THE GREEN BOOK OF MATHEMATICAL PROBLEMS

Kenneth Hardy and

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OF MATHEMATICAL PROBLEMS

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PREFACE

There is a famous set of fairy tale books, each volume of which is designated by the colour of its cover: *The Red Book, The Blue Book, The Yellow Book*, etc. We are not presenting you with *The Green Book* of fairy stories, but rather a book of mathematical problems. However, the conceptual idea of all fairy stories, that of mystery, search, and discovery is also found in our *Green Book*. It got its title simply because in its infancy it was contained and grew between two ordinary green file covers.

The book contains 100 problems for undergraduate students training for mathematics competitions, particularly the William Lowell Putnam Mathematical Competition. Along with the problems come useful hints, and in the end (just like in the fairy tales) the solutions to the problems. Although the book is written especially for students training for competitions, it will also be useful to anyone interested in the posing and solving of challenging mathematical problems at the undergraduate level.

Many of the problems were suggested by ideas originating in articles and problems in mathematical journals such as *Crux Mathematicorum, Mathematics Magazine*, and the *American Mathematical Monthly,* as well as problems from the Putnam competition itself. Where possible, acknowledgement to known sources is given at the end of the book.

We would, of course, be interested in your reaction to *The Green Book*, and invite comments, alternate solutions, and even corrections. We make no claims that our solutions are the "best possible" solutions, but we trust you will find them elegant enough, and that *The Green Book* will be a practical tool in the training of young competitors.

We wish to thank our publisher, Integer Press; our literary adviser; and our typist, David Conibear, for their invaluable assistance in this project.

Kenneth Hardy and Kenneth S. Williams
Ottawa, Canada
May, 1985

Dedicated to the contestants of the William Lowell Putnam Mathematical Competition

To Carole with love
KSW

CONTENTS

Page

Notation ix
The Problems 1
The Hints 25
The Solutions 41
Abbreviations 169
References 171

NOTATION

$[x]$	denotes the greatest integer $\leq x$, where x is a real number.
$\{x\}$	denotes the fractional part of the real number x, that is, $\{x\} = x - [x]$.
$\ln x$	denotes the natural logarithm of x.
$\exp x$	denotes the exponential function of x.
$\varphi(n)$	denotes Euler's totient function defined for any natural number n.
$GCD(a,b)$	denotes the greatest common divisor of the integers a and b.
$\binom{n}{k}$	denotes the binomial coefficient $n!/k!(n-k)!$, where n and k are non-negative integers (the symbol having value zero when $n < k$).
(a_{ij})	denotes a matrix with a_{ij} as the (i,j)th entry.
$\det A$	denotes the determinant of a square matrix A.

THE PROBLEMS

Problems, problems,
problems all day long.
Will my problems work out right or wrong?

The Everly Brothers

1. If $\{b_n : n = 0,1,2,\dots\}$ is a sequence of non-negative real numbers, prove that the series

$$\sum_{n=0}^{\infty} \frac{b_n}{(a+b_0+b_1+\dots+b_n)^{3/2}} \tag{1.0}$$

converges for every positive real number a.

2. Let a,b,c,d be positive real numbers, and let

$$Q_n(a,b,c,d) = \frac{a(a+b)(a+2b)\dots(a+(n-1)b)}{c(c+d)(c+2d)\dots(c+(n-1)d)} .$$

Evaluate the limit $L = \lim_{n\to\infty} Q_n(a,b,c,d)$.

3. Prove the following inequality:

$$\frac{\ln x}{x^3-1} < \frac{1}{3}\,\frac{(x+1)}{(x^3+x)} \quad , \quad x > 0,\ x \neq 1. \tag{3.0}$$

4. Do there exist non-constant polynomials $p(z)$ in the complex variable z such that $|p(z)| < R^n$ on $|z| = R$, where $R > 0$ and $p(z)$ is monic and of degree n?

5. Let $f(x)$ be a continuous function on $[0,a]$, where $a > 0$, such that $f(x) + f(a-x)$ does not vanish on $[0,a]$. Evaluate the integral

$$\int_0^a \frac{f(x)}{f(x) + f(a-x)}\,dx .$$

6. For $\varepsilon > 0$ evaluate the limit

$$\lim_{x \to \infty} x^{1-\varepsilon} \int_x^{x+1} \sin(t^2)\,dt .$$

7. Prove that the equation

$$x^4 + y^4 + z^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2 = 24 \tag{7.0}$$

has no solution in integers x,y,z.

8. Let a and k be positive numbers such that $a^2 > 2k$. Set $x_0 = a$ and define x_n recursively by

$$x_n = x_{n-1} + \frac{k}{x_{n-1}}, \quad n = 1,2,3,\ldots . \tag{8.0}$$

Prove that

$$\lim_{n \to \infty} \frac{x_n}{\sqrt{n}}$$

9. Let x_0 denote a fixed non-negative number, and let a and b be positive numbers satisfying

$$\sqrt{b} < a < 2\sqrt{b} .$$

Define x_n recursively by

(9.0) $$x_n = \frac{ax_{n-1} + b}{x_{n-1} + a}, \quad n = 1,2,3,\ldots .$$

Prove that $\lim_{n\to\infty} x_n$ exists and determine its value.

10. Let a,b,c be real numbers satisfying

$$a > 0,\ c > 0,\ b^2 > ac .$$

Evaluate

$$\max_{\substack{x,y \in R \\ x^2+y^2=1}} (ax^2 + 2bxy + cy^2) .$$

11. Evaluate the sum

(11.0) $$S = \sum_{r=0}^{[n/2]} \frac{n(n-1)\ldots(n-(2r-1))}{(r!)^2} 2^{n-2r}$$

for n a positive integer.

12. Prove that for $m = 0,1,2,\ldots$

(12.0) $$S_m(n) = 1^{2m+1} + 2^{2m+1} + \ldots + n^{2m+1}$$

is a polynomial in $n(n+1)$.

13. Let a,b,c be positive integers such that

$$\mathrm{GCD}(a,b) = \mathrm{GCD}(b,c) = \mathrm{GCD}(c,a) = 1 .$$

Show that $\ell = 2abc - (bc+ca+ab)$ is the largest integer such that

$$bc\,x + ca\,y + ab\,z = \ell$$

is insolvable in non-negative integers x,y,z.

14. Determine a function $f(n)$ such that the n^{th} term of the sequence

(14.0) $$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, \ldots$$

is given by $[f(n)]$.

