COBORDISM AND FORMAL GROUP LAWS

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ABSTRACT. There's an amazing connection between complex cobordism and the theory of formal group laws. In this talk I will try to describe:

- what I mean by a complex cobordism.
- what I mean by a formal group law.
- what the first two bullet points have to do with each other.

1. COMPLEX COBORDISM

1.1. **Relative stable normal bundles.** Let $f : M \to X$ be a map of smooth manifolds. Observe that for sufficiently large $n \in \mathbb{N}$ we may factor f through an embedding $\iota : M \to X \times \mathbb{R}^n$ over X fitting into:

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[d, "f"'] & X \times \mathbb{R}^n \arrow[d, "\pi"'] \\ X \arrow[r, equal] & X \end{tikzcd} \tag{1.1}$$

Indeed by the Whitney embedding theorem once $n \geq 2\dim M + 1$ we can find an embedding $g : M \to \mathbb{R}^n$, and crossing this with the map $f : M \to X$ will yield an embedding $\iota = f \times g : M \to X \times \mathbb{R}^n$ as above.

Remark 1.1. Thus every map in the category of smooth manifolds is affine!

Note that the tangent bundle of $X \times \mathbb{R}^n$ can be computed as $\tau_{X \times \mathbb{R}^n} = \tau_X \times \tau_{\mathbb{R}^n} = \pi^* \tau_X \oplus \epsilon^n$ where by ϵ^n I mean the trivial rank n real vector bundle over $X \times \mathbb{R}^n$. Over M we'll have a short exact sequence of real vector bundles

$$0 \to \tau_M \xrightarrow{d\iota} \pi^* \tau_X \oplus \epsilon^n|_M \to \nu_{M|X \times \mathbb{R}^n} \to 0$$

where by $\nu_{M|X\times\mathbb{R}^n}$ I mean the normal bundle of M in $X \times \mathbb{R}^n$. Notice that $\pi^*\tau_X \oplus \epsilon^n|_M = f^*\tau_M \oplus \epsilon^n$ and so we can rewrite this short exact sequence as

$$0 \to \tau_M \to f^* \tau_X \oplus \epsilon^n \to \nu_{M|X \times \mathbb{R}^n} \to 0$$

The idea is that this short exact sequence gives us an equation

$$\nu_{M|X\times\mathbb{R}^n} - \epsilon^n = f^*\tau_X - \tau_M$$

in the K-theory of real vector bundles over M, usually written as $KO^0(M)$. The left hand side is to be understood as the "stable normal bundle" of the map f (for instance, it really would be the class of the normal bundle of M in X, were f an embedding). The above discussion motivates the following definition:

Definition 1.2. The **relative stable normal bundle** of a smooth map $f : M \to X$ is the class $\nu_f := f^*\tau_X - \tau_M \in KO^0(M)$.

Date: September 2016.

Remark 1.3. "Rank" determines a well defined ring homomorphism $KO^0(M) \to \mathbb{Z}$, and we can see that

$$\mathrm{rank}\nu_f = \mathrm{rank} f^*\tau_X - \mathrm{rank}\tau_M = \dim X - \dim M$$

which should be understood as the **codimension of the map** f (for instance, if f were an embedding it would be the codimension of M in X).

One can also consider the K-theory of complex vector bundles over M-this is usually denoted by $K^0(M)$. Forgetting about complex structures (i.e. taking underlying real vector bundles) gives a homomorphism of abelian groups $K^0(M) \to KO^0(M)$ (it's not a ring homomorphism because things get wonky with tensor products and ranks).

Definition 1.4. A **complex orientation of a K-theory class** $\alpha \in KO^0(M)$ is a chosen class $\tilde{\alpha} \in K^0(X)$ mapping to α under the homomorphism $K^0(M) \to KO^0(M)$.

This leads to the following definition:

Definition 1.5. Let $f : M \to X$ be a smooth map. A **complex orientation of** f is a complex orientation of the relative stable normal bundle $\nu_f \in KO^0(M)$.

