THRIFTY RATIONAL RESOLUTIONS IN ARBITRARY CHARACTERISTIC

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ABSTRACT. After a brief discussion of rational singularities I introduce the key players of this paper, rational pairs. Rational singularities have always been more mysterious in positive characteristic - in particular, until recently it was not known that if X is a variety over a field k of characteristic $p > 0$, and X has one rational resolution of singularities $\pi : Y \to X$, then every other resolution of singularities $\pi' : Y' \to X$ is rational - a proof of this fact due to Chatzistamatiou and Rülling [CR11], using a cycle map from Chow theory to produce correspondences on Hodge theory, is presented in some detail. The analogous question for rational pairs is still open, and a strategy for answering it is proposed.

1. RATIONAL SINGULARITIES

Let k be a field and let X be a variety over k. Recall that a **resolution of singularities of X** is a proper birational morphism $\pi : Y \to X$ where Y is a smooth variety over k.

Definition 1. $\pi : Y \to X$ is a **rational resolution** if and only if the derived structure morphism $\mathscr{O}_X \to R\pi_* \mathscr{O}_Y$ is an isomorphism. Equivalently,

- $\mathscr{O}_X \simeq \pi_* \mathscr{O}_Y$ and
- $R^i\pi_* \mathscr{O}_Y = 0$ for $i > 0$.

If X has a resolution, we say X **has rational singularities**.

Remark 1. If $\pi : Y \to X$ is a rational resolution as above, and $\mathscr{E}$ is a locally free sheaf on X, then the natural map $H^i(X, \mathscr{E}) \xrightarrow{\simeq} H^i(Y, \pi^*\mathscr{E})$ is an isomorphism - this follows from the projection formula

$$R\pi_* \pi^* \mathscr{E} \simeq R\pi_* \mathscr{O}_Y \otimes \mathscr{E} \simeq \mathscr{E}$$

(and a Leray spectral sequence argument). Hence cohomology calculations on X can be carried out on Y.

For example: assume $\operatorname{char} k = 0$ and let $\mathscr{L}$ be an ample invertible sheaf on X. Then $\pi^*\mathscr{L}$ is big and nef, so by Kawamata-Viehwag vanishing

$$0 = H^i(Y, \pi^*\mathscr{L}^{-1}) = H^i(X, \mathscr{L}^{-1}) \text{ for } i < \dim X$$

2. Rational pairs

In birational geometry it has become standard to work with "pairs." A pair (X, Δ) is a (normal) variety X together with a $\mathbb{Q}$-Weil divisor Δ on X. Here are a couple of reasons why pairs naturally enter the picture:

- Add some bullet points here.

Suppose (X, Δ) is a pair, with Δ a reduced, effective divisor, and $\pi : Y \to X$ is a resolution of singularities such that $\Delta_Y := \pi_*^{-1}(\Delta)$ is an snc divisor.

Definition 2. π is a **rational resolution** if and only if

- the natural map $\mathscr{O}_X(-\Delta) \to R\pi_* \mathscr{O}_Y(-\Delta_Y)$ is an isomorphism.
- $R^i \pi_* \omega_Y(\Delta_Y) = 0$ for $i > 0$.

A good definition of rational singularities for pairs should involve more restrictions, as shown by the following example:

Example 1. Let (X, Δ) be a pair with X a smooth variety and Δ a reduced simple normal crossing divisor. Assume $Z \subset X$ is a non-empty stratum of Δ (i.e. if $\Delta = \sum_i D_i$, then $X = \bigcap_j D_{i_j}$ is an intersection of a subset of the D_i) with $\mathrm{codim}(Z, X) > 1$. Consider the blowup of Z:

$$\pi : Y := \mathrm{Blp}_Z X \to X$$

One can show that this is not a rational resolution of (X, Δ). Let $z \in Z$ be a closed point. By the theorem on formal functions we have an isomorphism

$$R^i \pi_* \widehat{\mathscr{O}_Y}(-\Delta_Y) \simeq \varprojlim H^i(Y_{z,n}, \mathscr{O}_Y(-\Delta_Y)_{z,n})$$

One can compute the right-hand-side (and show it's non-0 for positive values of i following the calculation on p. 387-388 of [Har77].

This motivates the following

Definition 3. A resolution $\pi : (Y, \Delta_Y) \to (X, \Delta)$ as above is **thrifty** if and only if the following equivalent conditions hold:

- π is an isomorphism over the generic point of every stratum of $snc(\Delta)$ and an isomorphism at the generic point of every stratum of Δ_Y.
- $\mathrm{Ex}(f)$ contains no stratum of Δ_Y and $f(\mathrm{Ex}(f))$ contains no stratum of $snc(\Delta)$.

Here $snc(\Delta)$ denotes the snc locus of Δ, defined to be $\Delta \cap V$ where $V \subset X$ is the largest open subset so that $\Delta \cap V$ is an snc divisor on V.

Definition 4. (X, Δ) **has rational singularities** if and only if it has a thrifty rational resolution.

Problem: Prove that if (X, Δ) has rational singularities, is *every* thrifty resolution of (X, Δ) a rational resolution.

In characteristic 0 this problem is not difficult to solve using resolutions of singularities:

Theorem 2.1 ([KK09])**.** *Let k be a field of characteristic 0 and let (X, Δ) be a pair as above. If (X, Δ) has a thrifty rational resolution, then every thrifty resolution of X is rational.*

Chatzistamatiou and Rülling have found at least 2 characteristic-independent proofs in the case where $\Delta = 0$ - one of their arguments ([CR11]) will be summarized momentarily. In positive characteristic with $\Delta \neq \emptyset$ the problem is still open.

2.1. **Connection with singularities of the MMP.** One source of interest in rational singularities stems from their relationship with the singularities that arise in the minimal model program and the moduli of higher dimensional varieties. In characteristic 0 we have the following facts:

Theorem 2.2 ([KK09])**.** *Let (X, Δ) be a dlt pair and let $\pi : Y \to X$ be a resolution of singularities, with $\pi_*^{-1} \lfloor \Delta \rfloor$ snc. Then the following are equivalent:*

- *Every exceptional divisor of f has discrepancy > -1.*
- *π is a thrifty resolution of $(X, \lfloor \Delta \rfloor)$.*
- *π is a rational resolution of $(X, \lfloor \Delta \rfloor)$.*

Corollary 2.3. *If (X, Δ) is a dlt pair and D is a reduced effective divisor on X with $D \leq \lfloor \Delta \rfloor$, then (X, D) is a rational pair.*

Note that built into the statement "(X, Δ) is a dlt pair" is the hypothesis that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

3. CHATZISTAMATIOU AND RÜLLING'S RESULTS

3.1. **Setup.** The main reference for this section is [CR11].

Let k be a perfect field and let S be a variety over k. Suppose $f : X \to S$ and $g : Y \to S$ are resolutions of singularities. By resolving the indeterminacies of the rational S-morphism $X \xrightarrow{g^{-1} \circ f} Y$ we can obtain a (possibly singular) variety Z over S together with proper birational morphisms $Z \to X$, $Z \to Y$ fitting into the following commutative diagram

(1)
$$
\begin{array}{ccc}
Z & \longrightarrow & Y \\
\downarrow & & g \downarrow \\
X & \xrightarrow{\ f\ } & S
\end{array}
$$

Let $Z_0 \subset X \times_k Y$ denote the image of Z (with its reduced closed subscheme structure). Associated to Z_0 is a class

$$[Z_0] \in A^n(X \times Y)$$

where $n = \dim S$. What Chatzistamatiou and Rülling show is that this cycle, together with a refined cycle morphism from Chow groups to Hodge cohomology compatible with correspondences, induces isomorphisms

$$R^i f_* \mathscr{O}_X \simeq R^i g_* \mathscr{O}_Y \text{ and } R^i f_* \omega_X \simeq R^i g_* \omega_Y$$

for all i. As a corollary: if f is a rational resolution, so is g.

There are several key pieces of machinery that go into this result:

- **Cohomology theories with supports**. Since X and Y are not assumed to be proper, we don't necessarily have pushforward morphisms

$$\pi_Y^* : A^*(X \times Y) \to A^*(Y) \text{ and } \pi_Y^* : H^*(X \times Y, \Omega^*) \to H^*(Y, \Omega^*)$$

However, the subscheme $Z_0 \subset X \times Y$ *is* proper over X and Y, so we should be able to define a correspondence $\rho : A^*(X) \to A^*(Y)$ of the form "$\rho(\alpha) = \pi_Y^*(\pi_X^*(\alpha) \cap [Z_0])$," and similarly for Hodge cohomology. To make this happen they introduce axioms for a cohomology theory with supports on the category of finite-type, separated k-schemes, and show that both Chow groups with supports and Hodge cohomology with supports satisfy those axioms.
- **Correspondences**. If X and Y are smooth, proper varieties over k, a class $\gamma \in A^d(X \times Y)$ defines a **correspondence**

$$A^i(X) \xrightarrow{\rho_\gamma} A^{i+d-\dim X}(Y) \text{ where } \rho_\gamma(\alpha) = \pi_{Y*}(\pi_X^*(\alpha) \cap \gamma)$$

They observe that correspondences make sense for any cohomology theory with supports - in particular one can consider correspondences in Hodge cohomology.
- **A cycle map from Chow groups with support to Hodge cohomology with support, compatible with correspondences.** They show that there is a natural transformation of cohomology theories with supports $A^* \to H^*$. In fact they prove that with the addition of one extra axiom ("purity") Chow groups become the universal cohomology theory with supports.

3.2. **Cohomology theories with supports.**

3.2.1. *Families of supports.*

Definition 5. Let k be a field and let X be a scheme of finite type over k. A **family of supports Φ on X** [Har66] is a non-empty collection Φ of closed subsets of X such that

- If $C \in \Phi$ and $D \subset C$ is a closed subset, then $D \in \Phi$.
- If $C, D \in \Phi$ then $C \cup D \in \Phi$.

Example 2. Φ = all closed subsets of X is a family of supports.

Example 3. Let $\mathcal{C}$ be a **Serre subcategory** [Sta18, Tag 02MN]] of $\mathrm{Coh}(X)$. Then the collection of supports

$$\Phi := \{\mathrm{Supp}\mathscr{F} \subset X \mid \mathscr{F} \text{ is a coherent sheaf in } \mathcal{C}\}$$

forms a family of supports on X.

Recall that "$\mathcal{C}$ is a Serre subcategory" means that

- it's non-empty.
- it's full.
- given an exact sequence $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$, if $\mathscr{F}', \mathscr{F}''$ are in $\mathcal{C}$ so is $\mathscr{F}$.

Immediate consequences are that 0 is in $\mathcal{C}$, $\mathcal{C}$ is "strictly full" (meaning it is a full subcategory closed under isomorphisms), and it is closed under subobjects, quotients and extensions. One can show that $\mathcal{C}$ is also an abelian category and the inclusion $\mathcal{C} \to \mathrm{Coh}(X)$ is a fully faithful exact functor.

Now, if $\mathscr{F}', \mathscr{F}''$ are in $\mathcal{C}$ so is $\mathscr{F} := \mathscr{F}' \oplus \mathscr{F}''$, and we have

$$\mathrm{Supp}\mathscr{F} := \mathrm{Supp}\mathscr{F}' \cup \mathrm{Supp}\mathscr{F}''$$

showing that Φ is closed under unions. If $\mathscr{F}$ is a sheaf in $\mathcal{C}$ and $Z \subset \mathrm{Supp}\mathscr{F}$ is a closed subset, consider the quotient

$$\mathscr{F} \to \mathscr{F}|_Z = \mathscr{F} \otimes \mathscr{O}_Z$$

Since $\mathcal{C}$ is closed under quotients, $\mathscr{F}|_Z$ is in $\mathcal{C}$, and $\mathrm{Supp}\mathscr{F}|_Z = Z$.

Conversely, one can show that given a family of supports Φ on X, the full subcategory of $\mathrm{Coh}(X)$ with objects the coherent sheaves $\mathscr{F}$ such that $\mathrm{Supp}\mathscr{F} \in \Phi$ is a Serre subcategory.

Let $f : X \to Y$ be a morphism of separated schemes of finite type over k and let Ψ be a family of supports on Y. Then $\{f^{-1}(Z) \,|\, Z \in \Psi\}$ is a family of closed subsets of X, and it's closed under unions, but *not* in general closed under taking closed subsets. However, we can let $f^{-1}(\Psi)$ be the smallest family of supports on X containing $\{f^{-1}(Z) \,|\, Z \in \Psi\}$.

