ALGEBRA PRELIM

FALL, 1972

Directions:

1. This exam is divided into four parts (A. Linear Algebra; B. Groups; C. Rings and Modules; D. Fields and Galois Theory). Do two problems in each part, a total of eight problems.

2. Mark clearly which eight problems you wish to have graded. Two half problems do not equal one problem.

3. You should write enough so that there is no doubt as to whether or not you do know what's going on. But use common sense. Don't write a book when a few lines are sufficient. When in doubt, ask.

4. Neatness counts in that illegible or very difficult-to-read exams will not be graded.

A. Linear Algebra

1. Let R be a commutative ring and let M and P be R-modules. Let $M^* = \mathrm{Hom}_R(M,R)$. If N is a submodule of M, let $A(N) = \{f \in M^* \mid f(x)=0$ for all $x \in N\}$.

   a) Let $\phi: M \to P$ be an R-module map. Define a map $\phi^t: P^* \to M^*$ by $(\phi^t(f))(x) = f(\phi(x))$ where $f \in P^*$ and $x \in M$. Prove that ϕ^t is an R-module map.

   b) With the notation of part a), prove that $\ker(\phi^t) = A(\mathrm{Im}(\phi))$.

   c) If R is a field and if $\dim M$, $\dim P < \infty$, prove that

   $$\dim \mathrm{Im}(\phi) = \dim \mathrm{Im}(\phi^t)$$

   and that

   $$\mathrm{Im}(\phi^t) = A(\ker \phi).$$

2. Let R be the field of real numbers and let V be the vector space of all polynomials $p \in R[x]$ such that $\deg p \leq 2$. Define $f_i \in V^* = \mathrm{Hom}_R(V,R)$, $i=1,2,3$, by

$$f_1(p) = \int_0^1 p(x)\,dx$$

$$f_2(p) = \int_0^2 p(x)\,dx$$

$$f_3(p) = \int_0^{-1} p(x)\,dx$$

Show that $\{f_1, f_2, f_3\}$ is a basis for V^*.

3. In each of the following either provide an example with the given properties or tell why no such example exists. In either case, justify your answer.

   a) A 3×3 matrix A such that $A \neq 0$, $A \neq I$ and $A^2 = A$.

   b) A linear transformation T on a three dimensional vector space V such that $T^4 = 0$, but $T^3 \neq 0$.

   c) A 3×3 matrix A such that A is singular but has no non-zero extries.

   d) A linear transformation T of a finite dimensional vector space V <u>onto</u> a finite dimensional vector space W such that T is not one-to-one.

   e) A 4×4 matrix whose minimal polynomial is of degree 5.

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A linear transformation T on a 5-dimensional vector space V satisfies the equation $x^2(x-1)^3 = 0$

and no polynomial of smaller degree. Let

$$M = \{v \in V \mid (T-1)^3(v) = 0\}.$$

a) Show that $T(M) \subseteq M$

b) Show that there is a subspace N of V such that

$$N \cap M = \{0\}$$
$$N + M = V$$
$$T(N) \subseteq N$$

c) If $v \in M$ then there is a polynomial $f(x)$ of minimal degree such that $vf(T) = 0$. Prove that this $f(x)$ must be a divisor of $(x-1)^3$.

B. Groups

1. Let Q be the rational numbers and let Z be the integers. For each prime p let

$$G_p = \{\frac{m}{p^n} \mid m,n \in Z\}.$$

Prove that $Q/Z \cong \bigoplus_p (G_p/Z)$.

2. Let p be a prime and let G be a group of order p^3. Prove that if G contains more than one normal subgroup of order p, then G is abelian and not cyclic.

3. a) State the Sylow theorems.

   Let G be a finite group and let S be a p-Sylow subgroup.

   b) Let N(S) denote the normalizer of S in G, i.e. $N = \{x \in G \mid x^{-1}Sx=S\}$. Prove that $N(S) = N(N(S))$.

   c) If H is a normal subgroup of G, then $H \cap S$ is a p-Sylow subgroup of H and HS/H is a p-Sylow subgroup of G/H.

4. a) Determine up to isomorphism all abelian groups A with the property: There is an infinite cyclic subgroup B of A such that the index of B in A is 6. State theorems to justify all assertions.

   b) Describe two abelian groups A and B such that

      (i) A and B are not cyclic

      (ii) A and B each have order 8

      (iii) A and B are not isomorphic.