15. Let $a_1, a_2, \ldots, a_n$ be given real numbers, which are not all zero. Determine the least value of

$$x_1^{\,2} + \ldots + x_n^{\,2} ,$$

where $x_1, \ldots, x_n$ are real numbers satisfying

$$a_1x_1 + \ldots + a_nx_n = 1 .$$

16. Evaluate the infinite series

$$S = 1 - \frac{2^3}{1!} + \frac{3^3}{2!} - \frac{4^3}{3!} + \ldots .$$

17. $F(x)$ is a differentiable function such that $F'(a-x) = F'(x)$ for all x satisfying $0 \leq x \leq a$. Evaluate $\int_0^a F(x)dx$ and give an example of such a function $F(x)$.

18. (a) Let r,s,t,u be the roots of the quartic equation

$$x^4 + Ax^3 + Bx^2 + Cx + D = 0 .$$

Prove that if $rs = tu$ then $A^2D = C^2$.

(b) Let a,b,c,d be the roots of the quartic equation

$$y^4 + py^2 + qy + r = 0 .$$

Use (a) to determine the cubic equation (in terms of p,q,r) whose roots are

$$\frac{ab - cd}{a + b - c - d}, \quad \frac{ac - bd}{a + c - b - d}, \quad \frac{ad - bc}{a + d - b - c} .$$

19. Let $p(x)$ be a monic polynomial of degree $m \geq 1$, and set

$$f_n(x) = e^{p(x)}D^n(e^{-p(x)}) ,$$

where n is a non-negative integer and $D \equiv \frac{d}{dx}$ denotes differentiation with respect to x.

Prove that $f_n(x)$ is a polynomial in x of degree $(mn - n)$. Determine the ratio of the coefficient of x^{mn-n} in $f_n(x)$ to the constant term in $f_n(x)$.

20. Determine the real function of x whose power series is

$$\frac{x^3}{3!} + \frac{x^9}{9!} + \frac{x^{15}}{15!} + \cdots .$$

21. Determine the value of the integral

$$I_n = \int_0^{\pi} \left(\frac{\sin nx}{\sin x}\right)^2 dx , \tag{21.0}$$

22. During the year 1985, a convenience store, which was open 7 days a week, sold at least one book each day, and a total of 600 books over the entire year. Must there have been a period of consecutive days when exactly 129 books were sold?

23. Find a polynomial $f(x,y)$ with rational coefficients such that as m and n run through all positive integral values, $f(m,n)$ takes on all positive integral values once and once only.

24. Let m be a positive squarefree integer. Let R,S be positive integers. Give a condition involving R,S,m which guarantees that there do not exist rational numbers x,y,z and w such that

$$R + 2S\sqrt{m} = (x + y\sqrt{m})^2 + (z + w\sqrt{m})^2 . \tag{24.0}$$

25. Let k and h be integers with $1 \leq k < h$. Evaluate the limit

$$L = \lim_{n \to \infty} \prod_{r=kn+1}^{hn} \left(1 - \frac{r}{n^2}\right) . \tag{25.0}$$

26. Let $f(x)$ be a continuous function on $[0,a]$ such that $f(x)f(a-x) = 1$, where $a > 0$. Prove that there exist infinitely many such functions $f(x)$, and evaluate

$$\int_0^a \frac{dx}{1 + f(x)} .$$

27. The positive numbers $a_1, a_2, a_3, \ldots$ satisfy

28. Let $p > 0$ be a real number and let n be a non-negative integer. Evaluate

(28.0) $$u_n(p) = \int_0^\infty e^{-px}\sin^n x\,dx \ .$$

29. Evaluate

(29.0) $$\sum_{r=0}^{n-2} 2^r \tan \frac{\pi}{2^{n-r}} ,$$

for integers $n \geq 2$.

30. Let $n \geq 2$ be an integer. A selection $\{s = a_i: i=1,2,\ldots,k\}$ of k $(2 \leq k \leq n)$ elements from the set $N = \{1,2,3,\ldots,n\}$ such that $a_1 < a_2 < \ldots < a_k$ is called a k-selection. For any k-selection S , define

$$W(S) = \min\,\{a_{i+1}-a_i: i = 1,2,\ldots,k-1\} .$$

If a k-selection S is chosen at random from N , what is the probability that

$$W(S) = r ,$$

where r is a natural number?

31. Let $k \geq 2$ be a fixed integer. For $n = 1,2,3,\ldots$ define

$$a_n = \begin{cases} 1 , & \text{if } n \text{ is not a multiple of } k , \\ -(k-1) , & \text{if } n \text{ is a multiple of } k . \end{cases}$$

Evaluate the series $$\sum_{n=1}^{\infty} \frac{a_n}{n} .$$

32. Prove that

$$\int_0^\infty x^m e^{-x} \sin x\, dx = \frac{m!}{2^{(m+2)/2}} \sin\ (m+1)\pi/4$$

for $m = 0,1,2,\ldots$.

33. For a real number u set

$$I(u) = \int_0^\pi \ln(1 - 2u\cos x + u^2)\, dx \quad . \tag{33.0}$$

Prove that

$$I(u) = I(-u) = \tfrac{1}{2}I(u^2) \ ,$$

and hence evaluate $I(u)$ for all values of u .

34. For each natural number $k \geq 2$ the set of natural numbers is partitioned into a sequence of sets $\{A_n(k) : n = 1,2,3,\ldots\}$ as follows: $A_1(k)$ consists of the first k natural numbers, $A_2(k)$ consists of the next $k+1$ natural numbers, $A_3(k)$ consists of the next $k+2$ natural numbers, etc. The sum of the natural numbers in $A_n(k)$ is denoted by $s_n(k)$. Determine the least value of $n = n(k)$ such that $s_n(k) > 3k^3 - 5k^2$, for $k = 2,3,\ldots$.

35. Let $\{p_n : n = 1,2,3,\ldots\}$ be a sequence of real numbers such that $p_n \geq 1$ for $n = 1,2,3,\ldots$. Does the series

$$\sum_{n=1}^{\infty} \frac{[p_n]-1}{([p_1]+1)([p_2]+1)\ldots([p_n]+1)} \tag{35.0}$$

converge?

leading coefficients A, B respectively. Evaluate

$$L = \lim_{x\to\infty} g(x)\int_0^x e^{f(t)-f(x)}\,dt ,$$

in terms of A, B and n.

37. The lengths of two altitudes of a triangle are h and k, where $h \neq k$. Determine upper and lower bounds for the length of the third altitude in terms of h and k.