On the other hand, let Φ be a family of supports on X. Say $f|_\Phi$ is proper if and only if $f|_C$ is proper for every $C \in \Phi$. In this case

$$f(C) \subset Y \text{ is closed for every } C \in \Phi$$

and in fact $\{f(C) \subset Y \mid C \in \Phi\}$ is a family of supports on Y. The key point here is that if $D \subset f(C)$ is closed, then $f^{-1}(D) \cap C \in \Phi$ and $D = f(f^{-1}(D) \cap C)$.

Definition 6. Denote by $\mathcal{V}^*$ the category with objects pairs (X, Φ) where X is a *smooth* separated scheme of finite type over k and Φ is a family of supports on X, with morphisms $f : (X, \Phi) \to (Y, \Psi)$ consisting of a morphism $f : X \to Y$ such that $f^{-1}(\Psi) \subset \Phi$.

Denote by $\mathcal{V}_*$ the category with objects pairs (X, Φ) as above, but with morphisms $f : (X, \Phi) \to (Y, \Psi)$ consisting of a morphism $f : X \to Y$ so that $f|_\Phi$ is proper and $f(\Phi) \subset \Psi$.

Remark 2. The categories $\mathcal{V}^*$ and $\mathcal{V}_*$ do have coproducts $((X, \Phi) \coprod (Y, \Psi) = (X \coprod Y, \Phi \coprod \Psi)$ where $\Phi \coprod \Psi = \{C \coprod D \,|\, C \in \Phi, D \in \Psi\})$, but they lack products. The natural candidate for a product is $(X \times Y, \Phi \times \Psi)$ where $\Phi \times \Psi$ is the smallest family of supports containing $\{C \times D \,|\, C \in \Phi, D \in \Psi\}$. While this does come with a natural transformation

$$\mathrm{Hom}((Z, \Theta), (X, \Phi)) \times \mathrm{Hom}((Z, \Theta), (Y, \Psi)) \to \mathrm{Hom}((Z, \Theta), (X \times Y, \Phi \times \Psi))$$

it doesn't come with natural projections to $(X, \Phi), (Y, \Psi)$.

On the plus side, while it's not a categorical product, setting $(X, \Phi) \otimes (Y, \Psi) := (X \times Y, \Phi \times \Psi)$ *does* provide a symmetric monoidal product on $\mathcal{V}^*$ and $\mathcal{V}_*$.

3.2.2. *Cohomology theories with supports.*

Remark 3. I've realized that this section should be rewritten in the "bivariant" formalism of Fulton-MacPherson. Unfortunately I didn't really "get" (read: appreciate the elegance of) that formalism until a few days ago.

Instead of using their bivariant setup I just phrased everything in terms of cohomology (as opposed to homology) - hopefully everything here is internally consistent and correct.

Definition 7. A **weak cohomology theory with supports** F^* is an assignment

$$(X, \Phi) \mapsto F^*(X, \Phi)$$

of a graded abelian group $F^*(X, \Phi)$ to each pair (X, Φ) with X a smooth separated k-scheme of finite type and Φ a family of supports on X, satisfying the following axioms:

(1) F^* is a *contravariant* functor on the category $\mathcal{V}^*$. That is, for every morphism $f : (X, \Phi) \to (Y, \Psi)$ in $\mathcal{V}^*$ there is a natural "pullback" homomorphism $f^* : F^*(Y, \Psi) \to F^*(X, \Phi)$.

(2) F^* is a *covariant* functor on the category $\mathcal{V}_*$. That is, for every $\mathcal{V}_*$-morphism $f : (X, \Phi) \to (Y, \Psi)$ there is a natural "pushforward" homomorphism $f_* : F^*(X, \Phi) \to F^*(Y, \Psi)$ (note that here it should be viewed only as a functor to abeliang groups, as the pushforwards general come with a degree shift).

(3) F^* comes with external products

$$F^*(X, \Phi) \otimes_{\mathbb{Z}} F^*(Y, \Psi) \xrightarrow{\times} F^*(X \times Y, \Phi \times \Psi)$$

that are functorial with respect to pullbacks.

(4) There is a distinguished element $1 \in F^0(\operatorname{Spec} k)$.

(5) Whenever (X_i, Φ_i) is a finite collection of separated k-schemes of finite type with families of supports, the natural maps

$$F^*(\coprod_i X_i, \coprod_i \Phi_i) \to \prod_i F^*(X_i, \Phi_i) \text{ and}$$

$$\bigoplus_i F^*(X_i, \Phi_i) \to F^*(\coprod_i X_i, \coprod_i \Phi_i)$$

are isomorphisms.

(6) If X is a separated k-scheme of finite type with 2 *disjoint* families of supports Φ_1 and Φ_2, and if

$$j_i : (X, \Phi_1 \cup \Phi_2) \to (X, \Phi_i), \text{ for } i = 1, 2$$

are the resulting $\mathcal{V}^*$-morphisms and

$$\iota_i : (X, \Phi_i) \to (X, \Phi_1 \cup \Phi_2)$$

the resulting $\mathcal{V}_*$-morphisms, the induced homomorphisms of groups

$$\bigoplus_{i=1}^2 F^*(X, \Phi_i) \xrightarrow{j_1^* \oplus j_2^* \text{ OR } \iota_{1*} \oplus \iota_{2*}} F^*(X, \Phi_1 \cup \Phi_2)$$

are isomorphisms. Note that this is probably the most important requirement.

(7) "F^* is a right lax symmetric monoidal functor." In words, the external products
$$F^*(X, \Phi) \otimes F^*(Y, \Psi) \to F^*(X \times Y, \Phi \times \Psi)$$
are associative in the obvious sense and commutative in the sense that

(2)
$$
\begin{array}{ccc}
F^*(X, \Phi) \otimes F^*(Y, \Psi) & \xrightarrow{\times} & F^*(X \times Y, \Phi \times \Psi) \\
\text{flip} \downarrow & & \text{flip}^* \downarrow \\
F^*(Y, \Psi) \otimes F^*(X, \Phi) & \xrightarrow{\times} & F^*(Y \times X, \Psi \times \Phi)
\end{array}
$$

is commutative. Here $F^*(X, \Phi) \otimes F^*(Y, \Psi) \xrightarrow{\text{flip}} F^*(Y, \Psi) \otimes F^*(X, \Phi)$ is the graded commutative flip, sending $a \otimes b$ to $(-1)^{\deg a \deg b} b \otimes a$. Moreover the element $1 \in F^0(\operatorname{Spec} k)$ serves as an identity in the sense that

(3)
$$
\begin{array}{ccc}
F^*(X) \otimes \mathbb{Z} & \xrightarrow{\text{id} \otimes 1} & F^*(X) \otimes F^*(\operatorname{Spec} k) \\
= \downarrow & & \times \downarrow \\
F^{(}X) & \xrightarrow{=} & F^*(X \times_k \operatorname{Spec} k)
\end{array}
$$

commutes, and similarly with 1 acting on the left.

(8) For every cartesian diagram

(4)
$$
\begin{array}{ccc}
(X', \Phi') & \xrightarrow{g_X} & (X, \Phi) \\
f'_X \downarrow & & f \downarrow \\
(Y', \Psi') & \xrightarrow{g_Y} & (Y, \Psi)
\end{array}
$$

(here cartesian just means the underlying diagram of schemes is cartesian) where g_X, g_Y are $\mathcal{V}^*$-morphisms and f, f' are $\mathcal{V}_*$-morphisms, such that either
- g_Y is smooth or
- g_Y is a closed immersion transversal to f
$$f'_* \circ g_X^* = g_Y^* \circ f_* : F^*(X, \Phi) \to F^*(Y', \Psi')$$

Remark 4. Both bullet points guarantee that X' is smooth, so it is indeed an object of $\mathcal{V}^*, \mathcal{V}_*$.

Let F^* be a weak cohomology theory with supports and let Φ_1, Φ_2 be families of support on X. Observe that the diagonal of X defines a $\mathcal{V}^*$-morphism
$$\Delta : (X, \Phi_1 \cap \Phi_2) \to (X \times X, \Phi_1 \times \Phi_2)$$
and hence we obtain a homomorphism
$$F^*(X, \Phi_1) \otimes F^*(X, \Phi_2) \xrightarrow{\times} F^*(X \times X, \Phi_1 \times \Phi_2) \xrightarrow{\Delta^*} F^*(X, \Phi_1 \cap \Phi_2)$$
called the **cup product**, and denoted by "$\smile$."

Suppose now that $f : X \to Y$ is a morphism of smooth, separated k-schemes of finite type. Let Φ_X, Ψ_X be families of supports on X and let Φ_Y, Ψ_Y be families of supports on Y so that

- f_{Φ_X} is proper and $f(\Phi_X) \subset \Phi_Y$ - hence $f : (X, \Phi_X) \to (Y, \Phi_Y)$ is a $\mathcal{V}_*$-morphism.
- $f^{-1}(\Psi_Y) \subset \Psi_X$, so that $f : (X, \Psi_X) \to (Y, \Psi_Y)$ is a $\mathcal{V}^*$-morphism.

It follows that $f|_{\Phi_X \cap \Psi_X}$ is proper and $f(\Phi_X \cap \Psi_X) \subset \Phi_Y \cap \Psi_Y$ and so we also have a $\mathcal{V}_*$-morphism $f : (X, \Phi_X \cap \Psi_X) \to (Y, \Phi_Y \cap \Psi_Y)$. The above axioms ensure that whenever $a \in F^*(Y, \Psi_Y)$ and $b \in F^*(X, \Phi_X)$,

$$f_*(f^*(a) \smile b) = a \smile f_*(b)$$

This will be referred to as the **projection formula**.

We've essentially shown that

Proposition 3.1. $F^*(X)$ *is a graded ring, with multiplication* $\smile$ *and identity the image of* $1 \in F^0(\mathrm{Spec}k)$. *The cup product* $F^*(X, \Phi) \otimes F^*(X) \xrightarrow{\smile} F^*(X, \Phi)$ *makes* $F^*(X, \Phi)$ *into a graded (right)* $F^*(X)$-*module.*

3.2.3. *Chow groups and Hodge cohomology with supports.* The main reference for all things intersection theory is [Ful98].

Let X be a smooth scheme of finite type over k and let Φ be a family of supports on X. Set

$$Z(X, \Phi) := \text{the free abelian group on the irreducible closed sets } C \in \Phi \text{ and set}$$

$$\mathrm{Rat}(X, \Phi) = \text{ the subgroup of } Z(X, \Phi) \text{ generated by}$$

$$\{\div f \in Z(X, \Phi) \,|\, W \in \Phi \text{ is an irreducible closed set and } f \in k(W)^\times\}$$

Elements of $Z(X, \Phi)$ are called **cycles**, and $\mathrm{Rat}(X, \Phi)$ is the subgroup of cycles **rationally equivalent to 0**. Finally let $A(X, \Phi) := Z(X, \Phi)/\mathrm{Rat}(X, \Phi)$ - this is the **Chow group with supports** of (X, Φ). The grading is given by codimension; more precisely, if X is irreducible then $A^{2d}(X, \Phi)$ is the summand generated by cycles $C \in \Phi$ with $\mathrm{codim}(C, X) = d$, and the odd degree summands are all 0. If X has multiple components, we define the grading on $A^*(X, \Phi)$ via the isomorphism $A^*(X, \Phi) = \bigoplus_i A^*(X_i, \iota_i^{-1}(\Phi))$ where the X_i are the components of X and the $\iota_i : X_i \to X$ are the inclusions.

Proposition 3.2. *The assignment* $(X, \Phi) \mapsto A^*(X, \Phi)$ *extends to a weak cohomology theory with supports.*

I should emphasize that this result makes essential use of Fulton-McPherson's "refined" intersection theory. While I won't go into the details, I feel morally obligated to provide one highlight of the theory: a technique called **deformation to the normal cone** that reduces calculations involving intersection classes to calculations involving characteristic classes.

Theorem 3.3. *Let X be a variety over a field k and let $Z \subset X$ be a closed subvariety. Then there is a variety M over k together with a flat morphism $\pi : M \to \mathbb{P}^1_t$ and a closed subvariety $W \subset M$, also flat over $\mathbb{P}^1_t$, with the following properties:*

- *Over $\mathbb{P}^1_t \setminus \{\infty\}$ there is an isomorphism $M_{t \neq \infty} \simeq X \times (\mathbb{P}^1_t \setminus \{\infty\})$ identifying $W_{t \neq \infty}$ with $Z \times (\mathbb{P}^1_t \setminus \{\infty\})$.*

- M_∞ is a union of the blowup $\mathrm{Bl}_Z X$ and the projective completion $\mathbb{P}(\mathscr{N}_{Z|X} \oplus \mathscr{O}_Z)$ of the normal cone $\mathscr{N}_{Z|X}$ of Z in X, glued together by the identification $E \simeq \mathbb{P}(\mathscr{N}_{Z|X})$ (where $E \subset \mathrm{Bl}_Z(X)$ is the exceptional divisor). $W_\infty \simeq Z$, embedded as the zero-section in $\mathbb{P}(\mathscr{N}_{Z|X} \oplus \mathscr{O}_Z)$.