      Prove explicitly (without using theorems) that A is not isomorphic to B.

5. Let G be a group of order 108.

   a) Cite a theorem which guarantees the existence of a subgroup H of G of order 27.

   b) Use a) to give a morphism of G into S_4 (the permutation group on 4 letters).

   c) Use b) to show that G has a normal subgroup of order 27 or 9.

C. Rings

1. Let R be a commutative ring with 1 and let R have exactly one maximal ideal M.
   a) Prove that $r \in M \Longleftrightarrow 1-r$ is an invertible element of R.
   b) If N is a finitely generated R-module such that $N = M \cdot N$, prove that $N=0$.

2. Definition The nil-radical of a ring R is the union of the nilpotent two-sided ideals of R. We denote it by $N(R)$. Prove the following:
   a) If A, B, and C are rings with identity such that $A = B \times C$ (ring product) and I is an ideal of A, then there are ideals I_1 and I_2 of B and C respectively such that $I = I_1 \times I_2$
   b) If A, B and C are rings such that $A = B \times C$, then $N(A) = N(B) \times N(C)$
   c) Let R be the ring of all 4×4 matrices of the form

$$\begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 0 & e \end{pmatrix}$$

   with a, b, c, d, e real numbers. Find the nil-radical N of R.
   d) Describe $\frac{R}{N(R)}$ where R is the ring in c).

3. Give examples of rings with the following properties. Justify your assertions about the rings by stating some theorems or well-known facts.
   a) R has no non-zero nilpotent left ideals but has non-zero nilpotent elements.
   b) R is an integral domain, but for any proper ideal I that is not maximal, R/I is not an integral domain.
   c) R has no proper two-sided ideals but has proper left ideals.
   d) R is artinian but is not Jacobson semi-simple.

4. Let M be a module with the property
   (π) Given a submodule X of M, there is a submodule Y of M such that $X + Y = M$ and $X \cap Y = (0)$.
   Suppose that K is a submodule of M. Show that <u>both</u> K and M/K have the property (π).

D. Fields and Galois Theory

1. a) State the Fundamental theorem of Galois theory.

   b) Find the Galois group of $x^4 - 2$ over Q. Show some work here. Don't use a magic machine for producing Galois groups (e.g. Kaplansky's).

   c) Illustrate the correspondences in the Fundamental theorem by finding the fixed field of each normal subgroup in the Galois group of $x^4 - 2$.

2. Suppse F is a field, $f(x) \in F[x]$ and $K = F(\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\{\alpha_i\}$ are all of the roots of $f(x)$. Show that if $g(x)$ is any irreducible polynomial in $F[x]$ with a root in K, then g has all of its roots in K. Note: This should be proved directly. Please don't cite powerful theorems.

3. a) Consider $x^3 - 2$ as a polynomial over $Z/(5)$. Construct a finite field F that contains $Z/(5)$ and at least two roots of $x^3 - 2$. By "construct" we mean give a list of the elements in F and an algorithm for adding and multiplying the elements.

   b) For any real number α, we will denote the smallest subfield of R (the reals) containing Q (the rationals) and α by $Q(\alpha)$. Either show that there is a Q isomorphism of $Q(e)$ onto $Q(\pi)$ or show that no such isomorphism can exist. (e and π are the usual real numbers denoted by these letters. You may use well-known algebraic facts about e and π.)

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Algebra Prelim

Spring 1972

1. This exam is divided into 5 parts:
   A. Groups, B. Linear Algebra, C. Fields, D. Rings, and
   E. Modules.
   Do a total of 6 problems including at least 1 from each part.

2. Indicate clearly which 6 problems you wish to have graded.
   2 half problems do not equal one problem.

3. You are encouraged to cite theorems from the "common knowledge" of mathematics unless the problem calls upon you to prove such a theorem.

4. You should write enough so that there is no doubt as to whether or not you do know what's going on. But use common sense. Don't write a book when a few lines are sufficient. When in doubt, ask.

5. Neatness counts. Illegible exams are hard to read.

A Groups

1. a. Prove that a group of order 15 is cyclic.

   b. Prove that a group of order 30 has a normal subgroup whose order is either 3 or 5.

   c. Prove that a group of order 30 has an element of order 15.

