NOTES ON GROTHENDIECK DUALITY

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1. Local cohomology

1.1. **Sheaves of families of supports.** Collected here are some useful basic facts about cohomology with supports - I won't be offended if you skip right over this section. First, let X be a topological space.

Definition 1.1. A **sheaf of families of supports $\underline{\Phi}$ on** X is a sheaf of sets on X with the following properties:

- for each open set $U \subset X$, $\underline{\Phi}(U)$ is a collection of closed subsets of U. In other words, $\underline{\Phi}$ is a subsheaf of the sheaf of sets $U \mapsto \{$ closed subsets of U $\}$.
- For each open set $U \subset X$, $\underline{\Phi}(U)$ is a family of supports on U. That is, if $Z, W \subset U$ are closed, $Z \subset W$ and $W \in \underline{\Phi}(U)$ then $Z \in \underline{\Phi}(U)$ and if $Z_1, Z_2 \in \Phi(U)$ then $Z_1 \cup Z_2 \in \Phi(U)$.

Remark 1.2. If Φ is a family of supports on X as above, then we obtain a pre-sheaf of families of supports by assigning $U \mapsto \{Z \cap U \mid Z \in \Phi\}$, which sheafifies to a sheaf of families of supports $\underline{\Phi}$. On the other hand if $\underline{\Phi}$ is a sheaf of families of supports on X then $\Gamma(X, \underline{\Phi}) = \underline{\Phi}(X)$ is a family of supports on X.

The functors $\Phi \mapsto \underline{\Phi}$ and $\underline{\Phi} \mapsto \Gamma(X, \underline{\Phi})$ need not be mutual inverses. For example if X is a locally compact Hausdorff space and $\Phi = \{Z \subset X \mid Z \text{ is compact }\}$ then $X \in \Gamma(X, \underline{\Phi})$ - if X isn't compact we see that $\Phi \neq \Gamma(X, \underline{\Phi})$ (however we do have an inclusion $\Phi \subset \Gamma(X, \underline{\Phi})$). For an example where $\underline{\Phi} \neq \underline{\Gamma(X, \underline{\Phi})}$ let $f : X \to Y$ be a morphism of finite type where Y is a noetherian scheme, and let p be a natural number. We can define a sheaf of families of supports $\underline{\Phi}$ on X by

$$\underline{\Phi}(U) := \{Z \subset U \text{ closed } \mid \operatorname{codim}(Z \cap X_y, X_y) \geq p \text{ for all } y \in f(U)\}$$

In general $\underline{\Gamma(X, \underline{\Phi})}$ will be *smaller* than Φ (this should happen essentially whenever f has varying fiber dimension - for a specific case take f to be the map $f : \mathbb{A}^2_k \times \mathbb{P}^1_k \to \mathbb{A}^2_k$ (here k is any field), and set $p = 1$. Then we have $\mathrm{Blp}_0 \mathbb{A}^2_k \subset \mathbb{A}^2_k \times \mathbb{P}^1_k$ - set $U := (\mathbb{A}^2_k \setminus \{0\}) \times \mathbb{P}^1_k \subset \mathbb{A}^2_k \times \mathbb{P}^1_k$ and set $Z = \mathrm{Blp}_0 \mathbb{A}^2_k \cap U$. Super-explicitly,

$$Z := \{((x_0, x_1), [x_0, x_1]) \in \mathbb{A}^2_k \times \mathbb{P}^1_k \mid (x_0, x_1) \neq 0\}$$

Evidently $Z \in \underline{\Phi}(U)$, but since the closure of Z in $\mathbb{A}^2_k \times \mathbb{P}^1_k$ is $\mathrm{Blp}_0 \mathbb{A}^2_k$, there's no $W \in \Gamma(X, \underline{\Phi})$ so that $W \cap U = Z$.

From here on out we'll assume X is a **Zariski space**, that is, a noetherian topological space in which every (non-$\emptyset$) irreducible closed subset has a unique generic point. A Hartshorne exercise shows that the underlying space of a noetherian scheme is a Zariski space.

Definition 1.3. A subset $Z \subset X$ is **specialization-closed** if and only if whenever $x \in Z$ and $y \in X$ is a specialization of x (meaning $y \in \bar{\{x\}}$), $y \in Z$ too.

Definition 1.4. For a specialization-closed subset $Z \subset X$, the associated family of supports Φ_Z is

$$\Phi_Z := \{\bigcup_{i=0}^{n} \bar{\{x_i\}} \subset X \mid x_0, \dots, x_n \in Z \text{ and } n \in \mathbb{N}\}$$

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Example 1.5. Recall that the *codimension* $\operatorname{codim}(x, X$ of a point $x \in X$ is

$$\sup\{n \in \mathbb{N} \mid \text{ there's a sequence of proper specializations } x_0 \mapsto x_1 \mapsto \cdots \mapsto x_n = x\}$$

(if X is a noetherian scheme, one can show $\operatorname{codim}(x, X) = \dim \mathcal{O}_{X,x}$). If y is a specialization of x then $\operatorname{codim}(y, X) \geq \operatorname{codim}(x, X)$ with strict inequality if $y \neq x$, and so we see that for every $p \in \mathbb{N}$,

$$Z_p := \{x \in X \mid \operatorname{codim}(x, X) \geq p\}$$

is specialization-closed.

Here's a philosophically interesting point:

Proposition 1.6. *Let X be a Zariski space. Then there's a natural one-to-one correspondence between families of supports Φ on X such that $\Gamma(X, \underline{\Phi}) = \Phi$ and specialization-closed subsets $Z \subset X$.*

Proof. On the one hand, given a specialization-closed set $Z \subset X$ we obtain a family of supports Φ_Z on X. Note that if $W \in \Gamma(X, \underline{\Phi}_Z)$ we can find an open cover $X = \bigcup_i U$ of X (and we can assume it's finite, since X is assumed to be Zariski, hence noetherian) so that for each i

$$W \cap U_i = V \cap U_i \text{ for some } V \in \Phi_Z$$

By the definition of Φ_Z, $V = \bigcup_j \{\bar{x_{ij}}\}$ and removing those x_{ij} with $x_{ij} \notin U_i$ if necessary, we have $W \cap U_i = (\bigcup_j \{\bar{x_{ij}}\}) \cap U_i$. The claim to make is that

$$W = \bigcup_{i,j} \{\bar{x_{ij}}\}$$

This boils down to the following foundational fact about Zariski spaces: if W is a closed subset of a Zariski space X and $x_0, \ldots, x_N$ are the generic points of its irreducible components $W_0, \ldots, W_N$, then $W = \bigcup_i \{\bar{x_i}\}$.

The upshot: $\Phi_Z = \Gamma(X, \underline{\Phi}_Z)$. On the other hand suppose Φ is a family of supports with $\Phi = \Gamma(X, \underline{\Phi})$, and set $Z_\Phi = \{x \in X \mid \{\bar{x}\} \in \Phi\}$. Since Φ is a family of supports, Z_Φ is specialization-closed. One must now check that our functions $Z \mapsto \Phi_Z$ and $\Phi \mapsto Z_\Phi$ are mutual inverses, but I'm going to omit the details.

$\square$

1.2. **Local cohomology.** Let X be a topological space and let $\underline{\Phi}$ be a sheaf of families of supports on X. Given a sheaf $\mathcal{F}$ of abelian groups on X, we get a *subsheaf* $\underline{\Gamma_\Phi}(\mathcal{F})$ of $\mathcal{F}$, defined by

$$\underline{\Gamma_\Phi}(\mathcal{F})(U) := \{\sigma \in \mathcal{F}(U) \mid \operatorname{supp}\sigma \in \underline{\Phi}(U)\}$$

Fact: $\underline{\Gamma_\Phi}$ is left exact. Its right derived functor will be denoted by $R\underline{\Gamma_\Phi}$. The cohomologies of $R\underline{\Gamma_\Phi}$ will be denoted by

$$\mathcal{H}^i_{\underline{\Phi}}(\mathcal{F}) = R^i\underline{\Gamma_\Phi}(\mathcal{F})$$

Remark 1.7. If $Z \subset X$ is a closed set and Φ is the collection of all closed subsets of Z, we'll abbreviate like $\underline{\Gamma}_Z := \underline{\Gamma_\Phi}$.

