K-theory of finite fields

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Abstract

To begin I will describe one accepted definition of the algebraic K-groups $K_i(R)$ of a ring R. The rest of the talk will consist of a sketch in broad strokes of Quillen's computation of the K-groups of a finite field k. The overarching strategy of the calculation is to relate the K-theory of k to the K-theory of $\mathbb{C}$ (which we know, due to Bott periodicity), and this involves creative use of the "Brauer lifting" construction from modular representation theory. Our discussion will end on a terrible cliffhanger.

1 K_0 of a ring

Let R be a topological ring (i.e. a ring in the category of spaces). Consider the isomorphism classes of finitely generated projective topological R-modules - Weibel denotes this set by $P(R)$.

Proposition 1.1. *The direct sum operation $\oplus$ makes $P(R)$ into an abelian monoid. If R is commutative, the tensor product operation $\otimes$ makes $P(R)$ into a commutative "semi-ring."*

Sketch. The only real subtlety: when R is commutative one must show the tensor product $M \otimes N$ of 2 finitely generated projectives M and N is finitely generated and projective (for the projective part use the $\otimes$-Hom adjunction). □

1.1 Grothendieck groups

Proposition 1.2. *The forgetful functor from abelian groups to abelian monoids has a left adjoint. So does the forgetful functor from commutative semi-rings to commutative rings. These functors are denoted*

$$K : \mathrm{AbMon} \to \mathrm{Ab} \text{ and similarly } K : \mathrm{ComSemiRng} \to \mathrm{ComRng}$$

$K(A)$ is known as the Grothendieck group (or ring) of A.

Sketch. Let A be an abelian monoid, and *define* $K(A)$ to be the quotient of $A \times A$ by the diagonal submonoid

$$\Delta = \{(a, a) \in A \times A \,|\, a \in A\}$$

I claim that $K(A)$ is a group, with inversion given by the "flip"

$$\tau : K(A) \to K(A) \text{ taking } [a, b] \mapsto [b, a]$$

Indeed, $[a, b] + [b, a] = [a + b, a + b] = [0, 0]$ (the idea is that elements of $K(A)$ are formal differences "$a - b$ of elements of A).

Now given a homomorphism of abelian monoids $\varphi : A \to B$ from A to an abelian *group* B, define a homomorphism of abelian groups

$$\tilde{\varphi} : K(A) \to B \text{ where } \tilde{\varphi}[a, b] = \varphi(a) - \varphi(b)$$

Check that this yields the required bijection

$$\mathrm{Hom}_{\mathrm{mon}}(A, B) \simeq \mathrm{Hom}_{\mathrm{grp}}(K(A), B)$$

□

Example 1. $K(\mathbb{N}) \simeq \mathbb{Z}$. In fact this isomorphism is adjoint to the usual inclusion $\mathbb{N} \subset \mathbb{Z}$.

Definition. Let R be a topological ring. The **Grothendieck group $K(R)$ of R** is the Grothendieck group of the abelian monoid $P(R)$. Thus (abusing notation here) "$K(R) = K(P(R))$." If R is commutative then $K(R)$ is a commutative ring with identity.

Example 2. Let F be a field. Then $K(F) \simeq \mathbb{Z}$, because $P(F) \simeq \mathbb{N}$ (every vector space is free!). Similarly if R is a PID then $K(R) \simeq \mathbb{Z}$ - this time one uses the structure theorem to show that every finitely generated projective R-module is already free.

Example 3. There are various reasons to allow for non-commutative rings R in the definition.

For instance, suppose G is a finite group and F is a field, and form the group ring FG. Suppose $|G| \neq 0 \in F$ (that is, the characteristic of k doesn't divide the order of G). In this situation a finitely generated projective FG-module is the same as a finite-dimensional representation of G over F. Certainly finitely generated FG-modules are the same as finite dimensional representations of G over F. The key point: by Maschke's theorem, our hypotheses on $|G|$ and $\mathrm{char} F$imply that every finitely generated FG-module is projective.

So, in this case the Grothendieck group $K(FG)$ coincides with the usual ring $R(G, F)$ of "virtual" finite dimensional representations of G over F.

1.2 Vector bundles

Remark. Let A be a noetherian commutative ring. Then it's a classic theorem that for a finitely generated A-module M the following are equivalent:

1. M is flat.

2. M is projective.

3. M is locally free.

Moreover the assignment $M \mapsto \tilde{M}$ gives an equivalence between the categories of finitely generated projective A-modules and locally free coherent sheaves on Spec$A$, compatible with direct sums and tensor products.

Note also that under these circumstances the assignment $E \to \Gamma(-, E)$ taking a geometric vector bundle $E \to \mathrm{Spec}A$ over the scheme Spec$A$ to its sheaf of sections gives an equivalence of categories between the geometric vector bundles over Spec$A$ and the locally free coherent sheaves on Spec$A$.

Hence $K(A)$ can be described as the Grothendieck ring of the isomorphism classes of locally free coherent sheaves on Spec$A$, or equivalently as the Grothendieck ring of the isomorphism classes of geometric vector bundles over Spec$A$.

Suppose in addition that A is regular. Then Exercise III.6.9 in Hartshorne's *Algebraic geometry* shows that the abelian group $K(A)$ we've defined above coincides with what people call "$G(A)$," the Grothendieck ring of the abelian monoid of isomorphism classes finitely generated (but not necessarily projective) A-modules.

This is a theorem due to Borel and Serre. They include the additional hypothesis that A is finite dimensional - not sure how we're supposed to get by without that. But Sandor says the theorem is true as stated in Hartshorne.

Motivated by the above discussion, we could say:

Definition. Let X be a noetherian scheme. Then the **Grothendieck ring** $K(X)$ **of** X is the Grothendieck ring of the commutative semi-ring of locally free coherent sheaves of $\mathscr{O}_X$ modules on X.

Remark. It's a theorem of Serre and Swan that if X is a compact Hausdorff space and $C(X, \mathbb{C})$ is the (topological) ring of continuous functions $f : X \to \mathbb{C}$, then sending a complex vector bundle $E \to X$ to its global sections $\Gamma(X, E)$ (this is a topological $C(X, \mathbb{C})$-module) defines an equivalence of categories between the complex vector bundles over X and the finitely generated projective topological $C(X, \mathbb{C})$-modules. Thus

$$K(C(X, \mathbb{C})) \simeq K(X)$$

where the right hand side is the classic Grothendieck ring of the abelian semi-ring of isomorphism classes of complex vector bundles over X.