2. If G is a group, $\mathrm{Aut}(G)$ denotes the group of automorphisms of G, $I(G)$ denotes the group of inner automorphisms of G, and $Z(G)$ denotes the center of G.

   a. Prove $I(G) \cong G/Z(G)$.

   b. Prove that $I(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

   c. If H is a normal subgroup of G prove that there is a homomorphism $\varphi : G \to \mathrm{Aut}(H)/I(H)$ whose kernel is $H \cdot C_G(H)$, where

$$C_G(H) = \{g \in G : h = g^{-1}hg \text{ for all } h \in H\}.$$

B Linear Algebra

3. Let A be an nxn matrix over the field k. Let $\lambda_1,\ldots,\lambda_n$ be the characteristic roots of A (possibly repeated) in the algebraic closure of k. Let p(x) be a polynomial in k[x].

   Prove that $p(\lambda_1),\ldots,p(\lambda_n)$ are the characteristic roots of p(A) in the algebraic closure of k.

   Hint: You may have a use for triangular forms.

4. If V is a vector space over the field k, $V^* = \mathrm{Hom}_k(V,k)$ denotes the dual space of V. For $T : V \to W$ a linear transformation between finite dimensional vector spaces the dual transformation $T^*: W^* \to V^*$ is defined by

$$\bigl(T^*(f)\bigr)(v) = f\bigl(T(v)\bigr) \qquad f \in W^*,\ v \in V.$$

   a. Prove that T is one to one if and only if T^* is onto.

   b. Prove that T is onto if and only if T^* is one to one.

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C Fields

5. Let k be the field of rational numbers. Let K be the splitting field of the polynomial x^n-1. Prove that the Galois group of K over k is abelian.

6. Give an example of each of the following. Justify each answer with a sentence or two.

   (a) An algebraic field extension which is an infinite extension.

   (b) An algebraic field extension which is normal but not separable.

   (c) An algebraic field extension which is separable but not normal.

   (d) An infinite field of characteristic $p \neq 0$.

   (e) A finite field with a non-prime number of elements.

   (f) A transcendental field extension.

7. A field K is called <u>algebraically closed</u> if every non-constant polynomial in K[x] has at least one root in K. Let K be algebraically closed and let $\sigma : F \to K$ be a homomorphism of fields. Let L be an algebraic extension of F. Prove that σ can be extended to a field homomorphism from L to K.

D Rings

8. Let R be a commutative domain with 1. An element $a \in R$ is said to be irreducible if for every factorization $a = bc$ in R, exactly one of b or c is invertable.

   a. State what it means for R to be Noetherian.

   b. Prove that if R is Noetherian every noninvertable element is divisible by an irreducible element.

   c. Prove that if R is Noetherian every noninvertable element is a finite product of irreducible elements.

9. Let R be a commutative domain with 1. R is said to be strongly Euclidian if there is a function d from $R-\{0\}$ to the positive integers satisfying:

   1) If $x, y \neq 0$ then $d(x) \leq d(xy)$.

   2) If $x, y \neq 0$ then $x = ty + r$ where $d(r) < d(y)$ or $r = 0$.

   3) If $x, y, x+y \neq 0$ then $d(x+y) \leq \mathrm{Max}\big(d(x), d(y)\big)$.

   Prove that a strongly Euclidian ring is either a field or isomorphic to the ring of polynomials in 1 variable over a field.

   Remark Without loss of generality it may be assumed that $d(1) = 0$ and $(\exists x)[d(x) > n] \rightarrow (\exists y)[d(y) = n+1]$.

10. Let k be a field and let A be an nxn matrix over k whose minimal polynomial factors into distinct irreducible polynomials. Let R be the ring of matrices of the form $p(A)$ where $p \in k[x]$ is a polynomial.

    a. Prove that R is commutative and has no nilpotent elements.

    b. Prove that R is isomorphic to a direct sum of fields.

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E Modules

11. Let K be a field and let $R \subseteq K$ be a subring whose quotient field is K. Let M be an R module.

 a. Give a K-vector space structure to $K \otimes_R M$.

 b. If $M = N \oplus P$ (as R-modules) prove $K \otimes_R M = (K \otimes_R N) \oplus (K \otimes_R P)$

 c. Let G be a finitely generated free abelian group. Prove that the number of summands in an expression of G as the direct sum of cyclic groups is uniquely determined.

