4.2 Linear Approx.

Ex1 $f(x) = \sqrt{1+x}$ $\quad f' = \frac{1}{2}(1+x)^{-1/2}$

$f(x) \approx f(a) + f'(a)(x-a)$

$a=0 \Rightarrow f(x) \approx \sqrt{1} + \frac{1}{2}\cdot 1^{-1/2}\cdot x = 1 + x/2 = L(x)$

$f(x) = (1+x)^k$

$f' = k(1+x)^{k-1}$

$f(x) \approx 1^k + k\cdot 1^{k-1}\cdot x = 1+kx$ ← tangent line at $(0,1)$

Ex2 $(122)^{2/3}$

note $(125)^{2/3} = (125^{1/3})^2 = 5^2 = 25$

$f = x^r \quad f(0)=0$
$f' = rx^{r-1} \quad f'(0)=0$
$f \approx 0+0=0$

$f(x) = x^{2/3} \approx 0$ for x near 0

$f(125) = 25$
$f' = \frac{2}{3}x^{-1/3} \Rightarrow f'(125) = \frac{2}{3}\cdot\frac{1}{5} = \frac{2}{15}$

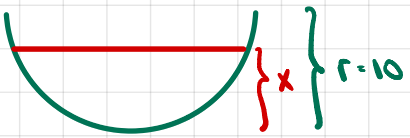
$f \approx 25 + \frac{2}{15}(x-125)$

$f(122) \approx 25 + \frac{2}{15}(-3) = 25 - \frac{2}{5} = \frac{123}{5} = 24.6$

$122^{2/3} \approx 24.6$

Example 3

$V = \frac{\pi}{3}(30x^2 - x^3)$



x is depth.

we measure x to be 5, maxim. error of $1/16$

max error in V, estimated

Note the formula for $V(x)$ is exact. the measurement of x is inexact. We measured $x=5$, which corresponds to $V(5)$ volume. However, if x is off by $\pm 1/16$, V is off by an estimated ± 14.73.
therefore we use $x=5$ and $V(5)$, knowing the maximum error relative to the estimate of volume is $dV/V = 2.25\%$, where $dV = 14.73$ is the differential of V calculated at $x=5$.

strategy: given x_0, we have a $V(x_0)$. we can look at the tangent at x_0 of V to estimate the variation in V for small variation in x_0

$\Rightarrow$ rel. error of measured depth $= \frac{\Delta x}{x} = \frac{1/16}{5} = 1.25\%$

$V(x) \approx V(x_0) + V'(x_0)(x-x_0)$

rel. error in estimated volume (note $V(5) = 654.5$)

$V'(x) = \frac{\pi}{3}(60x - 3x^2) = \pi(20x - x^2)$

$= \frac{dV}{V} \approx \frac{14.73}{654.50} = 2.25\%$

For $x_0 = 5$

$\Delta V = V(x) - V(5) \approx V'(5)\Delta x = \pi(100-25)\Delta x = 75\pi\Delta x$

$\Delta x = \pm 1/16 \Rightarrow \Delta V = \pm 75\pi/16 \approx \pm 14.73$

4 $y = 1/(x-\sqrt{x})$,

by def, $dy = f'(x)dx$

$$y' = \frac{-1}{(x-\sqrt{x})^2}\cdot\left(1-\frac{1}{2\sqrt{x}}\right) = \frac{1-2\sqrt{x}}{2\sqrt{x}(x-\sqrt{x})^2} \Rightarrow dy = \frac{1-2\sqrt{x}}{2\sqrt{x}(x-\sqrt{x})^2}dx$$

22 lin approx to $f(x) = e^{-x}$ near $a=0$

$f(a+\Delta x) \approx f(a) + f'(a)\cdot\Delta x$

$f(x) \approx f(0) + f'(0)\cdot x = 1 - 1\cdot x = 1-x$

$f(0) = e^{-0} = 1$

$f'(x) = -e^{-x}$, $f'(0) = -e^{-0} = -1$

$L(x) = 1-x$

30 Estimate $80^{3/4} = \sqrt[4]{80^3} = 8\sqrt[4]{125}$

We want a linear approxim. of $8\sqrt[4]{125}$. Let's use two different functions:

$f(x) = 8\sqrt[4]{x}$, near $(128, 32)$

$f(x) \approx f(128) + f'(128)(x-128)$

$f'(x) = 8\cdot\frac{1}{4}\cdot x^{-\frac{3}{4}} = 2x^{-3/4} = \frac{2}{\sqrt[4]{x^3}}$