The upshot is, we can think of finitely generated projective modules as vector bundles.

Note that $K(-)$ is a covariant functor on the category of topological rings. If $R \xrightarrow{\varphi} S$ is a homomorphism of topological rings, then extension of scalars defines a homomorphism of abelian monoids $P(R) \xrightarrow{-\otimes S} P(S)$ ($- \otimes S$ preserves projectives since it's left adjoint to an exact functor (restriction)). This induces a homomorphism of Grothendieck groups $K(R) \xrightarrow{\varphi_*} K(S)$.

Dualizing, $K(-)$ can be viewed as a contravariant functor on schemes, the induced maps corresponding to pullback.

2 The magic that happens with the rings $C(X, \mathbb{C})$

2.1 Classifying spaces

Definition. Let G be a topological group. A **classifying space** for G is a (pointed) topological space BG representing the "principal G-bundles" functor $\mathrm{Prin}(-G) : \mathrm{hCW} \to \mathrm{Set}_*$ from the homotopy category of CW-complexes to pointed sets assigning to a CW-complex X the isomorphism classes of principal G-bundles $P \to X$.

Proposition 2.1 (Milnor). *BG exists. In fact, there is a* functor $B : \mathrm{TopGrp} \to \mathrm{Top}$ *taking G to its classifying space BG. If G is a "countable CW group," this can be refined to a functor to CW complexes.*

In fact Milnor exhibited a functorial construction of the universal principal G-bundle $EG \to BG$ as $*^\infty G \to (*^\infty G)/G$ where $*^\infty G$ is the infinite join of G. When G is "countable CW" this coincides with the nerve of the topological category G.

A principal $GL(n, \mathbb{C})$-bundle is the same as a rank n complex vector bundles. For this reason $BGL(n, \mathbb{C})$ also represents the functor $\mathrm{Vect}^n(-, \mathbb{C})$. We have:

Theorem 2.2. *Let X be a finite CW complex (more generally it could be a compact Hausdorff space). Then pulling back the universal rank n complex vector bundle defines a natural bijection*

$$[X, BGL(n, \mathbb{C})] \xrightarrow{simeq} \mathrm{Vect}(n, \mathbb{C}) \simeq P(C(X, \mathbb{C}))_n$$

where the left hand side denotes homotopy classes of maps and the far right hand side denotes finitely generated projective topological $C(X, \mathbb{C})$-modules of rank n.

Now let $BGL(\mathbb{C})$ be the (homotopy) colimit of the classifying spaces $BGL(n,\mathbb{C})$ over the directed system of classifying maps $BGL(n,\mathbb{C}) \to BGL(n+1,\mathbb{C})$ corresponding to the identifications $\mathbb{C}^n \oplus \mathbb{C} \simeq \mathbb{C}^{n+1}$. Equivalently (but not obviously) this is the classifying space of the topological group $GL(\mathbb{C}) = \mathrm{colim}_n GL(n,\mathbb{C})$.

Theorem 2.3. *Let X be a finite CW complex (or even a compact Hausdorff space). Then there's a natural bijection*

$$[X, BGL(\mathbb{C}) \times \mathbb{Z}] \xrightarrow{simeq} K(X) \simeq K(C(X,\mathbb{C}))$$

Sketch. Suppose for simplicity that X is connected (otherwise, we can work one component at a time).

Say $f : X \to BGL(\mathbb{C}) \times \mathbb{Z}$ is a continuous map. Then the image of f lies in $BGL(\mathbb{C}) \times \{n\}$ for some $n \in \mathbb{Z}$. By compactness for suitably large $m \in \mathbb{N}$ f factors through some $BGL(m,\mathbb{C}) \times \{n\}$. Let $E(\gamma_m) \to BGL(m,\mathbb{C})$ be the universal m-plane bundle. Then pulling it back we obtain an m-plane bundle $f^*E(\gamma_m) \to X$, and from this we may define a class $[f^*E(\gamma_m)] - m + n \in K(X)$. Now define a function

$$[X, BGL(\mathbb{C}) \times \mathbb{Z}] \to K(X) \text{ sending } [f] \mapsto [f^*E(\gamma_m)] - m + n$$

One can show this is a bijection.

□

It's a fact that for any topological group G the space of loops ΩBG at a basepoint $b \in BG$ is (weakly) homotopy equivalent to G. The equivalence $\Omega BG \to G$ can be though of in terms of monodromy action of loops at b on the fiber $EG_b \simeq G$ of $EG \to BG$ (this can be precisified when we have enough structure on G). Now here's a crazy fact:

2.2 Extension to a cohomology theory

Theorem 2.4 (Bott periodicity)**.** *There's a canonical homotopy equivalence $BGL(\mathbb{C}) \times \mathbb{Z} \xrightarrow{\simeq} \Omega GL(\mathbb{C})$.*

For a proof that sheds some light on how Bott discovered this map, see the last section of Milnor's *Morse theory*. For a cleaner but far less enlightening proof, see the end of May's *More concise algebraic topology*. Here's a consequence (not claiming it's obvious, see section 4.3 of Hatcher's *Algebraic topology*).

Theorem 2.5 (Boosted up Bott periodicity)**.** $K(X)$ *extends to a (generalized) cohomology theory on the category of CW-complexes, which is periodic with period 2. That is, there's a generalized cohomology theory* K^* *on the category of CW-complexes so that* $K \simeq K^0$*, and* K^* *comes with natural periodicity isomorphisms* $K^* \simeq K^{*-2}$*.*

One way to think about this theorem: a restricted class of rings (namely those of the form $C(X, \mathbb{C})$ for some compact Hausdorff space X, which turn out to be exactly the commutative C^*-algebras over $\mathbb{C}$) we've found a sequence of abelian group valued functors K_n indexed over $n \in \mathbb{Z}$ (namely the functors $X \mapsto K_n C(X, \mathbb{C}) = K^n(X)$) so that K_0 coincides with the Grothendieck group functor K defined above.

The question is: to what extent can this be generalized to a broader class of rings? I.e. can one define "higher algebraic K-theory groups $K_n(R)$," say for $n \in \mathbb{N}$ or $\mathbb{Z}$, of a topological ring R?