12. If M is an R module let $\mathrm{End}(M)$ denote the endomorphism ring of M.

 a. Prove that if M is a simple R module then $\mathrm{End}(M)$ is a division ring.

 b. Prove that if M is indecomposable (not expressible as a nontrivial direct sum of submodules), then the only elements $a \in \mathrm{End}(M)$ satisfying $a^2 = a$ are 0 and 1.

 c. Prove the converse to part b.

 d. By considering the additive group of rational numbers, show that the converse to part a is false.

ALGEBRA PRELIM

FALL, 1971

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A. Groups

1. (a) If H is a normal subgroup of order p (p prime) in a finite p-group G, then prove that H is contained in the center of G.

   (b) Show that (a) may be false if H is not normal in G.

2. If G is a group which is not cyclic of order two, then prove that G has a nonidentity automorphism. [Thoroughly justify any claim that a particular map is in fact an automorphism.]

3. (a) Define "solvable group".

   (b) Using your definition of solvable, prove that every subgroup and every homomorphic image of a solvable group G is solvable.

   (c) If N is a normal subgroup of a group G and N and G/N are solvable, then prove that G is solvable.

4. Let $\mathcal{B}$ be a family of normal subgroups of a group G and suppose that G is generated by the groups in $\mathcal{B}$. If N is a normal subgroup of G such that $N \cap B$ is the trivial (identity) subgroup for every B in $\mathcal{B}$, then prove that N is in the center of G.

B. Rings and Modules

5. Let R be a ring with identity

   (a) Define the terms "free R-module" and "projective R-module". [Your definition of projective should be in terms of maps and should not mention free modules.]

   (b) Prove that every free R-module is projective.

   (c) Give an example of a projective module which is not free. Justify your answer.

6. Let A be a finite dimensional algebra with identity over a field K

   (a) Show that A satisfies both the ascending and descending chain conditions on right and left ideals.

   (b) What can be said about the structure of A in each of the following cases:

      (i) A has no nonzero nilpotent ideals;

      (ii) A has no nonzero nilpotent elements;

      (iii) A has no zero divisors.

7. Let R be a commutative ring with identity and J an ideal of R which is contained in every maximal ideal of R.

   (a) For each $j \in J$, prove that $1_R - j$ is a unit in R.

   (b) If A is a cyclic R-module such that $JA = A$, then prove that $A = 0$.

8. A nonempty subset S of a commutative ring R with identity is said to be <u>multiplicative</u> if $0 \notin S$ and $a \in S$, $b \in S \Longrightarrow ab \in S$.

   If S is a multiplicative subset of R, prove that there is a prime ideal of R which is disjoint from S.

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C. Linear Algebra

9. (a) If $\phi: E \to E$ is a linear transformation of a finite dimensional vector space E over a field K, then prove that there is a unique monic polynomial $f \in K[x]$ such that

   (i) $f(\phi) = 0$;

   (ii) $g \in K[x]$ and $g(\phi) = 0 \Rightarrow f$ divides g.

   (b) Give an example of a vector space E and linear transformation ϕ such that $K[x]/(f)$ is a field (where f is as in (a)). Give reasons for your answer.

10. (a) Prove that an $n \times n$ matrix A over a field K is similar to a diagonal matrix if and only if every elementary divisor of A is linear.

    (b) Classify under <u>similarity</u> all 5×5 nilpotent matrices A over a field K (i.e. state a finite list of matrices such that every nilpotent A is similar to exactly one matrix on the list). Justify your answers.

    (c) Which of the matrices on the list in (b) are <u>equivalent</u>? Justify your answer.

11. Let V and W be finite dimensional real inner product spaces and $f: V \to W$ a linear transformation. Prove that there is a unique linear transformation $f^*: W \to V$ such that

$$(f^*(w), v) = (w, f(v))$$

for all $v \in V$, $w \in W$ [where (a,b) denotes the inner product of a and b in either V or W as the case may be].

D. Fields and Galois Theory

12. Determine the Galois group of the polynomial $f = x^4 + x^2 - 6$ over the field Q of rational numbers. [Thoroughly justify any claim that a particular map is in fact an element of the Galois group of f.]

