POINCARE DUALITY VARIETIES

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1. Summary

If $X \subset \mathbb{P}^N$ is a smooth projective variety with dimension n and $C(X)$ is the projective cone over X, then if $C(X)$ satisfies Poincare duality over $\mathbb{Z}$ we must have $H^k(X;\mathbb{Z}) \simeq H^{k+2}(X;\mathbb{Z})$ for all k, and I think the multiplication by the class of a hyperplane gives the isomorphism. Similar statement for Poincare duality over $\mathbb{Q}$, with $\mathbb{Q}$-coefficients. When X is a hypersurface of degree $d > 1$ this is impossible, as is shown by an explicit calculation of the cohomology of X (or at least all of its Betti numbers).

However, if $d < N$, $C(X)$ has terminal singularities and when $N > 3$ X is $\mathbb{Q}$-factorial. Not sure about analytically $\mathbb{Q}$-factorial but I would guess so (we are only dealing with one isolated singulararity, and its a cone point...).

2. Poincare duality spaces

3. Intersection (co)homology

4. Example: (co)homology of cones

Let $X \subset \mathbb{P}^N$ be a smooth projective variety and let $C(X) \subset \mathbb{P}^{N+1}$ be the (projective) cone over X. We begin with a basic observation:

Proposition 1. The projective cone $C(X)$ is the Thom space of the geometric line bundle L on X associated to the invertible sheaf $\mathcal{O}_X(1)$.

Remark. I am following "Fulton" conventions for moving between locally free sheaves and vector bundles. This means that $\mathcal{O}_X(1)$ is the sheaf of local sections of L. If this irritates you ... sorry. In particular, L has a global section.

Proof. Recall that the Thom space $\mathrm{Th}(L)$ can be constructed as follows: start with the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_X(1) \oplus \mathcal{O}_X)$. It has 2 interesting global sections, σ_0, σ_∞ corresponding to the inclusions

$$X \simeq \mathbb{P}(\mathcal{O}_X) \subset \mathbb{P}(\mathcal{O}_X(1) \oplus \mathcal{O}_X) \text{ and}$$

$$X \simeq \mathbb{P}(\mathcal{O}_X(1)) \subset \mathbb{P}(\mathcal{O}_X(1) \oplus \mathcal{O}_X)$$

The difference between these global sections is that the normal bundle of $\sigma_0(X)$ can be identified with $\mathcal{O}_X(1)$ while the normal bundle of $\sigma_\infty(X)$ can be identified with $\mathcal{O}_X(-1)$. We have

$$\mathrm{Th}(L) = \mathbb{P}(\mathcal{O}_X(1) \oplus \mathcal{O}_X) / \mathbb{P}(\mathcal{O}_X(1))$$

(this may not be the most standard description, but see **CITE ATIYAH's K-THEORY HERE**). To see that this is the cone, blow up the vertex $p \in C(X)$ and observe that

- $\mathrm{Bl}_p C(X) \simeq \mathbb{P}(\mathcal{O}_X(1) \oplus \mathcal{O}_X)$ and
- The exceptional divisor $E \subset \mathrm{Bl}_p C(X)$ over p is exactly $\mathbb{P}(\mathcal{O}_X(1))$.

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This is just a projective version of the fact that the blowup of the *affine* cone $C_{\text{aff}}(X)$ at the vertex $p \subset C_{\text{aff}}(X)$ is the geometric line bundle $L^\vee$ associated to $\mathcal{O}_X(-1)$, with the exceptional divisor $E \subset C_{\text{aff}}(X)$ corresponding to the zero-divisor $X \subset L^\vee$.

□

Remark. Alternatively, view *points* $l \in X$ as *lines* $l \subset \mathbb{A}^{N+1}$. Then a vector in L_l is a linear functional $\lambda : l \to \mathbb{C}$. The *graph* of λ is a *line* $\lambda(l) \subset \mathbb{A}^{N+2}$, which we can view as a *point* $\lambda(l) \in \mathbb{P}^{N+1}$. Since omitting the last coordinate of $\lambda(l)$ gives back the line l, we see that in fact $\lambda(l) \subset C(X)$, and so we have a map

$$\varphi : L \to C(X)$$

At this point one checks that it's an isomorphism onto $C(X) \setminus \{p\}$, and as $\lambda \to \infty$, $\lambda(l) \to p$, so that φ extends to the one-point-compactification $\mathrm{Th}(L)$, yielding a homeomorphism $\mathrm{Th}(L) \simeq C(X)$.

