NOTES ON GROTHENDIECK DUALITY

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1. LOCAL COHOMOLOGY

1.1. **Sheaves of families of supports.** Collected here are some useful basic facts about cohomology with supports - I won't be offended if you skip right over this section. First, let X be a topological space.

Definition 1.1. A **sheaf of families of supports $\underline{\Phi}$ on** X is a sheaf of sets on X with the following properties:

- for each open set $U \subset X$, $\underline{\Phi}(U)$ is a collection of closed subsets of U. In other words, $\underline{\Phi}$ is a subsheaf of the sheaf of sets $U \mapsto \{$ closed subsets of U $\}$.
- For each open set $U \subset X$, $\underline{\Phi}(U)$ is a family of supports on U. That is, if $Z, W \subset U$ are closed, $Z \subset W$ and $W \in \underline{\Phi}(U)$ then $Z \in \underline{\Phi}(U)$ and if $Z_1, Z_2 \in \Phi(U)$ then $Z_1 \cup Z_2 \in \Phi(U)$.

Remark 1.2. If Φ is a family of supports on X as above, then we obtain a pre-sheaf of families of supports by assigning $U \mapsto \{Z \cap U \,|\, Z \in \Phi\}$, which sheafifies to a sheaf of families of supports $\underline{\Phi}$. On the other hand if $\underline{\Phi}$ is a sheaf of families of supports on X then $\Gamma(X, \underline{\Phi}) = \underline{\Phi}(X)$ is a family of supports on X.

The functors $\Phi \mapsto \underline{\Phi}$ and $\underline{\Phi} \mapsto \Gamma(X, \underline{\Phi})$ need not be mutual inverses. For example if X is a locally compact Hausdorff space and $\Phi = \{Z \subset X \,|\, Z \text{ is compact }\}$ then $X \in \Gamma(X, \underline{\Phi})$ - if X isn't compact we see that $\Phi \neq \Gamma(X, \underline{\Phi})$ (however we do have an inclusion $\Phi \subset \Gamma(X, \underline{\Phi})$. For an example where $\underline{\Phi} \neq \underline{\Gamma(X, \underline{\Phi})}$ let $f : X \to Y$ be a morphism of finite type where Y is a noetherian scheme, and let p be a natural number. We can define a sheaf of families of supports $\underline{\Phi}$ on X by

$$\underline{\Phi}(U) := \{Z \subset U \text{ closed } \,|\, \operatorname{codim}(Z \cap X_y, X_y) \geq p \text{ for all } y \in f(U)\}$$

In general $\underline{\Gamma(X, \underline{\Phi})}$ will be *smaller* than Φ (this should happen essentially whenever f has varying fiber dimension - for a specific case take f to be the map $f : \mathbb{A}^2_k \times \mathbb{P}^1_k \to \mathbb{A}^2_k$ (here k is any field), and set $p = 1$. Then we have $\mathrm{Blp}_0 \mathbb{A}^2_k \subset \mathbb{A}^2_k \times \mathbb{P}^1_k$ - set $U := (\mathbb{A}^2_k \setminus \{0\}) \times \mathbb{P}^1_k \subset \mathbb{A}^2_k \times \mathbb{P}^1_k$ and set $Z = \mathrm{Blp}_0 \mathbb{A}^2_k \cap U$. Super-explicitly,

$$Z := \{((x_0, x_1), [x_0, x_1]) \in \mathbb{A}^2_k \times \mathbb{P}^1_k \,|\, (x_0, x_1) \neq 0\}$$

Evidently $Z \in \underline{\Phi}(U)$, but since the closure of Z in $\mathbb{A}^2_k \times \mathbb{P}^1_k$ is $\mathrm{Blp}_0 \mathbb{A}^2_k$, there's no $W \in \Gamma(X, \underline{\Phi})$ so that $W \cap U = Z$.

From here on out we'll assume X is a **Zariski space**, that is, a noetherian topological space in which every (non-$\emptyset$) irreducible closed subset has a unique generic point. A Hartshorne exercise shows that the underlying space of a noetherian scheme is a Zariski space.

Definition 1.3. A subset $Z \subset X$ is **specialization-closed** if and only if whenever $x \in Z$ and $y \in X$ is a specialization of x (meaning $y \in \{\bar{x}\}$), $y \in Z$ too.

Definition 1.4. For a specialization-closed subset $Z \subset X$, the associated family of supports Φ_Z is

$$\Phi_Z := \{\bigcup_{i=0}^{n} \{\bar{x_i}\} \subset X \,|\, x_0, \ldots, x_n \in Z \text{ and } n \in \mathbb{N}\}$$

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Example 1.5. Recall that the *codimension* $\operatorname{codim}(x, X$ of a point $x \in X$ is

$$\sup\{n \in \mathbb{N} \mid \text{ there's a sequence of proper specializations } x_0 \mapsto x_1 \mapsto \cdots \mapsto x_n = x\}$$

(if X is a noetherian scheme, one can show $\operatorname{codim}(x, X) = \dim \mathscr{O}_{X,x}$). If y is a specialization of x then $\operatorname{codim}(y, X) \geq \operatorname{codim}(x, X)$ with strict inequality if $y \neq x$, and so we see that for every $p \in \mathbb{N}$,

$$Z_p := \{x \in X \mid \operatorname{codim}(x, X) \geq p\}$$

is specialization-closed.

Here's a philosophically interesting point:

Proposition 1.6. *Let X be a Zariski space. Then there's a natural one-to-one correspondence between families of supports Φ on X such that $\Gamma(X, \underline{\Phi}) = \Phi$ and specialization-closed subsets $Z \subset X$.*

Proof. On the one hand, given a specialization-closed set $Z \subset X$ we obtain a family of supports Φ_Z on X. Note that if $W \in \Gamma(X, \underline{\Phi}_Z)$ we can find an open cover $X = \bigcup_i U$ of X (and we can assume it's finite, since X is assumed to be Zariski, hence noetherian) so that for each i

$$W \cap U_i = V \cap U_i \text{ for some } V \in \Phi_Z$$

By the definition of Φ_Z, $V = \bigcup_j \bar{\{x_{ij}\}}$ and removing those x_{ij} with $x_{ij} \notin U_i$ if necessary, we have $W \cap U_i = (\bigcup_j \bar{\{x_{ij}\}}) \cap U_i$. The claim to make is that

$$W = \bigcup_{i,j} \bar{\{x_{ij}\}}$$

This boils down to the following foundational fact about Zariski spaces: if W is a closed subset of a Zariski space X and $x_0, \ldots, x_N$ are the generic points of its irreducible components $W_0, \ldots, W_N$, then $W = \bigcup_i \bar{\{x_i\}}$.

The upshot: $\Phi_Z = \Gamma(X, \underline{\Phi}_Z)$. On the other hand suppose Φ is a family of supports with $\Phi = \Gamma(X, \underline{\Phi})$, and set $Z_\Phi = \{x \in X \mid \bar{\{x\}} \in \Phi\}$. Since Φ is a family of supports, Z_Φ is specialization-closed. One must now check that our functions $Z \mapsto \Phi_Z$ and $\Phi \mapsto Z_\Phi$ are mutual inverses, but I'm going to omit the details.

$\square$

1.2. **Local cohomology.** Let X be a topological space and let $\underline{\Phi}$ be a sheaf of families of supports on X. Given a sheaf $\mathscr{F}$ of abelian groups on X, we get a *subsheaf* $\underline{\Gamma}_{\underline{\Phi}}(\mathscr{F})$ of $\mathscr{F}$, defined by

$$\underline{\Gamma}_{\underline{\Phi}}(\mathscr{F})(U) := \{\sigma \in \mathscr{F}(U) \mid \operatorname{supp}\sigma \in \underline{\Phi}(U)\}$$

Fact: $\underline{\Gamma}_{\underline{\Phi}}$ is left exact. Its right derived functor will be denoted by $R\underline{\Gamma}_{\underline{\Phi}}$. The cohomologies of $R\underline{\Gamma}_{\underline{\Phi}}$ will be denoted by

$$\mathscr{H}^i_{\underline{\Phi}}(\mathscr{F}) = R^i\underline{\Gamma}_{\underline{\Phi}}(\mathscr{F})$$

Remark 1.7. If $Z \subset X$ is a closed set and Φ is the collection of all closed subsets of Z, we'll abbreviate like $\underline{\Gamma}_Z := \underline{\Gamma}_{\underline{\Phi}}$.

