Equation Automata

Cellular Automaton Outputs an Equation

by

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Model: Equation Automaton[1]

Runs upon: PyCX 0.3 Realtime Visualization Template

PyCX 0.3 Realtime Visualization Template: Written by Chun Wong

Revised by Hiroki Sayama

Requires PyCX simulator to run

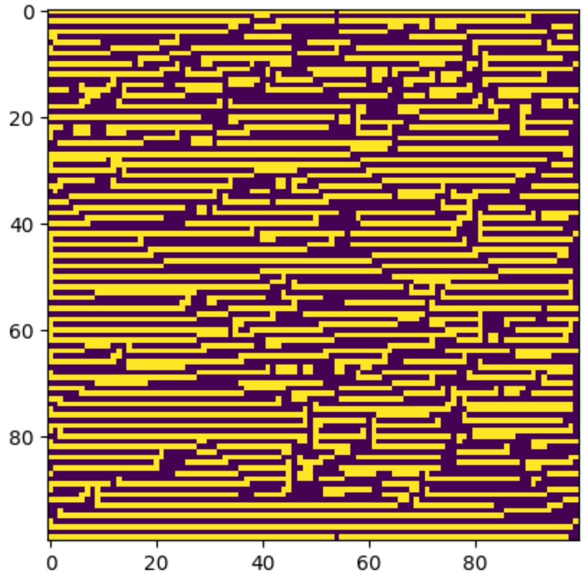
PyCX available from: https://github.com/hsayama/PyCX

1. https://github.com/goektug/Equation-Automata/

Idea

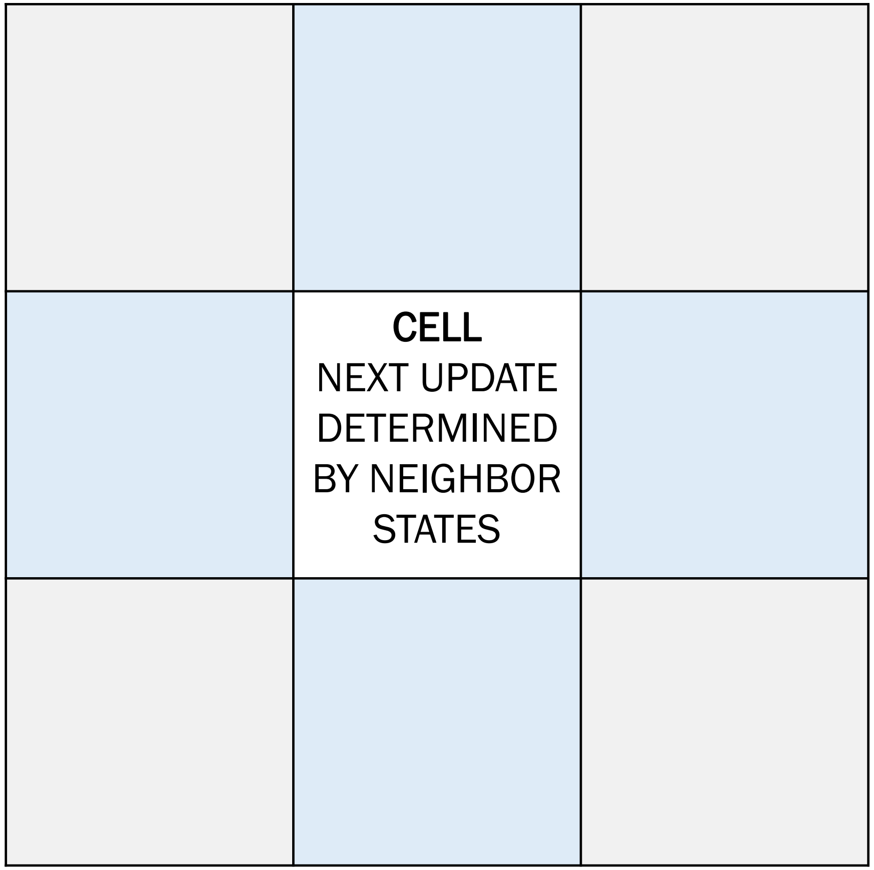
Critical behavior by extending cells of an automaton uniaxially

This is called the “driving motion”, aimed at maximizing cell count



Left: Maximum cell count of the automaton

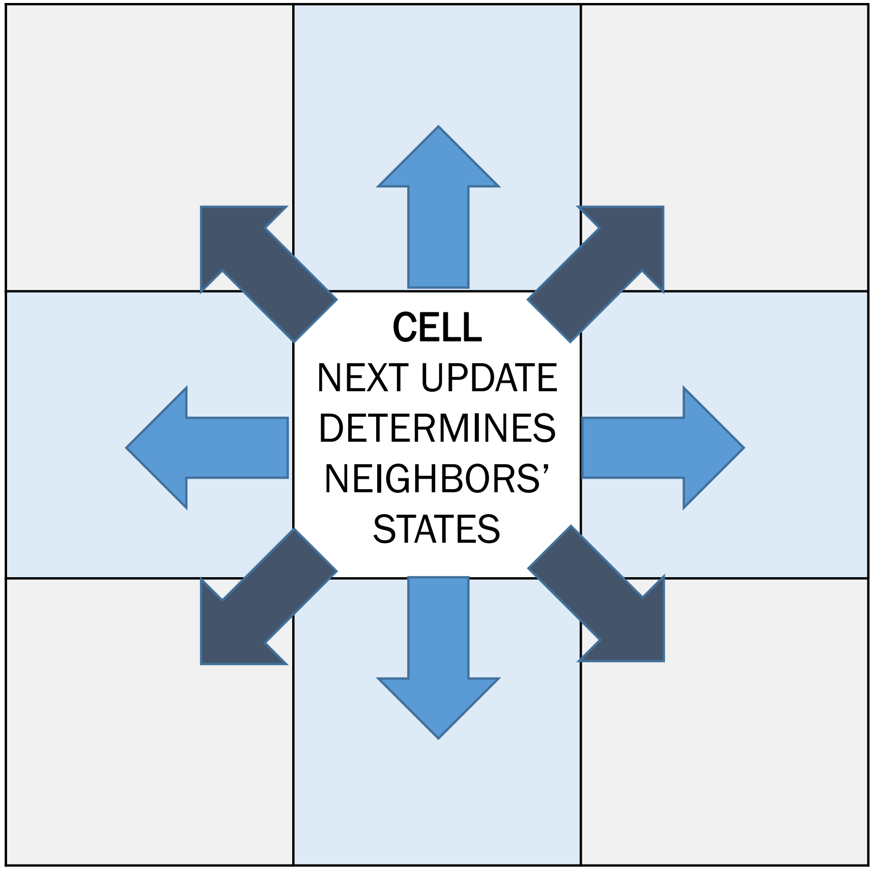
Conventional Cellular Automaton Update



A cellular automaton is a search function around a cell.

