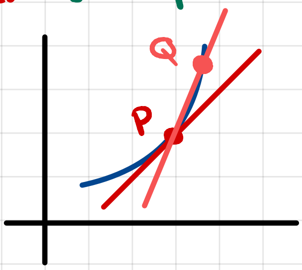
Def: $f'(x_0)$, derivative of f at x_0, slope of tangent line to $y=f(x_0)$ at P



$y-y_0=m(x-x_0)$

point $y_0=f(x_0)$

slope $m=f'(x_0)$

Tangent line: limit of secant lines PQ as $Q \to P$ (P fixed)

$$m=\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x} = \text{slope of tangent line}$$

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

$$\frac{d x^n}{dx}=nx^{n-1} \qquad \frac{d}{dx}\frac{1}{x}=-\frac{1}{x^2}$$

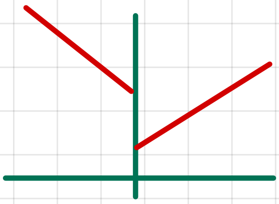
Derivative as instantaneous rate of change

Limits, Continuity

Def f continuous at x_0: $\lim_{x\to x_0} f(x)=f(x_0)$

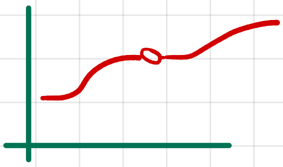
Jump discontinuity

lim left & right exist, but not equal



Removable discontinuity

equal from left and right

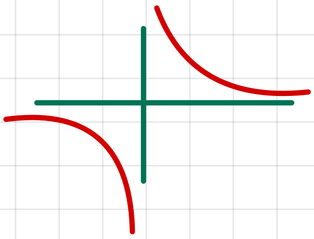


$$\lim \frac{\sin x}{x}=1 \quad \lim \frac{1-\cos x}{x}=0$$

Infinite discontinuity

* deriv. of odd function → even function

$y=\frac{1}{x}$

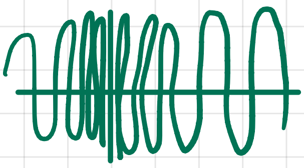


$$\lim_{x\to 0^+}\frac{1}{x}=+\infty$$

$$\lim_{x\to 0^-}\frac{1}{x}=-\infty$$

Other

$y=\sin\frac{1}{x}$ $\lim_{x\to 0}$



theorem f differentiable at $x_0 \Rightarrow f$ continuous at x_0

$$\lim_{x\to x_0} f(x)-f(x_0) = \lim_{x\to x_0} \frac{(f(x)-f(x_0))}{x-x_0}(x-x_0) = f'(x_0)\cdot 0 = 0$$

f is diff.

For reference
$\sin(a+b) = \sin a \cos b + \sin b \cos a$
$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$$\frac{d}{dx}\sin x = \lim_{\Delta x\to 0} \frac{\sin(x+\Delta x)-\sin x}{\Delta x} = \lim_{\Delta x\to 0} \frac{\sin x\cos\Delta x + \cos x \sin\Delta x}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\left[\sin x\underbrace{\left(\frac{\cos\Delta x-1}{\Delta x}\right)}_{\to 0} + \cos x\underbrace{\left(\frac{\sin\Delta x}{\Delta x}\right)}_{\to 1}\right]$$

$= \cos x$

to be proven later

$$\frac{d}{dx}\cos x = \lim_{\Delta x\to 0}\frac{\cos(x+\Delta x)-\cos(x)}{\Delta x} = \frac{\cos x\cos\Delta x - \sin x\sin\Delta x - \cos x}{\Delta x}$$

$$= \cos x\underbrace{\frac{(\cos\Delta x-1)}{\Delta x}}_{\to 0} - \sin x\underbrace{\frac{\sin\Delta x}{\Delta x}}_{\to 1}$$

$= -\sin x$

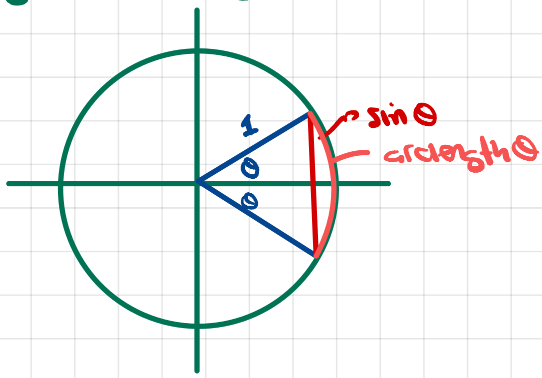
Now we want to show $\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x} = 1$ and $\lim_{\Delta x\to 0}\frac{\cos\Delta x-1}{\Delta x} = 0$

$$\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}=1 \quad = \frac{d}{dx}\sin(0)$$

$$\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin x}{\Delta x}=\cos x$$

$$\frac{d}{dx}\sin x\Big|_{x=0}=\lim_{\Delta x\to 0}\frac{\sin(0+\Delta x)-\sin 0}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$$

geometric argument:



$$\frac{2\sin\theta}{\theta} = \frac{\sin\theta}{\theta}$$

orange line approximates straight line as $\theta\to 0$

$$\lim_{\Delta x\to 0}\frac{\cos\Delta x-1}{\Delta x}=0$$

$$\frac{d}{dx}\cos x\Big|_{x=0}=\lim_{\Delta x\to 0}\frac{\cos(\Delta x)-\cos(0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\cos\Delta x-1}{\Delta x}$$

$$(\cos\Delta x-1)/\Delta x = -(1-\cos\Delta x)/\Delta x$$

If $\frac{d}{dx}f(x_0)=0$ then $\frac{d}{dx}(-f(x_0))=0$

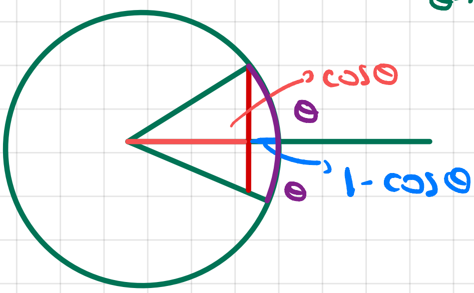
proof

Assume $\frac{d}{dx}f(x_0)=0$, $g(x)=-f(x)$

$$\frac{d}{dx}g(x_0)=\lim\frac{g(x_0+\Delta x)-g(x_0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{-f(x_0+\Delta x)+f(x_0)}{\Delta x}=$$

