CIRCUIT THEORY

VOl. 1

2nd Edition

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PREFACE

The present book in “Circuit Theory” is written for students in engineering who offers courses in electrical and electronics engineering. It provides an understanding of the fundamental principles of electrical engineering circuit. It is written in simple electrical terminology. It contains several worked examples. Some simple questions and their solutions are presented for each chapter at the end of the text.

The author presumes that the students are already familiar with electrical terminology, as such there is no separate chapter for introduction. Some of the basic unit and definitions are presented in chapter one before the main theme of the chapter.

Chapter one deals with the analysis of linear electric networks with constant voltage and current sources using basic circuit laws and theorems. Chapter three presents classical analogue network theorem, phasor notations series and parallel circuit and the resonance phenomena.

Chapter four represents transient processes in linear networks. The classical method, which involves the use of linear examples is presented at the end of this chapter. This volume is the Second Edition of the first volume. The next volume will deal with modern methods of electrical circuit analysis covering state variable concepts and their correlation with the classical differential equation view point and other topics in circuit theory.

Chapter five presents Laplace Transform Method of analysis of the transient processes in linear network. It contains some worked examples using the Laplace transform method.

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CHAPTER ONE

LINEAR CIRCUITS WITH CONSTANT VOLTAGE AND CURRENT SOURCES

1.1 Electrical quantities (Mks Units (S.I. Units)

In the beginning, we must be able to describe physical phenomena quantitatively in terms which will mean the same to every one. We need a standard set of units consistent among themselves and reproducible in any place in the world.
In Electrical Engineering we use the MKS (S.I. Units) system in which the metre is the unit of length, the kilogram the unit of mass, and the second the unit of time.

Table 1: Common quantities, symbols, and their unit abbreviation:

Quantity symbol	Unit	abbreviation	
Length	L	metre	m
Mass	m	kilogram kg	
Time	t	second	sec
Temperature	T	degree kelvin	°K
Current	I	Ampere	A

Another basic quantity is temperature, which in the MKS system is measured in degrees Kelvin. To define electrical quantities, an additional unit is needed, taking the ampere as the unit of electric current satisfies this requirement.

DEFINITIONS: For quantitative work in circuit theory, we have to define the various quantities we would encounter in the course of study.

FORCE: A force of 1 Newton is require to cause a mass of 1 kilogram to change its velocity at a rate of 1 metre per second per second.
In this text we are concerned with electric and magnetic forces.

ENERGY: Energy is the ability to do work. An object requiring a force of 1 Newton to hold it against the force of gravity receives 1 joules of potential energy when it is raised 1 metre. A mass of 1 kilogram moving with a velocity of 1 metre per second possesses 0.5 joules of kinetic energy.

POWER: Power is the rate at which energy is transformed. Power measures the rate at which energy is transferred. The transformation of 1 joules of energy in 1 second represents an average power of 1 watt. In general, instantaneous power is defined by:

$$p = \frac{dw}{dt} \quad (1)$$

CHARGE: Charge is created when there is surplus of protons in a given area. when a charge exists in one area, an unlike charge exists in another area somewhere because electrons gained or lost in one area are moved to or from another area. The integral of current with respect to time is electric charge, a concept useful in explaining physical phenomena. Charge is said to be conservative in that it can be neither created nor destroyed. It is said to be quantized because the charge on 1 electron (1.602×10^{-19} coulombs) is the smallest amount of charge that can exist. The coulomb can be defined as the charge on 6.24×10^{18} electrons, or as the charge experiencing a force of 1 Newton in an electric field of 1 volt per metre, or as the charge transferred in 1 second by a current of 1 ampere.

CURRENT: In order to have actual and possible current flow, there must be a charge. Electric field effects are due to the presence of charges; magnetic field effects are due to the motion of charges. The current through an area A is defined by the electric charge passing through per unit of time. In general, the charges may be positive and negative, moving through the area in both directions. The current is the net rate of flow of positive charges, a scalar quantity. In specific case of positive charges moving to the right and negative charges to the left, the net effect of both actions is positive charge moving to the right, the current to the right is

$$i = +\frac{dq^{+}}{dt} + \frac{dq^{-}}{dt} \quad (2)$$

An electron or current flow is possible if a suitable path exists.

VOLTAGE: The energy-transfer capability of a flow of electric charge is determined by the potential difference or voltage through which the charge moves. A charge of 1 Coulomb receives or delivers an energy of 1 joules in moving through a voltage of 1 volt or in general

$$V = \frac{dw}{dq} \quad (3)$$

ELECTRIC FIELD STRENGTH: The Electric Field Strength E, a vector, is defined by the magnitude and direction of the force f on a unit positive charge in the field. In vector notation the defining equation is

$$\overline{f} = q\,\overline{E} \quad (4)$$

where E could be measured in Newton per coulomb. However, bearing in mind the definition of energy and voltage, we note that;

$$\frac{Force}{Charge} = \frac{Force\ x\ distance}{Charge\ x\ distance} \quad 5$$

$$= \frac{energy}{Charge\ x\ distance} = \frac{Voltage}{distance} \quad 6$$

and electric field strength in Newton per coulomb is just equal and opposite to the voltage gradient or

$$E = -\frac{dV}{dl}\ in\ [V/m] \quad (5)$$

MAGNETIC FLUX DENSITY: Around a moving charge or current we visualize a region of influence called a "magnetic field". In a bar magnet the current consists of spinning electrons in the atoms of iron; the effect of this current on the spinning electrons of an un-magnetized piece of iron results in the similar force of attraction. The intensity of the magnetic effect is determined by the magnetic flux density B, a vector defined by the magnitude and direction of the force f exerted on a charge q moving in the field with a velocity U. In vector notation the defining equation is:

$$f = q \cdot UxB \quad (6)$$

A force of 1 Newton is experienced by a charge of 1 coulomb moving with a velocity of 1 metre per second normal to a magnetic flux density of 1 Tesla.

MAGNETIC FLUX: Magnetic flux in weber is a total quantity obtained by integrating magnetic flux density over an area. The equation is

$$\phi = \int B\ dA \quad 8 \qquad (7)$$

If the vector of the magnetic flux density B is perpendicular to the area A and the field is homogeneous, the magnetic flux is

$$\phi = B(S)\ ,\ A = BA \quad 9 \qquad (8)$$

Because of this background, magnetic flux density has another unit of measurement known as Weber per square meres.

ELECTRICAL POWER AND ENERGY: A common problem in electric circuits is to predict the power and energy transformations in terms of expected currents and voltages. Since by definition

$$V = \frac{dw}{dq}\ and \quad i = \frac{dq}{dt} \quad 10$$

$$P = \frac{dw}{dt} = \frac{dw}{dq}\,x\,\frac{dq}{dt} = V \bullet i \; 11 \qquad (9)$$

Therefore, total energy is:

$$= \int P\,dt = \int_0^T V \bullet i dt \; 12 \qquad (10)$$

1.2 BEHAVIOUR OF IDEAL CIRCUIT ELEMENTS

RESISTIVE ELEMENT: This is an element in which voltage and current are connected together with the relationship

$$V_r = R\cdot i_r \quad \text{or} \quad \_{r} = GV_r \qquad (11)$$

where R is called the resistance, G is called the conductance. the law relating voltage and current is called Ohm's law, in honour of George Ohm a German physicist, whose original experiments led to this simple relation.

We know that a metallic conductor such as copper contains many relatively free electrons. The application of a voltage increases the electric field which tends to accelerate these conduction electrons, and the resulting motion is superimposed on the random thermal motion of electrons at, say, room temperature. Electrons are accelerated by the field, collide with copper atoms, and give up their energy. They are accelerated again, gaining energy from the electric field, collide again, and give up their energy. Superimposed on the random motion due to thermal energy, there is an average net directed motion or drift due to the applied electric field. The speed of the drift is found to be directly proportional to the applied electric field. In a given conducting element, therefore the rate is directly proportional to the electric field which in turn, is directly proportional to the applied voltage.

CAPACITIVE ELEMENT: It is an element in which the voltage and current are related by the expression:

$$i_c = C\,\frac{dV_c}{dt} \qquad (12)$$

where C is a constant of proportionality called capacitance and measured in Farads.

$$q = C\cdot V \qquad (13)$$

INDUCTIVE ELEMENT: This is a linear element in which the voltage and current are related by the expression.

$$V_L = L \frac{d\, i_L}{dt} \quad 14 \qquad (14)$$

where L is a constant of proportionality called inductance and measured in henrys (H).

ENERGY STORAGE IN LINEAR ELEMENTS

Valuable insights into the behaviour of real circuit components can be obtained by considering the energy transformation which occur in the corresponding linear models.
Inductance, where V_L = di/dt and i = 0 at t = 0

$$W_L = \int_0^T L \frac{di}{dt}\, dt = \int_0^I L\, i\, di = \frac{1}{2} L\, I^2 \quad 15 \qquad (15)$$

The total energy input to an inductance is directly proportional to the square of the final current. Inductance is a measure of the ability of a device to store energy in the form of moving charge or in the form of magnetic field. The equation shows that the energy is stored rather than dissipated. If the current is increased from zero to a finite value and then decreased to zero, the upper limit of the integration becomes zero and the net energy input is zero, the energy input was stored in the field and then returned to the circuit.

CAPACITANCE: Where $i = C \frac{dv}{dt}$ 16 and V = 0 at t = 0

$$W_c = C \int_0^T V \bullet \frac{dv}{dt}\, dt = C \int_0^V V\, dV = \frac{1}{2} C\, V^2 \quad 17 \qquad (16)$$

In words, the total energy input to a capacitance is directly proportional to the square of the final voltage. The constant of proportionality is $C/_2$. Capacitance is a measure of the ability of a device to store energy in the form of separated charges or in the form of an electric field.

RESISTANCE: When a similar analysis is made of an electrical resistance, the results are quite different.
where V = R·i and i = 0 at t = 0

$$W_R = \int_0^T R\, i \bullet i\, dt = R \int_0^T i^2\, dt \quad 18 \qquad (17)$$

To evaluate the total energy supplied, we must know current _ as a function of time t. For the special case where the current is constant or i = I, 19 (18)

There is no possibility of controlling the current in such a way as to return any energy, to the circuit; the energy has been dissipated. In a real resistor, the dissipated energy appears in the form of heat, in describing the behaviour of the linear model we say that the energy has been dissipated in an irreversible transformation, irreversible because there is no way of heating an ordinary resistor and obtaining energy.

The rate of dissipation of energy or power is a useful characteristic of resistive elements.

$$P = \frac{dw}{dt} = P_R = \frac{dw_R}{dt} = R\, i^2 \quad 20 \qquad (19)$$

CONTINUITY OF STORED ENERGY

Since power is the time rate of change of energy, an instantaneous change in energy would require an infinite power. But the existence of an infinite power (physically) is contrary to our concept of a physical system, and we require that the energy stored in any element of a system, real or ideal, be a continuous function of time. Recalling that the energy of a moving mass is $\frac{mv^2}{2}$ 21.

We conclude that the velocity V cannot change instantaneously.

Following the same line of reasoning, we note that the energy stored in an inductance is $LI^2/2$ and, therefore, the current in an inductance cannot change instantaneously. Since energy stored in a capacitance is $CV^2/2$, the voltage across a capacitance cannot change instantaneously. Note that there is no such limitation on the rapidity with which inductance voltage or capacitance current change.

1.3 LINEAR ELECTRIC CIRCUITS WITH D.C. VOLTAGES AND CURRENTS SOURCES.

E.m.f.; Current, Voltage and their positive directions.

A common problem in electric circuit is to predict the power and energy in terms of the expected currents and voltages. To resolve this problem it is necessary to know the value of the e.m.f. and the resistances of the source; receiver; and the other parts of the circuit.

Let us have a simple electric circuit (fig 1.1), in which the source is connected to the load using conductors, whose resistance is negligible.

In the source there exist certain field of forces, that are acting internally in the source giving rise to a drift of charges. As a result one terminal marked with the positive sign has excess of positive charges and the other terminal marked with the negative sign (-) has excess of negative charges.

As a result of this division of charges internally in a source, an external electric field will exist in the electric circuit.

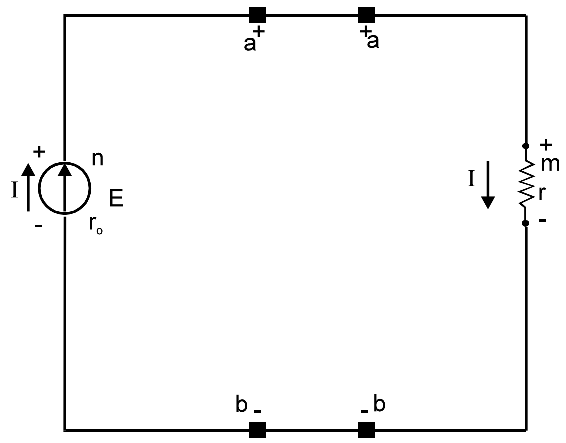


Fig. 1.1 A simple electrical circuit

The electric field acting between any two points e.g (a and b in fig 1.1) is characterized by the voltage or the potential difference between the given points $V_{ab} = V_a - V_b$. The voltage is equal to the work done by the electric field forces in moving a positive charge, equal to 1 coulomb, from one point to another point in the field.

Directed field of the source between the terminals is defined by the value E, called the electro-magnetic force (e.m.f) and is equal to the work done by the internal field forces in moving a positive unit charge from one terminal of the voltage source to the other terminal. Voltage and current exist in an electric circuit as a result of directed field.

The character of change of voltage and current depends on the character of change of the electro-motive force e.m.f. Therefore a constant current energy source has an unchanging e.m.f., hence the voltage and current in such electric circuits of constant current do not change.

The positive directions of voltages, current, and electro-motive force, play important role in circuit analysis. The positive directions is otherwise referred to as real direction of current and voltage in any given electric circuit. The positive directions are usually indicated with the aid of pointed arrows.
For current the conventional positive direction of flow is the direction of flow of positive charges.

For positive direction of voltage between two given points in an electric circuit is the direction in which the positive charges would have flown when an electric field is exerted between these two points, i.e the positive direction is taken to be from the point of higher potential to the point of lower potential.

The positive direction of voltage source (e.m.f.) is taken to be the direction of flow of positive charges when a directed field is exerted between these two terminals, e.e. the direction from the terminal of lower potential.

So, in the electric circuit given in fig. 1.1, potential of point a is greater than the potential of point b (Va - Vb), hence the voltage is directed from point a to point b, but the e.m.f - the positive direction is from b to point a.

Current-carrying conductors of different elements in electric circuits present a resistance to the movement of carriers, i.e. to electric current, and this is defined by the quantity called resistance.

If a conductor is made of the same type of material and a uniform area, then the resistance of the conductor is given:

$$r = \rho \frac{L}{s} \, [ohm] \, 22 \qquad (20)$$

Where,

ρ is called the resistivity of the material.
Ohm mm^2/m, l is the length of conductor,

S - area of the conductor.
The resistance of metallic conductors increases with increase in temperature. The law relating temperature and resistance together is given by the following equation.

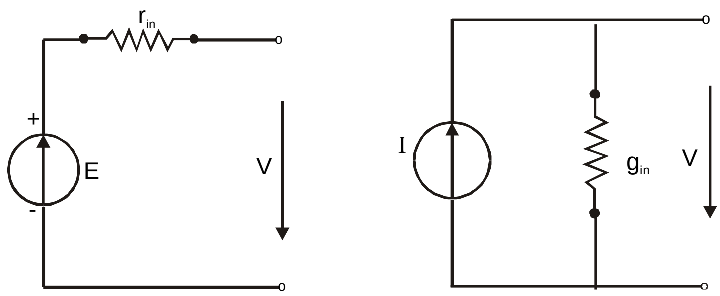
$$r_2 = r_1 \, [1 - \alpha \, (t_2 - t_1) \qquad (21)$$

where,

t_1 and t_2 - initial and final temperature in degree Celsius; r_1 and r_2 - resistances for temperatures t_1 and t_2 respectively. α - temperature coefficient of resistance, 1/°C.

An energy source of a linear electric circuit can be represented by two equivalent

circuits, the first is a voltage source in series with its internal resistance and the second is a current source in parallel with the internal conductance.



(a)Voltage source with its internal resistance

(b) Current source with its internal conductance

Fig. 1.2

In an ideal voltage source $r_{in} = 0$; it generates power

$$p = E \cdot I \tag{22}$$

and in an ideal current source $g_{in} = 0$; it generates a power equal to

$$P = V \cdot J \tag{23}$$

For the series circuit of fig. 1.2a

$$V = E - r_o I \tag{24}$$

and for the parallel circuit of fig. 1.2 (b)

$$I = J - g_o V \tag{25}$$

infact for the same energy source

$$E = r_o \cdot J \quad \text{and } g_o = 1/r_o$$

BASIC EQUATIONS OF CIRCUITS

(State Equations)

(1) The first law of Kirchhoff is known as kirchhoff's current law and it states that the algebraic sum of currents into a node is equal to zero, i.e.

$$\sum_k i_k = o \tag{26}$$

where the positive direction is taken to be direction going out of a node.

(2) The Second Law: Kirchhoff's second law is known as kirchhoff's voltage law and it states that the algebraic sum of voltages in any closed path is equal to zero, i.e

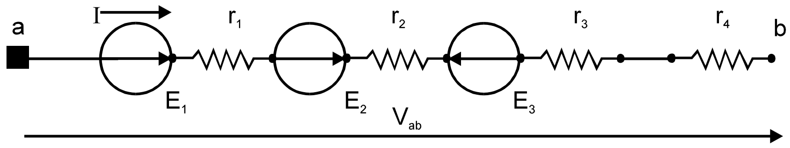
$$\sum_k V_k = 0\,23 \tag{27}$$

If you start from any point at one potential and come back to the same point and the same potential, the difference of potential must be zero.

In determining the algebraic signs for voltage terms, thus mark the polarity of each voltage. A convenient system then is: Go around any closed path and consider any voltage whose plus terminal is reached first as positive, and vice versa. This method applies to voltage drops and voltage sources. Also, direction can be

clockwise or counter clockwise. In any case if you come back to the starting point, the algebraic sum of all the voltage terms must be zero.
If you do not come back to the start, then the algebraic sum is the voltage between the start and terminal points.

You can follow any path. the reason is that the net voltage between any two points in a circuit is the same regardless of the path used in determining the potential difference.



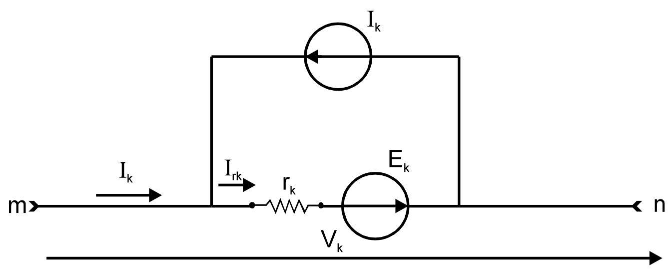
Using Ohm's law

$$I = \frac{V_{ab} + E_1 + E_2 - E_3}{r_1 + r_2 + r_3 + r_4} = \frac{V_{ab} + \sum_{ab} E}{r_{ab}} \quad 24$$

$$= \left(V_{ab} + \sum_{ab} E \right) g_{ab} \qquad (28)$$

where,

ΣE - is the algebraic sum of e.m.f in the branch, r_{ab} - sum of resistance in the branch and $g_{ab} = 1/r_{ab}$.



Algebraic methods of calculating electric circuits. Ohm's law for the above circuit is given thus:

$$V = r(I + J) - E \qquad (29)$$

CURRENT EQUATION:

For a circuit, refer to point C at the top of diagram in fig 1.3. the 6-A I_T into point C divides into the 2-A and 4-A $I_{4\text{-}5}$, both directed out. Note that $I_{4\text{-}5}$ is the current through R_4 and R_5. The algebraic equation is

$$I_T - I_5 - I_{4\text{-}5} = 0$$

substituting the values for these currents

$$6A - 2A - 4A = 0$$

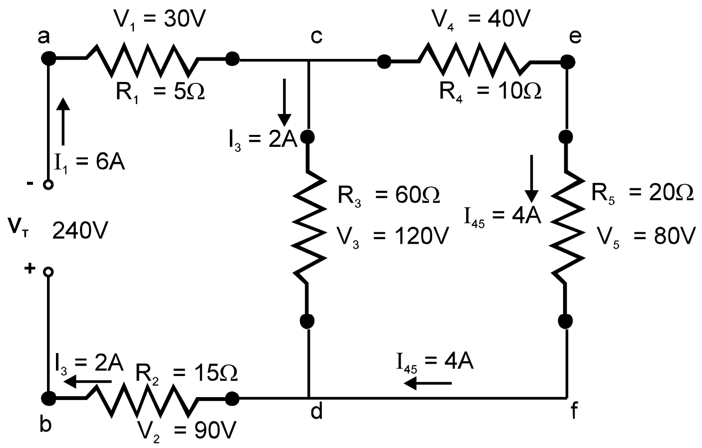


Fig. 1.3 Series - Parallel Current to illustrate Kerchhoff's laws

For the opposite directions refer to point d at the bottom of fig. 1.3. Here the branch currents into d combine to equal the main line current I_T returning to the voltage source. Now I_T is directed out from d, with I_3 and $I_{4\text{-}5}$ directed in. The algebraic equation is

$$- I_T + I_3 + I_{4\text{-}5} = 0$$

$$- 6A + 2A + 4A = 0$$

The $I_{in} = I_{out}$. Note that at either point C or point d in fig. 1.3, the sum of 2-A and 4-A branch currents must equal the 6-A total line current. Therefore, Kirchhoff's current law can also be stated as: $I_{in} = I_{out}$. For fig 1.3, the equation can be written.

At point C: $6A = 2A + 4A$

At point d: $2A + 4A = 6A$

Kirchhoff's current law is really the basis for the practical rule on parallel circuits that the total line current must equal the sum of the branch current.

KIRCHHOFF'S VOLTAGE LAW

The algebraic sum of the voltage around any closed path is zero. If you start from any point at one potential and come back to the same point and the same potential, the difference of potential must be zero.

LOOP EQUATIONS:

Any closed path is called a loop. A loop equation specifies the voltages around the loop. Fig. 1.3, has three loops. The outside loop, starting from point a at the top, through acefdb, and back to a, includes the voltage drops V_1, V_4, V_5 and V_2 and the source V_T.
The inside loop acdba includes V_1, V_3, V_2 and V_T. The other inside loop, cefdc with V_4, V_5, and V_3 does not include the voltage source.

Consider the voltage equation for the inside loop with V_T. In clockwise direction, starting from point a, the algebraic sum of the voltage is

$$- V_1 - V_3 - V_2 + V_T \text{ or } - 30V - 120V - 90V + 240V = 0$$

Voltages V_1, V_3 and V_2 have the negative sign, because for each of these voltages the negative terminal is reached first. However, the source V_T is a positive term because its plus terminal is reached first, going in the same direction.

For the opposite direction, going counter clockwise in the same loop from point b at the bottom, V_2, V_3, and V_1 have positive values and V_T is negative. The

$$V_2 + V_3 + V_1 - V_T = 0$$

Or, $$90v + 120V + 30V - 240V = 0$$

When we transpose the negative term of -240V, the equation becomes

$$90V + 120V + 30V = 240V$$

This equation states that the sum of the voltage drops equals the applied voltage. When a loop does not have any voltage source, the algebraic sum of the voltage drops on the resistors alone total zero. For instance, in fig. 1.3, for the loop cefdc without the source V_T, going clockwise from point C, the loop equation of voltage is

$$- V_4 - V_5 + V_3 = 0$$

- 40V - 80V + 120V = 0

METHOD OF BRANCH CURRENTS

Now we can use Kirchhoff's laws to analyze the circuit in fig. 1.4. The problem is to find the currents and voltages for the three resistors. For indicated current direction and mark the voltage polarity across each resistor consistent with the assumed current. Remember that electron flow in a resistor produces negative polarity where the current enters. In fig; 1.4, we assume that the source V_1 produces electron flow left to right through R_1, while V_2 produces electron flow from right to left through R_2.

The three different currents in R_1, R_2 abd R_3 are indicated as I_1, I_2 and I_3. However the unknown would require three equation for the solution.

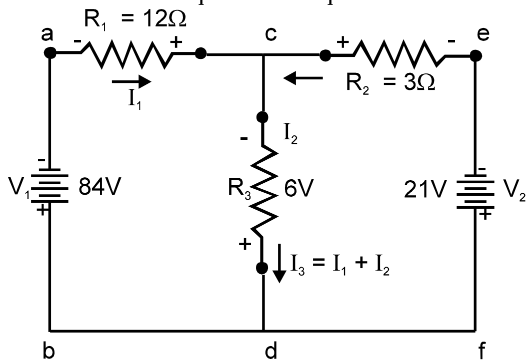


Fig. 1.4 Application of Kirchhoff's laws to a circuit with two sources on different branches

From Kirchhoff's law;

$I_3 = I_1 + I_2$, as the current out at point C must equal the current in. The current through R3, therefore, can be specified as $I_1 + I_2$. With two unknowns, two independent equations are needed to solve for I_1 and I_2. These equations are obtained by writing two Kirchhoff's voltage law equations around two loops. There are three loops in fig. 1.4, the outside loop and two inside loops, but we need only two. The inside loops are used for the solution here.

Writing the Loop Equations

For the loop with V_1, start at point b, at the bottom left, and go clockwise through V_1, VR_1 and VR_3. This equation for loop 1 is

$84 - VR_1 - VR_3 = 0$ $\qquad$ $84 - V_{R1} - V_{R3} = 0$

For the loop with V_2, start at point f, at the lower right, and go counter clockwise through V_2, VR_2, and VR_3. This equation for loop 2 is

$$21 - VR_2 - VR_3 = 0$$

Using the known values of R_1, R_2 and R_3 to specify the IR voltage drops.

$$VR_1 = I_1R_1 = I_1 \times 12 = 12I_1$$

$$VR_2 = I_2R_2 = I_2 \times 3 = 3I_2$$

$$VR_3 = (I_1 + I_2)R_3 = 6(I_1 + I_2)$$

Substituting the values in the voltage equation for loop 1

$$84 - 12I_1 - 6(I_1 + I_2) = 0$$

Also, in loop 2

$$21 - 3I_2 - 6(I_1 + I_2) = 0$$

Multiplying $(I_1 + I_2)$ by 6 and combining terms and transporting the two equations are

$$- 18I_1 - 6I_2 = -84$$

$$- 6I_1 - 9I_2 = -21$$

Divide the top equation by -6 and the bottom equation by -3 to make the coefficients smaller and to have all positive terms. The two equations in their simplest form then become

$$3I_1 + I_2 = 14$$

$$2I_1 + 3I_2 = 7$$

$$9I_1 + 3I_2 = 42$$

$$2I_1 + 3I_2 = 7$$

$$7I_1 = 35$$

$$I_1 = 5A$$

$$2(5) + 3I_2 = 7$$

$$3I_2 = 7 - 10$$

$3I_2 = -3$

$I_2 = -1A$

The negative sign for I_2 means this current is opposite to the assumed direction. Therefore, I_2 flows through R_2 from c to e instead of from e to c.

In fig. 1.4, I_2 was assumed to flow from point e to c through R_2 because V_2 produces electron flow in this direction. However, the other voltage source V_1 produces electron flow through R_2 in the opposite direction from point c to e. This solution of -1A from I_2 shows that the current through R_2 produced by V_1 is more than the current produced by V_2. Their net result is 1A through R_2 from c to e.

The actual direction of I_2 is shown in fig. 1.5 with all the values for the solution of this circuit. Notice that the polarity of V_{R2} is reversed from the assumed polarity in fig. 1.4 since the net electron flow through R_2 is actually from c to e, the end of R_2 at c is the negative end. However, the polarity of V_2 is the same in both diagrams because it is a voltage source which generates its own polarity.

To calculate I_3 through R_3

$I_3 = I_1 + I_2 = 5 + (-1) = 4A$

$I_3 = 4A$

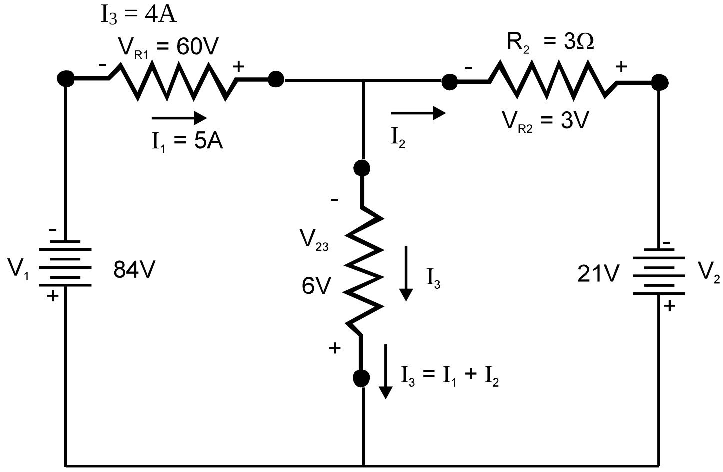


Fig. 1.5 Solution of Fig. 4 with all currents and voltages

The 4A for I_3 is the assumed direction from c to d. Although the negative sign from I_2 only, means a reversed direction, its algebraic value of -1 must be used for substitution in the algebraic equations written for the assumed direction. With all the currents known, the voltage across each resistor can be calculated as follows

$V_{R1} = I_1R_1 = 5 \times 12 = 60V$

$V_{R2} = I_2R_2 = 1 \times 3 = 3V$

$V_{R3} = I_3R_3 = 4 \times 6 = 24V$

All the currents are taken as positive in the correct direction, to calculate the voltages. The polarity of each IR drop as determined from the actual direction of current with electron flow into the negative end (fig. 1.5). Notice that the V_{R2} and V_{R3} have opposing polarities in loop 2. Then the sum of +3V and -24V equals the -21V for V_2.

Checking the solution

At point c; 5A = 4A + 1A

At point d; 4A + 1A = 5A

Around the loop with V_1: 84V - 60V - 24V = 0 clockwise from b.

Around the loop with V_2: 21 + 3V - 24V = 0 counter clockwise from f.

THE MESH CURRENT METHOD

A mesh is the simplest closed path possible. In the mesh current method, a mesh current is chosen for each mesh. The current direction in clockwise or counter clockwise direction is taken for all the mesh currents in the same direction. The Kirchhoff's voltage law is written for each of the meshes. The number of equations is equal to the number of meshes.

A system of equations is written whose unknowns are mesh currents. The circuit in fig. 1.6 has three meshes

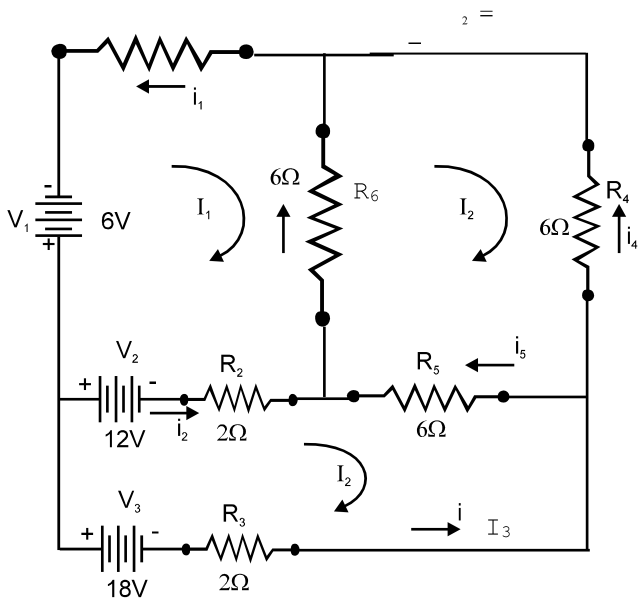


Fig. 1.6 The circuit for mesh circuit analysis

Three mesh currents are assumed to flow in a clockwise direction in all the three meshes. Mesh current I_1 flows through V_1, R_2, R_6, V_2. Mesh current I_2 flows through R_4, R_5, and R_6. Mesh current I_3 flows through V_2, R_2, R_5, R_3 and V_3. We write Kirchhoff's voltage law for the three meshes.

The fact that a mesh current does not divide at a branch point is the difference between mesh currents and branch currents. A mesh current is an assumed current, while a branch current is the actual current. However, when the mesh currents are known, all the individual currents and voltages can be determined.

Writing Mesh Equations

In writing the mesh equations according to Kirchhoff's voltage law, we shall define some quantities. We shall introduce the idea of total mesh resistance and common resistance. The total mesh resistance is the sum of the resistance going round a given mesh, while common mesh resistance is the resistance common to two meshes. The total mesh resistances are written with positive sign in the equation while common resistances are written with negative sign in the equations. Resistance R_{11} is the total mesh resistance for mesh 1, R_{22} for mesh 2 and so on. Common resistance $R_{12} = R_{21}$ is the common resistance between mesh 1 and 2, $R_{13} = R_{31}$ the resistance common to mesh 1 and 3 and so on.

Any resistance common to two meshes has two opposite mesh currents. As a result, a common resistance has two opposing voltage drops. One voltage is positive for the current of the mesh whose equation is being written. The opposing voltage is negative for the current of the adjacent mesh.

For mesh 1: $$R_{11}I_1 - R_{12}I_2 - R_{13}I_3 = \Sigma \text{e.m.f} \tag{1}$$

For mesh 2: $$-R_{21}I_1 + R_{22}I_2 - R_{23}I_3 = \Sigma \text{e.m.f} \tag{2}$$

For mesh 3: $$-R_{31}I_1 - R_{32}I_2 + R_{33}I_3 = \Sigma \text{e.m.f} \tag{3}$$

The sum of the e.m.f are algebraic sums for mesh 1 the right hand of the equation will be algebraic sum of e.m.f in mesh 1, which is equal to $V_1 - V_2$. Mesh 2 does not have a voltage source and therefore the right hand of equation 2 is zero. The right hand of equation 3 is equal to $V_2 - V_3$.

For resistances $R_{11} = R_1 + R_2 + R_6$;

$R_{12} = R_{21} = R_6$; $R_{22} = R_4 + R_5 + R_6$; $R_{13} = R_{31} = R_2$; R_{23}

$= R_{32} = R_5$; $R_{33} = R_2 + R_3 + R_5$.

$R_{11} = 10\Omega$; $R_{12} = R_{21} = 6\Omega$; $R_{22} = 6 + 6 + 6 = 18\Omega$

$R_{13} = R_{31} = R_2 = 2\Omega$; $R_{23} = R_{32} = 6\Omega$' $R_{33} = 10\Omega$

$V_1 - V_2 = 6 - 12 = -6V$
$V_2 - V_3 = 12 - 18 = -6V$

Therefore the final equation is given thus

$10I_1 - 6I_2 - 2I_3 = -6V$

$-6I_1 + 18I_2 - 6I_3 = 0V$

$-2I_1 - 6I_2 + 10I_3 = -6V$

We shall use the determinant method to solve the set of equations of three unknown currents I_1, I_2 and I_3.

$$\Delta = \begin{vmatrix} 10 & -6 & -2 \\ -6 & 18 & -6 \\ -2 & -6 & 10 \end{vmatrix} \quad (26)$$

$$= 10(180 - 36) - (-6)(-60 - 12) + (-2)(36 + 36) \quad (27)$$

$$= 1440 - 432 - 144 = 864$$

$$\Delta I_1 = \begin{vmatrix} -6 & -6 & -2 \\ 0 & 18 & -6 \\ -6 & -6 & 10 \end{vmatrix} = -1296 \quad (28)$$

$$\Delta I_2 = \begin{vmatrix} 10 & -6 & -2 \\ -6 & 0 & -6 \\ -2 & -6 & 10 \end{vmatrix} = -864 \quad (29)$$

$$\Delta I_3 = \begin{vmatrix} 10 & -6 & -6 \\ -6 & 18 & 0 \\ -2 & -6 & -6 \end{vmatrix} = -1296 \quad (30)$$

The mesh currents are determined as follows

$$I_1 = \frac{\Delta I_1}{\Delta} = -\frac{1296}{864} = -1.5\ A; \tag{31}$$

$$I_2 = \frac{\Delta I_2}{\Delta} = -\frac{864}{864} = -1.0\ A$$

$$I_3 = \frac{\_\, I_3}{\_} = -\frac{1296}{864} = -1.5\ A \tag{32}$$

$$I_1 = -1.5\ A;\ I_2 = -1.0\ A;\ I_3 = -1.5\ A \tag{33}$$

Finding the Branch currents and voltage drops.

Referring to fig 1.6, the -1.5A I_1, as the only current flowing through R_1. Therefore I_1 and i_1 are equal but opposite in direction i_2 is the algebraic sum of the mesh currents I_1 and I_3, i.e $i_2 = I_3 - I_1$, = -1.5A -(-1.5A) = 0A. The current i_3 is equal to the mesh current I_3, but the direction is opposite to the direction of the mesh current $i_3 = -I_3$ = -(-1.5A) = 1.5A. Branch current i_4 equal to mesh current I_2, but the assumed direction is opposite to the direction of the mesh current I_2, i.e. $i_4 = -I_2$ = -(-1·0A) = <u>1.0A</u>. Branch current I_5 and I_6 are calculated as algebraic sums of mesh current I_2 and I_3, I_1 and I_2 respectively, i.e. $i_5 = I_2 - I_3$ = -1.0 - (-1.5) = <u>0.5A</u>
Hence the branch currents are as follows:
i_1 = 1.5A; i_2 = 0A; i_3 = 1.5A; i_4 = 1.0A; i_5 = 0.5A; i_6 = 0.5A

NODE - VOLTAGE ANALYSIS (METHOD)

In the method of mesh currents, these are used to specify the voltage drops around the mesh. The mesh equations are written to satisfy Kirchhoff's voltage law. Solving the mesh equations, we can calculate the unknown branch currents.

Another methods uses the voltage drops to specify the currents at a branch point also called nodes. Then node equation of currents are written to satisfy Kirchhoff's current law. Solving the node equation, we can calculate the unknown node voltages. This method of node - voltage analysis often is shorter than the method of branch currents.

A node is simply a common connection for two or more components. A principal node has three or more connections in effect, a principal node is just a junction or branch point where currents can divide or combine. Therefore, we can always write an equation of currents of a principal node. In fig. 1.7, there are three principal nodes.

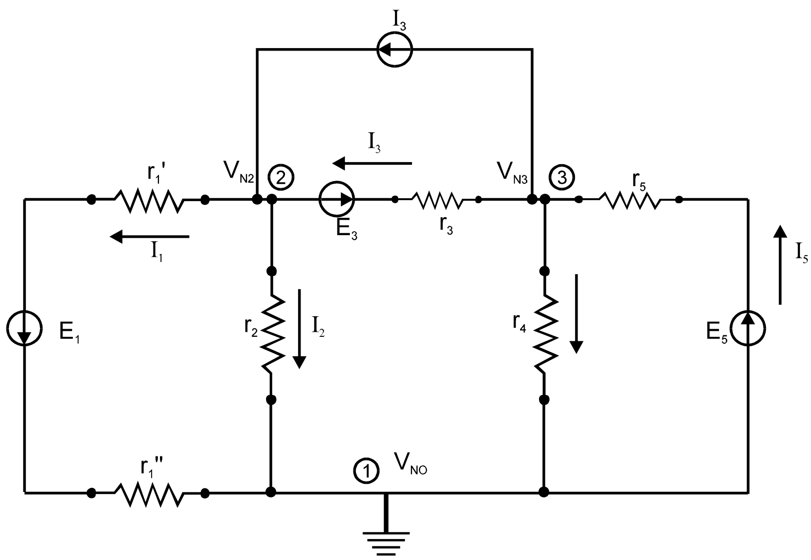


Fig. 1.7 The circuit for node - voltage analysis.

Points 1, 2 ,and 3 are those principal nodes.

However, one node must be reference for specifying the voltage at any other node. In fig 1.7, point 1 is connected to chassis ground at the reference node. Therefore, we need only write two current equations for the other two nodes (2 and 3). In general the number of current equations required to solve a circuit is one less than the number of principal nodes.

The problem here is to find the node voltages V_{N2} and V_{N3} from 2 to 1 and from 3 to 2. Once these voltages are known, all the other voltages and currents can be determined.

We shall write Kirchhoff's currents for nodes 2 and 3 as follows

Node 2: $I_1 + I_2 - I_3 - J_3 = 0$ going out positive

Node 3: $I_3 + I_4 + J_3 - I_5 = 0$ going in negative

$$I_1 + I_2 - I_3 - J_3 = 0 \qquad (1)$$

$$I_3 + I_4 + J_3 - I_5 = 0 \qquad (2)$$

We shall now write Ohm's law for currents in each branch in the circuit. Meanwhile V_{N1} is our reference voltage which has zero potential.

$$I_1 r_1 = V_N 2 - V_N 1 + E_1 = V_N 2 + E_1 \; 34$$

$$I_1 = \frac{V_N 2 + E_1}{r_1} \quad 35$$

$$I_2 r_2 = V_N 2 - V_N 1 = V_N 2\,;\; I_2 = \frac{V_N 2}{r_2} \quad 36$$

$$I_3 r_3 = V_N 3 - V_N 2 - E_3\,;\; I_3 = \frac{V_N 3 - V_N 2 - E_3}{r_3} \quad 37$$

$$I_4 = V_N 3 - V_N 1 = V_N 3\,;\; I_4 = \frac{V_N 3}{r_4} \quad 38$$

$$r_5 I_5 = V_N 1 - V_N 3 + E_5 = -V_N 3 + E_5 \quad 39$$

$$I_5 = \frac{-V_N 3 + E_5}{r_5} \quad 40$$

Note that $V_{N1} = 0$

Substitute the equations for currents in equations (1) and (2) above

$$\frac{V_N 2 + E_1}{r_1} + \frac{V_N 2}{r_2} - \frac{(V_N 3 - V_N 2 - E_3)}{r_3} - I_3 = 0 \quad 41 \qquad (3)$$

$$\frac{V_N 3 - V_N 2 - E_1}{r_3} + \frac{V_N 3}{r_4} - \frac{(-V_N 3 + E_5)}{r_5} + I \quad 42 \qquad (4)$$

$$V_N 2 + 4 + 4V_N 2 - 4V_N 3 + 4V_N 2 + 8 - 16 = 0 \quad 43$$

$$4V_N 3 - 4V_N 2 - 8 + V_N 3 + 8V_N 3 - 96 + 16 = 0 \quad 44$$

$$9V_N 2 - 4V_N 3 = 4 \quad 45$$

$$-4V_N 2 + 13V_N 3 = 88 \quad 46$$

$$36V_N 2 - 16V_N 3 = 16 \quad 47$$

$$\frac{-36V_N 2 + 117V_N 3 = 792}{101V_N 3 = 808} \quad 48$$

$$V_N 3 = \frac{808}{101} = \quad 49$$

$$V_N 3 = 8V \quad 50$$

$$\therefore\; V_N 2 = \frac{4 + 4 \text{ x } 8}{9} = \frac{32 + 4}{9} = \frac{36}{9} = 4V \quad 51$$

$$V_N 2 = 4V\,;\; V_N 3 = 8V \quad 52$$

Hence the currents in the branches are the computed using previously derived expressions for currents in a branch.

$$I_1 = \frac{V_N 2 + E_1}{r_1} = \frac{4 + 4}{8} = 1A \quad 53$$

$$I_2 = \frac{V_N2}{r_2} = \frac{4V}{2\Omega} = 2A \tag{54}$$

$$I_3 = \frac{V_N3 - V_N2 - E_3}{r_3} = \frac{8 - 4 - 2}{2} = \frac{2}{2} = 1A \tag{55}$$

$$I_4 = \frac{V_N3}{r_4} = \frac{8}{8} = \tag{56}$$

$$I_5 = \frac{-V_N3 + E_5}{r_5} = \frac{-8 + 12}{1} = 4A \tag{57}$$

Checking by Kirchhoff's current laws

$I_1 + I_2 - I_3 - J_3 = 0$; $1 + 2 - 1 - 2 = 0$

$I_3 + I_4 + J_3 - J_3 = 0$; $1 + 1 + 2 - 4 = 0$

SUPERPOSITION METHOD: The method of superposition states that the current or voltage in a particular branch can be determined as the algebraic sum of the effects produced by each source acting separately.

In order to use one source at a time all other sources are "killed" temporarily. This means disabling the source so that it cannot generate voltage or current without changing the resistance of the circuit. A voltage source such as a battery is killed by assuming a short circuit across its potential difference. The internal resistance remains. A current source is killed by assuming an open circuit between its terminals. In other words, assuming an infinite resistance across its terminal.

Let us illustrate this method with an example

For the circuit of fig. 1.8, Given the following parameters: $E_1 = 25V$, $j_2 = 125mA$; $r_1 = 100$ Ohms; $r_L = 500$ Ohms; $r_2 = 2000$ Ohms. Calculate the currents in all the branches of the circuit.

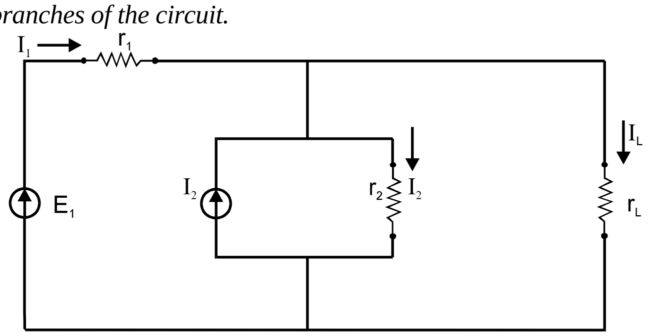


Fig. 1.8 Electric circuit for the illustration of the superposition

For the analysis of the circuit using the superposition method we shall calculate the currents in the branches with each source acting separating. We shall kill the other sources and replace them with their equivalent internal resistance. First we kill the current source and then calculate the current using the voltage source.

$$I_{1'} = \frac{E_1}{\left(r_1 + \frac{r_2 \bullet r_L}{r_2 + r_L}\right)} \quad 58$$

= 25/(100 + [2000 x 500/2500])

=25/(100 + 400) = 25/500

= 0.05A

$$I_{2'} = \frac{I_1 r_1}{r_2 + r_L} \quad 59$$

= 0.05x500/2500 = 0.01A

I_2' = 0.01A

I_L' = 0.05 - 0.01 = 0.04A

We shall now kill the voltage source and return the current source to the circuit. We then calculate the currents produced by the current source acting alone.

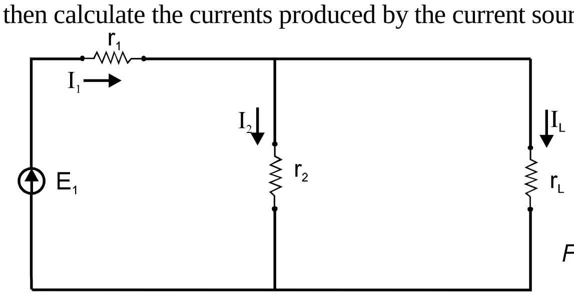


Fig. 1.8a

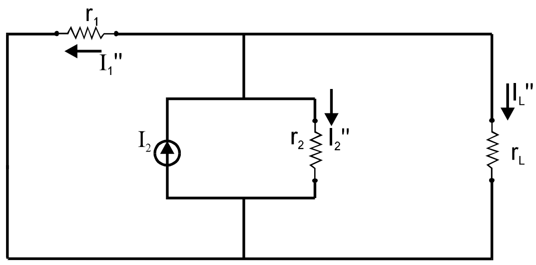


Fig. 1.8b

$$I_1'' = \frac{J_2 \bullet Req_1}{r_1 + Req_1} \quad 60$$

$$Req_1 = \frac{r_2 \bullet r_L}{r_2 + r_L} = \frac{2000\,x\,500}{2500} \quad 61$$

$$Req_1 = 400\Omega \quad 62$$

$$I_1'' = \frac{125 \, x \, 400}{100 + 400} = \frac{125 \, x \, 400}{500} = 100mA \quad 63$$

$$I_1'' = 100mA \quad 64$$

$$I_2'' = \frac{J_2 \bullet Req_2}{r_2 + Req_2}; \quad 65$$

$$Req_2 = \frac{r_1 \bullet r_L}{r_1 + r_L} = \frac{100 \, x \, 500}{100 + 500} = \frac{100 \, x \, 500}{600} = \frac{500}{6}\Omega \quad 66$$

$$I_2'' = 0.005A \quad 67$$

$$I_L'' = 0.02A; \; I_L'' = 125 - 100 - 5 = 20mA \quad 68$$

Therefore, the currents in the branches is the algebraic sum of the effects produced by each source acting separately.

$$I_1 = I_{1'} - I_1'' = 0.05A - 0.10A = -0.05A \quad 69$$

$$I_2 = I_1'' + I_{2'} = 0.01A + 0.005A = 0.015A \quad 70$$

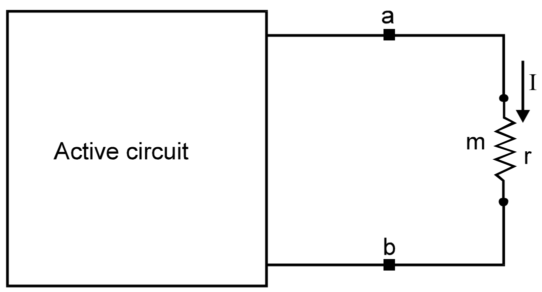
$$I_L = I_{L'} + I_L'' = 0.04A + 0.02A = 0.06A \quad 71$$

THEVENIN THEOREM

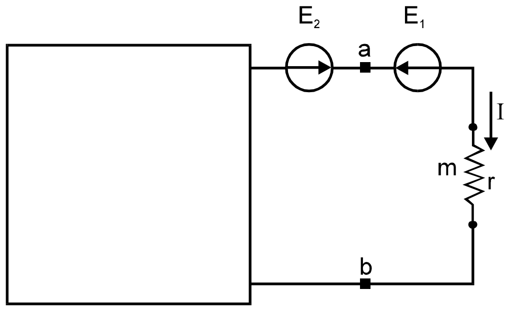
(METHOD OF EQUIVALENT GENERATORS)

Method of equivalent generators provides opportunity to simplify a complex electric circuit into a simple circuit for the purpose of analysis. Thevenin theorem named after M.L Thevenin a french engineer, this theorem is very useful in simplifying the voltages in a network. By Thevenin's theorem, many sources and components no matter how they are interconnected, can be represented by an equivalent series circuit with respect to any pair of terminals in the network fig. 1.9a.

Let us assume, that it is required to find the current I in the branch a m b of a given network (fig. 1.9a), in which the other elements of the circuit are presented in form of a black box. It is obvious, that when we introduce two e.m.f source E_1 and E_2 into



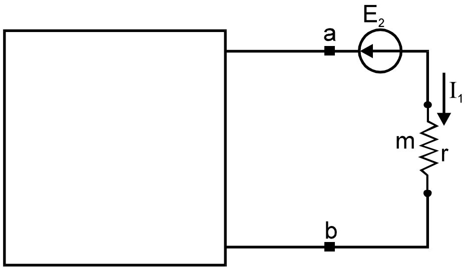
(a) Fig. 1.9a



(b) Fig. 1.9b

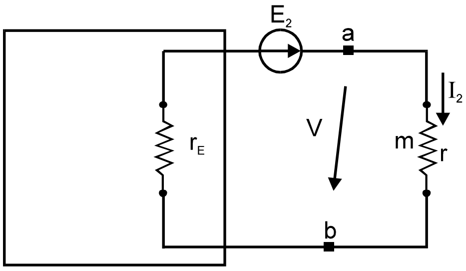
the branch, which are equal and directed in opposite direction, that the current I does not change in value. (fig. 1.9b)

According to the superposition method, the current I is given as the algebraic sum of currents produced by E_1 and E_2 acting separately (fig. 1.10)



(a)

Fig. 1.10



(b)

Fig. 1.10

$I = I_2 - I_1$, where I_1 - current, generated by all the voltage sources of the electric circuit and e.m.f. E_1 (Fig. 1.10a); I_2 - current, generated by the e.m.f. source E_2 (Fig 1.10b).

If we choose the e.m.f. E_1 in such a way, that we obtain a current $I_1 = 0$, then our current

$$I = I_2 = \frac{E_2}{r + r_E} \qquad 72$$

where r_E - the equivalent resistance of the network contained in the black box.

When $I_1 = 0$ (fig 1.10a), the given network will be operating in the open circuit regime with respect to the terminal a and b. The established open circuit voltage $V = V_{oc}$. As a result of this, by Kirchhoff's voltage law for the circuit of fig. 1.10a, we can write the following expression

$$E_1 = I_1 r + V_{OC} \qquad 73$$

from where we obtain that,

$$I_1 = \frac{E_1 - V_{oc}}{r} \qquad 74$$

To produce a current $I_1 = 0$, it is necessary to choose e.m.f E_1 equal to the open circuit voltage V_{oc}. But the condition set out initially $E_2 = E_1$, therefore $E_2 = V_{oc}$.

Taking this into account, we get,

$$I = \frac{V_{oc}}{r + r_E} \; 75 \qquad (1)$$

To determine the current I by the formula derived above, it is necessary to find the open circuit voltage V_{oc} and the equivalent resistance r_E, which can be calculated or measured by experiment.

EXAMPLE: *In the circuit of fig. 1;11a, $V = 100V$, $E = 40V$, $r_1 = 30\Omega$, $r_2 = r_3 = 20\Omega$, $r = 15\Omega$, $r_4 = 30\Omega$, $r_0 = 1\Omega$. Using the Thevenin theorem, calculate the current I and voltage across the resistor r.*

SOLUTION: when the branch with resistor r is removed from the circuit,

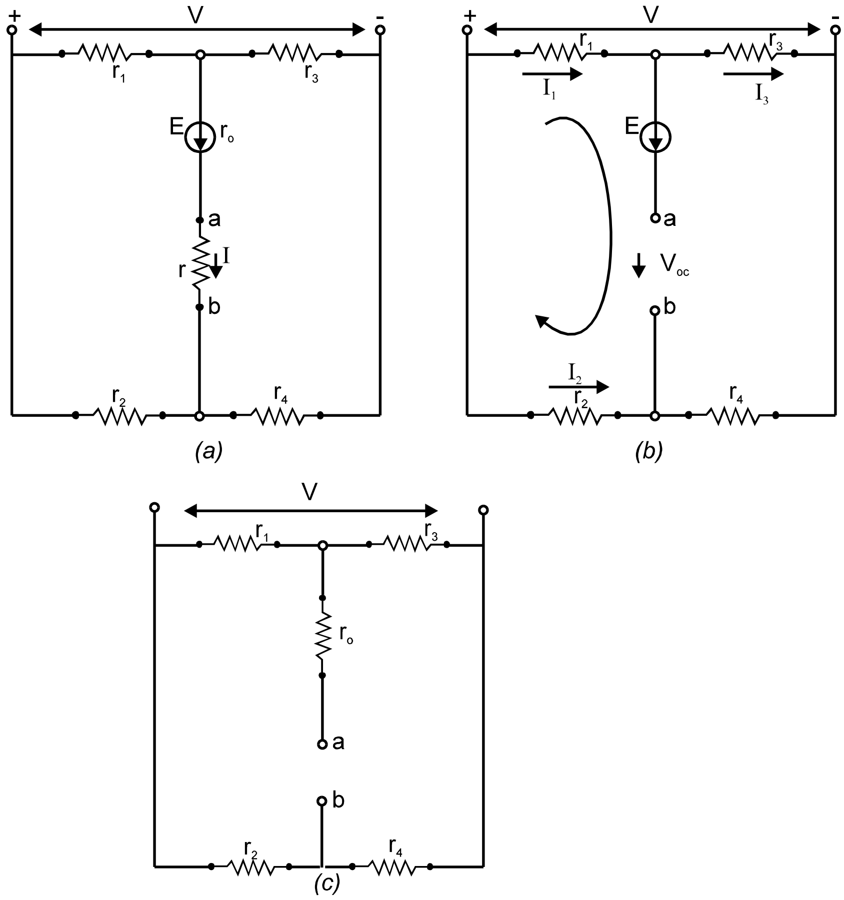


Fig. 1.11 Network analysis example using Thevenin theorem

we shall have an open circuit and the currents that flow will be determined thus

$$I_1 = \frac{V}{r_1 + r_3} = \frac{100}{30 + 20} = 2A; \quad 76$$

$$I_2 = \frac{V}{r_2 + r_4} = \frac{100}{20 + 30} = 2A \quad 77$$

On the basis of Kirchhoff's voltage law

$V_{oc} = E - I_1r_1 + I_2r_2 = 40 - 2 \times 30 + 2{\cdot}20 = \underline{\underline{20V}}$

After eliminating the voltage sources, we shall have a passive network containing only resistors fig 1.11c, whose equivalent resistance is given as follows;

$$r_E = \frac{r_1 \bullet r_3}{r_1 + r_3} + \frac{r_2 \bullet r_4}{r_2 + r_4} + r_0$$

$$= \frac{30 \times 20}{50} + \frac{20 \times 30}{50} + 1 \quad 78$$

$$= 12 + 12 + 1 = 25\,\Omega$$

The equivalent resistance is called the Thevenin equivalent resistance R_{Th}.

Therefore, $r_e = R_{Th} = 25\Omega$. By the current equation derived in equation (1) the current I flowing through the branch with resistor r is given thus

$$I = \frac{V_{oc}}{r + R_{Th}} = \frac{20}{r + R_{Th}} = \frac{20}{15 + 25} = 0.5A \quad 79$$

The voltage V_{ab}, we shall find using ohm's law.

$V_{ab} = I{\cdot}r = 0{\cdot}5 \times 15 = \underline{\underline{7.5V}}$

EXAMPLE 2: *In the network of fig 1.12 calculate the current I_6 using Thevenin theorem.*
Given r_1 = 120 ohms; r_2 = 30 ohms; r_3 = 15 ohms; r_4 = 36 ohms; r_5 = 18 ohms; r_6 = 10 ohms; r_7 = 5 ohms; E_1 = 225V; E_4 = 180V.

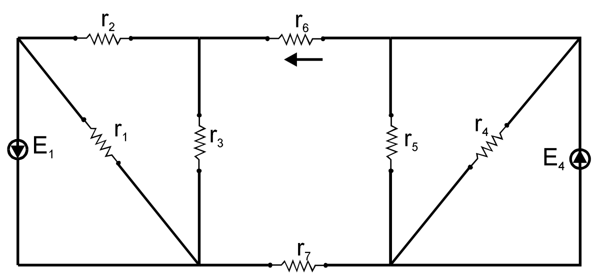


Fig. 1.12a

We shall open circuit the branch with resistor r_6 and we shall determine V_{Th} between point b and a

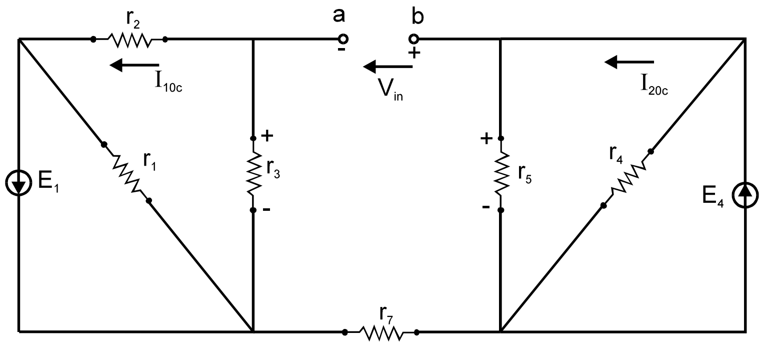


Fig. 1.12b

For this circuit fig. 1.12b, we determine the current I_{10c} and I_{20c}; open circuit currents.

$$I_{10C} = \frac{E_1}{(r_2 + r_3)} = \frac{225}{(30 + 15)} = 5A\ 80$$

$I_{20c} = E_4/r_5 = \underline{\underline{10A}}$

and the voltage V_{Th} is given as

$V_{Th} = r_3 I_{10c} + r_5 I_{20c}$

$V_{Th} = 15 \text{ x } 5 + 180 = 75 + 180 = \underline{\underline{255V}}$

The Thevenin equivalent resistance is found by eliminating (killing) the voltage sources and looking into one of the terminals (a) and (b) evaluate the total resistance. The equivalent circuit used for calculating R_{Th} is given below.

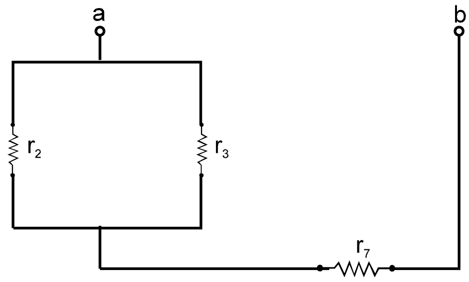


Fig. 1.12c

$$R_{Th} = r_7 + \frac{r_2 \bullet r_3}{r_2 + r_3} = 15\,\Omega \quad 81$$

Hence current I_6 according to Thevenin theorem is given as

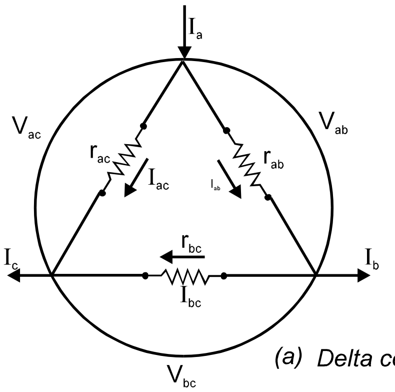
$$I_6 = \frac{V_{Th}}{R_{Th} + r_6} = \frac{255}{15 + 10} = \frac{255}{25} = 10.2\,A \quad 82$$

$\underline{\underline{I_6 = 10.2A}}$

ELECTRICAL NETWORK CONTAINING TRIANGULAR CONNECTED RESISTANCE.

