On Pricing Capped Options

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Abstract

This technical report examines the pricing of capped options, where the payoff is subject to a lower or upper limit. Under the Black-Scholes framework, we implement the implicit finite difference scheme and the Monte Carlo simulation to approximate the option prices numerically. The report provides a concise overview of the theory and implementation as a practical introduction to the pricing of capped options.

1 Capped Options

A vanilla option is a standard financial derivative that grants the holder the right, but not the obligation, to buy or sell a stock at a predetermined strike price on or before a specified expiration date. There are two types: a call option grants the right to buy, while a put option grants the right to sell.

A capped option is a variation of a vanilla option where the maximum possible payoff is limited by a predetermined cap. This means that even if the stock price moves significantly in a favorable direction, the option's profit is restricted. Capped options are often used to manage risk and reduce premium costs as they provide controlled exposure to price movements while capping potential profits. There are different variations of capped options. For example, a Taiwanese capped option consisting of barriers and resets which was studied by [3] or even the American capped options studied in [4]. In this report, we study a simple capped option without complex features.

We introduce some mathematics behind capped options. Let S be the stock price, K be the strike price, and C be the cap price. The current date is t while T is the maturity date. The stock price at the current date and maturity date are denoted by S_t and S_T, respectively.

Now, we take the payoff of vanilla options at maturity and limit them using the cap C. The payoff h_T of the capped call option and capped put option are given by:

$$h_T^{call} = \min(\max(S_T - K, 0), C), \tag{1}$$

$$h_T^{put} = \min(\max(K - S_T, 0), C). \tag{2}$$

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Figure 1 illustrates the payoff profiles for capped call and put options.

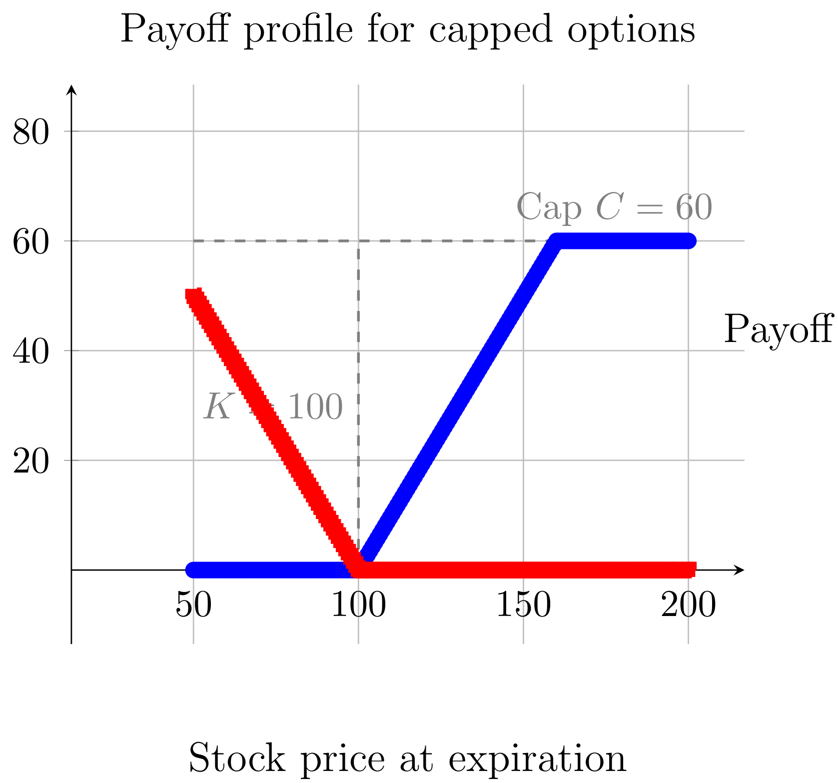

Figure 1: Payoff profile for capped call and capped put options with strike price $K = 100$ and cap level $C = 60$. We assume that $K > C$. The one in blue is a capped call payoff, and the one in red is a capped put payoff.

2 Methods

While it is possible to price capped options using the Black-Scholes (BS) model, in this report, we display the use of numerical methods to approximate the price of capped options. However, we use the analytical solutions obtained in the appendix as true values.

We employ two main approaches: (1) Implicit Finite Difference Method: this approach solves the Black-Scholes partial differential equation numerically. (2) Monte Carlo Simulation: this method estimates the expected option payoff by simulating a large number of stock price paths.

2.1 Implicit Finite Difference Method

For the implicit finite difference method, we start by writing the BS partial differential equation. Before we do that, we start by defining some important parameters needed to obtain the price for capped options.