3 Quillen's definition of higher algebraic K-theory

If R is a topological ring, then for each $n \in \mathbb{N}$ the general linear group $GL(n, R)$ is a topological group (it can be viewed as an open subset of $R^{n \times n}$), with an associated classifying space $BGL(n, R)$. In the limit as n goes to infinity we obtain the topological group $GL(R) = \mathrm{colim}_n GL(n, R)$ along with its classifying space $BGL(R)$. Proceeding by direct analogy with the case in which $R = \mathbb{C}$, we'd build the space $BGL(R) \times K_0(R)$ **NOTE: explain why this is silly but not that silly** and try to define:

$$''K_n(R) := \pi_n(BGL(R) \times K_0(R))''$$

Here's why that would be directly analogous: Since $\mathbb{C} = C(\mathrm{pt}, \mathbb{C})$,

$$K_n(\mathbb{C}) = K_n(C(\mathrm{pt}, \mathbb{C})) = K^{-n}(\mathrm{pt})$$

where I'm obligated to flip the n to a $-n$ to account for variance issues (apologies!). Now by the suspension isomorphism in K-theory

$$K^{-n}(\mathrm{pt}) = \tilde{K}^0(S^n) = \pi_n(BGL(\mathbb{C}) \times \mathbb{Z})$$

At this point we run into the following major road block, embodied in the following three propositions:

Proposition 3.1. *Let* X *be a pointed CW complex such that the functor* $[-, X]_*$ *on the homotopy category of pointed CW complexes takes values in the category of abelian groups (obviously this statement should be made more precise). Then* X *is an abelian H-group, i.e. an abelian group object in the homotopy category of CW complexes.*

Proposition 3.2. *Let X be a CW H-space (i.e. a magma in the category of CW complexes). Then the group structure on $\pi_1(X, e)$ (where $e \in X$ is the homotopy identity) comes from the H-space operation on X. Moreover $\pi_1(X, e)$ is abelian.*

Remark. On a quite unrelated note, if G is a compact Lie group $\pi_1(G, e)$ is a finitely generated abelian group with a neat description in terms of lattices of roots in the Lie algebra $\mathfrak{g}$.

Unfortunately, the weak homotopy equivalence $\Omega BGL(R) \simeq GL(R)$ shows that $\pi_1 BGL(R) \simeq \pi_0 GL(R)$, and there's no reason for this to be abelian - for instance if R is discrete this is just $GL(R)$, which tends to be all sorts of non-abelian!

3.1 The plus construction

The brilliant idea of Quillen is to abelianize the situation with what's now known as the "plus construction:"

Proposition 3.3. *Let X be a pointed CW complex, and let $N \subset \pi_1(X, x_0)$ be a perfect normal subgroup (i.e. N is normal and $[N, N] = N$). Then there is a universal pointed CW complex X^+ with a pointed map $f : X \to X^+$ such that $f_* : \pi_1(X, x_0) \to \pi_1(X^+, x_0)$ which is surjective with kernel N. By universal I mean given any other pointed map $g : X \to Y$ to a pointed CW complex Y with $N \subset \ker g_*$, there's a map $\tilde{g} : X^+ \to Y$, unique up to homotopy, making the following diagram commute:*

$$\begin{array}{ccc} X & \xrightarrow{f} & X^+ \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ Y & \xrightarrow{=} & Y \end{array} \tag{1}$$

Perhaps the most surprising thing is that we can take $X \xrightarrow{f} X^+$ to be the inclusion of X into a CW complex obtained from it by attaching cells of dimensions 2 and 3.

From here on out I'm going to restrict to the case of a discrete ring R. I hear people do think about analogues of the forthcoming definition of higher K-theory for rings with topologies - but I don't know how all that works.

Proposition 3.4 (Whitehead). *Let R be a ring, and let $E(R) \subset GL(R)$ be the subgroup generated by the* elementary *matrices, i.e. those which differ from the identity at precisely 1 off-diagonal entry.*

Then $[GL(R), GL(R)] = E(R)$. In particular, $[GL(R), GL(R)]$ is a perfect normal subgroup.

Sketch. One shows that every commutator in $GL(n,R)$ is a product of elementary matrices in $GL(2n,R)$, by direct calculation.

□

Thus if R is a (discrete) ring, the commutator $[GL(R), GL(R)] \subset GL(R) = \pi_1 BGL(R)$ is a perfect normal subgroup, and so one can form the plus construction $BGL(R)^+$ on $BGL(R)$ with respect to $[GL(R), GL(R)]$.

Theorem 3.5. *The maps $BGL(m,R) \times BGL(n,R) \to BGL(m+n,R)$ induce an operation $BGL(R)^+ \times BGL(R)^+ \to BGL(R)^+$ making $BGL(R)^+$ an abelian group object in the homotopy category. In fact one can show it's an infinite loop space.*

So, the plus construction obliterates the aforementioned roadblock.

Definition. The **higher algebraic K-groups** $K_i(R)$ of R are given by

$$K_i(R) := \pi_i(BGL(R)^+ \times K_0(R)) \text{ for } i \in \mathbb{N}$$

or equivalently by the previous definition of $K(R)$ when $i = 0$ and

$$K_i(R) := \pi_i BGL(R)^+ \text{ for } i \in \mathbb{N}, i > 0$$

One thing that's worth pointing out: before applying the plus construction we have

$$\pi_i BGL(R) = \begin{cases} BGL(R) & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

But the plus construction can have all sorts of interesting effects on homotopy groups.

On the other hand one can show that the canonical map $BGL(R) \to BGL(R)^+$ induces isomorphisms on any generalized (co)homology theory (basically because the universal mapping property ensures that for any H-space Y (in particular any infinite loopspace appearing in a spectrum representing a cohomology theory) pullback gives a natural bijection $[BGL(R), Y] \simeq [BGL(R)^+, Y]$). It follows that the singular homology of $BGL(R)^+$ with coefficients in any abelian group A is computed as

$$H_*(BGL(R)^+; A) \simeq H_*(GL(R); A), \text{ the group homology of } GL(R) \text{ with coefficients in } A$$

where A has trivial $GL(R)$-action. I think you can also allow for interesting $GL(R)$-action on A.

