Quantum Mechanics 2022

SC1. 203

Quiz-I

26/08/2022

Time 40 Mins

1) In the particle-in-a-box problem, a generic wave function $\Psi(x,t)$ at $t=0$ can be expressed as a linear combination of the steady states with coefficients c_n. What does one mean by the statement: **the probability of getting E_n is given by $|c_n|^2$?** Explain. Can you express $\Psi(x,t)$ in terms of c_n's? In terms of $\Psi(x,0)$? [3+1+1=5]
2) Is the operator for p^2 hermitian? Show by explicit integral. [5]
3) Evaluate $[\hat{x}, \hat{p}^2]$. [5]
4) Obtain the ground state wave function for a harmonic oscillator. [5]
5) For a free particle, $v_{\text{quantum}} = v_{\text{classical}}/2$. Explain. [5]

$\int e^{-x^2} dx.$

15.

Quantum Mechanics 2022

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Quiz-II

18/08/2022 || Time 40 Mins

1) Obtain the relation between k_F (the wave vector of the Fermi surface) and the electron density in the electron gas model. [5]
2) Assume three noninteracting electrons are in a one-dimensional infinite square well in the (one-particle) states ψ_2, ψ_5, ψ_7. Write the three-particle wave function. What is the total energy in the unit of $\pi^2\hbar^2/2m_e a^2$? [4+1=5]
3) How many ways can N identical bosons be put in a potential so that there are N_i particles in d_i (one-particle) states of energy E_i (where $i = 1, 2, 3, ...$)? [5]
4) Obtain the most probable occupation number for the case above. [5]
5) What is meant by the statement — *all electrons are identical*? Argue that it leads to the Pauli exclusion principle. [5]

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Quiz-III

11/11/2022 || Time 60 Mins

1) Suppose a delta-function bump appears in the centre of an infinite square well:

$$V(x) = \begin{cases} \alpha\delta(x - a/2) & \text{if } 0 \leq x \leq a \\ \infty & \text{otherwise} \end{cases}$$

where α is a constant. Find the first-order corrections to the allowed energies and explain why the energies are not perturbed for even n.

wave func 2) Three particles are in three distinct one-particle states $\psi_a(x)$, $\psi_b(x)$, and $\psi_c(x)$. Consider various possibilities and list the different three-particle states can you construct.

3) Place a hydrogen nucleus (proton) at the origin, calculate $\langle x \rangle$, $\langle x^2 \rangle$ in terms of the Bohr radius for the electron. What is the most probable value of r? Assume the electron to be in the ground state.

4) Find the spectrum and the eigenfunctions of the operator $\hat{Q} = i\frac{d}{d\phi}$ where ϕ is the usual polar coordinate. Is it hermitian?

5) Consider a three-dimensional harmonic oscillator $V(\mathbf{r}) = \frac{1}{2}m\omega^2 r^2$. What is the energy of the n^{th} state? What is the degeneracy of E_n?

INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY

Quantum Mechanics

Monsoon 2022 – CND Core – Credit 4

Instructor: Subhadip Mitra

Date: September 19, 2022

Time: 1 H 30 M (08:30 - 10:00)

Mid-Semester Examination

Total Marks: 50

Instructions:

- **Class notes or books are not permitted. But you may bring one A4 sheet of handwritten material (not photo-copy/printed).**
- **Calculators are allowed.**
- **Do not write anything** (except roll number, seat no. etc.) **on the first page of the answer book.**
- You may skip 'trivial' steps. However, unless the logic is clear, you will not get any credit for a problem.
- Illegible answers will not be graded.
- No 'benefit of doubt' because of bad notation/illegible hand-writing etc.

Q 1. Show that

(a) For any two observables represented by two operators, A and B,

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2$$

(where σ denotes the standard deviation) and that for $\hat{A}=\hat{x}$ and $\hat{B}=\hat{p}$, the above equation reduces to the Heisenberg's uncertainty principle.