Let $Z \subset X$ be a closed subset and let $U := X \setminus Z$. Let

$$\text{res} : \mathscr{F} \to \iota_*(\mathscr{F}|_U)$$

be the natural morphism, where $\iota : U \to X$ is the inclusion. Unraveling definitions we see that its kernel is $\underline{\Gamma}_Z(\mathscr{F})$, and in this way we obtain a left exact sequence

$$0 \to \underline{\Gamma}_Z(\mathscr{F}) \to \mathscr{F} \to \iota_*(\mathscr{F}|_U)$$

If $\mathscr{F}$ is flasque then this sequence is exact on the right as well.

So, if $\mathscr{I}^*$ is an injective (or even just flasque) resolution of $\mathscr{F}$, we obtain a short exact sequence of complexes

$$0 \to \underline{\Gamma}_Z(\mathscr{I}^*) \to \mathscr{I}^* \to \iota_*(\mathscr{I}|_U^*) \to 0$$

The resulting long exact sequence on cohomology looks like

$$\cdots \to \mathscr{H}^{i-1}(U, \mathscr{F}|_U) \xrightarrow{\delta} \mathscr{H}^i_Z(X, \mathscr{F}) \to \mathscr{H}^i(X, \mathscr{F}) \xrightarrow{\text{res}} \mathscr{H}^i(U, \mathscr{F}|_U) \xrightarrow{\delta} \mathscr{H}^{i+1}_Z(X, \mathscr{F}) \to \cdots$$

The thing that makes local cohomology *local* is the following "excision" property. Let $V \subset X$ be an open set containing Z. Then the natural restriction homomorphism $\underline{\Gamma}_Z(\mathscr{F})|_V \to \underline{\Gamma}_Z(j_*\mathscr{F}|_V)$ is an isomorphism, which induces a natural isomorphism $R\underline{\Gamma}_Z(\mathscr{F})|_V \to R\underline{\Gamma}_Z(j_*\mathscr{F}|_V)$ (here $j : V \to X$ is the inclusion). Taking cohomology yields isomorphisms

$$H^i_Z(X, \mathscr{F}) \simeq H^i_Z(V, \mathscr{F}|_V)$$

Note that if $\Psi \subset \Phi$ are families of supports, then there is a natural sub-sheaf inclusion

$$\underline{\Gamma}_\Psi(\mathscr{F}) \subset \underline{\Gamma}_\Phi(\mathscr{F})$$

so in particular if $Z \in \Phi$, then we have an inclusion $\underline{\Gamma}_Z(\mathscr{F}) \subset \underline{\Gamma}_\Phi(\mathscr{F})$, and taking the colimit over all $Z \in \Phi$ gives a natural morphism

$$\operatorname{co}\varinjlim_{Z \in \Phi} \underline{\Gamma}_Z(\mathscr{F}) \to \underline{\Gamma}_\Phi(\mathscr{F}); \text{ one can check it's an } \textit{isomorphism}$$

and this leads to the following

Proposition 1.8. *Let $\underline{\Phi}$ be a sheaf of families of supports such that $\underline{\Phi} = \underline{\Gamma(X, \underline{\Phi})}$. Then for any sheaf of abelian groups $\mathscr{F}$*

$$\mathscr{H}^i_{\underline{\Phi}}(\mathscr{F}) = \operatorname{colim}_{Z \in \underline{\Phi}(X)} \mathscr{H}^i_Z(\mathscr{F})$$

For this reason one can often reduce to the case where Φ consists of the closed subsets of a given subset $Z \subset X$.

If $\mathscr{F}$ is a sheaf of abelian groups on X, then $\Gamma(X, \underline{\Gamma}_{\underline{\Phi}}(\mathscr{F})) = \Gamma_{\underline{\Phi}(X)}(X, \mathscr{F})$. This extends to derived functors to yield

$$R\Gamma_{\underline{\Phi}(X)} = R\Gamma \circ R\underline{\Gamma}_{\underline{\Phi}}$$

Proposition 1.9. *There is a natural (composition of functors) spectral sequence*

$$E_2^{pq} = H^p(X, \mathscr{H}^q_{\underline{\Phi}}(\mathscr{F})) \implies H^{p+q}_{\underline{\Phi}(X)}(X, \mathscr{F})$$

Example 1.10. Suppose A is a noetherian commutative ring and let $I \subset A$ be an ideal, corresponding to a closed subscheme $Z \subset X := \operatorname{Spec} A$. Let M be an A-module and let $\tilde{M}$ be the corresponding quasi-coherent sheaf on X. Observe that $\sigma \in \Gamma_Z(X, \tilde{M})$ if and only if $V(\operatorname{ann}\sigma) = \operatorname{supp}\sigma \in Z$, or equivalently $\operatorname{rad} I \subset \operatorname{rad}\operatorname{ann}\sigma$. As A is noetherian (really, we just need I to be finitely generated), this occurs if and only if $I^r \subset \operatorname{ann}\sigma$ for $r \gg 0$, which is to say $I^r\sigma = 0$ for $r \gg 0$. Conclusion: $\Gamma_Z(X, \tilde{M}) = \Gamma_I(A, M)$, where by definition $\Gamma_I(A, M) := \{m \in M \mid I^r m = 0 \text{ for } r \gg 0\}$ is the I-torsion submodule of M.

There's a similar (but slightly more complicated) commutative algebraic description of $\Gamma_\Phi(X, \tilde{M})$ for a family of supports Φ on X. Here note that $\Phi^\vee := \{I \subset A \text{ an ideal } \mid V(I) \in \Phi\}$ is a family of ideals in A with the properties that

- if $I, J \in \Phi^\vee$ then $I \cap J \in \Phi^\vee$.
- if $I \in \Phi^\vee$ and $I \subset J$ then $J \in \Phi^\vee$.

If M is an A-module then we should have $\Gamma_\Phi(X, \tilde{M}) = \Gamma_{\Phi^\vee}(A, M)$ where

$$\Gamma_{\Phi^\vee}(A, M) := \{m \in M \mid I^r m = 0 \text{ for } r \gg 0 \text{ for some } I \in \Phi^\vee\}$$

Example 1.11. Let $X \subset \mathbb{P}^n_k$ be a projective variety of positive dimension over a field k, and let $C(X) \subset \mathbb{A}^{n+1}_k$ be the affine cone over X and let $p \in C(X)$ be the vertex of the cone. Let's describe the local cohomology of $\mathscr{O}_{C(X)}$ at p. Let $U := C(X) \setminus \{p\}$ and consider the long exact sequence

$$\cdots \to H^{i-1}(U, \mathscr{O}_U) \xrightarrow{\delta} H^i_p(C(X), \mathscr{O}_{C(X)}) \to H^i(C(X), \mathscr{O}_{C(X)}) \xrightarrow{res} H^i(U, \mathscr{O}_U) \xrightarrow{\delta} \cdots$$

Since $C(X)$ is *affine*, $H^i(C(X), \mathscr{O}_{C(X)}) = 0$ for $i > 0$, and so really we're looking at an exact sequence

$$0 \to H^0_p(C(X), \mathscr{O}_{C(X)}) \to H^0(C(X), \mathscr{O}_{C(X)}) \xrightarrow{res} H^0(U, \mathscr{O}_U) \xrightarrow{\delta} H^1_p(C(X), \mathscr{O}_{C(X)}) \to 0$$

$$\text{and isomorphisms } H^i(U, \mathscr{O}_U) \xrightarrow{\simeq} H^{i+1}_p(C(X), \mathscr{O}_{C(X)}) \text{ for } i > 0$$

