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ALGEBRA PRELIM

9:30-12:30
Monday, Sept.17/79

Directions: Answer as many of the following problems as you can.

Guideline: seven problems done completely will be considered a very reasonable performance.

1. Prove that any group G of order p^2q^2, where p and q are prime and $p > q \geq 3$, is solvable.

2. (a) If an abelian group G is generated by n elements, then prove that every subgroup of G is generated by n elements. [You may not assume that this theorem is true for free abelian groups].

   (b) Let G be a torsion-free abelian group and let H be a subgroup of G. Prove that G/H is torsion-free if and only if $nG \cap H = nH$ for every positive integer n.

3. Give an example of groups G_1 and G_2 where G_1 is abelian and G_2 is not abelian, but the groups of automorphisms of G_1 and G_2 are isomorphic.

4. Let R be an integral domain which is not a field, having the property that every proper ideal is the product of maximal ideals. Show that

   (a) If M is a maximal ideal of R, then there exists $a \in R$ and an ideal $K \neq \{0\}$ such that $MK = (a)$.

   (b) If I,J,M are ideals of R with M maximal, then $IM = JM$ implies $I = J$.

5. Let R be a ring with a two-sided ideal I. Suppose there exists $e \in I$ such that $ei = ie = i$ for all $i \in I$. Show that there exists a two-sided ideal J of R such that $R = I \oplus J$.

6. (a) Prove that a principal ideal domain satisfies the ascending chain condition on ideals.

   (b) Show that every non-unit in a principal ideal domain is a product of irreducible elements.

7. (a) Assume that $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \in \mathbb{Q}[x]$ has exactly one real root. Show that the Galois group of $f(x)$ over $\mathbb{Q}$ is either S_2 or S_3.

(b) Find the Galois group of $f(x) = x^3 - x + \frac{1}{3}$ over $\mathbb{Q}$.

(c) Find the degree of $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ over $\mathbb{Q}$.

8. Let $K \subset L \subset M$ be fields. Assume $\theta \in M$ is algebraic over K and that $K(\theta)$ is Galois over K.

(a) Why is $L(\theta)$ normal and separable over L?

(b) Show that the Galois group of $L(\theta)$ over L is isomorphic to the Galois group of $K(\theta)$ over $K(\theta) \cap L$.

9. List one representative of each similarity class of the $n \times n$ complex matrices J such that $J^2 = -I_n$, where I_n is the $n \times n$ identity matrix.

10. Suppose that A and B are $n \times n$ matrices over an algebraically closed field such that the $2n \times 2n$ matrices $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ and $\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$ are similar.

Show that A and B are similar.

ALGEBRA PRELIM Spring 1979

1. Prove that in a group G of order 30 each 3-Sylow subgroup is normal, each 5-Sylow subgroup is normal and G has a normal subgroup of order 15.

2. Show that a group of order 80 is solvable.

3. Let a be a positive real number and n an integer greater than 2. Show that the splitting field of $x^n - a$ in the complex field contains a real subfield of index 2.

4) Let p be a prime integer and F_{p^m} the field with p^m elements. For given n and m find all integers t for which there is a non-zero ring homomorphism $F_{p^n} \otimes_{F_p} F_{p^m} \to F_{p^t}$.

For which pairs (n,m) is $F_{p^n} \otimes_{F_p} F_{p^m}$ a field?

5. If R is a ring (perhaps without an identity) and I is a minimal left ideal of R with $I^2 \neq 0$, then $I = Re$ for some idempotent $e \in I$.

6. Let $M(n,F)$ be the ring of nxn matrices over the field F.
Find all the central idempotent elements in $M(n,F)$.
Show that, for $n \neq m$, $M(n,F)$ and $M(m,F)$ are not ring isomorphic.

7. Prove the following:
If R is a commutative ring with identity and J is an ideal of R such that every non-zero ideal of R is a power of J, then every ideal of R is principal.
If $\bigcap_n J^n = 0$ then there is an $x \in R$ such that each $0 \neq y \in R$ has the form $y = ux^k$ with u a unit.

