Pricing American Options using Monte Carlo simulation

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Abstract

Numerical methods such as binomial and finite difference methods can be used to price options however the problem is when the options have early exercise features. In this research project, we investigate the effectiveness and accuracy of Monte Carlo methods in pricing American options. We implement the Longstaff and Schwartz Least Squares Monte Carlo (LSM) algorithm and compare it with Bjerksund and Stensland closed form valuation and the Binomial method. We also analyze the effectiveness of the LSM using different basis functions.

Keywords: American option, European option, Early exercise, Least Squares Monte Carlo Simulation (LSM), Closed-form solution.

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1 Introduction

"It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment." —Carl Friedrich Gauss

1.1 Overview

An option is an agreement that gives the purchaser the privilege, however, not the obligation, to buy or sell an underlying asset at a particular price on or before a certain date [13]. By far most options are either European or American. Option pricing is a critical area of research in finance. A standout amongst the most vital problems in option pricing theory is the valuation of options with American features. American options can be exercised whenever up to and including the maturity date while the European options must be exercised only at the maturity date. It is a direct result of this difference that American options are more hard to price than European options [17]. The time at which we decide to exercise the option is known as the exercise time or the stopping time [12]. Subsequently, the American options are viewed as the optimal stopping time problems, and their exercise regions are quasi-continuous [17].

Albeit European options can be priced analytically, general closed-form analytical solutions for the pricing of American options are hard to obtain. To value the European options we can easily compute the expected value of the discounted payoffs at the maturity dates, but to value American options is difficult and it requires advanced methods. Different numerical methods were devised to attempt to price American options. Some of the standard methods that are usually used are finite difference methods and binomial lattice methods, these methods simply discretize the solution space into grids and then recursively solve the partial differential equation, usually Black-Scholes partial differential equation [12]. In general, Georgiadis demonstrated that binomial options pricing models do not have analytical solutions[1].

Binomial methods are less practical in pricing options with several sources of uncertainty

[1] Georgiadis, Evangelos. "Binomial options pricing has no closed-form solution." Algorithmic Finance 1.1 (2011): 13-16

due to several difficulties, and Monte Carlo option pricing models are instead employed [19]. Monte Carlo methods rely on reiterated random sampling to acquire numerical results. They are used mainly in finance and other fields to solve problems that might be deterministic in principle [15]. They are extremely flexible and have no restrictions to computational analysis. Finite difference methods aim at discretizing the Black-Scholes partial differential equation by finite difference equations and then approximate the solution.

Using Monte Carlo simulation in option pricing may offer various convincing and conclusive advantages. The first advantage is that the rate of convergence of the Monte Carlo approach is independent of the number of state factors and this makes the method computationally appealing for solving problems in high dimensions.

Furthermore the method is adaptable and flexible with respect to the evolution of the state factors offering the opportunity to value options with more intricate process dynamics [12]. It can also be used to value options with both path-dependent and American features [18]. Monte Carlo simulation method's ascendancy is that it can simulate the path by path of the underlying asset and then approximate the expected payoff of the option [13].

In higher dimensional spaces, Monte Carlo methods converge rapidly to the solution and they require less memory and are less demanding to program [16]. Another distinct method that we can use in the valuation of American options that is not well known is the Bjerksund-Stensland model [3]. A well-known and Nobel prize-winning paper by Fischer Black and Myron Scholes [4] in 1973 demonstrated that we can value the European options using the Black-Scholes model however this model does not hold for American options. One simple conclusion is that an American option has a bigger price than the related European option, as it gives greater chance to exercise.

1.2 American Option Pricing Theory

This section summarizes the essential tools that will be used in later chapters. The theory is borrowed from Glasserman [13].

1.2.1 General Formulation

Let $[0,T]$ be a finite time period. Suppose there is a continuous-time process $U(t)$ representing the discounted payoff at exercise time t for $t \in [0,T]$. The American option problem is to find $\tau \in \Gamma$ such that:

$$\sup_{\tau \in \Gamma} \mathbb{E}[U(\tau)],$$

where τ and Γ represent the stopping time and the class of admissible stopping times, respectively.

1.2.2 Dynamic Programming Formulation

Let $\tilde{h}_i$ be the payoff function at exercise time t_i. Let $\tilde{V}_i(x)$ represent the option price at time t_i given that $X_i = x$[2]. We want to find the value of the option at time zero i.e., $\tilde{V}_0(X_0)$. We do recursively as follows:

$$\tilde{V}_m(x) = \tilde{h}_m(x) \tag{1}$$

$$\tilde{V}_{i-1}(x) = \max\left\{\tilde{h}_{i-1}(x), \mathbb{E}[D_{i-1,i}(X_i)\tilde{V}_i(X_i)|X_{i-1} = x]\right\} \tag{2}$$

$\forall i = 1, 2, ..., m$ where $D_{i-1,i}(X_i)$ denote the discount factor from time t_{i-1} to t_i. Let $D_{0,j}(X_j)$ represent the discount factor from time 0 to t_j such that $D_{0,j}(X_j)$ is nonnegative and it satisfy $D_{0,0} = 1$ and $D_{0,i-1}(X_{i-1})D_{i-1,i}(X_i) = D_{0,i}(X_i)$. For any i in $0 \leq i \leq m$, define the following equations,

$$h_i = D_{0,j}(X_j)\tilde{h}_i(x), \tag{3}$$

$$V_i(x) = D_{0,i}(x)\tilde{V}_i(x). \tag{4}$$

[2] $\{X(t)$, $0 \leq t \leq T\}$ is d-dimensional real-valued Markov process. $X_i = X(t_i)$ is a state of the underlying Markov process at ith exercise opportunity [13].

Clearly at time zero $V_0 = \tilde{V}_0$ and the V_i satisfy,

$$
\begin{aligned}
V_m(x) &= h_m(x) \\
V_{i-1}(x) &= D_{0,i-1}(x)\tilde{V}_{i-1}(x) \\
&= D_{0,i-1} \max\left\{\tilde{h}_{i-1}(x), \mathbb{E}[D_{i-1,i}(X_i)\tilde{V}_i(X_i)|X_{i-1} = x]\right\} \\
&= \max\left\{\tilde{h}_{i-1}(x), \mathbb{E}[D_{0,i-1}(X_i)D_{i-1,i}(X_i)\tilde{V}_i(X_i)|X_{i-1} = x]\right\} \\
&= \max\left\{\tilde{h}_{i-1}(x), \mathbb{E}[\tilde{V}_i(X_i)|X_{i-1} = x]\right\}
\end{aligned}
$$

Hence we have shown that V_i satisfies dynamic programming of the form 3. The problem of pricing American options satisfies equations 1.

1.2.3 Stopping Strategy

As we have mentioned before American option problems can also be viewed as stopping times problems, we need to present the mathematical theory of this..
Define τ to be any stopping time (as before) for a Markov chain $X_0, X_1, ..., X_m$. Then the option value at time $t = 0$ is expressed as,

$$V_0^\tau(X_0) = \mathbb{E}[h_\tau(X_\tau)].$$

The aim is to find the stopping strategy and this can be accomplished by determining,

$$\hat{\tau} = \min\left\{k \in \left\{1, 2, ..., m\right\} : h_k(X_k) \geq \hat{V}(X_k)\right\},$$

where $\hat{V}_k$ is a price given by any stopping strategy at each state $x \in \mathbb{R}^d$ and exercise opportunity k. At the ith exercise time, the exercise region is determined by $Q = \left\{x : h_i \geq \hat{V}_i(x)\right\}$ and the complement of this is the continuation region.

