SOLUTION OF DIFFERENCE EQUATIONS USING Z-TRANSFORMS.

To solve the difference equation, first it is converted into algebraic equation by taking its Z-transform. The solution is obtained in z-domain and the time domain solution is obtained by taking its inverse Z-transform. The system response has two components. The source free response and the forced response. The response of the system due to input alone when the initial conditions are neglected is called the forced response of the system. It is also called the steady state response of the system. It represents the component of the response due to the driving force. The response of the system due to initial conditions alone when the input is neglected is called the free or natural response of the system. It is also called the transient response of the system. It represents the component of the response when the driving function is made zero. The response due to input and initial conditions considered simultaneously is called the total response of the system. For a stable system, the source free component always decays with time. In fact a stable system is one whose source free component decays with time. For this reason the source free component is also designated as the transient component and the component due to source is called the steady state component. When input is a unit impulse input, the response is called the impulse response of the system and when the input is a unit step input, the response is called the step response of the system.

EXAMPLE 1 A linear shift invariant system is described by the difference equation

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + x(n-1)$$

with $y(-1) = 0$ and $y(-2) = -1$.
Find (a) the natural response of the system (b) the forced response of the system for a step input and (c) the frequency response of the system.

Solution:

(a) The natural response is the response due to initial conditions only. So make $x(n) = 0$. Then the difference equation becomes

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = 0$$

Taking Z-transform on both sides, we have

$$Y(z) - \frac{3}{4}\,[z^{-1}\,Y(z) + y(-1)] + \frac{1}{8}\,[z^{-2}\,Y(z) + z^{-1}y(-1) + y(-2)] = 0$$

i.e.

$$Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) - \frac{1}{8} = 0$$

$$\therefore\ Y(z) = \frac{1/8}{1-(3/4)z^{-1}+(1/8)z^{-2}} = \frac{1/8z^2}{z^2-(3/4)z+(1/8)} = \frac{1/8z^2}{[z-(1/2)]\,[z-(1/4)]}$$

The partial fraction expansion of $Y(z)/z$ gives

$$\frac{Y(z)}{z} = \frac{(1/8)z}{[z-(1/2)]\,[z-(1/4)]} = \frac{A}{z-(1/2)} + \frac{B}{z-(1/4)} = \frac{1/4}{z-(1/2)} - \frac{1/8}{z-(1/4)}$$

$$Y(z) = \frac{1}{4}\,\frac{z}{z-(1/2)} - \frac{1}{8}\,\frac{z}{z-(1/4)}$$

Taking inverse Z-transform on both sides, we get the natural response as:

$$y(n) = \frac{1}{4}\left(\frac{1}{2}\right)^n u(n) - \frac{1}{8}\left(\frac{1}{4}\right)^n u(n)$$

(a) To find the forced response due to a step input, put $x(n) = u(n)$. So we have

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = u(n) + u(n-1)$$

We know that the forced response is due to input alone. So for forced response, the initial conditions are neglected. Taking Z-transform on both sides of the above equation and neglecting the initial conditions, we have

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = u(n) + u(n-1)$$

We know that the forced response is due to input alone. So for forced response, the initial conditions are neglected. Taking Z-transform on both sides of the above equation and neglecting the initial conditions, we have

$$Y(z) - \frac{3}{}z^{-1}Y(z) + \frac{1}{}z^{-2}Y(z) = U(z) + z^{-1}U(z) = 4\ 8$$

i.e. $$Y(z)\left(1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right)=\frac{z+1}{z-1}$$

$$\therefore \quad Y(z)=\frac{z+1}{(z-1)\,[1-(3/4)z^{-1}+(1/8)z^{-2})}=\frac{z^2(z+1)}{(z-1)[z^2-(3/4)z+(1/8)]}$$

$$=\frac{z^2(z+1)}{(z-1)\,[z-(1/2)]\,[z-(1/4)]}$$

Taking partial fractions of $Y(z)/z$, we have

$$\therefore \quad \frac{Y(z)}{z}=\frac{z(z+1)}{(z-1)\,[z-(1/2)]\,[z-(1/4)]}=\frac{A}{z-1}+\frac{B}{z-(1/2)}+\frac{C}{z-(1/4)}$$

$$=\frac{16/3}{z-1}-\frac{6}{z-(1/2)}+\frac{5/3}{z-(1/4)}$$

or $$Y(z)=\frac{16}{3}\left(\frac{z}{z-1}\right)-6\left[\frac{z}{z-(1/2)}\right]+\frac{5}{3}\left[\frac{z}{z-(1/4)}\right]$$

Taking the inverse Z-transform on both sides, we have the forced response for a step input.

$$y(n)=\frac{16}{3}u(n)-6\left(\frac{1}{2}\right)^n u(n)+\frac{5}{3}\left(\frac{1}{4}\right)^n u(n)$$

