}(W)$ may be realized as subcomplex inclusions; from the proof of (i), this can be done in such a way that $\mathcal{D}(f) \circ \mathcal{D}(f^{-1}) = \mathrm{id}_{\mathcal{D}(W)}$. In particular this implies $\mathcal{D}(f)$ is a surjective subcomplex inclusion, hence an isomorphism. □

Corollary 3.10. *Let (S, Δ_S) be a pair and let $f : X \to S$ be a proper birational morphism and set $\Delta_X := f_*^{-1}\Delta_S$. Then f induces morphisms of Δ-complexes*

$$\mathcal{D}(\operatorname{snc} \Delta_X) \xrightarrow{\mathcal{D}(f|_\Delta)} \mathcal{D}(\operatorname{snc} \Delta_S) \text{ and } \mathcal{D}(\operatorname{snc}(X, \Delta_X)) \xrightarrow{\mathcal{D}(f)} \mathcal{D}(\operatorname{snc}(S, \Delta_S))$$

which are isomorphisms *if f is thrifty.*

Proof. The induced morphisms come from Lemma 2.6; to see that they are isomorphisms when f is thrifty we may apply Definition 3.5 and Lemma 3.9. □

If S is a separated scheme of finite type over k and $f : X \to S$, $g : Y \to S$ are separated schemes of finite type over S, a **proper birational equivalence of X, Y over S** is a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "r"'] \arrow[dr, "s"] & \\ X \arrow[dr, "f"'] & \circlearrowleft & Y \arrow[dl, "g"] \\ & S & \end{tikzcd} \tag{3.11}$$

where r, s are proper birational morphisms.

Definition 3.12. Suppose (X, Δ_X), (Y, Δ_Y) are pairs over S, with X and Y normal and Δ_X, Δ_Y reduced and effective. A **thrifty proper birational equivalence of (X, Δ_X) and (Y, Δ_Y) over S** is a proper birational equivalence as in diagram (3.11), where $r_*^{-1}(\Delta_X) = s^{-1}(\Delta_Y)$ and r and s are thrifty.

Remark 3.13. By Corollary 3.10, a thrifty proper birational equivalence $X \xleftarrow{r} Z \xrightarrow{s} Y$ between (X, Δ_X) and (Y, Δ_Y) induces an isomorphism $\mathcal{D}(\Delta_X) \simeq \mathcal{D}(\Delta_Y)$.

Proposition 3.14. *Let (S, Δ_S) be a pair with Δ_S reduced and effective, and let $f : X \to S$, $g : Y \to S$ be 2 thrifty resolutions of (S, Δ_S). Then there is a thrifty proper birational equivalence of X and Y over S.*

Proof. Let $U \subset S$ be an open set such that both f and g are isomorphisms over U; then we have an isomorphism

$$g^{-1} \circ f : f^{-1}(U) \to g^{-1}(U)$$

Set

$$Z := \overline{\Gamma_{g^{-1} \circ f}} \subset X \times_S Y$$

and let $p : Z \to X$, $s : Z \to Y$ be the projections. The claim is that $X \xleftarrow{r} Z \xrightarrow{s} Y$ is a thrifty proper birational equivalence over S. It is birational by design, and proper since X, Y and hence $X \times_Y Z$ are proper over S and Z is closed in $X \times_S Y$. It remains to show that r, s are thrifty.

Lemma 3.15. *Let* $\mathrm{Ex}(r), \mathrm{Ex}(s) \subset Z$ *be the exceptional loci of r, s respectively; let* $\mathrm{Ex}(f) \subset X$, $\mathrm{Ex}(g) \subset Y$ *be the exceptional loci of f and g. Then*

$$r(\mathrm{Ex}(r)) \subset f^{-1}(g(\mathrm{Ex}(g))) \text{ and } s(\mathrm{Ex}(s)) \subset g^{-1}(f(\mathrm{Ex}(f)))$$

Proof of Lemma 3.15. Let $U \subset S$ and $V \subset Y$ be a maximal pair of open sets such that $g|_V : V \xrightarrow{\simeq} U$ is an isomorphism; note that since g is an honest morphism $\mathrm{Ex}(g) = Y \setminus V$ and $g(\mathrm{Ex}(g)) = S \setminus U$. Then $W := f^{-1}(U) \subset X$ is an open set such that $g^{-1} \circ f : X \dashrightarrow Y$ is defined on W. This implies the projection $\Gamma_{g^{-1} \circ f} \xrightarrow{r} X$ is an isomorphism over W, but what we need to know is that the same is true for $Z = \bar{\Gamma}_{g^{-1} \circ f} \xrightarrow{r} X$. For this, note that

$$\bar{\Gamma}_{g^{-1} \circ f} \cap r^{-1}(W) = \overline{\Gamma_{g^{-1} \circ f} \cap r^{-1}(W)} = \overline{\Gamma_{g^{-1} \circ f|_W}} \subset W \times_S Y$$

Since W and Y are both separated over S, the graph $\Gamma_{g^{-1} \circ f|_W}$ is already closed, so we conclude $\bar{\Gamma}_{g^{-1} \circ f} \cap r^{-1}(W) = \Gamma_{g^{-1} \circ f|_W}$. □

Now suppose $W \subset X$ is a stratum of (X, Δ_X) – we must show r is an isomorphism over the generic point $\eta \in W$. First, f is an isomorphism at η by hypothesis, and so by the proof of Lemma 3.9, $f(\eta)$ is the generic point of a stratum of $\mathrm{snc}(S, \Delta_S)$. Then g is an isomorphism over $f(\eta)$ by hypothesis, so in particular $f(\eta) \notin g(\mathrm{Ex}(g))$. By Lemma 3.15 we conclude that $\eta \notin r(\mathrm{Ex}(r))$, as desired.

Finally we show that s is an isomorphism at the generic point of every stratum of $\Delta_Z := r_*^{-1} f_*^{-1} \Delta_S$, using a more general lemma:

Lemma 3.16. *Let $r\colon (Z, \Delta_Z) \to (X, \Delta_X)$ be a proper birational morphism. If (X, Δ_X) is a simple normal crossing pair, then r is thrifty if and only if it satisfies condition (i) of Definition 3.5. Explicitly, r is thrifty if and only if it is an isomorphism over every stratum of Δ_X.*

Proof of Lemma 3.16. In this situation there is an honest morphism $\mathrm{snc}(\Delta_Z) \to \Delta_X$, so the hypotheses of Lemma 3.9 are satisfied. We then apply Lemma 3.9 (ii). □

□

Remark 3.17. In the case where the morphism $r\colon Z \to X$ of Lemma 3.16 is projective, [Har77, Thm. 7.17] implies that r is the blowup of some sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$ such that $V(\mathcal{I}) \subset X$ contains no stratum of Δ_X. If in addition $V(\mathcal{I})$ has *simple normal crossings* with Δ_X [Kol07, Def. 3.24], Lemma 3.16 can be obtained from known results on the effect of blowing up on dual complexes [Ste06, §2], [FKX17, §9], [Wlo16, Prop. 2.1.6].

4. Simplicial resolutions and descent spectral sequences

Let (X, Δ_X) be a simple normal crossing pair, where $\Delta_X = \bigcup_{i=1}^N D_i$ and each divisor $D_i \subset X$ is smooth and irreducible. We define an augmented semi-simplicial scheme $X_\bullet$ as follows: $X_{-1} = X$, $X_0 = \coprod_i D_i$ and for $k > 0$,

$$\begin{aligned} X_k &= \coprod_{I \subseteq \{1,\dots,N\} \,|\, |I|=k+1} D_I, \text{ where } D_I = \bigcap_{j \in I} D_j \\ &= \coprod_{\sigma \in \mathcal{D}(\Delta_X)^k} D(\sigma) \end{aligned}$$

The face maps are defined by various inclusions $d_k^j : D_I \hookrightarrow D_{I\setminus\{i_j\}}$ for $I = \{i_0, \dots, i_k\}$ and $0 \le j \le i$, as in Definition 2.3. For each k we have an augmentation map $\epsilon_p : X_k \to X$ obtained from the inclusions $D_I \subseteq X$. The X_k are smooth, so in particular the sheaves of differential forms $\Omega^1_{X_k}$ are locally free, and for each p the standard Čech construction applied to the co-semi-simplicial sheaf $\Omega^p_{X_\bullet}$ gives a cochain complex

$$R\epsilon_*\Omega^p_{X_\bullet} : \epsilon_{0*}\Omega^p_{X_0} \to \epsilon_{1*}\Omega^p_{X_1} \to \epsilon_{2*}\Omega^p_{X_2} \to \cdots$$

on X, together with a morphism $\Omega^p_X \to R\epsilon_*\Omega^p_{X_\bullet}$ induced by the augmentation — the shifted cone $\underline{\Omega}^p_{X,\Delta_X} := \mathrm{cone}(\Omega^p_X \to R\epsilon_*\Omega^p_{X_\bullet})[-1]$ is then represented by the following complex, with derived category degrees as indicated:[2]

$$\begin{array}{ccccccccc} \Omega^p_X & \longrightarrow & \epsilon_{0*}\Omega^p_{X_0} & \longrightarrow & \epsilon_{1*}\Omega^p_{X_1} & \longrightarrow & \epsilon_{2*}\Omega^p_{X_2} & \longrightarrow & \cdots \\ = \Omega^p_X & \to & \prod_{\sigma \in \mathcal{D}((\Delta_X))^0} \Omega^p_{D(\sigma)} & \to & \prod_{\sigma \in \mathcal{D}((\Delta_X))^1} \Omega^p_{D(\sigma)} & \to & \prod_{\sigma \in \mathcal{D}((\Delta_X))^2} \Omega^p_{D(\sigma)} & \to & \cdots \\ 0 & & 1 & & 2 & & 3 & & \end{array} \tag{4.1}$$

Lemma 4.2 (Cf. [Fri83, Prop. 1.5], [DI87, Rem. 4.2.2]). *The complex*

$$0 \to \Omega^p_X(\log \Delta_X)(-\Delta_X) \to \Omega^p_X \to \prod_{\sigma \in \mathcal{D}((\Delta_X))^0} \Omega^p_{D(\sigma)} \to \prod_{\sigma \in \mathcal{D}((\Delta_X))^1} \Omega^p_{D(\sigma)} \to \cdots$$

[2] This notation is chosen to align with the fact that over $\mathbb{C}$, the complex (4.1) represents the pth graded part of the Du Bois complex of the pair (X, Δ_X).

is exact. Equivalently, the complex (4.1) *is a resolution of the sheaf* $\Omega^p_X(\log \Delta_X)(-\Delta_X)$*. In particular (for* $p = 0$*) the complex*

$$\mathcal{O}_X \to \prod_{\sigma \in \mathcal{D}(\Delta_X)^0} \mathcal{O}_{D(\sigma)} \to \prod_{\sigma \in \mathcal{D}(\Delta_X)^1} \mathcal{O}_{D(\sigma)} \to \cdots$$

is a resolution of $\mathcal{O}_X(-\Delta_X)$*.*

We include a proof merely to make clear that the lemma is valid in arbitrary characteristic — the argument given follows [Fri83, Prop. 1.5] very closely.

Proof. We can check exactness on Zariski stalks over a point $x \in X$. We may also check exactness after renumbering the divisors D_i, and so we may assume that $x \in D_1, \dots, D_k$ and $x \notin D_i$ for $i > k$. By hypothesis, there are local coordinates $z_1, \dots, z_c \in \mathcal{O}_{X,x}$ such that in a Zariski neighborhood of x, $\Delta_X = V(\prod_{i=1}^k z_i)$ and $D_i = V(z_i)$ for $i = 1, \dots, k$.

We now proceed by simultaneous induction on k and $\dim X$. Letting $\Delta_{D_1} = \sum_{i=2}^k D_i \cap D_1$, we have $\dim D_1 < \dim X$ and $k - 1 < k$, so denoting by $\epsilon' : D_{1\bullet} \to D_1$ the semi-simplicial scheme associated to (D_1, Δ_{D_1}), by inductive hypothesis the complex

$$0 \to \Omega^p_{D_1}(\log \Delta_{D_1})(-\Delta_{D_1}) \to \Omega^p_{D_1} \to \epsilon'_{0*}\Omega^p_{D_{1,0}} \to \epsilon'_{1*}\Omega_{D^*_{1,1}} \to \cdots \tag{4.3}$$

is exact. On the other hand, letting $\Delta^{>1} = \sum_{i=2}^r D_i$ we obtain a divisor with $k-1 < k$ components, so denoting $\epsilon'' : X^{>1}_\bullet \to X$ the semi-simplicial scheme associated to $(X, \Delta^{>1})$, by inductive hypothesis the complex

$$0 \to \Omega^p_X(\log \Delta^{>1})(-\Delta^{>1}) \to \Omega^p_X \to \epsilon''_{0*}\Omega^p_{X^{>1}_0} \to \epsilon''_{1*}\Omega^p_{X^{>1}_1} \to \cdots$$

is exact. Moreover, there is a sequence of complexes

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Omega^p_{D_1} \arrow[r, "d'"] \arrow[d, "\alpha"] & \epsilon'_{0*}\Omega^p_{D_{1,0}} \arrow[r, "d'"] \arrow[d, "\alpha"] & \epsilon'_{1*}\Omega^p_{D_{1,1}} \arrow[r, "d'"] \arrow[d, "\alpha"] & \cdots \\
\Omega^p_X \arrow[r, "\epsilon^\sharp"] \arrow[d, equal] & \epsilon_{0*}\Omega^p_{X_0} \arrow[r, "d"] \arrow[d, "\beta"] & \epsilon_{1*}\Omega^p_{X_1} \arrow[r, "d"] \arrow[d, "\beta"] & \epsilon_{2*}\Omega^p_{X_2} \arrow[r, "d"] \arrow[d, "\beta"] & \cdots \\
\Omega^p_X \arrow[r, "\epsilon''^\sharp"] & \epsilon''_{0*}\Omega^p_{X^{>1}_0} \arrow[r, "d''"] & \epsilon''_{1*}\Omega^p_{X^{>1}_1} \arrow[r, "d''"] & \epsilon''_{2*}\Omega^p_{X^{>1}_2} \arrow[r, "d''"] & \cdots \\
0 & 1 & 2 & 3 &
\end{tikzcd} \tag{4.4}$$

and since for each k, $X_k = X^{>1}_k \coprod D_{1,k-1}$ the columns are (split) exact. Using the long exact sequence of cohomology sheaves, the inductive hypotheses show that $h^i(\underline{\Omega}^p_{X,\Delta_X}) = 0$ for $i > 1$, and in low degrees we have an exact sequence

$$0 \to \Omega^p_X(\log \Delta_X)(-\Delta_X) \to \Omega^p_X(\log \Delta^{>1})(-\Delta^{>1}) \to \Omega^p_{D_1}(\log \Delta_{D_1})(-\Delta_{D_1}) \to h^1(\underline{\Omega}^p_{X,\Delta_X}) \to 0$$

It remains to show $h^1(\underline{\Omega}^p_{X,\Delta_X}) = 0$. For this consider a local section

$$(\varphi_i) = (\varphi_i | i = 1, \dots, k) \in \ker d \subseteq \epsilon_{0*}\Omega^p_{X_0} = \prod_{i=1}^k \Omega^p_{D_i}$$

As $d''\beta(\varphi_i) = \beta d(\varphi_i) = 0$, by inductive hypothesis there is a local section $\omega \in \Omega^p_X$ such that $\beta(\varphi_i) = \epsilon''^{\sharp}\omega$. Unravelling, $\beta(\varphi_i) = (\varphi_2, \dots, \varphi_k)$ and $\omega|_{D_i} = \varphi_i$ for $i = 2, \dots, k$. Since

$$0 = d(\varphi_i) = (\varphi_i|_{D_i \cap D_j} - \varphi_i|_{D_i \cap D_j} | 1 \le i < j \le N), \text{ so in particular for } i = 1$$
$$0 = \varphi_1|_{D_1 \cap D_j} - \varphi_j|_{D_1 \cap D_j} = \varphi_1|_{D_1 \cap D_j} - \omega|_{D_1 \cap D_j} \text{ for } j = 2, \dots, k$$

we find that $\varphi_1 - \omega|_{D_1}$ vanishes on Δ_{D_1}, and applying exactness of (4.3) once more we see $\varphi_1 - \omega|_{D_1} \in \Omega^p_{D_1}(\log \Delta_{D_1})(-\Delta_{D_1})$. At x, $\Omega^p_{D_1}(\log \Delta_{D_1})(-\Delta_{D_1})$ is generated by the forms

$$(\prod_{i=2}^{k} z_i) \cdot \frac{dz_{i_1}}{z_{i_1}} \wedge \dots \wedge \frac{dz_{i_l}}{z_{i_l}} \wedge dz_{i_{l+1}} \wedge \dots \wedge dz_{i_p} \text{ where } 1 < i_1 < \dots < i_l \le k < i_{l+1} < \dots < i_p \le N$$

The key point is: each of these vanishes on D_i for $i > 1$ (since they each contain either a z_i or a dz_i for all $1 < i \le k$), and so we may find a local section $\xi \in \Omega^p_X$ with

(*i*) $\xi|_{D_1} = \varphi_1 - \omega|_{D_1}$;
(*ii*) $\xi|_{D_i} = 0$ for $i > 1$.

