ex4: $\int x^2e^{-x}dx = -x^2e^{-x} + \underbrace{\int e^{-x}\cdot 2x\,dx} = -x^2e^{-x} + 2[-e^{-x}(x+1)] + C$

$u = x^2 \qquad dv = e^{-x}dx$
$du = 2x\,dx \qquad v = -e^{-x}$

$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C$

$u = x \qquad dv = e^{-x}dx$
$du = dx \qquad v = -e^{-x}$

ex5 $\int e^{2x}\sin(3x)dx = \frac{-\cos(3x)}{3}\cdot e^{2x} - \underbrace{\int \frac{-\cos(3x)}{3}\cdot 2e^{2x}dx}$

$u = e^{2x} \qquad dv = \sin(3x)dx$
$du = 2e^{2x}dx \qquad v = -\frac{\cos(3x)}{3}$

$u = e^{2x} \qquad dv = \cos(3x)\,dx$
$du = 2e^{2x}dx \qquad v = \frac{\sin(3x)}{3}$

$= \frac{\sin(3x)}{3}e^{2x} - \int \frac{\sin(3x)}{3}\cdot 2e^{2x}dx$

$$\int e^{2x}\sin(3x)dx = \frac{-\cos(3x)e^{2x}}{3} + \frac{2}{3}\left[\frac{\sin(3x)e^{2x}}{3} - \frac{2}{3}\int \sin(3x)e^{2x}dx\right]$$

$$= \frac{-\cos(3x)e^{2x}}{3} + \frac{2}{9}\sin(3x)e^{2x} - \frac{4}{9}\int e^{2x}\sin(3x)dx$$

$$\Rightarrow \frac{13}{9}\int e^{2x}\sin(3x)dx = \frac{-\cos(3x)e^{2x}}{3} + \frac{2}{9}\sin(3x)e^{2x}$$

$$\int e^{2x}\sin(3x)dx = -\frac{3}{13}\cos(3x)e^{2x} + \frac{2}{13}\sin(3x)e^{2x}$$

1 $\int xe^{2x}dx$, $u = x$, $du = dx$, $dv = e^{2x}dt$, $v = \frac{1}{2}e^{2x}$, $= \frac{xe^{2x}}{2} - \int \frac{1}{2}e^{2x}dx = \frac{xe^{2x}}{2} - \frac{1}{4}e^{2x} + C$

2 $\int x^2e^{2x}dx$ $u = x^2$, $du = 2x\,dx$, $dv = e^{2x}dt$, $v = \frac{1}{2}e^{2x}$, $= \frac{x^2e^{2x}}{2} - \cancel{\frac{1}{2}}\int e^{2x}\cdot\cancel{2}x\,dx = \frac{x^2e^{2x}}{2} - \left[\frac{xe^{2x}}{2} - \frac{1}{4}e^{2x}\right] + C$

3 $\int t\sin t\,dt$, $u = t$, $du = dt$, $dv = \sin t\,dt$, $v = -\cos t$ $= -t\cos t + \int \cos t\,dt = -t\cos t + \sin t + C$

4 $\int t^2\sin t\,dt$, $u = t^2$, $du = 2t\,dt$, $dv = \sin t\,dt$, $v = -\cos t$, $= -t^2\cos t + 2\underbrace{\int t\cos t\,dt} = -t^2\cos t + 2t\sin t + 2\cos t + C$

$u = t \qquad dv = \cos t\,dt$
$du = dt \qquad v = \sin t$

$= t\sin t - \int \sin t\,dt = t\sin t + \cos t + C$

5 $\int x\cos 3x\,dx$, $u = x$, $du = dx$, $dv = \cos 3x\,dx$, $v = \frac{\sin 3x}{3}$, $= \frac{x\sin 3x}{3} - \frac{1}{3}\int \sin 3x\,dx = \frac{x\sin x}{3} + \frac{\cos 3x}{9}$

