ALGEBRA PRELIMINARY EXAMINATION

SEPTEMBER, 1989

Three complete and correct answers will guarantee a pass. In general, one whole solution is worth more than several partial solutions.

1. Consider the set $\mathcal{S}$ of $n \times n$ matrices over the complex numbers with minimal polynomial $(t - 1)^2$. Define an equivalence relation $\sim$ on $\mathcal{S}$ by $A \sim B$ if there is an invertible $n \times n$ matrix P such that $A = PBP^{-1}$. Give a formula for the number of equivalence classes. Give a representative of each class.

2. Define an action of S_n, the symmetric group on n letters, on $\mathbb{Q}^n$ by

$$\sigma(x_1, x_2, \ldots, x_n) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}), \text{ for all } \sigma \in S_n.$$

This gives $\mathbb{Q}^n$ the structure of a $\mathbb{Q}S_n$-module. Decompose $\mathbb{Q}^n$ as a direct sum of simple $\mathbb{Q}S_n$-modules.

3. Suppose that K is an extension field of $\mathbb{Q}$ of degree d, assumed to be finite.
   (a) Show that the number of field imbeddings $\sigma : K \to \mathbb{C}$ can be written as $r_1 + 2\, r_2$, where r_1 is the number of distinct field imbeddings of K into $\mathbb{R}$, and r_2 is a positive integer.
   (b) Show that $d = r_1 + 2r_2$.
   (c) Calculate r_1 and r_2 when $K = \mathbb{Q}(\sqrt[3]{2}, \sqrt{2})$.

4. Let K be an algebraically closed field of characteristic zero.
   (a) Show that the ring $K[x,y]/(y - x^2, y^2 - x^3)$ is finite dimensional over K. Calculate its dimension. <u>Hint</u>: View the ring as a quotient of $K[x,y]/(y - x^2)$.
   (b) Find all maximal ideals of $K[x,y]$ that contain the ideal $(y - x^2, y^2 - x^3)$.

5. Let $K = \mathbb{F}_{2^n}$ be the field of 2^n elements, $n \geq 1$. Consider the following rings:

$$R_1 = K[x]/(x^2) \qquad R_2 = \mathbb{Z}/2^{n+1}\mathbb{Z} \qquad R_3 = K[t]/(t^2 - 1)$$

$$R_4 = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in K\right\} \quad R_5 = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in K\right\}$$

$R_6 = KG$, where G is the group of order 2.

(a) Find all of the rings isomorphic to R_6.

(b) Is R_2 isomorphic to any of the other rings?

(c) Which of these rings is semisimple?

Give a reason to support each of your answers.

6. (a) Show that the abelian group $\mathbb{Q}/\mathbb{Z}$ is the direct sum of the groups $\mathbb{Z}[1/p]/\mathbb{Z}$, where p ranges over all prime numbers. Here $\mathbb{Z}[1/p]$ denotes the subgroup of $\mathbb{Q}$ consisting of all rational numbers of the form a/p^n, where $a \in \mathbb{Z}$.

(b) Show that every proper subgroup of $\mathbb{Z}[1/p]/\mathbb{Z}$ is finite, has order divisible by p, and that $\mathbb{Z}[1/p]/\mathbb{Z}$ has a unique subgroup of order p^n, for each positive integer n .

(c) Does every finite abelian group occur as a subgroup of $\mathbb{Q}/\mathbb{Z}$? Give a reason for your answer.

7. The integral group ring $\mathbb{Z}G$ of a group G is the set of all formal $\mathbb{Z}$-linear combinations $\Sigma n_g g$ of elements of G, where only finitely many of the integers n_g are non-zero. This is made into a ring in exactly the same way as is the group algebra KG of G over a field K; thus $\mathbb{Z}G$ is a subring of $\mathbb{Q}G$. Define a function $\phi : \mathbb{Z}G \to \mathbb{Z}$ by $\phi(\Sigma n_g g) = \Sigma n_g$.

(a) Show that ϕ is a ring homomorphism.

(b) Denote the kernel of ϕ by J. Show that J is a free $\mathbb{Z}$-module with basis $\{g - 1 : g \in G, g \neq 1\}$.

(c) Denote the square of the ideal J by J^2. Consider J/J^2 as an abelian group via the addition of $\mathbb{Z}G$. Define a function $\Psi : G \to J/J^2$ by $\Psi(g) = (g - 1) + J^2$. Show that Ψ is a group homomorphism.

