Problem set 2

1F-3 $y = x^{1/n} \Rightarrow y' = 1/n \cdot x^{1/n - 1}$

Altern., $\ln y = \frac{1}{n}\ln x \Rightarrow \frac{y'}{y} = \frac{1}{nx} \Rightarrow y' = \frac{1}{n}\cdot\frac{y}{x} = \frac{1}{n}x^{\frac{1}{n}-1}$

1F-5 $\sin x + \sin y = 1/2 \Rightarrow \sin y = \frac{1}{2} - \sin x \Rightarrow y = \sin^{-1}\left(\frac{1}{2} - \sin x\right) \Rightarrow -1 < \frac{1}{2} - \sin x < 1$

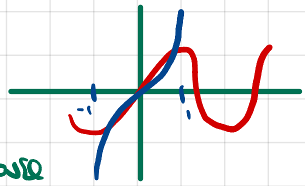
$\Rightarrow \sin x < 3/2$

$\sin x > -\frac{1}{2} \Rightarrow x \in \left(-\frac{k\pi}{6}, \frac{7k\pi}{6}\right)$

$\cos x + \cos y \cdot y' = 0 \Rightarrow y' = -\frac{\cos x}{\cos y}$

$y' = 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2} + 2\pi k,\ k \in Z$

(x can't be $\frac{3\pi}{2}$ because it's not in the domain of $y = f(x)$)



we can check in the first equation for solutions:

$\sin x + \sin y = \frac{1}{2}$

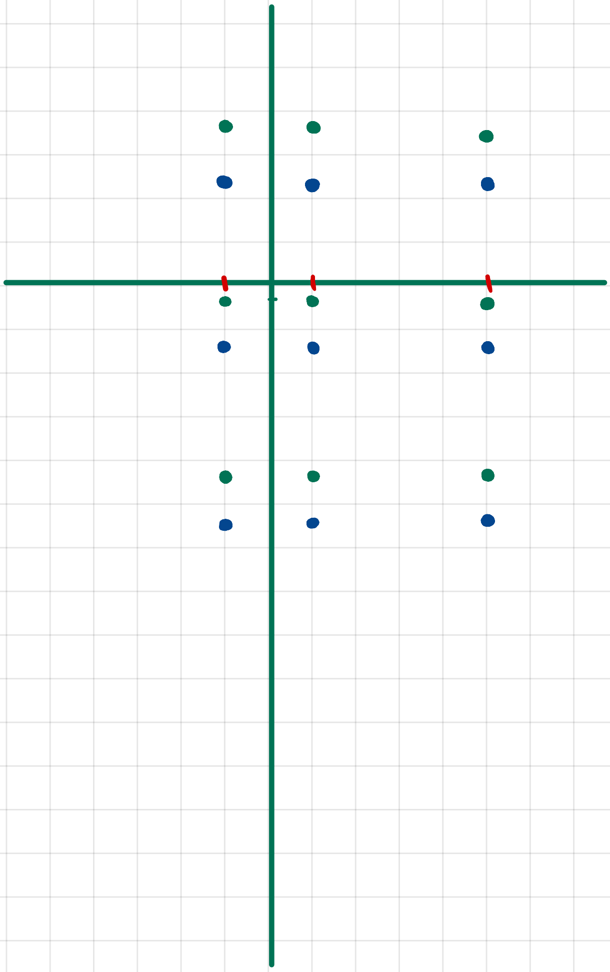
if $x = \frac{\pi}{2} + k\pi$, $\pm 1 + \sin y = 1/2 \Rightarrow \mp 1 + \frac{1}{2}$ $\nearrow$ $-\frac{1}{2} \Rightarrow$ k even are solutions

$\searrow$ $\frac{3}{2} \Rightarrow$ the case of k odd is not a solution

$\Rightarrow x = \frac{\pi}{2} + 2\pi k,\ k \in Z$

$\sin y = -\frac{1}{2} \Rightarrow y = -\frac{\pi}{6} + 2\pi k$ or $\frac{7\pi}{6} + 2\pi k,\ k \in Z$

So $\left(\frac{\pi}{2} + 2\pi k, -\frac{\pi}{6} + 2\pi n\right), \left(\frac{\pi}{2} + 2\pi k, \frac{7\pi}{6} + 2\pi n\right)$ $k, n \in Z$



