ALGEBRA PRELIM FALL 1975

Directions:

The exam is in five parts. Part I is a true-false question. You should spend about 75 minutes on this part, which should enable you to do about 10 of the 15 questions. Don't try to do them all. In each of Parts II, III, IV and V you should do one (1) of the 2 or three questions. Indicate clearly which one you have done.

You should write enough so that there is no doubt about whether you know what is going on. But use common sense. Don't write a book. If you wish to use a major theorem, do so, and indicate why the theorem is applicable, but do not feel compelled to state the theorem in detail.

Neatness counts.

One complete solution is better than two partial solutions.

PART I. This is a "true-false" question. State whether each statement is true or false and give a short proof or disproof (a disproof would most often be a counterexample). You should probably spend about 75 minutes on this part, and should do 8 to 10 of the 15 parts.

1. There is no polynomial of degree 4 with integer coefficients whose Galois group over Q is cyclic of order 12.

2. If G is a group with a normal subgroup N such that $|G/N| = 3$ and N is cyclic of order 25, then G is Abelian.

3. Every (unitary) module over a ring with identity has a maximal proper submodule.

4. Let R be a semisimple ring with minimum condition and let ${}_R R = L_1 \oplus \ldots \oplus L_n$ be a decomposition of R as a direct sum of minimal left ideals. Then R is commutative if and only if $i \neq j$ implies $L_i \not\cong L_j$.

5. All 4x4 matrices T over R (the field of real numbers) satisfying $T^2 = -1$ are similar.

6. There is a field k and an integer n, $n > 0$, and a set S in k^n such that S is an intersection of hypersurfaces, but not the intersection of a finite number of hypersurfaces. (Definition: a hypersurface is a set of the form $\{\underline{x} : f(\underline{x}) = 0\}$, for some fixed polynomial f.

7. There exists an epimorphism of Abelian groups $f : Z_8 \oplus Z_2 \to Z_4$ whose kernel contains an element of order 4.

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8. There is no Dedekind domain R such that for some ideal I in R, $R/I \cong Z_2[x,y]/(x^2,y^2,xy)$.

9. If F_0, F_5, and F_7 are the splitting fields of the polynomial x^3-2 over the fields Q, Z_5, and Z_7, then the degrees of these extension fields are all different.

10. If R is a commutative ring with 1 and $\phi: R^m \to R^n$ a surjective module homomorphism, then $m \geq n$.

11. If R is a commutative ring with identity such that every ideal except R itself is prime, then R is a field.

12. $x^3 - 9$ is irreducible in $Z_{31}[x]$.

13. For every quadratic extension K of a field k, there is an $\alpha \in k$ such that $K = k(\sqrt{\alpha})$.

14. For every nonconstant polynomial $g(x) \in Z[x]$, there are infinitely many primes p such that $g(x) \equiv 0 \pmod p$ has an integer solution.

15. If G is a group of order 216 then G has a normal subgroup N such that $8 \leq |N| \leq 60$.

PART II. Do one of the following 3 problems.

1. Prove that every prime ideal of $C[x,y,z]$ (where C is the ring of complex numbers) is an intersection of maximal ideals.

2. Construct explicitly (i.e. give the elements) a normal cubic extension K of Q such that $Q \subset K \subset Q(\xi)$, where ξ is a primitive 7th root of 1. (Hint: recall $Gal(Q(\xi)/Q \cong (Z/7Z)^*$.)

3. Let G be a subgroup of $GL(n,C)$ whose points are exactly those elements of $GL(n,C)$ which are the zeros of a certain family of polynomials. It is known that there is a unique irreducible component G_0 of G, in the Zariski topology, which contains the identity element of G. Prove

   (a) G_0 is a subgroup

   (b) G_0 is normal

   (c) G_0 has finite index in G.

PART III. Do one of the following problems

Let G be a finite group such that distinct subgroups have distinct orders. Prove that G is cyclic. (Hint: find an element of prime order and use induction.)

2. Give a complete classification for countable Abelian groups G such that $14G = 0$, (i.e. groups such that for all $x \in G$, $14x = 0$) (Hint: first try groups such that $2G = 0$.)

PART IV. Do one of the following problems

1. Let P be a finitely generated prime ideal of a commutative ring R with 1. Suppose $\mathrm{Ann}(P) = \{r \in R\colon rP = 0\} = 0$. Prove that $\mathrm{Ann}(P/P^2) = P$. (Hint: for any nxn matrix A, over a commutative ring, there is a matrix A^* such that $A^*A = (\det A)\, 1_n$. (A^* is the matrix of cofactors.).)

