Exam 1 Review

Power Rule

$$\frac{d}{dx} x^r = r x^{r-1}$$

Proof for $r \in \mathbb{Z}^+$

By the definition of $f'(x)$ if $f(x) = x^r$: $\lim_{\Delta x \to 0} \frac{(x+\Delta x)^r - x^r}{\Delta x}$

We find the limit indirectly, identifying it elsewhere:

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + b^{n-3}a^2 + \cdots + a^{n-1})$$

$$\frac{b^n - a^n}{b-a} = b^{n-1} + b^{n-2}a + \cdots + a^{n-1}$$

As $b \to a$, all n terms on rhs $\to a^{n-1}$ $\Rightarrow \lim_{b \to a} \frac{b^n - a^n}{b-a} = na^{n-1}$

But if $b = x+h$, $a = x$ then $\lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = f'(x) = nx^{n-1}$

Proof for negative integer exponent

$f(x) = x^n$, $n \in \mathbb{Z}^-$

Let $m = -n \Rightarrow m \in \mathbb{Z}^+ \Rightarrow \frac{d}{dx} x^m = mx^{m-1}$

$x^n = x^{-m} = 1/x^m$, $D_x 1/x^m = \frac{-mx^{m-1}}{(x^m)^2} = -mx^{m-1-2m} = -mx^{-m-1} = nx^{n-1}$

Generalized for functions as exponents (chain rule version of power rule)

We know $h = f \circ g \Rightarrow h'(x) = f'(g(x)) \cdot g'(x)$

$g(x) = u$
$g'(x) = du/dx$ $\Rightarrow \frac{d}{dx} f(u) = \frac{d}{du} f(u) \cdot \frac{du}{dx} = f'(u) \cdot \frac{du}{dx}$

$f(u) = u^n \Rightarrow \frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx}$, ie $\frac{d}{dx}(g(x))^n = ng(x)^{n-1} g'(x)$

Assumptions used: $u = g(x)$ differentiable at x, f diff at $g(x)$ $\Rightarrow h = f \circ g$ diff at x

Generalized to Rational exponents

$y = u^r$, u diff. fn of x, $r = p/q \in \mathbb{Q}$

$$\frac{dy}{du} = ru^{r-1}$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = ru^{r-1}\frac{du}{dx}$$

sin x

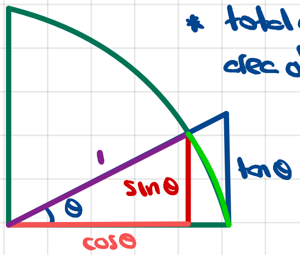
$f(x) = \sin x$

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} = \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim \frac{\cos x \sin h}{h}$$

$$= \sin x \cdot \underbrace{\lim_{h\to 0} \frac{(\cos h - 1)}{h}}_{0} + \cos x \cdot \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{1} = \cos x$$

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

$$\frac{\sin\theta \cdot \cos\theta}{2} < \frac{1}{2}\theta < \frac{1}{2}\frac{\sin\theta}{\cos\theta}$$



* total area $= \pi r^2$

area of sector w/ angle $\theta = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \frac{1}{2}\theta r^2$

$$\cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$\cos\theta < \frac{\sin\theta}{\theta} < \frac{1}{\cos\theta}$$

Squeeze law of limits

$f(\theta) = \cos\theta$
$g(\theta) = \sin\theta / \theta$
$h(\theta) = 1/\cos\theta$

$f(\theta) < g(\theta) < h(\theta)$ for all θ near 0

$\lim_{\theta\to 0} f(\theta) = 1 \qquad \lim_{\theta\to 0} h(\theta) = 1$

$$\Rightarrow \lim_{\theta\to 0} g(\theta) = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

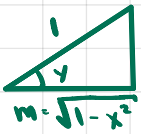
$$\frac{d\tan x}{dx} = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos^2 x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$y = \sin^{-1} x$

$y = f(x)$

$x = f^{-1}(y) \Rightarrow x = \sin y$

$$1 = \cos y \cdot y' \Rightarrow y' = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1-x^2}}$$



$\sin y = x$

$m^2 + x^2 = 1 \Rightarrow m = \sqrt{1-x^2} \Rightarrow \cos(y) = \cos(\sin^{-1} x) = \sqrt{1-x^2}$

$y = \ln^{-1} x$

$x = \ln y \Rightarrow 1 = \frac{1}{y} \cdot y' \Rightarrow y' = y = \ln^{-1} x$

$\ln x$ is a definition. If $y = e^x$ then $\ln y = x$

Therefore if $f(x) = e^x$, $f^{-1}(x) = \ln x$

$$\begin{tikzcd} x \arrow[r, bend left, "e^x"] & y \arrow[l, bend left, "\ln y"] \end{tikzcd}$$