1F-8

a) $V = \frac{1}{3}\pi r^2 h$

$0 = \frac{1}{3}\pi(2rr'h + r^2) \Rightarrow 2rr'h = -r^2 \Rightarrow r' = \frac{-r^2}{2rh} = \frac{-r}{2h}$

c) $c^2 = a^2 + b^2 - 2ab\cos\theta$

$0 = 2aa' + 2b - 2a'b\cos\theta - 2a\cos\theta$

$= a'(a - b\cos\theta) + b - a\cos\theta$

$a' = \frac{a\cos\theta - b}{a - b\cos\theta}$

1G-4

$y^{(n)}$ of $y = 1/(x+1) = (x+1)^{-1}$

$y' = -1(x+1)^{-2}$

$y'' = 2(x+1)^{-3}$

$y''' = -6(x+1)^{-4}$

$y^{(4)} = 24(x+1)^{-5} = (-1)^4 \cdot 4 \cdot 3 \cdot 2 \cdot 1 (x+1)^{-5}$

$y^{(n)} = (-1)^n n! (x+1)^{-(n+1)}$

1G-5

$y = u(x) \cdot v(x)$

a) $y' = u'v + uv'$

$y'' = u''v + u'v' + u'v' + uv'' = u''v + 2u'v' + uv''$

$y''' = u'''v + u''v' + 2u''v' + 2u'v'' + u'v'' + uv'''$

$= u'''v + 3u''v' + 3u'v'' + uv'''$

b) $y = x^p(1+x)^q$

$$y^{(p+q)} = \cancelto{0}{(x^p)^{(p+q)}}(1+x)^q + \binom{p+q}{1}\cancelto{0}{(x^p)^{(p+q-1)}}[(1+x)^q]^{(1)} + \binom{p+q}{2}\cancelto{0}{(x^p)^{(p+q-2)}}[(1+x)^q]^{(2)} + \cancelto{0}{(\cdot)}$$

$$+ \binom{p+q}{q}(x^p)^{(p)}[(1+x)^q]^{(q)} + \binom{p+q}{q+1}(x^p)^{(p-1)}\cancelto{0}{[(1+x)^q]^{(q+1)}} + \cancelto{0}{(\cdot)} + \binom{p+q}{p+q}(x^p)\cancelto{0}{[(1+x)^q]^{(p+q)}}$$

$$= \binom{p+q}{q}p!\,q! = (p+q)!$$

1H-1

$y = y_0 e^{-kt}$ $\qquad \lambda$ = half-life time

a) $\frac{y_0}{2} = y_0 e^{-kt} \Rightarrow e^{-kt} = \frac{1}{2} \Rightarrow e^{kt} = 2 \Rightarrow kt = \ln 2 \Rightarrow \lambda(k) = \frac{\ln 2}{k}$

b) $y(t_1) = y_1 = y_0 e^{-kt_1}$

$y(t_1 + \lambda) = y(t_1 + \frac{\ln 2}{k}) = y_0 e^{-k(t_1 + \frac{\ln 2}{k})} = y_0 e^{-kt_1} \cdot e^{-\ln 2} = y_1 \cdot 2^{-1} = y_1/2$

1H-2

solution with H^+ ions
dilution calculates
pH change

$pH = -\log_{10}(H^+ \text{concentr.})$

$pH(x) = -\log_{10} x$

$pH(x/2) = -\log_{10}(x/2) = -[\log_{10} x - \log_{10} 2] = pH(x) + \log_{10} 2$

1H-3

a) $\ln(y+1) + \ln(y-1) = 2x + \ln x$

$y - 1 > 0 \Rightarrow y > 1$
$y + 1 > 0 \Rightarrow y > -1$ $\Rightarrow y > 1$

$\ln[(y+1)(y-1)] = \ln(y^2 - 1) = 2x + \ln x$

$y^2 - 1 = e^{2x} \cdot x$

$y^2 = 1 + xe^{2x} \Rightarrow y = (1 + xe^{2x})^{1/2} \quad (y > 0)$

b) $\log(y+1) = x^2 + \log(y-1)$

$\log(y+1) - \log(y-1) = x^2 \Rightarrow \log\left(\frac{y+1}{y-1}\right) = x^2 \Rightarrow \frac{y+1}{y-1} = 10^{x^2}$

$y + 1 = 10^{x^2}y - 10^{x^2} \Rightarrow y(10^{x^2} - 1) = 1 + 10^{x^2} \Rightarrow y = \frac{1 + 10^{x^2}}{10^{x^2} - 1}$

c) $2\ln y = \ln(y+1) + x$

$y > 0$
$y+1>0 \Rightarrow y > -1 \Rightarrow y > 0$

$(e^{\ln y})^2 = y^2 = e^{\ln(y+1)} \cdot e^x = (y+1)e^x$

$y^2 = (y+1)e^x$

$y^2 - e^x y - e^x = 0$

$\Delta = e^{2x} + 4e^x$

$y = \dfrac{e^x \pm \sqrt{e^{2x}+4e^x}}{2}$

but $(e^{2x}+4e^x)^{1/2} = (e^x(e^x+4))^{1/2} > [(e^x)^2]^{1/2} = e^x$

so $y = \dfrac{e^x + \sqrt{e^{2x}+4e^x}}{2}$ (because $y>0$)

