Conventions on notation for this thesis

We will use the following conventions throughout the thesis:

- $\mathbb{N} := \{0, 1, \dots\}$
- $\mathbb{N}_{\geq 1} := \{1, 2, \dots\}$
- $\underline{n} := \{1, 2, \dots, n\}$
- $\underline{n}_0 := n \cup \{0\}$
- $A \setminus B = \{a \in A \mid a \notin B\}$
- Let R be a ring, $a_1, \dots, a_n \in R$. Then $(a_1, \dots, a_n)$ denotes the smallest ideal of R containing $a_1, \dots, a_n$.
- Let R be a difference ring, $a_1, \dots, a_n \in R$. Then $[a_1, \dots, a_n]$ denotes the smallest difference ideal of R containing $a_1, \dots, a_n$.
- Let R be a difference ring. Then $A \trianglelefteq_\sigma R$ means that A is a difference ideal of R.
- The variable y in (difference) polynomial rings will, in general, mean n variables $y_1, \dots, y_n$, i.e., we will write for example $k[y]$ for $k[y_1, \dots, y_n]$.

1 Basics of Difference Algebra

2 Difference Ideals

In this section we will study difference ideals more closely. A special emphasis will be given to certain difference ideals which mimic in many ways the properties of and relationship between radical and prime ideals on (general) rings. Much of the following is based on Section 1.2 of the lecture notes of M. Wibmer [1], where most of it is worked out for an analogous case.

There are numerous properties in which we are interested for studying difference ideals. We will start by defining some of them:

Definition 2.1. *Let $\mathfrak{a} \trianglelefteq_\sigma R$ be a σ-ideal of R.*

- *Then $\mathfrak{a}$ is called a* mixed σ-ideal *if for any $f, g \in R$ with $fg \in \mathfrak{a}$ it follows that $f\sigma(g) \in \mathfrak{a}$.*
- *$\mathfrak{a}$ is called* perfect*, if $\sigma^{i_1}(f) \cdots \sigma^{i_n}(f) \in \mathfrak{a}$ implies that $f \in \mathfrak{a}$, where $n \in \mathbb{N}_{\geq 1}, i_j \in \mathbb{N}$ for all $j \in \underline{n}$.*
- *$\mathfrak{a}$ is called* reflexive*, if $\sigma(f) \in \mathfrak{a}$ implies $f \in \mathfrak{a}$.*
- *$\mathfrak{a}$ is called* σ-prime*, if $\mathfrak{a}$ is a prime, reflexive σ-ideal.*
- *The ring R is called* well-mixed*, if the zero ideal $[0]$ is mixed, and* perfectly σ-reduced*, if it is perfect.*

Remark 2.2. It is easy to see from the definitions that σ-prime ideals are perfect, and that perfect σ-ideals are mixed, radical and reflexive. Prime σ-ideals are also mixed, but not necessarily perfect. Note that there is a difference between a prime σ-ideal, and a σ-prime ideal: the former does not necessarily have to be reflexive, as is the case with the latter. Both, prime σ-ideals and σ-prime ideals will be very important throughout this thesis. It is for that reason that the distinction between both is of utmost importance.

All the above properties behave well with respect to morphisms of σ-rings in the following sense:

Lemma 2.3. *Let $\varphi : R \to S$ be a morphism of σ-rings and $\mathfrak{a} \trianglelefteq_\sigma S$ a σ-ideal of S. Then $\varphi^{-1}(\mathfrak{a}) \trianglelefteq_\sigma R$ is a σ-ideal of R. Similarly, if $\mathfrak{a}$ is a mixed σ-ideal, then so is $\varphi^{-1}(\mathfrak{a})$. The same is true for perfect and for reflexive σ-ideals.*

Proof. Since $\mathfrak{a} \trianglelefteq S$ is an ideal, so is $\mathfrak{b} := \varphi^{-1}(\mathfrak{a}) \trianglelefteq R$. Let $b \in \mathfrak{b}$. Then $\varphi(b) =: a \in \mathfrak{a}$ by definition. Since $\mathfrak{a} \trianglelefteq_\sigma S$ is a σ-ideal, $\sigma(a) \in \mathfrak{a}$, and since φ is a morphism of σ-rings it follows that $\sigma(a) = \sigma(\varphi(b)) = \varphi(\sigma(b)) \in \mathfrak{a}$. Hence, $\sigma(b) \in \mathfrak{b}$ which implies that $\mathfrak{b}$ is a σ-ideal.

Now let $\mathfrak{a}$ be mixed and $fg \in \mathfrak{b}$. This means by definition of $\mathfrak{b}$, that $\varphi(fg) = \varphi(f)\varphi(g) \in \mathfrak{a}$. Since $\mathfrak{a}$ is mixed, this in turn implies that

$$\varphi(f)\sigma(\varphi(g)) = \varphi(f)\varphi(\sigma(g)) = \varphi(f\sigma(g)) \in \mathfrak{a},$$

which yields $f\sigma(g) \in \mathfrak{b}$, so that $\mathfrak{b}$ is also mixed. The proof for perfect and for reflexive difference ideals is analogous. $\square$

Remark 2.4. Let R be a σ-ring and $\mathfrak{a} \trianglelefteq_\sigma R$ a σ-ideal. We can define a canonical σ-ring structure on the quotient ring $R/\mathfrak{a}$ via $\sigma(r+\mathfrak{a}) := \sigma(r) + \mathfrak{a}$. This is well defined and in particular makes the canonical epimorphism $\tau : R \twoheadrightarrow R/\mathfrak{a}$ a morphism of σ-rings.

Proposition 2.5. *Let R be a σ-ring and $\mathfrak{a} \trianglelefteq_\sigma R$ a σ-ideal. The canonical epimorphism $\tau : R \twoheadrightarrow R/\mathfrak{a}$ induces, in the sense of Lemma 2.3, a bijection between the sets $\{\mathfrak{b} \trianglelefteq_\sigma R/\mathfrak{a}\}$ and $\{\mathfrak{a} \trianglelefteq_\sigma \mathfrak{b} \trianglelefteq_\sigma R\}$. The same holds true if we restrict both sets to prime, radical and mixed, σ-prime or perfect σ-ideals.*

Proof. See Proposition 1.2.8 of [1]. $\square$

Remark 2.6. Let R be a σ-ring, and $F \subseteq R$ be a subset of R. Any intersection of mixed, radical σ-ideals containing F is also a mixed, radical σ-ideal, which of course contains F. This means that there is a smallest (with respect to inclusion) mixed, radical σ-ideal $\mathfrak{a}$ containing F; namely, the intersection of all such σ-ideals:

$$\mathfrak{a} = \bigcap_{\substack{\mathfrak{b} \trianglelefteq_\sigma R, \\ \mathfrak{b} \text{ radical and mixed}}} \mathfrak{b}.$$

Proof. Let I be an index set and $\mathfrak{a}_i \trianglelefteq_\sigma R$ for all $i \in I$ be mixed, radical σ-ideals. Further let $\mathfrak{b} := \bigcap_{i \in I} \mathfrak{a}_i$ be the intersection of these. Obviously, $\mathfrak{b}$ is (algebraically) an ideal of R. We will show that it is also a σ-ideal, radical and mixed. If $a \in \mathfrak{a}_i$ for all $i \in I$, then $\sigma(a) \in \mathfrak{a}_i$ for all $i \in I$, since each $\mathfrak{a}_i$ is a σ-ideal. It follows that $\sigma(a) \in \mathfrak{b}$.

Similarly, if $aa' \in \mathfrak{a}_i$ for all $i \in I$, then $a\sigma(a') \in \mathfrak{a}_i$ for all $i \in I$, since each $\mathfrak{a}_i$ is mixed, which implies that $a\sigma(a') \in \mathfrak{b}$.

Finally, if $a \in \sqrt{(\mathfrak{b})}$ there exists an $n \in \mathbb{N}$ such that $a^n \in \mathfrak{b}$. This means that $a^n \in \mathfrak{a}_i$ for all $i \in I$, which implies that $a \in \sqrt{(\mathfrak{a}_i)}$ for all $i \in I$. Since every $\mathfrak{a}_i, i \in I$ is radical, this means that $a \in \mathfrak{a}_i$ for all $i \in I$, and thus $a \in \mathfrak{b}$. $\square$

Definition 2.7. *The σ-ideal $\mathfrak{a}$ from Remark 2.6 is called the radical, mixed closure of F, and we will denote it by $\{F\}_m$.*

Lemma 2.8. *Let R be a σ-ring and $\mathfrak{a} \trianglelefteq_\sigma R$ be a mixed σ-ideal. Then the radical of $\mathfrak{a}$, $\sqrt{\mathfrak{a}}$, is also mixed.*

Proof. Let $f, g \in R$ be such that $fg \in \sqrt{\mathfrak{a}}$. By definition there exists an $n \in \mathbb{N}_{\geq 1}$ such that $f^n g^n = (fg)^n \in \mathfrak{a}$. Since $\mathfrak{a}$ is mixed, this implies that $f^n\sigma(g^n) = f^n\sigma(g)^n = (f\sigma(g))^n \in \mathfrak{a}$. But this in turn implies that $f\sigma(g) \in \sqrt{\mathfrak{a}}$, which is what we wanted to show. $\square$

Example 2.9. *Let k be a constant σ-field and let $R := k\{y_1, y_2\}$. Consider the difference ideal $\mathfrak{a} := [y_1 y_2] \trianglelefteq_\sigma R$. We can inductively define a chain*

$$\mathfrak{a}^{\{0\}} := \mathfrak{a},\ \mathfrak{a}^{\{m+1\}} := [\{f\sigma(g) \mid f, g \in \mathfrak{a}^{\{m\}}\}]$$
$$= [\sigma^k(y_1)\sigma^l(y_2) \mid k, l = 0, 1, \ldots, m+1],\ \textit{for all } m \in \mathbb{N}_{\geq 1}.$$

This is an infinite properly ascending chain of difference ideals.

To try to find $\{\mathfrak{a}\}_m$ for a σ-ideal $\mathfrak{a} \trianglelefteq_\sigma R$ in a difference ring R, it might be tempting to consider $\mathfrak{a}' := \{f\sigma(g) \mid fg \in \mathfrak{a}\}$, or to ensure that is a difference ideal rather, $[\mathfrak{a}']$. The example above shows that this is not enough, as the ideal $[\mathfrak{a}']$ does not have to be mixed in general. However, by iteratively repeating this process and taking the union of σ-ideals obtained this way, we do get a mixed σ-ideal, as we will see in the following Lemma:

Lemma 2.10. *Let R be a σ-ring and $F \subseteq R$. Further let $F' := \{f\sigma(g) \mid fg \in F\}$, and set $F^{\{1\}} := [F']$, $F^{\{m\}} := [{F^{\{m-1\}}}']$ for all $m \in \mathbb{N}_{\geq 1}$. Then*

$$\{F\}_m = \sqrt{\bigcup_{n=1}^{\infty} F^{\{n\}}}. \tag{1}$$

This way of obtaining $\{F\}_m$ is called a shuffling processes and has an analog for perfect σ-ideals (see for example [6], p. 121f.)

