Algebra, Statistics and Probability:

A Mathematics Book for High Schools and Colleges

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Table of Contents

CHAPTER 1
NUMBER BASES

For general purposes we use numbers in base ten. The place value of each digit in a base ten number such as 816_{ten} can be expressed as follows:

8 1 6

Hundred	Tens	Unit
$(8 \times 10^2$ or $8 \times 100)$	$(1 \times 10^1$ or $1 \times 10)$	$(6 \times 10^0$ or $6 \times 1)$

Similarly, numbers may be expressed in other bases. For example 324_{seven} can be expressed as follows:

$3\,2\,4_{seven}$

(3×7^2) (2×7^1) (4×7^0)

The rule above is employed in converting numbers from one base to base ten.

Conversion of numbers from other bases to base ten

Examples

1. Convert the binary number 10111_{two} to base ten.

Solution

Each of the numbers is given a power starting from 0 on the right. This power is what the base digit will be raised to, when carrying out the expansion.

$1^4 0^3 1^2 1^1 1^0{}_{two} = (1x2^4) + (0x2^3) + (1x2^2) + (1x2^1) + (1x2^0) = 16 + 0 + 4 + 2 + 1 = 23_{ten}$

Note that any number raised to power zero is equal to 1. For example, $2^0 = 1$.

2. Convert 3042_{five} to a number in base ten.

Solution

$3^{3}0^{2}4^{1}2^{0}_{five} = (3x5^{3}) + (0x5^{2}) + (4x5^{1}) + (2x5^{0}) = (3x125) + (0x25) + 20 + (2x1) = 375 + 0 + 20 + 2 = 397_{ten}$

Conversion of numbers from base ten to other bases

Here the method of repeated division is employed. The base ten numbers will be divided by the new base digit, and the remainder will be written down. 'R' below, denotes remainder.

Examples

1. Convert 60_{ten} to a number in base two.

Solution

2	60	
2	30	R 0
2	15	R 0
2	7	R 1
2	3	R 1
2	1	R 1
	0	R 1

Read the remainders upwards

$\therefore \quad 60_{ten} = 111100_{two}$

2. Convert 587_{ten} to a number in base eight.

Solution

8	587	
8	73	R 3
8	9	R 1
8	1	R 1
	0	R 1

Read the remainders upwards

$\therefore$ $587_{ten} = 1113_{eight}$

Bicimals

Base two fractions are called bicimals. Bicimals can also be converted to decimals (base ten fractions). Similarly, fractions in other bases can be converted to base ten decimals.

Examples

1. Convert the bicimal 110.011_{two} to a decimal.

Solution
Powers given to the numbers after the decimal point should be negative.

$1^2 1^1 0^0 . 0^{-1} 1^{-2} 1^{-3}{}_{two} = (1 \text{x} 2^2) + (1 \text{x} 2^1) + (0 \text{x} 2^0) + (0 \text{x} 2^{-1}) + (1 \text{x} 2^{-2}) + (1 \text{x} 2^{-3}) = 4 + 2 + 0 + 0 + \frac{1}{2^2} + \frac{1}{2^3} = 6 + \frac{1}{4} + \frac{1}{8} = 6 + 0.25 + 0.125 = 6.375_{ten}$

2. Convert 223.32_{four} to base ten.

Solution

$2^2 2^1 3^0 . 3^{-1} 2^{-2}{}_{four} = (2 \text{x} 4^2) + (2 \text{x} 4^1) + (3 \text{x} 4^0) + (3 \text{x} 4^{-1}) + (2 \text{x} 4^{-2}) = (2 \text{x} 16) + 8 + (3 \text{x} 1) + \frac{3}{4^1} + \frac{2}{4^2} = 32 + 8 + 3 + \frac{3}{4} + \frac{2}{16} = 43 + \frac{3}{4} + \frac{1}{8} = 43 + 0.75 + 0.125 = 43.875_{ten}$

Conversion of decimals to numbers in other bases

Examples

1. Convert 61.875_{ten} to base two

Solution

The first step is to convert 61 to base two as follows:

$$\begin{array}{r|ll} 2 & 61 & \\ 2 & 30 & \text{R } 1 \\ 2 & 15 & \text{R } 0 \\ 2 & 7 & \text{R } 1 \end{array}$$

2	3	R 1
2	1	R 1
	0	R 1

$61_{ten} = 111101_{two}$

The decimal part is now converted as follows:

0.875: 2 x 0.875 = 1.750

2 x 0.750 = 1.500

2 x 0.500 = 1.000

Keep multiplying the decimal part by the base digit until you get to a whole number. You may stick to the original number of decimal places in the question.

Finally, the answer is obtained by taking only the digits before the decimal points, i.e. 111

$0.875_{ten} = 0.111_{two}$

$\therefore$ $61.875_{ten} = 111101.111_{two}$

2. Convert 127.75_{ten} to base six

<u>Solution</u>

The first step is to convert 127 to base six as follows:

6	127	
6	21	R 1
6	3	R 3
	0	R 3

$127_{ten} = 331_{six}$

The decimal part is now converted as follows:

0.75: 6 x 0.75 = 4.50

6 x 0.50 = 3.00

Multiply only the decimal part of each value by the base digit until you get to a whole number. Finally, the answer is obtained by taking only the digits before the decimal points, i.e. 43

$0.75_{ten} = 0.43_{six}$

$\therefore\ 127.75_{ten} = 331.43_{six}$

3. Convert the decimal 0.5625_{ten} to a number in base six

Solution

0.5625: $6 \times 0.5625 = 3.375$

$6 \times 0.375 = 2.25$

$6 \times 0.25 = 1.50$

$6 \times 0.50 = 3.00$

Taking only the integers of the values obtained after each multiplication gives:

$0.5625_{ten} = 0.3213_{six}$

Conversion of numbers from one base to another

Examples

1. Convert 110101_{two} to a number in base five.

Solution

The number 110101_{two} will first be converted to base ten before converting the base ten value to base five.

$1^5 1^4 0^3 1^2 0^1 1^0{}_{two} = (1 \times 2^5) + (1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0) = 32 + 16 + 0 + 4 + 0 + 1 = 53_{ten}$

Now, convert 53_{ten} to base five as follows:

$$\begin{array}{r|lll} 5 & 53 & & \\ 5 & 10 & R & 3 \\ 5 & 2 & R & 0 \\ & 0 & R & 2 \end{array}$$

$= 203_{five}$

$\therefore \quad 110101_{two} = 203_{five}$

2. Convert 317_{nine} to a number in base six.

Solution

$317_{nine} = (3x9^2) + (1x9^1) + (7x9^0) = (3x81) + 9 + (7x1) = 243 + 9 + 7 = 259_{ten}$

The next step is to convert 259_{ten} to base six. This gives:

$$\begin{array}{r|lll}
6 & 259 & & \\
6 & 43 & R & 1 \\
6 & 7 & R & 1 \\
6 & 1 & R & 1 \\
 & 0 & R & 1 \\
\end{array}$$

= 1111

$\therefore \quad 317_{nine} = 1111_{six}$

Addition and subtraction of numbers in other bases

Addition and subtraction of numbers in other bases are similar to that of base ten. Numbers equal to or greater than the base digit are not written down directly. Also, a larger number cannot be subtracted from a smaller number. The following examples illustrate how numbers are added and subtracted in other bases. Note that when arranging the numbers above each other, the place value system must be maintained. This means units under units, tens under tens, and so on.

Examples

1. Evaluate $11011_{two} + 111_{two}$

Solution

$$\begin{array}{rcccccc}
 & & 1 & 1 & 0 & 1 & 1 \\
+ & & & & 1 & 1 & 1 \\
\hline
 & 1 & 0 & 0 & 0 & 1 & 0_{two} \\
\end{array}$$

Workings: 1 + 1 = 2. Since 2 should not be written down in base two, it is evaluated as $\frac{2}{2\text{ (i.e the base digit)}}$ = 1 remainder 0. The 0 which is the remainder is written down, while 1 is added to the next column. So, the next column becomes 1 + 1 + 1 = 3. 3 divided by 2 is 1 remainder 1. Write down 1 which is the remainder, and take the other 1 to the third column. The third column becomes 1 + 0 + 1 = 2. As before this becomes 1 remainder 0. Write 0 and take 1 to the fourth column. This column gives 1 + 1 = 2. Write 0 and take 1 to the last column, which also adds up to 2. Write 0 down and take 1. Since there is no more columns left, write down the 1 at the end.

2. Evaluate $315.46_{eight} + 27.164_{eight}$

Solution

$$\begin{array}{r} 315.46 \\ +\ \ 27.164 \\ \hline 344.644_{eight} \\ \hline \end{array}$$

Workings: From the right, 4 + 0 = 4. Next, is 6 + 6 = 12. This 12 is greater than the base digit. This is now evaluated as $\frac{12}{8\text{(i.e base digit)}}$ = 1 remainder 4. The 4 is written down, while 1 is added to the next column, and so on. Note that the empty space is taken to contain 0.

3. Subtract 3178_{nine} from 6246_{nine}

Solution

$$\begin{array}{r} 6246 \\ -\ 3178 \\ \hline 3057_{nine} \\ \hline \end{array}$$

Workings: Since 6 - 8 will not go. Hence, 1 has to be borrowed from 4. That 1 borrowed is equal to 9 (i.e. the base digit). This 9 is added to 6 to give 15. So the first column becomes 15 – 8 = 7. Note that the 4 in the next column is now 3 since 1 has been borrowed from it. The next column becomes 3 – 7. This is impossible. 1 has to be borrowed from 2. That 1 borrowed is equal to 9 (i.e. the base digit). This 9 is now added to 3 to give 12. So it becomes 12 – 7 = 5. The 2 in the third column becomes 1. So, 1 – 1 = 0. Finally, the last column is 6 – 3 = 3.

4. Evaluate $100101_{two} - 1010_{two}$

Solution

$$\begin{array}{r} 100101 \\ -\quad 1010 \\ \hline 11011_{two} \\ \hline \end{array}$$

Note that any 1 borrowed is equal to 2 (i.e. the base digit), and it is added to the number that does the borrowing. In base two subtraction, it is always "0" that does the borrowing.

Multiplication of numbers in other bases

Multiplication is carried out in a similar way to addition. When numbers are multiplied and the result is greater than the base digit, the value obtained is divided by the base digit. The remainder is what is written down, while the answer is carried to the next stage.

Examples

1. Evaluate $1101_{two} \times 11_{two}$

Solution

$$\begin{array}{r} 1101 \\ \times \quad 11 \\ \hline 1101 \\ +\ 1101 \\ \hline 100111_{two} \\ \hline \end{array}$$

2. What is the total age of 253_{seven} girls whose average age is 31_{seven}. Express your answer in base seven.

Solution

This is a word problem that can be expressed as follows:

$$\frac{x}{253} = 31$$

Where all the values are in base seven

By cross multiplication, x is given by:

$x_{seven} = 253_{seven} \times 31_{seven}$

x =

$$
\begin{array}{r}
2\ 5\ 3 \\
\times\ \ \ 3\ 1 \\
\hline
2\ 5\ 3 \\
+\,1\ 1\ 2\ 2\ \ \ \\
\hline
1\ 1\ 5\ 0\ 3_{seven} \\
\hline
\end{array}
$$

∴ The total age is 11503_{seven}

<u>Division of numbers in other bases</u>

Division is carried out by using the usual long division method, but it should be carried out in the given base.

<u>Examples</u>

1. Evaluate $14201_{five} \div 314_{five}$

<u>Solution</u>

$$
\begin{array}{r|l}
 & \ \ 24 \\
\hline
314 & 1\ 4\ 2\ 0\ 1 \\
 & -1\ 1\ 3\ 3\ \ \ \\
\cline{2-2}
 & \ \ \ 2\ 3\ 2\ 1 \\
 & -2\ 3\ 2\ 1 \\
\cline{2-2}
 & \ \ -\ -\ -\ - \\
\end{array}
$$

∴ $14201_{five} \div 314_{five} = 24_{five}$

<u>Workings</u>: 1420 ÷ 314, which gives 2 with a remainder. The 2 is obtained by multiplying 314 by 1, 2, 3, etc, in base five until you obtain a value that is equal to, or just less than 1420. This 2 is written at the top of the bar and used to multiply 314 to get 1133. Then subtract 1133 from 1420 in base five to obtain 232 as the remainder. Then bring down the next digit in 14201 (i.e. 1) to it to obtain 2321. Then repeat the task

2. Divide 11111111_{two} by 101_{two}

Solution

The division is carried out just like the example explained above. However, since this is in base two, the only values we can obtain in the course of long division when an immediate higher number divides a number just lower than it is always 1. This makes division in base two to be very easy to solve. For example, $111_{two} \div 101_{two} = 1$. There will be a remainder.

Therefore, the example above is now solved as follows.

```
         110011
 101 | 1 1 1 1 1 1 1 1
      -1 0 1
         1 0 1
         1 0 1
             1 1 1
             1 0 1

               1 0 1
               1 0 1
               ----
```

∴ $11111111_{two} \div 101_{two} = 110011_{two}$

Workings: 111 ÷ 101 = 1. Write the 1 on top of the bar and use it to multiply 101. This gives the 101 which is written under the 111 and subtracted from it. The subtraction gives 10. Now bring down 1 from the original number to make the 10 to become 101. Repeat the process of division using this 101. When this is done, 101 is subtracted from 101 and this gives 0. Now bring down 1 from the original number. Then 1 ÷ 101 will not go, so write 0 on the bar and bring down another 1. This give 11 ÷ 101 which will also not go. So write another zero on the bar and bring down another 1. This gives 111 ÷ 101 = 1. Write the 1 on the bar and continue the division process.

Note that in division in other bases, the numbers involved can be converted to base ten. Then the division is carried out in base ten and the final answer is converted back to the original base. For example, $1100_{two} \div 100_{two} = 11_{two}$, can also be solved by converting 1100 and 100 to base ten to give 12 and 4 respectively. Then, 12 ÷ 4 = 3. When 3 is converted back to base two it gives 11 which is the required answer.

More examples on number bases

1. If $32_x = 122_{three}$, find the value of the base x

Solution

The numbers have to be converted to base ten. This gives:

$3^1 2^0{}_{x} = 1^2 2^1 2^0{}_{three}$

$(3 \times x^1) + (2 \times x^0) = (1 \times 3^2) + (2 \times 3^1) + (2 \times 3^0)$

$3x + 2 = 9 + 6 + 2$

$3x + 2 = 17$

$3x = 17 - 2 = 15$

$x = \frac{15}{3} = 5$

2. Solve for x if $23_x + 65_x - 71_x = 15$

Solution

Each number has to be converted to base ten except 15 which is already in base ten. This gives:

$(2 \times x^1) + (3 \times x^0) + (6 \times x^1) + (5 \times x^0) - [(7 \times x^1) + (1 \times x^0)] = 15$

$2x + 3 + 6x + 5 - (7x + 1) = 15$

$2x + 3 + 6x + 5 - 7x - 1 = 15$

$8x - 7x = 15 - 3 - 5 + 1$

$x = 8$

Exercises

1. Convert 101011_{two} to a number in base ten.
2. Convert 436_{ten} to base six
3. Convert 254_{eigh}t to a number in base ten.
4. Convert 1011.011_{two} to a decimal
5. Convert 3032_{four} to base seven

6. Convert 6136_{seven} to a number in base five

7. Convert 597_{ten} to an octadecimal (base 18) number. (Hint: Take A as 10, B as 11, C as 12, and so on).

8. Evaluate $405_{eight} - 217_{eight}$

9. Evaluate $1.101_{two} + 11.01_{two}$

10. Find the value of P in the equation: $101P_{three} + 11_{three} = 1100_{three}$

11. Evaluate $563_{eight} \times 62_{eight}$

12. If $23_n = 111_{two}$, find the value of the base n.

13. Find $(101_{two})^2$

14. Evaluate $425_{six} \div 11_{six}$

15. Evaluate $341_{five} \div 22_{five}$

16. Express the decimal $\frac{3}{4}$ in bicimal

17. Convert 239.68_{ten} to base five

18. Convert 47.625_{ten} to a number in base two

19. Convert the decimal 0.0625_{ten} to base six

20. If $198.921875_{ten} = m_{four}$, find the value of m.

CHAPTER 2
MODULAR ARITHMETIC

This is a kind of arithmetic in which remainder is of interest. For example, 9 = 1 (mod 4), since $\frac{9}{4}$ = 2 remainder 1. So, the remainder (i.e. 1) is the answer. Note that "mod" is short for modulo. The basic arithmetic operations can be carried out in modular arithmetic.

<u>Addition</u>

In modular arithmetic, addition is represented by $\oplus$, while subtraction is represented by $\ominus$, in order to differentiate them from the usual addition and subtraction signs.

<u>Examples</u>

Evaluate the following:

1. $5 \oplus 8$ (mod 5)

2. $67 \oplus 38$ (mod 7)

<u>Solutions</u>

1. $5 \oplus 8$ (mod 5) = 13 (mod 5). Since 13 is greater than 5 (i.e. the modulus), divide 13 by 5 to get the remainder which is the equivalent value. This gives:

13 (mod 5) = $\frac{13}{5}$ = 2 remainder 3

$\therefore$ $5 \oplus 8$ (mod 5) = 13 (mod 5) = 3 (mod 5)

2. $67 \oplus 38$ (mod 7) = 105 (mod 7) = $\frac{105}{7}$ = 15 remainder 0

$\therefore$ $67 \oplus 38$ (mod 7) = 105 (mod 7) = 0 (mod 7)

Subtraction

Examples

1. Find the simplest form of each of the following:

a. -5 (mod 6)

b. -52 (mod 11)

Solutions

a. -5 (mod 6) = -5 + (6x1) = -5 + 6 = 1 (mod 6). (This is obtained by adding a multiple of the modulus digit (i.e. 6) that is equal to or just greater than 5)

b. -52 (mod 11) = -52 + (11x5) = -52 + 55 = 3 (mod 11). (55 is the multiple of 11 that is just greater than 52)

2. Evaluate the following:

a. $21 \ominus 6$ (mod 8)

b. $8 \ominus 18$ (mod 3)

c. $21 \ominus 64$ (mod 9)

Solution

a. $21 \ominus 6$ (mod 8) = 15 (mod 8) = 7 (mod 8)

b. $8 \ominus 18$ (mod 3) = -10 (mod 3) = -10 + (3x4) = -10 + 12 = 2 (mod 3)

c. $21 \ominus 64$ (mod 9) = -43 (mod 9) = -43 + (9x5) = -43 + 45 = 2 (mod 9)

Multiplication

In modular arithmetic, multiplication is represented by $\otimes$, while division is represented by $\oplus$ in order to differentiate them from the usual multiplication and division signs.

Examples

Evaluate the following:

1. 5 ⊗ 7 (mod 4)

2. 21 ⊗ 65 (mod 6)

3. 3 (mod 5) ⊗ 4 (mod 5)

Solution

1. 5 ⊗ 7 (mod 4) = 35 (mod 4) = 3 (mod 4)

2. 21 ⊗ 65 (mod 6). This can be done easily by simplifying 21 and 65 in modulo 6. This gives:

21 (mod 6) = 3 (mod 6). And 65 (mod 6) = 5 (mod 6)

21 ⊗ 65 (mod 6) = 3 x 5 (mod 6) = 15 (mod 6) = 3 (mod 6)

3. 3 (mod 5) ⊗ 4 (mod 5) = 12 (mod 5) = 2 (mod 5)

Division

Evaluate the following:

1. 24 ⊜ 6 (mod 5)

2. 2 ⊜ 5 (mod 6)

3. 8 ⊜ 9 (mod 7)

Solutions

1. 24 ⊜ 6 (mod 5)

Let 24 ⊜ 6 (mod 5) be x. This gives:

24 ÷ 6 = x

$\frac{24}{6} = x$

$x = 4$

2. 2 ⊜ 5 (mod 6)

Let 2 ⊜ 5 (mod 6) be x

$2 \div 5 = x$

$\frac{2}{5} = x$

$5x = 2 \pmod{6}$

(Now, look for a multiple of 6 (i.e. the modulus) such that when it is added to 2 it gives a number that is divisible by 5). The multiple is 18 (i.e. 6 x 3)

$5x = 2 + (6 \times 3)$

$5x = 2 + 18 = 20$

$x = \frac{20}{5}$

$x = 4 \pmod{6}$

3. $8 \oslash 9 \pmod{7}$

Let $8 \oslash 9 \pmod{7}$ be x

$\frac{8}{9} = x$

$\therefore\ 9x = 8 \pmod{7}$ (The multiple of 7 that should be added to 8 to obtain a number divisible by 9 is 28)

$9x = 8 + (7 \times 4)$

$9x = 8 + 28 = 36$

$x = \frac{36}{9}$

$x = 4 \pmod{7}$

Simple equations in modular arithmetic

Examples

Solve the following equations:

1. $8 \oplus x = 0 \pmod{9}$

2. $2x = 3 \pmod{7}$

3. $5x \oplus 2 = 3 \pmod{11}$

4. $4x \oplus 8 = 2 \pmod{9}$

<u>Solutions</u>

1. $8 \oplus x = 0 \pmod{9}$

   $8 + x = 0$

   $x = -8 \pmod{9}$

   $= -8 + (9 \times 1)$

   $= -8 + 9$

   $x = 1 \pmod{9}$

2. $2x = 3 \pmod{7}$

   $2x = 3 + (7 \times 1) = 3 + 7$

   $2x = 10$

   $x = \frac{10}{2}$

   $x = 5 \pmod{7}$

3. $5x \oplus 2 = 3 \pmod{11}$

   $5x = 3 - 2 = 1$

   $5x = 1 + (11 \times 4)$

   $5x = 1 + 44$

   $5x = 45$

   $x = \frac{45}{5}$

   $x = 9 \pmod{11}$

4. $4x \oplus 8 = 2 \pmod{9}$

$4x = 2 – 8$

$4x = -6$

$4x = - 6 + (9 \times 2)$

$4x = - 6 + 18$

$4x = 12$

$x = \frac{12}{4}$

$x = 3 \pmod 9$

<u>Exercises</u>

1. Evaluate the following:

a. $28 \oplus 62 \pmod 5$

b. $39 \oplus 97 \pmod 8$

c. $39 \ominus 50 \pmod 7$

d. $7 \ominus 58 \pmod{14}$

2. Find the simplest positive form of each of the following:

a. $– 23 \pmod 5$

b. $– 81 \pmod{12}$

3. Evaluate the following:

a. $33 \otimes 74 \pmod 7$

b. $7 \pmod 8 \otimes 13 \pmod 8$

c. $42 \oslash 11 \pmod 2$

d. $11 \oslash 8 \pmod 5$

4. Solve the following equations:

a. $4x = 1 \pmod 7$

b. $2x + 3 = 1 \pmod 6$

5. Copy and complete the table below in modulo 8.

$\oplus$	5	6	7
1			
2			
3			
4			

6. Copy and complete the table below in modulo 7.

$\otimes$	4	5	6
2			
3			
4			
5			

CHAPTER 3
STANDARD FORM AND APPROXIMATION OF NUMBERS

A number is in standard form if it is expressed as follows: A x 10^n, where A is a number between 1 and 10 and n is either a positive or a negative whole number.

Examples

1. Express 8142 in standard form.

Solution

At the end of a whole number, there is a decimal point. This decimal point is moved towards the left, up to the right side of the first digit. The number of times that the point is moved becomes the power of 10.

8142 = 8.142 x 10^3

2. Express the following numbers in standard form:

a. 100000

b. 100008

c. 4562000

Solutions

a. 100000 = 1 x 10^5 (Note that the last zero(s) after a decimal point is/are irrelevant)

b. 100008 = 1.00008 x 10^5

c. 4562000 = 4.562 x 10^6

3. Express the following numbers in standard form:

a. 6510.248

b. 0.04381

c. 0.00000681

Solutions

a. $6510.248 = 6.510248 \times 10^3$

b. For numbers less than 1 and greater than 0, we move the decimal point to the right, not left like in other examples. The number of times moved is given as a negative power of 10.

$0.04381 = 4.381 \times 10^{-2}$

c. $0.00000681 = 6.81 \times 10^{-6}$

Approximation of numbers

Numbers can be approximated to the nearest hundred, ten, whole number or even to a given number decimal places or significant figures.

The digits 0, 1, 2, 3 and 4 are used to round down a number, while the digits 5, 6, 7, 8 and 9 are used to round up a number.

Examples

1. Approximate the number 1209849 to:

a. the nearest thousand

b. the nearest hundred

c. the nearest ten

Solutions

a. The thousand digit in the given number is 9. The number at its right side (i.e. 8) is large enough to round up 9 and make it 10. When this is done, all the numbers in front of 9 become zero. Note that when 9 becomes 10, it is written as 0, while 1 is added to the number next to it.

$\therefore$ 1209849 = 1210000 (To the nearest thousand)

b. Similarly, 1209849 = 1209800 (To the nearest hundred) Note that 4 is not large enough to round up 8 (i.e. the hundred digit), so 8 remains the same.

c. 1209849 = 1209850 (To the nearest ten)

2. Round off 24.65 to:

a. the nearest whole number

b. the nearest ten

Solution

a. To approximate a number to the nearest whole number means to either round up or round down the digit before the decimal point. In this example, 4 is the digit before the decimal point.

24.65 = 25 (To the nearest whole number) Note that 6 (i.e. the number to the right side of 4) is large enough to round up 4 to give 5.

b. 24.65 = 20 (To the nearest ten). Note that 2 is the tens digit and 4 cannot round it up.

3. Round off 381.256996 to:

a. 2 decimal places

b. 5 decimal places

Solutions

a. To approximate to two decimal places, count the first two numbers after the decimal point and see if the third number will be able to round up the second number or not. Here, 6 will round up 5 and make it 6.

381.256996 = 381.26 (To 2 decimal places)

b. 381.256996 = 381.25700 (To 5 decimal places)

Significant figures

It is important to know the first significant figure in a decimal fraction. For example, the first significant figure in 0.006045 is 6. The initial zeros are not regarded as significant figures.

Examples

1. Round off 4906997 to:

a. 1 significant figure

b. 6 significant figures

Solution

a. 4906997 = 5000000 (To 1 significant figure). Note that 9 has rounded up the first significant figure (i.e. 4) to give 5.

b. 4906997 = 4907000 (To 6 significant figures). Note that 7 has rounded up the 9 in 699 to make it 700, while the 7 itself becomes 0.

2. Approximate 0.000460794 to:

a. 1 significant figure

b. 4 significant figures.

Solutions

a. 0.000460794 = 0.0005 (To 1 significant figure) Note that the first zeros are not counted as significant figures.

b. 0.000460794 = 0.0004608 (To 4 significant figures). Note that a zero after a significant figure is counted as significant.

Exercises

1. Express 2189000 in standard form.

2. Express the following numbers in standard form:

a. 2500000

b. 12050800

c. 41102000

3. Express the following numbers in standard form:

a. 23700.212

b. 0.2170

c. 0.000026

4. Approximate 3840196 to:

a. 2 significant figures

b. 3 significant figures

5. Express the following numbers in standard form:

a. 814000

b. 41218004

c. 0.0001002

d. 642.42

6. Round off 149.00562 to:

a. the nearest ten

b. 5 significant figures

c. two decimal places

7. Approximate 0.005206798 to:

a. two significant figures

b. 6 significant figures

c. 2 decimal places

CHAPTER 4
LAWS OF INDICES

The following are the laws of indices. They are true for all values of a, b and $x \neq 0$

Law 1: $x^a \times x^b = x^{a+b}$

Law 2: $x^a \div x^b = x^{a-b}$

Law 3: $x^0 = 1$

Law 4: $x^{-a} = \frac{1}{x^a}$ or $bx^{-a} = \frac{b}{x^a}$ or $\left(\frac{b}{x}\right)^{-a} = \left(\frac{x}{b}\right)^a$

<u>Examples</u>

Simplify the following:

1. $10^5 \times 10^4$
2. $m^8 \div m^5$
3. $\frac{a^{-8}}{a^3}$
4. $5x^2 \times 4x^0 \times 2x^{-6}$
5. $y^{-5} \div b^0$

<u>Solution</u>

1. $10^5 \times 10^4 = 10^{5+4} = 10^9$
2. $m^8 \div m^5 = m^{8-5} = m^3$
3. $\frac{a^{-8}}{a^3} = a^{-8-3} = a^{-11} = \frac{1}{a^{11}}$
4. $5x^2 \times 4x^0 \times 2x^{-6} = (5 \times 4 \times 2)x^{2+0+(-6)} = 40x^{2-6} = 40x^{-4} = \frac{40}{x^4}$
5. $y^{-5} \div b^0 = y^{-5} \div 1 = y^{-5} = \frac{1}{y^5}$

Product of indices

In applying product of indices, the following are true:

$(x^a)^b = x^{ab}$

Similarly, $(x^a y^b)^c = x^{ac} y^{bc}$ and $(\frac{x}{y})^a = \frac{x^a}{y^a}$

Examples

Simplify the following:

1. $(h^4)^{-5}$
2. $(2^{-3})^2$
3. $(-c^3)^2$
4. $(-4u^2v)^3$

Solution

1. $(h^4)^{-5} = h^{4x(-5)} = h^{-20} = \frac{1}{h^{20}}$
2. $(2^{-3})^2 = 2^{-3x2} = 2^{-6} = \frac{1}{2^6} = \frac{1}{64}$
3. $(-c^3)^2 = -c^{3x2} = -c^6 = c^6$ (A negative number that is raised to an even number power will give a positive value).
4. $(-4u^2v)^3 = -4^{1x3}\ u^{2x3}\ v^{1x3} = -4^3u^6v^3 = -64u^6v^3$

Fractional indices

In applying fractional indices, the following are true:

$x^{1/a} = \sqrt[a]{x}$ and $x^{a/b} = \sqrt[b]{x^a}$ or $x^{a/b} = (\sqrt[b]{x}\ )^a$

In all cases, $x \neq 0$

Examples

Simplify the following:

1. $27^{⅓}$

2. $9^{-½}$

3. $(25a^2)^{½}$

4. $\sqrt{1\frac{9}{16}}$

5. $(\frac{16}{54})^{-2/3}$

Solutions

1. $27^{⅓} = \sqrt[3]{27} = 3$

2. $9^{-½} = \frac{1}{9^{½}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$ (Note that $\sqrt[2]{9}$ should be written as $\sqrt{9}$ since 2 is not usually written with the square root sign).

