Higher direct images of (log) structure sheaves

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Overview

Origin story

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Origin story

Problem (Grothendieck 1960, Problem B)

Let $f : X \longrightarrow Y$ be a proper birational morphism of non-singular varieties over a field k. Is $R^p f_* \mathcal{O}_X = 0$ for all $q > 0$?
Equivalently, are all of the natural maps $f^* : H^q(Y, \mathcal{O}_Y) \longrightarrow H^q(X, \mathcal{O}_X)$ isomorphisms?

Consequences of an affirmative answer:

- The Hodge numbers

$$\begin{aligned} &h^{0,q}(X) = \dim H^q(X, \mathcal{O}_X) \text{ and via Serre duality also} \\ &h^{n,n-q}(X) = \dim H^{n-q}(X, \omega_X) \end{aligned} \tag{1.1}$$

  are invariant under proper birational morphisms.

Solution when $\operatorname{char} k = 0$ (Corollary 2 in Hironaka 1964)

Rational singularities I

Definition

Let X be a variety over k. X has **rational singularities** if and only if for every resolution of singularities $\pi : \tilde{X} \to X$,

$$\begin{gathered} \pi_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X \text{ and } R^i \pi_* \mathcal{O}_{\tilde{X}} = 0 \text{ for } i > 0 \\ \text{and } \pi_* \omega_{\tilde{X}} = \omega_X \text{ and } R^i \pi_* \omega_{\tilde{X}} = 0 \text{ for } i > 0 \end{gathered} \tag{1.2}$$

Klt singularities are rational

Definition

Let (X, Δ) be a pair (so X is a normal variety, Δ is a $\mathbb{Q}$-Weil divisor on X, $K_X + \Delta$ is $\mathbb{Q}$-Cartier). Then, (X, Δ) **has klt singularities** if and only if ... :)

Why we care:

- Minimal Model Program,
- moduli of varieties

Related to rational singularities via:

Theorem (Elkik 1981, Kawamata, Matsuda, and Matsuki 1987)

If X is a variety over k, $\operatorname{char} k = 0$ with klt singularities, then X has rational singularities.

Idea (Kollár, Sándor): there should be a version of Elkik's theorem for pairs.

Rational resolutions of pairs I

Definition (Kollár 2013)

Let (X, Δ) be a pair over k, $\operatorname{char} k = 0$, let $\pi : \tilde{X} \to X$ be a log resolution, and let $\tilde{\Delta} = \pi_*^{-1}\Delta$ (strict transform). Say π is a **rational resolution** if and only if

1. $\pi_*\mathcal{O}_{\tilde{X}}(-\tilde{\Delta})$ and $R^i\pi_*\mathcal{O}_{\tilde{X}}(-\tilde{\Delta}) = 0$ for $i > 0$
2. $R^i\pi_*\omega_{\tilde{X}}(\tilde{\Delta}) = 0$ for $i > 0$

Issues: pairs that "should be rational" have non-rational resolutions.

Cautionary tales I

Example

Take $(X, \Delta) = (\mathbb{A}^2_{xy}, V(xy))$ and $\tilde{X} := \mathrm{Bl}_0 \mathbb{A}^2$. Then $R^1\pi_*\mathcal{O}_{\tilde{X}}(-\tilde{\Delta})$ corresponds to the module

$$H^1(\mathbb{P}^1_{xy}, \bigoplus_{d=0}^{\infty} \mathcal{O}_{\mathbb{P}^1}(d-2)) = H^1(\mathbb{P}^1_{xy}, \mathcal{O}_{\mathbb{P}^1}(-2)) = k \neq 0$$

Example

Take $X = C(\mathbb{P}^1 \times \mathbb{P}^1), \Delta = D_0 \cup D_\infty$, where $D_0 = C(\mathbb{P}^1 \times \{0\})$, $D_\infty = C(\mathbb{P}^1 \times \{\infty\})$ and $\tilde{X} = \mathrm{Bl}_{D_0} X$. Here $R^1\pi_*\mathcal{O}_{\tilde{X}}(-\tilde{\Delta})$ corresponds to the module

$$H^1(\mathbb{P}^1, \bigoplus_{d \geq 0} \mathrm{pr}_{1*}\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d, d-2)) = H^1(\mathbb{P}^1, \bigoplus_{d \geq 0} \mathcal{O}_{\mathbb{P}^1}(d) \otimes_k k[x,y]_{d-2}) = 0$$

Thriftiness

Definition (Kollár 2013; Kollár and Xu 2016)

Let (S, Δ_S) be a pair and let $f\colon X \to S$ be a proper birational morphism. The map f is **thrifty** if and only if

1. f is an isomorphism *over* the generic point of every stratum of $\operatorname{snc}(S, \Delta_S)$ and
2. f is an isomorphism *at* the generic point of every stratum of $\operatorname{snc}(X, \Delta_X)$.

Theorem (Kollár 2013 ($\operatorname{char} k = 0$))

If (X, Δ) has a thrifty rational resolution, then every thrifty resolution is rational, and if (X, Δ) is dlt and $\pi\colon \tilde{X} \to X$ is a log resolution,

$$\pi \text{ is thrifty} \iff \pi \text{ is rational.}$$

Higher direct images in arbitrary characteristic I

Challenges:

- Resolutions aren't known to exist (yet ...);
- Vanishing theorems are known to fail.

A weak replacement for resolution of singularities:

Theorem (Kawasaki 2000, Cesnavicius 2018)

Let X be a quasi-excellent noetherian scheme. Then there is a proper birational morphism $\tilde{X} \to X$ such that $\tilde{X}$ is Cohen-Macaulay.

