5.9 Numerical Integration

Elementary function: one that can be expressed in terms of polynomial, trigonometric, exponential, or logarithmic functions by means of finite combinations of sums, differences, products, quotients, roots, and function composition

FTC, $\int_a^b f(x)\,dx = [G(x)]_a^b$, can be used to evaluate an integral only if a convenient formula for the antiderivative G of f can be found.

Some simple functions have antiderivatives that are not elementary functions.

Elementary functions can have non-elementary anti-derivatives.

Example: $\int_0^1 e^{-x^2}dx$ ↗ cannot be calculated w/ FTC

↳ $f(x) = e^{-x^2}$

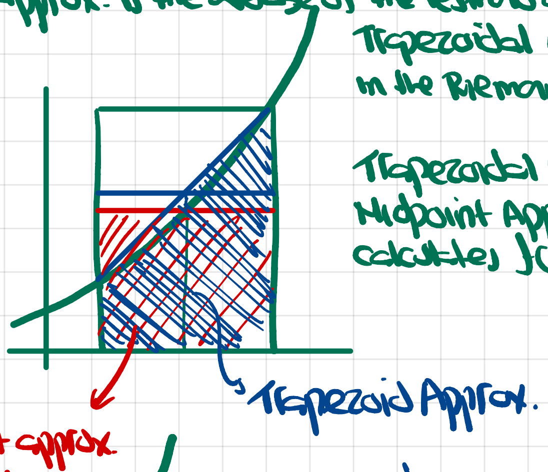
Given continuous f on $[a,b]$, we want to approximate an integral of f on the interval.

Partition $[a,b]$ into n subintervals $\Delta x = \frac{b-a}{n}$.

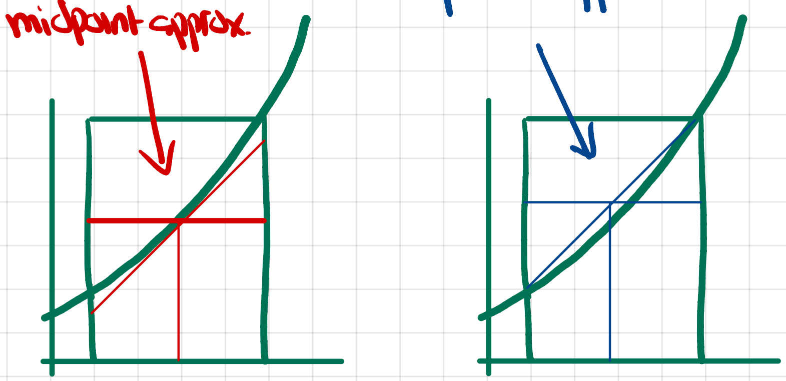
⇒ Any Riemann sum $S = \sum_{i=1}^{n} f(x_i^*)\Delta x$ can be used as approxim to $\int_a^b f(x)dx$.

Given n and choices for x_i^*, we can write out a formula for this approx, in particular with $x_i^* = x_{i-1}$ and $x_i^* = x_i$, the lefthand and righthand approxim. using Riemann sums.

Another approx. is the average of the lefthand and righthand Riemann approximations, called the Trapezoidal Approx. There is also another where we choose $x_i^* = \frac{x_i + x_{i-1}}{2}$ in the Riemann sum. Note the latter two approx are different.



Trapezoidal Approx. averages the f values of right and left.
Midpoint Approx averages two x_i values to obtain x_i^*, and then calculates $f(x_i^*)$.



midpoint approxim., aka, tangent-line approx.

Note

- Area of trapezoid associated to midpoint approx. generally better approx. to $\int_{x_{i-1}}^{x_i} f(x)\,dx$ than the one of trapezoid in trapezoidal approx.

Weighted average of M_n and T_n

M_n → midpoint; T_n → trapezoidal

- M_n weighted more heavily
- in particular, $S_{2n} = \frac{1}{3}(2M_n + T_n) = \frac{2}{3}M_n + \frac{1}{3}T_n$, is called **Simpson's approxim.**
- "$2n$" because we associate the approx. w/ a partition of $[a,b]$ into an even number of equal-length subintervals

$$a = x_0 < x_1 < \cdots < x_n$$

To get a formula for Simpson's Rule, we need to step back and get the formulas for midpoint and trapezoidal approxim.:

Recall: we are using $S = \sum_{i=1}^{n} f(x_i^*)\Delta x$ to approximate $\int_a^b f(x)\,dx$.