Update rule is the determination of a cell's state based on the states of its neighboring cells following a rule.

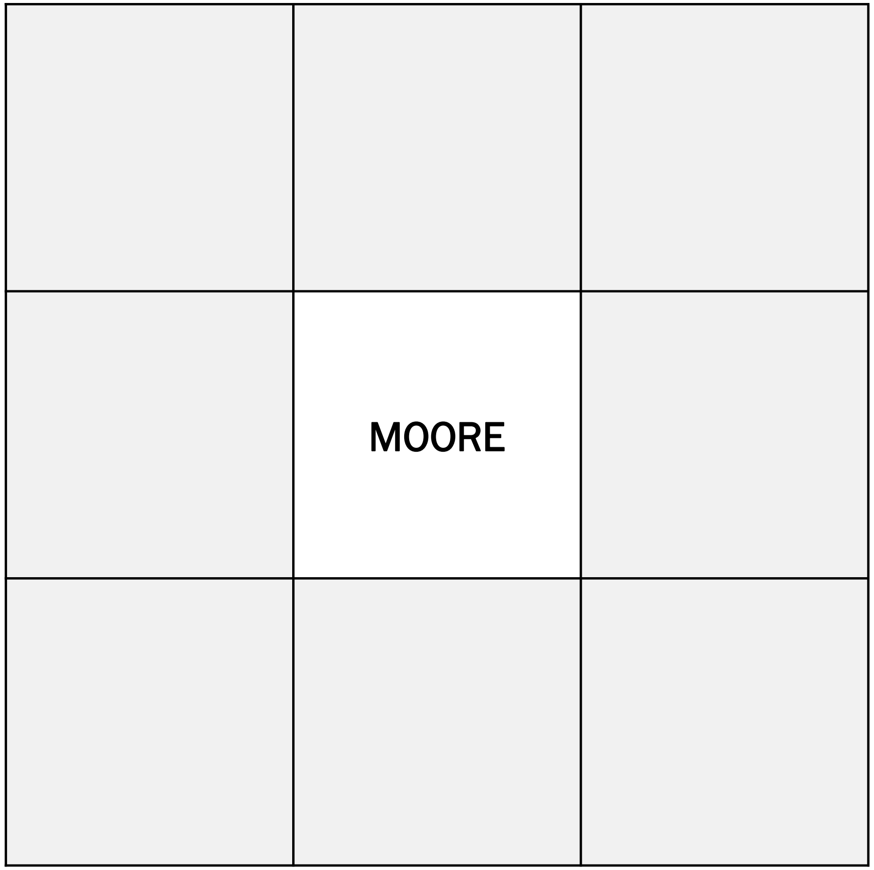
1st Rule: Inverse Cell Update

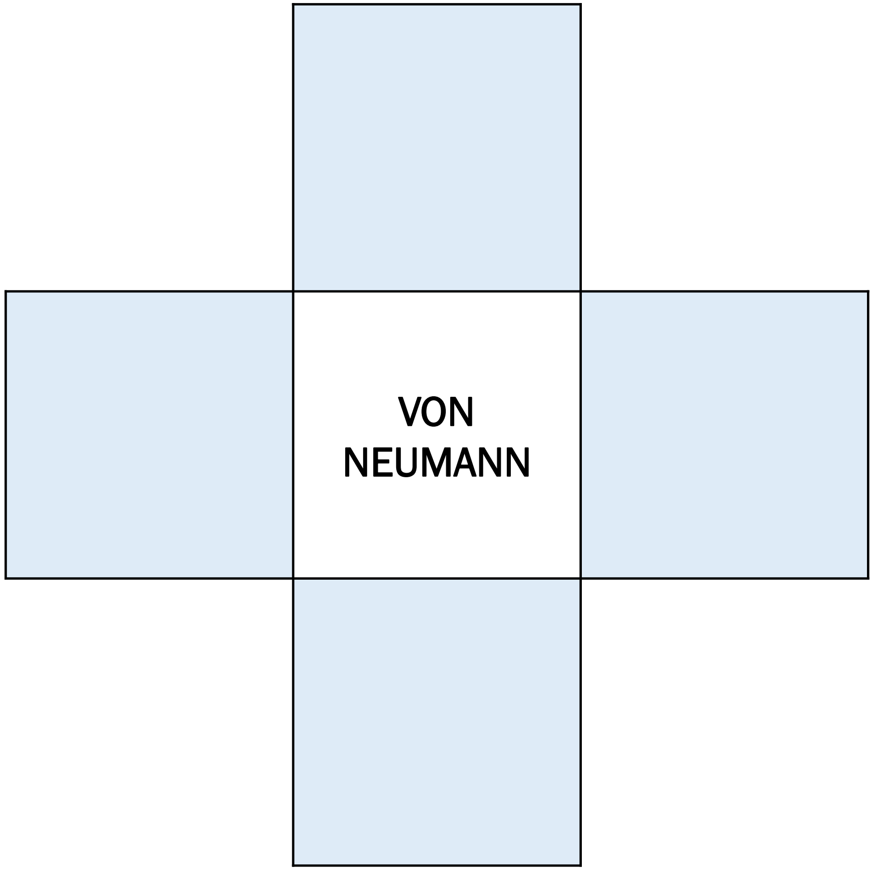


To achieve driving motion, update of a cell is reversed. Cell's state dictates its neighborhood.