3. $(25a^2)^{½} = 25^{½}a^{(2 \times ½)} = 25^{½}a^1 = (\sqrt{25}) \times a = 5a$

4. $\sqrt{1\frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$

5. $(\frac{16}{54})^{-2/3} = (\frac{8}{27})^{-2/3}$ (When the fraction is expressed in its lowest term)

$(\frac{8}{27})^{-2/3} = (\frac{27}{8})^{2/3} = \frac{27^{⅔}}{8^{⅔}} = \frac{(\sqrt[3]{27})^2}{(\sqrt[3]{8})^2} = \frac{3^2}{2^2} = \frac{9}{4}$ (Note that by taking the inverse of the term in the bracket, the negative power becomes positive)

Equations in indices

Examples

Solve the following equations:

1. $4^{x-1} = 64$

2. $n^{-⅔} = 9$

3. $2a^{-3} = -16$

4. $9^{x} = 27$

5. $5x = 40x^{-½}$

Solutions

1. $4^{x-1} = 64$ (This is solved by expressing both sides of the equation in the same base and then equating the powers. This gives:

$4^{x-1} = 4^{3}$

Equating the powers gives:

$x-1 = 3$

$x = 3 + 1$

$x = 4$

2. $n^{-⅔} = 9$ (In this case, the unknown letter is the base. To solve this, make the power of n to 1 by multiplying this power by its inverse and using the same sign of the power. The other side of the equation should also be raised to the same power). This gives:

$(n^{-⅔})^{-3/2} = 9^{-3/2}$

$n^{(-2/3 \text{ x } -3/2)} = \frac{1}{9^{3/2}}$

$n^{1} = \frac{1}{(\sqrt{9})^{3}} = \frac{1}{3^{3}} = \frac{1}{27}$

$n = \frac{1}{27}$

3. $2a^{-3} = -16$

Divide both sides by 2.

$a^{-3} = \frac{-16}{2}$

$a^{-3} = -8$

Now make the power of 'a' to be 1 by multiplying this power by its inverse. This gives:

$(a^{-3})^{-1/3} = (-8)^{-1/3}$

$a = \frac{1}{(-8)^{1/3}} = \frac{1}{\sqrt[3]{-8}} = \frac{1}{-2}$

$a = -\frac{1}{2}$

4. $9^x = 27$

Expressing both sides of the equation in the same base gives:

$(3^2)^x = 3^3$

$3^{2x} = 3^3$

Equating the powers gives

$2x = 3$

$x = \frac{3}{2}$

5. $5x = 40x^{-1/2}$

Divide both sides by 5.

$x = \frac{40x^{-1/2}}{5}$

$x = 8x^{-1/2}$ (Since 40 ÷ 5 gives 8)

Divide both sides by $x^{-½}$

$$\frac{x}{x^{-½}} = \frac{8x^{-½}}{x^{-½}}$$

Cancelling out $x^{-½}$ on the right hand side gives,

$x^{1-(-½)} = 8$ (Note that x can be expressed as x^1. Also, from the law of indices, $x \div x^{-½} = x^{1-(-½)}$)

$\therefore \quad x^{1+½} = 8$

$x^{3/2} = 8$

Make the power of x to be 1 by multiplying it by its inverse. Also raise the power of 8 to the same inverse. This gives:

$(x^{3/2})^{⅔} = 8^{⅔}$

$x = 8^{⅔}$

$x = (\sqrt[3]{8}\,)^2 = 2^2$

$x = 4$

<u>Exercises</u>

1. Simplify the following:

a. $-3(te^3)^4$

b. $(4ab^3)^3$

c. $\frac{(-a)^2 \ x \ a^7}{(-a)^5}$

d. $(-g^4)^5$

e. $\frac{(m^2)^3}{m^4 \ x \ (-m)}$

2. Simplify the following:

a. $(3a)^{-1}$

b. $(a^2)^{-1/2}$

c. $(49x^3)^{1/2}$

d. $(27x^{3/2})^{2/3}$

3. Solve the following equations:

a. $x^{-1/2} = 5$

b. $a^{-2} = 9$

c. $9^{x-2} = 27$

d. $\frac{4^{2x-1}}{16^2} = 64$

CHAPTER 5
LOGARITHMS OF NUMBERS GREATER THAN 1 – USE OF TABLES

Logarithm is another word for power. For example, $100 = 10^2$, and $\log_{10}100 = 2$. This means that the logarithm to base 10 of 100 is 2. Another example is $\log_3 81 = \log_3 3^4 = 4$.

Logarithm to base 10 which is called common logarithm will be used here.

Examples

1. Use the logarithm tables present in mathematical tables (commonly called four-figure tables) to find the logarithm of the following:

a. 6.2

b. 29.4

c. 8

d. 438.5

Solutions

a. When 6.2 is expressed in standard form it gives 6.2×10^0. The power of 10 which is 0 in this case is the "characteristic" of the logarithm of 6.2. This characteristic is the integer written down before checking the logarithm tables to get the fractional part.

Log 6.2 = 0.7924 (Note that 6.2 can be expressed as 6.200 to make it up to four digits. So, 0 is the characteristic of 6.2, while 7924 is obtained by using the logarithm tables to look up 62 under 0, 'difference' 0. In doing this, look up the first two numbers under the third number and add the 'difference' of the fourth number. Since the last number is 0, it gives a 'difference' of 0)

b. When 29.4 is expressed in standard form, it gives 2.94×10^1. This shows that the characteristic is 1 (i.e. the power of 10)

Log 29.4 = 1.4683 (Here, look up 29 under 4).

c. Log 8 = 0.9031 (8 can be expressed as 8.000. So, look up 80 under 0).

d. Log 438.5 = 2.6420 (Look up 43 under 8, 'difference' 5. The difference is obtained by using the 'difference' section of the logarithm tables. The 'difference' is added to the value obtained. Here, 43 under 8 gives a value of 6415, while 5 in the difference section gives a value of 5. Adding these values gives 6415 + 5 = 6420. So, log 438.5 = 2.6420. Note that the integer '2' before the decimal point is the characteristic of 438.5). The section of the logarithm tables in which log 438.5 was obtained is as shown below.

											Differences								
X	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
43	6335	63454	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9

Notice that all the examples above show that the characteristic of the logarithm of a number is one less than the number of digits before the decimal point in that number. For example, in 6.2, there is 1 number before the decimal point, so its characteristic is 0 (i.e. 1-1). Also, in 8 (or 8.000), there is also 1 number before the decimal point, so its characteristic is also 0. Similarly, in 438.5, there are 3 numbers before the decimal point, so its characteristic is 2 (i.e. 3-1). This method can be used to directly obtain the characteristic of a number.

2. Use the antilogarithm tables present in mathematical tables to find the number whose logarithm is:

a. 2.142

b. 0.6165

c. 4

Solutions

a. In finding the antilogarithm of a number, first ignore the integer part (i.e. the digit before the decimal point), and look up the fractional part using the antilogarithm tables. After that, you use the integer part to place the decimal point. When placing the decimal point, the number of digits before the decimal point is one greater than the integer (i.e. the opposite of what was done for logarithm).

The antilog of 2.142 = 138.7 (This is obtained by first ignoring the integer part i.e. 2, and looking up 14 under 2 in the antilogarithm tables to get 1387. The 2 (i.e. the integer part), that was initially ignored is now used to place the decimal point. The digit to count is obtained by adding 1 to the integer 2, to obtain 3 (i.e. 2 + 1 = 3). So, 3 digits should be counted in 1387 before placing the decimal point. This will give the final value of 138.7.

b. Antilog of 0.6165 = 4.135 (Look up 61 under 6, ‘difference’ 5. The difference is obtained by using the ‘difference’ section of the antilogarithm tables. The ‘difference’ is added to the value obtained. Here, 61 under 6 gives a value of 4130, while 5 in the difference section gives a value of 5. Adding the values gives 4130 + 5 = 4135. The final step involves the use of the integer part (i.e. 0) to place the decimal point. So count 1 digit (i.e. 0 + 1 = 1) in 4135 before placing the decimal point to obtain 4.135. So, the antilog of 0.6165 = 4.135. Note that the integer ‘0’ before the decimal point was initially ignored. The section of the antilogarithm tables in which the antilog of 0.6165 was obtained is as shown below.

											Differences								
X	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	6	8	9

c. Antilog of 4 = Antilog of 4.0000 = 10000. (Look up 00 under 0 to obtain 1000. Then count 5 digits (i.e. 4 + 1 = 5) before placing the decimal point. Notice that one zero has been added to 1000 to obtain 10000 in order to complete the 5 digits needed)

Multiplication and division of numbers by using mathematical tables

Using mathematical tables to evaluate calculations is based on the laws of indices.

Examples

1. Use antilogarithm tables to evaluate the following:

a. $10^{0.6112}$

b. $10^{1.24} \times 10^{2.1021}$

c. $10^{3.194} \div 10^{0.9317}$

Solutions

a. $10^{0.6112}$ = the antilog of 0.6112 = 4.085

b. $10^{1.24} \times 10^{2.1021} = 10^{1.24 + 2.1021} = 10^{3.3421} = 2198$ (From the antilog of 3.3421)

c. $10^{3.194} \div 10^{0.9317} = 10^{3.194 - 0.9317} = 10^{2.2623} = 182.9$ (From the antilog of 2.2623)

2. Use mathematical tables to evaluate the following:

a. 715.4 x 4.31

b. 216 x 28

c. 6214 ÷ 98.76

d. $\frac{62.4 \text{ x } 5.12}{12.04}$

e. $\frac{3.169 \text{ x } 92.1}{3.96 \text{ x } 6.72}$

<u>Solutions</u>

a. 715.4 x 4.31.

To do this, simply add the logarithm of the numbers and then find the antilogarithm of the value obtained.

No.	Log
715.4	2.8545
4.31	+ 0.6345
Antilog	3.4890
3083	

Note that the antilog of 3.4890 gives the answer, 3083

∴ 715.4 x 4.31 =3083

b. 216 x 28

No.	Log
216	2.3345
28	+ 1.4475
Antilog	3.7820
6049	

Note that the antilog of 3.7820 gives the answer, 6049

∴ 216 x 28 = 6049

c. 6214 ÷ 98.76 = $\frac{6214}{98.76}$

To do this, simply subtract the logarithm of the numbers and then find the antilogarithm of the value obtained.

No.	Log
6214	3.7934
98.76	- 1.9946
Antilog	1.7988
62.92	

Note that the antilog of 1.7988 gives the answer, 62.92

∴ 6214 ÷ 98.76 = 62.92

d. $\frac{62.4 \text{ x } 5.12}{12.04}$

In this case, add the logarithms of the numerator and subtract the logarithm of the denominator from it. The antilog of the value obtained gives the final answer

No.	Log
62.4	1.7952
5.12	+ 0.7093
	2.5045
12.04	- 1.0806
Antilog	1.4239
26.54	

∴ $\frac{62.4 \text{ x } 5.12}{12.04}$ = 26.54

e. $\frac{3.169 \text{ x } 92.1}{3.96 \text{ x } 6.72}$

In order to evaluate this, add the logarithms of the numerator and also add the logarithm of the denominator. Subtract the value obtained for the denominator from that obtained for the numerator. This gives a value whose antilog gives the final answer. This is as evaluated below.

No.	Log	
3.169	0.5009	
92.1	+ 1.9643	
	2.4652	2.4652
3.96	0.5977	
6.72	+0.8274	
	1.4251	- 1.4251
Antilog		1.0401
10.97		

Note that the antilog of 1.0401 gives 10.97

$$\therefore \quad \frac{3.169 \text{ x } 92.1}{3.96 \text{ x } 6.72} = 10.97$$

Calculations of powers and roots using mathematical tables

In carrying out calculations when numbers are in powers and roots, find the logarithm of the number and multiply it by its power. Note that fractional powers are known as roots.

Examples

Use mathematical tables to evaluate the following:

1. 84.14^2
2. $\sqrt[3]{31.2}$
3. $\left(\frac{403.4}{21.6}\right)^3$
4. $\left(\frac{1067}{29.4}\right)^{1/3}$
5. $\sqrt[4]{31.87 \text{ x } 1.863}$
6. $\sqrt[5]{(6.838)^3}$

7. $\sqrt[3]{\left(\frac{38.32\ x\ 2.964}{8.637\ x\ 6.285}\right)^3}$

8. $\frac{(17.2)^2 \text{ x } 4.93}{\sqrt[3]{675000}}$

<u>Solutions</u>

1. 84.14^2

Find the logarithm of 84.14 and multiply it by 2. Then find the antilog of the value obtained.

No	Log
84.14^2	1.9250 x 2
Antilog	3.8500
<u>7079</u>	

Note that the antilog of 3.8500 gives 7079

∴ $84.14^2 = 7079$

2. $\sqrt[3]{31.2}$

Find the logarithm of 31.2 and multiply it by ⅓, and then find the antilogarithm of the value obtained.

No	Log
$\sqrt[3]{31.2}$	1.4942 x ⅓
Antilog	0.4981
<u>3.148</u>	

3. $\left(\frac{403.4}{21.6}\right)^3$

This is evaluated by subtracting the logarithm of the denominator from the logarithm of the numerator. The antilogarithm of the value obtained gives the final answer. This is as evaluated below.

No.	Log
403.4	2.6057
21.6	<u>- 1.3345</u>
	1.2712 x 3
Antilog	3.8136
<u>6510</u>	

4. $\left(\frac{1067}{29.4}\right)^{1/3}$

No	Log
1067	3.0282
21.6	<u>- 1.4683</u>
	1.5599 x ⅓
Antilog	0.5200
3.311	

5. $\sqrt[4]{31.87 \, x \, 1.863}$

No	Log
31.87	1.5034
21.6	<u>+0.2702</u>
	1.7736 x ¼
Antilog	0.8864
<u>7.705</u>	

6. $\sqrt[5]{(6.838)^3}$

This can also be expressed as: $(6.838)^{3/5}$

No.	Log
6.838	0.8349 x ⅗
Antilog	0.5009
<u>3.169</u>	

7. $\sqrt[3]{\left(\frac{38.32 \text{ x } 2.964}{8.637 \text{ x } 6.285}\right)^2}$

This can also be expressed as $\left(\frac{38.32 \text{ x } 2.964}{8.637 \text{ x } 6.285}\right)^{⅔}$

No.	Log	
38.32	1.5834	
2.964	+ 0.4719	
	2.0553	2.0553
8.637	0.9364	
6.285	+ 0.7983	
	1.7347	- 1.7347
		0.3206 x ⅔
Antilog		0.2137
1.636		

8. $\frac{(17.2)^2 \text{ x } 4.93}{\sqrt[3]{675000}}$ Note that $\sqrt[3]{675000}$ can also be expressed as $(675000)^{⅓}$

No.	Log	
17.2^2	1.2355 x 2	
	2.4710	
4.93	+ 0.6928	
	3.1638	3.1638
$\sqrt[3]{675000}$	5.8293 x ⅓	
	1.9431	- 1.9431
Antilog		1.2207
16.62		

Note: The logarithm of a number can also be obtained directly by using calculators. In using a calculator, the antilogarithm of a number can be obtained by raising 10 to the power of that number. For example, the antilog of 0.3086 is given by $10^{0.3086}$ which is equal to 2.035.

Relationship between indices and logarithms

If $y = a^x$, then $\log_a y = x$. This means that x is the logarithm of y to the base a.

Similarly, if $\log_a y = x$, then $y = a^x$

Examples

1. Write the following in index form and find the values of x:

a. $\text{Log}_2 8 = x$

b. $\text{Log}_5 125 = x$

c. $\text{Log}_{10} 0.001 = x$

Solutions

a. $\text{Log}_2 8 = x$. Expressing this in index form will give:

$2^x = 8$

$2^x = 2^3$ (Since the base 2 are the same on both sides, they cancel out)

Equating the powers gives:

$x = 3$

b. $\text{Log}_5 125 = x$

$5^x = 125$

$5^x = 5^3$

The base will cancel out since they are equal. Equating the powers now gives:

$x = 3$

c. $\text{Log}_{10} 0.001 = x$

$10^x = 0.001$

$10^x = 10^{-3}$ (Note that 0.001 in standard form is 10^{-3})

$x = -3$

2. Solve the following equations:

a. $Log_a 3 = \frac{1}{4}$

b. $Log_y 0.25 = -\frac{1}{2}$

Solutions

a. $Log_a 3 = \frac{1}{4}$ Expressing this in index form gives:

$a^{¼} = 3$ (Since the base is the unknown, make its power to be 1 by multiplying this power by its inverse. Also raise the other side of the equation to the same power).

$(a^{¼})^4 = 3^4$ (Note that the inverse of $\frac{1}{4}$ is 4)

$a = 81$ (Note that $(a^{¼})^4 = a^{(¼ \times 4)} = a^1 = a$)

b. $Log_y 0.25 = -\frac{1}{2}$

$y^{-½} = 0.25$ Express 0.25 in fraction

$y^{-½} = \frac{1}{4}$ Making the power of y to be 1 gives:

$y^{(-½ \times -2)} = (\frac{1}{4})^{-2}$

$y = (\frac{4}{1})^2$

$y = 4^2$

$y = 16$

Exercises

1. Use mathematical tables to evaluate the following:

a. $10^{1.24} \times 10^{2.12}$

b. $\frac{10^{0.25} \text{ x } 10^{1.214}}{10^{0.715}}$

c. 615 x 30.04

d. 3.254 x 38.31 x 401.5

e. $\frac{81.6 \text{ x } 3.142}{12.2 \text{ x } 16}$

f. $\frac{713.4}{35 \text{ x } 4.95}$

g. $\frac{(314.5)^2}{84}$

h. $\sqrt{28 \text{ x } 5}$

i. $(10000)^{1/5}$

j. $\frac{\sqrt{960.5}}{14.02}$

k. $\frac{\sqrt[3]{6314000 \div 14.2}}{(2 \text{ x } 3.007)^2}$

l. $\sqrt[5]{28.5 \text{ x } 12 \text{ x } 3.14 \text{ x } 92}$

2. Solve the following equations:

a. $\text{Log}_4 64 = x$

b. $\text{Log}_a 1.5 = \frac{1}{3}$

c. $\text{Log}_{0.25} a = 4$

d. $\text{Log}_y \frac{1}{9} = - \frac{2}{3}$

CHAPTER 6
THEORY OF LOGARITHMS

Logarithm can be in bases other than base 10. For example $27=3^3$ means $\log_3 27=3$. For solved examples on the relationship between indices and logarithms, refer to chapter 5.

Laws of logarithms

The three basic laws of logarithms are:

1. $\text{Log}(XY) = \log X + \log Y$

2. $\text{Log}\left(\frac{X}{Y}\right) = \log X - \log Y$

3. $\text{Log}(X^Y) = Y\log X$

Examples

1. Simplify the following as far as possible:

a. $\text{Log}_{10}15 + \log_{10}6$

b. $\frac{3}{4}\log 81$

Solutions

a. $\text{Log}_{10}15 + \log_{10}6 = \log_{10}(15 \times 6) = \log_{10}90$

b. $\frac{3}{4}\log_{10}81 = \log_{10}81^{3/4} = \log_{10}(\sqrt[4]{81})^3 = \log_{10}(3)^3 = \log_{10}27$

2. Express the following as logarithms of single numbers:

a. $\text{Log}_{10}60 - \log_{10}3$

b. $2 - 2\log_{10}5$

Solutions

a. $\text{Log}_{10}60 - \log_{10}3 = \log_{10}\left(\frac{60}{3}\right) = \log_{10}20$

b. $2 - 2\log_{10}5 = \log_{10}100 - \log_{10}5^2 = \log_{10}100 - \log_{10}25 = \log_{10}(\frac{100}{25}) = \log_{10}4$

Note that 2 has been converted to $\log_{10}100$ as follows:

let $\log_{10}x = 2$

$x = 10^2 = 100$

$2 = \log_{10}100$ (This was substituted for 2 in example 2)

3. Given that $\log_{10}2 = 0.3010$, $\log_{10}3 = 0.4771$ and $\log_{10}7 = 0.8451$, evaluate the following:

a. $\log_{10}42$

b. $\log_{10}35$

Solutions

a. $\log_{10}42 = \log_{10}(2 \times 3 \times 7) = \log_{10}2 + \log_{10}3 + \log_{10}7 = 0.3010 + 0.4771 + 0.8451 = 1.6232$

b. $\log_{10}35 = \log_{10}(\frac{7 \times 10}{2}) = \log_{10}7 + \log_{10}10 - \log_{10}2 = 0.8451 + 1 + 0.3010 = 1.5441$ (Note that $\log_{10}10 = 1$, in a similar way that $\log_{10}100 = 2$)

4. Simplify $\frac{1}{2}\log_{10}\frac{25}{4} - 2\log_{10}\frac{4}{5} + \log_{10}\frac{320}{125}$

Solution

$$\frac{1}{2}\log_{10}\frac{25}{4} - 2\log_{10}\frac{4}{5} + \log_{10}\frac{320}{125} = \log_{10}(\frac{25}{4})^{½} - \log_{10}(\frac{4}{5})^2 + \log_{10}\frac{320}{125} = \log_{10}(\frac{\sqrt{\frac{25}{4}} \times \frac{320}{125}}{\frac{4^2}{5^2}}) = \log_{10}(\frac{\frac{5}{2} \times \frac{320}{125}}{\frac{16}{25}})$$

$\log_{10}(\frac{5}{2} \times \frac{320}{125} \times \frac{25}{16}) = \log_{10}(\frac{320}{125} \times \frac{125}{32})$ (Since 5 x 25=125, and 2 x 16=32)

Cancelling out 125 gives:

$\log_{10}(\frac{320}{32})$

$= \log_{10}10 = 1$

5. Simplify $1 + \log_{10}2$

Solution

Not that $1 = \log_{10}10$, since $10^1 = 10$ (This is similar to what was done in example 2)

$1 + \log_{10}2 = \log_{10}10 + \log_{10}2$ (When $\log_{10}10$ is substituted for 1)

$\log_{10}10 + \log_{10}2 = \log_{10}(10 \times 2) = \log_{10}20$

6. If $\log_{10}(2x + 1) – \log_{10}(3x - 2) = 1$, find x

Solution

$\log_{10}(2x + 1) – \log_{10}(3x - 2) = 1$

$\log_{10}(2x + 1) – \log_{10}(3x - 2) = \log_{10}10$ (Since converting 1 to logarithm in base 10 is given by $10^1 = 10$. Therefore, $1 = \log_{10}10$)

$\log_{10}(\frac{2x+1}{3x-2}) = \log_{10}10$

Comparing both sides of the equation above shows that:

$(\frac{2x+1}{3x-2}) = 10$

$10(3x - 2) = 2x + 1$ (By cross multiplication)

$30x - 20 = 2x + 1$

$30x - 2x = 1 + 20$

$28x = 21$

Dividing both sides by 28 gives:

$x = \frac{21}{28}$

$x = \frac{3}{4}$ (When simplified to its lowest term)

7. Without the use of tables, simplify the following:

a. $\frac{\log_3 16}{\log_3 8}$

b. $\frac{\log_{10} 8 + \log_{10} 4}{\log_{10} 8 - \log_{10} 4}$

c. $\frac{\log_8 \sqrt{27}}{\log_8 9}$

<u>Solutions</u>

a. $\frac{\log_3 16}{\log_3 8} = \frac{\log_3 2^4}{\log_3 2^3} = \frac{4 \log_3 2}{3 \log_3 2}$

$= \frac{4 \log_3 2}{3 \log_3 2}$

Cancelling out $\log_3 2$ gives:

$= \frac{3}{4}$

b. $\frac{\log_{10} 8 + \log_{10} 4}{\log_{10} 8 - \log_{10} 4} = \frac{\log_{10}(8 \text{ x } 4)}{\log_{10}(\frac{8}{4})} = \frac{\log_{10} 32}{\log_{10} 2}$

$= \frac{\log_{10} 2^5}{\log_{10} 2} = \frac{5 \log_{10} 2}{\log_{10} 2} = \frac{5 \log_{10} 2}{\log_{10} 2}$

$= 5$ (After cancelling out $\log_{10}2$)

c. $\frac{\log_8 \sqrt{27}}{\log_8 9} = \frac{\log_8 27^{1/2}}{\log_8 3^2} = \frac{\log_8 (3^3)^{1/2}}{\log_8 3^2} = \frac{\log_8 (3^{3 \text{ x } 1/2})}{\log_8 3^2}$

$= \frac{\log_8 3^{3/2}}{\log_8 3^2} = \frac{\frac{3}{2}\log_8 3}{2\log_8 3} = \frac{\frac{3}{2}}{2} = \frac{3}{2} \div 2 = \frac{3}{2} \text{ x } \frac{1}{2} = \frac{3}{4}$

8. Evaluate the following:

a. $\text{Log}_4 15$

b. $\text{Log}_7 21$

Solutions

a. $\text{Log}_4 15$

Let $\log_4 15 = x$

$4^x = 15$

Taking the logarithm to base 10 of both sides gives:

$\text{Log}_{10} 4^x = \log_{10} 15$

$x \log_{10} 4 = \log_{10} 15$

$x = \frac{\log_{10} 15}{\log_{10} 4} = \frac{1.1761}{0.6021}$ (From mathematical tables or the use of calculator)

$= 1.9533$

$\therefore$ $\text{Log}_4 15 = 1.95$

b. $\text{Log}_7 21$

Let $\log_7 21 = x$

$7^x = 21$

Taking the logarithm to base 10 of both sides gives:

$\text{Log}_{10} 7^x = \log_{10} 21$

$x \log_{10} 7 = \log_{10} 21$

$x = \frac{\log_{10} 21}{\log_{10} 7} = \frac{1.3222}{0.8451}$ (From mathematical tables or the use of calculator)

$= 1.5645$

$\therefore$ $\text{Log}_7 15 = 1.56$

Exercises

1. Simplify the following:

a. $2 – lo_{10}4$

b. $5\ lo_{10}2 + lo_{10}5 - lo_{10}1.6$

2. Simplify $lo_{10}\frac{75}{10} - 2log_{10}\frac{5}{9} + log_{10}\frac{160}{243}$

3. Given that log2 = 0.3010, log3 = 0.4771 and log5 = 0.6990, evaluate:

a. log45

b. log1.2

b. log3.6

3. Solve for x in the following equations:

a. $\log_{10}x – \log_{10}(2x – 1) = 2$

b. $2\log_x (3\frac{3}{8}) = 6$

4. Simplify the following without using tables:

a. $\frac{\log \sqrt{3}}{\log 3}$

b. $\frac{\log 256}{\log 16}$

c. $\frac{\log 40 - \log 5}{\log 16 - \log 8}$

5. Use logarithm tables to evaluate:

a. $\log_5 23.69$

b. $\log_2 40$

6. Simplify the following:

a. $\log_7 98 – \log_7 30 + \log_7 15$

b. $lo_3 24 + \log_3 15 – 2\log_3 10$

7. Given that $\log_5 2 = 0.431$ and $\log_5 3 = 0.681$, find the values of:

a. $\log_5 13.5$

b. $\log_5\frac{3}{8} + 2\log_5\frac{4}{5} - \log_5\frac{2}{5}$

CHAPTER 7
LINEAR EQUATIONS AND CHANGE OF SUBJECT OF FORMULAE

Linear equations

Linear equations can sometimes be expressed with brackets, with fractions or both. When solving linear equations with fractions, clear the fractions by multiplying each term of the equation by the L.C.M of the denominators of the equation.

