Polygonal Cannonball Numbers

Izaak van Dongen

April 9, 2019

Contents

1 Introduction

Recently I watched the video `https://www.youtube.com/watch?v=q6L06pyt9CA`, featuring Matt Parker. Being a huge fan of Matt and of Numperphile, and being rather full of myself, my first response was naturally one of doubt (also because of the spirit of mathematical enquiry and all that). I decided to have my own crack at the problem, since I my rough estimation for the complexity of this task did not seem to match with the claim that it could take a whole night to compute all cannonball numbers under 10^9, for $s \lesssim 31265$.

I reasoned that it should be a roughly $\mathcal{O}(1)$ operation as we can find the nth term of the base-s polygonal numbers $P(s, n)$, which will be quadratic in n, and solve it for n with the quadratic formula, so to check if some cannonball numbers $C(s, n_c)$ is polygonal we just see if the corresponding n_p is an integer. Now 10^9 is a fairly small number. Seeing as my CPU's clockspeed is in the range of gigahertz, and we're just checking a tiny fraction of those numbers as we're just computing the cannonball numbers under this limit, it seems reasonable that this should be doable fairly fast.

I've thought about the problem of higher-dimensional stacks of cannonballs (ie the ones formed by adding up the cannonball numbers), but I've not done anything about it.

2 The Maths

Indeed, this approach does seem to work. Almost by definition we have the recurrence in polygonal numbers

$$P(s, n) = P(s, n-1) + n(s-2) - (s-3)$$

so we can use

$$P(s, n) = \sum_{r=1}^{n} P(s, r) - P(s, r-1)$$

$$
\begin{aligned}
&= \sum_{r=1}^{n} (n(s-2) - (s-3)) \\
&= \frac{1}{2} n(n+1)(s-2) - n(s-3) \\
&= \frac{n^2(s-2) - n(s-4)}{2}
\end{aligned}
$$

Fortunately this seems to agree with what Wikipedia thinks. Now, we have

$$
\begin{aligned}
0 &= (s-2)n^2 - (s-4)n - 2P(s,n) \\
\implies n &= \frac{s - 4 + \sqrt{(s-4)^2 + 8(s-2)P(s,n)}}{2s-4}
\end{aligned}
$$

Wikipedia still seems to think we're on track.

Another result that I don't really use is that

$$
\begin{aligned}
C(s,n) &= \sum_{r=1}^{n} P(s,n) \\
&= \frac{1}{2} \sum_{r=1}^{n} (n^2(s-2) - n(s-4)) \\
&= \frac{1}{2} \left(\frac{n(n+1)(2n+1)(s-2)}{6} - \frac{n(n+1)(s-4)}{2} \right) \\
&= \frac{1}{12} n(n+1)[(2n+1)(s-2) - 3(s-4)]
\end{aligned}
$$

In fact I've only used this in verification of the results.

Regardless, now we need only work our way up the $C(s,n)$s using the recurrence $C(s,n) = P(s,n) + C(s,n-1)$, and check for each if the quadratic formula gives an integer result. This is most easily done by checking if the discriminant is a perfect square and then checking that the denominator divides the numerator.

3 The Programming

For speeeeeeed I implemented this in C (although there is a long abandoned parallel Python implementation). I used 128-bit integers to be on the safe side, as 10^{19} is a little small for my liking. This meant I had to do a lot of messing around to get things to actually display in base 10.

I did briefly consider either implementing or importing some kind of arbitrary precision integer arithmetic functionality, but then I decided I wasn't going to run it on anything fast enough to have to worry about that, and I have better things to do.

There's also a slick little progress update that gets printed to STDERR, and a number of zsh scripts to save me typing.

4 The Ugly

Table 1 lists all the solutions that I've found, so far. The TEXsource of the table is in `../src/tab.tex`, which is derived from `../src/c/solutions/*`.