Let $Z \subset X$ be a closed subset and let $U := X \setminus Z$. Let

$$\text{res} : \mathcal{F} \to \iota_*(\mathcal{F}|_U)$$

be the natural morphism, where $\iota : U \to X$ is the inclusion. Unraveling definitions we see that its kernel is $\underline{\Gamma}_Z(\mathcal{F})$, and in this way we obtain a left exact sequence

$$0 \to \underline{\Gamma}_Z(\mathcal{F}) \to \mathcal{F} \to \iota_*(\mathcal{F}|_U)$$

If $\mathcal{F}$ is flasque then this sequence is exact on the right as well.

So, if $\mathcal{I}^*$ is an injective (or even just flasque) resolution of $\mathcal{F}$, we obtain a short exact sequence of complexes

$$0 \to \underline{\Gamma}_Z(\mathcal{I}^*) \to \mathcal{I}^* \to \iota_*(\mathcal{I}|_U^*) \to 0$$

The resulting long exact sequence on cohomology looks like

$$\cdots \to \mathcal{H}^{i-1}(U,\mathcal{F}|_U) \xrightarrow{\delta} \mathcal{H}^i_Z(X,\mathcal{F}) \to \mathcal{H}^i(X,\mathcal{F}) \xrightarrow{\text{res}} \mathcal{H}^i(U,\mathcal{F}|_U) \xrightarrow{\delta} \mathcal{H}^{i+1}_Z(X,\mathcal{F}) \to \cdots$$

The thing that makes local cohomology *local* is the following "excision" property. Let $V \subset X$ be an open set containing Z. Then the natural restriction homomorphism $\underline{\Gamma}_Z(\mathcal{F})|_V \to \underline{\Gamma}_Z(j_*\mathcal{F}|_V)$ is an isomorphism, which induces a natural isomorphism $R\underline{\Gamma}_Z(\mathcal{F})|_V \to R\underline{\Gamma}_Z(j_*\mathcal{F}|_V)$ (here $j : V \to X$ is the inclusion). Taking cohomology yields isomorphisms

$$H^i_Z(X,\mathcal{F}) \simeq H^i_Z(V,\mathcal{F}|_V)$$

Note that if $\Psi \subset \Phi$ are families of supports, then there is a natural sub-sheaf inclusion

$$\underline{\Gamma}_\Psi(\mathcal{F}) \subset \underline{\Gamma}_\Phi(\mathcal{F})$$

so in particular if $Z \in \Phi$, then we have an inclusion $\underline{\Gamma}_Z(\mathcal{F}) \subset \underline{\Gamma}_\Phi(\mathcal{F})$, and taking the colimit over all $Z \in \Phi$ gives a natural morphism

$$\operatorname{co}\lim_{Z\in\Phi} \underline{\Gamma}_Z(\mathcal{F}) \to \underline{\Gamma}_\Phi(\mathcal{F});\ \text{one can check it's an } \textit{isomorphism}$$

and this leads to the following

Proposition 1.8. *Let $\underline{\Phi}$ be a sheaf of families of supports such that $\underline{\Phi} = \underline{\Gamma(X,\underline{\Phi})}$. Then for any sheaf of abelian groups $\mathcal{F}$*

$$\mathcal{H}^i_{\underline{\Phi}}(\mathcal{F}) = \operatorname{colim}_{Z\in\underline{\Phi}(X)}\mathcal{H}^i_Z(\mathcal{F})$$

For this reason one can often reduce to the case where Φ consists of the closed subsets of a given subset $Z \subset X$.

If $\mathcal{F}$ is a sheaf of abelian groups on X, then $\Gamma(X,\underline{\Gamma}_{\underline{\Phi}}(\mathcal{F})) = \Gamma_{\underline{\Phi}(X)}(X,\mathcal{F})$. This extends to derived functors to yield

$$R\Gamma_{\underline{\Phi}(X)} = R\Gamma \circ R\underline{\Gamma}_{\underline{\Phi}}$$

Proposition 1.9. *There is a natural (composition of functors) spectral sequence*

$$E_2^{pq} = H^p(X,\mathcal{H}^q_{\underline{\Phi}}(\mathcal{F})) \implies H^{p+q}_{\underline{\Phi}(X)}(X,\mathcal{F})$$

Example 1.10. Suppose A is a noetherian commutative ring and let $I \subset A$ be an ideal, corresponding to a closed subscheme $Z \subset X := \operatorname{Spec} A$. Let M be an A-module and let $\tilde{M}$ be the corresponding quasi-coherent sheaf on X. Observe that $\sigma \in \Gamma_Z(X,\tilde{M})$ if and only if $V(\operatorname{ann}\sigma) = \operatorname{supp}\sigma \in Z$, or equivalently $\operatorname{rad} I \subset \operatorname{rad}\operatorname{ann}\sigma$. As A is noetherian (really, we just need I to be finitely generated), this occurs if and only if $I^r \subset \operatorname{ann}\sigma$ for $r \gg 0$, which is to say $I^r\sigma = 0$ for $r \gg 0$. Conclusion: $\Gamma_Z(X,\tilde{M}) = \Gamma_I(A,M)$, where by definition $\Gamma_I(A,M) := \{m \in M \mid I^r m = 0 \text{ for } r \gg 0\}$ is the I-torsion submodule of M.

There's a similar (but slightly more complicated) commutative algebraic description of $\Gamma_\Phi(X,\tilde{M})$ for a family of supports Φ on X. Here note that $\Phi^\vee := \{I \subset A \text{ an ideal } \mid V(I) \in \Phi\}$ is a family of ideals in A with the properties that

- if $I, J \in \Phi^\vee$ then $I \cap J \in \Phi^\vee$.
- if $I \in \Phi^\vee$ and $I \subset J$ then $J \in \Phi^\vee$.

If M is an A-module then we should have $\Gamma_\Phi(X,\tilde{M}) = \Gamma_{\Phi^\vee}(A,M)$ where

$$\Gamma_{\Phi^\vee}(A,M) := \{m \in M \mid I^r m = 0 \text{ for } r \gg 0 \text{ for some } I \in \Phi^\vee\}$$

Example 1.11. Let $X \subset \mathbb{P}^n_k$ be a projective variety of positive dimension over a field k, and let $C(X) \subset \mathbb{A}^{n+1}_k$ be the affine cone over X and let $p \in C(X)$ be the vertex of the cone. Let's describe the local cohomology of $\mathcal{O}_{C(X)}$ at p. Let $U := C(X) \setminus \{p\}$ and consider the long exact sequence

$$\cdots \to H^{i-1}(U, \mathcal{O}_U) \xrightarrow{\delta} H^i_p(C(X), \mathcal{O}_{C(X)}) \to H^i(C(X), \mathcal{O}_{C(X)}) \xrightarrow{res} H^i(U, \mathcal{O}_U) \xrightarrow{\delta} \cdots$$

Since $C(X)$ is *affine*, $H^i(C(X), \mathcal{O}_{C(X)}) = 0$ for $i > 0$, and so really we're looking at an exact sequence

$$0 \to H^0_p(C(X), \mathcal{O}_{C(X)}) \to H^0(C(X), \mathcal{O}_{C(X)}) \xrightarrow{res} H^0(U, \mathcal{O}_U) \xrightarrow{\delta} H^1_p(C(X), \mathcal{O}_{C(X)}) \to 0$$

$$\text{and isomorphisms } H^i(U, \mathcal{O}_U) \xrightarrow{\simeq} H^{i+1}_p(C(X), \mathcal{O}_{C(X)}) \text{ for } i > 0$$