Rearranging shows $(\omega + \xi)|_{D_i} = \varphi_i$ for all i — in other words $(\varphi_i) = \epsilon^{\sharp}(\omega + \xi)$. □

Remark 4.5. As a byproduct we obtain an exact sequence

$$0 \to \Omega^p_X(\log \Delta_X)(-\Delta_X) \to \Omega^p_X(\log \Delta^{>1})(-\Delta^{>1}) \to \Omega^p_{D_1}(\log \Delta_{D_1})(-\Delta_{D_1}) \to 0,$$

and considering the snake-lemma definition of the connecting morphism shows this is, at least up to sign, restriction of log differential forms (see [EV92, §2])

The complex (4.1) comes with a descending filtration by truncations

$$\underline{\Omega}^p_{X,\Delta_X} = \sigma_{\ge 0}\underline{\Omega}^p_{X,\Delta_X} \supset \sigma_{\ge 1}\underline{\Omega}^p_{X,\Delta_X} \supset \sigma_{\ge 2}\underline{\Omega}^p_{X,\Delta_X} \supset \cdots$$

where

$$(\sigma_{\ge i}\underline{\Omega}^p_{X,\Delta_X})^j = \begin{cases} 0 & \text{if } j < i \\ (\underline{\Omega}^p_{X,\Delta_X})^j = \epsilon_{j-1*}\Omega^p_{X_{j-1}} = \prod_{\sigma \in \mathcal{D}(\Delta_X)^{j-1}} \Omega^p_{D(\sigma)} & \text{otherwise} \end{cases} \tag{4.6}$$

Using this filtration we obtain a spectral sequence for higher direct images.

Corollary 4.7. *Let S be a scheme of finite type over k and let $f : X \to S$ be a morphism. Then there is a filtered complex $(Rf_*\underline{\Omega}^p_{X,\Delta_X}, F)$ whose cohomology computes the higher direct images $R^{i+j}f_*\Omega^p_X(\log \Delta_X)(-\Delta_X)$. For each i there is a distinguished triangle*

$$F^{i+1}Rf_*\underline{\Omega}^p_{X,\Delta_X} \to F^iRf_*\underline{\Omega}^p_{X,\Delta_X} \to Rf_*\epsilon_{i-1*}\Omega^p_{X_{i-1}} = \prod_{\sigma \in \mathcal{D}(\Delta_X)^{i-1}} Rf_*\Omega^p_{D(\sigma)} \to \cdots$$

In particular, there is a spectral sequence

$$E_1^{ij} = R^jf_*(\epsilon_{i-1*}\Omega^p_{X_{i-1}}) = \prod_{\sigma \in \mathcal{D}(\Delta_X)^{i-1}} R^jf_*\Omega^p_{D(\sigma)} \implies R^{i+j}f_*\Omega^p_X(\log \Delta_X)(-\Delta_X)$$

The filtration F is defined as $F = Rf_*\sigma$. The resulting spectral sequence is just the usual hypercohomology spectral sequence.

Remark 4.8. Viewing $\epsilon : X_\bullet \to X$ as a sort of resolution of the pair (X, Δ_X), we can consider the spectral sequence of Corollary 4.7 as a sort of *descent* spectral sequence (see [SGA4II, Vbis], [Con03]).

Using Corollary 4.7 we can obtain a restricted form of Theorem 1.8, the case of a thrifty proper birational morphism of snc pairs.

Theorem 4.9. *Let (Y, Δ_Y) be an snc pair over a perfect field k and let $f : X \to Y$ be a thrifty proper birational equivalence. Assume X is smooth and $\Delta_X := f_*^{-1}\Delta_Y$ is snc. Then the natural map*

$$\mathcal{O}_Y(-\Delta_Y) \to Rf_*\mathcal{O}_X(-\Delta_X) \text{ is a quasi-isomorphism.}$$

Proof. By Corollary 3.10, the morphism f induces an isomorphism $\mathcal{D}(f) : \mathcal{D}(\Delta_X) \xrightarrow{\simeq} \mathcal{D}(\Delta_Y)$. Let $\mathcal{D}$ denote this dual complex, and for each i and each cell $\sigma \in \mathcal{D}^i$ denote the corresponding stratum on X (resp. Y) by $D_X(\sigma) \subset X$ (resp. $D_Y(\sigma) \subset Y$). Moreover in the morphism of semi-simplicial schemes

$$\begin{tikzcd} \cdots \arrow[r] & X_2 \arrow[r] \arrow[d, "f_2"] & X_1 \arrow[r] \arrow[d, "f_1"] & X_0 \arrow[r] \arrow[d, "f_2"] & X \arrow[d, "f"] \\ \cdots \arrow[r] & Y_2 \arrow[r] & Y_1 \arrow[r] & Y_0 \arrow[r] & Y \end{tikzcd} \tag{4.10}$$

for each i,

$$f_i : X_i = \coprod_{\sigma \in \mathcal{D}^i} D_X(\sigma) \to \coprod_{\sigma \in \mathcal{D}^i} D_Y(\sigma) = Y_i$$

is a proper birational morphism of smooth varieties over k. By [CR11, Cor. 3.2.10] (or [CR15, Thm. 1.1])

$$\mathcal{O}_{D_Y(\sigma)} = Rf_*\mathcal{O}_{D_X(\sigma)} \text{ for each } \sigma \in \mathcal{D}^i \tag{4.11}$$

The diagram (4.10) induces a morphism of *filtered* complexes $f^\sharp : \underline{\Omega}^0_{Y,\Delta_Y} \to Rf_*\underline{\Omega}^0_{X,\Delta_X}$, and by Lemma 4.2 and Corollary 4.7 it will suffice to show that the resulting map of descent spectral sequences

$$E_1^{ij}(Y) = \begin{cases} \prod_{\sigma \in \mathcal{D}(\Delta_Y)^{i-1}} \mathcal{O}_{D(\sigma)} & j = 0 \\ 0 & \text{otherwise} \end{cases} \to \prod_{\sigma \in \mathcal{D}(\Delta_X)^{i-1}} R^j f_*\mathcal{O}_{D(\sigma)} = E_1^{ij}(X)$$

is an isomorphism, and this last step is a consequence of (4.11). □

Suppose now that $(X, \Delta_X), (Y, \Delta_Y)$ are snc pairs over a finite-type k-scheme S with structure morphisms $X \xrightarrow{f} S \xleftarrow{g} Y$, related by a thrifty proper birational equivalence $X \xleftarrow{r} Z \xrightarrow{s} Y$ over S as in (3.11). If Z is smooth and $\Delta_Z = r_*^{-1}\Delta_X = s_*^{-1}\Delta_Y$ is snc, then Theorem 4.9 applied to both r and s shows

$$Rf_*\mathcal{O}_X(-\Delta_X) \simeq Rf_*Rr_*\mathcal{O}_Z(-\Delta_Z) = Rg_*Rs_*\mathcal{O}_Z(-\Delta_Z) \simeq Rg_*\mathcal{O}_Y(-\Delta_Y)$$

Of course, Z need not be smooth and in the absence of resolution of singularities in characteristic $p > 0$, we cannot replace it by a resolution — instead, we replace Z with a mildly singular (specifically Cohen-Macaulay) semi-simplicial scheme $Z_\bullet$ together with morphisms $X_\bullet \xleftarrow{r_\bullet} Z_\bullet \xrightarrow{s_\bullet} Y_\bullet$ over S which are term-by-term proper birational equivalences over S. This construction is made possible by the existence of Macaulayfications.

Theorem 4.12 ([Ces18, Thm. 1.6], cf. also [Kaw00, Thm. 1.1])**.** *For every a CM-quasi-excellent noetherian scheme X there exists a projective birational morphism $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is Cohen-Macaulay and π is an isomorphism over the Cohen-Macaulay locus* $\mathrm{CM}(X) \subset X$.

The usefulness of Macaulayfications for the problem at hand stems from an extension of the results of Chatzistamatiou-Rülling due to Kovács.

Theorem 4.13 ([Kov20, Thm. 1.4])**.** *Let $f : X \to Y$ be a locally projective birational morphism of excellent Cohen-Macaulay schemes. If Y has pseudo-rational singularities then*

$$\mathcal{O}_Y = Rf_*\mathcal{O}_X \text{ and } Rf_*\omega_X = \omega_Y.$$

By a result of Lipman-Teissier, if Y is regular (so in particular if it is smooth over k) then Y is pseudo-rational [LT81, §4].

Lemma 4.14. *Let (X, Δ_X) and (Y, Δ_Y) be simple normal crossing pairs over a finite-type k-scheme S, and let $X \xleftarrow{r} Z \xrightarrow{s} Y$ be a thrifty projective birational equivalence over S. Then their exists a semi-simplicial S-scheme $Z_\bullet$ and S-morphisms of semi-simplicial schemes $X_\bullet \xleftarrow{r_\bullet} Z_\bullet \xrightarrow{s_\bullet} Y_\bullet$ such that for all i,*

(i) Z_i is Cohen-Macaulay and

(ii) $X_i \xleftarrow{r_i} Z_i \xrightarrow{s_i} Y_i$ is a thrifty projective birational equivalence over S.

In (ii), thriftiness is with respect to the divisors Δ_{X_i} on X_i (resp. Δ_{Y_i} on Y_i) defined as in Definition 2.2. To prove Lemma 4.14 we need a few preliminary results, the first being a blowup lemma from the construction of Nagata compactifications.

Lemma 4.15 ([Con07, Cor. 2.10])**.** *Let S be a quasi-compact, quasi-separated scheme and let $j_i : U \to X_i$ be a finite collection of dense open immersions between finite type separated S-schemes. Then there exist U-admissible blowups $X'_i \to X_i$ and a separated finite type S-scheme X, together with open immersions $X'_i \hookrightarrow X$ over S, such that the X'_i cover X and the open immersions $U \hookrightarrow X'_i \hookrightarrow X$ are all the same.*

Proof of Lemma 4.14. To fix some notation, by Corollary 3.10 there are isomorphisms of dual complexes

$$\mathcal{D}(\Delta_X) \simeq \mathcal{D}(\mathrm{snc}(\Delta_Z)) \simeq \mathcal{D}(\Delta_Y) =: \mathcal{D},$$

For a cell $\sigma \subset \mathcal{D}$ let $D_X(\sigma)$ (resp. $D_Y(\sigma)$) denote the corresponding stratum of X (resp. Y) with generic point $\eta_X(\sigma)$ (resp. $\eta_Y(\sigma)$). Let η_X(resp. η_Y) be the generic point of X (resp. Y). Let $Z^0_\bullet$ be the semi-simplicial scheme associated to $\mathrm{snc}(Z, \Delta_Z)$. By hypothesis for each i, we have a thrifty birational equivalence $X_i \leftarrow Z^0_i \to Y_i$ over S — the problem is that Z^0_i need not be projective over X_i and Y_i. To remedy this we will build, by induction on i, projective Macaulayfications $Z^0_i \subseteq Z_i$ of Z^0_i over $X_i \times_S Y_i$.

In the case $i = -1$, by Theorem 4.12 there is a projective Macaulayfication $\pi : \tilde{Z} \to Z$ which is an isomorphism over $\mathrm{CM}(Z)$. For any $\sigma \in \mathcal{D}$, $r : Z \to X$ is an isomorphism over $\eta_X(\sigma)$ and so $r^{-1}(\eta_X(\sigma)) \subseteq \mathrm{CM}(Z)$, so π is an isomorphism over $r^{-1}(\eta_X(\sigma))$ and hence $r \circ \pi : \tilde{Z} \to X$ is an isomorphism over $\eta_X(\sigma)$. By a similar argument $s \circ \pi$ is an isomorphism over $\eta_Y(\sigma)$, and it follows that with $Z_{-1} = \tilde{Z}$, $r_{-1} = r \circ \pi$ and $s_{-1} = s \circ \pi$, (i), (ii) are satisfied.

When $i = 0$, for each $\sigma \in \mathcal{D}^0$ we have a diagram

$$\begin{tikzcd} & Z_{-1} \arrow[d, "\pi"] \\ D_{Z^0}(\sigma) \arrow[ur, dashed, "\varphi_X"] \arrow[r] \arrow[d, "r'"] & Z \arrow[d, "r"] \\ D_X(\sigma) \arrow[r] & X \end{tikzcd}$$

Here the dashed arrow φ_X denotes the rational map obtained from the fact that π is an isomorphism over the generic point $\eta_Z(\sigma)$.

The map φ_X is equivalent to a rational map $\psi_X : D_{Z^0}(\sigma) \dashrightarrow D_X(\sigma) \times_S Z_{-1}$ over $D_X(\sigma)$, and a similar construction with Y in place of X yields a rational map $\psi_X : D_{Z^0}(\sigma) \dashrightarrow D_Y(\sigma) \times_Y Z_{-1}$ over $D_Y(\sigma)$. Let $W_X(\sigma) \subseteq D_X(\sigma) \times_S Z_{-1}$ (resp. $W_Y(\sigma) \subseteq D_Y(\sigma) \times_Y Z_{-1}$) be the closure of the image of ψ_X

(resp. ψ_Y). The current situation is summarized in the picture below.

$$
\begin{tikzcd}
D_{Z^0}(\sigma) \arrow[r, dashed, "\psi_X"] \arrow[drr, dashed, "\psi_Y"] \arrow[ddr, "r^0"] \arrow[dddrr, bend right, "s^0"'] & W_X(\sigma) \subseteq D_X(\sigma) \times_S Z_{-1} \arrow[rr] \arrow[dd, "r'"] & & Z_{-1} \arrow[dr, equal] \arrow[dd, "r_{-1}"] & \\
& & W_Y(\sigma) \subseteq D_Y(\sigma) \times_Y Z_{-1} \arrow[rr] \arrow[dd, "s'"] & & Z_{-1} \arrow[dd, "s_{-1}"] \\
& D_X(\sigma) \arrow[rr, hook] & & X & \\
& & D_Y(\sigma) \arrow[rr, hook] & & Y
\end{tikzcd} \tag{4.16}
$$

Since $r^0 : D_{Z^0}(\sigma) \to D_X(\sigma)$ is birational (but not necessarily proper), it must be that $W_X(\sigma) \subseteq D_X(\sigma) \times_S Z_{-1}$ is the unique component of $D_X(\sigma) \times_S Z_{-1}$ dominating $D_X(\sigma)$. By the $i = -1$ case and base change, r' is an isomorphism over all strata of $\Delta_{D_X(\sigma)}$, and so there is a dense open $U_X \subseteq D_X(\sigma)$ such that r' is an isomorphism over U_X. Similarly, r^0 is an isomorphism over all strata of $\Delta_{D_X(\sigma)}$ so shrinking U_X if necessary we may additionally assume r^0 is an isomorphism over U_X. Analogous observations hold with Y in place of X; in particular, there is a dense open $U_Y \subseteq D_Y(\sigma)$ so that s' and s^0 are both isomorphisms over U_Y.

Remark 4.17. While it's not required for the construction, I think we may be able to simply require r' to be an isomorphism over U_X (more precisely I think it then follows that r^0 is too). Note that by definition we have a locally closed immersion $D_{Z^0}(\sigma) \subseteq Z$, adjoint to a morphism $D_{Z^0}(\sigma) \to D_X(\sigma) \times_S Z$ appearing in a commutative diagram

$$
\begin{tikzcd}
& D_{Z^0}(\sigma) \arrow[r, "r^0"] \arrow[dl, dashed, "\psi_X"'] \arrow[d] & D_X(\sigma) \arrow[d, equal] \\
D_X(\sigma) \times_S Z_{-1} \arrow[r, "\pi'"] \arrow[d, hook] & D_X(\sigma) \times_S Z \arrow[r, "r''"] \arrow[d, hook] & D_X(\sigma) \arrow[d, hook] \\
Z_{-1} \arrow[r, "\pi"] & Z \arrow[r, "r"] & X
\end{tikzcd}
$$

where the bottom 2 squares are cartesian. By hypothesis $r' = r'' \circ \pi'$ is an isomorphism over $U_X \subseteq D_X(\sigma)$, and by base change both π', r'' are proper and surjective, hence r'' is also an isomorphism over U_X. At this point it would suffice to verify $(r^0)^{-1}(U_X) \subseteq D_{Z^0}(\sigma)$ is open in $(r'')^{-1}(U_X) \subseteq D_X(\sigma) \times_S Z$ — I'm not sure if/why this is true or how to verify it.