Examples

1. Solve the following equations:

a. $2x + 4(3 - x) = 11$

b. $6(a - 3) - 2(5a - 8) = -4$

c. $a - 5(2 + a) - (3a - 4) = 2(2a - 1) - 7$

Solutions

a. $2x + 4(3 - x) = 11$ Expanding the bracket gives:

$2x + 12 - 4x = 11$

Collect like terms

$12 - 11 = 4x - 2x$

$1 = 2x$

Divide both sides by 2

$\therefore \quad x = \frac{1}{2}$

b. $6(a - 3) - 2(5a - 8) = -4$

Expanding the brackets gives:

$6a - 18 - 10a + 16 = -4$ (Note that $-2 \times -8 = +16$)

Collect like terms:

6a – 10a = -4 + 18 – 16

-4a = -2

Divide both sides by -4

$\frac{-4a}{-4} = \frac{-2}{-4}$

a = $\frac{-2}{-4}$

a = $\frac{1}{2}$

c. a – 5(2 + a) – (3a – 4) = 2(2a – 1) – 7

Expanding the brackets gives:

a – 10 – 5a – 3a + 4 = 4a – 2 – 7 (Note that -5 x (+a) = -5a, and - x -4 = -1 x -4 = +4)

Collect like terms:

7 + 2 + 4 – 10 = 5a – a + 3a + 4a

3 = 11a

Divide both sides by 11

$\frac{3}{11} = \frac{11a}{11}$

a = $\frac{3}{11}$

2. Solve the following equations:

a. $\frac{3}{4}x - \frac{1}{3}(x - 2) = \frac{5}{6} - (2x - 1)$

b. $\frac{1}{6}(5x - 2) - \frac{2}{3}(4 - x) = 1$

c. $\frac{5}{2m-3} - \frac{3}{4} = \frac{1}{6} + 7$

Solutions

a. $\frac{3}{4}x - \frac{1}{3}(x - 2) = \frac{5}{6} - (2x - 1)$

In order to clear the fractions, multiply each term in the equation by 12 which is the L.C.M of the denominators, i.e. 4, 3 and 6. This gives:

$12(\frac{3x}{4}) - 12 \text{ x } \frac{1}{3}(x - 2) = 12(\frac{5}{6}) - 12(2x - 1)$

Cancelling out by using the 12 to divide the various denominators gives:

$3(3x) - 4(x - 2) = 2(5) - 12(2x - 1)$

$9x - 4x + 8 = 10 - 24x + 12$

Collect like terms:

$9x - 4x + 24x = 10 + 12 – 8$

$29x = 14$

$x = \frac{14}{29}$

b. $\frac{1}{6}(5x - 2) - \frac{2}{3}(4 - x) = 1$

In order to clear the fractions, multiply each term in the equation by 6 which is the L.C.M of the denominators, i.e. 6 and 3. This gives:

$6 \text{ x } \frac{1}{6}(5x - 2) - 6 \text{ x } \frac{2}{3}(4 - x) = 6 \text{ x } 1$

Cancelling out by using the 6 to divide the various denominators gives:

$(5x - 2) - 2 \text{ x } 2(4 - x) = 6$

$5x - 2 - 4(4 - x) = 6$

$5x - 2 - 14 + 4x = 6$

$5x + 4x = 6 + 2 + 14$

$9x = 22$

$\therefore \quad x = \frac{22}{9}$

c. $\frac{5}{2m-3} - \frac{3}{4} = \frac{1}{6} + 7$

Multiply each term in the equation by 12(2m - 3) which is the L.C.M of the denominators, i.e. (2m - 3), 4 and 6. This gives:

$12(2m - 3)\frac{5}{2m-3} - 12(2m - 3)\frac{3}{4} = 12(2m - 3)\frac{1}{6} + 12(2m - 3) \times 7$

Cancelling out gives:

12(5) - 3(2m - 3)3 = 2(2m - 3) + 12(2m - 3)7

60 - 9(2m - 3) = 4m - 6 + 84(2m - 3)

60 - 18m + 27 = 4m - 6 + 168m - 252

60 + 27 + 252 + 6 = 4m + 168m +18m

345 = 190m

$m = \frac{345}{190}$

$\therefore \quad m = \frac{69}{38}$ (After equal division by 5)

<u>Change of subject of formulae</u>

If m = b + c, then m is the subject of the formula. If its rearranged to give b = m - c, then b is now the new subject of the formula.

In changing the subject of a formula, simply solve the equation for the letter which is to become the new subject.

<u>Examples</u>

1. Make h the subject of the formula: $s = \frac{wd}{h}(h - \frac{d}{2})$

Solution

$s = \frac{wd}{h}(h - \frac{d}{2})$

Expanding the bracket gives:

$s = \frac{wd}{h}(h) - \frac{wd}{h}(\frac{d}{2})$

Canceling out the h gives:

$s = wd - \frac{wd^2}{2h}$

To clear fractions, multiply throughout by 2h (LCM)

$2h(s) = 2h(wd) - 2h(\frac{wd^2}{2h})$

$2hs = 2hwd – wd^2$ (Note that the 2h at the end on the right side has cancelled out).

Collect terms in h

$wd^2 = 2hwd – 2hs$

Factorizing the right hand side gives:

$wd^2 = h(2wd - 2s)$

Divide both sides by (2wd - 2s)

$\frac{wd^2}{2wd - 2s} = \frac{h(2wd - 2s)}{2wd - 2s}$

Cancelling out the 2wd – 2s on the right hand side gives:

$h = \frac{wd^2}{2wd - 2s}$

2. Given that $I = \frac{E}{\sqrt{R^2 + W^2L^2}}$, express R in terms of I, E, W and L.

Solution

$I = \frac{E}{\sqrt{R^2 + W^2L^2}}$

Cross multiply

$I\sqrt{R^2 + W^2L^2} = E$

Square both sides to remove the square root sign.

$(I\sqrt{R^2 + W^2L^2})^2 = E^2$

$I^2(R^2 + W^2L^2) = E^2$

$I^2R^2 + I^2W^2L^2 = E^2$

$I^2R^2 = E^2 - I^2W^2L^2$

Divide both sides by I^2

$R^2 = \frac{E^2 - W^2I^2L^2}{I^2}$

This can also be simplified as follows:

$R^2 = \frac{E^2}{I^2} - \frac{I^2W^2L^2}{I^2}$

Canceling out I^2 on the right side gives:

$R^2 = \frac{E^2}{I^2} - W^2L^2$

Take the square root of both sides in order to remove the square sign on R^2.

$\therefore \quad R = \sqrt{\frac{E^2}{I^2} - W^2L^2}$

3. Make x the subject of the formula $R = \sqrt{\frac{ax - P}{Q + bx}}$

<u>Solution</u>

$R = \sqrt{\frac{ax - P}{Q + bx}}$

Square both sides to remove the square root sign

$R^2 = \dfrac{ax - P}{Q + bx}$

By cross multiplication it gives:

$R^2(Q + bx) = ax - P$

$R^2Q + R^2bx = ax - P$

Collecting terms in x gives

$ax - R^2bx = R^2Q + P$

Factorizing the left hand side gives:

$x(a - R^2b) = R^2Q + P$

Divide both sides by $(a - R^2b)$

$\dfrac{x(a - R^2b)}{(a - R^2b)} = \dfrac{R^2Q + P}{a - R^2b}$

$x = \dfrac{R^2Q + P}{a - R^2b}$

<u>Exercises</u>

1. Solve the following equations:

a. $5x + 2(3 - x) = 10$

b. $2(2a - 3) - 5(4a - 1) = -6$

c. $b - 4(1 + b) - (5b - 1) = -(b - 3) - 2$

2. Solve the following equations:

a. $\frac{1}{4}x - \frac{2}{3}(x - 1) = \frac{3}{4} - (5x - 2)$

b. $\frac{1}{6}(5x - 2) - \frac{5}{12}(3 - 2x) = 1$

c. $\dfrac{2}{2n-3} - \dfrac{3}{5} = 2\frac{1}{2}$

3. Make p the subject of the formula: tp = md(p - $\frac{d}{3}$)

4. Given that V = $\frac{P}{\sqrt{E^2 + I^2C^2}}$, express C in terms of I, V, E and P.

5. Make x the subject of the formula R = $\sqrt{\frac{bx - S}{T + ax}}$

CHAPTER 8
VARIATION

Direct variation

Direct variation involves the relationship between two quantities whereby an increase or decrease in one of them leads to an increase or decrease respectively in the other.

The symbol α means 'varies with' or is 'proportional to'.

Example

1. If x varies directly as y and x=30 when y=12, find:

a. the formula connecting x and y

b. x when y=10

c. y when x=20

Solution

a. $x \alpha y$ (This means x varies directly as y)

x = Ky (Replacing the proportionality sign with the equals sign introduces a constant K)

So, when x = 30 and y = 12, the equation above becomes:

30 = K x 12

30 = 12K

$K = \frac{30}{12} = \frac{5}{2}$

The formula connecting x and y is:

$x = \frac{5}{2}y$ (This is obtained by substituting $\frac{5}{2}$ for K in the equation above, i.e. x = Ky)

b. When y = 10, x is given by:

$x = \frac{5}{2}$ y

$x = \frac{5}{2} \times 10 = \frac{50}{2} = 25$

$\therefore\ x = 25$

c. When $x = 20$, y is given by:

$x = \frac{5}{2}y$

$20 = \frac{5}{2}y$

$5y = 40$

$y = \frac{40}{5}$

$y = 8$

2. If m varies as the square of n and m=27 when n=3, find:

a. the relationship between m and n

b. n when m = 48

c. m when n = $2\frac{1}{3}$

<u>Solution</u>

a. $m \propto n^2$ (This means m varies as n^2)

$m = Kn^2$ (Replacing the proportionality sign with the equals sign introduces a constant K)

So, when m = 27 and n = 3, the equation above becomes:

$27 = K \times 3^2$

$27 = 9K$

$K = \frac{27}{9} = 3$

The relationship between m and n is:

$m = 3n^2$ (This is obtained by substituting 3 for K in the equation $m = Kn^2$)

b. When m = 48, n is given by:

$m = 3n^2$

$48 = 3 \times n^2$

$48 = 3n^2$

$n^2 = \frac{48}{3} = 16$

$n = \sqrt{16}$

$n = 4$

c. When $n = 2\frac{1}{3}$, m is given by:

$m = 3n^2$

$m = 3 \times (\frac{7}{3})^2$ (Note that $2\frac{1}{3}$ has been converted to $\frac{7}{3}$)

$m = 3 \times \frac{49}{9}$

Canceling out gives:

$m = \frac{49}{3}$

<u>Inverse variation</u>

In inverse or indirect variation, as one quantity increases the other decreases.

<u>Examples</u>

1. If c varies inversely as d and c=18 when y=4, find:

a. the formula connecting c and d

b. c when d=10

c. d when c=12

Solution

a. $c \; \alpha \; \frac{1}{d}$ (This means c varies inversely as d)

$c = \frac{K}{d}$ (Replacing the proportionality sign with the equals sign introduces a constant K)

So, when c = 18 and d = 4, the equation above becomes:

$18 = \frac{K}{4}$

K = 18 x 4

K = 72

The formula connecting c and d is:

$c = \frac{72}{d}$ (This is obtained by substituting 72 for K in the equation above, i.e. $c = \frac{K}{d}$)

b. When d = 10, c is given by:

$c = \frac{72}{d}$

$c = \frac{72}{10}$

c = 7.2

c. When c = 12, d is given by:

$c = \frac{72}{d}$

$12 = \frac{72}{d}$

$12d = 72$

$d = \frac{72}{12}$

$d = 6$

2. If r varies inversely as the cube root of t and r=6 when t=64, find:

a. the relationship between r and t

b. t when r = 16

c. r when $t = \frac{8}{27}$

<u>Solution</u>

a. $r \,\alpha\, \frac{1}{\sqrt[3]{t}}$ (This means r varies inversely as the cube root of t)

$r = \frac{K}{\sqrt[3]{t}}$ (Replacing the proportionality sign with the equals sign introduces a constant K)

So, when r = 6 and t = 64, the equation above becomes:

$6 = \frac{K}{\sqrt[3]{64}}$

$6 = \frac{K}{4}$

$K = 6 \times 4 = 24$

The relationship between r and t is:

$r = \frac{24}{\sqrt[3]{t}}$ (This is obtained by substituting 24 for K in the equation $r = \frac{K}{\sqrt[3]{t}}$)

b. When r = 16, t is given by:

$r = \frac{24}{\sqrt[3]{t}}$

$16 = \frac{24}{\sqrt[3]{t}}$

$\sqrt[3]{t} = \frac{24}{16}$

$\sqrt[3]{t} = \frac{3}{2}$

In order to remove the cube root, take the cube of both sides. This gives:

$(\sqrt[3]{t})^3 = (\frac{3}{2})^3$

$\therefore\ t = \frac{27}{8}$

c. When $t = \frac{8}{27}$, r is given by:

$r = \frac{24}{\sqrt[3]{t}}$

$r = \frac{24}{\sqrt[3]{8/27}}$

$r = \frac{24}{2/3}$

$r = 24 \times \frac{3}{2}$

After equal division by 2, it gives:

$r = 12 \times 3$

$r = 36$

Joint variation

In joint variation, three or more quantities are related directly or inversely or both.

Examples

1. If m varies directly as the square of n and inversely as p, and m=3 when n=2 and p=8, find:

a. the relationship between m, n and p

b. m when n = 3 and p =27

c. p when m = $\frac{1}{2}$ and n =$\frac{3}{2}$

<u>Solutions</u>

m $\alpha \frac{n^2}{p}$ (This means m varies directly as the square of n and inversely as p)

m = $\frac{Kn^2}{p}$

So, when m = 3, n = 2, and p = 8, the equation above becomes:

3 = $\frac{K\ x\ 2^2}{8}$

3 = $\frac{4K}{8}$

4K = 3 x 8 = 24

K = $\frac{24}{4}$ = 6

The relationship between m, n and p is:

m = $\frac{6n^2}{p}$ (This is obtained by substituting 6 for K in the equation m = $\frac{Kn^2}{p}$)

b. When n = 3 and p = 27, then m is given by:

m = $\frac{6n^2}{p}$

m = $\frac{6 \text{ x } 3^2}{27}$

m = $\frac{6 \text{ x } 9}{27}$

m = $\frac{54}{27}$

m = 2

c. When $m = \frac{1}{2}$ and $n = \frac{3}{2}$, then p is given by:

$m = \frac{6n^2}{p}$

$\frac{1}{2} = \frac{6 \text{ x } \frac{3}{2}}{p}$

$\frac{1}{2} = \frac{9}{p}$

p = 9 x 2

P = 18

2. The weight w of a rod varies jointly as its length L and the square root of its density d. If w =12 when L = 5 and d = 9, find:

a. L in terms of w and d

b. w when L = 8 and d = 25

c. d when L = 20 and w = 4

<u>Solutions</u>

a. $w \propto L\sqrt{d}$ (This means w varies jointly as L and the square root of d)

$w = KL\sqrt{d}$

So, when w = 12, L = 5, and d = 9, the equation above becomes:

$12 = K \text{ x } 5 \text{ x } \sqrt{9}$

12 = 15K

$K = \frac{12}{15} = \frac{4}{5}$

The formula connecting w, L and d is:

$w = \frac{4}{5} L\sqrt{d}$ (This is obtained by substituting $\frac{4}{5}$ for K in the equation $w = KL\sqrt{d}$)

L can now be expressed in terms of w and d as follows:

$w = \frac{4}{5} L\sqrt{d}$

$5w = 4L\sqrt{d}$

Dividing both sides of the equation by $4\sqrt{d}$ gives:

$L = \frac{5w}{4\sqrt{d}}$

b. When L = 8 and d = 25, then w is given by:

$w = \frac{4}{5} L\sqrt{d}$

$w = \frac{4}{5} \times 8 \times \sqrt{25}$

$w = \frac{4}{5} \times 8 \times 5$

Cancelling out the 5 gives:

$w = 4 \times 8$

$w = 32$

c. When L = 20 and w = 4, then d is given by:

$w = \frac{4}{5} L\sqrt{d}$

$4 = \frac{4}{5} \times 20 \times \sqrt{d}$

$4 \times 5 = 4 \times 20 \times \sqrt{d}$

$20 = 80\sqrt{d}$

$\sqrt{d} = \frac{20}{80} = \frac{1}{4}$

Taking the square of both sides gives:

$(\sqrt{d})^2 = (\frac{1}{4})^2$

$d = \frac{1}{16}$

Partial variation

The fourth type of variation is called partial variation. In partial variation, one quantity is partly constant and partly varies with the other. Two constants are involved in partial variation.

Examples

1. x is partly constant and partly varies as y. When y=2, x=30, and when y=6, x=50.

a. Find the formula which connects x and y.

b. Find x when y=3

Solutions

a. From the first sentence, we have:

x = C + Ky (Let this be equation 1) where C and K are constants.

Substituting y=2 and x=30 in this equation gives:

30 = C + 2K (Let this be equation 2)

Similarly, when y=6 and x=50, we have:

50 = C + 6K (Let this be equation 3)

Bringing equation 2 and 3 together gives:

30 = C + 2K (Equation 2)

$\underline{50 = C + 6K}$ (Equation 3)

Equation 3 - Equation 2: 20 = 4K

Divide both sides by 4.

$K = \frac{20}{4} = 5$

Substitute 5 for K in equation 2.

$30 = C + 2K$

$30 = C + (2 \times 5)$

$30 = C + 10$

$30 - 10 = C$

$C = 20$

We now substitute the values of C and K into equation 1 in order to obtain the formula connecting x and y.

The formula connecting x and y is now given by:

$x = 20 + 5y$

b. When y=3, x is obtained by substituting 3 for y in the formula connecting x and y.

$x = 20 + 5y$

$x = 20 + (5 \times 3)$

$= 20 + 15 = 35$

$x = 35$

2. m is partly constant and partly varies as n. When n=4, m=5, and when n=12, m=14.

a. Find the formula which connects m and n.

b. Find m when n=16

c. Find n when m=9

Solutions

a. From the first sentence, we have:

$m = C + Kn$ (Let this be equation 1) where C and K are constants.

Substituting n=4 and m=5 in equation 1 gives:

$5 = C + 4K$ (Let this be equation 2)

Similarly, when n=12 and m=14, we have:

$14 = C + 12K$ (Let this be equation 3)

Bringing equation 2 and 3 together gives:

$$5 = C + 4K \qquad \text{(Equation 2)}$$

$$\underline{14 = C + 12K} \qquad \text{(Equation 3)}$$

Equation 3 - Equation 2: $9 = 8K$

Divide both sides by 8.

$$K = \frac{9}{8}$$

Substitute $\frac{9}{8}$ for K in equation 2.

$$5 = C + 4K$$

$$5 = C + (4 \times \frac{9}{8})$$

$$5 = C + \frac{9}{2}$$

$$5 - \frac{9}{2} = C$$

$$C = \frac{1}{2}$$

We now substitute the values of C and K into equation 1 in order to obtain the formula connecting m and n.

The formula connecting m and n is given by:

$$m = \frac{1}{2} + \frac{9}{8}n$$

b. When n=16, m is obtained by substituting 16 for n in the formula connecting m and n.

$$m = \frac{1}{2} + \frac{9}{8}n$$

$$m = \frac{1}{2} + (\frac{9}{8} \times 16)$$

$= \frac{1}{2} + 18 = \frac{37}{2}$

$m = 18\frac{1}{2}$

c. When m=9, n is obtained by substituting 9 for m in the formula connecting m and n.

$m = \frac{1}{2} + \frac{9}{8}n$

$9 = \frac{1}{2} + (\frac{9n}{8})$

$9 - \frac{1}{2} = \frac{9n}{8}$

$\frac{17}{2} = \frac{9n}{8}$

17 x 8 = 9n x 2

136 = 18n

$n = \frac{136}{18} = \frac{68}{9}$

$n = 7\frac{5}{9}$

Exercises

1. If x varies directly as y and x=10 when y=8, find:

a. the formula connecting x and y

b. x when y=10

c. y when x=16

2. If h varies as the square root of p and h=5 when p=9, find:

a. the relationship between h and p

b. p when h = 20

c. h when p = $6\frac{1}{4}$

3. If p varies inversely as q and p=12 when q=3, find:

a. the formula connecting p and q

b. q when p=20

c. p when q=5

4. If m varies inversely as the cube root of n and m=5 when n=27, find:

a. the relationship between m and n

b. m when n = 8

c. n when m = $\frac{64}{125}$

5. If a varies directly as the square of b and inversely as c, and when a=4 when b=3 and c=6, find:

a. the formula connecting a, b and c

b. a when b = 5 and c = 10

c. b when a = $\frac{1}{2}$ and c = 8

6. The height h of a box varies jointly as its length L and the square of its width w. If h = 20 when L = 4 and w = 3, find:

a. w in terms of h and L

b. w when h = 12 and w = 4

c. L when h = 8 and w = 5

7. x is partly constant and partly varies as y. When y=4, x=14, and when y=5, x=17.

a. Find the relationship between x and y.

b. Find x when y=8

8. E is partly constant and partly varies as F. When F=2, E=25, and when F=5, E=55.

a. Find the formula which connects E and F.

b. Find E when F=$2\frac{1}{2}$

c. Find F when E=40

CHAPTER 9
COLLECTION AND TABULATION OF DATA

When a large volume of data is obtained, it is necessary to present such data in frequency table. Sometimes the tally system which involves the use of vertical and horizontal strokes is applied.

Examples

1. A die is rolled 50 times and the following data is obtained. Represent the data in a frequency table.

4	6	4	3	5	3	1	4	6	5	6	4	2
6	4	5	6	2	1	6	4	3	4	6	1	5
1	3	6	2	2	4	3	4	5	3	4	1	2
3	1	2	1	5	3	4	3	4	2	5		

Solutions

The data which ranges from 1 to 6 is summarized as shown on the table below.

Number on die	Frequency
1	7
2	7
3	9
4	12
5	7
6	8

The data can also be represented on a horizontal table as shown below.

Number in die	1	2	3	4	5	6
Frequency	7	7	9	12	7	8

2. The scores of 40 students in a physics test are presented below. Prepare a frequency distribution table for the data.

64	66	68	63	70	63	67	64	70	69
66	64	65	70	62	70	66	69	67	64
61	63	67	62	68	64	63	69	70	63
63	61	68	67	68	63	61	67	69	68

Solution

The data which ranges from 61 to 70 is summarized as shown on the table below.

Score	61	62	63	64	65	66	67	68	69	70
Frequency	3	2	7	5	1	3	5	5	4	5

Exercise

1. The marks obtained in an examination by 40 students in a class are as shown below. Represent the data in a frequency table.

71	74	74	70	70	72	74	74	65	69
66	68	65	73	66	72	69	69	67	65
71	73	67	68	68	69	70	69	71	65
67	67	68	72	74	73	67	67	69	68

2. The score of 20 students in a test are as shown below. Represent the score in a frequency table.

6	6	8	9	5	6	7	5	6	9
8	6	7	5	9	5	9	6	6	5

3. The number of seeds in a sample of 40 cocoa pods are as shown below. Represent the information using a frequency table.

28	22	28	28	27	29	20	20	20	24
21	25	25	20	22	20	26	29	30	24
21	23	27	22	28	30	23	29	20	23
23	21	28	27	28	23	21	27	30	28

4. The ages of 30 students in a senior high school is represented below. Show the data using a frequency table.

14	15	12	13	10	13	11	14	10	12
15	11	15	10	12	10	11	13	14	14
15	13	12	11	14	14	13	12	10	15

CHAPTER 10
MEAN, MEDIAN AND MODE OF UNGROUPED DATA

The mean, median and mode are averages of sets of statistical data. They are called “measures of central tendency”.

MEAN

The mean of a set of data is obtained by adding all the data and then dividing the result by the total number in the data set.
For an ungrouped data given in a frequency table, the mean can be calculated by using the formula:

$$\bar{x} = \frac{\sum fx}{\sum f}$$

Where $\bar{x}$ is the mean, $\sum$ is a symbol representing summation, x is each number in the data, and f is the frequency.

MEDIAN

The median of a data is the middle number when the data is arranged in an increasing or decreasing order of size. For an odd number of data, the position of the middle number is obtained by the expression:

$$\text{Median} = \text{number in the } \left(\frac{N+1}{2}\right)\text{th position}$$

where N is the total number of data.
If there is an even number of data, the median is the average of the two middle numbers. In such a case the positions of the two middle numbers is obtained by the expression:

$$\text{Median} = \frac{Number\ \ in\ the\ \left(\frac{N}{2}\right)th\ position\ \ + Number\ \ in\ the\ \left(\frac{N+2}{2}\right)th\ position}{2}$$

However, for large data in a frequency table which has an even number of data, the median is given by:

$$\text{Median} = \frac{Number\ \ in\ the\ \left(\frac{\sum f}{2}\right)th\ position\ \ + Number\ \ in\ the\ \left(\frac{\sum f+2}{2}\right)th\ position}{2}$$

Where $\sum f$ is the total frequency of the data.
Note that for an odd number of data, only one number will be at the middle. However, for an even number of data, two numbers will be at the middle. The average of the two numbers gives the median of the data.

MODE

The mode is the most occurring number in a set of data. It is the number with the highest frequency. If a set of data has two modes, we say it is bimodal.

Examples

1. Find the mean, median and mode of the data below: 2, 5, 0, 3, 1, 6, 9, 7, 3

Solution

There are 9 numbers in the data. So, the mean is obtained as follows:

$$\text{Mean} = \frac{2+5+0+3+1+6+9+7+3}{9}$$

$$= \frac{36}{9}$$

$$= 4$$

∴ The mean is 4

In order to calculate the median, first arrange the numbers in ascending order as follows:

0, 1, 2, 3, 3, 5, 6, 7, 9

By inspection, the number that is at the middle of the data is 3

∴ The median is 3

Or, since there are 9 numbers in the data and 9 is an odd number, then the position of the middle number is obtained as follows:

Median = number in the $(\frac{N+1}{2})$th position

= number in the $(\frac{9+1}{2})$th position

= number in the $(\frac{10}{2})$th position

= number in the 5th position.

= 3 (Since 3 is in the 5th position in the data arranged above)

∴ The median is 3

The most occurring number in the data is 3 since it appears twice while every other number appears once.

∴ The mode is 3.

2. Find: a. the mean;

b. the median;

c. the mode of the data below.

6, 7, 10, 5, 11, 5, 9, 7, 10, 13, 5, 8, 7, 5, 12

<u>Solutions</u>

a. There are 15 numbers in the data. So, the mean is obtained as follows:

$$\text{Mean} = \frac{6 + 7 + 10 + 5 + 11 + 5 + 9 + 7 + 10 + 13 + 5 + 8 + 7 + 5 + 12}{15}$$

$$= \frac{120}{15}$$

$$= 8$$

∴ The mean is 8

b. In order to calculate the median, first arrange the numbers in ascending order as follows:

5, 5, 5, 5, 6, 7, 7, 7, 8, 9, 10, 10, 11, 12, 13

By inspection, the number that is at the middle of the data is 7

∴ The median is 7

Or, since there are 15 numbers in the data and 15 is an odd number, then the position of the middle number is obtained as follows:

Median = number in the $(\frac{N+1}{2})$th position

= number in the $(\frac{15+1}{2})$th position

= number in the $(\frac{16}{2})$th position

= number in the 8^{th} position.

= 7 (Since 7 is in the 8^{th} position of the data)

∴ The median is 7

c. The most occurring number in the data is 5. It occurs four times.

∴ The mode is 5.

3. Find the mean, median and mode of the data below:

11, 14, 10, 16, 18, 12, 11, 15, 10, 11, 15, 13

Solutions

There are 12 numbers in the data. So, the mean is obtained as follows:

$$\text{Mean} = \frac{11 + 14 + 10 + 16 + 18 + 12 + 11 + 15 + 10 + 11 + 15 + 13}{12}$$

$$= \frac{156}{12}$$

$$= 13$$

∴ The mean is 13

In order to calculate the median, first arrange the numbers in ascending order as follows:

10, 10, 11, 11, 11, 12, 13, 14, 15, 15, 16, 18

By inspection, the two numbers that are at the middle of the data are 12 and 13. So, we take their average.

$$\text{median} = \frac{12+13}{2}$$

$$= \frac{25}{2}$$

$$= 12.5$$

∴ The median is 12.5

Or, since there are 12 numbers in the data and 12 is an even number, then the positions of the two middle numbers and their average are obtained as follows:

$$\text{Median} = \frac{Number\ \ in\ the\ \left(\frac{N}{2}\right)th\ position\ \ + Number\ \ in\ the\ \left(\frac{N+2}{2}\right)th\ position}{2}$$

$$= \frac{Number\ \ in\ the\ \left(\frac{12}{2}\right)th\ position\ \ + Number\ \ in\ the\ \left(\frac{12+2}{2}\right)th\ position}{2}$$

$$= \frac{Number \ \ in \ the \ 6th \ position \ + Number \ \ in \ the \left(\frac{14}{2}\right)th \ position}{2}$$

$$= \frac{Number \ \ in \ the \ 6th \ position \ + Number \ \ in \ the \ 7th \ position}{2}$$

$= \frac{12 + 13}{2}$ (Note that 12 is in the 6th position while 13 is in the 7th position)

$= \frac{25}{2}$

= 12.5

∴ The median is 12.5

The most occurring number in the data is 11 since it appears three times.

∴ The mode is 11.