Note also that $H^0_p(C(X), \mathcal{O}_{C(X)}) = 0$ (since we're assuming $\dim X > 0$ and hence $\dim C(X) > 1$. So in fact the exact sequence in low degrees simplifies further to

$$0 \to H^0(C(X), \mathcal{O}_{C(X)}) \xrightarrow{res} H^0(U, \mathcal{O}_U) \xrightarrow{\delta} H^1_p(C(X), \mathcal{O}_{C(X)}) \to 0$$

Note that U is a principal $\mathbb{G}_m$-bundle over X. In fact it's the principal $\mathbb{G}_m$-bundle associated to the invertible sheaf $\mathcal{O}_X(-1)$, and from this we see that it's *affine* over X, with

$$\pi_* \mathcal{O}_U = \bigoplus_{k \in \mathbb{Z}} \mathcal{O}_X(k)$$

Letting $\pi : U \to X$ be the projection, we can calculate $H^i(U, \mathcal{O}_U)$ using the Leray spectral sequence for π, which collapses (since π is an affine map) to give

$$H^i(U, \mathcal{O}_U) = \bigoplus_{k \in \mathbb{Z}} H^i(X, \mathcal{O}_X(k))$$

To obtain some interesting conclusions, allow me to use the results of a few Hartshorne exercises **INCLUDE REFERNCE TO Section III.something**: first, if X is a noetherian scheme and $p \in X$ is a point, and if $\mathcal{F}$ is a quasi-coherent sheaf on X, then the natural maps $H^i_p(X, \mathcal{F}) \to H^i_p(\operatorname{Spec}\mathcal{O}_{X,p}, \mathcal{F}_p)$ are all isomorphisms, and if A is a noetherian local ring with maximal ideal m, then $H^i_m(\operatorname{Spec}A, A)$ vanishes unless $\operatorname{depth}A \leq i \leq \dim A$ (and it's guaranteed to be non-0 for $i = \operatorname{depth}A$ and $i = \dim A$). Also recall that A is **Cohen-Macaulay** if and only if $\operatorname{depth}A = \dim A$.

Recall also that a noetherian scheme X is normal if and only if it satisfies Serre's criteria R_1 and S_2 - that is, it's regular in codimension 1 and for every point $p \in X$,

$$\operatorname{depth}\mathcal{O}_{X,p} \geq \min 2, \operatorname{codim}(p, X)$$

Suppose first that X is *normal*. Then certainly U is normal (it's smooth over X. As X is assumed to be positive-dimensional, $\operatorname{codim}(p, C(X)) = \dim X + 1 > 1$, so $C(X)$ will be normal if and only if it's S_2 at p, i.e.

$$\operatorname{depth}\mathcal{O}_{C(X),p} \geq 2$$

Applying the above Hartshorne facts, we see that $C(X)$ will be normal if and only if $H^1_p(C(X), \mathcal{O}_{C(X)}) = 0$, which occurs if and only if the natural map

$$H^0(C(X), \mathcal{O}_{C(X)}) \xrightarrow{\simeq} \bigoplus_{k \in \mathbb{Z}} H^0(X, \mathcal{O}_X(k)) \text{ is an isomorphism.}$$

Unraveling some definitions, one sees that the composition

$$\bigoplus_k H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = H^0(\mathbb{A}^{n+1}, \mathcal{O}_{\mathbb{A}^{n+1}}) \to H^0(C(X), \mathcal{O}_{C(X)}) \to \bigoplus_{k \in \mathbb{Z}} H^0(X, \mathcal{O}_X(k))$$

factors the restriction $\bigoplus_k H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \to \bigoplus_{k \in \mathbb{Z}} H^0(X, \mathcal{O}_X(k))$ as a surjection followed by an injection, and so we recover the basic

Proposition 1.12. *X is projectively normal if and only if the restriction map on global sections*

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \to H^0(X, \mathcal{O}_X(k))$$

is an isomorphism for all k.

Now assume X is projectively normal - then

$$H^0_p(C(X), \mathcal{O}_{C(X)}) = H^1_p(C(X), \mathcal{O}_{C(X)}) = 0$$

$$\text{and } H^{i+1}_p(C(X), \mathcal{O}_{C(X)}) \simeq \bigoplus_k H^i(X, \mathcal{O}_X(k)) \text{ for } i > 0$$

From here it's not hard to show that $C(X)$ is Cohen-Macaulay if and only if X is Cohen-Macaulay and $H^i(X, \mathcal{O}_X(k)) = 0$ for all k when $0 < i < \dim X$.

For example let X be a general (hence smooth) complete intersection of type $d_1, \dots, d_r$. Then both $C(X)$ is also a complete intersection, and so both X and $C(X)$ are Cohen-Macaulay. In this case

$$H^i_p(C(X), \mathcal{O}_{C(X)}) = 0 \text{ for } i < \dim X + 1 \text{ and}$$

$$H^{\dim X+1}_p(C(X), \mathcal{O}_{C(X)}) = \bigoplus_k H^{\dim X}(X, \mathcal{O}_X(k)) = \bigoplus_k H^0(X, \mathcal{O}_X(\sum_i d_i - n - 1 - k))$$

where the last step uses Serre duality and the adjunction formual $\omega_X = \mathcal{O}_X(\sum_i d_i - n - 1)$. While this is abstractly isomorphic to $H^0(C(X), \mathcal{O}_{C(X)})$ the grading has been reversed and shifted by $\sum_i d_i - n - 1$.

On the other hand if X is an abelian variety of dimension g then $C(X)$ will never be Cohen-Macaulay, since $H^i(X, \mathcal{O}_X) \neq 0$ for $0 \leq i \leq g$.

1.3. **Cousin complexes.** Let $\underline{\Phi}$ and $\underline{\Psi}$ be 2 sheaves of families of supports on X and suppose $\underline{\Phi} \subset \underline{\Psi}$. Then for any sheaf of abelian groups $\mathcal{F}$ there's a natural inclusion $\underline{\Gamma}_{\underline{\Phi}}(\mathcal{F}) \subset \underline{\Gamma}_{\underline{\Psi}}(\mathcal{F})$ and this extends to a morphism of derived functors

$$R\underline{\Gamma}_{\underline{\Phi}} \to R\underline{\Gamma}_{\underline{\Psi}} \text{ which we can fit into an \textit{exact triangle}}$$

$$R\underline{\Gamma}_{\underline{\Phi}} \to R\underline{\Gamma}_{\underline{\Psi}} \to R\underline{\Gamma}_{\underline{\Psi}\setminus\underline{\Phi}} \xrightarrow{[+1]} R\underline{\Gamma}_{\underline{\Phi}} \to \dots$$

We can take this as the definition of the derived functor $R\underline{\Gamma}_{\underline{\Psi}\setminus\underline{\Phi}}$.

The following is an analogue of the "spectral sequence of a filtered space" from algebraic topology.

Proposition 1.13. *Let X be a topological space and let*

$$\underline{\Phi}_X = \underline{\Phi}_0 \supset \underline{\Phi}_1 \supset \underline{\Phi}_2 \supset \cdots$$

be a filtration of X by sheaves of families of supports, where $\underline{\Phi}_X$ is the maximal sheaf of families of supports (including all closed subsets). Let $\mathcal{F}^*$ be a bounded-below complex of abelian groups on X, i.e. an object of $D^+(\mathcal{A}\lfloor(X))$.. Then there's a spectral sequence

$$E_1^{pq} = \mathcal{H}^{p+q}_{\underline{\Phi}_p\setminus\underline{\Phi}_{p+1}}(\mathcal{F}) \implies \mathcal{H}^{p+q}(\mathcal{F})$$

which converges provided $\underline{\Phi}_n = \emptyset$ for $n \gg 0$.

Assume X is a locally Zariski space and let $X = Z_0 \supset Z_1 \supset Z_2 \supset \dots$ be a filtration of X by specialization-closed sets with the following properties:

- for each p, every point of $Z_p \setminus Z_{p+1}$ is maximal with respect to specialization - in other words, if $x \in Z_p$ and $y \in X$ is a proper specialization of x, then $y \in Z_{p+1}$.
- $Z_n = \emptyset$ for $n \gg 0$.