$f'(128) = 2/\sqrt[4]{128^3} = 2/4^3 = 1/32$

$f(125) = 32 + \frac{1}{32}(-3) =$

$= 1025/32 = 32.03125$

$f(x) = 8\sqrt[4]{128+x}$ near $(0, 32)$

$f(x) \approx f(0) + f'(0)(x-0)$

$f(0) = 32$

$f'(x) = 8(128+x)^{-\frac{3}{4}}\cdot\frac{1}{4} = \frac{2}{\sqrt[4]{(128+x)^3}}$

$f'(0) = \frac{2}{\sqrt[4]{4^{12}}} = \frac{2}{4^3}$

$= 1/32$

$f(x) \approx 32 + \frac{1}{32}(-3) = (\ldots) = \frac{1025}{32} = 32.03125$

Relative error is $\frac{dy(-3)}{y(0)} = \frac{-3/32}{32} = \frac{-3}{32^2} \sim 0.003$
$= 0.3\%$

Errors in this problem

lin approx error $= \Delta y - dy$

$dy = f'(0)(x-0) = x/32$

$\Delta y = f(x) - f(0) = 8\sqrt[4]{128+x} - 32$

$\Delta y - dy = 8\sqrt[4]{128+x} - 32 - x/32$, the error in the lin. approx. around $(0, 32)$

In our case, $x = -3 \Rightarrow \Delta y - dy = 8\sqrt[4]{125} - 32 + 3/32$

38 $x\ln y = 1 \Rightarrow \ln y = 1/x \quad y = e^{1/x}$

$$d(x\ln y) = (x\ln y)'\,dt = \left(x'\ln y + \frac{x}{y}\cdot y'\right)dt = 0 = \left(\frac{dx}{dt}\ln y + \frac{x}{y}\cdot\frac{dy}{dt}\right)dt$$

$$\Rightarrow dx\ln y + \frac{x}{y}dy = 0$$

$$\frac{dx}{dy} = \frac{x}{y\ln y} \qquad \frac{dy}{dx} = \frac{y\ln y}{x} = e^{1/x}/x^2$$

step-by-step

$x\ln y = 1$ $\quad$ x and y are functions of t

The differential of the dependent variable $z = x\ln y$ is $dz = d(x\ln y) = (x\ln y)'\,dt$

The differential of indep. var t is $dt = \Delta t$.

$(x\ln y)' = x'\ln y + \frac{x}{y}y'$ by prod. rule

$(x'\ln y + \frac{x}{y}y')\,dt = 0$, 0 the differential of the dep var of $w = g(t) = 1$ because $g'(t) = 0$

$\left(\frac{dx}{dt}\ln y + \frac{x}{y}\frac{dy}{dt}\right)dt = 0$

This is a slope times Δx (ie dx). Recalling defs of dx and dy we know $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$

where these are in fact ratios of differentials.

$\Rightarrow$ we can cancel the dt factor. $\Rightarrow dx\ln y + x\,dy = 0$

From here we have a relationship between different variables so we can find, for ex., $\frac{dy}{dx}$, the ratio of differentials.

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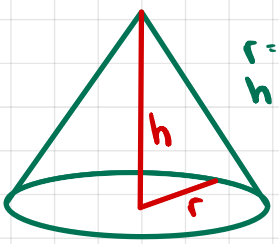


$r = 14$ in
$h = 7$ in $\to 7.1$ in

$V(h) = \frac{1}{3}\pi r^2 h \qquad V(7) = \frac{\pi\cdot 14^2\cdot 7}{3}$

$V'(h) = \frac{1}{3}\pi r^2 \qquad V'(7) = \frac{1}{3}\pi 14^2$

$dV = \frac{1}{3}\pi r^2\,dh$

$r = 14,\ dh = 0.1 \Rightarrow dV = \frac{196\pi}{3}\cdot 0.1 \approx 20.5251$

46

$R = \frac{v^2}{32}\sin(2\theta)$ = range of projectile

v = initial velocity

$v = 80$ ft/s

$R(\theta) = 200\sin(2\theta)$

$R(\theta) \approx R(\theta_0) + R'(\theta_0)(\theta - \theta_0)$

$\theta_0 = 60$

$R(\theta_0) = 200 \cdot \sin(120°) = 200 \cdot \frac{\sqrt{3}}{2}$

$R'(\theta) = 400\cos(2\theta)$

$R'(\theta_0) = 400\cos(120°) = 400 \cdot (-1/2) = -200$

$R(61) \approx R(60) + R'(60) \cdot 1$

$= 200\frac{\sqrt{3}}{2} + (-200) = 200\left(\frac{\sqrt{3}}{2} - 1\right) \approx 170$

* solution manual doing case of $\theta_0 = 45°$ $\theta = 46°$

$R(41) \approx R(45) + R'(45)$

$R(45) = 200$

$\Rightarrow R(41) = R(40)$

$R'(45) = 0$

49 $r = 10$ in, measured with max error of $1/16$ in. Max error in resulting volume?

