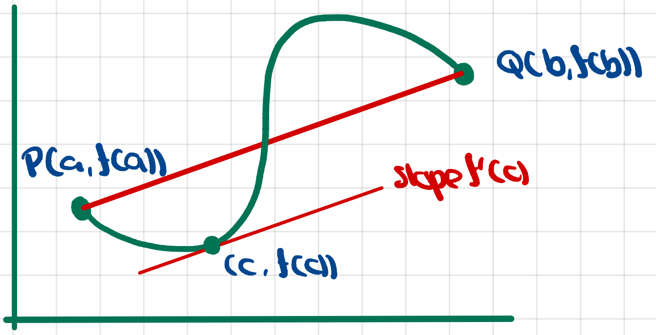
4.3 Mean Value Theorem

Def: f increasing on interval $I=(a,b)$: $x_1, x_2 \in I,\ x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

analogous decreasing

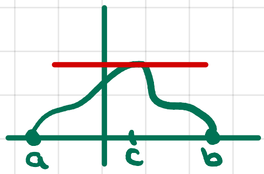
→ we have no proof yet of the significance of the sign of the derivative.

Geometric View



$$\frac{f(b)-f(a)}{b-a} = \text{average (mean) slope of } f \text{ over } [a,b]$$

Rolle's Theorem f continuous on $[a,b]$
differentiable in interior (a,b) $\Rightarrow \exists\, c \in (a,b),\ f'(c)=0$
$f(a)=f(b)=0$



MVT f continuous on $[a,b]$
differentiable in interior (a,b) $\Rightarrow f(b)-f(a)=f'(c)(b-a)$ for some c in (a,b)

Consequences of MVT

$f'(x) \equiv 0$ on $(a,b) \Rightarrow f$ is constant on $[a,b]$, ie $\exists C : f(x)=C$

proof: Consider f cont on $[a,b]$, diff on (a,b).
Apply MVT to $[a,x]$, $x \in (a,b] \Rightarrow \exists c$ in (a,x) s.t. $f(x)-f(a)=f'(c)(x-a)$
Since $f'(x)=0$ in (a,b) by assumption, $f'(c)=0 \Rightarrow f(x)=f(a)$, and this for all x in $(a,b]$
$\Rightarrow f$ constant on $[a,b]$

$f'(x)=g'(x)$ for all x in (a,b) $\Rightarrow \exists K$ constant: $f(x)=g(x)+K$, ie f and g differ by a constant, $\forall x$ in $[a,b]$

proof: $f'(x)=g'(x)\ \forall x$ in (a,b)
Let $h(x)=f(x)-g(x)$ $\Rightarrow h'(x)=0\ \forall x$ in $(a,b) \Rightarrow h(x)=K$ in $[a,b) \Rightarrow f(x)-g(x)=K \Rightarrow f(x)=g(x)+K$

$f'(x)>0\ \forall x \in (a,b)$ $\Rightarrow f$ is increasing fn on $[a,b]$
" < " " " " " decreasing " " "

proof: Let $[u,v] \subset [a,b]$. Apply MVT to f in $[u,v]$ $\Rightarrow f(v)-f(u)=f'(c)(v-u)$, but $f'(c)>0$
and $v-u>0$ by assumption $\Rightarrow f(v)-f(u)>0 \Rightarrow f(v)>f(u)$
Analogous for showing $f'(x)<0$ in $(a,b) \Rightarrow f(v)<f(u)$

First-Derivative Test For Global Extrema

f defined on open interval I, bounded or unbounded

f diff on each point in I except possibly at single critical point c where f is continuous

$f'(x) < 0 \ \forall x \in I : x < c$ and $f'(x) > 0 \ \forall x \in I : x > c \Rightarrow f(c)$ absolute min of f on I

$\Rightarrow$ " $>$ " " " " $<$ " " " $\Rightarrow$ " " max " " " "