1H-5

a) $\dfrac{e^x + e^{-x}}{e^x - e^{-x}} = y$

$u = e^x \Rightarrow \dfrac{u+u^{-1}}{u-u^{-1}} = y \Rightarrow u + u^{-1} = yu - yu^{-1}$

$u(1-y) = -u^{-1}(1+y)$

$\dfrac{u}{u^{-1}} = -\dfrac{(1+y)}{(1-y)} = u^2 \Rightarrow e^{2x} = -\dfrac{1+y}{1-y} \Rightarrow 2x = \ln(-1-y) - \ln(1-y)$

$\Rightarrow x = \dfrac{\ln(-1-y) - \ln(1-y)}{2} = \dfrac{1}{2}\ln\dfrac{y+1}{y-1}$

b) $y = e^x + e^{-x}$, $u = e^x$ $\Rightarrow y = u + u^{-1} = \dfrac{u^2+1}{u} \Rightarrow yu = u^2+1 \Rightarrow u^2 - yu + 1 = 0$ $\quad \Delta = y^2 - 4$ $\quad u = \dfrac{+y \pm \sqrt{y^2-4}}{2}$

$e^x = \dfrac{+y \pm \sqrt{y^2-4}}{2} \Rightarrow x = \ln\left[\dfrac{+y \pm \sqrt{y^2-4}}{2}\right]$

1I-1

c) $e^{-x^2} = f(x)$

$f'(x) = e^{-x^2} \cdot (-2x)$

d) $f = x\ln x - x$

$f' = \ln x + 1 - 1 = \ln x$

e) $f = \ln(x^2)$

$f' = \dfrac{1}{x^2} \cdot 2x = \dfrac{2}{x}$

f) $f = (\ln x)^2$

$f' = 2\ln x \cdot \dfrac{1}{x}$

m) $f = \dfrac{1-e^x}{1+e^x} = (1-e^x)(1+e^x)^{-1}$

$f' = -e^x(1+e^x)^{-1} + (1-e^x)(-1)(1+e^x)^{-2}e^x = \dfrac{-e^x(1+e^x) - e^x + e^{2x}}{(1+e^x)^2} = \dfrac{-e^x \cancel{-e^{2x}} - e^x \cancel{+e^{2x}}}{(1+e^x)^2} = \dfrac{-2e^x}{(1+e^x)^2}$

1I-4 $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$

a) $\lim_{n\to\infty}(1+\frac{1}{n})^{3n}$

$f(x) = (1+1/x)^x$

$g(x) = x^3$

$\lim_{n\to\infty} g(f(n)) = g[\lim_{n\to\infty} f(n)] = g(e) = e^3$

5A-1

a) $\tan^{-1}\sqrt{3}$ $\Rightarrow$ the angle θ with $\tan\theta = \sqrt{3}$

$\theta = \pi/3$

$$\frac{\sin \pi/3}{\cos \pi/3} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

b) $\sin^{-1}(\sqrt{3}/2)$ $\Rightarrow$ what θ has $\sin\theta = \sqrt{3}/2$

$\theta = \pi/3 + 2\pi k$
$\theta = 2\pi/3 + 2\pi k$

c) $\theta = \tan^{-1} 5$ $\Rightarrow$ $\tan\theta = 5$

$\sin\theta = 5\cos\theta$
$25\cos^2\theta + \cos^2\theta = 1$
$\cos^2\theta = 1/26$
$\cos\theta = \pm 1/\sqrt{26}$

$\sin\theta = 5/\sqrt{26}$

$\sec\theta = 1/\cos\theta = \sqrt{26}$

5A-3

f) $\sin^{-1}(a/x) = y = f(x)$

$$f'(x) = \frac{d}{dx}\sin^{-1}(a/x) = \frac{1}{\sqrt{1-(a/x)^2}}\cdot\left(\frac{-a}{x^2}\right) = \frac{-a}{x^2(1-(a/x)^2)^{1/2}}$$

g) $y = f(x) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ $\quad u(x) = \frac{x}{\sqrt{1-x^2}}$

$$f(x) = \tan^{-1}(u(x)) \Rightarrow f'(x) = \frac{1}{1+u^2}\cdot du/dx = \frac{1}{1+x^2/(1-x^2)}\cdot\frac{(1-x^2)^{1/2} - x\cdot\cancel{\frac{1}{2}}(1-x^2)^{-\frac{1}{2}}\cdot(-\cancel{2}x)}{(1-x^2)}$$