6 $\int x\ln x\,dx$, $u = x$, $du = dx$, $dv = \ln x\,dx$, $v = x\ln x - x$, $= x(x\ln x - x) - \int (x\ln x - x)\,dx$

$= x^2\ln x - x^2 - \int x\ln x\,dx + \int x\,dx$

$\Rightarrow 2\int x\ln x = x^2\ln x - x^2 + \frac{x^2}{2} + C$

$\int x\ln x = \frac{x^2\ln x}{2} - \frac{x^2}{4} + C$

* $\int \ln x\,dx$ $u = \ln x$, $du = \frac{dx}{x}$, $dv = dx$, $v = x$, $\Rightarrow \int \ln x\,dx = x\ln x - \int x \cdot \frac{dx}{x}$

$= x\ln x - x + C$

7 $\int x^3\ln x\,dx$ $u = x^3$, $du = 3x^2 dx$, $dv = \ln x\,dx$, $v = x\ln x - x$

$$= (x\ln x - x)x^3 - \underbrace{\int 3x^2(x\ln x - x)\,dx}_{3\int x^3\ln x\,dx - 3\int x^3\,dx}$$

$\Rightarrow 4\int x^3\ln x\,dx = x^4\ln x - x^4 + 3\frac{x^4}{4} + C$

$\int x^3\ln x\,dx = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C$

8 $\int e^{3z}\cos(3z)\,dz$ $u = e^{3z}$, $du = 3e^{3z}$, $dv = \cos 3z\,dz$, $v = \frac{1}{3}\sin 3z$

$= e^{3z}\cdot\frac{1}{3}\sin 3z - \int \frac{1}{3}\sin 3z \cdot 3e^{3z}\,dz = e^{3z}\cdot\frac{1}{3}\sin 3z - \int e^{3z}\sin 3z\,dz$

$\int e^{3z}\sin 3z\,dz = e^{3z}(-\frac{1}{3})\cos 3z - \int (-\frac{1}{3})\cos 3z\, e^{3z}\cdot 3\,dz = -\frac{e^{3z}\cos 3z}{3} + \int e^{3z}\cos 3z\,dz$

$u = e^{3z}$ $du = e^{3z}\cdot 3$
$dv = \sin 3z\,dz$ $v = -\frac{1}{3}\cos 3z$

$\Rightarrow \int e^{3z}\cos(3z)\,dz = e^{3z}\cdot\frac{1}{3}\sin 3z + \frac{e^{3z}\cos 3z}{3} - \int e^{3z}\cos 3z\,dz$

$\int e^{3z}\cos 3z\,dz = \frac{1}{2}\left[\frac{e^{3z}}{3}(\sin 3z + \cos 3z)\right]$

$= \frac{e^{3z}}{6}(\sin 3z + \cos 3z) + C$

9 $\int \arctan x \, dx$

$u = \arctan x$ $\quad dv = dx$

$du = (1/(1+x^2))\,dx$ $\quad v = x$

$y = \tan^{-1} x$

$\tan y = x$

$\sec^2 y \; y' = 1$

$y' = 1/\sec^2 y = \cos^2 y = \frac{1}{1+x^2}$

$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

$= \tan^{-1} x \cdot x - \int \frac{x}{1+x^2} dx$

$= (\tan^{-1} x)x - \frac{1}{2} \ln|1+x^2| + C$

10 $\int \frac{\ln x}{x^2} dx$ $\quad u = \ln x$ $\quad du = \frac{dx}{x}$ $\quad dv = x^{-2} dx$ $\quad v = -x^{-1}$

$= -\frac{\ln x}{x} + \int x^{-2} dx = -\frac{\ln x}{x} - x^{-1} + C$

11 $\int \sqrt{y} \ln y \, dy$ $\quad u = \ln y$ $\quad du = 1/y \, dy$ $\quad dv = \sqrt{y} \, dy$ $\quad v = \frac{2}{3} y^{3/2}$