Note also that $H^0_p(C(X), \mathscr{O}_{C(X)}) = 0$ (since we're assuming $\dim X > 0$ and hence $\dim C(X) > 1$. So in fact the exact sequence in low degrees simplifies further to

$$0 \to H^0(C(X), \mathscr{O}_{C(X)}) \xrightarrow{res} H^0(U, \mathscr{O}_U) \xrightarrow{\delta} H^1_p(C(X), \mathscr{O}_{C(X)}) \to 0$$

Note that U is a principal $\mathbb{G}_m$-bundle over X. In fact it's the principal $\mathbb{G}_m$-bundle associated to the invertible sheaf $\mathscr{O}_X(-1)$, and from this we see that it's *affine* over X, with

$$\pi_* \mathscr{O}_U = \bigoplus_{k \in \mathbb{Z}} \mathscr{O}_X(k)$$

Letting $\pi : U \to X$ be the projection, we can calculate $H^i(U, \mathscr{O}_U)$ using the Leray spectral sequence for π, which collapses (since π is an affine map) to give

$$H^i(U, \mathscr{O}_U) = \bigoplus_{k \in \mathbb{Z}} H^i(X, \mathscr{O}_X(k))$$

To obtain some interesting conclusions, allow me to use the results of a few Hartshorne exercises [Har77, §III]: first, if X is a noetherian scheme and $p \in X$ is a point, and if $\mathscr{F}$ is a quasi-coherent sheaf on X, then the natural maps $H^i_p(X, \mathscr{F}) \to H^i_p(\mathrm{Spec}\mathscr{O}_{X,p}, \mathscr{F}_p)$ are all isomorphisms, and if A is a noetherian local ring with maximal ideal m, then $H^i_m(\mathrm{Spec}A, A)$ vanishes unless $\mathrm{depth}A \leq i \leq \dim A$ (and it's guaranteed to be non-0 for $i = \mathrm{depth}A$ and $i = \dim A$). Also recall that A is **Cohen-Macaulay** if and only if $\mathrm{depth}A = \dim A$.

Recall also that a noetherian scheme X is normal if and only if it satisfies Serre's criteria R_1 and S_2 - that is, it's regular in codimension 1 and for every point $p \in X$,

$$\mathrm{depth}\mathscr{O}_{X,p} \geq \min 2, \mathrm{codim}(p, X)$$

Suppose first that X is *normal*. Then certainly U is normal (it's smooth over X. As X is assumed to be positive-dimensional, $\mathrm{codim}(p, C(X)) = \dim X + 1 > 1$, so $C(X)$ will be normal if and only if it's S_2 at p, i.e.

$$\mathrm{depth}\mathscr{O}_{C(X),p} \geq 2$$

Applying the above Hartshorne facts, we see that $C(X)$ will be normal if and only if $H^1_p(C(X), \mathscr{O}_{C(X)}) = 0$, which occurs if and only if the natural map

$$H^0(C(X), \mathscr{O}_{C(X)}) \xrightarrow{\simeq} \bigoplus_{k \in \mathbb{Z}} H^0(X, \mathscr{O}_X(k)) \text{ is an isomorphism.}$$

Unraveling some definitions, one sees that the composition

$$\bigoplus_k H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(k)) = H^0(\mathbb{A}^{n+1}, \mathscr{O}_{\mathbb{A}^{n+1}}) \to H^0(C(X), \mathscr{O}_{C(X)}) \to \bigoplus_{k \in \mathbb{Z}} H^0(X, \mathscr{O}_X(k))$$

factors the restriction $\bigoplus_k H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(k)) \to \bigoplus_{k \in \mathbb{Z}} H^0(X, \mathscr{O}_X(k))$ as a surjection followed by an injection, and so we recover the basic

Proposition 1.12. *X is projectively normal if and only if the restriction map on global sections*

$$H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(k)) \to H^0(X, \mathscr{O}_X(k))$$

is an isomorphism for all k.

Now assume X is projectively normal - then

$$H^0_p(C(X), \mathscr{O}_{C(X)}) = H^1_p(C(X), \mathscr{O}_{C(X)}) = 0$$
$$\text{and } H^{i+1}_p(C(X), \mathscr{O}_{C(X)}) \simeq \bigoplus_k H^i(X, \mathscr{O}_X(k)) \text{ for } i > 0$$

From here it's not hard to show that $C(X)$ is Cohen-Macaulay if and only if X is Cohen-Macaulay and $H^i(X, \mathscr{O}_X(k)) = 0$ for all k when $0 < i < \dim X$.

For example let X be a general (hence smooth) complete intersection of type $d_1, \dots, d_r$. Then both $C(X)$ is also a complete intersection, and so both X and $C(X)$ are Cohen-Macaulay. In this case

$$H^i_p(C(X), \mathscr{O}_{C(X)}) = 0 \text{ for } i < \dim X + 1 \text{ and}$$
$$H^{\dim X+1}_p(C(X), \mathscr{O}_{C(X)}) = \bigoplus_k H^{\dim X}(X, \mathscr{O}_X(k)) = \bigoplus_k H^0(X, \mathscr{O}_X(\sum_i d_i - n - 1 - k))$$

where the last step uses Serre duality and the adjunction formual $\omega_X = \mathscr{O}_X(\sum_i d_i - n - 1)$. While this is abstractly isomorphic to $H^0(C(X), \mathscr{O}_{C(X)})$ the grading has been reversed and shifted by $\sum_i d_i - n - 1$.

On the other hand if X is an abelian variety of dimension g then $C(X)$ will never be Cohen-Macaulay, since $H^i(X, \mathscr{O}_X) \neq 0$ for $0 \leq i \leq g$.

1.3. **Cousin complexes.** Let $\underline{\Phi}$ and $\underline{\Psi}$ be 2 sheaves of families of supports on X and suppose $\underline{\Phi} \subset \underline{\Psi}$. Then for any sheaf of abelian groups $\mathscr{F}$ there's a natural inclusion $\underline{\Gamma}_{\underline{\Phi}}(\mathscr{F}) \subset \underline{\Gamma}_{\underline{\Psi}}(\mathscr{F})$ and this extends to a morphism of derived functors

$$R\underline{\Gamma}_{\underline{\Phi}} \to R\underline{\Gamma}_{\underline{\Psi}} \text{ which we can fit into an \textit{exact triangle}}$$

$$R\underline{\Gamma}_{\underline{\Phi}} \to R\underline{\Gamma}_{\underline{\Psi}} \to R\underline{\Gamma}_{\underline{\Psi}\setminus\underline{\Phi}} \xrightarrow{[+1]} R\underline{\Gamma}_{\underline{\Phi}} \to \dots$$

We can take this as the definition of the derived functor $R\underline{\Gamma}_{\underline{\Psi}\setminus\underline{\Phi}}$.

The following is an analogue of the "spectral sequence of a filtered space" from algebraic topology.

Proposition 1.13. *Let X be a topological space and let*

$$\underline{\Phi}_X = \underline{\Phi}_0 \supset \underline{\Phi}_1 \supset \underline{\Phi}_2 \supset \cdots$$

be a filtration of X by sheaves of families of supports, where $\underline{\Phi}_X$ is the maximal sheaf of families of supports (including all closed subsets). Let $\mathscr{F}^*$ be a bounded-below complex of abelian groups on X, i.e. an object of $D^+(\mathcal{A}\lfloor(X))$.. Then there's a spectral sequence

$$E_1^{pq} = \mathscr{H}^{p+q}_{\underline{\Phi}_p\setminus\underline{\Phi}_{p+1}}(\mathscr{F}) \implies \mathscr{H}^{p+q}(\mathscr{F})$$

which converges provided $\underline{\Phi}_n = \emptyset$ for $n \gg 0$.