1.2.4 Continuation Values and Early Exercise Value

A continuation value of an American option is the value of holding rather than exercising the option. This value is computed using,

$$C_i(x) = \mathbb{E}[V_{i+1}(X_{i+1})|X_i = x], \quad \forall i = 0, ..., m-1$$

where x is a state and t_i is the ith time.

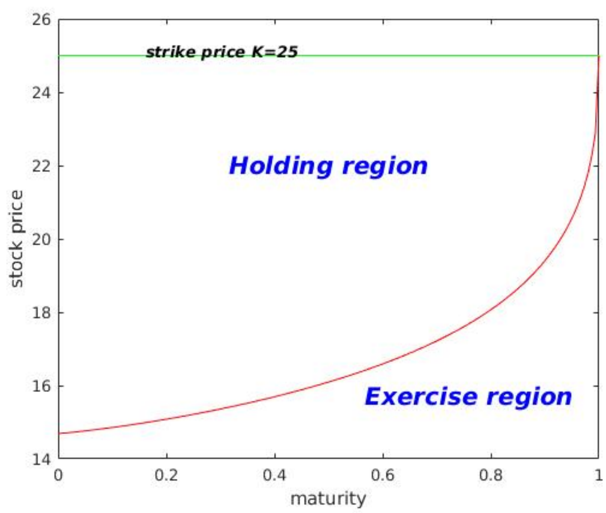
When $C_m \equiv 0$ and $C_i(x) = \mathbb{E}[\max\{h_{i+1}(X_{i+1}, C_{i+1}(X_{i+1})\}|X_i = x]$, the dynamic programming recursion is satisfied. At the time $t = 0$, we have that the option value is $C_0(X_0)$ which is the time zero continuation. To find the approximation of the continuation, say, $\hat{C}_i$ we use the stopping strategy through $\hat{\tau} = \min\{i \in 1, 2, ..., m : h_i(X_i) \geq \hat{V}(C_i)\}$.

The early exercise value is the difference between the American option price and the European price. That is to say,

$$\xi = V_{american} - V_{european}.$$

When the price of the underlying asset becomes large enough, the early exercise value tends to be zero. This means that at that point, the value of the American option coincides with that of the European option. This value ensures that an American option price is always greater than a European option price.

Figure 1: Early exercise boundary for an American put option.



1.3 The Geometric Brownian Motion

At this point, it is important to introduce the model for the underlying stock price. Thus we will introduce the geometric Brownian motion.

Definition 1.3.1. A stochastic process S_t[3] for each $t \geq 0$ is said to be a geometric Brownian motion if it satisfies the following stochastic differential equation,

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{5}$$

where S_0 is a constant; W_t is a Brownian motion[4]; $\mu > 0$ is a drift; and σ is the volatility.

This simply means that the stock price process follows the geometric Brownian motion. To obtain the solution of (5) we make use of Ito's formula and the assertion that; $dt^2 = 0$, $dtdW_t = 0$ and $dW_t^2 = dt$. Here we note that $dS^2(t) = \sigma^2 dW_t^2 = \sigma^2 dt$. Now set $f(t,x) := \ln x$. Then computing the partials derivatives with respect to t and x; $\frac{\partial f}{\partial x} = \frac{1}{x}$, $\frac{\partial f}{\partial t} = 0$, $\frac{\partial^2 f}{\partial x^2} = -\frac{1}{x^2}$. Then Ito's formula gives,

$$S_T = S_t e^{(\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma W_T}. \tag{6}$$

satisfies equation (5). This is vital in the simulation since it will enable us to generate stock prices at specific exercise times. Since the increments of Brownian motion are normally distributed then $W_t \sim \mathcal{N}(0,\, t)$ or we can write it as $z = \frac{W_t}{\sqrt{t}} \sim \mathcal{N}(0,1)$. In this way our fundamental underlying asset process becomes, $S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{T}z}$, where $z \sim \mathcal{N}(0,1)$ [21].

[3] S_t is an underlying asset of the option.

[4] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each sample path $\omega \in \Omega$ and $t \geq 0$, assume there is a continuous function W_t that depends on ω and satisfies $W_0 = 0$. We say that W_t is a Brownian motion if for each $0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n$,

1. $W_{t_1} = W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent increments,
2. For all $0 \leq t_j < t_{j+1}$, $W_{t_{j+1}} - W_{t_j}$ are normally distributed with mean zero and variance $t_{j+1} - t_j$ for each $j = 1, 2, \ldots, n$.

1.4 The Black-Scholes-Merton World

We assume the assumptions of the Black-Scholes-Merton (BSM) model [4]. The BSM model is usually used to price European options. A European call option is an agreement that gives the purchaser the right, yet not the obligation, to buy an underlying asset at the maturity date. Its payoff function is $(S(T)-K)^+$ where $(a-b)^+ = \max(0, a-b)$. A European put option is an agreement that gives the purchaser the right, yet not the obligation, to sell an underlying asset at the maturity date. Its payoff function is $(K-S(T))^+$.

We can simply price the European put option by taking the expectation of the discounted payoffs at maturity i.e., $e^{-rT}\mathbb{E}[(K-S(T))^+]$ or use the BSM model.

To detail the BSM model mathematically, let; $S = S(t)$, $V(t,S)$, t, r, σ, and K be the price of the underlying asset, the option price, exercise time, risk-free interest rate, volatility and strike price, respectively. Then the BSM partial differential equation describes the evolution of the option price V at times t and is given by:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV \tag{7}$$
$$V(T, S_T) = H(S_T)$$

where $H(S_T)$ is the derivative payout at maturity time T and $0 \leq t \leq T < \infty$.

If we want to price the European put and call option respectively, the following formulae immaculately give their prices:

$$P(t,S) = Ke^{-r\tau}\Phi(-d_2) - S\Phi(-d_1),$$
$$C(t,S) = -Ke^{-r\tau}\Phi(d_2) + S\Phi(d_1),$$

where

- $\tau = T - t$ and $\Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}dz$, the cumulative distribution function of the standard normal distribution.
- $d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$ and $d_2 = d_1 - \sigma\sqrt{\tau}$.

The BSM partial differential equation for the American put option is expressed as:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \leq rV$$

with the terminal condition:

$$V(T, S(T)) = (K - S)^+$$

and the boundary condition

$$V(t, S(t)) \geq (K - S)^+.$$

For European and American put options, there exists a far-field boundary condition saying that if the price of the underlying asset $S(t)$ becomes large enough, the put option tends to be worthless [25]. That is to say,

$$\lim_{S\to\infty} V(t, S) = 0.$$

1.5 Bjerksund and Stensland Closed Form Solution

This model assumes a complete continuous-time BSM model that we described above. In their paper [3], Bjerksund and Stensland considered two approximations, a flat exercise boundary, and a two-step exercise boundary. However, in this project, we will only consider the former due to its simplicity.

Let r, S_0, b, σ, K and W_t be the risk-free rate, current underlying price, cost of carry, volatility, strike price, and Brownian motion respectively. Then American call option is

$$C(S_0, K, T, r, b, \sigma) = \sup_{\tau\in[0,T]} \mathbb{E}_0[e^{-r\tau}(S_\tau - K)]$$

, where $S_t = S_0 \exp\left\{(b - \frac{1}{2}\sigma^2)t + \sigma W_t\right\}$, $b < r$.

This model can be used even if the underlying asset pays a constant dividend yield of q. This constant dividend yield is given by $q = r - b$.

