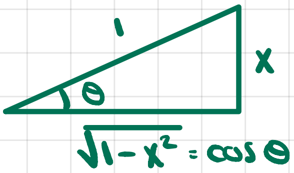
7.6 Trigonometric Substitution

Ex 1: $\displaystyle \int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^3\theta}{\cos\theta}\cos\theta\, d\theta = \int \sin^3\theta\, d\theta = \int (1-\cos^2\theta)\sin\theta\, d\theta = \int (1-u^2)(-du)$

$x = \sin\theta$
$dx = \cos\theta\, d\theta$

$u = \cos\theta$
$du = -\sin\theta\, d\theta$

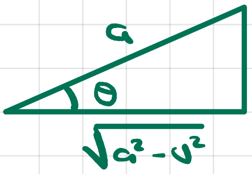


$$= -\left[u - \frac{u^3}{3}\right] + C = \frac{\cos^3\theta}{3} - \cos\theta + C$$

$$= \frac{(1-x^2)^{3/2}}{3} - (1-x^2)^{1/2} + C$$

Ex 2: $\displaystyle \int \sqrt{a^2-u^2}\,du = \int a\cos\theta \cdot a\cos\theta\, d\theta = a^2\int \cos^2\theta\, d\theta = a^2\left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right] + C$

$$\int \tfrac{1}{2}(1+\cos 2\theta)\, d\theta$$
$$= \tfrac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$
$$= \tfrac{1}{2}\left[\theta + \sin\theta\cos\theta\right]$$



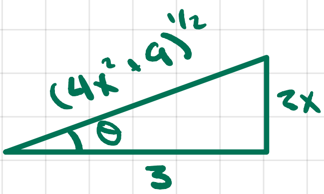
$u = a\sin\theta$
$\Rightarrow du = a\cos\theta\, d\theta$

$= \sqrt{a^2(1-\sin^2\theta)}$
$= \sqrt{a^2\cos^2\theta} = a\cos\theta$

$$= a^2\left[\frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2}\right] + C$$
$$= \frac{a^2\theta}{2} + \frac{a\sin\theta\, a\cos\theta}{2} + C$$
$$= \frac{a^2}{2}\cdot \sin^{-1}\left(\frac{u}{a}\right) + \frac{u\sqrt{a^2-u^2}}{2} + C$$

Ex 3 $\displaystyle \int \frac{1}{(4x^2+9)^2}\,dx = \int \frac{1}{[(2x)^2+3^2]^2}\,dx = \int \frac{(3/2)\sec^2\theta\, d\theta}{[9\tan^2\theta+9]^2} = \frac{3}{2}\int \frac{\sec^2\theta\, d\theta}{[9(\tan^2\theta+1)]^2}$

$2x = 3\tan\theta \qquad dx = \frac{3}{2}\cdot\sec^2\theta$



$$= \frac{3}{2}\int \frac{\sec^2\theta\, d\theta}{[9\sec^2\theta]^2}$$
$$= \frac{3}{2}\int \frac{1}{81\sec^2\theta}\, d\theta = \frac{1}{54}\int \frac{1}{\sec^2\theta}\, d\theta$$
$$= \frac{1}{54}\int \cos^2\theta\, d\theta$$

$\sin\theta = \dfrac{2x}{\sqrt{4x^2+9}} \qquad \theta = \arctan\left(\dfrac{2x}{3}\right)$

$\cos\theta = \dfrac{3}{\sqrt{4x^2+9}}$

$$= \frac{1}{54}\cdot\frac{1}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$
$$= \frac{1}{108}\cdot\left[\tan^{-1}\left[\frac{2x}{3}\right] + \frac{6x}{4x^2+9}\right] + C$$