2. Let V and W be finite dimensional vector spaces over a field k. Say that two elements f and g of Hom(V,W) are isomorphic if there are automorphisms α and β of V and W such that $f = \beta g \alpha$ (linear transformations act on the left). Say that f and g are V-isomorphic if there is an automorphism α of V such that $f = g\alpha$.

   (i) how many isomorphism classes are there? Is it finite or infinite, or does this depend on k?

   (ii) same question for V-isomorphism classes.

PART V. Do one problem

1. Let R be a commutative ring and I an ideal of R. Note that any R/I module can be regarded as an R-module, and if M is any R-module, and $M[I] = \{x \in M\colon rx = 0 \text{ for all } r \in I\}$ then $M[I]$ can be regarded as an R/I module.
Say whether each of the following statements is true or false, and give a proof or counterexample.

   (i) If M is an injective R/I module, then M is injective as an R-module.

   (ii) If M is an injective R-module, then $M[I]$ is injective as an R/I module.

PART V continued

Let θ be any root of x^3-2, $K = Q(\xi, \sqrt[3]{2}, \sqrt{7})$, where ξ is a primitive cube root of 1. Prove that $K \otimes_Q Q(\theta)$ is isomorphic as a ring to the direct sum of three copies of K. (Hint: the map $Q[x] \to Q(\theta)$ taking x to θ fits into an exact sequence.)

3. (a) Let R be a commutative ring with identity. Prove that the sum of a nilpotent element and a unit of R is a unit of R.

   (b) Let $S = R[x]$, the polynomial ring over a commutative ring R with identity. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n \in S$. Prove that f is a unit of S iff a_0 is a unit of R and the a_i $(1 < i \leq n)$ are nilpotent. (Hint: If $b_0 + \ldots + b_m x^m$ is the inverse, show $a_n^{r+1} b_{m-r} = 0$, for all r.)

ALGEBRA PRELIM

Spring 1975

Directions

1. This exam is divided into four parts (A. Linear Algebra; B. Groups; C. Rings and Modules; D. Fields and Galois Theory).

   Do two problems in each part, a total of eight problems.

2. Mark clearly which eight problems you wish to have graded. Two half problems do not equal one problem.

3. You should write enough so that there is no doubt as to whether or not you know what's going on. In particular, if you use a big standard theorem, explain why it is applicable. (Do not feel obliged to state it in detail.) But use common sense. Don't write a book when a few lines are sufficient. When in doubt, ask.

4. Neatness counts in that illegible or very difficult-to-read exams will not be graded.

A. LINEAR ALGEBRA

1. A linear transformation T on a finite dimensional vector space V is said to be semi-simple if and only if $V = \oplus \sum_{i=1}^{n} V_i$ where for each V_i

   (a) $T(V_i) \subseteq V_i$

   (b) V_i contains no proper subspace W such that $T(W) \subseteq W$.

   Prove: A linear transformation T on a finite dimensional vector space V over a field is semi-simple if and only if the minimal polynomial of T is the product of distinct irreducible polynomials.

2. (a) Suppose T is a linear transformation on a finite dimensional vector space V over a field such that the minimal polynomial of T factors completely into distinct linear factors. Show that there is a diagonal matrix representing T.

   (b) Give an example of a linear transformation S on a vector space V over a field such that there is a diagonal matrix representing S, but the minimal polynomial of S factors into linear factors with a factor being repeated.

   (c) Give an example of a linear transformation P on a vector space V over a field such that the minimal polynomial does not factor into linear factors.

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3. (a) Let V be a finite dimensional vector space over a field F, and suppose $V = X \oplus Y$ (for subspaces X and Y). Show that there is a <u>unique</u> linear transformation T on V such that $T|X = 1$ and $T(Y) = 0$. Such a map is called a <u>projection on X</u>.

   (b) Let V be a 2 dimensional vector space over the field R of real numbers (so that $V \cong R \oplus R$). Let X be the subspace spanned by $(1,-1)$. Give the matrices of <u>two distinct</u> projections on X.

   (c) Suppose $V = X_1 \oplus Y_1 = X_2 \oplus Y_2$ and T_i is the projection on X_i. Show that $X_1 \subseteq X_2$ if and only if $T_2 T_1 = T_1$.

4. Let V be a fininte dimensional vector space over a field F and let

$$V^* = \mathrm{Hom}(V,F)$$

For a linear transformation $T: V \to V$, define $T^* \in \mathrm{End}_F(V^*)$ by

$$T^*(f)(v) = f(T(v))$$

or, more picturesquely,

$$(v, T^*(f)) = (T(v), f)$$

Show that

(a) T^* is 1-1 if and only if T is onto

(b) T^* is onto if and only if T is 1-1.