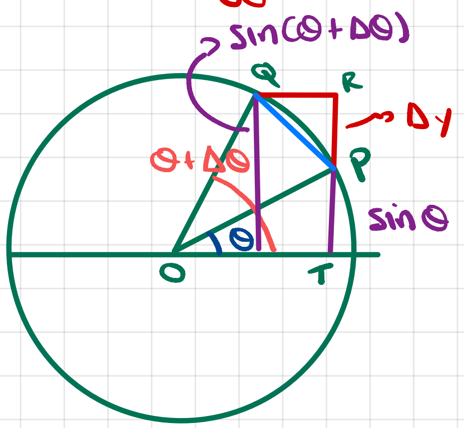
$$=\lim_{\Delta x\to 0}-1\left[\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\right]=0$$

geometric argument for $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=0$



As $\theta\to 0$ the blue line $1-\cos\theta$ goes to zero faster than the purple arc length.

Geometric Proof of $\frac{d}{d\theta}\sin\theta = \cos\theta$



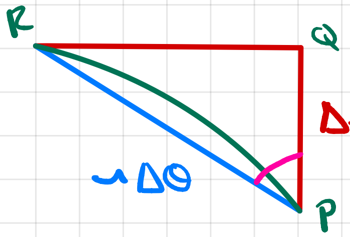
Δy is rate of change of $\sin\theta$

$PQ \approx \Delta\theta$

what is $\angle QPR$?

PQ almost $\perp$ to OP

PR is vertical



$\Delta y = \sin(\theta+\Delta\theta) - \sin(\theta)$

If we know $\angle QPR$ then using cosine we can create $\dfrac{\sin(\theta+\Delta\theta)-\sin(\theta)}{\Delta\theta}$

A geometric argument shows that OTP is similar to PQR, and rotation by 90° puts OTP in the same position as PQR.

Therefore $\angle QPR = \theta$

$$\Delta y = PR \approx \Delta\theta \cdot \cos\theta$$

$$\frac{\Delta y}{\Delta\theta} \approx \cos\theta$$

$$\lim_{\Delta\theta \to 0} \frac{\Delta y}{\Delta\theta} = \cos\theta$$

General Deriv. Rules

product rule $(uv)' = u'v + uv'$

quotient rule $\left(\frac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}, \; v \neq 0$

Lecture 4

proof product rule

$$\Delta(u(x)v(x)) = u(x+\Delta x)v(x+\Delta x) - u(x)v(x)$$
$$= [u(x+\Delta x)-u(x)]v(x+\Delta x) + u(x)v(x+\Delta x) - u(x)v(x)$$
$$= [u(x+\Delta x)-u(x)]v(x+\Delta x) + u(x)[v(x+\Delta x) - v(x)]$$

$$\frac{\Delta(u(x)v(x))}{\Delta x} = \left[\frac{u(x+\Delta x)-u(x)}{\Delta x}\right]\cdot v(x+\Delta x) + u(x)\left[\frac{v(x+\Delta x)-v(x)}{\Delta x}\right]$$

$$\lim \frac{\Delta u(x)v(x)}{\Delta x} = u'(x)\cdot v(x) + u(x)v'(x)$$

proof of quotient rule

$$f(x) = \frac{u(x)}{v(x)}$$

$$\Delta f(x) = \frac{u+\Delta u}{v+\Delta v} - \frac{u}{v} = \frac{v(u+\Delta u)-u(v+\Delta v)}{v(v+\Delta v)}$$

$$= \frac{v\Delta u - u\Delta v}{v(v+\Delta v)}$$

*Remember

$$\lim_{\Delta x\to 0}\frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$
$$= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$
and $\Delta x = x - x_0 \Rightarrow \lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

$$\frac{\Delta f(x)}{\Delta x} = \frac{v(x)\left[\frac{u(x+\Delta x)-u(x)}{\Delta x}\right] - u(x)\left[\frac{v(x+\Delta x)-v(x)}{\Delta x}\right]}{v(x)\left(v(x)+(v(x+\Delta x)-v(x))\right)}$$

$$\lim_{\Delta x\to 0}\frac{\Delta f(x)}{\Delta x} = \frac{v(x)u'(x) - u(x)v'(x)}{v\cdot v}$$

continuity of v assumption

$$\lim_{\Delta x\to 0} v(x+\Delta x) = \lim_{y\to x} v(y) = v(x)$$

Notation

$$u'' = \frac{d}{dx}\cdot\frac{du}{dx} = \frac{d}{dx}\cdot\frac{d}{dx}u = \left(\frac{d}{dx}\right)^2 u = \frac{d^2}{(dx)^2}u = \frac{d^2u}{dx^2}$$

↳ NOT $d(x^2)$

Lecture 5

Implicit Differentiation

(1) $y = x^{m/n}$ we don't know how to differentiate $x^{m/n}$ yet

(2) $y^n = x^m$ we do know how to diff. integer powers

we will derive an expression for $\frac{dy}{dx}$ using implicit diff. This gives us a formula for diff $x^{m/n}$.