(d) Show that Ψ induces an isomorphism $\Psi^{ab} : G/[G,G] \to J/J^2$, where $[G,G]$ denotes the commutator subgroup of G. Hint: Construct a homomorphism from J to $G/[G,G]$ using (b).

8. Show that there exists an irreducible polynomial $f(x)$ in $\mathbb{Q}[x]$ of degree 14 whose splitting field is of degree 14 over $\mathbb{Q}$. Hint: Begin by constructing a Galois extension of $\mathbb{Q}$ with Galois group $\mathbb{Z}/28\mathbb{Z}$.

ALGEBRA PRELIM

SPRING 1989

Directions: Do as many problems as you can. One complete solution is better than two partial solutions.

1. Let G be a group and X a finite set. Assume there is a group action of G on X. Let V be the vector space over $\mathbb{C}$ with X as basis and let each g in G act as a linear operator on V by acting on the basis X in the given way. Define Tr(g), the trace of g, to be the usual trace of g as a linear operator on V. Let $X_g = \{x \in X : g.x = x\}$.

   (i) Show that $\mathrm{Tr}(g) = \#X_g$.

   (ii) Show for g and h in G that $\#X_{gh} = \#X_{hg}$.

   (iii) Show that $\mathrm{Tr}(g^2) \geq \mathrm{Tr}(g)$. Is this true in general for the trace of a linear operator?

2. Let X be the closed interval [0,1] and let $R = \{f:X \to \mathbb{R} : f \text{ is continuous}\}$. Make R into a commutative ring by pointwise addition and multiplication. For a subset Y of X, let $I_Y = \{f \in R : f(y) = 0 \text{ for all } y \in Y.\}$.

   (i) Show that I_Y is a maximal ideal of R if and only if $Y = \{y\}$ for a point y of X.

   (ii) Show that R is not a noetherian ring.

3. Let $F = \mathbb{Q}(i)$ and let f(x) be an irreducible polynomial in F[x] of degree 4. Let K be a splitting field of f(x) over F.

   (i) Give an example of f(x) for which $[K:F] = 4$.

   (ii) Suppose the Galois group of K/F is the alternating group A_4. How many distinct fields L are there with $F \subseteq L \subseteq K$ such that $[L:F] = 3$? $= 4$? How many of these are normal extensions of F?

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4. Let A be the ring of 3x3 matrices with arbitrary entries from $\mathbb{R}$ in the indicated locations and 0 elsewhere:

$$\begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

Let V be the 3-dimensional space of column vectors on which elements of A acts on the left in the usual way. (This makes V into a left A-module.)

(i) Find two linearly independent vectors in V which are eigenvectors for A. (An eigenvector for A is a vector v such that $a.v \in \mathbb{R}.v$ for all $a \in A$.)

(ii) Using these vectors, find A-submodules $V_1 \subseteq V_2$ of V such that V_1, V_2/V_1, and V/V_2 are all simple A-modules. (An A-submodule is a subspace which is A-invariant. A simple module is one whose only submodules are 0 and itself.)

(iii) Show that no two of the three simple modules of (ii) are isomorphic A-modules.

5. Let $<\,,\,>$ be the standard inner product on $\mathbb{R}^n$. Given an nxn matrix M, let ${}^tM$ be its transpose.

(i) Show that $<Mv, w> = <v, {}^tMw>$ for any vectors v and w in $\mathbb{R}^n$.

(ii) Show that $<Mv, w> + <v, Mw> = 0$ for all v and w in $\mathbb{R}^n$ if and only if $M + {}^tM = 0$.

(iii) Suppose that M and N are matrices satisfying the condition of (ii). Must MN satisfy this condition? What about MN - NM?

6. Let G be a finite group of order 108.

(i) Why must G have a subgroup H of order 27?

(ii) Using the existence of H, find a set of four elements on which G acts and conclude that there is a non-trivial homomorphism from G into the group S_4 of permutations of a four element set.

(iii) Show that either H is a normal subgroup of G or G has a normal subgroup of order 9.

END

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ALGEBRA PRELIM

Fall 1988

Directions: Do as many problems as you can. One complete solution is better than two partial solutions.