Example 4. By Quillen's definition $K_1(R) = \pi_1 BGL(R)^+$, and by its defining property $BGL(R)^+$ comes with a map

$$f : BGL(R) \to BGL(R)^+ \text{ so that } f_* : \pi_1 BGL(R) \to \pi_1 BGL(R)^+$$

is surjective with kernel exactly $[GL(R), GL(R)]$. Thus

$$K_1(R) \simeq GL(R)_{\text{ab}} \text{ the abelianization of } GL(R)$$

The above definition defines a covariant functor from the category of discrete rings to abelian groups; if $\varphi : R \to S$ is a homomorphism of rings, it induces group homomorphisms $GL(n, R) \to GL(n, S)$ for all n and hence a homomorphism $GL(R) \to GL(S)$ (which necessarily takes commutators to commutators). This yields a classifying map $BGL(R) \to BGL(S)$, and applying the plus construction yields a map $BGL(R)^+ \to BGL(S)^+$, which finally induces homomorphisms of homotopy groups $\varphi_* : K_i(R) = \pi_i BGL(R)^+ \to \pi_i BGL(S)^+ = K_i(S)$. One can show that the functors K_i are compatible with filtered colimits.

4 The K-theory of finite fields

One (rare) case in which a complete (and beautiful) determination of higher algebraic K groups is possible occurs when R is a finite field. The following calculation is due to Quillen:

Theorem 4.1. *Let p be a prime, let q be a power of p and let k be a finite field with q elements. Then the algebraic K-groups of k are computed as follows: $K_0(k) = \mathbb{Z}$, and for $i > 0$*

$$K_{2i-1}(k) \simeq \mathbb{Z}/(q^i - 1) \text{ and } K_{2i}(k) = 0$$

He also shows that if $k \to k'$ is a homomorphism of finite fields the induced map $K_*(k) \to K_*(k')$ is injective, and that if $\mu : k \to k$ is the Frobenious automorphism sending $x \to x^p$, then for each i the induced automorphism

$$\mu_* : K_{2i-1}(k) \to K_{2i-1}(k) \text{ is multiplication by } p^i$$

The overarching strategy of this computation is to relate the K-theory of k to the K-theory of $\mathbb{C}$ (which we know, due to Bott periodicity). To achieve this, Quillen uses the "Brauer lift" of modular representation theory. In more detail:

4.1 Brauer lifting

Let $\bar{k}$ be an algebraic closure of k, and choose an embedding $\phi : \bar{k}^\times \to \mathbb{C}^\times$ identifying non-0 elements of $\bar{k}$ with roots of unity with order prime to p. According to Quillen, "nearly everything we do from now on will depend on this choice, but in a way that is well understood in the theory of etale cohomology."

Definition. Let G be a finite group and let V be a finite-dimensional representation of G over $\bar{k}$. The **modular character of V with respect to ϕ** is the complex function $\chi_V : G \to \mathbb{C}$ given by

$$\chi_V(g) := \sum_i \phi(\alpha_i) \text{ where}$$

$\alpha_i \in \bar{k}^\times$ are the eigenvalues of the automorphism $g : V \to V$, counted by multiplicity.

Remark. Here's one way to think about χ_V: taking the eigenvalues of the automorphisms $g : V \to V$ defines a function

$$G \to SP^n\bar{k}^\times \text{ taking } g \mapsto [\alpha_1, \ldots, \alpha_n]$$

where the α_i are the eigenvalues of g, counted by multiplicity. Here $n = \dim_k V$, and $SP^n\bar{k}^\times$ is the n-fold symmetric product of $\bar{k}^\times$. Now the embedding $\phi : \bar{k}^\times \to \mathbb{C}^\times$ induces an embedding $SP^n\phi : SP^n\bar{k}^\times \to SP^n\mathbb{C}^\times$, and of course addition defines a function $SP^n\mathbb{C}^\times \to \mathbb{C}$ taking $[c_1, \ldots, c_n] \mapsto \sum_i c_i$.

It is a theorem of Green (see his paper *The characters of the finite general linear groups*) that the function χ_V is in fact the character of a virtual complex representation of G, i.e. $\chi_V \in R(G, \mathbb{C})$. In fact the above construction defines a homomorphism of representation rings

$$R(G, \bar{k}) \to R(G, \mathbb{C}) \text{ taking } V \mapsto \chi_V$$

provided we define $R(G, \bar{k})$ to be the Grothendieck ring of the commutative semiring $\mathrm{Rep}(G, \bar{k})$ of finite-dimensional representations of G over $\bar{k}$, under direct sum and tensor product. **Note**: if $\mathrm{char} k \not| |G|$ (so that Maschke's theorem ensures every representation is projective) then this is just $K(\bar{k}G)$- but generally (in the modular case) the two will be different. To see this, suppose V, W are two finite-dimensional representations of G over k, say of dimensions m, n. If $g \in G$ and $[\alpha_i] \in SP^m\bar{k}^\times, [\beta_i] \in SP^n\bar{k}^\times$ are the eigenvalues of the automorphisms $V \xrightarrow{g} V, W \xrightarrow{g} W$ respectively, then it's not hard to show that the eigenvalues of $V \oplus W \xrightarrow{g \oplus g} V \oplus W$ and $V \otimes W \xrightarrow{g \otimes g} V \otimes W$

are given by $[\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n] \in SP^{m+n}\bar{k}^\times$ and $[\alpha_1\beta_1, \ldots, \alpha_m\beta_n] \in SP^{mn}\bar{k}^\times$. It follows that

$$\chi_{V\oplus W}(g) = \sum_i \phi(\alpha_i) + \sum_i \phi(\beta_i) = \chi_V(g) + \chi_W(g) \text{ and}$$

$$\chi_{V\otimes W}(g) = \sum_{i,j} \phi(\alpha_i\beta_j) = (\sum_i \phi(\alpha_i))(\sum_j \phi(\beta_j)) = \chi_V(g)\chi_W(g)$$

χ_V is generally known as the **Brauer lift of** V.

Now let V be a finite-dimensional representation of G over k; then $V \otimes_k \bar{k}$ is a finite-dimensional representation of G over $\bar{k}$, and the Brauer lifting construction produces a character $\chi_{V\otimes_k\bar{k}} \in R(G, \mathbb{C})$.

