HIGHER DIRECT IMAGES OF STRUCTURE SHEAVES OF WEAKLY ORDINARY VARIETIES IN EQUAL CHARACTERISTIC $p > 0$

CHARLIE GODFREY

Contents

1. Introduction

We begin by considering a theorem of Du Bois-Jarraud:

Theorem 1.1 ([Du 81, Thm. 4.6], see also [DJ74]). *If $f : X \to B$ is a flat proper morphism of schemes of finite type over $\mathbb{C}$, and if the geometric fibers of f are reduced with at worst Du Bois singularties, then the higher direct images of the structure sheaf $R^i f_* \mathcal{O}_X$ are locally free and compatible with arbitrary base change.*

While the definition of Du Bois singularties is notoriously technical, their usefulness stems from the fact that they include simultaneously normal crossing singularities and semi-log canonical singularities and enjoy some key Hodge-theoretic properties [Kol13, §6]. Theorem 1.1 has found various striking applications: for example, in [KK10, Thm. 1.8] it is used to show that for a family as above, the cohomology sheaves $h^i(\omega_f^\bullet)$ (including the relative dualizing sheaf $\omega_{X/B}$) are flat over B and compatible with base change. In a different direction, it was noticed by Kollár that [Du 81, Thm. 4.6] combined with a hypothetical strong form of semi-stable reduction would recover one of his theorems on higher direct images of dualizing sheaves [Kol86, Thm. 2.6 Rmk. 2.7].

Here variant of Theorem 1.1, which differs in 2 key aspects: first, it applies to flat proper families in characteristic $p > 0$, and second, it imposes global arithmetic restrictions (as opposed to local singularity restrictions) on the fibers.

Proposition 1.2. *Let B be a locally noetherian scheme of characteristic $p > 0$ and let $f : X \to B$ be a flat proper morphism. Assume that for every closed point $b \in B$ the natural morphism induced by Frobenius*

$$H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_* k(b) \twoheadrightarrow F^e_* H^i(X_b, \mathcal{O}_{X_b}) \text{ is surjective for all } e, i \in \mathbb{N}. \tag{1.3}$$

Then $R^i f_* \mathcal{O}_X$ is locally free and compatible with arbitrary base change for all $i \in \mathbb{N}$.

A few preliminary remarks:

Remark 1.4. In general, (1.3) is a map of $F^e_* k(b)$-vector spaces of the same finite dimension, so it is surjective if and only if it is an isomorphism. In the case $k(b)$ is *perfect*, (1.3) is equivalent to the condition that the adjoint morphisms

$$H^i(X_b, \mathcal{O}_{X_b}) \to F^e_* H^i(X_b, \mathcal{O}_{X_b})$$

Date: April 12, 2021.

are isomorphisms (or equivalently injective) for all $e, i \in \mathbb{N}$. This is exactly the weak ordinarity condition of [MS11]. The residue field $k(b)$ will be perfect if B is a scheme of finite type over a perfect field k (by the Nullstellensatz [Stacks, Tag 00FS] $k(b)$ is finite over k, hence also perfect). However, in what follows it seems $k(b)$ need not be perfect, or even F-finite.

Remark 1.5. The condition (1.3) can also be viewed as a global version of *F-full* [MQ18, Def. 2.3], which requires a similar surjectivity but for local cohomology modules.

2. Restriction maps from thickened fibers

Following the approach in [DJ74], we immediately apply [EGA$_2$, Prop. 7.7.10] which shows:

Proposition 2.1. *The sheaves $R^i f_*\mathscr{O}_X$ are locally free and compatible with arbitrary base change for all $i \in \mathbb{N}$ if and only if for every closed point $b \in B$ with associated maximal ideal $\mathfrak{m}_b \subseteq \mathscr{O}_X$, denoting $X_{b,n} := f^{-1}(V(\mathfrak{m}_b^{n+1})) \subseteq X$ the restriction morphisms*

$$H^i(X_{b,n}, \mathscr{O}_{X_{b,n}}) \twoheadrightarrow H^i(X_b, \mathscr{O}_{X_b}) \text{ are surjective for all } n, i \in \mathbb{N}. \tag{2.2}$$

It will be useful to consider not only the inclusion of a fiber X_b into its n-th thickening $X_{b,n}$, but the entire sequence of inclusions $X_{b,n-1} \subseteq X_{b,n}$. This not only decomposes the maps (2.2) but also yields useful long exact sequences.