Proof. Let $\mathfrak{a} := \bigcup_{n=1}^{\infty} F^{\{n\}}$. It is obvious from the construction that $F \subseteq \mathfrak{a}$. It also holds that $\mathfrak{a}$ is a mixed σ-ideal, since for any $f, g \in \mathfrak{a}$ there exists an $m \in \mathbb{N}_{\geq 1}$ such that $f, g \in F^{\{m\}}$. And hence $f + g, \sigma(f) \in F^{\{m\}} \subseteq \mathfrak{a}$, as well as $fh \in F^{\{m\}} \subseteq \mathfrak{a}$ for any $h \in R$. Furthermore, for $f, g \in R$ with $fg \in \mathfrak{a}$ there also exists an $m \in \mathbb{N}_{\geq 1}$ such that $fg \in F^{\{m\}}$. Then we have $f\sigma(g) \in F^{\{m+1\}} \subseteq \mathfrak{a}$.

On the other hand, by induction on the iterative steps $F^{\{n\}}$ it follows that for every mixed σ-ideal $\mathfrak{b}$ which contains F, $F^{\{n\}} \subseteq \mathfrak{b}$. Hence, $\mathfrak{a}$ is the smallest mixed σ-ideal containing F.

By Lemma 2.8 we know that $\sqrt{\mathfrak{a}}$ is mixed. This actually shows that $\sqrt{a}$ is indeed the smallest mixed, radical σ-ideal of R containing F, since every such ideal has to contain $\mathfrak{a}$, and thus $\sqrt{\mathfrak{a}}$ as well.

□

Example 2.11. *Let k be a σ-field, and consider $R = k\{y_1\}$. Then the σ-ideal $[y_1] \trianglelefteq_\sigma R$ is mixed, hence equal to its mixed closure. The mixed closure of $[y_1] \cdot [y_1]$ is $[y_1\sigma^i(y_1) \mid i \in \mathbb{N}] \not\ni y_1$. One could have expected, perhaps, to get an analog of the statement in algebraic geometry that $\sqrt{F_1 F_2} = \sqrt{F_1} \cap \sqrt{F_2}$, but this example shows it is not in general so for mixed ideals. It is however very noteworthy that the ideal $[y_1\sigma^i(y_1) \mid i \in \mathbb{N}] \trianglelefteq_\sigma R$ is not radical. For radical, mixed difference ideals we will in fact get such a statement later (Corollary 3.15).*

A very important result in commutative algebra is the fact that every radical ideal is the intersection of prime ideals. This has an analogue for perfect σ-ideals, as well as for mixed σ-ideals. We will prove the latter, but for this we need a few additional tools. We will first prove a weaker version of the statement, for which we need a few results from commutative algebra:

Lemma 2.12. *Let R be a ring.*

(a) If $S \geq R$ is an overring of R, and $\mathfrak{p}$ is a minimal prime ideal of R, then there exists a minimal prime ideal $\mathfrak{q}$ of S such that $\mathfrak{p} = \mathfrak{q} \cap R$.

(b) Every radical ideal of R is the intersection of prime ideals. If R is Noetherian, then every radical ideal of R is the intersection of finitely many prime ideals.

(c) If R is Noetherian and $\mathfrak{p} \trianglelefteq R$ is a minimal prime ideal of R, then there exists an element $a \in R$ such that $\mathfrak{p}$ is the annihilator ideal of a, i.e. $\mathfrak{p} = Ann(a) = \{r \in R \mid ra = 0\}$.

Proof.

(a) See Remark 2.9 of [7]

(b) See [8] Ch. 2, §2.6, Corollary 2 to Proposition 13 and Ch. 2, §4.3, Corollary 3 to Proposition 14.

(c) This is a special case of Theorem 3.1 of [3] for R as an R-module.

□

Definition 2.13. *Let R be a difference ring. We say R is* finitely σ-generated over $\mathbb{Z}$ *if there exists a finite set $A \subseteq R$ so that every $f \in R$ can be written as a finite $\mathbb{Z}$-linear combination of σ-powers of elements in A. In other words, for every $f \in R$ there exists an $n \in \mathbb{N}_{\geq 1}$ so that $f \in \mathbb{Z}[A, \sigma(A), \ldots, \sigma^n(A)]$.*

For any subset $F \subseteq R$ we denote by

$$\mathbb{Z}\{F\} = \{f \in R \mid \text{ there exists an } n \in \mathbb{N} : f \in \mathbb{Z}[F, \sigma(F), \ldots, \sigma^n(F)]\}$$

the set of all elements σ-generated by F over $\mathbb{Z}$.

Proposition 2.14. *Let R be a σ-ring finitely σ-generated over $\mathbb{Z}$. Then, every radical, mixed σ-ideal of R is the intersection of prime σ-ideals.*

Proof. Let $\mathfrak{a} \trianglelefteq_\sigma R$ be a mixed, radical σ-ideal. By Proposition 2.5 there is a bijection between the prime σ-ideals of R containing $\mathfrak{a}$ and those of $R/\mathfrak{a}$. Hence, we can assume without loss of generality that $\mathfrak{a} = [0] \trianglelefteq_\sigma R$, by replacing R with $R/\mathfrak{a}$. This means that we only have to show that the zero ideal $[0]$ of a well-mixed, reduced σ-ring R is the intersection of all its prime σ-ideals. Note that this does not change the fact that R is finitely σ-generated over $\mathbb{Z}$.

Let thus $f \in R$ be such that $f \in \mathfrak{q}$ for all $\mathfrak{q} \trianglelefteq_\sigma R$ prime. We assert that f then has to be 0. Assume this is not the case, i.e., $f \neq 0$. Then by assumption on R there is an $n \in \mathbb{N}$ such that $f \in \mathbb{Z}[A, \sigma(A), \ldots, \sigma^n(A)]$. We now use the special case for (algebraic) ideals: since $\mathbb{Z}[A, \sigma(A), \dots, \sigma^n(A)]$ is Noetherian and reduced, $(0) \trianglelefteq \mathbb{Z}[A, \dots, \sigma^n(A)]$ is the intersection of all prime ideals of R. In particular, there exist prime ideals which do not contain f. Let $\mathfrak{q}_0 \trianglelefteq \mathbb{Z}[A, \dots, \sigma^n(A)]$ be a minimal such prime ideal, i.e., with $f \notin \mathfrak{q}_0$. Since $f \in \mathbb{Z}[A, \sigma(A), \dots, \sigma^n(A)] \subset \mathbb{Z}[A, \sigma(A), \dots, \sigma^{n+1}(A)]$, again by Lemma 2.12, we can find a minimal prime ideal $\mathfrak{q}_1 \trianglelefteq \mathbb{Z}[A, \sigma(A), \dots, \sigma^{n+1}(A)]$ such that $\mathfrak{q}_1 \cap \mathbb{Z}[A, \sigma(A), \dots, \sigma^n(A)] = \mathfrak{q}_0$.

Inductively we find a chain of minimal prime ideals $\mathfrak{q}_i, i \in \mathbb{N}$, $\mathfrak{q}_i \trianglelefteq \mathbb{Z}[A, \sigma(A), \dots, \sigma^{n+i}(A)]$, with $\mathfrak{q}_{i+1} \cap \mathbb{Z}[A, \sigma(A), \dots, \sigma^{n+i}(A)] = \mathfrak{q}_i$ for all $i \in \mathbb{N}$. Then $\mathfrak{q} := \bigcup_{i=0}^{\infty} \mathfrak{q}_i$ is a prime ideal of R, with $f \notin \mathfrak{q}$. In fact, $\mathfrak{q}$ is a σ-ideal of R: Let $a \in \mathfrak{q}$. We want to show that $\sigma(a) \in \mathfrak{q}$. By construction of $\mathfrak{q}$ there exists an $i \in \mathbb{N}$ such, that $a \in \mathfrak{q}_{i-1} \subseteq \mathbb{Z}[A, \sigma(A), \dots, \sigma^{n+i-1}(A)]$, which implies that $\sigma(a) \in \mathbb{Z}[A, \sigma(A), \dots, \sigma^{n+i}(A)]$. Lemma 2.12 states then, that there is an $h \in \mathbb{Z}[A, \sigma(A), \dots, \sigma^{n+i}(A)]$ such that $\mathfrak{q}_i = \mathrm{Ann}(h)$. It follows that $ah = 0$, and since R is well-mixed, this implies that $\sigma(a)h = 0$, hence, $\sigma(a) \in \mathfrak{q}_i \subseteq \mathfrak{q}$. This means that $\mathfrak{q}$ is a prime σ-ideal of R not containing f, which contradicts the assumption on f, so that $f = 0$ has to follow. □

For the general case we need yet another tool, the concept of filters:

Definition 2.15. *Let U be a set, and let $F \subseteq Pot(U)$, where $Pot(U)$ denotes the power set on U. Then F is called a* filter *if it satisfies the following axioms:*

- *$U \in F$ and $\emptyset \notin F$.*
- *If $V, W \subseteq U$ with $V \subseteq W$ and $V \in F$ it holds that $W \in F$.*
- *For $V_1, \dots, V_n \in F$ it holds that*

$$\bigcap_{i=1}^{n} V_i \in F.$$

A filter F is called an ultrafilter, *if for any $V \subseteq U$ it holds that $V \in F$ or $U \setminus V \in F$. Note that the first and third axioms together imply that at most one of V and $U \setminus V$ can be in F.*

Remark 2.16. Let U be a set. Then, the set of filters on U is inductively ordered by inclusion. By Zorn's lemma, for every filter F on U there must exist a maximal filter G with respect to inclusion such that $F \subseteq G$. The maximality of the filter implies that G will be an ultrafilter, since we could otherwise find a new filter G' where G is properly included by adding one of the sets which contradict the ultrafilter property and considering the smallest filter containing this set.