1.5.1 A Flat Exercise Boundary method

Suppose an underlying asset hits $X > K$ from below, where X is a European up-and-out call knock-out barrier and K is a strike price. Define a rebate to be $X - K$ that is received at the knock-out date if the option is knocked out at maturity T. Then the American call option approximation is,

$$\begin{aligned}\bar{c}(S_0, K, T, r, b, \sigma; X) = \alpha(X)S_0^{\beta} - \alpha(X)\varphi(S_0, T|\beta, X, X) \\ + \varphi(S_0, T|1, X, X) - \varphi(S_0, T|1, K, X) \\ - K\varphi(S_0, T|0, X, X) + K\varphi(S_0, T|0, K, X)\end{aligned} \tag{8}$$

where $\alpha(X) = (X - K)X^{-\beta}$, $\beta = (\frac{1}{2} - \frac{b}{\sigma^2}) + \sqrt{(\frac{b}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}}$,
$\varphi(S_0, T|\gamma, H, X) = e^{\lambda T} S_0^{\gamma}\{\Phi(D_1) - (\frac{X}{S_0})^{\kappa}\Phi(D_2)\}$, $D_1 = -\frac{\ln(\frac{S_0}{H}) + (b + (\gamma - \frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}}$,
$D_2 = -\frac{\ln(\frac{X^2}{S_0 H}) + (b + (\gamma - \frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}}$, $\lambda = -r + \gamma b + \frac{1}{2}\gamma(\gamma - 1)\sigma^2$,
$\kappa = \frac{2b}{\sigma^2} + (2\gamma - 1)$.
And $X = B_0 + (B_\infty - B_0)(1 - \exp(h(T)))$, $h(T) = -(bT + 2\sigma\sqrt{T})(\frac{K^2}{B_0(B_\infty - B_0)})$,
$B_\infty = \frac{\beta K}{\beta - 1}$, $B_0 = \max\{K, \frac{rK}{r-b}\}$.
The American put option is approximated using,

$$\bar{p} = P(S_0, K, T, r, b, \sigma) = C(K, S_0, T, r - b, -b, \sigma). \tag{9}$$

Since we assume our underlying asset to pay no dividend, then we will assume that $r = b$ i.e., the risk-free interest rate coincides with the cost of carry.

2 Literature Review

"Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate." — Leonhard Euler

This section discusses the literature review of American option pricing. The theoretical background of finite difference methods, Binomial methods, and Monte Carlo methods are discussed.

The introduction of finite difference methods for pricing American options was first proposed by Brennan & Schwartz [7] when they priced an American put option. Their algorithm discretizes the variables in the partial differential equation of the evolution of the put option prices into both the solution and time-space. Although there are various finite difference methods that can be applied, Han & Wu [14] presented a method for pricing the American options governed by the Black-Scholes by using artificial boundary conditions for the Crank-Nicholson equation. The class of this method is a finite difference. The paper by Ehrhardt & Mickens [11] improves this method by introducing a sum-of-exponential approximation.
Wu & Kwok [24] employed the front-fixing transformation to change the unknown free boundary into a known and fixed one, they developed a finite difference procedure that produces numerous option values and the optimal exercise boundary, this method is however accurately comparable to the binomial method and is better than Brennan & Schwartz (1997) method.

The Binomial methods were first introduced by Cox et al [10] to pricing options due to their straightforwardness and adaptability. The model has been exceptionally useful for valuing claims contingent upon a single state variable which follows a well-known geometric Brownian Motion and it requires time discretizations. This model contains Black & Scholes model as a limiting case and it also makes use of no-arbitrage arguments. There have been a lot of extensions of Binomial methods as a way of improving the computational accuracy and efficiency of the model.

Broadie & Detemple [8] evaluates and develops lower and upper bounds on the prices of the American call and options with dividend-paying stocks. They gave two estimations of the prices based on the lower bound and upper bounded, and they went on to provide a modification of the well-known binomial methods (termed BBSR[5]) that is exceptionally simple to implement and has a good computational performance.

Bjerksund & Stensland [3] generalized their 2002 model by simply dividing the termination date into two dates where each one has a flat early exercise boundary. They devised a closed-form lower bound for the price of the American option and showed by numerical execution that this lower bound represents a precise and exceptionally computer estimation for the American put and call values.

Precise pricing methods were needed in light of the fact that binomial and finite difference methods are not remarkably accurate in higher dimensions and continuous space. That was when simulation came into play. Valuation of financial options by simulation dates back to the time of Boyle [5] when he developed a simulation method to price European options.

The first paper to show that simulation can be used in American options was due to Tilley [22]. The principle notion in this paper is to devise a strategy using Monte Carlo simulation to choose the early exercise boundary. Thereafter American option can be viewed as the knocked-and-exercised[6] option which belongs to a class of exotic options. Moreover, this paper has not managed complexities that emerge in determining exercise hold decision boundaries when multi-factor stochastic.

Tsitsiklis et al [23] proposed a simulation method by addressing the American option as a dynamic programming problem that approximates the value accurately and they also introduced a method that uses parametrization of the function value of the option.

[5] see Broadie & Detemple (1996)

[6] see [1]

Rogers [20] discussed the use of Lagrangian martingale in Monte Carlo path simulation in pricing American options and it is also based on the dual characterization of the optimal exercise problem. It generally comes up with an answer that is less than the true price value.

Longstaff and Schwartz [18] devised the simplest algorithm that uses simulation coupled with least squares techniques to approximate the value of the continuation. This algorithm can be used to value options that have early exercise features or exotic options. The main feature of this algorithm is that it considers only the in-the-money paths. Fu et al [12] compare simulation approaches in pricing American options.

3 Methodology

"If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?" —David Hilbert

This section derives the methodology of this research project and more specifically, the LSM algorithm. The valuation framework underlying the LSM algorithm depends on the general derivative valuation models of Black and Scholes (1973), Merton (1973), and others [18]. The last part of this section uses the same numerical example as the reference [18] to show how LSM works.

3.1 The valuation framework

Let $[0, T]$ be a finite time period. Suppose Ω is a set of all possible outcomes of the stochastic economy in the time horizon and define a sample path by ω. Denote the sigma field of events of Ω at time T by $\mathcal{F}$. Now assume the probability space to be $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbb{P}$ is the probability measure specified on the elements of $\mathcal{F}$. Additionally, because of the presence of no arbitrage opportunities, we assume that there exists an equivalent martingale measure $\mathbb{Q}$. We define the augmented filtration generated by the certain price processes for the derivatives in the economy to be $F = \{\mathcal{F}_t;\ t \in [0, T]\}$ with an additional condition that at maturity, $\mathcal{F} = \mathcal{F}_T$.

Our business is to find the optimal stopping time τ such that $\tau \in \Gamma$ for each path ω will maximize the price of the expected payoff. In Longstaff and Schwartz's framework, they assumed derivatives with payoffs that are elements of the space of quadratically integrable[7] or finite-variance functions denoted by $L^2(\Omega, \mathcal{F}, \mathbb{Q})$ [18]. The price of an American option is equivalent to the maximized price of the discounted cash flows from the option, where the maximum is assumed to be taken over all the admissible stopping time with respect to F. Now we propose the following notation $C(\omega, s; t, T)$ to represent the cash flows generated by the option with respect to the following stopping

[7] also called square-integrable

strategy $\forall s$, $0 \leqslant t < s < T$. Moreover, this cash flow $C(\omega, s; t, T)$ is conditional on the option not being purchased at or before time t [18].