1. Prove that any group of order 45 is abelian.

2. Let $\mathbb{F}$ and $\mathbb{F}'$ be finite fields. State and prove a necessary and sufficient condition that $\mathbb{F}'$ be a subfield of $\mathbb{F}$.

3. Consider

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

as an element of the space of 2x2 matrices over the field of complex numbers. If n is a positive integer, then compute A^n. Hint: think eigenvalues.

4. Let $\mathbb{F}_p$ be the field of p elements where p is a prime number (i.e. $\mathbb{F}_p = \mathbb{Z}/(p)$.) The *general linear group* $Gl_n(\mathbb{F}_p)$ is the group of all invertible $n \times n$ matrices over $\mathbb{F}_p$.

(a) Show that $Gl_3(\mathbb{F}_p)$ has $(p^3-1)(p^3-p)(p^3-p^2)$ elements.

(b) Show that a group H is a p-Sylow subgroup of $Gl_3(\mathbb{F}_p)$ iff with respect to a suitable basis it is precisely all elements of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$, where a, b and c are elements of $\mathbb{F}_p$.

5. Let L be an extension field of degree n of a field K of characteristic 0. Using Galois theory, find a finite upper bound B(n) for the number of intermediate fields T, $K < T < L$. Here B(n) is to depend only on n, but you should not try to give a relatively small B(n). Do not assume that the extension L/K is Galois.

6. If G is a finite group, a "chief series" for G is a maximal normal series--i.e. a series of subgroups

$$G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_n = \{1\}$$

all normal in G which has no nontrivial normal refinement. (That is, there is no subgroup N with $G_i > N > G_{i+1}$ such that N is normal in G.) If G is solvable show that the factors G_i/G_{i+1} of a chief series are "elementary abelian"--that is, vector spaces over $\mathbb{Z}/(p)$ for some primes p.

7. Let V be a finite dimensional vector space over a field k and let β be a bilinear form on V such that

(i) $\beta(x,y) = \beta(y,x)$ for all x and y,

(ii) $\beta(x,x) \neq 0$ unless $x = 0$.

Show that V has a basis with respect to which the matrix of β is diagonal.

8. Give an example, with a complete justification, of each of the following:

(i) A commutative Noetherian domain which is not a principal ideal domain.

(ii) A commutative ring which is not Noetherian.

(iii) A commutative domain in which not every nonzero prime ideal is maximal.

ALGEBRA PRELIM

Spring 1988

Directions: Do as many problems as you can.
One complete solution is better than two partial solutions.

1. Let K be a field and let $G = GL_2(K)$ be the group of invertible 2×2 matrices over K. Let $G_1 = \{g \in G \mid \det(g) = 1\}$, and $G_2 = \{g \in G \mid g$ has lower left entry zero$\}$.

   In the diagram

$$\begin{tikzcd}[column sep=small] & G \arrow[dl, dash] \arrow[dr, dash] & \\ G_1 \arrow[dr, dash] & & G_2 \arrow[dl, dash] \\ & G_1 \cap G_2 & \end{tikzcd}$$

   each of the four lines represents a subgroup included in a larger group. What are normal inclusions? Prove your answer.

2. Is it true that whenever R is a unique factorization domain (U.F.D.) and a is an irreducible element of R, then R/aR is also a U.F.D.? Explain.

3. Let A_5 be the alternating groups on five letters. Let H be a 5-Sylow subgroup of the simple group A_5.

   (a) How many elements does the normalizer of H in A_5 have?

   Let D_5 be the dihedral group of symmetries of the regular pentagon.

   (b) Construct a 1-1 homomorphism $D_5 \to A_5$.

   (c) How are your answers to (a) and (b) related?

4. Let $R = \mathbb{C}[X_1, X_2, \ldots, X_n]$ be the complex polynomial ring in n-variables. Let M be a finitely generated unitary R-module. Let $\varphi : R \to \mathbb{C}$ be a homomorphism of $\mathbb{C}$ - algebras.

   For each $j \geq 1$, let $M_j = \{m \in M \mid (a - \varphi(a)1)^j . m = 0$ for all $a \in R\}$ ($\varphi(a)1$ is a constant polynomial in R).

   Must there be an index k such that $M_j = M_k$ for all $j \geq k$? Explain.