13. Let F be an algebraic extension field of a field K. Prove that the following conditions are equivalent.

    (i) For every algebraic extension field E of K there exists a K-monomorphism $E \to F$.

    (ii) Every polynomial in $K[x]$ splits in $F[x]$.

    (iii) Every polynomial in $F[x]$ has a root in F.

    (iv) F has no proper algebraic extensions.

14. Let N be a finite-dimensional, normal separable extension field of a field K. Let G be the Galois group of N over K and let f be a polynomial in $K[x]$ which splits in $N[x]$ and has no multiple roots. Prove that f is irreducible if any only if G acts transitively on the roots of f in N (i.e. if u,v are roots of f, then there exists $\sigma \in G$ with $\sigma(u) = v$).

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MAR 26 1971

Ph.D. preliminary examination in Algebra

Spring, 1971

Instructions: There are four parts to this examination: (1) Linear algebra, (2) Rings and modules, (3) Groups, (4) Fields. Do two problems from each section. Two half-problems does not equal one problem.

1. Linear algebra

1.1 Suppose that L is a linear transformation of a four dimensional vector space V to itself such that L satisfies $(x-2)^2 \cdot (x-3)^2 = 0$ and no polynomial of smaller degree. Show directly that V can be decomposed into the sum of two L-invariant subspaces of dimension two.

$\ni \dim V < \infty$

1.2 Suppose L is a linear transformation from a vector space V to a vector space W. Prove that there exists a linear transformation L' from W to V such that $(L'L)(L'L) = L'L$.

$\ni L' \neq 0$

Suppose A is a linear transformation from V to V such that $A^j \neq 0$ for $j < m$ but $A^m = 0$ where m is the dimension of V. Prove that the matrix for A relative to some basis of V is

$$\begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & \text{\Large 0} & \\ & 1 & 0 & & & \\ & & 1 & 0 & & \\ & \text{\Large 0} & & \ddots & \ddots & \\ & & & & 0 & \\ & & & & 1 & 0 \end{pmatrix}$$

1.4 (a) Let A be a linear transformation of a finite dimensional vector space over a field K. Prove that if the K-minimal polynomial for A has non-zero constant term, then A has an inverse which is a linear combination of powers of A

(b) Let D be an algebra with identity over a field K such that (i) $[D : K] = n < \infty$ and (ii) If a and b are non-zero elements in D, then $ab \neq 0$. Show that D is a division ring. Hint: To find an inverse for $d \in D$, look at its "minimal polynomial".

2. Rings and modules

2.1 A commutative ring is called Noetherian if every ideal is finitely generated. It satisfies the ascending chain condition if for every infinite ascending sequence of ideals, $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \ldots$, there is some n such that $I_n = I_{n+k}$ for all $k \geqslant 1$. Show that these properties are equivalent.

2.2 (a) Show that the additive group of the rationals is an injective Z-module. (Z is the ring of integers.)

(b) Describe all injective indecomposable Z-modules.

2.3 (a) Prove: If K is an algebraically closed field and D is a division ring containing K in its center such that $[D : K] = n < \infty$, then D=K.

(b) Give a complete description of all simple K-algebras A satisfying $[A : K] = r < \infty$ where K is algebraically closed.

2.4 Let R be a ring and I an ideal of R. For each of the following, either prove or give a counterexample.

(a) If R is an integral domain, then so is R/I.

(b) If R is semi-simple with minimum condition, then so is R/I.

(c) If R has A.C.C. on left ideals, then so does R/I.

(d) If R has a unique maximal left ideal, then so does R/I.

(e) If R has no simple left ideals, then so does R/I.

(f) If R is indecomposable (not the ring direct product of two two-sided ideals), then so is R/I.

Groups

3.1 (a) Show that the additive group of the rational numbers is not finitely generated.

(b) Show that a finite non-cyclic abelian group contains a subgroup isomorphic to $Z(p) \times Z(p)$, for some prime p (where $Z(p)$ is the group of integers modulo p). State carefully any theorems you use.

3.2 Let G be a group of order 48.

(a) Show that G has a subgroup of index 3.

(b) Find a set X with three elements and a non-trivial homomorphism of G into the set of permutations of X.