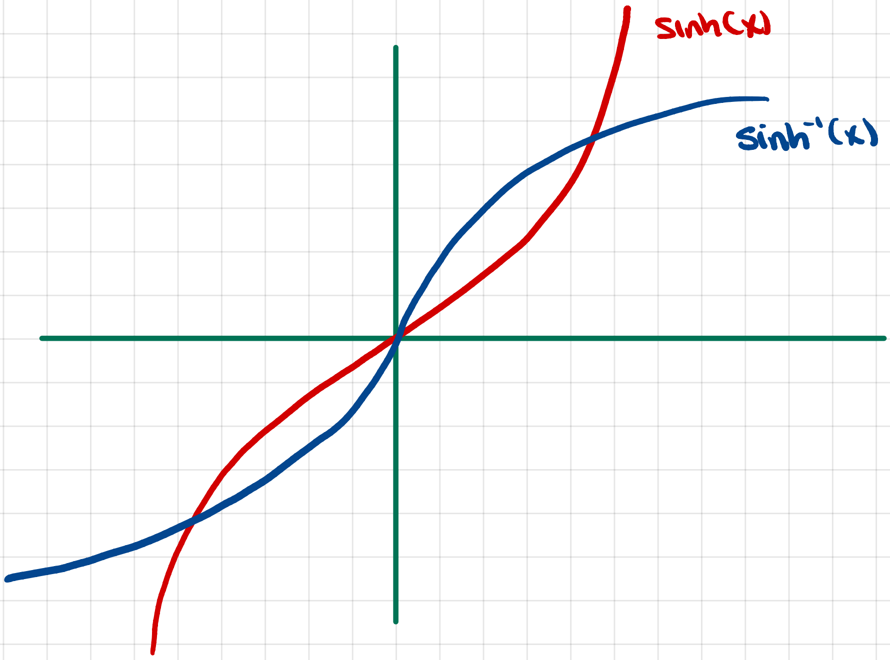
$$= \cancel{(1-x^2)}\cdot\frac{\left[(1-x^2)^{1/2} + x^2(1-x^2)^{-1/2}\right]}{\cancel{(1-x^2)}} = \frac{1-x^2+x^2}{(1-x^2)^{1/2}} = \frac{1}{\sqrt{1-x^2}}$$

5A-3

h) $y = \sin^{-1}\sqrt{1-x}$

$$y' = \frac{1}{(1-(1-x))} \cdot \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$$

$$= \frac{1}{x} \cdot \frac{-1}{2(1-x)^{1/2}} = \frac{-1}{2x(1-x)^{1/2}}$$



5A-5

a) $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$

$$y' = \frac{e^x - e^{-x}(-1)}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$y'' = \frac{e^x + e^{-x}(-1)}{2} = \sinh x$$

$y' = 0 \Rightarrow e^x + e^{-x} = 0$ but $e^x, e^{-x} > 0 \Rightarrow$ no critical points

$y'' = 0 \Rightarrow e^x - e^{-x} = 0 \Rightarrow e^x = \dfrac{1}{e^x} \Rightarrow (e^x)^2 = 1 \Rightarrow e^x = 1 \Rightarrow x = \ln 1 = 0$

$\Rightarrow$ inflection point at $X = 0$

$X = 0$ is also where $y = 0$

$\lim\limits_{x\to\infty} \sinh(x) = +\infty$ $\quad$ $\lim\limits_{x\to-\infty} \sinh = -\infty$

$y''(0) = (e^0 - e^0)/2 = 0$

$x < 0 \Rightarrow e^x < e^{-x} \Rightarrow e^x - e^{-x} < 0 \Rightarrow y'' < 0$

$x > 0 \Rightarrow e^x > e^{-x} \Rightarrow e^x - e^{-x} > 0 \Rightarrow y'' > 0$

$y = f(x) = \sinh(x)$

$f(-x) = \sinh(-x) = (e^{-x} - e^{-(-x)})/2 = (e^{-x} - e^{x})/2 = -(e^x - e^{-x})/2 = -f(x)$

$\Rightarrow$ odd function

b) $y = f(x) = \sinh^{-1}(x) \Rightarrow y$ is $\sinh^{-1} x$ if $\sinh y = x$

domain is $\mathbb{R}$, since Image of $\sinh(x)$ is $\mathbb{R}$

c) $\dfrac{d}{dy} \sinh y \cdot \dfrac{dy}{dx} = 1$

$\cosh y \cdot y' = 1 \Rightarrow \dfrac{d}{dx} \sinh(x) = \cosh(\sinh^{-1} x)$