8. If R is a commutative ring with identity and P is an ideal maximal among the ideals that are not finitely generated, then P is a prime ideal.

9. Suppose V is a finite dimensional vector space over a field F and $T:V \to V$ is linear. Show that there is a positive integer k such that $V = T^k(V) \oplus \mathrm{Ker}(T^k)$.

10. For σ an $n \times n$ complex matrix, $\overline{\sigma}$ is the matrix whose ij-th entry is the conjugate of the ij-th entry of σ and σ^t is the transpose of σ. σ is unitary if $\overline{\sigma}^t = \sigma^{-1}$ and A is Hermitian if $\overline{A}^t = A$.

    Explain how a Hermitian matrix may be diagonalized by some unitary matrix.

    Show that if B is a real symmetric matrix and σ a unitary matrix such that $\sigma B \sigma^{-1}$ is diagonal, then the entries of $\sigma B \sigma^{-1}$ are real.

ALGEBRA PRELIM AUTUMN 1978

Directions: Do Problem 1 and five (5) of the remaining problems. Complete solutions are much preferable to partial results. You should write enough so that there is no doubt that you know what is going on. But use common sense. Do not write a book when a few lines would do. Neatness counts--illegible exams will not be treated charitably.

1. This problem is required. It has 5 parts. Each is a statement, to be judged true or false, with a short proof or counterexample.

   (i) A finite ring with identity and with no nilpotent elements is commutative.

   (ii) The additive group of the rational numbers has no maximal proper subgroups.

   (iii) If R is a commutative ring and $R[x]$ is Noetherian, then R is Noetherian.

   (iv) Any subgroup of A_6 of index 6 is isomorphic to A_5.

   (v) If A and B are finite Abelian groups of orders n and m, then the tensor product $A \otimes B$ has exactly mn elements.

2. Let G be a group of order 80, which does not have a normal subgroup of order 16. Then G has at least two distinct normal subgroups (other than (1) and G).

3. G is a finite Abelian group with an element x of order 9 such that $G/(x)$ is cyclic of order 9. Up to isomorphism, what are the possibilities for G? (Be sure to specify with each group the element x you would use.)

4. Let V be a finite dimensional vector space and A and B two linear transformations of V into itself such that $A^2 = B^2 = 1$. Show that if $\operatorname{rank}(A-1) = \operatorname{rank}(B-1)$, then A and B are similar. (Here "1" denotes the identity matrix. Hint: first consider fields of characteristic not 2.)

5. Let ξ be a primitive nth root of unity $(n > 2)$. Show that the degree of $Q(\xi+\xi^{-1})$ over Q is $\frac{1}{2}\phi(n)$, where ϕ is the Euler function.

6. Let R be a ring with identity and let I be a minimal right ideal of R. Show that either (a) $I^2 = 0$, or (b) there is a right ideal K of R with $R = I \oplus K$.

7. Let R be a commutative ring with identity in which every non-empty collection of ideals has a minimal element.

(a) Show that every prime ideal of R is a maximal ideal.

(b) Show that R has only a finite number of prime ideals.

8. Let L be a finite normal extension of the field Q of rational numbers and let $f(x) \in Q[x]$ be an irreducible polynomial. Suppose that $f(x) = p_1(x)\ldots p_n(x)$ is a factorization of $f(x)$ into irreducible factors in $L[x]$. Show that the $p_i(x)$ all have the same degree.

9. Let V be a vector space (not necessarily finite dimensional) and s a linear transformation of V into itself. Show that there is a linear transformation t such that $sts = s$. Show that if V is finite dimensional, t may be chosen to be invertible, but that this is false if V is infinite dimensional.

ALGEBRA PRELIM Spring 1978

DO AS MANY PROBLEMS AS YOU CAN THOROUGHLY

1. P a p-Sylow subgroup of a finite group G. Denote the normalizer of a subgroup H of G by N(H). Show that N(N(P)) = N(P).

