# Du Bois singularities in families

(joint work with Takumi Murayama)

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# Properties of Du Bois singularities

Du Bois singularities are defined for schemes of finite type over fields of characteristic zero.

- Semi-log-canonical (slc) singularities are Du Bois (Kollár and Kovács 2010).
- If $f : X \to B$ is flat and projective with Du Bois fibers, $R^i f_* \mathcal{O}_X$ is locally free and compatible with base change for all $i$ (Du Bois and Jarraud 1974).

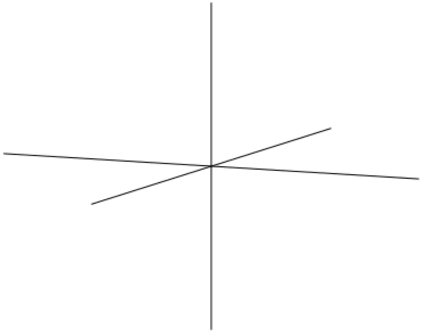
Application: If $f$ is a KSB-stable family, $\omega_{X/B}$ is flat over $B$ + compatible with base change.

# Examples of Du Bois singularities

- Du Bois singularities are semi-normal (Saito 2000).

For curves,

Du Bois $\iff$ $X$ semi-normal $\iff$ singularities analytic-locally isomorphic to unions of coordinate axes in $\mathbb{A}^n$



# Examples of Du Bois singularities – continued

- For *any* smooth projective variety $X$ over $\mathbb{C}$, there is an embedding

$$X \subseteq \mathbb{P}^N \text{ so that } C(X) \subseteq \mathbb{A}^{N+1}$$

  has Du Bois singularities (Bhatt, Schwede, and Takagi 2016, Lem. 2.14, Ma 2015, Thm. 4.4).
- Example: for $X$ a curve of genus $g > 1$, can use $\mathcal{O}_X(1) \simeq \omega_X^2$.
- $C(X)$ is only log canonical if $-K_X \sim_{\mathbb{Q}} rH$ for some $r \in \mathbb{Q}_{\geq 0}$ $(H \in |\mathcal{O}_X(1)|)$

# Du Bois and $F$-injective singularities

- Singularities of dense $F$-injective type are Du Bois (Schwede 2009).
- Cool fact: There is a "common definition" of Du Bois and $F$-injective singularities (Bhatt, Schwede, and Takagi 2016, Thm. 4.8)

# Common definition of $F$-injective and Du Bois singularities

## Definition (Bhatt, Schwede, and Takagi 2016, Thm. 4.8)

Let $x \in X$ be a point on a reduced scheme of finite type $X$ over $k$. For every proper hypercovering with smooth terms $\pi_\bullet : X_\bullet \to X$, there are natural maps

$$H^i_x(\mathcal{O}_X) \to \mathbb{H}^i_x(R\pi_{\bullet *}\mathcal{O}_{X_\bullet}) \text{ for } i \in \mathbb{N} \tag{1}$$

char $k = 0$ $X$ has Du Bois singularities at $x \iff$ the maps (1) are injective for all $\pi_\bullet$.

char $k = p > 0$, $k$ $F$-finite $X$ has $F$-injective singularities at $x \iff$ the maps (1) are injective for all $\pi_\bullet$.

# Permanence properties

Results that guarantee a class of singularities is preserved under some natural algebro-geometric construction. Example:

**Theorem (see e.g. Matsumura 1989, Thm. 23.7)**

*Let $f : Y \to X$ be a flat morphism of locally noetherian schemes.*

1. *If $f$ is* faithfully *flat and $Y$ is regular, then $X$ is regular, and*
2. *if $X$ and all of the fibers $Y_x := f^{-1}(x)$ are regular then $Y$ is regular.*

# Descent and ascent for Du Bois singularities

We can replace "regular" with "Du Bois."

**Theorem (G.-Murayama)**

*Let $f : Y \to X$ be a flat morphism of separated schemes of finite type over a field $k$ of characteristic zero.*

1. *If $f$ is* faithfully *flat and $Y$ has Du Bois singularities, then so does $X$.*
2. *If both $X$ and the fibers of $f$ have Du Bois singularities, then $Y$ has Du Bois singularities.*

Recovers a result of Doherty 2008: if $X, Z$ are Du Bois, then so is $Y \coloneqq X \times_k Z$ (special case of item 2).
Slogan: having Du Bois singularities is a *fppf-local* condition.

# Some essential ingredients

Splitting criteria Having Du Bois singularities is equivalent to the splitting of a certain map

$$\begin{tikzcd} \mathcal{O}_X \arrow[r] & \underline{\Omega}^0_X \arrow[l, bend right, dashed, "\sigma"'] \end{tikzcd} \tag{2}$$

Faithful flatness and splittings

## Lemma (Antieau and Datta 2020, Prop. 2.4.3, 2.4.7)

*Let $g\colon Y \to X$ be a faithfully flat morphism of affine schemes, with $X$ coherent, and let $\sigma\colon \mathcal{F} \to \mathcal{G}$ be a morphism in $D^b_{\mathrm{coh}}X$. Then $\sigma$ splits in $D^b_{\mathrm{coh}}X$ if and only if the induced morphism*

$$f^*\sigma\colon f^*\mathcal{F} \to f^*\mathcal{G} \text{ splits in } D^b_{\mathrm{coh}}Y$$

# Application: openness of the Du Bois locus

For a morphism $f : Y \to X$, define

$$U_{\mathrm{DB}}(f) \coloneqq \{x \in X \mid Y_x \text{ has Du Bois singularities}\} \subseteq X$$

**Question**

If $f$ is flat and proper, is $U_{\mathrm{DB}}(f)$ open?

- Known for $X$ smooth (Kovács and Schwede 2016, Cor. 4.2)
- Analogous result for *rational* singularities is a theorem of Elkik 1978 (generalized to pairs in Erickson 2014)

Can be proved in two steps (Hartshorne Ex. II.3.18):

$U_{\mathrm{DB}}(f)$ is constructible Follows proof of Kovács and Schwede 2016

$U_{\mathrm{DB}}(f)$ is stable under generization Uses the general framework in Murayama 2020 together with the above permanence properties

# Permanence properties $\rightsquigarrow$ stability under generization

## Theorem (Murayama 2020, Thm. A, paraphrased)

*Let* R *be a property of noetherian local rings satisfying*

- *ascent,*
- *descent,*
- *lifting from Cartier divisors and*
- *localization,*

*such that regular local rings satisfy* R*. Let $f : Y \to X$ be a flat morphism of noetherian schemes. If $f$ is closed and the local rings of $X$ have geometrically* R *formal fibers, then*

$$U_{\mathsf{R}}(f) := \{x \in X \,|\, f^{-1}(x) \textit{ is geometrically } \mathsf{R} \textit{ over } k(x)\}$$

*is stable under generization.*

# Openness of the Du Bois locus – continued

## Theorem (G.-Murayama)

*Let $f : Y \to X$ be a flat, proper morphism between separated schemes of finite type over a field of characteristic zero. Then, the locus*

$$U_{\mathrm{DB}}(f) := \{x \in X \,|\, Y_x \text{ has Du Bois singularities}\} \subseteq X$$

*is open.*

Thank you!

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