s	$C(s,n_c) = P(s,n_p)$	n_p	n_c

3	10	4	3
3	120	15	8
3	1540	55	20
3	7140	119	34
4	4900	70	24
6	946	22	11
8	1045	19	10
8	5985	45	18
8	123395663059845	6413415	49785
8	774611255177760	16068720	91839
10	175	7	5
10	368050005576	303336	6511
11	23725	73	25
11	1519937678700	581175	10044
11	7248070597636	1269127	16906
14	441	9	6
14	195661	181	46
17	975061	361	73
17	1580765544996	459096	8583
20	3578401	631	106
23	10680265	1009	145
26	27453385	1513	190
29	63016921	2161	241
30	23001	41	17
32	132361021	2971	298
35	258815701	3961	361
38	477132085	5149	430
41	55202400	1683	204
41	837244045	6553	505
43	245905	110	33
44	1408778281	8191	586
47	2286380881	10081	673
50	314755	115	34
50	3595928401	12241	766
53	5501691505	14689	865
56	8214519205	17443	970
59	12001111741	20521	1081
60	1785508245600	248132	5695
62	17194450141	23941	1198
65	24205450501	27721	1321
68	33535911025	31879	1450
71	45792819865	36433	1585
74	61704091801	41401	1726
77	82135801801	46801	1873
80	108110983501	52651	2026
83	140830060645	58969	2185
86	181692979525	65773	2350
88	48280	34	15
89	232323110461	73081	2521
92	294592986361	80911	2698
95	370651946401	89281	2881

98	462955752865	98209	3070
101	574298249185	107713	3265
104	707845127221	117811	3466
107	867169871821	128521	3673
110	1056291950701	139861	3886
113	1279717317685	151849	4105
116	1542481297345	164503	4330
119	1850193919081	177841	4561
122	2209087768681	191881	4798
125	2626068425401	206641	5041
128	3108767552605	222139	5290
131	3665598710005	238393	5545
134	4305815955541	255421	5806
137	5039575304941	273241	6073
140	5877999117001	291871	6346
143	6833243472625	311329	6625
145	101337426	1191	162
146	7918568615665	331633	6910
149	9148412523601	352801	7201
152	10538467676101	374851	7498
155	12105761089501	397801	7801
158	13868737685245	421669	8110
161	15847347060325	446473	8425
164	18063133727761	472231	8746
167	20539330895161	498961	9073
170	23300957849401	526681	9406
173	26374921015465	555409	9745
176	29790118757485	585163	10090
179	33577549990021	615961	10441
182	37770426667621	647821	10798
185	42404290220701	680761	11161
188	47517132005785	714799	11530
191	53149517838145	749953	11905
194	59344716674881	786241	12286
197	66148833516481	823681	12673
200	73610946594901	862291	13066
203	81783248916205	902089	13465
206	90721194225805	943093	13870
209	100483647464341	985321	14281
212	111133039782241	1028791	14698
215	122735528181001	1073521	15121
218	135361159849225	1119529	15550
221	149084041261465	1166833	15985
224	163982512107901	1215451	16426
227	180139324122901	1265401	16873
230	197641824880501	1316701	17326
233	216582146624845	1369369	17785
236	237057400203625	1423423	18250
239	259169874172561	1478881	18721
242	283027239138961	1535761	19198
245	308742757412401	1594081	19681

248	336435498030565	1653859	20170
251	366230557228285	1715113	20665
254	398259284417821	1777861	21166
257	432659513748421	1842121	21673
260	469575801313201	1907911	22186
263	509159668071385	1975249	22705
266	551569848553945	2044153	23230
269	596972545420681	2114641	23761
272	645541689936781	2186731	24298
275	697459208436901	2260441	24841
276	801801	77	26
278	752915294844805	2335789	25390
281	812108689316605	2412793	25945
284	875246963075641	2491471	26506
287	942546809507041	2571841	27073
290	1014234341580001	2653921	27646
293	1090545395665825	2737729	28225
296	1171725841819765	2823283	28810
299	1258031900594701	2910601	29401
302	1349730466454701	2999701	29998
305	1447099437856501	3090601	30601
308	1550428054066945	3183319	31210
311	1660017238784425	3277873	31825
314	1776179950632361	3374281	32446
317	1899241540592761	3472561	33073
320	2029540116447901	3572731	33706
322	1169686	86	28
323	2167426914298165	3674809	34345
326	2313266677224085	3778813	34990
329	2467438041160621	3884761	35641
332	2630333928051721	3992671	36298
335	2802361946353201	4102561	36961
338	2983944798951985	4214449	37630
341	3175520698569745	4328353	38305
344	3377543790718981	4444291	38986
347	3590484584279581	4562281	39673
350	3814830389763901	4682341	40366
353	4051085765338405	4804489	41065
356	4299772970669905	4928743	41770
359	4561432428664441	5055121	42481
362	4836623195166841	5183641	43198
365	5125923436689001	5314321	43921
368	5429930916234925	5447179	44650
371	5749263487290565	5582233	45385
374	15064335000	9000	624
374	6084559596046501	5719501	46126
377	6436478791921501	5859001	46873
380	6805702246455001	6000751	47626
383	7192933280636545	6144769	48385
386	7598897900740225	6291073	49150
389	8024345342732161	6439681	49921