(Δ -CONNECTION OF RESISTANCE).

By the word delta connected resistance, we understand that resistors are connected in such a way that the end of one resistor is connected to the beginning of the second, and the end of the second is connected to the beginning of the third and the end of the first resistor as shown in fig. 1.13, while the node points a, b and c are connected to the rest part of the network.



(a) Delta connection

To simplify the analysis of some networks consisting of delta connection, it is usually wise to replace the delta connected resistors with a star connected fig. 1.13b. Example of such a network is a bridge circuit shown in fig. 1.14, which is used frequently in automatic and measuring instruments. If we replace the delta connection with its equivalent star connection then we shall obtain the network in fig. 1.13b. The transformation is carried out only on the resistances, the currents I_a, I_b, I_c and voltages V_{ab}, V_{bc}, V_{ac} remain unchanged.

On the basis of Kirchhoff's voltage law for the delta connection

$$V_{ab} = I_{ac} r_{ac} - I_{bc} r_{bc} \qquad (1)$$

by Kirchhoff's current law and ohm's law

$$I_{ac} = I_a - I_{ab} = I_a - V_{ab}/r_{ab}$$

$$I_{bc} = I_{ab} - I_b = V_{ab}/r_{ab} - I_b$$

After substituting the above equations in equation (1), we shall obtain

$$V_{ab} = \frac{r_{ab} r_{ac}}{r_{ab} + r_{bc} + r_{ac}} \bullet I_a + \frac{r_{ab} r_{bc}}{r_{ab} + r_{bc} + r_{ac}} \bullet I_b \quad 83$$

By Kirchhoff's voltage law for the star connection

$$V_{ab} = I_a r_a + I_b r_b \qquad (2)$$

When we note that these two voltages are equal, then

$$V_{ab} = I_a r_a + I_b r_b = \frac{r_{ab} r_{ac}}{r_{ab} + r_{bc} + r_{ac}} \bullet I_a \quad 84$$

$$+ \frac{r_{ab} r_{bc}}{r_{ab} + r_{bc} + r_{ac}} \bullet I_b$$

Hence,

$$r_a = \frac{r_{ab} r_{ac}}{r_{ab} + r_{bc} + r_{ac}} ; r_b = \frac{r_{ab} r_{bc}}{r_{ab} + r_{bc} + r_{ac}} \quad 85 \qquad (3)$$

By analogy

$$r_c = \frac{r_{ac} r_{bc}}{r_{ab} + r_{bc} + r_{ac}} \quad 86 \qquad (4)$$

Sometimes, it is necessary to replace a star connected circuit with a delta

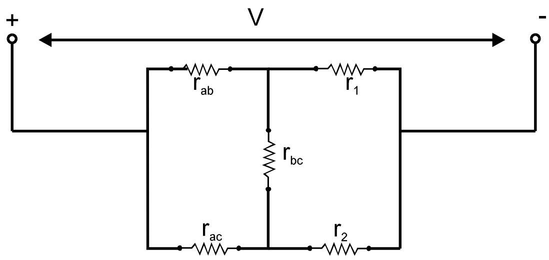
connected circuit. The formula for the transformation of a star connected to a delta is given as follows.

$$r_{ab} = r_a + r_b + \frac{r_a r_b}{r_c} ; r_{bc} + r_b + r_c + \frac{r_b r_c}{r_a} ; 87$$

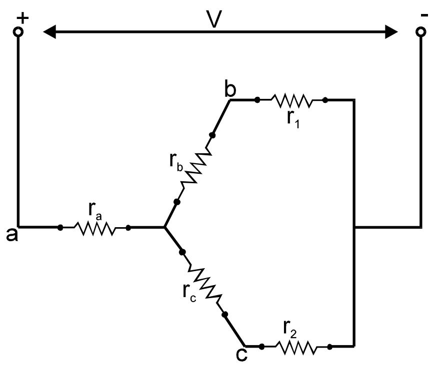
$$r_{ac} = r_a + r_c + \frac{r_a r_c}{r_b} \quad 88$$

These expressions are obtained by solving equation (3) and (4) together.

EXAMPLE: *Convert the delta to star connection*



(a) Delta connection



(b) Star connection (equivalent)

Fig. 1.14 Electric circuit

THE BASIC PROPERTIES AND TRANSFORMATION OF ELECTRICAL NETWORK

(1) The sum of power generated by voltage and current sources in any given linear network is equal to the sum of the power utilized by the network (power balance)

$$\sum_{k=1}^{b} E_k I_k + \sum_{k=1}^{b} V_{kJ_k} = \sum_{k=1}^{b} r_k I_k^{2} \quad 89$$

or in matrix form:

$$I_r^T R^{(B)} I_r = [E^{(B)}]^T \bullet I_r + [V^{(B)}]^T \bullet J^{(B)} \quad 90$$

(2) By superposition theorem the current and voltage in any branch of a linear electric circuit can be presented as the algebraic sum of the effects produced by each of the source acting separately.

$$I_h = \sum_{l=1}^{b} g_{hl} E_1 + \sum_{l=1}^{b} K_{hl}^{(i)} \bullet J_c \, , h = 1,2\, ; \qquad \text{(b)}$$

$$V_h = \sum_{l=1}^{b} r_{hl} \; J_l + \sum_{l=1}^{b} K_{hl} E_l \, , h = 1\,,2\,; \quad 92$$

(b)

where

$$g_{hl} = I_h / E_l \, ; \; K_{hl}^{(v)} = V_h / E_l \quad 93$$

(3) In the circuit of fig. 1.15 determine the power dissipated on the resistor with resistance r_3 = 3 ohms. Given r_1 = 10 ohms; r_4 = 2 ohms; r_5 = 5ohms; J_1 = 20A; J_2 = 20A

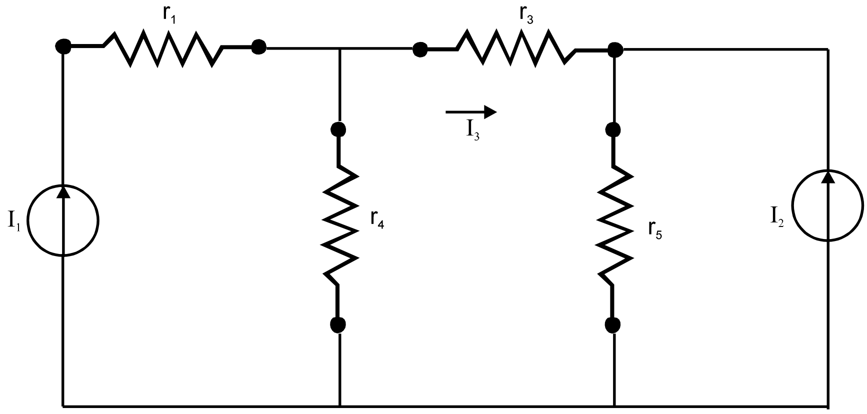


Fig. 1.15a

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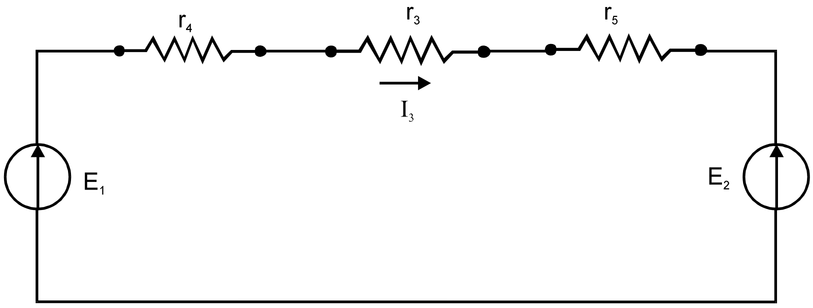


Fig. 1.5b

We convert the currents sources J_1 and J_2 into the equivalent voltage source to reduce the number of nodes and thereby simplify the computation.

The equivalent e.m.f. source is calculated thus

$$E_1 = r_4 J_1 = 40V\ ;\ E_2 = r_5 \bullet J_2 = 100V \quad 94$$

$$I_3 = (E_1 - E_2)/(r_3 + r_4 + r_5) = -6A \quad 95$$

The power dissipated on r_3 p_3 = $r_3 I_3^2 = 108Watt$ 96

P_3 = <u>108 watt</u>

THEVENIN - NORTON CONVERSION

Thevenin's theorem says that any network can be represented by a voltage source and series resistance, while Norton's theorem says that the same network can be represented by a current source and shunt resistance. It must be possible therefore to convert directly from a Thevenin form to a Norton form and vice versa. Such conversions are often useful.

NORTON FROM THEVENIN

Consider the Thevenin equivalent circuit in fig. 1.16. Just apply Norton theorem, the same as for any other circuits. The short-circuit current through terminals a and b is

$$I_N = \frac{V_{Th}}{R_{Th}} = \frac{15}{3} = 5A \quad 97$$

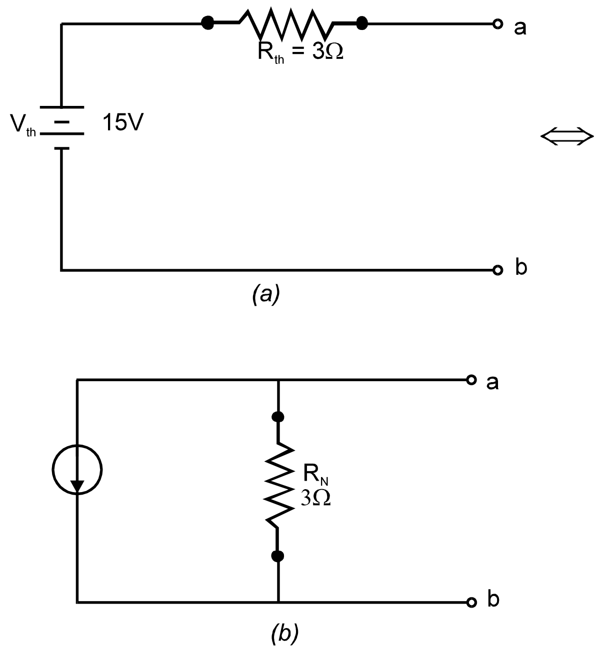


Fig. 1.16 Theveinin equivalent circuit in (a) corresponds to the Norton equivalence in (b)

The resistance looking back from open terminals a and b with the source V_{Th} short-circuited, is equal to the 3 ohms of R_{Th}. Therefore, the Norton equivalent consists of a current source that supplies the short-circuit current of 5A, shunted by the same 3 ohm resistance that is in series in the Thevenin circuit. The results are shown in fig. 1.16b.

CONVERSION FORMULAS: In summary, the following formulas can be used for these conversions Thevenin from Norton.

$R_{Th} = R_N$

$V_{Th} = I_N \times R_N$

Norton from Thevenin

$$R_N = R_{Th}$$

$$I_N = \frac{V_{Th}}{R_{Th}} \quad 98$$

APPLICATION OF THE MESH CURRENT METHOD

EXAMPLE 1: *In the circuit of fig 1.3 calculate the values of all the branches currents using the mesh current method.*

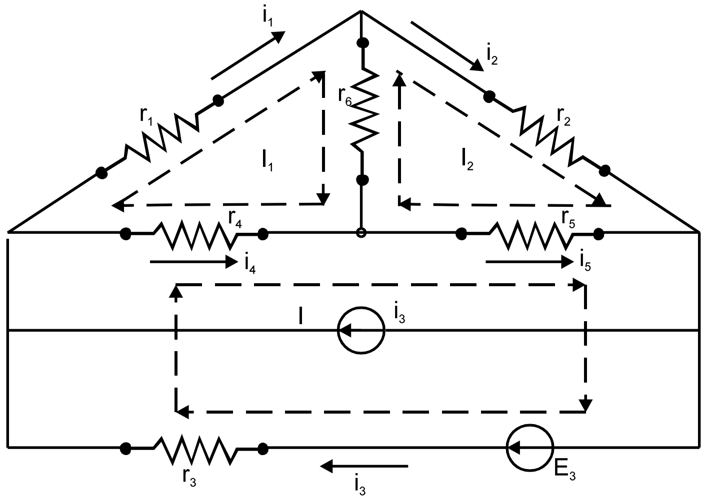


Fig. Q1.3

Given; $r_1 = 20\Omega$; $r_2 = 30\Omega$; $r3 = 40\Omega$; $r_4 = 80\Omega$; $r_5 = 20\Omega$; $r_6 = 20\Omega$; $E_3 = 16V$; $J = 0.3A$

SOLUTION: We shall choose three mesh currents I_1; I_2 and I_3 in a clockwise direction in all the three meshes. A mesh current is assume to flow through the resistors in a given mesh.

We now write equations (Kirchhoff's voltage equations) corresponding to the chosen mesh currents.

$$(r_1 + r_4 + r_6)I_1 - r_6 I_2 - r_4 I_3 = r_4 \bullet J \quad 99$$

$$-r_6 I_1 + (r_2 + r_5 + r_6)I_2 - r_5 I_3 = r_5 \bullet J \quad 100$$

$$-r_4 I_1 - r_5 I_2 + (r_3 + r_4 + r_5)I_3 = E_3 - (r_4 + r_5)J \quad 101$$

From here with the given parameters of the circuit we find $I_1 = 0.3A$; $I_2 = 0.2A$; $I_3 = 0.1A$.

The currents in the branches

$i_4 = - I_1 + I_3 + J = 0.1A;$

$i_5 = - I_2 + I_3 + J = 0.2A;$

$i_6 = I_1 - I_2 = 0.1A$

Hence the currents in the branches are as follows:

$i_1 = I_1 = 0.3A;$

$i_2 = I_2 = 0.2A;$

$i_3 = I_3 = 0.1A;$

$i_4 = 0.1A;$

$i_5 = 0.2A;$

$i_6 = 0.1A.$

APPLICATION OF THE NODE - VOLTAGE METHOD

EXAMPLE 2: *Using the node - voltage method calculate the currents in the branches of the circuit of fig. 1.3.*

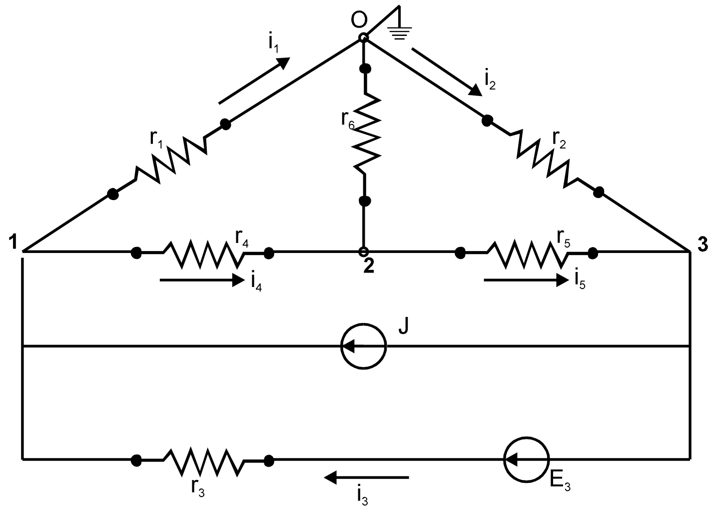


Fig. 1.3 Circuit for Node - Voltage analysis

SOLUTION: The positive direction of currents are indicated on the circuit diagram. We shall choose our reference principal node to be node O, with potential $V_{NO} = 0$. The remaining principal nodes 1, 2, and 3 will be used in determining the unknown potentials.

We write Kirchhoff's current law for nodes 1, 2, and 3;

Node 1: $\quad i_1 + i_4 - i_3 - J = 0 \qquad (1)$

Node 2: $\quad i_5 - i_4 - i_6 - J = 0 \qquad (2)$

Node 3: $\quad J + i_3 - i_5 - i_2 = 0 \qquad (3)$

We now derive expressions for the currents in the branches according to ohm's law:

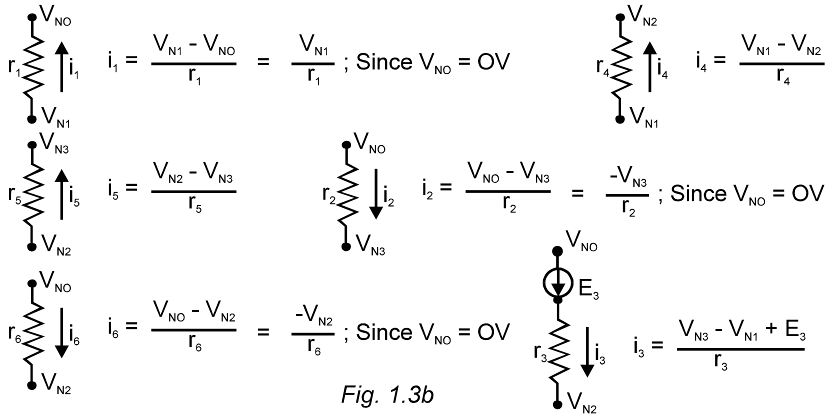


Fig. 1.3b

We shall substitute the values for the currents in equations 1, 2, 3 written according to Kirchhoff's current law. We shall obtain the following set of equations.

$$\frac{V_N 1}{r_1} + \frac{V_N 1 - V_N 2}{r_4} - \frac{V_N 3 - V_N 1 + E_3}{r_3} - J = 0 \; 102 \qquad (4)$$

$$\frac{V_2 - V_N 3}{r_5} - \frac{V_N 1 - V_N 2}{r_4} - \frac{(-V_N 2)}{r_6} = 0 \; 103 \qquad (5)$$

$$\frac{V_N 3 - V_N 1 + E_3}{r_3} - \frac{V_N 2 - V_N 3}{r_5} - \frac{(-V_N 3)}{r_2} + J = 0 \; 104 \qquad (6)$$

Substituting values of resistance e.m.f and current source into the set of equation above, we obtain the following:

$$7V_N 1 - V_N 2 - 2V_N 3 = 56 \; 105 \qquad (7)$$

$$-V_N 1 + 9V_N 2 - 4V_N 3 = 0 \; 106 \qquad (8)$$

$$-3V_N 1 + 6V_N 2 - 13V_N 3 = -84 \; 107 \qquad (9)$$

Solving the set of equations, we shall obtain the following values

$V_{N1} = 6V$; $V_{N2} = -2V$; $V_{N3} = -6V$

The current values are evaluated using the set of equations written for each branch of the circuit.

$$i_1 = \frac{V_N 1}{r_1} = \frac{6V}{20} = 0.3A \tag{108}$$

$$i_2 = \frac{-V_N 3}{r_2} = \frac{-(-6)}{3} = 0.2A \tag{109}$$

$$i_3 = \frac{V_N 3 + E_3 - V_N 1}{r_3} \tag{110}$$

$$= \frac{-6 + 16 - 6}{40} = \frac{4}{40} = 0.1A$$

$$i_4 = \frac{V_N 1 - V_N 2}{r_4} = \frac{6 - (-2)}{80} = \frac{8}{80} = 0.1A \tag{111}$$

$$i_5 = \frac{V_N 2 - V_N 3}{r_5} = \frac{-2 - (-6)}{20} = \frac{4}{20} = 0.1A \tag{112}$$

$$i_6 = \frac{-V_N 2}{r_6} = \frac{-(-2)}{20} = \frac{2}{20} = 0.1A \tag{113}$$

***EXAMPLE 3**: Determine the currents in the branches in the network of fig. 1.4, when $r_1 = 5\Omega$; $r_2 = 9\Omega$; $r_3 = 300\Omega$; $r_4 = 20\Omega$; $r_5 = 60\Omega$; $r_6 = 30\Omega$; $r_7 = 20\Omega$; $r_8 = 20\Omega$;*

$E_1 = 15V$; $E_2 = 105V$

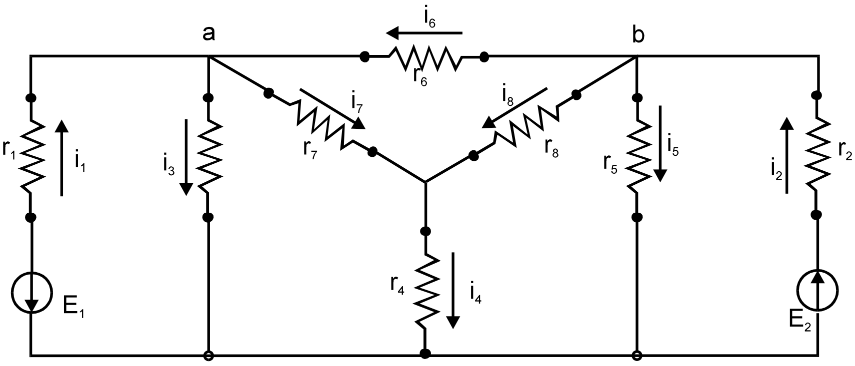


Fig. 1.4a

SOLUTION: To reduce the number of equations to be solved the star connected resistance r_7 - r_8 - r_4 replaced by its equivalent delta connection fig. 1.4b

$$r_{78} = r_7 + r_8 + r_7 \cdot r_8/r_4 = 60\Omega;$$

$$r_{47} = r_4 + r_7 + r_4 \cdot r_7/r_8 = 60\Omega;$$

$$r_{48} = r_4 + r_8 + r_4 \cdot r_8/r_7 = 60\Omega;$$

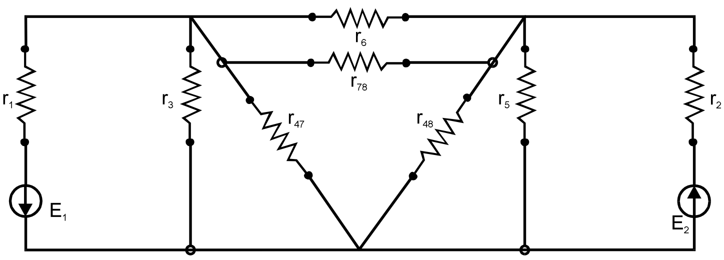


Fig. 1.4b

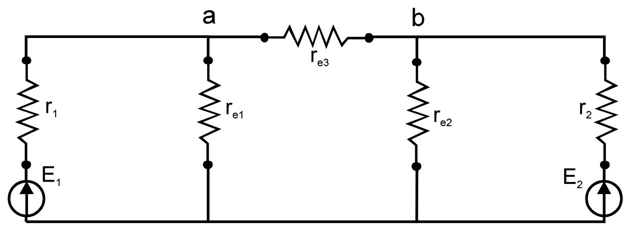


Fig. 1.4c

The obtained network is further reduced by replacing the parallel connection by their equivalent (Fig. 1.4c)

$$r_{e1} = r_3 \cdot r_{47}/(r_3 + r_{47}); \quad r_{e2} = r_5 \cdot r_{48}/(r_5 + r_{28}); \quad r_{e3} = r_6 \cdot r_{78}/(r_6 + r_{78}).$$

Finally, the delta connection r_{e1} - r_{e2} - r_{e3} shall be replaced by its equivalent star connection (fig 1.4d)

$$r_{12} = r_{e1} \cdot r_{e2}/(r_{e1} + r_{e2} + r_{e3}) = 15\Omega$$

$$r_{13} = r_{e1} \cdot r_{e3}/(r_{e1} + r_{e2} + r_{e3}) = 10\Omega$$

$$r_{23} = r_{e2} \cdot r_{e3}/(r_{e1} + r_{e2} + r_{e3}) = 6\Omega$$

The reduced network obtained contains two nodes and currents in the network is easy to determine using node - voltage method. It must be stated, that during these transformations, the potentials of point a, b, c do not change, i.e currents i_1 and i_2 of the branches, transformed also did not change.

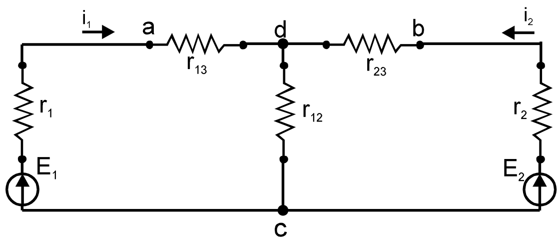


Fig. 1.4d

The voltage between node d and c

$$V_{dc} = \frac{-E_1/(r_1 + r_{13}) + E_2/(r_2 + r_{23})}{1/(r_1 + r_{13}) + 1/r_{12} + 1/(r_2 + r_{23})} = 30V \qquad 114$$

The current i_1 is given by the expression

$$i_1 = (V_{dc} - E_1)/(r_1 + r_{13}) = -3A \qquad 115$$

$$i_2 = (V_{cd} + E_2)/(r_2 + r_{23}) = 5A; \qquad 116$$

We choose, for example the potential of point C, $V_{NC} = 0$, find the potentials of node a, and b.

$V_{Na} = -E_1 - r_1 i_1 = 0$; $V_{Nb} = E_2 - r_2 i_2 = 60V$

In the given network (fig. 1.4), the currents

$i_3 = (V_{Na} - V_{Nc})/r_3 = 0$;

$i_5 = (V_{Nb} - V_{Nc})/r_5 = 1A$;

$i_6 = (V_{Nb} - V_{Na})/r_6 = 2A$.

The remaining currents are obtained using Kirchhoff's current law.

$i_7 = i_1 + i_6 - i_3 = -1A$; $\qquad i_8 = i_2 - i_5 - i_6 = -2A$.

CHAPTER TWO

AC CIRCUITS

2.1 ALTERNATING CURRENT

Alternating current is called any current which changes with time. The value of this current at any moment is called the instantaneous value. The direction of current in which its instantaneous value is positive is called positive direction, while the direction in which its instantaneous value is negative is called the negative direction.

Alternating current, whose its instantaneous value repeats at equal interval of time in the same manner, is called periodic alternating current. The smallest interval of time, through which its repetition is observed is called the period T. For a periodic current,

$$i = F(t) = F(t + T).$$

A D.C. current can be thought of as a special case of an alternating current in which the period of repetition is infinitely high and therefore the frequency is zero.

SINUSOIDAL CURRENT

Electric circuits, in which the magnitude and direction of e.m.f, voltage and current changes periodically with time by a sinusoidal law is called a circuit of sinusoidal current.

Electric network, in which the magnitude and direction of e.m.f, voltage and current changes periodically by a law other than sinusoidal law is called a circuit of non sinusoidal current.

The generators of electric station of alternating current are constructed in such a way, so that the resulting, from the coil windings, e.m.f changes follow a sinusoidal law. Sinusoidal e.m.f. in linear circuits, consisting of active resistances, inductance and capacitance, gives rise to a current, which also follows a sinusoidal law.

The originating e.m.f of self-induction in the coil and the voltage across the capacitor which arises from the expressions

$$e = - L\frac{di}{dt}, \quad i = C\frac{dVc}{dt} \quad 117$$

also have a sinusoidal character since the derivative of sinusoidal function is sinusoidal in nature. Any other periodic function has a derivative which defers from the original function. The voltage drop on the active resistor will also change in a sinusoidal manner, since

$$V_r = i \cdot r.$$

The engineering usefulness of a sinusoidal current is the fact that the efficiency obtained for generators, engines, transformers and transmission lines during sinusoidal e.m.f, voltage and current is a maximum when compared with non sinusoidal currents. Apart from this, during the former form changes of currents as a result of e.m.f of self-induction can give rise to high voltage surges in separate parts of the circuit.

Analysis and computations (calculations) of circuits involving sinusoidal e.m.f, voltages and currents are very simple. They are simpler than circuit involving non sinusoidal e.m.f., voltages and currents.

Let us discuss the mechanism of generation and basic relationship, characteristics of sinusoidal e.m.f. For this purpose it is convenient to use a simple model - coil rotating with a constant angular velocity (V) in uniform magnetic field (fig. 2.1a). The conductors of the coil moving in the magnetic field cuts the field and on the them, on the basis of law of electromagnetic induction, exist electromotive force. The magnitude of this e.m.f is proportional to the magnetic flux density B, the length of the conductor and the velocity of the moving conductor relative to the field V_t.

$$e = BLV_t$$

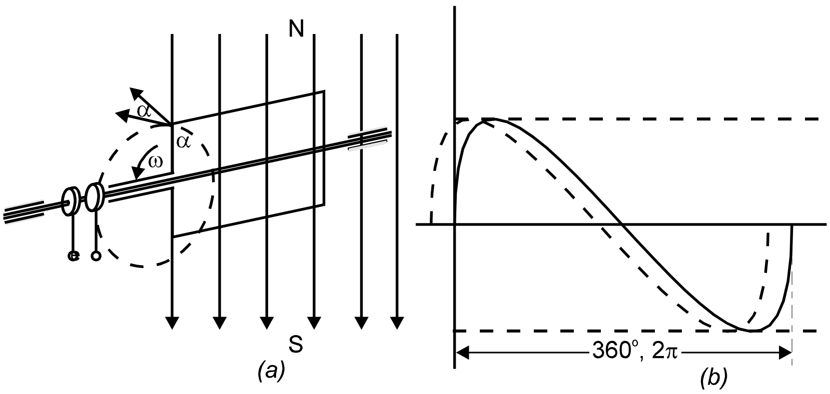


Fig. 2.1 Model explaining the geometry of sinusoidal e.m.f. (a) the graph of instantaneous values of e.m.f. (b).

We now express the tangential velocity V_t through the angular velocity V and the angle α, we get

$$e = BLV \sin\alpha \quad 118$$

The coefficient before sinα is the maximum value e.m.f obtained in the coil when $\alpha = 90°$

The angle α is the product of the angular velocity of rotation of the coil and the time t.

$$\alpha = \omega t \quad 119$$

Hence, e.m.f, generated in the coil will be

$$e = E_m \sin\alpha = E_m \sin \omega t.$$

In one rotation of the coil there is one cycle of change of e.m.f. If when t = 0, the e.m.f is not equal to zero, then the expression for e.m.f is written in the following form;

$$e = E_m \sin(\omega t + \Psi)$$

where e - the instantaneous value of the e.m.f t at any instant of time. E_m - maximum value of e.m.f or its amplitude, $(\omega t + \psi)$ - argument of the sine or the phase, characterizing the value of the e.m.f at the given instant of time; ψ -initial phase, defining the value of e.m.f when t = 0.

The time taken to complete one cycle of a process is called the period T, while period is in seconds, frequency f:

$$f = \frac{1}{T} \quad 120$$

is measured in hertz (Hz).

The quantity

$$\omega = \frac{\alpha}{t} = \frac{2\pi}{T} = 2\pi f \quad 121$$

is called the angular frequency and it is measured in radians per seconds.

The speed of rotation of the coil, n and the frequency of the e.m.f are related as follows:

$$\omega = 2\pi f = \frac{\pi n}{30} \quad 122$$

from where, we get

$$f = \frac{n}{60} \quad 123$$

Electrical energy for feeding users is generated using synchronous generators of electrical stations in the form of alternating (sinusoidal) current at frequency of 50Hz in Europe and at 60Hz in America.

The advantages of using energy generated by alternating current over the use of energy generated by DC current are so many.

First, energy sources of alternating current are cheap to produce. Synchronous generators are used. They are more reliable than the energy produced by D.C currents. They can be produced at high voltages and power.

Energy of an alternating current of one voltage can easily be transformed to another energy of different (higher or lower) voltage with the aid of a simple, cheap and reliable apparatus known as transformer, which is very important in the transmission of energy and power to long distances.

Consumers of electric energy, such as, lighting instruments and electric ovens, in which we use heaters of D.C. and A.C current, defers slightly in their economic rating, however A.C engines are cheaper, more reliable than D.C engines.

2.2 EFFECTIVE AND AVERAGE VALUES OF SINUSOIDAL CURRENT, E.M.F AND VOLTAGE.

Analysis and calculation of A.C with the use of instantaneous values of current, voltage and e.m.f requires complex and complicated computational work and time consuming. Therefore, the instantaneously values of current, voltage and e.m.f are usually replaced by equivalent unchanging values. Such a substitution simplifies the computation and helps to reduce the expression for the corresponding values.

The effective value is called that constant value of current which dissipates the same amount of energy on resistor in one period as a real changing sinusoidal current.

$$W = \int_0^T i^2 \; r\,dt = \int_0^T I_m^{\,2} \sin^2 \omega t dt \quad 124$$

with a D.C current, the energy

$$W = I^2 \cdot r \cdot T$$

when we equate the two expressions, we get;

$$I^2 \bullet r \bullet T = \int_0^T I_m^{\,2}\, r \, \sin^2 \; \omega t \; dt, \quad 125$$

we get the effective value of current:

$$I = \sqrt{\frac{1}{T}\int_0^T I_m^{\,2} \sin^2 \omega t dt} = \frac{I_m}{\sqrt{2}}\sqrt{\frac{I}{T}\int_0^T (1 - \cos 2\omega t) \; dt} \quad 126$$

$$= \frac{I_m}{\sqrt{2}} = 0.707 I_m \quad 127$$

Hence, the effective value of current is smaller than the amplitude by a factor of $\sqrt{2}$ 128.

Analogically, we can define the effective value of e.m.f and voltage.

$$E = E_m/\sqrt{2},\ V = V_m/\sqrt{2} \quad 129$$

The effective value of current is proportional to the force acting in a rotor of a.c. engine, moving a coil of a measuring instrument. When we talk of magnitude of voltages, emf and current in a.c. circuits we have in mind effective values or the root mean square values. The scales of measuring instruments are graduated correspondingly in effective values or root mean square values of current and voltage.

For example, if an instrument reads a current of 10A, then this means, that amplitude of the current is

$$I_m = \sqrt{2}I = 1.41 x 10 = 14.1A \quad 130$$

For the analysis and calculation of rectifying instruments we use average values of current e.m.f and voltage under which we understand average arithmetic value corresponding to half a period (average value in period, as we know, is zero).

$$E_m = \frac{2}{T}\int_0^{T/2} E_m \sin\omega t\, dt = \frac{2E_m}{T\omega}\int_0^{T/2} \sin\omega t\, dt \quad 131$$

$$\left|\frac{2E_m}{T\omega}\cos\omega t\right|_{\pi}^{0} \quad 132$$

since T‘ω = 2π, then

$$E_w = \frac{2E_m}{\pi} = 0.637 E_m \quad 133$$

Analogically, we can find the average values for current and voltage.

$$I_{av} = \frac{2I_m}{\pi};\quad V_{av} = \frac{2V_m}{\pi} \quad 134$$

The ratio of the effective value to the average value of any periodically changing quantity is called the form factor. For sinusoidal current

$$K_{av} = \frac{E}{E_{av}} = \frac{I}{I_{av}} = \frac{V}{V_{av}} = \frac{\pi}{2\sqrt{2}} = 1.11 \quad 135$$

2.3 SINUSOIDAL CURRENT IN A RESISTOR.

If a sinusoidal voltage $V = V_m \sin(\omega t + \Psi)$ is applied across a resistor r (fig. 2.2), then the current that will flow through the resistor is given by the expression.

$$i = \frac{V_m}{r}\sin(\omega t + \Psi) = I_m \sin(\omega t + \Psi) \quad 136$$

Hence, the voltage applied across the terminals of a resistor and the current flowing through the resistor have the same initial phase

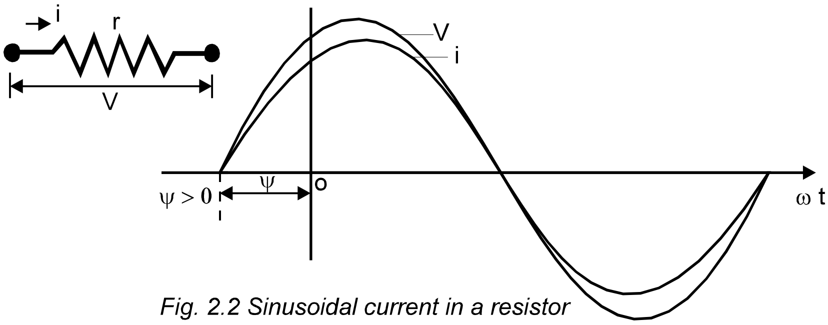


Fig. 2.2 Sinusoidal current in a resistor

The phase difference between the two sinusoidal quantities is called the phase shift between the two signals.

$\_ = \Psi_v - \Psi_i = 0$

Applying Ohm's law: $V_m = I_m \cdot r$; $V = I \cdot r$

Using conductance $g = \frac{1}{r}$ 137, we shall get

$I_m = gV_m$; $I = gV$.

The instantaneous power dissipated on the resistor

$P_r = i \cdot V \cdot = V_m I_m \sin^2(\omega t + \Psi)$
$= V \cdot I \cdot (1 - \cos 2(\omega t + \Psi))$

Changing the angular frequency, up to twice the frequency of the voltage and current, the value of the power is in the range between 0 and 2 V‘I.

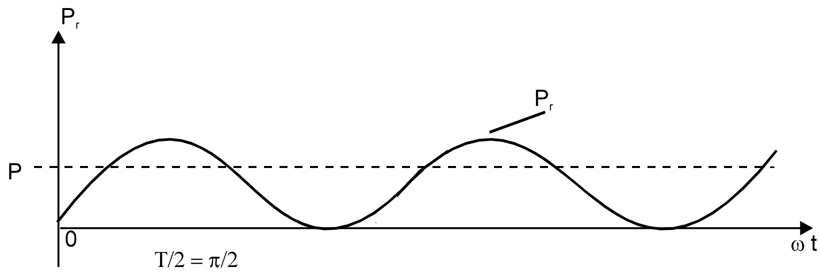


Fig. 2.3

In general cases phenomenon in a.c. circuits containing active resistance r inductance L and capacitor C, are by far much more complicated than D.C circuits the inductance and capacitance have their effects on the circuits during the time of disconnection, connection or as the current changes and there appears e.m.f of self-induction $e = -L\frac{di}{dt}$ 138 in the inductor and a voltage $V_c = \frac{1}{C}\int dt$ 139 appears across the capacitor.

In steady state condition the current in the circuit does not change and the emf equals zero, while the voltage across the capacitor V_c corresponds to a constant value.

In a.c. circuits, there is continuous change of voltage and current, which gives rise to a changing e.m.f *e* and voltage across the capacitor V_c.

As a result of this, the working regime of a.c. circuits is defined by the active resistance, the inductance L and the capacitance C.

The active resistance of a conductor of a.c current is higher than the resistance of D.C current. However, for very low frequency (about tens or hundreds Hertz) increase in the resistance is negligible and the active resistance is defined by the same formula as resistance of D.C current. For frequencies of about 100KHz and mega Hertz the active resistance can be much higher than the resistance of D.C. current and to determine this active resistance special methods are employed.

Instantaneous power in the circuit with active resistance is equal to the product of the instantaneous values of voltage and current.

$$P = V'i = V_m \sin\omega t \, I_m \sin\omega t$$

From the graph of fig 2.3, it is quite clear that the power changes from 0 to P_m, remaining positive all the time. This means, that in the circuit with active resistance the power is constantly fed into the consumer r and irreversibly transformed in it into heat, which heat up the resistance and is dissipated to the surrounding medium.

The average value of power in one period

$$P_{av} = \frac{1}{T}\int_0^T P\,dt = \frac{1}{T}\int_0^T V_m I_m \sin^2 \omega t\,dt \quad 140$$

$$= \frac{V_m I_m}{T}\int_0^T \frac{1-\cos 2\omega t}{2}\,dt = \frac{V_m I_m}{2} \quad 141$$

When we express the amplitude value of voltage and current through their root mean square value, we shall obtain

$$P_{av} = V‘I$$

After substitution of V = I‘r we will have

$$P_{av} = P = V‘I = I^2r. \qquad (2)$$

From the expression (2) it follows that average power is electrical power, which transforms into heat in an active resistance. This type of power is called active power and represented by the symbol P. A wattmeter measures active power when connected correspondingly to an a.c circuit.

2.4 CIRCUIT, CONTAINING AN INDUCTOR WITH INDUCTANCE L.

The coils of an electric machine, transformers magnetic amplifiers, electromagnets, relay, contactors, inductors, electrical heating instruments and a.c ovens all have a high value of inductance. In radioelectronics instruments inductance of coils are used for the formation of oscillating loops, electric filters etc. The parameters of coils are active resistance r and inductance L. Alternating current induces on the coils e.m.f of self-induction, which in most cases is much higher in value than the voltage drop on the active resistance.

Let us consider a coil, whose active resistance is so small, that it can be neglected.

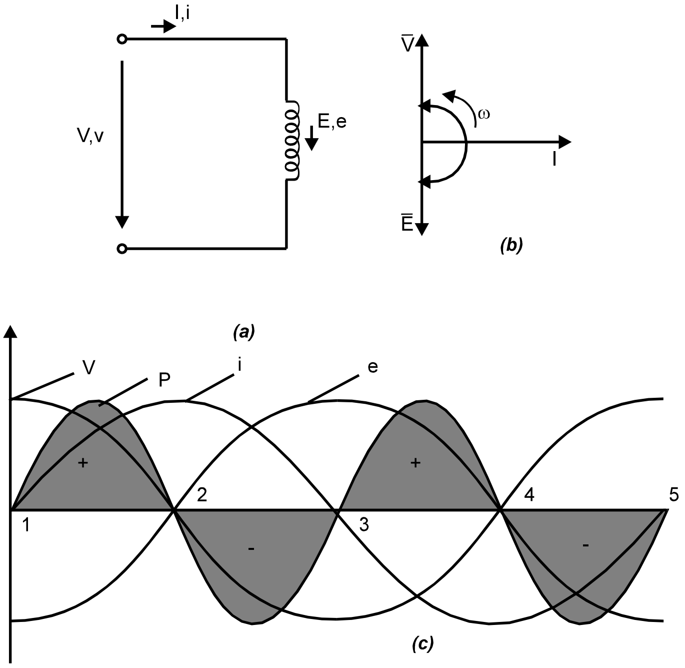


Fig. 2.4 Electric circuit containing inductance l (a) its vector diagram (b) and graphs for the instantaneous values of V, i, P (c)

For the explanation of processes taking pace in a circuit with inductance L, let us assume that the current through the inductor has a sinusoidal character and is given as:

$$i = I_m \sin\omega t \qquad (3)$$

This current will give rise to an e.m.f of self-induction in the inductor.

$$e_L = - L\frac{di}{dt} \; 142 \qquad (4)$$

Equation written in accordance to Kirchhoff's law for the given circuit have the following form:

$$e_L = - V \qquad (5)$$

When we express e_L and *i* through their value in equations (3) and (4), we find the voltage across the inductor:

$$V = L\frac{dI_m \sin\omega t}{dt} \; 143$$

We differentiate the above expression, we obtain the following;

$$V = \omega L I_m \cos\omega t = \omega L I_m \sin(\omega t + \pi/2) = V_m \sin(\omega t + \pi/2)$$

From the comparison of the expressions (5) and (4) we can conclude that, the current in a circuit with an inductor and the voltage across the inductor changes by a sinusoidal law, and the voltage leads the current in phase by an angle of 90°.

The vector diagram of the circuit with inductor is shown in fig. 2.4(b) While the graphs of the instantaneous values of current, voltage are shown in fig 2.4(c).

The voltage and current in a circuit with inductor, according to equation (5) are connected as follows

$V_m = \omega L I_m$

$I_m = V_m/\omega L$ (6)

When we divide both sides of equation (6) by $\sqrt{2}$ 144, we shall get Ohm's law for a.c circuit with inductors.

$$I = \frac{V}{\omega L} = \frac{V}{X_L}$$ 145

where $X_L = \omega L = 2\pi f L$ - inductive reactance (Ohms).

The instantaneous power in a circuit with an inductor is equal to

$P = V‘i‘ = I_m \sin 2\omega t‘V_m \,\mathrm{Sin}(\omega t + \pi/2)$

$$= \frac{V_m I_m}{2} \sin 2\omega t = V \bullet I,$$

146

$$since\ \frac{V_m}{\sqrt{2}} = V,\ \frac{I_m}{\sqrt{2}} = I$$

Instantaneous power changes with frequency, which is twice higher than the frequency of the current. The amplitude value of the power P_m is given as

$P_m = V‘I‘$

It is easy to show analytically and from the graph in fig. 2.4(c) that, the average power in one period (active power) is equal to zero.

$$P = \frac{1}{T}\int_0^T V \bullet i\, dt = 0$$ 147

To explain the energy processes taking place in the circuit with inductor we shall use the graph of fig 2.4c.

In the interval of time from t = 0 (point 1) to t = T/4 (point 2), when the current in the circuit increases from 0 to I_m, the electric energy is fed from the source into the circuit, it is transformed and stored in the inductance in the form of energy of magnetic field.

The maximum value of this energy in the form of magnetic field will be at the time corresponding to point 2 on the graph, when the current reaches it maximum value:

$$W_L = \frac{I_m^2 L}{2} \quad 148$$

We can show that this energy is equal to shaded area of the graph of P = f(t} in the time interval between points 1 and 2 (marked with (+) sign). Infact,

$$W_L = \int_0^{t/4} vi\,dt = \int_0^{T/4} \frac{V_m I_m}{2} \sin 2\omega t\,dt \quad 149$$

$$= \left| \frac{V_m I_m}{2.2\omega} (-\cos 2\omega t) \right|_0^{T/4} = \frac{V_m I_m}{2\omega} = \frac{I_m^2 X_L}{2\omega} \quad 150$$

$$= \frac{I_m^2 w_L}{2\omega} = \frac{I_m^2 L}{2\omega} \quad 151$$

In the time interval between 2 and 3 the current decreases. Energy of magnetic field is transformed into electric energy and is returned to the system (set). At the instant of time corresponding to point 3, the current and energy of the magnetic field are zero.

The energy, given to the set is equal to the shaded area of the graph P=f(t) in the time interval between points 2 and 3 (marked with (-) sign). From the graph of fig 2.4c it is obvious, that the area, defining the stored and returned energy are equal. Hence, the energy stored in the magnetic field of the inductor in the first quarter of the period is fully return to the set in the second quarter of the period.

In the next quarter of the period the direction of current and the magnetic flux changes. The process described above repeats itself.

As a result, in the circuit there is continuous periodic process of change of energy between the energy source and the inductor and during this process there is no loss of energy.

2.5 CIRCUIT, CONTAINING CAPACITOR WITH CAPACITANCE C.

In radio electronic devices the capacitor is the basic element of the oscillating circuits filters, coupling element between loops e.t.c In heavy electronic devices capacitors are used for the improvement of the power factor, as an element of higher frequency oscillating loop circuits. In any electronic device, capacitance exist and between elements of a current-carrying conductor.

For long length of wire the capacitance can be considerable and during computations of circuits at industrial frequency, the value of the capacitance can be negligible. In high frequency circuits, a small capacitance can have a considerable influence on the operating condition of the circuit and as such can not be neglected.

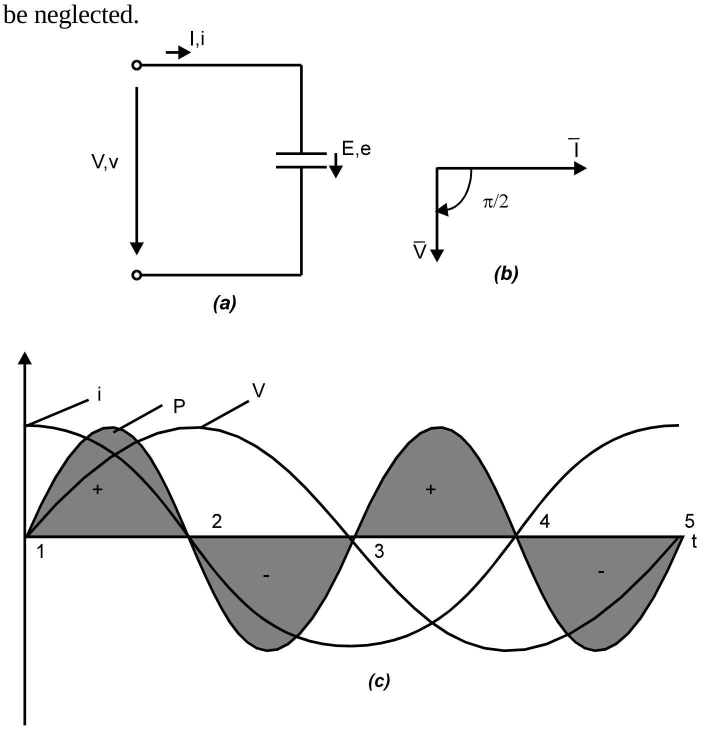


Fig. 2.5 Electric circuit containing capacitor with a capacitance C (a) vector diagram (b) Graphs of the instantaneous values of V, i, P (c).

The current in the circuit with capacitor fig. 2.5a is the movement of the charges towards the plates of the capacitor

$$i = dq/dt \qquad (6)$$

When we express the charge through capacitance C and voltage across the capacitor V_c, from the expression

$$C = q/Vc$$

we get

$$i = C\,\frac{dV_c}{dt} \quad 152$$

The voltage across the capacitor follows a sinusoidal law.

$V = V_c = V_m \sin\omega t$.

Then the current in the circuit is given by the expression:

$$i = C\ \frac{dV_m \sin\omega t}{dt}$$ 153

We take the first derivative, we get the instantaneous value of current in the circuit with capacitor:

$$i = \omega c V_m \cos\omega t = I_m \sin(\omega t + \pi/2) \qquad (7)$$

The current in a circuit with capacitor and the voltage across it are sinusoidal in nature, but the voltage lags behind the current by a phase angle of 90°. This means that if we move in a clockwise direction, we shall see the vector of current before that of the voltage.

The vector diagram and the graph of the instantaneous values of current, voltage and power are shown in fig. 2.5b and fig. 2.5c respectively.

The voltage and current in a capacitive circuit according to equation (7) are related by the expression:

$$I_m = \omega c V_m \qquad (8)$$

$$I_m = V_m/1/\omega c$$

When we divide both sides by $\sqrt{2}$ 154, we shall obtain Ohm's law for a capacitive circuit.

$$I = \frac{V}{1/\omega_c} = \frac{V}{X_c}$$ 155 (9)

where $X_c = 1/\omega c$ - capacitive reactance, Ohms.

The voltage across a capacitor can be expressed as a product of current and the capacitive reactance.

$$V = V_c = I \cdot X_c$$

The instantaneous power P in a capacitive circuit equals the product of the instantaneous values of voltage and current:

$$P = V \bullet I = V_m \sin\omega t\ I_m \sin(\omega t + \pi/2)$$ 156

$$= \frac{V_m I_m}{2} \sin 2\ \omega t = V.I. \sin 2\ \ \omega t$$ 157

$$= P_m \sin 2\omega t$$ 158

From the derived expression it follows that the instantaneous power follows a sinusoidal law with a frequency twice the frequency of the current, and its maximum value is given as:

$P_m = V \cdot I \cdot$

The average power in one period (active power), as shown in the graph (fig. 2.5c) is equal to zero.

$$P = \frac{1}{T}\int_0^T V \bullet i \, dt = 0 \quad 159$$

To explain the electric processes taking place in the circuit with capacitance we shall use the graph shown in fig. 2.5c. In the first quarter of the period, in the interval of time between points 1 and 2, the voltage across the capacitor increases, the capacitor is charging up: the electrical energy from the voltage source (set) enters the circuit, it is transformed and stored in the capacitor in the form of energy in the electric field. The stored energy is equal to the shaded area bounded by the curve P(t) (marked with (+) sign), and it remains.

$$W_c = \int_0^{T/4} Vi \, dt = \int_0^{T/4} \frac{V_m I_m}{2} \sin 2\omega t \, dt = \frac{V_m^{\ 2} C}{2} \quad 160$$

In the next quarter, in the interval of time between points 2 and 3, the current changes direction, while the voltage across the capacitor decreases. This is a process of discharge of the capacitor; electric energy stored in the electric field of the plates is returned to the set (voltage source). The energy returned to the set is equal to the area, bounded by the curve p(t) (marked with (-) sign).

From the graph of fig. 2.5c, it is clear that the area defining the stored energy are equal. Hence, the energy stored in the electric field of the capacitor in the first quarter of the period, is fully returned to the source in the second quarter of the period.

In next quarter, in the time interval between points 3 and 4, the polarity of the voltage on the plates changes. The capacitor is again charged; electrical energy enters the circuit and it is stored in it, in the form of energy of the electric field. In the last quarter, in the time interval between points 4 and 5 discharge of the capacitor takes place: energy in the form of electric field is returned to the source.

In conclusion, in a circuit with capacitor as in circuit with inductor there is continuous process of change of energy between the source and the capacitor. During this time there is no loss of energy.

2.6 CIRCUIT CONTAINING RESISTANCE AND INDUCTANCE.

A real coil of any electrical device has some amount of active resistance and inductance L. For ease of analysis these types of circuits with coils are usually presented in the form of two ideal elements of r and L connected in series (fig. 2.6a). Using the conclusions, derived from analysis of ideal circuits, a circuit branch with inductance L will be considered as a branch having an inductive reactance X_L. The voltage equation, written according to KVL for circuits with r and L, have the following form

$$V = V_r + V_L$$

when we express the voltage V_r and V_L through current $i = I_m \sin \omega t$ and the impedance of the branch r and X_L, we get

$$V = I_m r \sin \omega t + I_m X_L \sin(\omega t + \pi/2) \quad 161$$

where $V_r = I_m r\sin\omega t$ - voltage across the active resistance (active voltage) having the same phase as the current; $V_L = I_m X_L \sin(\omega t + \pi/2)$ - voltage across the inductive reactance (inductive voltage), which leads current by a phase angle of 90°(π/2).

On the vector diagram (fig. 2.6b), vector V_r is in phase with the current, and vector V_L leads the vector of current by a phase angle of 90° such a diagram is sometimes called the voltage triangle.

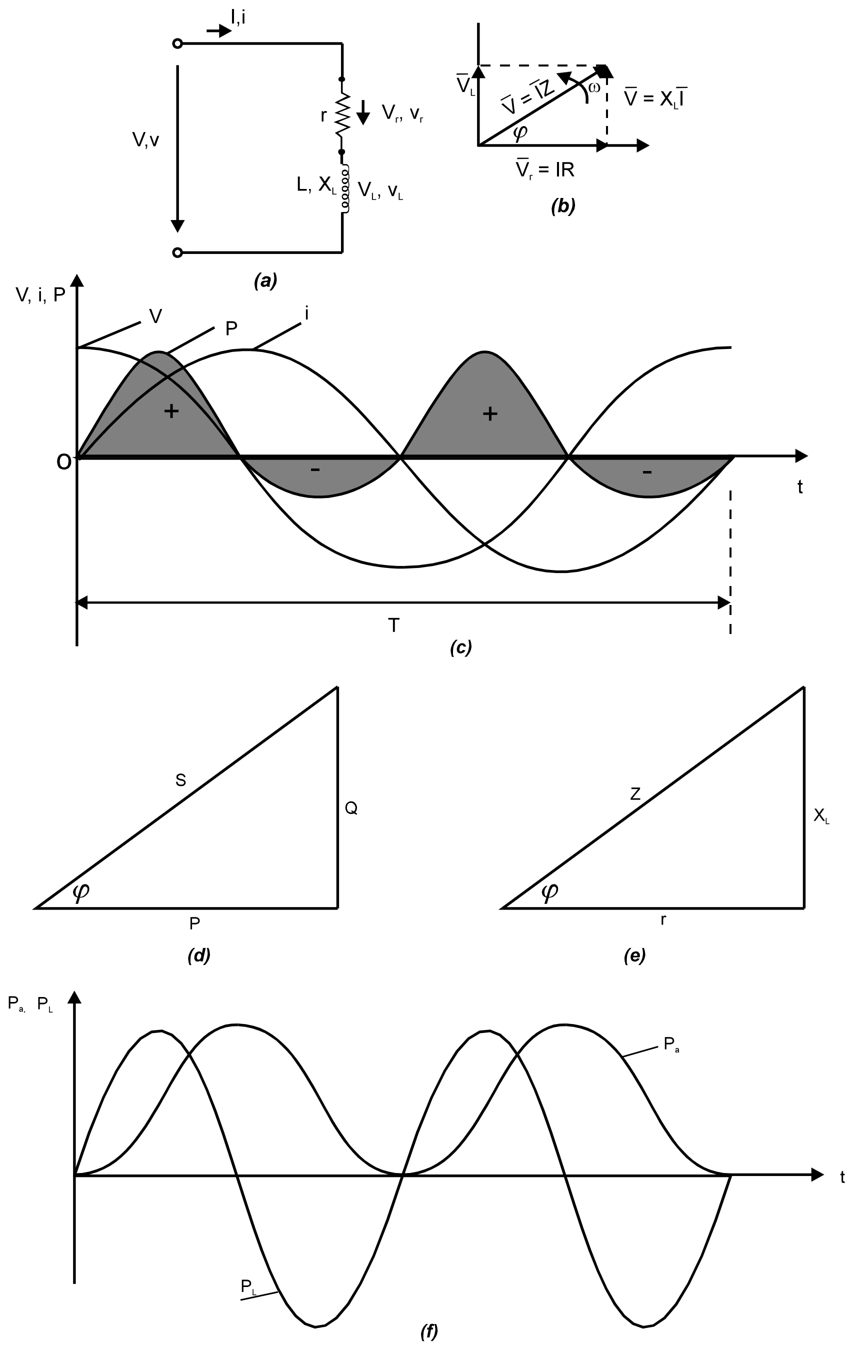


Fig. 2.6 Electric circuit containing r and I (a), Its vector diagram (b) Graphs of the instantaneous values of V, i, P (c), voltage (d) and power triangles (e), graphs of instantaneous values of active and reactive powers (f).

From the vector diagram it follows, that the vector of voltage of the set is equal to the geometric sum of vector

$$\vec{V} = \vec{V}_r + \vec{V}_L \quad 162$$

and its magnitude

$$V = \sqrt{V_r^{\,2} + V_L^{\,2}} \quad 163$$

when we express the voltage through the current and impedance, we shall obtain the following

$$V = \sqrt{(I \bullet r)^2 + (I \bullet X_L)^2} = I\sqrt{r^2 + X_L^2} \quad 164$$

The expression above is called Ohm's law for the circuit with r and L

$$I = \frac{V}{\sqrt{r^2 + X_L^{\,2}}} = \frac{V}{Z} \quad 165$$

where $Z = \sqrt{r^2 + X_L^{\,2}}$ 166 total impedance of the circuit, [Ohms].

From the vector diagram it follows, that the voltage of the r L circuit leads the current by a phase angle of _ and the instantaneous value

V = $V_m$sin(ωt + _)

The graphs of the instantaneous values of voltage and current of the circuit are shown in fig. 2.6c. Angle _ is determined from the relation.

$$\cos\varphi = \frac{V_r}{V} = \frac{I \bullet r}{I \bullet Z} = \frac{r}{Z} \quad 167$$

$$= \frac{r}{\sqrt{r^2 + X_L^2}} \quad 168 \qquad (1)$$

As seen from the equation above, $\cos\varphi$ 169, and hence angle _ depends only on the parameters of the circuit r and L.

Divide the sides of voltage triangle by current, we obtain triangle of impedance (fig. 2.6e). The sides of the triangle of resistance are ordinary lines and not vectors, since resistance is a constant, does not change by sine magnitude law.

The instantaneous power of rL circuit is the product of instantaneous values of voltage and current

$$P = V \bullet i = I_m \sin\omega t \bullet V_m \sin(\omega t + \varphi) \quad 170$$

Average power in one period,

$$P_a = \frac{1}{T}\int_0^T V \bullet i\, dt = \frac{1}{T}\int_0^T I_m V_m \sin \omega t \sin(\omega t + \varphi) \bullet dt \quad 171$$

we express product of sine through difference of cosines, after term by term integration, we obtain the following

$$P_a = \frac{1}{T}\int_0^T \frac{V_m I_m}{2} [\cos\varphi - \cos(2\cos\omega t + \varphi)]$$

172 (2)

$$= V\, I \cos\varphi$$

When we substitute in place of cos _ its value in equation (1), we shall obtain

$$P_a = P = VI\cos\varphi = VI \bullet \frac{r}{Z} = I^2 \bullet r\ldots\ldots\ldots\ldots\ldots(3) \quad 173$$

From (3) it follows that, the average power in the rL is the active power, which is dissipated in the active resistance in the form of heat.

The graph of instantaneous power is drawn in fig. 2.6c.

For the analysis of electrical processes taking place in the rL circuit, instantaneous power is convenient to present in the form of the sum of instantaneous values of the active power $P_a = V_r \cdot I$ and the reactive (inductive) power $P_L = V_L \cdot i$

$P = P_a + P_L$

The graphs of P_a; P_L (t) is drawn in fig. 2.6f. The graph of $P_a(t)$ is analogical to the graph for circuit with active resistance r (section 2.2), and the graph of $P_L(t)$ is similar to the graph for circuit with inductance L (section 2.3).

Hence, energy process in rL circuit can be considered as collection of processes taking place in circuits with active resistance r and circuit with inductance L.