38. Prove that

$$P_{n,r} = P_{n,r}(x) = \frac{(1-x^{n+1})(1-x^{n+2})\dots(1-x^{n+r})}{(1-x)(1-x^2)\dots(1-x^r)}$$

is a polynomial in x of degree nr, where n and r are non-negative integers. (When $r = 0$ the empty product is understood to be 1 and we have $P_{n,0} = 1$ for all $n \geq 0$.)

39. Let A, B, C, D, E be integers such that $B \neq 0$ and

$$F = AD^2 - BCD + B^2E \neq 0 .$$

Prove that the number N of pairs of integers (x,y) such that

$$Ax^2 + Bxy + Cx + Dy + E = 0 , \tag{39.0}$$

satisfies

$$N \leq 2d(|F|) ,$$

where, for integers $n \geq 1$, $d(n)$ denotes the number of positive divisors of n.

41. Let $P_m = P_m(n)$ denote the sum of all possible products of m different integers chosen from the set $\{1,2,\ldots,n\}$. Find formulae for $P_2(n)$ and $P_3(n)$.

42. For $a > b > 0$, evaluate the integral

$$\int_0^\infty \frac{e^{ax} - e^{bx}}{x(e^{ax}+1)(e^{bx}+1)}\,dx \,. \tag{42.0}$$

43. For integers $n \geq 1$, determine the sum of n terms of the series

$$\frac{2n}{2n-1} + \frac{2n(2n-2)}{(2n-1)(2n-3)} + \frac{2n(2n-2)(2n-4)}{(2n-1)(2n-3)(2n-5)} + \cdots \,. \tag{43.0}$$

44. Let m be a fixed positive integer and let $z_1, z_2, \ldots, z_k$ be k (≥ 1) complex numbers such that

$$z_1^s + z_2^s + \ldots + z_k^s = 0 \,, \tag{44.0}$$

for all $s = m, m+1, m+2, \ldots, m+k-1$. Must $z_i = 0$ for $i = 1,2,\ldots,k$?

45. Let $A_n = (a_{ij})$ be the $n \times n$ matrix where

$$a_{ij} = \begin{cases} x\,, & \text{if } i = j\,, \\ 1\,, & \text{if } |i-j| = 1\,, \\ 0\,, & \text{otherwise,} \end{cases}$$

where $x > 2$. Evaluate $D_n = \det A_n$.

46. Determine a necessary and sufficient condition for the equa-

$$(46.0)\qquad \begin{cases} x + y + z = A , \\ x^2 + y^2 + z^2 = B , \\ x^3 + y^3 + z^3 = C , \end{cases}$$

to have a solution with at least one of x,y,z equal to zero.

47. Let S be a set of k distinct integers chosen from $1,2,3,\ldots,10^n-1$, where n is a positive integer. Prove that if

$$(47.0)\qquad n < \ell n\left(\frac{(2^k-1)}{k} + \frac{(k+1)}{2}\right) \Big/ \ell n\ 10 ,$$

it is possible to find 2 disjoint subsets of S whose members have the same sum.

48. Let n be a positive integer. Is it possible for $6n$ distinct straight lines in the Euclidean plane to be situated so as to have at least $6n^2-3n$ points where exactly three of these lines intersect and at least $6n+1$ points where exactly two of these lines intersect?

49. Let S be a set with n (≥ 1) elements. Determine an explicit formula for the number $A(n)$ of subsets of S whose cardinality is a multiple of 3.

50. For each integer $n \geq 1$, prove that there is a polynomial $p_n(x)$ with integral coefficients such that

$$x^{4n}(1-x)^{4n} = (1+x^2)p_n(x) + (-1)^n 4^n .$$

Define the rational number a_n by

Prove that a_n satisfies the inequality

$$|\pi - a_n| < \frac{1}{4^{5n-1}}, \quad n = 1,2,\dots .$$

51. In last year's boxing contest, each of the 23 boxers from the blue team fought exactly one of the 23 boxers from the green team, in accordance with the contest regulation that opponents may only fight if the absolute difference of their weights is less than one kilogram.

Assuming that this year the members of both teams remain the same as last year and that their weights are unchanged, show that the contest regulation is satisfied if the lightest member of the blue team fights the lightest member of the green team, the next lightest member of the blue team fights the next lightest member of the green team, and so on.

52. Let S be the set of all composite positive odd integers less than 79 .

(a) Show that S may be written as the union of three (not necessarily disjoint) arithmetic progressions.

(b) Show that S cannot be written as the union of two arithmetic progressions.

53. For $b > 0$, prove that

$$\left| \int_0^b \frac{\sin x}{x}\, dx - \frac{\pi}{2} \right| < \frac{1}{b} ,$$

by first showing that

54. Let $a_1, a_2, \ldots, a_{44}$ be 44 natural numbers such that

$$0 < a_1 < a_2 < \ldots < a_{44} \le 125 .$$

Prove that at least one of the 43 differences $d_j = a_{j+1} - a_j$ occurs at least 10 times.

55. Show that for every natural number n there exists a prime p such that $p = a^2 + b^2$, where a and b are natural numbers both greater than n. (You may appeal to the following two theorems:

(A) If p is a prime of the form $4t+1$ then there exist integers a and b such that $p = a^2 + b^2$.

(B) If r and s are natural numbers such that $\mathrm{GCD}(r,s) = 1$, there exist infinitely many primes of the form $rk+s$, where k is a natural number.)

56. Let $a_1, a_2, \ldots, a_n$ be n (≥ 1) integers such that

(1) $0 < a_1 < a_2 < \ldots < a_n$,

(2) all the differences $a_i - a_j$ $(1 \le j < i \le n)$ are distinct,

(3) $a_i \equiv a \pmod{b}$ $(1 \le i \le n)$, where a and b are positive integers such that $1 \le a \le b-1$.

Prove that

$$\sum_{r=1}^{n} a_r \ge \frac{b}{6} n^3 + \left(a - \frac{b}{6}\right) n .$$

57. Let $A_n = (a_{ij})$ be the $n \times n$ matrix where

$$a_{ij} = \begin{cases} 2\cos t, & \text{if } i = j, \\ 1, & \text{if } |i - j| = 1, \\ 0, & \text{otherwise} \end{cases}$$

58. Let a and b be fixed positive integers. Find the general solution of the recurrence relation

$$x_{n+1} = x_n + a + \sqrt{b^2 + 4ax_n}\ , \quad n = 0,1,2,\ldots \ , \tag{58.0}$$

where $x_0 = 0$.