The reason why this concept is useful in our context is the following:

Lemma 2.17. *Let R be a σ-ring, and let M be the set of all σ-subrings of R which are finitely σ-generated over $\mathbb{Z}$. For any fixed subset $F \subseteq R$, consider the set $M_F := \{T \subseteq M \mid \{S \in M \mid F \subseteq S\} \subseteq T\} \subseteq Pot(M)$. Then,*

$$\mathcal{F} := \bigcup_{F \subseteq R \text{ finite}} M_F$$

defines a filter on M. If $\mathcal{G}$ is an ultrafilter containing $\mathcal{F}$, and $P := \prod_{S \in M} S$ with component-wise operations, then the ultrafilter $\mathcal{G}$ defines an equivalence relation on P via $(g_S)_{S\in M} \sim (h_S)_{S \in M} :\Leftrightarrow \{S \in M \mid g_S = h_S\} \in \mathcal{G}$. The set of equivalence classes $P/\mathcal{G} := P/{\sim}$ has a natural σ-ring structure and is called an ultraproduct.

Proof. Let us first show that $\mathcal{F}$ is a filter. For $F \subseteq R$ finite we have $\mathbb{Z}\{F\} \in \{S \in M \mid F \subseteq S\} \neq \emptyset$, and since $T \supseteq \{S \in M \mid F \subseteq S\}$ for all $T \in M_F$, $\emptyset \notin M_F$ (Note that $\mathbb{Z}\{\emptyset\} = (0)$, so it also holds for $F = \emptyset$). That $M \in M_F$ for any $F \subseteq R$ is obvious, as well as that for $T \subseteq U, T \in M_F$ it holds that $U \in M_F$.

We only need to show that $U, T \in \mathcal{F}$ implies that $U \cap T \in \mathcal{F}$. Let $\hat{U}, \hat{T} \subseteq R$ be finite, such that $U \in M_{\hat{U}}, T \in M_{\hat{T}}$. $\hat{U} \cup \hat{T} \subseteq R$ is also finite and it holds that $\{S \in M \mid \hat{U} \cup \hat{T} \subseteq S\} \subseteq \{S \in M \mid \hat{U} \subseteq S\} \subseteq U$, and similarly for T. This means that $U \cap T \in M_{\hat{U} \cup \hat{T}} \subseteq \mathcal{F}$, which finishes the proof that $\mathcal{F}$ is a filter.

Now, consider an ultrafilter $\mathcal{G} \supseteq \mathcal{F}$ and define $\sim$ on P as above. This is an equivalence relation: Let $f \sim g, g \sim h$ for $f, g, h \in P$. This means that $\{S \in M \mid f_S = g_S\} \in \mathcal{G}, \{S \in M \mid g_S = h_S\} \in \mathcal{G}$. But then

$$\{S \in M \mid f_S = g_S\} \cap \{S \in M \mid g_S = h_S\} \subseteq \{S \in M \mid f_S = h_S\} \in \mathcal{G},$$

since $\mathcal{G}$ is a filter. Reflexivity follows from the fact that $M \in \mathcal{G}$, and symmetry is obvious.

We now only need to show that we have a well-defined σ-ring structure on $P/{\sim}$. Consider $f, f' \in P$ with $f \sim f'$. We have that for all $S \in M$ with $f_S = f'_S$ it holds that $\sigma(f)_S = \sigma(f')_S$. But then $\{S \in M \mid \sigma(f)_S = \sigma(f')_S\} \supseteq \{S \in M \mid f_S = f'_S\} \in \mathcal{G}$ by assumption, and since $\mathcal{G}$ is a filter, this means that $\{S \in M \mid \sigma(f)_S = \sigma(f')_S\} \in \mathcal{G}$, hence $\sigma(f) \sim \sigma(f')$. That $+$ and $\cdot$ are also well-defined can be proven in an analogous fashion. $\square$

We can now turn our attention to the generalization of Proposition 2.14.

Theorem 2.18. *Let R be a σ-ring and $F \subseteq R$ be a subset of R. Then,*

$$\{F\}_m = \bigcap_{\substack{F \subseteq \mathfrak{p} \trianglelefteq_\sigma R \\ \mathfrak{p} \text{ prime}}} \mathfrak{p}$$

In particular, every radical, mixed σ-ideal of R is the intersection of prime σ-ideals.

Proof. It suffices to show that every radical, mixed σ-ideal of R is the intersection of prime σ-ideals. Indeed, since prime σ-ideals are radical and mixed, it is clear that $\{F\}_m \subseteq \mathfrak{p}$ for every prime $\mathfrak{p} \trianglelefteq_\sigma R$ with $F \subseteq \mathfrak{p}$, which together with the fact that every radical, mixed σ-ideal of R is the intersection of prime σ-ideals gives the representation

$$\{F\}_m = \bigcap_{\substack{F \subseteq \mathfrak{p} \trianglelefteq_\sigma R \\ \mathfrak{p} \text{ prime}}} \mathfrak{p}.$$

Now, by the same argument as in the beginning of the proof of Proposition 2.14, it is enough to prove in the case that R is well-mixed and reduced, that the intersection of all prime σ-ideals of R is $[0]$. Let $0 \neq f \in R$. We will construct a prime σ-ideal $\mathfrak{q}$ of R which does not contain f:

Let $P/\mathcal{G}$ be the difference ring as in Lemma 2.17. Consider the mapping $\varphi : R \to P/\mathcal{G}, g \mapsto (g_S)_{S \in M}$ with $(g_S) = g$ for all $S \in M$ with $g \in S$ and $(g_S) = 0$ for $g \notin S$. It is in fact $\{S \in M \mid g \in S\} \in M_{\{g\}}$ (with $M_{\{g\}}$ as in Lemma 2.17). It follows from this that the image of the mapping onto $P/\mathcal{G}$ is in fact independent of the (g_S) for $g \notin S$, as any other choice of these would be in the same $\sim$ class as the image described above. It follows that φ is a well-defined morphism of σ-rings.

From Proposition 2.14 we know that for every $S \in M$, there exists a prime σ-ideal $\mathfrak{p}_S \trianglelefteq_\sigma S$ such that $f \notin \mathfrak{p}_S$. We define $\mathfrak{p} \subseteq P/\mathcal{G}$ as the set of all equivalence classes of elements $(g_S)_{S \in M}$ such that $\{S \in M \mid g_S \in \mathfrak{p}_S\} \in \mathcal{G}$. For $[(g_S)_{S \in M}]_\sim, [(h_s)_{S \in M}]_\sim \in \mathfrak{p}$ we have

$$\mathcal{G} \ni \{S \in M \mid g_S \in \mathfrak{p}_S\} \cap \{S \in M \mid h_S \in \mathfrak{p}_S\} \subseteq \{S \in M \mid g_S + h_S \in \mathfrak{p}_S\} \in \mathcal{G},$$

since $\mathcal{G}$ is a filter. Similar arguments for $\sigma(g), gh$ for $h \in P/\mathcal{G}$ show that $\mathfrak{p}$ is indeed a σ-ideal. $\mathfrak{p}$ is also prime since $\mathcal{G}$ is an ultrafilter: Let $g, h \in P$ with $\{S \in M \mid g_S h_S \in \mathfrak{p}_S\} \in \mathcal{G}$. If $[g]_\sim \notin \mathfrak{p}$, then $V := \{S \in M \mid g_S \in \mathfrak{p}_S\} \notin \mathcal{G}$. Since $\mathcal{G}$ is an ultrafilter, this means that $M \setminus V \in \mathcal{G}$. But

$$\mathcal{G} \ni (M \setminus V) \cap \{S \in M \mid g_S h_S \in \mathfrak{p}_S\} \subseteq \{S \in M \mid h_S \in \mathfrak{p}_S\} \in \mathcal{G},$$

which means that $[h]_\sim \in \mathfrak{p}$. The preimage of a prime σ-ideal, $\mathfrak{q} := \varphi^{-1}(\mathfrak{p}) \trianglelefteq_\sigma R$ is also prime. By construction, $[\varphi(f)]_\sim \notin \mathfrak{p}$, which means that $f \notin \varphi^{-1}(\mathfrak{p})$, as desired.

□

2.1 An Analog of the Cohn Topology

Definition 2.19. *Let R be a σ-ring. We denote the set of all prime σ-ideals of R by $\sigma\text{-Spec}(R) := \{\mathfrak{p} \trianglelefteq_\sigma R \mid \mathfrak{p}$ prime $\}$. Similarly, we denote the set of σ-prime ideals by $\mathrm{Spec}^\sigma(R) := \{\mathfrak{p} \trianglelefteq_\sigma R \mid \mathfrak{p}$ σ-prime $\} \subseteq \sigma\text{-Spec}(R)$.*

Remark 2.20. As is the case with $\mathrm{Spec}^\sigma(R)$, it can be the case that $\sigma\text{-Spec}(R) = \emptyset$. For example, let R be a σ-ring, and consider the σ-ring $R \oplus R$, with

$\sigma((r,s)) := (\sigma(s), \sigma(r))$. We will show that this ring has no prime σ-ideals. Let $\mathfrak{p} \trianglelefteq R$ prime. Then $0 = (1,0)(0,1) \in \mathfrak{p}$, which means that either $(1,0) \in \mathfrak{p}$ or $(0,1) \in \mathfrak{p}$. But then $R \oplus 0 \subseteq \mathfrak{p}$ or $0 \oplus R \subseteq \mathfrak{p}$. If we assume that $\mathfrak{p}$ is a σ-ideal then this implies that $R \oplus R \subseteq \mathfrak{p}$, which cannot be, by definition.

In algebraic geometry, one usually considers $\mathrm{Spec}(R)$ as a topological space with a topology called the Zariski topology. This has an analog for $\mathrm{Spec}^{\sigma}(R)$, usually called the Cohn topology. Here we will develop a further analog of both, which we will define on $\sigma\text{-Spec}(R)$, and will be closely related to radical, mixed σ-ideals, as we shall see by its many properties.

Definition 2.21. *Let R be a σ-ring and $F \subseteq R$ be a subset of R. We set $\mathcal{V}_m(F) := \{\mathfrak{p} \in \sigma\text{-Spec}(R) \mid F \subseteq \mathfrak{p}\}$. If F has only one element f, we write $\mathcal{V}_m(f)$ for $\mathcal{V}_m(F)$.*

Lemma 2.22. *Let R be a σ-ring. Then we have:*

(a) $\mathcal{V}_m((0)) = \sigma\text{-Spec}(R)$, and $\mathcal{V}_m(R) = \emptyset$.

(b) For any two ideals $\mathfrak{a}, \mathfrak{b} \trianglelefteq R$ we have $\mathcal{V}_m(\mathfrak{a}) \cup \mathcal{V}(\mathfrak{b}) = \mathcal{V}_m(\mathfrak{a} \cap \mathfrak{b})$.

(c) For any family of ideals $(\mathfrak{a}_i)_{i\in I}$ for an index set I, we have

$$\bigcap_{i\in I} \mathcal{V}_m(\mathfrak{a}_i) = \mathcal{V}_m(\sum_{i\in I} \mathfrak{a}_i).$$

Proof.

(a) We have $(0) \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \sigma\text{-Spec}(R)$, as well as $R \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \sigma\text{-Spec}(R)$.