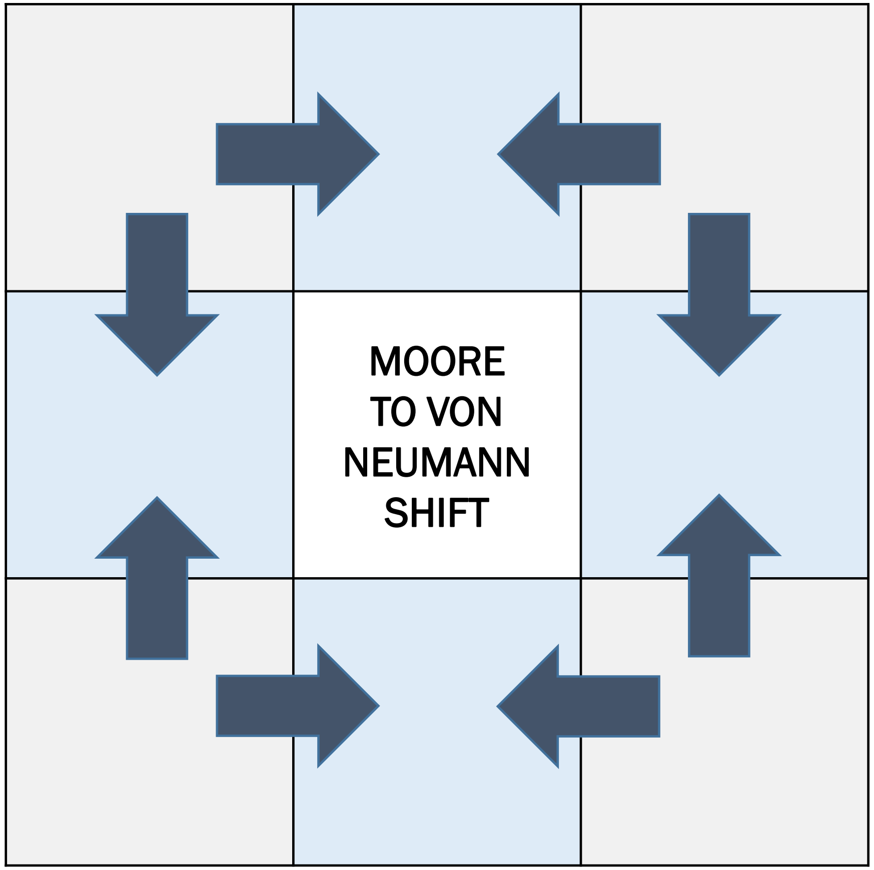
```
elif c[x, y] == 1:
    array1.append(c[x, y])
    for z in range(-1, 2):
        # block generation from randomly distributed points

        #neighbor updating from cell(x,y)
        m = number_of_upper_neighbors(x, y)
        if m == 1:
            nc[x, (y + 1) % L] = 1
```

Moore vs Von Neumann Neighborhoods





2nd Rule: Moore to Von Neumann Shift



To achieve criticality, competition between Moore cells and von Neumann cells are needed.

```
#neighbor updating from cell(x,y)
m = number_of_upper_neighbors(x, y)
if m == 1:
    nc[x, (y + 1) % L] = 1

n = number_of_lower_neighbors(x, y)
if n == 1:
    nc[x, (y - 1) % L] = 1

k = number_of_right_neighbors(x, y)
if k == 0 and (m <= 1 or n <= 1):
    nc[(x + 1) % L, (y + z) % L] = 1

l = number_of_left_neighbors(x, y)
if l == 1 and (m > 1 or n > 1):
    nc[(x - 1) % L, (y + z) % L] = 0
```

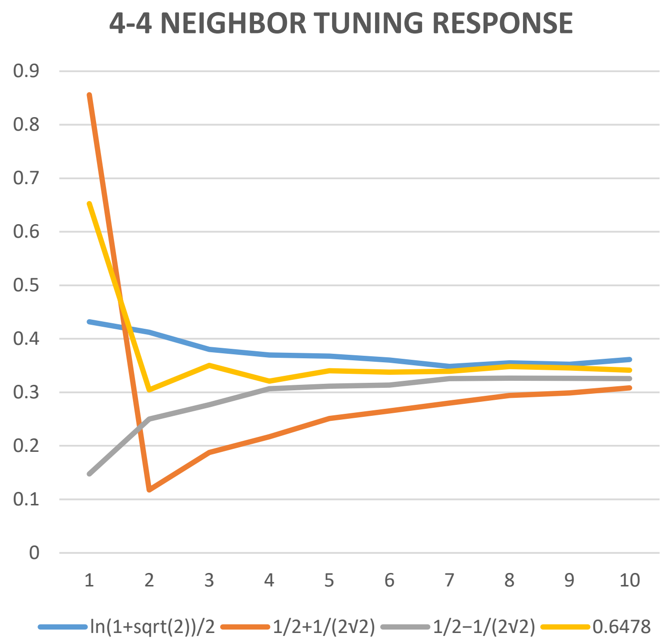
3rd Rule: Neighborhood Configuration

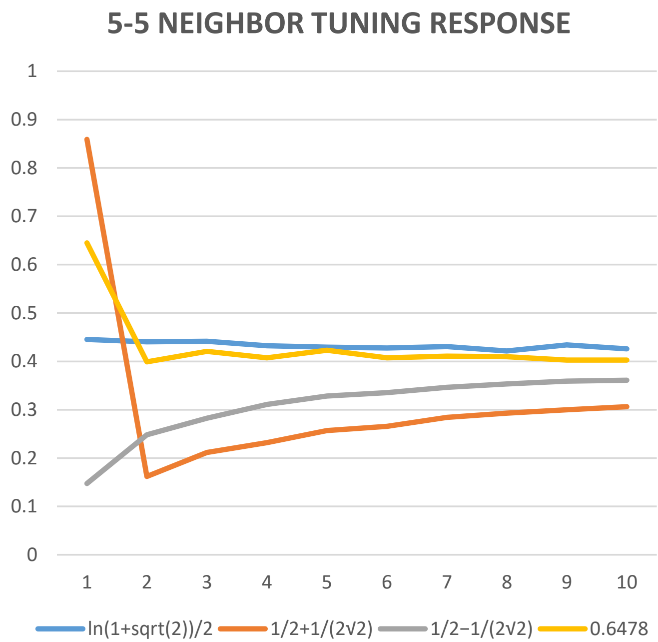
1	1	1
0	TUNING THE NUMBER OF NEIGHBORS WITH STATE 1	0
1	1	1