<u>B. GROUPS</u>

5. Prove or disprove each of the following statements.

   (a) If H is a subgroup of a finitely generated abelian group G and G is the internal direct sum of subgroups $G_1, \ldots, G_n$ (that is, $G = \oplus \sum_{i=1}^{n} G_i$), then H is the internal direct sum of subgroups $H_1, H_2, \ldots, H_n$ ($H = \oplus \sum_{i=1}^{n} H_i$), where each H_i is a subgroup of G_i.

   (b) If G is an abelian group and there is a monomorphism $\alpha: G \to G \times G$, then

$$\frac{G \times G}{\mathrm{Im}\ \alpha} \cong G.$$

6. (a) Exhibit a non-abelian group G with subgroups A and B such that
   - (i) G is generated by A and B;
   - (ii) A is normal in G;
   - (iii) $A \cap B = \langle e \rangle$;
   - (iv) A and B are abelian.

   (b) Show that for such a group G, there are homomorphisms

$$A \xrightarrow{\alpha} G \underset{\gamma}{\overset{\beta}{\rightleftarrows}} B$$

such that

$$\alpha \text{ is a monomorphism}$$
$$\beta \text{ is an epimorphism}$$
$$\beta\gamma = 1_B$$
$$\text{Ker } \beta = \text{Im } \alpha.$$

7. Let H be a subgroup of a finite p-group G. Let C(H) be the set of cosets Ha of H in G.
   (a) Show that right translation defines an action of H on C(H)

   (b) Show that an element Ha in C(H) is a fixed point of this action if and only if $a^{-1}Ha = H$

   (c) Use (b) and a counting argument to deduce that there is an element $x \in G$ such that $x \notin H$ and $x^{-1}Hx = H$.

8. Suppose G is a group with a subgroup H of order 25 having 22 conjugates (including itself). Suppose the index of H in its normalizer in G is 11.
   (a) What must be the order of G?
   (b) Why <u>can't</u> the above situation occur?

C. RINGS AND MODULES

Note: all modules over rings with identity are assumed to be unitary (that is, $1a = a$ for all a).

9. Suppose R is a ring with identity such that every left R-module is the direct sum of simple sub-modules. Suppose M is a left R-module. Show that there is a finite set of left ideals $L_1, L_2, \ldots, L_k$ in R such that

$$M \cong \oplus\sum_{J_1} L_1 \oplus \oplus\sum_{J_2} L_2 \oplus \cdots \oplus \oplus\sum_{J_k} L_k \quad \text{where} \quad \oplus\sum_{J_i} L_i \quad \text{is the sum of}$$

J_i copies of L_i.

10. (a) Suppose R is a principal ideal domain with no cyclic injective modules. Show that no injective left module over R has finitely many generators.

(b) Give an example of a ring S and an S-module M such that M is projective but not free.

11. (a) Let I be an ideal in a commutative ring R with identity. Show that

$\{x \mid x^n \in I \text{ for some } n\}$ = the intersection of all prime ideals containing I.

(b) Denoting the set given in (a) by $\sqrt{I}$, describe in detail the set of ideals I in the ring C[x] such that $I = \sqrt{I}$.
(C[x] is the ring of polynomials over the complex numbers).

12. What is the Jacobson radical of each of the following rings?

(a) The ring of formal power series with coefficients in a field K.

(b) The polynomial ring K[x]

(c) $K[x]/(x^7)$

(d) The ring T_5 of all those 5×5 matrices with entries in K which have zeroes in every entry <u>above</u> the main diagonal.

13. Prove or disprove: Let R be a ring with identity which is left Noetherian (that is, the ACC on left ideals holds). Then every direct sum of injective left R-modules is injective.

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D. FIELDS AND GALOIS THEORY

14. Let F be the field obtained by adjoining $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ to the field Q of rational numbers Determine the Galois group of F over Q.

15. A Fundamental Theorem of Galois theory states that if a field K is separable over a field F and a splitting field for a polynomial in F[x], then the Galois Group of K over F has order [K:F].

    (a) Exhibit a non-separable field extension where the Galois group has only one element.

    (b) Exhibit a nontrivial separable field extension where the Galois group has only one element.

16. Consider $x^6 - 7$ as a polynomial over the field Q of rational numbers.

    (a) Show that $x^6 - 7$ is irreducible in Q[x].

    (b) Compute the dimension of the splitting field of $x^6 - 7$ over Q.

    (c) Determine the Galois group of $x^6 - 7$.

ALGEBRA PRELIM

Fall 1974

Directions:

1. This exam is divided into four parts (A. Groups; B. Rings and Modules; C. Fields and Galois Theory; D. Linear Algebra).