The construction is explicit: one can just take $M = \mathrm{Bl}_{Z \times \{\infty\}} X \times \mathbb{P}^1_t$.

Remark 5. As Fulton himself points out, "deformation from the normal cone" would be more appropriate. Personally I would prefer "degeneration to the normal cone." Unfortunately the terminology is fixed at this point.

Remark 6. In the case where X is smooth and Z is the vanishing of a generic section $\sigma \in \Gamma(X, E)$ of a vector bundle E on X (more precisely, we should require that $\sigma(X)$ intersects the zero-section $X \subset E$ transversely), there's a cool interpretation of the deformation to the normal cone. The idea is that if we scale σ by $s \in \mathbb{A}^1_s$ to obtain the section $s\sigma$, and let $s \to \infty$, the section $\sigma(X)$ becomes steeper and steeper near $Z = V(\sigma)$, approaching the "vertical axis" $E|_Z \simeq N_{Z|X}$.

More precisely, define a rational morphism $\varphi : X \times \mathbb{P}^1_{[s,t]} \dashrightarrow \mathbb{P}(\mathscr{E} \oplus \mathscr{O}_X)$ by

$$\varphi(x, [s, t]) := [s\sigma(x), t]$$

This is defined away from $Z \times \{[1, 0]\} = Z \times \{\infty\} \subset X$. One can show that blowing up $X \times \mathbb{P}^1_{[s,t]}$ along $Z \times \{\infty\}$ resolves the indeterminacies of φ.

Chatzistamatiou and Rülling observe that A^* satisfies a couple additional useful properties, which they call **semipurity conditions**:

Definition 8. Let F^* be a weak cohomology theory with supports. F^* satisfies the **semipurity conditions** if and only if

(1) For a smooth, separated scheme X of finite type over k and a closed set $W \subset X$, $F^i(X, W) = 0$ for $i < \mathrm{codim}(W, X)$. Here $F^*(X, W)$ is short for $F^*(X, \Phi_W)$ where Φ_W is the smallest family of supports containing W, or in other words the collection of closed sets $C \subset W$.
(2) If $U \subset X$ is open and $\eta_i \in U$ for every generic point $\eta_i \in W_i$ of a an irreducible component $W_i \subset W$, then the natural pullback map

$$F^{2\mathrm{codim}(W,X)}(X, W) \to F^{2\mathrm{codim}(W,X)}(U, W \cap U)$$

is injective.

For Chow groups these can be seen as follows: if $C \subset W$ is an irreducible closed subset, then $\mathrm{codim}(C, X) \geq \mathrm{codim}(W, X)$ and hence $A^i(X, W) = 0$ for $i < 2\mathrm{codim}(W, X)$. For the second condition one uses the fact that for any variety (i.e. separated integral scheme of finity type) Z over k, $A^0(Z) \simeq \mathbb{Z}$, generated by the **fundamental class** $[Z]$ - see [Ful98].

It is a theorem that Chow groups with supports are universal among weak cohomology theories with supports satisfying the semipurity conditions, in the following sense:

Definition 9. Let F^* and G^* be weak cohomology theories with supports. A **morphism** $\tau : F^* \to G^*$ consists of homomorphisms of graded abelian groups

$$\tau_{(X,\Phi)} : F^*(X, \Phi) \to G^*(X, \Phi)$$

for every object (X, Φ) in $\mathcal{V}^*$ (hence every object in $\mathcal{V}_*$) satisfying the following compatibility conditions:

- τ is a natural transformation of functors on $\mathcal{V}^*$ *and* $\mathcal{V}_*$. Which is to say, it's compatible with both pullbacks and pushforwards.
- τ is a natural transformation of symmetric monoidal functors - in other words, it is compatible with external products and sends $1 \in F^0(\mathrm{Spec}\,k)$ to $1 \in G^0(\mathrm{Spec}\,k)$.

Theorem 3.4. *Assume k is a perfect field and let F^* be a weak cohomology theory with supports satisfying the semipurity conditions. Then there is at most 1 morphism $A^* \to F^*$. Such a morphism exists if and only if F^* satisfies a few extra conditions listed on p. 13 of [CR11].*

Remark 7. A morphism $A^* \to F^*$ will be referred to as a **cycle map**.

Definition 10. Let X be a separated scheme of finite type over k and let Φ be a family of supports on X. Let $\mathscr{F}$ be a quasi-coherent sheaf on X. Define

$$\Gamma_\Phi(X, \mathscr{F}) := \{\sigma \in \Gamma(X, \mathscr{F}) \mid \mathrm{supp}\,\sigma \in \Phi\}$$

Proposition 3.5. $\Gamma_\Phi(X, -)$ *is left exact.*

Proof. The global sections functor $\Gamma(X, -)$ is left exact, and we have a commutative diagram

(5)
$$
\begin{array}{ccccccc}
0 & \longrightarrow & \Gamma_\Phi(X, \mathscr{F}') & \xrightarrow{\ i\ } & \Gamma_\Phi(X, \mathscr{F}) & \xrightarrow{\ j\ } & \Gamma_\Phi(X, \mathscr{F}'') \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma(X, \mathscr{F}') & \xrightarrow{\ i\ } & \Gamma(X, \mathscr{F}) & \xrightarrow{\ j\ } & \Gamma(X, \mathscr{F}'')
\end{array}
$$

where the vertical arrows are inclusions. So, exactness of the top row is equivalent to: if $\sigma \in \Gamma(X, \mathscr{F}')$ and $i(\sigma) \in \Gamma_\Phi(X, \mathscr{F})$, then $\sigma \in \Gamma_\Phi(X, \mathscr{F}')$. This can be verified directly. $\square$

Definition 11 ([Har77]). The derived functors of $\Gamma_\Phi(X, -)$ are denoted by $H^i_\Phi(X, -)$ and called **sheaf cohomology with supports in** Φ, or **local cohomology for the family of supports** Φ.

Remark 8. When $W \subset X$ is a closed subset and Φ_W is the smallest family of supports containing W, $H_{\Phi_W}(X, \mathscr{F}) = H_W(X, \mathscr{F})$ where the right hand side is the usual local cohomology of $\mathscr{F}$ at the closed subset W.

Definition 12. Let X be a smooth, separated scheme of finite type over k and let Φ be a family of supports on X. The **Hodge cohomology of X with supports in** Φ is

$$H^*(X, \Phi) := H^*_\Phi(X, \Omega^*_X) = \bigoplus_{p,q} H^q_\Phi(X, \Omega^p_X)$$

Here Ω_X^p is the sheaf of differential p-forms on X. Since X is assumed to be smooth, it's locally free. The grading is defined by

$$H_\Phi^r(X, \Phi) := \bigoplus_{p+q=r} H_\Phi^q(X, \Omega_X^p) \text{ for all } r$$

Theorem 3.6. *Hodge cohomology with supports extends to a weak cohomology theory with supports satisfying the semipurity condition, and so there is a unique cycle map*

$$A^* \to H^*$$

Given a V^* morphism $f : (X, \Phi) \to (Y, \Psi)$, by differentiating f we obtain morphisms of sheaves $df^\vee : f^*\Omega_Y \to \Omega_X$; taking exterior powers gives morphisms $f^*\Omega_X^p \to \Omega_Y^p$. These induce homomorphisms

$$H_\Phi^q(Y, f^*\Omega_X^p) \to H_\Phi^q(Y, \Omega_Y^p)$$

which combined with the natural maps $H_\Psi^q(X, \Omega_X^p) \to H_{f^{-1}(\Psi)}^q(Y, f^*\Omega_X^p))$ and $H_{f^{-1}(\Psi)}^q(Y, f^*\Omega_X^p)) \to H_\Phi^q(Y, f^*\Omega_X^p))$ (recall that $f^{-1}(\Psi) \subset \Phi$) provide homomorphisms

$$f^* : H_\Psi^q(X, \Omega_X^p) \to H_\Phi^q(Y, \Omega_Y^p) \text{ for all } p, q$$

Given 2 objects (X, Φ) and (Y, Ψ) of V^*, there's a natural isomorphism

$$\pi_X^*\Omega_X \oplus \pi_Y^*\Omega_Y \xrightarrow{d\pi_X^\vee \oplus d\pi_Y^\vee} \Omega_{X \times Y} \text{ inducing an isomorphism of exterior algebras}$$

$$\pi_X^*\Omega_X^* \otimes \pi_Y^*\Omega_Y^* \xrightarrow{d\pi_X^\vee \wedge d\pi_Y^\vee} \Omega_{X \times Y}^*$$

From this we obtain a **Kunneth isomorphism**

$$H^*(X, \Omega_X^*) \otimes H^*(Y, \Omega_Y^*) \xrightarrow{\pi_X^* \wedge \pi_Y^*} H^*(X \times Y, \Omega_{X \times Y}^*)$$

Paying a bit more attention to supports, one obtains an external product $H_\Phi^*(X) \otimes H_\Psi^*(Y) \to H_{\Phi \times \Psi}(X \times Y)$.

On the other hand, the construction of pushforwards $f_* : H_\Phi^*(X) \to H_\Psi^*(Y)$ for V_*-morphisms $f : (X, \Phi) \to (Y, \Psi)$ is far from trivial. Describing these maps will occupy the remainder of this section. We will make extensive use of Grothendieck duality (see appendix B).

To begin let $f : X \to Y$ be a proper morphism of separated schemes of finite type over k. Then Grothendieck duality applied to the complex $\Omega_X^p[p]$ reads

$$Rf_* R\underline{Hom}_X(\Omega_X^p[p], \omega_X^*) \simeq R\underline{Hom}_Y(Rf_*\Omega_X^p, \omega_Y^*)$$

Now observe that differention of the map f provides a morphism

$$df^\vee \Omega_Y^p[p] \to Rf_*\Omega_X^p[p] \text{ and this yields an induced morphism}$$

$$R\underline{Hom}_Y(Rf_*\Omega_X^p[p], \omega_Y^*) \to R\underline{Hom}_Y(\Omega_Y^p[p], \omega_Y^*)$$

The upshot: we get a morphism

$$Rf_* D_X(\Omega_X^p[p]) \xrightarrow{f_*} D_Y(\Omega_Y^p[p])$$

Remark 9. When X and Y are smooth and equidimensional, $\omega_X^* \simeq \omega_X[n_X]$ and similarly for Y - here $n_X = \dim X$ and $n_Y = \dim Y$. There are natural isomorphisms

$$D_X(\Omega_X^p[p]) = R\underleftarrow{Hom}_X(\Omega_X^p[p], \omega_X[n_x])$$

$$= \underleftarrow{Hom}_X(\Omega_X^p, \omega_X)[n_X - p] \simeq \Omega_X^{n_X-p}[n_X - p] \text{ and similarly on } Y$$

(since X is smooth Ω_X^p is locally free, so we can drop the R in $R\underleftarrow{Hom}_X$, and the perfect pairing $\Omega_X^p \otimes \Omega_X^{n_X-p} \to \omega_X$ induces the last isomorphism). So, in the smooth case we get an isomorphism

$$Rf_*\Omega_X^{n_X-p}[n_X - p] \simeq \Omega_Y^{n_Y-p}[n_Y - p]$$

One can show that the morphism $Rf_*D_X(\Omega_X^p[p]) \xrightarrow{f_*} D_Y(\Omega_Y^p[p])$ is functorial in the way one would hope for - see [Con00].

Now let X be a proper separated scheme of finite type over k. Assume X is equidimensional of dimension d_X. Let Y be a smooth separated scheme of finite type over k, also equidimensional with dimension d_Y. Let $d = d_X + d_Y$.