Assume X is a locally Zariski space and let $X = Z_0 \supset Z_1 \supset Z_2 \supset \dots$ be a filtration of X by specialization-closed sets with the following properties:

- for each p, every point of $Z_p \setminus Z_{p+1}$ is maximal with respect to specialization - in other words, if $x \in Z_p$ and $y \in X$ is a proper specialization of x, then $y \in Z_{p+1}$.
- $Z_n = \emptyset$ for $n \gg 0$.

In this situation we obtain associated sheaves of families of supports $\underline{\Phi}_p := \underline{\Phi}_{Z_p}$ fitting into a filtration

$$\underline{\Phi}_X = \underline{\Phi}_0 \supset \underline{\Phi}_1 \supset \dots$$

with $\underline{\Phi}_n = \emptyset$ for $n \gg 0$. Since the notation is already fierce, let's write $\underline{\Gamma}_{Z_p \setminus Z_{p+1}}$ for $\underline{\Gamma}_{\underline{\Phi}_p \setminus \underline{\Phi}_{p+1}}$ and similarly for its derived functors.

Example 1.14. The most important case to keep in mind is where $Z_p = \{x \in X \mid \mathrm{codim}(x, X) \geq p\}$.

Proposition 1.15. *For any bounded-below complex of sheaves of abelian groups $\mathscr{F}^*$ on X and for any $p \in \mathbb{N}$, there are natural isomorphisms*

$$\mathscr{H}^i_{Z_p \setminus Z_{p+1}}(\mathscr{F}^*) \simeq \bigoplus_{x \in Z_p \setminus Z_{p+1}} \mathscr{H}^i_x(\mathscr{F}^*) \textit{ for all } i$$

where on the right hand side $\mathscr{H}^i_x(\mathscr{F}^*) := \mathscr{H}^i_{\{\bar{x}\}}(\mathscr{F}^*)_x$, viewed as the pushforward of the constant sheaf $\mathscr{H}^i_{\{\bar{x}\}}(\mathscr{F}^*)_x$ on $\{\bar{x}\}$ to X. So in particular it's flasque*.*

Combining this with the spectral sequence of a filtered space, we obtain:

Proposition 1.16. *With notation as above, there is a convergent spectral sequence*

$$E_1^{pq} = \bigoplus_{x \in Z_p \setminus Z_{p+1}} \mathscr{H}^{p+q}_x(\mathscr{F}^*) \implies \mathscr{H}^{p+q}(\mathscr{F}^*)$$

Let's assume for a moment that $\mathscr{F}^*$ is just a sheaf supported in degree 0. Then in fact

$$\mathscr{H}^i_{Z_p \setminus Z_{p+1}}(\mathscr{F}) = 0 \text{ for i > p}$$

(this boils down to Grothendieck vanishing applied on the local space of a point $x \in Z_p \setminus Z_{p+1}$).
Looking at the $q = 0$ axis of the spectral sequence, we obtain a complex

$$C^*(\mathscr{F}) : \dots \xrightarrow{d_1} \bigoplus_{x \in Z_{p-1} \setminus Z_p} \mathscr{H}^{p-1}_x(\mathscr{F}) \xrightarrow{d_1} \bigoplus_{x \in Z_p \setminus Z_{p+1}} \mathscr{H}^p_x(\mathscr{F}) \xrightarrow{d_1} \bigoplus_{x \in Z_{p+1} \setminus Z_{p+2}} \mathscr{H}^{p+1}_x(\mathscr{F}) \xrightarrow{d_1} \dots$$

together with an augmentation $\mathscr{F} \xrightarrow{\epsilon} \bigoplus_{x \in Z_0 \setminus Z_0} \mathscr{H}^0_x(\mathscr{F})$ (ϵ can be thought of as taking the germs of a section of $\mathscr{F}$ at the generic points of X)

Definition 1.17. $C^*(\mathscr{F})$ is the **Cousin complex of** $\mathscr{F}$.

Based on the above discussion, we see that

Proposition 1.18. $C^*(\mathscr{F})$ *is a flasque resolution of $\mathscr{F}$ if and only if*

$$\mathscr{H}^i_x(\mathscr{F}) = 0 \textit{ for } i \neq p \textit{ for } x \in Z_p \setminus Z_{p+1}$$

If either of these equivalent conditions are satisfied one says $\mathscr{F}$ is ***Cohen-Macaulay with respect to the filtration*** Z_*.

Similarly $C^*(\mathscr{F})$ is an injective resolution of $\mathscr{F}$ if and only if

$$\mathscr{H}^i_x(\mathscr{F}) = 0 \textit{ for } i \neq p \textit{ for } x \in Z_p \setminus Z_{p+1} \textit{ and}$$

$$\mathscr{H}^p_x(\mathscr{F}) \textit{ is injective for all } x \in Z_p \setminus Z_{p+1}$$

If either of these equivalent conditions are satisfied one says $\mathscr{F}$ is ***Gorenstein with respect to the filtration*** Z_*.

Remark 1.19. By the aforementioned Hartshorne exercises, this reduces to the usual notions of Cohen-Macaulay-ness and Gorenstein-ness when X is a noetherian scheme, Z_* is the codimension filtration and $\mathscr{F}$ is a quasi-coherent sheaf on X.

Example 1.20. Let X be a noetherian scheme of finite-dimension and let Z_* be the codimension filtration of X. Let $\mathscr{F}$ be a Cohen-Macaulay quasi-coherent sheaf on X (for instance if X itself is Cohen-Macaulay then $\mathscr{F}$ can be any locally free sheaf on X). Then the Cousin complex gives a canonical flasque resolution

$$0 \to \mathscr{F} \to \bigoplus_{\operatorname{codim}(x,X)=0} \mathscr{H}^0_x(\mathscr{F}) \xrightarrow{d} \bigoplus_{\operatorname{codim}(x,X)=1} \mathscr{H}^1_x(\mathscr{F}) \xrightarrow{d} \bigoplus_{\operatorname{codim}(x,X)=2} \mathscr{H}^2_x(\mathscr{F}) \xrightarrow{d} \dots$$

To be even more explicit, let X be a smooth projective curve over a field k (algebraically closed if you want). Then we get a complex of the form

$$0 \to k(X) \xrightarrow{d} \bigoplus_{x \in X \text{ closed}} \mathscr{H}^1_x(\mathscr{O}_X) \to 0$$

computing the cohomology of $\mathscr{O}_X$- here $k(X)$ is the function field of X. The differential can be interpreted as computing the "principal part" of a rational function $f \in k(X)$ at a point $x \in X$. To see this note that for a closed point $x \in X$ with an affine neighborhood $U \subset X$ excision gives isomorphisms $H^*_x(X, \mathscr{O}_X) \simeq H^*_x(U, \mathscr{O}_U)$. Now set $V := U \setminus \{x\}$ and consider the exact sequence

$$0 \to H^0_x(U, \mathscr{O}_U) \to H^0(U, \mathscr{O}_U) \to H^0(V, \mathscr{O}_V)$$

$$\xrightarrow{\delta} H^1_x(U, \mathscr{O}_U) \to H^1(U, \mathscr{O}_U) \to H^1(V, \mathscr{O}_V) \to 0$$

Note that $H^0_x(U, \mathscr{O}_U) = 0$ (since U is smooth, so in particular S_1) and also $H^1(U, \mathscr{O}_U) = H^1(V, \mathscr{O}_V) = 0$ - this is because U and V are affine (any finite set of points on a curve can be viewed as an ample divisor, so its complement is affine). So we have an exact sequence

$$0 \to H^0(U, \mathscr{O}_U) \to H^0(V, \mathscr{O}_V) \to H^1_x(U, \mathscr{O}_U) \to 0$$

This can be made even more explicit if $X = \mathbb{P}^1_k$ and $x = 0$. In that case $U = \mathbb{A}^1_k$ and $V = \mathbb{A}^1_k \setminus \{0\}$, so $H^0(U, \mathscr{O}_U) = k[x]$ and $H^0(V, \mathscr{O}_V) = k[x, x^{-1}]$; the map $H^0(U, \mathscr{O}_U) \to H^0(V, \mathscr{O}_V)$ is the usual inclusion $k[x] \subset k[x, x^{-1}]$. So, we see that

$$H^1_{\{0\}}(\mathbb{A}^1_k, \mathscr{O}) \simeq k[x, x^{-1}]/k[x]$$

and the map $H^0(V, \mathscr{O}_V) \to H^1_{\{0\}}(\mathbb{A}^1_k, \mathscr{O})$ corresponds to taking the "principal part" of a Laurent polynomial at the origin.