   Do a total of seven problems, including at least two problems in part A, two problems in part B, one problem in part C, and one problem in part D.

2. Mark clearly which seven problems you wish to have graded. Two half problems do not equal one problem.

3. You should write enough so that there is no doubt as to whether or not you know what's going on. In particular, if you use a big standard theorem, explain why it is applicable. (Do not feel obliged to state it in detail.) But use common sense. Don't write a book when a few lines are sufficient. When in doubt, ask.

4. Neatness counts in that illegible or very difficult-to-read exams will not be graded.

A. Groups

1. Find (up to isomorphism) all subgroups of order 8 of S_7.

   [Hint: You might want to notice that for each $n < 7$, you can consider S_n as the subgroup of S_7 which fixes all j with $n < j \leq 7$. Observe that if G is a subgroup of S_5, then there is a subgroup H of S_7 such that $|H| = 2$ and GH is the direct product of G and H.]

2. Let G be the group of all permutations of the set of positive integers. For each n identify the symmetric group S_n with the permutations of G which fix all $j > n$.

   Let A_n be the alternating group of degree n and let $A_\omega = \bigcup_n A_n$. Clearly A_ω is a subgroup of G.

   (a) Show that A_ω is simple. [Quote suitable theorems; no long proof needed.]

   (b) Show that A_ω has no nontrivial subgroups of finite index.

3. Prove that a group of order 48 has a normal subgroup of order 8 or 16.

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A. Groups (continued)

4. Let V be finite-dimensional vector space over a field K and let G be a group of linear automorphisms of V. A subspace W of V is G-stable if $g(W) \subseteq W$ for all $g \in G$. V is G-semi-simple if, for every G-stable subspace W of V there is a G-stable subspace W' with $V = W \oplus W'$. Finally a G-stable subspace W is G-simple if it does not properly contain a G-stable subspace of V, other than (0).

   Suppose V is G-semi-simple and that H is a normal subgroup of G.

   a) Show that V is the sum of H-stable subspaces which are H-simple.

   b) Show that V is H-semi-simple.

B. Rings and Modules

5. If A is a minimal right ideal of a ring R (with or without 1) and $A^2 \neq 0$, then prove there is an idempotent $e \in A$ with $A = eR$.

6. Let R be a commutative ring and $a \in R$.

   a) Show that the set S of all ideals in R not containing any power of a contains a maximal element (with respect to set theoretic inclusion).

   b) Show that a maximal element in S is a prime ideal.

7. Let M be a free Z-module of rank n and f an endomorphism of M. Relative to any basis of M f is represented by an nxn matrix with integer entries. The determinant of f is by definition the determinant of any matrix representing f and is denoted det f.

   (a) Show that f is an automorphism if and only if det f = $\pm$ 1

   (b) Show that M/f(M) is a finite abelian group if and only if det f $\neq$ 0.

   (c) Show that the order of M/f(M) is precisely |det f| if det f $\neq$ 0.

B. Rings and Modules (continued)

8. (a) Let R be a ring and I,J left ideals of R, which are isomorphic (as R-modules). If $\phi: R \to R$ is a ring automorphism, show that $\phi(I)$ and $\phi(J)$ are isomorphic as R-modules.

   (b) Assume that $L_1, L_2, \ldots, L_s$ are simple left ideals of R such that every simple left ideal of R is isomorphic (as an R-module) to one and only one of $L_1, L_2, \ldots L_s$. (In particular $L_i \cong L_j$ iff $i = j$).

   For each i, let R_i be the sum of all left ideals of R which are isomorphic (as R-modules) to L_i.

   If $\phi: R \to R$ is a ring isomorphism, then prove that ϕ permutes the set $\{R_1, R_2, \ldots, R_s\}$.

C. Fields and Galois Theory

9. Show that the Galois group over the field Q of rational numbers of $x^n - 1$ is isomorphic to the group of invertible elements in the ring Z/nZ.

   [You may assume that the n-th cyclotomic polynomial $g_n(x)$ is irreducible in $Q[x]$; you will recall that $g_n(x) = \Pi(x-\zeta_i)$, where ζ_i ranges over all the primitive n-th roots of unity.]

10. Let K be a field, G a finite group of field automorphisms of K and let $F = \{a \in K \mid \sigma a = a \text{ for all } \sigma \in G\}$. F is clearly a subfield

    (a) For $b \in K$, describe a non-zero polynomial $f \in F[x]$ such that $\deg f = |G|$ and $f(b) = 0$.

    (b) For $b \in K$, find a non-zero separable $f \in F[x]$ with $f(b) = 0$.

    (c) Using the above results and the existence of a primitive element for finite separable extensions, show that $[K:F] \leq |G|$. [Note: you must justify any claim that $[K:F]$ is finite.]