From the graph of $P_a = f(t)$ it is clear, that the active power is continuously released into the circuit from the source and is dissipated on the active resistance in the form of heat.

The power equals to

$$P = \frac{1}{T}\int_0^T V_{mr} I_m \sin\omega t \sin^2 \omega t\, dt = V_r \bullet I \quad 174$$

= VIcos_

The instantaneous power P_L, due to the inductance, is continuously circulating between the source and the coil. Its average value in one period equals to zero.

$$P_L = \frac{1}{T}\int_0^T V_{mL} I_m \sin\omega t \sin(\omega t + \pi/2) dt = 0 \quad 175$$

2.7 ACTIVE, REACTIVE AND FULL POWERS OF THE CIRCUIT.

When we multiply the sides of the voltage triangle by current, then we shall obtain power triangle (fig. 2.6d).

The sides of the power triangle are designated as follows

$P = V_r \cdot I \cdot = I^2 \cdot r$ - active power [watt] [k watt]

$Q = V_L I = I^2 X_L$ - reactive power, VAR, KVAR

$S = V \cdot I \cdot = I^2 Z$ - full (apparent) power of the circuit, VA, KVA.

cos_ = p/s - power factor of the circuit.

From the triangle of power we can establish the following between P,Q,S and Cos_;

$$P = S \cos\varphi = V\ I \cos\varphi \quad 176$$

$$Q = S \bullet \sin\varphi = V \bullet I \bullet \sin\varphi \quad 177$$

$$S = \sqrt{P^2 + Q^2} = V \bullet I \quad 178$$

The unit of measurement of active power is the watt or kilo watt; reactive power is the volt-ampere reactive or kilovolt-ampere reactive (KVAR), full power - is the volt-ampere (VA) or the kilovolt-ampere (KVA).

Reactive power, due to the energy of the magnetic field (inductance), does not perform any useful work, however it has a great influence on the working condition of the electric circuit. As it circulates along the conductors of the transformers, generators, engines, transmission lines, it loads them. Therefore calculation of conductors and other elements of devices of A.C circuits is accomplished using full power S, which takes into account the active and the reactive powers.

The power factor has a great practical value: It shows, which part of the full power that is active power. The full power and the power factor are calculated values and in short they are determined by overall dimensions of the transformers, generators, engines and other electrical devices.

The measurement of active, reactive full powers and cos_ and also parameters r,L can be done with the aid of a wattmeter, ammeter and voltmeter, connected in circuit as shown in fig. 2.7.

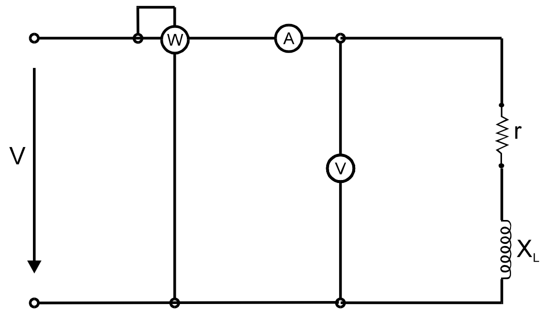


Fig. 2.7 Circuit of connection of instrument of active, reactive and full power, also its parameters.

The wattmeter measures the active power of the circuit. The full power of the circuit equals to the product of the indication of the voltmeter and the ammeter.

The reactive (inductive) power and power factor are determined by calculation through formula.

$$Q = \sqrt{S^2 - P^2}, \ \cos\varphi = \ 179$$

Active resistance is found from the formula

$P = I^2r$

whence $r = P/I^2$

The total impedance of the circuit

$Z = V/I$

The inductive reactance

$$X_L = \sqrt{Z^2 - r^2} \ 180$$

The inductance L is determined from the formula

$X_L = 2\pi fL$

whence $L = \dfrac{X_L}{2\pi f}$ 181

EXAMPLE 1*: The instruments connected in the circuit above indicate the following values: W → P = 500watt, A → I = 5A, V = U = 400V.*

Determine the active resistance r and the inductance L, if the frequency of the set *is 50Hz.*

SOLUTION: Active resistance of the circuit

r = P/I^2 = 500/5^2 = 20 Ohms

The inductive reactance

$$X_L = \sqrt{Z^2 - r^2} = \sqrt{\left(\frac{V}{I}\right)^2 - r^2} \quad 182$$

$$= \sqrt{\left(\frac{400}{5}\right)^2 - 20^2} = 77.5\ Ohms \quad 183$$

The inductance of the circuit L

$$L = \frac{X_L}{2\pi f} = \frac{77.5}{2\,x\,3.14\,x\,50} = 0.247\,H \quad 184$$

2.8 CIRCUIT, CONTAINING r and C

Using the derivations from the section on circuit with capacitor of capacitance C we shall present the circuit containing rC as one having capacitive reactance X_c. In this we use the voltage equation of the circuit (fig. 2.8a) will have the following form:

$V = V_r + V_c$

The voltage across the active resistance

$$V_r = I_m \bullet r \bullet \sin\omega t \quad 185$$

is in phase with the current. The voltage across the capacitor

$$V_c = I_m \bullet X_c \bullet \sin(\omega t - \pi/2) \quad 186$$

lags the current by a phase angle of π/2(90°)
Hence, the voltage V, applied across the circuit, will be equal to the following.

$$V = I_m \bullet r \bullet \sin(\omega t + I_m\ X_c\ \sin(\omega t - \pi/2) \quad 187$$

$$= V_{mr}\ \sin\ \omega t + V_{mc}\ \sin(\omega t - \pi/2) \quad 188$$

The vector diagram is shown in fig. 2.8b. The vector Vr is in phase with the current vector, vector $\vec{V}_r$ 189 is in phase with the current vector, vector $\vec{V}_c$ 190 lags the current vector by angle 90°. From the diagram, it follows, that the voltage vector, applied

across the circuit, equals to the geometric sum of the vectors of
$\vec{V}_r$ 191 and $\vec{V}_c$ 192.

$$V = \vec{V}_r + \vec{V}_c \quad 193$$

and its magnitude

$$V = \sqrt{{V_r}^2 + V_c^2} \quad 194$$

when we express Vr and Vc through current and impedance, we shall obtain

$$V = \sqrt{(I_r)^2 + (IX_c)^2}, \quad 195$$

whence we obtain

$$V = I\sqrt{r^2 + {X_c}^2}, \quad 196$$

The last expression is Ohm's law for r and c circuit.

$$I = \frac{V}{\sqrt{r^2 + {X_c}^2}} = \frac{V}{Z} \quad 197$$

where $Z = \sqrt{r^2 + {X_c}^2}$ 198 - total impedance, [Ohms]

From the vector diagram it follows, that the voltage of the rc circuit lags the current at an angle _ and its instantaneous value

$$V = V_m \sin(\omega t - \_)$$

The graphs V, i = f(t) are drawn on fig. 2.8c. When we divide the sides of the voltage triangle by current, we shall obtain triangle of impedance (fig. 2.8e), from which we can define cosine of the shift in phase between voltage and current.

$$\cos\varphi = \frac{r}{Z} = \frac{r}{\sqrt{r^2 + {X_c}^2}} \quad 199 \qquad (1)$$

The instantaneous power of the circuit

$$P = V \cdot i = I_m \sin\omega t V_m \sin(\omega t - \_)$$

The average power in one period

$$P_a = \frac{1}{T}\int_0^T V \bullet i dt = \frac{1}{T}\int_0^T I_m V_m \sin\omega t \sin(\omega t - \varphi) dt \quad 200$$

$$= VI\cos\_ \qquad (2)$$

Substitute in (2) in place of cos_ its value from (1), we shall obtain

$$P_a = P = VI\cos\varphi = V \bullet I \bullet \frac{r}{Z} = I^2 \bullet r \quad 201 \qquad (3)$$

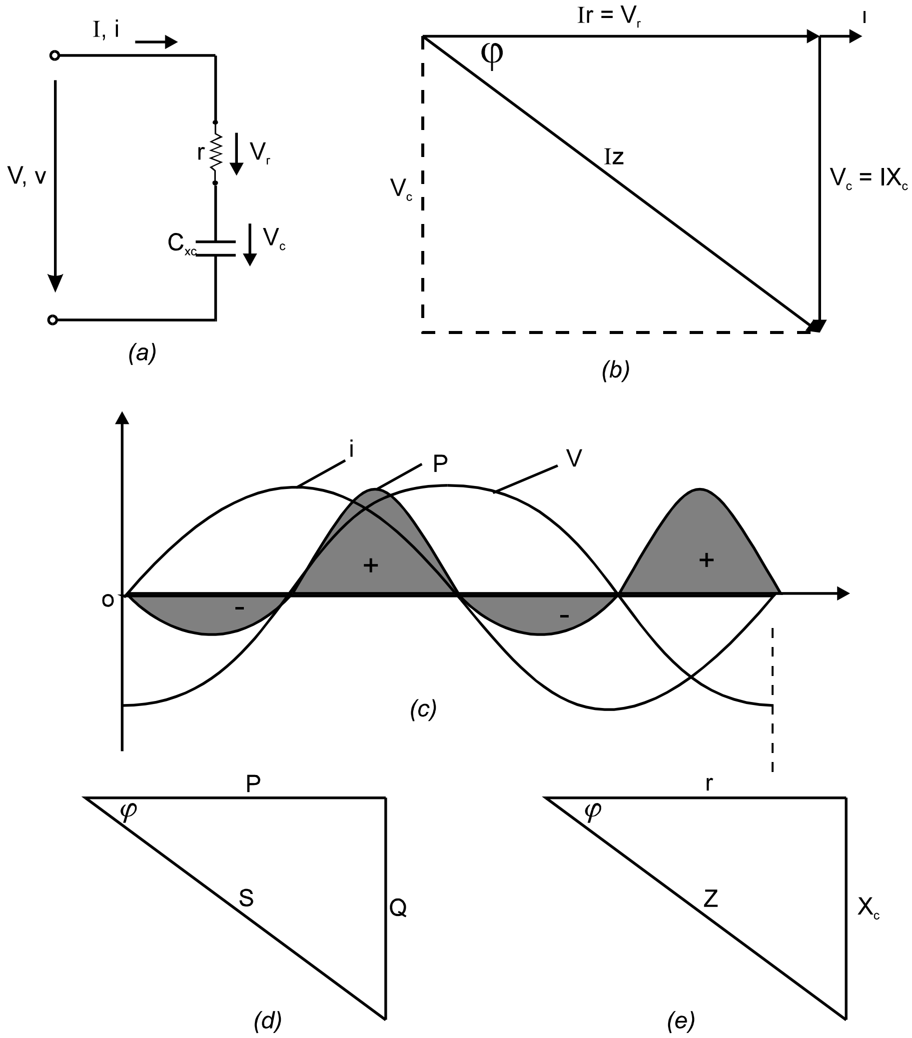


Fig. 2.8 Electric circuit containing r and C (a) Its vector diagram (b), graphs of instantaneousvalues of V, i, P (c), triangle of power and impedance (d) and (e).

As a result, the average power in r.C circuit, as in r, L circuit is presented as active power, which is dissipated on the active resistance in the form of heat.

The graph of the instantaneous power is shown in fig. 2.8c.

Energy processes taking place in an r.c circuit can be considered as a collection of processes taking place in separate circuits of r and c. From the source, there is continuous energy given to the circuit. There is active power given to the circuit from the source. The reactive power, due to the electric field of the capacitor, continuously circulates between the source and the circuit. Its average value in one period is zero.

The active; reactive; and full powers are given as follows

$P = V_r I = I^2 \cdot r \cdot$ - active power of the circuit [watt]

$Q = V_c I = I^2 Xc$ - reactive power (capacitive) of the circuit, [VAR]

$S = V \cdot I \cdot = I^2 Z$ - full power (apparent) of the circuit, [VA]

cos_ = p/s - power factor of the circuit.

The measurement of active, reactive, full power and cos_, and also parameters of the rC can be carried out with the aid of wattmeter, ammeter and voltmeter, connected as in fig 2.7, in which in place of L we should connect C.

2.9 A Series Circuit With r, L and C.

The voltage equation for an rLC circuit shown in fig. 2.9a is given below:

$$V = V_r + V_L + V_c \qquad (1)$$

The current in the circuit changes by a sinusoidal law.

$$i = I_m \sin \omega t \quad 202$$

when we express the voltages in equation (1) through current and resistance (reactance) of the elements, we shall obtain

$$V = I_m r \sin \omega t + I_m X_L \sin(\omega t$$

203

$$+ \pi/2) + I_m X_c \sin(\omega t - \pi/2)$$

$$= V_{mr} \sin \omega t + V_{mL} \sin(\omega t + \pi/2) + V_{mL} \sin(\omega t - \pi/2) \quad 204$$

The vector diagram for the circuit of fig. 2.9a is drawn in fig. 2.9b and c. The vector of voltage across the resistor is in phase with the vector of the current through the resistor, the vector of voltage across the inductor leads the current through the inductor by a phase angle of 90° while the vector of voltage across the capacitor lags the current through the capacitor by a phase angle of 90°.

Hence, between the vectors of voltages across the inductor and capacitor is a phase angle of 180°.

If $X_L > X_C$, then $V_L > V_C$ and vector diagram will have the shape, shown in fig 2.9b, while the triangle of impedance is shown in fig. 2.9d. If $X_C > X_L$, then $V_C > V_L$, and the vector diagram will have the shape shown in fig. 2.9c and the triangle of impedance is shown in fig. 2.9e. The voltage vector applied to the circuit is equal to the geometric sum of the voltage vectors across the elements separately.

$$\vec{V} = \vec{V}_r + \vec{V}_L + \vec{V}_c \quad 205$$

and its magnitude

$$V = \sqrt{(V_r)^2 + (V_L - V_c)^2} \quad 206$$

When we express the voltage through current and impedance, we shall obtain

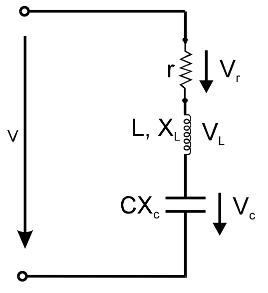
$$V = \sqrt{(Ir)^2 + (IX_L - IX_c)^2} = I\sqrt{r^2 + (X_L - X_C)^2} \quad 207$$

The last expression can be presented according to Ohm's law.

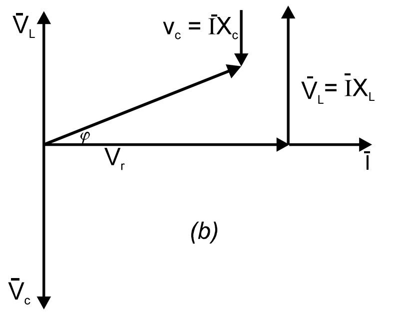
$$I = \frac{V}{\sqrt{r^2 + (X_L - X_c)^2}} = \frac{V}{Z} \quad 208$$

$$Z = \sqrt{r^2 + (X_L - X_c)^2} = \sqrt{r^2 + X^2} \quad 209$$

total impedance of the circuit, $X = X_L - X_C$ reactance of the circuit, in Ohms.



(a)



(b)

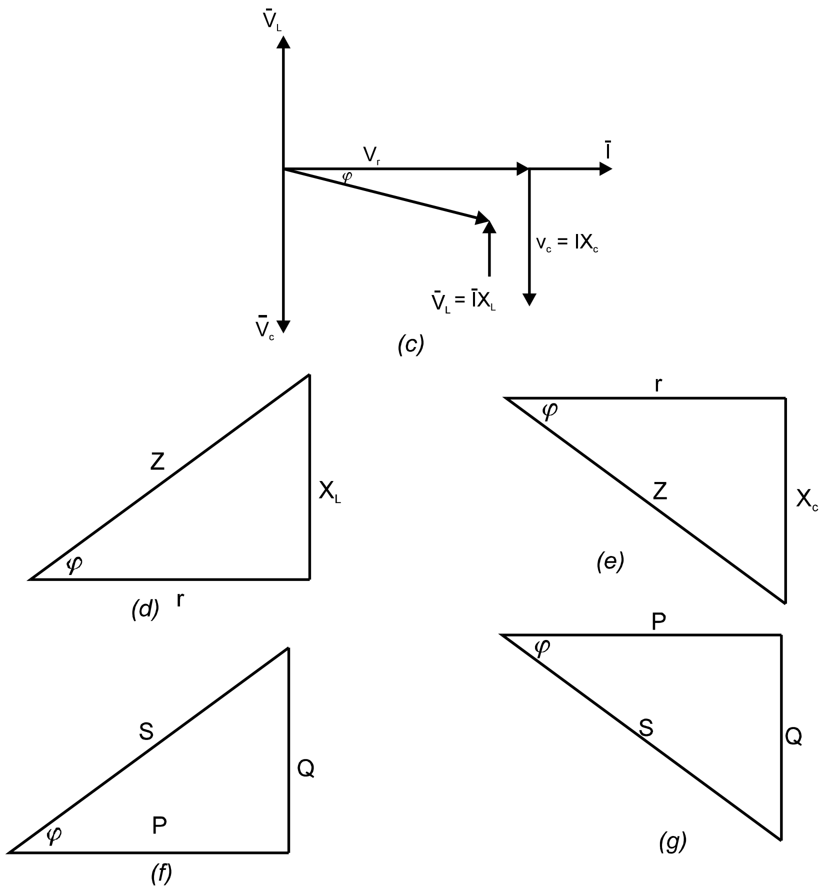


Fig. 2.9. Electric circuit containing a series r, L, and C elements (a), (a) Vector diagram for $X_L > X_C$ (c) Vector diagram for $X_c > X_L$ (d) and (e) Triangles of impedance (f), (g) Triangle of power

The power triangles for $X_L > X_C$ and $X_C > X_L$ are shown in fig. 2.9 (f) and (g) respectively.

The active power $P = V_r I = I^2 r$

The reactive power

$$Q = Q_L - Q_c = V_L \bullet I - V_c I = I^2 X_L - I^2 X_c \quad 210$$

The full power $S = V.I. = I^2 Z = \sqrt{P^2 + Q^2}$ 211

$\cos\varphi = \frac{r}{z} = \frac{P}{s}$ 212 - is the power factor of the circuit.

On the basis of the above analysis of a series r. L and C circuit one can make the following conclusion:

If $X_L > X_c$, then, the voltage of the set leads the current by a phase angle of Ψ

$$V = V_m \sin(\omega t + \varphi) \quad 213$$

The circuit has an inductive character, since $Q_L > Q_c$

If $X_c > X_L$ then: the voltage of the set lags the current by a phase angle of Ψ

$$V = V_m \sin(\omega t - \varphi) \quad 214$$

The current has a capacitive character, since $Q_c > Q_L$.

EXAMPLE 1: *Calculate the total impedance, current; active, and reactive power and also the voltage across the various elements.*

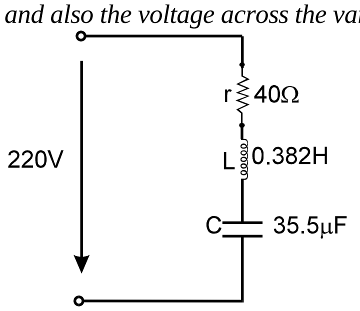


Fig. Q1

f = 50Hz

the inductive reactance

$$X_L = 2\pi f_c = 2 \times 3.14 \times 50 \times 0.382 = 120\Omega \quad 215$$

$$X_c = \frac{1}{2\pi f_c} = \frac{10^6}{2 \times 3.14 \times 50 \times 0.355} = 90\omega \quad 216$$

impedance

$$Z = \sqrt{r^2 + (X_L - X_c)^2} = \sqrt{40^2 + (120 - 90)^2} = 50\Omega \quad 217$$

I = V/Z = 220/50 = 4.4A

Active power = I^2r = 970 x 0.8 = $(4.4)^2$ x 40 = 775W

$$Reactive\ power = Q = S \bullet \sin\varphi \quad 218$$

= $(4.4)^2$ (120 - 90) = 580VAR

Voltage across each element of the circuit

V^r = I.r = 4.4 x 40 = 176V

V_L = I.X_L = 4.4 x 120 = 528V

V_c = I.X_C = 4.4 x 90 = 396V

EXAMPLE 2: *Calculate the current in the circuit in example 1 above when the frequency of the set f = 500Hz.*

SOLUTION: The active resistance of the circuit practically does not change r = 40Ω

The inductive reactance of the circuit

$$X_L = X_L 50Hz\ x\ \frac{f_1}{f_2} = 120\ x\ \frac{500}{50} = 1200\Omega \quad 219$$

Capacitive reactance

$$X_c = X_c 50Hz\ x\ \frac{f_2}{f_1} = 90\ x\ \frac{50}{500} = 9\Omega \quad 220$$

Total impedance of the circuit

$$Z = \sqrt{r^2 + (X_L - X_c)^2} = \sqrt{(40)^2 + (1200 - 9)^2} = 1201\Omega \quad 221$$

current in the circuit

I = V/Z = 220/1201 = 0.183A

EXAMPLE 3: *Determine the character of the load, the full active and reactive power of the circuit in which the instantaneous value of the voltage and current are given as follows:*

$V = 282\ sin(\omega t + 60°)$

$i = 141\ sin(cot\omega t + 30°)$

SOLUTION: the initial phase of voltage _1 = 60° is greater than the initial phase of the current _2 = 30°, as such the voltage leads the current by a phase angle of 60° - 30° = 30°, the circuit has an inductive character and the load has inductive character.

The full power

$$S = V\ I = \frac{V_m}{\sqrt{2}}, \frac{I_m}{\sqrt{2}}, = \frac{282}{\sqrt{2}}\ x\ \frac{141}{\sqrt{2}} = 2000VA \quad 222$$

The active power

P = Scosφ 223 = 2000cos30° = 2000 _3/2 = 1730W

The reactive power

Q = S.sinφ 224 = 2000sin30° = 1000VAR

2.10 RESONANCE (VOLTAGE RESONANCE)

It is known that in mechanical system resonance occur whenever the frequency of the mechanical system equals the frequency of the disturbing force acting on the system. Oscillation of mechanical system, e.g. oscillation of pendulum is accompanied by a periodic transition of kinetic energy into potential energy and vice-versa. During resonance of mechanical system a small disturbance can give rise to deep oscillation of the system, e.g a high amplitude oscillation of a pendulum.

In A.C circuits, where at the same time, there is an inductor and a capacitor the phenomenon of resonance can occur, which is analogical to the resonance in mechanical systems. However, a complete analogy: equality of the frequency of the disturbance (frequency of the voltage of the source) is possible only for ideal loop (loop without loss r = 0), the resonance frequency equals to the following.

$$f_o = \frac{1}{2\pi\sqrt{Lc}} \quad 225$$

In general, under resonance in electric circuit we understand a situation of the circuit, when the current and the voltage are in phase, and the equivalent circuit presents itself as active resistance. Resonance is a condition which occurs in an electrical circuit whereby the reactance of the circuit is equal to zero i.e, the inductive reactance is equal to the capacitive reactance or a situation whereby the total input reactive admittance is equal to zero. During resonance the current at the input of the circuit, if not equal to zero, is in phase with the voltage applied to the circuit.

Resonance in electric circuit is accompanied by a periodic transfer of energy of electric field in the capacitor into the energy of the magnetic field of the inductor and vice-versa. During resonance in electric circuit, small voltages, applied to the circuit can give rise to high currents and voltages in the separate branches of the circuit. A circuit where r, L, C are connected in series can give rise to voltage resonance.

Let us consider the phenomenon of voltage resonance in fig. 2.10a.

As already mentioned above, during resonance the current and voltage are in phase, i.e, the phase difference φ 226 = 0, and the total impedance of the circuit is equal to its active resistance.

$$r = \sqrt{r^2 + (X_L - X_c)^2} = r \quad 227$$

The equality will be valid, if $X_L = X_c$ i.e the reactance of the circuit is equal to zero

$$X_L - X_c = 0$$

When we express X_L and X_c correspondingly through L, C and f, we shall obtain

$$2\pi fL = \frac{1}{2\pi fC} \quad 228$$

whence $$f = \frac{1}{2\pi\sqrt{LC}} = f_o \quad 229$$

where f - frequency of the voltage source, applied to the loop, f_o - resonance frequency.

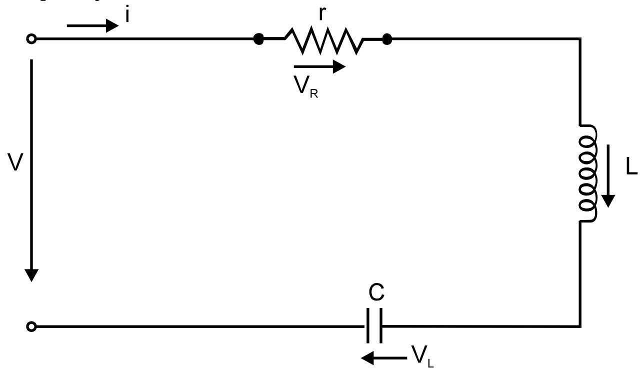


Fig. 2.10 Series r, L, C circuit with resonance.

Hence, when $X_L = X_c$, voltage resonance takes place, since the resonance frequency f_o equals the frequency of the voltage connected to the circuit.
From Ohm's law for a series circuit

$$I = \frac{V}{\sqrt{r^2 + (X_L - X_c)^2}} \quad 230$$

It arises that the current in the circuit during resonance is equal to the voltage, divided by the active resistance.

$$I = V/r$$

The current in the circuit can appear to be by far greater than current in the circuit without resonance.

During voltage resonance, the voltage across the inductor is equal to the voltage across the capacitor.

$$IX_L = IX_c = V_L = V_c$$

For higher values of X_L and X_c than r these voltages can be many times greater than the voltage of the source.

The voltage across the active resistance is equal to the voltage, applied to the circuit

$V_r = Ir = V$

Fig. 2:11 represents the vector diagram of the circuit of fig. 2:10 for voltage resonance.

The diagram confirms the fact, that the current is in phase with the voltage of the source and that the voltage across the active resistance is equal to the voltage of the source.

The reactive power during resonance is equal to zero.

$Q = Q_L - Q_c = V_L I - V_c I = 0$

since $V_L = V_c$

The full power is equal to the active power:

$$S = \sqrt{P^2 + Q^2} = P \quad 231,$$

since the reactive power is equal to zero. The power factor is equal to one.

$$\text{Cos}\,\varphi \quad 232 \quad = p/s = r/z = 1$$

As long as voltage resonance takes place, when the inductive reactance of the series circuit equals the capacitive reactance, and their values are determined by the inductance, capacitance of the circuit and the frequency of the set respectively:

$$X_L = 2\pi fL, \; X_c = \frac{1}{2\pi fC}, \quad 233,$$

resonance can be obtained either by choosing the parameters, of the circuit with a given frequency of the source or by selecting a frequency of the source with a given set of parameters of the circuit.

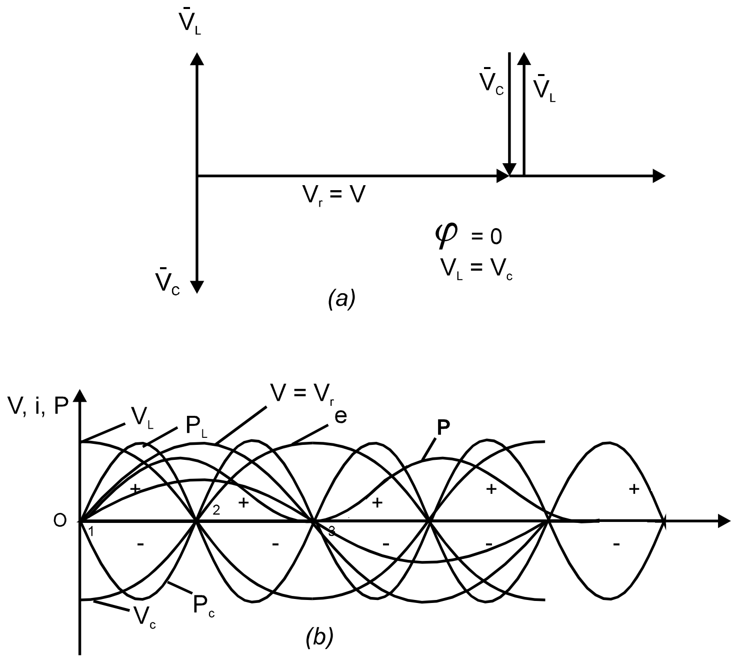


Fig. 2.11 (a) Vector diagram of a series R, L, C circuit
(b) graphs of instantaneous values of V, i, P of a series
R, L, C with resonance

Fig. 2:11 (b) represents the graphs of the instantaneous values of current i, voltage V of the source and the voltages V_L, V_c, V_r across the separate elements, and also the active power and reactive power P_L, P_c in one period, during voltage resonance, with the aid of these graphs we can follow the energy processes, taking place in the circuit during voltage resonance.

The active power P is positive all the time: It is delivered from the source to the active resistance and it is dissipated on the resistor in the form of heat. the reactive power P_L and P_c change in sign as seen from the graph, and their average values are equal to zero.

At the moment of time t =0 (point 1 on the graph fig. 2:11b) the current in the circuit i = 0 and the energy of the magnetic field W_L = 0. The voltage across the capacitor is equal to the amplitude value V_{mc}, the capacitor is charged and the energy of its electric field

$$W_c = \frac{V_{mc}^2 . C}{2} \quad 234$$

In the second quarter of the period, in the time interval between points 1 and 2 the voltage across the capacitor and hence, energy of the electric field decreases. The current in the circuit and the energy of the magnetic field increases.

At the end of the first quarter of the period (point 2) V_c =0; W_c =0; i =I_m,

$$W_L = \frac{I_m^{\,2} \bullet L}{2} \quad 235$$

In this manner, in the first quarter of the period, the energy of the electric field is transfer into energy of the magnetic field.

Since the area P_c =f(t) and P_L =f(t), expressing the stored energy in the electric field and magnetic field are equal, all the energy of the electric is transformed into energy of the magnetic field in the inductance. In the second quarter of the period, i.e the interval between points 2 and 3, the energy of the magnetic field is transformed into energy of the electric field.

Analogically the picture repeats itself even in the next quarter of the period.

In this manner, during resonance the reactive energy is circulating internally in the loop from the inductance to the capacitance and a reverse exchange of reactive energy between the source and the circuit does not take place. The current in the lines connecting the source and the circuit is due only to the current produced by the active power.

For the analysis of the circuit, we often make use of the frequency method, which helps us to explain in details the dependence of the parameters of the circuit and other quantities on the frequency.

In fig. 2:12 is shown the graphs of the dependence of V_r, V_c, V_L, I, r, X_c, X_L on the frequency when the voltage of the source is kept unchanged.

When f = 0, the inductive reactance $X_L = 2\pi fL = 0$, while the capacitive reactance $X_c = 1/2\pi fc = \infty$, the current I = 0, the voltage V_r = Ir = 0, $V_L = IX_L = 0$, V_c = V. When f = f_o, $X_L = X_c$, I = V/r, $V_L = V_c$, V_r = V. When $f \to \infty$, $X_L \to \infty$, $X_c \to 0$, $I \to 0$, $V_r \to 0$, $V_c \to 0$, $V_L \to V$.

In the frequency interval from f=0 to f=fo, the load has a capacitive character, the current leads the voltage of the source in phase. In the frequency interval from f = f_o to $f \to \infty$ the load impedance has an inductive character, the current lags behind the voltage of the source in phase.

The maximum value of voltage across the capacitor is obtained at frequency a bit less than resonance frequency, while across the inductor the maximum is obtained at frequency a bit greater than the resonance.

The phenomenon of resonance is widely used in radio electronics instruments and in industrial electronics equipments.

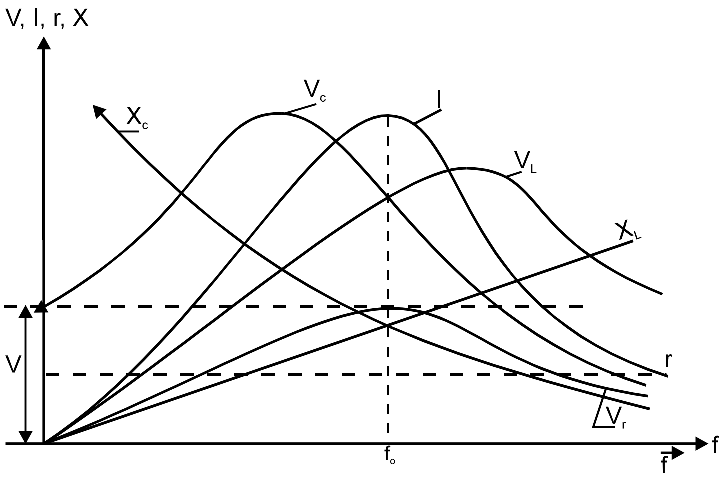


Fig. 2.12. The graphs of dependence of I, r, X_c, X_L, V_r, V_L, on the frequency for the circuit of Fig. 2.10 (a)

In a resonance circuit, we also use the parameter called quality factor Q. In general, quality factor Q is given by the expression:

$$Q = \omega_o \frac{\sum W_{\max}}{P} \quad 236$$

here, ω_o is the resonance frequency; ΣW_{max} is the sum of the maximum value of the energy, which is periodically delivered, during resonance, to the inductive (or capacitive elements), P is the active power on the terminals of the circuit during resonance.

The sum (Σ) sign relates to the case, when the number of inductive (or capacitive) elements is more than one.

For the circuit of Fig. 2.10 on the basis of the equation above, we obtain:

$$Q = \frac{\omega_o L}{r} = \frac{1}{\omega_o C r} = \frac{\sqrt{\frac{L}{C}}}{r} = \frac{P}{r} \quad 237$$

where

$$\rho = \sqrt{\frac{L}{C}} \quad 238$$

is called the characteristic resistance of the resonance circuit.

EXAMPLE 1: *Determine the frequency at which the circuit of fig. 2:10 goes into resonance (voltage). Determine also, by how many times is the voltage across the inductor greater than the voltage supplied to the circuit, if the circuit has the following parameters r = 20Ω, L = 0.1H, C = 5micro-Farad.*

SOLUTION: The resonance frequency

$$f_o = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\,x\,3.14\sqrt{0.1\,x\,5\,x\,10^{-6}}} = 224Hz \qquad 239$$

The inductive reactance of the circuit during resonance

$X_L = 2\pi f_o L = 6.28 \text{ x } 224{\cdot}0.1 = 140\Omega$

The voltage across the inductor during resonance

$$\frac{V_L}{V} = \frac{IX_L}{I \bullet r},\; V_L = \frac{V \bullet X_L}{r} = V \bullet \frac{140}{20} = 7V \qquad 240$$

Therefore the voltage across the inductor during resonance is seven times greater than the supply voltage.

2.11 CURRENT RESONANCE (PARALLEL RESONANCE)

Current resonance can take place in a parallel circuit (as shown in fig. 2:13a, one branch contains L and r, while the other branch contains C and r).

Current resonance is called such a state of the circuit, when the total current is in phase with the voltage supply, the reactive power is equal to zero and the circuit consumes only active power. In fig. 2:13d is shown the vector diagram of the circuit of fig. 2:13a during current resonance.

As seen from the vector diagram, the total current to the circuit is in phase with the voltage of the reactive components of the currents in the branches with the inductor and capacitor equal in value.

$I_1 r_1 = I_2 r_2$

The total reactive current of the circuit, is equal to the difference of the reactive currents in the branches, which in this case is equal to zero:

$I_1 r_1 - I_2 r_2 = 0$

The total current of the circuit has only an active component, which is equal to the sum of the active components of currents in the branches.

$I_a = I_{1a} + I_{2a}$.

If we express the reactive currents through voltages and reactive admittance, we shall obtain

$V{\cdot}b_L = V{\cdot}b_c$

from where

$b_L = b_c$

Therefore, during current resonance the reactive conductance of the branch with the inductor is equal to the reactive conductance of the branch with the capacitor.

If we express b_L and b_c through the impedance of the corresponding branches, we can determine the resonance frequency of the loop:

$$\frac{r_1}{r_1^2 + X_L^2} = \frac{r_2}{r_2^2 + X_c^2};\ \frac{r_1}{r_1^2 + (2\pi f L)^2} \qquad 241$$

$$= \frac{r_2}{r_2^2 + \left(\frac{1}{2\pi f C}\right)^2}$$

from where

$$f_o = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{\frac{L}{C} - r_1^2}{\frac{L}{C} - r_2^2}} \qquad 242$$

In ideal case, when $r_1 = r_2 = 0$

$$f_o = \frac{1}{2\pi\sqrt{LC}} \qquad 243$$

During current resonance, since this follows from definition, the power factor is equal to one

$$\cos\varphi = 1 \qquad 244$$

The full power is equal to the active power

$$S = P$$

The reactive power is equal to zero

$$Q = Q_L - Q_c = 0$$

The energy processes in the circuit during current resonance are similar to the processes, taking place during voltage resonance, which we have discussed above. The reactive energy circulates internally in the circuit. In one part of the period energy of the magnetic field of the inductance is transformed into energy of the electric field of the capacitance, in the next part of the period, energy of the electric field of the capacitance is transformed into energy of the magnetic field of the inductance.

Exchange of energy between the consumers of the circuit and the feeding source does not take place. The current in the lines connecting the circuit with the source is due only to the active power.

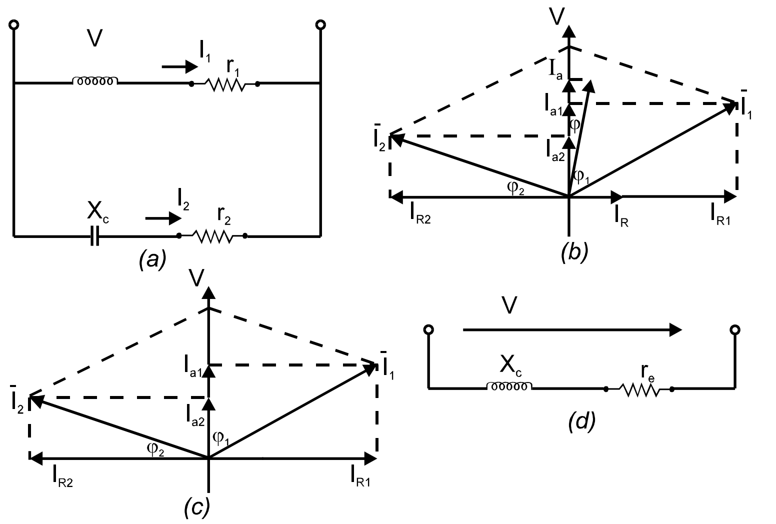


Fig. 2.13 (a) Electrical circuit, (b) its vector diagram, (c) vector diagram

For current resonance it is characteristic that the total current, can be by far smaller than the currents in each of the branches. For example, in an ideal circuit, when $r_1 = r_2 = 0$ (fig. 2.14), the total current is equal to zero, since the circuit does not consume active power, while currents in the branches with capacitance and inductance exist; they are equal in magnitude and have an angle shift between them of 180°. Resonance in a circuit when the loads are connected in parallel is called current resonance.

Current resonance can be obtained by selecting the parameters of the circuit with a given frequency of the source or by selecting the frequency of the source with a given set of parameters of the circuit.

Let us consider the influence of the frequency of the source on the magnitude of the current, for example in the circuit, shown in fig. 2.14(a).

The current in the branch with inductance is inversely proportional to the

frequency:
$I_L = V/2\pi fL$, while the current in the branch with capacitance is directly proportional to the frequency of the set.

$I_c = V \cdot 2\pi fc$

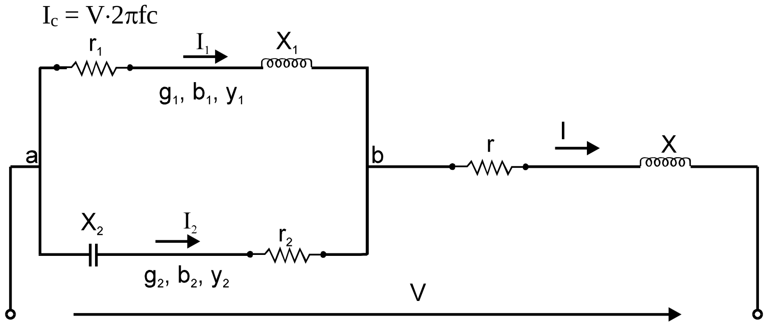


Fig. 2.14 A parallel - series network of R, L, C.

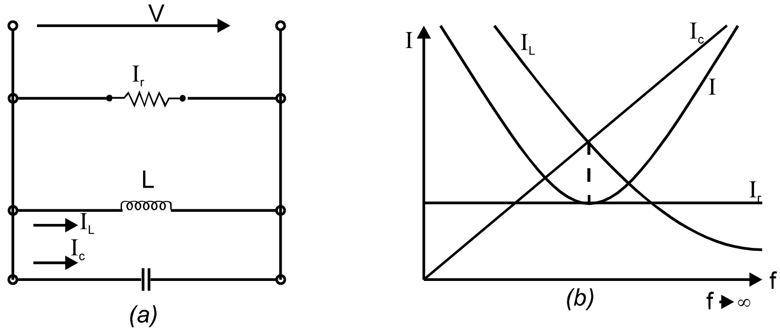


Fig. 2.15 Electric circuit for investigating the influence of the frequency on current in the circuit. (b) graphs of frequency versus I_r, I_L, I and I_c.

The current in the branch with active resistance does not depend on frequency.

$I_r = V/r.$

The vector of total current in the circuit is equal to the geometric sum of the vector of the current in the branches:

$$I = \bar{I}_L + \bar{I}_c + \bar{I}_r \quad 245$$

and it's magnitude

$$I = \sqrt{I_r^2 + (I_L - I_c)^2} \quad 246$$

when f = 0

$I_L = \infty$, $I_c = 0$; $I_r = v/r$; $I = \infty$

when $f = f_o$
$I_L = I_c$, $I = I_r = v/r$

when f $\to \infty$, $I_L \to 0$, $I_c \to \infty$, $I_r = V/r$, $I = \infty$

These graphs are shown in fig. 2.14b

Majority of industrial consumers of A.C current have active - inductive character; sum of them operate with low power factor and hence consume a high reactive power. Such consumers are asynchronous machine operating without a full load.

To reduce the reactive power and increase the power factor we can connect batteries of capacitor parallel to the consumers.

Reactive power of capacitive batteries decreases the total reactive power of the system, since

$$Q = Q_L - Q_c$$

and as such increases the power factor.

Increase in power factor leads to the decrease of the current in the lines, connecting the consumer with source of energy and the full power of the source.

EXAMPLE 3: *Calculate the capacitance of the capacitor, for which in the circuit of Fig. 2.14 exist current resonance if $X_L = 40\Omega$, $r_1 = 30\Omega$, $r_2 = 28\Omega$, $f = 100Hz$.*

SOLUTION: During current resonance the reactive power of the circuit is equal to zero.

$Q_L - Q_c$ or $Q_L = Q_c$

$$Q_L = I_1^2 X_L = \frac{V^2}{r_1^2 + X_L^2} \bullet X_L ; Q_c = I_2^2 X_c = \frac{V^2}{r_2^2 + X_c^2} \bullet X_c \quad 247$$

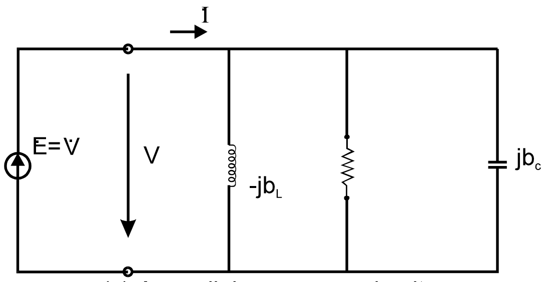
$$\frac{V^2}{30^2 + 40^2} \bullet 40 = \frac{V^2}{28^2 + X_c^2} \bullet X_c ; \therefore X_c = 17 \bullet 75\Omega \quad 248$$

The capacitance

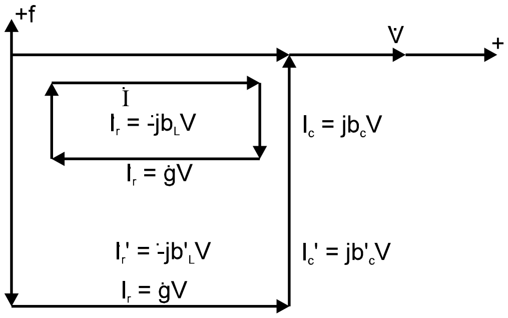
$$X_c = \frac{1}{2\pi fC} \quad 249$$

$$C = \frac{X_c}{2\pi f} = \frac{17.75 \ x \ 10^6}{2 \ x \ 3.14 \ x \ 1000} = 2.82\mu F \quad 250$$

This form of resonance may occur in a parallel resonance circuit, that is, one containing an inductive element, a capacitive element, and a resistive element connected in parallel.



(a) A parallel resonance circuit.



(b) Vector diagram

Fig. 2.16

Given the supply voltage

$$\dot{V} = \dot{V} < \varphi_v \quad 251$$

the common current is

$$\dot{I} = Y\dot{V} = y\exp(-j\,\varphi_v) \quad 252$$

where Y = g -jb = g - j (b_L - b_c) is the complete admittance of a parallel resonant circuit.

$$Y = \sqrt{g^2 + (b_L - b_c)^2} \quad 253$$

is the resonant circuit;

$$\psi = \psi_v - \psi_i = \arctan\frac{1/\omega L - \omega C}{g} \quad 254$$

is the phase difference between the voltage and the common current.

$$I = YV = \sqrt{g^2 + (b_L - b_c)^2}\,V \quad 255$$

is the rms current in the resonant circuit; and $-\varphi$ 256 is the argument of the complex admittance.

At $\omega_o / = 1/\sqrt{LC}$ 257, the inductive susceptance b_L = $1/\omega L$ and the capacitive susceptance b_c = $\omega_o C$ of the parallel branches are the same, the argument of the complex admittance $-\Psi$ is zero (that is, $\Psi_i = \Psi_v$), the admittance of the resonant circuit is a minimum (y = g) and the common current is a minimum, too; $I_o = gV$

The condition is a parallel resonant circuit in which the phase difference of the voltage and common current is zero is called parallel (or current) resonance.

At resonance, the rms currents in the inductive and capacitive elements are the same:

$$I_L = (1/\omega_o L)V = I_c = \omega_o CV$$

and the phase difference between the currents is π, because the current in the inductive elements lags behind the voltage across it by $\pi/2$, and the current in the capacitive element leads the voltage across it by the same angle $\pi/2$.

It is to be noted that, in contrast to series resonance, parallel resonance does not present any hazard in electric installations. In this case, heavy currents can arise in the branches only when the branches contain large reactances, that is, when their capacitors have a high capacitance or their inductors have a low inductance. This is not surprising because the circuits in the inductive and capacitive branches are independent of each other, and their values are determined by Ohm's law by the applied voltage.

CHAPTER THREE

METHOD OF ANALYSIS OF COMPLEX ELECTRICAL NETWORKS

3.1 APPLICATION OF KIRCHHOFF'S LAW FOR THE ANALYSIS OF COMPLEX NETWORK

In the case, when the electrical network is quite complex and cannot be transformed to a single loop circuit or circuit with two nodes, then when we apply more general method of analysis.

The method written below can be used both for the analysis of d.c circuits and a.c circuits;

In the general case, the unknown values of the electrical quantities and their relations can be determined as a result of the solution of a system of equations, written in accordance to first and second laws of Kirchhoff for the given electrical network.

Let us assume that, a circuit contains P branches and q nodes and some given voltage source, e.m.f and the unknown value is current in the branches. Hence, the number of unknown is equal to the number of branches.

According to Kirchhoff's current law (first law) the expression for equality to zero of the algebraic sum of currents in a node can be written as q-1 independent equations; the equation of the q-th node is a dependent node. Hence, one of the equation is a dependent one; i.e it arises from the remaining equation.

The nodes, for which we write the independent equations according to Kirchhoff's current law, can be called independent nodes. From the above, explanation it follows that, from the general q principal nodes any q-1 nodes are called independent nodes and the remaining node is called a dependent node.

According to Kirchhoff's voltage law (second law), the expression for equality of the algebraic sum of the voltage source in the loop (mesh) to the algebraic sum of voltage drop in the loop, can be written as p-1=1 independent equations. In the real sense of it, if we apply Ohm's law to each branch, then we will obtain p equations of the form

$$V_{ik} = \dot{V}_i - \dot{V}_k = -\dot{E}_n + Z_n I_n; \qquad (3.1)$$

Where V_{ik} - is the complex voltage between the nodes i and k; E_n, I_n - is the

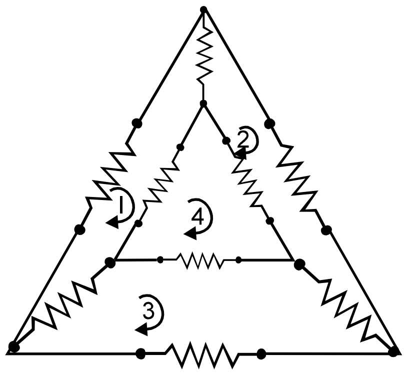
complex e.m.f and the current in the n-branch, directed from node i to node k; Z_n - is the complex impedance of the same branch.
In the system of equation of the form (3.1), there are p unknown currents I_n and q-1 unknown potentials V_i, V_k e.t.c (the potential of one of the nodes is assumed to be equal to zero). If we now eliminate the unknown potentials, the remaining p-q+1 equations, which relate the complex e.m.f with the voltages across the complex impedances; obtained in this way is called the Kirchhoff's voltage law for the network.

Therefore the analysis of electrical network with the use of Kirchhoff's current and voltages law leads to the solution of (q-1) + (p-q+1) = p equations - that the number of branches.

The loops, for which the equation, written according to Kirchhoff's voltage is independent are called independent loops.

In fig 3.1, in the form of an example we show an electric circuit containing nine (9) branches and six (6) nodes correspondingly, the number of equations according to Kirchhoff's current law is equal to 9-6 = 3 and the number of equations according to Kirchhoff's voltage law is equal to 9-6+1 = 4. In the circuit of fig 3.1 the independent loops are equal to four loops. In the fig 3.1, is shown one of the variant of choice of independent loops.

In order, that the equations written according to Kirchhoff's voltage law, and hence the loops to be independent. It is sufficient that, each succeeding loop differs from the previous one at least in one new branch.



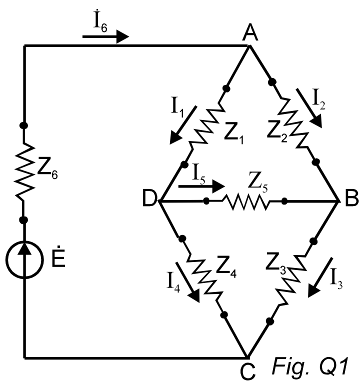


Fig. Q1

Fig. 3.1 Electrical network with 4 independent loops

EXAMPLE 1: *In the Bridge circuit, shown the circuit of fig Q1, is given all the complex impedance and the e.m.f E. It is required to determine the current I_5 in the branch with Z_5 (the current in the diagonal of the bridge).*

SOLUTION: The circuit contains four nodes and six branches. Hence, we can write three equations according to Kirchhoff's current law and three equations according to Kirchhoff's voltage law.

Node A: $-I_1 - I_3 + I_6 = 0$

Node B: $I_2 - I_3 + I_5 = 0$

Node C: $I_3 + I_4 - I_6 = 0$

For the loop ABDA

$-Z_1I_1 - Z_3I_3 - Z_5I_5 = 0$

For the loop BCDB

$-Z_3I_3 - Z_4I_4 - Z_5I_5 = 0$

For the loop ABCA

$-Z_2I_2 - Z_3I_3 - Z_6I_6 = E$

In obtained system from six equations the independent are the currents in the branches. When we solve the system of equation in term of the sought after current, we will find;

$$\dot{I}_5 = \frac{\dot{E}}{M}\,(Z_1 Z_4 - Z_1 Z_3)\ 259$$

where $M = Z_5 [(Z_1 + Z_4) (Z_2 + Z_3) + Z_6 (Z_1 + Z_2 + Z_3 + Z_4)] + Z_1 Z_4 (Z_2+ Z_3) + Z_2 Z_3 (Z_1 Z_4) + Z_6 (Z_1 + Z_2 (Z_3 + Z_4)$.

The obtained expression shows, that the current in the diagonal of the bridge circuit is equal to zero, if the condition $Z_1 Z_3 = Z_2 Z_4$ is fulfilled (the condition of a balanced bridge circuit).

3.2 THE MESH CURRENT METHOD.

The mesh current method is one of the fundamental methods of analysis of complex electrical network, which is widely used in practice. The method uses the fact that in place of the branch current, we determine mesh currents on the basis of Kirchhoff's voltage law and the currents are called mesh currents. In the circuit of fig. 3.2, we present a two mesh electric circuit in which I_1 and I_2 are the mesh currents. Currents in the impedances Z_1 and Z_2 are equal to the respective mesh currents; current in the impedance Z_3, which is common to both meshes is equal to the difference of currents I_1 and I_2, since these currents are directed in the branch Z_3 in opposite direction. (We must state here, that if we change the mesh current direction in any of the loops, then the current I_3 will be equal to the sum of the mesh currents I_1 and I_2). When we choose the positive direction of current I_3 to be in the same direction as the mesh current I_1, then the current in the branch Z_3 will be equal to $I_1 - I_2$. On the other hand, the current will be equal to $I_2 - I_1$.

The number of equations, written for mesh currents according to Kirchhoff's voltage law, is equal to the number of independent loops, i.e for electric circuits with number of nodes equal to q and the number of branches equal to p the problem of finding the mesh currents is reduced to solving a system of p-q+1 equations. So in the circuit of fig 3.2 q = 2, p = 3; hence, the number of equation is equal to 3 - 3 +1 = 2 (number of independent loops).

Let as assume that, the sum of complex impedances going in a mesh are called mesh impedances of the mesh, and the impedance belonging to two or more meshes is called the common impedance of the loops (meshes).

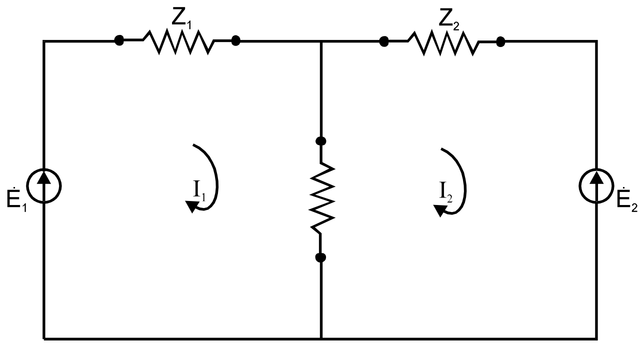


Fig. 3.2 Illustration of mesh current

The positive direction of the mesh current is given arbitrarily. The direction of going round a loop is usually given to coincide with the direction of mesh current in any given mesh. Therefore when writing Kirchhoff's voltage law for mesh the voltage drops on the mesh impedances is taken with positive sign. The drop on the common impedance belonging to two or more meshes is usually taken with negative sign if the mesh currents through this impedance are opposite in direction as is the case in fig. 3.2, where the direction of the mesh current in both meshes is taken in clockwise direction.

For the given electric circuit with two independent meshes, we can write two equation according to Kirchhoff's voltage law, and namely:
$E_1 = (Z_1 + Z_3) I_1 - Z_3 I_3$; $-E_2 = -Z_3 I_3 + (Z_2 + Z_3)I_2$, where $Z_1 + Z_3$ and $Z_2 + Z_3$ are the mesh impedances for mesh 1 and 2 respectively; Z_3 - is the
common impedance between mesh 1 and 2 (the sign minus in the equation is due to the chosen positive directions of the mesh currents).

If the given electric circuit contains n independent mesh, then on the basis of Kirchhoff's voltage law we will obtain a system of n equations;

$$\begin{aligned}
\dot{E}_1 &= Z_{11}\dot{I}_1 + Z_{12}\dot{I}_2 + \ldots\ldots + Z_{1n}\dot{I}_n \\
\dot{E}_2 &= Z_{21}\dot{I}_1 + Z_{22}\dot{I}_2 + \ldots\ldots + Z_{2n}\dot{I}_n \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\dot{E}_n &= Z_{n1}I_1 + Z_{n2}\dot{I}_2 + \ldots\ldots + Z_{nn}\dot{I}_n
\end{aligned} \qquad (3.2)$$

Here E_i - are the mesh e.m.f in mesh i (i = 1, 2, ... n), i.e. algebraic sum of the e.m.f, acting in the given mesh; e.m.f coinciding in direction with the direction of moving round the mesh is taken with positive sign, and when the direction is opposite - it is taken with negative sign.
Z_{ii} - are the mesh impedance in mesh i; Z_{ik} - are the common impedance of mesh i and k.

According to what we have earlier discussed the mesh impedances are taken with positive sign as long as the mesh current I_i direction coincide with the direction of moving round the mesh. The common impedances Z_{ik} are taken with negative sign, when currents I_i and I_k are in opposite direction.

The solution of equation 3.2 in terms of the unknown mesh currents can be obtained with the use of the determinant method.

$$I_1 = \frac{1}{\Delta_Z}\begin{vmatrix} \dot{E}_1 & Z_{12} & \ldots\ldots & Z_{1n} \\ \dot{E}_2 & Z_{22} & \ldots\ldots & Z_{2n} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ \dot{E}_n & Z_{n2} & \ldots\ldots & Z_{nn} \end{vmatrix} \quad 261$$

$$I_2 = \frac{1}{\Delta_Z}\begin{vmatrix} Z_{11} & \dot{E}_1 & \ldots\ldots & Z_{1n} \\ Z_{21} & \dot{E}_2 & \ldots\ldots & Z_{2n} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ Z_{n1} & Z_{n2} & \ldots\ldots & Z_{nn} \end{vmatrix} \quad 262$$

e.t.c, where the determinant of the system

$$\Delta_2 = \begin{vmatrix} Z_{11} & Z_{12} & \cdots\cdots & Z_{1n} \\ Z_{21} & Z_{22} & \cdots\cdots & Z_{2n} \\ \cdot\cdot & \cdot\cdot & \cdots\cdots & \cdot\cdot \\ Z_{n1} & Z_{n2} & \cdots\cdots & Z_{nn} \end{vmatrix} \tag{263}$$

EXAMPLE 2: *Using the mesh current method, determine the current in the diagonal of the bridge circuit of fig. Q2. The chosen positive direction of the mesh currents I_1, I_2 and I_3 are shown on the circuit with directed arrows.*

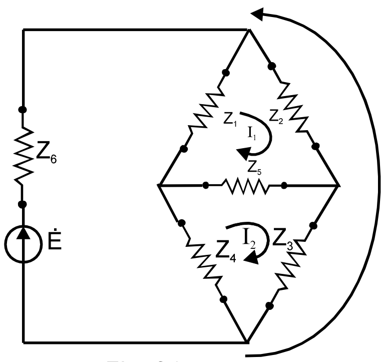


Fig. Q2

The number of equations written according to Kirchhoff's voltage are equal to three (according to the number of independent meshes).

$$\begin{aligned} 0 &= (Z_1 + Z_2 + Z_5) I_1 - Z_5 I_2 - Z_2 I_3 \\ 0 &= -Z_5 I_1 + (Z_3 + Z_4 + Z_5)\dot{I}_2 - Z_3 \dot{I}_3 \\ -\dot{E} &= -Z_2 I_1 - Z_3 I_2 + (Z_2 + Z_3 + Z_6)\dot{I}_3 \end{aligned} \tag{264}$$

The solution of the obtained system of equations in terms of the mesh current I_1 and I_2 will give

$$I_1 = -\frac{\dot{E}}{M}[Z_3 \; Z_5 + Z_2(Z_3 + Z_4 + z_5)]; \tag{265}$$

$$I_2 = -\frac{\dot{E}}{M}[Z_3 \; Z_5 + Z_3(Z_1 + Z_2 + Z_5)]; \tag{266}$$

where M has the same value as in example 1.

The current I_5 sought after which is the current in the diagonal of the bridge circuit is equal to the difference in the mesh currents.

$$\dot{I}_5 = I_2 - \dot{I}_1 = \frac{\dot{E}}{M}(Z_2 Z_4 - Z_1 Z_3) \quad 267$$

which is the same result as obtained in example 1.

3.3 THE NODE - VOLTAGE METHOD.

The node voltage method is hinged on the basis of Kirchhoff's current law. On this basis we determine the potentials in the nodes of the electric circuit relative to a chosen reference node, whose potentials is assumed to be equal to zero.

The voltage across any given branch is the difference in potential of the nodes at the end of the branches; the product of the voltage and the complex admittance of the given branch is equal to the current in this branch. Accordingly, if we know the nodal voltages in circuit, we can now find the current in the branch.

If we take the reference voltage to be equal to zero, then the voltage between the remaining nodes and reference node will be equal to the potential of these nodes. Therefore this method also called the node-potential method.