We claim that if $U := (r^0)^{-1}(U_X) \cap (s^0)^{-1}(U_Y) \subseteq D_{Z^0}(\sigma)$ then ψ_X and ψ_Y are defined on U and the morphisms

$$\psi_X|_U : U \to W_X(\sigma) \text{ and } \psi_Y|_U : U \to W_Y(\sigma)$$

are dense open immersions. Since r' is an isomorphism over U_X, it will suffice to verify that $r^0|_U : U \to D_X(\sigma)$ is an open immersion, and this is indeed the case as we have

$$U = (r^0)^{-1}(U_X) \cap (s^0)-1(U_Y) \xrightarrow{\text{open imm.}} (r^0)^{-1}(U_X) \xrightarrow{\simeq} U_X$$

Hence we may view U as a common dense open subscheme of $D_{Z^0}(\sigma), W_X(\sigma)$ and $W_Y(\sigma)$ — we now apply Lemma 4.15, *over* $X \times_S Y$ [3] — this gives in particular U-admissible blowups

[3] I think taking Z_{-1} as the base scheme would also work here.

$p : \tilde{W}_X(\sigma) \to W_X(\sigma)$ (resp. $q : \tilde{W}_Y(\sigma) \to W_Y(\sigma)$) and a scheme $W(\sigma)$ over $X \times_S Y$ together with open immersions $\tilde{W}_X(\sigma) \subseteq W(\sigma)$ over $X \times_S Y$ such that the composite open immersions

$$U \subseteq \tilde{W}_X(\sigma) \subseteq W(\sigma) \text{ and } U \subseteq \tilde{W}_Y(\sigma) \subseteq W(\sigma)$$

coincide. Since $\tilde{W}_X(\sigma)$ is proper over $X \times_S Y$ (its structure morphism factors as

$$\tilde{W}_X(\sigma) \xrightarrow{U-\text{admissible blowup}} W_X(\sigma) \xrightarrow{\text{closed immersion}} Z_{-1} \xrightarrow{\text{proper}} X \times_S Y)$$

the open immersion $\tilde{W}_X(\sigma) \subseteq W(\sigma)$ must be an isomorphism. Similarly $\tilde{W}_Y(\sigma) = W(\sigma)$, and we conclude $W(\sigma)$ is a common U-admissible blowup of $W_X(\sigma), W_Y(\sigma)$. By the choice of U and the U-admissibility of these blowups the composite morphisms $W(\sigma) = \tilde{W}_X(\sigma) \xrightarrow{p} W_X(\sigma) \xrightarrow{r'} D_X(\sigma)$ are isomorphisms over all generic points of strata of $\Delta_{D_X(\sigma)}$, and so in particular for each cell $\tau \subseteq \mathcal{D}$ with $\sigma \in \tau$, with corresponding generic point of stratum $\eta_X(\tau) \in D_X(\tau) \subseteq D_X(\sigma)$, we have $(r' \circ p)^1(\eta_X(\tau)) \subseteq \mathrm{CM}(W(\sigma))$. Similarly, $(s' \circ q)^{-1}(\eta_Y(\tau)) \subseteq \mathrm{CM}(W(\sigma))$. Hence if we define $\pi : Z(\sigma) \to W(\sigma)$ to be a Macaulayfication of $W(\sigma)$ of the form guaranteed by Theorem 4.12, the morphism

$$\begin{tikzcd} Z(\sigma) \arrow[r, "\pi"] \arrow[rrr, bend right, "r(\sigma)"'] & W(\sigma) \arrow[r, "p"] & W_X(\sigma) \arrow[r, "r'"] & D_X(\sigma) \end{tikzcd}$$

is an isomorphism over each $\eta_X(\tau) \in D_X(\sigma)$, hence thrifty, and it is also projective as each map in the composition defining $r(\sigma)$ is projective. Similarly $s(\sigma) := s' \circ q \circ \pi : Z(\sigma) \to D_Y(\sigma)$ is a thrifty projective birational map, and so $D_X(\sigma) \xleftarrow{r(\sigma)} Z(\sigma) \xrightarrow{s(\sigma)} D_Y(\sigma)$ is a thrifty projective birational equivalence over S. Defining $Z_0 := \coprod_{\sigma \in \mathcal{D}^0} Z(\sigma)$ and

$$r_0 := \coprod_{\sigma \in \mathcal{D}^0} r(\sigma) : Z_0 = \coprod_{\sigma \in \mathcal{D}^0} Z(\sigma) \to \coprod_{\sigma \in \mathcal{D}^0} D_X(\sigma) = X_0$$
$$s_0 := \coprod_{\sigma \in \mathcal{D}^0} s(\sigma) : Z_0 = \coprod_{\sigma \in \mathcal{D}^0} Z(\sigma) \to \coprod_{\sigma \in \mathcal{D}^0} D_Y(\sigma) = Y_0$$

then ensures properties (i) and (ii).

When $i > 0$, extra care must be taken to ensure compatibility with all semi-simplicial face maps. While somewhat ad-hoc as presented here, the construction that follows is modelled on [Stacks, Tag 018A]. Writing $[i] = \{0, \ldots, i\}$ and $[i]^2_< = \{j, k \in [i] \mid j < k\}$, we define $\delta_+, \delta_- : Z_{i-1}^{[i]} \to Z_{i-2}^{[i]^2_<}$ by

$$\delta_+(z_0, \ldots, z_i) = (d_j^{i-1} x_k \mid j < k) \text{and} \delta_-(z_0, \ldots, z_i) = (d_{k-1}^{i-1} x_j \mid j < k)$$

and we define the equalizer $E := \mathrm{Eq}(\delta_+, \delta_{-1}) \subset Z_{i-1}^{[i]}$ — here the equalizer is taken in the category of $\mathrm{Spec}(k)$, S or even $X_{i-2}^{[i]^2_<} \times_S Y_{i-2}^{[i]^2_<}$-schemes (they are all functorially isomorphic). For the purposes of this construction the key property of E is that if we view $Z_0, \ldots, Z_{i-1}$ as an $i-1$-truncated semi-simplicial scheme, then given an S-scheme Z_i and an S-morphism $Z_i \to E$, defining $d_j^Z : Z_i \to Z_{i-1}$ as the composition $Z_i \to E \subseteq Z_{i-1}^{[i]} \xrightarrow{\mathrm{pr}_j} Z_{i-1}$ (for each $0 \leq j \leq i$) makes $Z_0, \ldots, Z_i$ an i-truncated semi-simplicial scheme extending $Z_0, \ldots, Z_{i-1}$.

Fixing a $\sigma \in \mathcal{D}^i$, for $j = 0, \ldots, i$ we have closed immersions $d_j^X : D_X(\sigma) \to X_{i-1}$ and $d_j^Y : D_Y(\sigma) \to Y_{i-1}$. From all of this data we obtain a diagram similar to (4.16), but this time with separate fiber products for each j:

$$
\begin{tikzcd}
D_{Z^0}(\sigma) \arrow[r, dashed, "\psi_{X,j}"] \arrow[drr, dashed, "\psi_{Y,j}"] \arrow[ddr, "r^0"] \arrow[dddrr, bend right, "s^0"'] & W_X(\sigma)_j \subseteq D_X(\sigma) \; {}_{d_j^X}\!\times_{r_{i-1}\circ \mathrm{pr}_j} E \arrow[rr] \arrow[dd, "r'_j"] & & E \arrow[dr, equal] \arrow[dd, "r_{i-1}\circ \mathrm{pr}_j"] & \\
& & W_Y(\sigma)_j \subseteq D_Y(\sigma) \; {}_{d_j^Y}\!\times_{s_{i-1}\circ \mathrm{pr}_j} E \arrow[rr] \arrow[dd, "s'_j"] & & E \arrow[dd, "s_{i-1}\circ \mathrm{pr}_j"] \\
& D_X(\sigma) \arrow[rr, hook, "d_j^X"'] & & X_{i-1} & \\
& & D_Y(\sigma) \arrow[rr, hook, "d_j^Y"'] & & Y_{i-1}
\end{tikzcd}
\tag{4.18}
$$

In this case, the existence of the rational maps ψ_X, ψ_Y requires further justification, which we provide below in the case of ψ_X (the argument for ψ_Y is analogous). As before, the $W_X(\sigma)_j$ (resp. $W_Y(\sigma)_j$) are the scheme theoretic closures of the $\psi_{X,j}$ (resp. $\psi_{Y,j}$).

By adjunction and by definition of the equalizer E, a collection of rational maps $\psi_{X,j}$ as in (4.18) is equivalent to a rational map φ_X in a commutative diagram

$$
\begin{tikzcd}
D_{Z^0}(\sigma) \arrow[r, dashed, "\varphi_X"] \arrow[d, "r^0"] & Z^{[i]}_{i-1} \arrow[r, shift left, "\delta^Z_+"] \arrow[r, shift right, "\delta^Z_-"'] \arrow[d, "r^{[i]}_{i-1}"] & Z^{[i]^2_<}_{i-2} \arrow[d, "r^{[i]^2_<}_{i-2}"] \\
D_X(\sigma) \arrow[r, "(d^X_j)"] \arrow[d, equal] & X^{[i]}_{i-1} \arrow[r, shift left, "\delta^X_+"] \arrow[r, shift right, "\delta^X_-"'] \arrow[d, "\mathrm{pr}_j"] & X^{[i]^2_<}_{i-2} \\
D_X(\sigma) \arrow[r, "d^X_j"] & X_{i-1} &
\end{tikzcd}
$$

with the property that $\delta^Z_+ \circ \varphi_X = \delta^Z_- \circ \varphi_X$. By inductive hypothesis (ii) the morphism $r_{i-1} : Z_{i-1} \to X_{i-1}$ is thrifty, and hence for each j it is an isomorphism over $d^X_j(\eta_X(\sigma)) \in X_{i-1}$. More precisely, if $U_{X,i-1} \subseteq X_{i-1}$ is a dense open set such that $Z_{i-1} \xrightarrow{r_{i-1}} X_{i-1}$ is an isomorphism over $U_{X,i-1}$ then on $V_{X,j} := (d^X_j \circ r^0)^{-1}(U_{X,i-1}) \subseteq D_{Z^0}(\sigma)$ we have an honest morphism

$$
\begin{tikzcd}
V_{X,j} = (d^X_j \circ r^0)^{-1}(U_{X,i-1}) \arrow[r, "r^0"] \arrow[rrr, bend left, "\varphi_{X,i}"] & (d^X_j)^{-1}(U_{X,i-1}) \arrow[r, "d^X_j"] & U_{X,i-1} \arrow[r, "r^{-1}_{i-1}"] & Z_{i-1} \\
\subseteq D_{Z^0}(\sigma) & \subseteq D_X(\sigma) & \subseteq X_{i-1} &
\end{tikzcd}
$$

and hence on $V_X := \cap_{j=0}^{i} V_{X,j}$ we have an honest morphism $\varphi_X := (\varphi_{X,i}) : V_X \to Z^{[i]}_{i-1}$. Shrinking V_X further we can and will assume that $r^0 : V_X \to D_X(\sigma)$ is an isomorphism over its image. Since $Z^{[i]^2_<}_{i-2}$ is separated, it will suffice to verify $\delta^Z_+ \circ \varphi_X = \delta^Z_- \circ \varphi_X$ on any dense open subset of $D_{Z^0}(\sigma)$; moreover, it will suffice to verify

$$
\mathrm{pr}_{ij} \circ \delta^Z_+ \circ \varphi_X = \mathrm{pr}_{ij} \circ \delta^Z_- \circ \varphi_X : D_{Z^0}(\sigma) \dashrightarrow Z_{i-2}
$$

for each i, j and by inductive hypothesis (ii) again, $Z_{i-2} \xrightarrow{r_{i-2}} X_{i-2}$ is an isomorphism over some dense open $U_{X,i-2} \subseteq X_{i-2}$. Consideration of the middle row of (4) and the definition of the $\varphi_{X,i}$ shows that the $\mathrm{pr}_{ij} \circ \delta_+^Z \circ \varphi_X, \mathrm{pr}_{ij} \circ \delta_-^Z \circ \varphi_X$ *generically factor through* $r_{i-2}^{-1}(U_{X,i-2}) \subseteq Z_{i-2}$,[4] and in this way $\delta_+^Z \circ \varphi_X = \delta_-^Z \circ \varphi_X$ is a consequence of the semi-simplicial identity $\delta_+^X \circ (d_j^X) = \delta_-^X \circ (d_j^X)$.

To complete the inductive step, let $U \subseteq D_{Z^0}(\sigma)$ be a dense open set such that

- $U \subseteq (r^0)^{-1}(V_X) \cap (s^0)^{-1}(V_Y)$;
- $r^0|_U$ and $s^0|_U$ are isomorphisms;
- U contains the generic point of every stratum of $\Delta_{D_{Z^0}(\sigma)}$.[5]

Such a U can be viewed as a common dense open of $D_{Z^0}(\sigma)$, the $W_X(\sigma)_i$ and the $W_Y(\sigma)_i$ — we now apply Lemma 4.15 over $X_{i-1} \times_S Y_{i-1}$ to obtain a common U-admissible blowup $W(\sigma)$ of the $W_X(\sigma)_i$ and $W_Y(\sigma)_i$, say with structure morphisms $p_i : W(\sigma) \to W_X(\sigma)_i$ and $q_i : W(\sigma) \to W_Y(\sigma)_i$, then Theorem 4.12 to obtain a Macaulayfication $\pi : Z(\sigma) \to W(\sigma)$, and define morphisms $r(\sigma) : Z(\sigma) \to D_X(\sigma)$ as the compositions

$$\begin{tikzcd} Z(\sigma) \arrow[r, "\pi"] \arrow[rrr, bend right, "r(\sigma)"'] & W(\sigma) \arrow[r, "p_j"] & W_X(\sigma)_j \arrow[r, "r'_j"] & D_X(\sigma) \end{tikzcd}$$

Note that (despite appearances!) this composition *does not depend on* j, since $D_X(\sigma)$ is separated and the $r'_j \circ p_j$ all coincide on the dense open $U \subseteq W(\sigma)$. Similarly we define $s(\sigma) = s'_j \circ q_j \circ \pi$, and again the composition is independent of $0 \leq j \leq i$. Defining $Z_i := \coprod_{\sigma \in \mathcal{D}^i} Z(\sigma)$ and

$$r_i := \coprod_{\sigma \in \mathcal{D}^i} r(\sigma) : Z_i = \coprod_{\sigma \in \mathcal{D}^i} Z(\sigma) \to \coprod_{\sigma \in \mathcal{D}^i} D_X(\sigma) = X_i$$
$$s_i := \coprod_{\sigma \in \mathcal{D}^i} s(\sigma) : Z_i = \coprod_{\sigma \in \mathcal{D}^i} Z(\sigma) \to \coprod_{\sigma \in \mathcal{D}^i} D_Y(\sigma) = Y_i$$

then ensures properties (i) and (ii) and completes the inductive step of the construction. □

Alternative proof. We will construct $Z_\bullet$ by induction on i, with the auxiliary inductive hypothesis that the morphisms r_i, s_i of (ii) are thrifty with respect to the divisors $\Delta_{X_i} \subset X_i$, $\Delta_{Y_i} \subset Y_i$ defined as in Definition 2.2. To fix some notation, by Corollary 3.10 there are isomorphisms of dual complexes

$$\mathcal{D}(\Delta_X) \simeq \mathcal{D}(\mathrm{snc}(\Delta_Z)) \simeq \mathcal{D}(\Delta_Y) =: \mathcal{D},$$

For a cell $\sigma \subset \mathcal{D}$ let $D_X(\sigma)$ (resp. $D_Y(\sigma)$) denote the corresponding stratum of X (resp. Y) with generic point $\eta_X(\sigma)$ (resp. $\eta_Y(\sigma)$). Let η_X(resp. η_Y) be the generic point of X (resp. Y).

In the case $i = -1$, by Theorem 4.12 there is a projective Macaulayfication $\pi : \tilde{Z} \to Z$ which is an isomorphism over CM(Z). For any $\sigma \in \mathcal{D}$, $r : Z \to X$ is an isomorphism over $\eta_X(\sigma)$ and so $r^{-1}(\eta_X(\sigma)) \subseteq \mathrm{CM}(Z)$, so π is an isomorphism over $r^{-1}(\eta_X(\sigma))$ and hence $r \circ \pi : \tilde{Z} \to X$ is an isomorphism over $\eta_X(\sigma)$. By a similar argument $s \circ \pi$ is an isomorphism over $\eta_Y(\sigma)$, and it follows that with $Z_{-1} = \tilde{Z}$, $r_{-1} = r \circ \pi$ and $s_{-1} = s \circ \pi$, (i), (ii) and the auxiliary thriftiness hypothesis are satisfied.

---

[4]This is a critical point: if the preimage of $r_{i-2}^{-1}(U_{X,i-2})$ in $D_{Z^0}(\sigma)$ were empty this argument wouldn't make sense. So I should probably elaborate here.