Left-endpoint approx.: $\sum_{i=1}^{n} f(x_{i-1})\Delta x = \Delta x \sum_{i=1}^{n} y_{i-1} = L_n$

Right-endpoint approx.: $\sum_{i=1}^{n} f(x_i)\Delta x = \Delta x \sum_{i=1}^{n} y_i = R_n$

Trapezoidal approx.: $T_n = \frac{L_n + R_n}{2} = \frac{\Delta x}{2}\sum_{i=1}^{n}(y_{i-1} + y_i) = \frac{\Delta x}{2}\left[(y_0 + y_1) + (y_1 + y_2) + \cdots + (y_{n-2} + y_{n-1}) + (y_{n-1} + y_n)\right]$

1, 2, 2, …, 2, 1 pattern of coeff. $= \frac{\Delta x}{2}\left[y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right]$

Midpoint approx.: $M_n = \sum_{i=1}^{n} f(m_i)\Delta x = \Delta x\,(y_{1/2} + y_{3/2} + \cdots + y_{n-1/2})$

Simpson's approx.: n pairs of subintervals: $[x_0, x_2], [x_2, x_4], \ldots, [x_{2n-2}, x_{2n}]$, each w/ length $2\Delta x$

Midpoint approx. on this partition $= M_n = 2\Delta x \cdot (f(x_1) + f(x_3) + \cdots + f(2n-1))$

Trapezoidal approx. on this partition $= T_n = \frac{2\Delta x}{2}(f(x_0) + 2f(x_2) + 2f(x_4) + \cdots + 2f(x_{2n-2}) + f(x_{2n}))$

We now take a specific weighted average $S_{2n} = \frac{2}{3}M_n + \frac{1}{3}T_n$, to obtain:

$$S_{2n} = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{2n-2} + 4y_{2n-1} + y_{2n}\right)$$

We can also rewrite this for n subintervals, if n even:

$$S_n = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

1, 4, 2, 4, …, 2, 4 coeff. pattern

→ Simpson's Approx. has an important alternative interpretation in terms of parabolic approxim. to the curve $y = f(x)$.

$[a,b]$ is partitioned into $2n$ intervals, each Δx length

Define parabolic function $p_i(x) = A_i + B_i x + C_i x^2$ on $[x_{2i-2}, x_{2i}]$ as follows:

choose A_i, B_i, and C_i so that $p_i(x)$ agrees with $f(x)$ at three points x_{2i-2}, x_{2i-1}, and x_{2i}.

This is done solving equations:

$$A_i + B_i x_{2i-2} + C x_{2i-2}^2 = f(x_{2i-2})$$
$$A_i + B_i x_{2i-1} + C x_{2i-1} = f(x_{2i-1})$$
$$A_i + B_i x_{2i} + C x_{2i} = f(x_{2i})$$

For A_i, B_i, C_i.

A routine computation shows: $\displaystyle\int_{x_{2i-2}}^{x_{2i}} p_i(x)\,dx = \frac{\Delta x}{3}(y_{2i-2} + 4y_{2i-1} + y_{2i})$

If we now approxim. $\int_a^b f(x)\,dx$ by replacing $f(x)$ with $p_i(x)$ on $[x_{2i-2}, x_{2i}]$ for $i = 1, 2, 3, \ldots, n$ we get:

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{x_{2i-2}}^{x_{2i}} f(x)\,dx \approx \sum_{i=1}^{n} \int_{x_{2i-2}}^{x_{2i}} p_i(x)\,dx = \sum_{i=1}^{n} \frac{\Delta x}{3}(y_{2i-2} + 4y_{2i-1} + y_{2i})$$

$$= \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{2n-2} + 4y_{2n-1} + y_{2n}\right)$$

= Simpson's Approx.