D. Linear Algebra

11. Let V be a finite dimensional vector space over a field K of characteristic different from 2. Let β be a bilinear form $V \times V \to K$, which is symmetric (i.e. $\beta(x,y) = \beta(y,x)$ for all x and y).

    (a) Show that if $\beta \neq 0$, there is a vector v such that $\beta(v,v) \neq 0$.

    (b) Use (a) to show that there is a basis for V such that the matrix of β is diagonal.

12. Let V be a finite dimensional vector space over a field F and for each $W \subset V$, let

$$A(W) = \{f \in \mathrm{Hom}_F(V,F) \mid f(W) = 0\}.$$

If W_1 and W_2 are subspaces of V, prove that

$$A(W_1 \cap W_2) = A(W_1) + A(W_2).$$

ALGEBRA PRELIM

Spring 1974

Directions:

1. This exam is divided into four parts (A. Linear Algebra; B. Groups; C. Rings and Modules; D. Fields and Galois Theory).

   Do two problems in each part, a total of eight problems.

2. Mark clearly which eight problems you wish to have graded. Two half problems do not equal one problem.

3. You should write enough so that there is no doubt as to whether or not you know what's going on. In particular, if you use a big standard theorem, explain why it is applicable. (Do not feel obliged to state it in detail.) But use common sense. Don't write a book when a few lines are sufficient. When in doubt, ask.

4. Neatness counts in that illegible or very difficult-to-read exams will not be graded.

A. LINEAR ALGEBRA

1. Let ϕ and ψ be endomorphisms of a finite dimensional vector space E (over a field). Assume that E has a basis of eigenvectors of ϕ and also a basis of eigenvectors of ψ. Assume that $\phi\psi = \psi\phi$. Prove that E has a basis consisting of vectors which are eigenvectors for both ϕ and ψ.

2. Let A and B be idempotent matrices over a field (i.e. $A^2 = A$, $B^2 = B$). Prove that A is similar to B if and only if A is equivalent to B. [Recall that A is equivalent to B means that there exist invertible matrices P,Q such that $B = PAQ$.]

3. Let V be a finite dimensional vector space over a field F and let W be a subspace of V. Let $A(W) = \{f \in \mathrm{Hom}_F(V,F) \mid f(w) = 0 \text{ for all } w \in W\}$. If W_1, W_2 are subspaces of V, prove that

$$A(W_1 \cap W_2) = A(W_1) + A(W_2).$$

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B. GROUPS

4. If N is a normal subgroup of prime order p in the finite p-group G, then prove that N is contained in the center of G.

5. Determine up to isomorphism all groups of order 245 [i.e. exhibit (with proof) a list of groups of order 245, no two of which are isomorphic and such that every group of order 245 is isomorphic to some group on the list.]

6. (a) Prove that all automorphisms of S_3 are inner and that the group of all automorphisms of S_3 is actually isomorphic to S_3.

   (b) Construct an abelian group (of small order) whose automorphism group is isomorphic to S_3.

7. Prove that the additive group Q/Z (Q the rationals, Z the integers) is isomorphic to the group of all complex roots of unity.

C. RINGS AND MODULES

8. Determine all rings of order 2700 which are Jacobson semisimple. Give reasons for your answer.

9. Let R be a commutative ring with identity such that every submodule of every free R-module is free. Prove that R is a principal ideal domain.

10. Let $f(x) = \Sigma a_i x^i$ be a monic polynomial with coefficients in Z. For any $n > 1$, let $\bar{f}(x) = \Sigma \bar{a}_i x_i$, where $\bar{a}_i \varepsilon Z_n$ is the image of $a_i \varepsilon Z$ under the canonical map $Z \to Z_n$. Prove that if $\bar{f}(x)$ is irreducible over Z_n for some n, then f(x) is irreducible over the field of rational numbers.

11. Answer any 3 parts of this problem. True or false. If the statement is true, prove it. If it is false, give a counterexample (without proof).

   (a) If I is an ideal in a ring with identity such that every element of I is nilpotent, then I is a nilpotent ideal.

   (b) If M is a maximal ideal in a ring R with identity, then R/M is a field.

   (c) Every nonzero homomorphic image of a semisimple module A is semisimple.

   (d) Let F,A be unitary modules over a ring R with identity. If F is free and $F \otimes_R A = 0$, then $A = 0$.

D. FIELDS AND GALOIS THEORY

12. (a) If F is an extension field of dimension 2 over a field K, then prove that F is a normal extension of K.

    (b) If K is a finite field of characteristic p, prove that every element of K has a unique p-th root in K.