Recall that a smooth map $f : M \to X$ is **proper** if for every compact set $K \subset X$ the preimage $f^{-1}(K) \subset M$ is compact. The complex cobordism groups of X will consist of proper complex oriented maps $f : M \to X$ modulo the *cobordism relation* defined below:

Definition 1.6. A **cobordism** between two proper complex oriented maps $f_0 : M_0 \to X$ and $f_1 : M_1 \to X$ of codimension n is a proper complex oriented map $h : W \to X \times \mathbb{R}$ of codimension n which is transverse to both of the inclusions $\iota_i : X \simeq X \times \{i\} \subset X \times \mathbb{R}$ (for $i = 0, 1$) together with isomorphisms

$$\iota_0^* W \simeq M_0 \text{ and } \iota_1^* W \simeq \bar{M}_1$$

Here $\bar{M}_1$ denotes M_1 with the "opposite complex structure." I'll come back to this time permitting.

One can show that cobordism defines an equivalence relation on the proper, complex oriented smooth manifolds over X of codimension n.

Remark 1.7. In writing things like "$\iota_0^* W$" I'm implicitly appealing to the fact that proper maps are preserved under base change and if

$$\begin{array}{ccc} M' & \xrightarrow{g'} & M \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ X' & \xrightarrow{g} & X \end{array} \tag{1.2}$$

is a cartesian diagram of smooth manifolds with g transverse to f then $\nu_{f'} = g'^* \nu_f \in KO^0(M')$ - this is the sense in which "relative stable normal bundles pull back."

Definition 1.8. The n**th complex cobordism group of** X is the abelian group $\Omega_U^n(X)$ of cobordism classes of proper complex oriented smooth manifolds over X with codimension n. The group operation is disjoint union.

In fact the $\Omega_U^n(X)$ come together to form a graded commutative ring $\Omega_U^*(X)$, called the **complex cobordism ring of** X. The multiplication is, roughly, "fiber product over X," however one must make use of transversality to stay within the category of smooth manifolds.

Speaking of transversality: let $\varphi : Y \to X$ be a map of smooth manifolds. If $f : M \to X$ is a proper complex oriented smooth manifold over X, say with codimension n, then one can always find a smooth map $\tilde{\varphi} : Y \to X$ transverse to f, in which case the pullback $\tilde{\varphi}^* M$ fitting into the cartesian diagram

$$\begin{array}{ccc} \tilde{\varphi}^* M & \longrightarrow & M \\ \Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{\tilde{\varphi}} & X \end{array} \tag{1.3}$$

will be a proper complex oriented smooth manifold over Y of codimension n. One can show that its cobordism class depends only on the cobordism class of M over X and the homotopy class of φ, and in this way one obtains a homomorphism (of graded commutative rings)

$$\varphi^* : \Omega_U^*(X) \to \Omega_U^*(Y)$$

On the other hand, suppose $\varphi : Y \to X$ is itself a proper complex oriented smooth map, say of codimension d. Now if $f : M \to Y$ is a proper complex oriented smooth map of codimension n, I claim the composition

$$M \xrightarrow{f} Y \xrightarrow{\varphi} X$$

is a proper complex oriented smooth map of codimension $n + d$. Indeed, the codimension calculation is purely numerological, the composition of two proper maps is always proper, and the stable normal bundle of M over X can be calculated as

$$\begin{aligned} \nu_{\varphi \circ f} &= \tau_M - f^* \varphi^* \tau_X = \tau_M - f^* \tau_Y + f^* \tau_Y - f^* \varphi^* \tau_X \\ &= \tau_M - f^* \tau_Y + f^* (\tau_Y - \varphi^* \tau_X) = \nu_f + f^* \nu_\varphi \end{aligned}$$

So, the complex orientations of ν_f and ν_φ determine a complex orientation of $\nu_{\varphi \circ f}$. In this way we obtain a homomorphism of graded abelian groups

$$\varphi_* : \Omega_U^*(Y) \to \Omega_U^*(X)[d]$$

(what I'm trying to say is φ_* raises degrees by d). In fact this is a homomorphism of graded $\Omega_U^*(X)$-modules, and the composition

$$\Omega_U^*(X) \xrightarrow{\varphi^*} \Omega_U^*(Y) \xrightarrow{\varphi_*} \Omega_U^*(X)$$

is multiplication by the class of $\varphi : Y \to X$.