Apply $\frac{dy}{dx}$ to (2)

$$\frac{dy}{dx} y^n = \frac{dy}{dx} x^m$$

$$= \frac{d}{dy} y^n \cdot \frac{dy}{dx} = m x^{m-1}$$

$$n y^{n-1} \frac{dy}{dx} = m x^{m-1}$$

$$\frac{dy}{dx} = \frac{m x^{m-1}}{n y^{n-1}} = \frac{m x^{m-1}}{n (x^{m/n})^{n-1}} = a x^{m-1-(n-1)\frac{m}{n}} = a x^{a-1} \quad \text{with } a = \frac{m}{n}$$

Inverse Functions

→ Find deriv. of inverse functions

Ex: $y = \sqrt{x}$ $x \geq 0$, $y^2 = x$

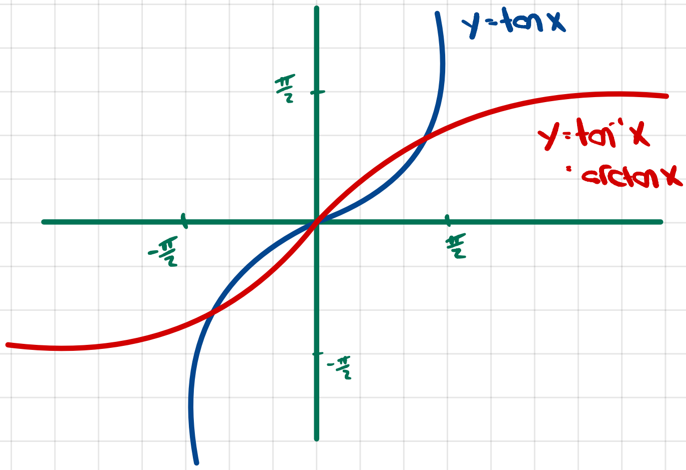
$f(x) = \sqrt{x}$, $g(y) = x$, $g(y) = y^2$

In general,

$y = f(x)$, $x = g(y)$, $g(f(x)) = x$, $g = f^{-1}$, $f = g^{-1}$

Implicit Diff allows us to find deriv. of any inverse fn, provided we know deriv of fn.

Ex: $y = f(x) = \tan^{-1}(x) \Rightarrow \tan y = x$



$$\frac{d}{dy} \tan y = \frac{d}{dy} \frac{\sin y}{\cos y} = (\ldots) = \frac{1}{\cos^2 y} = \sec^2 y$$

$$\frac{d}{dy} (\tan y = x)$$

$$\frac{d}{dy}(\tan y) \frac{dy}{dx} = 1 \Rightarrow \frac{1}{\cos^2 y} y' = 1 \Rightarrow y' = \cos^2 y = \cos^2(\tan^{-1} x)$$

$\tan y = x$

$\cos y = 1/\sqrt{1+x^2} \Rightarrow \cos^2 y = 1/(1+x^2)$

$$\Rightarrow \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

Lecture 6

Goal: $\frac{d}{dx}a^x$

$$\lim_{\Delta x\to 0}\frac{a^{x+\Delta x}-a^x}{\Delta x}=\lim_{\Delta x\to 0}a^x\frac{a^{\Delta x}-1}{\Delta x}=a^x.$$

$$M(a)=\lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}$$

$$\frac{d}{dx}a^x=M(a)\cdot a^x$$

Plug in $x=0$, $\frac{d}{dx}a^x\Big|_{x=0}=M(a)$

$\Rightarrow M(a)$ is slope of a^x at $x=0$. What is $M(a)$?

Define base e as unique number such that $M(e)=1$.

$$\Rightarrow \frac{d}{dx}e^x=e^x,\quad \frac{d}{dx}e^x\Big|_{x=0}=1$$

Why e exists?

$f(x)=2^x$, $f'(0)=M(2)$

stretch by k

$$f(kx)=2^{kx}=(2^k)^x=b^x$$

$$\frac{d}{dx}b^x=\frac{d}{dx}f(kx)=kf'(kx)$$

$$\frac{d}{dx}b^x\Big|_{x=0}=kf'(0)=kM(2)$$

$b=e$ when $k=1/M(2)$

Natural Logarithm

$w=\ln x$

$y=e^x \Leftrightarrow \ln y=x$, defines "ln"

To find $\frac{d}{dx}\ln x$, use implicit diff.

$w=\ln x$, which we don't know how to diff.
$e^w=x$, now we can diff.

$$\frac{d}{dx}e^w=1\Rightarrow \frac{d}{dw}e^w\cdot\frac{dw}{dx}=1\Rightarrow \frac{dw}{dx}=\frac{1}{e^w}=\frac{1}{x}$$

$$\Rightarrow \frac{d}{dx}\ln x=\frac{1}{x}$$

To diff. any exponential, two methods:

Method 1

$$\frac{d}{dx}a^x=?$$

use base e

$$a^x=(e^{\ln a})^x=e^{x\ln a}$$

differentiate

$$\frac{d}{dx}a^x=\frac{d}{dx}e^{x\ln a}=e^{x\ln a}\cdot\ln a=\ln a\cdot a^x$$

$$\Rightarrow M(a)=\ln a$$

Method 2, logarithmic different.