To understand the LSM algorithm, let us assume that American options are discretely[8] exercisable. Suppose that there are K discrete times, $0 < t_1 \leqslant t_2 \leqslant t_3 \leqslant \ldots \leqslant t_K = T$ and also take into consideration of the optimal stopping rule at each exercise time. If this option is continuously exercisable we simply take a very large K so that we can use the LSM to approximate the value of these options.

At the option's final maturity date, if the option is out of the money, the investor leaves it to expire or exercises it if it is in the money. However, at any exercise time t_i before the final maturity date, it is the investor's choice to choose whether to do an immediate exercise or continue the option's life.

If the exercise is immediate, then at time t_i where $i = 1, 2, \ldots, K-1$ the option holder will definitely know the value of the option but won't know the value of the continuation. So by no-arbitrage pricing theory, the value of the option can be valued by taking the discounted cash flows $C(\omega, s; t_i, T)$ up until the maturity date under the risk-neutral measure $\mathbb{Q}$. Precisely, at time t_i the price of the continuation can be given by

$$F(\omega, t_i) = \mathbb{E}^{\mathbb{Q}} \left[\sum_{q=i+1}^{K} \exp\left(- \int_{t_i}^{t_q} r(\omega, s) ds \right) C(\omega, t_q; t_i, T) \mid \mathcal{F}_{t_i} \right], \tag{10}$$

where $r(\omega, t)$ is the risk-free discount rate and at time t_q the expectation is conditional on the $\mathcal{F}_{t_q}$ [18].

3.2 The Least Squares Monte Carlo Simulation

The LSM algorithm merges least squares principles (regression analysis) and Monte Carlo techniques, to estimate the conditional expectation function at times $t_{K-1}, t_{K-2}, \ldots, t_1$. As we've defined the path of cash flows $C(\omega, s; t, T)$ generated by the option to be recursive then in our pricing process we will work backward by starting at the final

[8] That is to say it can be exercised only at the discrete times.

value. At the time t_{K-1}, the continuation defined in equation (10) can be expressed as a linear combination of a countable set of $\mathcal{F}_{t_{K-1}}$-measurable basis functions. This assumption is rationalized when the conditional expectation is an element of the L^2 space of quadratically integrable functions [18]. As the L^2 space is Hilbert [9], its basis is countable orthonormal and the linear function of the elements of this basis can be used to express the conditional expectation. Possible choices of basis functionals are Hermite, Laguerre, Jacobi, Gegenbauer, and Legendre polynomials. Nonetheless, in this paper we will employ Laguerre polynomials:

$$
\begin{aligned}
L_0(X) &= \exp(-X/2), \\
L_1(X) &= \exp(-X/2)(1-X), \\
L_2(X) &= \exp(-X/2)(1-2X+X^2/2), \\
L_n(X) &= \exp(-X/2)(1-X)\frac{e^X}{n!}\frac{d^n}{dX^n}(X^n e^{-X}).
\end{aligned}
$$

With this detail, $F(\omega, t_{K-1})$ can be expressed as

$$
F(\omega; t_{K-1}) = \sum_{j=0}^{\infty} b_j L_j(X),
$$

where X is the value of the asset underlying the option and b_j are regression constant coefficients.

Let $M < \infty$ be a finite number of basis functions. We implement the LSM method by estimating the continuation $F(\omega, t_{K-1})$ and we denote it by $F_M(\omega, t_{K-1})$ using the first M basis functions [18]. When estimating $F_M(\omega, t_{K-1})$ we only use the option with the in the money paths because it is suitable. After defining a subset of the basis functions to be used, we regress the discounted values of the cash flows $C(\omega, s; t_{K-1}, T)$. In doing so, fewer basis functions to approximate the conditional expectation functions will be required and the region of estimation of the conditional expectation will be limited.
After managing to estimate the conditional expectation function at time t_{K-1}, we would now be able to decide whether the immediate exercise is optimal for the in-the-money

[9]Hilbert space generalizes the idea of Euclidean space

path ω. When the exercise strategy is determined, the cash flow $C(\omega, s; t_{K-2}, T)$ can easily be estimated. Then we recursively continued in this way until all the exercise strategies are known at each time along each path. Subsequently, we obtained the final cash flow matrix $C(\omega, s; 0, T)$. We simply obtain an estimation of the American option by simply discounting all the cash flows in the obtained cash flow matrix and averaging over all paths [18].

The pseudo-algorithm that summarizes this is presented below.

3.2.1 The Main Algorithm

(a) Define M number of basis functions and simulate paths ω_j for $j = 1, 2, ..., N$.

(b) For each simulated path ω_j, compute the option value at the maturity date.

(c) By taking into consideration of only the in-the-money paths, use the cash flow matrix $C(\omega_j, s; t_{K-1}, T)$ to compute the conditional expectation of the continuation $F(\omega, t_{K-1})$.

(d) Decide whether the immediate exercise is optimal.

(e) Now compute the cash flow matrix $C(\omega_j, s; t_{K-2}, T)$ and continue this way recursively for each exercise date.

(f) To estimate the value of the option $F(\omega, 0)$, find the sum of cash flow and take their average.

3.3 Convergence Results

Longstaff and Schwartz (2001) presented two propositions on the convergence of LSM. These propositions are given below.

Proposition 3.3.1. *For any finite choice of M, K, and vector $\nu \in \mathbb{R}^{M\times(K-1)}$ representing the coefficients for the M basis functions at each of the $K-1$ early exercise dates, let $LSM(\omega; M, K)$ denote the discounted cash flow resulting from following the LSM rule of exercising when the immediate exercise value is positive and greater than or equal to $\widehat{F}_M(\omega_j; t_k)$ as defined by ν. Then the following inequality holds almost surely,*

$$V(X) \geq \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} LSM(w_j; M, K)$$

where X is the value of the asset underlying the option and it follows the Markov process and $\widehat{F}_M(\omega_j; t_k)$ is a value of the fitted regression at the jth path. $V(X)$ represents the true value of the American option.

Proposition 3.3.2. *Assume that the value of an American option depends on a single state variable X with support on $(0, \infty)$ which follows a Markov process. Assume further that the option can only be exercised at times t_1 and t_2, and that the conditional expectation function $F(\omega, t_1)$ is absolutely continuous and*

$$\int_0^\infty e^{-X} F^2(\omega, t_1) dX < \infty,$$

$$\int_0^\infty e^{-X} F_X^2(\omega, t_1) dX < \infty.$$

Then for any $\epsilon > 0$, there exists an $M < \infty$ such that

$$\lim_{N\to\infty} \mathbb{P}\left[\left| V(X) - \frac{1}{N} \sum_{j=1}^{N} LSM(w_j; M, K) \right| > \epsilon \right] = 0$$

Proposition 3.3.1 asserts that the value of an American option is based on the stopping rule that maximizes the value of the option and that all other stopping rules, including the one implied by the LSM algorithm results in values that are less than or equal to that implied by the optimal stopping rule. This ensures the objective criterion for convergence. Proposition 3.3.2 asserts that if one increases the number of basis functions (M to be large enough) and set $N \to \infty$, the LSM algorithm will result in a value for the American option within ϵ of the true value since ϵ is arbitrary.

3.4 A Numerical Example

Consider an American put option on a share of non-dividend-paying stock with pricing parameters; initial stock price $S_0 = 1.0$, strike price $K = 1.10$, maturity date $T = 3$ and risk-free interest rate $r = 6\%$. We illustrate the algorithm by using eight sample paths for the stock prices. These sample paths are generated under the risk-neutral measure and are shown in the following matrix table.

