COHOMOLOGY OF STRUCTURE SHEAVES OF GOBALLY F-FULL VARIETIES IN EQUAL CHARACTERISTIC $p > 0$

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1. Introduction

We begin by considering a theorem of Du Bois-Jarraud.

Theorem 1.1 ([Du 81, Thm. 4.6], see also [DJ74])**.** *If $f : X \to B$ is a flat proper morphism of schemes of finite type over $\mathbb{C}$, and if the geometric fibers of f are reduced with at worst Du Bois singularties, then the higher direct images of the structure sheaf $R^i f_* \mathscr{O}_X$ are locally free and compatible with arbitrary base change.*

All known characterizations of Du Bois singularities are somewhat technical, so rather than giving definitions we summarize a few of their properties. For *applications* of Theorem 1.1, the important facts are that both normal crossing and semi-log canonical schemes of finite type over $\mathbb{C}$ have Du Bois singularities (for proofs as well as the necessary definitions see [Kol13, §6] – the lc case is [KK10, Thm. 1.4]). On the other hand, the *proof* hinges on the fact that if X is a proper $\mathbb{C}$-scheme with Du Bois singularities then the natural maps

$$H^i(X, \mathbb{C}) \twoheadrightarrow H^i(X, \mathscr{O}_X)$$

are surjective for all i (when X is smooth this is an immediate consequence of degeneration of the Hodge-to-de-Rham spectral sequence). Theorem 1.1 has found various striking applications: for example, in [KK10, Thm. 1.8] it is used to show that for a family as above, the cohomology sheaves $h^i(\omega_f^\bullet)$ (including the relative dualizing sheaf $\omega_{X/B}$) are flat over B and compatible with base change. In a different direction, it was noticed by Kollár that Theorem 1.1 combined with a hypothetical strong form of semi-stable reduction would recover one of his theorems on higher direct images of dualizing sheaves [Kol86, Thm. 2.6 Rmk. 2.7].

As mentioned above, the proof of Theorem 1.1 makes essential use of Hodge theory, and moreover it is currently unknown how to even define Du Bois singularities away from characteristic 0. Below we present a variant of the theorem which differs in at least 2 aspects: first, it applies exclusively to flat proper families in characteristic $p > 0$, and second it replaces local singularity conditions on the closed fibers with global arithmetic restrictions.

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Definition 1.2. Let k be a field of characteristic $p > 0$. A proper k-scheme X is *globally F-full* if and only if the natural morphisms induced by Frobenius

$$H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_* k(b) \twoheadrightarrow F^e_* H^i(X_b, \mathcal{O}_{X_b}) \text{ is surjective for all } e, i \in \mathbb{N}. \tag{1.3}$$

Proposition 1.4. *Let B be a locally noetherian scheme of characteristic $p > 0$ and let $f : X \to B$ be a flat proper morphism. Assume that for every closed point $b \in B$, the fiber X_b is globally F-full over $k(b)$. Then $R^i f_* \mathcal{O}_X$ is locally free and compatible with arbitrary base change for all $i \in \mathbb{N}$.*

Remark 1.5. In general, (1.3) is a map of $F^e_* k(b)$-vector spaces of the same finite dimension, so it is surjective if and only if it is an isomorphism. In the case $k(b)$ is *perfect*, (1.3) is equivalent to the condition that the adjoint morphisms

$$H^i(X_b, \mathcal{O}_{X_b}) \to F^e_* H^i(X_b, \mathcal{O}_{X_b})$$

are isomorphisms (or equivalently injective) for all $e, i \in \mathbb{N}$. This is a strengthening of the weak ordinarity condition of [MS11], which would only require an injection $H^{\dim X_b}(X_b, \mathcal{O}_{X_b}) \hookrightarrow F^e_* H^{\dim X_b}(X_b, \mathcal{O}_{X_b})$.[1]

Moreover, (1.3) can be checked on perfect (or geometric) fibers.

Remark 1.6. The terminology "globally F-full" is chosen to mirror the notion of *F-full* defined in [MQ18, Def. 2.3], which requires a surjectivity similar to the one appearing in (1.3) but for local cohomology modules.