In this situation we obtain associated sheaves of families of supports $\underline{\Phi}_p := \underline{\Phi}_{Z_p}$ fitting into a filtration

$$\underline{\Phi}_X = \underline{\Phi}_0 \supset \underline{\Phi}_1 \supset \dots$$

with $\underline{\Phi}_n = \emptyset$ for $n \gg 0$. Since the notation is already fierce, let's write $\underline{\Gamma}_{Z_p \setminus Z_{p+1}}$ for $\underline{\Gamma}_{\underline{\Phi}_p \setminus \underline{\Phi}_{p+1}}$ and similarly for its derived functors.

Example 1.14. The most important case to keep in mind is where $Z_p = \{x \in X \mid \mathrm{codim}(x, X) \geq p\}$.

Proposition 1.15. *For any bounded-below complex of sheaves of abelian groups $\mathcal{F}^*$ on X and for any $p \in \mathbb{N}$, there are natural isomorphisms*

$$\mathcal{H}^i_{Z_p \setminus Z_{p+1}}(\mathcal{F}^*) \simeq \bigoplus_{x \in Z_p \setminus Z_{p+1}} \mathcal{H}^i_x(\mathcal{F}^*) \text{ for all } i$$

where on the right hand side $\mathcal{H}^i_x(\mathcal{F}^*) := \mathcal{H}^i_{\{\bar{x}\}}(\mathcal{F}^*)_x$, viewed as the pushforward of the constant sheaf $\mathcal{H}^i_{\{\bar{x}\}}(\mathcal{F}^*)_x$ on $\{\bar{x}\}$ to X. So in particular it's flasque*.*

Combining this with the spectral sequence of a filtered space, we obtain:

Proposition 1.16. *With notation as above, there is a convergent spectral sequence*

$$E_1^{pq} = \bigoplus_{x \in Z_p \setminus Z_{p+1}} \mathcal{H}^{p+q}_x(\mathcal{F}^*) \implies \mathcal{H}^{p+q}(\mathcal{F}^*)$$

Let's assume for a moment that $\mathcal{F}^*$ is just a sheaf supported in degree 0. Then in fact

$$\mathcal{H}^i_{Z_p \setminus Z_{p+1}}(\mathcal{F}) = 0 \text{ for i > p}$$

(this boils down to Grothendieck vanishing applied on the local space of a point $x \in Z_p \setminus Z_{p+1}$).

Looking at the $q = 0$ axis of the spectral sequence, we obtain a complex

$$C^*(\mathcal{F}) : \dots \xrightarrow{d_1} \bigoplus_{x \in Z_{p-1} \setminus Z_p} \mathcal{H}^{p-1}_x(\mathcal{F}) \xrightarrow{d_1} \bigoplus_{x \in Z_p \setminus Z_{p+1}} \mathcal{H}^p_x(\mathcal{F}) \xrightarrow{d_1} \bigoplus_{x \in Z_{p+1} \setminus Z_{p+2}} \mathcal{H}^{p+1}_x(\mathcal{F}) \xrightarrow{d_1} \dots$$

together with an augmentation $\mathcal{F} \xrightarrow{\epsilon} \bigoplus_{x \in Z_0 \setminus Z_0} \mathcal{H}^0_x(\mathcal{F})$ (ϵ can be thought of as taking the germs of a section of $\mathcal{F}$ at the generic points of X)

Definition 1.17. $C^*(\mathcal{F})$ is the **Cousin complex of** $\mathcal{F}$.

Based on the above discussion, we see that

Proposition 1.18. $C^*(\mathcal{F})$ *is a flasque resolution of $\mathcal{F}$ if and only if*

$$\mathcal{H}^i_x(\mathcal{F}) = 0 \textit{ for } i \neq p \textit{ for } x \in Z_p \setminus Z_{p+1}$$

*If either of these equivalent conditions are satisfied one says $\mathcal{F}$ is **Cohen-Macaulay with respect to the filtration** Z_*.*

Similarly $C^*(\mathcal{F})$ is an injective resolution of $\mathcal{F}$ if and only if

$$\mathcal{H}^i_x(\mathcal{F}) = 0 \textit{ for } i \neq p \textit{ for } x \in Z_p \setminus Z_{p+1} \textit{ and}$$

$$\mathcal{H}^p_x(\mathcal{F}) \textit{ is injective for all } x \in Z_p \setminus Z_{p+1}$$

*If either of these equivalent conditions are satisfied one says $\mathcal{F}$ is **Gorenstein with respect to the filtration** Z_*.*

Remark 1.19. By the aforementioned Hartshorne exercises, this reduces to the usual notions of Cohen-Macaulay-ness and Gorenstein-ness when X is a noetherian scheme, Z_* is the codimension filtration and $\mathcal{F}$ is a quasi-coherent sheaf on X.

Example 1.20. Let X be a noetherian scheme of finite-dimension and let Z_* be the codimension filtration of X. Let $\mathcal{F}$ be a Cohen-Macaulay quasi-coherent sheaf on X (for instance if X itself is Cohen-Macaulay then $\mathcal{F}$ can be any locally free sheaf on X). Then the Cousin complex gives a canonical flasque resolution

$$0 \to \mathcal{F} \to \bigoplus_{\operatorname{codim}(x,X)=0} \mathcal{H}^0_x(\mathcal{F}) \xrightarrow{d} \bigoplus_{\operatorname{codim}(x,X)=1} \mathcal{H}^1_x(\mathcal{F}) \xrightarrow{d} \bigoplus_{\operatorname{codim}(x,X)=2} \mathcal{H}^2_x(\mathcal{F}) \xrightarrow{d} \dots$$

To be even more explicit, let X be a smooth projective curve over a field k (algebraically closed if you want). Then we get a complex of the form

$$0 \to k(X) \xrightarrow{d} \bigoplus_{x \in X \text{ closed}} \mathcal{H}^1_x(\mathcal{O}_X) \to 0$$

computing the cohomology of $\mathcal{O}_X$- here $k(X)$ is the function field of X. The differential can be interpreted as computing the "principal part" of a rational function $f \in k(X)$ at a point $x \in X$. To see this note that for a closed point $x \in X$ with an affine neighborhood $U \subset X$ excision gives isomorphisms $H^*_x(X, \mathcal{O}_X) \simeq H^*_x(U, \mathcal{O}_U)$. Now set $V := U \setminus \{x\}$ and consider the exact sequence

$$0 \to H^0_x(U, \mathcal{O}_U) \to H^0(U, \mathcal{O}_U) \to H^0(V, \mathcal{O}_V)$$

$$\xrightarrow{\delta} H^1_x(U, \mathcal{O}_U) \to H^1(U, \mathcal{O}_U) \to H^1(V, \mathcal{O}_V) \to 0$$

Note that $H^0_x(U, \mathcal{O}_U) = 0$ (since U is smooth, so in particular S_1) and also $H^1(U, \mathcal{O}_U) = H^1(V, \mathcal{O}_V) = 0$ - this is because U and V are affine (any finite set of points on a curve can be viewed as an ample divisor, so its complement is affine). So we have an exact sequence

$$0 \to H^0(U, \mathcal{O}_U) \to H^0(V, \mathcal{O}_V) \to H^1_x(U, \mathcal{O}_U) \to 0$$

This can be made even more explicit if $X = \mathbb{P}^1_k$ and $x = 0$. In that case $U = \mathbb{A}^1_k$ and $V = \mathbb{A}^1_k \setminus \{0\}$, so $H^0(U, \mathcal{O}_U) = k[x]$ and $H^0(V, \mathcal{O}_V) = k[x, x^{-1}]$; the map $H^0(U, \mathcal{O}_U) \to H^0(V, \mathcal{O}_V)$ is the usual inclusion $k[x] \subset k[x, x^{-1}]$. So, we see that

$$H^1_{\{0\}}(\mathbb{A}^1_k, \mathcal{O}) \simeq k[x, x^{-1}]/k[x]$$

and the map $H^0(V, \mathcal{O}_V) \to H^1_{\{0\}}(\mathbb{A}^1_k, \mathcal{O})$ corresponds to taking the "principal part" of a Laurent polynomial at the origin.

