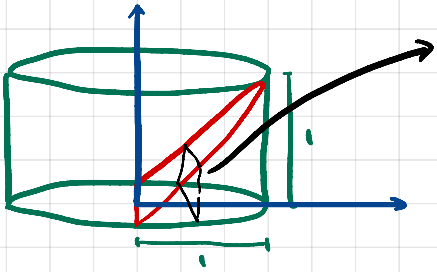
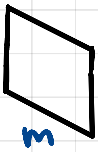
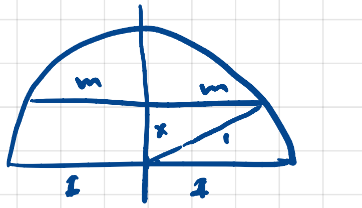
6.2 Volumes by Cross Section

Example 7







$m^2 + x^2 = 1 \Rightarrow m = \sqrt{1-x^2}$

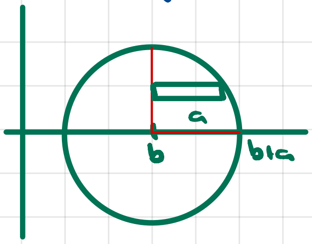
$$\int_0^1 2x\sqrt{1-x^2}\,dx = \int_0^1 -u\,du = -\frac{2}{3}(1-x^2)^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}$$

$u = 1-x^2$

$du = 2x\,dx$

49 $(x-b)^2 + y^2 \le a^2$ revolved around y-axis, $0 < a < b$.

Show $V = 2\pi^2 a^2 b$



$(x-b)^2 = a^2 - y^2$

$x - b = \pm\sqrt{a^2-y^2}$

$x = b \pm \sqrt{a^2-y^2}$ $\quad x^2 = b^2 + 2b\sqrt{a^2-y^2} + (a^2-y^2)$

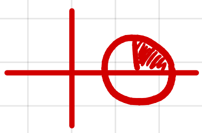
$x^2 - b^2 = 2b\sqrt{a^2-y^2} + (a^2-y^2)$

$$\int_0^a \pi(x^2-b^2)\,dy = \pi\int_0^a \left[2b\sqrt{a^2-y^2} + (a^2-y^2)\right]dy$$

$$= \pi\cdot 2b\int_0^a \sqrt{a^2-y^2}\,dy + \pi a^2\int_0^a dy - \pi\int_0^a y^2\,dy$$

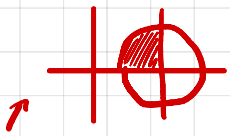
$$= \pi\cdot 2b\cdot\frac{\pi a^2}{4} + \pi a^2 a - \pi\frac{a^3}{3}$$

$$= \frac{\pi^2 a^2 b}{2} + \frac{2}{3}\pi a^3 \rightarrow$$



$$\int_0^a \pi(b^2-x^2)\,dy = \pi\cdot 2\cdot b\int_0^a \sqrt{a^2-y^2}\,dy + \pi\int_0^a y^2\,dy - \pi a^2\int_0^a dy$$

$$= \frac{\pi a^2 b}{2} + \frac{\pi a^3}{3} - \pi a^3 = \frac{\pi a^2 b}{2} - \frac{2}{3}\pi a^3$$



$x = b - \sqrt{a^2-y^2}$

$x^2 = b^2 - 2b\sqrt{a^2-y^2} + a^2 - y^2$

$b^2 - x^2 = 2b\sqrt{a^2-y^2} + y^2 - a^2$

$$\int\sqrt{a^2-y^2}\,dy = \int\sqrt{a^2(1-\sin^2\theta)}\,a\cos\theta\,d\theta$$

$y = a\sin\theta$ $\quad = \int a^2\cos^2\theta\,d\theta$

$dy = a\cos\theta\,d\theta$

$$\int\cos^2\theta\,d\theta = \frac{1}{2}\int(1+\cos 2\theta)\,d\theta = $$

$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)$$

$$\Rightarrow = a^2\cdot\frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) = \frac{a^2}{2}(\theta + \sin\theta\cos\theta)$$

$$= \frac{a^2}{2}\left[\sin^{-1}\left(\frac{y}{a}\right) + \frac{y}{a}\cdot\frac{\sqrt{a^2-y^2}}{a}\right]$$

from 0 to a:

$$\frac{a^2}{2}\left[(\sin^{-1}(1) + 0) - (\sin^{-1}(0) + 0)\right]$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi a^2}{4}$$

Total Volume $= 2\left[\frac{\pi a^2 b}{2} + \cancel{\frac{2}{3}\pi a^3}\right] + 2\left[\frac{\pi a^2 b}{2} - \cancel{\frac{2}{3}\pi a^3}\right]$

$= 2\pi a^2 b$

Note alternative easier solution

Area of single slice revolved around y-axis:

$$\pi\left((b+\sqrt{a^2-y^2})^2-(b-\sqrt{a^2-y^2})^2\right)$$

$$=\pi\left[\cancel{b^2}+2b\sqrt{a^2-y^2}+(\cancel{a^2}-\cancel{y^2})-(\cancel{b^2}-2b\sqrt{a^2-y^2}+(\cancel{a^2}-\cancel{y^2}))\right]$$

$$=4\pi b\sqrt{a^2-y^2}$$

$$V=\int_{-a}^{a}4\pi b\sqrt{a^2-y^2}\,dy=4\pi b\cdot\frac{1}{2}\pi a^2=2\pi^2a^2b$$