2. Cohomology of the structure sheaf

2.1. **Restriction maps from thickened fibers.** Following the approach in [DJ74], we immediately apply [EGA$_2$, Prop. 7.7.10] which shows:

Proposition 2.1. *The sheaves $R^i f_* \mathcal{O}_X$ are locally free and compatible with arbitrary base change for all $i \in \mathbb{N}$ if and only if for every closed point $b \in B$ with associated maximal ideal $\mathfrak{m}_b \subseteq \mathcal{O}_X$, denoting $X_{b,n} := f^{-1}(V(\mathfrak{m}_b^{n+1})) \subseteq X$ the restriction morphisms*

$$H^i(X_{b,n}, \mathcal{O}_{X_{b,n}}) \twoheadrightarrow H^i(X_b, \mathcal{O}_{X_b}) \text{ are surjective for all } n, i \in \mathbb{N}. \tag{2.2}$$

It will be useful to consider not only the inclusion of a fiber X_b into its n-th thickening $X_{b,n}$, but the entire sequence of inclusions $X_{b,n-1} \subseteq X_{b,n}$. This not only decomposes the maps (2.2) but also yields useful long exact sequences.

Lemma 2.3. *Let B be a locally noetherian scheme, let $f : X \to B$ be a proper morphism and let $\mathcal{F}$ be a coherent sheaf on X flat over B. For any closed point $b \in B$ and any $n \in \mathbb{N}$, let $\mathcal{F}_{b,n} := \mathcal{F}|_{X_{b,n}}$ with the exception that we write $\mathcal{F}_b := \mathcal{F}|_{X_b}$. Then, there are long exact sequences*

$$\cdots \longrightarrow H^i(X_b, \mathcal{F}_b) \otimes_{k(b)} (\mathfrak{m}_b^n / \mathfrak{m}_b^{n+1}) \longrightarrow H^i(X_{b,n}, \mathcal{F}_{b,n}) \longrightarrow H^i(X_{b,n-1}, \mathcal{F}_{b,n-1}) \longrightarrow \cdots \tag{2.4}$$

which are natural in the sense that if $g : Y \to B$ is another proper morphism and $\mathcal{G}$ is a coherent sheaf on Y flat over B, and if we are given a B-morphism $h : X \to Y$ together with a map of sheaves $\varphi : \mathcal{G} \to h_* \mathcal{F}$, there is a functorial morphism of long exact sequences (of modules over the local ring $\mathcal{O}_{B,b}$)

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(Y_b, \mathcal{G}_b) \otimes_{k(b)} (\mathfrak{m}_b^n / \mathfrak{m}_b^{n+1}) \arrow[r] \arrow[d] & H^i(Y_{b,n}, \mathcal{G}_{b,n}) \arrow[r] \arrow[d] & H^i(Y_{b,n-1}, \mathcal{G}_{b,n-1}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^i(X_b, \mathcal{F}_b) \otimes_{k(b)} (\mathfrak{m}_b^n / \mathfrak{m}_b^{n+1}) \arrow[r] & H^i(X_{b,n}, \mathcal{F}_{b,n}) \arrow[r] & H^i(X_{b,n-1}, \mathcal{F}_{b,n-1}) \arrow[r] & \cdots
\end{tikzcd} \tag{2.5}$$

[1] Hence globally F-full could have been called strongly weakly ordinary.

Proof. We derive (2.5) as it includes (2.4) as a special case (e.g. with $\varphi = \mathrm{id}$). By functoriality of derived pushforwards, we have a morphism $Rg_*\mathscr{G} \to Rf_*\mathscr{F}$ in $D^b_{\mathrm{coh}}(B)$. Taking the derived tensor product of this with the distinguished triangle $\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1} \to \mathscr{O}_B/\mathfrak{m}_b^{n+1} \to \mathscr{O}_B/\mathfrak{m}_b^{n+1}$ and applying the derived projection formula [Stacks, Tag 08ET] yields a morphism of distinguished triangles