Path	$t = 0$	$t = 1$	$t = 2$	$t = 3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34

Table 1: Stock prices matrix

Our goal is to solve the stopping policy that maximizes the value of the option at each point along each path. Firstly, we need to compute a number of intermediate matrices and cash flows on each intermediate step and then we decide whether to make an immediate exercise or continue with the life of the option until maturity. Note that the cash flows are similar to the ones that would be received if the option were European rather than American [10]. These cash flows at time $T = 3$ are given in Table 2:

Path	$t = 0$	$t = 1$	$t = 2$	$t = 3$
1	—	—	—	0.00
2	—	—	—	0.00
3	—	—	—	0.07
4	—	—	—	0.18
5	—	—	—	0.00
6	—	—	—	0.20
7	—	—	—	0.09
8	—	—	—	0.00

Table 2: Cash flow at time $t = 3$

At time 2 we have only five paths for which the option is in-the-money. That is paths

[10] This is because the European option can only be exercised at the maturity date.

one, three, four, six, and seven. To obtain the regression matrix; let X be the stock prices at time $t = 2$ and Y be the corresponding discounted cash flows (continuation value, discounted back to time 2) at time $T = 3$ if the is not exercised at time $t = 2$. Precisely, with t being the time step, Y is the mentioned payoff$\times e^{-rt}$. Thus we have $e^{-rt}\times$payoff $= 0.94176\times$payoff. We only use the in-the-money paths for good estimation. The regression matrix comprising of real-valued vectors X and Y at time $t = 2$ is presented below (Table 3).

Path	Y	X
1	0.00×0.94176	1.08
2	—	—
3	0.07×0.94176	1.07
4	0.18×0.94176	0.97
5	—	—
6	0.20×0.94176	0.77
7	0.09×0.94176	0.84
8	—	—

Table 3: Regression matrix at time 2

We need to find a_0, a_1, and a_2 such that $Y = a_0 + a_1 X + a_2 X^2$. This will enable us to estimate the expected cash flow continuing the life of the option conditional on the stock price at time $t = 2$. We will regress $G = \sum_{i=1}^{5}(Y_i - a_0 - a_1 X_i - a_2 X_i^2)^2$ such a_0, a_1, and a_2 minimize G. These values are given by $a_0 = -1.070$, $a_1 = 2.983$, and $a_2 = -1.813$. The following approximation of the conditional expectation is obtained:

$$\mathbb{E}[Y|X] = -1.070 + 2.983X - 1.813X^2$$

Now using this approximated conditional expectation we compare the value of immediate exercise at time $t = 2$ with the value from the continuation. We compute the immediate exercise value using $K - X = 1.10 - X$ and the value from the continuation by simply substituting X into the approximated expectation function. Table 4 gives the optimal early exercise decision at time $t = 2$.

We can see that it is optimal to exercise the option at time $t = 2$ for the fourth, sixth, and seventh paths, this is entailed by comparing the two columns of Table 4.

Path	Exercise	Continuation
1	0.02	0.0369
2	—	—
3	0.03	0.0461
4	0.13	0.1176
5	—	—
6	0.33	0.1520
7	0.26	0.1565
8	—	—

Table 4: Optimal early exercise decision at time 2

Path	$t = 1$	$t = 2$	$t = 3$
1	—	0.00	0.00
2	—	0.00	0.00
3	—	0.00	0.07
4	—	0.13	0.00
5	—	0.00	0.00
6	—	0.33	0.00
7	—	0.26	0.00
8	—	0.00	0.00

Table 5: Cash flow matrix at time $t = 2$

Table 5 shows the cash flows received by the option holder conditional on not exercising before time $t = 2$. We see that the cash flow in the final column becomes zeros if the option is exercised at time $t = 2$. Proceeding recursively, we now check whether the option should be exercised at time $t = 1$. Once again at time $t = 1$ there are five paths where the option is in-the-money. The future cash flows occur at either time $t = 2$ or time $t = 3$, but not both as the option can be exercised once. Using these five paths; let X denote the price of the stock at time $t = 1$ for the in-the-money paths and Y be the corresponding discounted value of the subsequent option cash flow. Table 6 gives the regression matrix at time $t = 1$ comprising of real-valued vectors X and Y at time 1 is given by:

Proceeding as before we need to find c_1, c_2, c_3 such that $Y = c_1 + c_2X + c_3X^2$. Thus we get $c_1 = 2.038$, $c_2 = -3.335$, and $c_3 = 1.356$. The approximated conditional expectation function at time $t = 1$ is: $\mathbb{E}[Y|X] = 2.038 - 3.335X + 1.356X^2$. Again we compute the immediate exercise value using $K - X = 1.10 - X$ and the value from the continuation by substituting X into $\mathbb{E}[Y|X] = 2.038 - 3.335X + 1.356X^2$. These

Path	Y	X
1	0.00×0.94176	1.09
2	—	—
3	—	—
4	0.13×0.94176	0.93
5	—	—
6	0.33×0.94176	0.76
7	0.26×0.94176	0.92
8	0.00×0.94176	0.88

Table 6: Regression matrix at time 1

estimated values are given below in Table 7

Path	Exercise	Continuation
1	0.01	0.0139
2	—	—
3	—	—
4	0.17	0.1092
5	—	—
6	0.34	0.2866
7	0.18	0.1175
8	0.22	0.1533

Table 7: Optimal early exercise decision at time 1

The exercise is clearly optimal at time $t = 1$ for the fourth, sixth, seventh, and eighth paths when we compare the two columns in Table 7.

Path	$t = 1$	$t = 2$	$t = 3$
1	0	0	0
2	0	0	0
3	0	0	1
4	1	0	0
5	0	0	0
6	1	0	0
7	1	0	0
8	1	0	0

Table 8: Stopping rule at $t = 1, 2, 3$.

Table 8 represents the stopping rule at time $t = 1, 2, 3$ and is obtained after establishing the exercise strategy. Here the zeros represent the dates at which the option is not exercised and the ones represent the exercise dates.

Lastly, we compute the option cash flow matrix. We make use of the cash flows obtained at time $t = 1, 2, 3$ and the stopping rule table. Below is the required option cash flow matrix.

Path	$t = 1$	$t = 2$	$t = 3$
1	0	0	0
2	0	0	0
3	0	0	0.07
4	0.17	0	0
5	0	0	0
6	0.34	0	0
7	0.18	0	0
8	0.22	0	0

Table 9: Final cash flow matrix

Now that we have established the cash flow matrix of the American put at each date along the paths, then we discount each cash flow back to time zero. Thereafter, we average over all the paths to get the price of the option. Thus the price of the option is $V = \frac{1}{8}(0.07e^{-3\times 6\%} + 0.17e^{-1\times 6\%} + 0.34e^{-1\times 6\%} + 0.18e^{-1\times 6\%} + 0.22e^{-1\times 6\%}) = 0.1144.$

4 Analysis and Key Results

"In mathematics the art of proposing a question must be held of higher value than solving it." —Georg Cantor

All the codes and graphs are implemented using MATLAB R2016b. In section 4.1 we test the computational effectiveness of the LSM algorithm using different polynomials (given in the appendix) with different orders and varying numbers of simulated paths. In section 4.2 we compare the Binomial method with the LSM algorithm. Section 4.3 compares the LSM and Bjerksund and Stensland closed form valuation. By LSM we mean the one using Laguerre polynomials unless stated otherwise.

