Real Analysis

Mid-Sem 2022

Full marks 50 (10 × 5)

1. a) If A and B are sets, then show that

$$(i)\mathcal{P}(A)\cup\mathcal{P}(B)\subseteq\mathcal{P}(A\cup B),\quad ii)\quad \mathcal{P}(A)\cap\mathcal{P}(B)=\mathcal{P}(A\cap B).$$

Here $\mathcal{P}$ denotes powerset.

b) Prove that a set and its powerset do not have the same cardinality.

2. Prove that for $p\in(1,\infty)$, we have $xy\leq\frac{x^p}{p}+\frac{y^q}{q}$, with $(x,y)\in\mathbb{R}^+$ and $\frac{1}{p}+\frac{1}{q}=1$.

3. Let S be a nonempty subset of $\mathbb{R}$ which is bounded above. Set $s = supS$. Show that there exists a sequence $\{x_n\}$ in S with $n\in\mathbb{N}$, which converges to s.

4. Show that $\{x_n\}$ defined by

$$x_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots\frac{1}{n}-log_e n,$$

is convergent.

5. Let $\{x_n\}$ be a sequence defined by

$$x_1 = 1 \text{ and } x_{n+1}=\sqrt{x_n^2+\frac{1}{2^n}}.$$

Show that the sequence is convergent.

$1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n} - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

$(1-x) + \frac{1+x^2}{2} + \frac{1-x^3}{3} + \cdots \frac{1 \pm x^n}{n}$

Q4 $\quad x_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n} - \ln n$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$x_m = 1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{m} - \ln m.$

WLOG $m>n$, $\quad |x_m - x_n| = \frac{1}{n+1}+\frac{1}{n+2}+\cdots\frac{1}{m} + \ln\frac{n}{m} < \varepsilon.$

$\frac{1}{x} - 1.$

$x_1 = 1$

$x_2 = \sqrt{3/2}$

$x_3 = \sqrt{\frac{3}{2}+\frac{1}{2^{\sqrt{x}}}}$

$\frac{m-n}{mn} + \ln\frac{n}{m}.$

$\frac{m-n}{n} + \ln\frac{n}{m}.$

$\ln x - x.$

$\frac{m}{\ }+ \ln n$

Real Analysis

End-Sem 2022

Full marks 100 (10×10) Time - 3 hours

1. Prove that $\sqrt{2}$ is not rational. sure

2. Consider the Fibonacci numbers $\{F_n\}$ defined by $F_1 = 1,\ F_2 = 1,\ and\ F_{n+2} = F_{n+1} + F_n$. Show that sure
$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}, \quad n = 1, 2, 3, \ldots.$$

3. Show that the sequence $\{x_n\}$ defined by $x_n = \int_1^n \frac{\cos t}{t^2}\, dt$ is Cauchy. sure

4. Discuss the convergence or divergence of
$$x_n = \frac{[\alpha] + [2\alpha] + [3\alpha] + \ldots + [n\alpha]}{n^2}, \quad n \in \mathbb{N},$$ sure

   where $[x]$ represents the greatest integer less that or equal to the x and α is an arbitrary real number.

5. Given $x \geq 1$, show that $\lim_{n\to\infty} \left(2x^{1/n} - 1\right)^n = x^2$.

6. Let $f(x) = [x]$ and $g(x) = x - [x]$. Sketch the plots for f and g. Find the points at which they are continuous. sure

7. Show that any function continuous and periodic on $\mathbb{R}$ must be uniformly continuous.

8. Show that there exists a continuous function $F : [0, 1] \to \mathbb{R}$ whose derivative exists and equals zero almost everywhere but which is not constant.

9. Let $f(x)$ is differentiable at a. Then find
$$\lim_{n\to\infty} \frac{a^n f(x) - x^n f(a)}{x - a}, \quad n \in \mathbb{N}.$$

10. Consider a function $f(x)$, whoose second derivative $f''(x)$ exists and continuous on (a, b) with $c \in (a, b)$. Show that
$$\lim_{h\to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$ sure

    Is the existence of the second derivative necessary to prove the existence of the above limit?

$$\frac{x \ln(2x^{1/n} - 1)}{1/n} = \ln y \qquad \frac{d(a^x)}{dx} = a^x \log a$$

$a^{1/n}$

$a^{1/n} \log a \ \frac{-1}{n^2}$ $\qquad \frac{2 \frac{1}{n} x^{1/n - 1}}{2x^{1/n} - 1}$ $\qquad x^{1/n} \log 1/n \qquad x$

$e^{1/n \ln a} \qquad \frac{d(e^{x \ln a})}{dx}$

$$\frac{2x^{1/n} \log x \ \frac{-1}{n^2}}{2x^{1/n} - 1} \left(\frac{-1}{n^2}\right) \qquad e^{x \ln a} \ln a$$