On Decompositions of Infinite Groups

A. Horn, D. Simms, et al.

17th February 2020

1 $(\mathbb{Z}, +)$

$\mathbb{Z}$ is not decomposable. Suppose we had $\mathbb{Z} \cong G \times H$ for nontrivial G, H. Then since the subgroups of $\mathbb{Z}$ are of the form $k\mathbb{Z}$, we have $G \cong n\mathbb{Z}$ and $H \cong m\mathbb{Z}$ for some n, m.

Since $k\mathbb{Z} \cong \mathbb{Z}$, this implies $\mathbb{Z} \cong G \times H \cong \mathbb{Z} \times \mathbb{Z}$. This is not possible since $\mathbb{Z}$ has $2\mathbb{Z}$ as its only subgroup of index two, but $\mathbb{Z} \times \mathbb{Z}$ has two subgroups of index two: $2\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z} \times 2\mathbb{Z}$.

Hence $\mathbb{Z}$ is not decomposable.

2 $(\mathbb{Q}, +)$

Suppose we had $\mathbb{Q} \cong G \times H$ for some nontrivial groups G, H. Both G and H have a subgroup isomorphic to $\mathbb{Z}$, since each contains a nonzero element, and each element of $\mathbb{Q}$ has infinite order.

Hence $\mathbb{Q}$ has a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Suppose $a/b, c/d \in \mathbb{Q}$ generate the corresponding subgroups isomorphic to $\mathbb{Z} \times 0\mathbb{Z}$ and $0\mathbb{Z} \times \mathbb{Z}$ respectively, so $\mathbb{Z} \times \mathbb{Z} \cong \langle a/b \rangle \times \langle c/d \rangle$.

Then $bd[\langle a/b \rangle \times \langle c/d \rangle] = \{mad + nbc : m, n \in \mathbb{Z}\}$, as $\mathbb{Q}$ is abelian. But by Bézout's Theorem, $mad + nbc$ can take on precisely the values in $[\gcd(ad, bc)]\mathbb{Z}$, so in fact

$$\langle a/b \rangle \times \langle c/d \rangle \cong \left[\frac{\gcd(ad, bc)}{bd}\right]\mathbb{Z} \cong \mathbb{Z}$$

giving $\mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z}$, which is a contradiction, as previously established.

So also $\mathbb{Q}$ is not decomposable.

3 $(\mathbb{R}, +)$

Let $\mathcal{H}$ be a Hamel basis for $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Then any partition of $\mathcal{H}$ as a disjoint union $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ gives rise to two subgroups $G_1 = \operatorname{span} \mathcal{H}_1, G_2 = \operatorname{span} \mathcal{H}_2 \leq \mathbb{R}$. These are subgroups since they are subspaces by definition.

Now we can apply the Direct Product Theorem, since

(i) By linear independence, if $x_1 \in G_1, x_2 \in G_2$ with $x_1 = x_2$, then $x_1 = x_2 = 0$, so $G_1 \cap G_2$ is precisely $\{0\}$.

(ii) $\mathbb{R}$ is abelian, so trivially, $x_1 + x_2 = x_2 + x_2 \,\forall\, x_1 \in G_1, x_2 \in G_2$.

(iii) Each $x \in \mathbb{R}$ can be written as a linear combination of elements of $\mathcal{H}$, by definition. Then simply splitting this linear combination into linear combinations of the elements in $\mathcal{H}_1$ and $\mathcal{H}_2$, we get $x_1 \in G_1, x_2 \in G_2$ with $x_1 + x_2 = x$.

So $\mathbb{R} \cong G_1 \times G_2$ for any such choice.

To be explicit, a possible choice here might be $\mathcal{H}_1 = \mathcal{H} \cap \mathbb{Q}$, $\mathcal{H}_2 = \mathcal{H} \setminus \mathbb{Q}$, or possibly replacing $\mathbb{Q}$ with some more imaginative subsets of $\mathbb{R}$. Here, we know $|\mathcal{H}_1| = 1$ precisely, as there must be a rational in $\mathcal{H}$, so neither of G_1, G_2 is trivial.

4 $(\mathbb{C}, +)$

$\mathbb{C}$ is quite straightforwardly isomorphic to $\mathbb{R} \times \mathbb{R}$ via $z \mapsto (\operatorname{Re} z, \operatorname{Im} z)$.

5 $(\mathbb{Q}^\times, \cdot)$

This is isomorphic to $C_2 \times (\mathbb{Q}^+, \cdot)$ via $x \mapsto (\operatorname{sgn} x, |x|)$ (where $\operatorname{sgn} x := x/|x|$).

6 $(\mathbb{R}^\times, \cdot)$

This is isomorphic to $C_2 \times (\mathbb{R}^+, \cdot)$ via $x \mapsto (\operatorname{sgn} x, |x|)$.

7 $(\mathbb{C}^\times, \cdot)$

This is isomorphic to $S^1 \times (\mathbb{R}^+, \cdot)$ via $re^{i\theta} \mapsto (e^{i\theta}, r)$.

8 $\big(\{a/2^n : a, n \in \mathbb{Z}\}, +\big)$

9 $(\mathbb{Q}^+, \cdot)$

10 $(\mathbb{R}^+, \cdot)$

This is isomorphic to $(\mathbb{R}, +)$ via $x \mapsto \log x$, so we can reuse the previous construction.

11 $S^1 \cong \mathbb{R}/\mathbb{Z}$

12 $\mathbb{Q}/\mathbb{Z}$