(b) Let $\mathfrak{a}, \mathfrak{b} \trianglelefteq R$ be two ideals in R. Then $\mathcal{V}_m(\mathfrak{a}) \cup \mathcal{V}_m(\mathfrak{b}) \subseteq \mathcal{V}_m(\mathfrak{a} \cap \mathfrak{b})$, since for $\mathfrak{p} \trianglelefteq_\sigma R$ prime, $\mathfrak{a} \subseteq \mathfrak{p}$ it follows that $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$, and similarly for $\mathfrak{b}$. On the other hand, let $\mathfrak{p} \trianglelefteq_\sigma R$ prime with $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$, and $\mathfrak{a} \not\subseteq \mathfrak{p}$ (otherwise $\mathfrak{p} \in \mathcal{V}_m(\mathfrak{a})$ and we are done). Then there exists an $f \in \mathfrak{a}$, $f \notin \mathfrak{p}$. For any $g \in \mathfrak{b}$, it follows that $fg \in \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime, this implies that $g \in \mathfrak{p}$. Hence, $\mathfrak{b} \subseteq \mathfrak{p}$, which concludes the proof.

(c) Let $(\mathfrak{a}_i)_{i\in I}$ be a family of ideals of R. Then

$$\mathfrak{p} \in \bigcap_{i\in I} \mathcal{V}_m(\mathfrak{a}_i) \Leftrightarrow \mathfrak{a}_i \subseteq \mathfrak{p} \text{ for all } i \in I \Leftrightarrow \mathfrak{p} \in \mathcal{V}_m(\sum_{i\in I} \mathfrak{a}_i).$$

∎

Remark 2.23. Since for a σ-ring R any prime σ-ideal of R is radical and mixed, it holds that for any $F \subseteq R$, and any prime σ-ideal $\mathfrak{p} \trianglelefteq_\sigma R$ with $F \subseteq \mathfrak{p}$ we have $(F) \subseteq [F] \subseteq \{F\}_m \subseteq \mathfrak{p}$. In particular, this means that $\mathcal{V}_m(F) = \mathcal{V}_m((F)) = \mathcal{V}_m([F]) = \mathcal{V}_m(\{F\}_m)$.

Definition 2.24. *Let R be a σ-ring. We define a topology on σ-$\mathrm{Spec}(R)$ by setting $A \subseteq \sigma$-$\mathrm{Spec}(R)$ closed if $A = \mathcal{V}_m(\mathfrak{a})$ for an ideal $\mathfrak{a} \trianglelefteq R$, or equivalently, by defining a set to be open, if it is a complement of such a $\mathcal{V}_m(\mathfrak{a})$. This is a well-defined topology thanks to Lemma 2.22. For $f \in R$ we set*

$$\sigma\text{-}\mathrm{D}(f) := \sigma\text{-}\mathrm{Spec}(R) \setminus \mathcal{V}_m(f).$$

σ-$\mathrm{D}(f)$ is the complement of a closed set, and hence, open. We call the sets of the form σ-$\mathrm{D}(f) \subseteq \sigma$-$\mathrm{Spec}(R)$ basic open subsets *of σ-$\mathrm{Spec}(R)$.*

From here on, if not explicitly stated otherwise, when referring to topological concepts on σ-$\mathrm{Spec}(R)$ we will be referring to the topology just defined.

Remark 2.25. From its definition it is clear that σ-$\mathrm{Spec}(R) \subseteq \mathrm{Spec}(R) := \{I \trianglelefteq R \mid I \text{ prime}\}$. Since Lemma 2.22 does not require the ideals to be σ-ideals, it is easy to conclude that in fact the topology on σ-$\mathrm{Spec}(R)$ is just the topology induced by restriction of the Zariski topology to σ-$\mathrm{Spec}(R)$. The same argument can be made to see that the Cohn topology in turn, defined on $\mathrm{Spec}^{\sigma}(R) = \{\mathfrak{p} \trianglelefteq_{\sigma} R \mid \mathfrak{p}\ \sigma\text{-prime}\ \} \subseteq \sigma$-$\mathrm{Spec}(R)$, is also the restriction of the topology defined on σ-$\mathrm{Spec}(R)$.

Definition 2.26. *Let X be a topological space.*

(a) We say that X is irreducible *if $X = X_1 \cup X_2$ with X_1, X_2 closed implies that $X = X_1$ or $X = X_2$. $X_1 \subseteq X$ is called* irreducible *if it is an irreducible topological space with the topology induced by the restriction to X_1.*

(b) Let $Y \subseteq X$ be closed. We say that a point $f \in Y$ is a generic point *of Y, if $\overline{\{f\}} = Y$, where for $A \subseteq X$, $\overline{A}$ denotes the closure of A.*

Proposition 2.27. *Let R be a σ-ring. We have:*

(a) The mapping

$$\{\mathfrak{a} \trianglelefteq_{\sigma} R \mid \mathfrak{a} \text{ mixed and radical } \} \to \{A \subseteq \sigma\text{-}\mathrm{Spec}(R) \mid A \text{ closed } \}, \mathfrak{a} \mapsto \mathcal{V}_m(\mathfrak{a})$$

is bijective and order-reversing.

(b) For $F \subseteq R$ it holds that $\mathcal{V}_m(F)$ is irreducible if and only if $\{F\}_m$ is prime.

(c) σ-$\mathrm{Spec}(R)$ is quasi-compact.

(d) The basic open sets $\{\sigma\text{-}\mathrm{D}(f) \mid f \in R\}$ form a basis for the topology on σ-$\mathrm{Spec}(R)$.

(e) Every irreducible closed subset Y of σ-$\mathrm{Spec}(R)$ has a unique generic point y.

Proof.

(a) That the mapping is order-reversing is obvious. The injectivity follows from the fact that by Theorem 2.18 $\mathfrak{a} = \bigcap_{\mathfrak{a}\subseteq\mathfrak{p}\in\sigma\text{-Spec}(R)} \mathfrak{p}$. By Remark 2.23 we obtain the surjectivity, since $\mathcal{V}_m(\mathfrak{a}) = \mathcal{V}_m(\{\mathfrak{a}\}_m)$.

(b) Since $\mathcal{V}_m(F) = \mathcal{V}_m(\{F\}_m)$, we can assume without loss of generality, that $F \trianglelefteq_\sigma R$ is a radical, mixed σ-ideal. For the first implication, "$\Leftarrow$", let $F \trianglelefteq_\sigma R$ be prime, and $\mathcal{V}_m(F) = \mathcal{V}_m(\mathfrak{a}) \cup \mathcal{V}_m(\mathfrak{b})$ with radical, mixed σ-ideals $\mathfrak{a}, \mathfrak{b}$. Assume that $\mathcal{V}_m(F) \not\subseteq \mathcal{V}_m(\mathfrak{a})$. Then by (a), $\mathfrak{a} \not\subseteq F$, so there exists an $a \in \mathfrak{a}$, with $a \notin F$. For any $b \in \mathfrak{b}$ we then have $ab \in \mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{V}(F) = \mathcal{V}_m(\mathfrak{a}) \cup \mathcal{V}_m(\mathfrak{b})$, and with Theorem 2.18 we get $ab \in F = \bigcap_{\mathfrak{p}\in\mathcal{V}_m(F)} \mathfrak{p}$. By assumption, F is prime and $a \notin F$, which implies that $b \in F$. But this means that $\mathfrak{b} \subseteq F$, and thus $\mathcal{V}_m(F) \subseteq \mathcal{V}_m(\mathfrak{b})$, which shows the irreducibility.
Now, for the other implication, "$\Rightarrow$", assume that $\mathcal{V}_m(F)$ is irreducible, and let $a, b \in R$ with $ab \in F$. Consider $F \subseteq \mathfrak{p} \in \mathcal{V}_m(F)$. Then $ab \in \mathfrak{p}$, which means that $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, since $\mathfrak{p}$ is prime. This implies that $\mathfrak{p} \in \mathcal{V}_m(\{a\}_m) \cup \mathcal{V}_m(\{b\}_m)$, which in turn implies that $\mathcal{V}_m(F) \subseteq \mathcal{V}_m(\{a\}_m) \cup \mathcal{V}_m(\{b\}_m)$. Now, by assumption, $\mathcal{V}_m(F)$ is irreducible, and thus it has to be that $\mathcal{V}_m(F) \subseteq \mathcal{V}_m(\{a\}_m)$ or $\mathcal{V}_m(F) \subseteq \mathcal{V}_m(\{b\}_m)$. By the bijectivity of the mapping in (a) this means that $a \in F$ or $b \in F$.

(c) Let $\mathcal{V}_m(\mathfrak{a}_i)_{i\in I}$ be a family of closed sets, $\mathfrak{a}_i \trianglelefteq_\sigma R$ mixed, radical for all $i \in I$, satisfying that $\bigcap_{i\in J} \mathcal{V}_m(a_i) \neq \emptyset$ for every finite $J \subseteq I$. By going to the complement of open sets, quasi-compactness is equivalent to the implication that $\bigcap_{i\in I} \mathcal{V}_m(a_i) \neq \emptyset$. By Lemma 2.22 we see that $\bigcap_{i\in I} \mathcal{V}_m(\mathfrak{a}_i) = \mathcal{V}_m(\sum_{i\in I} \mathfrak{a}_i)$. Assume that $\mathcal{V}_m(\sum_{i\in I} \mathfrak{a}_i) = \emptyset$. By Theorem 2.18 this means that $\{\sum_{i\in I} \mathfrak{a}_i\}_m = R$. In particular, $1 \in \{\sum_{i\in I} \mathfrak{a}_i\}_m$. By the construction in Lemma 2.10 (and with the notation used there), this means that there has to be an $n \in \mathbb{N}_{\geq 1}$, so that $1 \in (\sum_{i\in I} \mathfrak{a}_i)^{\{n\}}$. In particular, this means that 1 can be written as a finite R-linear combination $\sum_{k=1}^{l} r_k a_k$ with $a_k \in (\sum_{i\in I} \mathfrak{a}_i)^{\{n\}}, r_k \in R, k = 1, \ldots, l$. In particular, there exists a $J \subseteq I$ finite, such that $a_k \in (\sum_{i\in J} \mathfrak{a}_i)^{\{n\}}$ for all $k \in \{1, \ldots, l\}$. But this implies that $1 \in (\sum_{i\in J} \mathfrak{a}_i)^{\{n\}}$, meaning that $\mathcal{V}_m(\sum_{i\in J} \mathfrak{a}_i) = \cap_{i\in J} \mathcal{V}_m(\mathfrak{a}_i) = \emptyset$, a contradiction.