4.1 Effect of Basis Functions

The effectiveness of the LSM algorithm by using six different types of polynomials to compute the price of the standard American put option was investigated. The number of exercise dates used was 1000 and the number of simulations was 100 000. The price of this put option using the Binomial method with the same number of exercise dates has been approximated to be 2.677.

Table 10: The evolution of the LSM prices of the American put option using different basis functions. The pricing parameters are are: $S_0 = 48$, $K = 45$, $r = 6\%$, $\sigma = 0.20$ and $T = 3$.

Type of polynomial	1st degree	2nd degree	3rd degree	4th degree
Laguerre	2.607	2.644	2.653	2.679
Hermite (probabilists)	2.604	2.656	2.667	2.672
Hermite (physicists)	2.597	2.663	2.669	2.682
Chebyshev (1st kind)	2.603	2.665	2.671	2.680
Chebyshev (2nd kind)	2.606	2.649	2.664	2.689
Legendre	2.603	2.663	2.681	2.683
Univariate polynomial	2.604	2.649	2.662	2.691

Table 10 asserts that as we increase the number of polynomials the price of the option converges. From this table (considering the last column) we see that the LSM price using Laguerre is very close to the Binomial method price with an absolute error of 0.2%, this suggests that it is an effective polynomial to use in the LSM algorithm. As the LSM price by using a Univariate polynomial has an absolute error of 1.4% this suggests that it is not a good type of polynomial to use in the LSM algorithm.

Figure 2: Evolution of LSM put option prices against a varying number of simulated paths. $K = 45$, $r = 9\%$, $\sigma = 0.20$, $T = 3$, exercise dates $= 1000$ per year and the number of simulated paths is 1000.

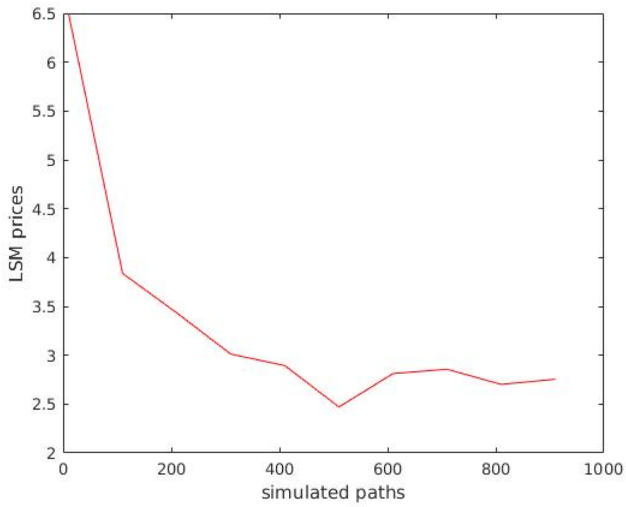


Figure 2 depicts the evolution of the LSM prices against a varying number of simulated paths. When the number of simulated paths is small, say, less than 200, the option price divaricates quickly but when the number of simulated paths is large enough, say, 800, the price of the put option converges and moves between 2.5 and 3.

This confirms that if we increase the number of simulated paths and the number of basis functions to be large enough, the option price will surely converge to the true value.

4.2 Comparison with Binomial Method

A vanilla American put option is priced and compared with the European put option. American (LSM) is the American put option price obtained by the LSM algorithm and American (Binomial) is obtained by the Binomial method and European (BSM) is the European price using the BSM formula.

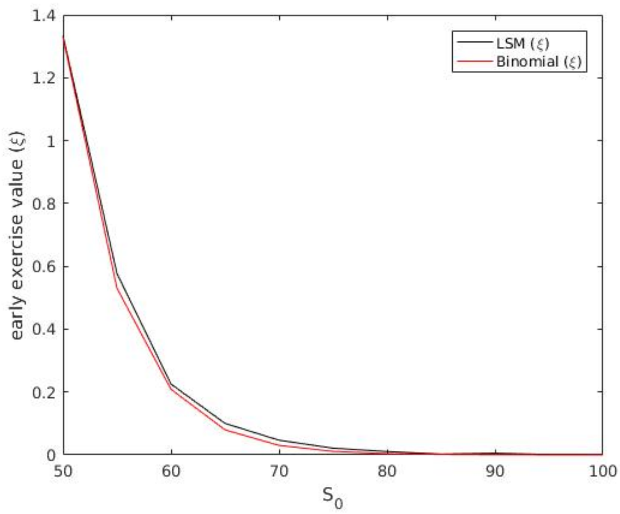
As seen in Table 11, when we increase the price of the stock S_0 the value of the option decreases and will eventually go to zero. This confirms that if we increase the price of the underlying asset to be large enough, the value becomes zero.

Table 11: Evolution of option prices with varying parameters. $K = 55$, $r = 9\%$, number of exercise dates = 300 and number of simulated paths = 10000. ξ_1 is the early exercise value, the difference between American (LSM) and European (BSM), and ξ_2 is the early exercise value, the difference between American (Binomial) and European (BSM).

S_0	σ	T	European (BSM)	American (LSM)	ξ_1	American (Binomial)	ξ_2
50	0.20	1	4.128	5.518	1.390	5.456	1.328
50	0.20	2	3.603	5.697	2.232	5.752	2.149
50	0.40	1	8.081	8.950	0.869	8.933	0.852
50	0.40	2	8.769	10.507	1.738	10.478	1.709
55	0.20	1	2.241	2.806	0.565	2.771	0.530
55	0.20	2	2.247	3.303	1.056	3.295	1.048
55	0.40	1	6.184	6.811	0.627	6.750	0.566
55	0.40	2	7.257	8.345	1.088	8.521	1.264
60	0.20	1	1.130	1.339	0.209	1.339	0.209
60	0.20	2	1.366	1.895	0.529	1.886	0.520
60	0.40	1	4.701	5.052	0.351	5.085	0.384
60	0.40	2	6.013	6.960	0.947	6.973	0.960
65	0.20	1	0.535	0.596	0.061	0.615	0.080
65	0.20	2	0.815	1.074	0.259	1.073	0.258
65	0.40	1	3.557	3.834	0.227	3.822	0.248
65	0.40	2	4.991	5.691	0.700	5.726	0.735
70	0.20	1	0.241	0.279	0.038	0.271	0.030
70	0.20	2	0.478	0.632	0.154	0.609	0.131
70	0.40	1	2.682	2.923	0.241	2.862	0.180
70	0.40	2	4.151	4.725	0.574	4.718	0.567
75	0.20	1	0.104	0.116	0.012	0.115	0.011
75	0.20	2	0.278	0.358	0.080	0.345	0.067
75	0.40	1	2.019	2.155	0.136	2.145	0.126
75	0.40	2	3.460	3.949	0.489	3.901	0.441
80	0.20	1	0.043	0.051	0.008	0.047	0.004
80	0.20	2	0.160	0.210	0.050	0.195	0.035
80	0.40	1	1.518	1.664	0.146	1.602	0.084
80	0.40	2	2.891	3.190	0.299	3.238	0.347
85	0.20	1	0.018	0.026	0.008	0.019	0.001
85	0.20	2	0.092	0.109	0.017	0.109	0.017
85	0.40	1	1.141	1.190	0.049	1.204	0.063
85	0.40	2	2.421	2.723	0.302	2.697	0.276
90	0.20	1	0.007	0.008	0.001	0.007	0.000
90	0.20	2	0.052	0.068	0.016	0.061	0.009
90	0.40	1	0.858	0.908	0.050	0.902	0.044
90	0.40	2	2.033	2.191	0.158	2.253	0.220
95	0.20	1	0.003	0.008	0.005	0.003	0.000
95	0.20	2	0.029	0.038	0.009	0.035	0.006
95	0.40	1	0.646	0.713	0.067	0.676	0.030
95	0.40	2	1.711	1.904	0.193	1.887	0.116
100	0.20	1	0.001	0.001	0.000	0.001	0.000
100	0.20	2	0.017	0.025	0.008	0.020	0.003
100	0.40	1	0.487	0.529	0.042	0.507	0.020
100	0.40	2	1.444	1.631	0.187	1.583	0.139

Another observation we can make is that of the early exercise values. When we increase the prices of the stocks S_0 the early exercise value goes to zero. This is depicted in figure 3. If the early exercise value is zero, then the value of the American option coincides with that of the European option. Another case where the early exercise is zero is when an American option is only exercised at maturity.