$$\frac{d}{dx}u=?$$

sometimes $\ln(u)$ is easier to differentiate

$$\frac{d}{dx}\ln u=\frac{d}{du}\ln u\cdot\frac{du}{dx}=\frac{u'}{u}$$

$\frac{d}{dx}a^x$, $u=a^x$ $\ln u=x\ln a$

$$\frac{u'}{u}=\ln a\Rightarrow u'=a^x\ln a$$

Ex 3 $\lim_{n\to\infty}(1+\frac{1}{n})^n$

$$\ln\left[(1+1/n)^n\right]=n\ln(1+1/n)$$

$\Delta x=1/n \Rightarrow n\to\infty\Rightarrow\Delta x\to 0$

$$n\ln(1+1/n)=\frac{1}{\Delta x}\ln(1+\Delta x)$$

$$\lim_{\Delta x\to 0}\frac{\ln(1+\Delta x)-\ln(1)}{\Delta x}=\frac{d}{dx}\ln(x)\Big|_{x=1}=\frac{1}{x}\Big|_{x=1}=1$$

so, $a_n=\lim_{n\to\infty}(1+1/n)^n$, $\ln a_n=\ln\left[\lim_{n\to\infty}(1+1/n)^n\right]=1$

$a_n=e$

$\Rightarrow$ we have a numerical way to calculate e.

Review

x^r
$\sin x$
$\cos x$
$\tan x$
$\sec x$
e^x
$\ln x$
$\tan^{-1} x$
$\sin^{-1} x$

From definition of derivative:

$1/x$, $\sin x$, $\cos x$, x^n, a^x, uv, u/v

Fundamental Limits

recognize limit as derivative

$$\lim_{u \to 0} \frac{e^u - 1}{u} = \frac{d}{du} e^u \Big|_{u=0}$$

Derive formulas for $(\sin^{-1} x)'$, $(\ln x)'$ by implic. diff. via $y = \sin^{-1} x \iff \sin y = x$

compute tangent lines

$\Rightarrow$ recognize differentiable functions

graph y'

check left/right tangents must be equal

Lecture 9 Applications of Differentiation

Linear Approximations

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

If you have a curve $y = f(x)$, it's approx. the same as its tangent line

same thing

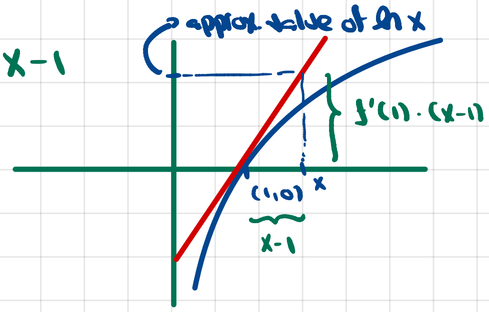
Ex:

$f(x) = \ln x \qquad f' = \frac{1}{x}$

$x_0 = 1, \quad f(1) = 0 \quad f'(1) = 1$

$\ln x \approx f(1) + f'(1)(x-1) = 0 + 1 \cdot (x-1)$

$\ln x \approx x - 1$



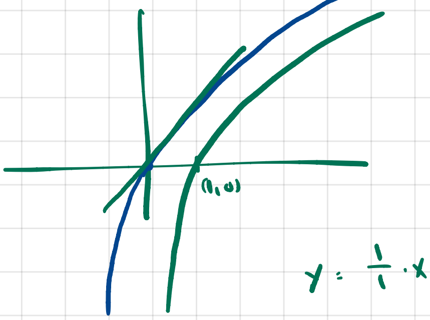
Recall Def. of derivative: $\lim\limits_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = f'(x_0)$

Now we're saying: $\frac{\Delta f}{\Delta x} \approx f'(x_0)$

$\Delta f \approx f'(x_0)\Delta x$

$f(x) - f(x_0) \approx f'(x_0)(x - x_0)$

$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$



Examples

For $x \approx 0$,

$f(x) \approx f(0) + f'(0)x$

$\sin x \approx \sin 0 + \cos 0 \cdot x = x$
$\approx x$

$\ln(1+x) \approx \ln 1 + \frac{1}{1+0}x = x$

$\cos x \approx \cos 0 - \sin 0 \; x = 1$
≈ 1

$(1+x)^r \approx 1^r + r \cdot 1^{r-1} x = 1 + rx$

$e^x \approx e^0 + e^0 x = 1 + x$
$\approx 1 + x$

$\ln 1.1 \approx 0.1$ because $\ln(1+x) \approx x$

$\swarrow$ HARD $\qquad \searrow$ EASY

linear approx. $\Rightarrow$ get s.th. much easier to deal with

Find lin. approx near $x=0$ of $\dfrac{e^{-3x}}{\sqrt{1+x}}$

remember

$(1+x)^r \approx 1+rx$

$e^x \approx 1+x$

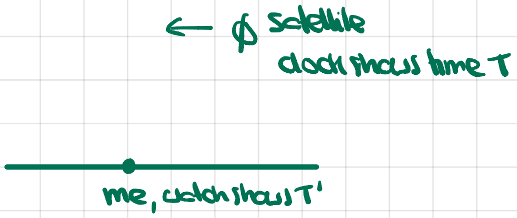
$$e^{-3x}(1+x)^{-\frac{1}{2}} \approx (1-3x)(1-\tfrac{1}{2}x) = 1-3x-\tfrac{1}{2}x + \underbrace{\tfrac{3}{2}x^2}$$

$$\approx 1-\tfrac{7}{2}x$$

throw away x^2 terms and higher

negligible

Example 4 (real life)



time dilation

$$T' = \frac{T}{\sqrt{1-v^2/c^2}}$$

linear approx.