$$\begin{tikzcd}
Rg_*(\mathscr{G} \otimes^L Lg^*(\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1})) \arrow[r] \arrow[d] & Rg_*(\mathscr{G} \otimes^L Lg^*(\mathscr{O}_B/\mathfrak{m}_b^{n+1})) \arrow[r] \arrow[d] \arrow[dr, phantom, "*"] & Rg_*(\mathscr{G} \otimes^L Lg^*(\mathscr{O}_B/\mathfrak{m}_b^{n})) \arrow[r] \arrow[d] & \cdots \\
Rf_*(\mathscr{F} \otimes^L Lf^*(\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1})) \arrow[r] & Rf_*(\mathscr{F} \otimes^L Lf^*(\mathscr{O}_B/\mathfrak{m}_b^{n+1})) \arrow[r] & Rf_*(\mathscr{F} \otimes^L Lf^*(\mathscr{O}_B/\mathfrak{m}_b^{n})) \arrow[r] & \cdots
\end{tikzcd} \tag{2.6}$$

Since $\mathscr{F}, \mathscr{G}$ are flat over B the derived pullbacks/tensor products simplify; we have

$$\mathscr{F} \otimes^L Lf^*(\mathscr{O}_B/\mathfrak{m}_b^{n+1}) \simeq \mathscr{F} \otimes_{\mathscr{O}_X} f^*(\mathscr{O}_B/\mathfrak{m}_b^{n+1}) \simeq \mathscr{F} \otimes_{f^{-1}\mathscr{O}_B} f^{-1}(\mathscr{O}_B/\mathfrak{m}_b^{n+1}) = \mathscr{F}_{b,n}$$

and similarly for the other terms on the corners of $(*)$ in (2.6). Moreover since $\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}$ is a $k(b)$-vector space a similar tensor product manipulation gives

$$\mathscr{F} \otimes^L Lf^*(\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) \simeq \mathscr{F} \otimes_{f^{-1}\mathscr{O}_B} f^{-1}(\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) \simeq \mathscr{F} \otimes_{f^{-1}\mathscr{O}_B} f^{-1}k(b) \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) = \mathscr{F}_b \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1})$$

Applying Künneth gives a natural isomorphism $Rf_*(\mathscr{F}_b \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1})) \simeq Rf_*\mathscr{F}_b \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1})$. Similarly for the top right corner of (2.6).

Hence the map of distinguished triangles (2.6) is isomorphic to

$$\begin{tikzcd}
Rg_*\mathscr{G}_b \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) \arrow[r] \arrow[d] & Rg_*(\mathscr{G}_{b,n}) \arrow[r] \arrow[d] & Rg_*(\mathscr{G}_{b,n-1}) \arrow[r] \arrow[d] & \cdots \\
Rf_*\mathscr{F}_b \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) \arrow[r] & Rf_*(\mathscr{F}_{b,n}) \arrow[r] & Rf_*(\mathscr{F}_{b,n-1}) \arrow[r] & \cdots
\end{tikzcd} \tag{2.7}$$

and taking cohomology yields (2.5). □

2.2. **Thickened fibers of Frobenius twists.** Let F_B^e be the e-th iterate of the absolute Frobenius of B (similarly for X) and form the diagram defining the e-th *relative* Frobenius of f (sometimes called the B-linear Frobenius of f), here denoted F_f^e [Stacks, Tag 0CC6].

$$\begin{tikzcd}
X \arrow[r, "F_f^e"] \arrow[dr, "f"'] & X^{(e)} \arrow[r] \arrow[d, "f^{(e)}"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
& B \arrow[r, "F_B^e"'] & B
\end{tikzcd} \tag{2.8}$$

Applying Lemma 2.3 to F_f^e (which automatically comes with a map of sheaves $\mathscr{O}_{X^{(e)}} \to F^e_{f*}\mathscr{O}_X$) gives us a map of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(X_b^{(e)}, \mathscr{O}_{X_b}) \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) \arrow[r] \arrow[d] & H^i(X_{b,n}^{(e)}, \mathscr{O}_{X_b,n}) \arrow[r] \arrow[d] & H^i(X_{b,n-1}^{(e)}, \mathscr{O}_{X_b,n-1}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^i(X_b, \mathscr{O}_{X_b}) \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) \arrow[r] & H^i(X_{b,n}, \mathscr{O}_{X_b,n}) \arrow[r] & H^i(X_{b,n-1}, \mathscr{O}_{X_b,n-1}) \arrow[r] & \cdots
\end{tikzcd} \tag{2.9}$$

For large e, the top row simplifies considerably.