[5]For example, we may shrink V_X, V_Y if necessary to ensure that r^0, s^0 are isomorphisms over V_X and V_Y respectively and then set $U = (r^0)^{-1}(V_X) \cap (s^0)^{-1}(V_Y)$.

In the case $i = 0$, for each 0-cell $\sigma \in \mathcal{D}^0$ consider the fiber product

$$\begin{tikzcd} W(\sigma) := D_X(\sigma) \times_S D_Y(\sigma) \times_{X \times_S Y} Z_{-1} \arrow[r] \arrow[d, "t"] & Z_{-1} \arrow[d, "r_{-1} \times s_{-1}"] \\ D_X(\sigma) \times_S D_Y(\sigma) \arrow[r] & X \times_S Y \end{tikzcd}$$

We claim that there is a closed subscheme of $W(\sigma)^* \subseteq W(\sigma)$ for which the projections to $D_X(\sigma)$ and $D_Y(\sigma)$ are projective and birational.[6] Setting $V(\sigma) = \mathrm{pr}_X^{-1}(\eta_X(\sigma)) \cap \mathrm{pr}_Y^{-1}(\eta_Y(\sigma))$, we see that $t^{-1}(V(\sigma)) = r_{-1}^{-1}(\eta_X(\sigma)) \cap s_{-1}^{-1}(\eta_Y(\sigma))$. Since r_{-1} and s_{-1} are thrifty they isomorphisms over $\eta_X(\sigma)$ and $\eta_Y(\sigma)$ respectively, and moreover the preimages of $\eta_X(\sigma)$ and $\eta_Y(\sigma)$ coincide — in other words, $r_{-1}^{-1}(\eta_X(\sigma)) \cap s_{-1}^{-1}(\eta_Y(\sigma))$ consists of a single point $\xi(\sigma)$. Let . Since $W(\sigma)^* \subseteq W(\sigma)$ is closed and $\mathrm{pr}_{D_X(\sigma)} \circ t : W(\sigma) \to D_X(\sigma)$ is projective (because r_{-1} is projective and (4) is cartesian) we see that $W(\sigma)^*$ is projective over $D_X(\sigma)$, and from the above discussion of generic points we see that the map $W(\sigma)^* \to D_X(\sigma)$ is birational. Similarly, $W(\sigma)^*$ is projective and birational over $D_Y(\sigma)$.

Let $\tau \subset \mathcal{D}$ be any cell containing σ, corresponding to strata $D_X(\tau) \subset D_X(\sigma)$ and $D_Y(\tau) \subset D_Y(\sigma)$. Letting $V(\tau) = \mathrm{pr}_X^{-1}(\eta_X(\tau)) \cap \mathrm{pr}_Y^{-1}(\eta_Y(\tau))$, we show $t|_{W(\sigma)^*}^{-1}(V(\tau))$ consists of a single point, say $\xi(\tau)$, and that $[k(\xi(\tau)) : k(\eta_X(\sigma))] = [k(\xi(\tau)) : k(\eta_Y(\sigma))] = 1$. Note that $t^{-1}(V(\tau)) = r_{-1}^{-1}(\eta_X(\tau)) \cap s_{-1}^{-1}(\eta_Y(\tau))$; by thriftiness of r_{-1}, s_{-1} this consists of a single point $\xi(\tau)$, and by going up/going down the specializations $\eta_X(\sigma) \rightsquigarrow \eta_X(\tau)$, $\eta_Y(\sigma) \rightsquigarrow \eta_Y(\tau)$ lift to a specialization $\xi(\sigma) \rightsquigarrow \xi(\tau)$, so that $\xi(\tau) \in \overline{\{\xi(\sigma)\}} = W(\sigma)^*$. Now using Theorem 4.12 to obtain a projective Macaulayfication $Z_0(\sigma) \to W(\sigma)^*$ which is an isomorphism over the Cohen-Macaulay locus and defining $Z_0 = \coprod_{\sigma \in \mathcal{D}^0} Z_0(\sigma)$ with the evident morphisms to X_0, Y_0 ensures (i), (ii) and the auxiliary thriftiness hypothesis.

When $i > 0$, extra care must be taken to ensure compatibility with all semi-simplicial face maps. While somewhat ad-hoc as presented here, the construction that follows is modelled on [Stacks, Tag 018A]. Writing $[i] = \{0, \dots, i\}$ and $[i]^2_< = \{j, k \in [i] \mid j < k\}$, we define $\delta_+, \delta_- : Z_{i-1}^{[i]} \to Z_{i-2}^{[i]^2_<}$ by

$$\delta_+(z_0, \dots, z_i) = (d_j^{i-1} x_k \mid j < k) \text{ and } \delta_-(z_0, \dots, z_i) = (d_{k-1}^{i-1} x_j \mid j < k)$$

and we define the equalizer $E := \mathrm{Eq}(\delta_+, \delta_{-1}) \subset Z_{i-1}^{[i]}$ — here the equalizer is taken in the category of $\mathrm{Spec}(k)$, S or even $X_{i-2}^{[i]^2_<} \times_S Y_{i-2}^{[i]^2_<}$-schemes (they are all functorially isomorphic). Next, consider the cartesian diagram

$$\begin{tikzcd} W \arrow[r, "u"] \arrow[d, "t"] \arrow[dr, phantom, "\square"] & E \arrow[d] \\ X_i \times_S Y_i \arrow[r] & X_{i-1}^{[i]} \times_S Y_{i-1}^{[i]} \end{tikzcd} \tag{4.19}$$

We claim that for each $\sigma \in \mathcal{D}^i$, the scheme W has exactly 1 point, say $\xi(\sigma) \in W$, over

$$V(\sigma) := \mathrm{pr}_{X_i}^{-1}(\eta_X(\sigma)) \cap \mathrm{pr}_{Y_i}^{-1}(\eta_Y(\sigma)) \subseteq X_i \times_S Y_i$$

sith $[k(\xi(\sigma)) : k(\eta_X(\sigma))] = [k(\xi(\sigma)) : k(\eta_Y(\sigma))] = 1$. *Given* this claim, defining $Z_i(\sigma)$ to be a Macaulayfication of $\overline{\xi(\sigma)} \subset W$ of the form guaranteed by Theorem 4.12 and defining $Z_i := \coprod_{\sigma \in \mathcal{D}^1} Z_i(\sigma)$ completes the inductive step.

[6]In words, this is the "common strict transform of $D_X(\sigma)$ and $D_Y(\sigma)$"

The cartesian diagram (4.19) decomposes as a disjoint union of the cartesian diagrams

$$\begin{tikzcd} W(\sigma) \arrow[rr, "u"] \arrow[d, "t"] \arrow[drr, phantom, "\square"] & & F \arrow[d] \\ D_X(\sigma)\times_S D_Y(\sigma) \arrow[r] & \prod_{j=0}^i D_X(d^i_j\sigma)\times_S D_Y(d^i_j\sigma) \end{tikzcd} \tag{4.20}$$

where $F = \prod_{j=0}^i D_X(d^i_j\sigma)\times_S D_Y(d^i_j\sigma)\times_{X^{[i]}_{i-1}\times_S Y^{[i]}_{i-1}} E \subseteq \prod_{j=0}^i Z_{i-1}(d^i_j\sigma)$, and $t^{-1}(V(\sigma)) \subseteq W(\sigma)$, so it will suffice to work with the smaller diagram (4.20). Suppose first $w \in t^{-1}(V(\sigma))$ and $[k(w) : k(\eta_X(\sigma))] = 1$. Then $u(w) \in F \subseteq \prod_{j=0}^i Z_{i-1}(d^i_j\sigma)$ is a point such that

$$(r_{i-1}\circ \mathrm{pr}_j)(u(w)) = d^i_j(\eta_X(\sigma)) \text{ for } 0 \le j \le i$$

Since $d^i_j(\eta_X(\sigma))$ is a generic point of a stratum of $\Delta_{D_X(d^i_j\sigma)} \subseteq D_X(d^i_j\sigma)$, by inductive hypothesis $r_{i-1} : Z_{i-1} \to X_{i-1}$ is an isomorphism over $d^i_j(\eta_X(\sigma))$ and so $\mathrm{pr}_j(u(w)) \in Z_{i-1}$ is the unique point of $r^{-1}_{i-1}(d^i_j\eta_X(\sigma)) \subseteq Z_{i-1}$. Thus there is at most 1 such w.

On the other hand, from the above discussion there are rational maps $\varphi_j : D_X(\sigma) \dashrightarrow Z_{i-1}(d^i_j\sigma)$ such that $r_{i-1}(\varphi_j) = d^i_j$, and together these give a rational map

$$\begin{tikzcd} & \prod_{j=0}^i Z_{i-1}(d^i_j\sigma) \arrow[d] \\ D_X(\sigma) \arrow[ur, dashed, "(\varphi_j)"] \arrow[r, "(d^i_j)"'] & \prod_{j=0}^i D_X(d^i_j\sigma) \end{tikzcd}$$

then (φ_j) is defined at $\eta_X(\sigma)$ and we obtain a point $v := (\varphi_j)(\eta_X(\sigma)) \in \prod_{j=0}^i Z_{i-1}(d^i_j\sigma)$, and it will suffice to show that

- $v \in E$;
- $(s_{i-1}\circ \mathrm{pr}_j)(v) = d^i_j(\eta_Y)$

The second point follows from the inductive hypothesis that r_{i-1}, s_{i-1} are thrifty.[7] For the first, we look 1 step further to Z_{i-2}:

$$\begin{tikzcd} & \prod_{j=0}^i Z_{i-1}(d^i_j\sigma) \arrow[r, shift left, "\delta^Z_+"] \arrow[r, shift right, "\delta^Z_-"'] \arrow[d] & \prod_{j<k} Z_{i-2}(d^{i-1}_j d^i_k\sigma) \arrow[d, "\rho"] \\ D_X(\sigma) \arrow[ur, dashed, "(\varphi_j)"] \arrow[r, "(d^i_j)"'] & \prod_{j=0}^i D_X(d^i_j\sigma) \arrow[r, shift left, "\delta^X_+"] \arrow[r, shift right, "\delta^X_-"'] & \prod_{j<k} X_{i-2}(d^{i-1}_j d^i_k\sigma) \end{tikzcd} \tag{4.21}$$

[7]This probably requires further detail and explanation to be believable/visible.

Let $U \subseteq \overline{w} \subseteq \prod_{j<k} Z_{i-2}(d_j^{i-1}d_k^i\sigma)$ be an open set such that (φ_j) is defined over U[8] Since the $\prod_{j<k} Z_{i-2}(d_j^{i-1}d_k^i\sigma)$ are separated, using [Stacks, Tag 01RH] it will suffice to show that

$$\delta_+^Z(\varphi_j) = \delta_-^Z(\varphi_j) \text{ on } \cap_j T_j \subset D_X(\sigma),$$

or for that matter on any dense open subset of $\cap_j T_j$. This follows from commutativity of (4.21), since in the bottom row (d_j^i) factors through $\mathrm{Eq}(\delta_+^X, \delta_-^X)$ (because $X_\bullet$ is a semi-simplicial scheme) and by inductive hypothesis in the right vertical map $\rho = (r_{i-2}^{jk})$, the factor r_{i-2}^{jk} is an isomorphism over $d_j^{i-1}d_k^i\eta_X(\sigma)$ for each j, k.

□

Corollary 4.22. *With the same hypotheses as Lemma 4.14, there exists a filtered complex $(\mathcal{K}, F)$ together with filtered quasi-isomorphisms $\underline{\Omega}^0_{X,\Delta_X} \simeq Rr_*\mathcal{K}$ and $\underline{\Omega}^0_{Y,\Delta_Y} \simeq Rs_*\mathcal{K}$. In particular there are quasi-isomorphisms $Rf_*\mathcal{O}_X(-\Delta_X) \simeq Rf_*Rr_*\mathcal{K} = Rg_*Rs_*\mathcal{K} \simeq Rg_*\mathcal{O}_Y(-\Delta_Y)$.*

.

Proof. By Lemma 4.14 there is a commutative diagram of augmented semi-simplicial schemes

$$\begin{tikzcd}
 & Z_\bullet \arrow[rr, "\epsilon^Z"] \arrow[ddl, "r_\bullet"'] \arrow[dr, "s_\bullet"] & & Z \arrow[dr, "s"] \arrow[ddl, "r"] & \\
 & & Y_\bullet \arrow[rr, "\epsilon^Y"] & & Y \\
X_\bullet \arrow[rr, "\epsilon^X"] & & X & &
\end{tikzcd} \tag{4.23}$$

□

such that for each i the maps $X_i \xleftarrow{r_i} Z_i \xrightarrow{s_i} Y_i$ define a projective birational equivalence over S. Defining $\mathcal{K} = \mathrm{cone}(\mathcal{O}_Z \to R\epsilon_*^Z\mathcal{O}_{Z_\bullet})[-1]$, filtered by its truncations $\sigma_{\geq i}\mathcal{K}$ as in (4.6), from (4.23) we obtain a map of filtered complexes $r^\sharp : \underline{\Omega}^0_{X,\Delta_X} \to Rr_*\mathcal{K}$ appearing in a map of distinguished triangles

$$\begin{tikzcd}
\underline{\Omega}^0_{X,\Delta_X} \arrow[r] \arrow[d, "r^\sharp"] & \mathcal{O}_X \arrow[r] \arrow[d] & R\epsilon_*\mathcal{O}_{X_\bullet} \arrow[r, "+1"] \arrow[d] & \cdots \\
Rr_*\mathcal{K} \arrow[r] & Rr_*\mathcal{O}_Z \arrow[r] & Rr_*\epsilon_{Z*}\mathcal{O}_{Z_\bullet} \arrow[r, "+1"] & \cdots
\end{tikzcd}$$

The map of spectral sequences induced by $r^\sharp$ then has E_1 term

$$E_1^{ij}(X) = \begin{cases} \epsilon_{X*}\mathcal{O}_{X_i-1} & \text{if } j = 0 \\ 0 & \text{else} \end{cases} \to R^j r_*\mathcal{O}_{Z_{i-1}} = E_1^{ij}(Z)$$

By [Kov20, Thm. 1.4] this is an isomorphism, and so $r^\sharp$ is a (filtered) quasi-isomorphism. Applying Rf_* and using Lemma 4.2 then gives a quasi-isomorphism

$$Rf_*\mathcal{O}_X(-\Delta_X) \simeq Rf_*\underline{\Omega}^0_{X,\Delta_X} \simeq Rf_*Rr_*\mathcal{K}.$$

A similar argument applied on the Y side gives the desired quasi-isomorphism $Rf_*\mathcal{O}_X(-\Delta_X) \simeq Rg_*Rs_*\mathcal{K}$.

[8] This uses the fact that if, say, φ_j is defined on a dense open $T_j \subseteq D_X(\sigma)$ for each j then the image of $\cap_j T_j \xrightarrow{(\varphi_j)} \prod_{j<k} Z_{i-2}(d_j^{i-1}d_k^i\sigma)$ is a constructible set, so it contains a dense open subset of its closure $\overline{(\varphi_j)(\cap_j T_j)} = \overline{w}$ [Stacks, Tag 005K], [Stacks, Tag 054K].

5. Cycle morphisms to Log Hodge cohomology

The original proof of [CR11, Thm. 3.2.8] makes use of a cycle morphism $\mathrm{cl} : CH^*(X) \to H^*(X, \Omega^*_X)$ from Chow cohomology to Hodge cohomology, which is ultimately applied to a cycle $Z \subset X \times Y$ obtained from a proper birational equivalence. That cycle morphism satisfies 2 key properties: the first is that it is compatible with *correspondences*: here Chow correspondences are homomorphisms

$$CH^*(X) \to CH^*(Y) \text{ of the form } \alpha \mapsto \mathrm{pr}_{Y*}(\mathrm{pr}_X^* \alpha \smile \gamma) \text{ for some } \gamma \in CH^*(X \times Y)$$

where $\smile$ is the cup product induced by intersecting cycles; Hodge correspondences are defined in a similar way. The second key property is a compatibility with the filtrations

$$CH^n(X \times Y) = F^0 CH^n(X \times Y) \supseteq F^1 CH^n(X \times Y) \supseteq \cdots \supseteq F^{\dim Y} CH^n(X \times Y) \supseteq 0$$

where $F^c CH^n(X \times Y)$ is the subgroup generated by cycles $Z \subseteq X \times Y$ such that $\mathrm{codim}(\mathrm{pr}_Y Z \subseteq Y) \geq c$, and

$$H^n(X \times Y, \Omega^m_{X \times Y}) = F^0 H^n(X \times Y, \Omega^m_{X \times Y}) \supseteq F^1 CH^*(X \times Y) \supseteq \cdots \supseteq F^{\dim Y} H^n(X \times Y, \Omega^m_{X \times Y}) \supseteq 0$$

where $F^c H^n(X \times Y, \Omega^m_{X \times Y})$ is the image of the map $H^n(X \times Y, \oplus_{j=c}^m \Omega_X^{m-j} \boxtimes \Omega_Y^j) \to H^n(X \times Y, \Omega^m_{X \times Y})$ coming from the Künneth decomposition.