59. Let a be a fixed real number satisfying $0 < a < \pi$, and set

$$I_r = \int_{-a}^{a} \frac{1 - r\cos u}{1 - 2r\cos u + r^2}\, du\ . \tag{59.0}$$

Prove that

$$I_1\ ,\quad \lim_{r\to 1^+} I_r\ ,\quad \lim_{r\to 1^-} I_r$$

all exist and are all distinct.

60. Let I denote the class of all isosceles triangles. For $\Delta \in I$, let h_Δ denote the length of each of the two equal altitudes of Δ and k_Δ the length of the third altitude. Prove that there does not exist a function f of h_Δ such that

$$k_\Delta \le f(h_\Delta)\ ,$$

for all $\Delta \in I$.

61. Find the minimum value of the expression

$$\left(x^2 + \frac{k^2}{x^2}\right) - 2\left((1+\cos t)x + \frac{k(1+\sin t)}{x}\right) + (3 + 2\cos t + 2\sin t)\ , \tag{61.0}$$

for $x > 0$ and $0 \le t \le 2\pi$, where $k > \frac{3}{2} + \sqrt{2}$ is a fixed real number.

62. Let $\varepsilon > 0$. Around every point in the xy-plane with integral co-ordinates draw a circle of radius ε . Prove that every straight line through the origin must intersect an infinity of these circles.

63. Let n be a positive integer. For $k = 0,1,2,\ldots,2n-2$ define

$$I_k = \int_0^\infty \frac{x^k}{x^{2n}+x^n+1}\,dx \quad . \tag{63.0}$$

Prove that $I_k \geq I_{n-1}$, $k = 0,1,2,\ldots,2n-2$.

64. Let D be the region in Euclidean n-space consisting of all n-tuples $(x_1,x_2,\ldots,x_n)$ satisfying

$$x_1 \geq 0 \ , \ x_2 \geq 0 \ , \ \ldots \ , \ x_n \geq 0 \ , \quad x_1 + x_2 + \ldots + x_n \leq 1 \ .$$

Evaluate the multiple integral

$$\iint_D \cdots \int x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \ (1-x_1-x_2-\ldots-x_n)^{k_{n+1}} dx_1 \ldots dx_n \ , \tag{64.0}$$

where $k_1,\ldots,k_{n+1}$ are positive integers.

65. Evaluate the limit

$$L = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\left[\frac{2\sqrt{n}}{\sqrt{k}}\right] - 2\left[\frac{\sqrt{n}}{\sqrt{k}}\right] \right) \ .$$

66. Let p and q be distinct primes. Let S be the sequence consisting of the members of the set

integers, give an explicit expression involving a, b, p and q for the position of $p^a q^b$ in the sequence S .

67. Let p denote an odd prime and let Z_p denote the finite field consisting of the p elements $0,1,2,\ldots,p-1$. For a an element of Z_p , determine the number $N(a)$ of 2×2 matrices X , with entries from Z_p , such that

$$(67.0) \qquad X^2 = A \ , \quad \text{where} \quad A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} .$$

68. Let n be a non-negative integer and let $f(x)$ be the unique differentiable function defined for all real x by

$$(68.0) \qquad (f(x))^{2n+1} + f(x) - x = 0 .$$

Evaluate the integral

$$\int_0^x f(t)\,dt ,$$

for $x \geq 0$.

69. Let $f(n)$ denote the number of zeros in the usual decimal representation of the positive integer n , so that for example, $f(1009) = 2$. For $a > 0$ and N a positive integer, evaluate the limit

$$L = \lim_{N\to\infty} \frac{\ell n\ S(N)}{\ell n\ N},$$

where

$$S(N) = \sum_{k=1}^{N} a^{f(k)} .$$

70. Let $n \geq 2$ be an integer and let k be an integer with $2 \leq k \leq n$. Evaluate

$$M = \max_{S} \left(\min_{1 \leq i \leq k-1} (a_{i+1} - a_i) \right) ,$$

where S runs over all selections $S = \{a_1, a_2, \ldots, a_k\}$ from $\{1,2,\ldots,n\}$ such that $a_1 < a_2 < \ldots < a_k$.

71. Let $az^2 + bz + c$ be a polynomial with complex coefficients such that a and b are nonzero. Prove that the zeros of this polynomial lie in the region

(71.0) $$|z| \leq \left|\frac{b}{a}\right| + \left|\frac{c}{b}\right| .$$

72. Determine a monic polynomial $f(x)$ with integral coefficients such that $f(x) \equiv 0 \pmod{p}$ is solvable for every prime p but $f(x) = 0$ is not solvable with x an integer.

73. Let n be a fixed positive integer. Determine

$$M = \max_{\substack{0 \leq x_k \leq 1 \\ k=1,2,\ldots,n}} \sum_{1 \leq i < j \leq n} |x_i - x_j| .$$

74. Let $\{x_i : i = 1,2,\ldots,n\}$ and $\{y_i : i = 1,2,\ldots,n\}$ be two sequences of real numbers with

$$x_1 \geq x_2 \geq \ldots \geq x_n .$$

How must $y_1, \ldots, y_n$ be rearranged so that the sum

(74.0) $$\sum^{n} (x_i - y_i)^2$$

75. Let p be an odd prime and let Z_p denote the finite field consisting of $0,1,2,\dots,p-1$. Let g be a given function on Z_p with values in Z_p. Determine all functions f on Z_p with values in Z_p, which satisfy the functional equation

$$f(x) + f(x+1) = g(x) \tag{75.0}$$

for all x in Z_p.

76. Evaluate the double integral

$$I = \int_0^1\int_0^1 \frac{dxdy}{1-xy}. \tag{76.0}$$

77. Let a and b be integers and m an integer > 1. Evaluate

$$\left[\frac{b}{m}\right] + \left[\frac{a+b}{m}\right] + \left[\frac{2a+b}{m}\right] + \dots + \left[\frac{(m-1)a+b}{m}\right].$$

78. Let $a_1,\dots,a_n$ be $n\ (>1)$ distinct real numbers. Set

$$S = a_1^2 + \dots + a_n^2, \quad M = \min_{1\le i<j\le n} (a_i - a_j)^2.$$

Prove that

$$\frac{S}{M} \ge \frac{n(n-1)(n+1)}{12}.$$

79. Let $x_1,\dots,x_n$ be n real numbers such that

$$\sum_{k=1}^n |x_k| = 1, \quad \sum_{k=1}^n x_k = 0.$$

80. Prove that the sum of two consecutive odd primes is the product of at least three (possibly repeated) prime factors.

81. Let $f(x)$ be an integrable function on the closed interval $[\pi/2,\pi]$ and suppose that

$$\int_{\pi/2}^{\pi} f(x)\sin kx\,dx = \begin{cases} 0\ , & 1 \le k \le n-1\ , \\ 1\ , & k = n\ . \end{cases}$$

Prove that $|f(x)| \ge \dfrac{1}{\pi \ln 2}$ on a set of positive measure.