Lemma 2.3. *Let B be a locally noetherian scheme, let $f : X \to B$ be a proper morphism and let $\mathscr{F}$ be a coherent sheaf on X flat over B. For any closed point $b \in B$ and any $n \in \mathbb{N}$, let $\mathscr{F}_{b,n} := \mathscr{F}|_{X_{b,n}}$ with the exception that we write $\mathscr{F}_b := \mathscr{F}|_{X_b}$. Then, there are long exact sequences*

$$\cdots \longrightarrow H^i(X_b, \mathscr{F}_b) \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) \longrightarrow H^i(X_{b,n}, \mathscr{F}_{b,n}) \longrightarrow H^i(X_{b,n-1}, \mathscr{F}_{b,n-1}) \longrightarrow \cdots \tag{2.4}$$

which are natural in the sense that if $g : Y \to B$ is another proper morphism and $\mathscr{G}$ is a coherent sheaf on Y flat over B, and if we are given a B-morphism $h : X \to Y$ together with a map of sheaves $\varphi : \mathscr{G} \to h_*\mathscr{F}$, there is a functorial morphism of long exact sequences (of modules over the local ring $\mathscr{O}_{B,b}$)

$$\begin{tikzcd} \cdots \arrow[r] & H^i(Y_b, \mathscr{G}_b) \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) \arrow[r] \arrow[d] & H^i(Y_{b,n}, \mathscr{G}_{b,n}) \arrow[r] \arrow[d] & H^i(Y_{b,n-1}, \mathscr{G}_{b,n-1}) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H^i(X_b, \mathscr{F}_b) \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) \arrow[r] & H^i(X_{b,n}, \mathscr{F}_{b,n}) \arrow[r] & H^i(X_{b,n-1}, \mathscr{F}_{b,n-1}) \arrow[r] & \cdots \end{tikzcd} \tag{2.5}$$

Proof. We derive (2.5) as it includes (2.4) as a special case (e.g. with $\varphi = \mathrm{id}$). By functoriality of derived pushforwards, we have a morphism $Rg_*\mathscr{G} \to Rf_*\mathscr{F}$ in $D^b_{\mathrm{coh}}(B)$. Taking the derived tensor product of this with the distinguished triangle $\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1} \to \mathscr{O}_B/\mathfrak{m}_b^{n+1} \to \mathscr{O}_B/\mathfrak{m}_b^{n+1}$ and applying the derived projection formula [Stacks, Tag 08ET] yields a morphism of distinguished triangles

$$\begin{tikzcd} Rg_*(\mathscr{G} \otimes^L Lg^*(\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1})) \arrow[r] \arrow[d] & Rg_*(\mathscr{G} \otimes^L Lg^*(\mathscr{O}_B/\mathfrak{m}_b^{n+1})) \arrow[r] \arrow[d] \arrow[dr, phantom, "*"] & Rg_*(\mathscr{G} \otimes^L Lg^*(\mathscr{O}_B/\mathfrak{m}_b^{n})) \arrow[r] \arrow[d] & \cdots \\ Rf_*(\mathscr{F} \otimes^L Lf^*(\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1})) \arrow[r] & Rf_*(\mathscr{F} \otimes^L Lf^*(\mathscr{O}_B/\mathfrak{m}_b^{n+1})) \arrow[r] & Rf_*(\mathscr{F} \otimes^L Lf^*(\mathscr{O}_B/\mathfrak{m}_b^{n})) \arrow[r] & \cdots \end{tikzcd} \tag{2.6}$$