Proposition 3.7. *For each j there is a natural morphism* [1]

$$(\pi_X^!\mathscr{O}_Y) \otimes \pi_Y^*\Omega_Y^j[j] \xrightarrow{\gamma} D_{X \times Y}(\pi_Y^*\Omega_Y^{d_Y-j}[d_Y - j])$$

making the following diagram commute:

$$(6) \quad \begin{array}{ccc}
R\pi_{Y*}((\pi_X^!\mathscr{O}_Y) \otimes \pi_Y^*\Omega_Y^j[j]) \longrightarrow \xrightarrow{\gamma_*} \longrightarrow & R\pi_{Y*}D_{X \times Y}(\pi_Y^*\Omega_Y^{d_Y-j}[d_Y - j]) \\
{\scriptstyle\text{projection formula}} \downarrow & \quad f_* \downarrow \\
(R\pi_{Y*}\pi_Y^!\mathscr{O}_Y) \otimes \Omega_Y^j[j] \xrightarrow{\text{tr}_f \otimes \text{id}} \Omega_Y^j[j] \xrightarrow{\simeq} & D_Y(\Omega_Y^{d_Y-j}[d_Y - j])
\end{array}$$

Furthermore if X is also smooth (or alternatively, on any smooth neighborhood $U \subset X$) the map

$$\pi_Y^!\mathscr{O}_Y \otimes \pi_Y^*\Omega_Y^j[j] \to D_{X \times Y}(\pi_Y^*\Omega_Y^{d_Y-j}[d_Y - j])$$

can be described as follows: first, since X is smooth $\pi_Y^!\mathscr{O}_Y \simeq \omega_{X \times Y|Y}[d_X]$ and so

$$\pi_Y^!\mathscr{O}_Y \otimes \pi_Y^*\Omega_Y^j[j] \simeq \omega_{X \times Y|Y}[d_X] \otimes \pi_Y^*\Omega_Y^j[j] \simeq \omega_{X \times Y|Y} \otimes \pi_Y^*\Omega_Y^j[d_X + j]$$

On the other hand, since $\omega_{X \times Y}[d] \simeq \pi_Y^*\omega_Y[d_Y] \otimes \omega_{X \times Y|Y}[d_X]$,

$$D_{X \times Y}(\pi_Y^*\Omega_Y^{d_Y-j}[d_Y - j]) = \underline{Hom}_{X \times Y}(\pi_Y^*\Omega_Y^{d_Y-j}[d_Y - j], \pi_Y^*\omega_Y[d_Y] \otimes \omega_{X \times Y|Y}[d_X])$$

$$\simeq \pi_Y^*\underline{Hom}_Y(\Omega_Y^{d_Y-j}[d_Y - j], \omega_Y[d_Y]) \otimes \omega_{X \times Y|Y}[d_X] \simeq \pi_Y^*\Omega_Y^j[j] \otimes \omega_{X \times Y|Y}[d_X]$$

$$\simeq \pi_Y^*\Omega_Y^j \otimes \omega_{X \times Y|Y}[d_X + j]$$

Combining these calculations gives an isomorphism

$$\gamma : \pi_Y^!\mathscr{O}_Y \otimes \pi_Y^*\Omega_Y^j[j] \xrightarrow{\simeq} D_{X \times Y}(\pi_Y^*\Omega_Y^{d_Y-j}[d_Y - j])$$

---

[1]At this point the indexing in the present paper diverges from that of [CR11]. Obviously that will make it hard to cross-reference but I prefer the present setup to an extent that it seemed worth it. Obviously I'm biased.

Now suppose $\iota : X \to Y$ is a closed immersion of *smooth* separated k-schemes of finite type. Assume X, Y are equidimensional of pure dimension d_X, d_Y respectively, so that ι has pure codimension $c = d_Y - d_X$.

Proposition 3.8. *there's an isomorphism*

$$\underline{R\Gamma}_X \Omega_Y^p[p] \simeq \mathscr{H}_X^c(\Omega_Y^p)[p - c]$$

Moreover if $t_1, \ldots, t_c \in H^0(Y, \mathscr{O}_Y)$ and $X = V(t_1, \ldots, t_c)$ (since X is smooth, it's a local complete intersection so this can always be arranged by getting local on Y) we can define a morphism

$$\iota_X^p : \iota_* \Omega_X^{p-c}[p - c] \to \mathscr{H}_X^c(\Omega_Y^p)[p - c] \ \ by \ \iota_X^p(\alpha) = (-1)^c \begin{bmatrix} (\wedge_i dt_i) \wedge \tilde{\alpha} \\ t \end{bmatrix}$$

and the following diagram commutes:

$$
\begin{array}{ccc}
\iota_* \Omega_X^{p-c}[p - c] \simeq \iota_* D_X(\Omega_X^{d_Y - p}[d_Y - p]) & \xrightarrow{\iota_*} & D_Y(\Omega_Y^{d_Y - p}[d_Y - p]) \simeq \Omega_Y^p[p] \\
\iota_X^p \downarrow & & \uparrow \\
\mathscr{H}_X^c(\Omega_Y^p)[p - c] & \xrightarrow{\simeq} & \underline{R\Gamma}_X(\Omega_Y^p[p])
\end{array}
$$

(7)

Remark 10. Note the (maybe) familiar special cases obtained by taking $p = 0$ and $p = n$:

$$\underline{R\Gamma}_X \mathscr{O}_Y \simeq \mathscr{H}_X^c(\mathscr{O}_Y)[-c] \ \text{and} \ \underline{R\Gamma}_X(\omega_Y[d_Y]) \simeq \mathscr{H}_X^c(\omega_Y)[d_X]$$

Remark 11. For the notation used in the definition of ι_X^p see the appendices.

Let X, Y be smooth separated schemes of finite type over k and let Φ, Ψ be families of support on X, Y respectively. Let $f : (X, \Phi) \to (Y, \Psi)$ be a V_*-morphism (recall this means $f|_\Phi$ is proper and $f(\Phi) \subset \Psi$).

Definition 13. A **compactification of** f is a commutative diagram of the form

(8)
$$
\begin{array}{ccc}
(X, \Phi) & \xrightarrow{\iota, \text{ open immersion}} & (\bar{X}, \Phi) \\
\downarrow & & \bar{f}, \text{ proper} \downarrow \\
(X, \Phi) & \xrightarrow{f} & (Y, \Psi)
\end{array}
$$

Here we are viewing Φ as a family of supports on $\bar{X}$ (this makes sense since $\bar{f}|_\Phi$ and $\bar{f} \circ \iota|_\Phi = f|_\Phi$ are both proper, which implies $\iota|_\Phi$ is proper. Hence in particular if $Z \in \Phi$, then $\iota(Z) \subset \bar{X}$ is closed. Note that this means Z is disjoint from the "boundary" $\bar{X} \setminus X$.

Theorem 3.9 ([Nag62], [Del10])**.** *A compactification of f exists. Moreover any two compactifications are dominated by a third.*

Finally, here's the definition of the pushforward morphism $f_* : H^*(X, \Phi) \to H^*(Y, \Psi)$: consider a V_* morphism f as above with a compactification $\bar{f}$ - for simplicity assume that X and Y are equidimensional, with dimensions d_X and d_Y respectively. For each $p \in \mathbb{N}$, we have a natural isomorphism

$$H_\Phi^q(X, \Omega_X^p[p]) \simeq H_\Phi^q(X, D_X(\Omega_X^{d_X - p}[d_X - p]))$$

Combining the condition $\Phi \subset f^{-1}(\Psi)$ with excision for local cohomology, we get a morphism

$$H_\Phi^q(X, D_X(\Omega_X^{d_X-p}[d_X-p])) \to H_{f^{-1}(\Psi)}^q(\bar{X}, D_{\bar{X}}(\Omega_{\bar{X}}^{d_X-p}[d_X-p])) \simeq H_\Psi^q(Y, Rf_* D_{\bar{X}}(\Omega_{\bar{X}}^{d_X-p}[d_X-p]))$$

(the last isomorphism is just derived composition of functors). Now recall that we have a morphism

$$Rf_* D_{\bar{X}}(\Omega_{\bar{X}}^{d_X-p}[d_X - p]) \xrightarrow{f_*} D_Y(\Omega_Y^{d_X-p}[d_X - p])$$

which induces a homomorphism

$$H_\Psi^q(Y, Rf_* D_{\bar{X}}(\Omega_{\bar{X}}^{d_X-p}[d_X - p])) \xrightarrow{f_*} H_\Psi^q(Y, D_Y(\Omega_Y^{d_X-p}[d_X - p]))$$

$$\simeq H_\Psi^q(Y, \Omega_Y^{d_Y-d_X+p}[d_Y - d_X + p])$$

The net result is a homomorphism

$$H_\Phi^q(X, \Omega_X^p[p]) \xrightarrow{f_*} H_\Psi^q(Y, \Omega_Y^{d_Y-d_X+p}[d_Y - d_X + p]) \text{ or getting rid of the shifts,}$$

$$H_\Phi^q(X, \Omega_X^p) \xrightarrow{f_*} H_\Psi^{q+c_f}(Y, \Omega_Y^{p+c_f}) \text{ where } c_f = d_Y - d_X \text{ is the codimension of } f$$

Remark 12. Suppose $f : X \to Y$ is a morphism of smooth proper varieties over k. Then the push-forwards $f_* : H^q(X, \Omega_X^p) \to H^{q+c_f}(Y, \Omega_Y^{p+c_f})$ have a comparatively simple description. Note that by Serre duality

$$H^q(X, \Omega_X^p) \simeq H^{d_X-q}(X, \Omega_X^{d_X-p})^\vee$$

The pull-back morphism $f^* : H^{q-d_X}(Y, \Omega_Y^{d_X-p}) \to H^{d_X-q}(X, \Omega_X^{d_X-p})$ induces a dual map

$$f^{*\vee} : H^{d_X-q}(X, \Omega_X^{d_X-p})^\vee \to H^{q-d_X}(Y, \Omega_Y^{d_X-p})^\vee$$

Applying Serre duality once more gives an isomorphism

$$H^{q-d_X}(Y, \Omega_Y^{d_X-p})^\vee \simeq H^{c_f-q}(Y, \Omega_Y^{c_f-p})$$

f_* is the composite of all these maps - in other words, f_* is Serre dual to f^*.

As an application of the push-forward, suppose X is a smooth separated scheme of finite type over k and let $D \xrightarrow{\iota} X$ be a smooth divisor on X. Let Φ be a family of supports on D - we can also view it as a family of supports on X, in which case ι is a V_*-morphism. There are homomorphisms

$$\iota_* : H_\Phi^q(D, \Omega_D^p) \to H_\Phi^{q+1}(X, \Omega_X^{p+1}) \text{ for all } p, q$$

Proposition 3.10. ι_* *is injective as long as* $q < \mathrm{codim}\Phi$.

Proof. Recall that there's a short exact sequence

$$0 \to \Omega_X^{p+1} \to \Omega_X^{p+1}(\log D) \xrightarrow{\mathrm{Res}} \Omega_D^p \to 0$$

where $\Omega_X^{p+1}(\log D)$ is the sheaf of log differential $p + 1$-forms on X with poles at D. The residue map Res can be described locally as follows: on a neighborhood where we have

a regular function t with $D = V(t)$, a section σ of $\Omega_X^{p+1}(\log D)$ looks like $\sigma = \frac{dt}{t} \wedge \nu + \eta$ where ν is a section of Ω_X^p and η is a section of Ω_X^{p+1}. Set [2]

$$\mathrm{Res}(\sigma) := \nu|_D$$

Taking local cohomology we get a long exact sequence

$$\cdots \to H_\Phi^q(X, \Omega_X^{p+1}) \to H_\Phi^q(X, \Omega_X^{p+1}(\log D)) \xrightarrow{\mathrm{Res}} H_\Phi^q(D, \Omega_D^p) \xrightarrow{\delta} H_\Phi^{q+1}(X, \Omega_X^{p+1}) \to \cdots$$

Fact: The connecting map δ coincides with ι_*. I'm not going to prove this here - see [CR11]. By exactness it now suffices to show:

$$H_\Phi^q(X, \Omega_X^{p+1}(\log D)) \simeq 0 \text{ for } q < \mathrm{codim}\,\Phi$$

For this we'll make use of the derived composition of functors $R\Gamma_\Phi = R\Gamma \circ \underline{R\Gamma}_\Phi$. We have a composition of functors spectral sequence of the form

$$H^i(X, \underline{H}_\Phi^j(\Omega_X^{p+1}(\log D))) \implies H^{i+j}(X, \Omega_X^{p+1}(\log D))$$

Hence it's enough to show that

$$\underline{H}_\Phi^j(\Omega_X^{p+1}(\log D)) = 0 \text{ for } j < \mathrm{codim}\,\Phi$$

This last fact holds since X, D are smooth and hence $\Omega_X^{p+1}(\log D)$ is locally free, so in particular Cohen-Macaulay.

$\square$

3.2.4. *Correspondences.* Let F be a cohomology theory with supports and say X_1, X_2, X_3 are separated smooth varieties of finite type over k. Lt Φ_{ij} be a family of supports on $X_i \times X_j$, for $ij = 12, 13, 23$. Assume

- $\pi_{13}|_{\pi_{12}^{-1}(\Phi_{12}) \cap \pi_{23}^{-1}(\Phi_{23})}$ is proper and
- $\pi_{13}(\pi_{12}^{-1}(\Phi_{12}) \cap \pi_{23}^{-1}(\Phi_{23})) \subset \Phi_{13}$.