    (c) If K is a finite field then there is a homomorphism of rings from $Z[x]$ onto K (where Z is the ring of integers).

13. Let F be a subfield of the complex numbers which is finite dimensional over the field Q of rationals. Assume $[F:Q]$ is odd. Let $p(x)$ be irreducible in $Q[x]$ and assume all roots of $p(x)$ lie in F. Prove that all the roots of $p(x)$ are real.

14. Compute the Galois group of the following polynomials over the field Q of rational numbers

    (a) $x^5 - x^4 - 3x^3 + 3x^2 - 10x + 10$

    (b) $x^4 + x^2 + 1$

ALGEBRA PRELIM

AUTUMN 1973

There are 5 sections, entitled: Groups, Linear Algebra, Rings, Fields, Mo[illegible]s. Do 5 problems, one from each section. Indicate which 5 problems are to be graded.

Partial credit will be given. If you cannot do part of a problem, it is acceptable for you to use that part for later parts of the same problem.

If you use a standard big theorem, be sure to explain why it is applicable.

You should write enough so that there is no doubt that you know what is going on. But use common sense. Don't write a book when a few lines would suffice. When in doubt, ask.

GROUPS

1. Determine how many mutually non-isomorphic groups of order 12 there are, and describe each one. Include a proof that your list is complete. HINT: Think semidirect product.

2. A subset P of a group G is called a positive cone if (i) $1 \notin P$; (ii) if $x \neq 1$ then either $x \in P$ or $x^{-1} \in P$, but not both; and (iii) if x and y are in P, then $xy \in P$. The cone is invariant if for all $x \in G$, $xPx^{-1} = P$. If G is a group with a positive cone P, define $g < h$ to mean $hg^{-1} \in P$ and $g \prec h$ to mean $g^{-1}h \in P$.
   (a) Show that these two orderings on G coincide if and only if P is invariant.
   (b) Show that if G contains any non-identity element of finite order, then G contains no positive cone.
   (c) Show that if G is generated by two elements x and y of infinite order satisfying $xyx^{-1} = y^{-1}$, then G contains no invariant positive cone.

3. Let G be an abelian group in which every element has finite order. For each prime p, let $G_p = \{x \in G : p^n x = 0 \text{ for some integer } n \geq 0\}$. Prove that each G_p is a subgroup of G, and that G is the direct ~~product~~ sum of the subgroups G_p. Do not assume that G itself has finite order.

LINEAR ALGEBRA

1. Let V be a finite dimensional vector space over an algebraically closed field k, and let T and S be linear transformations from V to itself such that ST = TS.

   (a) If $\alpha \in k$ and $V_\alpha = \{x \in V \mid T(x) = \alpha x\}$, show that V_α is S-invariant.

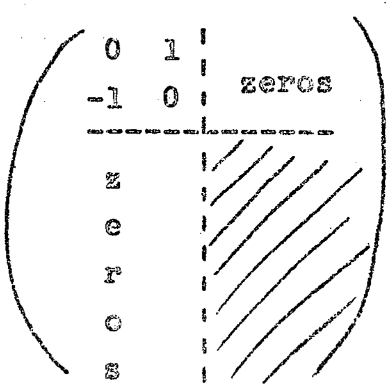
   (b) Show that S and T have a common eigenvector (with possibly different eigenvalues).

   (c) If G is a commutative group of linear transformations of V, show that V has a basis relative to which every element of G has a triangular matrix (i.e. a matrix with all entries below the main diagonal zero).

2. Let V be a finite dimensional vector space over a field k, and let $\beta: V \times V \to k$ be a bilinear form such that $\beta(x,x) = 0$ for all $x \in V$. (This implies that $\beta(x,y) = -\beta(y,x)$ for all $x,y \in V$.) Let $W = \{x \in V \mid \beta(x,y) = 0 \text{ for all } y \in V\}$.

   (a) If $x \notin W$, show there is some $y \in V$ with $\beta(x,y) = 1$.

   (b) Show there exists a basis for V relative to which the matrix of β has form:



assuming $W \neq V$.

   (c) Show that if β is non-degenerate, then V must be even dimensional.

3. Let V be a non-zero vector space. Prove that;

   (a) If S is any linearly independent subset of V, then there is a basis for V containing S.

   (b) If T is any subset of V which spans V, then some subset of T forms a basis for V.

   Do not assume V is finite dimensional.

RINGS

1. A <u>Boolean ring</u> is a ring R with more than one element such that $a^2 = a$ for all $a \in R$. You may assume every such ring is commutative. (Proof: expand $(a + a)^2$ and $(a + b)^2$.) Let R be a <u>finite</u> Boolean ring. Show that (a) R has an identity.