Since $\mathscr{F}, \mathscr{G}$ are flat over B the derived pullbacks/tensor products simplify; we have

$$\mathscr{F} \otimes^L Lf^*(\mathscr{O}_B/\mathfrak{m}_b^{n+1}) \simeq \mathscr{F} \otimes_{\mathscr{O}_X} f^*(\mathscr{O}_B/\mathfrak{m}_b^{n+1}) \simeq \mathscr{F} \otimes_{f^{-1}\mathscr{O}_B} f^{-1}(\mathscr{O}_B/\mathfrak{m}_b^{n+1}) = \mathscr{F}_{b,n}$$

and similarly for the other terms on the corners of (∗) in (2.6). Moreover since $\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}$ is a $k(b)$-vector space a similar tensor product manipulation gives

$$\mathscr{F}\otimes^L Lf^*(\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) \simeq \mathscr{F}\otimes_{f^{-1}\mathscr{O}_B} f^{-1}(\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) \simeq \mathscr{F}\otimes_{f^{-1}\mathscr{O}_B} f^{-1}k(b)\otimes_{k(b)}(\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) = \mathscr{F}_b\otimes_{k(b)}(\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1})$$

Applying Künneth gives a natural isomorphism $Rf_*(\mathscr{F}_b \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1})) \simeq Rf_*\mathscr{F}_b \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1})$. Similarly for the top right corner of (2.6).

Hence the map of distinguished triangles (2.6) is isomorphic to

$$\begin{array}{ccccccc} Rg_*\mathscr{G}_b \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) & \longrightarrow & Rg_*(\mathscr{G}_{b,n}) & \longrightarrow & Rg_*(\mathscr{G}_{b,n-1}) & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ Rf_*\mathscr{F}_b \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) & \longrightarrow & Rf_*(\mathscr{F}_{b,n}) & \longrightarrow & Rf_*(\mathscr{F}_{b,n-1}) & \longrightarrow & \cdots \end{array} \tag{2.7}$$

and taking cohomology yields (2.5). □

3. Interplay with relative Frobenii

Let F_B^e be the e-th iterate of the absolute Frobenius of B (similarly for X) and form the diagram defining the e-th *relative* Frobenius of f (sometimes called the B-linear Frobenius of f), here denoted F_f^e [Stacks, Tag 0CC6].

$$\begin{tikzcd} X \arrow[r, "F_f^e"] \arrow[dr, "f"'] & X^{(e)} \arrow[r] \arrow[d, "f^{(e)}"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ & B \arrow[r, "F_B^e"'] & B \end{tikzcd} \tag{3.1}$$

Applying Lemma 2.3 to F_f^e (which automatically comes with a map of sheaves $\mathscr{O}_{X^{(e)}} \to F^e_{f*}\mathscr{O}_X$) gives us a map of long exact sequences

$$\begin{array}{ccccccccc} \cdots \to & H^i(X_b^{(e)}, \mathscr{O}_{X_b}) \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) & \to & H^i(X_{b,n}^{(e)}, \mathscr{O}_{X_b,n}) & \to & H^i(X_{b,n-1}^{(e)}, \mathscr{O}_{X_b,n-1}) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & H^i(X_b, \mathscr{O}_{X_b}) \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) & \longrightarrow & H^i(X_{b,n}, \mathscr{O}_{X_b,n}) & \to & H^i(X_{b,n-1}, \mathscr{O}_{X_b,n-1}) & \to \cdots \end{array} \tag{3.2}$$

For large e, the top row simplifies considerably.