82. For $n = 0,1,2,\ldots$, let

$$s_n = \sqrt[3]{a_n + \sqrt[3]{a_{n-1} + \sqrt[3]{a_{n-2} + \ldots + \sqrt[3]{a_0}}}} \tag{82.0}$$

where $a_n = \dfrac{6n+1}{n+1}$. Show that $\lim\limits_{n\to\infty} s_n$ exists and determine its value.

83. Let $f(x)$ be a non-negative strictly increasing function on the interval $[a,b]$, where $a < b$. Let $A(x)$ denote the area below the curve $y = f(x)$ and above the interval $[a,x]$, where $a \le x \le b$, so that $A(a) = 0$.

Let $F(x)$ be a function such that $F(a) = 0$ and

$$(x' - x)f(x) < F(x') - F(x) < (x'-x)f(x') \tag{83.0}$$

for all $a \le x < x' \le b$. Prove that $A(x) = F(x)$ for $a \le x \le b$.

84. Let a and b be two given positive numbers with $a < b$. How should the number r be chosen in the interval $[a,b]$ in order

85. Let $\{a_n : n = 1,2,\dots\}$ be a sequence of positive real numbers with $\lim_{n\to\infty} a_n = 0$ and satisfying the condition $a_n - a_{n+1} > a_{n+1} - a_{n+2} > 0$. For any $\varepsilon > 0$, let N be a positive integer such that $a_N \le 2\varepsilon$. Prove that $L = \sum_{k=1}^{\infty} (-1)^{k+1} a_k$ satisfies the inequality

$$(85.0) \qquad \left| L - \sum_{k=1}^{N} (-1)^{k+1} a_k \right| < \varepsilon .$$

86. Determine all positive continuous functions $f(x)$ defined on the interval $[0,\pi]$ for which

$$(86.0) \qquad \int_0^{\pi} f(x) \cos nx \, dx = (-1)^n (2n+1) , \quad n = 0,1,2,3,4 .$$

87. Let P and P' be points on opposite sides of a non-circular ellipse E such that the tangents to E through P and P' respectively are parallel and such that the tangents and normals to E at P and P' determine a rectangle R of maximum area. Determine the equation of E with respect to a rectangular coordinate system, with origin at the centre of E and whose y-axis is parallel to the longer side of R.

88. If four distinct points lie in the plane such that any three of them can be covered by a disk of unit radius, prove that all four points may be covered by a disk of unit radius.

89. Evaluate the sum

90. If n is a positive integer which can be expressed in the form $n = a^2 + b^2 + c^2$, where a,b,c are positive integers, prove that, for each positive integer k, n^{2k} can be expressed in the form $A^2 + B^2 + C^2$, where A,B,C are positive integers.

91. Let G be the group generated by a and b subject to the relations $aba = b^3$ and $b^5 = 1$. Prove that G is abelian.

92. Let $\{a_n : n = 1,2,3,\dots\}$ be a sequence of real numbers satisfying $0 < a_n < 1$ for all n and such that $\sum_{n=1}^{\infty} a_n$ diverges while $\sum_{n=1}^{\infty} a_n^2$ converges. Let $f(x)$ be a function defined on $[0,1]$ such that $f''(x)$ exists and is bounded on $[0,1]$. If $\sum_{n=1}^{\infty} f(a_n)$ converges, prove that $\sum_{n=1}^{\infty} |f(a_n)|$ also converges.

93. Let a,b,c be real numbers such that the roots of the cubic equation

$$x^3 + ax^2 + bx + c = 0 \tag{93.0}$$

are all real. Prove that these roots are bounded above by $(2\sqrt{a^2-3b} - a)/3$.

94. Let $Z_5 = \{0,1,2,3,4\}$ denote the finite field with 5 elements. Let a,b,c,d be elements of Z_5 with $a \neq 0$. Prove that the number N of distinct solutions in Z_5 of the cubic equation

$$f(x) \equiv a + bx + cx^2 + dx^3 = 0$$

$$A = \begin{bmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{bmatrix} .$$

95. Prove that

$$S = \sum_{\substack{m,n=1 \\ (m,n)=1}}^{\infty} \frac{1}{(mn)^2} \tag{95.0}$$

is a rational number.

96. Prove that there does not exist a rational function $f(x)$ with real coefficients such that

$$f\left(\frac{x^2}{x+1}\right) = p(x) , \tag{96.0}$$

where $p(x)$ is a non-constant polynomial with real coefficients.

97. For n a positive integer, set

$$S(n) = \sum_{k=0}^{n} \frac{1}{\binom{n}{k}} .$$

Prove that

$$S(n) = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k} .$$

98. Let $u(x)$ be a non-trivial solution of the differential equation

on I . Prove that u has only finitely many zeros in any interval $[a,b]$, $1 \le a < b$.

(A zero of $u(x)$ is a point z , $1 \le z < \infty$, with $u(z) = 0$).

99. Let P_j $(j = 0,1,2,\ldots,n-1)$ be n (≥ 2) equally spaced points on a circle of unit radius. Evaluate the sum

$$S(n) = \sum_{0 \le j < k \le n-1} |P_j P_k|^2 ,$$

where $|PQ|$ denotes the distance between the points P and Q .

100. Let M be a 3×3 matrix with entries chosen at random from the finite field $Z_2 = \{0,1\}$. What is the probability that M is invertible?

THE HINTS

The little fishes of the sea,
  They sent an answer back to me.

The little fishes' answer was
  "We cannot do it, Sir, because ——,"

Lewis Carroll

1. Define

$$a_n = a + b_0 + b_1 + \dots + b_n , \quad n \geq 0,$$

and prove an inequality of the type

$$\frac{a_n - a_{n-1}}{a_n^{3/2}} \leq c\left(\frac{1}{a_{n-1}^{1/2}} - \frac{1}{a_n^{1/2}}\right), \quad n \geq 1 ,$$

where c is a constant.