Here $\pi_{ij} : X_1 \times X_2 \times X_3 \to X_1 \times X_j$ is the projection (note that these are all flat). Say $\alpha \in F(X_1 \times X_2, \Phi_{12})$ and $\beta \in F(X_2 \times X_3, \Phi_{23})$. Then (since the projections are flat)

$$\pi_{12}^*\alpha \in F(X_1 \times X_2 \times X_3, \pi_{12}^{-1}(\Phi_{12})) \text{ and } \pi_{23}^*\beta \in F(X_1 \times X_2 \times X_3, \pi_{23}^{-1}(\Phi_{23}))$$

$$\text{so that } \pi_{12}^*\alpha \smile \pi_{23}^*\beta \in F(X_1 \times X_2 \times X_3, \pi_{12}^{-1}(\Phi_{12}) \cap \pi_{23}^{-1}(\Phi_{23}))$$

Now (using the above 2 bullet points) we obtain

$$\beta \circ \alpha := \pi_{13*}(\pi_{12}^*\alpha \smile \pi_{23}^*\beta)$$

We've defined a homomorphism

$$F(X_1 \times X_2, \Phi_{12}) \otimes F(X_2 \times X_3, \Phi_{23}) \xrightarrow{\circ} F(X_1 \times X_3, \Phi_{13})$$

Now let X, Y be smooth separated schemes of finite type over k and let Φ, Ψ be families of support on X, Y respectively.

---

[2] There are probably multiple sign conventions in use - this is the one I've seen most frequently.

Definition 14. $P(\Phi, \Psi)$ is the family of supports on $X \times Y$ defined by

$$P(\Phi, \Psi) := \{Z \subset X \times Y \text{ closed } \mid Z \cap \pi_X^{-1}(W) \subset \pi_Y^{-1}(\Psi) \text{ for } W \in \Phi \text{ and } \pi_Y|_Z \text{ is proper}\}$$

Now suppose $\gamma \in F(X_1 \times X_2, P(\Phi, \Psi))$ and $\alpha \in F(X, \Phi)$. Then $\pi_X^* \alpha \in F(X \times Y, \pi_X^{-1}(\Phi))$ and so

$$\pi_X^* \alpha \smile \gamma \in F(X \times Y, \pi_X^{-1}\Phi \cap P(\Phi, \Psi))$$

At this point we observe that the definition of $P(\Phi, \Psi)$ guarantees that $\pi_Y|_{\pi_X^{-1}\Phi \cap P(\Phi, \Psi)}$ is proper and $\pi_Y(\pi_X^{-1}\Phi \cap P(\Phi, \Psi)) \subset \Psi$, and so we obtain

$$\gamma(\alpha) := \pi_{Y*}(\pi_X^* \alpha \smile \gamma) \in F(Y, \Psi)$$

In this way we get a pairing

$$F(X, \Phi) \otimes F(X \times Y, P(\Phi, \Psi)) \to F(Y, \Psi) \text{ sending}$$

$$(\alpha, \gamma) \mapsto \gamma(\alpha) \ (= \pi_{Y*}(\pi_X^* \alpha \smile \gamma))$$

Proposition 3.11. *If $X_1, \ldots, X_4$ are smooth separated schemes of finite type over k and Φ_i is a family of supports on X_i for $i = 1, \ldots, 4$, and $P_{ij} = P(\Phi_i, \Phi_j)$ for $1 \le i < j \le 4$, then*

- $\pi_{13}|_{\pi_{12}^{-1}(P_{12}) \cap \pi_{23}^{-1}(P_{23})}$ *is proper and* $\pi_{13}(\pi_{12}^{-1}(P_{12}) \cap \pi_{23}^{-1}(P_{23})) \subset P_{13}$. *Hence the pairing*

$$\circ : F(X_1 \times X_2, P_{12}) \otimes F(X_2 \times X_3, P_{23}) \to F(X_1 \times X_3, P_{23}) \to F(X_1 \times X_3, P_{13}) \text{ is well defined.}$$

- *If $\gamma_{ij} \in F(X_i \times X_j, P_{ij})$ for $ij = 12, 23, 34$ then*

$$\gamma_{34} \circ (\gamma_{23} \circ \gamma_{12}) = (\gamma_{34} \circ \gamma_{23}) \circ \gamma_{12} \in F(X_1 \times X_4, P_{14})$$

In other words, $\circ$ is associative.

Definition 15. The category Cor_F of F-correspondences has as objects the pairs (X, Φ) where X is a smooth separated scheme of finite type over k and Φ is a family of supports on X, and morphisms

$$\mathrm{Hom}_{\mathrm{Cor}_F}((X, \Phi), (Y, \Psi)) := F(X \times Y, P(\Phi, \Psi))$$

Composition is defined using the pairing $\circ$ above. The identity of (X, Φ) is given by the cycle class $[\Delta] \in F(X \times X, P(\Phi, \Phi))$ of the diagonal $X \xrightarrow{\Delta} X \times X$.

Remark 13. Cor_F is a symmetric monoidal category with product $(X, \Phi) \otimes (Y, \Psi) = (X \times Y, \Phi \times \Psi)$. It lacks products for the same reasons V_* and V^* do.

Remark 14. We can make Cor_F a category enriched in graded abelian groups (rather than just abelian groups) by defining a grading

$$\mathrm{Hom}((X, \Phi), (Y, \Psi))^d = \bigoplus_i F^{2 \dim X_i + d}(X_i \times Y, P(\Phi_i, \Psi))$$

where X_i are the connected components of X and Φ_i is the restriction of Φ to X_i.

To unpack this note that if X is connected and if $\alpha \in F^j(X, \Phi)$ then $\pi_X^* \alpha \in F^j(X \times Y, \pi_X^{-1}(\Phi))$. Now if $\gamma \in F^{2 \dim X + d}(X \times Y, P(\Phi, \Psi))$ we'll have $\pi_X^* \alpha \smile \gamma \in F^{2 \dim X + d + j}(X \times$

$Y, \pi_X^{-1}(\Phi) \cap P(\Phi, \Psi))$ and since the relative dimension of the map $\pi_Y : X \times Y \to Y$ is $\dim X$, π_{Y*} drops dimension by $2 \dim X$. So,

$$\gamma(\alpha) = \pi_{Y*}(\pi_X^* \alpha \smile \gamma) \in F^{d+j}(Y, \Psi)$$

In other words, the correspondence γ gives a morphism of graded abelian groups $F^*(X, \Phi) \to F^*(Y, \Psi)[d]$.

Let V denote the symmetric monoidal category enriched (in a trivial way) in graded abelian groups with objects the pairs (X, Φ) as above but with only identity morphisms. The symmetric monoidal product is $\otimes$ as above. Observe that Cor_F comes with a natural functor $V \to \mathrm{Cor}_F$, so it's an object in the functor category

$$V/\mathrm{Cat}_{\mathrm{GrAb},\otimes}$$

In fact if $\mathcal{T}$ is the category of (weak) cohomology theories with supports, we have a functor

$$\mathrm{Cor} : \mathcal{T} \to V/\mathrm{Cat}_{\mathrm{GrAb},\otimes}$$

Not surprisingly, it's fully faithful.

It should be emphasized that the functoriality of the Cor construction is actually very useful in practice. In particular, it implies that:

calculations with correspondences commute with cycle maps .

For example if $\gamma \in A(X \times Y, P(\Phi, \Psi))$ and $\eta(\gamma) \in H^*(X \times Y, P(\Phi, \Psi))$ is the Hodge cycle class of γ, then the following diagram commutes:

(9)
$$
\begin{array}{ccc}
A(X, \Phi) & \xrightarrow{\gamma} & A(Y, \Psi) \\
\eta \downarrow & & \eta \downarrow \\
H(X, \Phi) & \xrightarrow{\eta(\gamma)} & H(Y, \Psi)
\end{array}
$$

3.3. **Theorems.** Let S be a separated k-scheme of finite type and let X, Y be connected S-schemes of finite type that are smooth and separated over k - let $f : X \to S$ and $g : Y \to S$ be their structure morphisms. Let $Z \subset X \times_S Y$ be an integral closed subscheme of dimension $d_Y := \dim Y$ and assume that $\pi_Y|_Z$ is proper. Observe that in this situation $[Z] \in A^{d_X}(X \times_k Y, P(\Phi_X, \Phi_Y)) = \mathrm{Hom}_{\mathrm{Cor}_A}(X, Y)^0$ defines a Chow correspondence of degree 0 from X to Y. Here Φ_X, Φ_Y are the maximal families of supports (Φ_X is all the closed subsets of X and similarly for Y). Using the cycle class map $\eta : A \to H$ we get a Hodge correspondence

$$\eta(Z) \in H^{2d_X}(X \times Y, P(\Phi_X, \Phi_Y)) = \mathrm{Hom}_{\mathrm{Cor}_H}(X, Y)^0$$

Moreover if $U \subset S$ is open then we obtain an integral subvariety $Z_U := Z \times_S U \subset X \times_S Y \times_S U = X_U \times_U Y_U$, and from Z_U we get a Hodge correspondence $\eta(Z_U)$ inducing morphisms

$$\varphi_{Z_U} : H^q(X_U, \Omega_{X_U}^p) \to H^q(Y_U, \Omega_{Y_U}^p) \text{ for all } p, q$$

Proposition 3.12. *The φ_{Z_U} are $\mathcal{O}_U$ linear and compatible with localization, and thus form a morphism of pre-sheaves of $\mathcal{O}_X$-modules.*

Since the presheaf $U \mapsto H^q(X_U, \Omega^p_{X_U})$ sheafifies to the higher direct image $R^q f_* \Omega^p_X$ and similarly for Y, we obtain the following

corollary

An integral subvariety $Z \subset X \times_S Y$ as above induces morphisms of quasi-coherent sheaves $\phi_Z : R^q f_* \Omega^p_X \to R^q g_* \Omega^p_Y$ for all p, q.

corollary

Now let X, Y be separated, smooth connected varieties over k with dimensions d_X and d_Y respectively. Let

$$\alpha \in \mathrm{Hom}_{\mathrm{Cor}_A}(X, Y)^0 = A^{d_X}(X \times Y, P(\Phi_X, \Phi_Y))$$

be a Chow correspondence of degree 0. Recall that if $\alpha = \sum_i n_i[Z_i]$ where the Z_i are irreducible subvarieties of $X \times Y$ with codimension d_X,

$$\mathrm{supp}\,\alpha := \cup_i Z_i \subset X \times Y$$

Proposition 3.13 (the key technical lemma of [CR11]). *If $\pi_Y(\mathrm{supp}\,\alpha) \subset Y$ has codimension r, then the correspondence*

$$\eta(\alpha) \in \mathrm{Hom}_{\mathrm{Cor}_H}(X, Y)^0 = H^{2d_X}(X \times Y, P(\Phi_X, \Phi_Y))$$

vanishes on $H^q(X, \Omega^p_X)$ for $p < r$. If $\pi_X(\mathrm{supp}\,\alpha) \subset X$ has codimension r, then $\eta(\alpha)$ vanishes on $H^q(X, \Omega^p_X)$ for $p > d_X - r$. [3]

Proof. Since the map

$$A^{d_X}(X \times Y, P(\Phi_X, \Phi_Y)) \to \mathrm{Hom}_{\mathrm{GrAb}}(H^*(X), H^*(Y))$$

given by the cycle map and the correspondence construction is linear, we may assume that $\alpha = [V]$ where $V \subset X \times Y$ is an integral subvariety of codimension d_X, with the property that $\pi_Y|_V$ is proper, so that $V \in P(\Phi_X, \Phi_Y)$. Note that in this situation $[V] \in A^{d_X}(X \times Y, P(\Phi_X, \Phi_Y))$ is the image of a class in $A^{d_X}(X \times Y, \Phi_V)$.

Recall that the correspondence $\eta(V)$ takes a class $a \in H^q(X, \Omega^p_X)$ to

$$\pi_{Y*}(\pi_X^* a \smile \eta(V)) \in H^q(Y, \Omega^p_Y)$$

Note that $\pi_X^* a \in H^q(X \times Y, \Omega^p_{X \times Y})$. In fact with respect to the decomposition $\Omega^p_{X \times Y} \simeq \bigoplus_i \Omega^i_X \boxtimes \Omega^{p-i}_Y$ and the resulting decomposition

$$H^q(X \times Y, \Omega^p_{X \times Y}) \simeq \bigoplus_i H^q(X \times Y, \Omega^i_X \boxtimes \Omega^{p-i}_Y), \text{ we have } \pi_X^* a \in H^q(X, \pi_X^* \Omega^p_X)$$

(the $i = p$ summand). On the other hand,

$$\eta(V) \in H^{d_X}_V(X \times Y, \Omega^{d_X}_{X \times Y}) \simeq \bigoplus_i H^{d_X}_V(X \times Y, \Omega^i_X \boxtimes \Omega^{d_X-i}_Y)$$

Let $\eta(V)_i$ be the component of $\eta(V)$ in the ith summand. Now $\pi_X^* a \smile \eta(V)_i \in H^{d_X+q}_V(X \times Y, \Omega^{p+i}_X \boxtimes \Omega^{d_X-i}_Y)$.