Remark 1.9. This is what makes Ω_U^* a "complex-oriented cohomology theory." All cohomology theories come with pullback homomorphisms - the presence of pushforward homomorphisms for proper complex oriented maps says something special about Ω_U^*.

I should probably give a precise definition:

Definition 1.10. A **complex oriented cohomology theory** is a commutative multiplicative cohomology theory h^* together with functorial "transfer maps"

$$f_* : h^*(Y) \to h^*(X)[d]$$

of $h^*(X)$-modules for every proper complex oriented smooth map $f : Y \to X$ of codimension d (i.e. these transfers must make h^* a *covariant functor* on the category of smooth manifolds with proper complex oriented smooth maps), subject to the following additional restriction: if

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{g'} & Z \\ {\scriptstyle f'}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & X \end{array} \tag{1.4}$$

is a cartesian diagram in the category of smooth manifolds where f is proper and complex oriented of codimension d, g is transverse to f and f' is given the pulled-back complex orientation, then

$$g^* \circ f_* = f'_* g'^* : h^*(Z) \to h^*(Y)[d]$$

Theorem 1.11. *Let h^* be a complex oriented cohomology theory as above. Then there exists a unique morphism of complex oriented cohomology theories $\Omega_U^* \to h^*$.*

Thus complex cobordism is the universal complex oriented cohomology theory.

Idea (not a proof!) Let X be a smooth manifold and let $f : M \to X$ be a proper complex oriented map of codimension d, defining a class in $[f]\Omega_U^d(X)$. *If* there is a morphism of complex oriented cohomology theories $\Omega_U^* \to h^*$, the homomorphism $\Omega_U^d(X) \to h^d(X)$ *must* send $[f] \mapsto f_*1 \in h^d(X)$ where by 1 I mean the element $1 \in h^0(M)$.

Now one must argue that sending $[f] \mapsto f_*1$ actually does the trick. However I should emphasize that this whole discussion is highly non-standard and the better way to do all of this involves working in the stable homotopy category.

□

I should point out that based on what I've said so far the calculation of cobordism groups seems like a hopelessly difficult task, even when $X = \mathrm{pt}$, in which case one is attempting to classify smooth manifolds with stably complex normal bundles up to cobordism! It's a beautiful theorem that this task can be viewed as a problem in stable homotopy theory:

Let γ_n be the tautological rank n complex vector bundle over the usual classifying space $BU(n)$ for $U(n)$, by which I mean the Grassmannian $G_n\mathbb{C}^\infty$ of n-dimensional subspaces of $\mathbb{C}^\infty$. It's **Thom space** $MU(n) := \mathrm{Th}\gamma_n$ is obtained by adjoining a common point at infinity for all the vector space fibers (there are easy ways to make this precise, for instance take the disk bundle of γ_n and crush the sphere bundle boundary to a point.) For each n there is a canonical map of vector bundles $\gamma_n \oplus \epsilon \to \gamma_{n+1}$ giving isomorphisms on fibers, and from this one obtains an induced map of Thom spaces $\Sigma^2 MU(n) \to MU(n+1)$. One can show that these spaces $MU(n)$, together with the above "structure maps" between their suspensions, assemble to form a **spectrum**, known as MU.

Theorem 1.12. *Let X be a manifold. For each $n \in \mathbb{Z}$ there is a natural isomorphism*

$$\Omega_U^n(X) \simeq [\Sigma^\infty X_+, MU]_{-n}$$

The right hand side denotes degree $-n$ stable homotopy classes of maps from the suspension spectrum $\Sigma^\infty X_+$ of X_+ (X with a disjoint basepoint) to MU.