Lemma 2.10. *For fixed n and $e \gg 0$, the composite $V(\mathfrak{m}_b^n) \hookrightarrow B \xrightarrow{F_B^e} B$ factors through* $\operatorname{Spec} k(b)$. *Equivalently, for e in this range $F_*^e\mathscr{O}_{B,b}/\mathfrak{m}_b^n$ is a $k(b)$-algebra.*

Proof. We must show that the kernel I of $\mathcal{O}_{B,b} \xrightarrow{F^e} \mathcal{O}_{B,b} \to \mathcal{O}_{B,b}/\mathfrak{m}_b^n$ is $\mathfrak{m}_b$. Explicitly this kernel is

$$I = \{x \in \mathcal{O}_{B,b} \mid x^{p^e} \in \mathfrak{m}_b^n\}$$

from which we see $I = \mathfrak{m}_b$ for $p^e \geq n$. □

Remark 2.11. Lemma 2.10 is equivalent to the trivial inclusion $\mathfrak{m}_b^{[p^e]} \subseteq \mathfrak{m}_b^n$ for $p^e \geq n$.

Corollary 2.12. *For fixed n and $e \gg 0$, there is a natural isomorphism of finite-type $k(b)$-schemes $F^e_* X^{(e)}_{b,n-1} \simeq X_b \otimes_{k(b)} F^e_*(\mathcal{O}_{B,b}/\mathfrak{m}_b^n)$. Here $F^e_* X^{(e)}_{b,n-1}$ denotes the scheme $X^{(e)}_{b,n-1}$ together with the structure morphism $X^{(e)}_{b,n-1} \to V(\mathfrak{m}_b^n) \xrightarrow{F^e_B} \operatorname{Spec} k(b)$.*

We now apply Corollary 2.12 to rewrite the top row of (2.9). In order to keep track of all the Frobenii, we actually apply F^e_* to push forward (2.9), which is a diagram of modules over the local ring $\mathcal{O}_{B,b}$ in the *bottom left corner of* (2.8), to get a diagram over $\mathcal{O}_{B,b}$ in the *bottom right corner* of the form

$$\begin{tikzcd}
\cdots \arrow[r] & F^e_* H^i(X^{(e)}_b, \mathcal{O}_{X_b}) \otimes_{F^e_* k(b)} F^e_*(\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) \arrow[r] \arrow[d] & F^e_* H^i(X^{(e)}_{b,n}, \mathcal{O}_{X_b,n}) \arrow[r] \arrow[d] & F^e_* H^i(X^{(e)}_{b,n-1}, \mathcal{O}_{X_b,n-1}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & F^e_* H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) \arrow[r] & F^e_* H^i(X_{b,n}, \mathcal{O}_{X_b,n}) \arrow[r] & F^e_* H^i(X_{b,n-1}, \mathcal{O}_{X_b,n-1}) \arrow[r] & \cdots
\end{tikzcd} \tag{2.13}$$