(d) For an open subset $U \subseteq \sigma\text{-Spec}(R)$ there exists by definition an $\mathfrak{a} \trianglelefteq_\sigma R$ such that $U = \sigma\text{-Spec}(R) \setminus \mathcal{V}_m(\mathfrak{a})$. We can then write U as a union of basic open sets as follows:
$$U = \bigcup_{a\in\mathfrak{a}} \sigma\text{-}\mathrm{D}(a).$$

(e) By (b), an irreducible closed subset A of $\sigma\text{-Spec}(R)$ has the form $A = \mathcal{V}_m(\mathfrak{p})$, for $\mathfrak{p} \trianglelefteq_\sigma R$ prime. This prime σ-ideal $\mathfrak{p}$ is the unique generic point of A. To see this, consider the closure of $\mathfrak{p}$:
$$\overline{\{\mathfrak{p}\}} = \bigcap_{\{\mathfrak{p}\}\subseteq\mathcal{V}_m(F)} \mathcal{V}_m(F).$$

From the definition of $\mathcal{V}_m(F)$ it holds that $\{\mathfrak{p}\} \subseteq \mathcal{V}_m(F)$ if and only if $F \subseteq \mathfrak{p}$. By Lemma 2.22 (c), and the obvious fact that we can restrict the intersection to σ-ideals, we get thus

$$\overline{\{\mathfrak{p}\}} = \bigcap_{F \subseteq \mathfrak{p}} \mathcal{V}_m(F) = \mathcal{V}_m(\sum_{F \subseteq \mathfrak{p}, F \trianglelefteq_\sigma R} F) = \mathcal{V}_m(\mathfrak{p}) = A$$

□

3 Difference Varieties

In this section we will introduce difference varieties. We will do so in a way that they correspond with the topology on σ-Spec(R), which we defined in the previous section. It will be again based on M. Wibmer's lecture notes [1], where it is worked out for the analogous case of perfect σ-ideals.

Definition 3.1. *Let A be a σ-ring. If A is (algebraically) an integral domain, we call A an* integral σ-ring. *If additionally the endomorphism σ on A is injective, then we call A a* σ-domain.

Remark 3.2. Let A be a σ-domain. Then $k := \mathrm{Quot}(A)$ is a σ-field: For $\frac{r}{s} \in k$ we can define $\sigma(\frac{r}{s}) := \frac{\sigma(r)}{\sigma(s)}$. Since σ is injective, it holds that $\sigma(s) \neq 0$ for $s \neq 0$, which implies that σ is well defined on k. By this argument we see that in general for an integral σ-ring A, $\mathrm{Quot}(A)$ is a σ-field (in this natural way) if and only if A is a σ-domain.

Our main purpose, in a first instance at least, is to investigate the properties of solutions to difference equations. We will start with an integral σ-field k and look for solutions (zeros) of some σ-polynomial p over k, i.e. $p \in k\{y_1, \dots, y_n\}$. In general, rather, it will be a set of σ-polynomials $F \subseteq k\{y_1, \dots, y_n\}$ that we will study. For this we want to define σ-varieties; we cannot mimic the usual approach from algebraic geometry, where we would take the algebraic closure of k. The next remark shows why.

Remark 3.3. Consider the constant σ-field $\mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{2})$, with $\sigma(\sqrt{2}) = \sqrt{2}$; $L = \mathbb{Q}(\sqrt{2}), \sigma(\sqrt{2}) = -\sqrt{2}$. Both K and L are σ-field extensions of $\mathbb{Q}$, but there cannot be a further extension $\mathbb{Q} \leq M$ of σ-fields, such that $K, L \leq M$ are both (isomorphic to) σ-subfields of M. To see this, assume there was such an M. Then the set $\{a \in M \mid a^2 - 2 = 0\}$ has exactly two elements, which we will call $\sqrt{2}, -\sqrt{2}$ (since $\sqrt{2} + (-\sqrt{2}) = 0$). But $\sqrt{2} \in K$ has to be mapped to one of these two in any embedding, and the same for $\sqrt{2} \in L$, which already yields the contradiction, since in M either $\sigma(\sqrt{2}) = \sqrt{2}$ or $\sigma(\sqrt{2}) = -\sqrt{2}$.

To avoid this problem, we will define σ-varieties as functors. For this we will need a few category-theoretic definitions:

Definition 3.4. *Let k be a σ-field. The category of all σ-ring extensions A of k we denote by σ-$\mathbf{ring}_k$, where the morphisms are defined as follows: For $B, C \in \sigma$-$\mathbf{ring}_k$ we say that a morphism of σ-rings $\varphi : B \to C$ is a morphism of σ-ring extensions of k, if and only if, $\varphi_{|k} = id_k$. The subcategory which arises from restricting the object class to integral σ-rings, the category of integral σ-ring extensions of A, we denote by σ-$\mathbf{int}_k$.*

Now we are ready to define σ-varieties of σ-rings, with mixed σ-ideals in mind:

Definition 3.5. *Let k be a σ-field and $B \in \sigma$-$\mathbf{int}_k$ an integral σ-overring of k. Further let $F \subseteq k\{y_1, \dots, y_n\}$ be a set of σ-polynomials over k. Then we define*

$\mathbb{V}_B(F) := \{b \in B^n \mid f(b) = 0$ *for all* $f \in F\}$. *A functor* $X : \sigma\text{-}\mathbf{int}_k \to \mathbf{Set}$, *for which there exists a set* $F \subseteq k\{y_1, \dots, y_n\}$ *such that* $X(B) = \mathbb{V}_B(F)$ *for all* $B \in \sigma\text{-}\mathbf{int}_k$ *we denote as a* σ-variety over k, *or a* k-σ-variety. *Here,* **Set** *denotes the usual category of sets with mappings as morphisms. We also write* $X := \mathbb{V}(F)$ *as a short notation for this functor.*

Definition 3.6. *Let* k *be a* σ*-field and* $X : \sigma\text{-}\mathbf{int}_k \to \mathbf{Set}$ *be a* k*-*σ*-variety. We say a subfunctor* $Y \subseteq X$ *is a* σ-subvariety *of* X, *if* Y *is a* k*-*σ*-variety itself.*

Remark 3.7. Let k be a σ-field and X be a k-σ-variety. Not every subfunctor of X is a σ-subvariety. Consider the functor $X = \mathbb{V}(0)$, for $\{0\} \subset k\{y_1\}$. For $B \in \sigma\text{-}\mathbf{int}_k$ we denote by $B^* = \{b \in B \mid b \text{ invertible }\}$ the set of units of B. Then for $B \in \sigma\text{-}\mathbf{int}_k$, $B \mapsto B^*$ is a subfunctor Y of X (since $B^* \subset B$ for all $B \in \sigma\text{-}\mathbf{int}_k$ and morphisms of rings always map units to units). Y is not a σ-variety, however: there exists no $F \subseteq k\{y_1\}$ such that $\mathbb{V}_B(F) = B^*$ for all $B \in \sigma\text{-}\mathbf{int}_k$. Indeed, assume there was such an F, and let $0 \neq f \in F$. Then $f(b) = 0$ for all $b \in B^*$ and for all $B \in \sigma\text{-}\mathbf{int}_k$. In particular, for $B = k\langle y_1\rangle = \mathrm{Quot}(k\{y_1\}) \in \sigma\text{-}\mathbf{int}_k$ it holds that $f(y_1) = f = 0$, $y_1 \in k\langle y_1\rangle^*$, a contradiction.

Definition 3.8. *Let* $X = \mathbb{V}(F)$ *be a* σ*-variety over the* σ*-field* k, $F \subseteq k\{y_1, \dots, y_n\}$. *Then we set*

$$\mathbb{I}(X) := \{f \in k\{y_1, \dots, y_n\} \mid f(b) = 0 \textit{ for all } b \in \mathbb{V}_B(F),\ B \in \sigma\text{-}\mathbf{int}_k\}.$$

Example 3.9. *Let* k *be a* σ*-field and consider the set* $\{0\} = F \subseteq k\{y_1, \dots, y_n\}$. *Then the* σ*-variety* X *defined by* F, $X(B) := \mathbb{V}_B(F)$ *for all* $B \in \sigma\text{-}\mathbf{int}_k$ *is called the affine* n*-space, and is denoted by* $\mathbb{A}^n_k$, *or simply* $\mathbb{A}^n$, *whenever* k *is clear from the context. Then for every* $G \subseteq A\{y_1, \dots, y_n\}$ *the* σ*-variety given by* $Y : B \mapsto \mathbb{V}_B(G)$ *is a* σ*-subvariety of* $\mathbb{A}^n$, *and we write* $Y \subseteq \mathbb{A}^n$.

We note that 0 is in any (radical, mixed, difference) ideal, so it is not surprising that every σ-variety is a σ-subvariety of $\mathbb{V}(0)$. This “intuition” will be made more concrete later on.

Since we have this functorial definition, we have in principle a whole proper class of solutions for most systems of difference equations. It is obvious we want to have some sort of equivalence relation between solutions to group them up in a reasonable manner.

Definition 3.10. *Let* k *be a* σ*-field,* $B, C \in \sigma\text{-}\mathbf{int}_k$. *Further let* $F \subseteq k\{y_1, \dots, y_n\}$ *be a system of difference equations and* $b \in B^n, c \in C^n$ *be solutions of* F, *i.e.* $b \in \mathbb{V}_B(F), c \in \mathbb{V}_C(F)$. *We say that* b *and* c *are equivalent if the mapping* $b \mapsto c$ *is a well-defined isomorphism between the integral* σ*-rings* $k\{b\}$ *and* $k\{c\}$ *(as elements of* $\sigma\text{-}\mathbf{int}_k$*).*

Remark 3.11. The usual approach, for example in [5], Chapter 4 or [6], Section 2.6, is to restrict the definition of σ-varieties to σ-fields instead of allowing any integral σ-rings. With this concept, two solutions a, b of a system of difference equation over a difference field k are said to be equivalent if the σ-field extensions $k\langle a\rangle$ and $k\langle b\rangle$ are isomorphic as σ-field extensions of k via $a \mapsto b$. This is in accordance with Definition 3.10, i.e., solutions in difference field extensions are equivalent if and only if they are equivalent as solutions in integral σ-overrings in the sense of Definition 3.10.