   (b) R has order 2^n for some integer $n \geq 1$.

   (c) R is characterized up to isomorphism by the integer n.

   HINT: Think Wedderburn.

2. Prove or give a counterexample (without proof):

   (a) Every unique factorization domain is a principal ideal domain.

   (b) Every unique factorization domain is Noetherian.

   (c) Every minimal prime ideal in a unique factorisation domain is principal.

   (d) Every integral domain with 1 is a unique factorization domain.

3. Let R be a ring with 1.

   (a) If I is a nilpotent left ideal of R and $r \in R$, show that Ir is a nilpotent left ideal.

   (b) If I and J are nilpotent left ideals of R, show that I + J is nilpotent.

   (c) The nil radical of R, N(R), is defined to be the union of all the nilpotent left ideals of R. Show that N(R) is a two-sided ideal.

   (d) If R is left Noetherian, show that N(R) is a nilpotent ideal.

   (e) Show by example that N(R) need not contain all nilpotent elements of R.

FIELDS

1. (a) Prove that a field with p^n elements (p a prime) is the splitting field over $F_p = Z/pZ$ of the polynomial $x^{p^n} - x$.

   (b) If K is the extension field of F_2 (the integers modulo 2) generated by a primitive 15^{th} root of unity, z, (i.e. an element z such that $z^{15} = 1$ and $z^n \neq 1$ if $0 < n < 15$), is there an automorphism of K taking z to z^7? Justify your answer.

2. Let F be a field, F^* its algebraic closure, and K a subfield of F^* containing F, such that [K:F] is finite. We say K is a <u>normal</u> extension of F if it satisfies one of the following three equivalent conditions: (i) for every irreducible polynomial $f \in F[X]$, if K contains one root of f, it contains all roots of f (in F^*); (ii) every automorphism of F^* which fixes the elements of F, takes K into itself; (iii) K is the splitting field of a polynomial in F[X].

   The Fundamental Theorem of Galois theory states that if K is normal and separable over F, then the number of F-automorphisms of K is exactly the degree [K:F]. Show by example that this equality need not hold if the extension is

   (a) normal but not separable

   (b) separable but not normal.

   Prove your assertions.

3. Let F be the field of rationals and E the splitting field of the polynomial $x^4 + 2x^2 - 2$ over F.

   (a) What is the degree of E over F?

   (b) <u>Without</u> computing the Galois group, and <u>without</u> stating a great big theorem in detail, describe how information contained in the Galois group could be used to obtain information about subfields of E.

   (c) Compute the Galois group (as an abstract group)(but don't go back and give a lot of details in (b).)

MODULES

1. Let R be any ring, A be a right R-module, and B be a left R-module which is the direct sum of a family $\{B_i\}_{i \in I}$ of left R-modules. Prove that $A \otimes_R B$ is isomorphic to the direct sum of the family $\{A \otimes_R B_i\}_{i \in I}$.

2. Let R be any ring. An R-module Q is called <u>injective</u> if given any R-modules X,Y and any R-module map $f:Y \to Q$ and any injective (one-to-one) R-module map $g:Y \to X$, there is an R-module map $\phi:X \to Q$ such that $\phi g = f$.

$$\begin{tikzcd} & Q & \\ 0 \arrow[r] & Y \arrow[u, "f"] \arrow[r, "g"'] & X \arrow[ul, dashed, "\phi"'] \end{tikzcd}$$

   (a) Prove that if G is an abelian group which is an injective $\mathbb{Z}$-module, then G is <u>divisible</u>; i.e. given any $x \in G$ and any positive integer n, there is some $y \in G$ with $ny = x$.

   (b) Prove that if Q is an injective R-module and if $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is any exact sequence of R-modules, then the sequence $0 \to \mathrm{Hom}(C,Q) \to \mathrm{Hom}(B,Q) \to \mathrm{Hom}(A,Q) \to 0$ is also exact.

3. Let M be a left R-module satisfying the descending chain condition on submodules. Show that M has only a finite number of nonisomorphic, simple submodules. (i.e. there are a finite number of simple submodules such that any simple submodule is isomorphic to exactly one on this list.)

Algebra Prelim

SPRING 1973

Directions: Do your best to answer the required number of questions in each section. As a general rule it is better to answer fewer questions well rather than many questions partially or superficially.

Answer a total of 8 questions. Two of these must be in part I (Groups), two in part II (Fields and Galois Theory), and four in part III (Rings, Modules and Linear Algebra).

Part I. GROUPS

1. Show that every group of order 56 has a nontrivial normal subgroup. Do the same for groups of order 30.

2. (a) If p is the smallest prime dividing the order of a finite group G and H is a subgroup of index p in G, then prove that H is normal in G.

