Extending capillary pressure

The capillary pressure relation for drainage reads

$$p_c = p_{ce}\left(\frac{S_w - S_{wc}}{1 - S_{wc}}\right)^{-1/\lambda},$$

and the capillary pressure at zero water saturation is p_c^0, which is a large number, e.g., 10^9 Pa. We draw a tangent from $[S_w, p_c^0]$=$[0,p_c^0]$ to the capillary pressure curve which touches the curve at $[S_w^*, p_c^*]$, i.e.,

$$p_c^* - p_c^0 = \frac{-S_w^*}{\lambda}\frac{p_{ce}}{1 - S_w^c}\left(\frac{S_w^* - S_{wc}}{1 - S_{wc}}\right)^{-1/\lambda - 1},$$

$$p_c^* = p_{ce}\left(\frac{S_w^* - S_{wc}}{1 - S_{wc}}\right)^{-1/\lambda},$$

or by subtracting

$$p_{ce}\left(\frac{S_w^* - S_{wc}}{1 - S_{wc}}\right)^{-1/\lambda} + \frac{S_w^*}{\lambda}\frac{p_{ce}}{1 - S_w^c}\left(\frac{S_w^* - S_{wc}}{1 - S_{wc}}\right)^{-1/\lambda - 1} - p_c^0 = 0$$

$$p_{ce} + \frac{S_w^*}{\lambda}\frac{p_{ce}}{1 - S_w^c}\left(\frac{S_w^* - S_{wc}}{1 - S_{wc}}\right)^{-1} - p_c^0\left(\frac{S_w^* - S_{wc}}{1 - S_{wc}}\right)^{1/\lambda} = 0$$

If I divide the equations, I obtain:

$$\frac{p_c^* - p_c^0}{p_c^* S_w^*} = \frac{-1}{\lambda}\frac{1}{1 - S_w^c}\left(\frac{S_w^* - S_{wc}}{1 - S_{wc}}\right)^{-1}$$

Let's approach the problem this way:

$$(1 - S_{wc})\, exp\left(-\lambda \ln\left(\frac{p_c}{p_{ce}}\right)\right) + S_{wc} = S_w$$

Another one:

$$\frac{dln\,(p_c)}{dS_w} = -\frac{1}{\lambda}\frac{1}{S_w - S_{wc}},$$

$$\frac{dln\,(p_c)}{dS_w} = \frac{ln(p_c^*) - ln(p_c^0)}{S_w^*}.$$

$$ln(p_c^*) - ln(p_c^0) = -\frac{S_w^*}{\lambda}\frac{1}{S_w^* - S_{wc}}$$

$$lnp_c^* - lnp_{ce} = -\frac{1}{\lambda}\left(\frac{S_w^* - S_{wc}}{1 - S_{wc}}\right)$$

$$\frac{1}{(1 - S_{wc})}\left(S_w^* - S_{wc}\right)^2 + \left(-1 + \lambda ln\frac{p_c^0}{p_{ce}}\right)\left(S_w^* - S_{wc}\right) - S_{wc} = 0$$

$$\Delta = \left(-1 + \lambda ln\frac{p_c^0}{p_{ce}}\right)^2 + \frac{4S_{wc}}{1-S_{wc}}$$

$$S_w^* = \frac{+1 - \lambda ln\frac{p_c^0}{p_{ce}} \pm \sqrt{\Delta}}{2}\left(1 - S_{wc}\right)$$