Proof. Assume there exist σ-field extensions $k \leq A, B$, and elements $a \in A^n$, $b \in B^n$ such that a and b are equivalent as solutions in the sense of Definition 3.10. Since A, B are σ-fields, it means that $k\{a\}$ and $k\{b\}$ are σ-domains, and $k\langle a\rangle, k\langle b\rangle$ have the "canonical" difference structure induced by $k\{a\}, k\{b\}$ (see Remark 3.2). Let $\varphi : k\{a\} \to k\{b\}, a \mapsto b$ be an isomorphism of integral σ-ring extensions of k. Then we can define $\tilde{\varphi} : k\langle a\rangle \to k\langle b\rangle, \frac{x}{y} \mapsto \frac{\varphi(x)}{\varphi(y)}$. This is a well-defined isomorphism of σ-field extensions of k, since:

$$\tilde{\varphi}\left(\sigma\left(\frac{x}{y}\right)\right) = \tilde{\varphi}\left(\frac{\sigma(x)}{\sigma(y)}\right) = \frac{\varphi(\sigma(x))}{\varphi(\sigma(y))} = \frac{\sigma(\varphi(x))}{\sigma(\varphi(y))} = \sigma\left(\tilde{\varphi}\left(\frac{x}{y}\right)\right)$$

The inverse implication is obvious. □

Example 3.12. *In the two σ-field extensions of $\mathbb{Q}$ in Remark 3.3 we have two solutions of the (algebraic) polynomial $y^2 - 2$, which represent two different solutions in the difference algebraic sense, since the σ-fields $\mathbb{Q}(\sqrt{2}), \sigma(\sqrt{2}) = \sqrt{2}$ and $\mathbb{Q}(\sqrt{2}), \sigma(\sqrt{2}) = -\sqrt{2}$ are not isomorphic.*

Example 3.13. *Let k be a σ-field. The σ-variety X given by $\sigma(y) \in k\{y\}$, i.e. $X(B) = \mathbb{V}_B(\sigma(y))$ for all $B \in \sigma\text{-}\mathbf{int}_k$ has a single point in any σ-field extension of k, namely 0. However, in general integral σ-rings, this is not necessarily the case: Take, for example, $B := k\{y\}/[\sigma(y)] \in \sigma\text{-}\mathbf{int}_k$. In B we have $0 \neq ker(\sigma) = [y] \trianglelefteq_\sigma B$, which means that in particular, $[y + [\sigma(y)]] \subseteq \mathbb{V}_B(\sigma(y))$.*

It is not a coincidence that in the previous example we found more solutions on the σ-ring $B = k\{y\}/[\sigma(y)]$. The σ-ideal $[\sigma(y)]$ is radical and mixed, i.e., $[\sigma(y)] = \{[\sigma(y)]\}_m$. In fact, the ring B as we chose it plays an analogous role to that of the coordinate ring of an affine variety in the usual (algebraic) case.

The next proposition shows why our definition of σ-variety is "the right one" for mixed ideals:

Proposition 3.14. *Let k be a σ-field and $X = \mathbb{V}(F) \subseteq \mathbb{A}^n$ be a difference variety over k. Then $\mathbb{I}(X) = \{F\}_m \trianglelefteq_\sigma k\{y_1, \ldots, y_n\}$.*

Proof. We will first show that $\mathbb{I}(X)$ is a radical, mixed σ-ideal. Let $f, g \in \mathbb{I}(X)$, $h \in k\{y_1, \ldots, y_n\}$. Then, for every $B \in \sigma\text{-}\mathbf{int}_k$, $b \in \mathbb{V}_B(F)$, we have $f(b) = g(b) = 0$. It follows that $(f + g)(b) = f(b) + g(b) = 0$ as well as $(fh)(b) = f(b)h(b) = 0 \cdot h(b) = 0$ and $\sigma(f)(b) = \sigma(f(b)) = \sigma(0) = 0$, so that $\mathbb{I}(X)$ is a σ-ideal. It further follows that $h(b)^n = 0$ implies $h(b) = 0$, since B is an integral domain, and this means that $h^n \in \mathbb{I}(X)$ implies that $h \in \mathbb{I}(X)$.

It only remains to show that $\mathbb{I}(X)$ is mixed. Let now $f, g \in k\{y_1, \ldots, y_n\}$ be such that $fg \in \mathbb{I}(X)$. This means that for all $B \in \sigma\textbf{-int}_k$, $b \in \mathbb{V}_B(F)$ it holds $(fg)(b) = f(b)g(b) = 0$. Since B is an integral domain, this implies that $f(b) = 0$ or $g(b) = 0$. But that also implies that $\sigma(f(b)) = \sigma(0) = 0$, or $\sigma(g(b)) = 0$, so that in any case $(f\sigma(g))(b) = 0$, from which it follows that $f\sigma(g) \in \mathbb{I}(X)$. Note that it does not always have to be the same case, $f(b) = 0$ or $g(b) = 0$, as it depends on B. In particular, $\mathbb{I}(X)$ does not have to be prime in general. We thus see that $\mathbb{I}(X)$ is radical and mixed, hence $\{F\}_m \subseteq k\{y_1, \ldots, y_n\}$.

For the other inclusion, let $f \in \mathbb{I}(X)$. We will show that $f \in \{F\}_m$. Let $F \subseteq \mathfrak{p} \trianglelefteq_\sigma k\{y_1, \ldots, y_n\}$ be a prime σ-ideal. Then, consider $B := k\{y_1, \ldots, y_n\}/\mathfrak{p}$: this is an integral σ-ring. Since $F \subseteq \mathfrak{p}$, we know that $y + \mathfrak{p} \in \mathbb{V}_B(F)$. By assumption we have $f \in \mathbb{I}(\mathbb{V}(F))$, which means by definition that $f(y + \mathfrak{p}) = 0$, which in turn means that $f \in \mathfrak{p}$. But since this holds for any prime $\mathfrak{p} \trianglelefteq_\sigma R$, Theorem 2.18 implies that $f \in \{F\}_m$. $\square$

From this we immediately get a further result on radical, mixed ideals, which is analogous to the case for radical ideals in algebraic geometry.

Corollary 3.15. *Let $\mathfrak{a}, \mathfrak{b} \trianglelefteq_\sigma k\{y\}$ be two radical, mixed difference ideals. Then it holds that $\mathfrak{a} \cap \mathfrak{b} = \{\mathfrak{ab}\}_m$.*

Proof. We can assume that $\mathfrak{a}, \mathfrak{b} \neq \{0\}$, as the assertion is obvious otherwise. Since $\mathfrak{a}, \mathfrak{b}$ are radical and mixed, we know from Proposition 3.14 that $\mathfrak{a} = \mathbb{I}(\mathbb{V}(\mathfrak{a})), \mathfrak{b} = \mathbb{I}(\mathbb{V}(\mathfrak{b}))$, and $\{\mathfrak{ab}\}_m = \mathbb{I}(\mathbb{V}(\mathfrak{ab}))$. For any $B \in \sigma\textbf{-int}_k$, it holds that:

$$\mathbb{V}_B(\mathfrak{ab}) = \mathbb{V}_B(\mathfrak{a}) \cup \mathbb{V}_B(\mathfrak{b})$$

The inclusion "$\supseteq$" is obvious. For "$\subseteq$", let $p \in \mathbb{V}_B(\mathfrak{ab})$ and assume there exists an $f \in \mathfrak{a}$ such that $f(p) \neq 0$. Then, from the definition of $\mathbb{V}_B(\mathfrak{ab})$ it follows that $f(p)g(p) = 0$ for all $g \in \mathfrak{b}$. This means, however, that $p \in \mathbb{V}_B(\mathfrak{b})$ (since B is an integral domain). The other case is completely analogous. Since this holds for any B, the σ-varieties are also equal: $\mathbb{V}(\mathfrak{ab}) = \mathbb{V}(\mathfrak{a}) \cup \mathbb{V}(\mathfrak{b})$. Now,

$$\begin{aligned} \mathbb{I}(\mathbb{V}(\mathfrak{ab})) &= \mathbb{I}(\mathbb{V}(\mathfrak{a}) \cup \mathbb{V}(\mathfrak{b})) \\ &= \{f \in k\{y\} \mid f(p) = 0 \text{ for all } p \in \mathbb{V}_B(\mathfrak{a}) \cup \mathbb{V}_B(\mathfrak{b}),\ B \in \sigma\textbf{-int}_k\} \end{aligned}$$

And $f(p) = 0$ for all $p \in \mathbb{V}_B(\mathfrak{a}) \cup \mathbb{V}_B(\mathfrak{b}),\ B \in \sigma\textbf{-int}_k$, is equivalent to $\underbrace{f(p) = 0 \text{ for all } p \in \mathbb{V}_B(\mathfrak{a}),\ B \in \sigma\textbf{-int}_k}_{\Leftrightarrow f \in \mathbb{I}(\mathbb{V}(\mathfrak{a}))}$ and $\underbrace{f(p) = 0 \text{ for all } p \in \mathbb{V}_B(\mathfrak{b}),\ B \in \sigma\textbf{-int}_k}_{\Leftrightarrow f \in \mathbb{I}(\mathbb{V}(\mathfrak{b}))}$.
Hence, $\{\mathfrak{ab}\}_m = \mathbb{I}(\mathbb{V}(\mathfrak{ab})) = \mathbb{I}(\mathbb{V}(\mathfrak{a})) \cap \mathbb{I}(\mathbb{V}(\mathfrak{b})) = \{\mathfrak{a}\}_m \cap \{\mathfrak{b}\}_m = \mathfrak{a} \cap \mathfrak{b}$. $\square$

Definition 3.16. *Let k be a σ-field and let X be a σ-variety over k. Further let $F \subseteq k\{y_1, \ldots, y_n\}$ be a system of difference equations over k with $X(B) = \mathbb{V}_B(F)$ for all $B \in \sigma\textbf{-int}_k$. Then we consider the σ-ring $k\{y_1, \ldots, y_n\}/\{F\}_m = k\{y_1, \ldots, y_n\}/\mathbb{I}(X) =: k\{X\}$ and call it the* coordinate ring *of X. Since $\{F\}_m$ is a radical, mixed σ-ideal, $k\{X\}$ is reduced and well-mixed.*

Remark 3.17. Let k be a σ-field and X a k-σ-variety. Further let $b \in X(B)$, $B \in \sigma\textbf{-int}_k, f + \mathbb{I}(X) \in k\{X\}$. Then the value of $f(b) \in k$ is independent of the representative f, since for $f' + \mathbb{I}(X) = f + \mathbb{I}(X)$, we know that $f - f' \in \mathbb{I}(X)$, and thus by definition, $(f - f')(b) = 0$. By abuse of notation, we will sometimes use the representative f to refer to its equivalence class $f + \mathbb{I}(X)$ and we will simply write $f(b)$ to mean the well-defined value of evaluating b on any representative of the class.

We can now clarify what we meant after Example 3.9.