I'm not going to define these stable-homotopy-theoretic objects in any sort of generality. For today the following special case is plenty interesting: when $X = \mathrm{pt}$, one has

$$\Omega_U^n(\mathrm{pt}) = [\Sigma^\infty \mathrm{pt}_+, MU]_{-n} = \operatorname{co}\lim_i \pi_{-n+2i} MU(2i)$$

where the homotopy groups $\pi_{-n+2i}MU(2i)$ on the far right are just regular old homotopy groups. Thus the cobordism ring of a point can be computed as the stable homotopy ring of the spectrum MU.

In one of the first major applications of the **Adams spectral sequence** (a spectral sequence relating cohomology groups to stable homotopy groups) Milnor obtained the following beautiful result:

Theorem 1.13. *The cobordism ring $\Omega_U^*(\mathrm{pt})$ is a graded polynomial ring of the form*

$$\mathbb{Z}[x_i \,|\, i \in \mathbb{N}, i > 0] \text{ where } \deg x_i = -2i$$

I should hasten to point out that the x_i are *not* canonical (and to this day nobody has found a canonical set of generators). I'd now like to describe an even cooler form of the above computation. We'll need a few basic facts:

Proposition 1.14. *Let h^* be a complex oriented cohomology theory.*

Let $\mathbb{C}P^n$ denote n-dimensional complex projective space, and let $x \in h^2(\mathbb{C}P^n)$ denote the class of a linear embedding $\mathbb{C}P^{n-1} \to \mathbb{C}P^n$. Then $x^{n+1} = 0$ and the resulting homomorphism of rings

$$h^*(\mathrm{pt})[x]/(x^{n+1}) \to h^*(\mathbb{C}P^n) \text{ is an isomorphism.}$$

Furthermore the natural map

$$h^*(\mathbb{C}P^\infty) \to \lim \Omega_U^*(\mathbb{C}P^n) = \lim h^*(\mathrm{pt})[x]/(x^{n+1}) = h^*(\mathrm{pt})[[x]]$$

is an isomorphism, and there is a "Kunneth formula" isomorphism

$$h^*(\mathrm{pt})[[x,y]] \simeq h^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

Outline. The standard proof of this fact is a so-called "straightforward application of the Atiyah-Hirzebruch spectral sequence."

□

Next, observe that infinite-dimensional complex projective space $\mathbb{C}P^\infty$ is a topological abelian monoid. Here's one cool way to see it: identify $\mathbb{C}^\infty$ with the polynomials $\mathbb{C}[z]$ in a single variable z. We know that this is an integral domain, and so multiplication gives a map

$$(\mathbb{C}[z] - \{0\}) \times (\mathbb{C}[z] - \{0\}) \xrightarrow{\text{multiply}} \mathbb{C}[z] - \{0\}$$

Since it's bilinear it descends to a map $\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. Using the fact that $\mathbb{C}[x] - \{0\}$ is an abelian monoid under multiplication, it's not hard to show that $\mathbb{C}P^\infty$ is an abelian monoid under the operation μ.

With all notation as above consider the map of graded commutative rings $\mu^* : h^*(\mathbb{C}P^\infty) \to h^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$. Using proposition 1.3 one can identify this with a homomorphism $\mu^* : h^*(\mathrm{pt})[[x]] \to h^*(\mathrm{pt})[[x,y]]$ which of course is equivalent to a degree 2 formal power series

$$\mu(x,y) \in h^*(\mathrm{pt})[[x,y]]$$

The associativity, identity and commutativity of $\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ imply that:

- $\mu(x, \mu(y,z)) = \mu(\mu(x,y),z)$.
- $\mu(x,0) = x$ and $\mu(0,y) = y$.
- $\mu(x,y) = \mu(y,x)$.

Definition 1.15. Let R be a graded commutative ring. A (commutative 1-dimensional) **formal group law** over R is a degree 2 formal power series $\mu(x,y) \in R[[x,y]]$ (where $\deg x = \deg y = 2$) subject to the restrictions given in the above bullet points.