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5. For each of the fields $K = \mathbb{Q}$ (rational numbers) and $K = \mathbb{C}(X)$ (the extension of the complex numbers by an indeterminate X), give an example of an extension field L such that L/K is a galois extension whose galois group is the non-cyclic group of order 4. In each example, describe a primitive element α of the extension (i.e. an element so that $L = K(\alpha)$) and give its minimal polynomial. No proofs are required in this problem.

6. Let G be the group of unitary complex $n \times n$ matrices. Consider the set of conjugacy classes of G. In each conjugacy class there is an element in Jordan canonical form. Which Jordan form matrices arise in that way? Explain.

7. Let $G = \{\pm 1, \pm 1 \cdot i, \pm 1 \cdot j, \pm 1 \cdot k\}$ be the eight element group of units of the ring of integral quaternions. The group G may be described by the following: 1 is the identity element, $\{1, -1\}$ is the center, $i^2 = j^2 = k^2 = -1$, and $i \cdot j = k$, while $j \cdot i = -1 \cdot k$.

(It follows that $i \cdot k = (-1) \cdot j$, $k \cdot i = j$, $j \cdot k = i$, $k \cdot j = -1 \cdot i$)

(a) Identify the group of inner automorphisms of G.

(b) Find an outer automorphism of G of order 3.

End

ALGEBRA PRELIM

Fall 1987

Directions: Do as many of the problems as you can.
Try for complete solutions.
One complete solution is better than two partial solutions.

$\mathbb{Z}$ = ring of integers

$\mathbb{Q}$ = field of rational numbers

$\mathbb{F}_p$ = field with p elements

1. Let G be a finite group of order pqr, where p,q,r are distinct primes. Show that G is not a simple group.

2. Let G be a finitely generated group. Prove that for any $n \geq 1$, G can have only finitely many subgroups of index n.

3. a) Prove the existence of an irreducible polynomial over $\mathbb{Q}$ of degree 14 whose splitting field over $\mathbb{Q}$ also has degree 14.

   b) Prove the existence of an irreducible polynomial over $\mathbb{Q}$ of degree 10 whose splitting field over $\mathbb{Q}$ has degree 20.

4. Let R be a commutative ring with identity.

   a) Show that if $R[X]$ is a noetherian ring, then R must also be noetherian.

   b) Show that if $R[X]$ is a P.I.D., then R must be a field.

5. Prove that $I = (5, X^2 + 2)$ is a prime ideal in $\mathbb{Z}[X,Y]$ and that $\mathbb{Z}[X,Y]/I$ is a P.I.D.

6. Let M be a free abelian group of finite rank, and let A be an endomorphism of M. Show that M/AM is finite if and only if $\det A$ is nonzero. Show also that if $\det A \neq 0$, then M/AM has order $|\det A|$.

7. Let R be a finite dimensional algebra with identity over a field F. Prove that there are only finitely many nonisomorphic simple R-modules. Also show that there is at least one simple R-module.

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8. Let G be a finite p-group, p a prime, and let A be the group ring of G over $\mathbb{F}_p$ (thus A consists of formal linear combinations $\sum_{g\in G} c_g \cdot g$, with $c_g \in \mathbb{F}_p$; the multiplication in A comes from the multiplication in G).

(a) the <u>trivial</u> A-module is $\mathbb{F}_p$ itself with the A-module structure defined by $g\cdot x = x \quad \forall g \in G,\ x \in \mathbb{F}_p$. Show that the trivial module is the only simple A-module, up to isomorphism.

(Hint: A simple A-module is in particular a finite G-set).

(b) Let $I = \{\Sigma c_g\, g \in A : \ \Sigma c_g = 0\}$. Use (a) to show that I is a nilpotent two-sided ideal.

ALGEBRA PRELIM

Spring 1987

Directions: Do as many of the problems as you can.
Try for complete solutions. (One complete solution is better than two partial solutions.)

$\mathbb{Z}$ = ring of integers

$\mathbb{Q}$ = field of rational numbers

$\mathbb{C}$ = field of complex numbers

$\mathbb{F}_p$ = field with p elements

1. Show that A_5 (alternating group) has no subgroup of order 15. (You may use the fact that A_5 is a simple group.) Hence show that S_5 (symmetric group) has no subgroup of order 15.

2. Let p be a prime number. Let G be a finite group and suppose that the quotient of G by its center is a p-group. Show that G has only one p-Sylow subgroup.