In fig. 3.3 as an example we present an electric circuit with two current sources having three nodes 1, 2, and 3 we take as reference node, node 3 and represent node voltages 1 and 2 through V_{N1} and V_{N2}. According to the symbols for the complex admittances of the branches, we have that

$$Y_1 = \frac{1}{Z_1}; \quad Y_2 = \frac{1}{Z_2}; \quad Y_3 = \frac{1}{Z_3} \quad 268$$

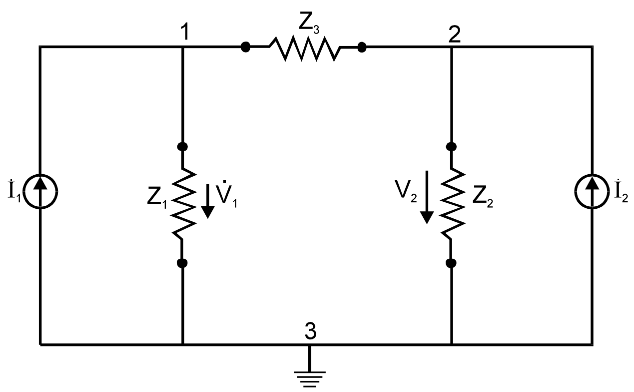


Fig. 3.3 Illustration of node - voltage method

For the given circuit with three node, we can write two equations according to

Kirchhoff's current law, and namely:
For node 1

$$\dot{I}_1 = Y_1 V_{N1} + Y_3(\dot{V}_{N1} - \dot{V}_{N2}) = 269$$

$$(Y_1 + Y_3)\dot{V}_{N1} - Y_3 V_{N2} \; 270$$

For Node 2: $\dot{I}_2 = Y_2 \dot{V}_{N2} + Y_3(\dot{V}_{N2} - \dot{V}_{N1}) = 271$

$$= - y_3 \dot{V}_{N1} + (Y_2 + Y_3)\dot{V}_{N2} \; 272$$

The value $y_1 + y_3$, is the sum of the complex admittance of the branches going into node 1 and is called the nodal admittance of node 1, the value y_3 is equal to the complex admittance of the branches between nodes 1 and 2, and it goes into the equation with a negative sign, and it is called the common admittance between nodes 1 and 2.

In the general case, if an electric circuit contains q nodes, then on the basis of Kirchhoff's current law we obtain q-1 system of equal (node q is taken as the reference node)

$$\dot{I}_1 = y_{11}\dot{V}_1 + Y_{12}\dot{V}_2 + ... + Y_{(1,q-1)}\dot{V}_{q-1}$$

$$\dot{I}_2 = Y_{21}\dot{V}_1 + Y_{22}\dot{V}_2 + ... + Y_{(2,q-1)}\dot{V}_{q-1} \qquad (3.4)$$

$$\dot{I}_{q-1} = Y(q-1)\dot{V}_1 + Y_{(q-1,2)}\dot{V}_2 + ... + Y_{(q-1,q-1)}\dot{V}_{q-1}$$

Here current sources, whose direction is towards the node are taken with positive sign; and going away from the node are taken with negative sign.

Y_{ii} - complex nodal admittance of all the branches going into the node i; y_{ik} - is the common admittance between nodes i and k, which is taken with negative sign for a chosen direction all the nodal voltages to the reference.

When we solve the system of equation 3.4, we will obtain formula for the k-th nodal voltage relative to the reference voltage.

$$\dot{V}_K = \frac{1}{\Delta_Y} \sum_{i=1}^{q-1} \dot{I}_i \Delta_{ik} ; \; 274$$

where Δy is the determinant of the system.

$$\Delta y = \begin{vmatrix} y_{11} & y_{12} & ..... & y_{i,q-1} \\ y_{21} & y_{22} & ..... & y_{2,q-1} \\ y_{q-1,1} & y_{q-11} & ..... & y_{q-1\,q-1} \end{vmatrix} \quad 2\,7\,5$$

where i is the column that is replaced. Δik is the algebraic signed minor, or cofactor.

The first index i of the algebraic sign minor, represent the row number that is struck out in the determinant, corresponding to the node number of the given current source which is multiplied by the given algebraic signed minor. The second index k, represent the number of the column, which is struck out in the system determinant, corresponding to the node, for which we are calculating the node voltage.

Equations (3.4), expressing Kirchhoff's current law are written on the assumption that in terms of energy source we have current sources. If the electric circuit contains e.m.f source, then the e.m.f should be replaced by its equivalent current source.

If in the circuit we have branches containing e.m.f without impedance, the complex admittance of the branch is equal to infinity then these branches must be considered as sources with unknown currents, which are then eliminated during the addition of the equations.

Additional connection between unknown nodal voltages will be known voltages between nodes, equal to the given e.m.f.

The node voltage method has advantages over the mesh current method in the case, when the number of equations written according to Kirchhoff's current law is less than the number of equations written according to Kirchhoff's voltage law. If the given circuit has q nodes and p branches, then in accordance with what we have discussed so far, the node voltage has advantages over the mesh current method when $q - 1 < p - q + 1$, or when $2(q - 1) < p$.

EXAMPLE 3: *Using the node-voltage method calculate the current in the diagonal of the bridge circuit of fig Q2 (redrawn in fig Q3).*

As result of the replacement of the e.m.f with its equivalent current source, we obtain the circuit of fig. Q3 contains four nodes. For this circuit by Kirchhoff's current we will write 4-1 = 3 independent equations. If we choose node 4 as our reference node and direct the node voltages towards the reference node, then the equation will have the form.
Node 1:

$$Y_6 \dot{E} = (Y_1 + Y_2 + Y_6)\dot{V}_1 - Y_2 \dot{V}_2 - Y_6 V_3 \quad 276$$

Node 2: $0 = - y_2 V_1 + (y_2 + y_3 + y_6) V_2 - y_3 V_3$

Node 3: $-y_6 E = y_6 V_1 - y_3 V_2 + (y_3 + y_4 + y_6) V_3$

The solution of the obtained system of equation in terms of V_2 will give

$$\dot{V}_2 = \frac{\dot{E}}{N} Y_6 (Y_2 Y_4 - Y_1 Y_3) \quad 277$$

where $N = y_5 [(y_1 + y_2)(y_3 + y_4) + y_6 (y_1 + y_2)(y_3 + y_4)] + y_1 y_4 (y_2 + y_3) + y_2 + y_3 (y_1 + y_4) + y_6 (y_1 + y_4)(y_2 + y_3)$

If we multiply the obtained result of V_2 by the admittance y_5 of the diagonal branch of the bridge circuit and we change the sign in accordance with the chosen direction (originally) of current I_5, we then obtain the unknown current I_5.

$$\dot{I}_5 = \frac{\dot{E}}{N} Y_5 Y_6 \ (Y_1 Y_3 - Y_2 Y_4) \quad 278$$

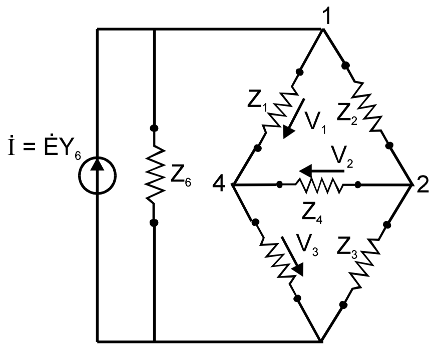


Fig. Q3

The physical meaning of this formula is that the current in any loop of a linear electric circuit can be obtained as the algebraic sum of the currents, obtained in this loop when each e.m.f is acting separately.

The method of computation of the currents, is based on the determination of currents in the same loop (or branch) when each e.m.f is acting alone and then algebraic summation of these currents; this is called superposition method.

When determining each component of the current by superposition method it is necessary to take into account the internal impedances of these emf, which is assumed to be absent during the computation of the terms; i.e the internal impedance of emf source is equal to zero then when determining currents induced by an emf, the remaining emf are short-circuited.

3.4 SUPERPOSITION THEOREM

In linear electric circuit containing emf, the mesh currents (and corresponding the currents in the branches) present themselves as linear functions of the mesh e.m.f. Mathematically they are expressed by the formula

$$\dot{I}_K = \frac{1}{\Delta_Z} \sum_{i=1}^{n} \dot{E}_i \Delta_{ik} \quad 279$$

The physical meaning of this formula is that the current in any loop of a linear electric circuit can be obtained as the algebraic sum of the currents, obtained in this loop when each emf is acting separately.

The method of computation of the currents is based on the determination of currents in the same loop (or branch) when each emf is acting alone and then algebraic summation of these currents; this is called superposition method.

When determining each component of the current by superposition method it is necessary to take into account the internal impedance of these emf, which is assumed to be absent during the computation of the terms, i.e the internal impedance of emf source is equal to zero then when determining currents induced by any emf, the remaining emf sources are short-circuited.

On the other hand, in a linear electric circuit containing current sources, the nodal voltages (and correspondingly the voltage across the branch present itself as a linear function of the given current sources. Mathematically they are expressed by the formula

$$\dot{V}_K = \frac{1}{\Delta_Y} \sum_{i=1}^{q-1} I_i \Delta_{ik} \quad 280 \qquad (3.5)$$

The physical meaning of this formula is that the nodal voltage for any node of a linear electric circuit can be obtained as the algebraic sum of the voltages, induced is this node by each given current source acting separately.

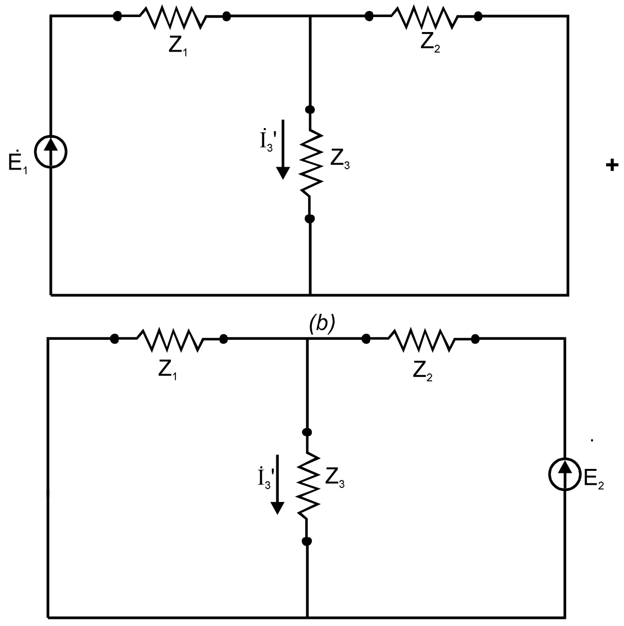
The formulae (3.5) and also (3.4) is the mathematic expression of super position method, which holds for linear electric circuits.

When determining the component of nodal voltages by the superposition method it necessary to take into account the internal admittance of these current sources which are taken to be absent when calculating each term of the nodal-voltages.

If the current source is given without an internal admittance, that is the admittances are equal to zero, then when using the superposition method the branch with current sources is open-circuited.

If in a linear electric circuit there are emf sources as well as current sources, then we can apply the superposition method for the this case too.

EXAMPLE 4: *Using the superposition method, determine the current in the branch with Z_3 in fig Q_3.*



(c)
Fig.Q4

The unknown current I_3 is determined as the sum of the currents $\dot{I}_{3'}$ 281 and $\dot{I}_{3''}$ 282 passing through the branch with Z_3 on the action of emf E_1 and E_2 acting separately.

Currents $\dot{I}_{3'}$ 283 and $\dot{I}_{3''}$ 284 are sum up, and not subtracted, since the positive directions of these currents were chosen to coincide

$$\dot{I}_{3'} = \frac{Z_2}{Z_2 + Z_3} \bullet \frac{\dot{E}_1}{Z_1 + \dfrac{Z_1 Z_3}{Z_2 + Z_3}} = \frac{Z_2 \dot{E}_1}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1} \quad 285$$

$$\dot{I}_{3''} = \frac{Z_1}{Z_1 + Z_3} \bullet \frac{\dot{E}_3}{Z_2 + \dfrac{Z_1 Z_3}{Z_1 + Z_3}} = \frac{Z_1 \dot{E}_2}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1} \quad 286$$

hence

$$I_3 = \dot{I}_{1'} + \dot{I}_{3''} = \frac{Z_2 \dot{E}_1 + Z_1 \dot{E}_2}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1} \quad 287$$

3.5: DUALITY THEOREM (RECIPROCITY THEOREM)

Passive linear electric circuits, have the ability to exhibit a property which is referred to as duality (reprocity). On the basis of this property is the duality (reciprocity) theorem which can be stated in two variant: applicable to emf sources and current sources.

We shall consider the first variant.

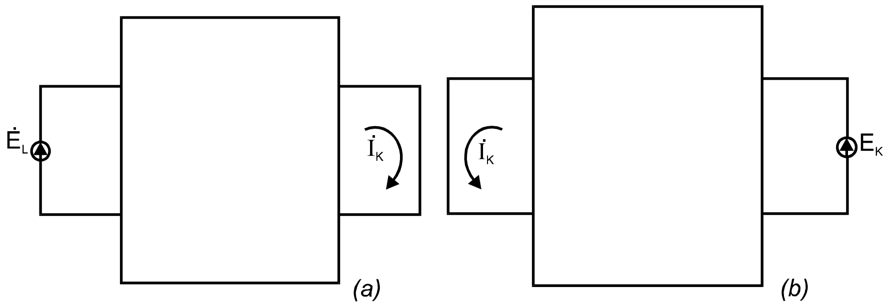


Fig. 3.4 Duality (Reciprocity) theorem (the variant with emf source)

THE VARIANT WITH EMF SOURCE: In Fig 3.4 we represent an electric circuit with a drawn out loops i and k. The electromotive force E_i in loop i induces a current in top loop k, which is equal to

$$I_k = Y_{ki} E_i$$

Correspondingly emf E_k in loop k also induces a current in loop i, which is given by the expression

$$I_i = Y_{ik} E_k$$

From whence, it follows that

$$\frac{\dot{I}_k}{I_i} = \frac{y_{ki}}{y_{ik}} \bullet \frac{\dot{E}_i}{\dot{E}_k} \quad 288$$

The algebraic signed minor or cofactor Δ_{ik} and Δ_{ki} 289 in the expression for y_{ki} and y_{ik} differs only in the fact that, in them the rows are replaced by columns such that their elements - that is their mesh impedances of the given circuit does not change with the transfer of the index.

Therefore $\Delta_{ik} = \Delta_{ki}$ 290 and hence

$$y_{ki} = y_{ik} \tag{3.6}$$

Electric circuit, for which the condition (3.6) is fulfilled are called reciprocity circuits. For such circuits we have that

$$\frac{\dot{I}_k}{I_i} = \frac{\dot{E}_i}{\dot{E}_k} \quad 291$$

If we take $E_i = E_k$, then $I_i = I_k$

As such, for duality circuits holds the following position: if some emf, situated in any loop of an electric circuit, induced current in another loop of the same given circuit, then that same emf, if when transferred to the second loop, it will induce in the first loop current with the same magnitude and phase.

During the choice of the respective mesh currents, the current in the branch is equal to the mesh current. Therefore this present theorem also holds for currents in branches.

EXAMPLE 5: *Making use of the duality theorem, calculate the current I in the circuit of fig Q5, and during any finite value of impedance Z. $E = 10 < 0°V$.*

The application of the reprocity theorem reduces the computation involved in this case, after transferring emf to branch z, we obtain in the circuit
in fig Q5(b) which is transfer Fig Q5(b) of current, in which the current source is equal to zero therefore, the unknown current is equal to the current in the

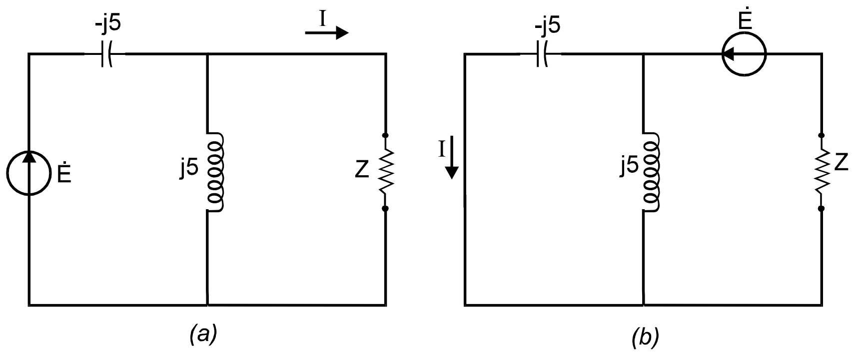


Fig. Q5

capacitor fig. Q5(b), which is obtained as the ratio of the emf to the reactance of the capacitance of the capacitive branch.

$$I = \frac{10 < 0°}{-j5} = j2A\ 292$$

3.6 THE COMPENSATION THEOREM

This compensation theorem states; that the current in an electric circuit does not change, if any portion of the circuit is replaced by the emf, which is equal to the voltage drop across this given portion of the circuit and is directed opposite the current, flowing through this given portion of the circuit.

The correctness of this position, which is called the compensation theorem arises from the fact that any of the terms of the voltage drops, entering the equation for Kirchhoff's voltage law can be transferred to the other side of the equation with opposite sign, i.e can be considered as an additional emf source in the opposite direction.

The illustration of the above statement is shown in Fig 3.5; Kirchhoff's voltage law written for Fig 3.5 is $E = Z_1I + ZI$, which can be presented in the form

$$E - ZI = Z_1I$$

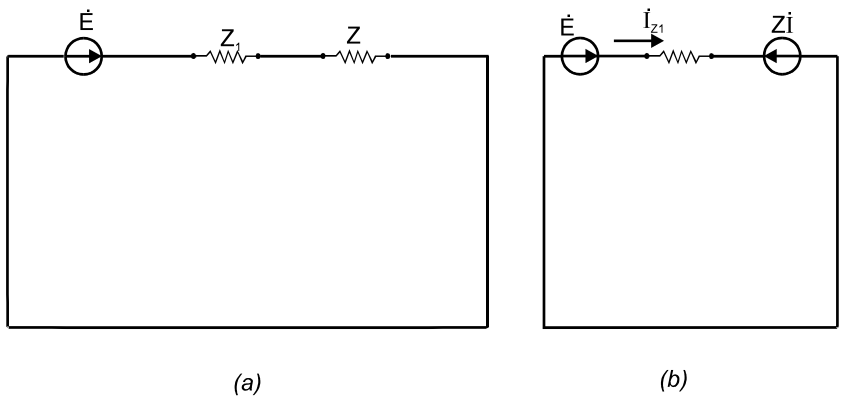


Fig. 3.5 The compensation Theorem

The last expression above corresponds to the circuit of Fig 3.5, in which in place of impedance Z we include emf ZI, directed opposite the current I.

This theorem also holds for distributed electric circuits.

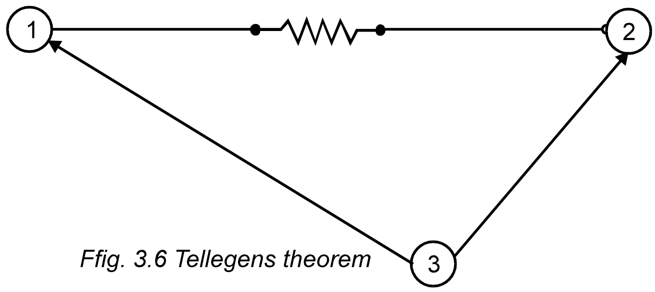
3.7 TELLEGEN'S THEOREM:

Tellegen's theorem applies to any linear network which obeys Kirchhoff's laws. It essentially shows that energy is conserved in a looped network.

The Theorem: Consider a network with n branches with branch currents $i_1, i_2 \ldots i_n$ corresponding to the branch voltages $V_1, V_2 \ldots V_n$, then Tellegen's theorem states that at all times,

$$\sum_{k=1}^{n} V_k I_k = 0 \quad 293$$

This means that (branch voltage) x (branch current) is equal to power delivered to the branch. The total power delivered to a circuit by a branch containing a source must therefore be absorbed by the remaining branches so that energy is conserved.
Proof.



Ffig. 3.6 Tellegens theorem

Consider the branch in a network connected between node (1) and (2). Node (3) is a reference node, so that the node voltages V_1 and V_2 may be defined relative to it as shown in the Fig 3.6.

Then $V_{12} = V_1 - V_2$ and the power delivered to the branch $V_{12}\, i_{12} = V_1\, i_{12} - V_2\, i_{12}$. The total delivered branch power for all the branches in the network is

$$\sum_{j=1}^{n}\sum_{k=1}^{n} V_{jk}\, i_{jk} = \sum_{j=1}^{n}\sum_{k=1}^{n} (V_1\, i_{j2} - V_2\, i_{j2}) \quad 294$$

$$\sum_{j=1}^{n} V_1 \sum_{k=1}^{n} i_{jk} - \sum_{K=1}^{n} V_2 \sum_{J=1}^{n} i_{jk} \quad 295$$

$\sum_{k=1}^{n} i_{jk}$ 296 is equal to the sum of current entering node 1 which is zero by KCL. Also

$\sum_{j=1}^{n} i_{jk}$ 297 is algebraic sum of all currents leaving node 2 and is equal to zero. Therefore the total power delivered to the network branches is zero. Note that the circuit element can be any of the passive elements of a network.

3.8 MAXIMUM POWER TRANSFER THEOREM

Let us assume that, it is required to choose a complex impedance load in such a way, that with the given impedance of the source, the source will transfer maximum active power to the load. We represent the impedance of voltage source and the load respectively as

$Z_o = r_o + jX_o$

The power, delivered to the load is equal to

$$P = rI^2 = \frac{rE^2}{|Z_o + Z|^2} = \frac{rE^2}{(r_o + r)^2 + (x_o + x)^2} \quad 298$$

We will first of all change the reactance x. It is obvious that for any value r the current and corresponding active power reaches its highest value when $x = -x_o$. For this condition

$$P = \frac{rE^2}{(r_o + r)^2} \quad 299$$

We will now find the condition of maximum of this obtained function with the assumption, that r - is a variable quantity, that is, from the condition that dp/dr = 0; this will give the following expressions:

$$(r_o + r)^2 - 2r(r_o + r) = 0,$$

whence

$$r = r_o$$

On the basis of this obtained equality, it follows that condition of maximum power (active) transfer from a source to a load is the fulfilment of the expression

$$Z = Z_o^* \quad 300 \qquad (3.8)$$

where Z_o^* 301 is the complex conjugate impedance of Z_o.

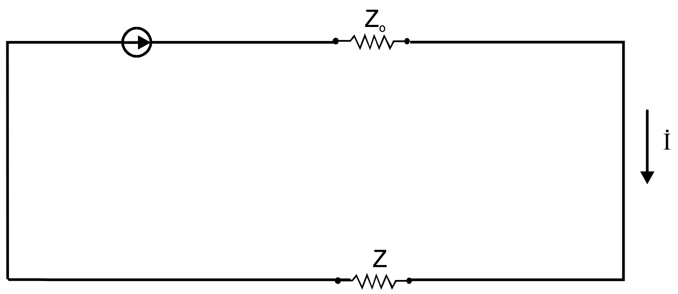


Fig. 3.7 Maximum power transfer from source to receiver

When this condition is fulfilled the power consumed (dissipated) by a load (receiver) is given by the expression

$$P_{\max} = \frac{E^2}{4r_0} \quad 302$$

which is defined as the ratio of the active efficiency power consumed by the load to the total power consumed (dissipated) on the active resistance of the circuit equals 0.5.

In the case, when the complex impedance of the source has an inductive character, the complex impedance of the load on the basis of eq (3.8) should have a capacitive character. This type of compensation of reactance of circuit accomplished in practice with the use of capacitors connected in series or parallel to the load.

3.9 MAXIMUM POWER TRANSFER FOR A GIVEN POWER FACTOR OF THE RECEIVER (LOAD)

In practice it is often necessary to choose the impedance of the load in such a way, so that with a given source impedance and power factor of the load there is maximum transfer of full power and corresponding active power from source to the receiver.

Let us use the same expression in the last paragraph; we will determine the full power on the terminals of the load;

$$S = ZI^2 = \frac{ZE^2}{|Z_O + Z|^2} = \frac{ZE^2}{Z_O^2}\left[1 + \frac{Z}{Z_O} < (\varphi - \varphi_O)\right]^2 \quad 303$$

where φ *and* φ_o 304 - are phases (angles) of the complex impedance Z and
Z_0.

After transformation we obtain the expression

$$S = \left(\frac{E}{Z_o}\right)^2 \frac{Z}{1 + 2\frac{Z}{Z_o}\cos(\varphi - \varphi_o) + \left(\frac{Z}{Z_o}\right)^2} \quad 305$$

If we take Z as the variable, we will write the condition maximum of the function S:

$$\frac{ds}{dz} = 0 \quad 306$$

whence

$$1 + \frac{2Z}{Z_o}\cos(\varphi - \varphi_o) + \left(\frac{Z}{Z_O}\right)^2 - Z\left[2\frac{1}{Z_O}\cos(\varphi - \varphi_O) + 2\frac{Z}{Z_o^2}\right] = 0; \quad 307$$

or

$$1 + \left(\frac{Z}{Z_o}\right)^2 - 2\left(\frac{Z}{Z_o}\right)^2 = 0 \quad 308$$

Hence,

$Z = Z_o$

When we substitute (3.10) in (3.9) we get

$$S_{\max} = \frac{E^2}{2 Z_o [1 + \cos(\varphi - \varphi_o)]} \quad 309$$

As such maximum power transfer delivered to a load with a given $\text{Cos}\,\varphi$ 310 (power factor) is achieved when the total impedance of the load and the source are equal. The higher the absolute value of the difference in angles of the load impedance and source impedance; $|\varphi - \varphi|$ 311; the higher the delivered power to the load.

The condition of maximum power transfer is widely used in electronic, automation and instrumentation. In energy systems, generating and utilizing high power, there is tendency to achieve high efficiency of the generators; therefore the impedance of the load is by far greater than the impedance of the generators.

CIRCUITS WITH INDUCTIVELY COUPLED ELEMENTS

If we have two inductors with turn w_1 and w_2 and with current i_1 and i_2 (fig. 3:8a) some of the magnetic lines of force of each coil will link with the turns of the other. Therefore in addition to the self-flux linkages ψ_{11} - ψ_{22} it is important to consider the additional flux linkages of the first coil with the turns of the seconds.

$$\psi_{12} = \sum_{k=1}^{w_1} Q_k 12 \quad 312$$

and of the second coil with the turns of the first,

$$\psi_{12} = \sum_{k=1}^{w_2} Q_k 21 \quad 313$$

where Q_{k12} is the flux threading the Kth turn of the first coil due to the current in the second coil, and Q_{k21} is the flux threading the Kth turn of the second coil due to the current in the first coil.

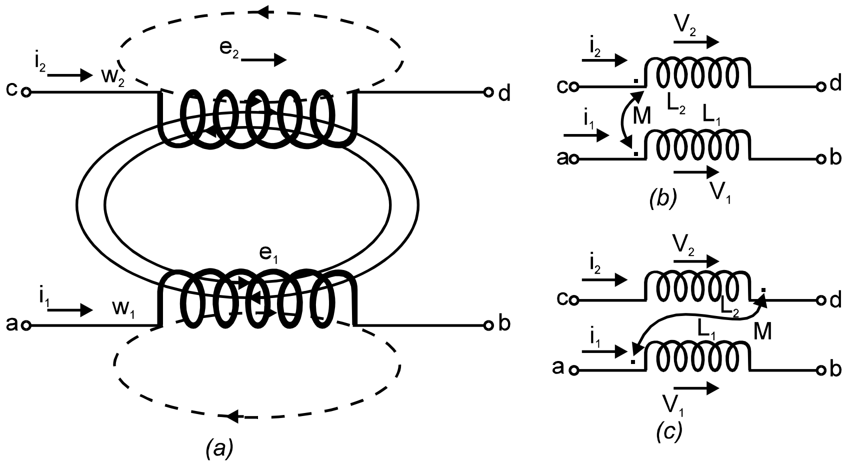


Fig. 3.8 (a) Mutual and self-induction (b) Coils connected together aiding(c) Coils connected together in opposition.

The ratio of the additional flux linkage of the first coil to the current i_2 in the second coil is called the mutual inductance of two coils.

$M_{12} = \psi_{12}/i_2$

Similarly, we can define the mutual inductance of the second and first coils.

$M_{12} = \psi_{21}/i_1$

Experience shows that $M_{21} = M_{12} = M$. A rigorous proof of the statement can be given by recourse of the theory of the electromagnetic field.

In linear electric circuits the mutual inductance is independent of the directions and values of the currents and is solely determined by the design and relative position of the coupled coils. The total flux linkage ψ 314 of each of the two inductively coupled coils has two components which can be added together or subtracted from each other, depending on the direction of the currents in the coils and their relative position. When the two components are added together, the coils are said to be connected aiding. When the two components are subtracted from each other, the two coils are said to be connected in opposition. In circuit diagrams the character of inductive coupling is identified by conventional signs. In equivalent circuits the "start" of the coils are labelled with dots (asterisks) fig. 3.8b and c. If the currents are flowing in the same direction relative to the like terminals (fig. 3.8b), the coils are connected aiding. In the circumstances, the self and mutual-flux linkages in each coil are added together, so the total flux linkage of the first coil is

$$\psi_1 = \psi_{11} + \psi_{12} \quad 315$$

and the total flux linkage of the second coil is

$$\psi_2 = \psi_{22} + \psi_{21} \quad 316$$

If the currents are flowing in different directions relative to the like terminals (fig. 3.8c), the coils are connected in opposition, and

$$\psi_1 = \psi_{11} - \psi 12 , \psi_2 = \psi_{22} - \psi_{21} \quad 317$$

In sinusoidally excited circuits, the direction of a current refers to the assumed positive direction.

By the law of electromagnetic induction, an emf is induced in each coil. The emf induced in the first coil is

$$e_1 = -d\,\psi_1 /dt = \frac{-d\,(\psi_{11} \pm \psi_{12})}{dt} = e_{1L} \pm e_{1m} \quad 318$$

and that in the second

$$e_2 = -d\,\psi_2 /dt = \frac{-d\,(\psi_{22} \pm \psi_{21})}{dt} = e_{2L} \pm e_{2m} \quad 319$$

where

$$e_{1L} = -d\,\psi_{11} /dt = -L_1 di_1 /dt \quad 320$$

$$e_{2L} = -d\,\psi_{22} /dt = -L_2 di_2 /dt \quad 321$$

are the emfs of self-induction in the first and second coils, respectively;

$$e_{1m} = -d\,\psi_{12} /dt = -M di_2 /dt \quad 322$$

$$e_{2m} = -d\psi_{21}/dt = -Mdi_1/dt \quad 323$$

are emfs of mutual induction in the first and second coils, respectively.

The voltage across an inductor is $V_L = V_{ab} = -e_L$
similarly, for inductively coupled coils,

$$V_1 = V_{ab} = -e_1 = -e_{1L} \pm e_{1m} = L_1 di_1/dt \pm Mdi_2/dt = V_{1L} \pm V_{1m} \quad 324$$

$$V_2 = V_{cd} = -e_2 = -e_{2L} \pm e_{2m}$$
$$= L_2 di_2/dt \pm Mdi_1/dt = V_{2L} \pm V_{2m} \quad 325$$

When inductors are series-connected, the leads meeting at the common junction may have the same or the opposite polarities. In the former case the inductors are said to be connected aiding and in the latter, in opposition.

If during a time internal t_1 the currents in two inductively coupled coils vary from zero to i_1 and i_2 respectively, their common magnetic field stores an amount of energy given by

$$W_m = \int_0^{t_1} V_1 i_1 dt + \int_0^{t_1} v_2 i_2 dt = L_1 \int_0^{i_1} i_1 di_1 + L_2 \int_0^{i_2} i_2 di_2$$
$$\pm M \int_0^{i_2} i_1 di_2 \pm M \int_0^{i_2} i_2 di_1 \quad 326$$

$$= L_1 i_1^2/2 + L_2 i_2^2/2 \pm M\, i_1 i_2$$

where we have used integration by parts

$$\int_0^{i_2} i_1 di_2 = i_1 i_2 - \int_0^{i_1} i_2 di_1 \quad 327$$

Thus, as compared with energy stored in the magnetic field due to two coils not coupled inductively, the energy in the common magnetic field due to two inductively coupled coils is increased or decreased by

$W_{mm} = Mi_1i_2$

Sinusoidally excited circuits containing inductively coupled coils may be analysed by the complex-number method in about the same manner as circuits which are not coupled inductively. By analogy with Ohms law in complex form for an inductive element we may write equation (1) in complex for as

$$\dot{V}_2 = \dot{V}_{2L} \pm \dot{V}_{2m} = j\omega L_2 \dot{I}_2 \pm jX_{1L}\dot{I}_1 \pm jX_m \dot{I}_1 \quad (2b)$$

$$\dot{V}_1 = \dot{V}_{1L} \pm \dot{V}_{1m} = j\omega L_1 \dot{I}_1 \pm j\omega M\dot{I}_2 = jX_{1L}\dot{I}_1 \pm jX_m \dot{I}_2 \quad (2a)$$

where $X_m = \omega M$ is the mutual induction impedance, and I_1 and I_2 are the respective complex currents.

The respective complex emfs of self and mutual induction are given by

$$\dot{E}_{1L} = -\dot{V}_{1L} = -j\omega L_1 I_1 = -jX_{1L} I_1 \quad 330$$

$$\dot{E}_{1m} = -\dot{V}_{1m} = -j\omega M I_2 = -jX_m I_{12} \quad 331$$

$$\dot{E}_{2L} = -\dot{V}_{2L} = -j\omega L_2 \dot{I}_2 = -jX_{2L} \dot{I}_2 \quad 332$$

$$\dot{E}_{2m} = -\dot{V}_{2m} = -j\omega M \dot{I}_1 = -jX_m \dot{I}_1 \quad 333$$

The complex power in each of the two inductively coupled coils is given by

334

335

(5a)

337 (5b)

where $Q_{12} = Q_{21}$ and $P_{12} = -P_{21}$, are respectively, the reactive and active powers transferred from the second coil to the first and from the first to the second.

In the general case of an electric circuit containing an inductively coupled coils, the voltage across each K^{th} coil is

$$\dot{V}_k = jX_{kL} \dot{I}_k + \_ \sum jX_{mkp} \dot{I}_p \quad 338$$

where $P \neq K$

REPRESENTATION OF SINUSOIDAL QUANTITIES AS ROTATING PHASORS

In order to represent a sinusoidal quantity $a = A_m \mathrm{Sin}(\omega t + \psi)$ with an initial phase ψ as a rotating phasor, we draw (Fig. 3.9a) a radius phasor Am such that its length (on the adopted scale) is equal to the amplitude Am and its angle with the horizontal axis is ψ. This is the initial position at time t =0. From the tip of the radius phasor Am in the initial position we drop a perpendicular of length Am sinψ on the horizontal axis.

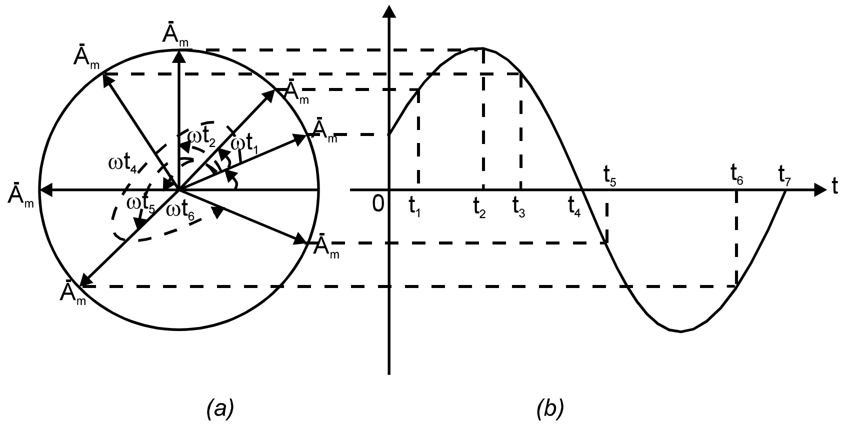


Fig. 3.9 Representation of sinusoidal quantities as rotating phasors

Suppose that the radius phasor rotates at a constant angular frequency $\omega = 2\pi/T = 2\pi f$ counter clockwise. Here is the period T and f is the cyclic frequency of rotation. At time t_1 the radius phasor A_m turns from the initial position by an angle ωt_1, and length of the perpendicular dropped from its tip is $A_m \sin(\omega t_1 + \psi)$.

Obviously, the length of the perpendicular dropped from the tip of the rotating radius phasor on the horizontal axis is a maximum at time t_2 when $\omega t_2 + \psi = \pi/2$.

$$A_m \sin(\omega t_2 + \psi) = A_m \sin(\pi/2) = A_m$$

Next to the circle described by the tip of the rotating radius phasor, we construct a cartesian plot of the sinusoidal quantity A_m x $\mathrm{Sin}(\omega t + \psi)$ as a function of the phase wt or of time (Fig. 3.9b). At time t_2 the sinusoidal quantity a is a maximum. As the radius phasor rotates, the sinusoidal quantity $a = A_m \sin(\omega t + \psi)$ decreases while remaining positive, then drops to zero at time t_4, changes sign to become negative at time t_5 and t_6 and again positive beginning at t_7 and so on.

Rotating phasors offer a compact means for representing a combination of several sinusoidal quantities varying at the same frequency on a single plot, which provides a special convenience in the analysis of complex electric networks.

REPRESENTATION OF SINUSOIDAL QUANTITIES AS COMPLEX NUMBERS:

It is easy to change from the rotating phasor to the complex number representation of sinusoidal quantities in order to represent a sinusoidal quantity specified in trigonometric form

$$a = A_m \sin(\omega t + \psi) \qquad (1)$$

with an initial phase ψ as a complex number, we draw a phasor on a complex plane (fig. 3.10) from the origin at the angle ψ to the real axis on the adopted scale, the length of the phasor is equal to the amplitude A_m of the sinusoidal quantity, and its tip is located at a point corresponding to a particular complex number the complex amplitude of the sinusoidal quantity.

$$A_m = A \exp(j\psi) = A_m\angle\psi.$$

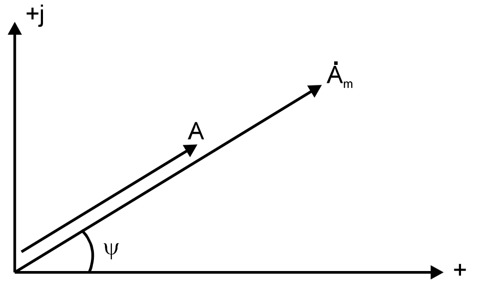


Fig. 3.10 The complex and complex amplitude

The same symbol is use to designate the phasor corresponding to the complex amplitude on the complex plane.

As the phase $\omega t + \psi$ of the sinusoidal quantity increases with time, the angle between the phasor and the real axis increases too, and so a rotating phasor results such that

$$A_m \exp j(\omega t + \psi) = A_m \cos(\omega t + \psi) + jA_m \sin(\omega t + \psi)$$

it is easy to see that the imaginary part of the rotating phasor is equal to the specified sinusoidal quantity (Eq. 1). Infact, the representation of a sinusoidal quantity as a complex amplitude A_m and the corresponding phasor on a complex plane is geometrically similar to the representation of the same quantity as a rotating radius phasor A_m at time =-0 (see fig. 3.9a). Therefore it might seem that the two representation are practically the same. But this is not so.

With sinusoidal quantities represented as complex numbers, we can use the effective complex method of sinusoidal A.C circuit analysis which has won general recognition now.
The phasor on a complex plane, whose length on the adopted scale is equal to the rms value of the sinusoidal quantity involved, and the corresponding complex number are referred to as the complex rms value of that sinusoidal quantity.

$$\dot{A} = \frac{A_m}{\sqrt{2}} = A\exp(j\psi) = A < \psi \quad 339$$

The symbol is used to designate the phasor itself on a complex plane (see fig. 3.10).

The complex value of a sinusoidal quantity can be written in any one of three forms, as follows:

(i) exponential

$$\dot{A} = \frac{A_m}{\sqrt{2}} = A\exp(j\psi) = A < \psi \quad 340$$

(ii) trigonometric

$$\dot{A} = A\cos\psi + jA\sin\psi \quad 341$$

(iii) and algebraic

$$\dot{A} = A' + j''A \quad 342$$

where $A' = A\cos\psi$ is the real part and $A'' = A\sin\psi$ is the imaginary part of the complex value of the sinusoidal quantity.

$$\dot{A} = \sqrt{(A')^2 + (A'')^2} \quad 343 \text{ and } \psi = \arctan(A''/A')$$

We can change from the exponential to the trigonometric form by invoking Euler's formula:

$$\exp(j\psi) = \cos\psi + j\sin\psi$$

when $\psi = \pi/2$ and $\psi = -\pi/2$, two frequently used relations follow immediately.

$$\exp(j\pi/2) = j \text{ and } \exp(-j\pi/2) = -j = 1/j.$$

In the analysis of a sinusoidal A.C circuits we mostly use the complex rms values of sinusoidal quantities. For short they are called the complex values of sinusoidal quantities, and the corresponding phasors on a complex plane are referred to as the phasors of complex values. For example, in the case of the sinusoidal current.

$$i = I_m\sin(\omega t + \psi_i) = 10\sin(\omega t + 45°)$$

The corresponding complex value is

$$\dot{I} = I\exp(j\psi_i) = \frac{10}{\sqrt{2}}\exp(j45°) = 7.07 < 45° \quad 344$$

The collection of phasors representing the complex values of sinusoidal quantities at the same frequency is called a phasor diagram. By reference to a phasor diagram, we can replace the addition and subtraction of complex values by the additions and subtraction of the respective phasors. This often simplifies the computation and makes them more instructive.

The relative position of the complex phasor on a phasor diagram will remain unchanged if the initial phases ψ of all the complex values are increased (decreased) by the same amount. This only implies that all the phasors are turned through the same angle. In the analysis of electric circuit, a phasor diagram is often constructed so that one of the complex phasors is directed along the real axis and is called the reference (or datum) phasor.

The sinusoidal quantities (current, voltage and the like) defining the conditions in an electric circuit periodically vary in direction, but one of them is assumed to be positive. This direction is chosen arbitrarily and indicated by an arrow in the circuit diagram. With the positive direction thus assumed, a sinusoidal quantity can be represented both by its instantaneous value defined as

$$a = A_m \sin(\omega t + \psi)$$

and by the respective complex value

$$A = A<\psi$$

Thus, the one-to-one correspondence between the representations of sinusoidal currents, voltages and other quantities implies the same positive direction (fig. 3.11)

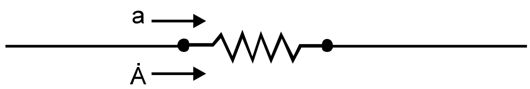


Fig. 3.11 Positive direction (assumed)

OHM'S LAW IN COMPLEX FORM FOR RESISTIVE INDUCTIVE AND CAPACITIVE ELEMENT

The relations between the currents and voltages associated with resistive, inductive and capacitive elements are governed by the physical processes taking place in them. A mathematical description of the physical processes in each of these elements depends on the choice of an analytical representation for sinusoidal quantities. In our subsequent exposition, when representing sinusoidal currents, voltages and the like, we will use both the respective trigonometric functions and plots, and the complex values.

A. **THE RESISTIVE ELEMENT**: Let the positive direction of the sinusoidal current

$$i_r = I_{rm} \sin(\omega t + \psi_i)$$

in a resistive element of fixed resistance r be the same as the positive direction of the voltage applied to the element (fig. 3.12). Then the instantaneous values of the voltage and current will be connected by a relation recognised as Ohm's law

$$V_r = r \cdot i_r$$

or $$V_r = rI_{rm} \sin(\omega t + \psi_i) = V_{rm} \sin(\omega t + \psi_v)$$

where the amplitude of current and voltage are connected by a relation of the form

$$V_r = r \cdot I_{rm} \tag{3a}$$

and their initial phases are the same

$$\psi_v = \psi_i$$

which means that in a resistive element the current and the voltage vary in phase as shown in fig. 3.13) for the initial phase $\psi_v = \psi_i > 0$

On dividing the right hand and the left hand of Equ. (3a) by $\sqrt{2}$ 345, we obtain the relation defining the rms values of voltage and current in a resistive element

$$V_r = rI_r$$

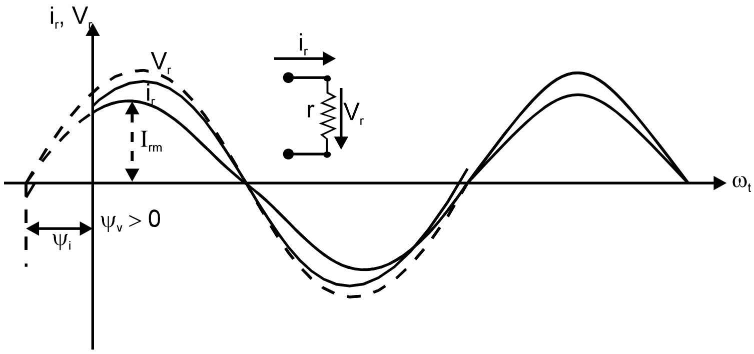


Fig. 3.12 Sinusoidal current and voltage in a resistance element.

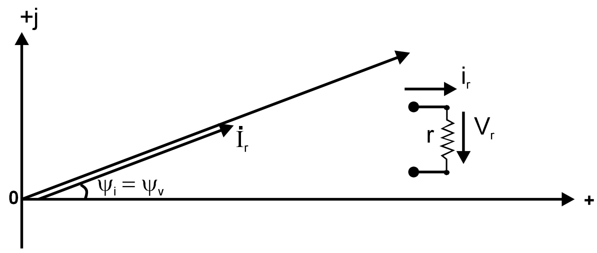


Fig. 3.13 Phasor diagram for current and voltage in resistance element.

Now we will represent the sinusoidal current in and the sinusoidal voltage across a resistive element in terms of their complex values

$$\dot{I}_r = I_r \exp(j\psi_i)$$ 346

and $V_r = V_r \exp(j\psi_v)$

Since $V_r = r \cdot I_r$ and $\psi_v = \psi_i$, it follows that the complex current I_r in and the complex voltage V_r across a resistive element obey Ohm's law in complex form.

$$V_r = rI_r$$

The relation between the complex current in and the complex voltage across a resistive element is illustrated in the phasor diagram of fig. 3.13. From the phasor diagram it is also seen that the complex current and voltage phasors for a resistive element are in phase.

B. **THE INDUCTIVE ELEMENT:** If an inductive element carries a sinusoidal current

$$i_L = I_{Lm}\sin(\omega t + \psi_i)$$

then in accordance with the law of electromagnetic induction, the voltage across the inductive element is given by

$$V_L = -e_L = Ldi_i/dt = \omega LI_{Lm}\cos(\omega t + \psi_i) = V_{Lm}\sin(\omega t + \psi_i + \pi/2) = V_{Lm}\sin(\omega t + \psi_v)$$

where the amplitudes of voltage and current are connected by a relation of the form

$$V_{Lm} = \omega LI_{Lm} \qquad (6a)$$

and their initial phases by the relation $\psi_v = \psi_i + \pi/2$

On dividing the right-hand and left-hand sides of Eq. (6a) by $\sqrt{2}$ 347, we obtain the relations for the rms voltage across the inductive element, and the current through the inductive element.

$$V_L = \omega LI_L = X_L \cdot I_L$$

The instantaneous values of sinusoidal current in and sinusoidal voltage across an inductive element appear in the plot of fig. 3.14 (constructed for $\psi_i>0$) from which it is seen that the sinusoidal current i_L lags behind the sinusoidal voltage V_L in phase by an angle $\psi = \psi_v - \psi_i = \pi/2$.

The quantity $X_L = \omega L$ has the units of Ohm and is called the inductive reactance, while its reciprocal $b_L = 1/\omega L$ is called the inductive susceptance. The values of X_L and b_L are the parameters of the inductive elements in sinusoidal A.C circuits.

Inductive reactance is proportional to the angular frequency of the sinusoidal currents, so it is zero for direct current ($\omega = 0$). For this reason many pieces of electric equipment designed for operation in a.c circuit may not be used in d.c. circuits. Their d.c. impedance is relatively small, and the heavy direct current might be destructive (say, for the primary of winding the transformer in a radio receiver. The sinusoidal current i_L in and the sinusoidal voltage V_L across an inductive element can be represented by the corresponding complex values as follows:

$$I_L = I_L \exp(j\psi_i)$$

and $$V_L = V_L \exp(j\psi_v$$

A phasor diagram for an inductive element is shown in fig. 3.15. As is seen the phasor of the complex current I_L lags behind the phasor of the complex voltage V_L in phase by a phase of $\pi/2$ (90°), so the phase shift between them is
$\psi = \pi/2$ (90°) as shown in fig. 3.14.

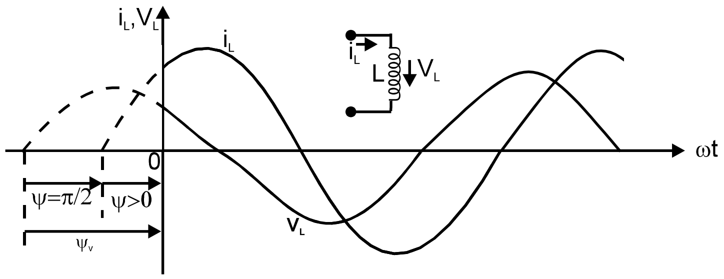


Fig. 3.14 The simusoidal current annd voltage in an inductance element.

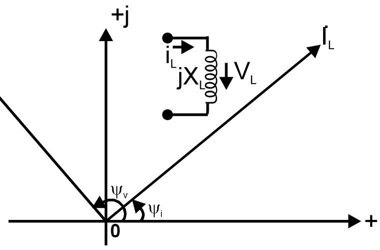


Fig. 3.15 The phasor for inductive element

We deduce Ohm's law in complex form for an inductive element

$$V_L = \omega LI \exp(j\psi_v) = \omega LI\mathrm{exp}j(\psi_i + \pi/2)$$

or

$$v_L = j\omega LI_L = jX_LI_L \tag{7}$$

The quantity $j\omega L = jX_L$ is called the complex inductive reactance, and its reciprocal $1/\omega L = -jb_L$ is called the complex inductive susceptance.

Alternatively, the complex voltage across an inductive element may be expressed in terms of the complex flux linkage. As follows

$$\dot{\psi} = L\,\dot{I}_L \quad 348$$

and on the basis of Eq. (7)

$$\dot{V}_L = -\dot{E}_L = j\omega\psi \quad 349 \tag{8}$$

Equation (8) is a mathematical statement of the law of electromagnetic induction in complex form.

Sometimes the phasor diagram of an inductive element may be explained to include the phasor E_L which is opposite in direction and equal in magnitude to V_L. This convention is for example, used in the analysis of transformers.

C. **THE CAPACITIVE ELEMENT:** If the voltage between the terminals of a capacitive element is sinusoidal.

$V_c = V_{cm}\sin(\omega t + \psi_V)$

then, the associated sinusoidal current is given by

$i_c = CdV_c/dt = \omega cV_{cm}\cos(\omega t + \psi_v)$
$= I_m \sin(\omega t + \psi_v + \pi/2) = I_{cm}\sin(\omega t + \psi_i)$

where the amplitude of current and voltage are connected by a relation of the form.

$I_{cm} = \omega cV_{cm}$

and their initial phases by the relation

$\psi_i = \psi_v + \pi/2$

on dividing the right-hand and left-hand sides of the above equation by $\sqrt{2}$ 350, we obtain the relation for the rms values of the voltage across and the current in a capacitive element

$V_c = (1/\omega c)\ I_c = I_c.bc$ (9)

The quantity $bc = \omega c$ in equation (9) has the units of siemens(s) and is called the capacitive susceptance and its reciprocal $X_c = 1/\omega c$ is called the capacitive reactance used in sinusoidal a.c circuits.

In contrast to inductive reactance capacitive reactance decreases with increasing frequency of a sinusoidal current. At constant voltage, the reactance of a capacitive element is infinite.

The instantaneous values of the sinusoidal voltage across and the sinusoidal current in a capacitive element are shown in the plot of fig. 3.16 (constructed for ψ_v>0). As is seen, V_c lags behind i_c in phase by an angle $\psi_i - \psi_v = \pi/2$. In other words, the phase shift between them is $\psi_v - \psi_i = -\pi/2$ (-90°).

In complex notation, the sinusoidal current i_c and the sinusoidal voltage V_c for a capacitive element will be written as

$\dot{I}_c = I_c \exp(j\psi_i)$ 351

and $\dot{V}_c = V_c \exp(j\psi_v)$ 352

A phasor diagram for a capacitive element is shown in fig. 3.17. As is seen, the complex voltage V_c lags behind the complex current I_c in phase by $\pi/2$ (90°).

In view of this, Ohm's law in complex form for a capacitive element

$\dot{V}_c = (1/j\omega c)\,\dot{I}_c = -\,jX_c \bullet \dot{I}_c$ 353

The quantity $1/j\omega c = -jX_c$ is called the complex reactance of a capacitive element, and its reciprocal $j\omega c = jbc$ is called the complex susceptance of a capacitive element.

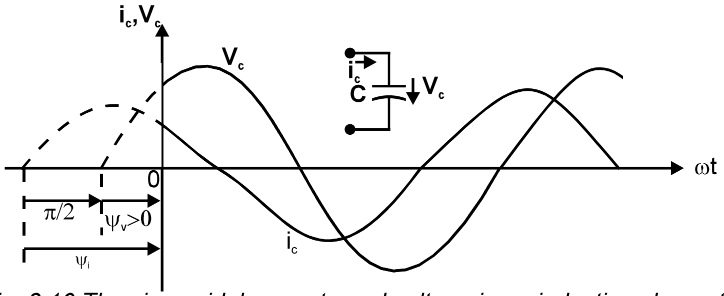


Fig. 3.16 The simusoidal current annd voltage in an inductive element.

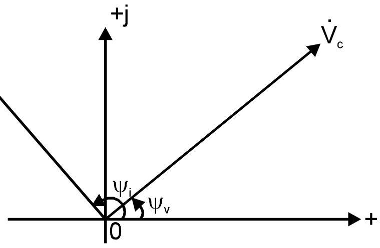


Fig. 3.17 Phasor diagram for current and voltage in an inductive element

KIRCHHOFF'S LAWS FOR SINUSOIDAL A.C CIRCUITS

The mathematical statement of Kirchhoff's two laws depends on the choice of a representation for sinusoidal quantities. We will use trigonometric functions and the corresponding complex values. In the trigonometric form of representation, Kirchhoff's laws relate the instantaneous values of the respective sinusoidal quantities (for any instant of time). In the complex-number form of representation, these laws relate the complex values of the same quantities.

A. KIRCHHOFF'S CURRENT (OR FIRST) LAW:

This law states that the algebraic sum of currents at any node of an electric circuit at any instant of time is equal to zero. For sinusoidal a.c circuits this implies that in the branches meeting at any node the algebraic sum of instantaneous currents is zero.

$$\sum_{k=1}^{n} i_k = 0 \qquad (10)$$

that is

$$\sum_{k=1}^{n} I_{mk} \sin(\omega t + \varphi_{ik}) = 0 \qquad (11)$$

where n is the number of branches meeting at a node. In our further discussion we will assign a "+" sign to the currents entering a node, and a "-" sign to the current leaving a node. As an example, fig. 3.18 shows a plot of the instantaneous values of three sinusoidal currents at one node.

$i_1 = I_{m1}\sin(\omega t + \psi_{i1})$

$i_2 = I_{m2}\sin(\omega t + \psi_{i2})$

$i_3 = I_{m3}\sin(\omega t + \psi_{i3})$

for the assumed positive direction. by Kirchhoff's current law

$$\sum_{k=1}^{3} i_k = i_1 + i_2 - i_3 = 0 \quad 356$$

for any instant of time.

If we wish to re-cast the mathematical statement of Kirchhoff's current law in complex form, we should represent all the currents in (11) with their complex values.

$$\dot{I}_k = I_k < \psi_{ik} \quad 357$$

In complex notation kirchhoff's current law is written as

$$\sum_{k=1}^{n} \dot{I}_k = 0 \quad 358$$

or in words, the algebraic sum of the complex currents in all the branches meeting at a node of a sinusoidal a.c circuit is zero. Here too, we assign a "+" sign to the currents entering the node, and a "-" sign to the currents leaving the node.

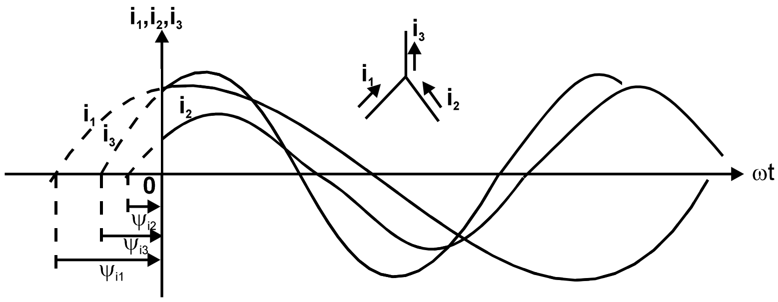


Fig. 3.18 Kirchhoff's current law in simusoidal form.

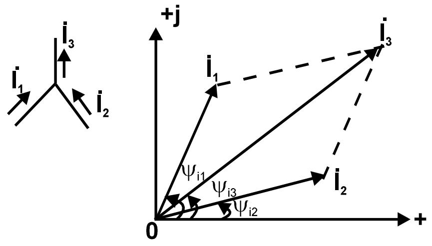


Fig. 3.19 Kirchhof,s current law in phasor diagram.

On the phasor diagram either of the two following equalities must be satisfied;

$$\sum_{k=1}^{3} \dot{I}_k = \dot{I}_1 + \dot{I}_2 - \dot{I}_3 = 0 \quad 359$$

$$or \quad \dot{I}_1 + \dot{I}_2 = \dot{I}_3 \quad 360$$

B. **KIRCHHOFF'S VOLTAGE (SECOND LAW):** This law states that the algebraic sum of the voltages across all the resistive, inductive and capacitive (that is, all passive) elements in any loop (mesh) of an electric circuit is equal at any instant of time to the algebraic sum of all the emfs in the loop (mesh). In sinusoidal a.c circuits the various emfs and the voltages across the passive elements in any loop (mesh) are varying all the time. Yet the algebraic sums of the voltages and of the emfs are same

$$\sum_{k=1}^{n} V_k = \sum_{k=1}^{n} e_k \qquad (13)$$

or

$$\sum_{k=1}^{n} V_{mk} \sin(\omega t + \psi_{Vk}) = \sum_{k=1}^{m} E_{mk} \sin(\omega t + \psi_{ek}) \quad 362$$

where n is the number of passive elements and m is the number of emfs in the loop (mesh). We assume that all the sinusoidal voltages V_K and emfs e_k in (13) for which the positive direction coincides with an arbitrarily chosen direction for traversal around the loop are written with a "+" sign otherwise they are written with a "-" sign. Taking fig. 3.20 as an example, the loop is traversed clockwise, so by Kirchhoff's voltage law.

$-V_L + V_r = e_1 - e_2$

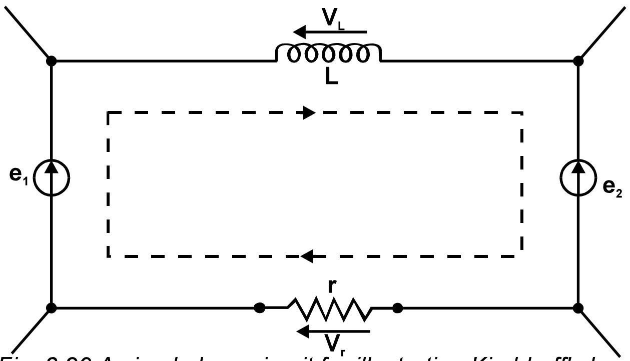


Fig. 3.20 A simple loop circuit for illustrating Kirchhoff's law.

If we wish to express Kirchhoff's voltage law in complex form, all the sinusoidal voltages V_k and emfs e_k in (13) should be replaced with their complex values

$$\dot{V}_k = V_k < \psi_{Vk} \quad and \quad 363$$

$$\dot{E}_k = E_k < \psi_{ek} \quad 364$$

In complex notation kirchhoff's voltage law is written as

$$\sum_{k=1}^{n} \dot{V}_k = \sum_{k=1}^{m} \dot{E}_k \quad 365$$

or in words, the algebraic sum of the complex voltages across all the passive (resistive, inductive, and capacitive) elements in any loop (mesh) of a sinusoidal a.c circuit is equal to the algebraic sum of all the complex emfs in that loop (mesh). Here we assign a "+" sign to the complex voltages and emfs whose positive directions are the same as an arbitrarily chosen direction by which we traverse the loop; otherwise we assign "-" sign.

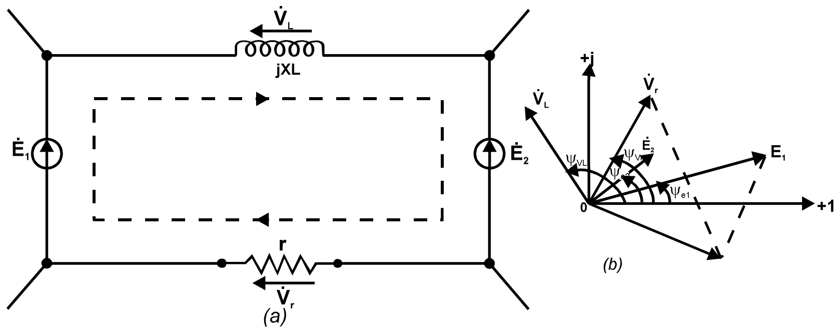


Fig. 3.21 (a) A simple loop containing passive elements to calculate Kirchhoff's voltage law (b) A corresponding phasor diagram.