---

[3]In [CR11] this appears as "$p > d_X - r + 1$" - however the argument shows vanishing for $p > d_X - r$.

It follows that

$$\pi_{Y*}(\pi_X^* a \smile \eta(V)) = \pi_{Y*}(\pi_X^* a \smile \eta(V)_{d_X-p})$$

Here's what they actually show: if $\operatorname{codim}(\pi_Y(V), Y) \geq r$ then $\eta(V)_{d_X-p} = 0$ for $p < r$ (in other words, $\eta(V)_i = 0$ for $i > d_X - r$) and if $\operatorname{codim}(\pi_X(V), X) \geq r$ then $\eta(V)_{d_X-p} = 0$ for $p > d_X - r + 1$ (in other words, $\eta(V)_i = 0$ for $i < r - 1$). Unwinding further: if $\operatorname{codim}(\pi_Y(V), Y) \geq r$, then the image of $\eta(V)$ in $H_V^{d_X}(X \times Y, \Omega_X^i \boxtimes \Omega_Y^{d_X-i})$ is 0 when $d_X - i > r$. If $\operatorname{codim}(\pi_X(V), X) \geq r$ then that projection of $\eta(V)$ is 0 when $i < r$. For the punchline I must ask you to see the original paper.

$\square$

Combining these results yields:

Proposition 3.14. *Let S be a separated k-scheme of finite type and let $f : X \to S, g : Y \to S$ be connected S-schemes of finite type that are smooth and separated over k. Let $Z \subset X \times_S Y$ be a integral subvariety of dimension $d_Y := \dim Y$. If $\operatorname{codim}(\pi_Y(Z), Y) \geq r$ then $\varphi_Z^{p;q} : R^q f_* \Omega_X^p \to R^q g_* \Omega_Y^p$ vanishes for $p < r$. If $\operatorname{codim}(\pi_X(Z), X) \geq r$ then $\varphi_Z^{p;q}$ vanishes for $p > d_X - r + 1$.*

Definition 16. Two integral schemes X, Y over a base scheme S are **properly birational over** S if and only if there's a third integral S-scheme Z together with proper birational S-morphisms

$$
\begin{array}{ccc}
Z & \longrightarrow & Y \\
\downarrow & & \\
X & &
\end{array}
$$

(10)

Now let S be a scheme over a (perfect) field k and let $f : X \to S$ and $g : Y \to S$ be integral S schemes which are smooth and separated over k. Let Z be another separated, integral S-scheme together with proper birational S-morphisms $p_X : Z \to X$ and $p_Y : Z \to Y$. Let $Z_0 \subset X \times_S Y$ be the image of $p_X \times_S p_Y : Z \to X \times_S Y$ (with its reduced subscheme structure).

Theorem 3.15. *Then the induced morphisms of sheaves*

$$\varphi_Z^{q,0} : R^q f_* \mathscr{O}_X \to R^q g_* \mathscr{O}_Y \ \text{and} \ \varphi_Z^{q,n} : R^q f_* \omega_X \to R^q g_* \omega_Y$$

are isomorphisms for all q.

Here $n = \dim X = \dim Y$

Sketch. It will suffice to prove that for every open set $U \subset S$ the homomorphisms

$$\varphi_{Z_U}^{q,0} : H^q(X_U, \mathscr{O}_{X_U}) \to H^q(Y_U, \mathscr{O}_{Y_U}) \ \text{and} \ \varphi_{Z_U}^{q,n} : H^q(X_U, \omega_{X_U}) \to H^q(Y_U, \omega_{Y_U})$$

are isomorphisms for all q. The desired result is then obtained after sheafification; in this way we reduce to the "absolute case" where $S = \operatorname{Spec} k$. We may also assume $Z = Z_0$ (otherwise just replace Z with Z_0).

Let $Z^t \subset Y \times_k X$ be the image of Z under the standard "flip" map $X \times Y \xrightarrow{\pi_Y \times \pi_X} Y \times X$. This is also a codimension n subvariety, defining a Chow correspondence

$$[Z^t] \in \mathrm{Hom}_{\mathrm{Cor}_A}(Y, X)^0 = A^n(Y \times X, P(\Phi_Y, \Phi_X))$$

The key observation is:

Lemma 3.16. *There exist decompositions*

$$[Z] \circ [Z^t] = \Delta_Y + E_Y \in \mathrm{Hom}_{\mathrm{Cor}_A}(Y, Y)^0 \ \text{and} \ [Z^t] \circ [Z] = \Delta_X + E_X \in \mathrm{Hom}_{\mathrm{Cor}_A}(X, X)^0$$

with the following property: if X', Y', Z' are open subsets of X, Y, Z respectively such that $\pi_X|_{Z'} : Z' \xrightarrow{\simeq} X'$ and $\pi_Y|_{Z'} : Z' \xrightarrow{\simeq} Y'$ are both isomorphisms, then

$$\mathrm{supp} E_X \subset (X \setminus X') \times (X \setminus X') \ \text{and} \ \mathrm{supp} E_Y \subset (Y \setminus Y') \times (Y \setminus Y')$$

In particular, the projection of $\mathrm{supp} E_X$ to either factor has codimension ≥ 1, and hence it induces the 0 map on $H^q(X, \mathscr{O}_X)$ and $H^q(X, \omega_X)$ by Proposition 3.14. Since the correspondence Δ_X acts as the idenity on Hodge cohomology, it follows that $[Z^t] \circ [Z]$ acts as the identity on $H^q(X, \mathscr{O}_X)$ and $H^q(X, \omega_X)$; similarly for Y. We conclude that

$$\phi_Z^{q,0} : H^q(X, \mathscr{O}_X) \to H^q(Y, \mathscr{O}_Y) \ \text{and} \ \phi_{Z^t}^{q,0} : H^q(Y, \mathscr{O}_Y) \to H^q(X, \mathscr{O}_X)$$

are mutually inverse isomorphisms; similarly for ω.

$\square$

4. Gysin homomorphisms

A natural strategy for generalizing the results of section 3.3 to pairs (X, Δ) is to reduce to the absolute case $(\Delta = 0)$ by relating the Hodge cohomology of (X, Δ) to that of X and Δ. Gysin homomorphisms and the Gysin exact sequence serve this purpose. But first I should explain what is meant by "the Hodge cohomology of (X, Δ)."

4.1. **Sheaves of log differentials.** Let X be a smooth variety over a field k, and let Δ be a reduced, effective snc divisor on X. Let $U := X \setminus \Delta$ and let $\iota : U \to X$ be the inclusion. If Ω_U^* is the de Rham complex on U, then $\iota_* \Omega_U^*$ is a complex on X.

Remark 15 (Caution). While the individual sheaves in the complex $\iota_* \Omega_U^*$ are quasi-coherent, the differentials (exterior differentiation) are only k-linear, satisfying the Leibniz rule

$$d(f \cdot \sigma) = df\sigma + fd\sigma \ \text{for} \ f \in \mathscr{O}_X(V), \ \sigma \in \Omega_U^p(V)$$

for all neighborhoods V in U. Staring at the Leibniz rule one sees that the cohomology sheaves $\mathscr{H}^i(\iota_* \Omega_U^*)$ are generally not even sheaves of $\mathscr{O}_X$-modules.

Since $\iota : U \to X$ is the inclusion of the complement of the divisor Δ, we have identifications

$$\mathrm{colim}_k \Omega_X^p(k\Delta) simeq \iota_* \Omega_U^p \ \text{for all} \ p$$

Definition 17. The **sheaf of log differentials on** X **with poles along** Δ, denoted $\Omega_X^p(\log\Delta)$, is the subsheaf of $\iota_*\Omega_U^p$ defined by

$$\Omega_X^p(\log\Delta) := \{\sigma \in \iota_*\Omega_U^p \mid \sigma \in \Omega_X^p(\Delta) \text{ and } d\sigma \in \Omega_X^{p+1}(\Delta)\}$$

which is to say, *both σ and $d\sigma$ have poles of order at most 1 along Δ.*

The definition guarantees that the $\Omega_X^p(\log\Delta)$ form a *subcomplex* of $\iota_*\Omega_U^*$.

Proposition 4.1 ([EV92]). *Write $\Delta = \sum_{i=1}^N D_i$, where the $D_i \subset X$ are smooth, connected divisors (intersecting transversely according to the snc condition). Then for each i and for all p there are short exact sequences of quasi-coherent sheaves*

$$0 \to \Omega_X^p(\log\Delta)(-D_i) \to \Omega_X^p(\log(\Delta - D_i)) \xrightarrow{\text{res}} \Omega_{D_i}^p(\log(\Delta - D_i)|_{D_i}) \to 0 \text{ and}$$

$$0 \to \Omega_X^p(\log(\Delta - D_i) \to \Omega_X^p(\log\Delta) \xrightarrow{\text{Res}} \Omega_{D_i}^{p-1}(\log(\Delta - D_i)|_{D_i}) \to 0$$

Implicit in these formulae is the fact that $(\Delta - D_i)|_{D_i}$ is a snc divisor on the smooth variety D_i (this is, like, why everyone loves snc divisors). In the first short exact sequence the map res is just restricting differential forms to D_i. In the second exact sequence the map Res is known as the residue map along D_i - it's a bit more complicated - see [EV92].

Remark 16. In particular we see that $\Omega_X^0(\log\Delta) = \mathcal{O}_X$ and $\Omega_X^n(\log\Delta) = \omega_X(\Delta)$ (here $n = \dim X$) - this will be quite useful later on.

Both the res and Res exact sequences can be obtained from an explicit description of $\Omega_X^p(\log\Delta)$:

Proposition 4.2 ([EV92]). *The sheaf $\Omega_X^1(\log\Delta)$ is locally free of rank $n = \dim X$. If $x \in X$ and $x \in D_1, \ldots, D_r$, and if $z_1, \ldots z_n$ are local coordinates on a neighborhood $W \subset X$ of x such that*

$$D_i \cap W = V(z_i) \text{ for } i = 1, \ldots r$$

then the sections

$$\frac{dz_1}{z_1}, \ldots, \frac{dz_r}{z_r}, dz_{r+1}, \ldots, dz_n$$

freely generate $\Omega_X^1(\log\Delta)$ over V. Moreover the natural map

$$\bigwedge^p \Omega_X^1(\log\Delta) \xrightarrow{\cong} \Omega_X^p(\log\Delta) \text{ is an isomorphism}$$

Consider the long exact sequence on cohomology associated to

$$0 \to \Omega_X^p(\log(\Delta - D_i)) \to \Omega_X^p(\log\Delta) \xrightarrow{\text{Res}_i} \Omega_{D_i}^{p-1}(\log(\Delta - D_i)|_{D_i}) \to 0$$

It takes the form

$$\cdots \to H^q(X, \Omega_X^p(\log(\Delta - D_i))) \to H^q(X, \Omega_X^p(\log\Delta)) \xrightarrow{\text{Res}_i} H^q(D_i, \Omega_{D_i}^{p-1}(\log(\Delta - D_i)|_{D_i}))$$

$$\xrightarrow{\delta} H^{q+1}(X, \Omega_X^p(\log(\Delta - D_i))) \to \cdots$$

As p, q vary these can be assembled to form a long exact sequence on Hodge cohomology; we need one more definition

Definition 18. Let (X, Δ) be a pair as above, with X a smooth variety over k and $\Delta \subset X$ a reduced, effective snc divisor. The **Hodge cohomology of** X, Δ is the graded k-algebra $H^*(X, \Delta)$ defined by

$$H^r(X, \Delta) := \bigoplus_{p+q=r} H^q(X, \Omega^p_X(\log \Delta)) \text{ for all } r$$

With this terminology, we have obtained an exact sequence

$$\cdots \to H^r(X, \Delta - D_i) \to H^r(X, \Delta) \xrightarrow{\text{Res}_i} H^{r-1}(D_i, (\Delta - D_i)|_{D_i})$$

$$\xrightarrow{\delta} H^{r+1}(X, \Delta - D_i) \to \cdots$$

referred to from now on as the **Gysin sequence** for Hodge cohomology.