$V(r) = \frac{4}{3}\pi r^3$

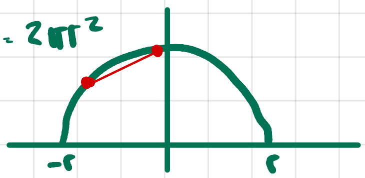
$\Delta V \approx 4\pi r^2 \cdot \Delta r$

$\Delta r = \pm\frac{1}{16} \Rightarrow \Delta V \approx 4\pi \cdot 100 \cdot \left(\pm\frac{1}{16}\right) = \pm 25\pi$

True value of ΔV = $V(10 + 1/16) - V(10)$

51 hemispherical dome, radius = 100 m $\pm$ 0.01 m

surface area = $2\pi r^2$



$y = f(x) = \sqrt{r^2 - x^2}$

$f' = \frac{-x}{\sqrt{r^2 - x^2}}$

$\int_{-r}^{r} 2\pi y\sqrt{1 + f'(x)^2}\,dx =$

$\int_{-r}^{r} 2\pi\sqrt{r^2 - x^2}\sqrt{1 + x^2/(r^2 - x^2)}\,dx$

$= 2\pi\int_{-r}^{r}\sqrt{r^2 - x^2 + x^2}\,dx$

$= 2\pi\int_{-r}^{r} r\,dx = 2\pi r(r - (-r))$

$= 2\pi r \cdot 2r = 4\pi r^2$

true max error in surface area $= 2\pi(100 \pm 0.01)^2 - 2\pi \cdot 100^2$

$= 4.0002\pi$

estimated max error $= \Delta A \approx 8\pi r\Delta r = 4\pi \cdot 100 \cdot (\pm 0.01)$

$= 4\pi$

error % $\approx \frac{\cancel{4}^{2}\cancel{\pi}}{\cancel{2}\cancel{\pi} \cdot 100^2} = 2 \cdot 10^{-4} = 0.0002$

$= 0.02\%$

52 error of at most 0.01% of calculated surface area

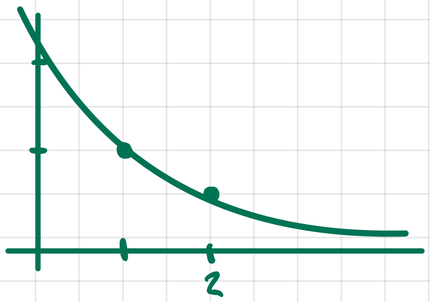
$$\frac{|\overset{2}{\cancel{4}}\cancel{\pi}\cancel{r}\,dr|}{|\cancel{2}\cancel{\pi}r^{\cancel{2}}|} < 0.0001$$

In ex 51, we had $\frac{4\pi\cdot 100\cdot 0.01}{2\pi\cdot 100^2} = \frac{4\pi}{2\pi\cdot 10^4} = \frac{2}{10^4} = \frac{0.02}{10^2}$

$= 0.02\%$

$$\frac{2\,dr}{10^2} < 10^{-4} \Rightarrow dr < \frac{0.5}{10^2} = 0.005\,m$$

55 $f(x) = \frac{1}{x}$ $a = 2$ $\epsilon = 0.01$ $f'(x) = -\frac{1}{x^2}$



$$L(x) \approx \frac{1}{2} - \left[\frac{x-2}{4}\right] = \frac{2-x+2}{4} = \frac{4-x}{4}$$

$\Delta = 16 - 4\cdot 4 = 0$ $x = \frac{4}{2} = 2$
$\Rightarrow \geq 0$

$$|f(x) - L(x)| = \left|\frac{1}{x} - \frac{4-x}{4}\right| = \left|\frac{4-4x+x^2}{4x}\right| < 0.01$$

≥ 0 around 2

$4 - 4x + x^2 < \frac{4}{100}x$

$x^2 - x\left(4 + \frac{4}{100}\right) + 4 < 0$

$x^2 - x\,\frac{101}{25} + 4 < 0$ $\Delta = \left(\frac{101}{25}\right)^2 - 16 > 0$ $x = \frac{\frac{101}{25} \pm \sqrt{\left(\frac{101}{25}\right)^2 - 16}}{2}$ 1.7365 , 2.3035

$\Rightarrow x \in (1.7365, 2.3035)$