It is natural to ask if a similar method can be applied to prove Theorem 1.8, by replacing the ordinary sheaves of differentials Ω_X appearing in Hodge cohomology with sheaves of differentials with log poles $\Omega_X(\log \Delta_X)$. Many of the preliminary results on Hodge cohomology in [CR11, §2] carry over without difficulty, however log poles add complications when one begins to deal with correspondences $H^*(X, \Omega_X(\log \Delta_X)) \to H^*(Y, \Omega_Y(\log \Delta_Y))$ associated to certain Hodge classes with log poles on $X \times Y$.

This section has substantial overlap with [BPØ20, §9], however in that article only *finite* correspondences are considered, with additional strictness (in the sense of logarithmic geometry) conditions. Such correspondences seem to be insufficient to deal with proper birational equivalences, which are generally not finite.

5.1. Functoriality properties of log Hodge cohomology with supports.

Let X be a noetherian scheme.

Definition 5.1 ([R&D], [CR11])**.** A **family of supports** Φ **on** X is a non-empty collection Φ of closed subsets of X such that

- If $C \in \Phi$ and $D \subset C$ is a closed subset, then $D \in \Phi$.
- If $C, D \in \Phi$ then $C \cup D \in \Phi$.

Example 5.2. $\Phi = \{$ all closed subsets of X $\}$ is a family of supports. More generally if $\mathcal{C}$ is any collection of closed subsets $C \subset X$, there's a *smallest* family of supports $\Phi(\mathcal{C})$ containing $\mathcal{C}$ (explicitly, $\Phi(\mathcal{C})$ consists of finite unions $\bigcup_i Z_i$ of closed subsets $Z_i \subset C_i$ of elements $C_i \in \mathcal{C}$). Taking $\Phi = \Phi(\{X\})$ recovers the previous example. For a closed subset $Z \subset X$ we will use the abbreviation $\Phi(Z) := \Phi(\{Z\})$.

There is a close relationship between families of supports on X and certain collections of specialization-closed subsets of points on X. One can also consider sheaves of families of supports. See [R&D].

If $f : X \to Y$ is a morphism of noetherian schemes and Ψ is a family of supports on Y, then $\{f^{-1}(Z) \mid Z \in \Psi\}$ is a family of closed subsets of X, and is closed under unions, but is *not* in general closed under taking closed subsets.

Definition 5.3. $f^{-1}(\Psi)$ be the smallest family of supports on X containing $\{f^{-1}(Z) \mid Z \in \Psi\}$.

Let Φ be a family of supports on X. The notation/terminology $f|_\Phi$ **is proper** will mean $f|_C$ is proper for every $C \in \Phi$. If $f|_\Phi$ is proper then $f(C) \subset Y$ is closed for every $C \in \Phi$ and in fact

$$f(\Phi) = \{f(C) \subset Y \mid C \in \Phi\} \tag{5.4}$$

is a family of supports on Y. The key point here is that if $D \subset f(C)$ is closed, then $f^{-1}(D) \cap C \in \Phi$ and $D = f(f^{-1}(D) \cap C)$.

Definition 5.5. A **scheme with supports** (X, Φ_X) is a scheme X together with a family of supports Φ_X on X.

When no confusion is likely to arise we will abbreviate (X, Φ_X) by simply X.

Definition 5.6. A **pushing morphism** $f : (X, \Phi_X) \to (Y, \Phi_Y)$ of schemes with supports is a morphism $f : X \to Y$ of underlying schemes such that $f|_{\Phi_X}$ is proper and $f(\Phi_X) \subset \Phi_Y$. A **pulling morphism** $f : X \to Y$ is a morphism $f : X \to Y$ such that $f^{-1}(\Phi_Y) \subset \Phi_X$.

These morphisms provide 2 different categories with underlying set of objects schemes with supports (X, Φ_X), and pushing/pulling morphisms respectively (the verification is elementary; for instance a composition of pushing morphisms is again a pushing morphism since compositions of proper morphisms are proper).

Schemes with supports provide a natural setting for local cohomology [R&D]. Let $\mathcal{F}$ be a sheaf of abelian groups on a scheme with supports (X, Φ_X) (more precisely $\mathcal{F}$ is just a sheaf of abelian groups on X).

Definition 5.7. The **sheaf of sections with supports** of $\mathcal{F}$, denoted $\underline{\Gamma}_\Phi(\mathcal{F})$, is obtained by setting

$$\underline{\Gamma}_\Phi(\mathcal{F})(U) = \{\sigma \in \mathcal{F}(U) \mid \operatorname{supp} \sigma \in \Phi_X|_U\} \tag{5.8}$$

for each open $U \subset X$ (here $\Phi_X|_U$ is short for $\iota^{-1}\Phi_X$ where $\iota : U \to X$ is the inclusion). More explicitly: for a local section $\sigma \in \mathcal{F}(U)$, $\sigma \in \underline{\Gamma}_\Phi(\mathcal{F})(U)$ means $\operatorname{supp} \sigma = C \cap U$ for a closed set $C \subset \Phi_X$.

The functor $\underline{\Gamma}_\Phi$ is right adjoint to an exact functor, for instance the inclusion of the subcategory $\mathbf{Ab}_\Phi(X) \subset \mathbf{Ab}(X)$ of abelian sheaves on X with supports in Φ; so, $\underline{\Gamma}_\Phi$ is left exact and preserves injectives (for the case $\Phi = \Phi(Z)$ for some closed $Z \subset X$, see [Stacks] §17.5 and §20.21). Its right derived functor will be denoted $R\underline{\Gamma}_\Phi$. Taking global sections on X gives the **sections with supports** of $\mathcal{F}$:

$$\Gamma_\Phi(\mathcal{F}) := \Gamma_X(\underline{\Gamma}_\Phi(\mathcal{F})) \tag{5.9}$$

This is also left exact, and (the cohomologies of) its derived functor give the **cohomology with supports in** Φ:

$$H^i_\Phi(X, \mathcal{F}) := R^i\Gamma_\Phi(\mathcal{F}) \tag{5.10}$$

Proposition 5.11. *Cohomology with supports enjoys the following functoriality properties*:

(*i*) *If $f : (X, \Phi_X) \to (Y, \Phi_Y)$ is a* pulling *morphism of schemes with supports, $\mathcal{F}, \mathcal{G}$ are sheaves of abelian groups on X, Y respectively, and if*

$$\varphi : \mathcal{G} \to f_*\mathcal{F} \text{ is a morphism of sheaves,} \tag{5.12}$$

then there is a natural morphism $R\underline{\Gamma}_\Phi\mathcal{G} \to Rf_*R\underline{\Gamma}_\Phi\mathcal{F}$. Similarly if $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent then there are natural morphisms $R\underline{\Gamma}_\Phi\mathcal{G} \to Rf_*R\underline{\Gamma}_\Phi\mathcal{F}$.

(*ii*) *If $f : (X, \Phi_X) \to (Y, \Phi_Y)$ is a* pushing *morphism, $\mathcal{F}, \mathcal{G}$ are sheaves of abelian groups on X, Y respectively, and*

$$\psi : Rf_*\mathcal{F} \to \mathcal{G} \text{ is a morphism in the derived category of } X, \tag{5.13}$$

then there is a natural morphism $Rf_*R\underline{\Gamma}_\Phi(\mathcal{F}) \to R\underline{\Gamma}_\Phi\mathcal{G}$.

Let k be a field.

Definition 5.14. A **snc pair with supports** (X, Δ_X, Φ_X) over k is a smooth scheme X over k with a family of supports Φ_X together with a $\mathbb{Q}$-divisor Δ_X on X such that $\operatorname{supp} \Delta_X$ has simple normal crossings. The **interior** U_X of a snc pair with supports (X, Δ_X, Φ_X) is

$$U_X := X \setminus \Delta_X \tag{5.15}$$

The inclusion of U_X in X is denoted by $\iota_X : U_X \to X$.

When no confusion is likely to arise we may abbreviate (X, Δ_X, Φ_X) to simply X, and drop subscripts. Here $\operatorname{supp} \Delta_X$ denotes the **support** of Δ_X (if $\Delta_X = \sum_i a_i D_i$ where the D_i are prime divisors, then $\operatorname{supp} \Delta_X = \cup_i D_i$). Similarly let $j_X : \operatorname{supp} \Delta_X \to X$ denote the evident inclusion.

Observation 5.16. U_X inherits a family of supports from X, namely

$$\Phi_{U_X} := \iota_X^{-1}(\Phi_X) \tag{5.17}$$

Moreover $\iota_X : (U_X, \Phi_{U_X}) \to (X, \Phi_X)$ is a *pulling* morphism (but generally not a pushing morphism) From now on we will promote the interior of X to the scheme with supports (U_X, Φ_{U_X}).

Definition 5.18. A **pulling morphism** $f : (X, \Delta_X, \Phi_X) \to (Y, \Delta_Y, \Phi_Y)$ **of snc pairs with supports** is a pulling morphism $f : X \to Y$ of underlying schemes with support such that[9] $f^{-1}(\operatorname{supp} \Delta_Y) \subset \operatorname{supp} \Delta_X$. A **pushing morphism** $f : (X, \Delta_X, \Phi_X) \to (Y, \Delta_Y, \Phi_Y)$ **of snc pairs with supports** is a pushing morphism of underlying schemes with support such that $f^*\Delta_Y = \Delta_X$.

Definition 5.19 (conventions)**.** A morphism of snc pairs with supports $f : (X, \Delta_X, \Phi_X) \to (Y, \Delta_Y, \Phi_Y)$ is flat, proper, an immersion, etc. if and only if the same is true of the induced morphism $f|_{U_X} : U_X \to U_Y$. A diagram of snc pairs with supports

$$\begin{tikzcd}
(X', \Delta_{X'}, \Phi_{X'}) \arrow[r, "g'"] \arrow[d, "f'"'] & (X, \Delta_X, \Phi_X) \arrow[d, "f"] \\
(Y', \Delta_{Y'}, \Phi_{Y'}) \arrow[r, "g"'] & (Y, \Delta_Y, \Phi_Y)
\end{tikzcd} \tag{5.20}$$

is **cartesian** if and only if the induced diagram of interiors

$$\begin{tikzcd}
U_{X'} \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & U_X \arrow[d, "f"] \\
U_{Y'} \arrow[r, "g"'] & U_Y
\end{tikzcd} \tag{5.21}$$

is cartesian.

The terminology is meant to suggest that pushing (resp. pulling) morphisms induce pushforward (resp. pullback) maps on log Hodge cohomology, as we now describe.

Let (X, Δ_X) be a log-smooth pair, let $U_X = X \setminus \Delta_X$ and let $\iota_X : U_X \to X$ be the inclusion. Let $\Omega_X^\bullet$ be the de Rham complex of X and recall that while each term Ω_X^p is a locally free coherent sheaf, $\Omega_X^\bullet$ is only a complex of sheaves of k-vector spaces (the differential d is k-linear and satisfies the Leibniz rule

$$d(f\sigma) = df \wedge \sigma + f d\sigma$$

where f and σ are local sections of $\mathcal{O}_X$ and Ω_X^p respectively). The same remarks apply to the de Rham complex $\Omega_{U_X}^\bullet$. Since $\Omega_{U_X}^\bullet$ is a complex on U_X, by functoriality $\iota_{X*}\Omega_{U_X}^\bullet$ is a complex on X and adjunction gives a natural morphism of complexes $d\iota^\vee : \Omega_X^\bullet \to \iota_{X*}\Omega_{U_X}^\bullet$

[9] In slogan form: "f maps the interior to the interior."

Proposition 5.22. *Let $\mathcal{F}$ be a sheaf on a noetherian normal scheme and let D be an effective Cartier divisor on X; let $U := X \setminus D$. Then there is a natural isomorphism*

$$\operatorname*{colim}_{r\to\infty} \mathcal{F}(rD) \xrightarrow{\simeq} \iota_{X*}(\mathcal{F}|_U) \tag{5.23}$$

Proposition 5.22 gives isomorphisms $\iota_{X*}\Omega^p_{U_X} \simeq \operatorname{colim} \Omega^p_X(r \operatorname{supp} \Delta_X)$ and so in particular there are natural morphisms of sheaves $\Omega^p_X(r \operatorname{supp} \Delta_X) \to \iota_{X*}\Omega^p_{U_X}$, for all p and all $r \geq 0$. At least in the context at hand, where X is smooth and Δ_X has simple normal crossings, these natural maps are injective.

Definition 5.24 (cf. [Del71])**.** The complex $\Omega^\bullet_X(\log \Delta_X)$ of **differential forms on X with log poles along Δ_X** is the *largest* subcomplex of $\iota_{X*}\Omega^\bullet_{U_X}$ such that

$$\Omega^p_X(\log \Delta_X) \subset \Omega^p_X(\operatorname{supp} \Delta_X) \text{ for all } p$$

More explicitly, on a neighborhood $W \subset X$ a local section $\sigma \in \Omega^p_X(\log \Delta)(W)$ is a section $\sigma \in \iota_{X*}\Omega^p_{U_X}(W)$ such that $\sigma \in \Omega^p_X(\operatorname{supp} \Delta_X)(W)$ *and* $d\,\sigma \in \Omega^{p+1}_X(\operatorname{supp} \Delta_X)(W)$ so that less formally but more memorably,

$$\Omega^p_X(\log \Delta) = \{\sigma \in \iota_{X*}\Omega^p_{U_X} \mid \sigma \in \Omega^p_X(\operatorname{supp} \Delta_X) \text{ and } d\,\sigma \in \Omega^{p+1}_X(\operatorname{supp} \Delta_X)\} \tag{5.25}$$

Let $z_1, z_2, \ldots, z_n$ be local coordinates at a point $x \in X$ such that

$$\operatorname{supp} \Delta_X = V(z_1 z_2 \cdots z_r)$$

in a neighborhood of x (the existence of such local coordinates it essentially the *definition* of the simple normal crossing condition given in [Kol13]). Recall that as X is smooth the differentials $d\,z_1, d\,z_2, \ldots, d\,z_n$ freely generate Ω_X on a neighborhood of x. In this situation we have the following useful description of $\Omega_X(\log \Delta)_X$):

Lemma 5.26 (see e.g. [EV92])**.** *The sections $\frac{d\,z_1}{z_1}, \ldots, \frac{d\,z_r}{z_r}, d\,z_{r+1}, \ldots, d\,z_n$ freely generate $\Omega_X(\log \Delta_X)$ on a neighborhood of x. For every p the natural map*

$$\wedge^p \Omega_X(\log \Delta_X) \to \Omega^p_X(\log \Delta_X)$$

is an isomorphism.

Definition 5.27. The **log-Hodge cohomology with supports** of a log-smooth pair with supports (X, Δ_X, Φ_X) is defined by

$$H^d(X, \Delta_X, \Phi_X) = \bigoplus_{p+q=d} H^q_\Phi(X, \Omega^p_X(\log \Delta_X)) \tag{5.28}$$

Here H^q_Φ denotes local cohomology with respect to the family of supports Φ_X. For connected X, we *define* $H_d(X, \Delta_X, \Phi_X) := H^{2\dim X - d}(X, \Delta_X, \Phi_X)$, and in general we set $H_d(X, \Delta_X, \Phi_X) = \bigoplus_i H_d(X_i, \Delta_{X_i}, \Phi_{X_i})$ where X_i are the connected components of X.

Let $f : (X, \Delta_X, \Phi_X) \to (Y, \Delta_Y, \Phi_Y)$ be pulling morphism of snc pairs with supports.

Lemma 5.29. *The map f induces a morphism of complexes of sheaves of k-vector spaces*

$$\begin{aligned} &f^*\Omega^\bullet_Y(\log \Delta_Y) \xrightarrow{d\,f^\vee} \Omega^\bullet_X(\log \Delta_X) \textit{ adjoint to a morphism} \\ &f^*\Omega^\bullet_Y(\log \Delta_Y) \xrightarrow{df^\vee} \Omega^\bullet_X(\log \Delta_X) \end{aligned} \tag{5.30}$$

fitting into the following commutative diagram:

$$\begin{tikzcd} f_*\iota_{X*}\Omega^\bullet_{U_X} & f_*\Omega^\bullet_X(\log\Delta_X) \arrow[l] & f_*\Omega^\bullet_X \arrow[l] \\ \iota_{Y*}\Omega^\bullet_{U_Y} \arrow[u, "df|_U^\vee"] \arrow[ur, phantom, "\circlearrowleft"] & \Omega^\bullet_Y(\log\Delta_Y) \arrow[l] \arrow[u, "df^\vee"] \arrow[ur, phantom, "\circlearrowleft"] & \Omega^\bullet_Y \arrow[l] \arrow[u, "df^\vee"'] \end{tikzcd} \tag{5.31}$$

of complexes of k-vector spaces on Y.