If you're weirded out by this use of the word "group" in the absence of inverses:

Proposition 1.16. *Let $\mu(x,y)$ be a formal group law over a graded commutative ring R. Then there exists a unique degree 2 formal power series $\iota(x) \in R[[x]]$ so that*

$$\mu(x, \iota(x)) = \mu(\iota(x), x) = 0$$

Definition 1.17. A **homomorphism** $\psi : \mu \to \mu'$ of formal group laws over a graded commutative ring R is a degree 2 formal power series $\mu(x) \in R[[x]]$ so that

$$\psi(\mu(x,y)) = \mu(\psi(x), \psi(y))$$

Such a homomorphism is invertible if and only if the coefficient $\psi'(0)$ on x is a unit in R. It's called a **strict isomorphism** if $\psi'(0) = 1$.

For a given graded commutative ring R, one can consider the set $FGL(R)$ of all formal group laws $\mu(x,y)$ over R. Given a homomorphism $\varphi : R \to R'$, one obtains a function $\varphi_* : FGL(R) \to FGL(R')$ by base change along φ, which in this situation has a very down-to-earth interpretation: just send $\mu(x,y) = \sum_{i,j} a_{ij} x^i y^j$ to $\varphi_* \mu(x,y) = \sum_{i,j} \varphi(a_{ij}) x^i y^j$. It's "obvious" that

Theorem 1.18. *The set-valued functor $FGL(-)$ on the category of graded commutative rings is co-representable.*

Proof. A formal group law $\mu(x,y)$ over R is entirely determined by its coefficients, and so it gives us a homomorphism

$$\varphi : \mathbb{Z}[a_{ij} \mid i,j \in \mathbb{N}] \to R$$

taking a_{ij} to the ijth coefficient of $\mu(x,y)$. The restrictions on $\mu(x,y)$ (associativity, identity and commutativity) now impose relations on the images $\varphi(a_{ij})$ of the a_{ij} - for instance, commutativity requires that $\varphi(a_i j) = \varphi(a_{ji})$, identity requires that $\varphi(a_{i0}) = 0$ unless $i = 1$ and $\varphi(a_{0j}) = 0$ unless $j = 1$ and associativity says - well, when we expand the formal power series $\mu(x, \mu(y,z)) = \mu(\mu(x,y),z)$ and compare the coefficients on $x^i y^j z^k$, we get a bunch of polynomial relations like $p_{ijk}(\varphi(a)) = q_{ijk}(\varphi(a))$. Let $I \subset \mathbb{Z}[a_{ij}]$ be the ideal generated by the relations outlined above, and set $L := \mathbb{Z}[a_{ij}]/I$. Then our homomorphism φ evidently factors through a homomorphism $\bar{\varphi} : L \to R$. From here it's not hard to conclude that L co-represents $FGL(-)$.

If you want to keep track of grading issues, you must declare $\deg a_{ij} = 2(1-i-j)$ and point out that all the relations generating I are homogeneous.

□

Remark 1.19. L is known as the **Lazard ring**.

In contrast, it's highly non-obvious that

Theorem 1.20 (Lazard)**.** *L is a polynomial ring of the form $\mathbb{Z}[x_i \mid i \in \mathbb{N}, i > 0]$ where $\deg x_i = -2i$.*

Again I should point out that there's no canonical choice of the generators x_i. I'm not even going to sketch the proof of theorem 1.6 (there are several key parts that I still don't understand after spending multiple days reading it!).

Recall that earlier we obtained an interesting formal group law $\mu(x,y) \in \Omega^*_U(\mathrm{pt})[[x,y]]$. We now know that this corresponds to a homomorphism of graded commutative rings $\varphi : L \to \Omega^*_U(\mathrm{pt})$. The punch line:

Theorem 1.21 (Quillen)**.** *The homomorphism $\varphi : L \to \Omega^*_U(\mathrm{pt})$ corresponding to the formal group law $\mu(x,y) \in \Omega^*_U(\mathrm{pt})[[x,y]]$ is an isomorphism.*

The beauty of this connection between formal group laws and complex cobordism is twofold. On the one hand, it gives an interesting algebraic interpretation of the ring $\Omega^*_U(\mathrm{pt})$, whose definition appeared to be entirely geometric. More importantly, it facilitates the application of algebraic results about formal group laws in cobordism theory. If you want examples of this I have some good ones. But we're probably out of time.