Now let's recall the classic **INCLUDE A REFERENCE HERE**

Theorem 1 (Thom)**.** Let X be a reasonable space (say with the homotopy type of a CW complex) and let $E \xrightarrow{\pi} X$ be an oriented real vector bundle. Then there is a class $\tau(E) \in \tilde{H}^r(\mathrm{Th}(E);\mathbb{Z})$ generating $\tilde{H}(\mathrm{Th}(E);\mathbb{Z})$ as a free $H^*(X;\mathbb{Z})$-module of rank 1.

There is a parallel Thom isomorphism identifying $H_i(X;\mathbb{Z}) \simeq \tilde{H}_{i+r}(\mathrm{Th}(E);\mathbb{Z})$.

Remark. The $H^*(X;\mathbb{Z})$-module structure comes from the indentification $\tilde{H}^*(\mathrm{Th}(E);\mathbb{Z}) \simeq H^*(E, E \setminus X;\mathbb{Z})$.

Applying this result, we obtain

Proposition 2. There is a class $\tau(L) \in \tilde{H}^2(C(X);\mathbb{Z})$ generating $\tilde{H}^*(C(X);\mathbb{Z})$ as a free $H^*(X;\mathbb{Z})$-module of rank 1. Similarly there are identifications $H_i(X;\mathbb{Z}) \simeq \tilde{H}_{i+2}(C(X);\mathbb{Z})$.

Remark. In the matter at hand, the tildes translate to:

$$H^k(C(X);\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{if } k = 1 \\ H^{k-2}(X;\mathbb{Z}) & \text{if } k > 1 \end{cases}$$

Now: assuming X is smooth, we have a fundamental class $[X] \in H_{2n}(X;\mathbb{Z})$ (here n is the complex dimension of X) and Poincare duality states that the cap product with the fundamental class

$$H^k(X;\mathbb{Z}) \to H_{2n-k}(X;\mathbb{Z}) \text{ sending } \alpha \mapsto \alpha \cap [X]$$

is an isomorphism. We also have the universal coefficient formula, which provides exact sequences

$$0 \to \mathrm{Ext}^1(H_{k-1}(X;\mathbb{Z}),\mathbb{Z}) \to H^k(X;\mathbb{Z}) \to \mathrm{Hom}(H_k(X;\mathbb{Z}),\mathbb{Z}) \to 0$$

Of course, we can say much more about the general structure of $H^*(X;\mathbb{Z})$, using e.g. the hard Lefschetz theorem - more on that later.

Suppose for a minute that Poincare duality also holds on $C(X)$. Which is to say, we have isomorphisms

$$H^k(C(X);\mathbb{Z}) \simeq H_{2(n+1)-k}(C(X);\mathbb{Z})$$

presumably given by capping with a fundamental class. Note that the obvious choice of fundamental class would be the image of $[X]$ under the isomorphism $H_{2n}(X;\mathbb{Z}) \simeq H_{2(n+1)}(C(X);\mathbb{Z})$. This will place serious restrictions on the (co)homology of X, since we must have

$$H^k(X;\mathbb{Z}) \simeq H^{k+2}(C(X);\mathbb{Z}) \simeq H_{2(n+1)-k-2}(C(X);\mathbb{Z}) \simeq H_{2n-k-2}(X;\mathbb{Z})$$