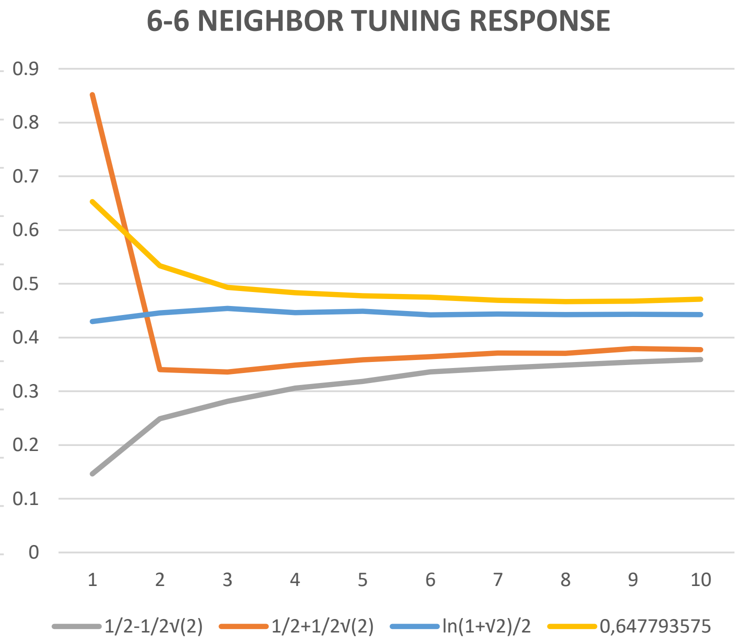
Based on cell's value, Moore and von Neumann neighborhoods are tuned. Critical values are 6 for both.

```
 g = number_of_Moore_neighbors(x, y) #CA tuning
 if c[x, y] == 0:
     nc[x, y] = 0 if g <= 6 else 1
     array0.append(c[x, y])

h = number_of_Neumann_neighbors(x, y) #CA tuning
if h >= 1:
    nc[x, y] = 1 if g <= 6 else 0
```

Neighbor Tuning for Ising Criticality







Tuning for **4** and **5** neighbors has **Inverse Ising critical temperature** as the highest state **1** cell count.

For **6** neighbors however, there is another maximum susceptibility, corresponding to ferromagnetism.

4th Rule: Coupled Cellular Automata

$$\rho(t+1) = (1-p)\varphi\big(\rho(t)\big)$$

Left: Stochastic coupling mechanism evolution equation.[2]

1/8: Inverse Moore Neighborhood

$$p(1-p) = \frac{1}{8}$$

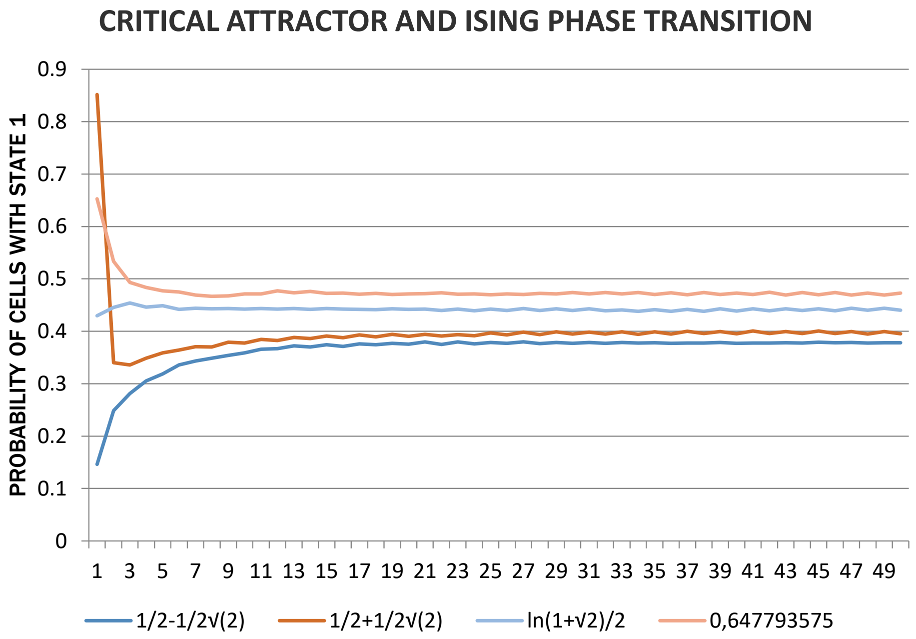
$$p^2 - p + \frac{1}{8} = 0$$

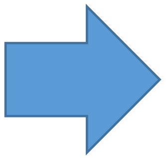
$$p = \frac{1}{2} \pm \frac{1}{2\sqrt{2}}$$

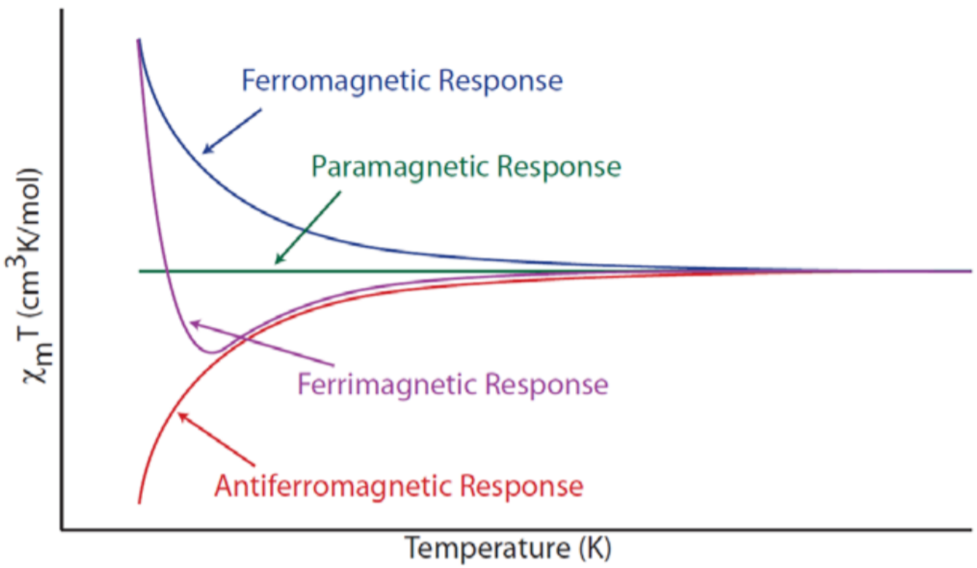
```
if g / 8 > (1 - p) * p:  # coupling function
    nc[(x + 1) % L, y] = 1
elif g / 8 < (1 - p) * p:
    nc[(x - 1) % L, y] = 1
else:
    nc[x, y] = 1
```

2. Synchronization of stochastically coupled cellular automata. Luis G. Morelli and Damian H. Zanette. Phys. Rev. E 58, R8(R) – Published 1 July 1988

Coupling Function – Magnetizing Automaton







2D Ising square lattice model’s critical inverse temperature is the **paramagnetic response**.

