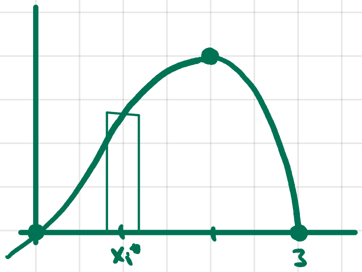
6.3 Volumes by Cylindrical Shells

Example 1

$y = 3x^2 - x^3$ $[0,3]$, revol. around y axis



$y' = 6x - 3x^2 = 3x(2-x) = 0 \Rightarrow x = 0, x = 2$

$y(2) = 3 \cdot 4 - 8 = 4$

$y'' = 6 - 6x = 6(1-x)$ + | − (1)

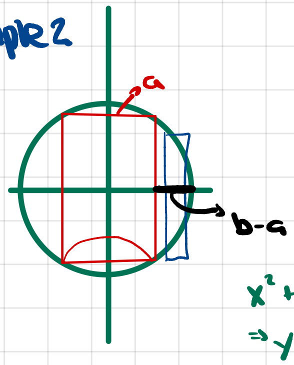
$\Delta V_i \approx 2\pi \cdot x_i^* \cdot f(x_i^*)\Delta x_i$

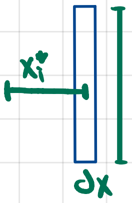
$V \approx \sum \Delta V_i$

$$V = \lim_{n\to\infty} \sum_i \Delta V_i = \int_0^3 2\pi x(3x^2 - x^3)dx = 2\pi \int_0^3 (3x^3 - x^4)dx = 2\pi\left(\frac{3}{4}x^4 - \frac{x^5}{5}\right)\Big|_0^3$$

$$= 2\pi\left[\frac{3}{4}\cdot 3^4 - 3^5/5\right] = 2\pi\left[\frac{5\cdot 3^5 - 4\cdot 3^5}{20}\right] = \frac{\pi \cdot 3^5}{10} = \frac{243\pi}{10}$$

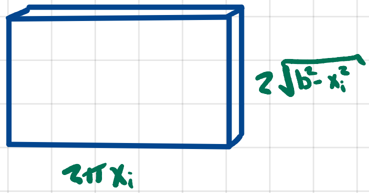
Example 2





$\sqrt{b^2-x^2} - (-\sqrt{b^2-x^2})$
$= 2\sqrt{b^2-x^2}$

$x^2 + y^2 = b^2$
$\Rightarrow y = \pm\sqrt{b^2-x^2}$



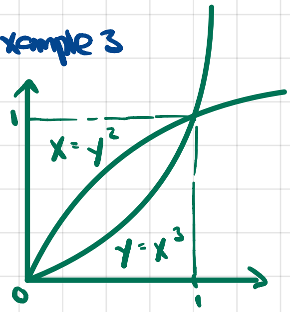
$dV_i \approx 2\pi x_i \cdot 2\sqrt{b^2-x_i^2} \cdot dx$

$V_i \approx \sum_{i=1}^{n} dV_i$

$$V = \lim_{n\to\infty} V_i = \int_a^b 2\pi x \cdot 2\sqrt{b^2-x^2}\,dx = 2\pi\int_a^b 2x\sqrt{b^2-x^2}\,dx = -\frac{2\pi(b^2-x^2)^{\frac{3}{2}}}{3/2}\Big|_a^b = -\frac{4\pi}{3}\left[\overset{0}{\cancel{(b^2-b^2)^{3/2}}} - (b^2-a^2)^{3/2}\right]$$

$$= \frac{4\pi}{3}(b^2-a^2)^{3/2}$$

Example 3



$y^2 = \sqrt[3]{y} \Rightarrow y^3 = y$

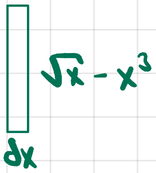
$y(y^2-1) = 0 = y(y+1)(y-1)$

x-axis revolution, cylindrical shell

$$\int_0^1 2\pi y(y^{1/3} - y^2)\,dy = 2\pi\int_0^1 (y^{4/3} - y^3)\,dy$$

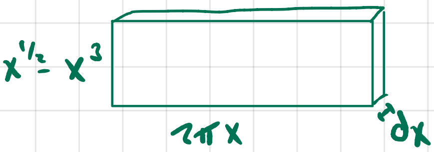
$$= 2\pi\left(y^{7/3}\cdot\frac{3}{7} - \frac{y^4}{4}\right)\Big|_0^1 = 2\pi\left(\frac{3}{7} - \frac{1}{4}\right) = \frac{2\pi\cdot 5}{28} = \frac{5\pi}{14}$$

x-axis revolution, cross sections



$$\int_0^1 \pi(x - x^6)\,dx = \pi\left(\frac{x^2}{2} - \frac{x^7}{7}\right)\Big|_0^1 = \pi\left(\frac{1}{2} - \frac{1}{7}\right) = \pi\left(\frac{7-2}{14}\right) = \frac{5\pi}{14}$$