Higher direct images in arbitrary characteristic II

Theorem (Chatzistamatiou and Rülling 2011; Chatzistamatiou and Rülling 2015; Kovács 2019)

Let S be a scheme and let X, Y be S-schemes which are noetherian, excellent, regular and properly birational over S:

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[dr, "f"'] \arrow[rr, phantom, "\circlearrowleft"] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd} \quad \textit{(with } p, q \textit{ proper and birational).} \tag{3.1}$$

Then, there is a quasi-isomorphism $Rf_*\mathcal{O}_X \simeq Rg_*\mathcal{O}_Y$.
Moreover if Z is Cohen-Macaulay then $Rp_*\mathcal{O}_Z = \mathcal{O}_X$ and $Rq_*\mathcal{O}_Z = \mathcal{O}_Y$.

- Note that this applies even in *mixed characteristic*.

Setup: thrifty proper birational equivalences

Question: What about pairs in characteristic $p > 0$?
From now on: k is a perfect field, (X, Δ_X), (Y, Δ_Y) are simple normal crossing pairs over k, and we have a **thrifty proper birational equivalence** over a base scheme S of finite type over k:

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[dr, "f"'] & \circlearrowleft & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

with p, q proper, birational and thrifty, and with $p_*^{-1}(\Delta_X) = q_*^{-1}(\Delta_Y)$.

Theorem (G. 2020)

There is a quasi-isomorphism $Rf_*\mathcal{O}_X(-\Delta_X) \simeq Rg_*\mathcal{O}_Y(-\Delta_Y)$.

A resolution of $\mathcal{O}(-\Delta)$

Write $\Delta_X = \cup_{i=1}^{N} D_i$. For each $I = \{i_1, \ldots, i_c\} \subseteq \{1, \ldots, N\}$,

$$X_I := \cap_{i \in I} D_i \subseteq X \text{ is smooth of codimension } |I|$$

Define $X_c := \cup_{|I|=c} X_I$ (still smooth of codimension c) and set $X_0 = X - X_\bullet$ is a **(semi-) simplicial scheme**. Set

$$\check{C}(X, \Delta_X) : \mathcal{O}_X = \mathcal{O}_{X_0} \xrightarrow{d^0} \mathcal{O}_{X_1} \xrightarrow{d^1} \mathcal{O}_{X_2} \to \cdots$$

Theorem (Friedman 1983)

The augmented complex $\mathcal{O}_X(-\Delta_X) \to \check{C}(X, \Delta_X)$ is exact. In particular, there is a quasi-isomorphism $\mathcal{O}_X(-\Delta_X) \simeq \check{C}(X, \Delta_X)$ in $D^b_{\mathrm{coh}}(X)$.

Thriftiness I

Recall: $Z \xrightarrow{p} X$ is an isomorphism over all generic points of all of the X_c (similarly for Y).

Lemma

There is an augemented semi-simplicial scheme $Z_\bullet \xrightarrow{\iota_\bullet} Z$, together with morphisms $X_\bullet \xleftarrow{p_\bullet} Z_\bullet \xrightarrow{q_\bullet} Y_\bullet$ over S, satisfying:

1. *Z_c is Cohen-Macaulay for all c, and*
2. *the morphisms $X_c \xleftarrow{p_\bullet} Z_c \xrightarrow{q_\bullet} Y_c$ are projective and birational for all c.*

A descent spectral sequence argument I

Let $\iota^Z_\bullet : Z_\bullet \to Z$ be the augmentation (similarly for X, Y). There is a *complex* $\mathcal{K}^\bullet := R\iota_{\bullet *}\mathcal{O}_{Z_\bullet}$ in $D(Z)$, and $p_\bullet : Z_\bullet \to X_\bullet$ induces

$$\tau_p : \mathcal{O}_X(-\Delta_X) \simeq R\iota^X_{\bullet *}\mathcal{O}_{X_\bullet} \to Rp_* R\iota^Z_{\bullet *}\mathcal{O}_{Z_\bullet} = Rp_*\mathcal{K}^\bullet \text{ in } D(X)$$

Theorem (G. 2020)

The maps τ_p and τ_q are quasi-isomorphism. Hence pushing forward along f, g we obtain

$$\begin{array}{rl} Rf_*\mathcal{O}_X(-\Delta_X) \xrightarrow{Rf_*\tau_p} Rf_*Rp_*\mathcal{K}^\bullet & \\ & Rg_*Rq_*\mathcal{K}^\bullet \xleftarrow{Rg_*\tau_q} Rg_*\mathcal{O}_Y(-\Delta_Y) \end{array} \tag{4.2}$$

Key ingredient: there is a “descent” spectral sequence

$$E_1^{ij} := R^j p_{i*}\mathcal{O}_{Z_i} \implies R^{i+j}Rp_*\mathcal{K}^\bullet \text{and by Kovács 2019, } Rp_{i*}\mathcal{O}_{Z_i} = \mathcal{O}_{X_i}$$

Open questions

- Relaxing the hypothesis that $(X, \Delta_X), (Y, \Delta_Y)$ are snc?
- What can we say about "(X, Δ_X) klt $\implies (X, \Delta_X)$ is rational" in positive characteristic? Best hope: true for large p for fixed $\dim X$.
- Consequences for counting points over $\mathbb{F}_q$? (Ekedahl 1983 proved that if X, Y are smooth, proper and birationally equivalent over $\mathbb{F}_q$ then $|X(\mathbb{F}_q)| \equiv |Y(\mathbb{F}_q)| \bmod q$).

Thanks!

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