2. Consider five cases according as

(a) $b > d$,

(b) $b = d$ and $a > c$,

(c) $b < d$,

(d) $b = d$ and $a < c$,

(e) $b = d$ and $a = c$.

In case (a) show that $L = +\infty$ by bounding $Q_n(a,b,c,d)$ from below by a multiple of $(\frac{b}{d})^n$. In case (b) show that $L = +\infty$ by estima-

Cases (c) and (d) are easily treated by considering $\frac{1}{Q_n(a,b,c,d)}$. The final case (e) is trivial.

3. A straightforward approach to this problem is to show that the function

$$F(x) = \frac{(x^3-1)(x+1)}{(x^3+x)} - 3\ln x \ , \quad x > 0 \ ,$$

suggested by the inequality (3.0), is increasing.

4. Apply Rouché's theorem to the polynomials $f(z) = -z^n$ and $g(z) = p(z)$. *Rouché's theorem states that if $f(z)$ and $g(z)$ are analytic within and on a simple closed contour C and satisfy $|g(z)| < |f(z)|$ on C, where $f(z)$ does not vanish, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C .*

5. Apply the change of variable $x = a - t$ to

$$I = \int_0^a \frac{f(x)}{f(x) + f(a-x)}\, dx \ .$$

6. Integrate

$$I = \int_x^{x+1} \frac{2t \sin(t^2)}{2t}\, dt$$

by parts and obtain an upper bound for $|I|$.

7. Consider (7.0) modulo 16.

9. Assume that the required limit exists, and use (9.0) to determine its value L. Again use (9.0) to estimate $|x_n - L|$.

10. Either set $x = \cos\theta$, $y = \sin\theta$ and maximize the resulting function of θ, or express $ax^2 + 2bxy + cy^2$ in the form $A(x^2 + y^2) - (Bx + Cy)^2$ for appropriate constants A,B,C.

11. Consider the coefficient of x^n in both sides of the identity

$$(1 + x)^{2n} \equiv ((1 + 2x) + x^2)^n \quad .$$

12. Express $(\ell(\ell + 1))^k - ((\ell - 1)\ell)^k$, $(k = 1,2,3,\dots)$ as a polynomial in ℓ, then sum over $\ell = 1, 2, \dots, n$ to obtain $(n(n + 1))^k$ as a linear combination of

$$S_{[k/2]}(n), \dots, S_{k-1}(n) \quad .$$

Complete the argument using induction.

13. Prove that the equation

$$bc\,x + ca\,y + ab\,z = 2abc - (bc + ca + ab) + k$$

is solvable in non-negative integers x,y,z for every integer $k \geqq 1$. Then show that the equation with $k = 0$ is insolvable in non-negative integers x,y,z.

14. Let u_n be the n^{th} term of the sequence (14.0) and show that $u_n = k$ for $n = \frac{(k-1)k}{2} + 1 + \ell$, $\ell = 0, 1, 2, \dots, k-1$, and deduce that $k \le \frac{1}{2}(1 + \sqrt{8n-7}) < k + 1$.

15. Use Cauchy's inequality to prove that

$$\sum_{i=1}^{n} x_i^2 \geq \left(\sum_{i=1}^{n} a_i^2 \right)^{-1} ,$$

and choose the x_i so that equality holds.

16. Express $(n + 1)^3$ in the form

$$An(n - 1)(n - 2) + Bn(n - 1) + Cn + D$$

for suitable constants A,B,C,D .

17. Integrate $F'(a-x) = F'(x)$ twice.

18. For part (b), find the quartic equation whose roots are $a - z, b - z, c - z, d - z,$ and use part (a) to ensure that the product of two of these roots is equal to the product of the other two.

19. Differentiate $f_n(x)$ to obtain the difference-differential equation

$$f_{n+1}(x) = f_n'(x) - p'(x) f_n(x) .$$

20. Consider $\sinh x + \sinh wx + \sinh w^2x$, where $w = \frac{1}{2}(-1 + \sqrt{-3})$.

21. Show that

$$I_n - I_{n-1} = \int_0^{\pi} \frac{\sin(2n - 1) x}{\sin x} dx , \quad n \geq 2 ,$$

22. Let a_i $(i = 1, 2, \ldots, 365)$ denote the number of books sold during the period from the first day to the i^{th} day inclusive. Apply Dirichlet's box principle to

$$a_1, a_2, \ldots, a_{365}, a_1+129, a_2+129, \ldots, a_{365}+129 .$$

23. Show that a polynomial of the required type is

$$f(x,y) = \frac{(x + y - 1)(x + y - 2)}{2} + x ,$$

by showing that $f(x,y) = k$, where k is a positive integer, has a unique solution in positive integers x and y which may be expressed in terms of the integers r and m defined by

$$\frac{(r - 1)(r - 2)}{2} < k \leq \frac{r(r - 1)}{2} , \quad m = k - \frac{(r - 1)(r - 2)}{2} .$$

24. Consider the complex conjugate of (24.0).

25. Consider

$$\ln \prod_{r=kn+1}^{hn} \left(1 - \frac{r}{n^2}\right)$$

and use the expansion

$$- \ln (1 - x) = \sum_{k=1}^{\infty} \frac{x^k}{k} , \quad |x| < 1 .$$

26. For the evaluation, set $x = a - y$ in the integral.

27. Use mathematical induction to prove that $a_r = r$ for all

28. Use integration by parts to establish the recurrence relation

$$u_n = \frac{n(n-1)}{n^2+p^2} u_{n-2}, \quad n \geq 2 .$$

29. The series may be summed by using the identity

$$\tan A = \cot A - 2\cot 2A .$$

30. Prove that the number of k-selections S from N such that $W(S) \geq r$, $r = 1,2,3,\ldots,$ is

$$\binom{n - (k-1)(r-1)}{k} .$$

31. For each $n \geq 1$ define integers q_n and r_n uniquely by $n = kq_n + r_n$, $0 \leq r_n < k$. Express the nth partial sum s_n of the series in terms of n and q_n, and determine $\lim_{n\to\infty} s_n$ by appealing to the result

$$\lim_{m\to\infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{m} - \ln m\right) = c ,$$

where c denotes Euler's constant.

32. Recognize the given integral as the imaginary part of the integral $\int_0^\infty x^m e^{(i-1)x} dx$. Evaluate the latter integral using integration by parts.

33. For the evaluation, iterate $I(u) = \frac{1}{2}I(u^2)$ to obtain

$$I(u) = \frac{1}{2^n} I(u^{2^n}) \quad (n = 1,2,3,\ldots)$$

34. Determine an exact expression for $s_n(k)$ and then compare the values of $s_{k-1}(k)$ and $s_k(k)$ with $3k^3 - 5k^2$.