2. Let q be a prime integer and let p be a prime integer which divides q - 1. For each integer $n \geq 1$, construct a non-abelian group of order $p^n \cdot q$.

3. Let G be a group of order n and let H be a subgroup of order m. Suppose that for each $g \in G - H$, $H \cap (g^{-1}Hg) = \{1\}$. Show that there are exactly $\frac{n}{m} - 1$ elements of G which lie in no conjugate of H.

4. Show that the Jacobson radical of a ring with 1 (commutative or not) contains no (non-zero) idempotents.

5. a) Let p be a prime in the ring Z of integers. Denote the localization of Z at the prime ideal (p) by $Z_{(p)}$. Show that $Z_{(p)}$ is not finitely generated as a Z-algebra.

   b) Let R be any commutative Noetherian ring. Show that the localization R_P of R at a prime ideal P is Noetherian.

6. Let M be a finitely generated Z-module (Z = ring of integers). Express $\dim_{F_p} M \otimes_Z F_p$, the dimension of $M \otimes_Z F_p$ over the field F_p with p elements, in terms of the invariants of M as a Z-module (i.e. as an abelian group.)

7. Let M_{nF} be the $n \times n$-matrix ring with entries in a field F. Show that all irreducible (simple), left modules for M_{nF} are isomorphic.

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Let V be a finite-dimensional real inner product space. Suppose that V is the orthogonal direct sum of subspaces W and W'. Let $p_1 : V \to W$ be the projection onto the summand W. Show that, relative to any orthonormal basis for V, p_1 has a symmetric matrix. (Regard p_1 as a linear transformation from V to V.)

9. Let n,m be positive integers. Classify up to similarity all $n \times n$ matrices A over the complex numbers that satisfy the relation $A^m = I$. (I = identity matrix.)

10. a) Let $f(x) = f_1(x)f_2(x)$ be the product of polynomials f_1, f_2 with rational coefficients. Suppose that f_2 is irreducible over the splitting field of f_1. Show that the Galois group of the splitting field of f is the direct product of the Galois groups of the splitting fields of f_1 and f_2. *False (see below)*

b) What are the Galois groups of $f(x) = (x^2 + 1)(x^2 + 4)$ and of $g(x) = (x^2 + 1)(x^2 + 2)$?

11. Let p be an odd prime integer and a an integer relatively prime to p. Show that the equation $x^2 \equiv a \pmod{p}$ has a solution in the ring of integers if and only if $a^{(p-1)/2} \equiv 1 \pmod{p}$.

Counter example for 10a

$f_1 = x^2+1$ Splitting field $Q(i)$

$f_2 = x^4 - 2$ (roots $\pm\sqrt[4]{2}, \pm\sqrt[4]{2}\,i$) spl. field $Q(\sqrt[4]{2}, i)$

f_2 is irred over $Q(i)$, but

$Q(\sqrt[4]{2}, i)$ is a spl. field of both f_2 and $f_1 f_2$.

If the circled hypothesis is replaced by "the intersection of the splitting fields of f_1 and of f_2 is Q" then the conclusion is true.

ALGEBRA PRELIM AUTUMN 1977

Directions: The exam is divided into four parts. Answer a total of seven questions, at least one in each part, and not more than two in each part. Indicate clearly which seven problems are to be graded (you cannot count two partially done problems as one).

Part I. GROUPS

1. (a) Prove that there is no simple group of order 56.

   (b) If C is the center of a group G and G/C is cyclic, then prove that G is abelian.

2. (a) If $m|n$ and $2\nmid n$, then prove that the automorphism group of $Z_m \oplus Z_n$ is nonabelian.

   (b) Prove that the automorphism group of a noncyclic finite abelian group (whose order is not divisible by 2) is nonabelian.

3. Prove that the multiplicative group of positive rational numbers is a free abelian group.

4. A subgroup H of G is special if for each pair $x,y \in G$ with $x \notin H$, there is a unique $u \in H$ with $y^{-1}xy = u^{-1}xu$.