4. Find the mean, median and mode of the data below:

50, 56, 58, 52, 55, 59, 51, 55

Solutions

There are 8 numbers in the data. So, the mean is obtained as follows:

Mean = $\frac{50 + 56 + 58 + 52 + 55 + 59 + 51 + 55}{8}$

$= \frac{436}{8}$

= 54.5

∴ The mean is 54.5

In order to calculate the median, first arrange the numbers in ascending order as follows:

50, 51, 52, 55, 55, 56, 58, 59.

By inspection, the two numbers that are at the middle of the data are 55 and 55. So, we take their average.

median = $\frac{55+55}{2}$

$= \frac{110}{2}$

$= 55$

∴ The median is 55

Or, since there are 8 numbers in the data and 8 is an even number, then the positions of the two middle numbers and their average are obtained as follows:

$$\text{Median} = \frac{Number\ \ in\ the\ \left(\frac{N}{2}\right)th\ position\ \ + Number\ \ in\ the\ \left(\frac{N+2}{2}\right)th\ position}{2}$$

$$= \frac{Number\ \ in\ the\ \left(\frac{8}{2}\right)th\ position\ \ + Number\ \ in\ the\ \left(\frac{8+2}{2}\right)th\ position}{2}$$

$$= \frac{Number\ \ in\ the\ 4th\ position\ \ + Number\ \ in\ the\ \left(\frac{10}{2}\right)th\ position}{2}$$

$$= \frac{Number\ \ in\ the\ 4th\ position\ \ + Number\ \ in\ the\ 5th\ position}{2}$$

$= \frac{55 + 55}{2}$ (Note that 55 is in the 4^{th} and the 5^{th} position)

$$= \frac{110}{2}$$

$= 55$

∴ The median is 55

The most occurring number in the data is 55.

∴ The mode is 55.

5. The table below shows the marks of 50 students in a test.

Mark	1	2	3	4	5	6	7
No of Student	8	16	10	5	3	6	2

Calculate: a. the mean b. the median c. the mode of the marks

<u>Solutions</u>

a. Note that the number of students is also the frequency. Presenting the table as shown below allows for easy calculation of the mean.

Mark (x)	No of student (f)	Fx
1	8	8
2	16	32
3	10	30
4	5	20
5	3	15
6	6	36
7	2	14
Total:	$\sum f$= 50	$\sum fx$= 155

Note that the column fx is obtained by multiplying the values of numbers in the column f by numbers in the column x.

Mean, $\bar{x} = \frac{\sum fx}{\sum f}$

$= \frac{155}{50}$

= 3.1

b. Since there are 50 students, i.e. the total frequency is 50, and 50 is an even number, then the positions of the two middle marks and their average are obtained as follows:

$$\text{Median} = \frac{Number\ \ in\ the\ \left(\frac{\sum f}{2}\right)th\ position\ \ + Number\ \ in\ the\ \left(\frac{\sum f\ +2}{2}\right)th\ position}{2}$$

$$= \frac{Number\ \ in\ the\ \left(\frac{50}{2}\right)th\ position\ \ + Number\ \ in\ the\ \left(\frac{50+2}{2}\right)th\ position}{2}$$

$$= \frac{Number\ \ in\ the\ 25th\ position\ \ + Number\ \ in\ the\ \left(\frac{52}{2}\right)th\ position}{2}$$

$$= \frac{Number\ \ in\ the\ 25th\ position\ \ + Number\ \ in\ the\ 26th\ positio\ n}{2}$$

$= \frac{3+3}{2}$ (Note that 3 is the mark in the 25th position and in the 26th position)

$= \frac{6}{2}$

= 3

∴ The median is 3

Use the frequency (number of students) to locate the marks in the 25^{th} and 26^{th} position as follows: The first frequency of 8 shows that mark 1 occupies the position of 1^{st} to 8^{th}. Adding the second frequency of 16 to the first frequency gives, 8 + 16 = 24. This shows that after the 8^{th} position occupied by the mark 1, the positions 9^{th} to 24^{th} is occupied by the mark 2. Adding the third frequency of 10 to the previous sum of frequencies gives, 10 + 24 = 34. This shows that after the 24^{th} position occupied by the mark 2, the positions 25^{th} to 34^{th} is occupied by the mark 3. Hence, 3 is the mark in the 25^{th} and 26^{th} position which are at the middle of the data.

c. The mode is the mark that has the highest frequency. From the table, the mark 2 has the highest frequency of 16. So, the mode is 2.

$\therefore$ The mode is 2.

Note that the mode is not the frequency itself, but that particular mark that has the highest frequency. Avoid the mistake of taking 16 (frequency) as the mode.

6. The table below shows the ages of 30 students in a school.

Age	10	11	12	13	14	15
No of Student	1	4	3	7	9	6

Calculate: a. the mean b. the median c. the mode of the ages

Solutions

a. Using the number of students as the frequency, the table can be presented for easy calculation of the mean as follows:

Age (x)	No of student (f)	Fx
10	1	10
11	4	44
12	3	36
13	7	91
14	9	126
15	6	90
Total:	$\sum f$= 30	$\sum fx$= 397

Note that the column fx is obtained by multiplying the values of numbers in the column f by numbers in the column x.

Mean, $\bar{x} = \frac{\sum fx}{\sum f}$

$= \frac{397}{30}$

$= 13.3$

b. Since there are 30 students, i.e. the total frequency is 30, and 30 is an even number, then the positions of the two middle ages and their average are obtained as follows:

$$\text{Median} = \frac{Age\ in\ the\ \left(\frac{\sum f}{2}\right)th\ position\ + Age\ in\ the\ \left(\frac{\sum f+2}{2}\right)th\ position}{2}$$

$$= \frac{Age\ in\ the\ \left(\frac{30}{2}\right)th\ position\ + Age\ in\ the\ \left(\frac{30+2}{2}\right)th\ position}{2}$$

$$= \frac{Age\ in\ the\ 15th\ position\ + Age\ in\ the\ \left(\frac{32}{2}\right)th\ position}{2}$$

$$= \frac{Age\ in\ the\ 15th\ position\ + Age\ in\ the\ 16th\ position}{2}$$

$$= \frac{13+14}{2}$$

$$= \frac{27}{2}$$

$= 13.5$

∴ The median is 13.5

Note that 13 is the age in the 15th position while 14 is the age in the 16th position.

The frequency (number of students) was used to locate the ages in the 15th and 16th position as follows:

The first frequency of 1 shows that age 10 occupies the 1st position. Adding the second frequency of 4 to the first frequency gives, 1 + 4 = 5. This shows that after the 1st position occupied by the age 10, the positions 2nd to 5th is occupied by the age 11. Adding the third frequency of 3 to the previous sum of frequencies gives, 3 + 5 = 8. This shows that after the 5th position occupied by the age 11, the positions 6th to 8th is occupied by the age 12. Adding the fourth frequency of 7 to the previous sum of frequencies gives, 7 + 8 = 15. This shows that after the 8th position occupied by the age 12, the positions 9th to 15th is occupied by the age 13. Adding the fifth frequency of 9 to the previous sum of frequencies gives, 9 + 15 = 24. This shows that after the 15th position occupied by the age 13, the positions 16^{th} to 24^{th} is occupied by the age 14. Hence, 13 is the age in the 15^{th} position while 14 is the age in the 16^{th} position.

c. The mode is the age that has the highest frequency. From the table, the age 14 has the highest frequency of 9.

∴ The mode is 14.

Note that 9 is the frequency. It should not be taken as the mode.

RANGE

Range is the difference between the highest and lowest values in a given set of data. It is a measure of dispersion or variation.

Examples

1. Find the range of the following set of numbers: 4, 8, 2, 5, 8, 3, 6, 4, 9, 2, 5

Solution

The highest number in the data set is 9, while the lowest number is 2.

∴ Range = Highest number – Lowest number

= 9 – 2 = 7

Range = 7

2. The monthly salaries of five workers in a company are: $845, $1205, $694, $626 and $864. What is the range of the salaries?

Solution

Range = Highest salary – Lowest salary

= 1205 – 626

Range = $579

Exercises

1. Find the mean, median and mode of the data below:

    1, 6, 10, 4, 1, 2, 5, 2, 3, 2, 8

2. Find: a. the mean;

    b. the median;

    c. the mode of the data below.

    20, 24, 21, 25, 22, 25, 28, 26, 20, 23, 25, 27 and 26

3. Find the mean, median and mode of the data below:

    101, 105, 120, 116, 109, 112, 118, 115, 105 and 111

4. Find the mean, median and mode of the data below:

    0, 6, 8, 2, 5, 9, 1, 5, 4, 7, 5, 2, 3, 3

5. The table below shows the marks of 40 students in a test.

Mark	3	4	5	6	7	8	9
No of Student	8	16	10	5	3	6	2

Calculate: a. the mean b. the median c. the mode of the marks

6. The table below shows the ages of 30 students in a school.

Age	10	11	12	13	14	15
No of Student	1	4	3	7	9	6

Calculate: a. the mean b. the median c. the mode of the ages

7. Find the range of the following set of data

a. 12, 17, 21, 15, 19, 13, 11, 16, 22, 12, 13

b. 231kg, 258kg, 213kg, 243kg, 216kg, 271kg, 262kg, 219kg, 238kg, 231kg.

CHAPTER 11
COLLECTION AND TABULATION OF GROUPED DATA

Statistical data containing numerous values is easier to work with when the values are grouped into class intervals.

Examples

1. The data below gives the marks of 30 students in an exam.

43	45	50	47	51	58	52	47	42	54
61	50	45	55	57	41	46	49	51	50
59	44	53	57	49	40	48	52	51	58

Taking class intervals 40 – 44, 45 – 49, ……, construct a frequency distribution for the data.

Solution

The data is summarized as shown on the table below. Note that the highest value in the given data falls within the range 60 – 64.

Class interval	40 - 44	45 - 49	50 - 54	55 - 59	60 - 64
Frequency	5	8	10	6	1

2. The data below gives the ages of lecturers in a university.

34	62	54	41	51	63	31	44	48	50
33	45	59	55	47	31	39	55	60	40
63	45	53	55	36	58	61	34	34	43
47	35	43	51	35	48	42	51	36	31

Taking class intervals 31 – 35, 36 – 40, ……, construct a frequency table for the data.

Solution

The data is summarized as shown on the table below. Note that the highest value in the given data falls within the range 61 – 65.

Age	31 - 35	36 - 40	41 - 45	46 - 50	51 - 55	56 - 60	61 - 65
Frequency	9	4	7	5	8	3	4

TERMS USED IN GROUPED DATA

The table below will be used to explain the terms used in grouped data.

Class interval	Frequency
8 – 14	3
15 – 21	5
22 – 28	8
29 – 35	18

1. Class limit: The end numbers in each class interval are called the class limits. 8 is the lower class limit, while 14 is the upper class limit of the first class interval.

2. Class boundaries: The class boundary for the second class interval is 14.5 – 21.5. The lower class boundary is 14.5 which is obtained by subtracting 0.5 from 15 (the lower class limit). The upper class boundary is 21.5 which is obtained by adding 0.5 to 21 (the upper class limit). Other class boundaries are obtained in a similar way.

3. Class width: For each class interval the difference between the upper class boundary and the lower class boundary gives the class width. From the table above the class width for the third class interval is 28.5 – 21.5 = 7

4. Class mid-value: This is half of the sum of the lower and upper class limit of a given class interval. The class-value of the first class interval is given by: $\frac{8+14}{2} = \frac{22}{2} = 11$.

Examples

1. Copy and complete the table below.

Class interval	Frequency	Class boundary	Class width	Class mid-value
55 – 59	3			
60 – 64	2			
65 – 69	5			
70 – 74	4			
75 – 79	1			

Solution

The completed table is as shown below

Class interval	Frequency	Class boundary	Class width	Class mid-value
55 – 59	3	54.5 – 59.5	5	57
60 – 64	2	59.5 – 64.5	5	62
65 – 69	5	64.5 – 69.5	5	67
70 – 74	4	69.5 – 74.5	5	72
75 – 79	1	74.5 – 79.5	5	77

Note that the class boundaries are obtained by subtracting and adding 0.5 to the lower and upper class limits respectively. This 0.5 is obtained by finding the difference between the lower class limit of one class and the upper class limit of the previous class and dividing the result by 2. This gives, for example (60 – 59)/2 = ½ = 0.5. The class width is the difference between the upper and lower class boundaries. The class mid-values are obtained by taking the mean of the upper and lower class limits.

2. Copy and complete the table below.

Class interval	Frequency	Class boundary	Class width	Class mid-value
0 – 19	2			
20 – 39	8			
40 – 59	3			
60 – 79	1			
80 – 99	4			

Solution

The completed table is as shown below

Class interval	Frequency	Class boundary	Class width	Class mid-value
0 – 19	2	-0.5 – 19.5	20	9.5
20 – 39	8	19.5 – 39.5	20	29.5
40 – 59	3	39.5 – 59.5	20	49.5
60 – 79	1	59.5 – 79.5	20	69.5
80 – 99	4	79.5 – 99.5	20	89.5

Exercise

1. The data below gives the scores of 50 students in an exam.

43	45	50	47	51	58	52	47	42	54
61	50	45	55	57	41	46	49	51	50
59	44	53	57	49	40	48	52	51	58
48	54	43	54	61	60	49	57	45	42
56	45	57	61	54	62	44	47	46	62

Taking class intervals 40 – 44, 45 – 49, …, construct a frequency distribution for the data.

2. The data below shows the weights in kg of students in a school.

24	32	44	51	31	23	51	34	48	40
53	45	29	35	27	51	29	35	50	30
43	55	53	35	26	28	41	44	54	43
27	45	33	41	55	28	32	51	26	39

Taking class intervals 21 – 25, 26 – 30, …, construct a frequency table for the data.

3. Copy and complete the table below.

Class interval	Frequency	Class boundary	Class width	Class mid-value
5 – 9	2			
10 – 14	5			
15 – 19	5			
20 – 24	7			
25 – 29	1			

4. Copy and complete the table below.

Class interval	Frequency	Class boundary	Class width	Class mid-value
1 – 20	1			
21 – 40	4			
41 – 60	7			
61 – 80	3			
81 – 100	5			

5. Copy and complete the table below.

Class interval	Frequency	Class boundary	Class width	Class mid-value
0 – 90	2			
100 – 190	4			
200 – 290	1			
300 – 390	7			
400 – 490	1			

CHAPTER 12
MEAN, MEDIAN AND MODE OF GROUPED DATA

MEAN

The mean of a grouped data can be calculated by substituting thre class mid value as the values of x in the formula given by:

$$\text{Mean } \bar{x} = \frac{\sum fx}{\sum f}$$

MEDIAN

The median of a grouped data can be estimated by:

$$\text{Median} = L + C\left(\frac{\frac{\sum f}{2} - CF_{bm}}{F_m}\right)$$

Where, $\frac{\sum f}{2}$ determines the median class
L = Lower class boundary of the median class
CF_{bm} = Cumulative frequency before the median class
F_m = Frequency of the median class
C = Class width

MODE

The mode of a grouped data can be calculated as follows:

$$\text{Mode} = L + C\left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)$$

Where, L = Lower class boundary of modal class
C = Class width
Δ_1 = Difference between the frequency of the modal class and the frequency before it
Δ_2 = Difference between the frequency of the modal class and the frequency after it

Examples

1. The following table shows the frequency distribution of ages, in years of 50 people at a bus stop.

Ages	10 - 19	20 - 29	30 - 39	40 - 49	50 - 59	60 - 69
Number of people	6	12	16	9	5	2

Calculate: a. the mean

b. the median

c. the mode of the distribution

Solution

Ages	Number of people (f)	Cumulative frequency	Class mid-value(x)	fx	Class boundary	Class width
10 – 19	6	6	14.5	87	9.5-19.5	10
20 – 29	12	6+12=18	24.5	294	19.5-29.5	10
30 – 39	16	18+16=34	34.5	552	29.5-39.5	10
40 – 49	9	34+9=43	44.5	400.5	39.5-49.5	10
50 – 59	5	43+5=48	54.5	272.5	49.5-59.5	10
60 – 69	2	48+2=50	64.5	129	59.5-69.5	10
	$\sum f$ = 50			$\sum f$= 1735		

a. Mean $\bar{x} = \frac{\sum fx}{\sum f}$

$= \frac{1735}{50} = 34.7$

b. Median = L + C($\frac{\frac{\sum f}{2} - CF_{bm}}{F_m}$)

$\frac{\sum f}{2} = \frac{50}{2} = 25$

This shows that the median class falls in the 25th position

∴ Median class = 30 – 39

Lower class boundary of median class, L = 29.5

Class width, C = 10

Cumulative frequency before the median class, CF_{bm} = 18

Frequency of the median class, F_m = 16

∴ Median = L + C($\frac{\frac{\sum f}{2} - CF_{bm}}{F_m}$)

$= 29.5 + 10(\frac{25 - 18}{16})$

$= 29.5 + 10(\frac{7}{16})$

$= 29.5 + 4.375 = 33.875$

$\therefore$ Median = 33.9 (To 1 d.p)

c. Mode = $L + C(\frac{\Delta_1}{\Delta_1 + \Delta_2})$

Modal class = 30 – 39

Lower class boundary of modal class, L = 29.5

Class width = 10

Δ_1 = Modal class frequency – frequency before it

= 16 - 12 = 4

Δ_2 = Modal class frequency – frequency after it

= 16 - 9 = 7

$\therefore$ Mode = $L + C(\frac{\Delta_1}{\Delta_1 + \Delta_2})$

$= 29.5 + 10(\frac{4}{4+7})$

$= 29.5 + 10(\frac{4}{11})$

$= 29.5 + 3.64 = 33.14$

$\therefore$ Mode = 33.1 (To 1 d.p)

2. The data below is the weight of students in a high school.

Weight	31 - 35	36 - 40	41 - 45	46 - 50	51 - 55	56 - 60	61 - 65
Number of student	2	9	7	5	8	3	6

Determine: a. the mean

b. the median

c. the mode of the weights

Solution

Weight	Number of student (f)	Cumulative frequency	Class mid-value(x)	fx	Class boundary	Class width
31 – 35	2	2	33	66	30.5-35.5	5
36 – 40	9	2+9=11	38	349	35.5-40.5	5
41 – 45	7	11+7=18	43	301	40.5-45.5	5
46 – 50	5	18+5=23	48	240	45.5-50.5	5
51 – 55	8	23+8=31	53	424	50.5-55.5	5
56 – 60	3	31+3=34	58	174	55.5-60.5	5
61 – 65	6	34+6=40	63	378	60.5-65.5	5
	$\sum f$ = 40			$\sum fx$= 1932		

a. Mean $\bar{x} = \frac{\sum fx}{\sum f}$

$= \frac{1932}{40}$ = 48.3

b. Median = L + C($\frac{\frac{\sum f}{2} - CF_{bm}}{F_m}$)

$\frac{\sum f}{2} = \frac{40}{2}$ = 20

This shows that the median class falls in the 20th position

∴ Median class = 46 – 50

Lower class boundary of median class, L = 45.5

Class width, C = 5

Cumulative frequency before the median class, CF_{bm} = 18

Frequency of the median class, F_m = 5

∴ Median = L + C($\frac{\frac{\sum f}{2} - CF_{bm}}{F_m}$)

= 45.5 + 5($\frac{20 - 18}{5}$)

= 45.5 + 5($\frac{2}{5}$)

= 45.5 + 2 = 47.5

∴ Median = 47.5

c. Mode = L + C$(\frac{\Delta_1}{\Delta_1 + \Delta_2})$

Modal class = 36 – 40

Lower class boundary of modal class, L = 35.5

Class width = 5

Δ_1 = Modal class frequency – frequency before it

= 9 - 2 = 7

Δ_2 = Modal class frequency – frequency after it

= 9 - 7 = 2

∴ Mode = L + C$(\frac{\Delta_1}{\Delta_1 + \Delta_2})$

$= 35.5 + 5(\frac{7}{7+2})$

$= 35.5 + 5(\frac{7}{9})$

= 35.5 + 3.89 = 39.39

∴ Mode = 39.4 (To 1 d.p)

Exercise

1. The following table shows the weights of 30 people at a company.

Weight	60 - 64	65 - 69	70 - 74	75 - 79	80 - 84	85 - 89
Number of people	1	12	7	5	3	2

Calculate: a. the mean

b. the median

c. the mode of the distribution

2. The data below is the load distribution in tones, a chain can support.

Load	83 - 85	86 - 88	89 - 91	92 - 94	95 - 97
Number of chain	2	8	5	14	1

Determine: a. the mean

b. the median

c. the mode of the weights

2. The data below is the ages, in years, of 50 people at a party.

Ages	1 - 20	21 - 40	41 - 60	61 - 80	81-100
Number of people	3	21	17	7	2

Determine: a. the mean

b. the median

c. the mode of the weights

CHAPTER 13
MEAN DEVIATION

The mean deviation of a set of data is the mean of the absolute deviation of the values from the mean of the group. The mean deviation for data not given in a frequency table is given by:

Mean deviation = $\frac{\sum |x - \bar{x}|}{N}$ where x is each value in the data, $\bar{x}$, is the mean and N is the number of values in the data.

For data given in a frequency table, the mean deviation is given by:

Mean deviation = $\frac{\sum f|x - \bar{x}|}{\sum f}$

Examples

1. Calculate the mean deviation of the following data: 2, 4, 1, 3, 0

Solution

Let us first calculate the mean of the data.

Mean, $\bar{x} = \frac{2 + 4 + 1 + 3 + 0}{5} = \frac{10}{5} = 2$

$\therefore$ $\bar{x} = 2$

The deviation from the mean $(x - \bar{x})$ is now tabulated as follows.

Data (x)	$x - \bar{x}$ ($\bar{x} = 2$)	$\lvert x - \bar{x} \rvert$
2	0	0
4	2	2
1	-1	1
3	1	1
0	-2	2
		$\sum \lvert x - \bar{x} \rvert = 6$

$\therefore$ Mean deviation = $\frac{\sum|x - \bar{x}|}{N} = \frac{6}{5} = 1.2$

2. Calculate the mean deviation of the following data: 6, 2, 5, 8, 3, 6, 4, 5, 7, 4.

Solution

Let us first calculate the mean of the data.

Mean, $\bar{x} = \frac{6+2+5+8+3+6+4+5+7+4}{10} = \frac{50}{10} = 5$

$\therefore \bar{x} = 5$

The deviation from the mean $(x - \bar{x})$ is now tabulated as follows.

Data (x)	$x - \bar{x}$ $(\bar{x} = 5)$	$\lvert x - \bar{x} \rvert$
6	1	1
2	-3	3
5	0	0
8	3	3
3	-2	2
6	1	1
4	-1	1
5	0	0
7	2	2
4	-1	1
		$\sum \lvert x - \bar{x} \rvert = 14$

$\therefore$ Mean deviation $= \frac{\sum|x-\bar{x}|}{N} = \frac{14}{10} = 1.4$

3. The marks obtained by 40 students in a mathematics test are as shown below. Calculate the mean deviation of the data.

Marks	31 - 40	41 - 50	51 - 60	61 - 70	71 - 80	81 - 90	91- 100
Number of student	1	2	8	11	8	6	4

Solution

The table below summarizes the determination of the mean and the values needed for the mean deviation. Note that the mean used on the table has been calculated below the table.

Mark	mid-value x	$x - \bar{x}$ $\bar{x}$ = 69.75	$\|x - \bar{x}\|$	No of student f	fx	$f\|x - \bar{x}\|$
31 – 40	35.5	-34.25	34.25	1	35.5	34.25
41 – 50	45.5	-24.25	24.25	2	91	48.50
51 – 60	55.5	-14.24	14.24	8	444	114
61 – 70	65.5	-4.25	4.25	11	720.5	46.75
71 – 80	75.5	5.75	5.75	8	604	46
81 – 90	85.5	15.75	15.75	6	513	94.5
91 – 100	95.5	25.75	25.75	4	382	103
				$\sum f$ = 40	$\sum$fx =2790	

Mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{2790}{40}$ = 69.75

Using the values from the table above, $\sum f|x - \bar{x}|$ = 34.25 + 48.50 + 114 + 46.75 + 46 + 94.5 + 103 = 487

∴ Mean deviation = $\frac{\sum f|x-\bar{x}|}{\sum f}$

= $\frac{487}{40}$

∴ Mean deviation = 12.2

4. The ages of 50 people in a hospital are as shown below. Calculate the mean deviation of the ages.

Age	1 - 5	6 - 10	11 - 15	16 - 20	21 - 25	26 - 30
Number of people	6	9	14	10	4	7

<u>Solution</u>

The table below summarizes the determination of the mean and the values needed for the mean deviation. The mean has been calculated below the table.

Age	mid-value x	$x - \bar{x}$ $\bar{x}$ = 14.8	$\lvert x - \bar{x} \rvert$	No of people f	fx	$f\lvert x - \bar{x} \rvert$
1 – 5	3	-11.8	11.8	6	18	70.8
6 – 10	8	-6.8	6.8	9	72	61.2
11 – 15	13	-1.8	1.8	14	182	25.2
16 – 20	18	3.2	3.2	10	180	32
21 – 25	23	8.2	8.2	4	92	32.8
26 – 30	28	13.2	13.2	7	196	92.4
				$\sum f$ = 50	$\sum$fx =740	

Mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{740}{50}$ = 14.8

Using the values from the table above, $\sum f\lvert x - \bar{x} \rvert$ = 70.8 + 61.2 + 25.2 + 32 + 32.8 + 92.4 = 314.4

∴ Mean deviation = $\frac{\sum f\lvert x - \bar{x}\rvert}{\sum f}$

$= \frac{314.4}{50}$

∴ Mean deviation = 6.29

5. The table below shows the number of cars owned by some political public office holders.

Number of cars	1	2	3	4	5	6
Number of politicians	9	15	11	7	3	5

Calculate the mean deviation of the data.

Solution

The table below summarises the calculations of the mean and the mean deviation.

Cars x	$x - \bar{x}$ $\bar{x}$ = 2.9	$\lvert x - \bar{x} \rvert$	No of politicians f	fx	$f\lvert x - \bar{x} \rvert$
1	-1.9	1.9	9	9	17.1
2	-0.9	0.9	15	30	13.5
3	0.1	0.1	11	33	1.1
4	1.1	1.1	7	28	7.7
5	2.1	2.1	3	15	6.3
6	3.1	3.1	5	30	15.5
			$\sum f$ = 50	$\sum$fx =145	

Mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{145}{50} = 2.9$

Using the values from the table above, $\sum f|x - \bar{x}|$ = 17.1 + 13.5 + 1.1 + 7.7 + 6.3 + 15.5 = 61.2

$\therefore$ Mean deviation = $\frac{\sum f|x-\bar{x}|}{\sum f}$

$= \frac{61.2}{50}$

$\therefore$ Mean deviation = 1.224

<u>EXERCISE</u>

1. Calculate the mean deviation of the following data: 0, 5, 7, 4, 5, 3

2. Calculate the mean deviation of the following data: 4, 6, 5, 9, 9, 5, 2, 4, 8, 6, 8

3. Calculate the mean deviation of the following data: 1, 3, 1, 4, 6

4. The marks obtained by 30 students in a physics test are as shown below. Calculate the mean deviation of the data.

Marks	0 - 9	10 - 19	20 - 29	30 - 39	40 - 49	50 - 59	60- 69
Number of student	4	1	5	8	3	2	7

5.

The ages of 100 people in a village are as shown below. Calculate the mean deviation of the ages.

Age	11 - 20	21 - 30	31 - 40	41 - 50	51 - 60	61 - 70
Number of people	12	9	15	24	29	11

6. The number of employees in 50 enterprises are as shown below. Calculate the mean deviation of the data.

Marks	0 - 4	5 - 9	10 - 14	15 - 19	20 - 24	25 - 29	30- 34
Number of student	2	11	15	3	4	2	13

7.

The table below shows the number of farms owned by some people in a city.

Number of farms	2	4	6	8	10	12
Number of people	3	5	10	6	8	8

Calculate the mean deviation of the data.

8. A die is rolled 50 times and the following data is obtained.

2	5	4	3	5	3	1	4	6	5	6	4	2
6	1	5	6	2	1	6	4	3	4	3	1	6
1	3	6	4	2	4	3	4	5	3	4	1	2
3	1	2	2	5	6	4	3	4	6	5		

a. Present the data in a frequency table

b. Calculate the mean deviation of the data.

CHAPTER 14
VARIANCE AND STANDARD DEVIATION

Variance is the mean of the squares of the deviations from the mean. Standard deviation is the positive square root of the variance.
Variance, standard deviation and mean deviation are also regarded as measures of dispersion or variation.
The variance of data not given on a frequency table is given by:

$$\text{Variance} = \frac{\sum(x - \bar{x})^2}{N}$$

For data given on a frequency table, the variance is given by:

$$\text{Variance} = \frac{\sum f\,(x - \bar{x})^2}{\sum f}$$

Standard deviation is the square root of variance.

Examples

1. Calculate the variance and standard deviation of the following data: 4, 2, 1, 5.

Solution

Let us first calculate the mean of the data.

$$\text{Mean, } \bar{x} = \frac{4 + 2 + 1 + 5}{4} = \frac{12}{4} = 3$$

We now present the deviation from the mean as follows.