Figure 3: Evolution of early exercise values with the pricing parameters are, $K = 55$, $r = 9\%$, $\sigma = 0.20$, $T = 1$, exercise dates = 300 per year and the number of simulated paths is 10000.



We now compare the prices computed using the LSM algorithm and the ones computed using the Binomial method. For simplicity, we consider the pricing parameters $K = 55$, $r = 9\%$, $T = 1$, $\sigma = .20$, exercise dates = 300 and simulated paths = 10000 (see Table 11). Figure 4 depicts the absolute errors between LSM and the Binomial method. When the price of the stock increases, the absolute errors tend to zero, the reason behind this is that increasing the stock price decreases the option price and thus LSM and Binomial method prices will tend to zero. So when both methods tend to zero, the difference between their errors will also tend to zero. As we can see, the absolute errors are between 0% and 3%. Thus we can conclude that the LSM algorithm is good for approximating the value of the option.

Lastly, both the prices of an American option computed by the LSM algorithm and the Binomial method are greater than the prices of the European options as we have claimed before.

Figure 4: Absolute errors (LSM and Binomial). The pricing parameters are, $K = 55$, $r = 9\%$, $\sigma = 0.20$, $T = 1$, exercise dates = 300 per year and the number of simulated paths is 10 000.

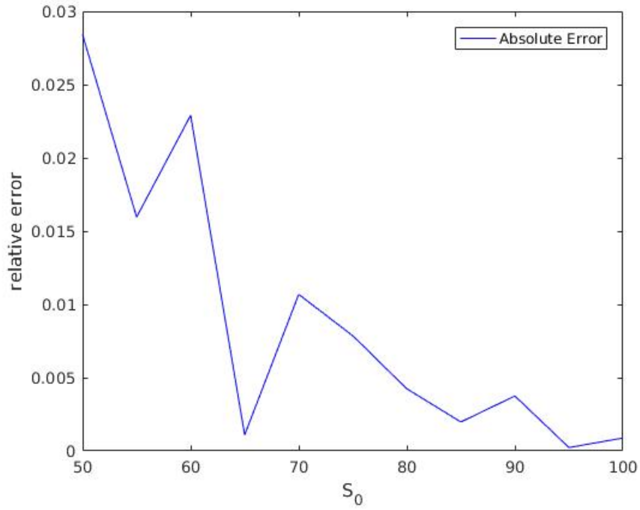
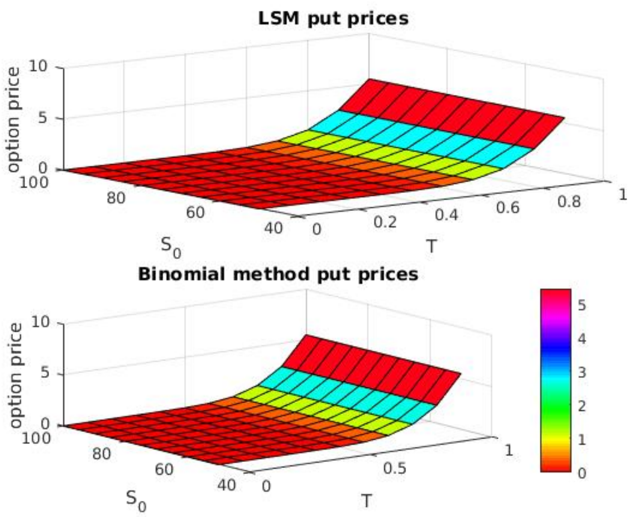


Figure 5: Surface of the evolution of American put option (LSM) prices and Binomial method. The pricing parameters are, $K = 55$, $r = 9\%$, $\sigma = 0.20$, $T = 1$, exercise dates = 300 per year and the number of simulated paths is 10 000.



4.3 Comparison with the closed form approximation

In this computational result, we consider the Bjerksund and Stensland closed-form formula that we described in the introductory chapter. As we have claimed before that if we pay no dividend then the cost of carry b coincide with the risk-free rate r (i.e., $r = b$). As we can see in Table 12 and Table 13, all the prices obtained using the LSM algorithm and closed-form approximation are very close.

Figure 6 shows the time consumption between LSM, Closed form, and Binomial method.

Table 12: Evaluating American put option using Closed form solution. The pricing parameters are $K = 80$, $r = 7.5\%$, $b = 7.5\%$, $\sigma = 0.40$.

Maturity	$S_0 = 85$	$S_0 = 90$	$S_0 = 95$	$S_0 = 100$	$S_0 = 105$	$S_0 = 110$
1.0	8.340	6.877	5.664	4.661	3.833	3.152
1.2	9.090	7.634	6.410	5.832	4.521	3.799
1.4	9.747	8.300	7.072	6.030	5.145	4.394
1.6	10.330	8.895	7.667	6.615	5.714	4.942
1.8	10.855	9.432	8.206	7.149	6.237	5.449
2.0	11.330	9.919	8.698	7.639	6.719	5.919
2.2	11.763	10.365	9.149	8.089	7.165	6.356
2.4	12.161	10.774	9.564	8.506	7.579	6.764
2.6	12.527	11.153	9.949	8.893	7.964	7.145
2.8	12.866	11.503	10.307	9.254	8.324	7.502
3.0	13.181	11.829	10.640	9.590	8.661	7.837
3.2	13.474	12.133	10.951	9.905	8.978	8.153
3.4	13.748	12.417	11.242	10.201	89.275	8.450
3.6	14.004	12.684	11.242	10.201	9.275	8.450
3.8	14.245	12.934	11.773	10.741	9.820	8.996
4.0	14.471	13.170	12.015	10.988	10.069	9.246

Table 13: Evaluating American put option using LSM algorithm. The pricing parameters are $K = 80$, $r = 7.5\%$, $\sigma = 0.40$. The number of exercise dates is 800 per year and the number of simulated paths is 10 000.

Maturity	$S_0 = 85$	$S_0 = 90$	$S_0 = 95$	$S_0 = 100$	$S_0 = 105$	$S_0 = 110$
1.0	8.416	7.034	5.703	4.802	3.842	3.238
1.2	9.190	7.768	6.446	5.564	4.424	3.938
1.4	9.915	8.514	7.172	6.201	5.141	4.425
1.6	10.261	9.025	7.735	6.714	5.776	5.108
1.8	11.042	9.319	8.832	7.155	6.475	5.508
2.0	11.586	10.162	8.870	7.828	6.805	5.982
2.2	11.730	10.354	9.403	8.368	7.673	6.340
2.4	12.304	10.940	9.694	8.602	8.195	6.805
2.6	12.717	11.309	10.095	9.021	8.488	7.143
2.8	13.111	11.835	10.231	9.324	8.729	7.589
3.0	13.295	11.884	10.784	9.735	9.031	7.870
3.2	13.380	12.361	11.231	10.347	9.312	8.219
3.4	13.796	12.614	11.331	10.360	89.275	8.528
3.6	13.924	12.714	11.491	10.531	9.557	8.965
3.8	14.191	13.231	11.869	11.027	9.992	9.087
4.0	14.477	13.294	11.985	11.098	10.137	9.506

Figure 8 shows the relative error[11] between LSM and Closed form approximation. We see that the relative error fluctuates between 0% and 1.8%. We can conclude that

[11]A relative error is given by $\frac{|V_{LSM} - V_{Closed}|}{|V_{LSM}|}$

Figure 6: The pricing parameters are $S_0 = 100$, $T = 1$, $K = 102$, $r = 4.8\%$, $b = 4.8\%$, $\sigma = 0.20$, and the number of simulated paths is 10 000.