(b) If $\hat{A}$ and $\hat{B}$ have a complete set of common eigenstates (which then can form a basis), then $[\hat{A},\hat{B}]\,|\psi\rangle = 0$ for any $|\psi\rangle$ in the Hilbert space.

(c) Eigenvalues of Hermitian operators are real, and the eigenstates corresponding to different eigenvalues of a Hermitian operators are orthogonal.

[5+2+3 = 10 CO: 1,2,5]

Q 2. For a simple harmonic oscillator, the ladder operators are given by

$$\hat{a}_\pm = \frac{1}{\sqrt{2m\omega\hbar}}(\mp i\hat{p}+m\omega\hat{x}).$$

(a) Show that the Hamiltonian operator can be written as

$$\hat{H} = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right) = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right).$$

(b) Obtain the normalized ground-state wave function. What is its energy?

proof (c) Let $\psi_n(x)$ be for the normalized (steady state) wavefunction of the n^{th} energy state. Find how $\psi_n(x)$ is related to $\psi_0(x)$.

[2+(3+1)+4 = 10 CO: 3]

Q 3. Let $|x\rangle$ denote the state (wave-function) at x. We can define an infinitesimal translation operator $\hat{T}(dx)$ such that

$$\hat{T}(dx')\,|x\rangle = |x+dx'\rangle.$$

(a) What properties should such an operator satisfy? In particular, argue for

(i) $\hat{T}^\dagger(dx')$,

(ii) $\hat{T}^{-1}(dx')$,

(iii) $\hat{T}(dx')\cdot\hat{T}(dx'')$ and

(iv) $\lim_{dx'\to 0}\hat{T}(dx')$.

(b) Show that $\hat{T}(dx') = 1 - i\,\hat{K}\,d\hat{x}'$ satisfies all the above properties if we ignore terms of second order or higher in dx'.

(c) Show that

$$[\hat{x},\hat{T}(dx')]\,|x'\rangle = d\hat{x}'\,|x'+d\hat{x}'\rangle \approx d\hat{x}'\,|x'\rangle$$

and obtain $[\hat{x},\hat{K}]$.

[4+2+(3+1) = 10 CO: 2,4]

Q 4. (a) Show that the time evolution because of the Schrödinger equation does not affect the normalization of a wave function.

(b) However, if we assume that a particle is in a potential with an imaginary part, i.e.,

$$V = V_0 - i\Gamma$$

(where V_0 is the true potential and Γ is a positive real constant), show that the probability of finding the particle at any point $\rho(x,t)$ decreases with time, i.e., the particle decays. What is the lifetime of this particle?

(c) If the potential is real, the probability is conserved and hence, in 3D, it satisfies the continuity equation,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

where $\vec{J}$ is the probability current. Write the expression for $\vec{J}$. [4 + 4 + 2 = 10 CO: 3,4]

Q 5. (a) For the general spinor $\chi = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ find the probability of getting $\pm\hbar/2$ if one measures $\hat{S}_x$. Also find $\langle S_x \rangle$.

(b) Obtain the operator to measure the component of spin of an electron in the direction making 45° with the x axis in the x-z plane?

(c) Argue that the eigenvalues of the operator $\hat{L}^2 - \hat{L}_z^2$ are always positive.

(d) Construct the $\hat{S}_z$ and $\hat{S}^2$ matrices and for a spin-1 particle. [2 + 2 + 2 + (2 + 2) = 10 CO: 1,3,4]

$$\frac{d^2x}{dt^2} = -kx$$

INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY HYDERABAD

QUANTUM MECHANICS

Monsoon 2022 – CND Core – Credit 4

Instructor: Subhadip Mitra **Date:** NOVEMBER 19, 2022

Time: 03 H 00 M

End Examination

Total Marks: 100

Instructions:

- Keep your answers **to the point**. You may skip 'trivial' steps. However, unless the logic is clear, you will **not** get any credit for a problem.
- Illegible answers will **not** be graded.
- **No** 'benefit of doubt' because of bad notation/illegible hand-writing etc.

