Lecture 4, lech A Grzelak e^{A+Br}

$$A(0,\tau) = \frac{-\eta^2}{4\lambda^3}\left(3+e^{-2\lambda\tau} - 4e^{-\lambda\tau} - 2\lambda\tau\right)$$
$$+\lambda\int_0^\tau \theta(T-z)\,B(0,z)\,dz \qquad (P1)$$

In terms of $B(t,T)$

$$= \frac{-\eta^2}{4\lambda^3}\left(3+e^{-2\lambda(T-t)} - 4e^{-\lambda(T-t)} - 2\lambda(T-t)\right)$$
$$= \frac{-\eta^2}{4\lambda^3}\left(\left(1-e^{-\lambda(T-t)}\right)^2 - 2e^{-\lambda(T-t)} + 2 - 2\lambda(T-t)\right)$$
$$= \frac{-\eta^2}{4\lambda}\left(B(t,T)\right)^2 - \frac{\eta^2}{4\lambda^3}\left(2-2e^{-\lambda(T-t)} - 2\lambda(T-t)\right)$$
$$= \frac{-\eta^2}{4\lambda}\left(B^2(t,T)\right) - \frac{\eta^2}{2\lambda^2}\left(-B(t,T) - (T-t)\right)$$

so $A(0,\tau)$ reduce to

$$\frac{-\eta^2}{4\lambda}B^2(t,T) - \frac{\eta^3}{2\lambda^2}\left(B(t,T) - (T-t)\right) +$$
$$\lambda\int_0^\tau \theta(T-z)\,B(0,z)\,dz$$

Consider $\int_t^T \frac{\partial f(0,u)}{\partial u} B(u,T)\, du$

$$= \int_t^T B(u,T)\, d f(0,u)$$

$$= f(0,u) B(u,T) \Big|_t^T - \int_t^T f(0,u)\, dB(u,T)$$

$$= f(0,u) B(u,T) \Big|_t^T - \int_t^T f(0,u) \frac{\partial B(u,T)}{\partial u}\, du$$

$\because B(u,T) = -\frac{1}{a}(1 - e^{-a(T-u)})$ / In the Book $-\frac{1}{a}(1-e^{-a(T-t)})$ using B def

$$\frac{\partial B(u,T)}{\partial u} = e^{-a(T-u)}$$

$$= f(0,u) B(u,T) \Big|_t^T - \int_t^T f(0,u) e^{-a(T-u)}\, du$$

$$= -f(0,t) B(t,T) - \int_t^T f(0,u) e^{-a(T-u)}\, du$$

$\because -\alpha B(u,T) = 1 - e^{-\alpha(T-u)}$

$$e^{-\alpha(T-u)} = 1 + \alpha B(u,T)$$

$$= -f(0,t) B(t,T) - \int_t^T f(0,u)\, du - \alpha \int_t^T f(0,u) B(u,T)\, du$$

$$\Leftrightarrow \int_t^T \frac{\partial f(0,u)}{\partial u} B(u,T)\, du + \alpha \int_t^T f(0,u) B(u,T)\, du$$

$$= -f(0,t) B(t,T) - \int_t^T f(0,u)\, du$$

$$\because \quad f(0,u) = \frac{-\partial \ln P(0,u)}{\partial u}$$

$$= -f(0,t)B(t,T) + \ln P(0,u)\Big|_t^T$$

$$= -f(0,t)\,B(t,T) + \ln P(0,T) + \ln P(0,t)$$

$$= -f(0,t)B(t,T) + \ln\frac{P(0,T)}{P(0,t)}$$

$$\Leftrightarrow \int_t^T \frac{\partial f(0,u)}{du} B(u,T)\,du + \alpha \int_t^T f(0,u)B(u,T)\,du$$

$$\boxed{\theta(t) = f^r(0,t) + \frac{1}{\lambda}\frac{\partial}{\partial t} f^r(0,t) + \frac{\eta^2}{2\lambda^2}\left(1 - e^{-2\lambda t}\right).}$$

computational fin book

$\alpha = \lambda$

$$\Rightarrow \lambda \int_t^T \left(f(0,u)B(u,T) + \frac{1}{\lambda}\frac{\partial f(0,u)}{\partial u} B(u,T) \right) du = -f(0,t)B(t,T) + \ln\frac{P(0,T)}{P(0,t)}$$