Proposition 4.2. *$\chi_{V\otimes_k\bar{k}}$ is fixed by the Adams operation $\psi^q : R(G, \mathbb{C}) \to R(G, \mathbb{C})$. Moreover the homomorphism $R(G, k) \to R(G, \mathbb{C})$ obtained by Brauer lifting has image precisely the subring $R(G, \mathbb{C})^{\psi^q} \subset R(G, \mathbb{C})$ of invariants of ψ^q.*

Remark. Here's one definition of the operations ψ^k, for $i \in \mathbb{N}$: observe that if V and W are representations of G over $\mathbb{C}$ then taking exterior powers we have the identities

$$\bigwedge^*(V \oplus W) \simeq \bigwedge^*(V) \otimes \bigwedge^*(W), \text{ i.e. } \bigwedge^k(V \oplus W) \simeq \bigoplus_{i+j=k} \bigwedge^i(V) \otimes \bigwedge^j(W) \text{ for all } k$$

This implies that the assignment $V \mapsto \lambda(V, z) := \sum_k [\bigwedge^k V]z^k \in R(G, \mathbb{C})[[z]]$ taking a representation V to its "exterior power series" induces a homomorphism $R(G, \mathbb{C}) \to 1+zR(G, \mathbb{C})[[z]] \subset R(G, \mathbb{C})[[z]]^\times$ (we land in the multiplicative group of units because every exterior power series has constant coefficient 1). Now it's a purely algebraic fact that the assignment $f(z) \mapsto -zd\log f(-z)$ (where we need to interpret $d\log$ formally, in terms of power series) gives a group homomorphism $1 + zR(G, \mathbb{C})[[z]] \to R(G, \mathbb{C})[[z]]$, where the left hand side is multiplicative, and the right hand side is additive. In this way the **Adams power series**

$$\psi(\alpha, z) := \dim\alpha - zd\log\lambda(\alpha, -z) \in R(G, \mathbb{C})[[z]]; \text{ for } \alpha \in R(G, \mathbb{C})$$

defines a homomorphism $\psi : R(G, \mathbb{C}) \to R(G, \mathbb{C})[[z]]$. The ψ^k are now defined implicitly by the expansion

$$\psi(\alpha, z) = \sum_k \psi^k(\alpha)z^k \text{ for } \alpha \in R(G, \mathbb{C})$$

One can check that the ψ^k are natural ring homomorphisms.

It's a useful fact that for a 1-dimensional representation L of G, we have $\psi^k L = L^k$, the k-fold tensor power.

The pleasant thing is that the above construction works perfectly well with complex vector bundles $E \to X$ on a compact Hausdorff space X in place of representations, and in this way one defines Adams operations on $K^0(X)$ - they're natural ring homomorphisms here too, and for a line bundle $L \to X$, we have $\psi^k L = L^k$.

Proposition 4.3. *Let V be a finite-dimensional complex representation of G, and let χ_V be its character, viewed as a complex function on G. Then*

$$\psi^k(\chi_V)(g) = \chi_V(g^k) \in \mathbb{C} \textit{ for } g \in G$$

so if we use characters to identify $R(G, \mathbb{C})$ with a subring of $C(G, \mathbb{C})$, ψ^k corresponds to pullback over the kth power map

$$G \xrightarrow{\wedge k} G \textit{ taking } g \mapsto g^k$$

Sketch. The idea is to restrict to an abelian subgroup $A \subset G$, where V splits up as a direct sum $V \simeq \oplus_i L_i$ of 1-dimensional representations L_i. As remarked above, on a 1-dimensional representation L of A, we have $\psi^k L = L^k$, the k-th tensor power. Now observe that $\chi_{L^k}(g) = \chi_L(g^k)$. □

Proof of proposition. Suppose $g \in G$. Since V is a representation over k, the eigenvalues $\alpha_i \in \bar{k}^\times$ of $g : V \otimes_k \bar{k} \to V \otimes_k \bar{k}$ are *permuted* by the Frobenius endomorphism $\mu : \bar{k} \to \bar{k}$ sending $x \to x^q$ (at a nitty gritty level, the α_i are roots of a characteristic polynomial with coefficients in k). Hence the eigenvalues α_i^q of g^q are simply a permutation of the α_i, and from this we see that

$$\psi^q \chi_{V \otimes \bar{k}}(g) = \chi_{V \otimes \bar{k}}(g^q) = \sum_i \phi(\alpha_i^q) = \sum_i \phi(\alpha_i) = \chi_{V \otimes \bar{k}}(g)$$

I won't prove the stronger statement.

□

4.2 From characters to classifying maps

Now let $n \in \mathbb{N}$ be a positive integer, take $G = GL(n, k)$. We have a standard representation of $GL(n, k)$ over k (namely, k^n) and extending scalars we obtain a representation $V = \bar{k}^n$ of $GL(n, k)$ over $\bar{k}$. Now the Brauer lift construction produces a character

$$\chi_n := \chi_{\bar{k}^n} \in R(GL(n, k), \mathbb{C})$$

The previous proposition shows that χ_n is fixed by the Adams operation ψ^q on $R(GL(n,k),\mathbb{C})$.

It's clear from functoriality that a representation of $GL(n,k)$ over $\mathbb{C}$, from which we can define a homomorphism $GL(n,k) \to GL(\mathbb{C})$, induces a map $BGL(n,k) \to BGL(\mathbb{C})$. The pleasant fact is that even *virtual representations* of $GL(n,k)$ induce maps $BGL(n,k) \to BGL(\mathbb{C})$. In fact, let's take a brief digression:

If G is any compact Lie group and V is an n-dimensional finite-dimensional complex representation of G, one can define a rank n complex vector bundle $EG \times_G V \to BG$ over a classifying space BG. In fact the construction gives a functor $\mathrm{Rep}(G,\mathbb{R}) \xrightarrow{EG_n \times_G -} \mathrm{Vect}(BG_n,\mathbb{R})$ compatible with direct sums and tensor products, and so it induces a ring homomorphism $R(G,\mathbb{C}) \xrightarrow{\varphi} K^0(BG)$ compatible with Adams operations. Let $I(G,\mathbb{C}) \subset R(G,\mathbb{C})$ be the augmentation ideal, i.e. the kernel of the homomorphism $R(G,\mathbb{C}) \to R(\{e\},\mathbb{C}) \simeq \mathbb{Z}$ induced by the inclusion of the identity $e \in G$. It's a special case of the Atiyah-Segal completion theorem that:

Theorem 4.4. *The homomorphism $\varphi : R(G,\mathbb{C}) \to K^0(BG)$ factors through an isomorphism of rings $\hat{R}(G,\mathbb{C}) \to K^0(BG)$ compatible with Adams operations, where $\hat{R}(G,\mathbb{C})$ denotes the $I(G,\mathbb{C})$-adic completion of $R(G,\mathbb{C})$. Also, $K^{-1}(BG) = 0$.*

See Atiyah and Segal's *Equivariant K-theory and completion* or Adams et. al.'s *A generalization of the Atiyah-Segal completion theorem.*

So, χ_n yields a class in $K^0(BGL(n,k))$, which by abuse of notation I'll also call χ_n, corresponding to a (homotopy class of) map $BGL(n,k) \to BGL(\mathbb{C}) \times \mathbb{Z}$ - forgetting about the copy of $\mathbb{Z}$ we can extract a map

$$f_n : BGL(n,k) \to BGL(\mathbb{C})$$

It's relatively straightforward to check that these maps f_n are compatible with the directed system

$$BGL(n,k) \to BGL(n+1,k) \text{ for } n \in \mathbb{N}$$

(again, these maps classify the usual inclusion $GL(n,k) \subset GL(n+1,k)$ as the upper left block), and so they define an element $(f_n) \in \lim_n [BGL(n,k), BGL(\mathbb{C})]$. To show that they define a unique homotopy class of maps $BGL(k) \xrightarrow{f} BGL(\mathbb{C})$, we may appeal to the following

Theorem 4.5 (Milnor)**.** *Let $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$ be a filtration of a pointed CW complex X, and let Y be another pointed CW complex. Then the map $[X,Y]_* \to \lim_n [X_n, Y]_*$ obtained by restricting maps to the subcomplexes fits into a natural short* *exact sequence of pointed sets*

$$0 \to \lim_n^1 [\Sigma X_n, Y]_* \to [X, Y]_* \to \lim_n [X_n, Y]_* \to 0$$

In particular taking $Y = E_i$ to be the ith term in a representing Ω-spectrum for a (reduced) generalized cohomology theory $\tilde{E}^*$ on the homotopy category of pointed CW complexes, we have a natural short exact sequence of abelian groups

$$0 \to \lim_n^1 \tilde{E}^{i-1}(X_n) \to \tilde{E}^i(X) \to \lim_n \tilde{E}^i(X_n) \to 0$$

There's also an analogous "unpointed" version of this theorem. Returning to the matter at hand, we have a Milnor exact sequence

$$\lim_n^1 K^{-1}(BGL(n,k)) \to K^0(BGL(k)) \to \lim_n K^0(BGL(n,k)) \to 0$$

and by the Atiyah-Segal theorem every group in the inverse system $K^{-1}(BGL(n,k))$ is 0. Hence the map $[BGL(k), BGL(\mathbb{C})] \to \lim_n [BGL(n,k), BGL(\mathbb{C})]$ is an isomorphism.

5 Homotopy fixed points and the space $F\psi^q$

Since the characters $\chi_n \in R(GL(n,k), \mathbb{C})$ and the associated classes in K-theory $\chi_n \in K^0(BGL(n,k))$ are fixed by the Adams operation ψ^q. The idea is that the maps $f_n : BGL(n,k) \to BGL(\mathbb{C})$ and their limit $f : BGL(k) \to BGL(\mathbb{C})$ should also be "fixed by ψ^q" in some sense, which will now be precisified.

Let X be spaces, say a CW complex, and let $\varphi : X \to X$ be a continuous self map of X. Then one can characterize the subspace of fixed points

$$X^\varphi = \{x \in X \mid \varphi(x) = x\} \subset X$$

by its position in the top left corner of the following cartesian diagram:

$$\begin{array}{ccc} X^\varphi & \longrightarrow & X \\ {\scriptstyle\iota}\big\downarrow & & {\scriptstyle\Delta}\big\downarrow \\ X & \xrightarrow{\mathrm{id}\times\varphi} & X \times X \end{array} \tag{2}$$

Here ι is the inclusion and Δ is the diagonal. The **homotopy fixed point space** hX^φ **of** φ is obtained by *first* replacing Δ with a homotopy equivalent fibration, and then forming the analogous cartesian diagram. An explicit fibration homotopy equivalent to Δ is given by the map

$$X^I \xrightarrow{\mathrm{ev}_0\times\mathrm{ev}_1} X\times X \text{ taking } \gamma\mapsto(\gamma(0),\gamma(1))$$

and so the resulting diagram looks like

$$\begin{tikzcd} hX^\varphi \arrow[r] \arrow[d] & X^I \arrow[d, "\mathrm{ev}_0\times\mathrm{ev}_1"'] \\ X \arrow[r, "\mathrm{id}\times\varphi"] & X\times X \end{tikzcd} \tag{3}$$

So, hX^φ has an explicit description as the space

$$\{(x,\gamma)\in X\times X^I \,|\, \gamma(0)=x, \gamma(1)=\varphi(x)\}\subset X\times X^I$$

of points $x\in X$ together with paths $\gamma: I\to X$ from x to $\varphi(x)$.

By the Yoneda lemma, the natural transformation $K^0\xrightarrow{\psi^q}K^0$ given by the Adams operation must be represented by a (unique up to homotopy) map $BGL(\mathbb{C})\times\mathbb{Z}\xrightarrow{\psi^q} BGL(\mathbb{C})\times\mathbb{Z}$, and as one can check this is the identity on the $\mathbb{Z}$ factor we can restrict our attention to the map $BGL(\mathbb{C})\xrightarrow{\psi^q}BGL(\mathbb{C})$. Now define $F\psi^q = hBGL(\mathbb{C})^{\psi^q}$ to be the homotopy fixed point space of ψ^q ($F\psi^q$ stands for "fixed points of ψ^q).