Data x	$x - \bar{x}$ ($\bar{x}$ = 3)	$(x - \bar{x})^2$
4	1	1
2	-1	1
1	-2	4
5	2	4
		$\sum(x - \bar{x})^2 = 10$

$$\text{Variance} = \frac{\sum(x - \bar{x})^2}{N} = = \frac{10}{4} = 2.5$$

∴ Standard deviation = $\sqrt{2.5}$ = 1.58

2. Calculate the variance and standard deviation of the data below:
   2, 5, 3, 2, 6, 5, 7, 2.

<u>Solution</u>

Let us first calculate the mean of the data as follows:

Mean, $\bar{x} = \frac{2+5+3+2+6+5+7+2}{8} = \frac{32}{8} = 4$

The deviation from the mean is as presented below.

Data x	$x - \bar{x}$ $(\bar{x} = 4)$	$(x - \bar{x})^2$
2	-2	4
5	1	1
3	-1	1
2	-2	4
6	2	4
5	1	1
7	3	9
2	-2	4
		$\sum(x - \bar{x})^2 = 28$

Variance $= \frac{\sum(x-\bar{x})^2}{N} = = \frac{28}{8} = 3.5$

∴ Standard deviation = $\sqrt{3.5}$ = 1.87

3. The distances in Km, from school to the homes of 30 students are as shown below. Calculate:

a. the variance
b. the standard deviation of the data

Distance (Km)	0 - 4	5 - 9	10 – 14	15 - 19	20 - 24	25 - 29
Number of students	2	10	8	6	3	1

Solutions

The working is set out as shown on the table below

Distance	mid-value x	No of student f	fx	$x - \bar{x}$ $\bar{x}$ = 12.2	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
0 – 4	2	2	4	-10.2	104.04	208.08
5 – 9	7	10	70	-5.2	27.04	270.4
10 – 14	12	8	96	-0.2	0.04	0.32
15 – 19	17	6	102	4.8	23.04	138.24
20 – 24	22	3	66	9.8	96.04	288.12
25 – 29	27	1	27	14.8	219.04	219.04
		$\sum f = 30$	$\sum fx = 365$			

Mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{365}{30} = 12.2$

a. Using the values from the table above, $\sum f(x - \bar{x})^2$ = 208.08 + 270.4 + 0.32 + 138.24 + 288.12 + 219.04 = 1124.2

∴ Variance = $\frac{\sum f(x - \bar{x})^2}{\sum f}$

$= \frac{1124.2}{30}$

∴ Variance = 37.5

b. Standard deviation = $\sqrt{Variance}$

$= \sqrt{37.5}$

∴ Standard deviation = 6.1

4. The projected population in millions, of 20 states in a country are as shown below. Calculate:

a. the variance

b. the standard deviation of the data

Population	1 - 5	6 - 10	11 - 15	16 - 20	21 - 25	26 - 30
Number of state	1	8	5	3	2	1

Solutions

The working is set out as shown on the table below

Population	mid-value x	No of states f	fx	$x - \bar{x}$ $\bar{x}$ = 13	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
1 – 5	3	1	3	-10	100	100
6 – 10	8	8	64	-5	25	200
11 – 15	13	5	65	0	0	0
16 – 20	18	3	54	5	25	75
21 – 25	23	2	46	10	100	200
26 – 30	28	1	28	15	225	225
		$\sum f$ = 20	$\sum$fx =260			

Mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{260}{20} = 13$

a. Using the values from the table above, $\sum f(x - \bar{x})^2 = 100 + 200 + 0 + 75 + 200 + 225 = 800$

∴ Variance = $\frac{\sum f(x - \bar{x})^2}{\sum f}$

$= \frac{800}{20}$

∴ Variance = 40

b. Standard deviation = $\sqrt{Variance}$

$= \sqrt{40}$

∴ Standard deviation = 6.3

5. The scores obtained by 100 students in a test are as shown below. Calculate:

a. the variance

b. the standard deviation of the scores

Scores	2	3	4	5	6	7
Number of student	10	22	18	30	12	8

Solutions

The working is set out as shown on the table below

Scores x	No of students f	fx	x - $\bar{x}$ $\bar{x}$ = 4.4	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
2	12	24	-2.4	5.76	69.12
3	18	54	-1.4	1.96	35.28
4	22	88	-0.4	0.16	3.52
5	24	120	0.6	0.36	8.64
6	14	84	1.6	2.56	35.84
7	10	70	2.6	6.76	67.6
	$\sum f$ = 100	$\sum$fx =440			

Mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{440}{100}$ = 4.4

a. Using the values from the table above, $\sum f(x - \bar{x})^2$ = 69.12 + 35.28 + 3.52 + 8.64 + 35.84 + 67.6 = 220

∴ Variance = $\frac{\sum f\,(x - \bar{x})^2}{\sum f}$

$= \frac{220}{100}$

∴ Variance = 2.2

b. Standard deviation = $\sqrt{Variance}$

$= \sqrt{2.2}$

∴ Standard deviation = 1.48

EXERCISE

1. Calculate the variance and standard deviation of the following data: 3, 5, 4, 7, 6.

2. Calculate the variance and standard deviation of the data below:
   1, 0, 4, 3, 5, 8, 6, 4, 7, 2.

3. The scores of 50 students in a test are as shown below. Calculate:

a. the variance
b. the standard deviation of the data

Scores	0 - 9	10 - 19	20 - 29	30 - 39	40 - 49	50 - 59
Number of students	5	12	6	18	5	4

4. The ages of employees in an organization are as shown below. Calculate:

a. the variance
b. the standard deviation of the data

Age	20 - 24	25 - 29	30 - 34	35 - 39	40 - 44
Number of empoyees	8	6	3	1	2

5. The scores obtained by 40 students in a test are as shown below. Calculate:

a. the variance
b. the standard deviation of the scores

Scores	5	6	7	8	9	10
Number of student	1	2	4	12	20	1

CHAPTER 15
QUARTILES AND PERCENTILES BY INTERPOLATION METHOD

When a distribution is divide into four equal parts, it is called a quartile. When it is divided into hundred equal parts, such a division is called percentile.

The first quartile is also called lower quartile, and it is denoted by Q_1.
The second quartile is also called median, and it is denoted by Q_2.
The third quartile is also called upper quartile, and it is denoted by Q_3.

The lower quartile is calculated as follows:

$$Q_1 = L_1 + C\left(\frac{\frac{\sum f}{4} - CF_{bQ1}}{F_{Q1}}\right)$$

Where, $\frac{\sum f}{4}$ determines the lower quartile class
L_1 = Lower class boundary of the lower quartile class
CF_{bQ_1} = Cumulative frequency before the lower quartile class
F_{Q_1} = Frequency of the lower quartile class
C = Class width

The median is calculated as follows:

$$Q_2 = L_2 + C\left(\frac{\frac{\sum f}{2} - CF_{bm}}{F_m}\right)$$

Where, $\frac{\sum f}{2}$ determines the median class
L_2 = Lower class boundary of the median class
CF_{bm} = Cumulative frequency before the median class
F_m = Frequency of the median class
C = Class width

The upper quartile is calculated as follows:

$$Q_3 = L_3 + C\left(\frac{\frac{3\sum f}{4} - CF_{bQ3}}{F_{Q3}}\right)$$

Where, $\frac{3\sum f}{4}$ determines the upper quartile class
L_3 = Lower class boundary of the upper quartile class
CF_{bQ_3} = Cumulative frequency before the upper quartile class
F_{Q_3} = Frequency of the upper quartile class
C = Class width

The interquartile range is given by:

Interquartile range = $Q_3 - Q_1$

The semi-interquartile range is also called quartile deviation, and it is given by:

$$\text{Semi-interquartile range} = \frac{Q_3 - Q_1}{2}$$

The percentile is calculated as follows:

$$P_N = L_N + C\left(\frac{\frac{N\sum f}{100} - CF_{bP_N}}{F_{P_N}}\right)$$

Where P_N is the N percentile
Where, $\frac{N\sum f}{100}$ determines the N percentile class
L_N = Lower class boundary of the N percentile class
CF_{bP_N} = Cumulative frequency before the N percentile class
F_{P_N} = Frequency of the N percentile class
C = Class width

Examples

1. The following is the record of marks of 40 students in an examination:

64 84 91 58 43 86 73 33 76 80 57 33 53 29 40 27 72 19 51
67 37 14 18 92 13 45 61 39 23 22 22 41 27 51 63 47 19 35
39 76

Using class interval 11 – 20, 21 – 30, …, prepare a frequency table for the distribution.
Hence calculate the:
a. median

b. lower quartile

c. upper quartile

d. interquartile range

e. quartile deviation/semi-interquartile range

f. 40^{th} percentile

g. 85^{th} percentile

Solutions

The frequency table is as shown below.

Class interval	Frequency
11 – 20	5
21 – 30	6
31 – 40	7
41 – 50	4
51 – 60	5
61 – 70	4
71 – 80	5
81 – 90	2
91 - 100	2

a. In order to calculate the median, a table of the class boundaries and cumulative frequency has to be drawn as shown below.

Class interval	Class boundary	Frequency	Cumulative frequency	Class width
11 – 20	10.5 – 20.5	5	5	10
21 – 30	20.5 – 30.5	6	11	10
31 – 40	30.5 – 40.5	7	18	10
41 – 50	40.5 – 50.5	4	22	10
51 – 60	50.5 – 60.5	5	27	10
61 – 70	60.5 – 70.5	4	31	10
71 – 80	70.5 – 80.5	5	36	10
81 – 90	80.5 – 90.5	2	38	10
91 - 100	90.5 – 100.5	2	40	10

The median is calculated as follows:

$$Q_2 = L_2 + C\left(\frac{\frac{\Sigma f}{2} - CF_{bm}}{F_m}\right)$$

$\frac{\sum f}{2} = \frac{40}{2} = 20$. This shows that the median class is at the 20th position. This is the class, 41 – 50. This position is obtained by counting the frequency to get to the 20th position. 5 + 6 + 7 = 18. This shows that the 18th position is occupied by the class 31 – 40. After this class, the next frequency is 4. When 4 positions are added to 18 positions, it gives 22. This means that these 4 positions are the 19th, 20th, 21st and 22nd positions. These 4 positions are occupied by the class 41 – 50 as shown on the table. Hence the class in the 20th position is 41 – 50. You can also look at the cumulative frequency and see where the 20th position class falls.

L_2 = Lower class boundary of the median class = 40.5

CF_{bm} = Cumulative frequency before the median class = 18

F_m = Frequency of the median class = 4

C = Class width = 10. The class limit is the difference between an upper class boundary and a lower class boundary. For example, 20.5 – 10.5 = 10.

$$\therefore \quad Q_2 = L_2 + C\left(\frac{\frac{\sum f}{2} - CF_{bm}}{F_m}\right)$$

$$= 40.5 + 10\left(\frac{\frac{40}{2} - 18}{4}\right)$$

$$= 40.5 + 10\left(\frac{20 - 18}{4}\right)$$

$$= 40.5 + 10\left(\frac{2}{4}\right)$$

$$= 40.5 + \left(\frac{10 \ x \ 2}{4}\right)$$

$$= 40.5 + 5$$

$$Q_2 = 45.5$$

b. The lower quartile is calculated as follows:

$$Q_1 = L_1 + C\left(\frac{\frac{\sum f}{4} - CF_{bQ_1}}{F_{Q_1}}\right)$$

$\frac{\sum f}{4} = \frac{40}{4} = 10$. Hence the lower quartile class is at the 10th position. This class is, 21 – 30.

L_1 = Lower class boundary of the lower quartile class = 20.5

CF_{bQ_1} = Cumulative frequency before the lower quartile class = 5

F_{Q_1} = Frequency of the lower quartile class = 6

C = Class width = 10

$$\therefore \quad Q_1 = L_1 + C\left(\frac{\frac{\sum f}{4} - CF_{bQ_1}}{F_{Q_1}}\right)$$

$$= 20.5 + 10\left(\frac{\frac{40}{4} - 5}{6}\right)$$

$$= 20.5 + 10\left(\frac{10 - 5}{6}\right)$$

$$= 20.5 + 10\left(\frac{5}{6}\right)$$

$$= 20.5 + \left(\frac{10 \ x \ 5}{6}\right)$$

$$= 20.5 + 8.3$$

$$Q_1 = 28.8$$

c. The upper quartile is calculated as follows:

$$Q_3 = L_3 + C\left(\frac{\frac{3\sum f}{4} - CF_{bQ_3}}{F_{Q_3}}\right)$$

$\frac{3\sum f}{4} = \frac{3 \ x \ 40}{4}$ = 30. Hence the upper quartile class is at the 30th position. This class is, 61 – 70.

L_3 = Lower class boundary of the upper quartile class = 60.5

CF_{bQ_3} = Cumulative frequency before the upper quartile class = 27

F_{Q_3} = Frequency of the upper quartile class = 4

C = Class width = 10

$$\therefore \quad Q_3 = L_3 + C\left(\frac{\frac{3\sum f}{4} - CF_{bQ_3}}{F_{Q_3}}\right)$$

$$= 60.5 + 10\left(\frac{\frac{3 \ x \ 40}{4} - 27}{4}\right)$$

$$= 60.5 + 10\left(\frac{30 - 27}{4}\right)$$

$$= 60.5 + 10\left(\frac{3}{4}\right)$$

$$= 60.5 + \left(\frac{10 \ x \ 3}{4}\right)$$

$$= 60.5 + 7.5$$

$$Q_3 = 68$$

d. Interquartile range = $Q_3 - Q_1$

$= 68 - 28.8$

$= 39.2$

e. Quartile deviation/semi-interquartile range, $Q = \frac{Q_3 - Q_1}{2}$

$= \frac{68 - 28.8}{2}$

$= \frac{39.2}{2}$

$Q = 19.6$

f. The 40th percentile is calculated as follows:

$$P_N = L_N + C\left(\frac{\frac{N\sum f}{100} - CF_{bP_N}}{F_{P_N}}\right)$$

$P_N = P_{40}$

$\frac{N\sum f}{100} = \frac{40\ x\ 40}{100}$ = 16. Hence the 40th percentile class is at the 16th position. This class is: 31 - 40

$L_N = L_{40}$ = Lower class boundary of the 40th percentile class = 30.5

$CF_{bP_N} = CF_{bP_{40}}$ = Cumulative frequency before the 40th percentile class = 11

$F_{P_N} = F_{P_{40}}$ = Frequency of the 40th percentile class = 7

C = Class width = 10

Hence, $$P_{40} = L_{40} + C\left(\frac{\frac{40\sum f}{100} - CF_{bP40}}{F_{P40}}\right)$$

$$= 30.5 + 10\left(\frac{\frac{40\ x\ 40}{100} - 11}{7}\right)$$

$$= 30.5 + 10\left(\frac{16 - 11}{7}\right)$$

$$= 30.5 + 10\left(\frac{5}{7}\right)$$

$$= 30.5 + \left(\frac{10\ x\ 5}{7}\right)$$

$$= 30.5 + 7.1$$

$P_{40} = 37.6$

g. The 85th percentile is calculated as follows:

$$P_N = L_N + C\left(\frac{\frac{N\sum f}{100} - CF_{bP_N}}{F_{P_N}}\right)$$

$P_N = P_{85}$

$\frac{N\sum f}{100} = \frac{85 \ x \ 40}{100}$ = 34. Hence the 85th percentile class is at the 34th position. This class is: 71 - 80

$L_N = L_{85}$ = Lower class boundary of the 85th percentile class = 70.5

$CF_{bP_N} = CF_{bP_{85}}$ = Cumulative frequency before the 85th percentile class = 31

$F_{P_N} = F_{P_{85}}$ = Frequency of the 85th percentile class = 5

C = Class width = 10

Hence, $P_{85} = L_{85} + C\left(\frac{\frac{85 \ x \ 40}{100} - CF_{bP_{85}}}{F_{P_{85}}}\right)$

$$= 70.5 + 10\left(\frac{\frac{85 \ x \ 40}{100} - 31}{5}\right)$$

$$= 70.5 + 10\left(\frac{34 - 31}{5}\right)$$

$$= 70.5 + 10\left(\frac{3}{5}\right)$$

$$= 70.5 + \left(\frac{10 \ x \ 3}{5}\right)$$

$$= 70.5 + 6$$

$P_{85} = 76.5$

2. The table below shows the distribution of marks scored by students in an examination.

Class interval	Frequency
60 – 64	2
65 – 69	4
70 – 74	7
75 – 79	13
80 – 84	10
85 – 89	8
90 – 94	5
95 – 99	1

From the data, calculate:

a. median

b. lower quartile
c. upper quartile
d. interquartile range
e. semi-interquartile range
f. 70th percentile
g. the pass mark if 25% of the students passed
h. the pass mark if it was later agreed that only 40% of the students should fail.

Solution

a. In order to calculate the median, a table of the class boundaries and cumulative frequency has to be drawn as shown below.

Class interval	Class boundary	Frequency	Cumulative frequency	Class width
60 – 64	59.5 – 64.5	2	2	5
65 – 69	64.5 – 69.5	4	6	5
70 – 74	69.5 – 74.5	7	13	5
75 – 79	74.5 – 79.5	13	26	5
80 – 84	79.5 – 84.5	10	36	5
85 – 89	84.5 – 89.5	8	44	5
90 – 94	89.5 – 94.5	5	49	5
95 – 99	94.5 – 99.5	1	50	5

The median is calculated as follows:

$$Q_2 = L_2 + C\left(\frac{\frac{\Sigma f}{2} - CF_{bm}}{F_m}\right)$$

$\frac{\Sigma f}{2} = \frac{50}{2} = 25$. This shows that the median class is at the 25th position. This is the class, 75 – 79. This is obtained by looking at the cumulative frequency to see where the 25th position class falls.

L_2 = Lower class boundary of the median class = 74.5

CF_{bm} = Cumulative frequency before the median class = 13

F_m = Frequency of the median class. This is also 13

C = Class width = 5

$$\therefore \quad Q_2 = L_2 + C\left(\frac{\frac{\Sigma f}{2} - CF_{bm}}{F_m}\right)$$

$$= 74.5 + 5(\frac{\frac{50}{2} - 13}{13})$$

$$= 74.5 + 5(\frac{25 - 13}{13})$$

$$= 74.5 + 5(\frac{12}{13})$$

$$= 74.5 + (\frac{5 \ x \ 12}{13})$$

$$= 74.5 + 4.6$$

$Q_2 = 79.1$

b. The lower quartile is calculated as follows:

$$Q_1 = L_1 + C(\frac{\frac{\sum f}{4} - CF_{bQ1}}{F_{Q1}})$$

$\frac{\sum f}{4} = \frac{50}{4} = 12.5$. Hence the lower quartile class is at the 12^{th} or 13^{th} position. This class is, 70 – 74.

L_1 = Lower class boundary of the lower quartile class = 69.5

CF_{bQ_1} = Cumulative frequency before the lower quartile class = 6

F_{Q_1} = Frequency of the lower quartile class = 7

C = Class width = 5

$$\therefore \quad Q_1 = L_1 + C(\frac{\frac{\sum f}{4} - CF_{bQ1}}{F_{Q1}})$$

$$= 69.5 + 5(\frac{\frac{50}{4} - 6}{7})$$

$$= 69.5 + 5(\frac{12.5 - 6}{7})$$

$$= 69.5 + 5(\frac{6.5}{7})$$

$$= 69.5 + (\frac{5 \ x \ 6.5}{7})$$

$$= 69.5 + 4.6$$

$Q_1 = 74.1$

c. The upper quartile is calculated as follows:

$$Q_3 = L_3 + C\left(\frac{\frac{3\sum f}{4} - CF_{bQ3}}{F_{Q3}}\right)$$

$\frac{3\sum f}{4} = \frac{3\ x\ 50}{4}$ = 37.5. Hence the upper quartile class is at the 37th or 38th position. This class is, 85 – 89.

L_3 = Lower class boundary of the upper quartile class = 84.5

CF_{bQ_3} = Cumulative frequency before the upper quartile class = 36

F_{Q_3} = Frequency of the upper quartile class = 8

C = Class width = 5

$$\therefore \quad Q_3 = L_3 + C\left(\frac{\frac{3\sum f}{4} - CF_{bQ3}}{F_{Q3}}\right)$$

$$= 84.5 + 5\left(\frac{\frac{3\ x\ 50}{4} - 36}{8}\right)$$

$$= 84.5 + 5\left(\frac{37.5 - 36}{8}\right)$$

$$= 84.5 + 5\left(\frac{1.5}{8}\right)$$

= 84.5 + 0.9

Q_3 = 85.4

d. Interquartile range = $Q_3 - Q_1$

= 85.4 – 74.1

= 11.3

e. Semi-interquartile range, Q = $\frac{Q_3 - Q_1}{2}$

$$= \frac{85.4 - 74.1}{2}$$

$$= \frac{11.3}{2}$$

Q = 5.65

f. The 70th percentile is calculated as follows:

$$P_N = L_N + C\left(\frac{\frac{N\sum f}{100} - CF_{bP_N}}{F_{P_N}}\right)$$

$P_N = P_{70}$

$\frac{N\sum f}{100} = \frac{70\ x\ 50}{100}$ = 35. Hence the 70th percentile class is at the 35th position. This class is: 80 - 84

$L_N = L_{70}$ = Lower class boundary of the 70th percentile class = 79.5

$CF_{bP_N} = CF_{bP_{70}}$ = Cumulative frequency before the 70th percentile class = 26

$F_{P_N} = F_{P_{70}}$ = Frequency of the 70th percentile class = 10

C = Class width = 5

Hence, $P_{70} = L_{70} + C\left(\frac{\frac{70\sum f}{100} - CF_{bP_{70}}}{F_{P_{70}}}\right)$

$= 79.5 + 5\left(\frac{\frac{70\ x\ 50}{100} - 26}{10}\right)$

$= 79.5 + 5\left(\frac{35 - 26}{10}\right)$

$= 79.5 + 5\left(\frac{9}{10}\right)$

$= 79.5 + 4.5$

$P_{70} = 84$

g. If 25% of the students passed, then the first 75% (i.e. 100 – 25) of the students failed. This means that the pass mark is at the 75th percentile.

Note that the pass mark is always at the failure percentile.

Hence the 75th percentile is calculated as follows:

$P_N = L_N + C\left(\frac{\frac{N\sum f}{100} - CF_{bP_N}}{F_{P_N}}\right)$

$P_N = P_{75}$

$\frac{N\sum f}{100} = \frac{75\ x\ 50}{100}$ = 37.5. Hence the 75th percentile class is at the 37.5th position. This class is: 85 - 89

$L_N = L_{75}$ = Lower class boundary of the 75th percentile class = 84.5

$CF_{bP_N} = CF_{bP_{75}}$ = Cumulative frequency before the 75th percentile class = 36

$F_{P_N} = F_{P_{75}}$ = Frequency of the 75th percentile class = 8

C = Class width = 5

Hence, $P_{75} = L_{75} + C\left(\frac{\frac{75\sum f}{100} - CF_{bP_{75}}}{F_{P_{75}}}\right)$

$= 84.5 + 5(\frac{\frac{75 \ x \ 50}{100} - 36}{8})$

$= 84.5 + 5(\frac{37.5 - 36}{8})$

$= 84.5 + 5(\frac{1.5}{8})$

$= 84.5 + 0.9$

$P_{75} = 85.4$

Hence the pass mark is 85.4

h. If 40% of the students should fail, then the pass mark is at the 40th percentile.

Hence the 40th percentile is calculated as follows:

$P_N = L_N + C(\frac{\frac{N\sum f}{100} - CF_{bP_N}}{F_{P_N}})$

$P_N = P_{40}$

$\frac{N\sum f}{100} = \frac{40 \ x \ 50}{100}$ = 20. Hence the 40th percentile class is at the 20th position. This class is: 75 - 79

$L_N = L_{40}$ = Lower class boundary of the 40th percentile class = 74.5

$CF_{bP_N} = CF_{bP_{40}}$ = Cumulative frequency before the 40th percentile class = 13

$F_{P_N} = F_{P_{40}}$ = Frequency of the 40th percentile class = 13

C = Class width = 5

Hence, $P_{40} = L_{40} + C(\frac{\frac{40\sum f}{100} - CF_{bP_{40}}}{F_{P_{40}}})$

$= 74.5 + 5(\frac{\frac{40 \ x \ 50}{100} - 13}{13})$

$= 74.5 + 5(\frac{20 - 13}{13})$

$= 74.5 + 5(\frac{7}{13})$

$= 74.5 + 2.7$

$P_{40} = 77.2$

Hence the pass mark is 77.2

3. The table below shows the masses of some items sold in a supermarket.

Mass	1.5 – 1.9	2.0 – 2.4	2.5 – 2.9	3.0 – 3.4	3.5 – 3.9	4.0 – 4.5
Number of Items	5	12	6	18	5	4

From the table given above, estimate:

a. median
b. lower quartile
c. upper quartile
d. the 55^{th} percentile

Solution

a. In order to calculate the median, a table of the class boundaries and cumulative frequency has to be drawn as shown below.

Class interval	Class boundary	Frequency	Cumulative frequency	Class width
1.5 – 1.9	1.45 – 1.95	5	5	0.5
2.0 – 2.4	1.95 – 2.45	12	17	0.5
2.5 – 2.9	2.45 – 2.95	6	23	0.5
3.0 – 3.4	2.95 – 3.45	18	41	0.5
3.5 – 3.9	3.45 – 3.95	5	46	0.5
4.0 – 4.4	3.95 – 4.45	4	50	0.5

Note that in computing the class boundaries, a difference between an upper class limit and a lower class limit, such as, 2.0 – 1.9 = 0.1, is first determined and then divided by 2 to give, 0.1/2 = 0.05. It is this 0.05 that is added and subtracted from the class limit values to obtain the class boundary values. This is the method applied in obtaining the class boundaries of any given grouped data.

The median is calculated as follows:

$$Q_2 = L_2 + C\left(\frac{\frac{\Sigma f}{2} - CF_{bm}}{F_m}\right)$$

$\frac{\Sigma f}{2} = \frac{50}{2} = 25$. Hence the median class is at the 25^{th} position. This is the class, 3.0 – 3.4. This is obtained by looking at the cumulative frequency to see where the 25^{th} position class falls.

L_2 = Lower class boundary of the median class = 2.95

CF_{bm} = Cumulative frequency before the median class = 23

F_m = Frequency of the median class = 18

C = Class width = 0.5, i.e. 1.95 – 1.45 = 0.5.

$\therefore \quad Q_2 = L_2 + C(\frac{\frac{\sum f}{2} - CF_{bm}}{F_m})$

$= 2.95 + 0.5(\frac{\frac{50}{2} - 23}{18})$

$= 2.95 + 0.5(\frac{25 - 23}{18})$

$= 2.95 + 0.5(\frac{2}{18})$

$= 2.95 + 0.06$

$Q_2 = 3.01$

b. The lower quartile is calculated as follows:

$Q_1 = L_1 + C(\frac{\frac{\sum f}{4} - CF_{bQ1}}{F_{Q1}})$

$\frac{\sum f}{4} = \frac{50}{4} = 12.5$. This shows that the lower quartile class is at the 12th or 13th position. This class is, 2.0 – 2.4.

L_1 = Lower class boundary of the lower quartile class = 1.95

CF_{bQ_1} = Cumulative frequency before the lower quartile class = 5

F_{Q_1} = Frequency of the lower quartile class = 12

C = Class width = 0.5

$\therefore \quad Q_1 = L_1 + C(\frac{\frac{\sum f}{4} - CF_{bQ1}}{F_{Q1}})$

$= 1.95 + 0.5(\frac{\frac{50}{4} - 5}{12})$

$= 1.95 + 0.5(\frac{12.5 - 5}{12})$

$= 1.95 + 0.5(\frac{7.5}{12})$

$= 1.95 + 0.31$

$Q_1 = 2.26$

c. The upper quartile is calculated as follows:

$$Q_3 = L_3 + C(\frac{\frac{3\sum f}{4} - CF_{bQ3}}{F_{Q3}})$$

$\frac{3\sum f}{4} = \frac{3\ x\ 50}{4}$ = 37.5. This shows that the upper quartile class is at the 37th or 38th position. This class is, 3.0 – 3.4.

