infinite sequence: $a_1, a_2, \ldots, a_n, a_{n+1}, \ldots = \{a_n\}_{n=1}^{\infty}$

limit of a sequence: $\lim_{n\to\infty} a_n = L$ means $\{a_n\}$ converges to the real number L, or has the limit L.

this occurs if given $\epsilon > 0$, $\exists N$ s.t. $|a_n - L| < \epsilon$ for $n \geq N$

infinite series: $\sum a_n = a_1 + a_2 + \cdots$

partial sum: $S_n = \sum_{k=1}^{n} a_k$

sum of infinite series

limit of the sequence of partial sums

$\sum a_n$ converges with sum S if $\lim_{n\to\infty} S_n$ exists and finite

geometric series $a_{n+1} = r a_n$ $n \geq 0$, $\sum_{n=0}^{\infty} a_n$

sum of geometric series

$|r| < 1 \Rightarrow S = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

$|r| \geq 1 \Rightarrow$ divergence

proof

$r = 1 \Rightarrow S_n = \sum a_n = a + a + \cdots = (n+1)a$
$\Rightarrow$ divergence as $n \to \infty$

$r = -1$ $a \neq 0 \Rightarrow S_n = a - a + a - a + \cdots$
$\Rightarrow$ sequence of partial sums is $a, 0, a, 0, a, \ldots$: no limit as $n \to \infty$

$|r| \neq 1 \Rightarrow S_n = a(1 + r + r^2 + \cdots + r^n) = a\left(\frac{1}{1-r} - \frac{r^{n+1}}{1-r}\right)$

$|r| < 1$ means $r^{n+1}/1-r$ term $\to 0$ so $\lim_{n\to\infty} S_n = \frac{a}{1-r}$

$|r| > 1 \Rightarrow \lim_{n\to\infty} S_n$ does not exist

n^{th} term test for divergence

$\lim_{n\to\infty} a_n \neq 0$ or limit does not exist $\Rightarrow \sum a_n$ diverges

note: $\lim a_n = 0$ is necessary but not sufficient for convergence.

proof: we show: $\sum a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$

$\sum a_n$ converges $\Rightarrow \lim_{n\to\infty} S_N = L$, $S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N$

$a_n = S_n - S_{n-1}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = L - L = 0$

$\Rightarrow \lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n$ diverges