A brief digression: suppose $\varphi: X\to Y$ is a continuous map of pointed CW complexes. The literal fiber $\varphi^{-1}(y_0)$ of φ over the basepoint $y_0\in Y$ fits into the cartesian diagram

$$\begin{tikzcd} \varphi^{-1}(y_0) \arrow[r] \arrow[d, "\varphi"'] & X \arrow[d, "\varphi"'] \\ \{y_0\} \arrow[r, "\iota"] & Y \end{tikzcd} \tag{4}$$

Proceeding along the lines of our construction of homotopy fixed points, we define the **homotopy fiber** $h\varphi^{-1}(b)$ **of** φ by first replacing the inclusion $\{y_0\}\xrightarrow{\iota}Y$ with an equivalent fibration, and then forming the analogous commutative diagram. An explicit fibration equivalent to ι is given by

$$Y^I_{y_0}\xrightarrow{\mathrm{ev}_1}Y \text{ sending } \gamma\mapsto\gamma(1)$$

where $Y^I_{y_0}$ is the space of paths $\gamma: I\to Y$ with $\gamma(0)=y_0$. So, the resulting cartesian diagram looks like

$$\begin{tikzcd} h\varphi^{-1}(y_0) \arrow[r] \arrow[d, "\varphi"'] & X \arrow[d, "\varphi"'] \\ Y^I_{y_0} \arrow[r, "\mathrm{ev}_1"] & Y \end{tikzcd} \tag{5}$$

So, $h\varphi^{-1}(y_0)$ has an explicit description as the space

$$\{(x,\gamma) \in X \times Y_{y_0}^I \mid \gamma(1) = \varphi(x)\} \subset X \times Y_{y_0}^I$$

of points $x \in X$ together with paths $\gamma : I \to Y$ from y_0 to $\varphi(x)$.

Quillen now provides an alternative description of $F\psi^q$ which will be more useful in forthcoming calculations. Observe that the operation $1-\psi^q$ also defines a natural transformation $K^0 \to K^0$, and moreover for any CW complex X

$$1 - \psi^q : K^0(X) \to K^0(X) \text{ sending } \alpha \mapsto \alpha - \psi^q\alpha$$

is an abelian group homomorphism (since id and ψ^q are). It follows (again by repeated application of the Yoneda lemma) that a self map $1 - \psi^q : BGL(\mathbb{C}) \to BGL(\mathbb{C})$ representing $1 - \psi^q$ is an H-map, i.e. it's compatible with the H-space structure of $BGL(\mathbb{C})$ (in fact, it should be an "infinite loop map").

Theorem 5.1 (Quillen)**.** *(i) There's a canonical homotopy equivalence $F\psi^q \simeq h(1-\psi^q)^{-1}(b)$. That is, $F\psi^q$ can also be described as the homotopy fiber of the map $1-\psi^q : BGL(\mathbb{C}) \to BGL(\mathbb{C})$.*

(ii) If X is a CW complex with $K^{-1}(X) = [X, GL(\mathbb{C})] = 0$, then the natural map

$$[X, F\psi^q] \to [X, BGL(\mathbb{C})]^{\psi^q} \textit{ is an isomorphism.}$$

Note that if X is connected then the right hand side is just $\tilde{K}^0(X)^{\psi^q}$.

(iii) $F\psi^q$ is a simple space (i.e. $\pi_1 F\psi^q$ acts trivially on the higher homotopy groups $\pi_* F\psi^q$), in fact an infinite loop space, and

$$\pi_{2i}F\psi^q = 0 \textit{ for } i \in \mathbb{N}, \pi_{2i-1}F\psi^q \simeq \mathbb{Z}/(q^i - 1) \textit{ for } i \in \mathbb{N}, i > 0$$

I'm not going to prove (i), (ii) or the first part of (iii). I'll sketch the computation of the homotopy groups of $F\psi^q$.

Computation. By part (i), $F\psi^q$ fits into a fibration

$$F\psi^q \xrightarrow{\iota} BGL(\mathbb{C}) \xrightarrow{1-\psi^q} BGL(\mathbb{C})$$

Associated to this fibration is a long exact sequence of homotopy groups

$$\ldots \xrightarrow{\partial} \pi_{i+1}F\psi^q \xrightarrow{\iota_*} \pi_{i+1}BGL(\mathbb{C}) \xrightarrow{1-\psi^q} \pi_{i+1}BGL(\mathbb{C}) \xrightarrow{\partial} \pi_i F\psi^q \xrightarrow{\iota_*} \pi_i BGL(\mathbb{C}) \xrightarrow{1-\psi^q} \ldots$$

Now observe that the homotopy groups of $BGL(\mathbb{C})$ are given by

$$\pi_i BGL(\mathbb{C}) = \tilde{K}^0(S^i) = \begin{cases} \mathbb{Z} & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases}$$

Thus the above long exact sequence breaks into short exact sequences

$$0 \to \pi_{2i} F\psi^q \xrightarrow{\iota_*} \pi_{2i} BGL(\mathbb{C}) \xrightarrow{1-\psi^q} \pi_{2i} BGL(\mathbb{C}) \to 0 \text{ and}$$

$$0 \to \pi_{2i} BGL(\mathbb{C}) \xrightarrow{1-\psi^q} \pi_{2i} BGL(\mathbb{C}) \xrightarrow{\partial} \pi_{2i-1} F\psi^q \to 0$$

where the homomorphisms $1 - \psi^q$ correspond to the maps

$$1 - \psi^q : \tilde{K}^0(S^{2i}) \to \tilde{K}^0(S^{2i})$$

We'll need the following facts: if $L \to \mathbb{C}P^1 \simeq S^2$ is the line bundle corresponding to $\mathscr{O}(1)$, then $L - 1$ serves as a canonical generator for $\tilde{K}^0(S^2)$. In fact there's a ring isomorphism $K^0(S^2) \simeq \mathbb{Z}[L]/(L-1)^2$, under which $\tilde{K}^0(S^2)$ corresponds to the ideal $(L-1) \subset \mathbb{Z}[L]/(L-1)^2$. The Bott periodicity isomorphisms $\beta : \tilde{K}^0(S^{2i}) \to \tilde{K}^0(S^{2(i+1)})$ can be described as "external multiplication by $L-1$, and so the i-fold external product of $L-1$, which I'll write like $\wedge^i(L-1) \in \tilde{K}^0(S^{2i})$, serves as a canonical generator.