Let P be the capped option price, r be the risk-free interest rate, and σ be the volatility of the stock. The partial differential equation is given by:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0. \tag{3}$$

Using central differences for the second derivative and backward difference for the time derivative, we approximate:

$$\frac{\partial P}{\partial t} \approx \frac{P_i^{n+1} - P_i^n}{\Delta t}, \tag{4}$$

$$\frac{\partial P}{\partial S} \approx \frac{P_{i+1}^n - P_{i-1}^n}{2\Delta S}, \tag{5}$$

$$\frac{\partial^2 P}{\partial S^2} \approx \frac{P_{i+1}^n - 2P_i^n + P_{i-1}^n}{\Delta S^2}. \tag{6}$$

Substituting these into Equation (3), we obtain the implicit finite difference scheme:

$$-\alpha_i P_{i-1}^{n+1} + (1+\beta_i)P_i^{n+1} - \gamma_i P_{i+1}^{n+1} = P_i^n, \tag{7}$$

where:

$$\alpha_i = \frac{\Delta t}{2}\left(\sigma^2 S_i^2 \frac{1}{\Delta S^2} - rS_i\frac{1}{2\Delta S}\right), \ \beta_i \ = 1 + \Delta t\left(\sigma^2 S_i^2 \frac{1}{\Delta S^2} + r\right), \ \gamma_i = \frac{\Delta t}{2}\left(\sigma^2 S_i^2 \frac{1}{\Delta S^2} + rS_i\frac{1}{2\Delta S}\right). \tag{8}$$

To solve this, we need to impose boundary (lower and upper stock prices) and terminal conditions. Terminal conditions are simply the payoffs at maturity. For the capped call option, when $S \to 0$, the price becomes $h_{S\to 0} = 0$ and when $S \to \infty$ the price becomes $h_{S\to\infty} = C \cdot e^{-r(T-t_n)}$. For the capped put option, when $S \to 0$, the price becomes $h_{S\to 0} = \min(K, C) \cdot e^{-r(T-t_n)}$ and when $S \to \infty$ the price becomes $h_{S\to\infty} = C \cdot e^{-r(T-t_n)}$. Another way to understand these boundary conditions is by looking at Figure 1.

2.2 The Monte Carlo Simulation Method

The Monte Carlo method is a numerical technique used to estimate the price of financial derivatives by simulating multiple paths of the underlying asset's price. We simulate multiple stock price paths under the risk-neutral measure. The stock price follows the BS process (aka geometric Brownian motion) given by:

$$S_{t+\Delta t} = S_t e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}w}, \tag{9}$$

where $w \sim N(0,1)$ is a standard normal random variable. For each simulated path, the payoff at maturity is:

$$h_T^{call} = \min(\max(S_T - K, 0), C), \quad h_T^{put} = \min(\max(K - S_T, 0), C). \tag{10}$$

The option price is then estimated as the discounted expectation:

$$P_0 \approx e^{-rT}\frac{1}{N}\sum_{i=1}^{N} h_T^{(i)}. \tag{11}$$

3 Results

The results of our numerical methods are summarized in Table 1.

As we can see, the prices obtained from the Monte Carlo simulation and finite difference are close to the analytical prices. They can be improved by increasing the number of simulations and time steps. Monte Carlo can also be improved by using variance reduction techniques.

Table 1: Capped option prices for different stock prices. The strike price is $K = 100$, cap price $C = 20$, risk-fre interest rate $r = 0.05$, volatility is $\sigma = 0.2$, $T = 1$ month. The implicit finite difference scheme used 100 time steps and 300 stock price steps. The Monte Carlo used 100 time steps and 100000 numbers of simulations.

S0	MC Call	Implicit Call	BSM Call	MC Put	Implicit Put	BSM Put
90	0.0897	0.0895	0.0878	9.6489	9.6422	9.6414
95	0.6518	0.6565	0.6557	5.2641	5.2389	5.2383
100	2.5102	2.5032	2.5103	2.0969	2.0895	2.0962
105	5.9867	5.9597	5.9620	0.5719	0.5738	0.5743
110	10.2983	10.2950	10.2956	0.1009	0.1053	0.1044

Another experiment can be conducted to check the stability of numerical methods when the capped option is deep in the money, or deep out of the money, or when the underlying parameters are extremely huge.

4 Appendix

Let $P_{\text{call}}(S, K)$ be the price of the vanilla call option and $P_{\text{put}}(S, K)$ be the price of the vanilla put option. The price of the capped call $P_{\text{capped call}}$ and put options $P_{\text{capped put}}$ are given by:

The price of a capped call option is given by:

$$P_{\text{capped call}} = P_{\text{call}}(S, K) - P_{\text{call}}(S, K + C),$$

where:

$$P_{\text{call}}(S, K) = S \cdot \Phi(d_1(K)) - K \cdot e^{-rT} \cdot \Phi(d_2(K)),$$

$$d_1(K) = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad d_2(K) = d_1(K) - \sigma\sqrt{T}.$$

The price of a capped put option is given by:

$$P_{\text{capped put}} = P_{\text{put}}(S, K) - P_{\text{put}}(S, K - C),$$

where:

$$P_{\text{put}}(S, K) = K \cdot e^{-rT} \cdot \Phi(-d_2(K)) - S \cdot \Phi(-d_1(K)),$$

$$d_1(K) = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad d_2(K) = d_1(K) - \sigma\sqrt{T}.$$

Here, Φ is the standard normal cumulative distribution function.

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