Lemma 3.3. *For fixed n and $e \gg 0$, the composite $V(\mathfrak{m}_b^n) \hookrightarrow B \xrightarrow{F_B^e} B$ factors through* $\operatorname{Spec} k(b)$*. Equivalently, for e in this range $F^e_*\mathscr{O}_{B,b}/\mathfrak{m}_b^n$ is a $k(b)$-algebra.*

Proof. We must show that the kernel I of $\mathscr{O}_{B,b} \xrightarrow{F^e} \mathscr{O}_{B,b} \to \mathscr{O}_{B,b}/\mathfrak{m}_b^n$ is $\mathfrak{m}_b$. Explicitly this kernel is

$$I = \{x \in \mathscr{O}_{B,b} \mid x^{p^e} \in \mathfrak{m}_b^n\}$$

from which we see $I = \mathfrak{m}_b$ for $p^e \geq n$. □

Remark 3.4. Lemma 3.3 is equivalent to the trivial inclusion $\mathfrak{m}_b^{[p^e]} \subseteq \mathfrak{m}_b^n$ for $p^e \geq n$.

Corollary 3.5. *For fixed n and $e \gg 0$, there is a natural isomorphism of finite-type $k(b)$-schemes $F^e_*X^{(e)}_{b,n-1} \simeq X_b \otimes_{k(b)} F^e_*(\mathscr{O}_{B,b}/\mathfrak{m}_b^n)$. Here $F^e_*X^{(e)}_{b,n-1}$ denotes the scheme $X^{(e)}_{b,n-1}$ together with the structure morphism $X^{(e)}_{b,n-1} \to V(\mathfrak{m}_b^n) \xrightarrow{F_B^e}$* $\operatorname{Spec} k(b)$*.*

We now apply Corollary 3.5 to rewrite the top row of (3.2). In order to keep track of all the Frobenii, we actually apply F^e_* to push forward (3.2), which is a diagram of modules over the local ring $\mathcal{O}_{B,b}$ in the *bottom left corner of* (3.1), to get a diagram over $\mathcal{O}_{B,b}$ in the *bottom right corner* of the form

$$\begin{tikzcd} \cdots \arrow[r] & F^e_* H^i(X^{(e)}_b, \mathcal{O}_{X_b}) \otimes_{F^e_* k(b)} F^e_*(\mathfrak{m}^n_b/\mathfrak{m}^{n+1}_b) \arrow[r] \arrow[d] & F^e_* H^i(X^{(e)}_{b,n}, \mathcal{O}_{X_b,n}) \arrow[r] \arrow[d] & F^e_* H^i(X^{(e)}_{b,n-1}, \mathcal{O}_{X_b,n-1}) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & F^e_* H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} (\mathfrak{m}^n_b/\mathfrak{m}^{n+1}_b) \arrow[r] & F^e_* H^i(X_{b,n}, \mathcal{O}_{X_b,n}) \arrow[r] & F^e_* H^i(X_{b,n-1}, \mathcal{O}_{X_b,n-1}) \arrow[r] & \cdots \end{tikzcd} \tag{3.6}$$

Note that since Frobenius is affine, F^e_* is equivalent to a restriction of scalars and so this has no effect on the underlying abelian groups; in particular the homomorphisms $F^e_* H^i(X_{b,n}, \mathcal{O}_{X_b,n}) \to F^e_* H^i(X_{b,n-1}, \mathcal{O}_{X_b,n-1})$ are surjective if and only if the $H^i(X_{b,n}, \mathcal{O}_{X_b,n}) \to H^i(X_{b,n-1}, \mathcal{O}_{X_b,n-1})$ are surjective. By Corollary 3.5, for $e \geq \log_p(n+1)$ there are isomorphisms

$$F^e_* H^i(X^{(e)}_{b,n-1}, \mathcal{O}_{X_b,n-1}) \simeq H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathcal{O}_{B,b}/\mathfrak{m}^n_b)$$