Taking the loop of fig. 3.20, also shown in fig. 3.21a as an example, we may write on the basis of Kirchhoff's voltage law.

$$\dot{V}_r - \dot{V}_L = \dot{E}_1 - \dot{E}_2 \quad 366$$

Fig. 3.21b shows a phasor diagram for the emfs and voltages in that loop. It convincingly illustrates Kirchhoff's voltage law in complex form.

ANALYSIS OF SINUSOIDAL A.C. CIRCUITS BY THE COMPLEX METHOD.

As has been in section 2.7, there is a one-to-one correspondence between the instantaneous values of sinusoidal quantities and their complex values. Therefore and form of representation for sinusoidal quantities may be used in order to define the conditions in a sinusoidal a.c circuit. When however, sinusoidal quantities are presented in complex notation elements and capacitive elements and the two Kirchhoff's law are set us as algebraic rather than differential equations. Simultaneous solution of algebraic equations in order to find the complex value of currents in and of voltages across all the elements of an electric circuit, that is, the use of the complex method analysis is a relatively simple task. Once the complex values are found, we can immediately write, if necessary the corresponding instantaneous values of the sinusoidal quantities involved.

The analysis of a sinusoidal a.c circuit by the complex method may conveniently be divided into several logical independent steps.

(1) The initial specifications of all the elements of the circuit involved are presented in complex form. To do so, firstly, the sinusoidal emf or currents of energy sources specified by their instantaneous values (in trigonometric form) are presented in complex form (see table 3.1) and secondly, the corresponding complex reactance or susceptance of all the inductive and capacitive elements of the circuit (table 3.2) are found.

(2) Positive direction are chosen for the currents on all the branches and labelled with arrow heads in the circuit diagram.

(3) Using Ohm's and Kirchhoff's laws in complex form and noting the chosen positive direction of current as the branches, a set

Table 3:1: Instantaneous and complex values of sinusoidal Emfs and currents

Source	Instantaneous value	Complex value	Symbol
EMF	$e = E_m \sin(\omega t + \psi_2)$	$(\dot{E}_m/\sqrt{2})\exp(j\psi_e)$ 367	
Current	$J(t) = J_m \sin(\omega t + \psi_i)$	$\dot{J} = (J_m/\sqrt{2})\exp(j\psi_i)$ 368	

Table 3:2: Complex impedances (Resistance and Reactance) and complex Admittance (conductance and susceptance) of passive elements.

Element	Parameter	Complex impedance	Complex admittance
Resistance	r	r	$1/r = g$
Inductance	L	$j\omega L = jX_c$	$1/j\omega L = -jb_L$
Capacitance	C	$1/j\omega C = -jX_c$	$j\omega C = jb_c$

of equations are set up to define the conditions in the circuit.

(4) The set of equations thus obtained is solved that is, the complex currents in the circuit branches and the complex voltages across the circuit elements are found. The complex values of currents and voltages thus determined uniquely define the corresponding instantaneous values of sinusoidal currents and voltages.

As an example, we will apply the above procedure to the sinusoidal a.c circuit of fig. 3.22a which contains a source of emf $e = E_m \sin(\omega t + \psi_e)$, a source of current $J(t) = J_m \sin(\omega t + \psi_i)$, a resistive element r an inductive element L, and a capacitive element c.

(1) To begin with, we re-write the sinusoidal emf and current specified by their instantaneous values in complex form on the basis of table 3.1

$$\dot{E} = E < \psi_e ; \dot{J} = J < \psi_i \quad 369$$

Next we find the complex reactance of the inductive element, jω_L - jX_L and of the capacitive element, 1/jω_c = -jX_c, of the circuit from table 3.2.

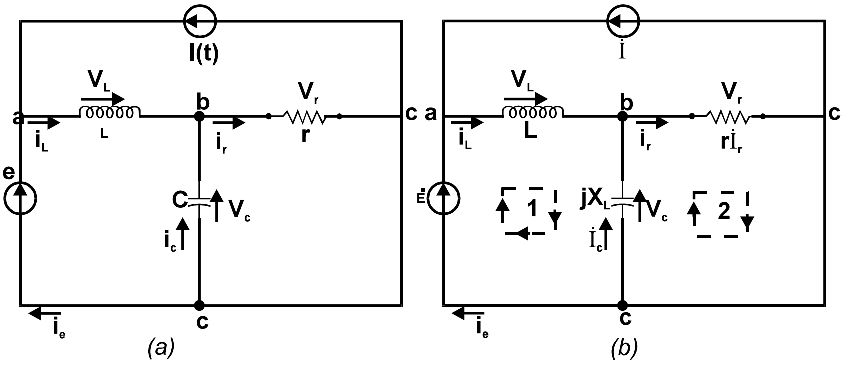


Fig. 3.22 Complex method of analysis of sinusoidal A.C. circuit.

The circuit of fig. 3.22a with its parameters represented in complex form is drawn up in fig. 3.22b.

(2) Now we assume positive directions for unknown currents in the branches (fig. 3.22a) and positive directions for the voltages across the circuit elements that they are the same as the directions of the currents. The positive directions for the respective complex values are the same (fig. 3.22b).

(3) Noting the assumed positive directions for the currents and voltages, we write a complete set of equations for circuit analysis. Since the circuit has three nodes (a,b and c), we use Kirchhoff's current law in complex form and set up an equation for two nodes, say a and b

$$\dot{J} + \dot{I}_e - \dot{I}_L = 0 \quad 370$$

$$\dot{I}_L + \dot{I}_c - \dot{I}_r = 0 \quad 371 \qquad (1)$$

The circuit has three loops (meshes), so in the general case three independent equations should have been written by Kirchhoff's voltage law. However, the upper loop contains a current source which implies that the current J in that loop is known, so we only need to write equations for the remaining two loops labelled 1 and 2 in fig 3.22b

$$\dot{V}_L - \dot{V}_c = \dot{E} \quad 372$$

$$\dot{V}_r + \dot{V}_c = 0 \quad 373$$

Applying Ohm's law in complex form for a resistive element, for an inductive element, and for a capacitive element, we write

$$\dot{V}_r = r\,\dot{I}_r \quad 374$$

$$\dot{V}_L = jX_L\dot{I}_L \quad 375$$

$$\dot{V}_c = -\,jX_c\dot{I}_c \quad 376$$

Hence, we may write the above equations as

$$jX_L\dot{I}_L + jX_C\dot{I}_C = \dot{E} \quad 377$$

$$rI_r - jX_C\dot{I}_C = 0 \quad 378$$

(4) By solving the set of four algebraic equations above in four unknown currents we find their complex values;

$\_{r}$ = $I_r \exp(j\psi_{ir})$

$\_{L}$ = $I_L \exp(j\psi_{iL})$

$\_{c}$ = $I_c \exp(j\psi_{ic})$

$\_{e}$ = $I_e \exp(j\psi_{ie})$

The instantaneous values corresponding to the above complex values are as follows;

$$i_r = \sqrt{2}I_r\sin(\omega t + \psi_{ir}) \quad 379$$

$$i_L = \sqrt{2}I_L\sin(\omega t + \psi_{iL}) \quad 380$$

$$i_c = \sqrt{2}I_c\sin(\omega t + \psi_{ic}) \quad 381$$

$$i_e = \sqrt{2}I_e\sin(\omega t + \psi_{ie}) \quad 382$$

The instantaneous values of voltages are found in the same manner, and their instantaneous values are written in the same manner as for currents.

As in the case with linear d.c circuits, linear sinusoidal a.c networks obey the principle of superposition. Therefore, the analysis of linear sinusoidal circuits can be simplified through the use of the various techniques applicable to linear d.c circuits, such as the circuit transformation method, the node-pair method the loop (mesh) current method, the equivalent generator method. As a rule, these analysis techniques are used in combination with complex method. Obviously, the mathematical statements involved in the analysis of d.c circuits remain in force for a.c circuits as well. We only need to replace all the emfs voltages and currents with the complex values of the respective sinusoidal quantities and the resistances of the circuit elements with their complex impedances (resistances and reactance).

In our further discussion, the complex values of emfs, voltages, currents, and the like will be referred to for brevity, as complex current or still simpler as emfs, voltages or currents etc.

THE SINGLE-LOOP SINUSOIDAL A.C. CIRCUIT

In a single-loop sinusoidal A.C circuit(fig. 3.23) containing a source of sinusoidal emf.

$$= E_m\sin(\omega t + e)$$

the current is also sinusoidal

$$i = I_m\sin(\omega t + \psi_i)$$

and the voltages across the resistive, inductive and capacitive elements are as follows:

$$V_r = V_{rm}\sin(\omega t + \psi_{vr})$$

$$V_L = V_{Lm}\sin(\omega t + \psi_{vL})$$

$$V_c = V_{cm}\sin(\omega t + \psi_{vc})$$

The conditions in a single-loop sinusoidal A.C. circuit can conveniently be analysed by the complex method. To this end, we write all the sinusoidal quantities in complex notation:

$$\_ = E < \psi_e$$

$$\_ = I < \psi_i$$

$$V_r = V_r < \psi_{vr}$$

$$V_L = V_L < \psi_{vL}$$

$$V_c = V_c < \psi_{vc}$$

The positive directions of the currents emfs and voltages are shown in fig. 3.23 by arrowheads. We choose to traverse the loop in the clockwise direction and write an equation by Kirchhoff's voltage law.

$$V_L + V_r + V_c = j\omega L\_ + r\_ - j/\omega c\_ = \_ \qquad (3)$$

which fakes into account Ohm's law for the resistive element, the inductive

element and the capacitive element.
From equation (3), we find the complex current in the circuit

$$\dot{I} = \frac{\dot{E}}{r + j(\omega L - 1/\omega c)}$$ 383

$$\dot{I} = \frac{\dot{V}}{r + j(\omega L - 1/\omega c)}$$ 384

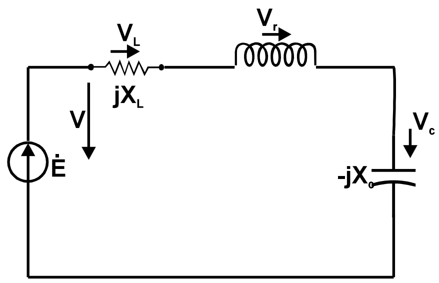


Fig. 3.23

where $\dot{V}$ 385 = Vexp(jψ_v) = _ = Eexp(jψ_e) is the voltage between the terminals of the single-loop circuit of fig. 3.23.

The quantity in the denominator of the above equation is called the complex impedance of a single-loop circuit

$$Z = r + j(\omega L - 1/\omega c) = r + j(X_L - X_c)$$

and its reciprocal

Y = 1/Z is called the complex admittance of a single-loop circuit. Each value of Z, that is, each complex number corresponds to a point in a complex plane. The point is uniquely located by a phasor in a complex plane (fig. 3.24). This phasor is a geometric interpretation at the complex impedance and is likewise symbolized by Z.

The phasor diagram in fig. 3.24 also shows the terms of the complex impedance as phasors or two cases: $X_L > X_c$ (fig. 3.24b). In the former case the complex impedance is inductive in its effect; in the latter case it is capacitive. On the basis of the geometric interpretation of complex

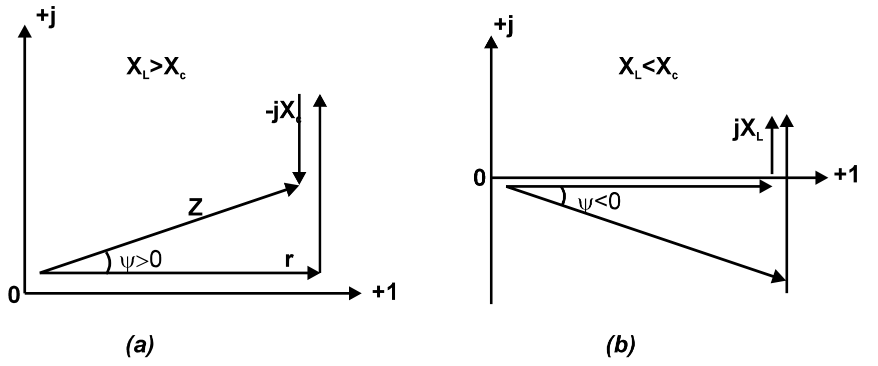


Fig. 3.24 Phasor diagram for $X_L>X_c$ (a)$X_L<X_c$ (b)

impedance we can readily go over from its algebraic representation in Eq. (14) to its trigonometric and exponential forms;

$$\underline{Z} = Z\cos\psi + jZ\sin\psi \quad 386$$

$$\underline{Z} = Z\exp(j\psi) = \psi \quad 387$$

= Z < ψ

where $Z = |Z| = \sqrt{r^2 + (X_L - X_c)^2}$ 388

is the magnitude of complex impedance or simply impedance, and

389

is the argument of the complex impedance. Depending on the sign of the difference (X_L - X_c), the argument of complex impedance can be either positive (lagging or inductive) or negative (leading or capacitive). In any case, however $|\varphi| = \pi/2$ 390

On substituting we obtain an expression for the current in the circuit on the basis of Ohm's law for a single-loop circuit.

$$\dot{I} = \dot{E}/\underline{Z} = (E/Z)\exp\dot{J}(\psi_e - \varphi) \quad 391$$

or $$\dot{I} = I\exp(J\psi_i)V/\underline{Z} = (V/Z)\exp\dot{J}(\psi_e - \varphi) \quad 392 \qquad (15)$$

that is

$$I = V/Z \text{ and } \psi_i = \psi_v - \psi \qquad (16)$$

If we know the values of the resistive, inductive and capacitive element and the voltage between the terminals of a single-loop circuit, then, by invoking Ohm's law for such a circuit as defined in Eq. (15), we can uniquely determine the complex current in the circuit. If we know the complex current in such a circuit, the complex voltages across the resistive, inductive and capacitive elements can be calculated.

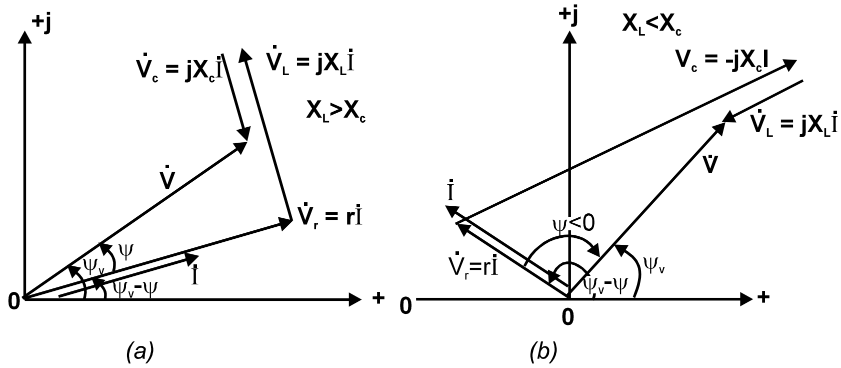


Fig. 3.25 (a), (b) Phasor diagram for current and voltage.

Fig. 3.25 shows phasor diagrams for the current and voltages in the single-loop circuit of fig. 3.23 for two cases: $X_L > X_c$ (fig. 3.25a) and $X_L < X_c$ (fig 3.25b) for the same specified voltage

$$V = V < \psi_v$$

When the complex impedance of a circuit is inductive in its effect, the current _ lags behind the voltage V in phase because _ > 0 (fig. 3.24a) and, as follows from Eq. (16), $\psi_i < \psi_v$. When the complex impedance of a circuit is capacitive in its effect, the current in the circuit leads the voltage in phase, because _ < 0 (fig. 3.24b) and as follow from Eq. (16), $\psi_i > \psi_v$. On the phasor diagram the positive values of the phase angle _ are counter clockwise from the complex current phasor I; its negative values are counter clockwise.

When a single-loop circuit contains several resistive, inductive and capacitive elements connected in series, the complex impedance is given by

$$\underline{Z} = \sum R + j(\sum X_L - \sum X_c) = r + jX \quad 393$$

where $r = \Sigma R$ is the resistance and $x = \Sigma X_L - \Sigma X_c$ is the reactance of this single-loop circuit. In the resistance, electric energy is irreversibly converted to other forms of energy; in the reactance such a conversion does not happen.

The concepts of resistance and reactance we have just introduced for single-loop circuit are equally applicable to the more elaborate circuits. In the general case, any passive, two-terminal sinusoidal a.c circuit has both a resistance and a reactance.

The voltage across the equivalent circuit element corresponding to the resistance or reactance of a circuit is called the voltage drop.

EXAMPLE 1: *Determine the currents in the branches in the circuit of Fig. Q1, if* $e = 220\sqrt{2}\sin(\omega t + 45°)V$; $r_1 = r_2 = 200\Omega$; $X^L = X_{C2} = 200\Omega$ *and* $X_{C1} = 100\Omega$.

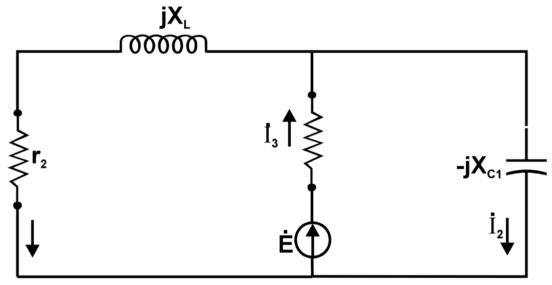


Fig. Q1

SOLUTION: First we shall transform the instantaneous value of voltage to the complex polar form and the root mean square value:

$$\dot{e} = 220\sqrt{2}\sin(\omega t + 45°) = 220e^{j45°}$$

394

$$= 220\angle 45°$$

$Z_1 = r_2 + jX_L$; $Z_2 = -jX_{C1}$; $Z_3 = r_1$

The current generated by the source is given thus:

$$\dot{I}_3 = \frac{\dot{E}}{r_1 + \dfrac{(R_2 + jX_L)(-jX_{C1})}{r_2 + jX_L - jX_{C1}}}$$

395

$$= \frac{220\angle 45°}{200 + \dfrac{(200 + j200)(-j100)}{200 + j200 - j100}}$$

$$= \frac{220\angle 45°(2 + j)}{600} = 0.82\angle 71.6° \quad 396$$

$$\dot{I}_3 = 0.82\angle 71.6° \quad 397$$

We use current divider rule to evaluate I_1 and I_2.

$$\dot{I}_1 = \frac{\dot{I}_3 \bullet (-jX_{C1})}{r_1 + jX_L - jX_{C1}}$$

398

$$= \frac{0.82\angle 71.6°(-j100)}{200 + j200 - j100}$$

$$= \frac{0.82\angle(71.6° - 90° - 26.6°)\, x\, \sqrt{5}}{5}$$

399

$$\therefore \quad \dot{I}_1 = 0.366\angle -45°$$

$$\dot{I}_2 = \frac{\dot{I}_3 \bullet (r_1 + jX_L)}{r_1 + jX_L - jX_C} = \frac{0.82\angle 71.6°(200 + j200)}{200 + j100} \qquad 400$$

$$\therefore \quad \dot{I}_1 = 1.04\angle 90° \qquad 401$$

We can now return to instantaneous values of currents in the branches

$$i_3 = 0.82\sqrt{2}\sin(\omega t + 71.6°) \qquad 402$$

$$i_2 = 1.04\sqrt{2}\sin(\omega t + 90°) \qquad 403$$

$$i_1 = 0.366\sqrt{2}\sin(\omega t - 45°) \qquad 404$$

EXAMPLE 2: *Determine the current in the branches in the circuit of Fig. Q2, if e = 220 √2sin(ωt + 45°), $r_1 = r_2 = 200\Omega$; $X_L = X_{C2} = 200\Omega$; and $X_{C1} = 100\Omega$.*

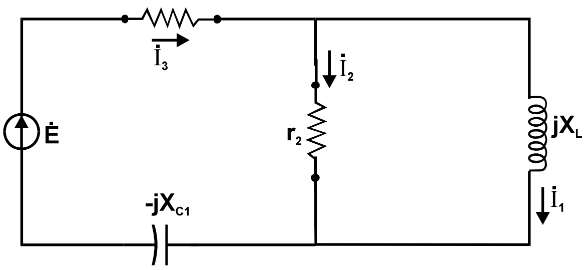


Fig. Q2

SOLUTION: Using the same approach as the one abovre, we obtain the root mean square value of the voltage.

$$\dot{E} = 220\angle 46° \,;\, Z_3 = r_1 - jX_{C1}\,;\, Z_2 = r_2\,;\, Z_1 = jX_L \qquad 405$$

$$\dot{I}_3 = \frac{\dot{E}}{r_1 - jX_{C1} + \dfrac{r_2(jX_L)}{r_2 + jX_L}} = \frac{220\angle 45°}{300} \qquad 406$$

$$\therefore \quad \dot{I}_3 = 0.733\angle 45° \qquad 407$$

$$\dot{I}_1 = \dot{i}_3 \bullet \frac{r_2}{r_2 + jX_L} = 0.73\angle 45°\; x \left(\frac{200}{200 + j200}\right) \qquad 408$$

$$\therefore \quad \dot{I}_1 = 0.52\, A \qquad 409$$

$$\dot{I}_2 = \dot{I}_3 \bullet \frac{jX_L}{r_2 + jX_L} = \frac{0.733\angle 45°(j200)}{200 + j200} = 0.52\angle 90° \quad 410$$

We now return to instantaneous values of currents in the branches.

$$i_1 = 0.52\sqrt{2}\sin\omega t \quad 411$$

$$i_2 = 0.52\sqrt{2}\sin(\omega t + 90°) \quad 412$$

$$i_3 = 0.733\sqrt{2}\sin(\omega t + 45°) \quad 413$$

EXAMPLE 3: *Determine the voltage $V_{i2}(t)$ across the capacitor C_2 in the circuit of Fig. Q3, if the parameters of the circuit are as follows: L_1 = 5mH; r_1 = 150Ω; C_1 = 0.667μF; L_2 = 19mH; r_2 = 200Ω; C_2 = 1μF; the voltage at the input V = 10 √2sin10^4t V.*

SOLUTION: For this circuit, we determine the impedance (reactance) of the various branches.

$$X_{L1} = j\omega L_1 = j10^4 \, x\, 5 \, x 10^{-3}\,\Omega = j50\,\Omega \quad 414$$

$$jX_{L2} = j\omega L_2 = j10^4 \, x\, 10 \, x 10^{-3} = j100\Omega \quad 415$$

$$jX_{C1} = -j\frac{1}{\omega C_1} = -j\frac{1}{10^4 \, x\, 0.667 \, x 10^{-6}} = -j150\Omega \quad 416$$

$$X_{C2} = -j\frac{1}{10^4 \, x\, 0^{-6}} = j100\,\Omega \quad 417$$

$$Z_1 = r_1 + j\omega L_1 = 150 + j50 \quad 418$$

$$Z_2 = \frac{jX_{L2} \bullet (r_2 - jX_{C2})}{r_2 + jX_{L2} - jX_{C2}} = \frac{j100(200 - j100)}{200 - j100 + j100}$$

419

$$= 50 + j100$$

$$Z_3 = -jX_{C1} = -j150\,\Omega \quad 420$$

$$Z = Z_1 + Z_2 + Z_3$$

421

$$= 150 + j50 + 50 + j100 + j150\, 200\Omega$$

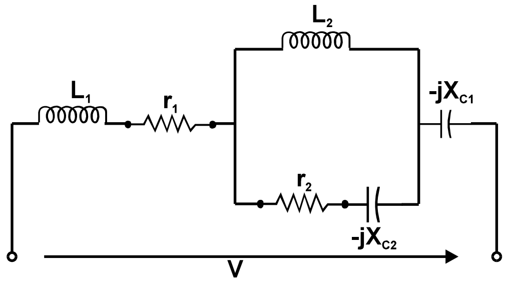


Fig. Q3

The current generated by the supply is given thus:

$$\dot{I} = \frac{10}{200} = 0.05\ A \qquad 422$$

$$\dot{I}_C = \frac{\dot{I} \bullet jX_{L2}}{r_2 - jX_{C2} + jX_{L2}} \qquad 423$$

$$= \frac{0.05 \bullet j100}{200 - j100 + j100} = \frac{0.05\ x\ j100}{200}$$

$$\dot{V}_{C2} = \frac{0.05\ x\ j100(-j100)}{200} = 2.5\ A \qquad 424$$

Returning to the instantaneous value of current and voltage,

$$V_{C2} = 2.5\sqrt{2}\sin(10^4 t)\ A \qquad 425$$

3.12 THE RESISTANCE, REACTANCE AND IMPEDANCE OF A PASSIVE TWO-TERMINAL NETWORK.

In fig. 3.26, a passive circuit external to an energy source is shown as a passive two-terminal network (or one-port). The parameter of such a one-port is its input complex impedance, that is, the complex impedance seen looking into terminals a and b;

$$\underline{Z} = \dot{E}/\dot{I} = \dot{V}/\dot{I} = Z < \varphi = Z\cos\varphi + jZ\sin\varphi = r + jX \qquad 426$$

where V = V < _v is the complex voltage across the one-part, _ = I < ψ_i is the complex current in the one-part, and $\psi = \psi_v - \psi_i$ is the argument of the complex impedance such that $|\varphi| \_ \pi/2(90^\circ)$ 427

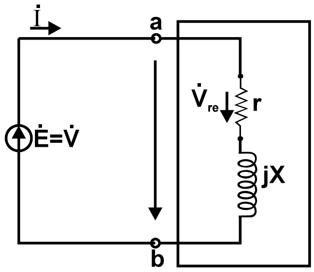


Fig. 3.26 A passive one-part network

From the above expression it follows that any passive one-port may be represented by an equivalent circuit consisting of a series combination of a resistance r and a reactance X. The impedance of a passive one-port is given by the equation:-

$$Z = \sqrt{r^2 + X^2} \quad 428$$

_ = arctan(x/r)

Depending on the sign of the reactance x, the complex impedance of a passive one-port may be inductive (X > 0) or capacitive (x < 0). Fig. 3.27 shows phasor diagrams plotted for the passive one-port of fig. 3.26, where Vres = r_ is the resistive component and V_{reac} = jX_ is the reactive component of the voltage V between the terminals of the one-port.

On a complex plane, the complex voltage phasors Vres, and V form a voltage triangle:

$$V = V_{res} + V_{reac}$$

The magnitude of the resistive voltage component phasor is Vres = V cos _ and this phasor runs in phase with current phasor _. The magnitude of the reactive voltage component phasor is $V_{reac} = V|\sin \_|$, and this phasor is shifted in phase relative to _ by $|\pi/2|$. Thus, the inductive reactive voltage leads the current _ in phase by π/2 (fig. 3.27a) and the capacitive reactive voltage lads behind the current _ in phase by π/2 (fig. 3.27b). From the voltage triangle it follows that

$$V = \sqrt{V_{res}^2 + V_{reac}^2} \quad 429$$

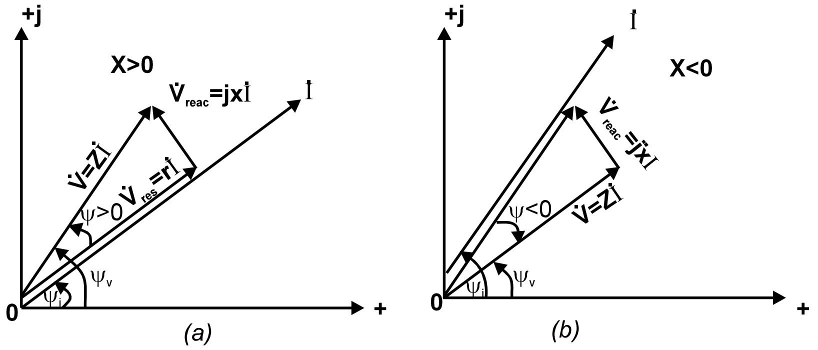


Fig. 3.27 Phasor diagram of current and voltages in a passive one-port (a) X>0, (b) X<0.

PARALLEL CIRCUITS: Fig. 3.28 shows an electric circuit which is a parallel combination of a resistive element, an inductive element and a capacitive element. Let the conductance g = 1/r of the resistive element, the complex inductive susceptive $-jb_L = 1/j\omega L$ and the capacitive susceptance $jb_c = j\omega c$. We assume that the voltage across each of the elements is the same and equal to

$V = \_ = V < \psi_v$

By Kirchhoff's current law, the complex value of the circuit current (equal to that of the emf source) is given by

$\_ = \_{r} + \_{L} + \_{c} = V(g - jb_L + jb_c)$ (1)

where, according to Ohm's law, $\_ = -jb_L V$, and $\_{c} = jb_c V$ are the complex currents in the resistive, inductive and capacitive elements.

The sum of the complex admittances of all parallel branches in Eq(1) is equal to the complex admittance of the circuit (in algebraic form);

$$\underline{Y} = 1/r - 1/j\omega L + j\omega c$$

430 (2)

$$= g - jb_L + jb_c = g - j(b_L - b_c)$$

The reciprocal of complex admittance

$1/Y = Z = Z\exp(j\varphi)$ 431 is the complex impedance. Therefore, complex admittance in exponential form is given by

$$\underline{Y} = 1/Z = 1/Z\exp(j\varphi) = Y\exp(-j\varphi)$$ 432 (3a)

and in trigonometric form by

$$\underline{Y} = y\cos\varphi - jy\sin\varphi \; 433 \qquad (3b)$$

where y = $|\underline{Y}|$ = $\sqrt{g^2 + (b_L - b_c)^2}$ 434 is the magnitude of the complex admittance or simply admittance, of a circuit, and _ = arctan [$(b_L - b_c)/g$] is its argument.

In fig. 3.29, the two components of complex admittance are shown as phasors on a

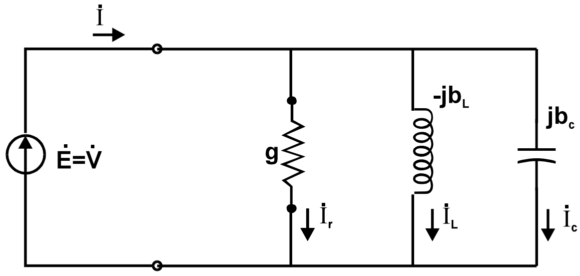


Fig. 3.28 A parallel RLC circuit

complex plane for two cases: $b_L > b_c$ (fig. 3.29a) and $b_L < b_c$ (fig. 3.29b). In the former case the complex admittance of the circuit is inductive; in the latter it is capacitive.

On substituting the complex admittance in exponential form by Eq. (3a) into equation (1) we obtain the complex current in the circuit as

_ = I exp $(j\psi_i)$ = $\underline{Y}$_ = $\underline{Y}$V = yVexpj$(\psi_v$ - _)

As follows from Eq.(4), the rms currents in the common part of the circuit is

$$I = yV = \sqrt{g^2 + (b_L - b_c)^2}\, V \; 435$$

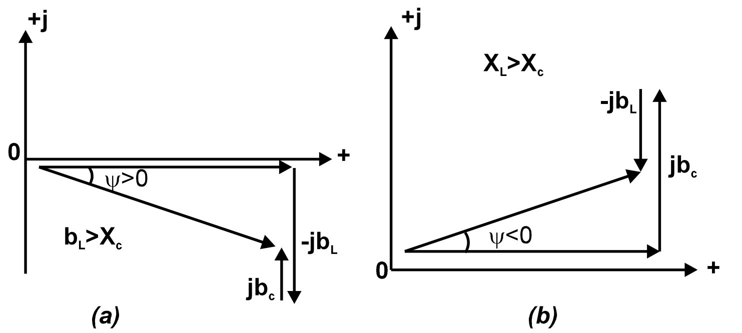


Fig. 3.29 Phasor diagram of admittance (a) $b_L > b_c$ (b) $b_L < b_c$.

If the complex admittance of the circuit is inductive, the common current lags behind the voltage because _ > 0, i.e, $\psi_v > \psi_i$. It is to be noted that as before, the positive values of the phase difference _ are counted counter clockwise from the complex current phasor _.

The complex power in the circuit is given by

$$\underline{S} = \dot{V}\dot{I}^* = \dot{V}(\dot{I}_r^* + \dot{I}_L^* + \dot{I}_c^*) = gV^2 + jb_{LV}^2 - jb_{cV}^2 \quad 436$$
$$= \underline{Y}^* V^2 = P + j(Q_L - Q_c) = P + jQ$$

If a circuit contains several resistive, inductive and capacitive elements connected in parallel, its complex admittance is given by

$$\underline{Y} = y\exp(-j\varphi) = \sum G - j\sum b_L + j\sum b_c \quad 437$$
$$= \sum G - j(\sum b_L - \sum b_c) = g - jb$$

where g = ΣG is the conductance of the circuit, B = $\Sigma b_L - \Sigma b_c$ is its susceptance.

THE POWER BALANCE AROUND A SINUSOIDAL A.C. CIRCUIT.

In a sinusoidal a.c circuit, the algebraic sum of the instantaneous powers of all the energy sources is equal to the algebraic sum of the instantaneous powers in all the loads at any instant. The same applies to the powers averaged over a period.

To begin with, we will consider energy sinks for loads, for which the equivalent circuits contain only resistive, inductive and capacitive elements. The power and energy relations in resistive, inductive and capacitive elements are different because the physical processes occurring in them are likewise different. In resistive elements input energy is irreversibly converted to other forms. The mean rate of the irreversible energy conversion in a resistive element is defined as active (true or real) power P_r. In inductive and capacitive elements, energy is periodically stored in the associated magnetic and electric fields and then returned to the external circuits. In these elements no irreversible energy conversion takes place, so active power P is zero.

The power balance around a sinusoidal a.c circuit containing an arbitrary number of energy (current and/or voltage) sources and of energy sinks or loads, such as resistive, inductive and capacitive elements, implies that, firstly, the algebraic sum of the active powers of all the energy sources is equal to the arithmetic sum of the powers in all the resistances.

$$\sum V_s I_s \mathrm{Cos}(\psi_v - \psi_i) = \sum r I_r^2 \quad 438 \qquad (6)$$

or $\Sigma P_s = \Sigma P_r$

Secondly, it implies that the algebraic sum of the reactive powers of all the energy is equal to the difference between the arithmetic sum of the reactive powers of all the inductive elements and the arithmetic sum of the reactive powers of all the capacitive elements.

$$\sum V_s I_s \mathrm{Sin}(\psi_v - \psi_i) = \sum X_L I_L^2 - \sum X_c I_c^2 \quad 439$$

or $\Sigma Q_s = \Sigma Q_L - \Sigma Q_c$

In the algebraic sum of active or reactive powers the terms representing an emf source (fig. 3.31a) takes a "+" sign if the positive direction of the current _ is the same as the direction of action of the emf E = V_{ab}, i.e., in the external circuit the current is flowing from terminal *a* which is at a higher potential than terminal *b*.

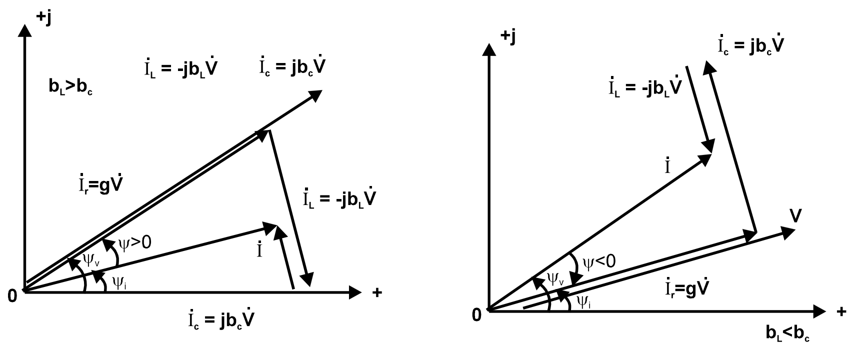


Fig. 3.30 Phasor diagrams for voltages and currents (a) $b_L > b_c$, (b) $b_L < b_c$.

(The positive direction of V_{ab} = $\_a$ - $\_b$ is from terminal *a* to terminal *b*) - otherwise (fig. 3.31b) the term associated with the emf source takes a "-" sign (now the sinusoidal emf generator is operating as a motor). Similarly the term is representing a current source j = _ (fig. 3.31c) takes a "+" sign; otherwise (fig. 3.31d) it takes a "-" sign.

The power balance around a sinusoidal a.c circuit may be stated in complex form. The algebraic sum of the complex powers of all energy sources is equal to the algebraic sum of the complex powers in all the loads.

$$\sum 440 \qquad (7)$$

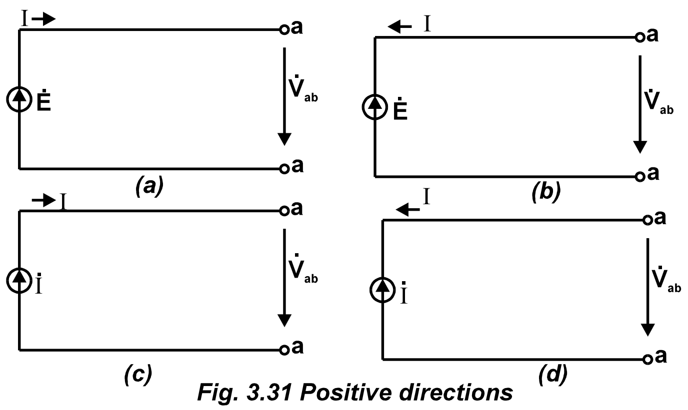


Fig. 3.31 Positive directions

We will write the power balance for the circuit for fig. 3.32.

$$\sum \underline{S}_s = \dot{V}_1 \dot{I}_1^* = \dot{V}_2 \dot{I}_2^* + \dot{V}_{ab} \dot{J} = \dot{E}_1 \dot{I}_1^* - \dot{E}_2 \dot{I}_2^* - V_L 3 \dot{J}^* = \sum P_s + j \sum Q_s \quad 44$$
1

$$\sum \underline{S}_{load} = \dot{V}_1 \dot{I}_1^* + \dot{V}_i 2 \dot{I}_2^* + \dot{V}_L 3 \dot{I}_3^* = r_1 I_1^2 - jX_c I_2^2 + jX_L I_3^2 = P_r + j(Q_L - Q_c)$$
442

$\Sigma P_s = P_r$

$\Sigma Q_s = Q_L - Q_c$

POWER FACTOR IMPROVEMENT:

Many industrial loads (such as phase shifters, electric motors and the like) have strong magnetic fields around them. As a result the current in such a load has a considerable reactive (inductive) component which implies a large positive (lagging) phase difference _ between voltage and current. This implies its power factor cosψ and, as a corollary, the power efficiency of the equipment or plant as a whole. A lagging power factor leads to an incomplete civilization of generators, power transmission lines and other facilities which are wastefully loaded by the reactive (inductive) current component. The reactive (inductive) current component is also responsible for increased power losses in all current carrying parts (motor windings, transformers, generators, power-line conductors, and so on.

One way to improve the power factor and to relieve the plant of the reactive (inductive) current is to connect a load with a lagging power factor in parallel with a bank of capacitors. Then the reactive (capacitive) current of the capacitor will balance out the reactive (inductive) current on the load.
To illustrate this method of power factor improvement, we will represent a load

with a lagging power factor as the equivalent circuit of a passive one-port (fig. 3.33a). The way in which the reactive (inductive) current in the load (the load current), $I_{reac\ load}$, is balanced out by a bank of capacitors is demonstrated in the phasor diagram of fig. 3.33b. As is seen, connection of the capacitors improves the power factor: Cos_ > Cos_$_{load}$. In most cases, an incomplete compensation of the phase difference will be sufficient. The small reactive (inductive) current, I_{reac}, at Cos_≥0.95 does not produce any additional power loss because $I = \sqrt{I_a^2 + I_{reac}^2} \approx I_a$ 443. Complete compensation of the inductive reactive current usually calls for the use of capacitors which is not always warranted economically.

It is usual to specify the value of the power factor that a plant should have after power factor correction. If the load current I_{load} and the power factor cos_ of the load are known and the desired value of cos_, that is the phase difference _, is specified, the required capacitance can be found by reference to the current phasor diagram such as shown in fig. 3.33b from which it follows that

$$I_a\tan\_{}_{load} - I_a\tan\_ = I_c = \omega cV$$

Hence,

$$C = (p/\omega V^2)(\tan\_{}_{load} - \tan\_)$$

where $P = I_aV$ is the active power in the load.

The improvement of the power factor by use of capacitors is usually referred to as power factor correction in contrast to the natural power factor improvement achieved through the complete utilization of the power developed by motors and through the use of synchronous motors in which the reactive current component is very small.

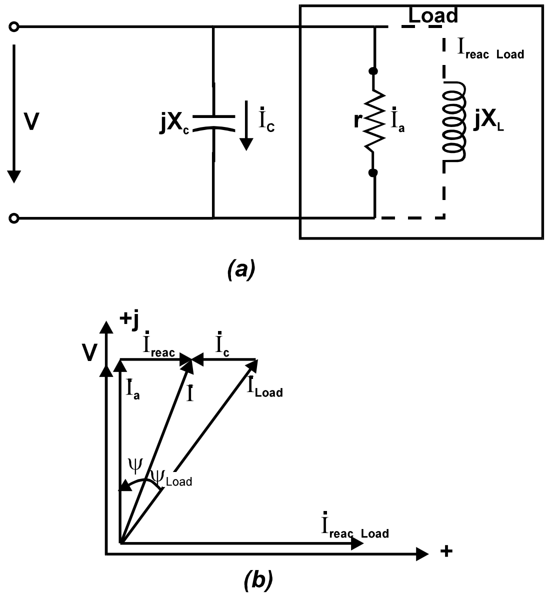


Fig. 3.33 power factor improvement (a) circuit diagram (b) phasor diagram.

CHAPTER FOUR

TRANSIENT PROCESS IN LINEAR NETWORKS

4.1 GENERAL QUESTIONS AND DEFINITIONS.

During the study of electrical networks it is assumed, that after switching the circuit to D.C. or an A.C. source, that the current and voltage in all the branches will attain instantaneous values immediately corresponding to its parameters and voltage of the set.

In reality, this happens only in circuits with active resistors, where there are no inductors and capacitors.

In an active resistor, the electrical energy is converted irreversibly into heat, which is dissipated into the surrounding medium. Inductors and capacitors are in themselves accumulators of electrical energy. In the inductor and capacitor electrical energy is transformed respectively into energy of the magnetic and electric fields. The stored energy is conserved, and it is not converted to heat, as in the case of the active resistor, as long as there is current in the inductor and there is voltage across the capacitor. In order words the energy is stored in the form of separated charges in the electric field of the capacitor and in the form of magnetic field of the inductor. The energy of the magnetic and electric field can only changed to electrical energy and it will return to the source or dissipated in the form of heat on the active resistor in the circuit.

The process of storing to a finite value or transfer to energy, cannot take place instantaneously, since in the contrary the power of the energy source should be equal to infinity

$$P = \omega/t = \omega/0 = \infty$$

and such energy sources do not exist. As long as energy cannot change instantaneously, hence, the defining quantities of energy in an inductor and a capacitor, which are current and voltage respectively cannot change instantaneously.

$$W_L = \frac{I^2 L}{2} \quad 444 \text{ energy in a magnetic field}$$

$$W_L = \frac{V_c^2 . C}{2} \quad 445 \text{ energy in an electric field}$$

The impossibility of instantaneous change of current in an inductor could also be understood from the comparison of processes taking place during the switching of inductors with the processes in mechanical system. It is known that the force acting on an object with a mass and acceleration is connected by Newton's law as

$$F = m\frac{dv}{dt} \text{ 446 }, \qquad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

From there it follows, that the constant acting force give rise to a motion of the object with an acceleration, equal to

$$\frac{dv}{dt} = F/m \text{ 447 } \ldots\ldots\ldots\ldots\ldots \quad (2)$$

The velocity of the object cannot change instantaneously, when dv/dt = ∞, since the force can have a finite value, and it is not infinitely large.

When an ideal coil with inductance L is connected to a source (as in fig. 4.1), under the influence of the applied voltage a current will flow in the circuit, and e.m.f of self-induction will be induced in the coil. From the expression, written according to Kirchhoff's voltage law

$$V = -e = L\frac{d\ i}{dt} \text{ 448 } \ldots\ldots\ldots\ldots\ldots \quad (3)$$

It is clear that the rate of growth of the current is equal to

di/dt = V/L ....(4)

which yields the expression;

$$i = \frac{V}{L}(t) + A \text{ 449}$$

When we compare and contrast (1) and (3), we can conclude, that the inductance by its action is analogical to mass in mechanical system. From expression (4) it follows that, for a defined finite value V the rate of change of current in the inductor has a finite value.

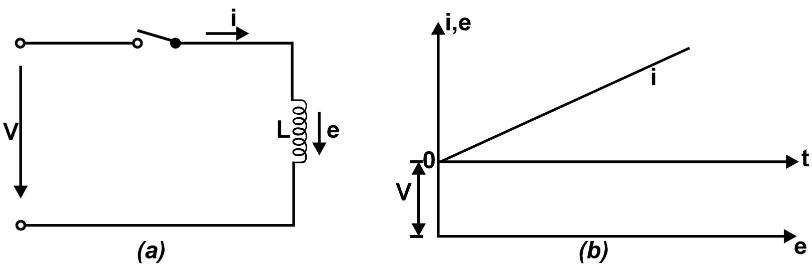


Fig. 4.1 A graph of i(t), e(t) for ideal coil with inductance L.

On the basis of the above discussion we can now formulate two very important laws, without which we cannot undertake the analysis and computation of transient processes.

1. **First law of commutation**: The current in a circuit with inductance after commutation (connexion, disconnexion or change of parameter of the circuit) when t = 0 has the same value, as before commutation.

2. **Second law of commutation:** The voltage across the capacitor after commutation when t = 0 has the same value, as before commutation.

In automatic control system it often occurs that the steady state regime is usually disturbed and these systems operate practically in transient conditions. In majority of automatic control systems we make use of electrical instruments, therefore it is necessary to consider transient processes if only for simple electric circuits.

The free, or transient, response is due to the difference between the energy stored by the capacitive and inductive elements of the circuit at the instant immediately preceding the change in the circuit steady state and the energy stored by the same elements when the circuit has reached a new steady state. The energy stored in these elements cannot change all at once; instead the change is gradual. Hence, the existence of transients in the circuit.

The transient state of a circuit is described by differential equations, generally non homogeneous (if the circuit contains voltage and current source) or homogeneous (if the circuit is source-free). In the case of a linear circuit, the transient state is described by linear differential equations, and in the case of a non linear circuit, by non-linear differential equations.

Let us consider the transient analysis of constant parameter linear circuits which are described by linear differential equations with constants coefficients. Several methods have been developed to solve such equations and to analyse the transient response of the circuits involved. We shall look at three methods of analysis namely, classical method which is physically instructive and is far simpler to apply to simple circuits; transform method; and the state variable approach to the analysis of transient response in circuits.

The transient analysis of a circuit by the classical method involves several steps.

(1) To begin with, a set of differential equations is written for the circuit in question. By elimination of variable, they are reduced to a single equation for the sought current *i* or voltage *V*. For simple circuits this procedure yields a first or second-order differential equation in which the unknown is either the current in the inductive element or the voltage across a capacitive element.

(2) The next step is to write the general solution of the differential equation thus obtained. It is the sum of a particular solution (or integral) of a non-

homogeneous differential equation and the general solution (or the complementary function) of the corresponding homogeneous equation.

As applied to electric circuits, the particular solution is the forced (or steady state) response of the circuit (if it exist), that is, direct currents and voltages if the circuit contains direct - current and direct - voltage sources or sinusoidal voltages and currents if the circuit contains sinusoidal voltage and current sources. The respective quantities are called steady-state currents and steady-state voltages and their symbols are i_{ss} and V_{ss}. The complementary function describes the source - free or transient response of the circuit, and the corresponding quantities are termed transient, their symbols being i_t and V_t. The expressions for the transient quantities must contain the constants of integration, of which there must be as many as is the order of the homogeneous equation.

(3) As the last step, the constants of integration for the general solution $i = i_{ss} + l_t$, are found. The integration constants are determined from the initial conditions, that is, the conditions existing in the circuit immediately after it is disturbed. We will assume that the switch performing the change in the state of the circuit is ideal which implies that the switch operates instantaneously at time *t*. In the circumstances, the current in an inductive element and the voltage across a capacitive element immediately after switching (t+) are the same as they are immediately before switching (t-). These conditions are derived from what are known as laws of commutation.

4.2 TRANSIENT IN A D.C. CIRCUIT WITH ONLY ONE INDUCTIVE ELEMENT.

RESPONSE OF A SERIES RL CIRCUIT TO A D.C. VOLTAGE EXCITATION:

As soon as the switch *S* (fig. 4.2) is closed at time *t* = 0, a current *i* begins to flow around the circuit, and voltages appear across the resistive element, $V_r = L\, di/dt$.

Using Kirchhoff's voltage law, we write a differential equation for the circuit

$$V_L + V_r = L\, di/dt + ri = E \quad \text{.......}(1)$$

which is recognised as a first-order non-homogeneous differential equation. The complete solution to Eq(1) is the sum of two solutions, one being the steady state (or forced) solution and the other being the source-free, (or transient) solution

$$i = i_{ss} + i_t \quad \text{.......}(2)$$

The solution holds for any instant after the closure of the switch, that is, beginning at time $t = 0+$.

The steady-state term in Eq. (2) is the direct current existing in the circuit after all transients have died out (which, theoretically goes on for an indefinitely long time)

$$i_{ss} = E/r \qquad (3)$$

By direct substitution it is an easy matter to verify that the above particular solution satisfies the non-homogeneous differential equation (1).

The second term on the right-hand side of Eq. (2) is the general solution of the homogeneous differential equation of the circuit.

$$L\frac{di_t}{dt} + ri_t = 0 \ . \qquad (4)$$

and is called the free solution because it is independent of the voltage and current sources in the circuit.

$$i_t = A \exp (kt) \qquad (5)$$

where $k = -r/L$ is the root of the characteristic equation

$$Lk + r = 0 \qquad (6)$$

Thus, subject to Eqs. (3) and (5), the complete solution (2) of the non-homogeneous differential equation (1) has the form

$$i = E/r + A \exp (-rt/L) \qquad (7)$$

It remains to find the integration constant *A*. To this end, we refer to the switching rule for an inductive element, at the time when the switch is closed, t = 0. Since the current in an inductive element cannot change instantaneously, and prior to the switching, that is at t = 0-. it was zero, it follows that

$$i(0-) = 0 = i(0+) = E/r + A$$

Hence,

$$A = -E/r \qquad (8)$$

On substituting for A in Eq.(8), we obtain an expression describing the rise of current in the circuit (fig. 4.2b)

$$i = \frac{E}{r}\left[1 - \exp\left(-\frac{t}{\tau}\right)\right] \qquad (9)$$

where T = L/r has the dimension of time

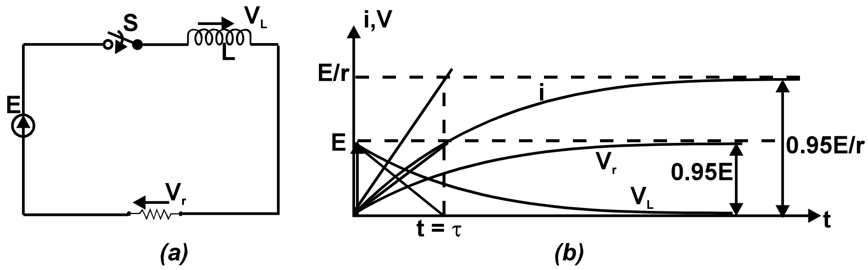


Fig. 4.2 (a) A series RL circuit. (b) response of a series i_{ss} circuit to a d.c. voltage excitation.

[H/Ω = S] and is called the time constant of the circuit in fig. 4.2a. The time constant defines the rate of current rise and is equal to the time during which the current *i* reaches its steady-state value, i_{ss} = E/r, if its rate of change remains constant and equal to the initial rate of change $di/dt\,|_{t = o+}$ = E/L.

It may be assumed that all transients practically die out after a time equal to 4.6T, reckoning from the instant when the switch is closed. At the end of that time interval the current will be *i*(4.6τ) = 0.95E/r. In other words, if closure of the switch takes place at t = 0, this interval is t = 4.6τ.

Now that we have found the time dependence of current, it is an easy matter to determine the time dependence of the voltage across the resistive and inductive elements.

V_r = ri = E(1 - exp(-t/τ)

V_L = L di/dt = E exp(-t/T)

For 0 ≤ t < T, the rate of current change may be taken approximately constant and equal to $di/dt\,|_{t=0-}$ = E/L. Therefore, during the time interval defined, the voltage across the resistive element is approximately equal to

$$V_r \approx \frac{rEt}{L} = \frac{r}{L}\int_0^t E\,dt \; 452$$

that is, directly proportional to the integral of the source emf E. This is what is usually called an integrating network.

When the excitation is a varying emf *e*, it may so happen that sometimes during the transient process $V_r >> V_L$. For such time intervals the current in the circuits is _e/r, and the voltage across the inductive element

V_L = L di/dt _ (L/r) de/dt

is approximately proportional to the rate of change of the source emf *e*. Accordingly, this is what is usually called a differentiating network.

THE SOURCE - FREE RESPONSE OF AN RL CIRCUIT UPON CLOSURE OF THE SWITCH:

This situation arises in the windings of electric machines and apparatus. Since in addition to inductance, the inductor of the circuit also has a resistance *r*, the equivalent circuit takes the form in fig. 4.3a.

The differential equation describing the transient process in the circuit upon switch closure has the form

$V_L + V_r$ = L di/dt + ri = 0...(10)

Since Eq. (10) is a homogeneous differential equation, its complete solution contains only a free component

i = i_t = A exp(-t/T)..(11)

where T = L/r is the time constant of the circuit.

It remains to find the constant A. As before we refer to the switching rule for an inductive element. Since prior to switch closure, that is at t = 0-, the inductor carried a direct current equal to E/(r+R), it follows that

i(0-) = E/(r+R) = i(0+) = A

On substituting the above expression for A in Eq. (11), we obtain the current in the inductor

$$i = \frac{E}{(r+R)} \bullet \exp\left(-\frac{t}{\tau} \right) \qquad (12)$$

Upon switch closure the current in the inductor is maintained by the energy stored in its magnetic field.

Now we can determine the time dependence of the voltages across the resistive and inductive elements (fig. 4.3b).

V_r = r.E/(r + R) exp (-t/T)

$$V_L = L\frac{di}{dt} = -\frac{rE}{(r+R)}\exp\left(-\frac{t}{\tau}\right) \quad 454$$

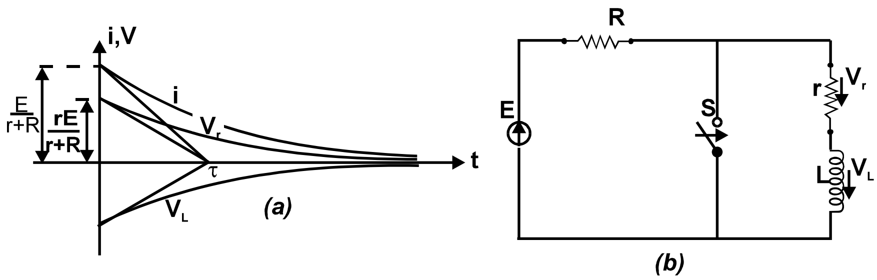


Fig. 4.3 The source - free respoonse of an RL circuit upon closure of the switch.

RESPONSE OF AN RL CIRCUIT FOLLOWING AN OPEN-CIRCUIT

When a simple series circuit containing an inductor is opened, an arc strikes between the parting contacts. This situation occurs, for example, at the trolleys of electric-traction vehicles. Arcing can be avoided by placing a resistor in parallel with the arc containing section of the circuit.

An applicable equivalent circuit is shown in fig. 4.4a. The inductor is replaced with a series combination of an inductance L and a resistor r, and the switch is replaced with a parallel combination of an ideal switch and a resistance R.

We may now apply Kirchhoff's second law and write the following differential equation for the network of fig. 4.4a after commutation:

$V_L + V_r + V_R$ = L di/dt + (r + R)i = E..(13)

This differential equation is the same as Eq. (1). In consequence, its complete solution is the same as that for Eq.(1), that is,

i = $i_{ss} + i_t$ = E/(r + R) + A exp(-t(r + R)/L) ....(14)

where i_{ss} = E/(r + R) is the steady-state component or the direct current flowing in the circuit after commutation (the switch is opened).

To determine the integration constant A we recall the laws of commutation the switching rule for an inductive element. Before the switch was opened at t = 0-, the inductor carried the direct current E/r. Therefore, by the switching rule,

$$i(0-) = E/r = i(0+) = E/(r + R) + A$$

Hence,

$$A = E/r - E/(r + R) = RE/r\,(r + R)$$

On substituting the above expression for A in Equation (14), we find the current in the inductor after the switch is opened (fig. 4.4b).

$$i = E/(r + R)\,[1 + (R/r)\exp(-t/T)] \quad \text{.........(15)}$$

where T = L/(r + R) is the time constant of the circuit.

Now that we know how the current varies in the circuit, Eq. (15), we can readily determine the time dependence of the voltages across the resistive and inductive elements (fig. 4.4b)

$$V_r = r\cdot i = [rE/(r + R)]\,[1 + (R/r)\exp(-t/T)]$$

$$V_R = Ri = [RE/(r + R)]\,[1 + (R/r)\exp(-t/T)]$$

$$V_L = L\,di/dt = -(R/r)\,E\exp(-t/T)$$

Just as the switch opens (t = 0+) the voltage across the resistive element R abruptly rises from zero,
$V_R(0-) = 0$ to $V_R(0+) = ER/r$. That is why for R>>r, a substantial voltage is produced between the contacts tips, and it may give rise to an arc discharge.

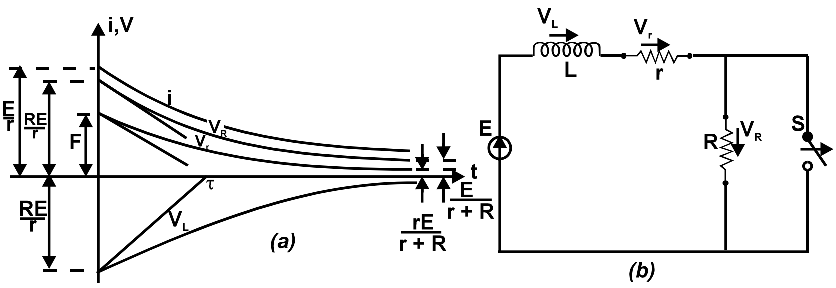


Fig. 4.4 Response of an RL following an open-circuit.

Example 1: *Calculate the currents in the second and third branches, when the*

switch in the first branch is suddenly opened.
The parameters of the circuit are as follows:
$E = 120V$; $J = 6A$; $r_1 = r_2 = r_3 = 10\Omega$; $L = 0.05H$

Solution: The current in an inductor before commutation (a d.c Current) we will obtain, using superposition method. There are two sources, a voltage and a current source.

$$i_3(0-) = I_{3'}(0-) + i_{3''}(0-) = 4A + 2A = 6A \quad 455$$

Kirchhoff's equations for the circuit after commutation is given thus:

$$-J - i_2 + i_3 = 0; \quad \ldots\ldots (1)$$

$$r_2 i_2 + L\frac{di_3}{dt} + r_3 i_3 = E \quad 456 \quad \ldots\ldots (2)$$

We will eliminate from equations (1) and (2) current i_2, we will obtain the differential equation for current i_3:

$$L\frac{di_3}{dt} + (r_2 + r_3)i_3 = E + r_2 J \quad 457 \quad \ldots\ldots (3)$$

The unknown currents are obtained in the form of steady state and transient components:

$$i_2 = i_{2ss} + i_{2t}; \quad i_3 = i_{3ss} + i_{3t}.$$

The steady state currents are obtained using superposition method.

$$i_{2ss} = E/(r_2 + r_3) - j.r_3/(r_2 + r_3) = 3A$$

$$i_{3ss} = E/(r_2 + r_3) + J\cdot r_2/(r_2 + r_3) = 9A$$

The characteristic equation is given as follows

$$LP + (r_2 + r_3) = 0$$

The roots $p_1 = -(r_2 + r_3)/L = -400\ S^{-1}$ and the transient component of currents:

$$i_{2t} = A\exp(P_1 t); \quad i_{3t} = B\exp(p_1 t).$$

Hence, the currents

$$i_2 = 3 + A\exp(P_1 t)\ A; \quad i_3 = 9 + B\exp(p_1 t) \ldots\ldots (4)$$

The equation for the determination of the constant of integration is obtained from (4) when $t = 0$

$i_2(0) = 3 + A$; $i_3(0) = 9 + B$

The independent initial condition - the current in the inductor $i_3(0)$ is obtained using the law of commutation:

$i_3(0) = i_3(0\text{-}) = 6A.$

The dependent initial condition $i_2(0)$, i.e the initial value is obtained from equation (1) at time $t = 0$, whence $i_2(0) = i_3(0) - j = 0$ and the constant of integration: A = $i_2(0) - 3 + -3A$; $B = i_3(0) - 9 = -3A$.
The currents are as follows:

$i_2 = 3 - 3\exp(-400t)$ A; $i_3 = 9 - 3\exp(-400t)$ A.

