Partition P of $[a,b]$: collection of subintervals $[x_0,x_1],[x_1,x_2],\dots,[x_{n-1},x_n]$ of $[a,b]$ such that
$a = x_0 < x_1 < \dots < x_n = b$

$\Delta x_i = x_i - x_{i-1}$ = length of i^{th} subinterval

Selection for partition P: collection of points $S = \{x_1^*, x_2^*, \dots, x_n^*\}$ with x_i^* in $[x_{i-1}, x_i]$

Definition (Riemann Sum)

f defined on $[a,b]$

P a partition of $[a,b]$ $\Rightarrow$ R = Riemann Sum = $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$

S a selection for P — we say this Riemann sum is associated with the partition P.

Def (Norm of a partition P): largest of the lengths Δx_i, denoted $\|P\|$

Def (Definite Integral):

The definite integral of f from a to b is the number $I = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$

provided the limit exists, in which case **f is integrable on $[a,b]$**

The eq. means for each $\epsilon > 0$, $\exists \delta > 0$ s.t. $\left| I - \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \right| < \epsilon$ for every Riemann sum associated with any partition P of $[a,b]$ for which $\|P\| < \delta$.

customary notation: $I = \int_a^b f(x)\,dx$

integrand

Def. for $a=b$ and $a>b$:

$$\int_a^a f(x)\,dx = 0 \qquad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Note:
$\Rightarrow$ not every function is integrable

Theorem f cont. on $[a,b]$ $\rightarrow$ f integrable on $[a,b]$

Theorem

f integrable on $[a,b]$ with integral I $\Leftrightarrow$ $\lim_{n\to\infty} R_n = I$ for every sequence $\{R_n\}_1^\infty$ of Riemann sums associated with a sequence of partitions $\{P_n\}_1^\infty$ of $[a,b]$ such that $\|P_n\| \to 0$ as $n \to \infty$

Def: Average Value of Function

f integrable on $[a,b]$ $\Rightarrow$ average value $\bar{y}$ of f on $[a,b]$ is

$\Updownarrow$

limit of Riemann sum exists on $[a,b]$

$$\bar{y} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Note

rewrite $\int_a^b f(x)\,dx = \bar{y}(b-a)$

Average Value Theorem

f cont. on $[a,b]$ $\Rightarrow$ $f(\bar{x}) = \frac{1}{b-a}\int_a^b f(x)\,dx$ for some $\bar{x}$ in $[a,b]$