and similarly $F^e_* H^i(X^{(e)}_{b,n}, \mathcal{O}_{X_b,n}) \simeq H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathcal{O}_{B,b}/\mathfrak{m}^{n+1}_b)$. In particular for $n = 0$ we have $F^e_* H^i(X^{(e)}_b, \mathcal{O}_{X_b}) \simeq H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_* k(b)$.[1]. Using these identifications, (3.6) becomes

$$\begin{tikzcd} \cdots \arrow[r] & H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathfrak{m}^n_b/\mathfrak{m}^{n+1}_b) \arrow[r] \arrow[d, "\psi^{(e),i}_n"] & H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathcal{O}_{B,b}/\mathfrak{m}^{n+1}_b) \arrow[r, "\rho^{(e),i}_n"] \arrow[d, "\varphi^{(e),i}_n"] & H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathcal{O}_{B,b}/\mathfrak{m}^n_b) \arrow[r] \arrow[d, "\varphi^{(e),i}_{n-1}"] & \cdots \\ \cdots \arrow[r] & F^e_* H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} (\mathfrak{m}^n_b/\mathfrak{m}^{n+1}_b) \arrow[r] & F^e_* H^i(X_{b,n}, \mathcal{O}_{X_b,n}) \arrow[r, "\rho^i_n"] & F^e_* H^i(X_{b,n-1}, \mathcal{O}_{X_b,n-1}) \arrow[r] & \cdots \end{tikzcd} \tag{3.7}$$

4. The key surjectivity

Proposition 4.1. *For fixed n and $e \gg 0$, the homomorphisms $\rho^{(e),i}_n$ and $\varphi^{(e),i}_{n-1}$ (and hence also ρ^i_n) are surjective for all $i \in \mathbb{N}$.*

Proof. Fixing n, choose $e \geq \log_p(n+1)$ (so $p^e \geq n+1$). Then the homomorphisms $\rho^{(e),i}_n$ are all surjective, since the reductions $\mathcal{O}_{B,b}/\mathfrak{m}^{n+1}_b \twoheadrightarrow \mathcal{O}_{B,b}/\mathfrak{m}^n_b$ are surjective, and because F^e_* and tensoring over $k(b)$ are both exact. Moreover the condition (1.3) guarantees the vertical maps $\psi^{(e),i}_n$ are all surjective (after choosing a basis for $\mathfrak{m}^n_b/\mathfrak{m}^{n+1}_b$, the map $\psi^{(e),i}_n$ can be written as a direct sum of maps of the type appearing in (1.3)).

We now show by induction on $m \leq n$ (with a subsidiary induction on i) that the $\varphi^{(e),i}_m$ and ρ^i_m are all surjective — the base case $m = 0$ is exactly (1.3). Now suppose $0 < m \leq n$ and consider

$$\begin{tikzcd} 0 \arrow[r] & H^0(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathfrak{m}^m_b/\mathfrak{m}^{m+1}_b) \arrow[r] \arrow[d, two heads, "\psi^{(e),0}_m"] & H^0(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathcal{O}_{B,b}/\mathfrak{m}^{m+1}_b) \arrow[r, two heads, "\rho^{(e),0}_m"] \arrow[d, "\varphi^{(e),0}_m"] & H^0(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathcal{O}_{B,b}/\mathfrak{m}^m_b) \arrow[r] \arrow[d, "\varphi^{(e),0}_{m-1}"] & 0 \\ 0 \arrow[r] & F^e_* H^0(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} (\mathfrak{m}^m_b/\mathfrak{m}^{m+1}_b) \arrow[r] & F^e_* H^0(X_{b,m}, \mathcal{O}_{X_b,m}) \arrow[r, "\rho^0_m"] & F^e_* H^0(X_{b,m-1}, \mathcal{O}_{X_b,m-1}) \arrow[r, "\delta^1_m"] & \cdots \end{tikzcd} \tag{4.2}$$

where in the top row we have applied the surjectivity of $\rho^{(e),0}_m$ mentioned above to obtain a short exact sequence, and in the left vertical map we have applied the surjectivity of $\psi^{(e),0}_n$. By inductive