The essential content of this lemma is that when we pull back a log differential form σ on (Y,Δ_Y), it doesn't *develop* poles of order ≥ 1 along Δ_X. To see why, it's illuminating to look at the following 2 examples:

Example 5.32. Consider the morphism of pairs $f:(\mathbb{A}^1_z,0)\to(\mathbb{A}^1_z,0)$ defined by $f(z)=z^n$, where $n\in\mathbb{Z}, n\neq 0$. When we pull back $\frac{dz}{z}$, we get

$$\frac{d\,(f(z))}{f(z)}=\frac{d(z^n)}{z^n}=n\cdot\frac{dz}{z} \tag{5.33}$$

Of course, if char $k|n$ this is 0, but regardless it has a pole of order ≤ 1 at $0\in\mathbb{A}^1$.

Example 5.34. Take the pair $(\mathbb{A}^2_x, L_1+L_2)$, where $L_i=V(x_i)$ for $i=1,2$ and blow up the origin to obtain $\mathrm{Bl}_0(\mathbb{A}^2)$; let $\pi:\mathrm{Bl}_0(\mathbb{A}^2)\to\mathbb{A}^2$ be the projection, let $E\subset\mathrm{Bl}_0(\mathbb{A}^2)$ be the exceptional divisor and let $\tilde{L}_1,\tilde{L}_2\subset\mathrm{Bl}_0(\mathbb{A}^2)$ be the strict transforms of L_1,L_2 respectively. We obtain a morphism of pairs

$$\pi:(\mathrm{Bl}_0(\mathbb{A}^2),\tilde{L}_1+\tilde{L}_2+E)\to(\mathbb{A}^2,L_1+L_2) \tag{5.35}$$

Note that with $\tilde{U}:=\mathrm{Bl}_0(\mathbb{A}^2)\setminus(\tilde{L}_1+\tilde{L}_2+E)$ and $U:=\mathbb{A}^2\setminus(L_1+L_2)$, we have $\pi(\tilde{U})\subset U$ (this would *not* hold if we didn't include E in the divisor on $\mathrm{Bl}_0(\mathbb{A}^2)$).

Now let's pull back $\frac{dx_1}{x_1}$: recall that

$$\mathrm{Bl}_0(\mathbb{A}^2)=V(x_1y_2-x_2y_1)\subset\mathbb{A}^2_x\times\mathbb{P}^1_y$$

On the $D(y_1)\subset\mathrm{Bl}_0(\mathbb{A}^2)$ affine neighborhood, π looks like

$$\begin{aligned}\mathbb{A}^2_{x_1,y_2}\simeq D(y_1)\xrightarrow{\pi}\mathbb{A}^2_{x_1,x_2}&\text{ sending}\\ (x_1,y_2)&\mapsto(x_1,x_1y_2)\end{aligned} \tag{5.36}$$

(note that the exceptional divisor corresponds to $V(x_1)\subset\mathbb{A}^2_{x_1,y_2}$, i.e. the y_2-axis). So, the pullback of $\frac{dx_1}{x_1}$ is still $\frac{dx_1}{x_1}$, but the pullback of $\frac{dx_2}{x_2}$ is

$$\frac{d(x_1y_2)}{x_1y_2}=\frac{dx_1}{x_1}+\frac{dy_2}{y_2}$$

We see that $d\pi^\vee(\frac{dx_2}{x_2})$ has a pole of order 1 along E.

Proof. Note that since $f(U_X)\subset U_Y$, $U_X\subset f^{-1}(U_Y)$.

Case 1 ($U_X=f^{-1}(U_Y)$): in this case we have a *cartesian* diagram

$$\begin{tikzcd} U_X \arrow[r, hook] \arrow[d, "f|_U"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "f"] \\ U_Y \arrow[r, hook] & Y \end{tikzcd} \tag{5.37}$$

First, functoriality of the de Rham complex yields morphisms

$$df|_{U_X}^\vee:\Omega^\bullet_{U_Y}\to f|_{U_X*}\Omega^\bullet_{U_X}\text{ and }df^\vee:\Omega^\bullet_Y\to f_*\Omega^\bullet_X \tag{5.38}$$

where $df|_{U_X}^\vee$ is the restriction of $df^\vee$ in the sense that applying ι_Y^* to $df^\vee$ and using the isomorphism

$$\iota_Y^* f_* \Omega_X^\bullet \simeq f_{U_X *} \iota_X^* \Omega_X^\bullet = f|_{U_X *} \Omega_{U_X}^\bullet$$

obtained from flat base change [10] yields $df|_U^\vee$. From this we obtain a commutative diagram

$$\begin{tikzcd} f_* \iota_{X*} \Omega_{U_X}^\bullet & f_* \Omega_X^\bullet \arrow[l] \\ \iota_{Y*} \Omega_{U_Y}^\bullet \arrow[u, "df|_{U_X}^\vee"] \arrow[ur, phantom, "\circlearrowleft"] & \Omega_Y^\bullet \arrow[l] \arrow[u, "df^\vee"'] \end{tikzcd} \tag{5.39}$$

Finally commutativity of diagram 5.37 provides an isomorphism

$$f_* \iota_{X*} \Omega_{U_X}^\bullet \simeq \iota_{Y*} f|_{U_X *} \Omega_{U_X}^\bullet \tag{5.40}$$

Case 2 ($U_X \subset f^{-1}(U_Y)$): Since $U_X \subset f^{-1}(U_Y)$ we have a natural restriction

$$\iota_{X*} \Omega_{f^{-1}(U_Y)}^\bullet \to \iota_{X,*} \Omega_{U_X}^\bullet$$

In either case, we obtain a commutative diagram of complexes of k-vector spaces as in equation 5.39. Finally we must check that the composition

$$\Omega_Y^p(\log \Delta_Y) \to \iota_{Y*} \Omega_{U_Y}^p \xrightarrow{df|_{U_X}^\vee} f_* \iota_{X*} \Omega_{U_X}^p \tag{5.41}$$

(where the second map $df|_{U_X}^\vee$ is taken from diagram 5.39) factors through $f_* \Omega_X^p(\log \Delta_X) \subset f_* \iota_{X*} \Omega_{U_X}^p$. This is a local calculation: say $x \in X$ is a closed point and let $y = f(x) \in Y$. From lemma 5.29, if $z_1, \dots, z_n$ are local coordinates at y so that $\Delta_Y = V(z_1 \cdot z_2 \cdots z_r)$ in a neighborhood of y, then the local sections

$$\frac{dz_1}{z_1}, \dots, \frac{dz_r}{z_r}, dz_{r+1}, \dots, dz_n \text{ freely generate } \Omega_Y^1(\log \Delta_Y) \text{ at } y.$$

From the same lemma, we know the natural maps

$$\bigwedge^p \Omega_Y^1(\log \Delta_Y) \xrightarrow{\simeq} \Omega_Y^p(\log \Delta_Y) \text{ and } \bigwedge^p \Omega_X^1(\log \Delta_X) \xrightarrow{\simeq} \Omega_X^p(\log \Delta_X) \tag{5.42}$$

are isomorphisms, and in this way we reduce to showing:

$$\text{For } i = 1, \dots, r, \text{ the local section } df|_U^\vee\left(\frac{dz_i}{z_i}\right) \text{ factors through } \Omega_X^1(\log \Delta_X) \tag{5.43}$$

Getting even more explicit, say $\tilde{z}_1, \dots, \tilde{z}_m$ are local coordinates at x such that $\Delta_X = V(\tilde{z}_1 \cdot \tilde{z}_2 \cdots \tilde{z}_q)$ in a neighborhood of x.

Claim 5.44.

$$f^*(z_i)(= z_i \circ f) = u \tilde{z}_1^{a_i} \cdot \tilde{z}_2^{a_2} \cdots \tilde{z}_q^{a_q} \tag{5.45}$$

where u is nowhere-vanishing on a neighborhood of x and the a_i are non-negative integers to be described below. *Given* claim 5.44, we obtain the following calculation:

$$df|_U^\vee \frac{dz_i}{z_i} = \frac{df^* z_i}{f^* z_i} = \frac{d\,(u \tilde{z}_1^{\nu_1} \cdots \tilde{z}_q^{\nu_q})}{(u \tilde{z}_1^{\nu_1} \cdots \tilde{z}_q^{\nu_q})} = \frac{du}{u} + \sum_{i=1}^q \nu_i \frac{d\tilde{z}_i}{z_i} \tag{5.46}$$

Since u is nowhere-vanishing at x, the first term $\frac{du}{u}$ has no poles near x, and appealing once more to lemma 5.26 we have verified equation 5.43.

[10] Here is where we use the fact that diagram 5.37 is cartesian and ι_X is flat (it's an open immersion)

Proof of (5.45). By hypothesis,

$$\operatorname{supp} f^{-1}(\Delta_Y) \subset \operatorname{supp}\Delta_X, \text{ so locally } \operatorname{supp} f^{-1}(V(\prod_{i=1}^{r} z_i)) \subset \operatorname{supp} V(\prod_{i=1}^{q} \tilde{z}_i)$$

Since $V(z_i) \subset \Delta_Y$, it must be that

$$\operatorname{supp} V(z_i \circ f) = \operatorname{supp} f^{-1}(V(z_i)) \subset \operatorname{supp} f^{-1}(V(\prod_{i=1}^{r} z_i)) \subset \operatorname{supp} V(\prod_{i=1}^{q} \tilde{z}_i) \tag{5.47}$$

So, $V(z_i \circ f)$ is a divisor with support contained in $\operatorname{supp} V(\prod_{i=1}^{q} \tilde{z}_i)$. For each j, let $\eta_j \in X$ be the generic point of $V(z_j)$, and recall O_{X,η_j} is a discrete valuation ring; let ν_j be its discrete valuation. Now set

$$a_j = \nu_j(z_i \circ f) \text{ for } ij = 1, 2, \ldots, q \tag{5.48}$$

Then by construction, $z_i \circ f$ and $\prod_j^q \tilde{z}_j^{a_j}$ are 2 local sections of O_X at x with the same associated divisor, so they must differ by a unit, say $u \in O_{X,x}^{\times}$. □

□

Combining the previous lemma with proposition 5.11 we find:

Proposition 5.49. *For every pulling morphism $f : (X, \Delta_X, \Phi_X) \to (Y, \Delta_Y, \Phi_Y)$ in PS^* there are natural morphisms*

$$R\underline{\Gamma}_{\Phi}\Omega_Y^p(\log \Delta_Y) \to Rf_* R\underline{\Gamma}_{\Phi}\Omega_Y^p(\log \Delta_Y) \textit{ for all } p \tag{5.50}$$

Proof. Combining the morphism $\Omega_Y^p(\log \Delta_Y) \to f_*\Omega_X^p(\log \Delta_X)$ of (5.30) with the natural map in the derived category $f_*\Omega_X^p(\log \Delta_X) \to Rf_*\Omega_X^p(\log \Delta_X)$ (coming from the fact that $f_*\Omega_X^p(\log \Delta_X)$ is the bottom non-0 cohomology sheaf of $Rf_*\Omega_X^p(\log \Delta_X)$) gives a functorial morphism $\Omega_Y^p(\log \Delta_Y) \to Rf_*\Omega_X^p(\log \Delta_X)$. Taking sections with support along Φ_Y we obtain

$$R\underline{\Gamma}_{\Phi_Y}\Omega_Y^p(\log \Delta_Y) \to R\underline{\Gamma}_{\Phi_Y} Rf_*\Omega_X^p(\log \Delta_X)$$

Composing with the natural morphism

$$R\underline{\Gamma}_{\Phi_Y} Rf_*\Omega_X^p(\log \Delta_X) \to Rf_* R\underline{\Gamma}_{\Phi_X}\Omega_X^p(\log \Delta_X)$$

obtained from the inclusion $f^{-1}(\Phi_Y) \subset \Phi_X$ completes the proof. □

Corollary 5.51. *For each p there are functorial homomorphisms*

$$f^* : H_{\Phi}^q(Y, \Omega_Y^p(\log \Delta_Y)) \to H_{\Phi}^q(X, \Omega_X^p(\log \Delta_X)) \tag{5.52}$$

and hence (summing over $p + q = d$) functorial homomorphisms

$$f^* : H^d(X, \Delta_X, \Phi_X) \to H^d(Y, \Delta_Y, \Phi_Y) \tag{5.53}$$

The maps $f_* : H_d(X, \Delta_X, \Phi_X) \to H_d(Y, \Delta_Y, \Phi_Y)$ induced by a pushing morphism $f : (X, \Delta_X, \Phi_X) \to (Y, \Delta_Y, \Phi_Y)$ will be obtained from a combination of Nagata compactification and Grothendieck duality.

Theorem 5.54 (Grothendieck duality, [R&D], [Con00])**.** *Let $f : X \to Y$ be a proper morphism of finite-dimensional noetherian schemes admitting dualizing complexes $\omega_X^\bullet$ and $\omega_Y^\bullet$ respectively (for example X and Y could be schemes of finite type over k). Then for any object $\mathcal{F}^*$ in the bounded derived category $D_c^b(X)$ of X there is a natural isomorphism*

$$Rf_* R\underline{Hom}_X(\mathcal{F}^*, \omega_X^\bullet) \simeq R\underline{Hom}_Y(Rf_*\mathcal{F}^*, \omega_Y^\bullet) \textit{ in } D_c^b(Y)$$

Lemma 5.55. *Let $f : (X, \Delta_X) \to (Y, \Delta_Y)$ be a morphism of equidimensional log-smooth pairs such that the map $X \xrightarrow{f} Y$ of underlying schemes is proper. Then for each p there are natural morphisms of complexes of coherent sheaves*

$$Rf_*(\Omega_X^{\dim X-p}(\log \Delta_X)(f^*\Delta_Y - \Delta_X)) \to \Omega_Y^{\dim Y-p}(\log \Delta_Y)[\operatorname{codim} f] \tag{5.56}$$

where $\operatorname{codim} f := \dim Y - \dim X$, *inducing maps on cohomology*

$$f_* : H^q(X, \Omega_X^{\dim X-p}(\log \Delta_X)(-\Delta_X + f^*\Delta_Y)) \to H^{q+\operatorname{codim} f}(Y, \Omega_Y^{\dim Y-p}(\log \Delta_Y)) \tag{5.57}$$

for all q. Alternatively, reindexing like $p \leftarrow \dim X - p$, we can write these as

$$\begin{aligned} &Rf_*(\Omega_X^p(\log \Delta_X)(f^*\Delta_Y - \Delta_X)) \to \Omega_Y^{p+\operatorname{codim} f}(\log \Delta_Y)[\operatorname{codim} f] \text{ and} \\ &H^q(X, \Omega_X^p(\log \Delta_X)(-\Delta_X + f^*\Delta_Y)) \to H^{q+\operatorname{codim} f}(Y, \Omega_Y^{p+\operatorname{codim} f}(\log \Delta_Y)) \end{aligned} \tag{5.58}$$

In the proof, it will be convenient to work with objects of the form $\Omega_X^p(\log \Delta_X)[p]$ in $D(X)$ — this is not at all essential but it makes the indexing as symmetric as possible.

Proof. Since X and Y are smooth, we have

$$\omega_X^\bullet \simeq \omega_X[\dim X] \text{ and } \omega_Y^\bullet \simeq \omega_Y[\dim Y] \tag{5.59}$$

Grothendieck duality for the object $\Omega_X^p(\log \Delta_X)[p]$ in $D(X)$ says that

$$Rf_*\, R\underline{Hom}_X(\Omega_X^p(\log \Delta_X)[p], \omega_X[\dim X]) \simeq R\underline{Hom}_Y(Rf_*\Omega_X^p(\log \Delta_X)[p], \omega_Y[\dim Y]) \tag{5.60}$$

We now make a couple observations. Focusing first on the left hand side of equation 5.60 note that by lemma 5.26

- $\Omega_X^{\dim X}(\log \Delta_X) \simeq \omega_X(\Delta_X)$ and
- The pairing $\Omega_X^p(\log \Delta_X) \otimes \Omega_X^{\dim X-p}(\log \Delta_X) \to \omega_X(\Delta_X)$ is perfect. Equivalently (twisting by $-\Delta_X$) $\Omega_X^p(\log \Delta_X) \otimes \Omega_X^{\dim X-p}(\log \Delta_X)(-\Delta_X) \to \omega_X$ is perfect.

