7.4 Trigonometric Integrals

integrand: power of trigonometric function or product of two such powers

commonly used identities

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1 \Rightarrow \boxed{\cos^2\theta = \frac{1+\cos(2\theta)}{2}}$$

$$\cos(2\theta) = (1-\sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta \Rightarrow \boxed{\sin^2\theta = \frac{1-\cos(2\theta)}{2}}$$

$$\sec^2\theta = \frac{1}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = 1 + \tan^2\theta \Rightarrow \boxed{\sec^2\theta = 1+\tan^2\theta}$$

$$\csc^2\theta = \frac{1}{\sin^2\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = 1 + \cot^2\theta \Rightarrow \boxed{\csc^2\theta = 1+\cot^2\theta}$$

products of sines and cosines

For $\int \sin^m x \cos^n x \, dx$ where

1) m and/or n is odd positive integer
2) m and n nonnegative even integers

we can use

$u = \sin x,\ du = \cos x\, dx$ or $u = \cos x,\ du = -\sin x\, dx$

steps, case 1)

$\int \sin^m x \cos^n x \, dx$, $m = 2k+1$ = odd positive integer

1) separate out a $\sin x$ factor; remaining $\sin^{m-1} x$ has even exponent; use $\sin^2 x = 1-\cos^2 x$

$$\int \sin^{m-1} x \cdot \cos^n x \cdot \sin x \, dx = \int (\sin^2 x)^k \cos^n x \cdot \sin x \, dx$$

$$= \int (1-\cos^2 x)^k \cos^n x \sin x \, dx, \quad u = \cos x, du = -\sin x\, dx$$

$$= \int (1-u^2)^k u^n (-du)$$

polynomial, "easy" to integrate

Analogous when the $\cos x$ exponent is odd.

case 2) m and n nonneg. even integers

→ use half-angle formulas to halve the even powers

→ we can repeat with resulting powers of cos(2x) if necessary until we get odd powers, then apply case 1).

Integrals of Products of Secants and Tangents

First, some integration results:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos x| + C = \ln|\sec x| + C$$

$u = \cos x,\ du = -\sin x\,dx$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}dx = \int \frac{du}{u} = \ln|\sin x| + C = -\ln|\csc x| + C$$

$u = \sin x,\ du = \cos x\,dx$

lengthy derivation

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

similarly, $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

Now, to solve $\int \tan^m x \sec^n x\,dx$ we can routinely solve in two cases:

1 m odd positive int.
2 n even posit. int.

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{-1}{\cos^2 x}\cdot(-\sin x) = \frac{\sin}{\cos}\cdot\frac{1}{\cos} = \tan x \cdot \sec x$$

case 1 separate $\sec x \tan x$ factor to form differential $\sec x \tan x\,dx$ of $\sec x$
we are left with even exponent on $\tan x$: use $\tan^2 x = \sec^2 x - 1$ and obtain expression w/ only $\sec x$ powers.

case 2