2. Local Grothendieck duality

2.1. **Dualizing complexes.** Let X be a locally noetherian scheme and let ω^* be a complex of quasi-coherent sheaves on X such that

- the cohomology of ω^* is bounded and coherent and
- ω^* has finite injective dimension.

In this situation ω^* is quasi-isomorphic to a bounded complex of injective quasi-coherent sheaves (it's an object of $D^b_c(X)$) and so $R\underline{Hom}_X(-, \omega^*)$ gives a contravariant derived functor

$$D^+_c(X) \to D^-_c(X) \text{ preserving } D^b_c(X)$$

Definition 2.1. An object $\mathscr{F}^*$ of $D^+_c(X)$ is ω^***-reflexive** if and only if the natural map

$$\mathscr{F}^* \to R\underline{Hom}_X(R\underline{Hom}_X(\mathscr{F}^*, \omega^*), \omega^*)$$

is a quasi-isomorphism.

Proposition 2.2. *If $\mathscr{O}_X$ (viewed as a complex concentrated in degree 0) is ω^*-reflexive then every $\mathscr{F}^*$ as above is ω^*-reflexive.*

Definition 2.3. If either of the equivalent conditions in the above proposition are satisfied, ω^* is called a **dualizing complex on** X.

Example 2.4. Let $X = \mathrm{Spec}\mathbb{Z}$. We will show that the complex $\mathbb{Z}$ (concentrated in degree 0) is dualizing. Note that

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

is an injective resolution of $\mathbb{Z}$ and so $\mathbb{Z}$ has finite injective dimension. By the above proposition we only need to check that the natural map

$$\mathbb{Z} \to RHom_{\mathbb{Z}}(RHom_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}),\mathbb{Z})$$

is an isomorphism, which follows from the calculation $\mathbb{Z} \simeq RHom_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})$ (since $Ext^i_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) = 0$, $\mathbb{Z}$ being free).

Example 2.5. More generally if X is a regular noetherian scheme then $\mathscr{O}_X$ (viewed as a complex concentrated in degree 0) is dualizing. Once we know $\mathscr{O}_X$ has finite injective dimension, the calculation $R\underline{Hom}_X(\mathscr{O}_X, \mathscr{O}_X) \simeq \mathscr{O}_X$ will show $\mathscr{O}_X$ is indeed dualizing. The fact that it has finite injective dimension is essentially Serre's theorem that a noetherian local ring is regular if and only if it has finite homological dimension!

Here are some local criteria for a complex to be dualizing:

Proposition 2.6. *Let ω^* be an object of $D^b_c(X)$ with finite injective dimension. Then the following are equivalent:*

- *ω^* is a dualizing complex.*
- *ω^*_x is a dualizing complex on $\mathrm{Spec}\mathscr{O}_{X,x}$ for all $x \in X$.*
- *$\hat{\omega}^*_x$ is a dualizing complex on $\mathrm{Spec}\hat{\mathscr{O}}_{X,x}$ for all $x \in X$.*
- *$\iota_{x*}k(x)$ is ω^*-reflexive for all $x \in X$ - here $k(x)$ is the residue field of x and $\iota_x : \{x\} \to X$ is the inclusion.*

We also have a global uniqueness statement:

Proposition 2.7. *Let X be a connected locally noetherian scheme and let ω^* be a dualizing complex on X. If ν^* is another complex on X then ν^* is dualizing if and only if there is an invertible sheaf $\mathscr{L}$ on X and an integer $k \in \mathbb{Z}$ so that*

$$\nu^* \simeq \omega^* \otimes \mathscr{L}[k]$$

Remark 2.8. Most of the content of this proof is wrapped up in the following fact: if $\mathscr{M}_1^*$ is an object of $D^b_c(X)$ such that there's another object $\mathscr{M}_2^*$ and an isomorphism

$$\mathscr{M}_1^* \otimes^L \mathscr{M}_2^* \simeq \mathscr{O}_X,$$

then there's an invertible sheaf $\mathscr{L}$ on X, an integer k and a quasi-isomorphism $\mathscr{M}_1^* \simeq \mathscr{L}[k]$ and it follows necessarily that $\mathscr{M}_2^* \simeq \mathscr{L}^\vee[-k]$.

Before returning to local cohomological considerations, we need one more lemma:

Proposition 2.9. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let ω^* be an object of $D^b_c(A)$. Then ω^* is dualizing if and only if there's a $d \in \mathbb{Z}$ so that*

$$\mathrm{Ext}^i_A(k,\omega^*) = \begin{cases} k & \text{if } i = d \\ 0 & \text{otherwise} \end{cases}$$

*In this situation ω^* is said to be **normalized** if $d = 0$.*

2.2. **Local duality.** Let X be a locally noetherian scheme and let $Z \subset X$ be a closed subscheme, corresponding to a coherent sheaf of ideals $\mathscr{I}_Z \subset \mathscr{O}_X$. Let Z_n be the n-th thickening of Z, corresponding to the ideal sheaf $\mathscr{I}_Z^n$ (so that $Z = Z_1$). Observe that the section 1 of $\mathscr{O}_{Z_n}$ is supported on Z, and so we have natural homomorphisms

$$\underline{Hom}_X(\mathscr{O}_{Z_n}, \mathscr{F}) \to \underline{\Gamma}_Z(\mathscr{F}) \text{ for all } n$$

compatible with the natural maps $\underline{Hom}_X(\mathscr{O}_{Z_n}, \mathscr{F}) \to \underline{Hom}_X(\mathscr{O}_{Z_{n+1}}, \mathscr{F})$. In this way we obtain a natural *isomorphism*

$$\operatorname{co}\lim_{n\to\infty} \underline{Hom}_X(\mathscr{O}_{Z_n}, \mathscr{F}) \to \underline{\Gamma}_Z(\mathscr{F})$$

That this natural map is indeed an isomorphism can be checked affine locally on X - the affine local verification is a Hartshorne exercise.

The upshot is that there's a natural isomorphism of derived functors

$$R\operatorname{co}\lim_{n\to\infty} \underline{Hom}_X(\mathscr{O}_{Z_n}, -) \to R\underline{\Gamma}_Z$$

Remark 2.10. While it's tempting to pull the colimit to the left of the R, that only makes sense after taking cohomology. So, it is the case that there are natural isomorphisms

$$\operatorname{co}\lim_{n\to\infty} \underline{Ext}^i_X(\mathscr{O}_{Z_n}, \mathscr{F}) \simeq \mathscr{H}^i_Z(\mathscr{F})$$

Remark 2.11. A related fact is that the natural map

$$j_*(\mathscr{F}|_U) \xrightarrow{\simeq} \operatorname{co}\lim_{n\to\infty} \underline{Hom}_X(\mathscr{I}_Z^n, \mathscr{F})$$

(where $U := X \setminus Z$ and $j : U \to X$ is the inclusion) is also an isomorphism. We obtain a quasi-isomorphism

$$j_*(\mathscr{F}|_U) \xrightarrow{\simeq} R\operatorname{co}\lim_{n\to\infty} \underline{Hom}_X(\mathscr{I}_Z^n, \mathscr{F})$$

Note that from this perspective the long exact sequence

$$\cdots \to H^i_Z(\mathscr{F}) \to H^i(X, \mathscr{F}) \to H^i(U, \mathscr{F}|_U) \to \dots$$

can be obtained from the short exact sequences

$$0 \to \mathscr{I}_Z^n \to \mathscr{O}_X \to \mathscr{O}_{Z_n} \to 0$$

applying $\mathrm{Hom}_X(-, \mathscr{F})$ to get an Ext long exact sequence, and then taking the colimit over n.