Q 1. Consider a finite square well,

$$V(x) = \begin{cases} -V_0 & \text{for } -a < x < a \quad (V_0 > 0) \\ 0 & \text{otherwise,} \end{cases}$$

with a particle of energy $E > 0$ (scattering state).

(a) Show that the probability of the particle reflecting back is nonzero in general.

(b) What happens if $E \gg V_0$ or $E \to 0$? Show that there are some energies for perfect transmission (transmission resonance, this is why you get a very large transmission when you scatter low-energy electrons through noble-gas atoms).

(c) We say that *the absolute value of potential does not matter; only the difference matters. Hence, if we add a constant to the overall potential, nothing changes.* Is this true in Quantum Mechanics? If so, how do we see that? If not, why not?

[3+3+4=10] **CO:** 1,4,5

Q 2. (a) Show with the momentum-space wave function $\Phi(p,t)$ that

$$\langle x \rangle = \int \Phi^* \left(-\frac{\hbar}{i} \frac{\partial}{\partial p} \right) \Phi \, dp.$$

(b) Prove the Virial theorem:

$$\frac{d}{dt} \langle xp \rangle = 2 \langle T \rangle - \left\langle x \frac{dV}{dx} \right\rangle,$$

where T is the kinetic energy.

(c) Consider a periodic potential, i.e., $V(x+\lambda) = V(x)$. Show that the wave function at $(x_0 + \lambda)$ is proportional to $\psi(x_0)$ up to a constant (i.e., x-independent) phase.

(d) Explain how one gets dynamic solutions out of the stationary states for the time-independent potential.

(e) Show that for a simple harmonic oscillator $\langle \hat{V} \rangle = \langle \hat{T} \rangle$.

[2+3+3+2+5=15] **CO:** 1,3,4,5

Q 3. A spinning electron constitutes a magnetic dipole. Its dipole moment is proportional to the spin,

$$\vec{\mu} = \gamma \vec{S}$$

where γ is the gyromagnetic ratio. If you put it in a magnetic field $\vec{B}$, it feels a torque. The energy associated with the torque is $-\vec{\mu} \cdot \vec{B}$.

(a) If the magnetic field is constant $\vec{B} = B_0 \hat{z}$, then show that $< \vec{S} >$ gets titled and it precesses about the field with a constant frequency.

(b) If $\vec{B} = B_0 \cos(\omega t) \hat{z}$ (where ω is a constant) and the electron starts out in the spin-up state in the x direction, i.e.,

$$\chi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

then obtain $\chi(t)$ by solving the time dependent Schrödinger equation

$$i\hbar \frac{\partial \chi}{\partial t} = \mathbf{H} \chi,$$

where $\mathbf{H}$ is the Hamiltonian matrix.

[7+8=15] **CO:** 2,3,4

Q 4. (a) Let, for a system of interest $\{|a_i\rangle\}$ be the set of eigenstates of an Hermitian operator A. Show that

(i) the matrix $A_{ij} = \langle a_i|A|a_j\rangle$ is diagonal,

(ii) the matrix $B_{ij} = \langle a_i|B|a_j\rangle$ is also diagonal where A and B are compatible observables.

(iii) the transformation from the basis $\{|a_i\rangle\}$ to another basis $\{|c_i\rangle\}$ is unitary, where $\{|c_i\rangle\}$ are the eigenstates of another Hermitian operator C incompatible with A or B.

(b) In the case of perturbation theory with degenerate states, why does one first look for some operator that commutes with the perturbed Hamiltonian?

proof

(c) If the lowest-order relativistic correction to the Hamiltonian is given as

$$H' = -\frac{p^4}{8m^3c^2},$$

partially

find the lowest-order relativistic correction to the energy levels of the one-dimensional harmonic oscillator.

[(1+2+2)+3+7=15] **CO:** 1,2,4,5

Q 5. Use a Gaussian trial function, $\psi(x) = \left(\frac{2b}{\pi}\right)^{1/4} e^{-bx^2}$ to obtain the lowest upper bound on the ground state energy of

(a) the linear potential: $V(x) = \alpha|x|$,

(b) the quartic potential: $V(x) = \alpha x^4$.