In this way we obtain an isomorphism

$$R\underline{Hom}_X(\Omega_X^p(\log \Delta_X), \omega_X) \xrightarrow{\simeq} \Omega_X^{\dim X-p}(\log \Delta_X)(-\Delta_X) \tag{5.61}$$

and hence introducing shifts on both sides an isomorphism

$$R\underline{Hom}_X(\Omega_X^p(\log \Delta_X)[p], \omega_X[\dim X]) \xrightarrow{\simeq} \Omega_X^{\dim X-p}(\log \Delta_X)(-\Delta_X)[\dim X - p] \tag{5.62}$$

Turning to the right hand side, note that the differential $f^*\Omega_Y^p(\log \Delta_Y) \to \Omega_X^p(\log \Delta_X)$ from lemma 5.29 is adjoint to a morphism $\Omega_Y^p(\log \Delta_Y) \to Rf_*\Omega_X^p(\log \Delta_X)$. Shifting by $[p]$ and applying $R\underline{Hom}_Y(-, \omega_Y[\dim Y])$ yields a morphism

$$\begin{aligned} R\underline{Hom}_Y(Rf_*\Omega_X(\log \Delta_X)[p], \omega_Y[\dim Y]) &\to R\underline{Hom}_Y(\Omega_Y(\log \Delta_Y)[p], \omega_Y[\dim Y]) \\ &\simeq \Omega_Y^{\dim Y-p}(\log \Delta_Y)(-\Delta_Y)[\dim Y - p] \end{aligned} \tag{5.63}$$

Putting everything together, we obtain a natural morphism

$$Rf_*(\Omega_X^{\dim X-p}(\log \Delta_X)(-\Delta_X)[\dim X - p]) \to \Omega_Y^{\dim Y-p}(\log \Delta_Y)(-\Delta_Y)[\dim Y - p] \tag{5.64}$$

Twisting by Δ_Y, applying the projection formula and shifting by $p - \dim X$ gives

$$Rf_*(\Omega_X^{\dim X-p}(\log \Delta_X)(f^*\Delta_Y - \Delta_X)) \to \Omega_Y^{\dim Y-p}(\log \Delta_Y)[\dim Y - \dim X] = \Omega_Y^{\dim Y-p}(\log \Delta_Y)[\operatorname{codim} f] \tag{5.65}$$

which is (5.56); the remaining statements of the lemma follow from taking global sections and reindexing. ☐

Lemma 5.66. *Suppose in addition that $f^*\Delta_Y - \Delta_X$ is effective. Then there is a natural morphism of complexes*

$$Rf_*(\Omega_X^{\dim X-p}(\log \Delta_X)) \to \Omega_Y^{\dim Y-p}(\log \Delta_Y)[\operatorname{codim} f] \tag{5.67}$$

inducing maps on cohomology

$$f_* : H^q(X, \Omega_X^p(\log \Delta_X)) \to H^{q+\operatorname{codim} f}(Y, \Omega_Y^{p+\operatorname{codim} f}(\log \Delta_Y)) \tag{5.68}$$

Proof. When $f^*(\Delta_Y) - \Delta_Y$ is effective, there's an inclusion

$$\Omega_X^{\dim X-p}(\log \Delta_X)(-\Delta_X + f^*\Delta_Y) \subseteq \Omega_X^{\dim X-p}(\log(\Delta_X))$$

□

The pushforward/pullback morphisms f_*/f^* satisfy a *projection formula*.

Lemma 5.69. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle f'} & \square & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of snc pairs with supports, where f, f' (resp. g, g') are pushing (resp. pulling) morphisms and g is either flat or a closed immersion transverse to f. Then

$$g^* f_* = f'_* g'^* : H^*(X, \Delta_X, \Phi_X) \to H^*(Y', \Delta_{Y'}, \Phi_{Y'}).$$

Proof. **Under construction. (follows along the lines of [CR11, Prop. 2.3.7])** □

Following the approach of [CR11], the next step would be to construct a cycle class $\mathrm{cl}(Z) \in H^*_{\Phi_X}(X, \Omega^*_X(\log \Delta_X))$ for a subvariety $Z \subset X$ with $Z \in \Phi_X$. This is possible, and is carried out in [BPØ20, §9], however it seems that for compatibility with correspondences in the absence of additional finiteness/strictness conditions, a more refined cycle class would be needed. For this reason we turn now to log Hodge correspondences and then return to the issue of cycle classes.

5.2. **Correspondences.** Given snc pairs with familes of supports (X, Δ_X, Φ_X) and (Y, Δ_Y, Φ_Y) with dimensions d_X and d_Y, as in [CR11, §1.3] we may define a family of supports $P(\Phi_X, \Phi_Y)$ on $X \times Y$ by

$$\begin{aligned} P(\Phi_X, \Phi_Y) := \{&\text{closed subsets } Z \subseteq X \times Y \mid \mathrm{pr}_Y|_Z \text{ is proper and for all } W \in \Phi_X, \\ &\mathrm{pr}_Y(\mathrm{pr}_X^{-1}(W) \cap Z) \in \Phi_Y\} \end{aligned}$$

(the conditions of Definition 5.1 are straightforward to verify). For convenience we will let $\Delta_{X\times Y} := \mathrm{pr}_X^*\Delta_X + \mathrm{pr}_Y^*\Delta_Y$.

Lemma 5.70. *A class $\gamma \in H^j_{P(\Phi_X,\Phi_Y)}(X \times Y, \Omega^i_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X))$ defines homomorphisms*

$$\mathrm{cor}(\gamma) : H^q_{\Phi_X}(X, \Omega^p_X(\log \Delta_X)) \to H^{q+j-d_X}_{\Phi_Y}(Y, \Omega_Y^{p+i-d_X}(\log \Delta_Y))$$

by the formula $\mathrm{cor}(\gamma)(\alpha) := \mathrm{pr}_{Y*}(\mathrm{pr}_X^*(\alpha) \smile \gamma)$. Moreover if (Z, Δ_Z, Φ_Z) is another snc pair with supports and $\delta \in H^{j'}_{P(\Phi_Y,\Phi_Z)}(Y \times Z, \Omega^{i'}_{Y\times Z}(\log \Delta_{Y\times Z})(-\mathrm{pr}_Y^*\Delta_Y))$, then

$$\mathrm{pr}_{X\times Z*}(\mathrm{pr}^*_{X\times Y}(\gamma) \smile \mathrm{pr}^*_{Y\times Z}(\delta)) \in H^{j+j'-d_Y}_{P(\Phi_X,\Phi_Z)}(X \times Z, \Omega^{i+i'-d_Y}_{X\times Z}(\log \Delta_{X\times Z})(-\mathrm{pr}_X^*\Delta_X)) \text{ and}$$

$$\mathrm{cor}(\mathrm{pr}_{X\times Z*}(\mathrm{pr}^*_{X\times Y}(\gamma) \smile \mathrm{pr}^*_{Y\times Z}(\delta))) = \mathrm{cor}(\delta) \circ \mathrm{cor}(\gamma)$$

as homomorphisms $H^q_{\Phi_X}(X, \Omega^p_X(\log \Delta_X)) \to H^{q+j+j'-d_X-d_Y}_{\Phi_Z}(Z, \Omega_Z^{p+i+i'-d_X-d_Y}(\log \Delta_Z))$.

Proof. We make two observations: first, there are natural wedge product pairings[11]

$$\Omega^p_{X\times Y}(\log \Delta_{X\times Y}) \otimes \Omega^i_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X) \xrightarrow{\wedge} \Omega^{p+i}_{X\times Y}(\log \Delta_Y)$$

Second, essentially by the definition of $P(\Phi_X, \Phi_Y)$ the Künneth morphism on cohomology for the tensor product $\Omega^p_{X\times Y}(\log \Delta_{X\times Y}) \otimes \Omega^i_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X)$ can be enhanced with supports as

$$\begin{aligned} &H^q_{\mathrm{pr}_X^{-1}(\Phi_X)}(X\times Y, \Omega^p_{X\times Y}(\log \Delta_{X\times Y})) \otimes H^j_{P(\Phi_X,\Phi_Y)}(X\times Y, \Omega^i_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X)) \\ &\qquad \to H^{p+j}_{\Psi}(X\times Y, \Omega^p_{X\times Y}(\log \Delta_{X\times Y}) \otimes \Omega^i_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X)) \end{aligned}$$

where $\Psi := \{\text{closed subsets } Z \in X\times Y \mid \mathrm{pr}_Y|_Z \text{ is proper and } \mathrm{pr}_Y(Z) \in \Phi_Z\}$. Combining these 2 observations gives a pairing

$$\begin{aligned} &H^q_{\mathrm{pr}_X^{-1}(\Phi_X)}(X\times Y, \Omega^p_{X\times Y}(\log \Delta_{X\times Y})) \otimes H^j_{P(\Phi_X,\Phi_Y)}(X\times Y, \Omega^i_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X)) \\ &\qquad \xrightarrow{\smile} H^{p+j}_{\Psi}(X\times Y, \Omega^{p+i}_{X\times Y}(\log \Delta_Y)) \end{aligned}$$

Now note that $\mathrm{pr}_X : (X\times Y, \Delta_{X\times Y}, \mathrm{pr}_X^{-1}(\Phi_X)) \to (X, \Delta_X, \Phi_X)$ is a pulling morphism, so by Corollary 5.51 there is an induced map $\mathrm{pr}_X^* : H^q_{\Phi_X}(X, \Omega^p_X(\log \Delta_X)) \to H^q_{\mathrm{pr}_X^{-1}(\Phi_X)}(X\times Y, \Omega^p_{X\times Y}(\log \Delta_{X\times Y}))$. On the other hand since $\mathrm{pr}_Y : (X\times Y, \Delta_Y, \Psi) \to (Y, \Delta_Y, \Phi_Y)$ is a pushing morphism, Lemma 5.66 provides a morphism $\mathrm{pr}_{Y*} : H^{p+j}_\Psi(X\times Y, \Omega^{p+i}_{X\times Y}(\log \Delta_Y)) \to H^{q+j-d_X}_{\Phi_Y}(Y, \Omega^{p+i-d_X}_Y(\log \Delta_Y))$. Composing, we obtain the desired homomorphism

$$\begin{aligned} H^q_{\Phi_X}(X, \Omega^p_X(\log \Delta_X)) &\xrightarrow{\mathrm{pr}_X^*} H^q_{\mathrm{pr}_X^{-1}(\Phi_X)}(X\times Y, \Omega^p_{X\times Y}(\log \Delta_{X\times Y})) \\ &\xrightarrow{\smile\gamma} H^{p+j}_\Psi(X\times Y, \Omega^{p+i}_{X\times Y}(\log \Delta_Y)) \\ &\xrightarrow{\mathrm{pr}_{Y*}} H^{q+j-d_X}_{\Phi_Y}(Y, \Omega^{p+i-d_X}_Y(\log \Delta_Y)) \end{aligned}$$

For the "moreover" half of the lemma, we again begin with a certain wedge product pairing, this time on $X\times Y\times Z$:

$$\begin{aligned} &\Omega^i_{X\times Y\times Z}(\log \mathrm{pr}_{X\times Y}^*\Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X) \otimes \Omega^{i'}_{X\times Y\times Z}(\log \mathrm{pr}_{Y\times Z}^*\Delta_{Y\times Z})(-\mathrm{pr}_Y^*\Delta_Y) \\ &\xrightarrow{\wedge} \Omega^{i+i'}_{X\times Y\times Z}(\log \mathrm{pr}_{X\times Z}^*\Delta_{X\times Z})(-\mathrm{pr}_X^*\Delta_X) \end{aligned} \tag{5.71}$$

If $V \in P(\Phi_X, \Phi_Y), W \in P(\Phi_Y, \Phi_Z)$ then unravelling definitions we find:

- $\mathrm{pr}_{X\times Z}|_{\mathrm{pr}_{X\times Y}^{-1}(V)\cap \mathrm{pr}_{Y\times Z}^{-1}(W)}$ is proper and
- $\mathrm{pr}_{X\times Z}(\mathrm{pr}_{X\times Y}^{-1}(V) \cap \mathrm{pr}_{Y\times Z}^{-1}(W)) \in P(\Phi_X, \Phi_Z)$

so that the Künneth morphism on cohomology associated to the middle term of (5.71) can be enhanced with supports like

$$\begin{aligned} &H^j_{\mathrm{pr}_{X\times Y}^{-1}(P(\Phi_X,\Phi_Y))}(X\times Y\times Z, \Omega^i_{X\times Y\times Z}(\log \mathrm{pr}_{X\times Y}^*\Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X)) \\ &\otimes H^{j'}_{\mathrm{pr}_{Y\times Z}^{-1}(P(\Phi_Y,\Phi_Z))}(X\times Y\times Z, \Omega^{i'}_{X\times Y\times Z}(\log \mathrm{pr}_{Y\times Z}^*\Delta_{Y\times Z})(-\mathrm{pr}_Y^*\Delta_Y)) \\ &\to H^{j+j'}_\Sigma(X\times Y\times Z, \Omega^i_{X\times Y\times Z}(\log \mathrm{pr}_{X\times Y}^*\Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X) \otimes \Omega^{i'}_{X\times Y\times Z}(\log \mathrm{pr}_{Y\times Z}^*\Delta_{Y\times Z})(-\mathrm{pr}_Y^*\Delta_Y)) \end{aligned}$$

where $\Sigma := \{\text{closed sets } W \subseteq X\times Y\times Z \mid \mathrm{pr}_{X\times Z}|_W \text{is proper and } \mathrm{pr}_{X\times Z}(W) \in P(\Phi_X, \Phi_Z)\}$.

[11] This is perhaps easiest to see by a verification in local coordinates.

Since $\mathrm{pr}_{X\times Y} : (X\times Y\times Z, \mathrm{pr}^*_{X\times Y}\Delta_{X\times Y}, \mathrm{pr}^{-1}_{X\times Y}(P(\Phi_X,\Phi_Y))) \to (X\times Y, \Delta_{X\times Y}, P(\Phi_X,\Phi_Y))$ is a pulling morphism, Corollary 5.51 gives an induced morphism $\Omega^i_{X\times Y}(\log\Delta_{X\times Y}) \to Rf_*\Omega^i_{X\times Y\times Z}(\log \mathrm{pr}^*_{X\times Y}\Delta_{X\times Y})$; twisting by $-\Delta_{X\times Y}$ and applying the projection formula gives a morphism

$$\Omega^i_{X\times Y}(\log\Delta_{X\times Y})(-\Delta_{X\times Y}) \to Rf_*\big(\Omega^i_{X\times Y\times Z}(\log \mathrm{pr}^*_{X\times Y}\Delta_{X\times Y})(-\mathrm{pr}^*_{X\times Y}\Delta_{X\times Y})\big)$$

and then taking cohomology with supports along $P(\Phi_X,\Phi_Y)$ and using Proposition 5.11 gives a modified pullback map

$$H^j_{P(\Phi_X,\Phi_Y)}(X\times Y, \Omega^i_{X\times Y}(\log\Delta_{X\times Y})(-\Delta_{X\times Y})) \to H^j_{\mathrm{pr}^{-1}_{X\times Y}(P(\Phi_X,\Phi_Y))}(X\times Y\times Z, \Omega^i_{X\times Y\times Z}(\log \mathrm{pr}^*_{X\times Y}\Delta_{X\times Y})(-\mathrm{pr}^*_X\Delta_X))$$

and a similar argument gives a modified pullback

$$H^{j'}_{P(\Phi_Y,\Phi_Z)}(Y\times Z, \Omega^{i'}_{Y\times Z}(\log\Delta_{Y\times Z})(-\Delta_{Y\times Z})) \to H^{j'}_{\mathrm{pr}^{-1}_{Y\times Z}(P(\Phi_Y,\Phi_Z))}(X\times Y\times Z, \Omega^{i'}_{X\times Y\times Z}(\log \mathrm{pr}^*_{Y\times Z}\Delta_{Y\times Z})(-\mathrm{pr}^*_X\Delta_Y))$$

On the other hand, $\mathrm{pr}_{X\times Z} : (X\times Y\times Z, \mathrm{pr}^*_{X\times Z}\Delta_{X\times Y}, \Sigma) \to (X\times Z, \Delta_{X\times Z}, P(\Phi_X,\Phi_Z))$ is a pushing morphism and hence by Lemma 5.66 induces morphisms

$$R\mathrm{pr}_{X\times Z*}R\underline{\Gamma}_\Sigma(\Omega^{\dim X\times Y\times Z-k}_{X\times Y\times Z}(\log \mathrm{pr}^*_{X\times Z}\Delta_{X\times Y})) \to R\underline{\Gamma}_{P(\Phi_X,\Phi_Z)}\Omega^{\dim X\times Z-k}_{X\times Z}(\log\Delta_{X\times Z})[-\dim Z]$$

for all k; twisting by $-\mathrm{pr}^*_X\Delta_X$ and applying the projection formula this becomes

$$R\mathrm{pr}_{X\times Z*}R\underline{\Gamma}_\Sigma(\Omega^{\dim X\times Y\times Z-k}_{X\times Y\times Z}(\log \mathrm{pr}^*_{X\times Z}\Delta_{X\times Y})(-\mathrm{pr}^*_X\Delta_X)) \to R\underline{\Gamma}_{P(\Phi_X,\Phi_Z)}\Omega^{\dim X\times Z-k}_{X\times Z}(\log\Delta_{X\times Z})(-\mathrm{pr}^*_X\Delta_X)[-\dim Z]$$

