1 Bases

Definition 1.1. Let R be a σ-ring, $F \subseteq R$ be a subset of R.

- Then we say that F has a perfect basis if there exisits a finite subset $B \subseteq F$ such, that $\{F\} = \{B\}$.
- We say that F has a mixed, radical basis (or just mixed, for short), if there exisits a finite $B \subseteq F$ such that $\{F\}_m = \{B\}_m$
- We say that F has a weak perfect basis, if there exists a finite $B \subseteq R$ (not necessarily $B \subseteq F$) such that $\{B\} = \{F\}$.
- We say that F has a weak mixed, radical basis, - a w.m. basis for short - if there exists a finite $B \subseteq R$ such that $\{B\}_m = \{F\}_m$.

Lemma 1.2. Let R be a σ-ring, $F \subseteq R$ be a finite subset of R. If F has mixed basis B, then B is also a perfect basis for F. Additionaly, F has a perfect basis iff F has a weak perfect basis, and F has a mixed, radical basis iff F has a weak mixed basis.

Proof. Let $B \subseteq F$ be a mixed basis of F. Then we know that $\{B\}_m = \{F\}_m$. Since for a mixed ideal $\mathfrak{a}$ it holds that $\{\mathfrak{a}\} = \sqrt{\mathfrak{a}}^*$, we know that

$$\{\{B\}_m\} = (\{B\}_m)^* = (\{F\}_m)^* = \{F\}$$

It thus sufficies to show that $\{B\} = \{\{B\}\}_m$. The inclusion $\{B\} \subseteq \{\{B\}\}_m$ is trivial. For the other inclusion we know:

$$\{B\}_m \subseteq \{B\} \Rightarrow \{\{B\}_m\} \subseteq \{\{B\}\} = \{B\}$$

In particular this means that $\{B\} = \{F\}$, so that B is a perfect basis for F. We now turn our attention to the equivalence of having a perfect basis and a weak perfect one. That every perfect basis is also a weak perfect one is obvious. Let thus $F \subseteq R$ have a weak perfect basis $B = b_1, \ldots, b_n \subseteq R$. This means that $\{F\} = \{B\}$. Consider now $b_1 \in B$. By the shuffling process, there exisits an $m \in \mathbb{N}$ such that $b_1 \in F^{\{m\}}$. By the definition of $F^{\{m\}}$ there are only finite many $f \in F$ "involved" in b_1, i.e., there exist $f_{1,1}, \ldots, f_{1,k_1} \in F$ such that $b_1 \in (f_{1,1}, \ldots, f_{1,k_1})^{\{m\}} \subseteq \{f_{1,1}, \ldots, f_{1,k_1}\}$. In a similar way we get $f_{2,1}, \ldots, f_{2,k_2} \in F$ such that $b_2 \in \{f_{2,1}, \ldots, f_{2,k_2}\}$, and so forth. In particular, this means that $B \subseteq \{f_{1,1}, \ldots, f_{1,k_1}, f_{2,1}, \ldots, f_{2,k_2}, \ldots, f_{n,k_n}\} \subseteq \{F\}$. But that means that $\{F\} = \{B\} \subseteq \{f_{1,1} \ldots, f_{n,k_n}\} = \{F\}$, so that $f_{1,1} \ldots, f_{n,k_n}$ is a a perfect basis of F. The equivalence for having mixed and weak mixed bases is obtained in an analogous fashion. $\square$

Remark 1.3. Let R be a σ-ring and let $F \subseteq R$ be a subset. Then, the last direction of Lemma 1.2 does not hold in general, i.e. a set that has a perfect basis does not necesarilly have a mixed basis.

Proof. Let k be a σ-field and consider the ring $R := k\{y_1\}^*$ (reflexive closure). Then the perfect ideal $\mathfrak{a} := \{y_1\} = [y_1]^* = [\sigma^{-i}(y_1) \mid i \in \mathbb{N}] \trianglelefteq_\sigma R$ has a perfect basis but does not have a mixed basis. y_1 is obviously a perfect basis for $\mathfrak{a}$. Now let $B \subset \mathfrak{a}$ be a finite subset of $\mathfrak{a}$. For every element $b \in B$, there is a maximal $i \in \mathbb{N}$ such that $\sigma^{-i}(y_1)$ appears in a monomial of b. Let then

$$i := \max_{i \in \mathbb{N}} \sigma^{-i}(y_1) \text{ appears in } b, \text{ for a } b \in B$$

Then $\sigma^{-i-1}(y_1) \notin \{B\}_m$. $\square$