3. Let (a,b), (c,d) be two elements of the free abelian group $\mathbb{Z}^2$ of rank 2. Show that these two elements generate $\mathbb{Z}^2$ if and only if $ad - bc = \pm 1$.

4. Let p be a prime number different from 2,3.
Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$. Let ζ be a primitive cube root of 1 in $\overline{\mathbb{F}}_p$. Let $\alpha = \zeta - \zeta^{-1} \in \overline{\mathbb{F}}_p$.

   (a) Show that $\alpha \in \mathbb{F}_p$ if and only if $p \equiv 1 \pmod 3$.
   You may use standard facts about the Galois theory of finite fields.

   (b) Show that $\alpha^2 = -3$. (Hint: First show that $1 + \zeta + \zeta^2 = 0$.)

   Putting (a) and (b) together, we find that -3 is a square in $\mathbb{F}_p$ if and only if $p \equiv 1 \pmod 3$.

5. Let n,k be two positive integers. We will say that two $\mathbb{C}$-algebra homomorphisms f_1, f_2 from $M_n(\mathbb{C})$ (n x n complex matrices) to $M_k(\mathbb{C})$ are equivalent if there exists an invertible matrix A in $M_k(\mathbb{C})$ such that
$f_1(x) = Af_2(x)A^{-1}$ for all $x \in M_n(\mathbb{C})$. How many equivalence classes of such homomorphisms are there?
(Hint: Use such a homomorphism to view $\mathbb{C}^k$ as a module over $M_n(\mathbb{C})$.)

(a) Let R be the following subring of $M_2(\mathbb{C})$:

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \;\middle|\; a \in \mathbb{Q},\quad b,d \in \mathbb{C} \right\}.$$

Show that R is right artinian but not left artinian. (Recall: R is right artinian if there is no infinite descending chain of right ideals.)
(Hint: Look at the nilpotent radical as a left R-module.)

(b) Give an example of a commutative ring which is not noetherian. Prove that your example is not noetherian.

7. (a) The numbers 20604, 20927, 53227, 78421, 25755 are all divisible by 17. Using elementary column operations, prove that 17 divides the determinant of the 5×5 matrix

$$\begin{pmatrix} 2 & 0 & 6 & 0 & 4 \\ 2 & 0 & 9 & 2 & 7 \\ 5 & 3 & 2 & 2 & 7 \\ 7 & 8 & 4 & 2 & 1 \\ 2 & 5 & 7 & 5 & 5 \end{pmatrix}.$$

(b) Let $S : U \to V$, $T : V \to W$ be linear maps of complex vector spaces.

Put $n = \dim U$, $m = \dim V$.

Write $\nu(T) = \text{nullity}(T) = \dim(\ker T)$,

$\nu(S) = \text{nullity}(S)$,

$\nu(TS) = \text{nullity}(TS)$.

Prove Sylvester's law of nullity:

$$\max\{n - m + \nu(T),\ \nu(S)\} \le \nu(TS) \le \min\{n,\ \nu(S) + \nu(T)\}.$$

8. Let K/F be a finite Galois extension of fields and suppose that the Galois group of K/F is isomorphic to the symmetric group S_4. How many intermediate fields L $(K \supset L \supset F)$ are there such that $[L : F] = 8$?

Show that any two such fields L are isomorphic over F.

ALGEBRA PRELIM
FALL 1986

Directions: Do as many of the problems as you can. Complete solutions are much preferable to partial results. You should write enough so that there is no doubt that you know what is going on, but do not write a book when a few lines suffices. Be neat. You should not reprove major theorems, but if you may use a major result (e.g. the Sylow theorems or Hilbert's basis theorem), you should state what you are using clearly. In this exam, all rings have identity.

1. Let G be a group with a normal subgroup H of finite index p, for p a prime number. For an element x of H, let $C_G(x)$ be the conjugacy class of x in G and let $C_H(x)$ be the conjugacy class of x in H.

   (i) Show for $g \in G$ and $x \in H$ that $g^{-1}C_H(x)\,g$ is again a conjugacy class of H and conclude that G acts on the set of conjugacy classes of H.

   (ii) Show for $x \in H$ that $C_G(x)$ lies in H.

   (iii) Show for $x \in H$ that either $C_G(x) = C_H(x)$ or $C_G(x)$ is the disjoint union of p-many conjugacy classes of H.

