7.3 Integration by Parts

Integral $\xrightarrow{\text{transformation}}$ Another integral

-> by substitution

-> integration by parts

-> by parts: consequence of product rule for derivatives

$$D_x(uv) = \frac{du}{dx}v + u\cdot\frac{dv}{dx} = u'(x)v(x) + u(x)v'(x)$$

$$\Rightarrow u(x)v'(x) = D_x[u(x)v(x)] - u'(x)v(x)$$

Formula for integr. by parts

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

Procedure

1 Factor integrand into two parts, u and dv.

Two principles guide this factorization: $v = \int dv$ easy to find, and $\int v\,du$ easier to compute than original $\int u\,dv$.

can be applied to definite integrals:

$$\int_a^b u(x)v'(x)\,dx = \int_a^b D_x[u(x)v(x)]\,dx - \int_a^b v(x)u'(x)\,dx$$

$$= [u(x)v(x)]_a^b - \int_a^b v(x)u'(x)\,dx \Rightarrow \int_{x=a}^{x=b} u\,dv = [uv]_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,du$$

Reduction Formula for $\int \sec^n x\,dx = \int \sec^{n-2}x \cdot \sec^2 x\,dx$

$u = \sec^{n-2}x$ $\qquad$ $dv = \sec^2 x\,dx$

$du = (n-2)\sec^{n-3}(x)\cdot\tan x\cdot\sec x = (n-2)\sec^{n-2}(x)\tan x$ $\qquad$ $v = \tan x$

$$\int \sec^n x\,dx = \sec^{n-2}(x)\cdot\tan(x) - \underbrace{\int \tan^2 x\,(n-2)\sec^{n-2}(x)\,dx}$$

$$(n-2)\int(\sec^2 x - 1)\sec^{n-2}x\,dx$$

$$= (n-2)\int(\sec^n x) - (n-2)\int \sec^{n-2}x\,dx$$

$$= \sec^{n-2}x\tan x - (n-2)\int\sec^n x\,dx + (n-2)\int\sec^{n-2}x\,dx$$

$$\Rightarrow (n-1)\int\sec^n x\,dx = \sec^{n-2}x\tan x + (n-2)\int\sec^{n-2}x\,dx$$

$$\Rightarrow \int\sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int\sec^{n-2}x\,dx$$

$$* \frac{d}{dx}\sec x = \frac{d}{dx}\cos^{-1}(x) = \frac{-(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \tan x\cdot\sec x$$

Note

- we derived an expression for $\int \sec^n x \, dx$, where n is a large number
- given $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

  and $\int \sec^2 x \, dx = \tan x + C$

  the term $\int \sec^{n-2} x \, dx$ will eventually yield one of the two integrals above, by repeated use of the reduction formula.