(c) Show that G has a normal subgroup of index 3 or 6.

3.3 Describe up to isomorphism all groups G satisfying both of the following properties:

(a) $|G| = 28$, (b) There exists an element g in G of order 4.

3.4 Suppose G is a finite group which is the direct product of its Sylow p-subgroups. Show that any subgroup H of G has the same property.

4. Fields

4.1 (a) Show that $f(x) = x^3-3x+3$ has exactly one real root and two complex roots.

(b) Describe the Galois group of $f(x)$ over the rationals, giving reasons for your description.

4.2 Let L be an algebraic extension of the field M and let F be an algebraic closure of L. An element u in L will be called separable over M if its minimal polynomial over M has no multiple roots in F. Using this definition of separable do the following:

(a) Show that u in L is separable if and only if the number of monomorphisms of M(u) into ~~L~~ F, leaving M pointwise fixed, is equal to $[M(u):M]$.

(b) Show that if u in L is separable over M, then every element in M(u) is separable over M.

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4.3 Let p be a prime number and F_p the field with p elements. Construct field extensions K and L of F_p such that

(i) $F_p \subseteq K \subseteq L$.

(ii) $K \neq L$.

(iii) K contains an element a such that the polynomial $x^p - a$ is irreducible over K.

(iv) $L = K(\beta)$ where β is a root of $x^p - a$.

Use basic definitions and justify all of your assertions and constructions.

4.4 An extension L of a field M is said to be normal if for some set S of polynomials over M, every polynomial in S splits over L and L is generated over M by roots of the polynomials in S.

Using this definition of "normal", show that if L is a normal extension of M and E is a field such that $M \subseteq E \subseteq L$, then E is a normal extension of M if and only if every automorphism of L that leaves ~~K~~ M pointwise fixed carries ~~M~~ E into itself.

Algebra Prelim - Fall 1970

The exam is divided into four parts, which can be described roughly as 1. linear algebra, 2. rings and ideals, 3. groups, 4. fields. The problems are numbered 1.1, 1.2, ..., 2.1,... Do at least one problem from each section, and at least six problems in all.

1.1. Let $\{a_{ij}\}$ $1 \leq i \leq n$, $1 \leq j \leq m$, be elements of a field k. Using the elementary theory of linear transformations of vector spaces, prove that the following two conditions are equivalent:
(i) $m \geq n$, and the number of independent solutions of the equations

$$\Sigma_{j=1}^{m} a_{ij}x_j = 0 \quad (i = 1,\dots,n)$$

is exactly $m - n$.
(ii) For any choice of elements b_i, $1 \leq i \leq n$, there is a solution of the set of equations

$$\Sigma_{j=1}^{m} a_{ij}x_j = b_i \quad (i = 1,\dots,n).$$

1.2. Let V be a finite dimensional vector space over the field C of complex numbers, with an inner product (,). [That is, for any vectors x and y there is an element (x,y) of C, satisfying the rules $(x + z,y) = (x,y) + (z,y)$, $(rx,y) = r(x,y)$ for all $r \in C$, $(x,y) = \overline{(y,x)}$ (complex conjugate), and $(x,x) \geq 0$ for all x, with equality if and only if $x = 0$.] Let T be a linear transformation of V into itself satisfying the rule $(x,T(y)) = (T(x),y)$.

(a) Show that if W is a subspace of V such that $T(W) \subseteq W$, then $T(W^{\perp}) \subseteq W^{\perp}$, (where $W^{\perp}$ is the orthogonal complement of W).

(b) Use this to show that there exists an orthonormal basis for V such that the matrix for T with respect to this basis is diagonal.

1.3. (a) Show that if k is a field and f a polynomial in $k[X]$, there is a k-vector-space V and a linear transformation T of V into itself whose minimal polynomial is f.

(b) Describe (in terms of the rational canonical form) the similarity classes of 6×6 matrices over the field of real numbers, whose minimal polynomial is $x^4 + x^2$. How many similarity classes are there?