   Suppose H is a special subgroup of G. Then show:

   (a) $G = C(x)H$ and $C(x) \cap H = \{1\}$ where $C(x)$ is the centralizer of x in G.

   (b) H is normal in G.

Part II. RINGS AND MODULES

5. Let R be a ring with identity. Assume R has a unique maximal left ideal. Prove that

   (a) R/J is a division ring, where J is the Jacobson radical of R.

   (b) The characteristic of R is either 0 or a prime power.

6. Let R be a principal ideal domain (with identity) that is not a field. Prove that R is Jacobson semisimple if and only if R contains an infinite number of distinct, nonassociate irreducible elements.

7. Let R be a ring with identity with Jacobson radical J.

   (a) Prove that if M is a finitely generated left R-module such that $JM = M$, then $M = 0$. [Prove this - don't just quote a famous theorem.]

   (b) Use part (a) to prove that if R is Artinian, then J is nilpotent.

Part III. FIELDS AND GALOIS THEORY

8. (a) Find the Galois group of $x^4 - 2x^2 - 15$ over the field Q of rationals.

   (b) Use the Fundamental Theorem of Abelian Groups to prove that the multiplicative group of a finite field is cyclic.

9. An algebraic closure of a field K is an algebraic extension field F of K such that every polynomial in K[x] has a root in F. Prove that an algebraic extension field F of K is an algebraic closure of K if and only if for every extension field E of a field K_1 and isomorphism $\sigma: K_1 \to K$, σ extends to a monomorphism $E \to F$.

10. For any field E of characteristic $p \neq 0$, let $E^{p^n} = \{u^{p^n} | u \in E\}$. E^{p^n} is known to be a subfield of E. If L and M are subfields of E, then LM denotes the smallest subfield of E containing both L and M.

    Let F be a finite dimensional extension field of K, with char $K = p \neq 0$. Assume that $KF^p = F$. Prove that F is separable over K.
    [Hints: Let S be the maximal separable subfield of F. Show that $S = KF^{p^m}$ for some m. Then show that $F = KF^p = KF^{p^2} = \cdots = KF^{p^m}$.]

ALGEBRA PRELIM SPRING 1977

All rings R are commutative with identity.

1\. The parts of this problem are to be answered true or false. If true, give a short proof; if false give a counterexample.

(a) The only field automorphism of the field of real numbers $\mathbb{R}$ is the identity.

(b) Each group in column (A) can be paired with an isomorphic group in column (B). Here, $Z_n = Z/nZ$, $Z_n^{\times}$ = group of units with respect to multiplication in Z_n, S_n = symmetric group on n letters; and A_n = alternating group on n letters.

(A)	(B)
$Z_6 \times Z_2$	$Z_{15} \times Z_{12}^{\times}$
$Z_{15}^{\times}$	$Z_5^{\times} \times (263Z/526Z)$
$Z_{30} \times Z_2$	$Z_2 \times A_3 \times (S_{20}/A_{20})$
$Z_8^{\times} \times Z_{15}$	$Z_{11}^{\times} \times Z_7^{\times}$
Z_8	$Z_{64}/8Z_{64}$

(c) Let $K = k(x)$ be the field of rational functions in the indeterminate x over the field k. Let $\{f_i \mid i \in I\}$ be the set of all monic irreducible polynomials in $k[x]$. Then $K = k[x][1/f_i]$, $i \in I$.

(d) The group of nonzero rational numbers is a (discrete) direct product of cyclic groups.

(e) If S is a multiplicatively closed subset of a ring R, and $\{A_i\}$ is a collection of ideals of R, then $S^{-1}(\bigcap_i A_i) = \bigcap_i (S^{-1}A_i)$.

(f) If $\phi : M \to M$ is a surjective homomorphism of a noetherian R-module M, then ϕ is an isomorphism

(g) If for every element x in a ring R, there is an integer $n \geq 2$ so that $x^n = x$, then every prime ideal of R is maximal.