$$T' \approx T\left(1+\frac{1}{2}\frac{v^2}{c^2}\right)$$

$$(1-u)^{-\frac{1}{2}} \approx 1+\tfrac{1}{2}u$$

$$u = v^2/c^2$$

real life

$v = 4\,km/s$

$c = 3\cdot 10^5\,km/s$ $\quad = \dfrac{v^2}{c^2} \approx 10^{-10}$

note that $u^2 \approx 10^{-20}$, negligible

Quadratic Approximation

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2$$

examples

→ $f = \ln(1+x)$, $f' = 1/1+x$, $f'' = -1/(1+x)^2$

$$\ln(1+x) \approx \underbrace{\underbrace{\ln(1+x_0) + \frac{x-x_0}{1+x_0}}_{\text{linear approx}} + \frac{1}{2}(-1/(1+x_0)^2)(x-x_0)^2}_{\text{quadratic approx. to } f(x) \text{ near } x_0}$$

near $x_0 = 0$

$$\ln(1+x) = x - \frac{1}{2}x^2$$

→ $\ln(1.1) \approx 0.1 - 0.5 \cdot 0.01 = 0.095$

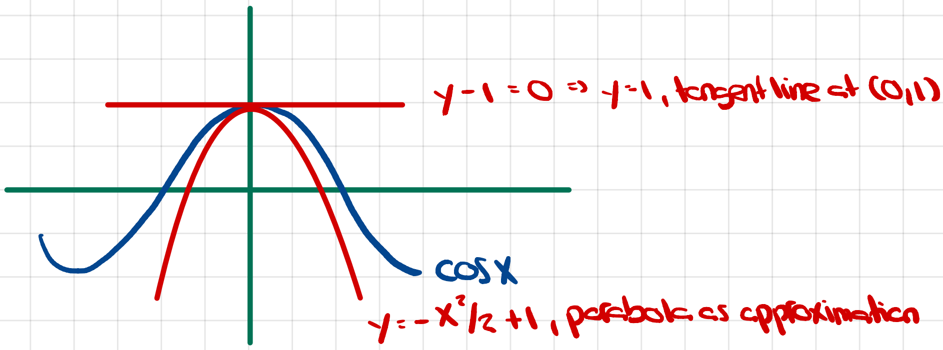
→ quadratic approx. near $x_0 = 0$: $\cos \to -\sin \to -\cos 0$

$$\sin x \approx x + \frac{1}{2}\cdot(\cancel{-\sin 0}^{\,0})x^2 = x$$

$$\cos x \approx 1 + \frac{1}{2}(-\cos 0)x^2 = 1 - x^2/2$$

$$e^x \approx 1 + x + \frac{1}{2}e^0 \cdot x^2 = 1 + x + \frac{x^2}{2}$$

Geometric Significance of Quadratic Term



$f = \cos x$
$f' = -\sin x$
$f'' = -\cos x$

linear approx. near $x=0$: $\cos 0 - \sin 0 \cdot x = 1$

quadratic approx near $x=0$: $1 + \frac{1}{2}(-\cos 0)x^2 = -\frac{x^2}{2} + 1$

Lecture 10

Approximations (cont'd)

$$T' = T(1 - v^2/c^2)^{-1/2} \approx T(1 + \tfrac{1}{2}v^2/c^2)$$

→ exact   → lin approx.

$$\Delta T = T' - T = T[1 + \tfrac{1}{2}v^2/c^2 - 1] = T \cdot \tfrac{1}{2}v^2/c^2$$

$\frac{\Delta T}{T} = v^2/2c^2$ = lin approx error rel to calculation of exact value.

error fraction is proportional to v^2/c^2 with factor $1/2$.

Quadratic Approxim.

use when lin approx. is not enough

$$f(x) \approx f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2$$

why the factor "$1/2$"?

$f(x) = ax^2 + bx + c$ Approximate $f(x)$ using a parabola, ie quadratic approxim.

$f' = 2ax + b$
$f'' = 2a$

$f(0) = c$
$f'(0) = b$
$f''(0) = 2a$

$\Rightarrow f(x) = c + bx + 2a \cdot x \cdot \frac{1}{2}$

these need to cancel

note

$$a_k = \left(1 + \frac{1}{k}\right)^k \xrightarrow[k \to \infty]{} e$$

$$\ln a_k = k \ln(1 + 1/k)$$

$$\ln(1+x) \approx x \Rightarrow \ln a_k \approx k \cdot \frac{1}{k} = 1.$$

Note $x = 1/k$ here. $x \to 0$ as $k \to \infty$

rate of convergence (how fast $a_k \to 1$)

$$\underbrace{\ln a_k - 1}_{} \longrightarrow 0$$

how big is this? use quadratic approxim. to answer this.

$$e^{-3x}(1+x)^{-\frac{1}{2}} \approx \left[1+(-3x)+\frac{(-3x)^2}{2}\right]\cdot\left[1+\left(-\tfrac{1}{2}\right)x+\frac{(-1/2)(-1/2-1)}{2}x^2\right]$$
$x \approx 0$

we can throw away higher order terms. Multiply terms that generate up to power 2 of x; keep power 2 terms intact

$$\approx \left[1-3x-\tfrac{1}{2}x+\tfrac{3}{8}x^2\right]+\tfrac{9}{2}x^2+\tfrac{3}{2}x^2$$ (drop all x^3, x^4, etc terms)

* note: we are approximating near 0 ⇒ $\Delta x = x$ itself. we are considering very small x therefore, e.g. 0.01. $(0.01)^3 = 10^{-6}$, negligible.

$$= 1-\tfrac{7}{2}x+\tfrac{51}{8}x^2$$

Recall, near $x=0$: $\sin x \approx x$
$\cos x \approx 1 - x^2/2$
$e^x \approx 1 + x + x^2/2$

next we derive $\ln x \approx x - x^2/2$
$(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2$