35. Show that (35.0) converges by comparison with $\sum_{n=1}^{\infty} \frac{p_n-1}{p_1\cdots p_n}$.

36. Apply l'Hôpital's rule.

37. Relate h,k,ℓ to the lengths of the sides of the triangle, and then use the triangle inequality.

38. Obtain the recurrence relation $P_{n+1,r} = P_{n+1,r-1} + x^r P_{n,r}$, and apply the principle of mathematical induction.

39. Show that $Bx + D$ is a divisor of F .

40. The first few terms of the series are

$$\frac{1}{3} = \frac{1}{2}\left(1 - \frac{1}{3}\right) ,$$

$$\frac{2}{21} = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{7}\right) ,$$

$$\frac{3}{91} = \frac{1}{2}\left(\frac{1}{7} - \frac{1}{13}\right) .$$

41. Use the identity

$$(1-x)(1-2x)\dots(1-nx) = 1 - P_1x + P_2x^2 - P_3x^3 + \dots + (-1)^n P_n x^n .$$

42. For a suitable constant C , set $f(x) = \dfrac{e^x}{e^x + 1} + C$, and show that for $t > 0$

$$\int_0^t \frac{(e^{ax} - e^{bx})}{x(e^{ax}+1)(e^{bx}+1)}\,dx = \int_0^t \frac{f(ax)}{x}\,dx - \int_0^t \frac{f(bx)}{x}\,dx .$$

43. Let S_n denote the sum of n terms of (43.0). Calculate the first few values of S_n, conjecture the value of S_n in general, and prove it by mathematical induction.

44. Consider the polynomial whose roots are $z_1, z_2, \ldots, z_k$, and use (44.0) to show that its constant term is zero.

45. Obtain a recurrence relation for D_n by expanding D_n by its first row.

46. Express xyz in terms of A, B and C.

47. Consider the sums of the integers in subsets of S and apply Dirichlet's box principle.

48. Count pairs of lines in the proposed configuration.

49. Show that $A(n) = \sum_{\substack{k=0 \\ k\equiv 0 \pmod 3}}^{n} \binom{n}{k}$ and evaluate this sum by considering $(1+1)^n + (1+w)^n + (1+w^2)^n$, where w is a complex cube root of unity.

50. To prove the required inequality, replace $p_n(x)$ by $\dfrac{x^{4n}(1-x)^{4n} - (-1)^n 4^n}{1 + x^2}$ in (50.0), and then use the inequalities

51. Let $B_1, B_2, \ldots, B_{23}$ (resp. $G_1, G_2, \ldots, G_{23}$) be the members of the blue (resp. green) team, ordered with respect to increasing weight. For each r $(1 \leqq r \leqq 23)$ consider last year's opponents of $B_{r+1}, \ldots, B_{23}$ or $G_{r+1}, \ldots, G_{23}$ according as B_r is heavier or lighter than G_r.

52. Each member of S can be written in the form $(2r+1)(2r+2s+1)$, for suitable integers $r \geqq 1$ and $s \geqq 0$. Use this fact to construct the three arithmetic progressions.

53. For $y > 0$ prove that

$$\int_0^y \left(\int_0^b e^{-ux} \sin x \, dx \right) du = \int_0^b (1-e^{-xy}) \frac{\sin x}{x} \, dx$$

and then show that

$$\lim_{y \to \infty} \int_0^b (1-e^{-xy}) \frac{\sin x}{x} \, dx = \int_0^b \frac{\sin x}{x} \, dx \; .$$

54. Consider $\sum_{j=1}^{43} d_j$.

55. For any natural number n, construct a prime p of the form

$$p = 4k \prod_{r=1}^{n} (r^2+q)^2 - q \; ,$$

where k is a natural number and $q > n$ is a prime of the form $4t + 3$, so that $p = a^2 + b^2$, $0 < a < b$. Then, assuming $a \leqq n$, obtain a contradiction by considering the factor a^2+q of b^2.

57. Evaluate D_1, D_2, D_3 and conjecture the value of D_n for all n. Prove your conjecture by using the recurrence relation which may be obtained by expanding D_n by its first row.

58. Prove that

$$x_n = x_{n+1} + a - \sqrt{b^2 + 4ax_{n+1}}$$

and use this to obtain the recurrence relation

$$x_{n+1} - 2x_n + x_{n-1} = 2a .$$

59. For $r > 0$ and $r \neq 1$ show that

$$I_r = a + \frac{1}{2}(1-r^2)\int_{-a}^{a} \frac{du}{1 - 2r\cos u + r^2} ,$$

and evaluate the integral using the transformation $t = \tan u/2$.

60. Construct a class of isosceles triangles whose members have two equal altitudes of fixed length h , while their third altitudes are arbitrarily long.

61. Recognize the expression in (61.0) as the square of the distance between a point on a certain circle and a point on another plane curve.

62. When the line L through the origin has irrational slope, use Hurwitz's theorem to obtain an infinity of lattice points whose distances from L are suitably small.

In 1881 Hurwitz proved the following basic result: If b is

$$\left| b - \frac{m}{n} \right| < \frac{1}{\sqrt{5}n^2} \ .$$

This inequality is best possible in the sense that the result becomes false if $\sqrt{5}$ is replaced by any larger constant.

63. Show that $I_k = I_{2n-k-2}$ and use the arithmetic-geometric mean inequality.

64. Express the multiple integral (64.0) as a repeated integral and use the value of $\int_0^a x^r (a-x)^s dx$, where r and s are positive integers and a is a positive real number, successively in the repeated integral.

65. Show that for a suitable integer $f(n)$

$$\sum_{k=1}^{n} \left(\left[\frac{2\sqrt{n}}{\sqrt{k}} \right] - 2\left[\frac{\sqrt{n}}{\sqrt{k}} \right] \right) = \sum_{s=1}^{f(n)} \left(\left[\frac{4n}{(2s+1)^2} \right] - \left[\frac{4n}{(2s+2)^2} \right] \right) ,$$

and thus compute L in terms of well-known series.

66. $p^a q^b$ is the n^{th} term of the sequence S, where n is the number of pairs of integers (r,s) such that $p^r q^s \leq p^a q^b$, $r \geq 0$, $s \geq 0$.

67. A straightforward approach is to determine explicitly all matrices X such that $X^2 = A$. The form of X depends on whether or not a is a square in Z_p.