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(h) Two idempotent linear transformations of a finite dimensional vector space are similar if and only if they have the same rank.

(i) For any field $K \supseteq Q$ and any integer $n \geq 20$, there exist $A, B \in M_n(K)$ such that $AB - BA = 1$.

(j) Let J be the intersection of all maximal ideals in a ring R. Then $x \in J$ if and only if $1 - xy$ is a unit of R for all $y \in R$.

(k) Recall that a local ring is a ring with only one maximal ideal. The only idempotents of a local ring are 0 and 1.

II

1. (a) Let H and K be two subgroups of a group G, each of finite index in G. Prove that $H \cap K$ has finite index in G and give an upper bound for this index.

(b) Using the result of (a), show that if a subgroup H of G has finite index in G, then H contains a normal subgroup of G which has finite index in G.

2. Prove that the polynomial $1 + x + x^2 + x^3 + x^4 + x^5 + x^6$ is irreducible over the field of rational numbers Q, but is reducible over Z_{127}.

3. Find the Galois group of $x^4 + 2 = 0$ over Q, and express this group as a permutation group on the zeros of $x^4 + 2$.

4. (a) Let V be a finite dimensional vector space, and let T be an endomorphism of V. Further, suppose that $V = \bigoplus_i V_i$, where the V_i are T-stable subspaces of V, and that the corresponding projections $V \to V_i$ are polynomials in T. Prove that if W is any T-stable subspace of V, then $W = \bigoplus_i (W \cap V_i)$.

(b) Let $V = C^{(4)}$, where C is the field of complex numbers,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Find all of the subspaces of V that are invariant under both A and B.

(a) Let P be a prime ideal in a commutative ring with identity R. Show that the field of fractions of $R/P \cong R_P/PR_P$, where R_P denotes the localization of R at P.

(b) Give the ring structure of $Q(\sqrt{-1}) \otimes_Q C$ in terms of simple rings.

ALGEBRA PRELIM

Fall 1976

The exam is in four parts. Justify all your answers.

I. Do 1 out of 2

II. Do 2 out of 3

III. Do 2 out of 3

IV. Do 2 out of 3

I. Linear Algebra

1. V a finite-dimensional vector space over a field k. Let $f \in \mathrm{End}_k(V)$.

(i) Show that $V_1 = \bigcap_{n=1}^{\infty} f^n(V)$ and $V_2 = \bigcup_{n=1}^{\infty} \ker(f^n)$ are subspaces of V invariant under f, with $V = V_1 \oplus V_2$.

(ii) Show that $f|V_1$ is an automorphism and $f|V_2$ is nilpotent.

2. Consider a form $ax^2 + bxy + cy^2$, where the coefficients are integers, and the variables x,y are to be given integer values. The discriminant of the form is the number $\Delta = b^2 - 4ac$.

(i) If $\Delta < 0$, show that the form is positive or negative definite as $a > 0$ or $a < 0$.

(ii) The form $ax^2 + bxy + cy^2$ is said to be equivalent to the form $AX^2 + BXY + CY^2$ if the first is carried to the second by a unimodular substitution, i.e.

$$x = pX + qY$$
$$y = rX + sY$$

where p,q,r,s are integers with $ps - qr = 1$.
Show that two equivalent forms have the same discriminant. Also show that the forms $X^2 + 6Y^2$ and $2X^2 + 3Y^2$ are inequivalent.

(iii) A number n is said to be properly represented by the form if

$$n = ax^2 + bxy + cy^2$$

for some relatively prime pair (x,y). Show that n is properly represented by a form $ax^2 + bxy + cy^2 \Longleftrightarrow$ there is an equivalent form $AX^2 + BXY + CY^2$, with $A = n$.

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II. Group Theory

1. Let G be a finite group and $H \subset G$ a subgroup. Suppose that $G = \bigcup_{g \in G} g^{-1} Hg$.

   Show that if a prime number p divides the order of G, then p also divides the order of H.