© The frequency response of the system $H(\ )$ is obtained by putting $z=e^{jw}$ in $H(z)$.

$$H(z)=\frac{Y(z)}{X(z)}=\frac{z(z+1)}{z^2-(3/4)z+(1/8)}$$

$$H(\omega)=\frac{e^{j\omega}(e^{j\omega}+1)}{(e^{j\omega})^2-(3/4)e^{j\omega}+(1/8)}$$

EXAMPLE 2 (a) Determine the free response of the system described by the difference equation

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n) \quad \text{with } y(-1) = 1 \text{ and } y(-2) = 0$$

(b) Determine the forced response for an input

Solution:

(a) The free response, also called the natural response or transient response is the response due to initial conditions only [i.e. make $x(n) = 0$]. So, the difference equation is:

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = 0$$

Taking Z-transform on both sides, we get

$$Y(z) - \frac{5}{6}[z^{-1}Y(z) + y(-1)] + \frac{1}{6}[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)] = 0$$

$$Y(z)\left(1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}\right) - \frac{5}{6} + \frac{1}{6}z^{-1} = 0$$

$$\therefore \quad Y(z) = \frac{(5/6) - (1/6)z^{-1}}{1 - (5/6)z^{-1} + (1/6)z^{-2}} = \frac{5/6[z - (1/5)]z}{z^2 - (5/6)z + (1/6)} = \frac{(5/6)z[z - (1/5)]}{[z - (1/2)][z - (1/3)]}$$

Taking partial fractions of $Y(z)/z$, we have

$$\frac{Y(z)}{z} = \frac{5/6[z - (1/5)]}{[z - (1/2)][z - (1/3)]} = \frac{A}{z - (1/2)} + \frac{B}{z - (1/3)} = \frac{3/2}{z - (1/2)} - \frac{2/3}{z - (1/3)}$$

$$\therefore \quad Y(z) = \frac{3}{2}\frac{z}{z - (1/2)} - \frac{2}{3}\frac{z}{z - (1/3)}$$

Taking inverse Z-transform on both sides, we get the free response of the system as:

(a) To determine the forced response, i.e. the steady state response, the initial conditions are to be neglected.

The given difference equation is:

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n) = \left(\frac{1}{4}\right)^n u(n)$$

Taking Z-transform on both sides and neglecting the initial conditions, we have

$$Y(z) - \frac{5}{6}z^{-1}Y(z) + \frac{1}{6}z^{-2}Y(z) = \frac{z}{z - (1/4)}$$

$$Y(z) - \frac{5}{6}z^{-1}Y(z) + \frac{1}{6}z^{-2}Y(z) = \frac{z}{z-(1/4)}$$

i.e.,
$$Y(z)\left(1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}\right) = \frac{z}{z-(1/4)}$$

$$\therefore \quad Y(z) = \frac{z}{z-(1/4)}\,\frac{1}{1-(5/6)z^{-1}+(1/6)z^{-2}} = \frac{z^3}{[z-(1/4)][z-(1/2)][z-(1/3)]}$$

Partial fraction expansion of $Y(z)/z$ gives

$$\frac{Y(z)}{z} = \frac{z^2}{[z-(1/4)][z-(1/3)][z-(1/2)]} = \frac{A}{z-(1/4)} + \frac{B}{z-(1/3)} + \frac{C}{z-(1/2)}$$

$$= \frac{3}{z-(1/4)} - \frac{8}{z-(1/3)} + \frac{6}{z-(1/2)}$$

Multiplying both sides by z, we get

$$Y(z) = 3\frac{z}{z-(1/4)} - 8\frac{z}{z-(1/3)} + 6\frac{z}{z-(1/2)}$$

Taking inverse Z-transform on both sides, the forced response of the system is:

$$y(n) = 3\left(\frac{1}{4}\right)^n u(n) - 8\left(\frac{1}{3}\right)^n u(n) + 6\left(\frac{1}{2}\right)^n u(n)$$

EXAMPLE 3 Find the impulse and step response of the system

$$y(n) = 2x(n) - 3x(n-1) + x(n-2) - 4x(n-3)$$

Solution: For impulse response, $x(n) = \delta(n)$

The impulse response of the system is:

$$y(n) = 2\delta(n) - 3\delta(n-1) + \delta(n-2) - 4\delta(n-3)$$

For step response, $x(n) = u(n)$

The step response of the system is:

$$y(n) = 2u(n) - 3u(n\ 1) + u(n\ 2) - 4u(n\ 3)$$

EXAMPLE 4 Solve the following difference equation

$$y(n) + 2y(n-1) = x(n)$$

with $x(n) = (1/3)^n u(n)$ and the initial condition $y(-1) = 1$.