Injective hulls. Let $\mathcal{A}$ be an abelian category. An **essential injection** $\iota : M \to N$ in $\mathcal{A}$ is an injection with the property that for all non-zero subobjects $N' \subset N$, $M \cap N' \neq 0$.

Lemma 2.12. *Let A be a commutative ring and let $M \subset N$ be A-modules. Then $\iota : M \to N$ is an essential injection if and only if for each non-zero $n \in N$ there's a $f \in A$ so that $fn \in M$ and $fn \neq 0$.*

Definition 2.13. Let $\mathcal{A}$ be an abelian category and let M be an object of $\mathcal{A}$. An injective hull of M is an essential injection $\iota : M \to I$ where I is an injective object of $\mathcal{A}$.

Let A be a commutative ring.

Lemma 2.14. *Every A-module has an injective hull.*

The proof above lemma makes use of the following souped-up version of Baer's criterion (well, along with Zorn's lemma and the fact that A-mod has enough injectives): an A-module I is injective if and only if every essential injection $I \to J$ is an isomorphism.

Taking injective hulls is *almost* functorial, in the following sense:

Proposition 2.15. *Let M, N be A-modules with injective hulls E_M, E_N respectively, and let $\varphi : M \to N$ be a homomorphism. Then there's a homomorphism $\psi : E_M \to E_N$ making the diagram*

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \downarrow & & \downarrow \\ E_M & \xrightarrow{\psi} & E_N \end{array} \tag{2.1}$$

commute and with the properties that

- *if φ is injective so is ψ.*
- *if φ is an essential injection then ψ is an isomorphism.*

In particular we see that E_M is unique up to (not necessarily unique) isomorphism.

Proposition 2.16. *Let $\mathfrak{p} \subset A$ be a prime ideal and let $E_\mathfrak{p}$ be an injective hull for $A/\mathfrak{p}$. Then*

- *$E_\mathfrak{p}$ is indecomposable.*
- *It is also the injective hull of $k(p)$ (over both A and $A_\mathfrak{p}$). Here $k(p)$ is the residue field of $\mathfrak{p}$.*

Moreover if A is noetherian then every indecomposable injective A-module arises in this way, so that every injective A-module I decomposes like $I \simeq \bigoplus_i E_{\mathfrak{p}_i}$ where the $\mathfrak{p}_i$ are an indexed set of primes in A (possibly with repetitions).

In the last statement we're using the (trivial) fact that every short exact sequence of injectives splits.

Here's the fun fact:

Proposition 2.17. *Let A be a noetherian local ring with maximal ideal m and residue field k, and let ω^* be a normalized dualizing complex on A. Recall this means*

$$\mathrm{Ext}^i_A(k, \omega^*) = \begin{cases} k & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $R\underline{\Gamma}_x(\omega^*)$ is quasi-isomorphic to an injective hull of k over A.

I'll just sketch the argument: let $k \to R\underline{\Gamma}_x(\omega^*)$ be the natural map, and recall that

$$R\underline{\Gamma}_x(\omega^*) \simeq R\mathrm{co}\lim_{n\to\infty} Hom_A(A/\mathfrak{m}^n, \omega^*)$$

so that $\mathscr{H}^i_x(\omega^*) \simeq \mathrm{co}\lim_{n\to\infty} \mathrm{Ext}^i_A(A/\mathfrak{m}^n, \omega^*)$ for all i. Now using the fact that $\mathrm{Ext}^i_A(k, \omega^*) = 0$ for $i \neq 0$ as a base case one shows that

$$\mathrm{Ext}^i_A(A/\mathfrak{m}^n, \omega^*) = 0 \text{ for all } n \text{ when } i \neq 0$$

The hard part is showing that $\mathrm{co}\lim_{n\to\infty} \mathrm{Hom}_A(A/\mathfrak{m}^n, \omega^*)$ is an injective hull for k over A. This is a theorem from Grothendieck's local cohomology paper.

Now let M be an A-module. Observe that there's a natural map

$$\Gamma_m M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M, \omega^*), \Gamma_m \omega^*)$$

(if $\sigma \in M$ is $\mathfrak{m}$-torsion then evaluating at σ sends a homomorphism $\varphi \in \mathrm{Hom}_A(M, \omega^*)$ to an "$\mathfrak{m}$-torsion element" $\varphi(\sigma)$ in $\Gamma_m \omega^*$. This extends to a natural transformation of derived functors

$$R\Gamma_m M \to \mathrm{Hom}_A(R\mathrm{Hom}_A(M, \omega^*), R\Gamma_m \omega^*)$$

Note that I've omitted an R on the Hom on the right hand side. This is because we know $\Gamma_m \omega^*$ is an injective hull for k- so in particular it's injective. The content of the local duality theorem is that this is an isomorphism:

Theorem 2.18. *The natural map $R\Gamma_m M \to \mathrm{Hom}_A(R\mathrm{Hom}_A(M, \omega^*), R\Gamma_m \omega^*)$ is a quasi-isomorphism.*

The local duality isomorphism above can be written in a more compact form if we write $D(M) := R\mathrm{Hom}_A(M, \omega^*)$ (so D is the dualizing functor associated to ω^*) and $I := R\Gamma_m \omega^*$ (so I is an injective hull of k). Then we have

$$R\Gamma_m M \simeq \mathrm{Hom}_A(D(M), I)$$

Taking cohomology we obtain isomorphisms

$$H^i_{\mathfrak{m}}(M) \simeq \mathrm{Hom}_A(\mathrm{Ext}^{-i}_A(M, \omega^*), I) \text{ for all } i$$

Applying $\mathrm{Hom}_A(-, I)$ one more time yields isomorphisms

$$\begin{aligned}\mathrm{Hom}_A(H^{-i}_{\mathfrak{m}}(M), I) &\simeq \mathrm{Hom}_A(\mathrm{Hom}_A(\mathrm{Ext}^i_A(M, \omega^*), I), I) \\ &\simeq \mathrm{Ext}^i_A(M, \omega^*)^{\wedge}\end{aligned}$$

where the $\wedge$ denotes $\mathfrak{m}$-adic completion. To see where the completion comes from, recall that

$$I \simeq R\Gamma_m \omega^* \simeq R\mathrm{co}\lim \mathrm{Hom}_A(A/\mathfrak{m}^n, \omega^*)$$

so that

$$\mathrm{Hom}_A(\mathrm{Hom}_A(\mathrm{Ext}^{-i}_A(M, \omega^*), I), I)$$
$$\simeq \mathrm{Hom}_A(\mathrm{Hom}_A(\mathrm{Ext}^{-i}_A(M, \omega^*), R\mathrm{co}\lim \mathrm{Hom}_A(A/\mathfrak{m}^n, \omega^*)), R\mathrm{co}\lim \mathrm{Hom}_A(A/\mathfrak{m}^n, \omega^*))$$

Supposing for a moment that we could just drop the R on $R\mathrm{co}\lim \mathrm{Hom}_A(A/\mathfrak{m}^n, \omega^*)$ (which isn't unreasonable, as this complex has cohomology only in degree 0) (one has to be more cautious than I'm being right now) we'd be left with

$$\mathrm{co}\lim_{n'} \lim_{n} \mathrm{Hom}_A(\mathrm{Hom}_A(\mathrm{Ext}^i_A(M, \omega^*), \mathrm{Hom}_A(A/\mathfrak{m}^n, \omega^*)), \mathrm{Hom}_A(A/\mathfrak{m}^{n'}, \omega^*)))$$

$$\simeq \mathrm{co}\lim_{n'} \lim_{n} \mathrm{Hom}_A(\mathrm{Hom}_A(\mathrm{Ext}^i_A(M, \omega^*) \otimes A/\mathfrak{m}^n, \omega^*) \otimes A/\mathfrak{m}^{n'}, \omega^*) \text{ by tensor-hom adjunction}$$

$$\simeq \mathrm{co}\lim_{n'} \lim_{n} \mathrm{Ext}^i_A(M, \omega^*) \otimes A/\mathfrak{m}^{n+n'} \simeq \mathrm{Ext}^i_A(M, \omega^*)^{\wedge} \text{ since } \omega^* \text{ is dualizing}$$