Now letting $k = \dim X\times Y\times Z - i - i'$, the induced morphisms of cohomology with supports are

$$H^{j+j'}_\Sigma(X\times Y\times Z, \Omega^{i+i'}_{X\times Y\times Z}(\log \mathrm{pr}^*_{X\times Z}\Delta_{X\times Y})(-\mathrm{pr}^*_X\Delta_X)) \to H^{j+j'-\dim Z}_{P(\Phi_X,\Phi_Z)}(X\times Z, \Omega^{i+i'-\dim Z}_{X\times Z}(\log\Delta_{X\times Z})(-\mathrm{pr}^*_X\Delta_X))$$

Combining the above ingredients, we obtain a bilinear pairing

$$\begin{aligned}&H^j_{P(\Phi_X,\Phi_Y)}(X\times Y, \Omega^i_{X\times Y}(\log\Delta_{X\times Y})(-\Delta_{X\times Y})) \otimes H^{j'}_{P(\Phi_Y,\Phi_Z)}(Y\times Z, \Omega^{i'}_{Y\times Z}(\log\Delta_{Y\times Z})(-\Delta_{Y\times Z}))\\ &\to H^{j+j'-\dim Z}_{P(\Phi_X,\Phi_Z)}(X\times Z, \Omega^{i+i'-\dim Z}_{X\times Z}(\log\Delta_{X\times Z})(-\mathrm{pr}^*_X\Delta_X))\end{aligned}$$

sending $\gamma\otimes\delta \mapsto \mathrm{pr}_{X\times Z*}(\mathrm{pr}^*_{X\times Y}(\gamma) \smile \mathrm{pr}^*_{Y\times Z}(\delta))$. It remains to be seen that

$$\mathrm{cor}(\mathrm{pr}_{X\times Z*}(\mathrm{pr}^*_{X\times Y}(\gamma) \smile \mathrm{pr}^*_{Y\times Z}(\delta))) = \mathrm{cor}(\delta)\circ\mathrm{cor}(\gamma)$$

and for this we will make repeated use of Lemma 5.69. Consider the diagram of smooth schemes

$$\begin{tikzcd}
 & & X\times Y\times Z \arrow[dl] \arrow[dr] & & \\
 & X\times Y \arrow[dl] \arrow[dr] \arrow[rr, phantom, "*"] & & Y\times Z \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}$$

where all morphisms are projections. There are various ways to enhance this to include supports; here we add the family of supports Ψ on $X\times Y$ defined above. Then in the cartesian diagram $(*)$, $\mathrm{pr}_Y : (X\times Y, \Psi) \to (Y, \Phi_Y)$ and $\mathrm{pr}_{Y\times Z} : (X\times Y\times Z, \mathrm{pr}^{-1}_{X\times Y}\Psi) \to (Y\times Z, \mathrm{pr}^{-1}_Y\Phi_Y)$ are pushing morphisms, whereas $\mathrm{pr}_{X\times Y}$ and pr_Y are pulling morphisms. At the same time, we have a pulling morphism $\mathrm{pr}_{X\times Z} : (X\times Y\times Z, \mathrm{pr}^{-1}_{X\times Z}(P(\Phi_Y,\Phi_Z))) \to (Y\times Z, P(\Phi_Y,\Phi_Z))$. To be precise in what follows, whenever ambiguity is possible we will use notation like $\mathrm{pr}^{X\times Y}_X$ to denote the projection $X\times Y\to X$, $\mathrm{pr}^{X\times Y\times Z}_X$ to denote the projection $X\times Y\times Z\to X$ and so on.

Applying the projection formula first to $\mathrm{pr}_{X\times Z}$ we see that

$$\mathrm{pr}_{Y\times Z*}(\mathrm{pr}^*_{X\times Y}(\mathrm{pr}^{X\times Y*}_X\alpha \smile \gamma) \smile \mathrm{pr}^*_{Y\times Z}\delta) = \mathrm{pr}_{Y\times Z*}(\mathrm{pr}^*_{X\times Y}(\mathrm{pr}^{X\times Y*}_X\alpha \smile \gamma)) \smile \delta$$

and then applying the projection formula to $(*)$ shows

$$\mathrm{pr}_{Y\times Z*}(\mathrm{pr}^*_{X\times Y}(\mathrm{pr}_X^{X\times Y*}\alpha \smile \gamma)) = \mathrm{pr}_Y^{Y\times Z*}(\mathrm{pr}^{X\times Y}_{Y*}(\mathrm{pr}_X^{X\times Y*}\alpha \smile \gamma)) = \mathrm{pr}_Y^{Y\times Z*}\operatorname{cor}(\gamma)(\alpha)$$

so that

$$\mathrm{pr}_{Y\times Z*}(\mathrm{pr}^*_{X\times Y}(\mathrm{pr}_X^{X\times Y*}\alpha \smile \gamma) \smile \mathrm{pr}^*_{Y\times Z}\delta) = \mathrm{pr}_Y^{Y\times Z*}\operatorname{cor}(\gamma)(\alpha) \smile \delta$$

Applying $\mathrm{pr}^{Y\times Z}_{Z*}$ we conclude that

$$(\operatorname{cor}\delta \circ \operatorname{cor}\gamma)(\alpha) = \mathrm{pr}^{X\times Y\times Z}_{Z*}(\mathrm{pr}_X^{X\times Y\times Z*}\alpha \smile \mathrm{pr}^*_{X\times Y}\gamma \smile \mathrm{pr}^*_{Y\times Z}\delta) \tag{5.72}$$

Finally, we rewrite the right hand side as

$$\mathrm{pr}^{X\times Z}_{Z*}\mathrm{pr}_{X\times Z*}(\mathrm{pr}^*_{X\times Z}\mathrm{pr}_X^{X\times Z*}\alpha \smile \mathrm{pr}^*_{X\times Y}\gamma \smile \mathrm{pr}^*_{Y\times Z}\delta)$$

and apply the projection formula to $\mathrm{pr}_{X\times Z}$ (with the pushing morphism $(X\times Y\times Z, \Sigma) \to (X\times Z, P(\Phi_X, \Phi_Z))$ and pulling morphism $(X\times Y\times Z, \mathrm{pr}_X^{X\times Y\times Z-1}(\Phi_X)) \to (X\times Z, \mathrm{pr}_X^{X\times Z-1}(\Phi_X)))$ to arrive at

$$\mathrm{pr}_{X\times Z*}(\mathrm{pr}^*_{X\times Z}\mathrm{pr}_X^{X\times Z*}\alpha \smile \mathrm{pr}^*_{X\times Y}\gamma \smile \mathrm{pr}^*_{Y\times Z}\delta) = \mathrm{pr}_X^{X\times Z*}\alpha \smile \mathrm{pr}_{X\times Z*}(\mathrm{pr}^*_{X\times Y}\gamma \smile \mathrm{pr}^*_{Y\times Z}\delta)$$

Applying $\mathrm{pr}^{X\times Z}_{Z*}$ on both sides shows that the right hand side of (5.72) is $\operatorname{cor}(\mathrm{pr}_{X\times Z*}(\mathrm{pr}^*_{X\times Y}\gamma \smile \mathrm{pr}^*_{Y\times Z}\delta)(\alpha)$, as desired. ◻

Remark 5.73. There is a Grothendieck-Serre dual approach to such correspondences, where classes $\gamma \in H^j_{P(\Phi_X,\Phi_Y)}(X\times Y, \Omega^i_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}^*_Y\Delta_Y))$ define homomorphisms

$$H^q(X, \Omega^p_X(\log \Delta_X)(-\Delta_X)) \to H^{q+j-d_X}(Y, \Omega_Y^{p+i-d_X}(\log \Delta_Y)(-\Delta_Y)).$$

The construction is formally similar.

5.3. **Attempts to construct a fundamental class of a thrifty birational equivalence.** Let $(X, \Delta_X), (Y, \Delta_Y)$ be simple normal crossing pairs, and assume in addition that X, Y are connected and proper. Let $Z \subseteq X\times Y$ be a smooth closed subvariety with codimension c. In this situation the fundamental class of $\mathrm{cl}(Z) \in H^c(X\times Y, \Omega^c_{X\times Y})$ (no log poles yet) can be described using only Serre duality, as follows: the composition

$$H^{\dim Z}(X\times Y, \Omega^{\dim Z}_{X\times Y}) \to H^{\dim Z}(Z, \Omega_Z^{\dim Z}) \xrightarrow{\mathrm{tr}} k \tag{5.74}$$

(where tr is the trace map of Serre duality) is an element of

$$H^{\dim Z}(X\times Y, \Omega^{\dim Z}_{X\times Y})^\vee \simeq H^c(X\times Y, \Omega^c_{X\times Y}) \tag{5.75}$$

which we may *define* to be $\mathrm{cl}(Z)$.[12] In light of Lemma 5.70 one might hope to modify eqs. (5.74) and (5.75) to obtain a class in $H^c(X\times Y, \Omega^c_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}^*_X\Delta_X))$. Let us focus on the case where

- $\mathrm{pr}_X|_Z : Z \to X$, $\mathrm{pr}_Y|_Z : Z \to Y$ are both thrifty and birational, so in particular $c = \dim X = \dim Y =: d$ and
- $(\mathrm{pr}_X|_Z)^{-1}_*\Delta_X = (\mathrm{pr}_Y|_Z)^{-1}_*\Delta_Y =: \Delta_Z$

To keep the notation under control, set $\pi_X := \mathrm{pr}_X|_Z$ and $\pi_Y := \mathrm{pr}_Y|_Z$.

In this situation letting $\iota : Z \to X\times Y$ be the inclusion there is a natural map

$d\iota^\vee : \Omega^d_{X\times Y}(\log \Delta_{X\times Y}) \to \iota_*\Omega^d_Z(\log \Delta_{X\times Y}|_Z)$ and twisting by $-\mathrm{pr}^*_Y\Delta_Y$ gives a map

$$\Omega^d_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}^*_Y\Delta_Y) \to \iota_*\Omega^d_Z(\log \Delta_{X\times Y}|_Z)(-\mathrm{pr}^*_Y\Delta_Y|_Z) = \iota_*\Omega^d_Z(\log \Delta_{X\times Y}|_Z)(-\pi^*_Y\Delta_Y)$$

[12]It may then be non-trivial to verify this agrees with other definitions, especially if one cares about signs, but we will not need that level of detail for what follows.

To identify $\Omega^d_Z(\log \Delta_{X\times Y}|_Z)(-\mathrm{pr}_X^*\Delta_X|_Z)$, write

$$(\pi_X)^*\Delta_X = (\pi_X)^{-1}_*\Delta_X + E_X = \Delta_Z + E_X \text{ and}$$
$$(\pi_Y)^*\Delta_Y = (\pi_Y)^{-1}_*\Delta_Y + E_Y = \Delta_Z + E_Y$$

so that $\Delta_{X\times Y}|_Z = (\pi_X)^*\Delta_X + (\pi_Y)^*\Delta_Y = 2\Delta_Z + E_X + E_Y$. While the hypotheses guarantee Δ_Z is reduced it may be that E_X, E_Y are non-reduced — however something can be said about their multiplicities. If $E_X = \sum_i a^i_X E^i_X, E_Y = \sum_i a^i_Y E^i_Y$ where the E^i_X, E^i_Y are irreducible, then by a generalization of [Har77, Prop. 3.6],

$$a^i_X = \mathrm{mlt}(\pi_X(E^i_X) \subseteq \Delta_X)$$

and since Δ_X is a reduced effective simple normal crossing divisor, if in addition we write $\Delta_X = \sum_i D^i_X$ $\mathrm{mlt}(\pi_X(E^i_X) \subseteq \Delta_X) = |\{i \mid \pi_X(E^i_X) \subseteq D^i_X\}|$. The thriftiness hypothesis that $\pi_X(E^i_X)$ is not a stratum then implies $a^i_X = \mathrm{mlt}(\pi_X(E^i_X) \subseteq \Delta_X) < \mathrm{codim}(\pi_X(E^i_X) \subset X)$. Since differentials with log poles are insensitive to multiplicities, we have

$$\Omega^d_Z(\log \Delta_{X\times Y}|_Z) = \omega_Z(\Delta_Z + E^{\mathrm{red}}_X + E^{\mathrm{red}}_Y)$$

where $-^{\mathrm{red}}$ denotes the associated reduced effective divisor. Then

$$\begin{aligned}\Omega^d_Z(\log \Delta_{X\times Y}|_Z)(-\pi_Y^*\Delta_Y) &= \omega_Z(\Delta_Z + E^{\mathrm{red}}_X + E^{\mathrm{red}}_Y - \Delta_Z - E_Y)\\ \omega_Z(E^{\mathrm{red}}_X + (E^{\mathrm{red}}_Y - E_Y)) &= \omega_Z(\sum_i E^i_X + \sum_i (1 - a^i_Y)E^i_Y)\end{aligned}$$

The upshot is that we have an induced map

$$H^d(X\times Y, \Omega^d_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_Y^*\Delta_Y)) \to H^d(Z, \omega_Z(E^{\mathrm{red}}_X + (E^{\mathrm{red}}_Y - E_Y))) \tag{5.76}$$

Here the left hand side is Serre dual to $H^d(X\times Y, \Omega^d_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X))$, so the k-linear dual of (5.76) is a morphism

$$H^d(Z, \omega_Z(E^{\mathrm{red}}_X + (E^{\mathrm{red}}_Y - E_Y)))^\vee \to H^d(X\times Y, \Omega^d_{X\times Y}(\log \Delta_{X\times Y})(-\mathrm{pr}_X^*\Delta_X))$$

Unfortunately[13] $H^d(Z, \omega_Z(E^{\mathrm{red}}_X + (E^{\mathrm{red}}_Y - E_Y)))$ is often 0. If E_X and E_Y are both reduced (an explicit example where this holds will be given below), then $H^d(Z, \omega_Z(E^{\mathrm{red}}_X + (E^{\mathrm{red}}_Y - E_Y))) = H^d(Z, \omega_Z(E_X))$. If in addition $E_X \neq 0$, we obtain $H^d(Z, \omega_Z(E_X)) = 0$ by an extremely weak (but characteristic independent) sort of Kodaira vanishing:

Lemma 5.77. *Let Z be a proper variety over a field k with dimension d, and assume Z is normal and Cohen-Macaulay. If $D \subset Z$ is a non-0 effective Cartier divisor on Z then $H^d(Z, \omega_Z(D)) = 0$.*

Proof. By Serre duality $H^d(Z, \omega_Z(D)) = H^0(Z, \mathcal{O}_Z(-D))$, which vanishes by the classic fact that "a nontrivial line bundle and its inverse can't both have non-0 global sections." Since I am not aware of a reference, here is a proof:

Suppose towards contraditction that there is a non-0 global section $\sigma \in H^0(Z, \mathcal{O}_Z(-D))$ — then the composition

$$\begin{tikzcd} \mathcal{O}_Z \arrow[r, "\sigma"] \arrow[rr, bend right, "\tau"'] & \mathcal{O}_Z(-D) \arrow[r, hook] & \mathcal{O}_Z \end{tikzcd}$$

is non-0. By [Stacks, Tag 0358] $H^0(Z, \mathcal{O}_Z)$ is a (normal) domain, and since it's also a finite dimensional k-vector space it must be an extension field of k. But then $\tau \in H^0(Z, \mathcal{O}_Z)$ is invertible hence surjective, so $\mathcal{O}_Z(-D) \hookrightarrow \mathcal{O}_Z$ is surjective, which is a contradiction since by hypothesis the cokernel $\mathcal{O}_D \neq 0$. □

[13] at least for the purposes of constructing log Hodge cohomology classes of subvarieties ...

Example 5.78. Let $X = \mathbb{P}^2$ and let $\Delta_X \subset X$ be a line. Let $p \in L$ be a k-point, let $Y = \mathrm{Bl}_p X$ and let $\Delta_Y = \tilde{L}$ = the strict transform of L. Finally let $f : Y \to X$ be the blowup map and let $Z = (f \times \mathrm{id})(Y) \subset X \times Y$. In this case (with all notation as above) $\pi_X \circ (f \times \mathrm{id}) = f$ and $\pi_Y \circ (f \times \mathrm{id}) = \mathrm{id}_Y$, so under the isomorphism $f \times \mathrm{id} : Y \simeq Z$, E_X is the exceptional divisor of f (with multiplicity 1). On the other hand $E_Y = 0$. In particular E_X and E_Y are reduced and $E_X \neq 0$ so from the above discussion $H^2(Z, \omega_Z(E_X)) = 0$.

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