Upper coupling is slightly striped, which is weak ferromagnetism (**ferrimagnetism**).

Lower coupling is vortex shaped, which is **antiferromagnetic** behavior.

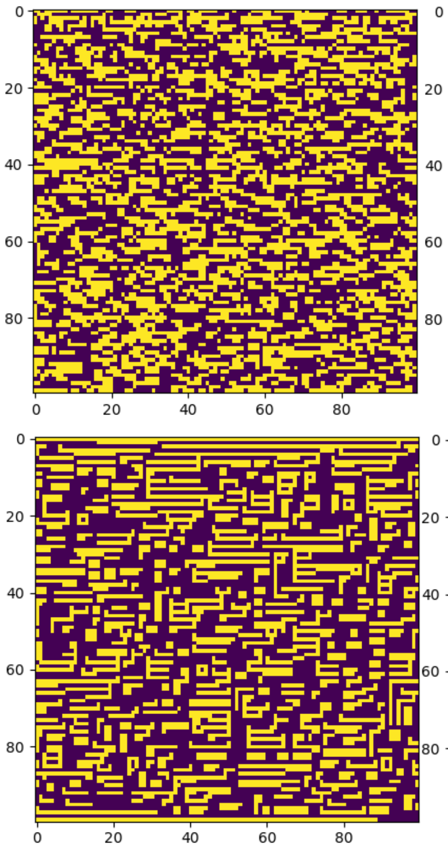
Right Figure: http://bh.knu.ac.kr/~leehi/index.files/MPMS_HIL.pdf

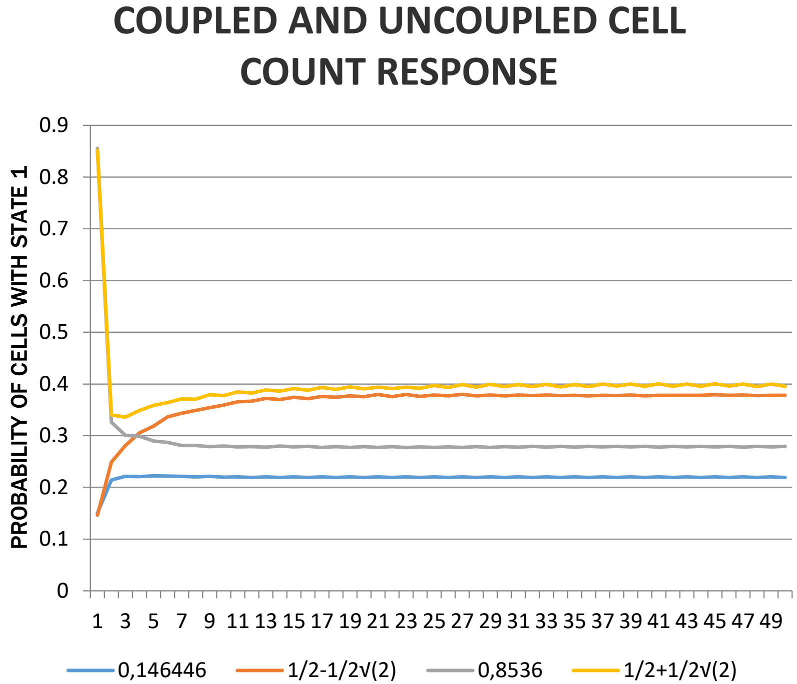
Evolution of Coupled CAs

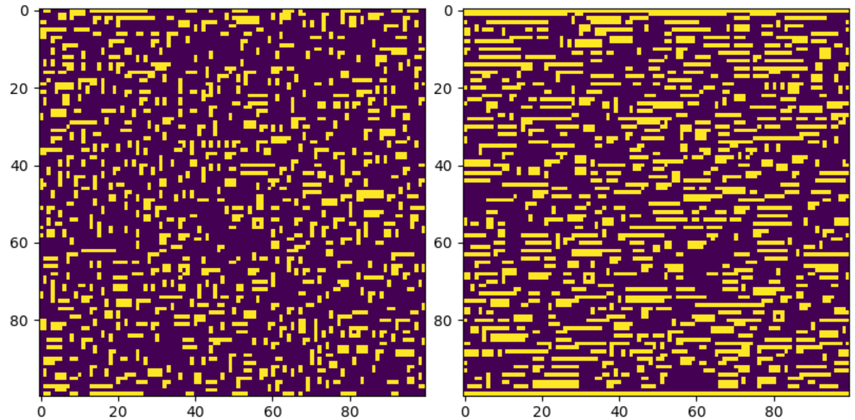
$$p(1-p)=\frac{1}{8}$$

$$p^2-p+\frac{1}{8}=0$$

$$p=\frac{1}{2}\pm\frac{1}{2\sqrt{2}}$$







CLOCKWISE: Neighbor update without coupling; Coupling without update; $\frac{1}{2}-\frac{1}{2\sqrt{2}}$ coupling, $\frac{1}{2}+\frac{1}{2\sqrt{2}}$ coupling

RIGHT TOP: Cell count of coupled states and uncoupled states.

RIGHT BOTTOM: Uncoupled states.