2. Local Grothendieck duality

2.1. **Dualizing complexes.** Let X be a locally noetherian scheme and let ω^* be a complex of quasi-coherent sheaves on X such that

- the cohomology of ω^* is bounded and coherent and
- ω^* has finite injective dimension.

In this situation ω^* is quasi-isomorphic to a bounded complex of injective quasi-coherent sheaves (it's an object of $D^b_c(X)$) and so $R\underline{Hom}_X(-, \omega^*)$ gives a contravariant derived functor

$$D^+_c(X) \to D^-_c(X) \text{ preserving } D^b_c(X)$$

Definition 2.1. An object $\mathcal{F}^*$ of $D^+_c(X)$ is ω^***-reflexive** if and only if the natural map

$$\mathcal{F}^* \to R\underline{Hom}_X(R\underline{Hom}_X(\mathcal{F}^*, \omega^*), \omega^*)$$

is a quasi-isomorphism.

Proposition 2.2. *If $\mathcal{O}_X$ (viewed as a complex concentrated in degree 0) is ω^*-reflexive then every $\mathcal{F}^*$ as above is ω^*-reflexive.*

Definition 2.3. If either of the equivalent conditions in the above proposition are satisfied, ω^* is called a **dualizing complex on** X.

Example 2.4. Let $X = \mathrm{Spec}\mathbb{Z}$. We will show that the complex $\mathbb{Z}$ (concentrated in degree 0) is dualizing. Note that

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

is an injective resolution of $\mathbb{Z}$ and so $\mathbb{Z}$ has finite injective dimension. By the above proposition we only need to check that the natural map

$$\mathbb{Z} \to RHom_{\mathbb{Z}}(RHom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}), \mathbb{Z})$$

is an isomorphism, which follows from the calculation $\mathbb{Z} \simeq RHom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ (since $Ext^i_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = 0$, $\mathbb{Z}$ being free).

Example 2.5. More generally if X is a regular noetherian scheme then $\mathcal{O}_X$ (viewed as a complex concentrated in degree 0) is dualizing. Once we know $\mathcal{O}_X$ has finite injective dimension, the calculation $R\underline{Hom}_X(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathcal{O}_X$ will show $\mathcal{O}_X$ is indeed dualizing. The fact that it has finite injective dimension is essentially Serre's theorem that a noetherian local ring is regular if and only if it has finite homological dimension!

Here are some local criteria for a complex to be dualizing:

Proposition 2.6. *Let ω^* be an object of $D^b_c(X)$ with finite injective dimension. Then the following are equivalent:*

- *ω^* is a dualizing complex.*
- *ω^*_x is a dualizing complex on* $\mathrm{Spec}\mathcal{O}_{X,x}$ *for all $x \in X$.*
- *$\hat{\omega^*_x}$ is a dualizing complex on* $\mathrm{Spec}\hat{\mathcal{O}_{X,x}}$ *for all $x \in X$.*
- *$\iota_{x*}k(x)$ is ω^*-reflexive for all $x \in X$ - here $k(x)$ is the residue field of x and $\iota_x : \{x\} \to X$ is the inclusion.*

We also have a global uniqueness statement:

Proposition 2.7. *Let X be a connected locally noetherian scheme and let ω^* be a dualizing complex on X. If ν^* is another complex on X then ν^* is dualizing if and only if there is an invertible sheaf $\mathcal{L}$ on X and an integer $k \in \mathbb{Z}$ so that*

$$\nu^* \simeq \omega^* \otimes \mathcal{L}[k]$$

Remark 2.8. Most of the content of this proof is wrapped up in the following fact: if $\mathcal{M}^*_1$ is an object of $D^b_c(X)$ such that there's another object $\mathcal{M}^*_2$ and an isomorphism

$$\mathcal{M}^*_1 \otimes^L \mathcal{M}^*_2 \simeq \mathcal{O}_X,$$

then there's an invertible sheaf $\mathcal{L}$ on X, an integer k and a quasi-isomorphism $\mathcal{M}^*_1 \simeq \mathcal{L}[k]$ and it follows necessarily that $\mathcal{M}^*_2 \simeq \mathcal{L}^\vee[-k]$.

Before returning to local cohomological considerations, we need one more lemma:

Proposition 2.9. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let ω^* be an object of $D^b_c(A)$. Then ω^* is dualizing if and only if there's a $d \in \mathbb{Z}$ so that*

$$\mathrm{Ext}^i_A(k, \omega^*) = \begin{cases} k & \text{if } i = d \\ 0 & \text{otherwise} \end{cases}$$

In this situation ω^* is said to be ***normalized*** *if $d = 0$.*

2.2. **Local duality.** Let X be a locally noetherian scheme and let $Z \subset X$ be a closed subscheme, corresponding to a coherent sheaf of ideals $\mathcal{I}_Z \subset \mathcal{O}_X$. Let Z_n be the n-th thickening of Z, corresponding to the ideal sheaf $\mathcal{I}_Z^n$ (so that $Z = Z_1$). Observe that the section 1 of $\mathcal{O}_{Z_n}$ is supported on Z, and so we have natural homomorphisms

$$\underline{Hom}_X(\mathcal{O}_{Z_n}, \mathcal{F}) \to \underline{\Gamma}_Z(\mathcal{F}) \text{ for all } n$$

compatible with the natural maps $\underline{Hom}_X(\mathcal{O}_{Z_n}, \mathcal{F}) \to \underline{Hom}_X(\mathcal{O}_{Z_{n+1}}, \mathcal{F})$. In this way we obtain a natural *isomorphism*

$$\operatorname{co}\lim_{n\to\infty} \underline{Hom}_X(\mathcal{O}_{Z_n}, \mathcal{F}) \to \underline{\Gamma}_Z(\mathcal{F})$$

That this natural map is indeed an isomorphism can be checked affine locally on X - the affine local verification is Hartshorne exercise **INCLUDE THE REFERENCE**.

The upshot is that there's a natural isomorphism of derived functors

$$R\operatorname{co}\lim_{n\to\infty} \underline{Hom}_X(\mathcal{O}_{Z_n}, -) \to R\underline{\Gamma}_Z$$

Remark 2.10. While it's tempting to pull the colimit to the left of the R, that only makes sense after taking cohomology. See **INCLUDE A REFERENCE**. So, it is the case that there are natural isomorphisms

$$\operatorname{co}\lim_{n\to\infty} \underline{Ext}^i_X(\mathcal{O}_{Z_n}, \mathcal{F}) \simeq \mathcal{H}^i_Z(\mathcal{F})$$

Remark 2.11. A related fact is that the natural map

$$j_*(\mathcal{F}|_U) \xrightarrow{\simeq} \operatorname{co}\lim_{n\to\infty} \underline{Hom}_X(\mathcal{I}_Z^n, \mathcal{F})$$

(where $U := X \setminus Z$ and $j : U \to X$ is the inclusion) is also an isomorphism. We obtain a quasi-isomorphism

$$j_*(\mathcal{F}|_U) \xrightarrow{\simeq} R\operatorname{co}\lim_{n\to\infty} \underline{Hom}_X(\mathcal{I}_Z^n, \mathcal{F})$$

Note that from this perspective the long exact sequence

$$\cdots \to H^i_Z(\mathcal{F}) \to H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}|_U) \to \ldots$$

can be obtained from the short exact sequences

$$0 \to \mathcal{I}_Z^n \to \mathcal{O}_X \to \mathcal{O}_{Z_n} \to 0$$

applying $\mathrm{Hom}_X(-, \mathcal{F})$ to get an Ext long exact sequence, and then taking the colimit over n.

