JLT model

- Based on Jarrow and Turnbull (1995) model, and characterizes the bankruptcy process as finite state Markov process in firm credit ratings (1997)

- Discrete time, (R,W), time-homogeneous finite state space Markov chain, for K Crediting State, K-th state is default, 1 is highest rating

$$Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1K} \\ q_{21} & q_{22} & \cdots & q_{2K} \\ \vdots & & & \\ q_{K-1,1} & q_{K-1,2} & \cdots & q_{K-1,K} \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Diagonal: $q_{ii} = 1 - \sum_{j \neq i} q_{ij}$

- Risk neutral transition Prob

$$\tilde{q}_{ij}(t,t+1) = \pi_i(t)\, q_{ij} \quad i \neq j$$

For $i=j$

$$\begin{aligned} & 1 - \sum_{i \neq j} \tilde{q}_{ij} \\ &= 1 - \sum_{i \neq j} \pi_i(t) q_{ij} \\ &= 1 - \pi_i(t) \sum_{i \neq j} q_{ij} \\ &= 1 - \pi_i(t)(1 - q_{ii}) \\ &= 1 + \pi_i(t)(q_{ii} - 1) \end{aligned}$$

$$\Rightarrow \tilde{Q}_{t,t+1} = I + \underbrace{\pi(t)}_{\text{Risk premiums}} \underbrace{[Q - I]}_{\text{time homogeneous}}$$

- Bond Pricing of Credit Risky zero coupon bond

(under assumption that: bankruptcy process and default free spot are statistically independent)

$$\begin{aligned} &= E^{Q}\left(\frac{B(t)}{B(T)}\left(\delta I_{\{\tau^* \leq T\}} + I_{\{\tau^* > T\}}\right)\right) \\ &= P(t,T)\left(\delta \tilde{Q}_t(\tau^* \leq T) + \tilde{Q}_t(\tau^* > T)\right) \\ &= P(t,T)\left(\delta\left(1 - \tilde{Q}_t(\tau^* > T)\right) + \tilde{Q}_t(\tau^* > T)\right) \\ &= P(t,T)\left(\delta + (1-\delta)\underline{\tilde{Q}_t(\tau^* > T)}\right) \end{aligned}$$

↳ default after T

↳ if annual transition:

= Not default within this year

• Generator Matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & \lambda_{12} & \lambda_{13} & \dots \lambda_{1,K-1} & \lambda_{1K} \\ \lambda_{21} & \lambda_2 & \lambda_{23} & \dots \lambda_{2,K-1} & \lambda_{2K} \\ \vdots \\ \lambda_{K-1,1} & \lambda_{K-1,2} & \lambda_{K-1,2} & \dots \lambda_{K-1} & \lambda_{K-1,K} \\ 0 & 0 & 0 & \dots 0 & 0 \end{pmatrix}$$

→ continuous time, time homogeneous Markov chain

→ Diagonal $\lambda_i = \lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$

• Risk neutral :

$$\tilde{\Lambda} = \Pi \Lambda$$

Π ↓ Diagonal matrix

• Transition Probability from Generator Matrix :

$$Q(t,T) = \exp(\Lambda(T-t))$$

$\because P(t,t) = I$

for small h

$i \neq j : \lambda_{ij} h + o(h)$

$i = j : 1 + \lambda_i + o(h)$

$$\frac{\partial \tilde{Q}(t,T)}{\partial T} = \lim_{h \to 0} \frac{\tilde{Q}(t,T+h) - \tilde{Q}(t,T)}{h}$$

$$= \lim_{h \to 0} \frac{\tilde{Q}(t,T)\tilde{Q}(T,T+h) - \tilde{Q}(t,T)}{h}$$

$$= \tilde{Q}(t,T) \lim_{h \to 0} \left(\frac{\tilde{Q}(T,T+h) - I}{h} \right)$$

$$\approx \tilde{Q}(t,T)\Pi(T)\Lambda$$

$$= \tilde{Q}(t,T)\tilde{\Lambda}(T)$$

the sol to above

$$\tilde{Q}(t,T) = \exp(\Pi(T)\Lambda(T-t))$$

⇒ If $T = t+1$

$$\Rightarrow \tilde{Q}(t,t+1) = \exp(\Pi(T)\Lambda)$$

Extended JLT

- Q Real world Historical transition Prob

  Consider $Q = \exp(\Lambda)$

  → Assume Q satisfied

  $Q = PDP^{-1}$, D is diagonal

  $\exp(Q) = I + \Lambda + \frac{1}{2}\Lambda^2 + \cdots$

  $\exp(PDP^{-1}) = I + PDP^{-1} + \frac{1}{2} PDP^{-1}PDP^{-1} + \cdots$

  $= I + PDP^{-1} + \frac{1}{2} PD^2P^{-1} + \cdots$

  $= P(\exp(D))P^{-1}$

  → let Λ be $\ln Q$

  consider $\ln Q = P(\ln D)P^{-1}$

  $\exp(\ln Q) = P\exp(\ln D)P^{-1} = PDP^{-1} = Q$

The extended part:

- Assume $\pi(t)$ follow CIR process

  → Mean-reverting

  → Non-negative

  → Affine Term Model, imply

  $E\left(\exp\left(-\int_t^T \pi(u)\,du\right)\right) = \exp\left(A(t,T) - B(t,T)\pi_t\right)$

$\Rightarrow d\pi_t = k(\theta - \pi_t)dt + \sigma\sqrt{\pi_t}\,dW_t$

From Brigo, Mercurio, section 3 :

The price at time t of a zero-coupon bond with maturity T is

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)},$$

where

$$A(t,T) = \left[\frac{2h\exp\{(k+h)(T-t)/2\}}{2h + (k+h)(\exp\{(T-t)h\}-1)}\right]^{2k\theta/\sigma^2},$$

$$B(t,T) = \frac{2(\exp\{(T-t)h\}-1)}{2h + (k+h)(\exp\{(T-t)h\}-1)},$$

$$h = \sqrt{k^2 + 2\sigma^2}.$$

- Under Stochastic $\pi(T)$, RN transition Probability:

$$\tilde{Q}(t,T) \triangleq E\left(\exp\left(\int_t^T \pi(u)\Lambda\, du\right)\right)$$

$$= P\, E\left(\exp\left[\underline{\left(\int_t^T \pi(u)du\right)(\ln D)}\right]\right)P^{-1}$$

Both are Diagonal matrix

$$= P X P^{-1}$$

where $X_{ii} = E\left(\exp\left(\int_t^T \ln D_i \pi_u^i du\right)\right)$

where D_i, π_u^i is the diagonal element of D, π

$$X = \begin{bmatrix} X_{11} & & & \\ & X_{22} & & \\ & & \ddots & \\ & & & X_{KK} \end{bmatrix}$$

to use the property of affine term structure, write X_{ii} as follow

$$X_{ii} = E\left(\exp\left(-\int_t^T -\ln D_i \pi_u^i du\right)\right)$$

Note that $Y_u^i = -\ln D_i \pi_u^i$ also follow CIR process

$$dY_u^i = -\ln D_i\, d\pi_u^i$$

$$= -\ln D_i\left[k(\theta - \pi_u)du + \sigma\sqrt{\pi_u}\, dW_u\right]$$

$$= K(-\theta \ln D_i - Y_u^i)du + \sigma\sqrt{-\ln D_i}\sqrt{Y_u^i}\, dW_u$$

$$= K(\theta' - Y_u^i)du + \sigma'\sqrt{Y_u^i}\, dW_u$$

Note: $\frac{2K\theta'}{\sigma'^2} = \frac{-2K\theta \ln D_i}{-\sigma^2 \ln D_i} = \frac{2K\theta}{\sigma^2}$

$$X_{ii} = \exp\left(A(t,T) - B(t,T)Y_t^i\right) = \exp\left(A(t,T) + B(t,T)\pi_t^i \ln D_i\right)$$

To further simplify the expression, if we assume annual transition

Note in implementation:

1. $\exp(A - B\pi)$ instead of $A\exp(-B\pi)$ in textbook
2. parameters
   - $K \to \alpha$
   - $\theta \to \pi_\infty$
   - $\pi_t \to \pi_{initial}$
   - $h \to \gamma$
3. We generate π_t for each $t \in$ projection period and use $\exp\left(A(t,T) - B'(t,T)\pi_t^i\right)$ where $B'(t,T) = -B(t,T)\ln D_i$

Result

$$A(t,t+1) = \frac{2\alpha\pi_\infty}{\sigma^2} \ln\left[\frac{2\gamma \exp\{(\alpha+\gamma)/2\}}{2\gamma + (\alpha+\gamma)(\exp\{h\}-1)}\right]$$

$$B(t,t+1) = \frac{-2(\ln D_i)(\exp\{\gamma\}-1)}{2\gamma+(\alpha+\gamma)(\exp\{\gamma\}-1)}$$

$$\gamma = \sqrt{\alpha^2 + 2\sigma'^2}$$
$$= \sqrt{\alpha^2 - 2(\ln D_i)(\sigma^2)}$$

$$\tilde{Q}(t,t+1) = P X P^{-1}$$
where $X_{ii} = \exp(A(t,t+1) - B(t,t+1)\pi^i_{initial})$