These are represented in fig. Q2B

4.4 TRANSIENTS IN A D.C. RC CIRCUIT:

THE FORCED RESPONSE OF AN RC NETWORK

As an example we will consider the network in fig. 4.5a initially the capacitor is uncharged and the switch S is open. At t = 0 the switch is closed, and a D.C. voltage source is inserted in the circuit. This causes the capacitor to charge.

Recalling that for a resistive element $V_r = r \cdot i$, and for a capacitive element $i = C\, dV_c/dt$, we can use Kirchhoff's voltage law and write the following differential equation for the circuit after switch closure at time t = 0;

$V_r + V_c = ri + v_c$

$r_c\, dV_c/dt + V_c = E$ ....(16)

The complete solution of the non homogeneous differential equation for the voltage across the capacitive element is the sum of the steady-state solution and the transient solution

$V_c = V_{css} + V_{c.t}$

The steady-state term is

$V_{css} = E$ ....(17)

because the capacitive element will cease charging when V_c has reached the value of the source voltage.

Since Eq.(16) is a first-order differential equation, its transient (or free) form is

$V_{c.t} = A \exp(kt)$

where k = -1/rc is the root of the characteristic equation

$rck + 1 = 0$

Thus, the complete solution is

$V_c = V_{css} + V_{c.t} = E + A \exp(-t/rc)$ ....(19)

In order to determine the constant in Eq.(19), we invoke the laws of commutation for a capacitive element. We have assumed that before the switch was closed, that is, at t = 0-, the capacitive element was uncharged. Therefore,

$V_c(0-) = 0 = V_c(0+) = E + A$

Hence, A = -E

On substituting this for A in eq.(19), we find the voltage across the capacitive element while it is charging.

$V_c = E[1 - \exp(-t/T)]$ ....(20)

where T = rc has the dimension of time [Ω x F = Ω x A x S x V^{-1} = S] and is called the time constant of the circuit. As with the network of fig. 4.2, it is a measure of the rate at which the voltage tends to its final value.

Now that we know the time dependence for the voltage across the capacitive element, Eq.(20), it is easy to establish the time dependence for the charging current and for the voltage across the resistive element.

$$i = c dv_c/dt = (E/r) \exp(-t/T)$$

$$V_r = r \cdot i = E \exp(-t/T)$$

It is to be noted that just as the switch is closed, or at t = 0+, the current in the circuit is

$$i(0+) = E/r$$

and the capacitive element is acting as a short circuit. Therefore at low values of *r* a substantial current surge may occur in the circuit.

For $0 \leq t < T$, the rate of change for the voltage across the capacitive element may approximately be taken as constant and equal to $dV_c/dt\big|_{t=0+} = E/(rc)$
and the voltage across the capacitive element is

$$V_c \_ \left(\frac{E}{rc}\right)t = \frac{1}{rc}\int_0^t E\,dt \quad 458$$

that is, proportional to the integral of the source voltage E.

If an RC network is excited by varying emf *e*, it may so happen that at some instants during the transient process when $V_r \ll V_c$, the voltage across the capacitive element will be approximately equal to the source emf, $V_c \_ e$, and the voltage across the resistive element will be proportional to the rate of change of the source voltage

$$V_r = r \cdot i = rc\, dV_c/dt = rc\, de/dt$$

Thus, like the series RL circuit examined earlier, a series RC network may, given certain conditions act as an integrator or as a differentiator.

In most cases, the capacitive element of an RC network may be assumed to be fully charged in a time equal to 4.6T. This time interval may be rather be too long (the value of T increases with the values of r and c), and this property is utilized in time relays which are devices used to open or closed a circuit (or circuits) at one or more predetermined times.

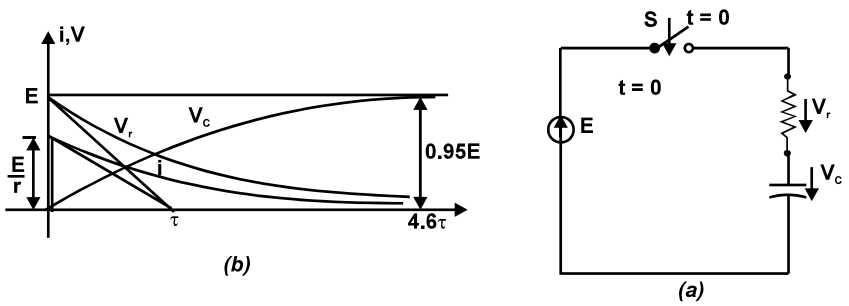


Fig. 4.5 Growth of charge in an RC circuit

Examples 2: *A capacitor C_2 of capacitance 2 microfarad in series with resistor r = 1 kilo ohms is connected to a charge capacitor c_1 of capacitance 4 micro Farad and the voltage 120V. (fig Q1). Calculate the voltages V_1 and V_2 across the capacitors and the current i in the circuit during the time of transient process.*

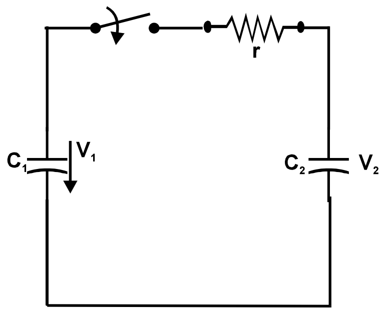


Fig. Q1

Solution: The voltage across the capacitors before commutation

$V_1(0-) = V_1 = 120V$; $V_2 - (0-) = 0$

The differential equation of the circuit after commutation is given thus:

$-V_1 + V_2 + ri = 0$, ...(1)

where

$$i = c_2 \frac{dv_2}{dt} = - c_1 \frac{dv_1}{dt} \quad 459 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

We eliminate from (1) and (2) all the variables except V_2, we will get

$$\left(1 + \frac{c_2}{c_1}\right)V_2 + rc_2 \frac{dv_2}{dt} = v_1 \quad 460 \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

The voltages and current we write in form of steady state and transient components:

$V_1 = V_{1ss} + V_{1t}$; $V_2 = V_{2ss} + V_{2t}$;

$i = i_t$.

where it is observed that in the steady state condition, the current does not exist.

The steady state voltages across the capacitor can be obtained from the law of conservation of charge. The sum of charges before commutation is equal to the sum of charges in any moment after commutation and in this case, in the steady state condition:

$$C_1V_1(0-) + (C_2V_2(0-) = C_1V_{1ss} + C_2V_{2ss}$$

Apart from this, it is clear that in the steady state condition $V_{1ss} = V_{2ss}$.

From the above two equations we find:

$$V_{1ss} = V_{2ss} = 80V.$$

For the differential equation (3) the characteristic equation is given thus:

$$\left(1+\frac{C_2}{C_1}\right)+rC_2\,P=0 \quad 461$$

it has one root $P_1 = -750\ S^{-1}$
Hence, the solution for the voltages

$$V_1 = 80 + A\exp(-p_1t);\ \ V_2 = 80 + B\exp(-p_1t)$$

$i = C\exp(-p_1t)$, and when $t = 0$

$$V_1(0) = 80 + A;\ \ V_2(0) = 80 + B;\ \ i(0) = C$$

where the initial values of voltages across the capacitors according to the law of commutation are given thus:

$$V_1(0) = V_1(0-) = 120V;\ \ V_2(0) = V_2(0-) = 0$$

and the initial value of current

$$i(0) = [V_1(0) - V_2(0)]/r = 0.12A.$$

Hence, A = 40V; B = -80V; C = 0.12A and

$V_1 = 80 + 40\exp(-750t)$ V; $V_2 = 80 - 80\exp(-750t)$ V_1; $i = 0.12\exp(-750t)$A.

4.5 THE DISCHARGE OF A CAPACITOR THROUGH A RESISTOR

The electric field of a charged capacitive element stores an amount of energy, owing to which the element can act as an energy source for some time.

As an illustration we will analyse the circuit of fig. 4.6a which contains a capacitive element charged to $V_c = E$. Just as the switch S is closed and the capacitive element is placed across the resistive element of resistance r, a current begins to flow around the circuit due to the change of the charge *q* on the capacitive element.

$$i = -dq/dt = -cdV_c/dt \quad \text{...(21)}$$

where the "-" sign indicates that the current *i* is the discharge current around the loop labelled by a dashed arrow in the figure; its direction is opposite to that of the voltage across the capacitor.

Noting Eq.(21), Kirchhoff's voltage law gives the following differential equation for the loop labelled in fig 4.6a;

$$VV_r - V_c = r{\cdot}i - V_c = rc\, dV_c/dt + V_c = 0 \quad \text{...(22)}$$

Since the discharge circuit of the capacitor contains no voltage source, Eq.(22) is a homogeneous differential equation, and its complete solution consists solely of a free (transient) term.

$$V_c = V_{c.t} = A \exp(-t/rc) \quad \text{...(23)}$$

In order to determine the constant of integration A, we invoke the switching rule for a capacitive element (laws of commutation). Since before the switch was closed, that is, at t = 0-, the capacitor was charged to the source voltage, it follows that

$$V_c(0-) = E = V_c(0+) = A$$

On substituting for A in Eq. (23), we derive the relation defining the manner in which the voltage across the capacitor varies on discharge (fig. 4.6b)

$$V_c = E \exp(-t/T)$$

where T = rc is the time constant of the circuit.

The discharge current is given by Eq.(21)

$$i = -cdV_c/dt = (E/r) \exp(-t/T)$$

The current rises stepwise from zero to i(0+) = E/r, then decreases exponentially (fig. 4.6b).

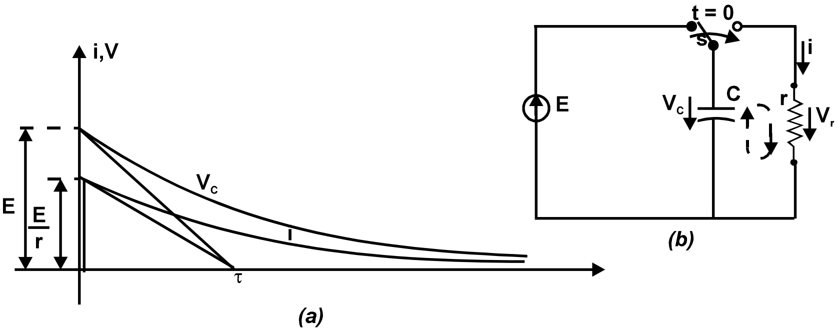


Fig. 4.6 Discharge of a capacitor through a resistor.

4.6 DISCHARGE OF A CAPACITIVE ELEMENT IN AN RLC CIRCUIT

The discharge of a capacitive element through a series combination of an inductive and a resistive element is an important process in pulse generators in which a capacitor acts as an energy source.

As an illustration we will consider the circuit of fig. 4.7. Suppose that initially, when the switch S is in position 1, the capacitive element C is charged by a D.C. source to a voltage equal to E. Then the switch is thrown into position 2, and the capacitor is connected to a series combination of an inductive element L and a resistive element r. Now the capacitive element begins to discharge, so its charge q and this voltage V_c diminish. In the process, the energy stored in the electric field of the capacitive element is converted to the energy stored in the magnetic field of the inductive element and is partly dissipated in the resistive element. This process may be a periodic (damped) or oscillatory, depending on the choice of the element values.

For the loop labelled with a dashed arrow in the figure, Kirchhoff's voltage loop establishes.

$$-V_c + V_r + V_L = V_c + r\cdot i + L\, di/dt = 0 \quad \text{..(24)}$$

The positive direction of the current in the capacitive element is opposite to that for the voltage across that element, because *i* is a discharge current. Therefore for the circuit of fig. 4.7a

$$i = -C\, dV_c/dt \quad \text{..(25)}$$

On substituting, we obtain a second-order homogeneous differential equation

$$LC\, d^2V_c/dt^2 + rc\, dV_c/dt + V_c = 0 \quad \text{..(26)}$$

for which the characteristic equation is

$$LCS^2 + rCS + 1 = 0 \quad\text{...(27)}$$

The complete solution contains only one free (or transient) term

$$V_c = V_{c.t} = A_1 \exp(k_1 t) + A_2 \exp(k_2 t) \quad\text{...(28)}$$

where

$$S_{1,2} = -\frac{r}{2L} \pm \sqrt{\frac{r^2}{4L^2} - \frac{1}{Lc}} \quad 462$$

are the characteristic equation (27). For $r^2/4L^2 > 1/Lc$, both roots of the characteristic equation are real, and the capacitive element discharges aperiodically (the overdamped case). For $r^2/4L^2 < 1/Lc$, the roots are complex conjugate, and the capacitor produces an oscillatory discharge (the underdamped case).

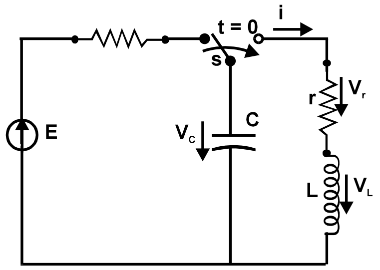


Fig. 4.7 Discharge of a capacitive Element in an RLC circuit.

Consider the case when the characteristic equation has complex conjugate roots

$$S_{1,2} = -\delta \pm j\omega\delta \quad\text{...(29)}$$

where $\delta = r/2L$ is the damping constant and $\omega_o = \sqrt{\frac{1}{Lc} - \delta^2}$ 463 is the natural angular frequency of the oscillatory response.

On substituting the complex values of the roots in Eq.(28), we obtain the following time dependencies for the voltage across the capacitive element and for the discharge current in the oscillatory (under damped) case.

$$V_c = \exp(-\delta t)\,[A_1 \exp(j\omega_o t) + A_2 \exp(-j\omega_o t)] \quad\text{....................................(30a)}$$

$$i = -cdV_c/dt = -C \exp(-\delta t)\,\{-\delta[A_1 \exp(j\omega_o t) + A_2 \exp(-j\omega_o t)] + j\omega_o[A_1 \exp(j\omega_o t) - A_2 \exp(-j\omega_o t)]\} \quad\text{....(30b)}$$

As in previous cases, the constants of integration A_1 and A_2 can be evaluated by reference to the switching rules for an inductive element, and a capacitive element. Prior to and at t = 0-, i.e., immediately before operation of the switch, the voltage across the capacitive element was equal to the source emf E, and no current was flowing in the inductive element.

Therefore,

$$V_c(0-) + E = V_c(0+) = A_1 + A_2$$

$$i(0-) = i(0+) = c[\delta(A_1 + A_2) - j\omega_o (A_1 - A_2)]$$

Hence

$$i(0-) = 0 = i(0+) \quad A_1 = E(\delta + j\omega_o)/2j\omega_o$$

Recalling that by Euler's formula $\exp(\pm j\omega_o)t = \cos\omega_o t \pm j \sin \omega_o t$ and substituting the above expressions in Eq.(30a), we obtain the time dependence of the voltage across the capacitive element in the form

$$V_c = (E/\omega_o) \exp(-\delta t) (\omega_o \cos \omega_o t + \delta \sin \omega_o t) \quad \text{.......................................(31)}$$

The sum of a cosine and a sine may be replaced with a single sine. To this end, we set $\omega_o/\delta = \tan \psi$, i.e., we take ω_o and δ as the sides of a right triangle (fig. 4.8) whose hypotenuse is

$$\sqrt{\omega_o^2 + \delta^2} = \sqrt{\frac{1}{Lc} - \delta^2 + \delta^2} = \frac{1}{\sqrt{Lc}}$$ 464

On dividing and multiplying Eq. (31) by $\frac{1}{\sqrt{Lc}}$ 465, we get

$$V_c = \frac{E}{\omega_o \sqrt{Lc}} \exp(-\delta t) \sin(\omega_o t + \psi)$$ 466 .... (32)

and by Eq.(25), the discharge current is

$$i = -CdV_c/dt = (E/\omega_o L) \exp(-\delta t) \sin \omega_o t \quad \text{...(33)}$$

Equation (32) and (33) show that the voltage across the capacitive element and the discharge current may be regarded as time-varying sine waves with their amplitudes decreasing exponentially with a time constant $T = 1/\delta = 2L/r$.

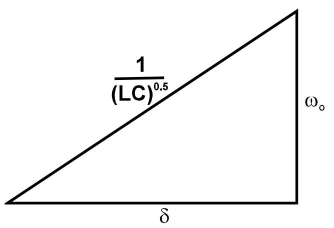


Fig. 4.8

Example 1: *In the circuit of fig Q1 the switch S is suddenly closed. The parameters of the circuit are as follows: E = 200V; J = 1A; r_1 = 100Ω, r_2 = 100Ω; L = 0.5H; C = 400μF. Find the time dependence of the currents in the branches: $i_1(t)$, $i_2(t)$, $i_3(t)$.*

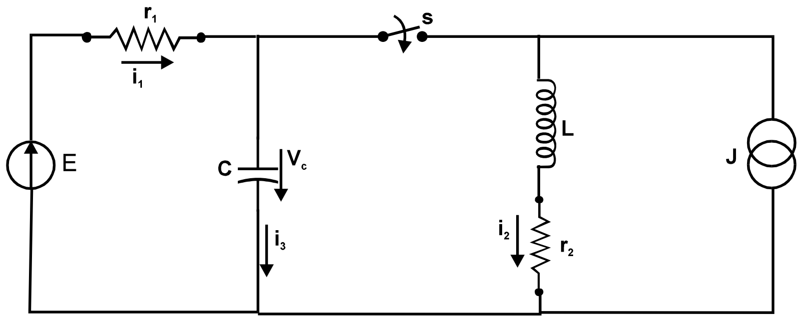


Fig. Q1(a)

Solution:- For the order of solution we use the following order. Initially we determine the voltage across the capacitor V_c. Then we find i_1 according to Ohm's law $i_1 = (E - V_c)/r_1$, current through the capacitor $i_3 = cdV_c/dt$, while current i_2 we find using Kirchhoff's current law $i_2 = J + i_1 - i_3$.

The current in the inductor and the voltage across the capacitor before commutation:

$$i_2(0\_) = J = 1A; \quad V_c(0\_) = E = 200V.$$

Using the superposition method, we calculate the currents and voltage of the steady-state regime of the circuit after commutation.

$$i_{1ss} = \frac{E}{r_1 + r_2} - \frac{j \cdot r_2}{r_1 + r_2} = 0.5A \quad 467$$

$$i_{2ss} = \frac{E}{r_1 + r_2} - \frac{j \cdot r_1}{r_1 + r_2} = 1.5A \quad 468$$

$$i_{3ss} = 0; \quad V_{css} = r_2 \, i_{2ss} = 150V.$$

For deriving the characteristic equation, we write the input impedance looking into the circuit from the branch with the inductor open-circuited and equate this expression to zero.

$$Z(s) = r_2 + SL + \frac{r_1\left(\frac{1}{Sc}\right)}{\left(r_1 + \frac{1}{Sc}\right)} = 0 \quad 469$$

From where for the given parameters of the circuit we obtain the characteristic equation

$S^2 + 225S + 10^4 = 0,$

the roots which are $S_1 = 61S^{-1}$, $S_2 = -164S^{-11}$.
Hence, the transient (free) term solution of the voltage is given

$$V_{ct} = A_1 e^{(S_{1t})} + A_2 e^{(S_{2t})} \quad 470$$

The voltage across the capacitor is presented as the sum of two component; the steady-state $V_{css} + V_{c.t}$

For the evaluation of the constant of integration A_1 and A_2 additionally we write the expression for the first derivative of the voltage V_c:

$$\frac{dV_c}{dt} = S_1 A_1 e^{S_1 t} + S_2 A_2 e^{S_2 t} \quad 471$$

$$For\ t = 0 \quad 472$$

$$V_{ct}(0) = A_1 + A_2 \quad 473$$

$$\frac{dV_c}{dt}\Big|_0 = S_1 A_1 + S_2 A_2 \quad 474$$

This is the system of equation for the definition of the constants of integration. For the computation of the constants of integrations it is necessary to find the initial conditions - the initials values of the transient components of voltage across the capacitor and its derivative.

According to the laws of commutation current in the inductor and voltage across the capacitor cannot change instantaneously, i.e. $1_2(0) = i_2(0\text{-}) = 1A$; $V_c(0) = V_c(0\text{-}) = 200V$.

Therefore, the initial values of the transient components of current and voltage in and across the inductor and capacitor respectively are given thus:

$i_{2t}(0) = i_2(0) - i_{2ss}(0) = 1 - 1.5 = - 0.5A$

$V_{c.t}(0) = V_c(0) - V_{css}(0) = 200 - 150 = 50V.$

We write equations according to Kirchhoff's laws for the transient (free) terms in the circuit after commutation. During this time the ideal voltage source we replace with a short-circuit and the ideal current source replace with an open circuit.

$$-i_{1t} + i_{2t} + i_{3t} = 0 \quad 475$$
$$r_1 i_{1t} + V_{ct} = 0; \quad 476$$
$$L\frac{di_{2t}}{dt} + r_2 i_{2t} - V_{ct} = 0 \quad 477$$
$$For\ t = 0 \quad 478$$
$$-i_{1t}(0) + i_{2t}(0) + i_{3t}(0) = 0 \quad 479$$
$$r_1 i_{1t}(0) + V_{ct}(0) = 0 \quad 480$$

In the last two equations there are two known values $i_{2t}(0)$ and $V_{ct}(0)$; the others we find

$i_{1t} = -V_{ct}(0)/r_1 = -0.5A$

$i_{3t}(0) = i_{1t}(0) - i_{2t}(0) = 0$

The first derivative of the transient term of voltage across the capacitor $dV_{ct}/dt = i_{3t}/C$ at the time $t = 0$ is also equal to zero;

$dV_{ct}/dt/0 = i_{3t}(0)/C = 0$

with the calculated values $V_{ct}(0)$ and dV_{ct}/dt from the above equations, we obtain the constants of integration $A_1 = 79.6V$; $A_2 = -29.6V$.

Now the solution for the voltage across the capacitor is obtained to be

$V_c = 150 + 79.6 \exp(-61t) - 29.6 \exp(-164t)V.$

we determine the currents

$i_1 = (E - V_c)/r_1 = 0.5 - 0.796 \exp(-61t) + 0.296 \exp(-164t)A.$

$i_3 = CdV_c/dt = -1.942 \exp(-61t) + 1.942 \exp(-164t)A.$

$i_2 = J + i_1 - i_3 = 1.5 + 1.146 \exp(-61t) - 1.646 \exp(-164t)A.$

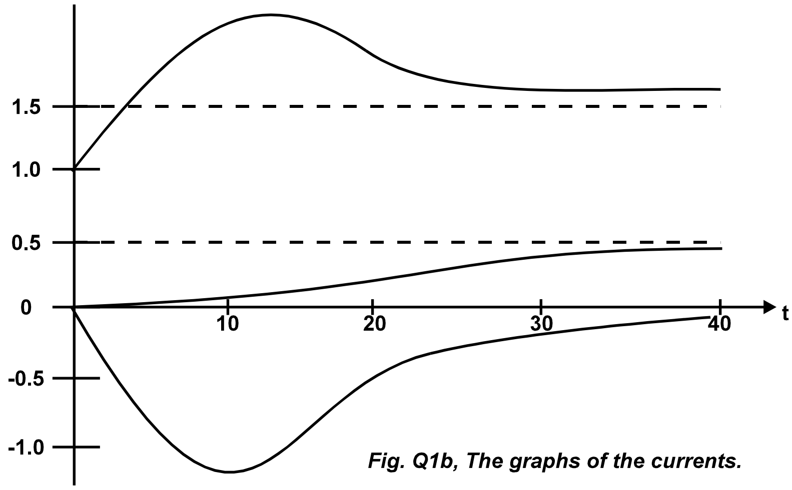


Fig. Q1b, The graphs of the currents.

Examples 2: *In the circuit of fig. Q2 the switch is suddenly opened. The parameters of the circuit are J(t) = 2 sin(2500t + 30°)A; r = 100Ω; c = 1μF; L = 0.2H. Determine the time dependence of voltage $V_c(t)$ and current $1_2(t)$.*

Solution: The voltage across the capacitor and the current in the inductor before commutation is first determined: $V_c(0\text{-}) = 0$; $i_1(0\text{-}) = o$. The unknown quantities we present in the form of sum of the steady-state and the transient components.

$$V_c = V_{css} + V_{ct};\ i_2 = i_{2ss} + i_{2t}$$

We find the complex amplitudes and instantaneous values of currents and voltages across the capacitor for the steady-state regime after commutation:

$$\dot{I}_{1m} = \dot{J}_m \frac{r}{r + jX_L - jX_c} \qquad 481$$

$$= 2\_30° \frac{100}{100 + j500 - j400}$$

$$\sqrt{2} < -15°A; \qquad 482$$

$$\dot{I}_{2m} = \dot{J}_m \frac{jX_L - jX_c}{r + jX_L - jX_c} = \sqrt{2}\ \_75°A; \qquad 483$$

$$\dot{V}_{cm} = \dot{I}_{1m}(-jX_c) = \sqrt{2}\ \_-15°(-j400) \qquad 484$$

$$400\sqrt{2}\ \_-105°V. \qquad 485$$

$$i_{1ss} = \sqrt{2}\ \sin(2500t - 15°)A. \qquad 486$$

$$i_{2ss} = \sqrt{2}\ \sin(2500t + 75°)A \quad 487$$

$$V_{css} = 400\sqrt{2}\ \sin(2500t - 105°)A. \quad 488$$

By method of input impedance, we derive an expression for the characteristic equation

$$Z(s) = r + SL + 1/sc = 0$$

This characteristic equation

$$S^2 + 500s + 5 \text{ x } 10^6 = 0$$

has complex conjugate roots $S_{1,2} = -250 \pm j2220$
Hence, the transient component can be written as follows:

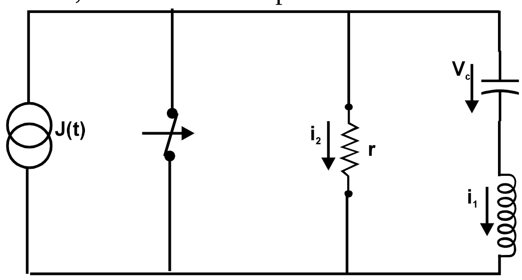


Fig. Q2

$$V_{ct} = Ae^{-250t} \sin(2220t + \alpha);$$

$$i_{2t} = Be^{-250t} \sin(2220t + \beta).$$

Let us find the transient components. We write a system of equations for the determination of the constants of integration A_1, α_1, B_1, β. For the definition of the two constants of integration, each unknown we state the first derivative.

dV_{ct}/dt = -250A exp(-250t) sin (2220t + α) + 2220A exp(-250t) cos (2220t + α);

di_{2t}/dt = -250B exp(-250t) sin (22220t + β) + 2220B exp(-250t) cos (2220t + β).

The equations for the definition of constants of integration (when t = 0) will have the following form.

$$V_c(0) = A \sin \alpha$$

$dV_{ct}/dt|_0 = -250A \sin\alpha + 2220A \cos\alpha$

$i_{2t}(0) = B \sin \beta$

$di_{2t}/dt|_0 = -250B \sin \beta + 2220B \cos \beta.$

For these computation we derive the initial conditions for the transient (free) components. From the laws of commutation it follows that, $V_c(0) = V_c(0\text{-}) = 0$; $i_1(0) = i_1(0\text{-}) = 0$. Therefore, the initial values for the transient components are given as follows:

$$i_{1t}(0) = i_1(0) - i_{1ss}(0)$$ 489

$$= -\sqrt{2}\ \sin(-15°) = 0.366A$$

$$V_{ct}(0) = V_c(0) - V_{css}(0)$$ 490

$$= -400\sqrt{2}\ \sin(-105°) = 546V$$

We write Kirchhoff's laws for the transient component (terms), and exclude the branch with the ideal current source

$$-i_{2t} - i_{1t} = 0;$$ 491

$$V_{ct} + L\frac{di_{1t}}{dt} - ri_{2t} =$$ 492

where 493

$$i_{1t} = c\frac{dV_{ct}}{dt}$$ 494

and in special case when t = 0

$-i_{2t}(0) - i_{1t}(0) = 0$

$V_{ct}(0) + L\, di_{1t}/dt|_0 - ri_{2t}(0) = 0$

$i_{1t}(0) = c dV_{ct}/dt|_0$

in these equations we know $V_{ct}(0)$ and $i_{1t}(0)$, therefore we find

$i_{2t}(0) = -0.366A$; $di_{1t}/dt|_0 = -2913A/s$; $dV_{ct}/dt|_0 = 366 \times 10^3 V/s$.
For definition of the second current we differentiate equation $di_{2t}/dt = -di_{1t}/dt$, from where it follows, that for t = 0 the derivative $di_{2t}/dt = -di_{1t}/dt = 2913A/s$.

From the equations for the definition of the constants of integration, we obtain the follows values A = 591V; α = 67.5°; B = 1.321A, β = -16.08. Finally we write the unknown quantities

$$V_c(t) = 400\sqrt{2}\,495\sin(2500t - 105°) + 591\exp(-250t)\sin(2220t + 67.5°)V.$$

$$i_2 = \sqrt{2}\,496\sin(2500t + 75°) + 1.321\exp(-250t) \times \sin(2220t - 16.08°)A.$$

Example 3: *Calculate the current i_1 of the emf source in fig. Q3 with parameters E = 80V and r_1 = 200Ω after commutation if $r_2 = r_1$; C = 40μF, L = 0.1H.*

Solution: The voltage across the capacitor and current in the inductor before commutation

$$V_c(0-) = E = 80V;\; i_3(0-) = 0$$

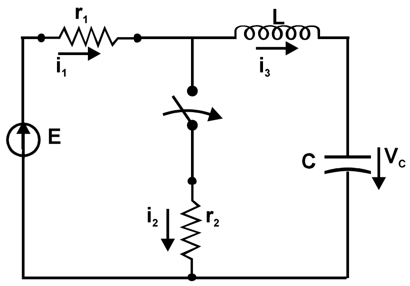


Fig. Q3

The unknown current will be presented in the form of sum of two components.

$$i_1 = i_{1ss} + i_{1t},$$

where after commutation

$$i_{1ss} = E/(r_1 + r_2) = 0.2A$$

Also $i_{3ss} = 0$; $V_{css} = Er_2/(r_1 + r_2) = 40V$.

The characteristic equation is the input impedance of the circuit equated to zero

$$Z(s) = r_1 + \frac{r_2\left(SL + \frac{1}{sc}\right)}{\left(r_2 + SL + \frac{1}{sc}\right)} = 0 \; 4\,9\,7$$

we obtain the characteristic equation

$$S^2 + 1000S + 250 \times 10^3 = 0$$

the roots of the equation are real: $S_1 = S_2 = -500S^{-1}$.

Therefore the solution for the transient term of the unknown current can be written as follows:

$$i_{1t} = A_1e^{-500t} + A_2te^{-500t}$$

To define the two constants of integration A_1 and A_2 we will evaluate the first derivative of the transient term.

$$di_{1t}/dt = -500A_1\exp(-500t)+A_2\exp(-500t)-500A_2t\exp(-500t)$$

for t = 0; we obtain

$$i_{1t}(0) = A_1 \text{ and } di_{1t}/dt\,|_0 = -500A_1 + A_2$$

For the computation of A_1 and A_2 from the system of equation above we find the initial values for the transient current and its derivative. According to the law of commutation, the voltage across the capacitor and the current in the inductor cannot change instantaneously, i.e.

$$V_c(0) = V_c(0-) = 80V;\ i_3(0-) = 0$$

$$V_c(0) = V_c(0) - V_{css}(0) = 40V;\ i_{3t}(0) = i_3(0) - i_{3ss}(0) = 0$$

We write Kirchhoff's laws for the transient components in the circuit after commutation (in the branch with emf source will have only r_1 emf source eliminated)

$$-i_{1t} + i_{2t} + i_{3t} = 0$$

$$r_1i_{1t} + r_2i_{2t} = 0$$

$$Ldi_{3t}/dt + V_{ct} - r_2i_{2t} = 0$$

For when t = 0, we obtain the following

$$-i_{1t}(0) + i_{2t}(0) + i_{3t}(0) = 0$$

$$r_1 i_{1t}(0) + r_2 i_{2t}(0) = 0$$ 498

$$\frac{Ldi_{3t}}{dt}\Big|_0 + V_{ct}(0) - r_2 i_{2t}(0) = 0$$

In these equations already we know $V_{ct}(0)$ and $i_{3t}(0)$, we find: $i_{1t}(0) = 0$; $i_{2t}(0) = 0$;

$di_{3t}/dt|_o = -400A/s$

For the definition of the first derivative of the current, we differentiate the equation

$-di_{1t}/dt + di_{2t}/dt + di_{3t}/dt + 0$

$r_1 di_{1t}/dt + r_2\, di_{2t}/dt = 0$

for t = 0,

$di_{1t}/dt|_0 + di_{2t}/dt|_0 + di_{3t}/dt|_0 = 0$

$r_2\, di_{1t}/dt|_0 + r_2 di_{2t}/dt|_0 = 0$

From the two equations above, $di_{3t}dt|_0$ is known, we find $di_{1t}/dt|_0 = -200A/s$. Now from the system of equation for the constants of integration we obtain $A_1 = 0$; $A_2 = -200A$

The unknown current i_1

$i_1 = i_{1ss} + i_{1t} = 0.2 - 200te^{-500t}A$

$i_1 = 0.2 - 200t \exp(-500t)$

Example 4: *commutation takes place in the circuit of fig. Q4. The parameters of the circuit are E = 100V; $r_1 = r_2 = 100\Omega$; L = 0.1H; C=10μF.*
Determine the initial values: $i_{1t}(0)$, $i_{2t}(0)$, $i_{3t}(0)$, $V_{ct}(0)$, $V_{Lt}(0)$, $di_{1t}\backslash dt/_0$, $di_{3t}\backslash dt/_0$, $dV_{ct}\backslash dt/_0$, $dV_{Lt}\backslash dt/_0$

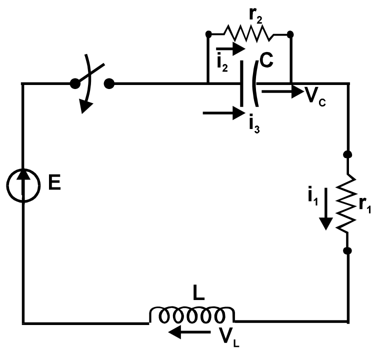


Fig. Q4

Solution: We determine the initial values of $V_c(0-)$ and $i_1(0)$

$V_c(0-) = V_c(0) = 0;$

$i_1(0-) = i_1(0) = 0$

The steady state values of i_{1ss} and V_{css} are given thus:

$$i_{1ss} = \frac{E}{(r_1 + r_2)} = \frac{100}{(100+100)} \quad 499$$

$$V_{css} = \frac{E \bullet r_2}{(r_1 + r_2)} = 50V \quad 500$$

$i_2(0) = 0; i_3(0) = 0$

$i_{2ss} = i_{1ss} = 0.5A; i_{3ss}(0) = 0$

To calculate the transient components we eliminate the emf source from the circuit

$-i_{1t} + i_{2t} + i_{3t} = 0$

$r_2 i_2 - V_{ct} = 0$

$r_1 i_{1t} + V_{ct} + V_{Lt} = 0$

$i_1(0) = i_{1ss}(0) + i_{1t}(0); i_{1t}(0) = i_1(0) - i_{1ss}(0)$

$= 0 - 0.5 = -0.5A.$ $\qquad$ $i_{1t}(0) = -0.5A$

$V_c(0) = V_{css}(0) + V_{c\text{-}t}(0);$

$V_{ct}(0) = V_c(0) - V_{css}(0) = 0 - 50 = -50V;$ *$V_{ct}(0) = -50V$*

$-i_{1t}(0) + i_{2t}(0) + i_{3t}(0) = 0$

$r_2 i_{2t}(0) - V_{ct}(0) = 0;\ i_{2t}(0) = V_{ct}(0)/r_2$

$= -50/100 = -\ 0.5A.$ *$i_{2t}(0) = -0.5A$*

$1_{3t}(0) = i_{1t}(0) - 1_{2t}(0) = -0.5 + 0.5 = 0A$ *$i_{3t}(0) = 0A$*

$r_1 i_{1t}(0) + V_{ct}(0) + V_{Lt}(0) = 0$

$100(-0.5) + (-50) + V_{Lt}(0)$

$-50 - 50 + V_{Lt}(0);$ *$V_{Lt}(0) = 100V$*

$Ldi_{1t}/dt + V_{ct} + r_1 i_{1t} = 0;$

$-i_{1t} + i_{2t} + i_{3t} = 0$

$r_2 i_{2t} - V_{ct} = 0$

For t = 0, differentiate the above equation, we obtain:

$Ldi_{1t}/dt|_0 + V_{ct}(0) + r_1 i_{1t}(0) = 0;$

$-di_{1t}/dt|_0 + di_{2t}/dt|_0 + di_{3t}/dt|_0 = 0$

$r_2 di_{2t}/dt|_0 - dV_{ct}/dt|_0 = 0$

$i_{3t} = CdV_{ct}/dt;$

$i_{3t}(0)/C = dV_{ct}(0)/dt|_0$

$$\frac{di_{1t}}{dt}\Big|_0 = \frac{-V_{ct}(0) - r_1 i_{1t}(0)}{L} = \frac{-(-50)-(-50)}{(0.1)} = \frac{100}{0.1}\ 501$$

$= 1000$ A/S

$di_{1t}/dt|_0 = 1000A/S$

$r_2 di_{2t}/dt|_0 - i_{3t}(0)/C = 0;\ \ di_{2t}/dt|_0 = i_{3t}(0)/r_2C = 0$

$di_{2t}/dt\,/_0 = 0;$

$-di_{1t}/dt|_0 + di_{2t}/dt|_0 = di_{3t}/dt|_0 = 0$

$-1000 + 0 + di_{3t}/dt|_0 = 0$ $\qquad$ $di_{3t}/dt|_0 = 1000A/S$

$dV_{ct}/dt/_0 = i_{3t}(0)/r_2C = 0;$

$r_1i_{1t} + V_{ct} + V_{Lt} = 0$

$r_1di_{1t}/dt|_0 + dV_{ct}/dt|_0 + dV_{Lt}/dt|_0 = 0$

$100(1000) + 0 + dV_{Lt}/dt|_0 = 0$

$dV_{Lt}/dt/_0 = -10^5$ V/S

Answers: *-0.5A; -0.5A; -50V; 100V; 1000A/s; 1000A/s; 0V/s;* -10^5*V/s*

Example 5: *An emf source with an internal resistance r = 20Ω is connected to an additional branch with a coil of inductance L = 0.05H and resistance* r_2 *= 30Ω.*
Calculate the currents i, i_2 *and voltage* V_C *across the capacitor whose capacitance C = 50μF. (fig. Q5)*

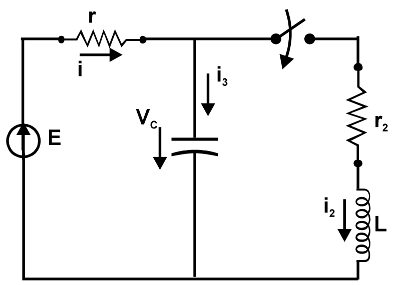


Fig. Q5

Solution: The initial conditions i(0) = 0; $i_2(0-) = 0$; $i_3(0-) = 0$; $V_c(0-) = E = 100V$.
These are the initial values according to laws of commutation. The steady - state values of currents and voltage across the capacitor.

$i_{ss} = i_{2ss} = E/(r + r_2) = 100/20 + 30) = 2A$

$i_{3ss} = 0$; $V_{css} = E.r_2/(r + r_2) = 2 \times 30 = 60V.$

We write Kirchhoff's equation for the circuit after commutation for the transient terms (meanwhile the emf source is eliminated).

$i_{1t} + i_{2t} + i_{3t} = 0$

$r_2 i_{2t} + L di_{2t}/dt - V_{ct} = 0$

$i_{3t} = C dV_{ct}/dt$

$i_2(0) = i_{2ss}(0) + i_{2t}(0)$; $i_{2t}(0) = i_2(0) - i_{2ss}(0)$

$= 0 - 2 = -2A$; $\quad$ *$i_{2t}(0) = -2A$*

$V_c(0) = V_{css}(0) + V_{ct}(0)$; $V_{ct}(0) = V_c(0) - V_{css}(0)$

$= 100 - 60 = 40V$; $\quad$ *$V_{ct}(0) = 40V$*

For t = 0

$-i_t(0) + i_{3t}(0) + i_{2t}(0)$

$= r i_t(0) + V_{ct}(0) = 0$; $i_t = -V_{ct}(0)/r = -40/20 = -2A$

$i_t(0) = -2A$

$-i_t(0) + i_{3t}(0) + i_{2t}(0) = 0$

$-(-2) + i_{3t}(0) + (-2) = 0$; $\quad$ *$i_{3t}(0) = 0A$*

The characteristic equation is the impedance of the circuit equated to zero

$$Z(s) = r + \frac{(r_2 + SL)\left(\frac{1}{sc}\right)}{\left(r_2 + SL + \frac{1}{sc}\right)} = 0 \quad 502$$

$rLcs^2 + rr_2sc + sL + r + r_2 = 0$

The roots of the characteristics equation are complex conjugate roots $S_{1,2} = -800 \pm j600$

Hence, the solution for the transient component will be written as follows

$V_{ct} = A \exp(-800t) \sin(600t + \alpha)$

$i_t = C \exp(-800t) \sin(600t + \gamma)$

$i_{2t} = B\exp(-800t)\sin(600t + \beta)$

To calculate the constants of integration we obtain the first derivatives of Kirchhoff's equations.

$-di_t/dt + di_{3t}/dt + di_{2t}/dt = 0$

$rdi_t/dt + dV_{ct}/dt = 0$

$r_2di_{2t}/dt + Ld^2i_{2t}/dt^2 - dV_{ct}/dt = 0$

$i_{3t} = CdV_{ct}/dt;\ \ dV_{ct}/dt|_0 = i_{3t}(0)/C = 0$

$rdi_t/dt|_0 + dV_{ct}/dt|_0 = 0$

$dV_{ct}/dt /_0 = 0\ ;\quad di_t/dt /_0 = 0$

$di_{2t}/dt|_0 + r_2i_{2t}(0) - V_{ct}(0) = 0$

$Ldi_{2t}/dt|_0 + 30(-2) - 40 = 0$

$di_{2t}/dt|_0 = (60 + 40)/0.05 = 100/0.05$

$di_{2t}/dt /_0 = 2000\ A/s$

$-di_t/dt|_0 + di_{3t}/dt|_0 + di_{2t}/dt|_0 = 0$

$0 + di_{3t}/dt|_0 + 2000 = 0$

$di_{3t}/dt /_0 = -2000\ A/s$

we differentiate the solutions V_{ct}, i_t, i_{2t}.

$dV_{ct}/dt = -800A\exp(-800t)\sin(600t + \alpha) + 600A\exp(-800t)\cos(600t + \alpha)$

$di_t/dt = -800C\exp(-800t)\sin(600t + \gamma) + 600C\exp(-800t)\cos(600t + \gamma)$

$di_{2t}/dt = -800B\exp(-800t)\sin(600t + \beta) + 600B\exp(-800t)\cos(600t + \beta)$

$dV_{ct}/dt|_0 = 0 = -800A\sin\alpha + 600A\cos\alpha$

$di_t/dt|_0 = 0 = -800C \sin\gamma + 600C \cos\gamma$

$di_{2t}/dt|_0 = 2000 = -800B \sin\beta + 600B \cos\beta$

$$A = \frac{40}{\sin\alpha} \quad 503$$

$V_{ct}(0) = 40 = A \sin\alpha$

$dV_{ct}/dt|_0 = 0 = -800A \sin\alpha + 600A \cos\alpha$

$$-800 \ x \ \frac{40}{\sin\alpha} \ x \ \sin\alpha + 600 \ x \ \frac{40}{\sin\alpha}.\cos\alpha \quad 504$$

$$\cot\alpha = \frac{800}{600}; \quad \tan\alpha = \frac{600}{800} \quad 505$$

$\alpha = 37°$

$$A = \frac{40}{\sin\alpha} = 66.67V \quad 506$$

$V_{ct} = 66.67 \exp(-800t) \sin(600t + 37^o)$

$i_t(0) = C \sin\gamma = -2$

$di_t/dt|_0 = 0 -800C \sin\gamma + 600C \cos\gamma$

$$C = \frac{-2}{\sin\gamma}; C = \quad 507$$

$$0 = -800 \ x \ \frac{(-2)}{\sin\gamma}\sin\gamma + 600\frac{(-2)}{\sin\gamma}\cos\gamma \quad 508$$

$$\cot\gamma = \frac{-800 \ x \ 2}{-600 \ x \ 2} = \frac{800}{600}; \quad 509$$

$$\text{Tan } \gamma = \frac{600}{800} = 0.75 \quad 510$$

$\gamma = -144$

$$C = \frac{-2}{\sin 37°} = -3.33A \quad 511$$

$$i_t = -3.33 \ \exp \ (-800t)\sin(600t + 37°) \quad 512$$

$$i_{2t}(0) = -2 = B\sin\beta; \qquad B = \frac{-2}{\sin\beta} \quad 513$$

$$\frac{di_{2t}}{dt}|_0 = 2000 = -800B\sin\beta + 600B\cos\beta \quad 514$$

$$2000 = -800\frac{(-2)}{\sin\beta}\,sinB + 600\frac{(-2)}{\sin\beta}\cos\beta \quad 515$$

$$2000 = 1600 - 1200\cot\beta \quad 516$$

$$2000 - 1600 = -1200\cot\beta \quad 517$$

$$400 - \quad 518$$

$$\cot\beta = \frac{-400}{1200}; \qquad \tan\beta = \frac{-1200}{400} \quad 519$$

$\beta = -71.6°$

$$B = \frac{-2}{-\sin 71.6°} = \frac{2}{\sin 71.6°} = 2.11A \quad 520$$

$i_{2t} = 2.11\exp(-800t)\sin(600t - 71.6°)$

$V_c = V_{css} + V_{ct} = 60 + 66.67\exp(-800t)\sin(600t + 37°)$

$i = i_{ss} + i_t = 2 - 3.33\exp(-800t)\sin(600t - 144)A$

$i_2 = i_{2ss} + i_t = 2 + 2.11\exp(-800t)\sin(600t - 71.6°)A$

Example 6*: The switch in the circuit of fig. Q6 is suddenly open. The parameters of the circuit are as follows: E = 100V; J = 1A; $r_1 = r_2 = 10\Omega$; L = 0.1H; C = 1000μF. Find the time dependence of the current $i_1(t)$ and voltage $V_c(t)$.*

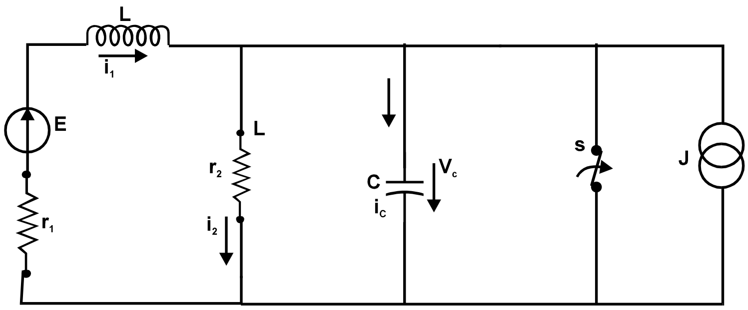


Fig. Q6

Solution: We begin by first determining the initial values according laws of commutation.

$i_1(0) = E/r_1 = 100/10 = 10A$

$V_c(0-) = 0$; $i_2(0-) = 0$; $i_c(0-) = 0$

We determine the steady-state component of currents and voltages V_{css}.

$$i_{1ss} = \frac{E}{(r_1 + r_2)} - \frac{j \cdot r_2}{r_1 + r_2}$$

521

$$= \frac{100}{(10+10)} - \frac{1\,x\,10}{10+10}$$

$$= 5 - 0.5 = 4.5A \quad 522$$

$$i_{2ss} = \frac{E}{(r_1 + r_2)} + \frac{j \cdot r_1}{(r_1 + r_2)} = 5 + 0.5 = 5.5A \quad 523$$

$$i_{css} = 0; \quad V_{css} = i_{2ss} \cdot r_2 = 55V \quad 524$$

We write Kirchhoff's laws for the circuit component with the voltage and current sources eliminated.

$i_{1t} + i_{2t} + i_{ct} = 0$

$r_1 i_{1t} + L di_{1t}/dt + r_2 i_{2t} = 0$

$-r_2 i_{2t} + V_{ct} = 0$

We derive the characteristic equation as the input impedance equated to zero

525

$$10^{-3} S^2 + 0.25 + 20 = 0 \quad 526$$

$$S^2 + 200s + 20\,x\,10^3 = 0 \quad 527$$

the roots are complex conjugate roots

$S_{1,2} = -100 \pm j100$

The solution of the transient terms will be presented as follows

$i_{1t} = A \exp(-100t) \sin(100t + \alpha) A$

$V_{ct} = B \exp(-100t) \sin(100t + \beta) V.$

To calculate the constants of integration we will obtain the first derivative of Kirchhoff's laws for the circuit after commutation and determine the transient terms $V_{ct}(0)$ and $i_{1t}(0)$

$i_1(0) = i_{1ss}(0) + i_{1t}(0)$; $i_{1t}(0) = i_1(0) - i_{1ss}(0)$

$= 10 - 4.5 = 5.5A;$ $i_{1t}(0) = 5.5A$

$i_{2t} = i_2(0) - i_{2ss}(0) = 0 - 5.5 - -5.5A$

$i_{2t}(0) = -5.5A$

$i_{ct}(0) = i_{1t}(0) - i_{2t}(0) = 5.5 - (-5.5) = 11A$

$V_{ct}(0) = V_c(0) - V_{css}(0) = 0 - 55V = -55V$

$-di_{1t}/dt + di_{2t}/dt + di_{ct}/dt = 0$

$r_1 di_{1t}/dt + Ld^2 i_{1t}/dt^2 + r_2 di_{2t}/dt = 0$

$-r_2 di_{2t}/dt + dV_{ct}/dt = 0;$ $i_{ct} = CdV_{ct}/dt$

$$\frac{di_{1t}}{dt}\Big|_0 = \frac{-r_1 i_{1t}(0) - r_2 i_{2t}(0)}{L}$$ 528

$$= \frac{-10(5.5) - 10(-5.5)}{0.1} = 0A/s$$

$i_{ct}(0)/C = dV_{ct}/dt\,/_0$ $di_{1t}/dt\,/_0 = 0$

$dV_{ct}/dt\,|_0 = i_{ct}(0)/C = 11/1000 \times 10^{-6} = 11 \times 10^3$ V/s

$$r_2 \frac{di_{2t}}{dt}\Big|_0 + dV_{ct}/dt = 0$$ 529

$$-r_2 \frac{di_{2t}}{dt}\Big|_0 + \frac{dV_{ct}}{dt}\Big|_0 = 0$$ 530

$$r_2 \frac{di_{2t}}{dt}\Big|_0 = \frac{dV_{ct}}{dt}\Big|_0 = 11 \times 10^3$$ 531

$$\frac{di_{2t}}{dt}\Big|_0 = \frac{11 \times 10^3}{r_2}$$ 532

$$= \frac{11 \times 10^3}{10} = 1100A/s$$

$di_{2t}/dt\,/_0 = -1100$ A/s

$di_{ct}/dt\,/_0 = -1100$ A/s

$i_{1t}(0) = 5.5 = A \sin\alpha$

$di_{1t}/dt|_0 = 0 = -100A \exp(-100t) \sin(100t + \alpha) + 100A \exp(-100t) \cos(100t + \alpha)$

$= -100A \sin\alpha + 100A \cos\alpha$

$5.5 = A \sin\alpha$; $A = 5.5/\sin\alpha$

$0 = -100A \sin\alpha + 100A \cos\alpha$

$$0 = -100\frac{5.5}{\sin\alpha}.\sin\alpha + 100\frac{5.5}{\sin\alpha}\cos\alpha \quad 533$$

$$550 = 550\cot\alpha; \quad 534$$

$$\tan\alpha = \frac{550}{550} = 1 \;; \qquad \alpha = 45^\circ \quad 535$$

$$A = \frac{5.5}{\sin 45} = 7.778A \quad 536$$

Hence;

$$i_{1t} = 7.778\exp(-100t)\sin(100t + 45^\circ) \qquad 537$$

$i_1 = i_{1ss} + i_{1t} = 4.5 + 7.78 \exp(-100t) \sin(100t + 45^o) A$

$V_{ct}(0) = -55 = B\sin\beta$; $B = \dfrac{-55}{\sin\beta}$ 538

$$\frac{dV_{ct}}{dt}\Big|_0 = 11 \; x \; 10^3 = -100B\sin\beta + 100B\cos\beta \quad 539$$

$$11000 = -100 \; x \; \frac{(-55)}{\sin\beta}\sin\beta + 100\frac{(-55)}{\sin\beta}\cos\beta \quad 540$$

$$11000 = 5500 - 5500\cot\beta \quad 541$$

$$\tan\beta = \frac{-5500}{5500} = -1; \qquad \beta = -45^\circ \quad 542$$

$$B = \frac{-55}{\sin(-45)} = \frac{-55}{-\sin 45} = 77.78V \quad 543$$

$$B = 55\sqrt{2} = 77.78V \quad 544$$

$$V_{ct} = V_{css} + V_{ct}$$
$$= 55 + 55\sqrt{2}\exp(-100t)\sin(100t - 45^\circ) \quad 545$$

$V_c = V_{css} + V_{ct} = 55 + 55\sqrt{2}$ 546 $\exp(-100t) \sin(100t - 45^o)$

Example 7: *In the circuit of fig Q7 a capacitor of capacitance C = 200μF is* *connected in parallel to a resistor $r_2 = 5\Omega$. Calculate the current i and voltage V_2 across the resistor if the emf source E = 100V, $r = 3\Omega$; L = 0.1H.*

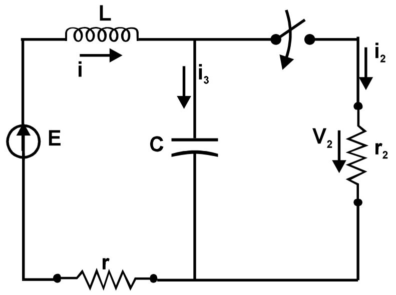


Fig. Q7

Solution: We calculate the initial value of current in the inductor and voltage across the capacitor before commutation

$$i(0-) = 0$$

$$V_c(0-) = E = 100V$$

$$i_3(0-) = 0; \; i_2(0-) = 0$$

The unknown current *i* and voltage V_2 will be determined as sum of two components

$$i = i_{ss} + i_t$$

$$V_2 = V_{2ss} + V_{2t}$$

The steady state term are determined as follows:

$$i_{ss} = i_{2ss} = E/(r + r_2) = 12.5A$$

$$V_{css} = i_{2ss}.r_2 = 62.5V$$

$$i_{3ss} = 0$$

The characteristic equation is the input impedance of the circuit equated to zero

$$r_2LCS^2 + rr_2sc + sL + r + r_2 = 0$$

$$10^{-4}S^2 + 0.103\ S + 8 = 0$$

The roots of the characteristic equation are real and different $S_1 = -945\ S^{-1}$, $S_2 = -85\ S^{-1}$.
Therefore the solution for the transient terms of current *i* and voltage V_2 will be presented as follows

$$i_t = A_1 \exp(-945t) + A_2 \exp(-85t)$$

$$V_{2t} = B_1 \exp(-945t) + B_2 \exp(-85t)$$

We write kirchhoff's laws for the transient terms without the emf source

$$-i_t + 1_{3t} + 1_{2t} = 0$$

$$ri_t + Ldi_t/dt + V_{ct} = 0$$

$$r_2 i_{2t} - V_{ct} = 0$$

Using the initial conditions, we obtain the zero conditions of the transient terms

$$i(0) = i_{ss}(0) + i_t(0);\ \ i_t(0) = i(0) - i_{ss}(0)$$

$$i_t(0) = 0 - 12.5 = -12.5A$$

$$V_c(0) = V_{css}(0) + V_{ct}(0);\ V_{ct}(0) = V_c(0) - V_{css}(0)$$

$$= 100 - 62.5 = 37.5V$$

We determine $i_{2t}(0)$ and $i_{3t}(0)$ using the above values

$$r_2 i_{2t}(0) - V_{ct}(0)$$

$$i_{2t}(0) = V_{ct}(0)/r_2 = 7.5A$$

$$-i_t(0) + i_{3t}(0) + i_{2t}(0) = 0$$

$$-(-12.5) + i_{3t}(0) + 7.5 = 0$$

$$i_{3t}(0) = -20A$$

We differentiate Kirchhoff's equations

$-di_t/dt + di_{3t}/dt + di_{2t}/dt = 0$

$rdi_t/dt + Ld^2i_t/dt^2 + dV_{ct}/dt = 0$

$r_2di_{2t}/dt - dV_{ct}/dt = 0$

$i_{3t} = cdV_{ct}/dt;\ dV_{ct}/dt|_0 = i_{3t}(0)/C = -20/2 \times 10^{-4}$

$dV_{ct}/dt/_0 = -10^5\ V/S$

$$\frac{di_t}{dt} = \frac{-(ri_t + V_{ct})}{L} = \frac{-(3(-12.5) + 37.5)}{0.1} = 0 \quad 547$$

$di_t/dt/o = 0$

$r_2di_{2t}/dt = dV_{ct}/dt/_0 = -10^5\ V/S$

To find the constants of integration A_1, A_2; B_1, B_2 we differentiate the solutions for current i_t and voltage V_{2t} and determine their initial values when t = 0

$di_t/dt|_0 = 0 = -945A_1 \exp(0) - 85A_2 \exp(0)$

$dV_{2t}/dt|_0 = -10^5 = -945B_1 \exp(0) - 85B_2 \exp(0)$

$di_t/dt|_0 = 0 = -945A_1 - 85A_2$

$dV_{2t}/dt|_0 = -10^5 = -945B_1 - 85B_2$

$V_{2t}(0) = 37 = B_1 + B_2$

$i_t(0) = -12.5 = A_1 + A_2$

We now have a system of simultaneous equations

$A_1 + A_2 = -12.5$
$-945A_1 - 85A_2 = 0$

$A_1 = 1.236A;\quad A_2 = -13.73A$

$B_1 + B_2 = 37.5$
$-945B_1 - 85B_2 = -10^5$

$B_1 = 112.7V;\quad B_2 = -75.1V$

Therefore the current i_t and voltage V_{2t} are given as follows

$$i_t = 1.236 \exp(-945t) - 13.73 \exp(-85t)$$

$$V_{2t} = 112.7 \exp(-945t) - 75.1 \exp(-85t)$$

finally the current $i = i_{ss} + i_t$

$$= 12.5 - 13.73 \exp(-85t) + 1.236 \exp(-945t)$$

$$V_2 = V_{2ss}+V_{2t} = 62.5 - 75.1\exp(-85t) + 112.7\exp(-945t).$$

EXample 8: *Determine the current i, the voltage V_L across the inductor, and the voltage V_c across the capacitor in the circuit of fig. Q8. The parameters of the circuits are $r = 5\Omega$, $r_2 = 3\Omega$, $L = 0.1H$, $C = 200\mu F$, $E = 100V$.*

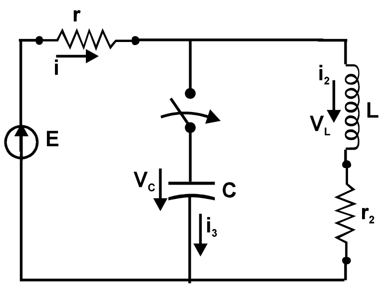


Fig. Q8

Solution: We calculate the initial current in the inductor and the voltage across the capacitor

$$i_2(0-) = E/(r + r_2) = 100/(3 + 5)$$

$$= 12.5A; \quad V(0-) = 0$$

The unknown quantities are presented as consisting of two components, the steady state and the transient components:

$$i = i_{ss} + i_t$$

$$V_L = V_{Lss} + V_{Lt}$$

$$V_c = V_{css} + V_{ct}$$

The steady state components are determined using basic circuit laws.