Solution: The solution of the difference equation considering the initial condition and input simultaneously gives the total response of the system.

The given difference equation is:

$$y(n) + 2y(n-1) = x(n) = \left(\frac{1}{3}\right)^n u(n) \text{ with } y(-1) = 1$$

Taking Z-transform on both sides, we get

$$Y(z) + 2[z^{-1}Y(z) + y(-1)] = X(z) = \frac{1}{1-(1/3)z^{-1}}$$

Substituting the initial conditions, we have

$$Y(z)\,(1+2z^{-1}) = -2(1) + \frac{1}{1-(1/3)\,z^{-1}}$$

$$\therefore \quad Y(z) = \frac{-2}{1+2z^{-1}} + \frac{1}{[1-(1/3)\,z^{-1}][1+2z^{-1}]}$$

$$= \frac{-2z}{z+2} + \frac{z^2}{[z-(1/3)]\,(z+2)}$$

$$\text{Let} \quad Y_1(z) = \frac{z^2}{[z-(1/3)]\,(z+2)}$$

Taking partial fractions of $Y_1(z)/z$, we have

$$\frac{Y_1(z)}{z} = \frac{z}{[z-(1/3)]\,(z+2)} = \frac{A}{z-(1/3)} + \frac{B}{z+2} = \frac{1/7}{z-(1/3)} + \frac{6/7}{z+2}$$

Multiplying both sides by z, we have

$$Y_1(z) = \frac{1}{7}\frac{z}{z-(1/3)} + \frac{6}{7}\frac{z}{z+2}$$

$$\therefore \quad Y(z) = -\frac{2z}{z+2} + \frac{6}{7}\frac{z}{z+2} + \frac{1}{7}\frac{z}{z-(1/3)} = -\frac{8}{7}\frac{z}{z+2} + \frac{1}{7}\frac{z}{z-(1/3)}$$

Taking inverse Z-transform on both sides, the solution of the difference equation is:

$$y(n) = -\frac{8}{7}(-2)^n u(n) + \frac{1}{7}\left(\frac{1}{3}\right)^n u(n)$$

EXAMPLE 5 Solve the following difference equation using unilateral Z-transform. with initial conditions

$$y(n)-\frac{7}{12}y(n-1)+\frac{1}{12}y(n-2)=x(n) \text{ for } n\geq 0$$

$$y(-1)=2,\ y(-2)=4 \text{ and } x(n)=\left(\frac{1}{5}\right)^n u(n)$$

Solution: The solution of the difference equation gives the total response of the system (i.e., the sum of the natural (free) response and the forced response)

The given difference equation is:

$$y(n)-\frac{7}{12}y(n-1)+\frac{1}{12}y(n-2)=x(n)=\left(\frac{1}{5}\right)^n u(n)$$

with initial conditions $y(-1)=2$ and $y(-2)=4$. Taking Z-transform on both sides, we have

$$Y(z)-\frac{7}{12}[z^{-1}Y(z)+y(-1)]+\frac{1}{12}[z^{-2}Y(z)+z^{-1}y(-1)+y(-2)]=\frac{1}{1-(1/5)z^{-1}}$$

i.e. $$Y(z)\left(1-\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}\right)=\frac{7}{12}(2)-\frac{1}{12}(2z^{-1})-\frac{1}{12}(4)+\frac{1}{1-(1/5)z^{-1}}$$

i.e. $$Y(z)\left(1-\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}\right)=\frac{5}{6}\left(1-\frac{1}{5}z^{-1}\right)+\frac{1}{1-(1/5)z^{-1}}$$

$$\therefore\ Y(z)=\frac{(5/6)[1-(1/5)z^{-1}]}{[1-(7/12)z^{-1}+(1/12)z^{-2}]}+\frac{1}{[1-(1/5)z^{-1}][1-(7/12)z^{-1}+(1/12)z^{-2}]}$$

$$\therefore\ Y(z)=\frac{(5/6)[1-(1/5)z^{-1}]}{[1-(7/12)z^{-1}+(1/12)z^{-2}]}+\frac{1}{[1-(1/5)z^{-1}][1-(7/12)z^{-1}+(1/12)z^{-2}]}$$

$$=\frac{(5/6)[z-(1/5)]z}{[z-(1/4)][z-(1/3)]}+\frac{z^3}{[z-(1/5)][z-(1/4)][z-(1/3)]}$$

$$=\frac{z[(11/6)z^2-(1/3)z+(1/30)]}{[z-(1/5)][z-(1/4)][z-(1/3)]}$$

Taking partial fractions of $Y(z)/z$, we have

$$\frac{Y(z)}{z}=\frac{A}{z-(1/5)}+\frac{B}{z-(1/4)}+\frac{C}{z-(1/3)}=\frac{6}{5}\frac{1}{z-(1/5)}+\frac{1}{8}\frac{1}{z-(1/4)}+\frac{100}{27}\frac{1}{z-(1/3)}$$

