Exam 1

1a) $f = \dfrac{x}{1-x^2}$ $\quad f' = \dfrac{1-x^2 - x(-2x)}{(1-x^2)^2} = \dfrac{1-x^2+2x^2}{(1-x^2)^2} = \dfrac{x^2+1}{(1-x^2)^2}$

b) $f = \ln(\cos x) - \frac{1}{2}\sin^2(x)$

$f' = \dfrac{-\sin x}{\cos x} - \frac{1}{2}\cdot 2\sin(x)\cdot\cos(x) = -\tan x - \dfrac{\sin(2x)}{2}$

c) $f = xe^x$ $\quad f' = e^x + xe^x = e^x(x+1)$

$f'' = e^x + e^x + xe^x = e^x(x+2)$

$f^{(5)} = e^x(x+5)$

2

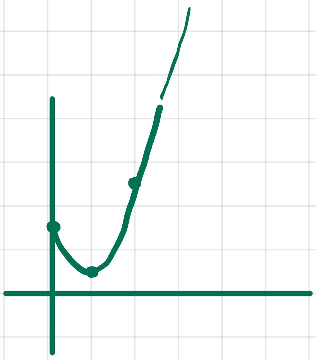
$x^{2/3} + y^{2/3} = 4$ tangent line equation at point $(-\sqrt{27}, 1)$

$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$

$y' = -y^{1/3}x^{-1/3} = -\left(\dfrac{y}{x}\right)^{1/3}$

$y' = -\left(\dfrac{1}{-\sqrt{27}}\right)^{1/3} = \dfrac{-1}{-1\sqrt[6]{27}} = \dfrac{1}{\sqrt{3}}$

$y - 1 = \frac{1}{\sqrt{3}}(x + \sqrt{27})$



3 $\quad y(t) = t^3 - 3t + 3 \quad t \geq 0$

$t \in [0,3]$ $\quad y(0) = 3$

$y(1) = 1$

$y(2) = 8 - 6 + 3 = 5$

$y(3) = 27 - 9 + 3 = 21$

$y' = 3t^2 - 3 = 3(t^2-1) = 3(t+1)(t-1) \Rightarrow t = 1$ critical point

−	−	+	
−	+	+	t+1
	−1	1	
	−	+	y'

Total distance $= |y(1)-y(0)| + |y(2)-y(1)| + |y(3)-y(2)|$

$= 2 + 4 + 16 = 22$

4 $\frac{d}{dx}(f(x)g(x)) = \frac{df(x)}{dx} g(x) + f(x)\frac{d}{dx}g(x)$

$(fg)' = f'g + fg'$

$D_x(fg) = (D_x f)g + f(D_x g)$

Let $h(x) = f(x)\cdot g(x)$

Def. of derivative

$$h'(x) = \frac{d}{dx}(f(x)g(x)) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

sum and subtract term

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - g(x+h)\cdot f(x) + g(x+h)f(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{g(x+h)(f(x+h)-f(x)) + f(x)(g(x+h)-g(x))}{h}$$

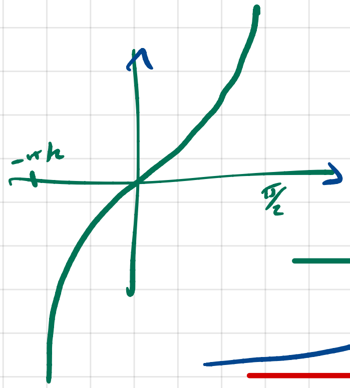
limit rules

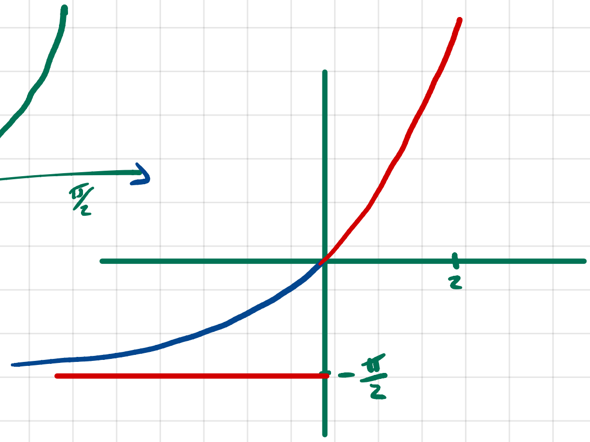
$$\lim_{h\to 0} g(x+h) \cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} f(x) \cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

continuity of g

$$g(x)\cdot f'(x) = f(x)\cdot g'(x)$$

5
$$f(x) = \begin{cases} \tan^{-1} x & x \le 0 \\ ax^2+bx+c & 0<x<2 \\ x^3 - \frac{x^2}{4} + 5 & x \ge 2 \end{cases}$$





continuity

at 0 $f(0) = \tan^{-1}(0) = 0$

$\lim_{x\to 0^+} f(x) = 0 \Rightarrow c = 0$

$a = \frac{5}{2}, b = 1, c = 0 \Rightarrow f$ is diffbdev. $\forall x \in R$

at 2 $f(2) = 8 - 1 + 5 = 12$

$\lim_{x\to 2^-} f(x) = 12 \Rightarrow 4a + 2b = 12 \Rightarrow 2a + b = 6$

differentiable

at 0 $f'(0^-) = \frac{1}{1+x^2}\Big|_{x=0} = 1$

$f'(0^+) = 1 \Rightarrow (2ax+b)\big|_{x=0^+} = 1 \Rightarrow b = 1 \Rightarrow a = \frac{5}{2}$, so $f = \frac{5x^2}{2} + x$ for $x \in (0,2)$

at 2 $f'(2^+) = (3x^2 - x/2)\big|_{x=2^+} = 3\cdot 4 - 2/2 = 11$

$f'(2^-) = 11 \Rightarrow (5x+1)\big|_{x=2^-} = 11$

6

$f(x+y) = f(x) + f(y) + x^2 y + x y^2 \quad \forall x, y \in \mathbb{R}$

$\lim_{x \to 0} \frac{f(x)}{x} = 1$

a) $f(0)$

$x \to 0 \Rightarrow f(y) = f(0) + f(y) \Rightarrow f(0) = 0$

b) $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h} = 1$

c) $\frac{f(x+y) - f(x)}{y} = \frac{f(y)}{y} + x^2 + xy$

$\lim_{y \to 0} \frac{f(x+y) - f(x)}{y} = \lim_{y \to 0} \left[\frac{f(y)}{y} + x^2 + xy \right]$

$f'(x) = 1 + x^2$