[5+5=10] **CO:** 3,4

Q 6. (a) Show that the x, y and z components of the angular momentum operator $(\hat{L}_x, \hat{L}_y, \hat{L}_z)$ are mutually incompatible but all of them commute with $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ (it is sufficient to show that $\hat{L}^2$ commutes with any one component, say $\hat{L}_z$, the rest can be argued similarly).

5.

(b) Since $\hat{L}^2$ and $\hat{L}_z$ commute, lets denote their common eigenstates as $|\lambda,\mu\rangle$ where

$$\hat{L}^2|\lambda,\mu\rangle = \lambda|\lambda,\mu\rangle \quad \text{and} \quad \hat{L}_z|\lambda,\mu\rangle = \mu|\lambda,\mu\rangle.$$

1

Now, with the following operators

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$$

show that

$$[\hat{L}_z, \hat{L}_\pm] = \pm\hbar\hat{L}_\pm \quad ; \quad [\hat{L}^2, \hat{L}_\pm] = 0 \quad ; \quad \hat{L}^2 = \hat{L}_\pm\hat{L}_\mp + \hat{L}_z^2 \mp \hbar\hat{L}_z \quad \text{and}$$

(c) the operators $\hat{L}_\pm$ take one eigenstate to another eigenstate as:

$$\hat{L}_\pm|\lambda,\mu\rangle \propto |\lambda,\mu\pm\hbar\rangle,$$

4

i.e., they act like ladder operators. In other words, show that

$$\begin{aligned}
\hat{L}^2\left(\hat{L}_\pm|\lambda,\mu\rangle\right) &= \lambda\left(\hat{L}_\pm|\lambda,\mu\rangle\right), \\
\hat{L}_z\left(\hat{L}_\pm|\lambda,\mu\rangle\right) &= (\mu\pm\hbar)\left(\hat{L}_\pm|\lambda,\mu\rangle\right).
\end{aligned}$$

(d) Now, there will be a μ_{max} and a μ_{min}, i.e., if we start with some $|\lambda,\mu\rangle$ and keep on applying $\hat{L}_+$ on it, the process will terminate when we apply $\hat{L}_+$ on $|\lambda,\mu_{max}\rangle$ and, similarly, $\hat{L}_-|\lambda,\mu_{min}\rangle = 0$. Show that λ for the μ_{max} state will be given as

4,

should be 1.

$$\lambda = \mu_{max}(\mu_{max} + \hbar) \quad \text{and} \quad \mu_{min} = -\mu_{max}.$$

(e) Finally show

$$\hat{L}_\pm|\lambda,\mu\rangle = \sqrt{\mu_{max}(\mu_{max}+\hbar) - \mu(\mu\pm\hbar)}\,|\lambda,\mu\pm\hbar\rangle.$$

[5+4+(2+2)+(2+2)+3=20] **CO:** 1,2,3

Q 7. Consider a box of volume V containing free electron gas (assume the total number of atoms to be N with each one contributing q electrons). The normalized wave functions are given as

$$\psi_{n_x,n_y,n_z} = \sqrt{\frac{8}{V}}\sin\left(\frac{n_x\pi}{l_x}x\right)\sin\left(\frac{n_y\pi}{l_y}y\right)\sin\left(\frac{n_z\pi}{l_z}z\right)$$

where $V = l_x l_y l_z$. The allowed energies are

$$E_{n_x,n_y,n_z} = \frac{\hbar^2k^2}{2m} = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2)$$

where the wave vector $\vec{k} = (k_x, k_y, k_z)$ with $k_i = n_i^2/l_i^2$.

(a) Show that the Fermi energy is $E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3}$ where ρ is the free electron density. How is it related to the chemical potential?

(b) The total energy $E_{tot} \propto V^{-2/3}$. Find the proportionality constant and the degeneracy pressure.

(c) Covalent bonding between two electrons requires the two to be in the singlet state. Explain.

[4+6+5=15] CO: 1,4,5