[1] this last isomorphism of course doesn't need restrictions on e.

hypothesis we may assume the right vertical arrow $\varphi_{m-1}^{(e),0}$ is surjective. Now the snake lemma [Stacks, Tag 07JV] gives us an exact sequence

$$0 = \operatorname{coker} \psi_n^{(e),0} \to \operatorname{coker} \varphi_m^{(e),0} \to \varphi_{m-1}^{(e),0} = 0$$

and hence $\operatorname{coker} \varphi_m^{(e),0} = 0$.

We also conclude from surjectivity of $\rho_m^{(e),0}$ and $\varphi_{m-1}^{(e),0}$ that ρ_n^0 is surjective, and so the connecting map $\delta_m^1 = 0$. This means that for $i > 0$, we obtain a diagram

$$\begin{tikzcd}
0 \arrow[r] & H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathfrak{m}_b^m/\mathfrak{m}_b^{m+1}) \arrow[r] \arrow[d, "\psi_m^{(e),i}"] & H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathcal{O}_{B,b}/\mathfrak{m}_b^{m+1}) \arrow[r, two heads, "\rho_m^{(e),i}"] \arrow[d, "\varphi_m^{(e),i}"] & H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathcal{O}_{B,b}/\mathfrak{m}_b^m) \arrow[r] \arrow[d, "\varphi_{m-1}^{(e),i}"] & 0 \\
0 \arrow[r] & F^e_* H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} (\mathfrak{m}_b^m/\mathfrak{m}_b^{m+1}) \arrow[r] & F^e_* H^i(X_{b,m}, \mathcal{O}_{X_b,m}) \arrow[r, "\rho_m^i"] & F^e_* H^i(X_{b,m-1}, \mathcal{O}_{X_b,m-1}) \arrow[r, "\delta_m^{i+1}"] & \cdots
\end{tikzcd} \tag{4.3}$$

where now exactness on the left is obtained the inductive hypothesis that $\rho_m^{(e),i-1}$ and ρ_m^{i-1} are surjective. Again we may assume by inductive hypothesis that the vertical map $\varphi_{m-1}^{(e),i}$ on the right is surjective, and then the snake lemma shows $\varphi_m^{(e),i}$ is surjective. Since $\rho_m^{(e),i}$ and $\varphi_{m-1}^{(e),i}$ are both surjective we conclude ρ_m^i is surjective, completing the inductive step. □

Proof of Proposition 1.2. Proposition 4.1 shows that the restriction maps

$$\rho_n^i : H^i(X_{b,n}, \mathcal{O}_{X_b,n}) \to H^i(X_{b,n-1}, \mathcal{O}_{X_b,n-1})$$

are surjective for all $n, i \in \mathbb{N}$, and so the composite

$$H^i(X_{b,n}, \mathcal{O}_{X_b,n}) \xrightarrow{\rho_n^i} H^i(X_{b,n-1}, \mathcal{O}_{X_b,n-1}) \to \cdots \to H^i(X_{b,n-1}, \mathcal{O}_{X_b,1}) \xrightarrow{\rho_1^i} H^i(X_b, \mathcal{O}_{X_b})$$

is surjective. This is precisely the restriction morphism (2.2). □

5. Examples

Example 5.1 (Suggested by A.J. de Jong; shows (1.3) is sufficient but not necessary). Let k be an algebraically closed field of characteristic $p > 2$[2], let $B = \mathbb{A}^1_\lambda$ and let $X = V(y^2z - x(x-z)(x-\lambda z)) \subseteq \mathbb{A}^1_\lambda \times \mathbb{P}^2_{xyz}$. Let $f : X \to B$ be the projection.