$i_{ss} = i_{2ss} = E/(r + r_2) = 100/8 = 12.5A$

$V_{css} = i_{2ss}.r_2 = 12.5 \text{ x } 3 = 37.5V$; $V_{Lss} = 0$

We obtain the characteristic equation from the input impedance of the circuit looking into the circuit from the branch with the emf source

$$Z(s) = r + \frac{(r_2 + SL)\left(\frac{1}{sc}\right)}{\left(r_2 + SL + \frac{1}{sc}\right)} = 0 \quad 548$$

$$rLcs^2 + rr_2 sc + SL + r + r_2 = 0 \quad 549$$

$$10^{-4} s^2 + 0.103s + 8 = 0 \quad 550$$

$$S^2 + 1030s + 8x10^4 = 0 \quad 551$$

$$S_1 = -945\, S^{-1}, \qquad S_2 = -85\, S^{-1} \quad 552$$

The roots of the characteristic are real and different. Therefore the solution for the transient component will be presented in the form as follows

$$i_t = A_1 \exp(-945t) + A_2 \exp(-85t) A \;. \quad 553$$

$$V_{Lt} = B_1 \exp(-945t) + B_2 \exp(-85t) V \quad 554$$

$$V_{ct} = C_1 \exp(-945t) + C_2 \exp(-85t) V \;. \quad 555$$

To determine the constants of integration we will write Kirchhoff's equations for the circuits and also obtain the first derivatives of these equations

$$-i_t + i_{3t} + i_{2t} = 0 \quad 556$$

$$ri_t + V_{ct} = 0 \quad 557$$

$$L\frac{di_{2t}}{dt} + r_2 i_{2t} - V_{ct} = 0 \quad 558$$

$$i_{3t} = C\frac{dV_{ct}}{dt} \quad 559$$

Using the initial conditions and the initial values of $i_2(0\text{-})$ and $V_c(0\text{-})$, we obtain the values of the transient components.

$i_2(0\text{-}) = i_{2ss}(0) + i_{2t}(0)$; $12.5 = 12.5 + i_{2t}(0)$ $\qquad$ $i_{2t}(0) = 0$

$V_c(0) = V_{css}(0) + V_{ct}(0)$; $V_{ct}(0) = V_c(0) - V_{css} = 0 = 0 - 37.5V$
$= -37.5V$;

$V_{ct}(0) = -37.5V$

$ri_t(0) + V_{ct}(0) = 0$

$i_t(0) = -V_{ct}(0)/r = -(-37.5)/5 = 7.5A$ $\qquad i_t(0) = 7.5A$

$$L\frac{di_{2t}}{dt} + r_2 i_{2t} - V_{Ct} = 0 \quad 560$$

$$\frac{di_{2t}}{dt}\Big|_0 = \frac{V_{Ct}(0) - r_2 i_{2t}(0)}{L} = \frac{-37.5 - 3(0)}{0.1} \quad 561$$

$$= -375\ A/s$$

$$\therefore \quad \frac{di_{2t}}{dt}\Big|_0 = -375\ A/s \quad 562$$

$$-i_t(0) + I_{3t}(0) + i_{2t}(0) = 0 \quad 563$$

$$-7.5 + i_{3t}(0) + 0 = 0 \quad 564$$

$$\therefore \quad i_{3t}(0) = 7.5\ A \quad 565$$

$$C\frac{dV_{Ct}}{dt} = i_{3t}\ ;\ i_{3t}(0) = C\frac{dV_{Ct}}{dt}\Big|_0 \quad 566$$

$$\frac{dV_{Ct}}{dt}\Big|_0 = \frac{i_{3t}(0)}{C} = \frac{7.5\ A}{2\ x\, 10^{-4}} = 3.75\ x\, 10^4\ V/s \quad 567$$

$$\therefore \quad \frac{dV_{Ct}}{dt}\Big|_0 = 3.75\ x\, 10^4\ V/s \quad 568$$

now to obtain the constants of integration, we differentiate the solutions.

$$\frac{di_t}{dt} = -945\, A_1 \exp(-945t) - 85\, A_2 \exp(-35t) \quad 569$$

$$i_t = A_1 \exp(-945t) + A_2 \exp(-85t) \quad 570$$

$$i_t = 7.5 = A_1 + A_2 \quad 571$$

$$\frac{di_t}{dt}\Big|_0 = -7{,}5\ x\, 10^3 = -945\, A_1 - 85\, A_2 \quad 572$$

$$85\, A_1 + 85\, A_2 = 637.5$$

$$\underline{-945\, A_1 - 85\, A_2 = -7.5\ x\, 10^3} \quad 573$$

$$-860\, A_1 \qquad = -6862.5$$

$$\therefore \quad A_1 = 7.98\ A\ ,\ A_2 = 7.5 - 7.98 = -0.48\ A \quad 574$$

$$i_t = 7.98 \exp(-945t) - 0.48 \exp(-85t) \quad 575$$

$$V_{Lt} = B_1 \exp(-945t) + B)2 \exp(-85t) \quad 576$$

$$\frac{dV_{Lt}}{dt} = -945B_1\exp(-945t) - 85B_2\exp(-85t) \quad 577$$

$$V_{Lt}(0) = L\frac{di_{2t}}{dt}\Big|_0 = -37.5 = B_1 + B_2 \quad 578$$

$$\frac{dV_{Lt}}{dt} = L\frac{d^2 i_{2t}}{dt^2}\Big|_0 = 38625 = -945B_1 - 85B_2 \quad 579$$

$$\begin{aligned} B_1 + B_2 &= -37.5 \\ -945B_1 - 85B_2 &= 38625 \\ 85B_1 + 85B_2 &= -3187.5 \end{aligned} \quad 580$$

$$\underline{-945B_1 - 85B_2 = 38625} \quad 581$$

$$-860B_1 = 35437.5$$

$$\therefore \; B_1 = -41.2V \quad 582$$

$$B_1 + B_2 = -37.5$$

$$583$$

$$-41.2 + B_2; \; B_2 = -37.5 + 41.2 = 3.71V$$

$$V_{Lt} = 3.71\exp(-85t) - 41.2\exp(-945t) \quad 584$$

$$V_{Ct} = C_1\exp(-945t) + C_2\exp(-85t) \quad 585$$

$$\frac{dV_{Ct}}{dt} = -945C_1\exp(-945t) - 85C_2\exp(-85t) \quad 586$$

$$V_{Ct}(0) = -37.5 = C_1 + C_2 \quad 587$$

$$\frac{dV_{Ct}}{dt}\Big|_0 = 3.75 \times 10^4 = -945C_1 - 85C_2 \quad 588$$

$$\begin{aligned} C_1 + C_2 &= -37.5 \\ -945C_1 - 85C_2 &= 3.75 \times 10^4 \\ 85C_1 + 85C_2 &= -3187.5 \end{aligned} \quad 589$$

$$\underline{-945C_1 - 85C_2 = 37500} \quad 590$$

$$-860C_1 = 34312.5$$

$$\therefore \; C_1 = -39.9V \quad 591$$

$$C_1 + C_2 = -37.5 \quad 592$$

$$-39.9 + C_2 = -37.5\,;\; C_2 = 2.4V$$ 593

$$V_{Ct} = 2.4\exp(-85t) - 39.9\exp(-945t)$$ 594

$$i = i_{ss} + i_t$$

595

$$= 12.5 - 0.48\exp(-85t) + 7.98\exp(-945t)$$

$$V_{Lt} = 3.71\exp(-85t) - 41.2\exp(-945t)$$ 596

$$V_{Ct} = 37.5 - 39.9\exp(-945t) + 2.4\exp(-85t)$$ 597

Example 9: *In the circuit of fig.Q9 the switch is closed suddenly. The parameters of the circuits are as follows: E = 80V, $r_1 = r_2 = 40\Omega$; L = 0.05H; C= 5μF. Find the initial values: $i_{1t}(0)$; $i_{2t}(0)$; $i_{3t}(0)$; $i_{4t}(0)$; $V_{ct}(0)$, $V_{Lt}(0)$, $di_{1t}/dt\,/_0$, $di_{2t}/dt\,/_0$, $di_{3t}/dt\,/_0$, $di_{4t}/dt\,/_0$, $dV_{ct}/dt\,/_0$, $dV_{Lt}/dt\,/_0$.*

Solution: We should obtain the voltage across the capacitor before commutation and the current in the inductor before commutation.

$i_3(0) = 0$; $V_c(0) = i_1(0)\,(r_1 + r_2) = i_2(0)(r_1 + r_2)$

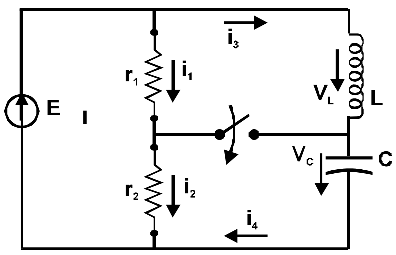


Fig. Q9

$$i_1(0) = i_2(0) = \frac{E}{r_1 + r_2} = 1\,A$$ 598

$V_c(0) = 1(40 + 40) = 80V$

The working circuit diagram is given below

$$i_{3ss} = i_{2ss} = \frac{E}{r_2} = \frac{80}{40}$$ 599

$i_{2ss} = 2A$; $i_{3ss} = 2A = 2A$

$i_3 = i_{3ss} + i_{3t}$; $i_3(0) = i_{3ss}(0) + i_{3t}(0)$

$= i_{3t}(0) = i_3(0) - i_{3ss}(0) = 0 - 2 = -2A$

$V_{css} + i_{2ss}.r_2 = 2 \times 40 = 80V$

$V_c = V_{css} + V_{ct}$; $V_c(0) = V_{css}(0) + V_{ct}(0)$

$V_{ct}(0) = V_c(0) - V_{css}(0) = 80 - 80 = 0V$;

We write Kirchhoff's laws for the circuit after commutation for the transient components

$i_{1t} + i_{3t} - i_{2t} - i_{4t} = 0$

$Ldi_{3t}/dt - r_1 i_{1t} = 0$

$V_{ct} - r_2 i_{2t} = 0$

$r_1 i_{1t} + i_{2t} r_2 = 0$

$i_{4t} = CdV_{ct}/dt$

From the above equations, we can obtain i_{2t} from the following expression:

$$V_{ct}(0) = r_2 i_{2t}(0); \qquad i_{2t}(0) = \frac{V_{ct}(0)}{r_2} \quad 600$$

$i_{2t}(0) = 0A$

$$r_1 i_{1t}(0) + i_{3t}(0) - i_{2t}(0) = \frac{r_2 i_2 t(0)}{r_1} = 0 \quad 601$$

$i_{1t}(0) = 0A$

$i_{1t}(0) + i_{3t}(0) - i_{2t}(0) - i_{4t}(0) = 0$

$0 + (-2) - 0 - i_{4t}(0) = 0$

$i_{4t}(0) = -2A$

To obtain the initial values of the first derivatives we differentiate the equations written according to Kirchhoff.

$di_{1t}/dt + di_{3t}/dt - di_{2t}/dt - di_{4t}/dt = 0$

$Ld^2 i_{3t}/dt^2 - r_1 di_{1t}/dt = 0$

$V_{ct} - r_2 i_{2t} = 0$

$r_1 i_{1t} + i_{2t}.r_2 = 0$; $dV_{ct}/dt - r_2 di_{2t}/dt = 0$

$i_{4t} = CdV_{ct}/dt$; $r_1 di_{1t}/dt + r_2 di_{2t}/dt = 0$

$V_{Lt}(0) = Ldi_{3t}/dt|_0 = r_1 i_{1t} = 40 \times 0$; $V_{Lt}(0) = 0V$

$dV_{ct}/dt|_0 = i_{4t}(0)/C - -2/5 \times 10^{-6} = -4 \times 10^{-5}$ V/S

$dV_{ct}/dt|_0 = -4 \times 10^{-5}$ V/S

$$\frac{di_{2t}}{dt}\Big|_0 = \frac{1}{r_2}\frac{dV_{ct}}{dt}\Big|_0 = \frac{-4x10^5}{40} = -10^4 \; A/S \qquad 602$$

$di_{2t}/dt|_0 = -10^4$ A/S; $rdi_{1t}/dt|_0 + r_2 di_{2t}/dt|_0 = 0$

$$\frac{di_{1t}}{dt}\Big|_0 = \frac{r_2}{r_1}\frac{di_{2t}}{dt} = \frac{-40}{40}.(-10^4) = 10^4 \;\; A/S \qquad 603$$

$di_{1t}/dt|_0 = -10^4$ A/S; $di_{3t}/dt = \dfrac{r_1 i_{1t}(0)}{L} = 0$ 604

$di_{3t}/dt|_0 = 0$ A/S;

$di_{1t}/dt|_0 + di_{3t}/dt|_0 - di_{2t}/dt|_0 - di_{4t}/dt = 0$

$10^4 + 0 - (10^4) - di_{4t}/dt|_0 = 0$;

$dV_{Lt}/dt = Ld^2 i_{3t}/dt^2|_0 = r_1 di_{1t}/dt$

$di_{4t}/dt|_0 = 2 \times 10^4$ A/S; $= 40(10^4) = 4 \times 10^5$

$dV_{Lt}/dt/o = 4 \times 10^5$ V/S

SUMMARY

TRANSIENT PROCESS IN LINEAR CIRCUITS WITH CONCENTRATED PARAMETERS

For the analysis of transient processes in electric circuits by classical method we can recommend, for example, the following order of computation of unknown quantities.

(1) From the calculation of the circuit before commutation, we find the current in the inductors $i_L(-)$ and the voltage across the capacitor $V_c(0+)$ at the moment of commutation t = 0; i.e. the independent initial conditions.

(2) Obtain a differential equation of the circuit after commutation and present the unknowns quantities as a sum of the steady state and transient components.

(3) Compute the steady state regime in the circuit after commutation using the basic laws and the network theorems.

(4) Write the characteristic equation and determine its roots.

(5) Depending on the form of the roots of characteristics equation write the solutions for transient components.

(6) Write a system of equations for the definition of constants of integration.

(7) Find the initial values of the transient currents in the inductors $i_t(0+)$ and the transient voltage across the capacitor $V_{ct}(0+)$ after commutation.

(8) Write Kirchhoff's equations for the transient components.

(9) Define the initial conditions for the transient components of the unknown quantities

(10) Calculate the constants of integration.

(11) The unknown quantities is written as the sum of the steady state and transient components.

(1) DERIVATION OF THE CHARACTERISTIC EQUATION:

For the circuit after commutation write a system of equations according to Kirchhoff's for the instantaneous values. Solve the system of equations according to Kirchhoff's in terms of the unknown quantities (or any other unknown variable), obtain the differential equation in terms of these quantities. The characteristic equation is obtained after substituting the symbol of differentiation d/dt by symbol S in the corresponding homogeneous differential equation and then equate the obtained expression to zero.

(2) In the circuit after commutation, open - circuit the branch with unknown variable (or any other branch). Find the input impedance of the circuit $Z(j\omega)$ looking from the open - circuit ends. The characteristic equation is obtained after substituting the symbol $j\omega$ with the operator S and then equating the input impedance to zero; $Z(s) = 0$

(3) For the circuit after commutation write the complex equations using the mesh current method (for sinusoidal current). Symbol $j\omega$ is substituted by the S operator and write the main determinant for the system of equations, which is equated to zero in order to obtain the characteristic equation.

SOLUTION FOR THE TRANSIENT COMPONENTS: For example, current, is written in different forms depending on the nature of the roots of characteristic equation.

If the roots S_1, S_2, ... are real and different, then

$$i_t = A_1e^{st} + A_2te^{st} + A_3t^2e^{st} + ...$$

For each pair of complex conjugate roots $S_1 = \alpha + j\omega_t$ 605 and $P_2^* = \alpha - j\omega$ 606, the transient component will take the following form.

$$i_t = A_1e^{\alpha t}\sin\omega_t t + A_2e^{\alpha t}\cos\omega_t t = Be^{\alpha t}\sin(\omega_t t + \beta)$$

The initial values (when t = 0+) of the transient components of current in the inductor and voltage across the capacitor after commutation (independent initial conditions) are determined from the expression:

$$i_{Lt}(0+) = i_L(0+) - i_{Lss}(0+)$$

where $i_{Lt}(0+)$ and $i_{Lss}(0+)$ are the transient and steady state components of current $i_L(0+)$.

$V_{ct}(0+) = V_c(0+) - V_{css}(0+)$

In the above expression the initial values of current through the inductor and voltage across the capacitor are determined using the laws of commutation.

$i_L(0+) = i_L(0-) = i_L(0)$

$V_c(0+) = V_c(0-) = V_c(0)$

In the rest unknowns the initial values are dependent and are determined from Kirchhoff equations at time t = 0+. During analysis of transient processes in electric circuits using the Laplace transform method, the following order of computation is recommended:

(1) From calculations of the circuit before commutation, we find the currents in the inductors $i_L(0-)$ and voltage across the capacitor $V_c(0-)$.

(2) By the nature of the circuit by being investigated, after commutation derive the equivalent circuit with Laplace variable if necessary, derive this circuit for the transient components with this working diagram using known methods for circuit analysis find the Laplace transform of the unknown variables.
The transform of the unknowns can be obtained otherwise. For the circuit after commutation write a system of equations according to Kirchhoff for the instantaneous values, then present the Laplace transform of all these variables and solve this system of equations in terms of the unknown variables.

(3) Determine the original function in the time domain using the inverse transform methods. If the given function is a real variable f(t) then the corresponding function in the complex S - plane is obtained using the direct Laplace transform method

$$F(s) = \int_0^{\infty} f(t) e^{-st}\, dt \quad 607$$

The equivalent operator diagrams of circuit elements are shown in fig. 4.9. The operator diagram for inductor contains operator reactance SL and an e.m.f source Li(0), having the same direction as the current *i*, while operator diagram for the capacitor contains the reactance 1/sc and an emf source $V_C(0)/s$, which has a direction opposite that of the voltage across the capacitor.

THE INVERSE TRANSFORM METHODS

(1) APPLICATION OF EXPANSION THEOREM: If the transform of the variables has the form of rational fractional polynomial.

$$\frac{F_1(s)}{F_2(s)} = \frac{a_m S^m + a_{m-1} S^{m-1} + ... + a_k S^k + ... + a_1 S + a_0}{b_n S^n + b_{n-1} S^{n-1} + ... + b_k S^k + ... b_1 S + b_0} \quad 608$$

$$= \frac{a_m S^m + a_{m-1} S^{m-1} + ... + a_k S^k + ... + a_1 S + a_0}{b_n(S - S_1)(S - S_2)....(S - S_k)....(S - S_n)}$$

where a_k and b_k are real numbers; s_1, s_2 - are real and different roots of the characteristic equation $F_2(s) = 0$, then

$$\frac{F_1(s)}{F_2(s)} \; implies \; f(t) = \sum_{k=1}^{n} \frac{F_1(s_k)}{F_2^1(s_k)} e^{s_k t} \quad 609$$

If the polynomial $F_2(s)$ has one zero root, i,e $F_2(s) = SF_3(s)$, then

$$\frac{F_1(s)}{SF_3(s)} \; implies \; f(t) = \frac{F_1(0)}{F_3(0)} + \sum_{k=1}^{n} \frac{F_1(s_k)}{S_k F_3^1(s_k)} e^{s_k t} \quad 610$$

If the polynomial $F_2(s)$ has n pairs of complex and conjugates roots, then

$$\frac{F_1(s)}{F_2(s)} - f(t) = \sum_{k=1}^{n} 2Re \frac{F_1(s_k)}{F_2^1(s_k)} e^{s_k t} \quad 611$$

and if there is one zero root.

$$\frac{F_1(s)}{SF_3(s)} - f(t) = \frac{F_1(0)}{F_3(0)} + \sum_{k=1}^{n} 2Re \frac{F_1(s_k)}{S_k F_3^1(s_k)} e^{s_k t} \quad 612$$

Examples of application of expansion methods are shown in ***Volume 2***.

If the roots S_k repeats in times, then from theory of function of complex variables it is known, that if the function $\frac{F_1(s)}{F_2(s)}e^{st}$ 613 has at the point S_K a range of the order of m, the residue of the function at this point is equal to:

$$Res\ \frac{F_1(s)}{F_2(s)}e^{st} = \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dS^{m-1}}\ \frac{F_1(s)}{F_2(s)}(S - S_k)^m e^{st}\right]_{s=sk}$$

614

$$Res\ \frac{F_1(s)}{F_2(s)}e^{st} = \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{ds^m - 1}\ \frac{F_1(s)}{F_2(s)}\ (S - S_k)^m e^{st}\right]_{s=sk}$$

The whole operation is carried out after cancelling out by the multiplier $(S - S_k)^m$.

Let us represent $A(s) = \frac{F_1(s)}{F_2(s)}\ (S - S_k)^m$, 615

we differentiate $A(s)e^{st}$ S_{m-1} times:

$$\frac{d}{ds}A(s)\,e^{st} = e^{st}\left[tA(s) + A'(s)\right]\ ;\ 616$$

$$\frac{d^2}{ds^2}A(s)e^{st} = e^{st}\left[t^2 A(s) + 2t'A(s) + A''(s)\right]\ ;\ 617$$

....;

$$\frac{d^{m-1}}{ds^{m-1}} A(s) e^{st} = e^{st}\left[t^{m-1} A(s) + (m-1)t^{m-2} A'(s)\right]$$

618

$$e^{st}\left[+\frac{(m-1)(m-2)}{2!} t^{m-3} A''(s) + \dots A^{(m-1)}(s)\right]$$

Hence

$$REs\ \frac{F_2(s)}{F_2(s)} e^{st} = e^{s_k t}\left[\frac{t^{m-1} A(s)}{(m-1)!} + \frac{t^{m-2} A'(s)}{m-2)!} + \frac{tm-3A''(s)}{(m-3)!2!} + \dots + \frac{A(m-1)(s)}{(m-1)!}\right]_{s=s_k}$$

$$= e^{s_k t} \sum_{i=1}^{m} \frac{tm-i A^{(i-1)}(s_k)}{(m-i)!\ (i-1)!}$$

619

The above equation is universal, the equation is defined for any roots. Here is assumed, that 0! = 1. Therefore if in equation (4) we take m = 1, then we obtain the following expression

$$= \frac{e^{s_k t} \bullet t^o A^o (s_K)}{0!\ 0!} = \frac{F_1(s_K) e^{s_k t}}{\left[\frac{F_2(s)}{S-Sk}\right]_{s=s_k}} \quad 620$$

CHAPTER FIVE

APPLICATION OF LAPLACE TRANSFORM METHOD TO THE ANALYSIS OF TRANSIENT PROCESSES

5.1 LAPLACE TRANSFORMATION:

The ability to obtain linear approximations of physical systems allows the analyst to consider the use of the Lap-lace transformation. The Lap-lace transform method substitutes the relatively easily solved algebraic equations for the more difficult differential equations. The time response solution is obtained by the following operations:

(1) Obtain the differential equations;

(2) Obtain the lap-lace transformation of the differential equation;

(3) Solve the resulting algebraic transform of the variable of interest. These notes are introductory and are intended only to illustrate some of the methods of the Lap-lace Transformation (LT). We shall omit many fundamental points associated with the mathematics.

The Laplace Transform exists for linear differential equations for which the transformation integral converges. Therefore, in order that f(t) be transformable, it is sufficient that

$$\int_{o}^{\infty} / f(t) / e^{\delta_1 \tau} \; dt < \infty$$

For some real positive δ_1. If the magnitude of f (t) is /f(t)/ < Me^{at} for all possible t, the integral will converge for $\delta_1 >$ a. The region of convergence is therefore given by : > δ_1 > a, and δ_1 is known as the abscissa of absolute convergence. Signals that are physically possible always have a Laplace transform. The Laplace transformation for a function of time f (t) is

$$F(S) = \int_{o}^{\infty} f(t) e^{-st} dt = L\{f(t)\} \text{.........................} 5.1$$

The inverse Laplace transform is written as

$$f(t) = \tfrac{1}{2\pi j} \int_{\delta - j\infty}^{\delta + j\infty} f(s) e^{+st} ds \text{.........................} 5.2$$

The transformation integrals have been use to derive tables of Laplace transform which are ordinarily used for the great majority of problems.

The Lap-lace variable S can be consider to be the differential operator so that

S = d/dt 5.3

Then we also have the integral operator.

$$\tfrac{1}{S} = \int_{o+}^{t} dt \text{...} 5.4$$

The inverse Lap-lace transformation is usually obtained by using the Heaviside Partial fraction expansion. Sometimes the function f (t) is determined otherwise

$$\varphi(s) = s\int_o^\infty e^{-st} f(t)dt$$

This expression is called linear transform of Heaviside.

It is clear that:

$$L\{f(t)\} = F(s)$$
$$L\{f'(t)\} = SF(s) - f(0+);$$
$$L\{f''(t)\} = S^2\left[F(s) - \frac{f(0+)}{S} - \frac{f'^{(0+)}}{S^2}\right]$$

We should note here that, if f (t) and its derivatives are not continuous and changes instantaneously when t = 0, then we should include the values with the plus sign to the right as in the equation above.

If the initial values of the function and its differentials is equal to zero, when t =0+, then the expression above will be written thus.

$$L\left\{f^1(t)\right\} = SF(s)$$
$$L\left\{f^{11}(t)\right\} = S^2F(s)$$

For integral equations, we have

$$L\left\{\int_o^t f(t)dt\right\} = \frac{F(s)}{S}$$
$$L\left\{\int_a^t f(t)dt\right\} = \frac{F(s)}{S} + \frac{1}{S}\int_a^{o+} dt, a < 0$$

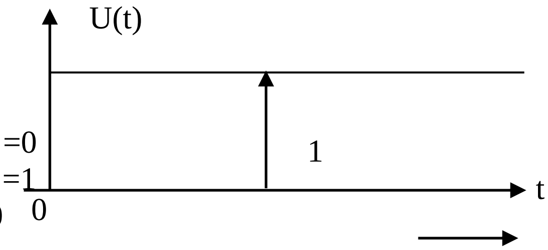
We should take note that the use of the Laplace transform for functions and their differentials and integral is not only to obtain the Laplace transformation, but also to obtain the inverse Laplace transform in order to obtain the original function f (t) in the time domain. For this purpose most of the known functions and their Lap-lace transformation have already been determined and are well documented in tables.

(i) Examples: f (t) = e^{-at} (a is a constant, a > 0)

$$F(s) = \int_o^\infty \{e^{-st}\ e^{-at}\}dt = \int_o^\infty e^{-(s+a)dt}$$
$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{s+a}$$

(ii) Laplace transform of unit step u(t)

$$F(s) = L\{u(t)\} = \int_o^\infty \{e^{-st}u(t)\}dt =$$

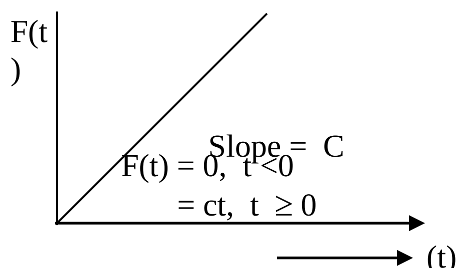


U(t) at t = 0 - =0
U(t) at t = 0 + =1
U(t) = 1, t > 0
= 0, t < 0

$$= \int_{o}^{\infty} e^{-st} dt = \left[\frac{e^{-st}}{S} \right]_{o}^{\infty} = \frac{1}{S}$$

$$if \quad f\ (t) = K.u(t);\ F\ (S) = \frac{K}{S}$$

(iii) Laplace Transform of a Ramp function
F(t) = ct



$$F(S) = \int_{o}^{\infty} \left\{ e^{-st} (ct) \right\} dt$$

$$= C \int_{o}^{\infty} e^{-st} t dt$$

$$= C \left\{ \frac{te^{-st}}{-S} \Big|_{o}^{\infty} - \int_{o}^{\infty} \frac{-e^{-st}}{S} dt \right\}$$

$$= C \left\{ [0] + \frac{1}{S} \int_{o}^{\infty} e^{-st\,dt} \right\} = \frac{C}{S^2}$$

(iv) Laplace Transform of Harmonic function
F (t) = cosωt.

$$F(S)=\int_o^\infty e^{-st}\cos\omega t\,dt=\left[\cos\omega t\frac{e^{-st}}{S}\right]_o^\infty-\int_o^\infty\left[\frac{e^{-st}}{S}\omega\sin\omega t\right]dt$$

$$=\frac{1}{S}-\left\{\left[\omega\sin\omega t\frac{e^{-st}}{-S^2}\right]_o^\infty-\int_o^\infty\left\{\frac{e^{-st}}{S^2}\omega^2\cos\omega t\right\}dt\right\}$$

$$=\frac{1}{S}-\left\{[0]+\left(\int_o^\infty e^{-st}\cos\omega t dt\right)\frac{\omega^2}{S^2}\right\}$$

$$=$$

$$\frac{1}{S}-\frac{\omega^2}{S^2}F(S)$$

$$F(S)\left(1+\frac{\omega^2}{S^2}=\frac{1}{S}\right)$$

$$F(s)=\frac{1}{s}\frac{(S^2)}{S^2+\omega^2}=\frac{S}{S+\omega^2}$$

(b) F (t) = sinωt

In a similar way

$$L\{\sin\omega t\}=\frac{\omega}{S^2+\omega^2}$$

Another way is to use cosωt = $\frac{e^{j\omega t}+e^{-j\omega t}}{2}$

$$F(S)=\frac{1}{2}\left[\frac{1}{S-j\omega}+\frac{1}{S+j\omega}\right]$$

$$=\frac{S}{S^2+\omega^2}$$

Transforms of Operators

(1) Differentiation of f(t)

$$L\left\{\frac{df(t)}{dt}\right\}=\int_o^\infty \left\{\frac{dt}{dt}e^{-st}\right\}dt$$

$$=\left[e^{-st}f(t)\right]_o^\infty-\int_o^\infty -Se^{st}f(t)dt$$

$$=S\int_o^\infty \left\{e^{-st}f(t)\right\}dt-f(0)$$

$$=SF(S)-F(0)$$

So the equivalent of differentiating a function in the time – domain is multiplying that function's Laplace transform by S in the S – domain (assuming zero initial condition).

$$L\left\{\frac{d^2f}{dt^2}\right\}=\int_o^\infty \left\{\frac{d^2f}{dt^2}e^{-st}\right\}dt$$

$$=\left[e^{-st}\frac{df}{dt}\right]_o^\infty-\int_o^\infty \left\{-Se^{-st}\frac{df}{dt}\right\}dt$$

$$=S\int_o^\infty \frac{dt}{dt}e^{-st}dt-\left[\frac{df}{dt}\right]_{t=0})$$

$$(f'=\frac{df}{dt};\ f''=\frac{d^2f}{dt^{2'}}\ \frac{d^{nf}}{dt^n}=f^{(n)}$$

$$=S[SF(S)-f(0)]-f'(0)$$

$$=S^2F(S)-Sf(0)-f'(0)$$

Generally we state that

If f (t) = F(s)

$$L\left\{\frac{d^nf}{dt^n}\right\}=S^nF(S)-S^{n-1}f(0)-S^{n-2}F^1(0)-SF^{n-2}(0)\ -F^{n-1}(0)$$

With zero initial conditions

$$L\left\{\frac{d^nf}{dt^n}\right\}=S^nF(S)$$

Sometimes the transform is presented in the form of a rational function

$$\frac{f_1(S)}{F_2(S)}=\frac{b_oS^m+b_1S^{m-1}+\ldots+b_m}{a_oS^n+a_1S^{n-1}+\ldots+a_m}$$

Where m < n, also $\frac{F_1(S)}{F_2(S)} = F(S)$ is non divisible, i.e. $\frac{F_1(S)}{F_2(S)}$ does not have a general root, and

a_k, b_k – are real numbers.

The original function f (t) using inverse Laplace transform is given by the equation.

$$L^{-1}\left\{\frac{F_1(S)}{F_2(S)}\right\} = f(t) = \sum_{k=1}^{n} \frac{F_1(S_k)}{F_2(S_k)} e^{s_k t};$$

This represent the sum of the function $F(s)e^{st}$, S_K – are the simple roots of the characteristic equation. $F_2(s) = 0$, where one of the roots could be equal to zero.

If the characteristic equation has a zero root, then the inverse Lap-lace transform of the polynomial is presented in the form below.

$$L^{-1}\left\{\frac{F_1(S)}{F_2(S)}\right\} = f(t) = \frac{F_1(0)}{F_2(0)} + \sum_{k=1}^{n} \frac{F_1(S_k)}{S_k F_3^{'}(S_k)} e^{s_k t};$$

Where $F_2(s) = Sf_3(s)$

If the equation $F_2(s) = 0$ has n pairs of complex conjugate roots, then the inverse Laplace transform of the polynomial is given as follows.

$$L^{-1}\left\{\frac{F_1(S)}{F_2(S)}\right\} = 2\sum_{k=1}^{n} \mathrm{Re}\frac{F_1(S_k)}{F_2^1(S_k)} e^{skt};$$

If the roots of the characteristic equation $F_2(s) = 0$ has a zero root and a pair of complex conjugate roots, then the inverse Laplace transform is presented as follows.

$$L^{-1}\left\{\frac{F_1(S)}{SF_3(S)}\right\} = \frac{F_1(0)}{F_3(0)} + 2\sum_{k=1}^{n} \mathrm{Re}\frac{F_1(S_k)}{S_k F_3^1(S_k)}$$

Where $F_2(s) = SF_3(s)$.

5.2 OHM'S AND KIRCHHOFF'S LAWS IN LAPLACE TRANSFORM FORM.

Let us investigate an RLC circuit that is connected to emf source $e_1(t)$ and at time $t = 0$, the switch is transferred from 1 to 2 to an emf source e(t) (Fig 5.1)

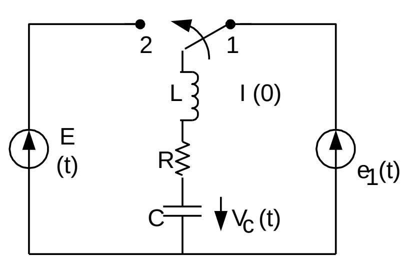


Fig. 5.1. An RLC Circuit

Ohm's law for the instantaneous value after commutation is written thus

$$Ri + L\,di/dt + \frac{1}{c}\int_{-\infty}^{\infty} idt = e(t)$$

The lower limit of the integral is taken only if before commutation (t = 0) the regime in the circuit has attained a steady-state. If before commutation the regime has not attained a steady state, then the lower limit of the integral will be taken as $(-t_1)$, where t_1 is the time to the moment before commutation (t = 0).

At this time the circuit was connected to the source $e_1(t)$.

The purpose for using the lower limit of $-\infty$ or $-t_1$ is to show that before commutation the capacitor has already been charged through $e_1(t)$.

$$\frac{1}{C}\int_{-\infty}^{t} idt = \frac{1}{C}\int_{-\infty}^{o} idt + \frac{1}{C}\int_{o}^{t} idt$$

$$= V_c(0) + \frac{1}{C}\int_{o}^{t} idt$$

where Vo (0) - is the voltage across the capacitor at the moment of commutation, i.e t = 0

To convert from 0hm's law for instantaneous values to an expression in Laplace form we must multiply both sides of the equation by e^{-st} dt and then integrate from 0 to ∞. Thus we have

$$R\int_{o}^{\infty} e^{-st}dt + L\int_{o}^{\infty} e^{-st}\frac{di}{dt}dt + \frac{1}{C}\int_{o}^{\infty} e^{-st}\int_{-\infty}^{t} idtdt$$

$$= \int_{o}^{\infty} e^{-st}e(t)dt$$

$$Let\ \ I(S) = L\{i(t)\};\ E(S) = L\{e(t)\}$$

Using this in the above equation, we have

$$RI(S) + LSI(S) - Li(0) + \frac{V_c(0)}{S} + \frac{I(S)}{SC} = E(s).$$

From here we obtain

$$I(S) = \frac{E(S) + Li(0) - V_c(0)/S}{R + SL + 1/SC}$$

This same expression can be obtained even when the system has not attained a steady state condition. In general i (0) and $V_c(0)$ stand for the current and voltage in the inductor and capacitor respectively before commutation.

It must be noted that in accordance with the law of commutation we should write i (0+) and V_0 (0+). But since current in an inductor and voltage across a capacitor does not change instantaneously when t = 0, we will continue to write i(0) and $V_c(0)$. The expression at the denominator of the equation.

$$I(S) = \frac{E(S) + Li(0) - V_c(0)/S}{R + SL + 1/SC}$$

Is called the total impedance in Laplace form

Z(s) = R+SL + 1/SC. The reciprocal of the value Z (s) is called the admittance

Y(s) = 1/Z(s)

The Laplace transform of the emf standing at the numerator of the equation of current I (S) is made up of E(s) and two other constant values, which are determined by the initial conditions of the circuit.

$$I(S)=\frac{E(S)+Li(0)-V_c(0)/S}{R+SL+1/SC}$$

The other two terms of the numerator shows that current i (0) in the inductive element, and voltage in a capacitive element cannot change instantaneously. In other words, the presence of the two additional voltage sources Li (0) and $V_c(0)/s$, which we can call internal voltage sources, indicates that in the magnetic field of the inductor and in the electric field of the capacitor energy had been stored into it, before commutation took place. Positive direction of these voltages was taken to coincide with the direction of flow of the current in the particular branch of the circuit.

(a) Capacitor. (i) Impedance

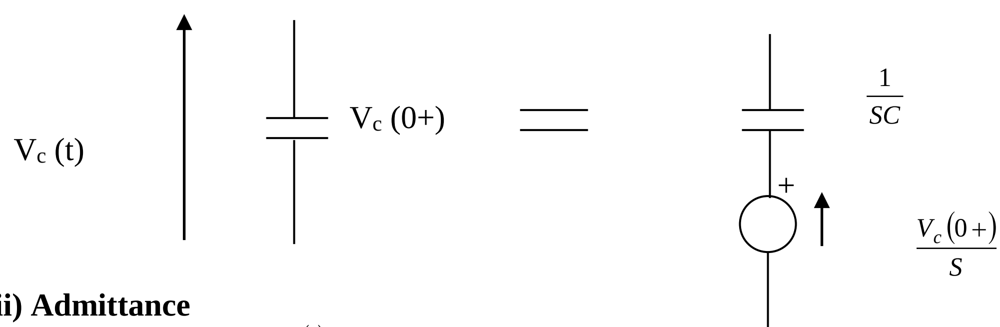
$$V(t)=\frac{1}{C}\int i(t)\,dt = V(S)=\frac{I(S)}{CS}+\frac{\int_{-\infty}^{o} idt}{CS}$$

$$=\frac{I(S)}{CS}+\frac{V_c(0+)}{S}$$

$$\left[V(S)-\frac{V_c(0+)}{CS}\right]=\frac{I(S)}{CS}$$

Hence we have $\frac{V_c(0+)}{S}$ as a voltage generator in opposition to V(S) and the

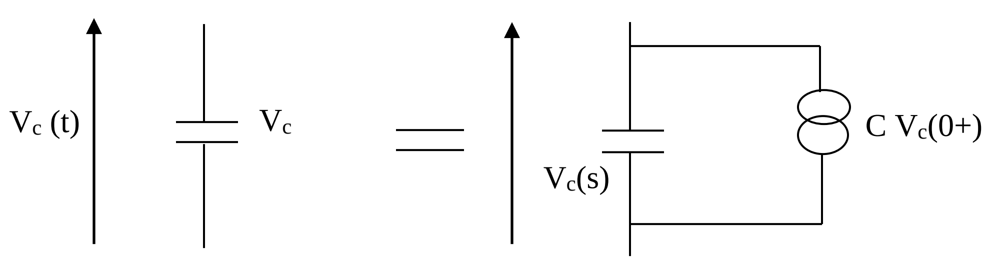
Impedance $\frac{1}{SC}$



(ii) Admittance

$$i(t)=C\frac{dV_c(t)}{dt}=I(S)=SCV_c(S)-CV_C(0+)$$

CV_c (0+) is a current source

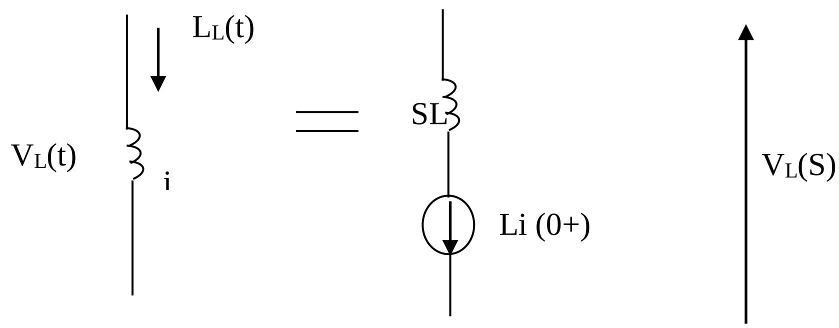


(b) Inductor

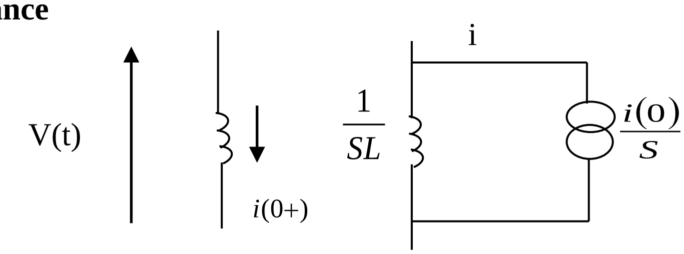
$$V_L(t) = L\frac{di_L}{dt} \Rightarrow V_L(S)$$

(i) Impedance

$$= SLI(S) - Li(0+)$$



(ii) Admittance



$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} Vdt \Rightarrow I_L(S) = \frac{V[S]}{SL} - + \frac{\int_{-\infty}^{o} +Vdt}{SL}$$

$$= \frac{V(S)}{SL} + \frac{i(0+}{S}$$

KICHHOFF'S LAW:- According to Kirchhoff's current law; for every node of a linear network:

$i_1 + i_2 + \ldots + i_n = 0$, therefore let

$I_k(S) = i_k(t)$, on this basis we obtain Kirchhoff's current law in operator form:

$$I_1(S) + I_2(S) + \ldots + I_n(S) = \sum_{k=1}^{n} I_k(S) = 0$$

For any closed loop containing n branches we can then write Kirchhoff's law in this form:

$$\sum_{k=1}^{n} I_k i_k + \sum_{k=1}^{n} L_k\frac{di_k}{dt} + \sum_{k=1}^{n} \frac{1}{C_k}\int_{-\infty}^{t} i_k dt = \sum_{k=1}^{n} e_k$$

Where $I_k(S) = L\{i_k(t)\}$; $E_k(S) = L\{e_k(t)\}$

And using the same arguments as we used in the case of ohm's law, we obtain Kirchhoff's voltage law in operator form;

$$\sum_{k=1}^{n} r_k I_k(S) + \sum_{k=1}^{n} L_k[SI_k(S) - i_k(0)] + \sum_{k=1}^{n} \frac{V_{ck}(0)}{S} + \sum_{k=1}^{n} \frac{I_k(S)}{C_k S} = \sum_{k=1}^{n} E_k(S)$$

we can rewrite it in the form below

$$\sum_{k-1}^{n} Z_k(S) I_k(S) = \sum_{k-1}^{n} \left[E_k(S) + L_k i_k(0) - \frac{V_{ck}(0)}{S} \right]$$

Where i_k (0) and V_{ck} (0) are the initial values of current and voltage in the inductor and across the capacitor respectively.

However, when $i_k(0) = 0$ and $V_{ck}(0) = 0$, the equation above becomes very simple.

$$\sum_{k=1}^{n} Z_k(S) I_k = \sum_{k=1}^{n} E_k(S)$$

We must take note that in every k branch of the circuit when $i_k(0) \neq 0$ and $V_{ck}(0) \neq 0$, exist an internal emf sources which is due to $L_k i_k(0)$; $-V_{ck}(0)/s$, apart from the external emf sources due to E_k (S). The positive direction of these internal sources is in the same direction of the current that flows in the branch of the circuit. Secondly, in terms of resistance Laplace transformation of the resistance.

EXAMPLE: Calculate the current i when the RLC circuit is disconnected from the voltage source $e_1(t)$ $E_{1m}\mathrm{Sin}(\omega t + \Psi)$ and then connected to voltage source whose emf e(t) = E

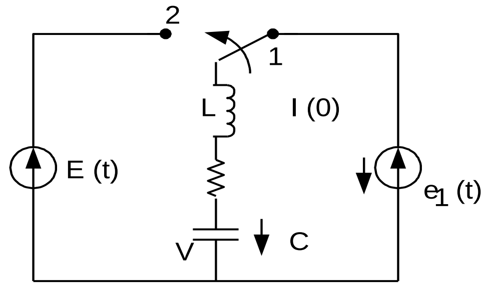


Fig. Q 1.

$$I(s) = \frac{E/S + Li(0) - V_c(0)/S}{r + SL + 1/sc}$$

$$\frac{F_1(S)}{F_2(S)} = \frac{C[E + SLi(0) - V_c(0)]}{1 + rcs + LCS^2}$$

Where i (0) and $V_c(0)$ are evaluated from the circuit in the steady state condition (regime) before commutation.

The original of i (s), e_1. E current i, will be determined using the method of partial fraction expansion. First we obtain the roots S_1 and S_2 from F_2 (s), which will contain the real and imaginary parts.

$$S_{1,2} = \frac{-r}{2L} + \sqrt{\frac{r^2}{4L^2} - \frac{1}{Lc}}$$

Then we find

$$F_1(S_1) = C[E + S_1 Li(0) - V_c(0)];$$
$$F_2(S_2) = C[E + S_2 Li(0) - V_c(0)];$$
$$F_2'(S_1) = (r + 2LS_1)C; F_2'(S_2) = (r + 2LS_2)C$$

Substituting these values in the equation

$$i = \sum_{k=1}^{n} \frac{F_1(S_k)}{F_2^1(S_K)}$$

$$e^{s1t} \frac{\lfloor E + S_1 Li(0) - V_c(0) \rfloor}{r + 2LS_1} - \frac{E + S_2 Li(0) - V_c(0)}{r + 2LS_2} e^{s_2 t}$$

After simple transformation, we obtain the current i

$$i = \frac{E - V_c(0)}{\sqrt{r^2 - 4L/C}} \left(e^{s_1 t} - e^{s_2 t}\right) + \frac{Li(0)}{\sqrt{R^2 - 4L/C}} \left(S_1 e^{s_1 t} - S_2 e^{s_2 t}\right)$$

Laplace transform of the emf $E_{1m}\mathrm{Sin}(\omega t + \psi)$ is given as

$$\frac{e_m s \, Sin\,\psi + \omega_o \cos\psi}{S^2 + \omega_o{}^2}$$

This is the complex. We know that this emf is the imaginary part of the complex function

$$e = \lfloor E_m e^{j(\omega_o t + \psi)} \rfloor$$

$$E(t) = E_m \left[\cos(\omega_o t + \psi) + j \sin(\omega_o t + \psi)\right] =$$

$$E_m e^{j(\omega_o t + \psi)} = \dot{E}_m e^{j\omega_o t}$$

5.3 THE INVERSE TRANSFORM METHODS

(1) APPLICATION OF EXPANSION THEOREM: If the transform of the variables has the form of rational fractional polynomial.

$$\frac{F_1(S)}{F_2(S)} = \frac{a_m S^m + a_{m-1} S^{m-1} + ... + a_k S^k + ... + a_1 S + a_o}{b_n S^n + b_{n-1} S^{n-1} + ... + b_k S^k + ... b_1 S + b_o}$$

$$= \frac{a_m S^m + a_{m-1} S^{m-1} + ... + a_k S^k + ... + a_1 S + a_o}{b_n (S - S_1)(S - S_2) .... (S - S_k) .... (S - S_n)}$$

where a_k and b_k are real numbers; s_1, s_2 - are real and different roots of the characteristic equation $F_2(s) = 0$, then the inverse Laplace transform is given as

$$L^{-1}\left\{ \frac{F_1(S)}{F_2(S)} \right\} = f(t) = \sum_{k=1}^{n} \frac{F_1(S_k) e^{s_k t}}{F_2'(S_k)}$$

If the polynomial $F_2(s)$ has one zero root, i.e $F_2(s) = SF_3(s)$, then

$$L^{-1}\left\{ \frac{F_1(S)}{SF_3(S)} \right\} = f(t) = \frac{F_1(0)}{F_3(0)} + \sum_{k=1}^{n} \frac{F_1(S_k) e^{skt}}{S_k F_3'(S_k)}$$

Examples of application of expansion methods are shown in 1, 2, 3.

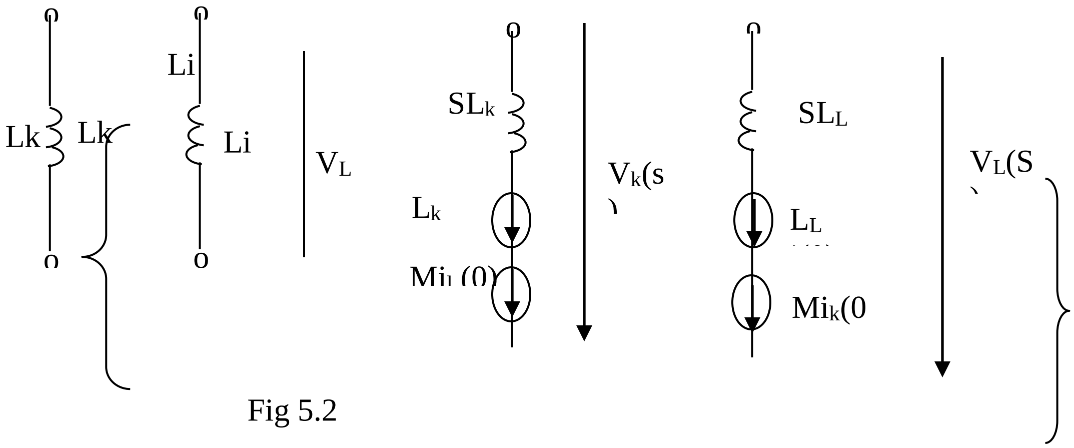


Fig 5.2

Fig. 5.2: Circuit elements and their equivalent Laplace transform.

If the roots S_k repeats n times, then from theory of function of complex variables it is known, that if the function $\frac{F_1(S)}{F_1(S)}e^{st}$ has at the point S_k a range of m, the residue of the function at this point is equal to:

Hence

$$RES\ \frac{F_2(S)}{F_2(S)}\quad e^{st} = e^{skt}\left[\frac{t^{m-1}A(S)}{(m-1)!}\right]+\frac{t^{m-2}A^1(S)}{(m-2)!}+\frac{tm-3A^{11}(S)}{(m-3)!\,2!}+..+\frac{A(m-1)}{(m-1)}$$

$$=e^{skt}\left\{\sum_{i-1}^{m}\frac{t^{m-1}iA^{(i-1)}(S_k)}{(m-i)!\,(i-1)!}\right\}$$

The above equation is universal, the equation is defined for any roots. Here is assumed, that 0! = 1. Therefore if in equation (4) we take m = 1, then we obtain the following expression

$$=\frac{e^{skt.}\,t^{\,o}A^{o}\left(S_k\right)}{0!\,0!}=\frac{F_1\left(S_k\right)e^{skt}}{\left[\dfrac{F_2(S)}{S-S_K}\right]S=S_K}$$

5.4 WORKED EXAMPLES: For the worked example below; these symbols must be noted: V_c – voltage across the capacitor; V_{ct} (0) – is the transient component of the voltage across the capacitor at t = 0; V_{css} – is the steady state component of the same voltage; i_L – current flowing through the inductor; i_{Lt} – is the transient component of the voltage; i_{Lss} – is the steady state component.

EXAMPLE 1: The branch with r_1L is connected in parallel with the branch r_2C (fig. Q1). Determine the dependence of i_1 (t) and V_c if the parameters of the branches are as follows

r_1 = 300Ω

r_2 = 200Ω

L = 1.25H

C = 10μF

J = 0.125A.

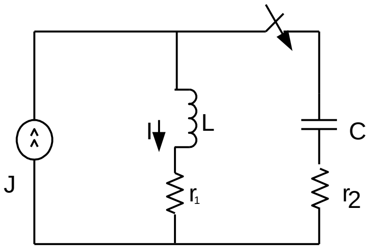


Fig. Q1

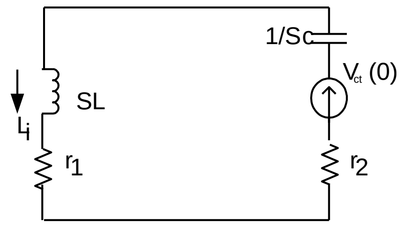


Fig Q 4b Working diagram

SOLUTION

Initial condition based on the laws of commutation. $i_1(0) = 0.125A$; $Vo(0-) = 0$

The steady state and transient values at t = 0

$i_{1ss} = 0.125A$; $V_{css} = J.\, r_1 = 0.125 \ x \ 300 = 37.5V$

$i_1 = i_{1ss} + i_{1t}$; $i_1(0)$ $i_{1ss}(0) + i_{1t}(0)$; $i_{1t}(0) = 0$

$V_c(0) = V_{css}(0) + V_{ct}(0)$; $V_{ct}(0) = -37.5V$

The transient current

$$i_{it} s) = \frac{V_{ct}(0)}{s(\frac{r_1 + r_2 + sL + 1}{Sc})}$$

$$= \frac{cv_{ct}(0)}{(r_1 + r_2)sc + Lcs^2 + 1)}$$

$$\frac{10x10^{-6}(-37.5)}{1.25x10^{-5}s^2 + 5x10^{-3}s + 1} = \frac{F_1(s)}{F_2(s)}$$

$$F_2(S) = 1.25x10^{-5}\, S^2 + 5x10^{-3}\, S + 1 = 0$$

$$S_{1,2} = -200 \pm j200$$

$$i_{1t}(t) = 2\operatorname{Re}\frac{F_1(S_1)}{F_2^1(S_1)}e^{s_1 t} = 2\operatorname{Re}\frac{10^{-5}(-37.5)}{2.5x10^{-5}(-200 + j200) + 5x10^{-3}}$$

$$V_{ct}(S) = -i_{1t}(S)\ x\frac{1}{SC} = \frac{-(10^{-5})(-37.5)}{1.25x10^{-5}\, S^2 + 400S + 1}x\frac{1}{10^{-5}}S$$

$$= \frac{37.5x10^{-5}}{(1.25x10^{-5}\ S^2 + 5x10^{-3}S + S + 1)}x\frac{1}{10^{-5}}.S$$

$$= \frac{37.5}{S(1.25x10^{-5}\, S^2 + 5x10^{-3}\ S + 1)}$$

$$V_{ct} = \frac{F_1(0)}{F_3(0)} + 2\operatorname{Re}\frac{F_1(S_1)}{S_1F_3(S_1)}e^{s_1 t}$$

$$= 37.5 + 53e^{-200t}\sin(200t - 135^o)V$$

EXAMPLE 2: A coil inductance L = 0.05H and resistance r_2 = 30Ω is connected to constant voltage source of emf E= 100V. Calculate the current i; i_2 and voltage Vc across the capacitor connected in parallel whose capacitance C = 50μF.

E = 100V; r = 20Ω; r_2 = 30Ω; L = 0.05H; C = 50μF

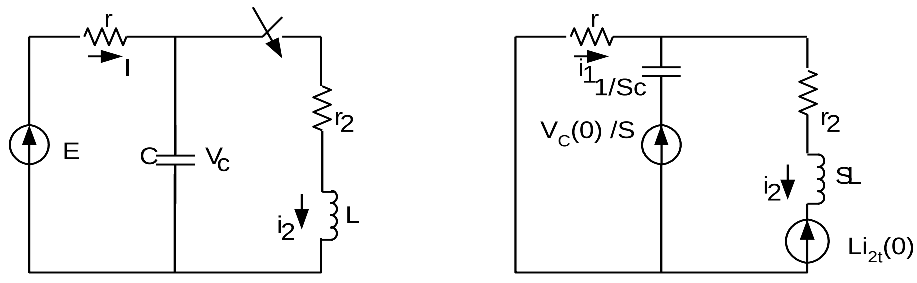


Fig. Q2 Fig. Q2b Working diagram

Initial condition based on the laws of commutation.

Before commutation no current flows through the inductor since it is an open circuit. The voltage across the capacitor before commutation is equal to the applied voltage of 100v

i(0-) = 0

$V_c(0-)$ = E = 100V; $i_2(0-)$ = 0

The steady state and transient values at t = 0After commutation the steady state current is = $\frac{E}{r+r_2}$

$$i_{ss} = i_{2ss} = \frac{E}{r+r_2} = 2A$$

the steady state voltage across the capacitor is equal to the voltage drop across r_2.

$$V_{css} = \frac{E.r_2}{r+r_2}.r_2 = 60V$$

$i_2(0) = i_{2ss}(0) + i_{2t}(0)$; $i_{2t}(0)$ = -2A

$V_c(0) = V_{css}(0) + V_{ct}(0)$; $V_{ct}(0)$ = 40V

We have two voltage sources in the working diagram. One is due to the transient current i2t(0) and the other due to the transient voltage $V_{ct}(0)$. We can apply superposition theorem to calculate the transient components of the unknown current i_1; i_2 and the voltage V_{ct}. We first “kill” the voltage source Li (0) and calculate the current i_2t due to $V_c(0)$

$$i_{2t}^{1}(S)=\frac{V_{ct}(0)}{S\left(\frac{1}{SC}+\frac{r(r_2+SL)}{r+r_2+SL}\right.}$$

Then return Li(0) and calculated i_{2t}

$$i''_{2t}=\frac{Li_{2t}(0)}{\frac{r_2+SL+r.\frac{1}{SC}}{r+\frac{1}{SC}}}\quad =\frac{Li_{2t}(0)(rsc+1)}{rLCS^2+(rr_2\ c\ +L)S+r+r_2}$$

i2t is the sum of the two current i'_{2t} and i''_{2t}

$$i_{2t}(S)=i_{2t}^{'}(S)+i_{2t}^{''}(S)\ =\frac{CVct(0).r+Li_{2t}\ (0)\ (rsc+1)}{rLcS^2+(rr_2\ c+L)S+r+r_2}$$

$$=\frac{-(10^{-4}\,S+0.06)}{0.05x10^{-3}\,S^2+0.08S+50}=\frac{F_1(S)}{F_2(S)}=$$

$$\frac{-(10^{-4}\,S+0.06)}{0.05x10^{-3}\,S^2+0.08S+50}=\frac{F_1(S)}{F_2(S)}$$

$$F_2(S)=0.05x10^{-3}\,S^2+0.08S+50=0;S_{1,2}=-800\pm j600$$

$$i_{2t}(t)=2\,\mathrm{Re}\frac{F_1(S_1)e^{s_1 t}}{F_1^1(S_1)}=2.11e^{-800t}\sin(600t-71^o)$$

The final current i2(t) is given as the sum of the steady state and transcent components

$$i_2(t)=i_{2ss}(t)+i_{2t}=2.+2.11e^{-800t}\sin(600t-71^o)$$

Using superposition method we also determine it the transient component of the current i_t

$$i_t'(S) = \frac{-V_{ct}(0)}{S\left(\frac{1}{SC} + \frac{r(r_2+SL)}{r+r_2+SL}\right)} x \frac{r_2+SL}{r+r_2+SL} = \frac{-CV_c(0)(r_2+SL)}{rIcs^2+(rr_2c+L)s+r+r_2}$$

$$i_t''(S) = \frac{Li_{2t}(0)}{r.\dfrac{1}{r_2+SL+\dfrac{SC}{r+\dfrac{1}{SC}}}}$$

$$i_t(S) = i_t(S) + i_t''(S) = \frac{-CV_{ct}(0)(r_2+SL)+Li_{2t}(0)}{rLcs^2+(rr_2c+L)S+r+r_2}$$

$$= \frac{-10^{-4}S-0.16}{0.05x10^{-3}S^2+0.08S+50} = \frac{F_1(S)}{F_2(S)}$$

$$i_t(t) = 2\operatorname{Re}\frac{F_1(S_1)e^{s_1t}}{F_2^1(S_1)} = 3.33e-^{800t}\sin(600t-144^o)$$

$$i(t) = i_{ss} + i_t = 2 + 3.33e^{-800t}\sin(600t-144^o)$$

The transient component of the voltage across capacitor is given by the expression, using node voltage method

$$V_{ct}(S) = \frac{-Li_{2t}(0)x\dfrac{1}{r_2+SL} + \dfrac{V_{ct}(0)}{S}xSC}{\dfrac{1}{r} + \dfrac{1}{r_1+SL} + SC}$$

$$= \frac{[CV_{ct}(0)(r_2+SL) - Li_{2t}(0)]r}{rLcs^2 + rr_2SC + SL + r + r_2}$$

$$=\frac{\left[50\,X\,10^{-3}\;X\;40(30+0.5S)-0.05(-2)\right]X\,20}{0.05\,X\,10^{-3}\;S^{2}+0.08S+50}$$

$$=\frac{2\,X\,10^{-3}\;S+3.2}{0.05S^{2}\;0.08S+50}=\frac{F_1(S)}{F_2(S)}$$

$$S_{1,2}=-800+j600$$

$$V_{ct}(t)=2\,\mathrm{Re}\,\frac{\mathrm{F}_1(s)}{\mathrm{F}_2{}^{1}(s)}e^{s_1t}$$

$$=2\,\mathrm{Re}\,\frac{1.6+j1.2}{j0.06}e^{-800t}.e^{j600t}$$

$$V_{ct}(t)=66.67e^{-800t}\sin\left(600t+37^{o}\right)$$

$$V_{ct}(t)=V_{css}+V_{ct}(t)=60+66.67e^{-800t}\sin\left(600t+37^{o}\right)$$

EXAMPLE 3: Calculate the current i and voltage Vc in the circuit (fig. Q3), after commutation. The parameters of the circuit are as follows:
E = 150V; r = 50Ω; r_2 = 30Ω; L = 0.05H; C = 100μF

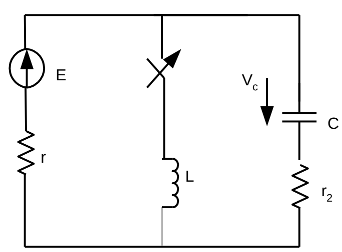


Fig. Q3.

