3.3 Chain Rule

10 $y = (1-x^2)^3 / (4+5x+6x^2)^2$

$= (1-x^2)^3 \cdot (4+5x+6x^2)^{-2}$

$y' = 3(1-x^2)(-2x)(4+5x+6x^2)^{-2} + (1-x^2)^3(-2)(4+5x+6x^2)^{-3}(5+12x)$

$= -6x(1-x^2)(4+5x+6x^2)^{-2} - (10+24x)(1-x^2)^3(4+5x+6x^2)^{-3}$

12 $y = [x+(x+x^2)^{-3}]^{-5}$

$y' = -5[x+(x+x^2)^{-3}]^{-6}[1+(-3)(x+x^2)^{-4}(1+2x)]$

13 $y = (u+1)^3 \quad u = 1/x^2$

$\frac{dy}{dx} = 3(u+1)^2 \cdot (-2)x^{-3} = -6(1+1/x^2)^2 x^{-3} = \frac{-6(x^2+1)^2}{x^7}$

17 $y = u(1-u)^3 \quad u = 1/x^4$

$y' = u'(1-u)^3 + u \cdot 3(1-u)^2(-1)u'$

$= -4x^{-5}(1-x^{-4})^3 - 3 \cdot x^{-4}(1-x^{-4})^2 \cdot (-4x^{-5})$

$= -4x^{-5}(1-x^{-4})^3 + 12x^{-9}(1-x^{-4})^2$

$= \frac{-4x^{-5} + 12x^{-9}(1-x^{-4})}{(1-x^{-4})^3} = \frac{-4x^{-5} + 12x^{-9} - 12x^{-13}}{(1-x^{-4})^3}$

19 $y = u^2(u-u^4)^3 \quad u = x^{-2}$

$y' = 2uu'(u-u^4)^3 + u^2 \cdot 3(u-u^4)^2 \cdot (u' - 4u^3u')$

$= 2x^{-2}(-2x^{-3})(x^{-2}-x^{-8})^3 + x^{-8} \cdot 3(x^{-2}-x^{-8})^2(-2x^{-3} - 4x^{-6} \cdot (-2x^{-3}))$

$= -4x^{-5}(x^{-2}-x^{-8})^3 + 3x^{-8}(x^{-2}-x^{-8})^2(-2x^{-3}+8x^{-9})$

$= (x^{-2}-x^{-8})^2[-4x^{-7}+4x^{-13}-6x^{-11}+24x^{-17}]$

$= [x^{-4}-2x^{-10}+x^{-16}][-4x^{-7}+4x^{-13}-6x^{-11}+24x^{-17}]$

$= -4x^{-11}+4x^{-13}-6x^{-15}+24x^{-21}+8x^{-17}-8x^{-23}+12x^{-21}-48x^{-27}-4x^{-23}+4x^{-29}-6x^{-27}+24x^{-33}$

50 $A(t) = \pi r(t)^2$

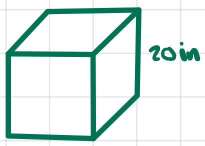
$\frac{dA}{dt}(t_*) = -2\pi \text{ cm}^2/\text{s}$

$A(t_*) = 75\pi = \pi r(t_*)^2 \Rightarrow r(t_*)^2 = 75 \Rightarrow r(t_*) = 5\sqrt{3}$

$\frac{dA}{dt} = \pi \cdot 2 r r' \Rightarrow r' = \frac{A'}{2\pi r}$

$r'(t_*) = \frac{-2\pi}{2\pi \cdot 5\sqrt{3}} = -1/5\sqrt{3} = -\sqrt{3}/15$

62



melting starts 8a.m.

$e(t)$ - edge length, time t, inches $= 20 - kt$

$e'(t) = -k$, $k > 0$, inches/h

$e(0) = 20$

$e(8) = 20 - k \cdot 8 = 8 \Rightarrow k = 12/8 = 3/2 \Rightarrow e'(t) = -\frac{3}{2}$

$V(t) = e(t)^3$

$V'(t) = 3e(t)^2 e'(t)$

$e(4) = 20 - \frac{3}{2} \cdot 4 = 14$

$e'(4) = -3/2$

$\Rightarrow V'(4) = 3 \cdot 14^2 \cdot \left(-\frac{3}{2}\right) = -3 \cdot 196 \cdot \frac{3}{2} = -9 \cdot 98 = -882 \text{ in}^3/\text{h}$

57 spherical balloon

$V = \frac{4}{3}\pi r^3$

$\frac{dr}{dt} = 1 \text{ cm/s}$

$\frac{dV}{dt} = \frac{4\pi}{3} \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 r'$

$\frac{dV}{dt}$ when $r = 10$ cm?

$V' = 4\pi \cdot (100 \text{ cm}^2) \cdot 1 \text{ cm/s}$

$= 400\pi \text{ cm}^3/\text{s}$

58 Air pumped in at $200\pi \text{ cm}^3/\text{s}$

$\frac{dr}{dt}$ when $r = 5$cm?