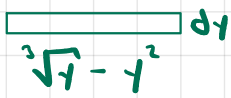
y-axis revolution, cylindrical shells



$$\int_0^1 2\pi x(x^{1/2} - x^3)\,dx = 2\pi\int_0^1 (x^{3/2} - x^4)\,dx = 2\pi\left(x^{5/2}\cdot\frac{2}{5} - \frac{x^5}{5}\right)\Big|_0^1$$

$$= 2\pi\left(\frac{2}{5} - \frac{1}{5}\right) = \frac{2\pi}{5}$$

y-axis revolution, cross sections



$$\int_0^1 \pi(y^{2/3} - y^4)\,dy = \pi\left(y^{5/3}\cdot\frac{3}{5} - \frac{y^5}{5}\right)\Big|_0^1 = \pi\left(\frac{3}{5} - \frac{1}{5}\right) = \frac{2\pi}{5}$$

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$$y_1 = 1 + \frac{x^2}{5} - \frac{x^4}{500} \qquad y_2 = \frac{x^4}{10^4}$$

a) $$\int_0^m 2\pi x\left[1 + \frac{x^2}{5} - \frac{x^4}{500} - \frac{x^4}{10^4}\right]dx = \int_0^m 2\pi x\left[1 + \frac{x^2}{5} - \frac{21x^4}{10^4}\right]dx = 2\pi\int_0^m \left[x + \frac{x^3}{5} - \frac{21x^5}{10^4}\right]dx$$

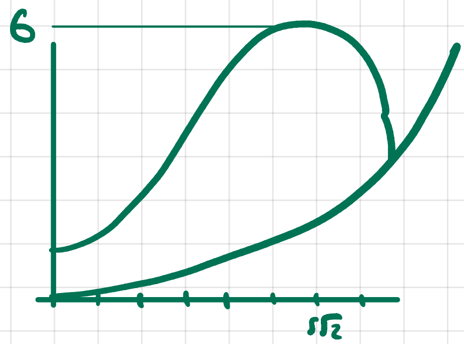
$$= 2\pi\left[\frac{x^2}{2} + \frac{x^4}{20} - \frac{21}{6\cdot 10^4}x^6\right]_0^m = 2\pi\left[\frac{m^2}{2} + \frac{m^4}{20} - \frac{21m^6}{6\cdot 10^4}\right] = 400\pi \text{ for } m = 10$$

Using Maple, we can find m by finding the roots of $y_1 - y_2$. The one we want is $x = 10$.

b) $$f_1'(x) = \frac{2x}{5} - \frac{4x^3}{500} = 0 \Rightarrow 200x - 4x^3 = 0 \Rightarrow x(200 - 4x^2) = 0 \begin{cases} x = 0 \\ x^2 = 50 \Rightarrow x = \sqrt{50} = 5\sqrt{2} \end{cases}$$

$$f_1(5\sqrt{2}) = 1 + \frac{25\cdot 2}{5} - \frac{5^4\cdot 2^2}{500} = 1 + 10 - \frac{125\cdot 4}{100} =$$

$$= 11 - 5 = 6$$



$$\int_0^{5\sqrt{2}} 2\pi x\left(6 - 1 - \frac{x^2}{5} + \frac{x^4}{500}\right)dx$$

$$2\pi\int_0^{5\sqrt{2}} \left[5x - \frac{x^3}{5} + \frac{x^5}{500}\right]dx$$

$$= 2\pi\left[\frac{5x^2}{2} - \frac{x^4}{20} + \frac{x^6}{3000}\right]_0^{5\sqrt{2}}$$

$$= 2\pi\left[\frac{5\cdot 25\cdot 2}{2} - \frac{5^4\cdot 2^2}{20} + \frac{5^6\cdot 2^3}{3000}\right] = \frac{250\pi}{3} \text{ (Maple)}$$

$$\Rightarrow 1+\frac{x^2}{5}-\frac{x^4}{500}=\frac{x^4}{10^4} \Rightarrow \frac{21x^4}{10^4}-\frac{x^2}{5}-1=0$$

$$\Delta=\frac{10^8}{105^2}+4\,\frac{10^4}{21}=\frac{10^8+4\cdot10^4\cdot5^2\cdot21}{(21\cdot5)^2}=\frac{10^8+2^2\cdot10^4\cdot5^2\cdot21}{105^2}$$

$$\Rightarrow x^4-\frac{10^4}{21\cdot5}\cdot x^2-1=0 \Rightarrow m^2-\frac{10^4}{105}m-\frac{10^4}{21}$$

$$m=\frac{\frac{10^4}{105}\pm\sqrt{\frac{10^8+2^2\cdot10^4\cdot5^2\cdot21}{105^2}}}{2}$$