$= \frac{2}{3} y^{3/2} \ln y - \underbrace{\int \frac{2}{3} y^{3/2} \cdot \frac{1}{y} dy}_{\frac{2}{3} \int y^{1/2} dy = \frac{2}{3} \cdot \frac{2}{3} y^{3/2}} = \frac{2}{3} y^{3/2} \ln y - \frac{4}{9} y^{3/2}$

12 $\int x \sec^2 x \, dx$ $\quad u = x$ $\quad du = dx$ $\quad dv = \sec^2 x \, dx$ $\quad v = \tan x$

$= x \tan x - \underbrace{\int \tan x \, dx}_{\int \frac{\sin x}{\cos x} dx} = x \tan x + \ln|\cos x| + C$

$u = \cos x$ $\quad du = -\sin x \, dx$

$= \int \frac{-du}{u} = -\ln|\cos x|$

13 $\int (\ln t)^2 dt$ $\quad u = (\ln t)^2$ $\quad du = \frac{2 \ln t}{t} dt$ $\quad v = t$

$= t(\ln t)^2 - \int \frac{2 \ln t}{\cancel{t}} \cdot \cancel{t} \, dt = t(\ln t)^2 - 2(t \ln t - t) + C$

14 $\int t (\ln t)^2 dt$ $\quad u = (\ln t)^2$ $\quad du = \frac{2 \ln t}{t} dt$ $\quad dv = t\,dt$ $\quad v = \frac{t^2}{2}$

$= (\ln t)^2 \cdot \frac{t^2}{2} - \int \frac{t^{\cancel{2}}}{\cancel{2}} \cdot \frac{\cancel{2} \ln t}{\cancel{t}} dt = (\ln t)^2 \cdot \frac{t^2}{2} - \underbrace{\int t \ln t \, dt}$ $\quad u = \ln t$ $\quad du = \frac{dt}{t}$

$dv = t\,dt$ $\quad v = t^2/2$

$= \ln t \cdot t^2/2 - \int \frac{t^{\cancel{2}}}{2} \frac{dt}{\cancel{t}} = \frac{t^2}{2} \ln t - \frac{1}{2} \frac{t^2}{2}$

$= (\ln t)^2 \cdot \frac{t^2}{2} - \frac{t^2 \ln t}{2} + \frac{t^4}{4} + C$

Note

$\int (\ln t)^n dt \quad , n \geq 2 \qquad = (\ln t)^n \cdot t - \int t \, \frac{n}{t} (\ln t)^{n-1} = (\ln t)^n t - n \int (\ln t)^{n-1}$

$u = (\ln t)^n \quad du = \frac{n (\ln t)^{n-1}}{t}$

$dv = dt \quad v = t$

↙ reduction formula

$\Rightarrow \int (\ln t)^n dt = (\ln t)^n t - n \int (\ln t)^{n-1} dt = (\ln t)^n t - n F_{n-1}(t)$

$F_n(t) = \int (\ln t)^n dt \quad , n \geq 2$

$F_2(t) = (\ln t)^2 t - 2 \int \ln t \, dt = (\ln t)^2 t - 2(t \ln t - t) + C$

$F_3(t) = (\ln t)^3 t - 3 [(\ln t)^2 t - 2(t \ln t - t)] + C$

53 $\int \sin^n x\,dx = \int \sin x \sin^{n-1} x\,dx$

$u = \sin^{n-1} x \quad du = (n-1)\sin^{n-2}(x)\cos(x)\,dx$

$dv = \sin x\,dx \quad v = -\cos x$

$= \sin^{n-1}(x)\cdot(-\cos(x)) + (n-1)\int \cos^2(x)\sin^{n-2}(x)\,dx$

$= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)(1-\sin^2(x))\,dx$

$= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)\int \sin^n x\,dx$

$\Rightarrow \left[\int \sin^n(x)\,dx\right](1+n-1) = -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx$

$$\Rightarrow \int \sin^n x\,dx = \frac{-\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$