4.1.1. *The situation over* $\mathbb{C}$: *some guiding principles.* It is a theorem of Grothendieck that when Y is a smooth projective variety over $\mathbb{C}$, the natural map $\underline{\mathbb{C}} \to \Omega^*_Y$ is a quasi-isomorphism of sheaves of $\mathbb{C}$-vector spaces on Y. Here $\underline{\mathbb{C}}$ is the locally constant sheaf associated to $\mathbb{C}$. Together with the collapse on page 1 of the Hodge-to-de-Rham spectral sequence

$$E_1^{pq} = H^q(X, \Omega^p_X) \implies \mathbb{H}^{p+q}(X, \Omega^*_X)$$

this provides isomorphisms

$$H^r(X) \simeq \mathbb{H}^r(X, \Omega^*_X) \simeq H^r(X, \underline{\mathbb{C}}) \text{ for all } r$$

See [PS08] for a further discussion.

Now let X be a smooth projective variety with a reduced effective snc divisor $\Delta \subset X$ - let $U := X \setminus \Delta$. From the above discussion we get a quasi-isomorphism $\underline{\mathbb{C}} \simeq \Omega^*_U$ and pushing forward along the inclusion $\iota : U \to X$ gives a quasi-isomorphism $\iota_* \underline{\mathbb{C}} \simeq \iota_* \Omega^*_U$ on X (yes, I know I am cheating a little bit here, as U is only quasi-projective - let us not worry about it).

Now the cool theorem (see [PS08] (or [DB81] for a souped up version)) is that in fact the inclusion $\Omega^*_X(\log \Delta) \subset \iota_* \Omega^*_U$ is a quasi-isomorphism, and the hypercohomology spectral sequence for $\Omega^*_X(\log \Delta)$ collapses on page 1. In this way we obtain identifications

$$H^r(X, \Delta) \simeq \mathbb{H}^r(X, \Omega^*_X(\log \Delta)) \simeq \mathbb{H}^r(X, \iota_* \Omega^*_U) \simeq H^r(U, \underline{\mathbb{C}})$$

In the case where Δ is smooth (in other words, its components are disjoint), so that $\Delta \subset X$ is a smooth submanifold of codimension 2, this matches up our Gysin sequence for Hodge cohomology with the Gysin sequence of classical topology:

$$\cdots \to H^{r-2}(\Delta, \underline{\mathbb{C}}) \xrightarrow{j_*} H^r(X, \underline{\mathbb{C}}) \xrightarrow{\iota^*} H^r(U, \underline{\mathbb{C}})$$

$$\xrightarrow{\delta} H^{r-1}(\Delta, \underline{\mathbb{C}}) \to \cdots$$

Here $j : \Delta \to X$ is the inclusion and j_* is the Gysin map for sheaf (or singular) cohomology.

As an example application of these comparison tools, allow me to sketch a proof of Kodaira-Akizuki-Nakano vanishing, inspired by the presentation in [VS03].

Theorem 4.3 (Kodaira-Akizuki-Nakano vanishing). *Let X be a smooth projective variety of dimension n over $\underline{\mathbb{C}}$ and let $\mathscr{L}$ be an ample line bundle on X. Then*

$$H^q(X, \Omega_X^p \otimes \mathscr{L}) = 0 \text{ for } p + q > n$$

I will assume the following wonderful theorem of Andreotti and Frankel [Mil63]:

Theorem 4.4. *Let $Y \subset \mathbb{C}^N$ be a smooth, affine algebraic variety over $\mathbb{C}$ of dimension n (hence real dimension $2n$). Then Y has the homotopy type of a (finite) CW complex of dimension $\leq n$.*

Now let $\mathscr{L}$ be a very ample line bundle on a smooth projective variety X over $\mathbb{C}$, and let $\sigma \in \Gamma(X, \mathscr{L})$ be a generic section. By the Bertini theorem, $\Delta := V(\sigma)$ is a smooth divisor on X. Take the exact sequence

$$0 \to \Omega_X^p(\log \Delta)(-\Delta) \to \Omega_X^p \to \Omega_\Delta^p \to 0,$$

twist by Δ and take cohomology to obtain an exact sequence

$$\cdots \to H^q(X, \Omega_X^p(\log \Delta)) \to H^q(X, \Omega_X^p(\Delta)) \to H^q(\Delta, \Omega_\Delta^p(\Delta|_\Delta)) \to \cdots$$

Since $\mathscr{L}$ is very ample, $U = X \setminus \Delta$ is affine. Hence by Theorem 4.4, $H^r(U, \underline{\mathbb{C}}) = 0$ for $r > n$ and by our comparison theorem,

$$H^q(X, \Omega_X^p(\log \Delta)) = 0 \text{ for } p + q > n$$

Since $\Delta|_\Delta$ is a very ample divisor on the smooth projective variety Δ (of dimension $n - 1$) by induction on dimension we can assume

$$H^q(\Delta, \Omega_\Delta^p(\Delta|_\Delta)) = 0 \text{ for } p + q > n - 1$$

and it follows that $H^q(X, \Omega_X^p \otimes \mathscr{L}) = 0$ for $p + q > n$.

One can now reduce to the case where $\mathscr{L}$ is merely ample (instead of very ample) using a cyclic covering construction along the lines of the proof of Kawamata-Viehwag vanishing in [KM98].

4.2. **Gysin exact sequences for Chow?** A natural question is whether there is a corresponding Gysin sequence for Chow groups, i.e. if $\Delta \subset X$ is a smooth divisor on a smooth variety X, with complement $U := X \setminus \Delta$, is there a long exact sequence of Chow groups

$$"\cdots \to A^{r-2}(\Delta) \xrightarrow{j_*} A^r(X) \xrightarrow{i^*} A^r(U) \xrightarrow{\delta} A^{r-1}(\Delta) \to \cdots"$$

The answer is: it's complicated. For one thing we'd need to extend $A^*(X)$ by adding terms in odd degrees.

Here's what we can say: the restriction maps $\iota^* : A^r(X) \to A^r(U)$ are always surjective (given a cycle $Z \subset U$, its closure $\bar{Z} \subset X$ is a cycle on X with $\iota^*\bar{Z} = \bar{Z} \cap U = Z$). One can show that any cycle $\alpha \in \ker \iota^*$ is supported on Δ, so that the pushforward $j_* : A^{r-2}(\Delta) \to A^r(X)$ surjects onto $\ker \iota^*$ (recall here that we have arranged the grading of A^* so that A^{2r} corresponds to codimension r cycles, and the odd degree components of A^* are all 0). So, we have right-exact sequences

$$A^{r-2}(\Delta) \xrightarrow{j_*} A^r(X) \xrightarrow{i^*} A^r(U) \to 0$$

for all r.

A natural question is whether these may be extended to the left. They can, but all of the extensions I have seen involve things like motivic cohomology and sheaves of higher K-theory spectra - see the last section of [Ful98]. Hopefully we can get by with right exactness.

5. THRIFTY RATIONAL RESOLUTIONS REVISITED

In this section I will outline one strategy for proving the following conjecture:

Conjecture 1. *Let S be a normal variety over a perfect field k and let $\Delta_S \subset S$ be a reduced, effective Weil divisor. Let $f : X \to S$ and $g : Y \to S$ be two thrifty resolutions, and let $\Delta_X := f_*^{-1}(\Delta_S)$, $\Delta_Y := g_*^{-1}(\Delta_S)$. Then there are quasi-isomorphisms*

$$Rf_*\mathscr{O}_X(-\Delta_X) \simeq Rg_*\mathscr{O}_Y(-\Delta_Y) \text{ and } Rf_*\omega_X(\Delta_X) \simeq Rg_*\omega_Y(\Delta_Y)$$

In particular, if f is a rational resolution so is g.

One basic observation is that

Proposition 5.1.

$$Rf_*\mathscr{O}_X(-\Delta_X) \simeq Rg_*\mathscr{O}_Y(-\Delta_Y) \text{ if and only if } Rf_*\omega_X(\Delta_X) \simeq Rg_*\omega_Y(\Delta_Y)$$

so it is enough to obtain one of the desired quasi-isomorphisms.

Proof. By Grothendieck duality, there is a natural isomorphism

$$Rf_*\underline{Hom}_X(\mathscr{O}_X(-\Delta_X), \omega_X^*) \simeq R\underline{Hom}_S(Rf_*\mathscr{O}_X(-\Delta_X), \omega_S^*)$$

Since X is smooth, $\omega_X^* \simeq \omega_X[n]$ where $n = \dim X$, and so

$$Rf_*\underline{Hom}_X(\mathscr{O}_X(-\Delta_X), \omega_X^*) \simeq Rf_*\omega_X(\Delta_X)[n]$$

On the other side of the equation, recall that

$$D_S := R\underline{Hom}_S(-, \omega_S^*)$$

is the **dualizing functor of** S. We have obtained an identification

$$Rf_*\omega_X(\Delta_X) \simeq D_S(Rf_*\mathscr{O}_X(-\Delta_X))[-n]$$

Similarly $Rg_*\omega_Y(\Delta_Y) \simeq D_S(Rg_*\mathscr{O}_Y(-\Delta_Y))[-n]$. The desired result now follows from the fact that D_s gives an equivalence $D_c^b(S)^{op} \simeq D_c^b(S)$.

$\square$

5.1. **Graph constructions for pairs.** Recall now that by hypothesis X is smooth and Δ_X is a reduced, effective snc divisor. Similarly for (Y, Δ_Y).

Thriftiness ensures a correspondence between the strata of Δ_X and the strata of $snc(\Delta_S)$:

Lemma 5.2. *Let $f : X \to S$ be a thrifty resolution of $(S, \Delta_S$. Then the assignment $W \mapsto f_*^{-1}(W)$ yields a one-to-one correspondence between the strata of $snc(\Delta_S)$ and the strata of Δ_X.*

Proof. Suppose $W \subset snc(\Delta_S)$ is a stratum of the snc locus - in other words, W is a component of an intersection $\bigcap_j D_{S,i_j}$ of a subset of the $D_{S,i}$ and the generic point of W lies in the snc locus. Then by hypothesis f is an isomorphism over the generic point of W, and so the strict transform

$$Z := f_*^{-1}(W) \subset \bigcap_j f_*^{-1} D_{S,i_j} = \bigcap_j D_{X,i_j}$$

(where $D_{X,i} = f_*^{-1}(D_{S,i})$) is a stratum of Δ_X. Furthermore $f|_Z$ is birational, with image the closure of W in S, and from this we see that

$$W \mapsto f_*^{-1}(W) \text{ gives an } injection$$

$$\{\text{strata of } snc(\Delta_S)\} \to \{\text{strata of } \Delta_X\}$$

Evidently its image is precisely the strata $Z \subset \Delta_X$ such that

$$f(Z) \cap snc(\Delta_S) \neq \emptyset$$

Alternatively, $Z \not\subset f^{-1}(nonsnc(\Delta_S))$ where $nonsnc(\Delta_S) := \Delta_S \setminus snc(\Delta_S)$. The claim is that these are actually all the strata of Δ_X.

Now observe that since X is smooth and $\Delta_X = f_*^{-1}(\Delta_S)$ is an snc divisor, f *cannot* be an isomorphism over $nonsnc(\Delta_S)$, and by hypothesis f is an isomorphism at the generic point of every stratum $Z \subset \Delta_X$. So, there can be no stratum Z of Δ_X contained in $f^{-1}(nonsnc(\Delta_S))$. $\square$

Now let $\varphi := g^{-1} \circ f : X \dashrightarrow Y$ be the induced birational equivalence between X and Y. From the lemma we obtain

Corollary 5.3. φ *is defined at the generic point of every stratum* $Z \subset \Delta_X$, *and the assignment* $Z \mapsto \varphi_*(Z)$ *gives a one-to-one correspondence between the strata of* Δ_X *and* Δ_Y. *In fact the restriction* $f|_{\Delta_X} : \Delta_X \dashrightarrow \Delta_Y$ *is a birational equivalence.*

Let

$$Z := \bar{\Gamma}_\varphi \subset X \times_S Y \text{ be the closure of the graph of } \varphi.$$

We have a commutative diagram

(11)
$$
\begin{array}{ccc}
Z & \xrightarrow{q} & Y \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{f} &
\end{array}
$$

where p and q are proper birational maps. In general, Z will be singular. If $U_X \subset X$ is the maximal domain of φ and $U_Y \subset Y$ is the maximal domain of φ^{-1}, then we have the following basic description of the exceptional loci of p and q.

Lemma 5.4. $p(\text{Ex}(p)) = X \setminus U_X$ *and* $q(\text{Ex}(q)) = Y \setminus U_Y$. *Also, for any exceptional fiber* $p^{-1}(x)$ *over a point* $x \in X \setminus U_X$, $q|_{p^{-1}(x)} : p^{-1}(x) \to Y$ *must be non-constant, and* $q(p^{-1}(x)) \subset \text{Ex}(g)$.