2. Classify all isomorphism classes of groups of order 55.

3. Construct a non-abelian group of order 44 by means of the semi-direct product. Give a complete and explicit description of the group.

III. Fields, Rings, Modules

1. Let k be a field, and R a finite-dimensional commutative k-algebra with no non-zero nilpotent elements. Show that R is the direct product of fields.

Let S be a set, k a field and R a subring of the ring k^S of all functions from S to k where the multiplication on this ring is defined (pointwise) as

$$(f \cdot g)\ (s) = f(s) \cdot g\ (s).$$

a) Let T be a k-algebra. Show that $R \otimes_k T$ can also be viewed as a ring of functions from S to T under pointwise multiplication.

b) Show that if R and T are integral domains, then so is $R \otimes_k T$.

3. Let R be any ring, and let T be a non-split extension of a simple R-module M by a simple R-module N:

$$0 \to N \to T \to M \to 0.$$

Suppose that $N \not\cong M$. Show that every non-zero R-endomorphism of T is an isomorphism.

IV. Fields and Galois Theory

1. Let k be a field of characteristic $p \neq 0$, and let $\bar{k}$ be an algebraic closure of k. Define

$$k(1/p^{\infty}) = \{a \in \bar{k} \mid \exists n \text{ with } a^{p^n} \in k\}$$

Show that $k(1/p^{\infty})$ is a perfect field.

2. Let k be a field, $\bar{k}$ an algebraic closure. Let $k \subset L,\ T \subset \bar{k}$ be two intermediate fields. Let M be the subfield of $\bar{k}$ generated by L and T.
   (i) Show that if L and T are finite Galois extensions of k, so is M.
   (ii) Suppose that one of the following two (equivalent) conditions holds on L and T
      (*) L and T are linearly disjoint over k, i.e. $L \otimes_k T \xrightarrow{\text{mult}} \bar{k}$ is an injection.
      (**) $\dim_k M = \dim_k L \cdot \dim_k T$ Show that

$$\text{Gal}\ (M/k) \cong \text{Gal}\ (L/k) \oplus \text{Gal}\ (T/k).$$

3. Give examples of Galois field extensions K of the rationals $\mathbb{Q}$, where
   (i) $\text{Gal}\ (K/\mathbb{Q})$ = symmetric group on 3 letters.
   (ii) $\text{Gal}\ (K/\mathbb{Q})$ = cyclic group of order 6.

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ALGEBRA PRELIM SPRING 1976

Directions

1. The exam is in five parts. In part I you should answer as many questions as you can, but at least 4 of them. Be sure to allow plenty of time for the other four parts.

2. In parts II-V, where you have a choice of problems, mark clearly the problems you wish to have graded. Two half problems do not equal one problem.

3. You should write enough so that there is no doubt as to whether or not you know what's going on. In particular, if you use a big standard theorem, explain why it is applicable. (Do not feel obliged to state it in detail.) But use common sense. Don't write a book when a few lines are sufficient. When in doubt, ask.

4. Neatness counts in that illegible or very difficult-to-read exams will not be graded.

Part I State whether each of the following statements is true or false. If the statement is true give a short proof. If it is false, give a counterexample or otherwise disprove it.

1. There exists an Artinian integral domain which is not a field.

2. If N is a normal subgroup of a group G and $N \cap G' = \langle e \rangle$, where G' is the commutator subgroup and e the identity element of G, then N is contained in the center of G.

3. The Galois group of $x^3 + 3x^2 + 3x + 2$ over the field Q of rational numbers is Z_2.

4. If R is a Dedekind domain, then so is the polynomial ring R[x].

5. There exist 11-dimensional subspaces U and V of a 20-dimensional vector space W such that $U \cap V = 0$.

5. Let C be the field of complex numbers. If $f \in C[x_1, \ldots, x_n]$ and $V = \{p \in C^n | f(p) = 0\}$, then f is irreducible if and only if V is irreducible.