392	8470048625319061	6590611	50698
395	8936805111705901	6743881	51481
398	9425437080130765	6899509	52270
401	9936792303244885	7057513	53065
404	10471744636405921	7217911	53866
407	11031194614952521	7380721	54673
410	11616070060528201	7545961	55486
413	12227326696522585	7713649	56305
416	12865948772698045	7883803	57130
419	13532949699069781	8056441	57961
422	14229372689107381	8231581	58798
425	14956291412325901	8409241	59641
428	15714810656334505	8589439	60490
431	16506066998410705	8772193	61345
434	17331229486668241	8957521	62206
437	18191500330886641	9145441	63073
440	19088115603070501	9335971	63946
443	20022345947806525	9529129	64825
446	20995497302486365	9724933	65710
449	22008911627463301	9923401	66601
452	23063967646210801	10124551	67498
455	24162081595551001	10328401	68401
458	25304707986021145	10534969	69310
461	26493340372446025	10744273	70225
464	27729512134784461	10956331	71146
467	29014797269317861	11171161	72073
470	30350811190248901	11388781	73006
473	31739211541778365	11609209	73945
476	33181699020728185	11832463	74890
479	34680018209778721	12058561	75841
482	36235958421388321	12287521	76798
485	37851354552463201	12519361	77761
488	39528087949845685	12754099	78730
491	41268087286688845	12991753	79705
494	43073329449785581	13232341	80686
497	44945840437920181	13475881	81673
500	46887696271310401	13722391	82666
503	48901023912208105	13971889	83665
506	50988002196726505	14224393	84670
509	53150862777962041	14479921	85681
512	55391891080478941	14738491	86698
515	57713427266224501	15000121	87721
518	60117867211943125	15264829	88750
521	62607663498157165	15532633	89785
524	65185326409782601	15803551	90826
527	67853424948447601	16077601	91873
530	70614587856582001	16354801	92926
533	73471504653345745	16635169	93985
536	76426926682464325	16918723	95050
539	79483668172039261	17205481	96121
542	82644607306401661	17495461	97198

545	85912687310076901	17788681	98281
548	89290917543928465	18085159	99370
551	92782374613548985	18384913	100465
554	96390203489966521	18687961	101566
557	100117618642734121	18994321	102673
560	103967905185470701	19304011	103786
563	107944420033921285	19617049	104905
566	112050593076604645	19933453	106030
569	116289928358116381	20253241	107161
572	120666005275155481	20576431	108298
575	125182479785342401	20903041	109441
578	129843085628896705	21233089	110590
581	134651635563242305	21566593	111745
584	139612022610608341	21903571	112906
587	144728221318693741	22244041	114073
590	150004289034463501	22588021	115246
593	155444367191144725	22935529	116425
596	161052682608490465	23286583	117610
599	166833548806379401	23641201	118801
602	172791367331819401	23999401	119998
605	178930629099423001	24361201	121201
608	185255915745422845	24726619	122410
611	191771900995295125	25095673	123625
614	198483352045059061	25468381	124846
617	205395130956320461	25844761	126073
620	212512196065127401	26224831	127306
623	219839603404706065	26608609	128545
626	227382508142144785	26996113	129790
629	235146166029094321	27387361	131041
632	243135934866552421	27782371	132298
635	251357275983800701	28181161	133561
638	259815755731561885	28583749	134830
641	268517046989445445	28990153	136105
644	277466930687749681	29400391	137386
647	286671297343688281	29814481	138673
650	296136148612109401	30232441	139966
653	305867598850775305	30654289	141265
656	315871876700270605	31080043	142570
659	326155326678607141	31509721	143881
823	197427385	694	113
2378	432684460	604	103
2386	29437553530	4970	420
31265	90525801730	2407	259

Table 1: Polygonal Cannonball Numbers