9.2 Polar Coordinates

→ give position relative to fixed reference point O (the pole), and to a given ray (the polar axis) beginning at O.

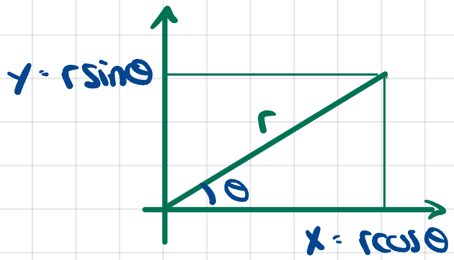
→ any point has an infinite number of reps. in polar coord.

ex: (r,θ), $(-r, \theta+\pi)$

→ polar to rectangular conversion

$x = a\cos\theta$

$y = a\sin\theta$



→ rectangular to polar conversion

$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x} \quad x \neq 0$

Note:

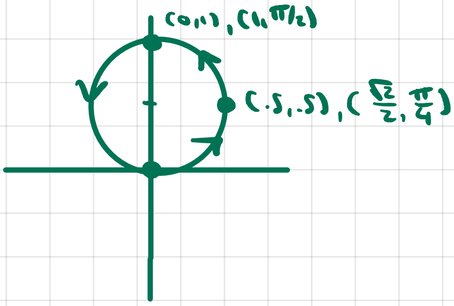
if $x>0$ then $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, range of $\tan^{-1}$ $\Rightarrow$ $\theta = \tan^{-1}(y/x)$

if $x<0$, we know the point is in the second or third quadrant.

$\theta \neq \tan^{-1}(y/x)$ because the image of $\tan^{-1}$ is $(-\pi/2, \pi/2)$, which represent points in quadrants 1 and 4.

we can pick θ as $\theta = \pi + \tan(y/x)$.

Ex: $r = 2\sin\theta$



$\theta = 0 \Rightarrow r = 0 \quad (0,0)$

θ from 0 to $\frac{\pi}{2}$ $\Rightarrow$ r from 0 to 1, $(1, \frac{\pi}{2})$ is $(0,1)$

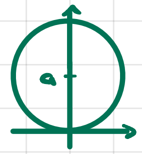
$\theta = \pi/4 \Rightarrow r = \frac{\sqrt{2}}{2} \quad x = \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} = 0.5 \quad y = 0.5$

$x^2 + y^2 = r^2 = 2r\sin\theta = 2y$

$x^2 + y^2 - 2y + 1 = 1 \Rightarrow x^2 + (y-1)^2 = 1$

circle centered $(0,1)$

Circles in Polar Coordinates



$\rightarrow$ $r = 2a\sin\theta$

$r^2 = 2ar\sin\theta = 2ay$

$x^2 + y^2 = 2ar\sin\theta = 2ay$

circle, radius a, centered (0,a)

$x^2 + y^2 - 2ay + a^2 = a^2 \rightarrow x^2 + (y-a)^2 = a^2$

$\rightarrow$ $r = \sin\theta$

$x^2 + y^2 = r\sin\theta = y$

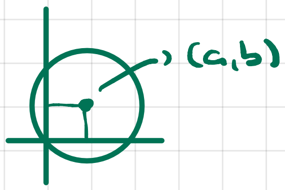
$x^2 + y^2 - y + \left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$

$x^2 + y^2 - 2\cdot\frac{1}{2}\cdot y + \left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$

$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$ $\rightarrow$ radius $\frac{1}{2}$, center $(0, 1/2)$

centered (a,b), radius (a^2+b^2), ie rim touches origin

$(x-a)^2 + (y-b)^2 = R^2$



$x^2 - 2ax + \cancel{a^2} + y^2 - 2by + \cancel{b^2} = \cancel{R^2}$

$x^2 + y^2 - 2ax - 2by = 0$

$r^2 - 2ar\cos\theta - 2br\sin\theta = 0$

$r(r - 2a\cos\theta - 2b\sin\theta) = 0$

$r = 0$ or $r = 2a\cos\theta + 2b\sin\theta$

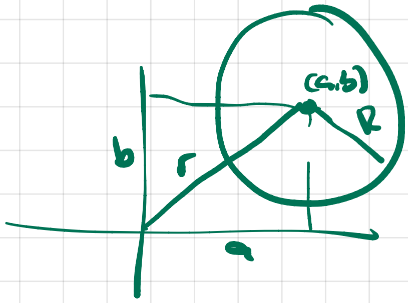
centered (a, b), radius R

$(x-a)^2 + (y-b)^2 = R^2$

$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = R^2$

$x^2 + y^2 - 2ax - 2by + a^2 + b^2 = R^2$

$r^2 - 2ar\cos\theta - 2br\sin\theta + a^2 + b^2 = R^2$



$r^2 - r(2a\cos\theta + 2b\sin\theta) + (a^2 + b^2 - R^2) = 0$

$\Delta = (2a\cos\theta + 2b\sin\theta)^2 - 4(a^2 + b^2 - R^2)$

$= 4a^2\cos^2\theta + 8ab\sin\theta\cos\theta + 4b^2\sin^2\theta - 4a^2 - 4b^2 + 4R^2$

$= 4a^2(\cos^2\theta - 1) + 4b^2(\sin^2\theta - 1) + ab \cdot 4\sin(2\theta) + 4R^2$

$= -4a^2\sin^2\theta - 4b^2\cos^2\theta + 4ab\sin(2\theta) + 4R^2$

$= -4[(a\sin\theta)^2 + (b\cos\theta)^2] + 4ab\sin(2\theta) + 4R^2$

to be continued...