Multiplying both sides by z, we have

$$Y(z)=\frac{6}{5}\frac{z}{z-(1/5)}+\frac{1}{8}\frac{z}{z-(1/4)}+\frac{102}{27}\frac{z}{z-(1/3)}$$

Taking inverse Z-transform on both sides, the solution of the difference equation is:

$$y(n)=\frac{6}{5}\left(\frac{1}{5}\right)^n u(n)+\frac{1}{8}\left(\frac{1}{4}\right)^n u(n)+\frac{102}{27}\left(\frac{1}{3}\right)^n u(n)$$

EXAMPLE 6 Using Z-transform determine the response of the LTI system described by $y(n) - 2r\cos y(n-1) + r^2 y(n-2) = x(n)$ to an excitation $x(n) = a^n u(n)$.

Solution: Taking Z-transform on both sides of the difference equation, we have

$$Y(z) - 2r\cos\theta\,[z^{-1}Y(z) + y(-1)] + r^2[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)] = X(z)$$

i.e. $$Y(z)\,[1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}] = \frac{z}{z-a}$$

$$\therefore \quad Y(z) = \frac{z^3}{(z-a)(z-re^{j\theta})(z-re^{-j\theta})}$$

$$= \frac{a^2}{a^2-2ar\cos\theta+r^2}\frac{z}{z-a} + \frac{r^2e^{j2\theta}}{(re^{j\theta}-a)(re^{j\theta}-re^{-j\theta})}\frac{z}{z-re^{j\theta}} + \frac{r^2e^{-j2\theta}}{(re^{-j\theta}-a)(re^{-j\theta}-re^{j\theta})}\frac{z}{z-re^{-j\theta}}$$

$$\therefore \quad y(n) = \frac{a^2}{a^2-2ar\cos\theta+r^2}a^n u(n) + \frac{r^{n+1}}{\sin\theta}\left[\frac{r\sin(n+1)\theta - a\sin(n+2)\theta}{a^2-2ar\cos\theta+r^2}\right]u(n)$$

EXAMPLE 7 Determine the step response of an LTI system whose impulse response $h(n)$ is given by $h(n) = a^{-n}u(-n); 0 < a < 1$.

Solution: The impulse response is

$$h(n) = a^{-n}u(-n); 0 < a < 1$$

$$H(z) = \frac{1}{1-az} = -\frac{1}{a}\frac{1}{z-(1/a)}$$

We have to find the step response

$$x(n) = u(n) \text{ and } H(z) = \frac{z}{z-1}$$

The step response of the system is given by

$$y(n) = x(n) * h(n)$$

$$Y(z) = X(z)\,H(z) = \left(-\frac{1}{a}\right)\frac{z}{z-1}\frac{1}{z-(1/a)} = \frac{1}{1-a}\left[\frac{z}{z-1} - \frac{z}{z-(1/a)}\right]$$

So the step response is

$$y(n) = \frac{1}{1-a}\left[u(n) - \left(\frac{1}{a}\right)^n u(n)\right]$$

EXAMPLE 8 The step response of an LTI system is

Solution: We have $s(n) = h(n) * u(n)$

$$\therefore \qquad S(z) = H(z)U(z) = H(z)\frac{z}{z-1}$$

Given

$$s(n) = \left(\frac{1}{3}\right)^{n-2} u(n+2)$$

$$S(z) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{3}\right)^{n-2} u(n+2)\,z^{-n} = 3^2\sum_{n=-2}^{\infty}\left(\frac{1}{3z}\right)^n$$

$$= 3^2\frac{\left(\frac{1}{3z}\right)^{-2}}{1-\frac{1}{3z}} = \frac{3^4 z^2}{1-\frac{1}{3}z^{-1}} = \frac{81z^3}{\left(z-\frac{1}{3}\right)}$$

The system function $H(z)$ is

The system function $H(z)$ is

$$H(z) = S(z)\frac{z-1}{z} = \frac{81z^3}{\left(z-\frac{1}{3}\right)}\frac{z-1}{z} = \frac{81z^2(z-1)}{\left(z-\frac{1}{3}\right)} = \frac{81z^3}{z-\frac{1}{3}} - \frac{81z^2}{z-\frac{1}{3}} = 81z^2\frac{z}{z-\frac{1}{3}} - 81z\frac{z}{z-\frac{1}{3}}$$

The impulse response of the system is

$$h(n) = 81\left(\frac{1}{3}\right)^{n+2} u(n+2) - 81\left(\frac{1}{3}\right)^{n+1} u(n+1) = 9\left(\frac{1}{3}\right)^{n} u(n+2) - 27\left(\frac{1}{3}\right)^{n} u(n+1)$$