Now Poincare duality on X provides an isomorphism

$$H_{2n-k-2}(X;\mathbb{Z}) \simeq H^{k+2}(X;\mathbb{Z})$$

and in this way we see that $H^k(X;\mathbb{Z}) \simeq H^{k+2}(X;\mathbb{Z})$ for all k. Also, it should be noted that since $H^1(C(X);\mathbb{Z}) = 0$ we must have $H_{2(n+1)-1}(C(X);\mathbb{Z}) = 0$ and hence $H_{2n-1}(X;\mathbb{Z}) = 0$, and so $H^1(X;\mathbb{Z}) = 0$. Since $H^0(X;\mathbb{Z}) = \mathbb{Z}$ we conclude that

$$H^k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

Remark. I am pretty sure that the isomorphism $H_{2n-k-2}(X;\mathbb{Z}) \simeq H^{k+2}(X;\mathbb{Z})$ obtained above coincides with multiplication by the Chern class $c_1(\mathcal{O}_X(1))$. Given $\alpha \in H^k(X;\mathbb{Z})$, we obtain $\alpha \smile \tau \in H^{k+2}(C(X);\mathbb{Z})$. From this we obtain $\alpha \smile \tau \cap [C(X)] \in H_{2(n+1)-k-2}(C(X);\mathbb{Z})$ and ... see here's where I really need to know the homology version of the Thom isomorphism. (Idea: this is the pullback of τ along the usual inclusion $X \subset C(X)$). Knowing this would put even further restrictions on X.

The basic example of this phenomenon is when $X \subset \mathbb{P}^n$ is a linear subspace, hence so is $C(X) \subset \mathbb{P}^{n+1}$. It's a little difficult to think of other such examples.

I'd like to also observe that our conditions on $H^*(X;\mathbb{Z})$ are not sufficient to guarantee Poincare duality for $H^*(C(X);\mathbb{Z}$. To see this, let $X \subset \mathbb{P}^2$ be a conic. Assuming the remark, Poincare duality for $C(X)$ would imply that multiplication by $c_1(\mathcal{O}_X(1))$ gives an isomorphism $\mathbb{Z} \simeq H^0(X;\mathbb{Z}) \simeq H^2(X;\mathbb{Z}) \simeq \mathbb{Z}$ which is false (it acts as multiplication by 2). Note however that if we worked over $\mathbb{Q}$ or a finite field k of characteristic not 2 (instead of $\mathbb{Z}$, multiplication by c_1 actually would give an isomorphism. The reason one should expect some funny business at the prime 2 in this example is that $C(X)$ is isomorphic to the quotient of $\mathbb{P}^2$ by the involution (a.k.a. $\mathbb{Z}/2$-action

$$t : \mathbb{P}^2 \to \mathbb{P}^2 \text{ sending } [x,y,z] \mapsto [-x,-y,z]$$

Similar remarks hold for rational normal curves of degree d, Veronese embeddings of $\mathbb{P}^n$, etc.

4.1. **The singularity class of a cone point.** I recall a simplified form of the criteria in Lemma 3.1 of *Singularities of the MMP*:

Proposition 3. Let $X \subset \mathbb{P}^N$ be a smooth projective variety. Then the projective cone $C(X) \subset \mathbb{P}^{N+1}$ is $\mathbb{Q}$-Gorenstein if and only if $r \cdot c_1(\mathcal{O}_X(1)) = K_X$ for some $r \in \mathbb{Q}$, and in this situation $C(X)$ is

- terminal if and only if $r < -1$,
- canonical if and only if $r \leq -1$,
- klt if and only if $r < 0$ and
- lc if and only if $r \leq 0$.

More precisely, if we resolve the singularities of $C(X)$ by blowing up the vertex, the discrepancy of the exceptional divisor $E \subset \mathrm{Bl}_0 C(X)$ is $-1-r$.

Some relevant corollaries, in no particular order:

Example 1. Suppose X is a degree d hypersurface. Then $\omega_X \simeq \mathcal{O}_X(d-N-1)$, and so we have

$$r \cdot c_1(\mathcal{O}_X(1)) = K_X \text{ with } r = d - N - 1$$

Hence we see that $C(X)$ is terminal when $d < N$, canonical when $d = N$ and lc when $d = N+1$. When $d > N+1$ it's not even lc.

One can generalize this example to complete intersections.

Example 2. More generally, a cone over an anti-canonically embedded Fano varieity is always at least klt. A cone over a variety with trivial canonical (e.g. a Calabi-Yau variety) is always at least lc.