The negative sign gives a negative value for S_w^*, therefore the final answer is

$$S_w^* = S_{wc} + \frac{+1 - \lambda ln\frac{p_c^0}{p_{ce}} + \sqrt{\left(-1 + \lambda ln\frac{p_c^0}{p_{ce}}\right)^2 + \frac{4S_{wc}}{1-S_{wc}}}}{2}\left(1 - S_{wc}\right)$$

$$p_c = \begin{cases} p_{ce}\left(\frac{S_w - S_{wc}}{1-S_{wc}}\right)^{-1/\lambda}, & S_w > S_w^* \\ \exp\left(\frac{\ln p_c^* - \ln p_c^0}{S_w^*}\left(S_w - S_w^*\right) + \ln p_c^*\right), & S_w \leq S_w^* \end{cases}$$

$$\frac{lnp_c^* - lnp_c^0}{S_w^*} = \frac{lnp_c^* - lnp_c}{S_w^* - S_w}$$

$$lnp_c = \frac{lnp_c^* - lnp_c^0}{S_w^*}\left(S_w - S_w^*\right) + lnp_c^*.$$

The imbibition capillary pressure function is defined as

$$p_c = p_{ce}\left[\left(\frac{1+\cos\theta}{2}\right)^b\left(\frac{S_w - S_{wc}}{1-S_{wc}}\right)^{-1/\lambda} - \left(\frac{1-\cos\theta}{2}\right)^b\left(\frac{1 - S_w - S_{wc}}{1-S_{or}}\right)^{-1/\lambda}\right].$$

The liquid saturation at which the capillary pressure is zero can be calculated by

$$\left(1+\cos\theta\right)^b\left(\frac{S_w - S_{wc}}{1-S_{wc}}\right)^{-1/\lambda} = \left(1-\cos\theta\right)^b\left(\frac{1 - S_w - S_{wc}}{1-S_{or}}\right)^{-1/\lambda}$$

$$\left(1 - S_w - S_{wc}\right)\left(1 - S_{wc}\right) - \left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{b\lambda}\left(S_w - S_{wc}\right)\left(1 - S_{or}\right) = 0$$

$$S_w\left[-\left(1 - S_{wc}\right) - \left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{b\lambda}\left(1 - S_{or}\right)\right] + \left(1 - S_{wc}\right)^2 + S_{wc}\left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{b\lambda}\left(1 - S_{or}\right)$$

which gives

$$S_{w,p_c=0} = \frac{\left(1 - S_{wc}\right)^2 + S_{wc}\left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{b\lambda}\left(1 - S_{or}\right)}{\left(1 - S_{wc}\right) + \left[\frac{1-\cos\theta}{1+\cos\theta}\right]^{b\lambda}\left(1 - S_{or}\right)}.$$

0.1 Another approach (#2)

Let's assume the curve reaches a maximum value of capillary pressure, called p_c^*,from which a tangent is drawn to intersect the $p_c$axis at $p_{c,0}$. We then write:

$$p_c^* = p_{ce}\left(\frac{S_w^* - S_{wc}}{1 - S_{wc}}\right)^{-1/\lambda},$$

$$\ln\frac{p_c^*}{p_{ce}} = -\frac{1}{\lambda}\ln\left(\frac{S_w^* - S_{wc}}{1 - S_{wc}}\right),$$

$$\frac{S_w^* - S_{wc}}{1 - S_{wc}} = \exp\left(-\lambda\ln\frac{p_c^*}{p_{ce}}\right),$$

$$S_w^* = (1 - S_{wc})\exp\left(-\lambda\ln\frac{p_c^*}{p_{ce}}\right) + S_{wc}$$

Now we assume that the slope of $\ln p_c$ curve at $[S_w^*,\, p_c^*]$ is equal to the slope of the line connecting $[S_w^*,\, p_c^*]$ to $[0,\, p_{c,0}]$, i.e.,

$$\frac{\ln p_c^* - \ln p_{c,0}}{S_w^* - 0} = \left(\frac{\mathrm{d}\ln p_c}{\mathrm{d}S_w}\right)_{S_w^*} = -\frac{1}{\lambda\,(1 - S_{wc})}$$

$$\ln p_{c,0} = \ln p_c^* + \frac{S_w^*}{\lambda\,(1 - S_{wc})}$$

and finally

$$\frac{\ln p_c^* - \ln p_c}{S_w^* - S_w} = -\frac{1}{\lambda\,(1 - S_{wc})},$$

$$p_c = p_c^*\exp\left(\frac{S_w^* - S_w}{\lambda\,(1 - S_{wc})}\right)$$

$$\frac{\mathrm{d}p_c}{\mathrm{d}S_w} = -\frac{p_c^*}{S_w^* - S_w} = \frac{p_c^*}{S_w - S_w^*}$$