7. A field F is an algebraic closure of a field K if and only if F is algebraic over K and for every algebraic extension E of K there is a K-monomorphism $E \to F$.

Part II Answer one of the following questions.

8. Let A and B be idempotent matrices over a field (that is $A^2 = A$ and $B^2 = B$). Prove: A and B are similar if and only if A and B have the same rank.

9. Let K be the field $Z_2(y)$ of all rational functions of y over the field of two elements. Find the Galois group over K of the polynomial

$$f(x) = (x^2 + y)(x^2 + yx + y) \in K[x].$$

Justify your answer.

10. Describe the structure of all rings R which have no nilpotent elements and satisfy the descending chain condition on left ideals.

Part III Answer one of the following questions.

11. If H is a subgroup of G with $H \neq G$ and $[G:H]$ finite, then there exists a subgroup N of H such that N is normal in G and $[G:N]$ is finite.

12. Let H be a proper normal subgroup of a finite group G. Suppose p is a prime which divides the order of H and T is a Sylow p-subgroup of H. Prove that

$$G = HN,$$

where N is the normalizer of T in G.

13. Prove or disprove: there is a simple group of order 48.

Part IV Answer one of the following questions.

14. A square matrix A over a field is diagonalizable if there exists a nonsingular matrix Q such that QAQ^{-1} is a diagonal matrix. Two square matrices A and B over a field are simultaneously diagonalizable if there exists a nonsingular matrix P such that both PAP^{-1} and PBP^{-1} are diagonal matrices.

Prove: If A and B are each diagonalizable and $AB = BA$, then A and B are simultaneously diagonalizable.

(continued on next page)

Part IV continued

15. Let K and F be fields with $K \subset F$. Let V be an n-dimensional vector space over K and $f: V \to V$ a K-linear transformation. Prove:
   (a) $F \otimes_K V$ is an n-dimensional vector space over F.
   (b) $1_F \otimes f: F \otimes_K V \to F \otimes_K V$ is an F-linear transformation.
   (c) The characteristic polynomial of $1_F \otimes f$ in $F[x]$ actually lies in $K[x]$ and is equal to the characteristic polynomial of f in $K[x]$.

Part V Answer two of the following questions.

16. Let G be the multiplicative group of units in the ring Z_{77}. Is G a cyclic group? What is the order of G? Justify your answers.

17. Find the Jacobson radical of the following rings.
   (a) The endomorphism ring of the abelian group $Z \oplus Z_2$.
   (b) $Q[x]/(x(x-1)^3)$, where Q = rational numbers.
   (c) The ring of $n \times n$ upper triangular matrices over a field (that is, matrices (a_{ij}) with $a_{ij} = 0$ for all $i > j$).

18. Let R be the field of real numbers. In $R[x,y]$, for what polynomials f is the ideal (f) a radical ideal? In terms of the prime factorization of f, how many irreducible components appear in a decomposition of the curve $f(x,y) = 0$?

19. If $K \subset E \subset F$ are fields with F Galois over K, then there exists a unique field L such that
   (i) $E \subset L \subset F$;
   (ii) L is Galois over K;
   (iii) if $E \subset M \subset F$ with M Galois over K, then $L \subset M$;
   (iv) the Galois group of F over L is
   $$\bigcap_{\sigma \in G} \sigma H \sigma^{-1}$$
   where H is the Galois group of F over E and G is the Galois group of F over K.

(continued on next page)

Part V continued

20. Let R be a left Artinian ring with identity. Prove:

(a) If M is a maximal ideal of R, then M is the intersection of all maximal <u>left</u> ideals of R which contain M.

(b) If L is a maximal left ideal of R, then there exists a maximal ideal M with $M \subseteq L$.
[<u>Hint</u>: let M be the annihilator of R/L.]

(c) The Jacobson radical J of R is the intersection of all the maximal ideals of R.

(d) R has only finitely many maximal ideals.