Note that since Frobenius is affine, F^e_* is equivalent to a restriction of scalars and so this has no effect on the underlying abelian groups; in particular the homomorphisms $F^e_* H^i(X_{b,n}, \mathcal{O}_{X_b,n}) \to F^e_* H^i(X_{b,n-1}, \mathcal{O}_{X_b,n-1})$ are surjective if and only if the $H^i(X_{b,n}, \mathcal{O}_{X_b,n}) \to H^i(X_{b,n-1}, \mathcal{O}_{X_b,n-1})$ are surjective. By Corollary 2.12, for $e \geq \log_p(n+1)$ there are isomorphisms

$$F^e_* H^i(X^{(e)}_{b,n-1}, \mathcal{O}_{X_b,n-1}) \simeq H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathcal{O}_{B,b}/\mathfrak{m}_b^n)$$

and similarly $F^e_* H^i(X^{(e)}_{b,n}, \mathcal{O}_{X_b,n}) \simeq H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathcal{O}_{B,b}/\mathfrak{m}_b^{n+1})$. In particular for $n = 0$ we have $F^e_* H^i(X^{(e)}_b, \mathcal{O}_{X_b}) \simeq H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_* k(b)$.[2]. Using these identifications, (2.13) becomes

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) \arrow[r] \arrow[d, "\psi_n^{(e),i}"] & H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathcal{O}_{B,b}/\mathfrak{m}_b^{n+1}) \arrow[r, "\rho_n^{(e),i}"] \arrow[d, "\varphi_n^{(e),i}"] & H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathcal{O}_{B,b}/\mathfrak{m}_b^n) \arrow[r] \arrow[d, "\varphi_{n-1}^{(e),i}"] & \cdots \\
\cdots \arrow[r] & F^e_* H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} (\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}) \arrow[r] & F^e_* H^i(X_{b,n}, \mathcal{O}_{X_b,n}) \arrow[r, "\rho_n^i"] & F^e_* H^i(X_{b,n-1}, \mathcal{O}_{X_b,n-1}) \arrow[r] & \cdots
\end{tikzcd} \tag{2.14}$$

2.3. **Surjectivity of relative Frobenii.**

Proposition 2.15. *If X_b is globally F-full then for fixed n and $e \gg 0$, the homomorphisms $\rho_n^{(e),i}$ and $\varphi_{n-1}^{(e),i}$ (and hence also ρ_n^i) are surjective for all $i \in \mathbb{N}$.*

Proof. Fixing n, choose $e \geq \log_p(n+1)$ (so $p^e \geq n+1$). Then the homomorphisms $\rho_n^{(e),i}$ are all surjective, since the reductions $\mathcal{O}_{B,b}/\mathfrak{m}_b^{n+1} \twoheadrightarrow \mathcal{O}_{B,b}/\mathfrak{m}_b^n$ are surjective, and because F^e_* and tensoring over $k(b)$ are both exact. Moreover global F-fullness of X_b guarantees the vertical maps $\psi_n^{(e),i}$ are all surjective (after choosing a basis for $\mathfrak{m}_b^n/\mathfrak{m}_b^{n+1}$, the map $\psi_n^{(e),i}$ can be written as a direct sum of maps of the type appearing in (1.3)).

[2]this last isomorphism of course doesn't need restrictions on e.

We now show by induction on $m \leq n$ (with a subsidiary induction on i) that the $\varphi_m^{(e),i}$ and ρ_m^i are all surjective — the base case $m = 0$ is exactly global F-fullness of X_b. Now suppose $0 < m \leq n$ and consider

$$\begin{tikzcd}
0 \arrow[r] & H^0(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathfrak{m}_b^m/\mathfrak{m}_b^{m+1}) \arrow[r] \arrow[d, two heads, "\psi_m^{(e),0}"] & H^0(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathcal{O}_{B,b}/\mathfrak{m}_b^{m+1}) \arrow[r, two heads, "\rho_m^{(e),0}"] \arrow[d, "\varphi_m^{(e),0}"] & H^0(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathcal{O}_{B,b}/\mathfrak{m}_b^m) \arrow[r] \arrow[d, two heads, "\varphi_{m-1}^{(e),0}"] & 0 \\
0 \arrow[r] & F^e_* H^0(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} (\mathfrak{m}_b^m/\mathfrak{m}_b^{m+1}) \arrow[r] & F^e_* H^0(X_{b,m}, \mathcal{O}_{X_b,m}) \arrow[r, "\rho_m^0"] & F^e_* H^0(X_{b,m-1}, \mathcal{O}_{X_b,m-1}) \arrow[r, "\delta_m^1"] & \cdots
\end{tikzcd} \tag{2.16}$$