Example 2.19. Suppose A is Gorenstein of dimension d (e.g. it might be regular). Then $A[d]$ is a dualizing complex for A. Applying the above machinery we see that

$$R\Gamma_{\mathfrak{m}} A[d] \simeq \mathrm{co}\lim_n \mathrm{Ext}^d_A(A/\mathfrak{m}^n, A)$$

is an injective hull for k - call it I for short. Now for any A-module M with there are natural isomorphisms

$$H^i_{\mathfrak{m}}(M) \simeq \mathrm{Hom}_A(\mathrm{Ext}^{d-i}_A(M, A), I) \text{ and}$$
$$\mathrm{Ext}^i_A(M, A)^{\wedge} \simeq \mathrm{Hom}_A(H^{d-i}_{\mathfrak{m}}(M), I)$$

Example 2.20. Let's get even more explicit - let X be a smooth curve over a field k and let $x \in X$ be a closed point. Let $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ be the maximal ideal and let $k(x)$ be the residue field. Then we see that

$$\mathscr{H}^1_x(\mathscr{O}_X) \simeq \mathrm{co}\lim_n Ext^1_X(\mathscr{O}_{X,x}/\mathfrak{m}^n_x, \mathscr{O}_{X,x})$$

To calculate the right hand side let $t \in \mathfrak{m}$ be a local parameter, and observe that

$$0 \to \mathscr{O}_{X,x} \xrightarrow{t^n} \mathscr{O}_{X,x} \to \mathscr{O}_{X,x}/\mathfrak{m}^n \to 0$$

is a free resolution of $\mathscr{O}_{X,x}/\mathfrak{m}^n$, and so $Ext^1_X(\mathscr{O}_{X,x}/\mathfrak{m}^n_x, \mathscr{O}_{X,x})$ is the cokernel of $\mathscr{O}_{X,x} \xrightarrow{t^n} \mathscr{O}_{X,x}$, namely $\mathscr{O}_{X,x}/\mathfrak{m}^n$. Hence we see that

$$Ext^1_X(\mathscr{O}_{X,x}/\mathfrak{m}^n_x, \mathscr{O}_{X,x}) \simeq \mathrm{co}\lim_n \mathscr{O}_{X,x}/\mathfrak{m}^n$$

the colimit being taken over the maps $\mathscr{O}_{X,x}/\mathfrak{m}^n \xrightarrow{t} \mathscr{O}_{X,x}/\mathfrak{m}^{n+1}$.

Remark 2.21. The above discussion is even interesting on Spec$\mathbb{Z}$- if p is a prime, we see that

$$H^1_p(\mathbb{Z}) \simeq \operatorname{co}\lim_n \mathbb{Z}/p^n, \text{ a.k.a. the Prufer group.}$$

The right hand side can be viewed as the elements of $\mathbb{Q}/\mathbb{Z}$ which are annihilated by some power of p (or alternatively as the union of the p^n-th roots of unity over all n).

Now let X be a locally noetherian scheme with a dualizing complex ω^*. For each point $x \in X$, ω^* is a dualizing complex on Spec$\mathscr{O}_{X,x}$, and so there's a $d(x) \in \mathbb{Z}$ so

$$\operatorname{Ext}^i_{\mathscr{O}_{X,x}}(k(x), \omega^*) = \begin{cases} k(x) & \text{if } i = d(x) \\ 0 & \text{otherwise} \end{cases}$$

In this way we obtain a function $d : X \to \mathbb{Z}$. One can show that if $x \rightsquigarrow y$ is an immediate specialization, then $d(y) = d(x) + 1$, which means d is an example of

Definition 2.22. A **codimension function** on a scheme X is a function $d : X \to \mathbb{Z}$ with the property that whenever $x, y \in X$ and $x \rightsquigarrow y$ is an immediate specialization,

$$d(y) = d(x) + 1$$

It's not hard to show that if X admits a codimension function then it is *catenary*, which is to say that whenever $x, y \in X$ and $x \rightsquigarrow y$ is a specialization, *every* sequence if immediate specializations

$$x = x_0 \rightsquigarrow x_1 \rightsquigarrow \cdots x_r = y$$

has the same length. The interesting upshot here is

Proposition 2.23. *If X is a locally noetherian scheme that admits a dualizing complex, then X is catenary.*

Using the codimension function d associated to a dualizing complex ω^* on X we can define a descending filtration

$$X = Z^0_\omega \supset Z^1_\omega \supset Z^2_\omega \supset \dots$$

of X by specialization closed sets with the property that for every i, each point of $Z^i_\omega \setminus Z^{i+1}_\omega$ is maximal (with respect to specialization) - just set

$$Z^i_\omega := \{x \in X \,|\, d(x) \geq i\}$$

Essentially by the definition of d we obtain

Proposition 2.24. *The complex ω^* is Gorenstein for the filtration Z^*_ω.*

Definition 2.25. Let X be a locally noetherian scheme. A dualizing complex ω^* on X is **normalized** if and only if the associated codimension function d on X satisfies

$$d(x) = \operatorname{codim}(x, X)$$

which is to say, *d is the* codimension function on X. Hopefully you know what I am trying to say here. .

To see that this notion of Gorenstein-ness coincides with more traditional definitions, we can appeal to the following pleasant "TFAE":

Proposition 2.26. *Let A be a noetherian local ring with maximal ideal m and residue field k. Then the following are equivalent:*

(1) A is a dualizing complex on A.
(2) A has finite injective dimension.
(3) There's a $d \in \mathbb{Z}$ so that

$$\operatorname{Ext}^i_A(k, A) = \begin{cases} k & \text{if } i = d \\ 0 & \text{otherwise} \end{cases}$$

(4) There's a $d \in \mathbb{Z}$ so that

$$H^i_m(A) = \begin{cases} \textit{an injective } A-\textit{module} & \text{if} i = d \\ 0 & \textit{otherwise} \end{cases}$$

Definition 2.27. A locally noetherian scheme is **Gorenstein** if and only if all of its local rings are.

Putting things together we obtain:

Proposition 2.28. *Let X be a locally noetherian scheme with a dualizing complex ω^*. Then X is Gorenstein if and only if ω^* is an invertible sheaf (possibly shifted).*

Remark 2.29. Recall that when X is connected, the condition that ω^* is a shifted invertible sheaf is equivalent to the condition that ω^* is a tensor-invertible object of $D^b_c(X)$.

Proposition 2.30. *Let Y be a locally noetherian scheme and let $f : X \to Y$ be a flat morphisms of finite type. Recall that the functor $f^! : D^b_c(Y) \to D^b_C(X)$ is given by*

$$f^!\mathscr{F}^* := D_X(Lf^*D_Y\mathscr{F}^*) \textit{ where } D_X, D_Y \textit{ are the dualizing functors of } X, Y \textit{ respectively.}$$

The complex $f^!$ is supported in one degree if and only if all fibers X_y of f are Cohen-Macaulay. It's and invertible sheaf up to a shift if and only if all fibers are Gorenstein.

Remark 2.31. Using the above proposition, together with functoriality/base change properties of the shriek functor , I think we can prove an "inversion-of-adjunction" statement for both Cohen-Macaulay and Gorenstein singularities. Which is to say, if $f : X \to Y$ is a proper, flat morphism of finite type and Y is locally noetherian, then if X_y is a Cohen-Macaulay fiber of f there's a neighborhood $U \subset Y$ of y so that for all $z \in U$, X_z is Cohen-Macaulay. Similarly for the Gorenstein condition.

Residual complexes.