2.2.1. *Injective hulls.* Let $\mathcal{A}$ be an abelian category. An **essential injection** $\iota : M \to N$ in $\mathcal{A}$ is an injection with the property that for all non-zero subobjects $N' \subset N$, $M \cap N' \neq 0$.

Lemma 2.12. *Let A be a commutative ring and let $M \subset N$ be A-modules. Then $\iota : M \to N$ is an essential injection if and only if for each non-zero $n \in N$ there's a $f \in A$ so that $fn \in M$ and $fn \neq 0$.*

Definition 2.13. Let $\mathcal{A}$ be an abelian category and let M be an object of $\mathcal{A}$. An injective hull of M is an essential injection $\iota : M \to I$ where I is an injective object of $\mathcal{A}$.

Let A be a commutative ring.

Lemma 2.14. *Every A-module has an injective hull.*

The proof above lemma makes use of the following souped-up version of Baer's criterion (well, along with Zorn's lemma and the fact that A-mod has enough injectives): an A-module I is injective if and only if every essential injection $I \to J$ is an isomorphism.

Taking injective hulls is *almost* functorial, in the following sense:

Proposition 2.15. *Let M, N be A-modules with injective hulls E_M, E_N respectively, and let $\varphi : M \to N$ be a homomorphism. Then there's a homomorphism $\psi : E_M \to E_N$ making the diagram*

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \downarrow & & \downarrow \\ E_M & \xrightarrow{\psi} & E_N \end{array} \tag{2.1}$$

commute and with the properties that

- *if φ is injective so is ψ.*
- *if φ is an essential injection then ψ is an isomorphism.*

In particular we see that E_M is unique up to (not necessarily unique) isomorphism.

Proposition 2.16. *Let $\mathfrak{p} \subset A$ be a prime ideal and let $E_{\mathfrak{p}}$ be an injective hull for $A/\mathfrak{p}$. Then*

- *$E_{\mathfrak{p}}$ is indecomposable.*
- *It is also the injective hull of $k(p)$ (over both A and $A_{\mathfrak{p}}$). Here $k(p)$ is the residue field of $\mathfrak{p}$.*

Moreover if A is noetherian then every indecomposable injective A-module arises in this way, so that every injective A-module I decomposes like $I \simeq \bigoplus_i E_{\mathfrak{p}_i}$ where the $\mathfrak{p}_i$ are an indexed set of primes in A (possibly with repetitions).

In the last statement we're using the (trivial) fact that every short exact sequence of injectives splits.

Here's the fun fact:

Proposition 2.17. *Let A be a noetherian local ring with maximal ideal m and residue field k, and let ω^* be a normalized dualizing complex on A. Recall this means*

$$\mathrm{Ext}^i_A(k, \omega^*) = \begin{cases} k & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $R\underline{\Gamma}_x(\omega^*)$ is quasi-isomorphic to an injective hull of k over A.

I'll just sketch the argument: let $k \to R\underline{\Gamma}_x(\omega^*)$ be the natural map, and recall that

$$R\underline{\Gamma}_x(\omega^*) \simeq R\mathrm{co}\lim_{n \to \infty} Hom_A(A/\mathfrak{m}^n, \omega^*)$$

so that $\mathscr{H}^i_x(\omega^*) \simeq \mathrm{co}\lim_{n\to\infty} \mathrm{Ext}^i_A(A/\mathfrak{m}^n, \omega^*)$ for all i. Now using the fact that $\mathrm{Ext}^i_A(k, \omega^*) = 0$ for $i \neq 0$ as a base case one shows that

$$\mathrm{Ext}^i_A(A/\mathfrak{m}^n, \omega^*) = 0 \text{ for all } n \text{ when } i \neq 0$$

The hard part is showing that $\mathrm{co}\lim_{n\to\infty} \mathrm{Hom}_A(A/\mathfrak{m}^n, \omega^*)$ is an injective hull for k over A. This is a theorem from Grothendieck's local cohomology paper **INCLUDE REFERENCE**.

Now let M be an A-module. Observe that there's a natural map

$$\Gamma_m M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M, \omega^*), \Gamma_m \omega^*)$$

(if $\sigma \in M$ is $\mathfrak{m}$-torsion then evaluating at σ sends a homomorphism $\varphi \in \mathrm{Hom}_A(M, \omega^*)$ to an "$\mathfrak{m}$-torsion element" $\varphi(\sigma)$ in $\Gamma_m \omega^*$. This extends to a natural transformation of derived functors

$$R\Gamma_m M \to \mathrm{Hom}_A(R\mathrm{Hom}_A(M, \omega^*), R\Gamma_m \omega^*)$$

Note that I've omitted an R on the Hom on the right hand side. This is because we know $\Gamma_m \omega^*$ is an injective hull for k- so in particular it's injective. The content of the local duality theorem is that this is an isomorphism:

Theorem 2.18. *The natural map $R\Gamma_m M \to \mathrm{Hom}_A(R\mathrm{Hom}_A(M, \omega^*), R\Gamma_m \omega^*)$ is a quasi-isomorphism.*

The local duality isomorphism above can be written in a more compact form if we write $D(M) := R\mathrm{Hom}_A(M, \omega^*)$ (so D is the dualizing functor associated to ω^*) and $I := R\Gamma_m \omega^*$ (so I is an injective hull of k). Then we have

$$R\Gamma_m M \simeq \mathrm{Hom}_A(D(M), I)$$

Taking cohomology we obtain isomorphisms

$$H^i_{\mathfrak{m}}(M) \simeq \mathrm{Hom}_A(\mathrm{Ext}^{-i}_A(M, \omega^*), I) \text{ for all } i$$

Applying $\mathrm{Hom}_A(-, I)$ one more time yields isomorphisms

$$\begin{aligned} \mathrm{Hom}_A(H^{-i}_{\mathfrak{m}}(M), I) &\simeq \mathrm{Hom}_A(\mathrm{Hom}_A(\mathrm{Ext}^i_A(M, \omega^*), I), I) \\ &\simeq \mathrm{Ext}^i_A(M, \omega^*)^\wedge \end{aligned}$$

where the $\wedge$ denotes $\mathfrak{m}$-adic completion. To see where the completion comes from, recall that

$$I \simeq R\Gamma_m \omega^* \simeq R\mathrm{co}\lim \mathrm{Hom}_A(A/\mathfrak{m}^n, \omega^*)$$

so that

$$\begin{aligned} &\mathrm{Hom}_A(\mathrm{Hom}_A(\mathrm{Ext}^{-i}_A(M, \omega^*), I), I) \\ \simeq\ &\mathrm{Hom}_A(\mathrm{Hom}_A(\mathrm{Ext}^{-i}_A(M, \omega^*), R\mathrm{co}\lim \mathrm{Hom}_A(A/\mathfrak{m}^n, \omega^*)), R\mathrm{co}\lim \mathrm{Hom}_A(A/\mathfrak{m}^n, \omega^*)) \end{aligned}$$

Supposing for a moment that we could just drop the R on $R\mathrm{co}\lim \mathrm{Hom}_A(A/\mathfrak{m}^n, \omega^*)$ (which isn't unreasonable, as this complex has cohomology only in degree 0) (one has to be more cautious than I'm being right now) we'd be left with

$$\begin{aligned} &\mathrm{co}\lim_{n'} \lim_n \mathrm{Hom}_A(\mathrm{Hom}_A(\mathrm{Ext}^i_A(M, \omega^*), \mathrm{Hom}_A(A/\mathfrak{m}^n, \omega^*)), \mathrm{Hom}_A(A/\mathfrak{m}^{n'}, \omega^*))) \\ \simeq\ &\mathrm{co}\lim_{n'} \lim_n \mathrm{Hom}_A(\mathrm{Hom}_A(\mathrm{Ext}^i_A(M, \omega^*) \otimes A/\mathfrak{m}^n, \omega^*) \otimes A/\mathfrak{m}^{n'}, \omega^*) \text{ by tensor-hom adjunction} \\ \simeq\ &\mathrm{co}\lim_{n'} \lim_n \mathrm{Ext}^i_A(M, \omega^*) \otimes A/\mathfrak{m}^{n+n'} \simeq \mathrm{Ext}^i_A(M, \omega^*)^\wedge \text{ since } \omega^* \text{ is dualizing} \end{aligned}$$