where in the top row we have applied the surjectivity of $\rho_m^{(e),0}$ mentioned above to obtain a short exact sequence, and in the left vertical map we have applied the surjectivity of $\psi_n^{(e),0}$. By inductive hypothesis we may assume the right vertical arrow $\varphi_{m-1}^{(e),0}$ is surjective. Now the snake lemma [Stacks, Tag 07JV] gives us an exact sequence

$$0 = \operatorname{coker} \psi_n^{(e),0} \to \operatorname{coker} \varphi_m^{(e),0} \to \varphi_{m-1}^{(e),0} = 0$$

and hence $\operatorname{coker} \varphi_m^{(e),0} = 0$.

We also conclude from surjectivity of $\rho_m^{(e),0}$ and $\varphi_{m-1}^{(e),0}$ that ρ_n^0 is surjective, and so the connecting map $\delta_m^1 = 0$. This means that for $i > 0$, we obtain a diagram

$$\begin{tikzcd}
0 \arrow[r] & H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathfrak{m}_b^m/\mathfrak{m}_b^{m+1}) \arrow[r] \arrow[d, two heads, "\psi_m^{(e),i}"] & H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathcal{O}_{B,b}/\mathfrak{m}_b^{m+1}) \arrow[r, two heads, "\rho_m^{(e),i}"] \arrow[d, "\varphi_m^{(e),i}"] & H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_*(\mathcal{O}_{B,b}/\mathfrak{m}_b^m) \arrow[r] \arrow[d, two heads, "\varphi_{m-1}^{(e),i}"] & 0 \\
0 \arrow[r] & F^e_* H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} (\mathfrak{m}_b^m/\mathfrak{m}_b^{m+1}) \arrow[r] & F^e_* H^i(X_{b,m}, \mathcal{O}_{X_b,m}) \arrow[r, "\rho_m^i"] & F^e_* H^i(X_{b,m-1}, \mathcal{O}_{X_b,m-1}) \arrow[r, "\delta_m^{i+1}"] & \cdots
\end{tikzcd} \tag{2.17}$$

where now exactness on the left is obtained the inductive hypothesis that $\rho_m^{(e),i-1}$ and ρ_m^{i-1} are surjective. Again we may assume by inductive hypothesis that the vertical map $\varphi_{m-1}^{(e),i}$ on the right is surjective, and then the snake lemma shows $\varphi_m^{(e),i}$ is surjective. Since $\rho_m^{(e),i}$ and $\varphi_{m-1}^{(e),i}$ are both surjective we conclude ρ_m^i is surjective, completing the inductive step. □

Proof of Proposition 1.4. Proposition 2.15 shows that the restriction maps

$$\rho_n^i : H^i(X_{b,n}, \mathcal{O}_{X_b,n}) \to H^i(X_{b,n-1}, \mathcal{O}_{X_b,n-1})$$

are surjective for all $n, i \in \mathbb{N}$, and so the composite

$$H^i(X_{b,n}, \mathcal{O}_{X_b,n}) \xrightarrow{\rho_n^i} H^i(X_{b,n-1}, \mathcal{O}_{X_b,n-1}) \to \cdots \to H^i(X_{b,n-1}, \mathcal{O}_{X_b,1}) \xrightarrow{\rho_1^i} H^i(X_b, \mathcal{O}_{X_b})$$

is surjective. This is precisely the restriction morphism (2.2). □

Corollary 2.18. *The set of points $b \in B$ such that X_b is globally F-full is open.*

Proof. If X_b is globally F-full then by Proposition 1.4 there is a neighborhood $U \subseteq B$ such that the sheaves $R^i f_* \mathcal{O}_X|_U$ are locally free and compatible with base change — replacing B with U we can assume that the $R^i f_* \mathcal{O}_X$ themselves are locally free and compatible with base change.