L_3 = Lower class boundary of the upper quartile class = 2.95

CF_{bQ_3} = Cumulative frequency before the upper quartile class = 23

F_{Q_3} = Frequency of the upper quartile class = 18

C = Class width = 0.5

$$\therefore \quad Q_3 = L_3 + C(\frac{\frac{3\sum f}{4} - CF_{bQ3}}{F_{Q3}})$$

$$= 2.95 + 0.5(\frac{\frac{3\ x\ 50}{4} - 23}{18})$$

$$= 2.95 + 0.5(\frac{37.5 - 23}{18})$$

$$= 2.95 + 0.5(\frac{14.5}{18})$$

$$= 2.95 + 0.4$$

$$Q_3 = 3.35$$

d. The 55th percentile is calculated as follows:

$$P_N = L_N + C(\frac{\frac{N\sum f}{100} - CF_{bP_N}}{F_{P_N}})$$

$P_N = P_{55}$

$\frac{N\sum f}{100} = \frac{55\ x\ 50}{100}$ = 27.5. Hence the 55th percentile class is at the 27th and 28th position. This class is: 3.0 – 3.4

$L_N = L_{55}$ = Lower class boundary of the 55th percentile class = 2.95

$CF_{bP_N} = CF_{bP_{55}}$ = Cumulative frequency before the 55th percentile class = 23

$F_{P_N} = F_{P_{55}}$ = Frequency of the 55th percentile class = 18

C = Class width = 0.5

Hence, $$P_{55} = L_{55} + C(\frac{\frac{55\sum f}{100} - CF_{bP_{55}}}{F_{P_{55}}})$$

$$= 2.95 + 0.5\left(\frac{\frac{55 \, x \, 50}{100} - 23}{18}\right)$$

$$= 2.95 + 0.5\left(\frac{27.5 - 23}{18}\right)$$

$$= 2.95 + 0.5\left(\frac{4.5}{18}\right)$$

$= 2.95 + 0.13$

$P_{55} = 3.08$

Exercise

1. The following is the record of marks of 40 students in an examination:

34 74 92 58 46 76 73 23 66 70 57 43 53 39
50 37 82 29 54 77 67 19 18 96 15 55 41 29
33 52 22 81 77 81 58 27 20 55 49 96

Using class interval 11 – 20, 21 – 30, …, prepare a frequency table for the distribution. Hence calculate the:

a. median
b. lower quartile
c. upper quartile
d. interquartile range
e. quartile deviation/semi-interquartile range
f. 30th percentile
g. 68th percentile

2. The table below shows the distribution of marks scored by students in an examination.

Class interval	Frequency
10 – 14	1
15 – 19	3
20 – 24	8
25 – 29	11
30 – 34	7

35 – 39	9
40 – 44	10
45 – 49	5

From the data, calculate:

a. median

b. lower quartile

c. upper quartile

d. interquartile range

e. semi-interquartile range

f. 80^{th} percentile

g. the pass mark if 35% of the students passed

h. the pass mark if 15% the students should fail.

3. The table below shows the height of some flowers sold in a farm.

Mass	0.5 – 0.9	1.0 – 1.4	1.5 – 1.9	2.0 – 2.4	2.5 – 2.9	3.0 – 3.5
No of Items	4	15	12	9	7	3

From the table given above, estimate:

a. median

b. lower quartile

c. upper quartile

d. the 45^{th} percentile

e. pass mark if 90% of the students passed

4. The table below shows the distribution of marks scored by students in an test.

Class interval	Frequency
0 – 4	1
5 – 9	4
10 – 14	7
15 – 19	5
20 – 24	9
25 – 29	1

30 – 34	2
35 – 39	1

From the data, calculate:

a. median

b. lower quartile

c. upper quartile

d. interquartile range

e. semi-interquartile range

f. 60^{th} percentile

g. the pass mark if 10% of the students passed

5. The table below shows the weight in gram of some seeds found in some cocoa pods.

Mass	0 – 0.4	0.5 – 0.9	1.0 – 1.4	1.5 – 1.9	2.0 – 2.4	2.5 – 2.9
No of Items	9	21	16	22	28	4

From the table given above, estimate:

a. median

b. lower quartile

c. upper quartile

d. the 25^{th} percentile

e. pass mark if 68% of the students passed

CHAPTER 16
THE BASIC THEORY OF PROBABILITY

Probability is the likelihood of an event happening. Mathematically probability is given by:

$$\text{Probability} = \frac{\text{number of required outcome}}{\text{number of total or possible outcome}}$$

If the probability of an event happening is x, then the probability that it will not happen will be given by:

$$1 - x$$

Probability must lie between the values of 0 and 1. If an event cannot happen, then its probably is 0. If an event is certain to happen, then its probability is 1.

Mutually Exclusive Events

When there is no member/element common between two or more similar events, then we say they are mutually exclusive events. For example the event of odd numbers or even numbers are mutually exclusive. They are disjoint sets.

Addition Law of Probability

If two events are mutually exclusive, then the probability of one or the other happening is the sum of their individual probabilities.

Independent Events

When a die is thrown, and a coin is tossed, these two events have no effect on each other. Such events are called independent events

Product law of probability

If two events are independent, then the probability of both events happening is the product is the product (multiplication) of their individual probabilities.

CHAPTER 17
PROBABILITY ON SIMPLE EVENTS

Examples

1. The table below give the number of students in each age group in a class.

Age (Years)	12	13	14	15	16	17
number of students	6	3	10	4	2	5

If a student is chosen at random, find the probability that the student is:
(a) 13 years old
(b) 15 years old or less
(c) at least 16 years old
(d) most 13 years old
(e) not 17 years old

Solution

(a) Pr. (13 years old) $= \frac{\text{Number of students who are 13 years old}}{\text{Total number of students}}$

$= \frac{3}{30}$

$= \frac{1}{10}$ (when $\frac{3}{30}$ is express in its lowest term, it gives $\frac{1}{10}$)

(b) Pr. (15 years or less) = $\frac{\text{Students who are 15 years and below}}{\text{Total number of students}}$

$= \frac{4+10+3+6}{30}$

$= \frac{23}{30}$

(c) Pr. (At least 16 years old) = $\frac{\text{Students who are 16 years and above}}{\text{Total number of students}}$

$= \frac{2+5}{30}$

$= \frac{7}{30}$

(d) Pr. (At most 13 years) = $\frac{\text{Students who are 13 years and below}}{\text{Total number of students}}$

$= \frac{3+6}{30}$

$= \frac{9}{30}$

$= \frac{3}{10}$ (When expressed in its lowest term)

(e) Pr. (17 years old) = $\frac{\text{number of students who are 17 years old}}{\text{total number of students}}$

$= \frac{5}{30}$

$= \frac{1}{6}$

Therefore, Pr. (Not 17 years old) = 1 - Pr. (17 years old)

$= 1 - \frac{1}{6}$

$= \frac{5}{6}$

2. The probability that a seed will germinate is $\frac{2}{5}$. What is the probability that it will not germinate?

<u>Solution</u>

Pr. (It will germinate) = $\frac{2}{5}$

Pr. (It will not germinate) = $1 - \frac{3}{5}$

$= \frac{2}{5}$

3. A letter is chosen at random from the alphabet. Find the probability that it is one of the letters of the word: PROBABILITY.

<u>Solution</u>

In this case a letter should not be counted more than once. Avoiding repetition, the word can now be written as:

PROBALITY (i.e. 9 letters). Note that there are 26 letters of the alphabet.

Therefore, Pr. (one letter from PROBABILITY) = $\frac{9}{26}$

4. The probability that a boy gains admission into a higher institution is $\frac{3}{7}$. What is the probability that he does not gain admission into the institution?

Solution

Pr. (He gains admission) = $\frac{3}{7}$

Pr. (He does not gain admissions) = 1 - $\frac{3}{7}$

$= \frac{4}{7}$

5. Out of every 100 cars, 4 develop mechanical fault within 6 months of purchase. What is the probability of buying a car which will not develop a mechanical fault within 6 months of purchase?

Solution

Total number of cars is 100. Number of cars with fault within 6 months is 4. Number of cars without fault within an months of purchase is going is 96, (i.e. 100 - 4 = 96).

Therefore, Pr. (Buying a car that will not develop fault) = $\frac{\text{Number of cars without fault}}{\text{Total number of cars}}$

$= \frac{96}{100}$

$= \frac{24}{25}$ (In its lowest term after equal division by 4)

6. In Mr. Smith's extended family, the number of males is 16, while the number of females is 14. Find the probability that Mr. Smith has:
(a) a male child
(b) a female child

Solution
(a) Total number of family members = 16 + 14 = 30

Therefore, Pr. (a male child) = $\frac{\text{Family members who are males}}{\text{Total number of family members}}$

$= \frac{16}{30}$

$= \frac{8}{15}$

(b) Pr. (a female child) = $\frac{\text{Family members who are females}}{\text{Total number of family members}}$

$= \frac{14}{30}$

$= \frac{7}{15}$

7. A survey shows that 36% of all women take size 8 shoes. What is the probability that Khan's grandmother takes size 8 shoes?

Solution

Pr. (Khan's grandmother takes size 8 shoes) = $\frac{36}{100}$ (Note that the total percentage is always 100%)

$= \frac{9}{25}$ (In its lowest term)

8. In a secondary school, 46 out of every 50 students are at least 130cm tall. What is the probability that a student chosen at random from the school is less than 130cm tall?

Solution

Total number of students for the sample = 50

Number of students who are at least 130cm tall = 46

Number of students who are less than 130cm tall = 50 - 46 = 4

Therefore, Pr. (a student less than 130cm tall) = $\frac{\text{Number of students less than 130cm tall}}{\text{Total number of students in the sample}}$

$= \frac{4}{50}$

$= \frac{2}{25}$

9. A number is chosen at random between 1 and 16, both inclusive. What is the probability that it is:
(a) even
(b) prime
(c) odd or prime
(d) divisible by 4
(e) a perfect square or a perfect cube

Solution
(a) Total numbers in all from 1 to 16 = 16

The even numbers are 2, 4, 6, 8, 10, 12, 14, 16

Therefore the number of even numbers is 8

Hence Pr. (even number selected) = $\frac{\text{Number of even numbers}}{\text{Total numbers in all}}$

$= \frac{8}{16}$

$= \frac{1}{2}$

(b) The prime numbers are 2, 3, 5, 7, 11, 13

Therefore the number of prime numbers is 6

Hence Pr. (prime number selected) = $\frac{\text{Number of prime numbers}}{\text{Total numbers in all}}$

$= \frac{6}{16}$

$= \frac{3}{8}$

(c) The odd numbers are 1, 3, 5, 7, 9, 11, 13, 15

The prime numbers are 1, 3, 5, 7, 11, 13

Since OR in probability means addition, then we add all the odd and prime numbers together, but we must not count any number twice. This gives 1, 3, 5, 7, 9, 11, 13, 15, which is a total of 8 numbers.

Hence Pr. (odd or prime number selected) = $\frac{\text{Number of odd and even numbers}}{\text{Total numbers in all}}$

$= \frac{8}{16}$

$= \frac{1}{8}$

(d) The numbers divisible by 4 are 4, 8, 12, 16

This gives a total of 4 numbers

Hence Pr. (a number divisible by 4) = $\frac{\text{The four numbers divisible by 4}}{\text{Total numbers in all}}$

$= \frac{4}{16}$

$= \frac{1}{4}$

(e) The perfect square numbers are 1, 4, 9, 16

The perfect cube numbers are 1, 8

Since OR in probability means addition, then we add all the set of values above without counting any number twice. This gives 1, 4, 8, 9, 15, which is a total of 5 numbers.

Hence Pr. (perfect square or perfect cube selected) = $\frac{15}{16}$

10. A letter is chosen at random from the alphabet. Find the probability that it is:
(a) T
(b) E or P
(c) not B or G
(d) either D, J, N, U, W or Y
(e) one of the letters of the word REJECTED

Solution
(a) There are 26 letters of the alphabet, out of which there is 1 T.

Therefore, Pr. (T) = $\frac{\text{Number of Ts}}{\text{Total numbers of alphabets}}$

$= \frac{1}{26}$

(b) Pr. (E or P) = $\frac{\text{Number of Es and Ps}}{\text{Total numbers of alphabets}}$

$= \frac{6}{26}$

$= \frac{1}{13}$

(c) Pr. (B or G) = $\frac{2}{26} = \frac{1}{13}$

Therefore, Pr. (not B or G) = 1 - Pr. (B or G)

$= 1 - (\frac{6}{13})$

$= \frac{12}{13}$

(d) The letters D, J, N, U, W and Y makes a total of 6 letters.

Pr. (D, J, N, U, W or Y) = $\frac{6}{26}$

$= \frac{3}{13}$

(e) Writing the letters of the word REJECTED without repeating a letter gives REJCTD. This gives a total of 6 letters

Therefore Pr. (one of the letters of REJECTED) = $\frac{6}{26}$

$= \frac{3}{13}$

11. A letter is selected at random from the word PROBABILITY. What is the probability of selecting the letter B.

Solution

In this case the total letters of the word PROBABILITY gives 11. The repeated letters should be counted more than once since this is not a case of letter from the alphabet. In the 26 alphabet each letter appears once, that is why they are counted once. But in PROBABILITY (or other words that might be given) some letters appear more than once, hence they should be counted as many times as they appear.

In PROBABILITY, B appears 2 times.

Therefore, Pr. (selecting B) = $\frac{2}{11}$

Exercises

1. The table below give the number of students in each mark group in a class.

Mark	5	6	7	8	9	10
Number of students	3	6	2	4	1	4

If a student is chosen at random, find the probability that the student scored:
(a) 7 marks
(b) 6 marks or less
(c) at least 9 marks
(d) at most 8 marks
(e) 5 or 8 maks

2. The probability that a seed will germinate is $\frac{3}{4}$. What is the probability that it will not germinate?

3. A letter is chosen at random from the alphabet. Find the probability that it is one of the letters of the word: MATHEMATICS.

4. The probability that a man wins an election is $\frac{3}{5}$. What is the probability that he does not win.

5. Out of every 10 bulbs, 2 do not last long. What is the probability that a bulb will last long when lit?

6. In family, the number of males is 3, while the number of females is 2. Find the probability that another child born into the family is:
(a) a male child
(b) a female child

7. A survey shows that 44% of all women take size 7 shoes. What is the probability that a mother of two takes size 7 shoes?

8. In a secondary school, 30 out of every 100 students are at least 160cm tall. What is the probability that a student chosen at random from the school is less than 160cm tall?

9. A number is chosen at random between 1 and 20, both inclusive. What is the probability that it is:
(a) prime

(b) odd

(c) even or prime

(d) divisible by 3

(e) a number less than 10 or a perfect cube

10. A letter is chosen at random from the alphabet. Find the probability that it is:

(a) F

(b) M or Q or Y

(c) in the word COME

(d) either in the word BUT or in REMOVE

(e) one of the letters of the word SURPRISED

11. A letter is selected at random from the word RESPIRATION. What is the probability of selecting the letter I.

CHAPTER 18
PROBABILITY ON PACK OF PLAYING CARDS

A pack of playing cards contains 52 cards of 4 types. There are 13 clubs, 13 diamonds, 13 hearts and 13 spades. Each of the set of 13 cards contains Ace (A), 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack (J), Queen (Q), and King (K). This means that out of the 52 cards, each card is four in number, i.e. Aces are 4 in number, 1s are 4 in number, 2s are 4 in number, 3s are 4 in number, 4s are 4 in number, 5s are 4 in number, 6s are 4 in number, 7s are 4 in number, 8s are 4 in number, 9s are 4 in number, 10s are 4 in number, Jacks are 4 in number, Queens are 4 in number, and Kings are 4 in number. Clubs and spades are black, diamonds and hearts are red. This means that there are 26 black cards and 26 red cards. This also means that out of the 4 Aces cards, 2 are black and 2 are red. Out of the four cards that are 1s, two are black and two are red, out of the four cards that are 2, two are black and two are red, and so on.

Examples

1. A card is picked at random from a pack of playing cards. Find the probability of picking a spade.

Solution

There are 13 spades in a pack of playing cards.

Therefore, Pr. (picking a spade) = $\frac{\text{Number of Spades}}{\text{Total numbers of cards}}$

$= \frac{13}{52}$

$= \frac{1}{4}$ (In its lowest term)

2. A card is picked at random from a pack of playing cards. Find the probability of picking a red card.

Solution

There are 26 red cards in a pack of playing cards.

Therefore, Pr. (picking a red card) = $\frac{\text{Number of red cards}}{\text{Total numbers of cards}}$

$= \frac{26}{52}$

$= \frac{1}{2}$ (In its lowest term)

3. A card is picked at random from a pack of playing cards. Find the probability of picking a red 5.

<u>Solution</u>

There are 2 red 5 cards in a pack of playing cards.

Therefore, Pr. (picking a red 5) = $\frac{\text{Number of red 5}}{\text{Total numbers of cards}}$

$= \frac{2}{52}$

$= \frac{1}{26}$ (In its lowest term)

4. A card is picked at random from a pack of playing cards. Find the probability of picking a 3.

<u>Solution</u>

There are 4 cards that are 3 in a pack of playing cards.

Therefore, Pr. (picking a 3) = $\frac{\text{Number of cards that are 3}}{\text{Total numbers of cards}}$

$= \frac{4}{52}$

$= \frac{1}{13}$ (In its lowest term)

5. A card is picked at random from a pack of playing cards. Find the probability of picking a black Ace.

<u>Solution</u>

There are 2 cards that are black ace in a pack of playing cards.

Therefore, Pr. (picking a black ace) = $\frac{\text{Number of cards that are black ace}}{\text{Total numbers of cards}}$

$= \frac{2}{52}$

$= \frac{1}{26}$ (In its lowest term)

6. A card is picked at random from a pack of playing cards. Find the probability of picking a card that is not a Jack.

Solution

(a) There are 4 cards that are Jacks.

Therefore, Pr. (picking a Jack) = $\frac{\text{Number of jacks}}{\text{Total numbers of cards}}$

$= \frac{4}{52}$

$= \frac{1}{13}$

Hence, Pr. (picking a card that in not a Jack) = 1 - Pr. (picking a Jack)

$= 1 - \frac{1}{13}$

$= \frac{12}{13}$

7. A card is picked at random from a pack of playing cards. Find the probability of picking

(a) a black or red card

(b) a 2 or a 5

(c) either a heart or the king of spades

(d) a club or a red Queen

(e) a diamond or a 9

(f) a 6 or a black card

Solution

(a) There are 26 black cards and 26 red card

Since or in probability means plus, then we have to add the numbers. This gives a total of: 26 + 26 = 52

Therefore, Pr. (picking a black or red card) = $\frac{\text{Number of black and red cards}}{\text{Total numbers of cards}}$

$= \frac{52}{52}$

$= 1$

(b) There are 4 cards that are 2, and 4 cards that are 5. This gives a total of 8 cards.

Therefore, Pr. (picking a 2 or a 5) = $\frac{8}{52}$

$= \frac{2}{13}$

(c) There are 13 cards that are Hearts, and 1 king that is a spade. This gives a total of 14 cards.

Therefore, Pr. (picking either a heart or the king of spades) = $\frac{14}{52}$

$= \frac{7}{26}$

(d) There are 13 cards that are club, and 2 cards that are red Queen, (i.e. the Queen of hearts and the queen of diamond). This gives a total of 15 cards.

Therefore, Pr. (picking a club or a red Queen) = $\frac{15}{52}$

(e) There are 13 cards that are diamonds, and 4 cards that are 9. But one of the 9 is in diamond and has already been counted among the 13 diamonds. So it must not be counted twice. Hence we count the other three 9 (each from clubs, hearts and spades). This will give a total of 16 (13 + 3) cards.

Therefore, Pr. (picking a diamond or a 9) = $\frac{16}{52}$

$= \frac{4}{13}$

(f) There are 4 cards that are 6, and 26 cards that are black. But two of the 26 black cards are among the four cards that are 6, and these two black 6 cards have already been counted among the 26 black cards. So they must not be counted twice. Hence we count the other two 6 cards that are red. This will give a total of 28 (26 + 2) cards.

Therefore, Pr. (picking a 6 or a back card) = $\frac{28}{52}$

$= \frac{7}{13}$

8. A card is picked at random from a pack of playing cards and then replaced. A second card is picked. What is the probability of picking:
(a) a 3 and a 10
(b) a queen and an ace
(c) two kings
(d) two red cards
(e) two cards of different colour
(f) two cards of the same colours

Solution

In probability problems, when two items are selected, it is important to logically analyse the situation when solving the problem. This will help you to know if addition (use of OR) is involved or multiplication (use of AND) is involved. For example, for a queen and a king to be selected, it simply means that, either the queen is selected first and then the king, or the king is selected first and then the queen. When this logical analysis is understood, then most questions in probability become easy to solve.

(a) There are four cards that are 3, and four cards that are 10

Therefore, Pr. (picking a 3) = $\frac{4}{52}$

$= \frac{1}{13}$

Similarly, Pr. (picking a 10) = $\frac{4}{52}$

$= \frac{1}{13}$

Recall that "and" in probability means multiplication.

The probability of picking a 3 and a 10 means that:

Either the first is a 3 AND the second is a 10, OR the first is a 10 AND the second is a 3.

This can be calculated by putting x in place of AND and + in place of OR in the above statement as follows:

Pr. (picking a 3) x Pr. (picking a 10) + Pr. (picking a 10) x Pr. (picking a 3)

$= (\frac{1}{13} \times \frac{1}{13}) + (\frac{1}{13} \times \frac{1}{13})$

$= \frac{1}{169} + \frac{1}{169}$

$= \frac{2}{169}$

Therefore, Pr. (picking a 3 and a 10) = $\frac{2}{169}$

(b) There are 4 cards that are queen, and 4 cards that are ace

Therefore, Pr. (picking a queen) = $\frac{4}{52}$

$= \frac{1}{13}$

Similarly, Pr. (picking an ace) = $\frac{4}{52}$

$= \frac{1}{13}$

The probability of picking a queen and an ace means that:

Either you first pick a queen AND then an ace, OR you first pick an ace AND then a queen.

This can be calculated by putting x in place of AND and + in place of OR in the above statement as follows:

Pr. (picking a queen) x Pr. (picking an ace) + Pr. (picking an ace) x Pr. (picking a queen)

$= (\frac{1}{13} \times \frac{1}{13}) + (\frac{1}{13} \times \frac{1}{13})$

$= \frac{1}{169} + \frac{1}{169}$

$= \frac{1}{169}$

Therefore, Pr. (picking a queen and an ace) = $\frac{2}{169}$

(c) There are four cards that are King

Therefore, Pr. (picking a king) = $\frac{4}{52}$

$= \frac{1}{13}$

The probability of picking two kings means that:

The first is a king AND the second is a king

= Pr. (picking a king) x Pr. (picking a king)

= $\frac{1}{13}$ x $\frac{1}{13}$

= $\frac{1}{169}$

Therefore, Pr. (picking two kings) = $\frac{1}{169}$

(d) There are 26 cards that are red

Therefore, Pr. (picking a red card) = $\frac{26}{52}$

= $\frac{1}{2}$

The probability of picking two red cards means that:

The first is a red card AND the second is a red card

= Pr. (picking a red card) x Pr. (picking a red card)

= $\frac{1}{2}$ x $\frac{1}{2}$

= $\frac{1}{4}$

Therefore, Pr. (picking two red cards) = $\frac{1}{4}$

(e) There are two colours of cards, red and black.

Therefore, Pr. (picking a red card) = $\frac{1}{2}$ (i.e from $\frac{26}{52}$ since there are 26 red cards)

Similarly, Pr. (picking a black card) = $\frac{1}{2}$ (i.e from $\frac{26}{52}$ since there are also 26 black cards)

The probability of picking two cards of different colours means that:

Either the first is a black card AND the second is a red card, OR the first is a red card AND the second is a black card.

This can be calculated by putting x in place of AND and + in place of OR in the above statement as follows:

Pr. (picking a black card) x Pr. (picking a red card) + Pr. (picking a red card) x Pr. (picking a black card)

$= (\frac{1}{2} \times \frac{1}{2}) + (\frac{1}{2} \times \frac{1}{2})$

$= \frac{1}{4} + \frac{1}{4}$

$= \frac{2}{4}$

$= \frac{1}{2}$

Therefore, Pr. (picking two cards of different colours) = $\frac{1}{2}$

(f) Pr. (picking two cards of the same colours) = 1 - Pr. (picking two cards of different colours)

$= 1 - \frac{1}{2}$

$= \frac{1}{2}$

Note that this can also be solved by using the logical process which is:

Either the first is red AND the second is red OR the first is black AND the second is black. This will also give $\frac{1}{2}$

9. Two cards are picked at random one after the other without replacement from a pack of playing cards. What is the probability of picking:
(a) a 5 and a 7
(b) a king and a jack
(c) two aces
(d) two diamond cards
(e) two black cards
(f) a red and a black card
(g) two cards of the same colours

Solution

This problem involves picking a card without replacement. This means that when one card is picked out, the total number of cards remaining in the pack become reduced to 51. That number of that particular type of card also reduces by 1.

(a) There are four cards that are 5. There are also four cards that are 7.

Hence the probability of picking a 5 and a 7 means that:

Either first picking a 5 AND then a 7, OR first picking a 7 AND then a 5.

Now, let us calculate each of the probabilities as follows:

Pr. (first card is a 5) = $\frac{4}{52}$ (There are four cards that are 5)

= $\frac{1}{13}$ (In its lowest term)

We now have 51 cards left in the pack.

Therefore, Pr. (second card is a 7) = $\frac{4}{51}$ (There are four cards that are 7, and a total of 51 cards remaining in the pack)

Or,

Pr. (first card is a 7) = $\frac{4}{52}$ (There are four cards that are 7)

= $\frac{1}{13}$ (In its lowest term)

We now have 51 cards left in the pack.

Therefore, Pr. (second card is a 5) = $\frac{4}{51}$ (There are four cards that are 5, and a total of 51 cards remaining in the pack)

Hence the probability of picking a 5 and a 7 means that:

Either first picking a 5 AND then a 7, OR first picking a 7 AND then a 5. Which is computed as:

Pr. (picking a 5 and a 7) = Pr. (first card is a 5) x Pr. (second card is a 7) + Pr. (first card is a 7) x Pr. (second card is a 5)

$$= \left(\frac{1}{13} \times \frac{4}{51}\right) + \left(\frac{1}{13} \times \frac{4}{51}\right)$$

$$= \frac{4}{663} + \frac{4}{663}$$

$$= \frac{8}{663}$$

(b) There are four cards that are kings. There are also four cards that are jacks.

Now, let us calculate each of the probabilities as follows:

Pr. (first card is a king) = $\frac{4}{52}$ (There are four cards that are kings)

$= \frac{1}{13}$ (In its lowest term)

We now have 51 cards left in the pack.

Therefore, Pr. (second card is a jack) = $\frac{4}{51}$ (There are four cards that are jack, and a total of 51 cards remaining in the pack)

Or,

Pr. (first card is a jack) = $\frac{4}{52}$ (There are four cards that are jack)

$= \frac{1}{13}$ (In its lowest term)

We now have 51 cards left in the pack.

Therefore, Pr. (second card is a king) = $\frac{4}{51}$ (There are four cards that are king, and a total of 51 cards remaining in the pack)

Hence the probability of picking a king and a jack means that:

Either first picking a king AND then a jack, OR first picking a jack AND then a king. This is computed as:

Pr. (picking a king and a queen) = Pr. (first card is a king) x Pr. (second card is a jack) + Pr. (first card is a jack) x Pr. (second card is a king)

$$= \left(\frac{1}{13} \times \frac{4}{51}\right) + \left(\frac{1}{13} \times \frac{4}{51}\right)$$

$$= \frac{4}{663} + \frac{4}{663}$$

$= \frac{8}{663}$

(c) There are 4 cards that are aces.

Hence the probability of picking two aces means that:

The first is an ace, and the second is an ace.

Now, let us calculate each of the probabilities as follows:

Pr. (first card is an ace) = $\frac{4}{52}$ (There are 4 cards that are aces)

$= \frac{1}{13}$ (In its lowest term)

We now have 3 aces left in the pack, and a total of 51 cards left in the pack.

Therefore, Pr. (second card is an ace) = $\frac{3}{51}$

Hence the probability of picking two aces is given by:

Pr. (picking two aces) = Pr. (first card is an ace) x Pr. (second card is an ace)

$= \frac{1}{13} \times \frac{3}{51}$

$= \frac{3}{663}$

(d) There are 13 cards that are diamonds.

Hence the probability of picking two diamonds means that:

The first is a diamond, and the second is a diamond.

Now, let us calculate each of the probabilities as follows:

Pr. (first card is a diamond) = $\frac{13}{52}$ (There are 13 cards that are diamonds)

$= \frac{1}{4}$ (In its lowest term)

We now have 12 diamonds left in the pack, and a total of 51 cards left in the pack.

Therefore, Pr. (second card is a diamond) = $\frac{1}{13}$

$= \frac{4}{17}$ (In its lowest term)

Hence the probability of picking two diamonds is given by:

Pr. (picking two diamonds) = Pr. (first card is a diamond) x Pr. (second card is a diamond)

$= \frac{1}{4} \times \frac{4}{17}$

$= \frac{4}{68}$

$= \frac{1}{17}$ (In its lowest term)

(e) There are 26 black cards.

Hence the probability of picking two black cards means that:

The first is a black card, and the second is a black card.

Now, let us calculate each of the probabilities as follows:

Pr. (first card is a black card) = $\frac{26}{52}$

$= \frac{1}{2}$ (In its lowest term)

We now have 25 black cards left in the pack, and a total of 51 cards left in the pack.

Therefore, Pr. (second card is a black card) = $\frac{25}{51}$

Hence the probability of picking two black cards is given by:

Pr. (picking two black cards) = Pr. (first card is a black card) x Pr. (second card is a black card)

$= \frac{1}{2} \times \frac{25}{51}$

$= \frac{25}{102}$

(f) The logical explanation for this situation is that:

Either the first card is red AND the second is black OR the first card is black and the second is red.

There are 26 red cards and also 26 black cards.