$f(x) = \ln(1+x)$, $f(0) = 0$

$f'(x) = 1/(1+x)$, $f'(0) = 1$ ⇒ $\ln(1+x) \approx 0 + 1\cdot x - \frac{1}{2}x^2 = x - \frac{x^2}{2}$

$f''(x) = -1/(1+x)^2$, $f''(0) = -1$

$f(x) = (1+x)^r$ $f(0) = 1$

$f'(x) = r(1+x)^{r-1}$ $f'(0) = r$ ⇒ $(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x$

$f''(x) = r(r-1)(1+x)^{r-2}$ $f''(0) = r(r-1)$

Curve sketching

Goal: draw graph of using f', f''

$f' > 0 \Rightarrow f$ increasing

$f' < 0 \Rightarrow f$ decreasing

$f'' > 0 \Rightarrow f'$ increasing, f concave up

$f'' < 0 \Rightarrow f'$ decreasing, f concave down

Definition

$f'(x_0) = 0 \Rightarrow x_0$ called critical point

$y_0 = f(x_0)$ called critical value

Ex 1 $f = 3x - x^3$

$f' = 3 - 3x^2 = 3(1-x^2) = 3(1+x)(1-x)$

critical points

$f' = 0 \Rightarrow x = 1, x = -1$

$f(1) = 2$

$f(-1) = -2$

the "ends" $x \to \pm\infty$

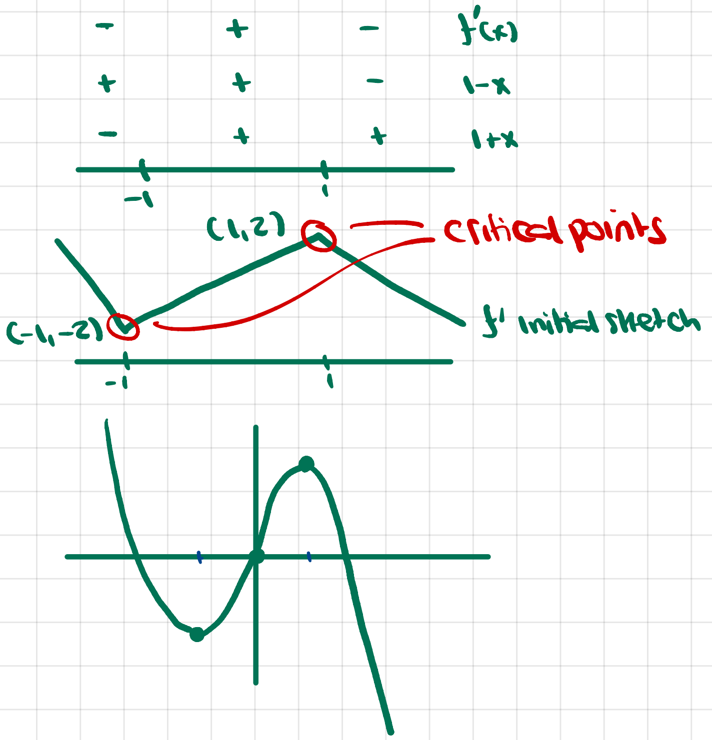
$x \to \infty \quad f(x) \to -\infty$

$x \to -\infty \quad f(x) \to \infty$

$f'' = -6x \Rightarrow \; > 0$ for $x < 0$

< 0 for $x > 0$

$x = 0$ is inflection point



Lecture 11

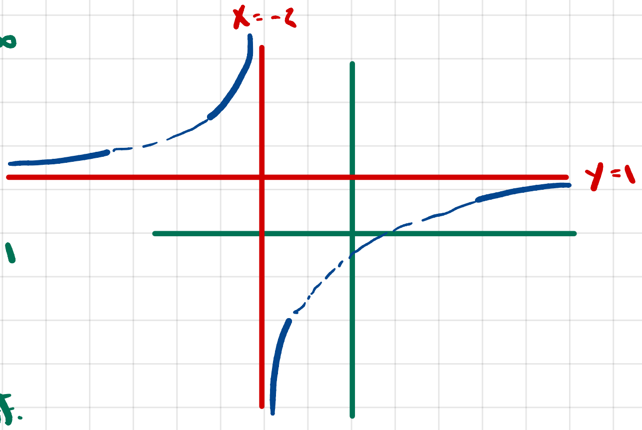
Sketching (cont'd)

$f(x) = \frac{x+1}{x+2}$ $\quad f'(x) = \frac{1}{(x+2)^2} \neq 0$ $\Rightarrow$ no critical points

Plot some points, in particular $x = -2$

$f(-2^+) = \frac{-2^+ + 1}{-2^+ + 2} = \frac{-1}{0^+} = -\infty$

$f(-2^-) = \frac{-2^- + 1}{-2^- + 2} = \frac{-1}{0^-} = +\infty$



Ends: $x \to \pm\infty$

$f(x) = \frac{x+1}{x+2} = \frac{1 + 1/x}{1 + 2/x} \to 1$

$f(\pm\infty) = 1$

Doublecheck

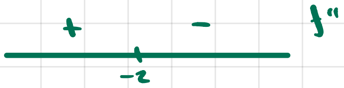
$f = \frac{x+2-1}{x+2} = 1 - \underbrace{\frac{1}{x+2}}$ easier to diff.

note: $f' \neq 0 \Rightarrow$ the graph doesn't curve back on itself, so in between the blue lines (ends, limit $\to 0$), it is predictable

$f' = 1/(x+2)^2 > 0$, as expected

$\Rightarrow f$ increasing on $(-\infty, -2), (2, +\infty)$

$f'' = -2/(x+2)^3 \quad x \neq -2$



, as expected

General Strategy

1 a) Plot discontinuities, esp. infinite
b) plot ends ($x \to \pm\infty$)
c) plot any easy points (optional)