By [Har77, Cor. 4.22] the locus of closed points $b \in \mathbb{A}^1_\lambda$ where (1.3) holds is the *non-vanishing* $D(h_p)$ of the polynomial

$$h_p(\lambda) = \sum_{i=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{i} \lambda^i$$

so in particular it is a *proper* open subset. However in this case the higher direct images $R^i f_* \mathcal{O}_X$ are still locally free: identifying them with the $k[\lambda]$-modules $H^i(X, \mathcal{O}_X)$ and using the exact sequence

$$\cdots \longrightarrow H^i(\mathbb{A}^1_\lambda \times \mathbb{P}^2_{xyz}, \mathcal{O}(-3)) \longrightarrow H^i(\mathbb{A}^1_\lambda \times \mathbb{P}^2_{xyz}, \mathcal{O}) \longrightarrow H^i(X, \mathcal{O}_X) \longrightarrow \cdots \tag{5.2}$$

induced by the section $y^2z - x(x-z)(x-\lambda z) \in H^0(\mathbb{A}^1_\lambda \times \mathbb{P}^2_{xyz}, \mathcal{O}(3))$ we get isomorphisms

$$H^0(X, \mathcal{O}_X) \simeq H^0(\mathbb{A}^1_\lambda \times \mathbb{P}^2_{xyz}, \mathcal{O}) \text{ and } H^1(X, \mathcal{O}_X) \simeq H^2(\mathbb{A}^1_\lambda \times \mathbb{P}^2_{xyz}, \mathcal{O}(-3))$$

and the latter 2 modules are free of rank 1 by [Har77, Thm. III.5.1].

[2] I think this works with k perfect, but it references [Har77, Ch. IV] which begins with a blanket assumption that the ground field is algebraically closed ...

References

[DJ74] Philippe Dubois and Pierre Jarraud. "Une propriété de commutation au changement de base des images directes supérieures du faisceau structural". In: *C. R. Acad. Sci. Paris Sér. A* 279 (1974), pp. 745–747. ISSN: 0302-8429.

[Du 81] Philippe Du Bois. "Complexe de de Rham filtré d'une variété singulière". In: *Bull. Soc. Math. France* 109.1 (1981), pp. 41–81. ISSN: 0037-9484.

[EGA_2] A. Grothendieck. "Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. II". In: *Inst. Hautes Études Sci. Publ. Math.* 17 (1963), p. 91. ISSN: 0073-8301.

[Har77] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9.

[KK10] János Kollár and Sándor J. Kovács. "Log canonical singularities are Du Bois". In: *J. Amer. Math. Soc.* 23.3 (2010), pp. 791–813. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-10-00663-6.

[Kol13] János Kollár. *Singularities of the minimal model program*. Vol. 200. Cambridge Tracts in Mathematics. With a collaboration of Sándor Kovács. Cambridge University Press, Cambridge, 2013, pp. x+370. ISBN: 978-1-107-03534-8. DOI: 10.1017/CBO9781139547895.

[Kol86] János Kollár. "Higher direct images of dualizing sheaves. II". In: *Ann. of Math. (2)* 124.1 (1986), pp. 171–202. ISSN: 0003-486X. DOI: 10.2307/1971390.

[MQ18] Linquan Ma and Pham Hung Quy. "Frobenius actions on local cohomology modules and deformation". In: *Nagoya Math. J.* 232 (2018), pp. 55–75. ISSN: 0027-7630. DOI: 10.1017/nmj.2017.20.

[MS11] Mircea Mustaţă and Vasudevan Srinivas. "Ordinary varieties and the comparison between multiplier ideals and test ideals". In: *Nagoya Math. J.* 204 (2011), pp. 125–157. ISSN: 0027-7630. DOI: 10.1215/00277630-1431849.

[Stacks] The Stacks project authors. *The Stacks project*. 2021. URL: https://stacks.math.columbia.edu.

Department of Mathematics, University of Washington, Seattle, WA 98105, USA
Email address: cgodfrey@uw.edu
URL: https://math.washington.edu/~cgodfrey