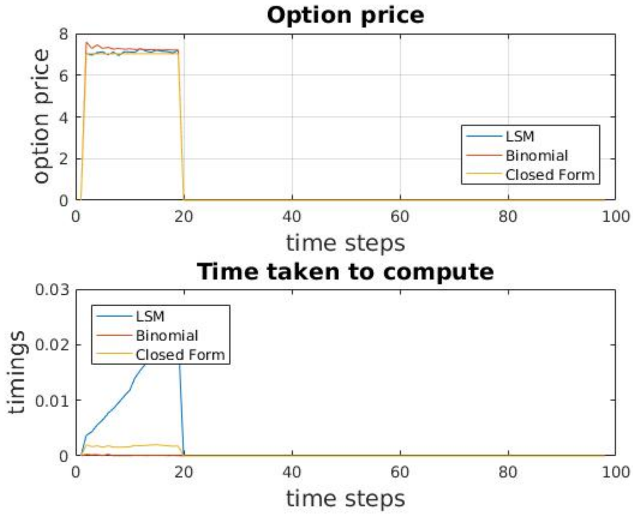
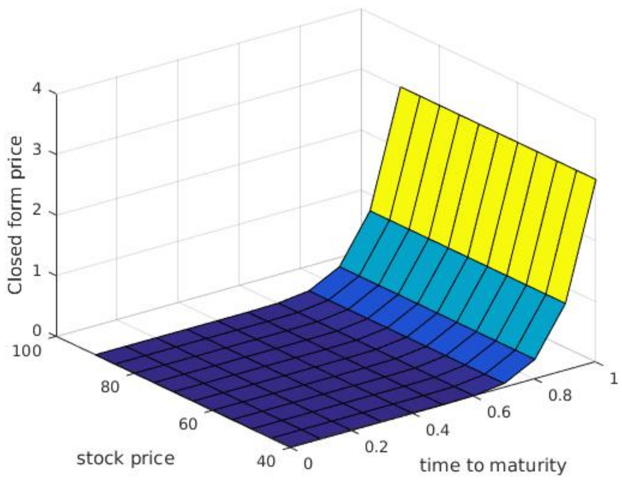


Figure 7: Surface of the evolution of American option prices via closed-form approximation. The pricing parameters are, $K = 42$, $S_0 = 40$, $r = 9\%$, $b = 9\%$, $\sigma = 0.20$, exercise dates $=$ 1000 per year and the number of simulated paths is 1000.



the LSM algorithm is effective for pricing American options relative to the closed-form approximation.

Another observation we can make here is that increasing the volatility, increases the price of the option.

Figure 9 shows the effect of increasing the strike price. As shown, as the strike price increases, the option price increase. We note that when the strike price becomes large enough the option price computed using the closed-form approximation starts to diverge.

Figure 8: Relative errors between LSM and Closed form solution. The pricing parameters are $S_0 = 10$, $K = 12$, $r = 4.8\%$, $b = 4.8\%$, $T = 1$, exercise dates = 150 per year and simulated paths = 15 000.

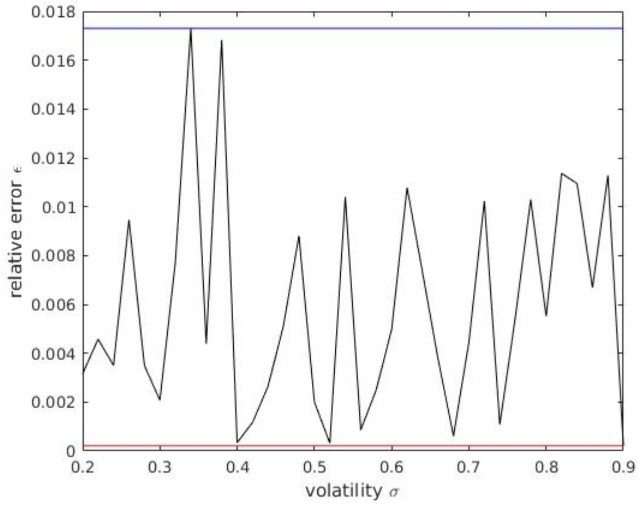
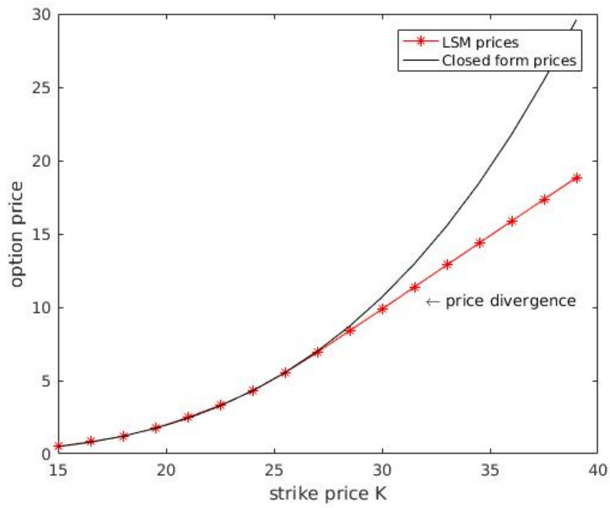


Figure 9: Comparison of LSM and Closed form approximation with varying K. The pricing parameters are $S_0 = 20$, $r = 12.75\%$, $b = 12.75\%$, $T = 3$, $\sigma = 0.3$, exercise dates = 100 per year and simulated paths = 10 000.



It seems like the closed-form approximation does not work perfectly when the strike price is much greater than the initial stock price.

5 Conclusion

5.1 Conclusion

European options can only be exercised at maturity while American options can be exercised whenever up to and including the maturity date. This difference makes it difficult to price American options. In this project, we considered one of the simplest simulation models, the least squares Monte Carlo simulation by Longstaff and Schwartz. The computational results confirm that this algorithm provides greater precision of computations when compared to the Binomial method and Bjerksund and Stensland (2002) model. This algorithm is also able to handle options with early exercise features and path dependence. Different basis functions give different prices, our results have shown that Laguerre polynomials are effective and they give accurate prices.

5.2 Further Work and Recommendations

Further work that we can conduct is of improving the LSM algorithm. We can do that by using Quasi-Monte Carlo instead of standard Monte Carlo. Quasi-monte Carlo methods use low-discrepancy sequences (they are not random at all but are deterministic) which makes the estimator give more accurate prices.

Another way we can improve the price of the estimator is to reduce its variance. We can use the variance reduction techniques such as antithetic methods and control variates. The aim of these techniques is to reduce the variance of the estimator. Thus reducing the variance of the estimator will give accurate computational results.

We can also expand the research by looking at pricing American swaptions or path-dependent or early exercise options using the LSM.

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