2 Solve $f'(x) = 0$, plot critical points and values.

3 Decide whether f' is positive or negative in each interval between critical points / discontinuities
(must be consistent with steps 1 and 2)
(this step double checks previous step, e.g. if there was some arithmetic mistake)

4 $f'' \gtrless 0$, concave up/down

$f''(x_0) = 0 \iff$ inflection point

Example

$$f(x) = \frac{x}{\ln x}, \quad x > 0$$

1a) $f(1^+) = \frac{1}{\ln 1^+} = \frac{1}{0^+} = +\infty$

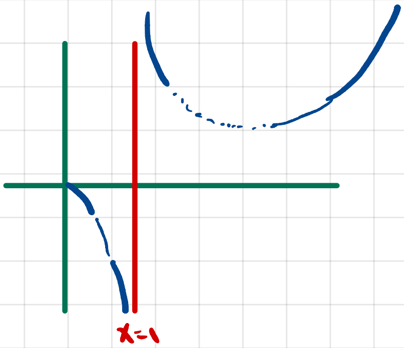
$f(1^-) = -\infty$

1b) "ends"

$f(0^+) = \frac{0^+}{\ln 0^+} = \frac{0^+}{-\infty} = 0$

$f(10^{10}) = \frac{10^{10}}{10 \ln 10}$, we surmise $f(+\infty) = +\infty$

2) $f'(x) = \frac{\ln x - 1}{(\ln x)^2}$



$f'(x) = 0 \Rightarrow x = e$

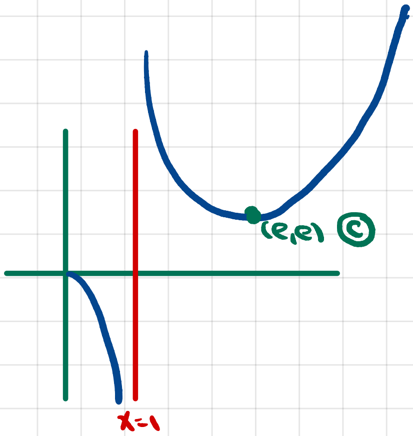
$f(e) = e$

only one critical point $\Rightarrow$ at this point we are confident in the shape

3) Double check

What I know so far

f decreas. in $(0,1)$
" " " $(1,e)$
" increasing in $(e,+\infty)$



Now actually check the sign of f'

	$(0,1)$	$(1,e)$	$(e,+\infty)$
f'	$-$	$-$	$+$

, as expected

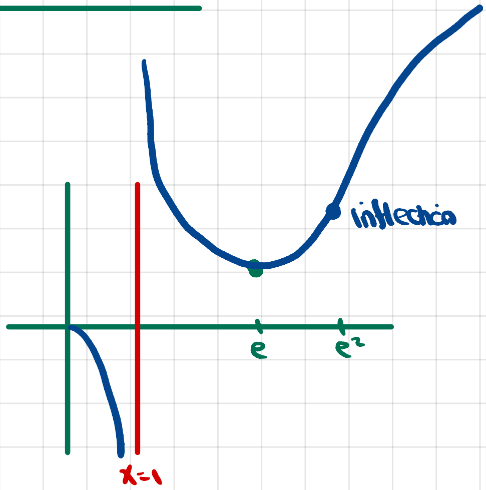
Rewrite $f'(x)$

$f' = \frac{1}{\ln x} - \frac{1}{(\ln x)^2}$, it's easier to see that $f'(0^+) = \frac{1}{-\infty} - \frac{1}{\infty} = 0$

$\Rightarrow$ slope approaches 0 near 0^+

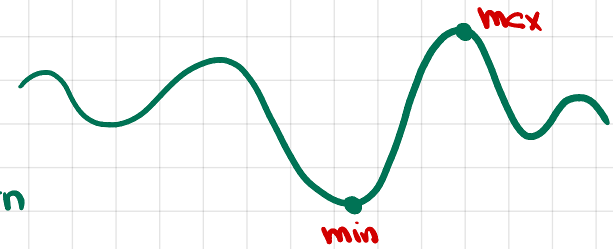
4) $f''(x) = \frac{2 - \ln x}{x(\ln x)^3}$

	$(0,1)$	$(1,e^2)$	$(e^2,+\infty)$
$f''(x)$	$-$	$+$	$-$
$\ln x$	$-$	$+$	$+$
$2-\ln x$	$+$	$+$	$-$



Max/Min Problems

suppose you have fn



Find max and min of fn

goal: use shortcuts

try to avoid using f''

key: only need to look at critical points, endpoints, and points of discontinuity.

Lecture 12 - max/min (cont'd)

wire length 1, cut into two pieces.

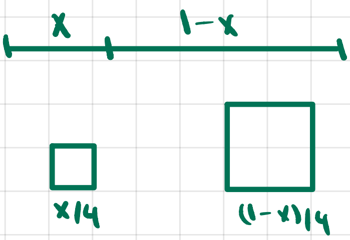
each piece encloses a square

find largest area enclosed → $\frac{1}{16}$, but only when we don't cut the wire at all

max reached in limit as $x \to 1^-$ or $x \to 0^+$

Draw diagram

Name variables



$$A = \left(\frac{x}{4}\right)^2 + \left(\frac{1-x}{4}\right)^2$$

Find critical points $\quad A' = \frac{x}{8} - \frac{(1-x)}{8} = \frac{x-1+x}{8} = \frac{2x-1}{8} = 0 \Rightarrow x = \frac{1}{2}$