68. Recall that if $y = g(x)$ is differentiable with positive

69. Evaluate $S(10^m-1)$ exactly and use it to estimate $S(N)$.

70. Show that $M = \left[\frac{n-1}{k-1}\right]$.

71. Express the roots of $az^2 + bz + c$ in terms of a, b and c and estimate the moduli of these roots.

72. Choose integers a, b and c such that $x^2 + a \equiv 0 \pmod p$ is solvable for primes $p \equiv 1 \pmod 4$ and $p = 2$; $x^2 + b \equiv 0 \pmod p$ is solvable for $p \equiv 3 \pmod 8$; $x^2 + c \equiv 0 \pmod p$ is solvable for $p \equiv 7 \pmod 8$; and set

$$f(x) = (x^2+a)(x^2+b)(x^2+c) .$$

73. Assume without loss of generality that $0 \le x_1 \le x_2 \le \ldots \le x_n \le 1$ and show that

$$S = \sum_{1 \le i < j \le n} |x_i - x_j| = \sum_{k=1}^{n} x_k(2k-n-1) .$$

Consider those terms in the sum for which $k \ge \frac{1}{2}(n+1)$ and deduce that $M = [n^2/4]$.

74. Show that the smallest sum (74.0) is obtained when the y_i are arranged in decreasing order.

75. Replace x by $x+k$ $(k = 0,1,2,\ldots,p-1)$ in (75.0) and form the alternating sum

$$\sum_{k=0}^{p-1} (-1)^k g(x+k) .$$

76. Express the improper double integral I as a limit of proper double integrals over appropriate subregions of the unit square and use standard methods to show that $I = \pi^2/6$.

77. Use the identity

$$\sum_{x=0}^{k-1}\left[\frac{x}{k} + e\right] = [ek] ,$$

where k is any positive integer and e is any real number.

78. Reorder the a's in ascending order and define $\min_{1\le i\le n} a_i^2 = a_j^2$, for a fixed subscript j .

Set $b_i = a_j + \sqrt{M}(i - j)$ $(i = 1,2,\ldots,n)$ and prove that $a_i^2 \ge b_i^2$. Deduce the required inequality from $S \ge \sum_{i=1}^{n} b_i^2$.

79. Establish and use the inequality

$$\left|\frac{2}{k} - 1 - \frac{1}{n}\right| \le 1 - \frac{1}{n} , \quad 1 \le k \le n .$$

80. Denote the n^{th} prime by p_n , and show that if $p_n + p_{n+1} = 2^k p^\ell$, for some odd prime p , then $k + \ell \ge 3$.

81. Estimate the integral

$$\int_{\pi/2}^{\pi} |f(x)| \left|\sum_{k=1}^{n} \sin kx\right| dx$$

from above under the assumption that $|f(x)| < \frac{1}{\pi \ell n 2}$ on $[\frac{\pi}{2},\pi]$ except for a set of measure 0. Use (81.0) to obtain a lower bound

82. Show that s_n is non-decreasing and bounded above.

83. Assume that $A(x)$ and $F(x)$ differ at some point c in $(a,b]$ and obtain a contradiction by partitioning $[a,c]$ and using (83.0) on each subinterval.

84. A direct approach recognizes $M(r)$ as $\max(\frac{r}{a} - 1, 1 - \frac{r}{b})$ and then minimizes $M(r)$ with an appropriate choice of r.

85. Let $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$ and show that $|S_n - L| < |S_{n-1} - L|$ and $a_n = |S_n - L| + |S_{n-1} - L|$.

86. Express $(\cos^2 x + \cos x + 1)^2$ as a linear combination of $\cos nx$ $(n = 0,1,2,3,4)$ and consider

$$\int_0^{\pi} f(x)(\cos^2 x + \cos x + 1)^2 \, dx .$$

87. Begin by determining R when the ellipse is in standard position and then rotate the axes through an appropriate angle.

88. Recall *Helly's theorem: Given $n(\geq 4)$ convex regions in the plane such that any three have non-empty intersection, then all n regions have non-empty intersection.*

89. Use partial fractions and the result

$$\lim \left(\sum^{N} \frac{1}{\ } - \ell n N \right) =$$

where c is Euler's constant, to evaluate

$$\lim_{N\to\infty} \sum_{\substack{n=1 \\ n\neq m}}^{N} \frac{1}{m^2-n^2} .$$

90. Use the identity

$$(x^2+y^2+z^2)^2 = (x^2+y^2-z^2)^2 + (2xz)^2 + (2yz)^2 .$$

91. Prove that a and b commute by using the relation $aba = b^3$ in the form $b^{-1}ab = b^2a^{-1}$ to deduce $ab^4 = b^4a$.

92. Start by applying the extended mean value theorem to f on $[0,a_n]$.

93. Let p be the largest root of (93.0). Consider the discriminant of $(x^3+ax^2+bx+c)/(x-p)$.

94. Let B be the Vandermonde matrix given by $B = \begin{bmatrix} 1 & 1 & 1^2 & 1^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{bmatrix}$, and consider the rank of BA .

95. Collect together terms having the same value for $GCD(r,s)$ in $\sum_{r,s=1}^{\infty} \frac{1}{(rs)^2}$.

96. Suppose that such a rational function $f(x)$ exists and use the decomposition of its numerator and denominator into linear factors to obtain a contradiction.

97. Sum the identity

$$(n+1)!\left(\frac{2}{(n+1)\binom{n}{k}} - \frac{1}{n\binom{n-1}{k}}\right) = k!(n-k)! - (k+1)!(n-k-1)! \ ,$$

for $k = 0,1,2,\ldots,n-1$.

98. Assume that the set of zeros of $u(x)$ on $[a,b]$, $1 \le a < b$, is infinite. Deduce the existence of an accumulation point c in $[a,b]$ with $u(c) = u'(c) = 0$, and then show that $u(x) \equiv 0$ on $[a,b]$.

99. Take P_j $(j = 0,1,2,\ldots,n-1)$ to be the point $\exp(2\pi ji/n)$ on the unit circle $|z| = 1$ in the complex plane, and express $|P_jP_k|^2$ in terms of $\exp(2\pi(k-j)i/n)$.

100. Let $M = (a_{ij})$ $(1 \le i,j \le 3)$ and with the usual notation let $\det M = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$. Begin by counting the number of triples (a_{11},a_{12},a_{13}) for which $\det M = 0$, distinguishing two cases according as $(A_{11},A_{12},A_{13}) = (0,0,0)$ or not.