   (b) If G is a finite group which contains a subgroup of order n for each n dividing the order of G, then G is solvable.

3. Definition. A group is nilpotent if it is the direct product of its Sylow subgroups.

   Suppose that G is a finite group with the property that for each $g \neq 1$ in G,

$$C_G(g) = \{x \mid gx = xg\} \text{ is a nilpotent group.}$$

Let p,q be distinct primes and suppose that P is a p-Sylow subgroup of G and Q is a q-Sylow subgroup of G. Suppose that

$$x \in P \text{ and } y \in Q \text{ are non-identity elements such that}$$

$xy = yx$.

(a) Prove that $xz = zx$ for every z in the center $Z(Q)$ of Q.

(b) Prove that x commutes with every element of Q.

(c) Prove that if $a \in P$ and $b \in Q$ then $ab = ba$.

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4. We say that a non-trivial normal subgroup M of G is a <u>minimal normal</u> subgroup of G if

$$1 \subseteq N \subseteq M \text{ and } N \text{ normal in } G \text{ imply } N = 1 \text{ or } N = M.$$

Prove that if M is a minimal normal subgroup of a <u>solvable</u> group G, then M is an abelian p-group for some prime p.

Part II

FIELDS AND GALOIS THEORY

5. Let F be a splitting field of the polynomial $x^5 - 5$ over the field Q of rational numbers. What is the dimension of F over Q? Why?

6. Suppose that $a_1, \ldots a_n$ are the roots of the polynomial $x^n - 1$ over the field Q of rational numbers. Assume that $a_1 + a_2 + \cdots + a_n$ is an integer. Prove that whenever k is relatively prime to n, then

$$a_1^k + a_2^k + \cdots + a_n^k = a_1 + a_2 + \cdots + a_n .$$

[<u>Hint</u>: you may assume that all primitive n-th roots of unity are roots of the same irreducible polynomial $g(x) \in Q[x]$.]

7. (a) <u>Prove</u> or <u>disprove</u>: If f(x) and g(x) belong to K[x] where K is a field, <u>and if</u> the splitting fields of f(x) and g(x) over K are isomorphic, then f(x) and g(x) have a non-trivial common factor.

   (b) <u>Give examples</u>, <u>without proof</u>, of polynomials whose Galois groups are respectively the cyclic and the noncyclic group of order 6.

   (c) Let F be an algebraic extension field of K and $u_1, \ldots, u_n \in F$. <u>State without proof</u> conditions which imply that there exists $u \in F$ such that $K(u_1, \ldots, u_n) = K(u)$.

   (d) Let F be an extension field of K and let E be an intermediate field. Suppose σ is a homomorphism of E into F which fixes K elementwise. <u>State without proof</u> conditions which imply that σ can be extended to an automorphism of F?

Part III RINGS, MODULES AND LINEAR ALGEBRA

8. Definitions:

1. An element r of a ring R is nilpotent if $r^n = 0$ for some n.
2. N(R) denotes the set of all nilpotent elements of R.

(a) If R is commutative, show that N(R) is an ideal.

(b) If R is commutative, then

$$N(R/N(R)) = 0.$$

(c) Give an example of a noncommutative ring R such that N(R) is not an ideal.

9. (a) Let R be a ring with identity and I the set of all elements in R which do not have multiplicative inverses. Suppose that I is an ideal. Characterize R/I.

(b) Suppose that R is a commutative ring with identity such that for each $r \in R$ either r or $1 - r$ has a multiplicative inverse. Prove that if J is the Jacobson radical of R, then R/J is a field

10. Suppose that A,B are nxn matrices with entries in a field F. Prove that $A^{-1}BA$ and B have the same characteristic polynomial and the same trace.

11. Suppose R is a ring without zero divisors which has a field F in its center. Prove that if the vector space dimension of R over F is finite then R is a division ring.

12. (a) Suppose L is a minimal left ideal in a ring R. Show: Either $xy = 0$ for all $x,y \in L$ or L contains a nonzero idempotent e (i.e. $e^2 = e$).

(b) Suppose $e \in R$ is idempotent and R has an identity. Show that $R = I \oplus J$ where I and J are nontrivial left ideals.

(c) Give an example of a commutative ring R, not a field, that has no minimal left ideal.

13. Let A be a module which is generated by a family of simple submodules. Prove that A is a direct sum of simple submodules.

14. A ring R is said to be reductive if: For any left ideal I or R, R/I is a projective R-module.
Show that any finitely generated module M over a reductive ring R is projective.
[Hint: If M is generated by n elements consider M/K where K is generated by $(n-1)$ of them.]