Now, let us calculate each of the probabilities as follows:

Pr. (first card is a red card) = $\frac{26}{52}$

$= \frac{1}{2}$ (In its lowest term)

We now have 51 cards left in the pack.

Therefore, Pr. (second card is a black card) = $\frac{26}{51}$ (There are 26 black cards, and a total of 51 cards remaining in the pack)

Or,

Pr. (first card is a black card) = $\frac{26}{52}$

$= \frac{1}{2}$ (In its lowest term)

We now have 51 cards left in the pack.

Therefore, Pr. (second card is a red card) = $\frac{26}{51}$ (There are 26 red cards, and a total of 51 cards remaining in the pack)

Hence the probability of picking a red card and a black card means that:

Either first picking a red card AND then a black card, OR first picking a black card AND then a red card. This is computed as:

Pr. (picking a red and black cards) = Pr. (first card is a red card) x Pr. (second card is a black card) + Pr. (first card is a black card) x Pr. (second card is a red card)

$$= (\frac{1}{2} \times \frac{26}{51}) + (\frac{1}{2} \times \frac{26}{51})$$

$$= \frac{26}{102} + \frac{26}{102}$$

$$= \frac{52}{102}$$

$$= \frac{26}{51}$$

(g) The logical explanation for this situation is that:

Either the first card is red AND the second is red OR the first card is black and the second is black.

There are 26 red cards and also 26 black cards.

Now, let us calculate each of the probabilities as follows:

Pr. (first card is a red card) = $\frac{26}{52}$

$= \frac{1}{2}$ (In its lowest term)

We now have 25 red cards left and a total of 51 cards left in the pack.

Therefore, Pr. (second card is a red card) = $\frac{25}{51}$

Or,

Pr. (first card is a black card) = $\frac{26}{102}$

$= \frac{1}{2}$ (In its lowest term)

We now have 25 black cards left and a total of 51 cards left in the pack.

Therefore, Pr. (second card is a black card) = $\frac{25}{51}$

Hence the probability of picking two cards of the same colour means that:

Either picking a red card AND then another red card, OR picking a black card AND then another black card. Which is computed as:

Pr. (picking two cards of the same colour) = Pr. (first card is a red card) x Pr. (second card is a red card) + Pr. (first card is a black card) x Pr. (second card is a black card)

$$= (\frac{1}{2} \times \frac{25}{51}) + (\frac{1}{2} \times \frac{25}{51})$$

$$= \frac{25}{102} + \frac{25}{102}$$

$$= \frac{50}{102}$$

$$= \frac{25}{51}$$

Alternatively, this question can also be solved as follows:

Recall that question (f) above gives the probability of picking a red and a black card. This also means the probability of picking two cards of different colours.

Hence the probability of picking two cards of different colours as given in (f) above = $\frac{26}{51}$

Therefore, Pr. (picking two cards of the same colour) = 1 - Pr. (picking two cards of different colours) (Note that they are opposite statements)

$$= 1 - \frac{26}{51}$$

$$= \frac{51-26}{51}$$

$$= \frac{25}{51}$$ (As obtained before)

10. If three cards are picked from a pack of playing cards with replacement, what is the probability if getting:
(a) at least two clubs
(b) at most two clubs

<u>Solution</u>

I am going to be using a special type of tree diagram without actually drawing the diagram.

Now, the total outcome in a selection of three items involving two events (i.e. a club or not a club) is given by:

$$2^n,$$

where n is the number of selection made.

In the question, n = 3, since three cards were picked.

Hence total outcome = $2^3 = 2 \times 2 \times 2$

$= 8.$

Now, in order to write out the outcomes, let us use the letter C to represent a club and letter N to represent not a club. Note that in tree diagrams like this, only two letters should be used in writing the outcomes since the question involves the picking of only one type of item (club).

Hence the outcome is written as follows:

(CCC), (CCN), (CNC), (CNN), (NCC), (NCN), (NNC), (NNN)

Note that there are 8 ways of arranging the two letters in the brackets. There is no fast rule in carrying out the arrangement. You just have to make sure that no two brackets have the same arrangement of the letters. Also make sure the number of brackets is complete.

(a) In order to determine the probability of getting at least two clubs, we need to compute the probabilities of the brackets that contain at least 2 clubs. They are, (CCC), (CCN), (CNC), and (NCC). Note that at least two, means two and above, (i.e. two and three clubs in this case).

Hence the probability of getting at least two clubs = (CCC) or (CCN) or (CNC) or (NCC)

Now, let us compute each of the probabilities.

There are 13 clubs in a pack of cards, and there are 39 cards that are not club. Note that this is a case of with replacement, which means that the total number of cards in the pack is always complete. Hence:

(CCC) = Pr. (first card is a club) x Pr. (second card is a club) x Pr. (third card is a club)

$= \frac{13}{52} \times \frac{13}{52} \times \frac{13}{52}$

$= \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4}$

$= \frac{1}{64}$

(CCN) = Pr. (first card is a club) x Pr. (second card is a club) x Pr. (third card is not a club)

$= \frac{13}{52} \times \frac{13}{52} \times \frac{39}{52}$ (Note that there are 39 cards that are not club)

$= \frac{1}{4} \times \frac{1}{4} \times \frac{3}{4}$

$= \frac{3}{64}$

(CNC) = Pr. (first card is a club) x Pr. (second card is not a club) x Pr. (third card is a club)

$$= \frac{13}{52} \text{ x } \frac{39}{52} \text{ x } \frac{13}{52}$$

$$= \frac{1}{4} \text{ x } \frac{3}{4} \text{ x } \frac{1}{4}$$

$$= \frac{3}{64}$$

(NCC) = Pr. (first card is not a club) x Pr. (second card is a club) x Pr. (third card is a club)

$$= \frac{39}{52} \text{ x } \frac{13}{52} \text{ x } \frac{13}{52}$$

$$= \frac{3}{4} \text{ x } \frac{1}{4} \text{ x } \frac{1}{4}$$

$$= \frac{3}{64}$$

Therefore, Pr. (getting at least two clubs) = (CCC) or (CCN) or (CNC) or (NCC)

= (CCC) + (CCN) + (CNC) + (NCC)

$$= \frac{1}{64} + \frac{3}{64} + \frac{3}{64} + \frac{3}{64}$$

$$= \frac{10}{64}$$

$$= \frac{5}{32}$$

(b) In order to determine the probability of getting at most two clubs, we need to compute the probabilities of the brackets that contain at most 2 clubs. From the outcome brackets given above, the ones that contain at most two clubs are, (CCN), (CNC), (CNN), (NCC), (NCN), (NNC), (NNN). Note that at most two, means two and below, (i.e. two, one and zero clubs in this case).

Hence the probability of getting at most two clubs = (CCN) or (CNC) or (CNN) or (NCC) or (NCN) or (NNC) or (NNN)

Now, let us compute each of the probabilities. Hence:

(CCN) = $\frac{3}{64}$ (As calculated in (a) above)

(CNC) = $\frac{3}{64}$ (As calculated in (a) above)

(CNN) = Pr. (first card is a club) x Pr. (second card is not a club) x Pr. (third card is not a club)

$= \frac{13}{52} \times \frac{39}{52} \times \frac{39}{52}$

$= \frac{1}{4} \times \frac{3}{4} \times \frac{3}{4})$

$= \frac{9}{64}$

(NCC) = $\frac{3}{64}$ (As calculated in (a) above)

(NCN) = Pr. (first card is not a club) x Pr. (second card is a club) x Pr. (third card is not a club)

$= \frac{39}{52} \times \frac{13}{52} \times \frac{39}{52}$

$= \frac{3}{4} \times \frac{1}{4} \times \frac{3}{4}$

$= \frac{9}{64}$

(NNC) = Pr. (first card is not a club) x Pr. (second card is not a club) x Pr. (third card is a club)

$= \frac{39}{52} \times \frac{39}{52} \times \frac{13}{52}$

$= \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4}$

$= \frac{9}{64}$

(NNN) = Pr. (first card is not a club) x Pr. (second card is not a club) x Pr. (third card is not a club)

$= \frac{39}{52} \times \frac{39}{52} \times \frac{39}{52}$

$= \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}$

$= \frac{27}{64}$

Therefore, Pr. (getting at most two clubs) = (CCN) or (CNC) or (CNN) or (NCC) or (NCN) or (NNC) or (NNN)

= (CCN) + (CNC) + (CNN) + (NCC) + (NCN) + (NNC) + (NNN)

$= \frac{3}{64} + \frac{3}{64} + \frac{9}{64} + \frac{3}{64} + \frac{9}{64} + \frac{9}{64} + \frac{27}{64}$

$= \frac{63}{64}$

Alternatively, a shorter method of solving this problem is as follows:

Pr. (getting at most two clubs) = 1 - Pr. (getting three clubs) (This is because the only item not in (b) is CCC. All 8 outcomes in brackets make a total probability of 1, but 7 of the outcomes are in (b), hence 1 minus the outcome not in (b) gives (b). This can also be expressed in fraction as, (b) = $\frac{7}{8} = \frac{8}{8} - \frac{1}{8}$). Therefore:

Pr. (getting at most two clubs) = 1 - Pr. (getting three clubs)

= 1 - CCC

$= 1 - \frac{1}{64}$ (Note that CCC computed in (a) = $\frac{1}{64}$)

$= \frac{63}{64}$ (As obtained before)

11. If three cards are chosen from a pack of playing cards without replacement, what is the probability of getting:
(a) at least two diamonds
(b) at most one diamond?

<u>Solution</u>

Now, in order to write out the outcomes, let us use the letter D to represent a diamond and letter N to represent not a diamond.

Hence the outcomes are written as follows:

(DDD), (DDN), (DND), (DNN), (NDD), (NDN), (NND), (NNN)

(a) In order to determine the probability of getting at least two diamonds, we need to compute the probabilities of the brackets that contain at least 2 diamonds. They are, (DDD), (DDN), (DND), and (NDD).

Hence the probability of getting at least two diamonds = (DDD) or (DDN) or (DND) or (NDD)

Now, let us compute each of the probabilities.

There are 13 diamonds in a pack of cards, and there are 39 cards that are not diamonds. Note that this is a case of without replacement, which means that after each selection, both the total number of cards left and the number of the particular card picked, are reduced by 1. Hence:

(DDD) = Pr. (first card is a diamond) x Pr. (second card is a diamond) x Pr. (third card is a diamond)

$= \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50}$ (Note that the number of diamond and the total number of card left, keep reducing by 1 after each selection)

$= \frac{1}{4} \times \frac{12}{51} \times \frac{11}{50}$

$= \frac{132}{10200}$

$= \frac{11}{850}$ (In its lowest term, after equal division by 12)

(DDN) = Pr. (first card is a diamond) x Pr. (second card is a diamond) x Pr. (third card is not a diamond)

$= \frac{13}{52} \times \frac{12}{51} \times \frac{39}{50}$ (Note that there are 39 cards that are not diamond)

$= \frac{1}{4} \times \frac{4}{17} \times \frac{39}{50}$

$= \frac{156}{3400}$

$= \frac{39}{850}$ (After equal division by 4)

(DND) = Pr. (first card is a diamond) x Pr. (second card is not a diamond) x Pr. (third card is a diamond)

$= \frac{13}{52} \times \frac{39}{51} \times \frac{12}{50}$

$= \frac{1}{4} \times \frac{13}{17} \times \frac{6}{25}$

$= \frac{78}{1700}$

$= \frac{39}{850}$

(NDD) = Pr. (first card is not a diamond) x Pr. (second card is a diamond) x Pr. (third card is a diamond)

$= \frac{39}{52} \times \frac{13}{51} \times \frac{12}{50}$

$= \frac{3}{4} \times \frac{13}{51} \times \frac{6}{25}$

$= \frac{234}{5100}$

$= \frac{39}{850}$

Therefore, Pr. (getting at least two diamonds) = (DDD) or (DDN) or (DND) or (NDD)

= (DDD) + (DDN) + (DND) + (NDD)

$= \frac{11}{850} + \frac{39}{850} + \frac{39}{850} + \frac{39}{850}$

$= \frac{128}{850}$

$= \frac{64}{425}$

(b) In order to determine the probability of getting at most one diamond, we need to compute the probabilities of the brackets that contain at most one diamond. From the outcome brackets given above, the ones that contain at most one diamond are, (DNN), (NDN), (NND), (NNN). Note that at most one, means one and below, (i.e. one and zero diamond in this case).

Hence the probability of getting at most one diamond = (DNN) + (NDN) + (NND) + (NNN)

Now, let us compute each of the probabilities. Hence:

(DNN) = Pr. (first card is a diamond) x Pr. (second card is not a diamond) x Pr. (third card is not a diamond)

$= \frac{13}{52} \times \frac{39}{51} \times \frac{38}{50}$

$= \frac{1}{4} \times \frac{13}{17} \times \frac{19}{25}$

$= \frac{247}{1700}$

(NDN) = Pr. (first card is not a diamond) x Pr. (second card is a diamond) x Pr. (third card is not a diamond)

$= \frac{39}{52} \times \frac{13}{51} \times \frac{38}{50}$

$= \frac{3}{4} \times \frac{13}{51} \times \frac{19}{25}$

$= \frac{741}{5100}$

$= \frac{247}{1700}$ (In its lowest term after equal division by 3)

(NND) = Pr. (first card is not a diamond) x Pr. (second card is not a diamond) x Pr. (third card is a diamond)

$= \frac{39}{52} \times \frac{38}{51} \times \frac{13}{50}$

$= \frac{3}{4} \times \frac{38}{51} \times \frac{13}{50}$

$= \frac{1482}{10200}$

$= \frac{247}{1700}$ (After equal division by 6)

(NNN) = Pr. (first card is not a diamond) x Pr. (second card is not a diamond) x Pr. (third card is not a diamond)

$= \frac{39}{52} \times \frac{38}{51} \times \frac{37}{50}$

$= \frac{3}{4} \times \frac{38}{51} \times \frac{37}{50}$

$= \frac{4218}{10200}$

$= \frac{703}{1700}$ (After equal division by 6)

Therefore, Pr. (getting at most one diamond) = (DNN) or (NDN) or (NND) or (NNN)

= (DNN) + (NDN) + (NND) + (NNN)

$$= \frac{247}{1700} + \frac{247}{1700} + \frac{247}{1700} + \frac{703}{1700}$$

$$= \frac{1444}{1700}$$

$$= \frac{361}{425}$$

Exercise

1. A card is picked at random from a pack of playing cards. Find the probability of picking a jack.

2. A card is picked at random from a pack of playing cards. Find the probability of picking a black 4.

3. A card is picked at random from a pack of playing cards. Find the probability of picking a red king.

4. A card is picked at random from a pack of playing cards. Find the probability of picking a either a black or red card.

5. A card is picked at random from a pack of playing cards. Find the probability of picking a black Queen.

6. A card is picked at random from a pack of playing cards. Find the probability of picking a card that is not an Ace.

7. A card is picked at random from a pack of playing cards. Find the probability of picking
(a) a queen or a king
(b) a 3 or a 9
(c) either a jack or the queen of diamonds
(d) a spade or a black 7
(e) a club or a red king
(f) a 2 or a red card

8. A card is picked at random from a pack of playing cards and then replaced. A second card is picked. What is the probability of picking:
(a) an 8 and a 5
(b) a black card and a 4
(c) two cards between 2 and 9 that have odd numbers
(d) two black cards
(e) two cards with the same number on them
(f) two cards with different number on them

9. Two cards are picked at random one after the other without replacement from a pack of playing cards. What is the probability of picking:

(a) a 4 and an ace

(b) a 2 and a 7

(c) two 8s

(d) two clubs

(e) two red cards

(f) a club and a diamond

(g) two cards that are queens

10. If three cards are picked from a pack of playing cards with replacement, what is the probability if getting:

(a) at least two 9s

(b) at most two 9s

11. If three cards are chosen from a pack of playing cards without replacement, what is the probability of getting:

(a) at least two kings

(b) at most one king?

CHAPTER 19
PROBABILITY ON TOSSING OF COINS

When a coin is tossed, the outcome can either be a head or a tail. However when two or more coins are tossed, the total outcome is obtained from 2^n, where n is the number of times the coin is tossed, or the number of coins tossed together.

Note that 'head' is the part of the coin that shows the person drawn on the coin, while the opposite side of the coin is called the 'tail'

Examples

1. A fair coin is tossed. What is the probability of getting:
(a) a head
(b) a tail

Solution

(a) There are only two possible outcomes. Head or tail.

Therefore, Pr. (getting a head) = $\frac{\text{Number of heads}}{\text{Total outcomes}}$

$= \frac{1}{2}$

(b) Pr. (getting a tail) = $\frac{\text{Number of tails}}{\text{Total outcomes}}$

$= \frac{1}{2}$

2. A coin is tossed two times. What is the probability of getting:
(a) a head and a tail
(b) at least a tail
(c) two heads
(d) two tails
(e) a head on the first toss, and a tail on the second toss.

Solution

The outcomes are written by using H for head and T for tail. The total number of outcomes will be 2^2 = 4 (i.e. from 2^n, and n = 2 in this case)

The outcomes are: (HH), (HT), (TH), (TT).

(a) The outcomes with head and tail are (HT) and (TH). This gives 2 outcomes.

Therefore, Pr. (getting a head and tail) = $\frac{\text{Number of outcome with head and tail}}{\text{Total outcomes}}$

$= \frac{2}{4}$

$= \frac{1}{2}$

(b) The outcomes with at least a tail are (HT), (TH) and (TT). This gives 3 outcomes.

Therefore, Pr. (getting at least a tail) = $\frac{\text{Number of outcomes with at least a tail}}{\text{Total number of outcomes}}$

$= \frac{3}{4}$

(c) The outcome with two heads is (HH). This gives 1 outcome.

Therefore, Pr. (getting two heads) = $\frac{\text{Number of outcomes with heads}}{\text{Total number of outcomes}}$

$= \frac{1}{4}$

(d) The outcome with two tails is (TT). This gives 1 outcome.

Therefore, Pr. (getting two tails) = $\frac{\text{Number of outcomes with two tails}}{\text{Total number of outcomes}}$

$= \frac{1}{4}$

(e) The outcome with a head on the first toss, and a tail on the second toss is (HT). This gives 1 outcome

Therefore, Pr. (getting a head on the first toss, and a tail on the second toss) = $\frac{1}{4}$

3. A coin is tossed three times. What is the probability of getting:
(a) two heads and one tail
(b) at least one head
(c) three tails

(d) at least two heads
(e) a tail, a head and a tail

Solution
(a) The total number of outcomes will be $2^3 = 8$ (i.e. from 2^n, and n = 3 in this case)

The outcomes are: (HHH), (HTH), (HTT), (HHT), (THH), (THT), (TTH), (TTT). This gives a total of 8 outcomes

(a) The outcomes with two heads and one tail are (HTH), (HHT) and (THH). This gives 3 outcomes.

Therefore, Pr. (getting two heads and one tail) = $\frac{\text{Number of outcomes with two heads and one tail}}{\text{Total number of outcomes}}$

$= \frac{3}{8}$

(b) The outcomes with at least one head are (HHH), (HTH), (HTT), (HHT), (THH), (THT) and (TTH). This gives 7 outcomes.

Therefore, Pr. (getting at least one head) = $\frac{\text{Number of outcomes with at least one head}}{\text{Total number of outcomes}}$

$= \frac{7}{8}$

(c) The outcome with three tails is (TTT). This gives 1 outcome.

Therefore, Pr. (getting three tails) = $\frac{1}{8}$

(d) The outcomes with at least two heads are (HHH), (HTH), (HHT) and (THH). This gives 4 outcomes.

Therefore, Pr. (getting at least two heads) = $\frac{\text{Number of outcomes with at least two heads}}{\text{Total number of outcomes}}$

$= \frac{4}{8}$

$= \frac{1}{2}$

(e) The outcome with a tail, a head and a tail is (THT). This is 1 outcome

Hence, Pr. (getting a tail, a head and a tail) = $\frac{1}{8}$

4. Four coins are tossed together. Find the probability of getting:
(a) two heads and two tails
(b) four tails
(c) at least three heads
(d) at least two heads
(e) one head

<u>Solution</u>

The total number of outcomes will be $2^4 = 16$ (i.e. from 2^n, and n = 4 in this case)

The outcomes are: (HHHH), (HHHT), (HHTT), (HTTT), (THHH), (TTHH), (TTTH), (THTH), (HTHT), (HHTH), (THHT), HTTH), (TTHT), (THTT), (HTHH), (TTTT). This gives a total of 16 outcomes.

(a) The outcomes with two heads and two tails are (HHTT), (TTHH), (THTH), (HTHT), (THHT), (HTTH). This gives 6 outcomes.

Therefore, Pr. (getting two heads and two tails) = $\frac{\text{Number of outcomes with two heads and two tails}}{\text{Total number of outcomes}}$

$$= \frac{6}{16}$$

$$= \frac{3}{8}$$

(b) The outcome with four tails is (TTTT). This gives 1 outcomes.

Therefore, Pr. (getting four tails) = $\frac{1}{16}$

(c) The outcomes with at least three heads are (HHHH), (HHHT), (THHH), (HHTH), (HTHH). This gives 5 outcomes.

Therefore, Pr. (getting at least three heads) = $\frac{\text{Number of outcomes with at least three heads}}{\text{Total number of outcomes}}$

$$= \frac{5}{16}$$

(d) The outcomes with at least two heads are (HHHH), (HHHT), (HHTT), (THHH), (TTHH), (THTH), (HTHT) (HHTH), (THHT), (HTTH), (HTHH). This gives 11 outcomes.

Therefore, Pr. (getting at least two heads) = $\frac{\text{Number of outcomes with at least two heads}}{\text{Total number of outcomes}}$

$$= \frac{11}{16}$$

(e) The outcomes with one head are, (HTTT), (TTTH), (TTHT), (THTT), . This gives 4 outcomes.

Therefore, Pr. (getting one head) = $\frac{\text{Number of outcomes with one head}}{\text{Total number of outcomes}}$

$$= \frac{4}{16}$$

$$= \frac{1}{4}$$

(5) A coin is tossed five times. Find the probability of getting at least one tail.

Solution

The total number of outcomes will be $2^5 = 32$

The only outcome without a tail is (HHHHH). This is an outcome of 1

Pr. (getting no tail, i.e. all head) = $\frac{1}{32}$

Therefore, Pr. (getting at least one tail) = 1 - Pr. (getting no tail)

$$= 1 - \frac{1}{32}$$

$$= \frac{31}{32}$$

Exercises

1. A fair coin is tossed. What is the probability of getting:
(a) a tail
(b) a head
(c) a tail or a head

2. A coin is tossed two times. What is the probability of getting:
(a) a tail and then a head
(b) at least a head
(c) two tails
(d) at least a tail
(e) a head on the first toss, and a tail on the second toss.

3. Three coins are tossed. What is the probability of getting:
(a) three heads
(b) at least one tail
(c) a head, a tail and then a head
(d) at least one head
(e) at least two heads
(f) at most two tails

4. Four coins are tossed together. Find the probability of getting:
(a) at least one head
(b) four heads
(c) at least two heads
(d) at most three tails
(e) two heads

(5) A coin is tossed five times. Find the probability of getting at least one head.

CHAPTER 20
PROBABILITY ON THROWING OF DICE

Examples

1. A fair die is thrown. Find the probability of getting:
(a) a 2
(b) a 5
(c) a 7
(d) a 4 or a 5
(e) a number less than 4
(f) an odd number

Solution
Note that a die has six faces numbered 1 to 6. That means that each number appears once.

(a) Pr. (getting a 2) = $\frac{\text{Number of faces having 2}}{\text{total number of faces}}$

$= \frac{1}{6}$

(b) Pr. (getting a 5) = $\frac{\text{Number of faces having 5}}{\text{total number of faces}}$

$= \frac{1}{6}$

(c) Pr. (getting a 7) = $\frac{\text{Number of faces having 7}}{\text{total number of faces}}$

$= \frac{0}{6}$

= 0 (This is a case of an impossible event)

(d) Pr. (getting a 4 or a 5) = $\frac{\text{Number of faces having 4 and having 5}}{\text{total number of faces}}$

$= \frac{2}{6}$

$= \frac{1}{3}$

(e) Pr. (getting a number less than 4) = $\frac{\text{Number of faces having numbers less than 4}}{\text{total number of faces}}$

$= \frac{3}{6}$ (Note that the faces with numbers less than 4 are 3, 2, and 1. This makes a total of 3 faces)

$= \frac{1}{2}$

(f) Pr. (getting an odd number) = $\frac{\text{Number of faces having odd numbers}}{\text{total number of faces}}$

$= \frac{3}{6}$ (Faces with odd numbers are 1, 3 and 5, i.e. three faces)

$= \frac{1}{2}$

2. A fair die is rolled once. What is the probability of getting:
(a) a number divisible by 3
(b) a multiple of 2
(c) at least 5
(d) at most 2
(e) a prime number or an even number
(f) either a number greater that 2 or a multiple of 4

Solution

(a) Pr. (getting a number divisible by 3) = $\frac{\text{Number of faces having numbers divis ible by 3}}{\text{total number of faces}}$

$= \frac{2}{6}$ (Faces with numbers divisible by 3 are 3 and 6, i.e. 2 faces)

$= \frac{1}{3}$

(b) Pr. (getting a multiple of 2) = $\frac{\text{Number of faces having numbers that are multiple of 2}}{\text{total number of faces}}$

$= \frac{3}{6}$ (Faces with numbers that are multiple of 2 are 2, 4 and 6, i.e. 3 faces)

$= \frac{1}{2}$

(c) Pr. (getting at least 5) = $\frac{\text{Number of faces having number s that are at least 5}}{\text{Total number of faces}}$

$= \frac{2}{6}$ (Faces with numbers that are at least 5 are 5 and 6, i.e. 2 faces)

$= \frac{1}{3}$

(d) Pr. (getting at most 2) = $\frac{\text{Number of faces having numbers that are at least 2}}{\text{Total number of faces}}$

$= \frac{2}{6}$ (Faces with numbers that are at most 2 are 1 and 2, i.e. 2 faces)

$= \frac{1}{3}$

(e) Pr. (getting a prime number or an even number) =

$\frac{\text{Number of faces having prime numbers and even numbers}}{\text{Total number of faces}}$

$= \frac{5}{6}$ (Faces with prime numbers are 2, 3, and 5. Faces with even numbers are 2, 4, 6. This will give a total of 5 faces because 2 which is both a prime and even number should be counted once)

Therefore, Pr. (getting a prime number or an even number) = $\frac{5}{6}$

(f) Pr. (getting either a number greater that 2 or a multiple of 4) =

$\frac{\begin{array}{l}\text{Number of faces having numbers greater}\\ \text{than 2 and numbers that are multiple of 4}\end{array}}{\text{Total number of faces}}$

$= \frac{4}{6}$ (Faces with numbers greater than 2 are 3, 4, 5 and 6. Faces with multiple of 4 is 4. This will give a total of 4 faces because 4 which appear in both events should be counted once)

Therefore, Pr. (getting either a number greater that 2 or a multiple of 4) = $\frac{4}{6}$

$= \frac{2}{3}$

3. A die is thrown and a coin is tossed. What is the probability of getting:

(a) a 3 and a head

(b) a tail and a prime number

<u>Solution</u>

(a) From the die, Pr. (getting a 3) = $\frac{1}{6}$

From the coin, Pr. (getting a head) = $\frac{1}{2}$

Since AND means multiplication in probability:

Therefore, Pr. (getting a 3 and a head) = Pr. (getting a 3) x Pr. (getting a head)

$$= \frac{1}{6} \times \frac{1}{2}$$

$$= \frac{1}{12}$$

(b) (a) From the coin, Pr. (getting a tail) = $\frac{1}{2}$

From the die, Pr. (getting a prime number) = $\frac{3}{6}$ (The prime numbers are 3, i.e. 2, 3 and 5)

$$= \frac{1}{2}$$

Therefore, Pr. (getting a tail and a prime number) = Pr. (getting a tail) x Pr. (getting a prime number)

$$= \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{4}$$

4. Two fair dice are thrown at the same time. Find the probability of getting:
(a) at least one six
(b) a sum of at least 10
(c) a sum of at most 5
(d) a sum less than 3
(e) a total of seven
(f) a sum that is either a prime number or a multiple of 3
(g) a sum that is either divisible by 3 or a multiple of 2

Solution

The outcome table is as shown below. The numbers in the bracket give the outcome from the first and second die respectively. Adding the numbers in the bracket will give the respective sum that will be obtained.

Number on second die

+	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
Number on 2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
first die 3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

The outcome table above can be presented in a more direct form by adding the values in the brackets above to obtain the sum. This is as shown below. In the table below, the numbers in the brackets represent the numbers on each die. The numbers that are not in bracket are the outcomes from the sum of numbers on first and second dice.

Number on second die

+	(1)	(2)	(3)	(4)	(5)	(6)
(1)	2	3	4	5	6	7
Number on (2)	3	4	5	6	7	8
first die (3)	4	5	6	7	8	9
(4)	5	6	7	8	9	10
(5)	6	7	8	9	10	11
(6)	7	8	9	10	11	12

Note that any of the tables above can be used to answer the questions asked above.