Trigonometric Expression of Coupling

$$\cos^2\left({}^{\pi}/_{8}\right) = \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

$$\sin^2\left({}^{\pi}/_{8}\right) = \frac{1}{2} - \frac{1}{2\sqrt{2}}$$

$$\cos^2\left({}^{\pi}/_{8}\right) - \sin^2\left({}^{\pi}/_{8}\right) = \frac{1}{\sqrt{2}} = 2\sin\left({}^{\pi}/_{8}\right)\cos\left({}^{\pi}/_{8}\right)$$

General Expression:

$$a\cos^2 x - b\sin^2 x = \sin x \cos x$$

5th Rule: Transformation

$a \cos^2 x - b \sin^2 x = \sin x \cos x$

when above equation is divided by $\cos^2 x$:

$\therefore\ b \tan^2 x + \tan x - a = 0$

When the following transformations are applied:

5th Rule: Transformation

count0 = cell with state 0 count

count1 = cell with state 1 count

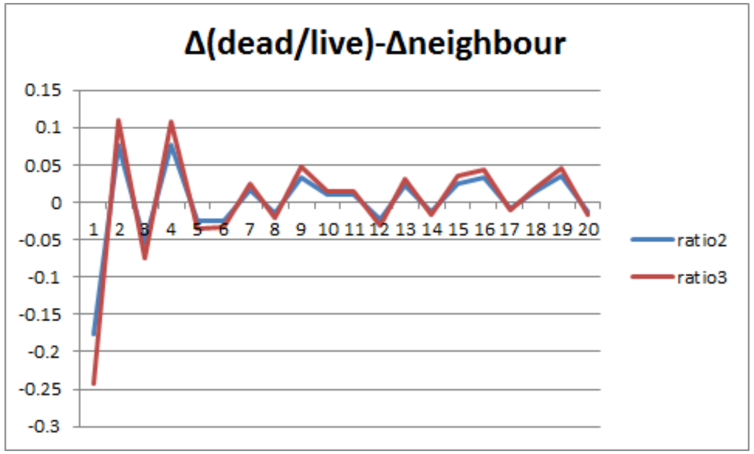
j = neighbor count, i = count0 + count1

$$\cos x = \frac{j}{i}$$

$$\sin x = \frac{count0}{count1}$$

$-\Delta \cos x \approx \Delta \sin x$ which is mathematically correct for $\frac{\pi}{4}$

$$-\Delta \cos x = ratio3$$
$$\Delta \sin x = ratio2$$



5th Rule: Transformation

$$\cos^2\left({}^{\pi}/_{8}\right) - \sin^2\left({}^{\pi}/_{8}\right) = \frac{1}{\sqrt{2}} = 2\sin\left({}^{\pi}/_{8}\right)\cos\left({}^{\pi}/_{8}\right) = \sin\frac{\pi}{4}$$

$$\frac{j}{i} = \frac{count0}{count1}$$

$$\frac{1}{ratio1} = \frac{{}^{count0}/_{count1}}{{}^{j}/_{i}} = \tan x$$

5th Rule: Transformation

$$-\Delta \cos x \cdot \sin^2 x + \ \sin x \cdot \cos x - \Delta \sin x \cdot \cos^2 x = 0$$

transforms into:

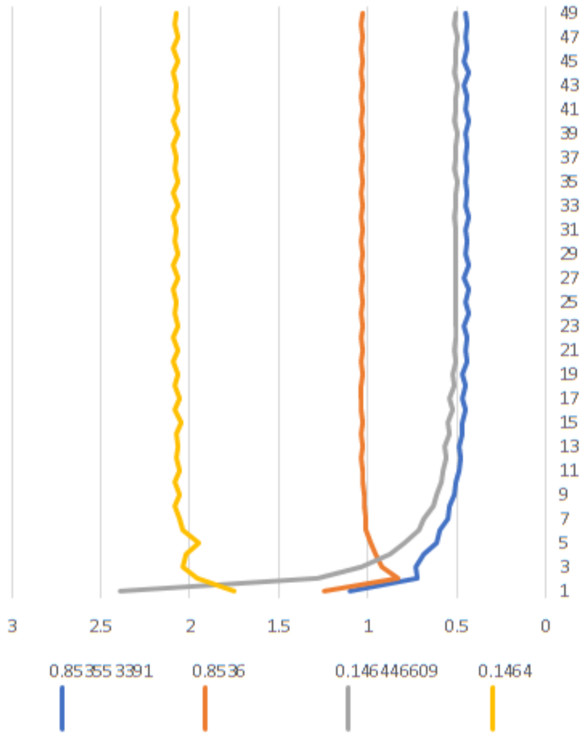
$$\therefore \frac{ratio3}{ratio1 * ratio1} + \frac{1}{ratio1} - ratio2$$

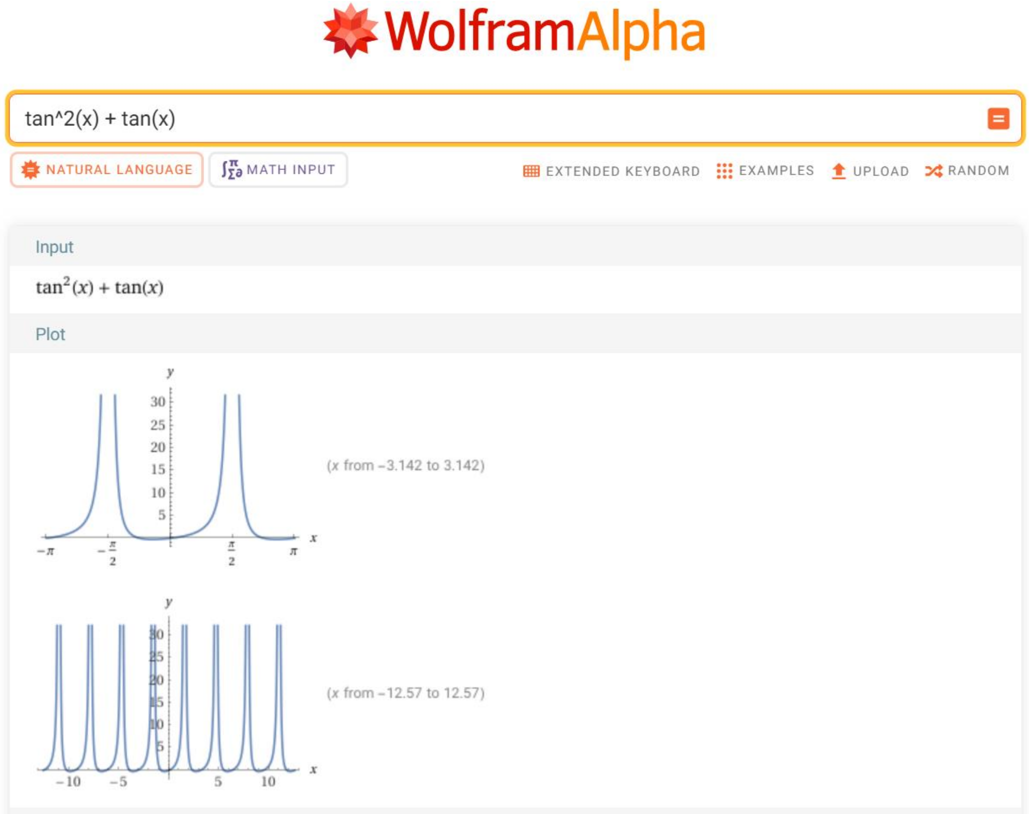
which plots the tangent graph:

Cotangent Graph Output of CA

Coupled and uncoupled values are arms of the tangent graph.

