# Building Quantum Computers (Cont.)

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These are notes on the lecture about building quantum computers, as taught by Seth Lloyd in 8.370 Quantum Computation at MIT during Fall 2016.

## I. THE PHYSICS

Remember that we have the Schrödinger equation as follows:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

Where $H = H^\dagger$ is the Hamiltonian for the quantum system. We then have that time evolution of $\psi$ to be

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$$

For example (where we set $\hbar = 1$), we have that H generates the dynamics of the system, and it is an observable corresponding to the energy of the system (i.e. $H|E_j\rangle = E_j|E_j\rangle$). We therefore have that

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \frac{-\hbar\omega}{2}\sigma_z|\psi(t)\rangle$$

$$|\psi(t)\rangle = e^{-i\omega t\sigma_z/2}|\psi(0)\rangle$$

where $\omega$ is the Larmor frequency. Now suppose that we can apply two different Hamiltonians $H_1$ and $H_2$ (e.g. $H_1 = \frac{-\hbar\omega}{2}\sigma_x$ and $H_2 = \frac{-\hbar\omega}{2}\sigma_y$, we can make almost any transformation we want. If we apply the following to state $\psi$:

$$e^{+iH_2\Delta t}e^{+iH_1\Delta t}e^{+-H_2\Delta t}e^{-iH_1\Delta t}|\psi\rangle$$

then we get

$$(I - \frac{[H_2, H_1]\Delta t^2}{2})|\psi\rangle + O(\Delta t^3)$$

**HW Problem 10.1** Show that this is true, by expanding the operation.

This is a pretty useful fact; for example if $H_1 = \frac{-\hbar\omega}{2}\sigma_x$ and $H_2 = \frac{-\hbar\omega}{2}\sigma_y$, then

$$[H_2, H_1] = \frac{\hbar^2\omega^2}{4}[\sigma_y, \sigma_x] = \frac{\hbar^2\omega^2}{4}(-2i\sigma_z)$$

This tells us that an infinitesimal rotation by $\Delta\varphi = \omega\Delta t$ about the -x axis, -y, +x, +y gives an (even more) infinitesimal rotation by $\frac{\Delta\varphi^2}{2}$ about the Z-axis.

More generally, the ability to apply $H_1$ and $H_2$ translates to into the ability to apply $\pm i[H_1, H_2]$.

**HW Problem 10.2** Show that $(i[H_1, H_2])^\dagger = i[H_1, H_2]$, which means that this operator is *Hermitian.* It also turns out that $[H_1, H_2]$ is anti-Hermitian, or $[H_1, H_2]^\dagger = -[H_1, H_2]$.

This is cool in that we can now chain these commutators together to get interesting transformations, e.g. $\pm i[i[\sigma_x, \sigma_y], \sigma_x]$. This leads to a fact: if we have a d-dimensional quantum system ($|\psi\rangle \in C^d$), then generically the operators $H_1, H_2, i[H_1, H_2], i[i[H_1, H_2], H_1], ...$ span the space of all Hermitian operators for the system; any $H = \sum_k a_k H_k$. Since any unitary $U = e^{-iHt}$ for some $H$,$t$, then by applying $H_1$, $H_2$ in suitable sequences, we can build up any transformation. Even further, in the discrete case with unitary operators $U_1$ and $U_2$, we can approximate any desired $U$ by sequences $...U_1U_2U_2U_1U_2...$ to any desired degree of accuracy (this comes from the **Solovay-Kitaev theorem**).

## II. GATE EXAMPLE: SWAP

The SWAP operator takes two registers and swaps them, i.e $S|\psi\rangle \otimes |\varphi\rangle = |\varphi\rangle|\psi\rangle$

However, we can't use this as a universal gate for the whole Hilbert space; for one, it preserves the number of zeros and ones. It can, on the other hand, be universal on the symmetric subspace.

## III. ROTATIONS CONTINUED

Suppose that we can apply single qubit rotations, such as $e^{-i\sigma_z^1 \otimes \sigma_z^2 \Delta t}$, then we have $(\sigma_z^1 \otimes \sigma_z^2)|\uparrow\rangle_1 \otimes |\uparrow\rangle_2 = +1|\uparrow\rangle_1 \otimes |\uparrow\rangle_2$ and also that $(\sigma_z^1 \otimes \sigma_z^2)|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 = -1|\uparrow\rangle_1 \otimes |\downarrow\rangle_2$

**HW Problem 10.3** Show that $\sigma\hat{j}^1 \otimes I^2, I^1 \otimes \sigma_{\hat{j}}^2, \sigma_z^1 \otimes \sigma_z^2$ are universal: they can generate any Hamiltonian containing, for example, $\sigma_x^1 \otimes \sigma_y^2$. *Hint: to generate $\sigma_x^1 \otimes \sigma_z^2$, see images. Also use the fact that any two qubit Hamiltonian can be written as* $H = a_{00}I^1 \otimes I^2 + a_{01}I^1 \otimes \sigma_x^2 + a_{02}I^1 \otimes \sigma_y^2 + a_{11}\sigma_x^1 \otimes \sigma_x^2 + ...$

## IV. CONCLUSION

If we have the ability to control small transformation very accurately, then we can construct a universal quantum machine. If we have only single qubit operations, with the addition of only one two qubit operation, then we can do this. Surprisingly, this leads directly into an algorithm for quantum simulation.

Suppose we have a Hamiltonian for a system where $H = \sum_k H_k$, where $H_k$ is a one or two qubit/spin interaction. Suppose we have control over pairs of qubits, which means that we can simulate / apply $e^{-iH_k \Delta t}$. The claim is that we can approximate $e^{-iHt}|\psi\rangle$. The method for doing this is to do things infinitesimally, i.e.

$$e^{-iH\Delta t} = I - i \sum_k H_k \Delta t + O(\Delta t^3)$$

When we expand this, we find that we can complete each component by just applying two qubit operations.