Indeterminate Form: "certain type of expression with a limit that is not evident by inspection"

→ there are several types:

→ $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \Rightarrow f(x)/g(x)$ has indet. form $0/0$ at $x=a$, ie $\lim_{x\to a} \frac{f(x)}{g(x)}$ is $\frac{0}{0}$, where a can be $\pm\infty$

→ $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty \Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)}$ is $\frac{\pm\infty}{\pm\infty}$

L'Hôpital's Rule

f and g differentiable

$g'(x)$ nonzero in some neighborhood of a, except possibly a itself

$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$

$\Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ provided the latter limit exists (as a finite real number) or is $\pm\infty$.

ex: $\lim_{x\to 1} \frac{1-x+\ln x}{1+\cos(\pi x)}$

$\frac{1-1+0}{-\pi\cdot 0} = \frac{0}{0}$

→ $(1-x+\ln x)$ and $(-\pi\cos\pi x)$ both diff.

→ $g'(x) = -\pi\sin(\pi x)$ which is nonzero in neighborhood of $x=1$

→ $\lim_{x\to 1} g = \lim_{x\to 1} f = 0$

$\Rightarrow = \lim_{x\to 1} \frac{-1+1/x}{-\pi\sin(\pi x)} = \lim_{x\to 1} \frac{-x+1}{-\pi x\sin\pi x} = \lim_{x\to 1} \frac{x-1}{\pi x\sin(\pi x)}$

$\frac{0}{0}$

$f = x-1 \to 1$

$g = \pi x\sin(\pi x) \to \pi\sin(\pi x) + \pi x\cos(\pi x)\cdot\pi$

both diff., $\lim_{x\to 1} f = \lim_{x\to 1} g = 0$

$g'(1) = \pi^2\cdot(-1) \neq 0$, also $\neq$ in neighborhood of 1

$= \lim_{x\to 1} \frac{1}{\pi^2 x\cos(\pi x)} = -\frac{1}{\pi^2}$

Note

Because the last limit exists, it means the first limit exists and is equal to the result of the last limit.

Other important cases of indeterminate forms

→ $\lim_{x\to a} f(x)g(x) = 0\cdot\infty$ → $\lim_{x\to a} \dfrac{f(x)}{1/g(x)} = \dfrac{0}{0}$ OR $\lim \dfrac{g(x)}{1/f(x)} = \dfrac{\infty}{\infty}$

→ $\lim_{x\to a} [f(x) - g(x)] = \infty - \infty$

technique: algebraic manipulation to get $\frac{0}{0}$ or $\frac{\infty}{\infty}$

e.g. common denominator, factorization

→ $y = [f(x)]^{g(x)}$

$\lim_{x\to a} y = 0^0$ or ∞^0 or 1^∞

technique: $\ln y = g(x)\ln f(x) = 0\cdot\infty$

$\lim [f(x)]^{g(x)} = \lim y = \lim e^{\ln y} = e^{\lim \ln y} = e^L$