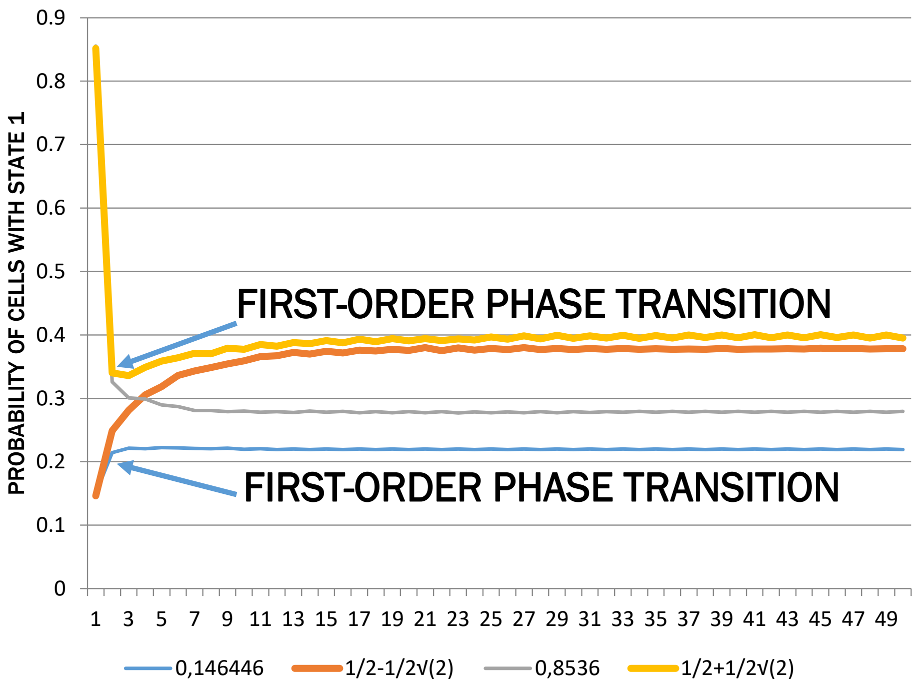
tan^{2}x + tanx Graph





First-Order Phase Transition

Ferrimagnetic phase (**first-order phase transition – top, bold)**
Antiferromagnetic phase (**first-order phase transition - bottom)**

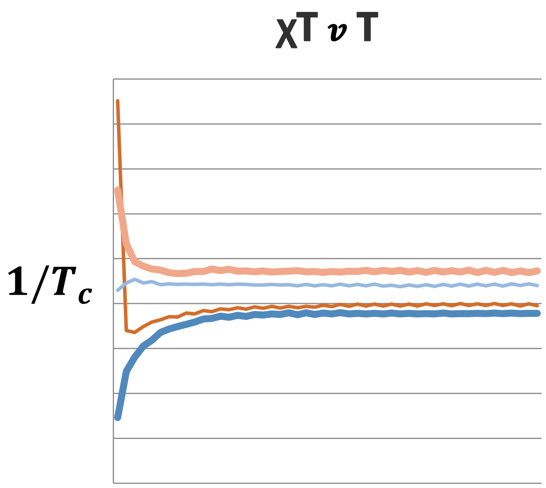


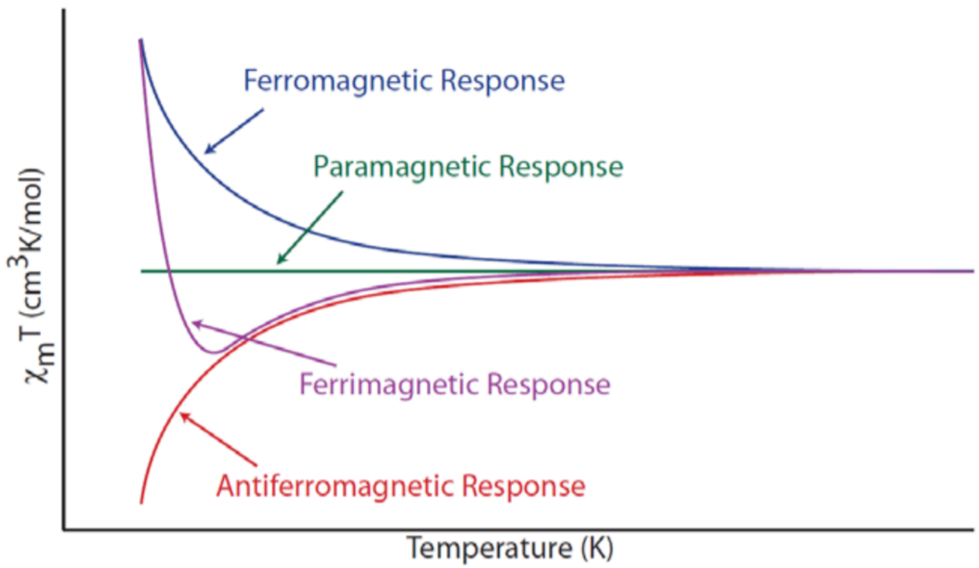
$$p = \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx \mathbf{0.8535; 0.8536}$$