4.2. **The link at a cone point.** Looking into any of the standard proofs of Poincare duality one sees that a key property of a manifold M exploited at various stages is that for any point $p \in M$,

$$H^k(M, M \setminus \{p\}; \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k = \dim M \\ 0 & \text{otherwise} \end{cases}$$

This property is axiomatized as follows: let X be a reasonable topological space (e.g. a CW-complex).

Definition 1. X is a **homology n-manifold** if and only if for every point $p \in X$,

$$H^k(X, X \setminus \{p\}; \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

If Y is an n-dimensional complex variety, then it is generically smooth, so it could only be a homology $2n$-manifold. Furthermore if $p \in Y$ is a point with a neighborhood $U \subset X$ that deformation-retracts onto p, then by excision $H^k(Y, Y \setminus \{p\}; \mathbb{Z}) \simeq H^k(U, U \setminus \{p\}; \mathbb{Z})$ and by the relative cohomology exact sequence $H^k(U, U \setminus \{p\}; \mathbb{Z}) \simeq H^{k-1}(U \setminus \{p\}; \mathbb{Z})$. If Y is an affine variety sitting in $\mathbb{C}^N$ (always the case locally) and $S_\epsilon(p)$ is a sphere of radius ϵ centered at p, then for suitably small U and ϵ one has $U \setminus \{p\} \approx S_\epsilon(p)$ where here $\approx$ denotes homotopy equivalence. . In this way we see that

$$H^k(Y, Y \setminus \{p\}; \mathbb{Z}) \simeq H^{k-1}(S_\epsilon(p); \mathbb{Z}) \text{ for all } k$$

Definition 2. The space $S_\epsilon(p)$ is called the **link of X at p**.

To justify the terminology "the" one shows that it is independent of ϵ for sufficiently small ϵ (up to homeomorphism, say).

Proposition 4. If $X \subset \mathbb{P}^N$ is a smooth projective variety and $C_a(X) \subset \mathbb{P}^{N+1}$ is the affine cone over X, with vertex $p \in C(X)$, then the link $S_\epsilon(p)$ is the S^1-bundel (a.k.a. circle bundle) associated to the invertible sheaf $\mathcal{O}_X(-1)$.

Proof. Let $\pi : \mathrm{Bl}_p C_a(X) \to C(X)$ be the blow-up of $C_a(X)$ at p. Recall that $\mathrm{Bl}_p C_a(X) \simeq L^\vee$, the geometric line bundle associated to $\mathcal{O}_X(-1)$, with exceptional divisor $E \simeq X$ corresponding to the 0-section. The preimage of a ϵ-sphere $S_\epsilon(p) \subset C_a(X)$ at p is the ϵ-sphere *bundle* of $L^\vee$. ☐

To relate the topology of $S_\epsilon(p)$ to that of X, we can use the long exact sequence on homotopy groups

$$\cdots \to \pi_i(S^1) \to \pi_i(S_\epsilon(p)) \to \pi_i(X) \xrightarrow{\partial} \pi_{i-1}(S^1) \to \cdots$$

Since $\pi_i(S^1) = 0$ for $i > 1$ and all the spaces are connected, this reduces to an exact sequence

$$0 \to \pi_2(S_\epsilon(p)) \to \pi_2(X) \to \mathbf{Z}$$

$$\to \pi_1(S_\epsilon(p)) \to \pi_1(X) \to \pi_0(S^1) \to 0$$

together with isomorphisms $\pi_i(S_\epsilon(p)) \simeq \pi_i(X)$ for $i > 2$. As for cohomology, we have a Gysin sequence of the form

$$\cdots \to H^{k-2}(X; \mathbb{Z}) \xrightarrow{-c_1} H^k(X; \mathbb{Z}) \xrightarrow{\pi^*} H^k(S_\epsilon(p); \mathbb{Z})$$

$$\to H^{k-1}(X; \mathbb{Z}) \to \cdots$$

where c_1 is the first Chern class of $\mathcal{O}_X(1)$ and $\pi : S_\epsilon(p) \to X$ is the projection.

Now let's recall a variant of the hard Lefschetz theorem **NOTE: MAKE SURE THE INDICES ARE EXACTLY RIGHT**:

Theorem 2 (Lefschetz)**.** Let X be a smooth projective variety of dimension n and let c_1 be its first Chern class. Then multiplication by c_1

$$H^k(X;\mathbb{Q}) \to H^{k+2}(X;\mathbb{Q})$$

is *injective* for $k < n$, and *surjective* for $k > n$.