Example 2.19. Suppose A is Gorenstein of dimension d (e.g. it might be regular). Then $A[d]$ is a dualizing complex for A. Applying the above machinery we see that

$$R\Gamma_{\mathfrak{m}} A[d] \simeq \mathrm{co}\lim_n \mathrm{Ext}^d_A(A/\mathfrak{m}^n, A)$$

is an injective hull for k - call it I for short. Now for any A-module M with there are natural isomorphisms

$$\begin{aligned} H^i_{\mathfrak{m}}(M) &\simeq \mathrm{Hom}_A(\mathrm{Ext}^{d-i}_A(M, A), I) \text{ and} \\ \mathrm{Ext}^i_A(M, A)^\wedge &\simeq \mathrm{Hom}_A(H^{d-i}_{\mathfrak{m}}(M), I) \end{aligned}$$

Example 2.20. Let's get even more explicit - let X be a smooth curve over a field k and let $x \in X$ be a closed point. Let $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ be the maximal ideal and let $k(x)$ be the residue field. Then we see that

$$\mathcal{H}^1_x(\mathcal{O}_X) \simeq \mathrm{co}\lim_n Ext^1_X(\mathcal{O}_{X,x}/\mathfrak{m}^n_x, \mathcal{O}_{X,x})$$

To calculate the right hand side let $t \in \mathfrak{m}$ be a local parameter, and observe that

$$0 \to \mathcal{O}_{X,x} \xrightarrow{t^n} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}^n \to 0$$

is a free resolution of $\mathcal{O}_{X,x}/\mathfrak{m}^n$, and so $Ext^1_X(\mathcal{O}_{X,x}/\mathfrak{m}^n_x, \mathcal{O}_{X,x})$ is the cokernel of $\mathcal{O}_{X,x} \xrightarrow{t^n} \mathcal{O}_{X,x}$, namely $\mathcal{O}_{X,x}/\mathfrak{m}^n$. Hence we see that

$$Ext^1_X(\mathcal{O}_{X,x}/\mathfrak{m}^n_x, \mathcal{O}_{X,x}) \simeq \mathrm{co}\lim_n \mathcal{O}_{X,x}/\mathfrak{m}^n$$

the colimit being taken over the maps $\mathcal{O}_{X,x}/\mathfrak{m}^n \xrightarrow{t} \mathcal{O}_{X,x}/\mathfrak{m}^{n+1}$.

Remark 2.21. The above discussion is even interesting on Spec$\mathbb{Z}$- if p is a prime, we see that

$$H^1_p(\mathbb{Z}) \simeq \operatorname{co}\lim_n \mathbb{Z}/p^n, \text{ a.k.a. the Prufer group.}$$

The right hand side can be viewed as the elements of $\mathbb{Q}/\mathbb{Z}$ which are annihilated by some power of p (or alternatively as the union of the p^n-th roots of unity over all n).

Now let X be a locally noetherian scheme with a dualizing complex ω^*. For each point $x \in X$, ω^* is a dualizing complex on Spec$\mathcal{O}_{X,x}$, and so there's a $d(x) \in \mathbb{Z}$ so

$$\operatorname{Ext}^i_{\mathcal{O}_{X,x}}(k(x), \omega^*) = \begin{cases} k(x) & \text{if } i = d(x) \\ 0 & \text{otherwise} \end{cases}$$

In this way we obtain a function $d : X \to \mathbb{Z}$. One can show that if $x \rightsquigarrow y$ is an immediate specialization, then $d(y) = d(x) + 1$, which means d is an example of

Definition 2.22. A **codimension function** on a scheme X is a function $d : X \to \mathbb{Z}$ with the property that whenever $x, y \in X$ and $x \rightsquigarrow y$ is an immediate specialization,

$$d(y) = d(x) + 1$$

It's not hard to show that if X admits a codimension function then it is *catenary*, which is to say that whenever $x, y \in X$ and $x \rightsquigarrow y$ is a specialization, *every* sequence if immediate specializations

$$x = x_0 \rightsquigarrow x_1 \rightsquigarrow \cdots x_r = y$$

has the same length. The interesting upshot here is

Proposition 2.23. *If X is a locally noetherian scheme that admits a dualizing complex, then X is catenary.*

Using the codimension function d associated to a dualizing complex ω^* on X we can define a descending filtration

$$X = Z^0_\omega \supset Z^1_\omega \supset Z^2_\omega \supset \dots$$

of X by specialization closed sets with the property that for every i, each point of $Z^i_\omega \setminus Z^{i+1}_\omega$ is maximal (with respect to specialization) - just set

$$Z^i_\omega := \{x \in X \mid d(x) \geq i\}$$

Essentially by the definition of d we obtain

Proposition 2.24. *The complex ω^* is Gorenstein for the filtration Z^*_ω.*

Definition 2.25. Let X be a locally noetherian scheme. A dualizing complex ω^* on X is **normalized** if and only if the associated codimension function d on X satisfies

$$d(x) = \operatorname{codim}(x, X)$$

which is to say, *d is the* codimension function on X. Hopefully you know what I am trying to say here. .

To see that this notion of Gorenstein-ness coincides with more traditional definitions, we can appeal to the following pleasant "TFAE": **INCLUDE REFERENCE to R and D or Grothendieck's local cohomology paper.**

Proposition 2.26. *Let A be a noetherian local ring with maximal ideal m and residue field k. Then the following are equivalent:*

(1) A is a dualizing complex on A.
(2) A has finite injective dimension.

(3) *There's a $d \in \mathbb{Z}$ so that*

$$\mathrm{Ext}^i_A(k, A) = \begin{cases} k & \text{if } i = d \\ 0 & \text{otherwise} \end{cases}$$

(4) *There's a $d \in \mathbb{Z}$ so that*

$$H^i_m(A) = \begin{cases} \text{an injective } A - \text{module} & \text{if} i = d \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.27. A locally noetherian scheme is **Gorenstein** if and only if all of its local rings are.

Putting things together we obtain:

Proposition 2.28. *Let X be a locally noetherian scheme with a dualizing complex ω^*. Then X is Gorenstein if and only if ω^* is an invertible sheaf (possibly shifted).*

Remark 2.29. Recall that when X is connected, the condition that ω^* is a shifted invertible sheaf is equivalent to the condition that ω^* is a tensor-invertible object of $D^b_c(X)$.

Proposition 2.30. *Let Y be a locally noetherian scheme and let $f : X \to Y$ be a flat morphisms of finite type. Recall that the functor $f^! : D^b_c(Y) \to D^b_C(X)$ is given by*

$$f^!\mathcal{F}^* := D_X(Lf^* D_Y \mathcal{F}^*) \text{ where } D_X, D_Y \text{ are the dualizing functors of } X, Y \text{ respectively.}$$

The complex $f^!$ is supported in one degree if and only if all fibers X_y of f are Cohen-Macaulay. It's and invertible sheaf up to a shift if and only if all fibers are Gorenstein.

Remark 2.31. Using the above proposition, together with functoriality/base change properties of the shriek functor **CITE CONRAD's BOOK HERE**, I think we can prove an "inversion-of-adjunction" statement for both Cohen-Macaulay and Gorenstein singularities. Which is to say, if $f : X \to Y$ is a proper, flat morphism of finite type and Y is locally noetherian, then if X_y is a Cohen-Macaulay fiber of f there's a neighborhood $U \subset Y$ of y so that for all $z \in U$, X_z is Cohen-Macaulay. Similarly for the Gorenstein condition.