Lemma 3.18. *Let k be a σ-field. Then the maps $X \mapsto \mathbb{I}(X)$ and $\mathfrak{a} \mapsto \mathbb{V}(\mathfrak{a})$ define inclusion-reversing bijections between the set of all σ-subvarieties of $\mathbb{A}^n$ and the radical, mixed ideals of $k\{y_1, \dots, y_n\}$.*

Proof. From Proposition 3.14 we know that $\mathbb{I}(\mathbb{V}(\mathfrak{a})) = \mathfrak{a}$ for all $\mathfrak{a} \trianglelefteq_\sigma k\{y_1, \dots, y_n\}$ radial, mixed. Conversely, for a σ-variety $X = \mathbb{V}(F) \subseteq \mathbb{A}^n$ we know $\mathbb{V}(\mathbb{I}(X)) = \mathbb{V}(\mathbb{I}(\mathbb{V}(F))) \subseteq \mathbb{V}(F) = X$, since $F \subseteq \mathbb{I}(X)$. On the other hand it is clear from the definitions of $\mathbb{V}$ and $\mathbb{I}$, that $X \subseteq \mathbb{V}(\mathbb{I}(X))$, so that $X = \mathbb{V}(\mathbb{I}(X))$. This proves the bijectivity of both mappings. That both mappings are inclusion-reversing follows directly from the definitions. $\square$

Note that since every σ-variety (as defined in this thesis) is a σ-subvariety of $\mathbb{A}^n$ for an $n \in \mathbb{N}_{\geq 1}$, it is no restriction to consider $\mathbb{A}^n$ instead of an arbitrary σ-variety, as we can see in the following corollary:

Corollary 3.19. *Let X be a σ-variety over the σ-field k. Then there is a bijection between the radical, mixed σ-ideals of $k\{X\}$ and the σ-subvarieties of X via*

$$X \supseteq Y \mapsto \{f \in k\{X\} \mid f(b) = 0 \textit{ for all } b \in Y(B), \textit{ for all } B \in \sigma\textbf{-int}_k\} =: \mathbb{I}_{k\{X\}}(Y)$$

Proof. If we identify the radical, mixed ideals of $k\{X\}$ with the radical, mixed ideals of $k\{y\}$ which contain $\mathbb{I}(X)$ (see Proposition 2.5), then this is just the restriction of the mapping described in Lemma 3.18. $\square$

A further very interesting bijection can also help us better understand equivalence classes of solutions:

Proposition 3.20. *Let $X = \mathbb{V}(F)$ be a σ-variety over the σ-field k. The equivalence classes of solutions of F are in bijection with the σ-spectrum of the coordinate ring σ-*Spec$(k\{X\})$

Proof. Let $B \in \sigma\textbf{-int}_k$, $b \in B^n$ be a solution of $F \subseteq k\{y_1, \dots, y_n\}$, i.e. $f(b) = 0$ for all $f \in F$. Consider the mapping

$$\varphi : k\{y_1, \dots, y_n\} \to B, y \mapsto b.$$

Then $F \subseteq \ker(\varphi) \trianglelefteq_\sigma R$. Since (forgetting the difference structure for a moment), B is an integral domain, the ideal $\ker(\varphi)$ has to be prime. It follows from this that $\{F\}_m = \mathbb{I}(X) \subseteq$ker$(\varphi)$. In particular, this implies that the mapping φ factors over $\mathbb{I}(X)$, and it induces a morphism of σ-rings $\tilde{\varphi} : k\{X\} \to B$. By the same argument as above, the kernel of this induced morphism, $\mathfrak{p}_b :=$ker$(\tilde{\varphi}) \trianglelefteq_\sigma k\{X\}$ is a prime σ-ideal of $k\{X\}$. The kernel of the mapping constructed this way is always the same for equivalent solutions. To see this, let $b' \in B^n$ such that $k\{b\} \cong k\{b'\}$ via $\iota : b \mapsto b'$. then it holds for the mapping $\varphi' : k\{y_1, \ldots, y_n\} \to B', y \mapsto b$ that $\varphi' = \iota \circ \varphi$ (which is well-defined since $\mathrm{Im}(\varphi) \subseteq k\{b\}$). In particular, since ι is an isomorphism, ker$(\varphi) =$ker(φ'). We define the mapping Ψ from the equivalence classes of solutions of F to σ-Spec$(k\{X\})$ via $b \mapsto \mathfrak{p}_b$.

On the other hand, for $\mathfrak{p} \in \sigma$-Spec$(k\{X\})$, which we identify with $\mathbb{I}(X) \subseteq \tilde{\mathfrak{p}} \in \sigma$-Spec$(k\{y_1, \ldots, y_n\})$ (see Proposition 2.5), consider the integral σ-ring $B(\mathfrak{p}) := k\{y_1, \ldots, y_n\}/\tilde{\mathfrak{p}}$. Since $\tilde{\mathfrak{p}}$ is a prime σ-ideal, $B(\mathfrak{p})$ is an integral σ-ring. Set $b(\mathfrak{p}) := \bar{y} \in B(\mathfrak{p})$, as the image of y in $B(\mathfrak{p})$. Then, because $F \subseteq \mathbb{I}(X) \subseteq \tilde{\mathfrak{p}}$ we know that $b(\mathfrak{p})$ is a solution of F. We define $\Psi^{-1}(\mathfrak{p})$ as the equivalence class of $b(\mathfrak{p})$. Then Ψ and Ψ^{-1} are inverses of each other, and hence, are both bijections. $\square$

From Proposition 3.20 we see that it is a good idea to concentrate on σ-Spec$(k\{X\})$ for a σ-variety X over a σ-field k. From here on, we will speak of the "topology on/of X" to refer to the topology on σ-Spec$(k\{X\})$, as in Definition 2.24. We will also use the convention $x \in X$ to mean $x \in \sigma$-Spec$(k\{X\})$, or $T \subseteq X$ closed to speak of a closed subset of σ-Spec$(k\{X\})$, and so forth.

3.1 Morphisms of Difference Varieties

So far we have only studied difference varieties themselves, but not really a way to relate them with each other; we have yet to properly define the category of difference varieties over a fixed σ-field k: we still have to define what the morphisms in this category shall be.

Definition 3.21. *Let k be a σ-field, $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$ σ-varieties over k. Then, a morphism of functors $f : X \to Y$ is called a* morphism of σ-varieties over k *or* σ-polynomial map *if there exist σ-polynomials $f_1, \ldots, f_m \in k\{y_1, \ldots, y_n\}$ such that $f(b) = (f_1(b), \ldots, f_m(b))$ for all $b \in X(B), B \in \sigma$-$\mathbf{int}_k$.*

Example 3.22. *For two σ-varieties $X \subseteq Y = \mathbb{A}^n_k$, over the σ-field k, the inclusion mapping $\iota : X \hookrightarrow Y$ is a morphism of σ-varieties over k, since we can choose $f_1 = y_1, f_2 = y_2, \ldots, f_n = y_n$. Similarly, for $m \geq n$ and $X \subseteq \mathbb{A}^m_k, Y \subseteq \mathbb{A}^n_k$ the "projection onto $\mathbb{A}^n$" is also a morphism of σ-varieties over k (with the same choice of f_i as the example above).*

Remark 3.23. Let $f : X \to Y$ be a morphism of σ-varieties over the σ-field k, $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$. Then by definition there exist $f_1, \ldots, f_m \in k\{y_1, \ldots, y_n\}$ such that $f(b) = (f_1(b), \ldots, f_m(b))$ for all $b \in B$, $B \in \sigma$-$\mathbf{int}_k$. Modulo $\mathbb{I}(X)$, these f_i are unique: If there is $f'_1, \ldots, f'_m \in k\{y_1, \ldots, y_n\}$ such that $f_i(b) =$

$f_i'(b)$ for all $b \in B,\ B \in \sigma\textbf{-int}_k$, and for all $i \in \underline{m}$, then it follows that $(f_i - f_i')(b) = 0$ for all $b \in B,\ B \in \sigma\textbf{-int}_k$, which implies that $f_i - f_i' \in \mathbb{I}(X)$ by definition, for all $i \in \underline{m}$.

Now, consider the mapping

$$\phi : k\{z_1, \dots, z_m\} \to k\{X\},\ z_i \mapsto f_i + \mathbb{I}(X) =: \overline{f_i}$$

This mapping factors over $\mathbb{I}(Y)$, since for $h \in \mathbb{I}(Y) \subseteq k\{z_1, \dots, z_m\},\ b \in X(B),\ B \in \sigma\textbf{-int}_k$, we have that

$$(\phi(h))(b) = h(\overline{f}_1(b), \dots, \overline{f}_m(b)) = h(f(b))$$

But since ϕ is a morphism of σ-varieties over k, it follows that $f(b) \in Y(B)$, which implies that $h(f(b)) = 0$, by choice of h, hence $h \in \ker(\phi)$. Altogether, this yields a mapping

$$f^* : k\{Y\} \to k\{X\},\ z_i + \mathbb{I}(Y) \mapsto y_i + \mathbb{I}(X)$$

This mapping is a morphism of integral σ-rings over k, and is called the *dual mapping* or *dual morphism* to f . It holds that

$$f^*(h)(b) = h(f(b)) \text{ for all } h \in k\{Y\}, b \in X(B),\ B \in \sigma\textbf{-int}_k.$$

From the definition it follows that for morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of σ-varieties over k, it holds that $(f \circ g)^* = g^* \circ f^*$. We thus get a contravariant functor $-^*$ from the category of difference varieties over k to $\sigma\textbf{-ring}_k$.

Proposition 3.24. *Let k be a σ-field. Then $-^*$ as defined in Remark 3.23 is an anti-equivalence between the category of σ-varieties over k and the subcategory of $\sigma\textbf{-ring}_k$ which arises by restricting the object class to reduced, well-mixed, finitely σ-generated σ-overrings of k. In particular, a morphism $f : X \to Y$ of σ-varieties over k is an isomorphism if and only $f^* : k\{Y\} \to k\{X\}$ is an isomorphism.*

Proof. Since for a σ-variety X over k, $\mathbb{I}(X)$ is radical and mixed, $k\{X\}$ is always a reduced and well-mixed σ-overring of k, and finitely σ-generated since σ-varieties are defined only for equations with finitely many difference variables. From this it follows that the functor $-^*$ from Remark 3.23 is well defined.
It suffices to show that it is surjective on the skeleton of the categories and bijective on morphisms. Let B be a finitely σ-generated, well-mixed and reduced σ-overring of k. We can then write $B \cong k\{y_1, \dots, y_n\}/\mathfrak{a}$, for an $\mathfrak{a} \trianglelefteq_\sigma k\{y_1, \dots, y_n\}$ radical and mixed. The σ-variety $X = \mathbb{V}(\mathfrak{a}) \subseteq \mathbb{A}^n$ is then a preimage of the isomorphism class of B, since $\mathbb{I}(X) = \mathbb{I}(\mathbb{V}(\mathfrak{a})) = \mathfrak{a}$, because of Proposition 3.14. Thus, $B \cong k\{X\}$.