2. Let R be the ring $\mathbb{Z}[x]$.

   (i) Show that there is no ring homomorphism of R **onto** the field $\mathbb{Q}$, or **onto** any field extension of $\mathbb{Q}$.

   (ii) Show that any maximal ideal of R contains a non-zero element of $\mathbb{Z}$.

   (iii) Using (ii), describe all fields K for which there is a ring homomorphism of R **onto** K.

3. Let K be the subfield of $\mathbb{C}$ obtained from $\mathbb{Q}[i]$ by adjoining $\sqrt[4]{7}$, the positive real fourth root of 7. Call this number α.

   (i) Show that K is a Galois extension of $\mathbb{Q}$, and calculate its dimension over $\mathbb{Q}$.

   (ii) Determine explicitly the Galois group of K over $\mathbb{Q}$; that is, describe what group it is and how its elements act on K or on generators of K over $\mathbb{Q}$.

4. Let A be the subring of $M_2(\mathbb{C})$ consisting of all matrices $\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| \ a, b, c \in \mathbb{C} \right\}$. Let V be the left A-module of column vectors $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \ x, y \in \mathbb{C} \right\}$, with elements of A acting by the usual operation, on the left, of matrices on column vectors.

(i) Show that V has a unique submodule W with $0 \neq W \neq V$, and that W is simple.

(ii) Show that V/W is also a simple A-module.

(iii) Are V and W isomorphic as A-modules?

5. Let L be an 84-dimensional field extension of a field F, with L **Galois** over F.

   (i) Which of the following is true and why?

   (a) There is an intermediate field extension K of dimension 21 over F.
   (b) There is no intermediate field extension K of dimension 21 over F.
   (c) The existence of such a K depends on the choice of L and F.

   (ii) Which of the following is true and why?

   (a) There is an intermediate field extension K of dimension 12 over F which is **Galois.**
   (b) There is no intermediate field extension K of dimension 12 over F which is **Galois.**
   (c) The existence of such a K depends on the choice of L and F.

6. The group of invertible 5x5 complex matrices acts on $M_5(\mathbb{C})$ by conjugation, and the orbits are called similarity classes (or conjugacy classes).

   (i) Show that all matrices in a similarity class have the same eigenvalues.

   (ii) Given a similarity class of matrices, all of which have exactly 4 distinct eigenvalues, is there a diagonal matrix in this similarity class? (yes? no? depends on the class? Justify your answer.)

   (iii) How many similarity classes are there of matrices having only the single eigenvalue 2? (Explain how you get your number.)

   (iv) How many of the classes in part (iii) have the property that for every matrix in the class there is a basis of $\mathbb{C}^5$ consisting of eigenvectors for that matrix?

The End

ALGEBRA PRELIM Monday, March 24, 1986

DIRECTIONS: Do as many problems as you can. Try for complete solutions. All rings have an identity 1, and all modules are unital ($1x = x$).

1. Suppose G is a nilpotent group of order $p_1p_2\cdots p_n$, where the p_i are distinct primes. Show that G is cyclic.

2. Let G be a finite p-group, H a subgroup of index p. Show that H is normal in G.

3. Suppose R is a principal ideal domain and x is a nonzero element of R. Show that the Jacobson radical of $R/(x)$ is zero if and only if x is square-free. (Here (x) is the ideal generated by x and square-free means that x factors as a product of distinct primes.)

4. Let M be a finitely generated module over a commutative Noetherian ring R, and let $\varphi : M \to M$ be a homomorphism of R-modules. Show that if φ is onto, then φ is an isomorphism. Give an example showing that this assertion is false if one does not assume M is finitely generated.

5. Show that if A is an $n \times n$ orthogonal matrix over $\mathbb{R}$ (i.e. $AA^T = I$, where A^T is the transpose of A), then $\det(I-A) \geq 0$. (I is the identity matrix.)

6. Suppose K is a field and E is a separable algebraic extension of K. Assume that there is an integer n such that for all $\alpha \in E$, the degree of the minimal polynomial $p_{\alpha,K}$ is less than or equal to n. Show that $\dim_K E$ is finite.

7. Let E be an algebraic extension of the field $\mathbb{F}_p$ (the field with p elements, p a prime), and assume that for every integer $n \geq 1$, E contains a subfield with p^n elements. Show that E is algebraically closed.