Remark. This is only true with $\mathbb{Q}$ coefficients, as one can see by considering a rational normal curve of degree $d > 1$ (or more generally a Veronese embedding of degree $d > 1$). However via the universal coefficient theorem one obtains a statement about integral cohomology (below the middle dimension the kernel of c_1 is torsion, above the middle dimension the cokernel is torsion).

Remark. It's because of this theorem that the Hodge diamond is, well, a diamond.

Applying this theorem we see that after tensoring with $\mathbb{Q}$, for $k-2 < n$ the Gysin sequence breaks up into short exact sequence

$$0 \to H^{k-2}(X;\mathbb{Q}) \xrightarrow{c_1} H^k(X;\mathbb{Q}) \to H^k(S_\epsilon(p);\mathbb{Q}) \to 0$$

Similarly for $k-2 > n$ we have short exact sequences

$$0 \to H^{k-1}(S_\epsilon(p);\mathbb{Q}) \to H^{k-2}(X;\mathbb{Q}) \to H^k(X;\mathbb{Q}) \to 0$$

Example 3. Let's actually take a closer look at cone over a Veronese. Let $X \subset \mathbb{P}^N$ be the image of $\mathbb{P}^n$ under the d-th Veronese embedding, and let $C(X)$ be the cone over X, with vertex p. Then $\mathcal{O}_X(1) \simeq \mathcal{O}_{\mathbb{P}^n}(d)$ and so $c_1(\mathcal{O}_X(d) = dh$, where $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1)$. Hence the Gysin exact sequence looks like

$$\cdots \to H^{k-2}(\mathbb{P}^n;\mathbb{Z}) \xrightarrow{-dh} H^k(\mathbb{P}^n;\mathbb{Z}) \xrightarrow{\pi^*} H^k(S_\epsilon(p);\mathbb{Z})$$
$$\to H^{k-1}(\mathbb{P}^n;\mathbb{Z}) \to \cdots$$

Since $H^k(\mathbb{P}^n;\mathbb{Z}) = \mathbb{Z}$ generated by $h^{\frac{k}{2}}$ if k is even and 0 otherwise, and since multiplication by $-dh$ is always injective, we see that $H^k(S_\epsilon(p);\mathbb{Z}) = 0$ for k odd and we obtain short exact sequences

$$0 \to \mathbb{Z} \xrightarrow{-d} \mathbb{Z} \to H^k(S_\epsilon(p);\mathbb{Z}) \to 0$$

for k even, showing that $H^k(S_\epsilon(p);\mathbb{Z}) \simeq \mathbb{Z}/d$ for even k. This is not surprising since the description of $C(X)$ as a quotient of $\mathbb{P}^{n+1}$ by an action of μ_d (if $\zeta \in \mu_d$ is a primitive root, then it acts on $[x_0, \dots, x_{n+1}]$ like

$$\zeta \cdot [x_0, \dots, x_{n+1}] = [\zeta x_0, \dots, \zeta x_n, x_{n+1}];$$

the fixed point $[0, \dots, 0, 1]$ corresponds to the cone point) identifies $S_\epsilon(p)$ with a lens space obtained as the quotient of a free action of μ_d on S^{2n+1}!

4.3. **Singular cohomology of hypersurfaces.** To see how the above discussion plays out in some specific cases it will be nice to know the singular cohomology of smooth hypersurfaces (and more generally complete intersections). I actually don't know a reference for the ensuing calculations so I will just go for it.

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface and let $\iota : X \to \mathbb{P}^{n+1}$ be the inclusion. Recall

Theorem 3 (Lefschetz)**.** The restriction map $\iota^* H^k(\mathbb{P}^{n+1};\mathbb{Z}) \to H^k(X;\mathbb{Z})$ is injective for $k \leq n$ and an isomorphism for $k < n$.