Now, for the morphisms: First, let X, Y be σ-varieties over k and $f, g \in \mathrm{Hom}(X, Y)$ with $f^* = g^*$. Then we know that for every $h \in k\{X\}$, and every $b \in B,\ B \in \sigma\textbf{-int}_k$ it holds that:

$$h(f(b)) = f^*(h(b)) = g^*(h(b)) = h(g(b)).$$

In particular, $f(b) = g(b)$ for all $b \in B$, $B \in \sigma\textbf{-int}_k$, which implies that $f = g$, and $-^*$ is injective. On the other hand, consider $\varphi : k\{Y\} \to k\{X\}$ a morphism of σ-overrings of k. There exist $n, m \in \mathbb{N}_{\geq 1}$ such that $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$, which means that $k\{X\} = k\{z_1, \ldots, z_n\}/\mathbb{I}(X), k\{Y\} = k\{y_1, \ldots, y_m\}/\mathbb{I}(Y)$. We will construct a preimage of φ: Choose $f_1, \ldots, f_m \in k\{z_1, \ldots, z_n\}$ such that $\varphi(y_i + \mathbb{I}(Y)) = f_i + \mathbb{I}(X)$ for all $i \in \underline{m}$. Then we define a morphism $f : X \to Y$ of σ-varieties over k as follows: $f(b) := (f_1(b), \ldots, f_m(b))$ for all $b \in B$, $B \in \sigma\textbf{-int}_k$. This is well-defined: Let $h \in \mathbb{I}(Y)$. Then, by definition, $h(y_1 + \mathbb{I}(Y), \ldots, y_n + I(Y)) = 0 + \mathbb{I}(Y)$. This implies that $h(f_1 + \mathbb{I}(X), \ldots, f_m + \mathbb{I}(X)) = 0 + \mathbb{I}(X)$, since φ is a morphism of σ-overrings of k. But this in turn implies that $h(f(b)) = 0$ for all $b \in X(B)$, $B \in \sigma\textbf{-int}_k$, which means that f maps indeed onto Y and $f^* = \varphi$ by construction. $\square$

This gives us a pretty good idea about the importance of the coordinate ring in difference algebra. Having defined a category for σ-varieties, we can now see how this new category-theoretic language helps us better understand the topological aspects of difference varieties.

Lemma 3.25. *Let R, S, T be σ-rings, and $\varphi : R \to S, \psi : S \to T$ morphisms of σ-rings. Then the mapping*

$$\tilde{\varphi} : \sigma\text{-Spec}(S) \to \sigma\text{-Spec}(R), \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$$

induced by φ is continuous. In fact, it holds that $\widetilde{\psi \circ \varphi} = \tilde{\varphi} \circ \tilde{\psi}$, and in particular, $R \mapsto \sigma\text{-Spec}(R)$ with $\psi \mapsto \tilde{\psi}$ is a contravariant functor from the category of σ-rings to **Top***, the category of topological spaces.*

Proof. Let $A = \mathcal{V}(F) \subseteq \sigma\text{-Spec}(R)$ be closed. We have to show that $\tilde{\varphi}^{-1}(A) \subseteq \sigma\text{-Spec}(S)$ is closed. But

$$\begin{aligned} \tilde{\varphi}^{-1}(A) = \tilde{\varphi}^{-1}(\mathcal{V}(F)) &= \{\mathfrak{p} \in \sigma\text{-Spec}(S) \mid F \subseteq \varphi^{-1}(\mathfrak{p})\} \\ &= \{\mathfrak{p} \in \sigma\text{-Spec}(S) \mid \varphi(F) \subseteq \mathfrak{p}\} = \mathcal{V}(\varphi(F)). \end{aligned}$$

That $\widetilde{\psi \circ \varphi} = \tilde{\varphi} \circ \tilde{\psi}$ is immediately clear from definition. $\square$

We see thus how radical, mixed σ-ideals and the definition of σ-varieties as functors from $\sigma\textbf{-int}_k$, for an integral σ-ring A, as well as the topology on $\sigma\text{-Spec}(A\{X\})$ all fit together well. These are all in analogous relations to the case for perfect σ-ideals, where σ-varieties are defined from the category of σ-overfields of a σ-field k, and a topology called the Cohn topology is defined on $\text{Spec}^\sigma(k\{X\})$ of σ-prime ideals (see Ch. 1 & 2 of [1]). We will try to shed some light on the choice of the category $\sigma\textbf{-int}_k$ here:

Definition 3.26. *Let k be a σ-field.*

(a) We denote by $\sigma\textbf{-VarField}_k$ the category which has functors of the form $B \mapsto \mathbb{V}_B(F)$ as objects, where B is a finitely σ-generated σ-field extension of k, and as morphisms σ-polynomial maps defined in a fashion analogous to Definition 3.21. We define $\mathbb{I}_{\text{Field}}(X)$ for $X \in \sigma\textbf{-VarField}_k$ and $\mathbb{V}_{\text{Field}}(F)$ analogous to Definitions 3.5 and 3.8.

(b) *Similarly, we denote by* σ**-VarDomain**$_k$ *the category which has functors of the form* $B \mapsto \mathbb{V}_B(F)$ *as objects, where* B *is a finitely* σ*-generated* σ*-domain extension of* k*, and as morphisms* σ*-polynomial maps defined in a fashion analogous to Definition 3.21. Again we define* $\mathbb{I}_{\mathrm{Domain}}(X)$ *for* $X \in \sigma$**-VarDomain**$_k$ *and* $\mathbb{V}_{\mathrm{Domain}}(F)$ *analogous to Definitions 3.5 and 3.8*

(c) *Finally, we denote by* σ**-VarRing**$_k$ *the category which has functors of the form* $B \mapsto \mathbb{V}_B(F)$ *as objects, where* $B \supseteq k$ *is a perfectly* σ*-reduced, finitely* σ*-generated ring over* k*, and as morphisms* σ*-polynomial maps defined in a fashion analogous to Definition 3.21. We also define* $\mathbb{I}_{\mathrm{Ring}}(X)$ *for* $X \in \sigma$**-VarRing**$_k$ *and* $\mathbb{V}_{\mathrm{Ring}}(F)$ *analogous to Definitions 3.5 and 3.8*

In all three cases $F \subseteq k\{y\}$ *denotes a set of* σ*-polynomials on finitely many difference variables* $y = y_1, \ldots, y_2$*.*

Proposition 3.27. *Let* k *be a* σ*-field. The three categories* σ**-VarField**$_k$*,* σ**-VarDomain**$_k$ *and* σ**-VarRing**$_k$ *are equivalent.*

Proof. Similar to Proposition 3.24, the category σ**-VarField**$_k$ is anti-equivalent to the category of perfectly σ-reduced σ-overrings of k which are finitely σ-generated over k (see [1], p. 30). It suffices to show that the other two categories are also anti-equivalent to it. From the proof of Proposition 3.24 we can see that it is enough to show that $\mathbb{I}_{\mathrm{Domain}}(\mathbb{V}_{\mathrm{Domain}}(\mathfrak{a})) = \{\mathfrak{a}\}$, and $\mathbb{I}_{\mathrm{Ring}}(\mathbb{V}_{\mathrm{Ring}}(\mathfrak{a})) = \{\mathfrak{a}\}$, where $\mathfrak{a} \trianglelefteq_\sigma k\{y\}$ is a σ-ideal and $\{\mathfrak{a}\}$ its perfect closure.

Let first $X = \mathbb{V}_{\mathrm{Ring}}(F) \in \sigma$**-VarRing**$_k$ be a σ-variety in this sense of perfectly reduced σ-fields. We first show that

$$\begin{aligned} \mathbb{I}_{\mathrm{Ring}}(X) = \{f \in k\{y\} \mid f(b) = 0 \text{ for all } b \in \mathbb{V}_B(F), \\ B \supseteq k \text{ perfectly } \sigma\text{-reduced and finitely } \sigma\text{-generated over } k\} \end{aligned}$$

is a perfect σ-ideal. Similar to Proposition 3.14, we know that $\mathbb{I}_{\mathrm{Ring}}(X)$ is a difference ideal. Let $f \in k\{y\}$ with $\sigma^{i_1}(f) \cdots \sigma^{i_r}(f) \in \mathbb{I}(X)$. This means that for all $b \in V_B(F)$, B perfectly σ-reduced and finitely σ-generated over k: $\sigma^{i_1}(f)(b) \cdots \sigma^{i_r}(f)(b) = 0$. Since B is perfectly σ-reduced, this means that $f(b) = 0$ for all such b, which in turn, by definition, means that $f \in \mathbb{I}_{\mathrm{Ring}}(X)$. Since every σ-domain is perfectly σ-reduced, the argument works the same for $X \in \sigma$**-VarDomain**$_k$ with $\mathbb{I}_{\mathrm{Domain}}$ instead of $\mathbb{I}_{\mathrm{Ring}}$.

For the other inclusion we shall consider first $X \in \sigma$**-VarDomain**$_k$. For $F \subseteq k\{y\}$, it holds that $\{F\} \subseteq \mathbb{I}_{\mathrm{Domain}}(\mathbb{V}_{\mathrm{Domain}}(F))$. To show this, let $f \in \mathbb{I}_{\mathrm{Domain}}(\mathbb{V}_{\mathrm{Domain}}(F))$. It holds that $\{F\}$ is the intersection of all σ-prime ideals of $k\{y\}$ which contain F (see for example Proposition 1.2.22 of [1]), so it is enough to show that $f \in \mathfrak{p}$ for each σ-prime $\mathfrak{p} \trianglelefteq_\sigma k\{y\}$ with $F \subseteq \mathfrak{p}$. We define the σ-domain $B := k\{y\}/\mathfrak{p} =: k\{a\}$, with $a := y + \mathfrak{p} \in k\{y\}/\mathfrak{p}$. Since $F \subseteq \mathfrak{p}$, it holds that $a \in \mathbb{V}_B(F)$, which, by definition of $\mathbb{I}_{\mathrm{Domain}}(\mathbb{V}_{\mathrm{Domain}}(F))$ means that $f(a) = 0$. Hence, $f \in \mathfrak{p}$, for all σ prime $\mathfrak{p}$ with $F \subseteq \mathfrak{p}$, which in turn implies that $f \in \{F\}$. Since every σ-domain B is perfectly σ-reduced, this works for the category σ**-VarRing**$_k$ as well, with $\mathbb{I}_{\mathrm{Ring}}, \mathbb{V}_{\mathrm{Ring}}$ instead of $\mathbb{I}_{\mathrm{Domain}}, \mathbb{V}_{\mathrm{Domain}}$. $\square$

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