Definition 2.32. Let X be a locally noetherian scheme and let $x \in X$ be a point. Let I_x be the injective hull of $k(x)$ as a module over $\mathscr{O}_{X,x}$. Then set

$$J(x) := \iota_{x*}I_x \text{ where } \iota_x : \{x\} \to X \text{ is the inclusion}$$

Remark 2.33. $J(x)$ is an injective quasi-coherent sheaf supported on $\{\bar{x}\}$.

Applying local Grothendieck duality, we see that

Proposition 2.34. *If X is a locally noetherian scheme with a* normalized *dualizing complex ω^*, then for each $x \in X$,*

$$H^0_x(\omega^*) \simeq I_x, \textit{ and hence } R\underline{\Gamma}_x\omega^* \simeq J(x)$$

Generalizing this a bit,

Definition 2.35. A **residual complex** on X is a bounded-below complex K^* of injective quasi-coherent sheaves, with coherent cohomology, such that

$$\bigoplus_{p \in \mathbb{Z}} K^p \simeq \bigoplus_{x \in X} J(x)$$

Remark 2.36. Note that we are just comparing the K^p and the $J(x)$ at the level of quasi-coherent sheaves on X- we are not worrying about grading of complexes on either side of the above equation.

Example 2.37. A mild generalization of the above proposition shows that if ω^* is a dualizing complex on X and Z^*_ω is the induced codimension filtration of X, then the Cousin complex of ω^* with respect to Z^*_ω is a residual complex.

Residual complexes are an important technical tool in the "standard" developments of Grothendieck duality theory, and their usefulness stems from the following result.

Proposition 2.38. *Assume X is a locally noetherian scheme with a dualizing complex. Then the functor*

$$E : \{\textit{dualizing complexes } \omega^* \textit{ in } D^b_c(X)\} \to \{\textit{residual complexes } K^* \textit{ in } \mathcal{K}(I)\}$$

(where $\mathcal{K}(I)$ denotes the homotopy category of bounded below complexes of injectives) defined by

$$E(\omega^*) = \textit{ the Cousin complex of } \omega^* \textit{ with respect to the filtration } Z^*_\omega$$

is an equivalence of full subcategories. The inverse equivalence is obtained by restricting the usual functor $\mathcal{K}(I) \to D^+(X)$ to the full subcategory of residual complexes.

Moreover if $K^*, \tilde{K}^*$ are two residual complexes on X then there is a one-to-one correspondence

$$\mathrm{Hom}(K^*, \tilde{K}^*) \simeq \{\textit{specialization-compatible collections of homomorphisms } \psi_x : K^*_x \to \tilde{K}^*_x \textit{ in } D^+_c(\mathrm{Spec}\,\mathscr{O}_{X,x})\}$$

Here "specialization-compatible" means that whenever $x \rightsquigarrow y$ is a specialization, the map ψ_y is obtained from ψ_x by localization.

Remark 2.39. It's not totally clear to me what the induced map of cousin complexes $E(\omega^*) \to E(\tilde{\omega}^*)$ is supposed to be, as we can expect the filtrations Z^*_ω and $Z^*_{\tilde{\omega}}$ to differ by a shift... one would have to think about how the two filtrations relate and what exactly is the correct functoriality statement for the spectral sequence of a filtered space.

Finally, I will conclude this section with a fun example.

Example 2.40. The normalized dualizing complex of $\mathrm{Spec}\mathbb{Z}$ is $\mathbb{Z}[1]$. Indeed, $\mathrm{Spec}\mathbb{Z}$ is regular, so we know that its structure sheaf is a pointwise dualizing complex. Since $\mathrm{Spec}\mathbb{Z}$ is 1-dimensional, to normalize we must shift by 1; the end result is $\omega^* = \mathbb{Z}[1]$.

Note that for a point $p \in \mathrm{Spec}\mathbb{Z}$ we have

$$\mathrm{Ext}^i_{\mathbb{Z}_p}(k(p), \mathbb{Z}[1]) = \mathrm{Ext}^{i+1}_{\mathbb{Z}_p}(k(p), \mathbb{Z})$$

and so the codimension function d associated to $\mathbb{Z}[1]$ is

$$d(p) := \mathrm{codim}(p, \mathrm{Spec}\mathbb{Z}) - 1$$

It follows that the Cousin complex of ω^* looks like

$$0 \to H^0_\eta(\mathbb{Z}) \to \bigoplus_p H^1_p(\mathbb{Z}) \to 0$$

where $\eta \in \mathrm{Spec}\mathbb{Z}$ is the generic point and p runs over the closed points, aka the prime numbers, and the term $H^0_\eta(\mathbb{Z})$ sits in degree -1. Now, clearly $H^0_\eta(\mathbb{Z}) = \mathbb{Q}$ and as discussed above we have

$$H^1_p(\mathbb{Z}) = \mathrm{co}\lim_{n\to\infty} \mathbb{Z}/p^n$$

and so we can write the Cousin complex as

$$0 \to \mathbb{Q} \to \bigoplus_p \mathrm{co}\lim_{n\to\infty} \mathbb{Z}/p^n \to 0$$

Under the isomorphism

$$\mathbb{Q}/\mathbb{Z} \simeq \bigoplus_p \mathrm{co}\lim_{n\to\infty} \mathbb{Z}/p^n$$

(obtained by decomposing $\mathbb{Q}/\mathbb{Z}$ into the direct sum of its p-torsion components), this is just the usual injective resolution $0 \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ of $\mathbb{Z}$, shifted by 1.

3. Grothendieck duality

The first global question to address is when a (locally noetherian) scheme X admits a dualizing complex. We have seen that when X is Gorenstein it has a dualizing complex (e.g. the structure sheaf).

Let $f : X \to Y$ be a finite morphism, with Y a locally noetherian scheme. Recall that since f is a finite morphism, f_* has a *right* adjoint $f^\flat$, defined as

$$f^\flat \mathscr{G} := \underline{Hom}_Y(f_*\mathscr{O}_X, \mathscr{G})$$

More precisely, $f^\flat \mathscr{G}$ is the quasi-coherent sheaf on X associated to the quasi-coherent sheaf of $f_*\mathscr{O}_X$ modules $\underline{Hom}_Y(f_*\mathscr{O}_X, \mathscr{G})$. The derived adjunction statement is that for coherent sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y,

$$R\underline{Hom}_Y(f_*\mathscr{F}, \mathscr{G}) \simeq f_* R\underline{Hom}_Y(\mathscr{F}, Rf^\flat \mathscr{G})$$

Proposition 3.1. *If ω_Y^* is a dualizing complex on Y then $f^\flat \omega_Y^*$ is a dualizing complex on X.*

Proof. We must show that the natural map

$$\mathscr{O}_X \to R\underline{Hom}_X(R\underline{Hom}_X(\mathscr{O}_X, f^\flat \omega_Y^*), \omega_Y^*)$$

is an isomorphism. Applying the exact functor f_* we can view these as complexes on Y, and then adjunction gives

$$f_* R\underline{Hom}_X(R\underline{Hom}_X(\mathscr{O}_X, f^\flat \omega_Y^*), \omega_Y^*) \simeq$$

□

Theorem 3.2 (Grothendieck duality)**.** *Let $f : X \to Y$ be a proper morphism of finite-dimensional noetherian schemes admitting dualizing complexes ω_X^* and ω_Y^* respectively (for example X and Y could be schemes of finite type over k). Then for any object $\mathscr{F}^*$ in the bounded derived category $D_c^b(X)$ of X there is a natural isomorphism*

$$Rf_* R\underline{Hom}_X(\mathscr{F}^*, \omega_X^*) \simeq R\underline{Hom}_Y(Rf_*\mathscr{F}^*, \omega_Y^*) \text{ in } D_c^b(Y)$$

Remark 3.3. It is a theorem of Kawasaki [Kaw00] that a scheme X admits a dualizing complex if and only if it is locally embeddable in a Gorenstein scheme. From this perspective it's clear that any scheme of finite type over k has a dualizing complex (since locally it can be embedded in affine space).

References

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