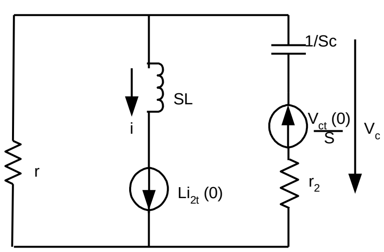


Fig. Q3b Working Diagram

Initial condition based on the laws of commutation.
i_L (0-) = 0
V_c (0-) = E = 150V

The steady state and transient values at t = 0

i_{ss} = i_{Lss} = E/r = 150/50 = 3A; V_{css} = 0V
$i_L(0)$ = $i_L(0)$ + i_{Lt} (0) $i_{Lt}(0)$ = 3A

V_c(0) = V_{css}(0); V_{ct} + (0) = <u>150V</u>

The transient current is computed using superposition method.

$$i_t^1(S)=\frac{Li_t(0)}{SL\frac{r\left(r_2+\frac{1}{SC}\right)}{r+r_2+\frac{1}{SC}}}x\frac{r_2+\frac{1}{SC}}{r+r_2+\frac{1}{SC}}$$

$$=\frac{Li_t(0)(r_2\,SC+1)}{(r+r^2)LCS^2+rr_2\,SC+SL+r}$$

$$\mathrm{i}_{\mathrm{t}}''=\frac{Vct(0)}{\frac{S(r_2+\frac{1}{SC}+r+SL}{r+S2}}.\frac{SL}{r+SL}$$

$$=\frac{-CV_{ct}(0)SL}{(r+r_2)LCS^2+rr_2\,SC+SL+r}$$

$$i_t=i_t^1+i_t^{"}=\frac{Li_t(0)(r_2\,SC+1)-CV_{ct}(0)SL}{(r+r_2)LCS^2+rr_2\,SC+SL+r}$$

$$=\frac{-1.2x10^{-3}S-0.15}{0.4x10^{-3}S+0.25+50}=\frac{F_1(S)}{F_2(S)}$$

$$S_1=-250\pm j250$$

$$i_t=2\,\mathrm{Re}\frac{F_1(S_1)\,e^{es_1t}}{f_2(S_1)}=2\,\mathrm{Re}\frac{0.15-j0.30}{j0.2}e^{-250t.}e^{j250t}$$

$$\mathrm{Re}\frac{2x0.335}{0.2}e^{-j90.}e^{63}.\!:e^{-250t.}e^{j250t.}$$

$$=3.5e^{-250t}Sin(250t-63^0)$$

$$i=i_{ss}+i_t=3+3.335e^{-250t}Sin(250t-63^0)$$

The voltage Vc(t) using superposition method

$$V_{ct}(0) = \frac{\dfrac{V_{ct}(0)}{S}}{\dfrac{1}{r} + \dfrac{1}{r_2 + \dfrac{1}{SC}} + \dfrac{1}{SC}} x \frac{1}{r_2 + \dfrac{1}{SC}} \left(-Li_t(0)\right) x \frac{1}{SL}$$

$$= \frac{\dfrac{V_{ct}(0)}{S} x \dfrac{SC}{r_2 SC+1} - Li_t(0) x \dfrac{1}{SL}}{\dfrac{1}{r} + \dfrac{SC}{r_2 SC+1} + \dfrac{1}{SL}}$$

$$= \frac{\dfrac{CV_{ct}(0).SL - Li_t(0)(r_2 SC+1)}{(r_2 SC+1)SL}}{\dfrac{SL(r2SC+1) + rLCS^2 + r_2 rSC + r}{r(r_2 SC+1)SL}}$$

$$= \frac{[CV_{ct}(0).SL - Li_t(0)(r_2 SC+)]r}{(r+r_2) + LCS^2 + rr_2 SC + SL + r}$$

$$= \frac{[10^{-4} x 150 x 0.05 S - -.05(-3)(30 x 10^{-4} S + 1)]}{0.4 x 10^{-3} S^2 + 0.2s + 50}$$

$$= \frac{[0.75 x 10^{-3} S + 0.15(30 x 10^{-4} S + 1)]50}{0.4 x 10^{-3} S^2 + 0.2s + 50}$$

$$= \frac{[0.75 x 10^{-3} S + 0.45(30 x 10^{-3} S + 1.15)] x 50}{0.4 x 10^{-3} S^2 + 0.2s + 50}$$

$$= \frac{[1.2 X 10^{-3} S + 0.15] X 50}{0.4 x 10^{-3} S^2 + 0.2s + 50}$$

$$= \frac{0.06S+7.5}{0.4X10^{-3}S^2+0.2s+50} = \frac{F_1(S)}{F_2(S)}$$

$$S_1, = -250 \pm j250$$

$$V_{ct}(t) = 2\operatorname{Re}\frac{(0.06(-250+j250)+7.5)}{0.8X10^{-3}[-250+j250]+0.2}e^{-250t}\cdot e^{j250t}$$

$$= 2\operatorname{Re}\frac{-15+j15\ \ )+7.5\ \ )}{j0.2}e^{-250t}\cdot e^{j250t}$$

$$= 2\operatorname{Re}\frac{16.77}{0.2}e^{-j90}\cdot e^{-j63^o}\cdot e^{j250t}$$

$$= \frac{2X167.7}{2}e^{-250t}\sin(250t-63^o)$$

$$=167.7e^{-250t}\ \sin(250t-63^o)$$

$$V_c(t) = 167.7e^{-250t}\ \sin(250t-63^o)$$

EXAMPLE 4: In fig. Q4, the branch with r_2C is connected in parallel with resistance r_1. Determine the dependence of ¿L(t) and V_c

E = 10V; r_1 = r_2 = 100Ω; L = 1.0HC = 10μF

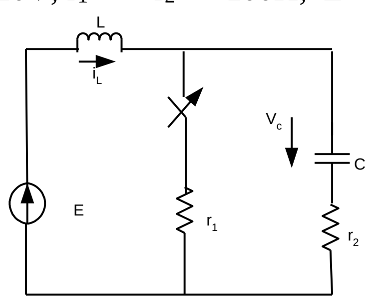


Fig. Q4.

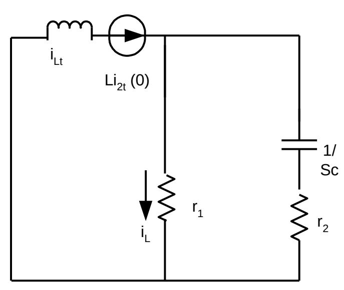


Fig. Q4b. Working Diagram,

i_1 - ? Vc - ?

Initial condition based on the laws of commutation.

i_L (0-) = 0

V_c(0-) = E = 10V

The steady state and transient values at t = 0

i_{lss} = E/r_1 = 10/100 = 0.1A

V_{cc} = i_{Lss} . r_1 = 10V i_{1ss} = 0.1A

$i_L(0) = i_{Lss}(0) + i_{Lt}(0) = -0.1A$
$V_c(0) = V_{cc}(0) + V_{ct}(0)$; $V_{ct}(0) = V_c(0) - V_{css}(0) = 10 - 10 = 0V$
The transient current

$$i_1 t(S) = \frac{\dfrac{L i_{Lt}(0)}{SL + r_1 (r_2 + \dfrac{1}{SC}}}{r_1 + r_2 + \dfrac{1}{SC}} X \frac{r_2 + \dfrac{1}{SC}}{r_1 + r_2 + \dfrac{1}{SC}}$$

$$= \frac{L i_{Lt}(0)(r_2 SC + 1)}{(r_1 + r_2) LCS^2 + r_1 r_2 SC + SL + r_1}$$

$$\frac{1x(-0.11)\,(100x10^{-5} S + 1)}{2x10^{-3} S^2 + 1.1S + 100} = \frac{-10^{-4} S - 0.1}{2x10^{-3} S^2 + 1.1S + 100} = \frac{F_1(S)}{F_2(S)}$$

$$= F_2^1(S) = 4x10^{-3} S + 1.1 = 0$$

$$S_{1.2} = -435; S_2 = -115.$$

The residue method for Laplace polynomial

$$i_{it}\ (t) = \frac{F_1(S_1)e^{s_1 t}}{F_2^1(S_1)} + \frac{F_{1(S_2)}}{F_2^1(S_2)} e^{s2t}$$

$$= \frac{-10^{-4}(-435) - 0.1}{4x10^{-3}(-435) + 1.1} e^{435t} + \frac{-10^{-4}(-115) - 0.1}{4x10^{-3}(-115) + 1.1} e^{-115t}$$

$$= \frac{-0.0565}{-0.64} e^{-435t} + \frac{-0.0885}{+0.64} e^{-115t}$$

$$i_t(t) = 0.138 e^{-11st} + 0.088 e^{-435t}$$

$$\underline{i_1(t) = i_{1ss}(t) + i_{1t}(t) = 0.1 + 0.138 e^{-115t} + 0.088 e^{-435t}}$$

Voltage across the capacitor is given by the expression

$$V_{ct}(S) = \frac{Li_t(0)\,x\,r_1}{(r_1+r_2)\,LCS^2 + r_1 r_2 SC + SL + r_1} = \frac{1(-0.1)\,x100}{2x10^{-3}S^2 + 1.1S + 100}$$

$$= \frac{-10}{(S+115)(S+435)} = \frac{F_1(S)}{F_2(S)}$$

$$V_{ct}(t) = \frac{-10}{-0.64}e^{-435t} + \frac{-10}{-0.46+1.1}e^{-115t}$$

$$=15.6e^{435t} - 15.6e^{-115t}$$

$$\underline{V_c = V_{css} + V_{ct} = 10 + 15.6e^{-435t} - 15.6e^{-115t}}$$

EXAMPLE 5: A parallel loop with parameter L = 1.0H. C = 100μF, r_1 = 80Ω, r_2 = 120Ω is connected to a current source J = 1A (fig Q5). Determine i_1 (t) and V_2(t).

L = 1.0H; C = 100μF; r_1 = 80Ω; r_2 = 120Ω; i_1(t)? V_2(t) - ?

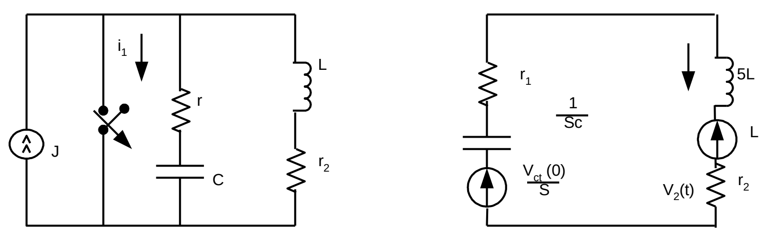


Fig. Q5b. Working Diagram

Initial condition based on the laws of commutation.
i_L (O-) = 0

V_c(0-) = 0
The steady state and transient values at t = 0
V_{css} = J.r_2 = 120V i_{Lss} = J = 1A
$i_L(0) = i_{Lss}(0) + i_{Lt}(0)$; $0 = 1 + i_{Lt}(0)$; $i_{Lt}(0) = -1A$
$V_c(0) = V_{css}(0) + V_{ct}(0)$; $0 = 120 + V_{ct}(0)$; $V_{ct}(0)$; = -120V

The transient current is computed using the Kirchhoff,s voltage law for closed loop. The algebraic sum of the emf divided by the total impedance of the circuit

$$I_{it}(S) = \frac{-\left(\dfrac{V_{ct}(0) + Li_{it}(0)}{S}\right)}{r_1 + r_2 + SL\dfrac{1}{SC}}$$

$$= -\left(\frac{\left[CV_{ct}(0) + LCi_{it}(0)S\right]}{LCS^2 + (r_1 + r_2)SC + 1} = \frac{10^{-4}[120 + S]}{10^{-4}S^2 + 0.02S + 1}\right)$$

$$= \frac{F_1(S)}{F_2(S)} = \frac{120 + S}{(S + 100)^2} = \frac{20}{(S + 100)^2} + \frac{1}{(S + 100)}$$

$$= 20te^{-100t} + e^{-100t}$$

$$\underline{i_{iss} = 0;\ I_1 = I_{iss} + i_{it} = e^{-100t} + 20t\,e^{-100t}}$$

$$i_{it} = -i_{it} = e^{-100t} - 20t\,e^{-100t}$$

$$\underline{I_{Lss} = 1A;\ i_L = i_{Lss} + i_{Lt} = 1 - e^{-100t} - 20t\,e^{-100t}}$$

$$V_2(t) = I_l(t)r_2 = \left(1 - e^{-100t} - 20t\,e^{-100t}\right) X\,120$$

$$V_2(t) = 120 - 120e^{-100t} - 2400t\,e^{-100t}$$

Example 6: Determine the current i_2 and the voltage V_1 in the circuit of fig Q6 after commutation. The parameters are as follows: E = 10; $r_1 = r_2 = 100\Omega$; L = 1.0H; C = 10 μF.

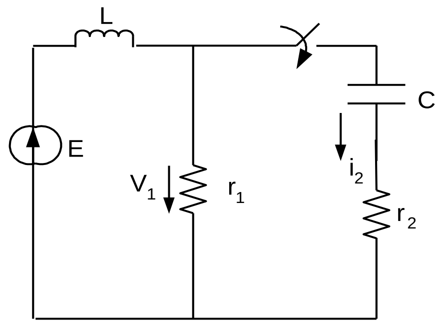


Fig. Q6a

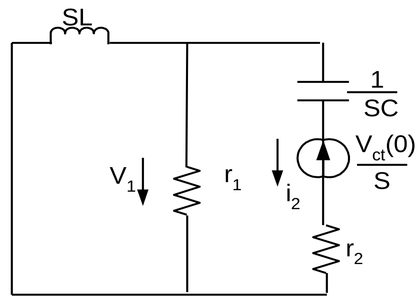


Fig. Q6b. Working Diagram

$i_{Lss} = i_{Lss} = E/r_1 = 10/100 = 0.1A$
$V_{css} = i_{Lss}$ x r_1 E/r_1 x $r_1 = 10V$
$i_L(0) = I_{Lss}(0) + i_{Lt}(0)$; $0.1 = 0.1 + i_{Lt}(0)$;
$i_{Lt}(0) = 0A$; $V_c(0) = V_{css}(0) + V_{ct}(0)$;
$0 = 10 + V_{ct}(0)$; $V_{ct}(0) = - 10V$
The transient current i_{2t} is calculated as the voltage divided by the total impedance. Hence,

$$i_{2t}(S) = \frac{-V_{ct}(0)}{S\left(r_2 + \frac{1}{SC} + \frac{r_1 SL}{r_1 + SL}\right)}$$

$$= \frac{-CV_{ct}(0)\ (r_1 + SL)}{(r_1 + r_2)\,LCS^2 + r_2 r_1\ SC + SL + r_1}$$

The minus will disappear since $V_{ct}(0) = - 10V$

$$= \frac{10^{-4}\ (100 + S)}{0.2x10^{-2} + S^2 + 1.1S + 100_1} = \frac{F_1(S)}{F_2(S)}$$

$F_2(S) = 0.2$ x $10^{-2}\ S^2 + 1.1S + 100$;
is the characteristic equation $F_2'(s) = 0.4$ x $10^{-2}s + 1.1$ is the first derivative of $F_2(s)$
$F_2^1(s) = 09\ 0.4x10^{-2}S + 1.1 = 0$
$S_1 = - 435$; $S_2 = - 115$
The residue method for Laplace polynomial

$$i_{2t}(t) = \frac{F_1(S_1)e^{s_1 t}}{F_2^1(S_1)} + \frac{F_1(_2)e^{s_2 t}}{F_1^2(S_2)}$$

$$\frac{10^{-4}(100 - 435)}{0.4X10^{-2}(-435) + 1.1} e -^{4351} + \frac{-1.5X10^{-4}}{-0.46 + 1.1} e^{-115t}$$

$$= \frac{-1335X10^{-4}}{-0.64} e -^{4351} + \frac{-15X10^{-4}}{0.64} e^{-115t}$$

$= 0.0525e^{-435t} - 0.0025e^{-115t}$
$\underline{i_2(t) = i_{2ss} + i_{2t} = 0.0525e^{435t} - 0.0025e^{-115t}}$
$V_{it}(s)$ is computed by current divider rule to obtains the current flowing through r_1 and then multiplying by r1 to obtain the voltage $V_{it}(t)$.

$$=\frac{-10^{-4}(100+s)}{0.2X10^{-2}S^2+1.1S+100}x\frac{SL}{r_1+SL}xr_1$$

$$=\frac{-10^{-4}(100+S)}{0.2X10^{-2}S^2+1.1S+100}x\frac{S}{100+S}x100$$

$$=\frac{-10X-0^{-4}S)}{0.2X10^{-2}S^2+1.1S+100}=\frac{-10^{-2}S}{0.2X10^{-2}S^2+1.1S+100}=\frac{F_1(S)}{F_2(S)}$$

$$V_{ct}(t)=\frac{F_1(S_1)e^{s_1t}}{F_2^1(S_2)}+\frac{F_1(S_2)e^{s_2t}}{F_2^1(S_2)}$$

$$=\frac{-10^{-2}(-115)}{0.64}e^{-115t}+\frac{-4.35}{0.64}e^{-435t}$$

$= 1.8e^{-115t} – 6.9e^{-435t}$

$\underline{V_1=V_{1ss} + V_{1t} = 10 + 1.8e^{-115t} – 6.9e^{-435t}}$

EXAMPLE 7: Determine the current i and the voltage V_L in the circuit of fig Q7 after commutation. The parameters of the circuit are given thus: E = 100V; r =5Ω, r_2 = 3Ω , L = 0.1H; C = 200µF.

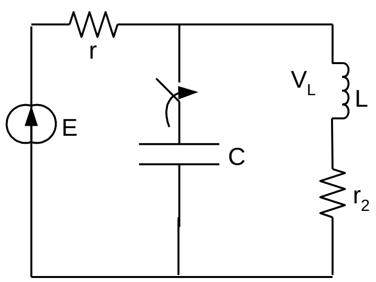


Fig. Q7

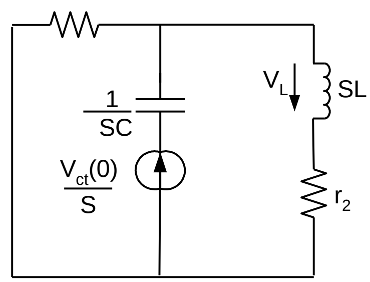


Fig. Q7b. Working Diagram

Initial conditions before communication is given as follows:

$$i(0)=\frac{E}{r+r_2}=\frac{100}{8}=12.5A$$

$$V_c(0)=0$$

The steady state and transient values at t = 0

$i_{ss} = \mathrm{i}_{Lss} = \dfrac{E}{r + r_2} = 12.5A$

$V_{css} = \mathrm{i}_{Lss}\, x\, r_2 = 12.5\, x\, 3 = 37.5V$

$i(0) = i_{ss}(0) + i_t(0) : 12.5 = 12.5 + \mathrm{i}_{\mathrm{t}}(0)$

$\mathrm{i}_{\mathrm{L}t}(0) = 0; V_c(0) = V_{css}(0) + V_{ct}(0);$

$V_{ct}(0) = -37.5V.$

The transient current $i_t(t)$

$$i_t(S) = \frac{-V_{ct}(0)}{\left(S\dfrac{1}{SC} + \dfrac{r(r_{2+SL})}{r + r_2 + SL}\right)} X \frac{r_2 + SL}{r + r_2 + SL}$$

$$\frac{-CV_{ct}(0)(r_2 SL)}{rLCS^2 rr_2 SC + SL + r + r_2}$$

$$\frac{10 -^{-4}(225 + 7.5s)}{10^{-4}S^2 + 0.103S + 8} = \frac{F_1(S)}{F_2(S)}$$

$S_1 = 85; \quad S_2 = -945$

$$i_t(t) = \frac{F_1(S_1)e^{s1t}}{F_2^1(S_1)} + \frac{F_1(S_2)e^{s_2 t}}{F_2^1(S_2)}$$

$$-0.486e^{-85t} + 7.98e^{-945t}$$

Now let us determine the transient current i_{Lt} (t)

$$i_{Lt}(S) = \frac{CV_{ct}(0)}{\left(S\frac{1}{SC} + \frac{r(r_2+SL)}{r+r_2+SL}\right)} x \frac{r}{r+r_2+SL}$$

$$\frac{-CV_{ct}(0).r}{rLCS^2 rr_2SC + SL + r + r_2}$$

$$\frac{2x(-37.5)\,x5x10^{-4}}{10^{-4}S^2 + 0.103S + 8} = \frac{F_1(S)}{F_2(S)}; \quad S_1 = -945;\ S_2 = -85$$

$$i_{Lt}(t) = \frac{F_1(S_1)}{F_2^1(S_1)}e^{s_1t} + \frac{F_1(S_2)}{F_2^1(S_2)}e^{s_2t}$$

$$\frac{-75.X\,5\,X10^{-4}}{2\,X\,10^{-4}(-945) + 0.103}e^{-945t} + \frac{-75.0X\,5X10^{-4}}{2X10^{-4}[-85) + 0.103}e^{-85t}$$

$$0.436e^{-945t} - 0.436e^{-85t}$$

$$i_{ct} = 0.436e^{-945t} - 0.436e^{-85t}$$

$$i_L(t) = i_{Lss} + i_{Lt} = 12.5 + 0.436e^{-945t} - 0.436e^{-85t}$$

$$i_L = 12.5 - 0.436e^{-85t} - 0.436e^{-945t}$$

$$V_L = L\frac{di_L}{d_t} = 3.710e^{-85t} - 41.2e^{-945t}$$

$$\underline{V_L = 3.710e^{-85t} - 41.2e^{-945t}}$$

$$V_L = 3.71e^{-85t} - 41.2e^{-945t}$$

$$0.13(13.71e^{-85t} - 41.2e^{-945t}$$

$$\underline{V_L = 3.71e^{-85t} - 41.2e^{-945t}}$$

EXAMPLE 8: In the circuit of fig.Q8, C = 200μF is connected in parallel with the resistor whose resistance $r_2 = 5\Omega$. Determine the current i and the voltage V_2 across the resistor if E = 100V
; $r = 3\Omega$; L = 0. 1H.

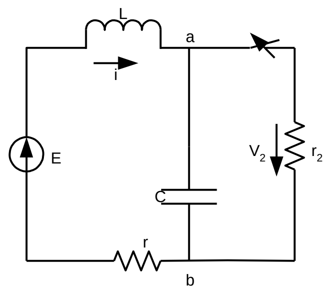


Fig. Q8a

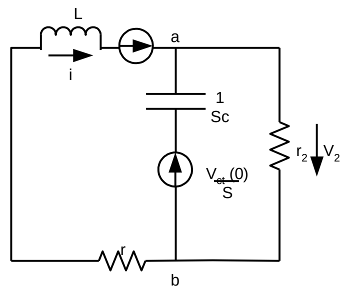


Fig. Q8b. Working Diagram

Initial condition
i (0-) = 0
Vc)-) = E = 100V
Steady state condition at t = 0
i_{ss} = E/r + r_2 = 100/8 = 12.5A
$V_{css} = i_{ss}$ x r_2 = 12.5 x 5 = 62.5V
$V_c = V_{css} + V_{ct}$; $V_c(0) = V_{css} + V_{ct}$;
$V_{ct}(0) = V_c(0) - V_{css}(0) = 100 - 62.5 = 37.5V$
$i = i_{ss} + i_t$; $i(0) = i_{ss}(0) + i_t(0)$; $i_t(0) = 0 - 12.5 = -12.5A$

Transient current is computed using superposition method

$$i_t(S) = \frac{L i_t(0)}{r+SL+\frac{r_2\frac{1}{SC}}{r_2+\frac{1}{SC}}} - \frac{V_{ct}(0)}{S\frac{1}{SC}+\frac{r_2(r+SL)}{r+r_2+SL}} x \frac{r_{21}}{r+r_2+SL}$$

$$= \frac{L i_t(0)\ (r_{sc+1}) - VC_{ct}(0).r_2}{r_2 LCS^2 + rr_2SC + SL + r + r_2}$$

$$\frac{(0.1)\ (-12.5)\ (5\,x\,2\,x10^{-4}S+1)+(0.0375)}{10^{-4}S^2\,0.103S+8} = \frac{-12.510^{-4}S-1.2875}{10^{-4}S^2+0.103S+8}$$

$$\frac{F_1(S)}{F_2(S)}; \qquad F_2(S) = 10^{-4}S^2 + 0.103S + 8 = 0$$

$$S_1 = -945$$

$$S_2 = -85$$

$$i_t(t) = \frac{F_1(S_1)}{F_2^1(S_1)} e^{s_1 t} + \frac{F_2(S_2)}{F_2^1(S_2)} e^{s_2 t}$$

$$= 1.236e^{-945t} - 13.73e^{-85t}$$

$$\underline{i = i_{ss} + i_t = 12.5 - 13.73e^{-85t} + 1.236e^{-945t} A}$$

Using the node-voltage $V_{ab}(S)$ is given as: The current into the node divided by the admittance.

$$V_{ab}(S) = \frac{Li_t(0)\frac{1}{r+SL} + \frac{V_{ct}(0)}{S} x\, SC}{\frac{1}{r_2} + \frac{1}{r+SL} + SC}$$

$$\frac{[Li_t(0) + CV_{ct}(0)(r+SL)]r^2}{r_2 LCS^2 + rr_2 CS + SL + r + r_2}$$

$$= \frac{[0.75x10^{-3} S - 1.227]s}{10^{-4} S^2 + 0.103S + 8} = \frac{F_2(S)}{F_2(S)}$$

$$S_1 = -945 \qquad V_{ab}(t) = \frac{F_1(S_1)}{F_2^1(S_2)} e^{s_2 t}$$

$$S_2 = -85$$

$$= 112.5e^{-945t} - 75.07e^{-85t}$$

$$V_{ab} = 112.5e^{-945t} - 75.07e^{-85t}$$

$$\underline{V_2(t) = V_{2SS} + V_{2t} = 62.5 - 75.1e^{-85t} + 112.5e^{-945t}}$$

EXAMPLE 9: In the fig.Q9, there is a sudden disconnexion of the switch. The parameters of the circuit are given thus: E = 100V; J = 1A; $r_1 = r_2 = 10\Omega$; L = 0.1H; C = 1000μF. Determine the following $i_1(t)$; and $V_c(t)$

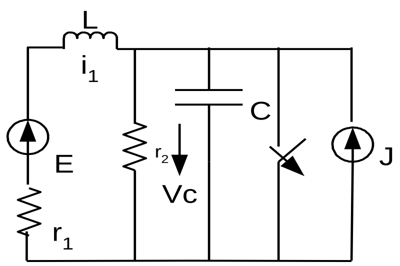


Fig. Q9a

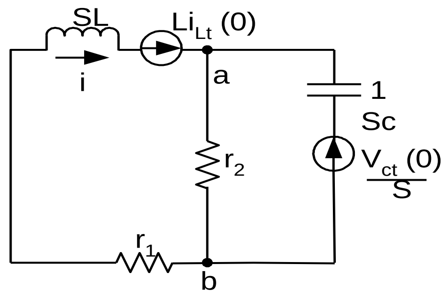


Fig. Q9b. Working Diagram

Initial condition based on the laws of commutation

$$i_1(0) = \frac{E}{r_1} = \frac{100}{10} = 10A$$

$Vc(0-) = 0$

Steady state and transient values at t = 0

$$i_{1ss} = \frac{E}{r_1 + r_2} - \frac{J.r_2}{r_1 + r_2} =$$
$$\frac{100}{10+10} - \frac{1x10}{10+10} = 5 - 0.5 = 4.5A$$

$i_2(0)=0$

$$V_{css} = \left(\frac{E}{r_1 + r_2} + \frac{J.r_1}{r_1 + r_2} \right).r_2 = 5.5x10 = \underline{55V}$$

$i_{1t} = i_{1ss} + i_{1t}$; $i_1(0) = i_{1ss}(0) + i_{1t}(0)$; $i_{it}(0) = 10 - 4.5 = 5.5$
$V_c = V_{css} + V_{ct}$: $V_c(0) = V_{css}(0) + V_{ct}(0)$; $V_{ct}(0) = 0 - 55 = -55V$.

The transient current using is computed superposition theorem

$$i_t(S) \frac{Li_{1t}(0)}{r_2 \dfrac{1}{r+sL+\dfrac{Sc}{r_2+\dfrac{1}{Sc}}}} - \frac{V_{ct}(0)}{S\left(\dfrac{1}{SC} + \dfrac{r_2(r+sL)}{r+r_2+sL}\right)} x \frac{r_2}{r+r_2+sL}$$

$$= \frac{Li_{1t}(0)(r_2 SC+1) - CV_{ct}(0).r_2}{r_2 LCS^2 + r_1 r_2 \; SC + SL + r_1 + r_2}$$

$$= \frac{0.55(10^2 S+1) - 0.55}{10^{-3} \; S^2 + 0.2S + 20} = \frac{0.55x10^{-2} S}{10^{-3} S^2 + 0.2S + 20}$$

$$= \frac{F_1(S)}{F_2(S)} \qquad S_{1,2} \; = -100 \pm j100$$

$$i_{1t}(t) = 2\,\mathrm{Re}\,\frac{F_1(S_1)e^{s_1 t}}{F_2^1(S_1)} \; = 2\,\mathrm{Re}\,\frac{0.55x10^{-2}(-100 \pm j100)}{2x10^{-3}(-100 \pm j100) + 0.2} e^{-100t.} \; e^{+j100}$$

$$= 2\,\mathrm{Re}\,\frac{-0.55 + j0.55}{j0.2} e^{-100t.} e^{j100t}$$

$$= 2\,\mathrm{Re}\,\frac{0.778}{0.2} e^{-j90.} e^{-100t.} e^{j100t}$$

$$i_{1t}(t) = 7.78\sin(100t + 45^o)e^{-100t}$$

$$\underline{i_1 = i_{1ss} + i_{1t}(t) = 4.5 + 7.78e^{-100t} \; \sin \; (100 \; t + 45^0)}$$

The voltage V_c(t) using the Node voltage method, is determined by the following set of equations:

$$V_{ct}(S) = V_{ab}(S) = \frac{Li_t(0)x\frac{1}{r+SL} + \frac{V_{ct}(0)}{S}xSC}{\frac{1}{r_2} + \frac{1}{r+SL} + Sc}$$

The current into node divided by the admittance

$$= \frac{\left[Li_{1t}(0) + CV_{ct}(0)(r_1 + SL)\right]r_2}{r_2 LCS^2 + r_1 r_2 SC + SL + r_1 + r_2}$$

$$= \frac{\left[(0.1)(5.5) + 10^{-3}(-55)(10 + 0.1S)\right]10}{10^{-3}S^2 + 0.2S + 20}$$

$$= \frac{\left[0.55 - 0.55 - 5.5x10^{-3}S\right]10}{10^{-3}S^2 + 0.2S + 20}$$

$$= \frac{-5.5x10^{-2}S}{10^{-3}S^2 + 0.2S + 20} = \frac{F_1(S)}{F_2(S)}$$

$$V_{ct}(t) = 2\,\mathrm{Re}\,\frac{-5.5x10^{-2}(-100 + j100)}{2x10^{-3}(-100 + j100) + 0.2} e^{-100t} \cdot e^{j100}$$

$$= 2\,\mathrm{Re}\,\frac{5.5 - j5.5}{j0.2} e^{-100t} \cdot e^{j100t}$$

$$= 2\,\mathrm{Re}\,\frac{0.778}{0.2} e^{-j45^0} \cdot e^{-j90^0} \cdot e^{-100t} \cdot e^{j100t}$$

$$= 2\,\mathrm{Re}\,\frac{55\sqrt{2}}{2} e^{-j45^0} \cdot e^{-j90^0} \cdot e^{-100t} \cdot e^{j100t}$$

$$V_{ct}(t) = 55\sqrt{2}e^{-100t}\cos\left(100t - 135^0\right)$$

$$V_{ct}(t) = 55\sqrt{2}e^{-100t} Sin\left(100t - 45^0\right)$$

$$\underline{V_c = V_{css} + V_{ct} = 55 + 55\sqrt{2}e^{-100t} Sin\left(100t - 45^0\right)}$$

EXAMPLE 10: Determine the current i; i and the voltage V_c in circuit of fig. Q10 after commutation if the parameters are given thus: E = 100V; J = 1A; $r_1 = r_2 = 200\Omega$; L = 0.5H; C = 10μF.

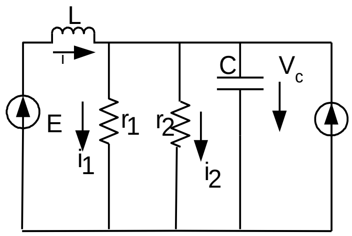


Fig. Q10a

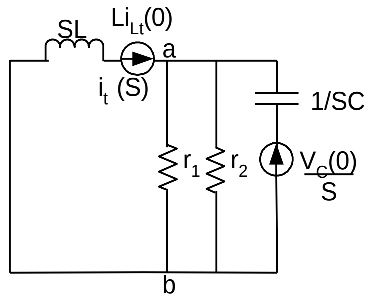


Fig. Q10b. Working Diagram

Initial conditions

$$i(0) \quad = \frac{E}{r_1} = \frac{100}{200} = 0.5A$$

steady state and transient values at t = 0

Vc(0-) = J. r_2 = 200 x 1 = 200V

$$i_{ss} \; = \; \frac{E}{r_1} + \frac{E}{r_2} = 1A; \qquad i_{2ss} = \frac{E}{r_1} + \frac{E}{r_2} - J = 0.5$$

i_{2ss} = 0.5A

V_{css} = 0.5 x 200 = 100V; i(0-) = $i_{ss}(0)$ + ¿t(0)

0.5 = 0 + $i_t(0)$; $i_t(0)$ = 0.5A

$V_c(0)$ = $V_{css}(0)$ + $V_{ct}(0)$; 200 = 100 + $V_{ct}(0)$

$V_{ct}(0)$ = 100V

The current i_t, is the algebraic sum of the current produced by voltages Li_t (0) and $V_{ct}(0)$ acting separately

$$i_t(S) = \frac{L¿t(0)}{SL + \dfrac{r_1 r_2}{r_1 + r_2 + r_1 r_2 SC}} - \frac{V_{ct}(0)}{\dfrac{S\left(\dfrac{1}{SC} + r_1 r_2 SL\right)}{(r_1 + r_2)SL + r_1 r_2} +} \quad x \quad \frac{\dfrac{r_1 r_2}{r_1 + r_2}}{SL + \dfrac{r_1 r_2}{r_1 + r_2}}$$

$$= \frac{L\mathrm{it}(0).\,(r_1 r_2 SC + r_1 + r_2) - CV_{ct}(0) r_1 r_2}{r_1 r_2 LCS^2 + (r_1 + r_2)SL + r_1 r_2}$$

$$= \frac{0.25(0.4S + 400) - 40}{0.2S^2 + 200S + 40000} = \frac{0.1S + 60}{0.2S^2 + 200S + 4000}$$

$$= \frac{F_1(S)}{F_2(S)}; \quad F_2(S) = 0.2S^2 + 200S + 40000;$$

$F_2^1(S) = 0.4S^2+200=0$

$$S_1 = -723; \quad S_2 = -277.$$

$$¿_t(t) = \frac{F_1(S_1)}{F_2^1(S_1)}e^{s1t} + \frac{F_1(S_2)}{F_2(S_2)}e^{s2t}$$

$$= \frac{0.1(-723)+60}{0.4(-723)+200}e^{-723t} + \frac{0.1(-277)+60}{0.4(-277)+200}e^{-277t}$$

$$\underline{i(t) = 0.138e^{-723t} + 0.362e^{-277t}}$$

$$¿_{it}(0) = i_{1t}^1(S) + i_{1t}^{11}(S)$$

$$= \frac{L_{1t}(0)r_2 + CV_{ct}(0).r_2SL}{r_1r_2LCS^2 + (r_1 + r_2)SL + r_1r_2}$$

$$= \frac{0.5(0.5)x200+10^{-5}(100)\,x\,200x\,0.5S}{0.2S^2 + 200S + 40000}$$

$$i_{1t}(t) = \frac{F_1(S_1)}{F_2^1(S_1)}e^{s1t} + \frac{F_1(S_2)}{F_2^1(S_2)}e^{s2t}$$

$$= \frac{-22.3}{-89.2}e^{-723t} + \frac{22.3}{89.2}e^{-277t}$$

$$0.25e^{-723t} + 0.25e^{-277t}$$

$$\underline{i_1(t) = i_{1ss} + i_{1t} = 0.5 + 0.25e^{-723t} + 0.25e^{-277t}}$$

The voltage across the capacitor is vab; the voltage drop between the nodes a and b. we apply node voltage method in determining this voltage. By the node voltage method – the voltage vab is given as the current going into either of the nodes divided by the total admittance.

$$V_{ab}(S) = \frac{Li_t(0)\frac{1}{r+SL} + \frac{V_{ct}(0)}{S} xSC}{\frac{1}{r_2} + \frac{1}{r+SL} + SC}$$

$$= \frac{Li_t(0) + CV_{ct}(0)SL}{\frac{SL\left(r_1 r_2 + r_2 SL + r_1 r_2 LCS^2\right)}{r_1 r_2 SL}}$$

$$= \frac{\left[Li_t(0) + CV_{ct}(0)SL\right] r_1 r_2}{r_1 r LCS^2 + (r_1 + r_2)SL + r_1 r_2}$$

$$= \frac{\left[Li_t(0) + CV_{ct}(0)SL\right] r_1 r_2}{r_1 r_2 LCS^2 + (r_1 + r_2)SL + r_1 r_2}$$

$$= \frac{\left[(0.5)[0.5) + 10^{-5}(100)0.5S\right]40000}{0.2S^2 + 200S + 40000}$$

$$= \frac{\left[0.5(0.5) + 10^{-3}S\right]40000}{0.2S^2 + 200S + 40000}$$

$$= \frac{20S + 10000}{0.2S^2 + 200S + 40000} = \frac{F_1(S)}{F_2(S)}$$

$$V_{ct}(t) = \frac{F_1(S_1)}{F_2^1(S_1)} e^{s1t} + \frac{F_1(S_2)}{F_2^1(S_2)} e^{s2t}$$

$$= \frac{20(-723) + 10000}{0.4(-723) + 200} e^{-723t} + \frac{20(-277) + 10000}{0.4(-277) + 200} e^{-277t}$$

$$V_{ct} = 50e^{-723t} + 50e^{-277t}$$

$$\underline{\underline{V_c = V_{css} + V_{ct} = 100 + 50e^{-723t} + 50e^{-277t}}}$$

5.5 EXERCISE 5 (Laplace transform application to transient processes)

Q5.1. As a result of the increase in the resistance in one of the branches of the circuit of fig Q5.1; there is a transient process in the circuit. Determine the voltage Vc and the currents i_2, i_2 if the parameters of the circuit are as follows:
E = 180V; $r_1 = 60\Omega$; $r_2 = 20\Omega$; $r_3 = 100\Omega$; C = 25 μ F.

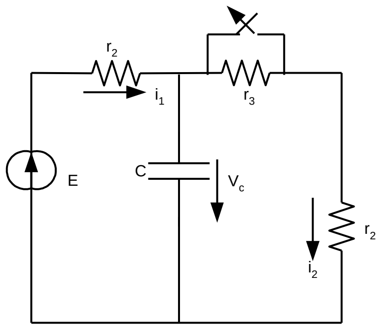


Fig Q 5.1

Ans: $\boldsymbol{Vc(t) = 120 - 75e^{-1000t} V}$
$\boldsymbol{i_1 = 1.0 + 1.25\, e^{-1000t} A;}$
$\boldsymbol{i_2 = 1.00 - 0.625\, e^{-1000t} A}$

Q5.2. In the circuit of Fig Q5.1, the switch suddenly opened. The parameters of the circuit are as follows: E = 80V; $r_1 = r_3$ 100Ω; $r_2 = 50\Omega$; C= 12.5 μ F. Find the dependence Vc(t) and i_3(t).

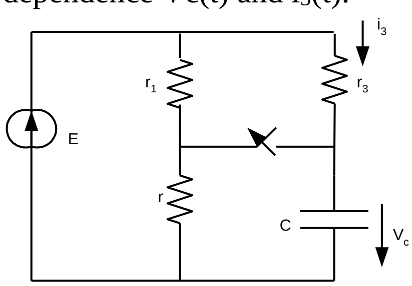


Fig Q5.2

Ans: $\boldsymbol{Vc(t) = 80 - 40\, e^{-800t} V}$
$\boldsymbol{i_3(t) = 0.4\, e^{-800t} A}$

Q5.3. In the circuit of fig 5.3, a resistor r_2 with resistance $r_2 = 5\Omega$ is connected in parallel to the capacitor whose capacitance c = 200 μ F. Calculate the current i and the voltage V_2 across the resistor r_2 if E = 100V; r = 3 Ω; L= 0.1H.

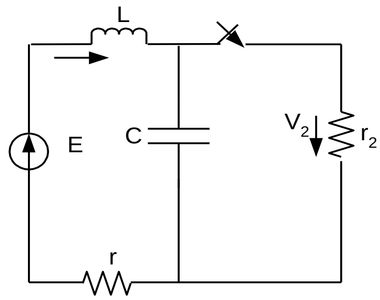


Fig Q 5.3

Ans:
$i = 12.5 – 13.73e^{-85t} + 1.23\ e^{–945t}$ A;
$V_2(t) = 62.5 + 41.21\ e^{-85t} – 3.69\ e^{–945t}$ V.

Q5.4. Determine the current is and the voltage V_c across the capacitor in the circuit of figQ5.4 after commutation. Given; e = 100sin10^5t V; $r_1 = r_2 = 20\Omega$; L = 0.1H; C = 10 μF.

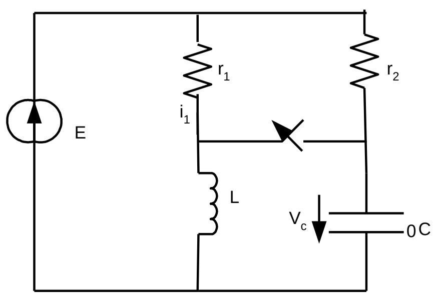


Fig Q5.4

Ans:
$i_1 = 3\sin(10^5t - 53^0) +$
$(4.4 – 2.16x10^6t)\ e^{-2x10^5t}$ A.
$V_c(t) = 80\sin(10^5t+37^0)+(- 88 + 43.2x10^6t)\ e^{-2x10^5t}$

Q5.6. A transformer on load with parameters of L_1 = 0.5H; $r_1 = 20\Omega$; L_2 = 0.6 H; M = 0.5H is connected to a voltage source of sinusoidal emf e = 100sin100tV. The resistance of the load $r_2 = 100\Omega$. Calculate the currents i_1 and i_2.

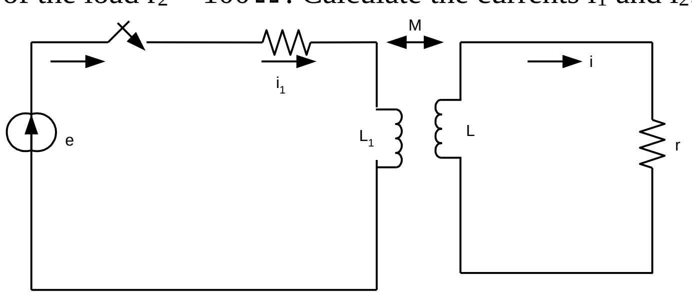


Fig Q 5.6

Ans : i_1 =1.83 sin (100t – 45.5^0) + 1.22 $e^{–30t}$ + 0.07$e^{– 1270t}$ A;
i_2 = 0.782 sin (100t + 13.5^0) – 0.238e^{-30t} + 0.065$e^{–7270t}$ A.

Q5.7. A circuit of rLc connected in series is switched on to a constant voltage source E

(Fig Q5.7). Determine the current i and the voltage Vc(t) and also their maximum values during the time of transient process, if E = 120V; r = 20Ω; L = 0.1H; L = 2.49 μF.

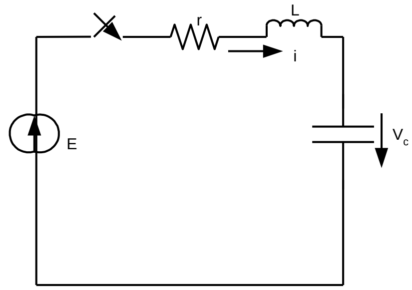


Fig Q 5.7

Ans:
$i = 0.6\ e^{-100t} \sin 2000t$ A;
$V_c(t) = 120 - 120\ e^{-100t} \sin(2000t + 87.1^0)$V
$I_{max} = 0.555$A; $V_{c\,max} = 222.5$V

Q5.8. In the circuit of fig Q5.8, at time t = 0 the switch is disconnected. The parameter of the circuits are E = 100v; r = 500Ω; L = 0.1H; C = 10 μF. Determine the values of the following quantities with respect to time: Vc (t) and i(t).

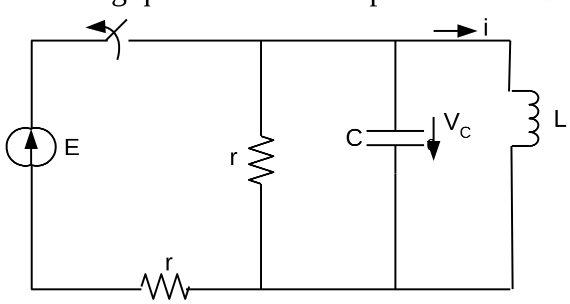


Fig Q5.8

Ans:
$V_c = -20.10^3 te^{-1000t}$V;
$i(t) = 0.2\ e^{-1000}t + 200t\ e^{-1000t}$ A.

Q5.9. A circuit made up of two parallel branches with parameters r_1 = 10Ω;
C = 80 μF and r_2 = 40Ω; L = 0.25H is disconnected from a voltage source of emf E = 50V (fig Q 5.9) after commutation. Calculate the voltage across the capacitor and the current following in the coil i_2

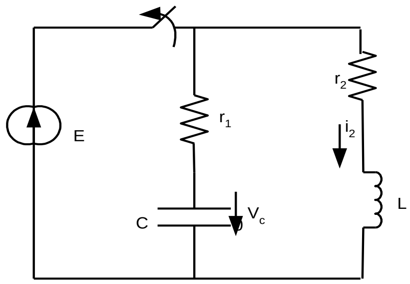


Fig Q5.9

Ans:
V_c -72.95e $^{-100t}$ sin (200t –43.26) V;
i_1 =1.305 e^{-100t} sin (100t +733^0)A

Q5.10 The switch in the circuit of fig Q5.10 is opened all of a sudden. Determine the voltage Vc(t) and current i_3 (t)

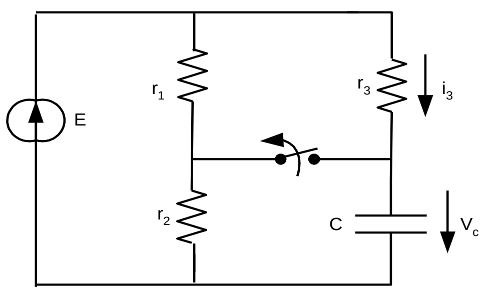


Fig Q5.10

Ans: 80 – 40e^{-800t}V
0.4e^{-800t}A

Q5.11 A capacitor initially charged ti a voltage Vc (0-) = 40V is connected r = 20Ω; L = 0.2H fig Q5.11. Determine the voltage Vc (t) and current i (t).

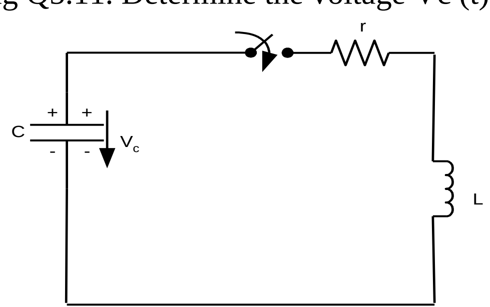


Fig Q5.11

Ans: 42.16e^{-50t} sin (150t + 71.56^0)V;
1.33e^{-50t} sin 150t A

Q5.12 A circuit consisting of two parallel branches with parameters r_1 = 10Ω; C = 80 μ f and r_2 40Ω; L = 0.25 H, is disconnected from a source of emf E = 50bv (fig Q4.16). Calculate the voltage across the capacitor and the current through the coil i_2.

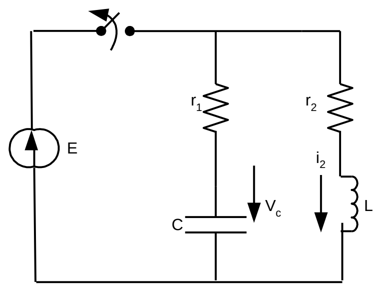


Fig Q5.12

***Ans:** $-72 - 95e^{-100t}\sin(200t - 43.26^0)$ V;*
$1.205e^{-100t}\sin(200t + 73.3^0)$A

EQ5.13. In the circuit of fig Q5.13, to a transformer fielding from a sinusoidal voltage source of emf e. = 100 sin 100t V, is connected a resistor with resistance $r_2 = 50\Omega$. The parameters of the transformer are as follows: $L_1 = 0.5$H; $r_1 = 20\Omega$; $L_2 = 0.6$ H, M = 0.5H. Evaluate the currents $i_1(t)$ and $i_2(t)$.

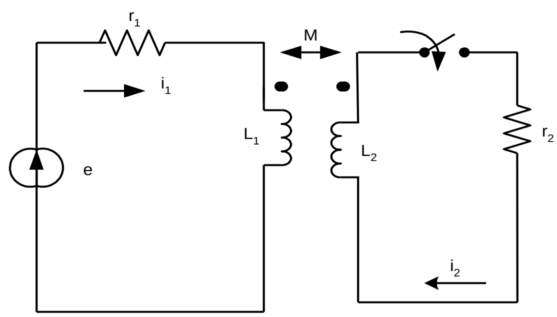


Fig Q5.13

***Ans:** $2.08\sin(100t - 32^0) - 0.176e^{-28t} - 0.264e^{-712t}$A*
$1.30\sin(100t + 8^0) + 0.0717e^{-28t} - 0.2517e^{-712t}$A

Q5.14. Calculate, the current i_1 and voltage Vc(t) in the circuit of fig Q5.14 after commutation. Given: e = 100 sin 10^5t V; $r_1 = r_2 = 20\Omega$; L = 0.1H; C = 0.25 μ F.

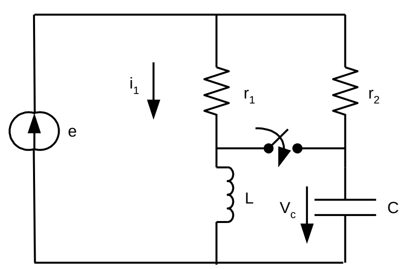


Fig Q 5.14

***Ans:** $3\sin(10^5t - 53^0) + (4.4 - 2.16 \times 10^6t)\,e^{-2x10^5t}$ A*
$80\sin(10^5t + 37^0) + (-88 + 43.2 \times 10^6t)\,e^{-2x10^5t}$ V.

CHAPTER SIX

THE STATE VARIABLE METHOD OF ANALYSIS OF TRANSIENT PROCESSES IN LINEAR CIRCUIT.

The state variable method has its basis on two main equations. Written matrix form. The structure of the first equation is defined by the fact, the combines the nmatrix of the first derivative. With respect to time, of the s, variable $X^{(1)}$ with state variable X and the external excitation V, which normally an emf or a current source.

The second equation in its structure is algebraic and combines that matrix of the output variable Y with the matrix of the state variable X and the excitation (input variables). V.

Defining state variables, we should note the following properties.

(1) The state variable in electric circuits are current i_L in an inductive element and voltage V_c across a capacitive element. However we consider only independent inductors and capacitors i.e such inductors and capacitors which define the order of the differential equation of the circuit.

(2) Differential equations of the circuit of the state variables are written in canonic form that is, they are presented resolved in terms of the first derivative of the state variables.

It is only when we take the state variable as current i_L in the independent inductors and voltage across the independent capacitors, that the first equation will have the structured required.

(3) Number of state variables is equal to the order of the system differential equation of the circuit under consideration.

(4) The choice of i_L and V_c as state variables of the circuit is comfortable because it is these quantities that dol not change instantaneously during commutation when t = 0-.

(5) The state variables i_L and V_c are so called because at any instant of time these two quantities determine the energy state of the electric circuit since the energy state of the circuit is defined by the sun of the expression $L_{iL}^2/2$ and $CV_c^2/2$

DERIVATION OF STATE VARIABLE EQUATION

Let us show with an example in fig. 2.1, how to derive equations using state variable method.

First we obtain a system of differential equation corresponding to the first matrix equations , and then we write matrix from of this equation. Algorithm of deriving such equations is as follows: First write equation according to Kirchhoff's laws or use mesh current method, then choose the state variable and obtain equations by method of state variables.

For fig. 2.1 according to Kirchhoff's laws we obtain.

$$-i(t) - i_L - ir + ic \quad \text{...}(1)$$

$$ri_L + {}_L di_L/dt + rir \; a(t) \quad \text{...}(2)$$

$$V_c - r\,i\,r = 0 \quad \text{...}(3)$$

We obtain ir from the first equation and substitute in the third equation and present ic = CdVc/dt and produce the equation thus obtained with dVc/dt as the subject of the equation.

$$V_c^{(1)} = \frac{dV_c}{dt} = \frac{1}{C} i_L - \frac{1}{rC} V_c + \frac{1}{C} i(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

We solve the second equation with respect to Ldi_L/dt; replacing ir with its value and ic = Cd_{Vc}/dt, we obtain

$$Ldi_L/dt = -2ri_L + rcd_{Vc}/dt + e(t) - ri(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Adding equation (5) with equation (4) multiplied by rc, and then obtain di_L/dt from the expression, we get

$$i_L^{(1)} = \frac{di_L}{dt} = -\frac{r}{L} V_c + \frac{1}{L} e(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

We now rewrite equation (6) and (4) in matric form.

$$\begin{matrix} i_L^{(1)} \\ V_c^{(1)} \end{matrix} \quad \begin{matrix} -\frac{r}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{rC} \end{matrix} \quad \begin{matrix} i_L \\ V_c \end{matrix} \; + \; \begin{matrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{C} \end{matrix} \quad \begin{matrix} e_{(t)} \\ i(t) \end{matrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

$$\text{Or } X^{(1)} = AX + BV \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Where for the circuit under investigation, we have

$$X^{(1)} = \begin{matrix} X_1^{(1)} \\ X_2^{(1)} \end{matrix} = \begin{matrix} i_L^{(11)} \\ V_c^{(1)} \end{matrix} ; \quad X = \begin{matrix} X_1 \\ X_2 \end{matrix} = \begin{matrix} i_L \\ V_c \end{matrix} ;$$

$$A = \begin{bmatrix} -\frac{1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{rc} \end{bmatrix} ; \quad B = \begin{bmatrix} \frac{1}{L} & \frac{0}{1} \\ 0 & C \end{bmatrix} \qquad =$$

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} e(t) \\ i(t) \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

In general cases the first equation of the state variable method in matrix is written in the form below:

$$X_{(1)} = \begin{matrix} X_1^{(1)} \\ X_2^{(1)} \\ \vdots \\ X_n^{(1)} \end{matrix} = A \begin{matrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{matrix} + B \begin{matrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{matrix} = AX + BV \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Where A and B are matrices of the linear circuits which depends only on parameters of the circuit r, i.e they are constants values. A is a square matrice of the order of n and is called the basic matrice of the circuit; matrix B is a rectangular matrice, whose size xs n m is called the matrix connection of the input function and the state variables; matrix x and V column matrice or state variables vectors (size n x 1) and external excitatise (size m x 1) respectively.

In the present example B is a 2 x 2 matrix, since the number of state variable (i_L, V_c) is equal to the number of external excitation (e (t), i(t).

We now determine the second equation for output function (quantities) we can choose any of the unknown quantities. For example, let us take ir, ic and as output unknowns.

We shall write their values through the state variables (i_L, Vc) and the external excitation

e (t), i(t)).

$$Ir = 0.\, i_L + V_c/r + 0.e(t) + 0.i(t)$$

$$ic = i(t) + i_L - ir = i_L - V_c/r + 0.e(t) + 1.i(t) \quad \text{............(11)}$$

$$V_L = -ri_L - V_c + 1.e(t) + 0.i(t)$$

or in matrix from

$$\begin{bmatrix} i_1 \\ 1_c \\ V_L \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{r} \\ 1 & -\frac{1}{r} \\ -r & -1 \end{bmatrix} \begin{bmatrix} V_L \\ V_c \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ i(t) \end{bmatrix} \quad \text{...(12)}$$

or in short

Y = CX + DV ....(13)

Where

$$Y=\begin{bmatrix} y_1 & i_r \\ y_2 = i_c \\ y_3 & V_L \end{bmatrix};\ C=\begin{bmatrix} 0 & \frac{1}{r} \\ 1 & -1/r \\ -r & -1 \end{bmatrix}\ D=\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Or in general

$$Y=\begin{bmatrix} y_1 \\ y_2 \\ : \\ y_n \end{bmatrix} = \begin{bmatrix} X_1 & & V_1 \\ X_2 & & V_2 \\ C & + & D \\ \quad : & & : \\ X_n & & V_m \end{bmatrix} = CX + DV \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15$$

)

C and D depends only on the parameters of the circuit r, L, c. C is matrice of state variables connected with the output circuit with the output circuit.

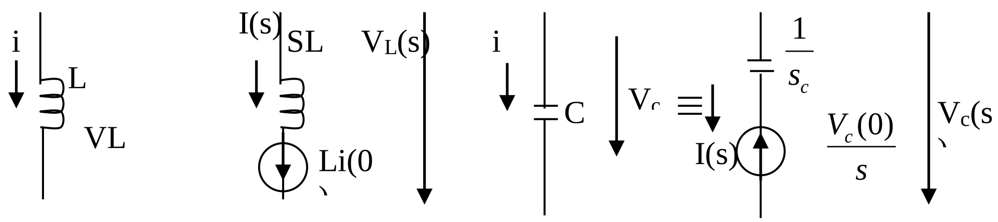
Let us stake a look at the solution of a linear non homogeneous differential equation in matrix form. For this purpose, we write equation (10) in Laplace transform. For this transform the equation in this number.

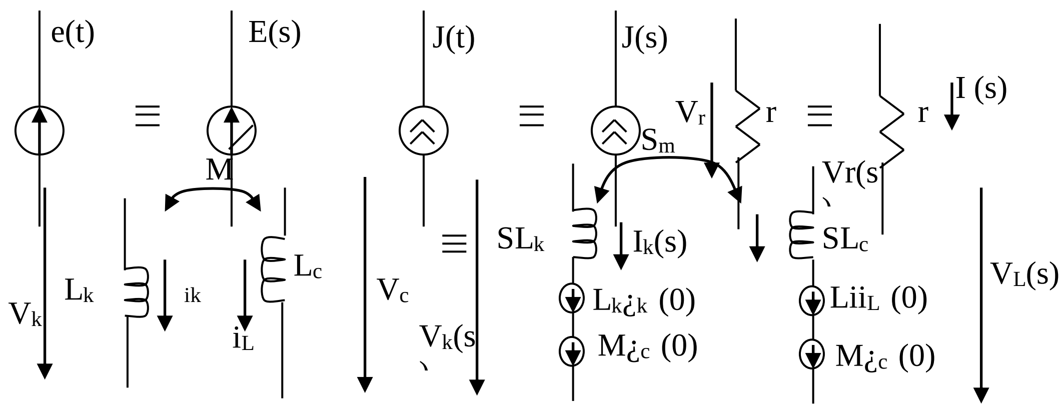
$$SX(s) - X(0) = A X(s) + BV(s)$$

......................................(16)

Where S (s) = L {x (t) }; V (s) = L{V (t) }; X(0+) = X (0-) = X(0), where also X(0) is the column matrix of the initial values of the state variables.

$$X(0) = \begin{bmatrix} X_1(0) \\ X_2(0) \\ \vdots \\ X_n(0) \end{bmatrix} \qquad \text{...(17)}$$





If the Polynimial F_2 (s) has n pairs of complex conjugate roots, then the inverse Laplace transform is given as:

$$L^{-1}\left\{\frac{F_1(s)}{F_2(s)}\right\} = f(t) = \sum_{k=1}^{n} 2\,\mathrm{Re}\left(\frac{F_1(Sk)}{F_2^{'}(Sk)}e^{skt}\right)$$

Where Re represent the real part of the complex solution. And if there is zero root; there we have

$$L^{-1}\left\{\frac{F_1(s)}{SF_3(s)}\right\} = f(t) = \frac{F_1(0)}{F_3(0)}\sum_{k=1}^{n} 2\,\mathrm{Re}\left(\frac{F_1(Sk)}{S_k F_3^{'}(Sk)}e^{skt}\right)$$