1.4. If A, B, and C are modules over a commutative ring R, show that there is a natural isomorphism

$$\mathrm{Hom}(A \otimes B, C) = \mathrm{Hom}(A, \mathrm{Hom}(B, C)).$$

If V is a finite dimensional vector space over a field k, and V^* its dual space, the above formula specializes to

$$(V \otimes V)^* = \mathrm{Hom}(V, V^*).$$

Show that a bilinear form on V can be regarded as an element of $(V \otimes V)^*$, and that the bilinear form is nondegenerate if and only if the corresponding element of $\mathrm{Hom}(V, V^*)$ is an isomorphism.

2.1. (a) Let A be a finite dimensional algebra with identity over a field k and suppose that A has no nilpotent ideals. Describe as fully as you can the structure of A.

(b) What simplifications occur if either (i) k is algebraically closed, or (ii) A has no zero divisors? Justify your answers in part (b) fully.

2.2. Let R be a commutative ring such that every ideal of R is finitely generated. Show that if M is a finitely generated R-module, then every submodule of M is finitely generated.

2.3. (a) Let k be an algebraically closed field. Define an irreducible algebraic variety in k^n.

(b) Let $f_1, \ldots, f_k$ be polynomials in n variables over k and let V be the set of elements $(a_1, \ldots, a_n)$ in k^n such that $f_i(a_1, \ldots, a_n) = 0$, $(1 \leq i \leq k)$. Let J be the ideal of $k[X_1, \ldots, X_n]$ generated by the elements f_i and let I be the set of all elements in $k[X_1, \ldots, X_n]$ which vanish on V. State carefully a theorem which describes the relation between the ideals I and J.

(c) Show carefully that if I is a prime ideal then V is an irreducible algebraic variety.

3.1. (a) Describe the isomorphism classes of Abelian groups of order 40, giving an example of one group in each class.

(b) What does it mean to say that a group is solvable?

(c) Show that every nontrivial finite p-group has a center containing more than one element.

3.2. Let G be a nonabelian group of order 20 which contains a cyclic subgroup of order 4. How many elements of order 10 does G contain? Explain your answer. (Hint: determine the number of elements of orders 1,2,4,5, and 20.) (Bonus: show that there is exactly one group, up to isomorphism, satisfying these conditions.)

4.1. (a) State carefully the fundamental theorem of Galois theory for a finite normal extension of a field of characteristic zero.

(b) Suppose K is a normal extension of F whose Galois group is S_3 (the symmetric group on three letters). How many distinct fields E are there satisfying $K \supsetneq E \supsetneq F$, and what are their degrees? How many of them are normal extensions of F and what are the degrees of the normal extensions?

4.2. Consider the polynomial $x^7 - 2$. What is the degree of the splitting field of this polynomial over the field of rational numbers? What is the order of the Galois group? Is it Abelian? Write down two automorphisms which, together, generate the Galois group.

4.3. Let K be a finite normal extension of a field F, let G be the Galois group of K over F, and N a normal subgroup of G. Let E be the fixed field of N. Show that if f is an irreducible polynomial over F and f has a root in E, then all roots of f are in E. (You may use the fact that if α and β are two roots of f in K, then there is a $g \in G$, such that $g(\alpha) = \beta$.)

4.4. Let α, β, and γ be the three roots (in the field of complex numbers) of the equation $x^3 - 2$. (a) Show carefully that the field $Q(\alpha)$ is not a normal extension of Q. (Q is the field of rational numbers.) (b) Show without computation that $(\alpha+1)(\beta+1)(\gamma+1)$ is rational. (You may use Galois theory.)

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ALGEBRA PRELIM

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Choose two problems from each section of the exam. If you should attempt more than two problems in a given section, be sure to mark clearly which two problems you wish to have corrected. Two halves of two problems does not equal one problem.

Linear Algebra

L.1 Let V be a vector space with a finite basis over a field F. Prove that any other basis for V over F is also finite and contains the same number of elements.

L.2 Let M be a module over a ring R and let $E(M) = \mathrm{Hom}_R(M,M)$ be the R-endomorphism ring of M. Let ε be an idempotent in $E(M)$, i.e. $\varepsilon^2 = \varepsilon$ and $\varepsilon \neq 0$. Prove that there are submodules M_1 and M_2 of M with $M = M_1 \oplus M_2$ and $\varepsilon(m_1 + m_2) = m_1$ for all $m_1 \in M_1$ and $m_2 \in M_2$.