(a) The outcomes that can be obtained from getting at least a six are (6,1), (6,2), (6,3), (6,4), (6,5), (6,6), (1,6), (2,6), (3,6), (4,6), (5,6). They are from the first table. They are the outcomes from the 6 on the first die, and 6 on the second die respectively. The number of brackets from this outcome is 11 (when the brackets are counted). Note that the total outcomes from any of the two outcome tables above is 36. This is easier obtained from the second table by counting the numbers that are not in bracket.

Therefore, Pr. (getting at least a six) = $\frac{\text{Number of outcomes obtained when at least a six shows}}{\text{Total number of outcomes on the table}}$

$= \frac{11}{36}$

(b) A sum of at least 10 as shown on the second table above are, 10, 10, 10, 11, 11, and 12. This gives a total of 6 outcomes.

Therefore, Pr. (getting a sum of at least 10) = $\frac{6}{36}$ (Note that 36 is the total outcome)

$= \frac{1}{6}$

(c) A sum of at most 5 as shown on the second table above are, 5, 5, 5, 5, 4, 4, 4, 3, 3, and 2. This gives a total of 10 outcomes.

Therefore, Pr. (getting a sum of at most 5) = $\frac{10}{36}$

$= \frac{5}{18}$

(d) A sum less than 3 as shown on the second table above 2 only. This gives a total of 1 outcome.

Therefore, Pr. (getting a sum less than 3) = $\frac{1}{36}$

(e) A total of 7 as shown on the second table above appears 6 times. This gives a total of 6 outcomes.

Therefore, Pr. (getting a total of 7) = $\frac{6}{36}$

$= \frac{1}{6}$

(f) Sums which are prime numbers are 2, 3, 5, 7 and 11. Sums which are multiple of 3 are 3, 6, 9 and 12. Hence we are to count the outcomes from 2, 3, 5, 6, 7, 9, 11, and 12 (3 should be counted once). Hence, from the table, 2 appears 1 time, 3 appears 2 times, 5 appears 4 times, 6 appears 5 times, 7 appears 6 times, 9 appears 4 times, 11 appears 2 times, 12 appears 1 time. This gives a total outcome of 1 time + 2 times + 4 times + 5 times + 6 times + 4 times + 2 times + 1 time = 25. This is easier done on the table by counting all 2, 3, 5, 6, 7, 9, 11 and 12. It will also give a total of 25 outcomes.

Therefore, Pr. (getting a sum that is either a prime number or a multiple of 3) = $\frac{25}{36}$

(g) Sums which are divisible by 3 are 3, 6, 9 and 12. Sums which are multiples of 2 are 2, 4, 6, 8, 10 and 12. Hence we are to count the outcomes from 2, 3, 4, 6, 8, 9, 10 and 12 (6 and 12 which appear in both events should be counted once each). Hence, we go to the second table above and count all 2, 3, 4, 6, 8, 9, 10 and 12. It will give a total of 24 outcomes.

Therefore, Pr. (getting a sum that is either divisible by 3 or a multiple of 2) = $\frac{24}{36}$

$= \frac{2}{3}$

5. An unbiased die with faces numbered 1 to 6 is rolled twice. Find the probability that the product of the numbers obtained is:
(a) odd
(b) even
(c) 12
(d) prime
(e) either odd or a multiple of 5

<u>Solution</u>
The outcome table is as shown below. The numbers in brackets are the numbers on the die.

		Number on second die					
	x	(1)	(2)	(3)	(4)	(5)	(6)
	(1)	1	2	3	4	5	6
Number on	(2)	2	4	6	8	10	12
first die	(3)	3	6	9	12	15	18
	(4)	4	8	12	16	20	24
	(5)	5	10	15	20	25	30
	(6)	6	12	18	24	30	36

(a) All the odd numbers from the outcome table above are 1, 3, 3, 5, 5, 9, 15, 15, 25. This gives a total of 9 outcomes.

Therefore, Pr. (product of numbers is odd) = $\frac{9}{36}$ (Note that the total outcomes is 36)

$= \frac{1}{4}$

(b) Pr. (product of numbers is even) = 1 - Pr. (product of numbers is odd)

$= 1 - \frac{1}{4}$

$= \frac{3}{4}$

This can also be obtained by counting all the outcomes that are even numbers in the table above. Total even numbers is 27.

Hence, Pr. (product of numbers is even) = $\frac{27}{36}$

$= \frac{3}{4}$ (As obtained before)

(c) Pr. (product of numbers is 12) = $\frac{4}{36}$ (12 appears 4 times in the table)

$= \frac{1}{9}$

(d) All the prime numbers are, 2, 2, 3, 3, 5, 5. This gives a total outcomes of 6.

Therefore, Pr. (product of numbers is prime) = $\frac{6}{36}$

$= \frac{1}{6}$

(e) All products that are odd numbers are, 1, 3, 3, 5, 5, 9, 15, 15, 25. All products that are multiples of 5 are, 5, 5, 10, 10, 15, 15, 20, 20, 25, 30, 30.

They will both give a total outcome of 15. Note that, 5, 5, 15, 15, 25 are counted once under odd number. They should not be counted under multiples of 5, as this will result to double counting. Hence with this total outcome of 15,

Pr. (product of numbers is either odd or a multiple of 5) = $\frac{15}{36}$

$= \frac{5}{12}$

6. Three dice are thrown together. What is the probability of getting a total score of 10?

Solution.
If a die is thrown once, the total outcome is given by $6^1 = 6$. If two dice are thrown, the total outcome is $6^2 = 36$. Similarly, if three dice are thrown, the total outcome will be $6^3 = 216$.

Now, for us to draw a table with 216 outcomes will be very tedious. So, a direct way of solving this problem will be to select the outcomes from each die that will result in a total score of 10. These outcomes are:

(6, 3, 1), (6, 2, 2), (5, 4, 1), (5, 3, 2), (4, 4, 2), (4, 3, 3)

Each of the brackets above can give us 6 outcomes. For example, the first bracket above can give us the following 6 outcomes:

(6, 3, 1): which means - First die shows 6, second die shows 3, third die shows 1

(6, 1, 3): which means - First die shows 6, second die shows 1, third die shows 3

(1, 6, 3): which means - First die shows 1, second die shows 6, third die shows 3

(1, 3, 6): which means - First die shows 1, second die shows 3, third die shows 6

(3, 1, 6): which means - First die shows 3, second die shows 1, third die shows 6

(3, 6, 1): which means - First die shows 3, second die shows 6, third die shows 1

Similarly, each of the other brackets can give us 6 outcomes.

Let us write out our outcome brackets again. They are, (6, 3, 1), (6, 2, 2), (5, 4, 1), (5, 3, 2), (4, 4, 2), (4, 3, 3)

When each of these brackets give us 6 outcomes, then we will obtain a total of 36 (i.e. 6 x 6) outcomes. Recall that our overall outcome table will give us a total of 216 (i.e. 6^3) outcomes.

Therefore, Pr. (getting a total score of 10) = $\frac{36}{216}$

$= \frac{1}{6}$

Exercises

1. A fair die is thrown once. Find the probability of getting:
(a) a 5
(b) a 1
(c) a 9
(d) a 2 or 3 or 6
(e) a number less than 6
(f) a prime or an even number

2. A fair die is rolled once. What is the probability of getting:
(a) a number divisible by 2
(b) a multiple of 3
(c) at least 2
(d) at most 3
(e) a perfect square or an odd number
(f) either a number greater that 5 or a multiple of 3

3. A die is thrown and a coin is tossed. What is the probability of getting:
(a) a 5 and a head
(b) a tail and a perfect cube

4. Two fair dice are thrown at the same time. Find the probability of getting:
(a) at least one four
(b) a sum of at least 6
(c) a sum of at most 10
(d) a sum less than 8
(e) a total of 12
(f) a sum that is either a perfect square or a multiple of 5
(g) a sum that is either divisible by 6 or a multiple of 4

5. An unbiased die with faces numbered 1 to 6 is rolled twice. Find the probability that the product of the numbers obtained is:
(a) prime
(b) divisible by 6
(c) 9
(d) a factor of 10
(e) either perfect cube or a multiple of 8

6. Three dice are thrown together. What is the probability of getting a total score of 11?

CHAPTER 21
MISCELLANEOUS PROBLEMS ON PROBABILITY

Examples

1. A box contains two green balls, three yellow balls and four white balls. A ball is picked at random from the box. What is the probability that it is:
(a) green
(b) yellow
(c) white
(d) blue
(e) not white
(f) either yellow or green

<u>Solution</u>

Total number of balls in the box = 2 + 3 + 4 = 9

(a) Pr. (that it is green) = $\frac{\text{Number of green balls}}{\text{Total number of balls in the box}}$

$= \frac{2}{9}$

(b) Pr. (that it is yellow) = $\frac{\text{Number of yellow balls}}{\text{Total number of balls in the box}}$

$= \frac{3}{9}$

$= \frac{1}{3}$

(c) Pr. (that it is white) = $\frac{\text{Number of white balls}}{\text{Total number of balls in the box}}$

$= \frac{4}{9}$

(d) There is no blue ball in the box.

Therefore, Pr. (that it is blue) = 0

(e) Pr. (that it is not white) = 1 - Pr. (that it is white)

$= 1 - \frac{4}{9}$

$= \frac{5}{9}$

(f) Pr. (that it is either yellow or green) = $\frac{\text{Number of yellow and green balls}}{\text{Total number of balls in the box}}$

$= \frac{3+2}{9}$

$= \frac{5}{9}$

Or,

Pr. (that it is either yellow or green) = Pr. (that it is yellow) + Pr. (that it is green) (Since OR means addition)

$= \frac{1}{3} + \frac{2}{9}$

$= \frac{3+2}{9}$

$= \frac{5}{9}$ (As obtained before)

2. A letter is chosen at random from the word COMPUTER. What is the probability that it is:
(a) either in the word MORE or in the word CUT
(b) either in the word COPE or in the word CUTE
(c) neither in the word ROT nor in the word CUP

<u>Solution</u>
(a) The total number of letters in COMPUTER is 8 letters.

In the word MORE, the number of letters is 4, while in the word CUT, the number of letters is 3. They both give a total of 7 letters.

Therefore, Pr. (that it is either in the word MORE or in the word CUT) = $\frac{7}{8}$

(b) In the word COPE, the number of letters is 4, while in the word CUTE, the number of letters is 4. Without counting any letter twice (i.e. C and E), the two words give a total of 6 letters (i.e. C, O, P, E, U, T).

Therefore, Pr. (that it is either in the word COPE or in the word CUTE) = $\frac{6}{8}$ (The total number of letters in COMPUTER is 8 letters).

$= \frac{3}{4}$

(c) Out of the 8 letters in COMPUTER, the letters that are neither in the word ROT nor in the word CUP are letters M and E. They are 2 letters.

Therefore, Pr. (that it is neither in the word ROT nor in the word CUP) = $\frac{2}{8}$

$= \frac{1}{4}$

(3) In a college 80% of the boys and 45% of the girls can drive a car. If a boy and a girl are chosen at random, what is the probability that:
(a) both of then can drive a car |
(b) the boy cannot drive a car and the girl can drive a car
(c) neither of them can drive a car?
(d) one of them can drive a car

Solution
The probabilities are given in percentage. Hence the total for each probability is 100%

Therefore, Pr. (a boy can drive a car) = $\frac{80}{100}$

$= \frac{4}{5}$

Pr. (a boy cannot drive a car) = $\frac{20}{100}$ (i.e. 100 - 80 = 20)

$= \frac{1}{5}$ (Can also be obtained from $1 - \frac{4}{5}$)

Similarly, Pr. (a girl can drive a car) = $\frac{45}{100}$

$= \frac{9}{20}$ (After equal division by 5)

Pr. (a girl cannot drive a car) = $1 - \frac{9}{20})$

$= \frac{11}{20}$

(a) Therefore, Pr. (both of them can drive a car) = Pr. (a boy can drive a car) AND Pr. (a girl can drive a car)

= Pr. (a boy can drive a car) x Pr. (a girl can drive a car)

$= \frac{4}{5} \times \frac{9}{20}$

$= \frac{36}{100}$

$= \frac{9}{25}$

(b) Pr. (the boy cannot drive a car and the girl can drive a car) = Pr. (a boy cannot drive a car) AND Pr. (a girl can drive a car)

= Pr. (a boy cannot drive a car) x Pr. (a girl can drive a car)

$= \frac{1}{5} \times \frac{9}{20}$

$= \frac{9}{100}$

(c) Pr. (neither of them can drive a car) = Pr. (a boy cannot drive a car) AND Pr. (a girl cannot drive a car)

= Pr. (a boy cannot drive a car) x Pr. (a girl cannot drive a car)

$= \frac{1}{5} \times \frac{11}{20}$

$= \frac{11}{100}$

(d) Since we do not know which of then can drive a car, then this case is logically explained as follows:

Pr. (one of them can drive a car) = either the boy can drive a car AND the girl cannot drive a car OR the girl can drive a car AND the boy cannot drive a car.

This in now calculated as follows:

Pr. (one of them can drive a car) = Pr. (the boy can drive a car) x Pr. (the girl cannot drive a car) + Pr. (the girl can drive a car) x Pr. (the boy cannot drive a car)

$$= (\frac{4}{5} \times \frac{11}{20}) + (\frac{9}{20} \times \frac{1}{5})$$

$$= \frac{11}{25} + \frac{9}{100}$$

$$= \frac{44 + 9}{100}$$

$$= \frac{53}{100}$$

4. The probability of a seed germinating is $\frac{2}{5}$. If three of the seeds are planted, what is the probability that:
(a) none will germinate
(b) at least one will germinate
(c) at least one will not germinate
(d) only one will germinate

<u>Solution</u>

This is a case of selection of three items from two possible events. We are going to write our outcomes in bracket like a tree diagram method. In order to write out the outcomes, let us use the letter G to represent germinate and letter N to represent not germinate.

Hence the outcomes are written as follows:

(GGG), (GGN), (GNG), (GNN), (NGG), (NGN), (NNG), (NNN)

(a) The probability that none will germinate is given by (NNN).

From the question, the probability that a seed germinate, G = $\frac{2}{5}$. Therefore the probability that it will not germinate, N = 1 - G = 1 - $\frac{2}{5}$ = $\frac{3}{5}$

Hence, G = $\frac{2}{5}$, N = $\frac{3}{5}$

Therefore, Pr. (that none will germinate) = (NNN)

$= \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5}$

$= \frac{27}{125}$

(b) The outcomes of the probability that at least one will germinate are, (GGG), (GGN), (GNG), (GNN), (NGG), (NGN), (NNG). Hence we can compute each of the outcomes and add them together. But this will be tedious. An easier way of solving this problem is as explained below.

The difference between the outcome in question (a) and (b) is (NNN). This shows that subtracting (NNN) from the total probability will give us the outcomes in question (b). Recall that the total of any probability is 1. Therefore, 1 - (NNN) = outcomes in (b)

Hence, Pr. (that at least one will germinate) = 1 - (NNN)

$= 1 - \frac{27}{125}$ [Note that (NNN) = $\frac{27}{125}$ as calculated in question (a)]

$= \frac{108}{125}$

(c) The outcomes of the probability that at least one will not germinate are, (GGN), (GNG), (GNN), (NGG), (NGN), (NNG), (NNN). Similar to (b) above, the difference between this outcomes of this question and the overall outcomes is (GGG).

Therefore, Pr. (that at least one will not germinate) = 1 - (GGG)

Let us calculate (GGG) as follows:

Pr. [that all three will germinate, i.e. (GGG)] = $\frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}$

$= \frac{8}{125}$

Therefore, Pr. (that at least one will not germinate) = 1 - (GGG)

$= 1 - \frac{8}{125}$

$= \frac{117}{125}$

(d) The outcomes of the probability that only one will germinate are, (GNN), (NGN), (NNG). Hence we will calculate each of these outcomes and add them together.

(GNN) = Pr. (that the first will germinate) x Pr. (that the second will not germinate) x Pr. (that the third will not germinate)

$$= \frac{2}{5} \times \frac{3}{5} \times \frac{3}{5}$$

$$= \frac{18}{125}$$

(NGN) = $\frac{3}{5} \times \frac{2}{5} \times \frac{3}{5}$

$$= \frac{18}{125}$$

(NNG) = $\frac{3}{5} \times \frac{3}{5} \times \frac{2}{5}$

$$= \frac{18}{125}$$

Therefore, Pr. (that only one will germinate) = $\frac{18}{125} + \frac{18}{125} + \frac{18}{125}$

$$= \frac{54}{125}$$

5. When children are born, they are equally likely to be boys or girls. What is the probability that in a family of four children:
(a) three are boys and one is a girl
(b) at least two are girls
(c) two are boys and two are girls
(d) the first and second born are girls

<u>Solution</u>

Since children are equally likely to be boys or girls, it means that the probability of having a boy is $\frac{1}{2}$, and the probability of having a girl is also $\frac{1}{2}$. This is similar to the case of tossing a coin (i.e. $\frac{1}{2}$ for head and $\frac{1}{2}$ for tail).

Therefore, the case of a family of four children is like when four coins are tossed. Refer to the example on tossing four coins in chapter 4.

Let us use B for boy and G for girl to write out the total outcomes of 16 (i.e. $2^4 = 16$) as shown below.

The outcomes are: (BBBB), (BBBG), (BBGG), (BGGG), (GBBB), (GGBB), (GGGB), (GBGB), (BGBG), (BBGB), (GBBG), (BGGB), (GGBG), (GBGG), (BGBB), (GGGG). This gives a total of 16 outcomes.

(a) The outcomes that the children are three boys and one girl are, (BBBG), (GBBB), (BBGB), (BGBB). This gives 4 outcomes.

Therefore, Pr. (three are boys and one is a girl) = $\frac{4}{16}$

$= \frac{1}{4}$

(b) The outcomes that the children are at least two girls are, (BBGG), (BGGG), (GGBB), (GGGB), (GBGB), (BGBG), (GBBG), (BGGB), (GGBG), (GBGG), (GGGG). This gives 11 outcomes.

Therefore, Pr. (at least two are girls) = $\frac{11}{16}$

(c) The outcomes that the children are two boys and two girls are, (BBGG), (GGBB), (GBGB), (BGBG), (GBBG), (BGGB). This gives 6 outcomes.

Therefore, Pr. (two are boys and two are girls) = $\frac{6}{16}$

$= \frac{3}{8}$

(d) The outcomes that the first and second born are girls are, (GGBB), (GGGB), (GGBG), (GGGG). This gives 4 outcomes.

Therefore, Pr. (the first and second born are girls) = $\frac{4}{16}$

$= \frac{1}{4}$

6. A bag contains three blue balls, four red balls and five white balls. Three balls are removed from the bag without replacement. What is the probability of getting:
(a) a white, blue and red balls in that order
(a) one of each colour
(c) at least two white balls

Solution

The total number of balls in the bag = 3 + 4 + 5 = 12

(a) A white, blue and red balls in that order means that the first is white, the second is blue and the third is red. This can be represented as (WBR).

Note that this is a case of without replacement. Hence after each ball is removed, the total number of ball remaining and the number of the particular ball removed are both reduced by one.

Therefore, Pr. (getting a white, blue and red balls, i.e. WBR) = $\frac{5}{12} \times \frac{3}{11} \times \frac{4}{10}$. (Notice how the total balls is reduced by 1 after each ball is removed from the bag.

$= \frac{60}{1320}$

$= \frac{1}{22}$ (After equal division by 60)

(b) Let B represent blue, R represent red and W represent white. Then the outcomes for getting one of each colour are given by: (BRW), (BWR), (RBW), (RWB), (WBR), (WRB).

Let us now calculate each of them.

(BRW) = Pr. (First is blue) x Pr. (Second is red) x Pr. (Third is white)

$= \frac{3}{12} \times \frac{4}{11} \times \frac{5}{10}$

$= \frac{1}{4} \times \frac{4}{11} \times \frac{1}{2}$

$= \frac{4}{88}$

$= \frac{1}{22}$

Similarly, each of the other five outcomes, i.e. (BWR), (RBW), (RWB), (WBR), (WRB), will each give us a value of $\frac{1}{22}$ when calculated. This is because each is obtained by multiplying 3 x 4 x 5, to give the numerator, and 12 x 11 x 10, to give the denominator, which simplifies to $\frac{1}{22}$.

Therefore, Pr. (getting one of each colour) = $\frac{1}{22} + \frac{1}{22} + \frac{1}{22} + \frac{1}{22} + \frac{1}{22} + \frac{1}{22}$

$= \frac{6}{22}$

$= \frac{3}{11}$

(c) Let us write out a different outcome for this problem. Since we are concerned about one colour, we are going to use W to represent white colour, and N to represent not a white colour. This will give us 8 outcomes in brackets as usual. The outcomes are:

(WWW), (WWN), (WNW), (WNN), (NWW), (NWN), (NNW), (NNN).

The outcomes representing at least two white balls are: (WWW), (WWN), (WNW), (NWW).

Number of white balls is 5. Therefore number of balls that are not white = 12 - 5 = 7, or blue + red = 3 + 4 = 7. (Blue and red ball are the balls that are not white balls).

Let us now calculate each of the outcomes above as follows:

(WWW) = Pr. (first is white) x Pr. (second is white) x Pr. (third is white)

$= \frac{5}{12} \text{ x } \frac{4}{11} \text{ x } \frac{3}{10}$ (Take note of the reduction in the white balls and total number of balls as each ball is removed from the bag)

$= \frac{60}{1320}$

$= \frac{1}{22}$

(WWN) $= \frac{5}{12} \text{ x } \frac{4}{11} \text{ x } \frac{7}{10}$ (Note that there are 7 balls that are not white)

$= \frac{140}{1320}$

$= \frac{7}{66}$

(WNW) $= \frac{5}{12} \text{ x } \frac{7}{11} \text{ x } \frac{4}{10}$

$= \frac{140}{1320}$

$= \frac{7}{66}$

(NWW) $= \frac{7}{12} \text{ x } \frac{5}{11} \text{ x } \frac{4}{10}$

$= \frac{140}{1320}$

$= \frac{7}{66}$

Therefore, Pr. (getting at least two white balls) = (WWW) or (WWN) or (WNW) or (NWW)

= (WWW) + (WWN) + (WNW) + (NWW)

$= \frac{1}{22} + \frac{7}{66} + \frac{7}{66} + \frac{7}{66}$

$= \frac{3 + 7 + 7 + 7}{66}$

$= \frac{24}{66}$

$= \frac{4}{11}$

7. A committee consist of 6 men and 4 women. A subcommittee made up of three members is randomly chosen from the committee members. What is the probability that:
(a) they are all men
(b) two of them are women?

Solution
Let us write out the outcome for this problem. Let M represent man, and W represent woman. This will give us 8 outcomes in brackets as usual. The outcomes are:

(WWW), (WWM), (WMW), (WMM), (MWW), (MWM), (MMW), (MMM).

(a) The total members in the committee are: 6 + 4 = 10.

The outcomes representing all men is (MMM)

Therefore, Pr. (they are all men, i.e. MMM) = Pr. (first is a man) x Pr. (second is a man) x Pr. (third is a man)

$= \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8}$ (Notice the reduction in the number of men and people left, as each member is chosen from the committee).

$= \frac{130}{720}$

$= \frac{13}{72}$

(b) The outcomes showing that two of them are women are: (WWM), (WMW), (MWW)

Let us calculate each of them as follows:

(WWM) = Pr. (the first is a woman) x Pr. (the second is a woman) x Pr. (the third is a man)

$$= \frac{4}{10} \times \frac{3}{9} \times \frac{6}{8}$$

$$= \frac{72}{720}$$

$$= \frac{1}{10}$$

(WMW) = $\frac{4}{10} \times \frac{6}{9} \times \frac{3}{8}$

$$= \frac{72}{720}$$

$$= \frac{1}{10}$$

(MWW) = $\frac{6}{10} \times \frac{4}{9} \times \frac{3}{8}$

$$= \frac{72}{720}$$

$$= \frac{1}{10}$$

Therefore, Pr. (two of them are women) = (WWM) or (WMW) or (MWW)

= (WWM) + (WMW) + (MWW)

$$= \frac{1}{10} + \frac{1}{10} + \frac{1}{10}$$

$$= \frac{3}{10}$$

8. A box contains seven blue pens and three red pens. Three pens are picked one after the other without replacement. Find the probability of picking:

(a) two blue pens

(b) at least two red pens

(c) at most two blue pens

<u>Solution</u>

Let B represent blue pen, and R represent red pen. The outcomes are:

(BBB), (BBR), (BRB), (BRR), (RBB), (RBR), (RRB), (RRR).

The total number of pens = 7 + 3 = 10

(a) The outcomes showing two blue pens are: (BBR), (BRB), (RBB)

Let us calculate each of them as follows:

(BBR) = Pr. (the first is a blue pen) x Pr. (the second is a blue pen) x Pr. (the third is a red pen)

$= \frac{7}{10} \times \frac{6}{9} \times \frac{3}{8}$

$= \frac{126}{720}$

$= \frac{7}{40}$ (In its lowest term after equal division by 18)

(BRB) = $\frac{7}{10} \times \frac{3}{9} \times \frac{6}{8}$

$= \frac{126}{720}$

$= \frac{7}{40}$

Also, (RBB) = $\frac{7}{40}$ (Similar to the once above)

Therefore, Pr. (picking two blue pens) = $\frac{7}{40} \times \frac{7}{40} \times \frac{7}{40}$

$= \frac{21}{40}$

(b) The outcomes representing at least two red pens are: (RRR), (RRB), (RBR), (BRR)

Let us now calculate each of the outcomes as follows:

(RRR) = Pr. (first is a red pen) x Pr. (second is a red pen) x Pr. (third is a red pen)

$= \frac{3}{10} \text{ x } \frac{2}{9} \text{ x } \frac{1}{8}$ (Take note of the reduction in the red pens and total number of pens as each pen is picked from the box)

$= \frac{6}{720}$

$= \frac{1}{120}$

(RRB) = $\frac{3}{10} \text{ x } \frac{2}{9} \text{ x } \frac{7}{8}$

$= \frac{42}{720}$

$= \frac{7}{120}$

Hence, (RBR) = $\frac{7}{120}$ (This is similar to the one above)

And, (BRR) = $\frac{7}{120}$ (Same reason as above)

Therefore, Pr. (picking at least two red pens) = $\frac{1}{120} + \frac{7}{120} + \frac{7}{120} + \frac{7}{120}$

$= \frac{1 + 7 + 7 + 7}{120}$

$= \frac{22}{120}$

$= \frac{11}{60}$

(c) The outcomes that represent picking at most two blue pens are: (BBR), (BRB), (BRR), (RBB), (RBR), (RRB), (RRR). Note that at most two blue pens means 2, 1 or 0 blue pens.

Notice that there is only (BBB) missing from this outcome. This shows that it can be obtained by: total probability - (BBB). Which is: 1 - (BBB).

Let us calculate (BBB) as follows:

(BBB) = Pr. (first is a blue pen) x Pr. (second is a blue pen) x Pr. (third is a blue pen)

$= \frac{7}{10} \text{ x } \frac{6}{9} \text{ x } \frac{5}{8}$

$= \frac{210}{720}$

$= \frac{7}{24}$ (After equal division by 30)

Therefore, Pr. (picking at most two blue pens) = 1 - (BBB)

$= 1 - \frac{7}{24}$

$= \frac{17}{24}$

Exercises

1. A box contains 5 green balls, 8 yellow balls and 7 white balls. A ball is picked at random from the box. What is the probability that it is:
(a) green
(b) yellow
(c) white
(d) blue
(e) not white
(f) either yellow or green

2. A letter is chosen at random from the word NORMADIC. What is the probability that it is:
(a) either in the word MAD or in the word CORN
(b) either in the word NORM or in the word DAM
(c) neither in the word RID nor in the word CAN

(3) In a college 20% of the boys and 8% of the girls who had graduated from the college, graduated with distinction since the inception of the college. If a boy and a girl are chosen at random, what is the probability that:
(a) both of them will graduate with distinction
(b) the boy will not and the girl will graduate with distinction
(c) neither of them will graduate with distinction?
(d) one of them will graduate with distinction

4. The probability of a seed germinating is $\frac{1}{4}$. If three of the seeds are planted, what is the probability that:

(a) none will germinate
(b) at least one will germinate
(c) at least one will not germinate
(d) only one will germinate

5. When parents who are carriers of sickle cell disorder get married, they are equally likely to give birth to normal child and sick child. What is the probability that in a family of three children:
(a) two are normal and one is sick
(b) at least two are sick
(c) one is normal and two are sick
(d) the first is sick
(e) at most one is normal

6. A box contains six blue balls, three red balls and five white balls. Three balls are removed from the bag without replacement. What is the probability of getting:
(a) a white, blue and red balls in that order
(a) one of each colour
(c) at least two white balls

7. A committee consist of 4 men and 2 women. A subcommittee made up of two members is randomly chosen from the committee members. What is the probability that:
(a) they are all men
(b) one of them is a woman?

8. A bag contains 5 blue balls and seven red balls. Three balls are picked one after the other without replacement. Find the probability of picking:
(a) two blue balls
(b) at least two red balls
(c) at most two blue balls

If you have any enquiries, suggestions or information concerning this book, please contact the author through the email below.

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