$$p = \frac{1}{2} - \frac{1}{2\sqrt{2}} \approx \mathbf{0.1466; 0.1465}$$

Magnetic Susceptibility v. Temperature

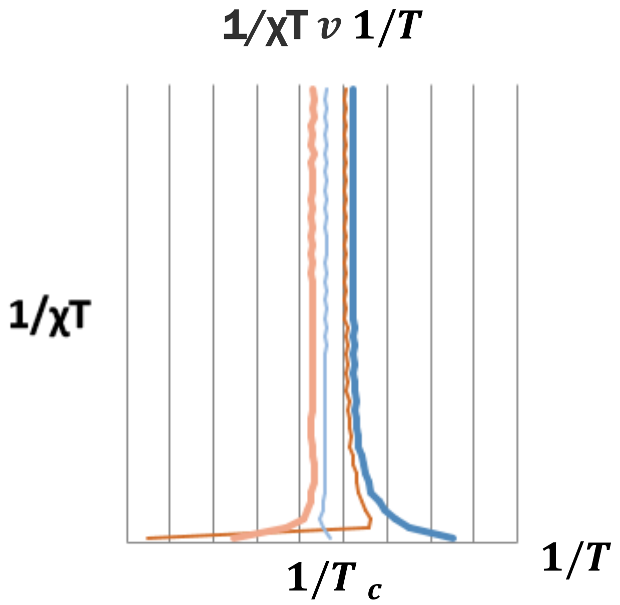
The transition from antiferromagnetic to ferromagnetic phase is very similar to molar magnetic susceptibility to temperature response.

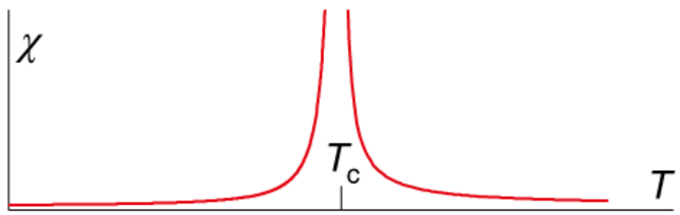




Second-Order Phase Transition

Inverse critical temperature and inverse magnetic susceptibility

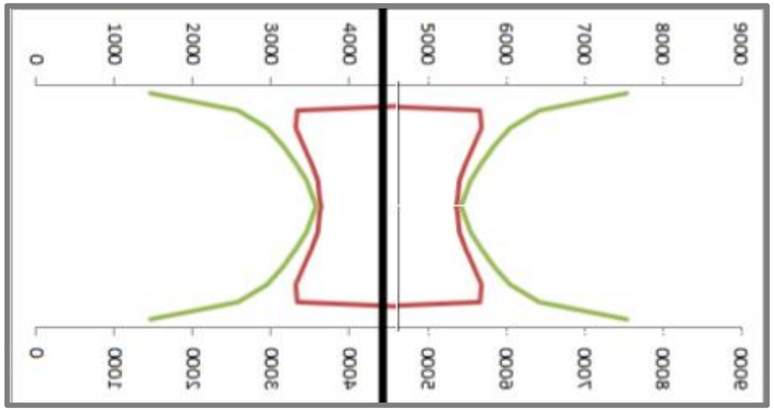




SECOND-ORDER PHASE TRANSITION

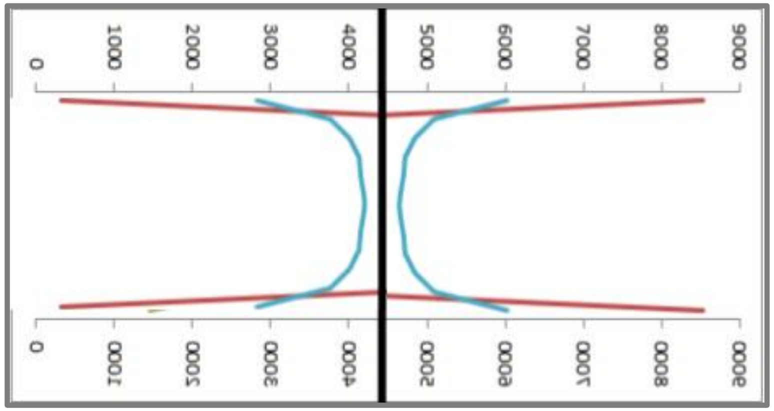
Symmetry Axes of the Magnetic Phases

There are two symmetry axes that correspond to different phases[3].



Polarization: Catenoid around p = 1/2

FIRST-ORDER
PHASE TRANSITION



Ising criticality: Pseudosphere around p = ln(1+√2)/2

SECOND-ORDER
PHASE TRANSITION

3. https://www.researchgate.net/publication/348521157_A_2D_Ising_Model_Cellular_Automaton_Mapped_Onto_Catenary_Involute

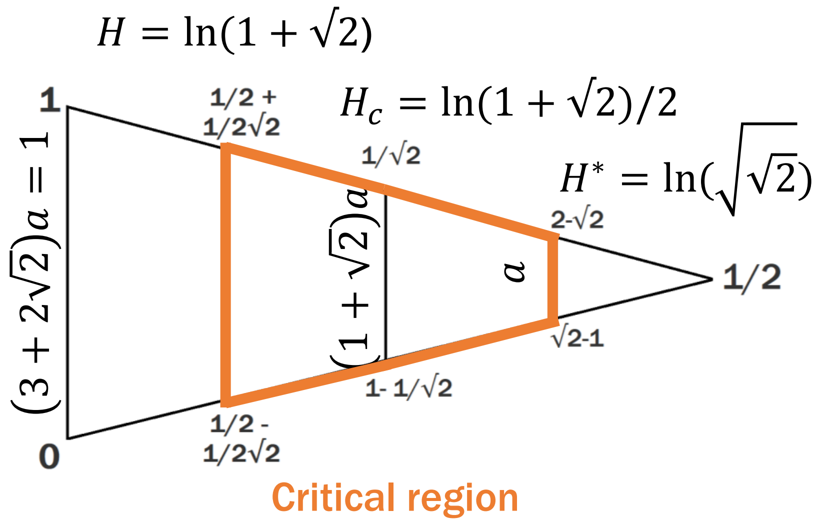
Magnetic Fields Visualized

H, H_c and H* are magnetic fields.

Maximum value of cell counts

= maximum ferromagnetism:

$$sigmoid\left(\frac{1}{\sqrt{2}}\right) \approx 0.6697$$



Susceptibility – Curie Law

$\chi = \frac{c}{T \pm \theta} = \frac{1}{4 \pm 2\sqrt{2}}$ for ferrimagnetism and antiferromagnetism

$\chi = \frac{c}{T_c} = \frac{\ln(1+\sqrt{2})}{2}$ for paramagnetism

$\chi = \frac{c}{T_f} = \frac{1}{1+ e^{-1/\sqrt{2}}}$ for global maximum of ferromagnetism