$u = T - z \qquad z = \tau \Rightarrow u = T - T + t = t$

$du = -dz \qquad z = 0 \Rightarrow u = T = T$

Orignal equation further to:

$$\lambda \int_t^T \theta(u)\,B(u,T)\,du$$

$$\frac{\eta^2}{2\lambda}\left(1 - e^{-2\lambda u}\right)B(u,T) = \frac{\eta^2}{2\lambda^2}\left(1 - e^{-2\lambda u}\right)\left(e^{-\lambda(T-u)} - 1\right)$$

$$= \frac{\eta^2}{2\lambda^2}\left(1 - e^{-2\lambda u}\right)\left(e^{-\lambda(T-u)} - 1\right)$$

$$= \frac{\eta^2}{2\lambda^2}\left(e^{-\lambda(T-u)} + e^{-2\lambda u} - e^{-\lambda(T+u)} - 1\right)$$

$$= \frac{\eta^2}{2\lambda^2}\left(\frac{e^{-\lambda(T-u)}}{\lambda}\Bigg|_t^T - \frac{1}{2\lambda} e^{-2\lambda u}\Bigg|_t^T + \frac{e^{-\lambda(T+u)}}{\lambda}\Bigg|_t^T - (T-t) \right)$$

$$= \frac{\eta^2}{2\lambda^2}\left(\frac{1-e^{-\lambda(T-t)}}{\lambda} - \frac{1}{2\lambda}\left(e^{-2\lambda T} - e^{-2\lambda t}\right) + \frac{1}{\lambda}\left(e^{-2\lambda T} - e^{-\lambda(T+t)}\right) - (T-t) \right)$$

$$= \frac{\eta^2}{2\lambda^2}\left(B(t,T) - (T-t) + \frac{1}{2\lambda} e^{-2\lambda T} + \frac{1}{2\lambda} e^{-2\lambda t} - \frac{2e^{-\lambda(T+t)}}{2\lambda} \right)$$

$$= \frac{\eta^2}{2\lambda^2}\left(B(t,T) - (T-t) + \frac{1}{2\lambda} e^{-2\lambda t}\left(e^{-2\lambda(T-t)} + 1 - 2e^{-\lambda(T-t)} \right) \right)$$

$$= \frac{\eta^2}{2\lambda^2}\left(B(t,T) - (T-t) \right) + \frac{\eta^2}{4\lambda} e^{-2\lambda t}\left(B^2(t,T) \right)$$

$$\lambda \int_t^T \theta(u)\, B(u,T)\, du$$

$$= \frac{\eta^2}{2\lambda^2}\left(B(t,T) - (T-t)\right) + \frac{\eta^2}{4\lambda} e^{-2\lambda t}\left(B^2(t,T)\right) - f(0,t)B(t,T) + \ln\frac{P(0,T)}{P(0,t)}$$

$A(0,T)$

$$= \frac{-\eta^2}{4\lambda} B^2(t,T) \cancel{- \frac{\eta^2}{2\lambda^2}\left(B(t,T) - (T-t)\right)}$$

$$\cancel{+ \frac{\eta^2}{2\lambda^2}\left(B(t,T) - (T-t)\right)} + \frac{\eta^2}{4\lambda} e^{-2\lambda t} B^2(t,T)$$

$$- f(0,t)B(t,T) + \ln\frac{P(0,T)}{P(0,t)}$$

$$\boxed{= -f(0,t)B(t,T) + \ln\frac{P(0,T)}{P(0,t)} - \frac{\eta^2}{4\lambda} B^2(t,T)\left(1 - e^{-2\lambda t}\right)}$$

Wiki:

$$\frac{\eta^2}{4\lambda^3}\left(e^{-\lambda T} - e^{-\lambda t}\right)^2\left(e^{2\lambda t} - 1\right)$$

$$\frac{\eta^2}{4\lambda^3}\left(e^{-2\lambda t}\left(e^{-\lambda(T-t)} - 1\right)^2\right)\left(e^{2\lambda t} - 1\right)$$

$$= \frac{\eta^2}{4\lambda}\left(B^2(t,T)\right)\left(1 - e^{-2\lambda t}\right)$$

matched