Proof omitted for now. We obtain a couple helpful corollaries:

Lemma 5.5. *Let $Z \subset \Delta_X$ be a stratum. Then $Z \cap U_X \neq \emptyset$. Similarly for Y.*

Proof. Indeed, we have seen that φ is defined at the generic point of Z.

$\square$

Lemma 5.6. *Let $p^{-1}(x) \subset Z$ be an exceptional fiber over a point $x \in p(\mathrm{Ex}(p)) \subset X$. Then $q(p^{-1}(x))$ contains no stratum of Δ_Y.*

Proof. This is because $q(p^{-1}(x)) \subset \mathrm{Ex}(g)$, and as g is thrifty $\mathrm{Ex}(g)$ contains no stratum of Δ_Y.

$\square$

Take the first components $D_{X,1}$, $D_{Y,1}$ of Δ_X, Δ_Y respectively. Note that $(D_{X,1}, (\Delta_X - D_{X,1})|_{D_{X,1}})$ and $(D_{Y,1}, (\Delta_Y - D_{Y,1})|_{D_{Y,1}})$ are snc pairs.

Question 1. Does it make sense to say the resolutions $D_{X,1} \to D_{S,1}$ and $D_{Y,1} \to D_{S,1}$ are thrifty, say with respect to the divisor $(\Delta_S - D_{S,1})|_{D_{S,1}}$? The only cause for concern here is that $D_{S,1}$ need not be normal. In fact, this seems to be one reason to relax the conditions on S, since the hope is to simultaneously induct on dimension and the number of components of the divisors Δ.

By restricting φ we obtain a birational equivalence $D_{X,1} \dashrightarrow D_{Y,1}$. The closure of its graph $W := \Gamma_{\varphi|_{D_{X,1}}} \subset D_{X,1} \times_{D_{S,1}} D_{Y,1}$ coincides with $D_{Z,1} := p_*^{-1}(D_{X,1}) = q_*^{-1}(D_{Y,1})$ (they are both closed subvarieties of $D_{X,1} \times_S D_{Y,1}$ and they are equal over the dense open U_X (or U_Y). However to apply Chatzistamatiou and Rülling's work on correspondences, we will be more interested in the cycle $Z \cap (D_{X,1} \times D_{Y,1}) \subset D_{X,1} \times D_{Y,1}$.

Question 2. What is the exact relationship between W and $Z \cap (D_{X,1} \times D_{Y,1})$?

One containment $W \subset Z \cap (D_{X,1} \times D_{Y,1})$ is clear - I am not sure when the two are equal.

5.2. **Gysin sequences, correspondences and dimension reduction.** Import all notation from the previous section.

Consider the exact sequence of sheaves

$$0 \to \Omega_X^p(\log(\Delta_X - D_{X,1})) \to \Omega_X^p(\log \Delta_X) \xrightarrow{Res_1} \Omega_{D_{X,1}}^{p-1}(\log(\Delta_X - D_{X,1})|_{D_{X,1}}) \to 0$$

and apply Rf_*. The result is an exact triangle

$$Rf_*\Omega_X^p(\log(\Delta_X - D_{X,1})) \to Rf_*\Omega_X^p(\log \Delta_X) \xrightarrow{Res_1} Rf_*\Omega_{D_{X,1}}^{p-1}(\log(\Delta_X - D_{X,1})|_{D_{X,1}}) \xrightarrow{[+1]} \cdots$$

Similarly, we obtain an exact triangle

$$Rg_*\Omega_Y^p(\log(\Delta_Y - D_{Y,1})) \to Rg_*\Omega_Y^p(\log \Delta_Y) \xrightarrow{Res_1} Rg_*\Omega_{D_{Y,1}}^{p-1}(\log(\Delta_Y - D_{Y,1})|_{D_{Y,1}}) \xrightarrow{[+1]} \cdots$$

Observation 1. Suppose that we have morphisms $Rf_*\Omega_X^p(\log(\Delta_X - D_{X,1})) \xrightarrow{\rho_Z} Rg_*\Omega_Y^p(\log(\Delta_Y - D_{Y,1}))$ and $Rf_*\Omega_{D_{X,1}}^{p-1}(\log(\Delta_X - D_{X,1})|_{D_{X,1}}) \xrightarrow{\rho_W} Rg_*\Omega_{D_{Y,1}}^{p-1}(\log(\Delta_Y - D_{Y,1})|_{D_{Y,1}})$ fitting into

a commutative diagram

(12)
$$
\begin{array}{ccc}
Rf_*\Omega_{D_{X,1}}^{p-1}(\log(\Delta_X - D_{X,1})|_{D_{X,1}}) & \xrightarrow{[+1]} & Rf_*\Omega_X^p(\log(\Delta_X - D_{X,1})) \\
\rho_W \downarrow & & \rho_Z \downarrow \\
Rg_*\Omega_{D_{Y,1}}^{p-1}(\log(\Delta_Y - D_{Y,1})|_{D_{Y,1}}) & \xrightarrow{[+1]} & Rg_*\Omega_Y^p(\log(\Delta_Y - D_{Y,1}))
\end{array}
$$

Then there will be a unique morphism (in the derived category)

$$
\tilde{rho}_Z : \Omega_X^p(\log \Delta_X) \to Rf_*\Omega_X^p(\log \Delta_X)
$$

filling in a morphism of triangles
(13)
$$
\begin{array}{ccccccc}
Rf_*\Omega_X^p(\log(\Delta_X - D_{X,1})) & \longrightarrow & Rf_*\Omega_X^p(\log \Delta_X) & \xrightarrow{\mathrm{Res}_1} & Rf_*\Omega_{D_{X,1}}^{p-1}(\log(\Delta_X - D_{X,1})|_{D_{X,1}}) & \xrightarrow{[+1]} & \cdots \\
\rho_Z \downarrow & & \tilde{\rho}_Z \downarrow & & \rho_W \downarrow & & \\
Rg_*\Omega_Y^p(\log(\Delta_Y - D_{Y,1})) & \longrightarrow & Rg_*\Omega_Y^p(\log \Delta_Y) & \xrightarrow{\mathrm{Res}_1} & Rg_*\Omega_{D_{Y,1}}^{p-1}(\log(\Delta_Y - D_{Y,1})|_{D_{Y,1}}) & \xrightarrow{[+1]} & \cdots
\end{array}
$$

Let's be optimistic and see what this would give us: since $\dim D_{X,1} = \dim X - 1$ and $\Delta_X - D_{X,1}$ has one fewer component than Δ_X, we may assume by inductive hypothesis that ρ_Z and ρ_W are quasi-isomorphisms for $p = 0$ and $p = n = \dim X$ - the results of section 5 serve as a base case (where $\Delta_X = D_{X,1}$, i.e. Δ_X is a smooth divisor (and similarly on Y)).

Remark 17. Okay, they don't really serve as a base case since they only provide isomorphisms on the higher direct images (the cohomology sheaves of these complexes). One would have to show that those isomorphisms on higher direct images are induced by an underlying morphism of complexes (which would automatically be a quasi-isomorphism).

The case $p = 0$ is not helpful (since $\Omega_X^0(\log \Delta_X) = \mathscr{O}_X$ (similarly on Y) and the terms on the right are both 0) but taking $p = n$ we have a morphism of triangles

(14)
$$
\begin{array}{ccccccc}
Rf_*\omega_X(\Delta_X - D_{X,1}) & \longrightarrow & Rf_*\omega_X(\Delta_X) & \xrightarrow{\mathrm{Res}_1} & Rf_*\omega_{D_{X,1}}((\Delta_X - D_{X,1})|_{D_{X,1}}) & \xrightarrow{[+1]} & \cdots \\
\rho_{Z,\simeq} \downarrow & & \tilde{\rho}_Z \downarrow & & \rho_W \simeq \downarrow & & \\
Rg_*\omega_Y(\Delta_Y - D_{Y,1}) & \longrightarrow & Rg_*\omega_Y(\Delta_Y) & \xrightarrow{\mathrm{Res}_1} & Rg_*\omega_{D_{Y,1}}((\Delta_Y - D_{Y,1})|_{D_{Y,1}}) & \xrightarrow{[+1]} & \cdots
\end{array}
$$

By the derived version of the 5-lemma we could conclude that

$$
\tilde{\rho}_Z : Rf_*\omega_X(\Delta_X) \xrightarrow{\simeq} Rg_*\omega_Y(\Delta_Y)
$$

and as discussed above, the Grothendieck dual of this isomorphism would give an isomorphism $Rf_*\mathscr{O}_X(-\Delta_X) \simeq Rg_*\mathscr{O}_Y(-\Delta_Y)$.

Here is some circumstantial evidence that one can find compatible morphisms ρ_Z, ρ_W as above:

Lemma 5.7. *Let F be a weak cohomology theory with supports, and let $(X, \Phi), (Y, \Psi)$ be varieties with families of supports. Let $\gamma \in F(X \times Y, P(\Phi, \Psi))$ be an F-correspondence. Let $A \subset X$, $B \subset Y$ be subvarieties and let $\Phi|_A, \Psi|_B$ be the restricted families of supports. Then $P(\Phi|_A, \Psi|_B) = P(\Phi, \Psi)|_{A \times B}$ so that if $j_A : A \to X$, $j_B : B \to Y$ are the inclusions, the inclusion $j := j_A \times j_B : A \times B \to X \times Y$ is a morphism in both V_* and V^*, and for any $\alpha \in F(A, \Phi_A)$,*

$$j_{B*}(\pi_{B*}(\pi_A^* \alpha \smile j^* \gamma)) = \pi_{Y*}(\pi_X^* j_{A*} \alpha \smile \gamma)$$

or more concisely/suggestively,

$$"j_{B*}(j^* \gamma(\alpha)) = \gamma(j_{A*}\alpha)"$$

Proof. Recall that by definition

$$P(\Phi, \Psi) := \{Z \subset X \times Y \text{ closed } \mid Z \cap \pi_X^{-1}(W) \subset \pi_Y^{-1}(\Psi) \text{ for } W \in \Phi \text{ and } \pi_Y|_{Z \cap \pi_X^{-1}(W)} \text{ is proper}\}$$

Similarly for $P(\Phi|_A, \Psi|_B)$. On the other hand

$$P(\Phi, \Psi)|_{A \times B} = \{Z \subset A \times B \text{ closed } \mid Z \cap \pi_X^{-1}(W) \subset \pi_Y^{-1}(\Psi) \text{ for } W \in \Phi \text{ and } \pi_Y|_{Z \cap \pi_X^{-1}(W)} \text{ is proper}\}$$

So the only things to check are: for $Z \subset A \times B$ closed,

- $Z \cap \pi_X^{-1}(W) \subset \pi_Y^{-1}(\Psi)$ for $W \in \Phi$ if and only if $Z \cap \pi_X^{-1}(W) \subset \pi_B^{-1}(\Psi|_B)$ for all $W \in \Phi|_A$. This is because when $Z \subset A \times B$, $Z \cap \pi_X^{-1}(W) = Z \cap \pi_A^{-1}(W \cap A)$.
- $\pi_Y|_{Z \cap \pi_X^{-1}(W)}$ is proper for all $W \in \Phi$ if and only if $\pi_B|_{Z \cap \pi_A^{-1}(W)}$ for all $W \in \Phi|_A$. This is because when $Z \subset A \times B$, $Z \cap \pi_X^{-1}(W) = Z \cap \pi_A^{-1}(W \cap A)$ as mentioned above and $\pi_Y|_{Z \cap \pi_A^{-1}(W \cap A)} = j_B(\pi_B|_{Z \cap \pi_A^{-1}(W \cap A)}$ (and of course j_B is proper).

Now for the more exciting part: by functoriality, $j_{B*} \circ \pi_{B*} = \pi_{Y*} \circ j_*$, and so

$$j_{B*}\pi_{B*}(\pi_A^* \alpha \smile j^* \gamma)) = \pi_{Y*}j_*(\pi_A^* \alpha \smile j^* \gamma)$$

By the projection formula from 7 $j_*(\pi_A^* \alpha \smile j^* \gamma) = (j_* \pi_A^* \alpha) \smile \gamma$, and so

$$\pi_{Y*}j_*(\pi_A^* \alpha \smile j^* \gamma) = \pi_{Y*}((j_* \pi_A^* \alpha) \smile \gamma)$$

Now by the "base change" (or "Riemann-Roch") formula, $j_* \pi_A^* = \pi_X^* j_{A*}$, and this shows that

$$\pi_{Y*}((j_* \pi_A^* \alpha) \smile \gamma) = \pi_{Y*}(\pi_X^*(j_{A*}\alpha) \smile \gamma)$$

as desired.

$\square$

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