L.3 Let V and W be finite dimensional vector spaces over a field F and let V^* and W^* be the dual spaces. Let $T : V \to W$ be an F-linear transformation and let $T^* : W^* \to V^*$ be the adjoint transformation. If bases for V and W are given, prove that the matrix of T^* relative to the dual bases is the transpose of the matrix of T relative to the given bases.

1.4 Let V be a finite dimensional vector space over a field K and let $T : V \to V$ be a K-linear transformation.

a) By considering polynomials in T, show that V is a $K[X]$ - module (X a 'variable').

b) Using the structure theorem for finitely generated modules over a principal ideal domain, describe the decomposition of V as a $K[X]$ - module.

c) Describe how this decomposition can be used to obtain the rational canonical form of the matrix of T.

Groups

2.1 Let S be a p-Sylow subgroup of a group G and let $N_G(S)$ denote the normalizer of S in G.

a) Show that $N_G(S) = N_G(N_G(S))$.

b) Let H be a normal subgroup of G. Prove that $H \cap S$ is a p-Sylow subgroup of H and that HS/H is a p-Sylow subgroup of G/H.

2.2 a) Prove that the number of elements in any conjugacy class of a finite group G divides the order of G.

b) Prove that the center of any finite p-group, p a prime, contains at least p elements.

c) Prove that any group of order p^2 is abelian.

2.3 If G is a group of order p^3, p a prime, and if G has more than one normal subgroup of order p, then G is abelian and not cyclic. Prove it.

2.4 a) Prove that a group of order 35 is abelian.

b) Find primes p and q such that there is a nonabelian group of order pq and construct this group.

Rings and Modules

3.1 a) State some form of the Wedderburn structure theorem for rings.

b) Let A be an algebra over a field k with $[A:k] < \infty$. Prove that A is isomorphic to a direct product of division algebras if and only if A has no nonzero nilpotent elements.

3.2 Let R be a ring such that $R^2 \neq \{0\}$ and the only right ideals of R are $\{0\}$ and R. Prove that R is a division ring. (Do not assume that R has an identity. Note that the following are all right ideals of R: $\{x \in R : xR = \{0\}\}$, aR, $\{x \in R : ax = 0\}$.)

3.3 Let R be a ring (with identity if you wish).

a) Define the nilradical N of R and the Jacobson radical J of R.

b) Show that $N = J$ if R satisfies the descending chain condition on left ideals (R is left Artinian).

c) Find an example where $N \neq J$.

3.4 Let R be a commutative ring with identity and let M, N, and P be R-modules. Let $M \otimes_R N$ denote the tensor product of M and N over R and let $\mathrm{Hom}_R(M,N)$ denote the R-module of R-homomorphisms from M to N

a) Prove that $\mathrm{Hom}_R(M \otimes_R N, P)$ is isomorphic as an R-module to $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P))$.

b) Find an example where $M \otimes_R N = \{0\}$ but neither M nor N are zero.

Fields and Galois Theory

4.1 State the Fundamental theorem of Galois theory and illustrate the correspondences with the splitting field of $X^3 - 2$ over Q, the field of rational numbers.

4.2 Let K be an extension field of a field F. If $x \in K$, let $F(x)$ be the subfield of K generated by F and x.

a) Prove that an element $x \in K$ is algebraic over F if and only if $F(x)$ is a finite extension of F.

b) Prove that the set of all elements of K which are algebraic over F forms a subfield of K.

4.3 Definition: A Galois extension K of a field k is the splitting field of a family of separable polynomials in $k[X]$.

Let K be a Galois extension of a field k and let G be the group of all k - automorphisms of K. Prove that

$k = K^G = \{x \in K : \sigma(x) = x \text{ for all } \sigma \in G\}$,

(Note that $[K:k]$ need not be finite.)

4.4 Let F be a finite field of characteristic p.

a) Prove that F has p^m elements for some integer m.

b) What is the structure of the additive group of F?

c) What is the structure of the multiplicative group of nonzero elements of F?

d) Describe the automorphisms of F.

5. (b) continued

(2) The ring of all continuous real valued functions on the unit interval

(3) $R(x)/(x^2+1)^5$ the real polynomial ring in x factored by the ideal generated by $(x^2+1)^5$.

6. Prove that <u>every</u> vector space over a field has a basis.

7. Show that if H is a subgroup of the finite group G then the order of H divides the order of G.