One more try (this time slope itself, not the slope of the logarithm)

$$\frac{p_c^* - p_{c,0}}{S_w^*} = -\frac{1}{\lambda}\frac{p_{ce}}{1 - S_{wc}}\left(\frac{S_w^* - S_{wc}}{1 - S_{wc}}\right)^{-1/\lambda - 1}$$

$$p_{c,0} = p_c^* + \frac{1}{\lambda}\frac{S_w^*}{1 - S_{wc}}\left(\frac{S_w^* - S_{wc}}{1 - S_{wc}}\right)^{-1/\lambda - 1}$$

$$\frac{p_c^* - p_c}{S_w^* - S_w} = -\frac{1}{\lambda}\frac{p_{ce}}{1 - S_{wc}}\left(\frac{S_w^* - S_{wc}}{1 - S_{wc}}\right)^{-1/\lambda - 1}$$

0.2 Approach 3 (and hopefully final)

For $S_w < S_w^*$, I define the following function:

$$p_c = a - b\exp\left(S_w^* - S_w + c\right).$$

The unknowns a, b, and c are calculated by solving the following system of equations:

$$\begin{aligned} S_w &= 0, \quad p_c = p_{c,0} \\ S_w &= S_w^*, \quad p_c = p_c^* \\ S_w &= S_{wc}, \quad \text{slopes are equal} \end{aligned}$$

We can now write:

$$p_{c,0} = a - b\exp\left(S_w^* + c\right) = a - b\exp\left(S_w^*\right)\exp\left(c\right)$$

$$p_c^* = a - b\exp\left(c\right)$$

$$-\frac{1}{\lambda}\frac{p_{ce}}{1-S_{wc}}\left(\frac{S_w^* - S_{wc}}{1-S_{wc}}\right)^{-1/\lambda-1} = \frac{b}{c}$$

which gives

$$a = \frac{p_{c,0} - p_c^*\exp\left(S_w^*\right)}{1-\exp\left(S_w^*\right)},$$

but difficult to solve for other variables. We assume $\exp(c) = c + 1 + h.o.t$, then c can be estimated by

$$c = \frac{-1 + \sqrt{1 - \frac{p_{c,0} - p_c^*}{m(\exp(S_w^*)-1)}}}{2},$$

where m is defined by

$$m = -\frac{1}{\lambda}\frac{p_{ce}}{1-S_{wc}}\left(\frac{S_w^* - S_{wc}}{1-S_{wc}}\right)^{-1/\lambda-1}.$$

One simple way is to assign a small value to variable c that makes the function monotonically decreasing, i.e.,

$$\frac{\mathrm{d}p_c}{\mathrm{d}S_w} < 0.$$

After some algebraic operations and assuming $b < 0$, we find that

$$c > 0.$$

Here, I assume that $c = S_w^*/100$. Then,

$$b = \frac{1}{\lambda}\frac{p_{ce}}{1-S_{wc}}\left(\frac{S_w^* - S_{wc}}{1-S_{wc}}\right)^{-1/\lambda-1}\left(\frac{S_w^*}{100}\right),$$

$$a = p_c^* + \frac{1}{\lambda}\frac{p_{ce}}{1-S_{wc}}\left(\frac{S_w^* - S_{wc}}{1-S_{wc}}\right)^{-1/\lambda-1}\left(\frac{S_w^*}{100}\right)\exp\left(-\frac{S_w^*}{100}\right)$$