$V'(t) = 200\pi \text{ cm}^3/\text{s}$

$V = \frac{4\pi r^3}{3}$

$\Rightarrow V'(t) = 4\pi r^2 r'(t)$

$200\pi = 4\pi \cdot 25 r' \Rightarrow r' = 2 \text{ cm/s}$

Alternatively

$V = 4\pi r^3/3 \Rightarrow r = (3V/4\pi)^{1/3}$

$V(5) = 4\pi \cdot 5^3/3$

$\Rightarrow r' = \frac{1}{\cancel{3}}\left[3V/4\pi\right]^{-\frac{2}{3}} \cdot \frac{\cancel{3}}{4\pi} \cdot V' = \frac{V'}{4\pi}\left[3V/4\pi\right]^{-\frac{2}{3}}$

For $t = t_*$, where $r(t_*) = 5$, $V(t_*) = 4\pi \cdot 5^3/3$, $V'(t_*) = 200\pi$

$r'(t_*) = \frac{200\cancel{\pi}}{4\cancel{\pi}} \cdot \left[\frac{\cancel{3} \cdot 4\cancel{\pi} \cdot 5^3}{\cancel{3}} \cdot \frac{1}{4\cancel{\pi}}\right]^{-\frac{2}{3}} = \frac{200}{4 \cdot 5^2} = \frac{200}{100} = 2$

63 $u = f(v)$
$v = g(w)$
$w = h(x)$

f, g, h differentiable $\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists for all x in domain

Explain

$$\frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$$

The chain rule deals with compositions of functions.

$u(w) = f(g(w)) = f \circ g$

$v(x) = g(h(x)) = g \circ h$

Diff $v(x)$ applying chain rule: $\frac{dv}{dx} = \frac{dv}{dw} \frac{dw}{dx}$

$u(x) = f(g(x)) = u(v(x))$

$$\frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$$

64 f differentiable
$f(1) = 1$

$F(x) = f(x^n)$

$G(x) = [f(x)]^n$

n fixed integer

Show $F(1) = G(1)$

$F(1) = f(1^n) = f(1) = 1$
$G(1) = [f(1)]^n = 1^n = 1$ $\Rightarrow F(1) = G(1)$

Show $F'(1) = G'(1)$

$F'(x) = f'(x^n) \cdot nx^{n-1}$
$F'(1) = f'(1) \cdot n$

$\Rightarrow F'(1) = G'(1)$

$G'(x) = n[f(x)]^{n-1} \cdot f'(x)$
$G'(1) = n[f(1)]^{n-1} \cdot f'(1)$
$= nf'(1)$

Intuition

$f(x) = x^3$

$F(x) = f(x^2) = x^6$ ie, F gives f a function as argument.
The arg is a monomial.

If $x = 1$, the monomial is 1, so the input to f is simply 1, so $F(1) = f(1)$.
Differentiating F means using the chain rule. If we have a general monomial ax^r then $(ax^r)' = arx^{r-1}$, and at $x = 1$ this is ar. So at $x = 1$ $F'(1)$ is just $f'(1)$ times the deriv. at 1 of the monom.

As for G, it simply raises f to a power. It gives f as input to a power function. Now, $G(1)$ means take $f(1)$ and raise it to n^{th} power.
Specifically at $x = 1$ with $f(1) = 1$, this outer function makes no difference, because $1^n = 1$.
When we diff. G, it's f' times the deriv. of a general power function. At $x = 1$, again that outer deriv. is simply 1, so G has no effect.

Let's change the parameters of 64:

f differentiable

$f(1)=7$

$F(x)=f(x^n)$

$G(x)=[f(x)]^n$

n fixed integer

$\left.\begin{array}{l} F(1)=f(1^n)=f(1)=7 \\ G(1)=[f(1)]^n=7^n \end{array}\right\}$ no longer the same

$F'(x)=f'(x^n)nx^{n-1}$

$\left.\begin{array}{l} F'(1)=f'(1)\cdot n \\ G'(x)=n[f(x)]^{n-1}\cdot f'(x) \\ G'(1)=n[f(1)]^{n-1}\cdot f'(1) = nf'(1)\cdot 7^{n-1} \end{array}\right\}$ no longer the same

65 $h(x)=\sqrt{x+4}$

$h'(x)=\frac{1}{2}(x+4)^{-\frac{1}{2}}$

66 $h(x)=x^{3/2}=(\sqrt{x})^3$

$h'(x)=3(\sqrt{x})^2\cdot\frac{1}{2}x^{-\frac{1}{2}} = \frac{3}{2}\cdot x^{\frac{1}{2}}$

67 $h(x)=(x^2+4)^{3/2}$

$h'=\frac{3}{2}(x^2+4)^{1/2}\cdot 2x = 3x(x^2+4)^{1/2}$

68 $h(x)=|x|=\sqrt{x^2}$

$h'(x)=\frac{1}{2}(x^2)^{-\frac{1}{2}}\cdot 2x = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}$

$\Rightarrow h'(1)=1 \quad h'(-1)=-1$