Lemma 5.2. *$\psi^q(L-1) = q(L-1) \in \tilde{K}^0(S^2)$. More generally, $\psi^q \wedge^i (L-1) = q^i \wedge^i (L-1) \in \tilde{K}^0(S^{2i})$.*

Proof of lemma. First observe that the relation $(L-1)^2 = 0$ can be written as $L(L-1) = L-1$, and from this we obtain the relations $L^k(L-1) = L-1$ for all $k \in \mathbb{N}$. Now write

$$\psi^q(L-1) = (L^q - 1) = (L-1)\sum_{k=0}^{q-1} L^k$$

$$= \sum_{k=0}^{q-1} L^k(L-1) = q(L-1)$$

Now, assuming that the Adams operations are compatible with exterior products, one can write

$$\psi^q \wedge^i (L-1) = \wedge^i \psi^q(L-1) = \wedge^i q(L-1) = q^i \wedge^i (L-1)$$

□

Thus relative to the isomorphisms $\mathbb{Z} \simeq \tilde{K}^0(S^{2i})$ obtained by choosing the canonical generator $\wedge^i(L-1)$, the homomorphisms

$$1 - \psi^q : \tilde{K}^0(S^{2i}) \to \tilde{K}^0(S^{2i}) \text{ look like } \mathbb{Z} \xrightarrow{1-q^i} \mathbb{Z}$$

and this yields the desired result.

$\square$

6 Identifying $BGL(k)^+$ with $F\psi^q$

Recall that the maps $f_n : BGL(n,k) \to BGL(\mathbb{C})$ corresponding to the Brauer lifted characters $\chi_n \in R(GL(n,k),\mathbb{C})$ of the standard representations are fixed by ψ^q. Since $[BGL(n,k),U] = K^{-1}(BGL(n,k)) = 0$ by the Atiyah-Segal completion theorem, part (ii) of Quillen's theorem 5.11 shows that the f_n *lift* to maps

$$\tilde{f}_n : BGL(n,k) \to F\psi^q$$

One can show that these are compatible with the directed system $BGL(n,k) \to BGL(n+1,k)$, and so they yield an element $(\tilde{f}_n) \in \lim_n [BGL(n,k), F\psi^q]$. To obtain a (unique up to homotopy) map $\tilde{f} : BGL(k) \to F\psi^q$ lifting our map $f : BGL(k) \to BGL(\mathbb{C})$, one can show that the natural map

$$[BGL(k), F\psi^q] \to \lim_n [BGL(n,k), F\psi^q] \text{ is an isomorphism}$$

In fact, using Milnor's exact sequence and theorem 5.1 (ii), one reduces to showing the natural map $[BGL(k), BGL(\mathbb{C})]^{\psi^q} \to \lim_n [BGL(n,k), BGL(\mathbb{C})]^{\psi^q}$ is an isomorphism, and this is comparatively straightforward.

Now recall that $F\psi^q$ is an H-space, in fact an infinite loop space. In particular its fundamental group is abelian, and so $[GL(k), GL(k)] \subset GL(k) = \pi_1 BGL(k)$ lies in the kernel of

$$\tilde{f}_* : \pi_1 BGL(k) \to \pi_1 F\psi^q$$

Thus by the defining property of the plus construction $BGL(k)^+$, $\tilde{f}$ induces a map $\tilde{f}^+ : BGL(k)^+ \to F\psi^q$ fitting into a commutative diagram

$$\begin{array}{ccc} BGL(k) & \xrightarrow{=} & BGL(k) \\ \downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ BGL(k)^+ & \xrightarrow{\tilde{f}^+} & F\psi^q \end{array} \tag{6}$$

Theorem 6.1 (Quillen)**.** *The map $\tilde{f}^+ : BGL(k)^+ \to F\psi^q$ is a homotopy equivalence.*

Note that this immediately implies

$$K_i(k) := \pi_i BGL(k)^+ \simeq \pi_i F\psi^q$$

and hence the higher algebraic k-groups of k are computed in part (iii) of theorem 5.1.

I'll only sketch in broad strokes the proof of theorem 6.1. The first observation is that by the homology version of Whitehead's theorem (which applies since $BGL(k)^+$ and $F\psi^q$ are simple, being H-spaces (and infinite loop spaces)) and universal coefficient theorem arguments, one reduces to showing:

Theorem 6.2. *For any prime $l \in \mathbb{Z}$, the induced homomorphism*

$$\tilde{f}^+_* : H_*(BGL(k)^+; \mathbb{F}_l) \to H_*(F\psi^q; \mathbb{F}_l)$$

on singular homology with coefficients in $\mathbb{F}_l$ is an isomorphism.

Now recall that for any prime $l \in \mathbb{Z}$ the canonical map $BGL(k) \to BGL(k)^+$ induces an isomorphism $H_*(BGL(k); \mathbb{F}_l) \simeq H_*(BGL(k)^+; \mathbb{F}_l)$; hence we'll have a commutative diagram

$$\begin{array}{ccc} H_*(BGL(k); \mathbb{F}_l) & \xrightarrow{=} & H_*(BGL(k); \mathbb{F}_l) \\ \simeq\Big\downarrow & & \tilde{f}_*\Big\downarrow \\ H_*(BGL(k)^+; \mathbb{F}_l) & \xrightarrow{\tilde{f}^+_*} & H_*(F\psi^q; \mathbb{F}_l) \end{array} \tag{7}$$

and so it will suffice to show

Theorem 6.3. *For any prime $l \in \mathbb{Z}$, the induced homomorphism*

$$\tilde{f}_* : H_*(BGL(k); \mathbb{F}_l) \to H_*(F\psi^q; \mathbb{F}_l)$$

on singular homology with coefficients in $\mathbb{F}_l$ is an isomorphism.

In fact the proof of this theorem occupies the majority of Quillen's paper *On the cohomology and K-theory of the general linear groups over a finite field*. I'm basically going to leave this as a "black box." But it is worth pointing out some extra structure that on these homologies that drives Quillens calculations of these homologies: both $H_*(BGL(k); \mathbb{F}_l)$ and $H_*(F\psi^q; \mathbb{F}_l)$ are Hopf algebras over $\mathbb{F}_l$ and $\tilde{f}_*$

is a homomorphism of Hopf algebras. In the first case the Hopf algebra structure comes from the homomorphisms (coefficients suppressed)

$$H_*BGL(m,k) \otimes H_*BGL(n,k) \to H_*BGL(m+n,k)$$

induced by the usual "direct sum" inclusions $GL(m,k) \times GL(n,k) \to GL(m+n,k)$. In the second case the Hopf algebra structure comes from the homomorphisms

$$H_*F\psi^q \otimes H_*F\psi^q \to H_*F\psi^q$$

induced by the H-space multiplication $F\psi^q \times F\psi^q \to F\psi^q$.