In particular applying compatibility with base change to (2.8) gives morphisms

$$LF_B^{e*} R f_* \mathcal{O}_X = R f_*^{(e)} \mathcal{O}_{X^{(e)}} \xrightarrow{\varphi^{(e)}} R f_* \mathcal{O}_X \text{ in } D^b_{\mathrm{coh}}(B) \tag{2.19}$$

where the latter map $\varphi^{(e)}$ is induced by F_f^e. We claim $\varphi^{(e)}$ is a quasi-isomorphism on a neighborhood of b: the first equality in (2.19) shows that the sheaves $R^i f_*^{(e)} \mathcal{O}_{X^{(e)}} = F_B^{e*} R^i f_* \mathcal{O}_X$ are locally free. Now for each i the induced morphism

$$R^i f_*^{(e)} \mathcal{O}_{X^{(e)}} \xrightarrow{\varphi^{(e)}} R^i f_* \mathcal{O}_X \tag{2.20}$$

is a map of locally free sheaves whose reduction mod $\mathfrak{m}_b$ is $H^i(X_b, \mathcal{O}_{X_b}) \otimes_{k(b)} F^e_* k(b) \to F^e_* H^i(X_b, \mathcal{O}_{X_b})$, by hypothesis an isomorphism. By Nakayama's lemma (2.20) is an isomorphism on a neighborhood of b. Chooosing $b \in U \subseteq B$ small enough so that (2.20) is an isomorphism for all i, for any $b' \in U$ tensoring with $k(b')$ gives

$$\begin{array}{ccc} R^i f_*^{(e)} \mathcal{O}_{X^{(e)}} \otimes k(b') & \xrightarrow[\simeq]{\varphi^{(e)} \otimes \mathrm{id}} & R^i f_* \mathcal{O}_X \otimes k(b') \\ \big\downarrow \simeq & & \big\downarrow \simeq \\ H^i(X_{b'}, \mathcal{O}_{X_{b'}}) \otimes_{k(b')} F^e_* k(b') & \longrightarrow & F^e_* H^i(X_{b'}, \mathcal{O}_{X_{b'}}) \end{array} \tag{2.21}$$

□

2.4. **Examples.**

Example 2.22 (Suggested by A.J. de Jong; shows (1.3) is sufficient but not necessary). Let k be an algebraically closed field of characteristic $p > 2$[3], let $B = \mathbb{A}^1_\lambda$ and let $X = V(y^2 z - x(x-z)(x-\lambda z)) \subseteq \mathbb{A}^1_\lambda \times \mathbb{P}^2_{xyz}$. Let $f : X \to B$ be the projection.

By [Har77, Cor. 4.22] the locus of closed points $b \in \mathbb{A}^1_\lambda$ where (1.3) holds is the *non-vanishing* $D(h_p)$ of the polynomial

$$h_p(\lambda) = \sum_{i=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{i} \lambda^i$$

so in particular it is a *proper* open subset. However in this case the higher direct images $R^i f_* \mathcal{O}_X$ are still locally free: identifying them with the $k[\lambda]$-modules $H^i(X, \mathcal{O}_X)$ and using the exact sequence

$$\cdots \longrightarrow H^i(\mathbb{A}^1_\lambda \times \mathbb{P}^2_{xyz}, \mathcal{O}(-3)) \longrightarrow H^i(\mathbb{A}^1_\lambda \times \mathbb{P}^2_{xyz}, \mathcal{O}) \longrightarrow H^i(X, \mathcal{O}_X) \longrightarrow \cdots \tag{2.23}$$

induced by the section $y^2 z - x(x-z)(x-\lambda z) \in H^0(\mathbb{A}^1_\lambda \times \mathbb{P}^2_{xyz}, \mathcal{O}(3))$ we get isomorphisms

$$H^0(X, \mathcal{O}_X) \simeq H^0(\mathbb{A}^1_\lambda \times \mathbb{P}^2_{xyz}, \mathcal{O}) \text{ and } H^1(X, \mathcal{O}_X) \simeq H^2(\mathbb{A}^1_\lambda \times \mathbb{P}^2_{xyz}, \mathcal{O}(-3))$$

and the latter 2 modules are free of rank 1 by [Har77, Thm. III.5.1].

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