Knowledge of the cohomology of $\mathbb{P}^{n+1}$ shows that for $k < n$

$$H^k(X;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

For simplicity I will assume n is (the case where n is odd is slightly more complicated). In that case we have an injection $\mathbb{Z} \to H^n(X;\mathbb{Z})$. Poincare duality together with the universal coefficient theorem then shows that $H^k(X;\mathbb{Z})$ is torsion-free for all k and for $k > n$,

$$H^k(X;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

The only thing left to do is compute the rank of $H^n(X;\mathbb{Z})$ (of course one might also want to know about the intersection form - maybe another day). The preceding discussion shows

$$\chi(X) = \sum_k \mathrm{rk} H^k(X;\mathbb{Z}) = n + \mathrm{rk} H^n(X;\mathbb{Z})$$

and so we just need to calculate $\chi(X)$. For this we can use the formula

$$\chi(X) = \int_X c_n(\tau_X)$$

the integral of the top Chern class of the tangent bundle. To get going on this integral, note that there is a short exact sequence of vector bundles on X

$$0 \to \tau_X \to \iota^* \tau_{\mathbb{P}^{n+1}} \to \mathcal{N}_{X|\mathbb{P}^{n+1}} \to 0$$

and hence

$$c(\tau_X) = \frac{\iota^* c(\tau_{\mathbb{P}^{n+1}})}{c(\mathcal{N}_{X|\mathbb{P}^{n+1}})}$$

From the Euler exact sequence on $\mathbb{P}^{n+1}$ we find that

$$c(\tau_{\mathbb{P}^{n+1}} = c(\mathcal{O}_{\mathbb{P}^{n+1}}(1))^{n+2} = (1+h)^{n+2}$$

and since $\mathcal{N}_{X|\mathbb{P}^{n+1}} \simeq \mathcal{O}_X(d)$ where $d = \deg X$, we compute

$$c(\tau_X) = \frac{(1+h)^{n+2}}{1+dh}$$

(where I am abusively dropping the ι^* in $\iota^* h$). We need to expand this as a power series in h:

$$\frac{(1+h)^{n+2}}{1+dh} = (\sum_j (-1)^j d^j h^j) \cdot (\sum_k \binom{n+2}{k} h^k)$$

$$= \sum_{j,k} (-1)^j d^j \binom{n+2}{k} h^{j+k}$$

and now recall that the integral will only pick off the degree n term: so, we find

$$\chi(X) = \sum_{j+k=n} (-1)^j d^j \binom{n+2}{k} \int_X h^n$$

and since $\int_X h^n = d$ this is just

$$\sum_{j+k=n} (-1)^j d^{j+1} \binom{n+2}{k} = \sum_{k=0}^{n} (-1)^{n-k} d^{n-k+1} \binom{n+2}{k}$$

$$= \frac{1}{d}((1-d)^{n+2} + (n+2)d - 1)$$

after a little bit of rearranging. Combining this with the formula $\chi(X) = n + \mathrm{rk} H^n(X;\mathbb{Z})$ we obtain

$$\mathrm{rk} H^n(X;\mathbb{Z}) = \frac{1}{d}((d-1)^{n+2} + (n+2)d - 1) - n$$

$$= \frac{(d-1)^{n+2}-1}{d} + n + 2 - n$$
$$= \frac{(d-1)^{n+2}-1}{d} + 2$$

If n is odd, the Chern class calculation is identical, but we have $\chi(X) = n + 1 - \mathrm{rk} H^n(X;\mathbb{Z})$, and so

$$\mathrm{rk} H^n(X;\mathbb{Z}) = n + 1 - \frac{1}{d}((1-d)^{n+2} + (n+2)d - 1)$$
$$= \frac{(d-1)^{n+2}+1}{d} - 1$$

as a reality check, note that when $n = 1$ we recover the classic formula for the genus g of a plane curve X in terms of its degree: for in that situation

$$2g = \mathrm{rk} H^1(X;\mathbb{Z}) = \frac{(d-1)^3+1}{d} - 1$$
$$= d^2 - 3d + 2 = (d-1)(d-2)$$

so that $g = \frac{(d-1)(d-2)}{2}$. Lovely! Note also that all the formulas for the rank output 1 when $d = 1$ (so $X = \mathbb{P}^n$), as they must.