2.2.2. *Residual complexes.*

Definition 2.32. Let X be a locally noetherian scheme and let $x \in X$ be a point. Let I_x be the injective hull of $k(x)$ as a module over $\mathcal{O}_{X,x}$. Then set

$$J(x) := \iota_{x*} I_x \text{ where } \iota_x : \{x\} \to X \text{ is the inclusion}$$

Remark 2.33. $J(x)$ is an injective quasi-coherent sheaf supported on $\{\bar{x}\}$.

Applying local Grothendieck duality, we see that

Proposition 2.34. *If X is a locally noetherian scheme with a* normalized *dualizing complex ω^*, then for each $x \in X$,*

$$H^0_x(\omega^*) \simeq I_x, \text{ and hence } R\underline{\Gamma}_x \omega^* \simeq J(x)$$

Generalizing this a bit,

Definition 2.35. A **residual complex** on X is a bounded-below complex K^* of injective quasi-coherent sheaves, with coherent cohomology, such that

$$\bigoplus_{p \in \mathbb{Z}} K^p \simeq \bigoplus_{x \in X} J(x)$$

Remark 2.36. Note that we are just comparing the K^p and the $J(x)$ at the level of quasi-coherent sheaves on X- we are not worrying about grading of complexes on either side of the above equation.

Example 2.37. A mild generalization of the above proposition shows that if ω^* is a dualizing complex on X and Z^*_ω is the induced codimension filtration of X, then the Cousin complex of ω^* with respect to Z^*_ω is a residual complex.

Residual complexes are an important technical tool in the "standard" developments of Grothendieck duality theory, and their usefulness stems from the following result.

Proposition 2.38. *Assume X is a locally noetherian scheme with a dualizing complex. Then the functor*

$$E : \{\text{dualizing complexes } \omega^* \text{ in } D^b_c(X)\} \to \{\text{residual complexes } K^* \text{ in } \mathcal{K}(I)\}$$

(where $\mathcal{K}(I)$ denotes the homotopy category of bounded below complexes of injectives) defined by

$$E(\omega^*) = \text{ the Cousin complex of } \omega^* \text{ with respect to the filtration } Z^*_\omega$$

is an equivalence of full subcategories. The inverse equivalence is obtained by restricting the usual functor $\mathcal{K}(I) \to D^+(X)$ to the full subcategory of residual complexes.

Moreover if $K^*, \tilde{K}^*$ are two residual complexes on X then there is a one-to-one correspondence

$$\mathrm{Hom}(K^*, \tilde{K}^*) \simeq \{\text{specialization-compatible collections of homomorphisms } \psi_x : K^*_x \to \tilde{K}^*_x \text{ in } D^+_c(\mathrm{Spec}\mathcal{O}_{X,x})\}$$

Here "specialization-compatible" means that whenever $x \rightsquigarrow y$ is a specialization, the map ψ_y is obtained from ψ_x by localization.

Remark 2.39. It's not totally clear to me what the induced map of cousin complexes $E(\omega^*) \to E(\tilde{\omega}^*)$ is supposed to be, as we can expect the filtrations Z^*_ω and $Z^*_{\tilde{\omega}}$ to differ by a shift... one would have to think about how the two filtrations relate and what exactly is the correct functoriality statement for the spectral sequence of a filtered space.

Finally, I will conclude this section with a fun example.

Example 2.40. The normalized dualizing complex of $\mathrm{Spec}\mathbb{Z}$ is $\mathbb{Z}[1]$. Indeed, $\mathrm{Spec}\mathbb{Z}$ is regular, so we know that its structure sheaf is a pointwise dualizing complex. Since $\mathrm{Spec}\mathbb{Z}$ is 1-dimensional, to normalize we must shift by 1; the end result is $\omega^* = \mathbb{Z}[1]$.

Note that for a point $p \in \mathrm{Spec}\mathbb{Z}$ we have

$$\mathrm{Ext}^i_{\mathbb{Z}_p}(k(p), \mathbb{Z}[1]) = \mathrm{Ext}^{i+1}_{\mathbb{Z}_p}(k(p), \mathbb{Z})$$

and so the codimension function d associated to $\mathbb{Z}[1]$ is

$$d(p) := \mathrm{codim}(p, \mathrm{Spec}\mathbb{Z}) - 1$$

It follows that the Cousin complex of ω^* looks like

$$0 \to H^0_\eta(\mathbb{Z}) \to \bigoplus_p H^1_p(\mathbb{Z}) \to 0$$

where $\eta \in \mathrm{Spec}\mathbb{Z}$ is the generic point and p runs over the closed points, aka the prime numbers, and the term $H^0_\eta(\mathbb{Z})$ sits in degree -1. Now, clearly $H^0_\eta(\mathbb{Z}) = \mathbb{Q}$ and as discussed above we have

$$H^1_p(\mathbb{Z}) = \mathrm{co}\lim_{n\to\infty} \mathbb{Z}/p^n$$

and so we can write the Cousin complex as

$$0 \to \mathbb{Q} \to \bigoplus_p \mathrm{co}\lim_{n\to\infty} \mathbb{Z}/p^n \to 0$$

Under the isomorphism

$$\mathbb{Q}/\mathbb{Z} \simeq \bigoplus_p \mathrm{co}\lim_{n\to\infty} \mathbb{Z}/p^n$$

(obtained by decomposing $\mathbb{Q}/\mathbb{Z}$ into the direct sum of its p-torsion components), this is just the usual injective resolution $0 \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ of $\mathbb{Z}$, shifted by 1.

3. Grothendieck duality

The first global question to address is when a (locally noetherian) scheme X admits a dualizing complex. We have seen that when X is Gorenstein it has a dualizing complex (e.g. the structure sheaf).

Let $f : X \to Y$ be a finite morphism, with Y a locally noetherian scheme. Recall that since f is a finite morphism, f_* has a *right* adjoint $f^\flat$, defined as

$$f^\flat \mathcal{G} := \underline{Hom}_Y(f_*\mathcal{O}_X, \mathcal{G})$$

More precisely, $f^\flat \mathcal{G}$ is the quasi-coherent sheaf on X associated to the quasi-coherent sheaf of $f_*\mathcal{O}_X$ modules $\underline{Hom}_Y(f_*\mathcal{O}_X, \mathcal{G})$. The derived adjunction statement is that for coherent sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y,

$$R\underline{Hom}_Y(f_*\mathcal{F}, \mathcal{G}) \simeq f_* R\underline{Hom}_Y(\mathcal{F}, Rf^\flat \mathcal{G})$$

Proposition 3.1. *If ω_Y^* is a dualizing complex on Y then $f^\flat \omega_Y^*$ is a dualizing complex on X.*

Proof. We must show that the natural map

$$\mathcal{O}_X \to R\underline{Hom}_X(R\underline{Hom}_X(\mathcal{O}_X, f^\flat \omega_Y^*), \omega_Y^*)$$

is an isomorphism. Applying the exact functor f_* we can view these as complexes on Y, and then adjunction gives

$$f_* R\underline{Hom}_X(R\underline{Hom}_X(\mathcal{O}_X, f^\flat \omega_Y^*), \omega_Y^*) \simeq$$

□

Theorem 3.2 (Grothendieck duality)**.** *Let $f : X \to Y$ be a proper morphism of finite-dimensional noetherian schemes admitting dualizing complexes ω_X^* and ω_Y^* respectively (for example X and Y could be schemes of finite type over k). Then for any object $\mathcal{F}^*$ in the bounded derived category $D_c^b(X)$ of X there is a natural isomorphism*

$$Rf_* R\underline{Hom}_X(\mathcal{F}^*, \omega_X^*) \simeq R\underline{Hom}_Y(Rf_*\mathcal{F}^*, \omega_Y^*) \text{ in } D_c^b(Y)$$

Remark 3.3. It is a theorem of Kawasaki **INCLUDE REFERENCE** that a scheme X admits a dualizing complex if and only if it is locally embeddable in a Gorenstein scheme. From this perspective it's clear that any scheme of finite type over k has a dualizing complex (since locally it can be embedded in affine space).