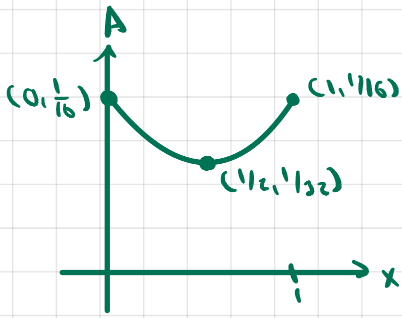
$$A\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{8}\right)^2 + \left(\tfrac{1}{8}\right)^2 = \frac{2}{64} = 1/32$$

check endpoints

$0 < x < 1$

$A(0^+) = 0 + \frac{1}{16} = \frac{1}{16}$

$A(1^-) = \frac{1}{16} + 0 = \frac{1}{16}$



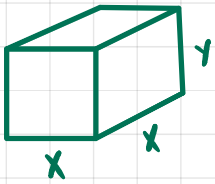
what is the minimum? (min value) $1/32$

where " " ? (min point) $1/2 = x$, or $(1/2, 1/32)$

Example 2

Consider box without a top, with least surface area for fixed volume.

Diagram, variables



$V = x^2 y$ (constraint) ↗ relationship between y and x is fixed

$A = x^2 + 4xy$

$y = V/x^2$

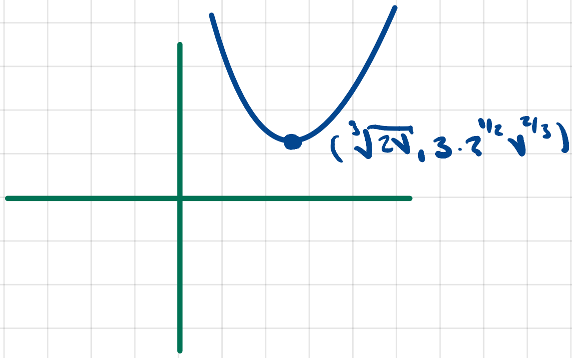
$A = x^2 + \dfrac{4V}{x}$

Critical points

critical point

$A' = 2x - 4V/x^2 = 0 \Rightarrow 2x = 4V/x^2 \Rightarrow x^3 = 2V \Rightarrow x = \sqrt[3]{2V}$

Ends



$x \in (0, \infty)$

$A(0^+) = \dfrac{4V}{0^+} \to +\infty$

$A(\infty) = \infty$

Alternatively, check A''

$A''(x) = 2 + 2\cdot 4V/x^3 = 2 + \dfrac{8V}{x^3} > 0$ for $x \in (0,\infty)$

$\Rightarrow$ critical point is min.

min value

$$A(\sqrt[3]{2V}) = (2V)^{2/3} + \frac{4V}{(2V)^{1/3}} = \frac{2V + 4V}{(2V)^{1/3}} = \frac{6V}{(2V)^{1/3}} = \frac{2\cdot 3V}{2^{1/3}V^{1/3}} = 3\cdot 2^{2/3}\cdot V^{2/3}$$

$$y(\sqrt[3]{2V}) = V/2^{2/3}V^{2/3}$$

more meaningful answers

dimensionless variables: e.g. $\dfrac{A}{V^{2/3}}$ ↗ m^2, ↘ m^3

$$\frac{x}{y} = \frac{2^{1/3}V^{1/3}}{V\cdot 2^{-2/3}V^{-2/3}} = \frac{2\cdot V}{V} = 2$$

best answer: the optimal shape has a 2 to 1 ratio of base length to height

Example 2 by implicit diff.

$V = x^2 y$

$A = x^2 + 4xy$

$\frac{d}{dx}(V = x^2 y) = 0 = 2xy + x^2 y' \Rightarrow y' = \frac{-2xy}{x^2} = \frac{-2y}{x}$

$\frac{dA}{dx} = 2x + 4y + 4xy'$

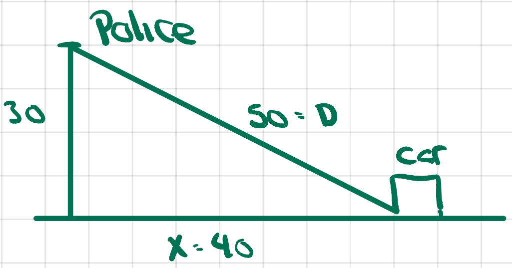
$= 2x + 4y + 4x \cdot \left(-\frac{2y}{x}\right) = 0$

$2x + 4y - 8y = 0 \Rightarrow 2x = 4y \Rightarrow x/y = 2$

Faster method, but did not check whether critical point is max/min or neither.

Still need to check ends.

Related Rates



car approaching at 80 ft/s : $\frac{dD}{dt} = -80$ ft/s

is the car above 95 ft/s along the road? $\frac{dx}{dt} > 95$?

t : time

$x^2 + 30^2 = D^2$

$\frac{dD}{dt} = -80$

could solve for x, but much more complicated. Better to implicitly differentiate.

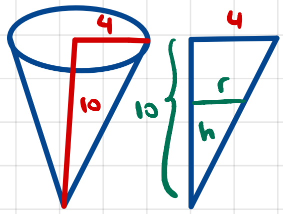
$2xx' = 2DD' \Rightarrow$ now we can plug in values

$2 \cdot 40\, x' = 2 \cdot 50 \cdot (-80) \Rightarrow x' = -100$ ft/s

Ex 2 Conical tank with top of radius 4ft, depth 10ft, is being filled at 2 ft^3/min.

How fast is the water rising when at depth 5ft?

1 Diagram, variables



$\frac{r}{h} = \frac{4}{10}$ similar triangles

$V = \frac{1}{3}\pi \cdot r^2 \cdot h$

$\frac{dV}{dt} = 2$

$\frac{dh}{dt}$ when $h = 5$?

$r = \frac{2h}{5}$

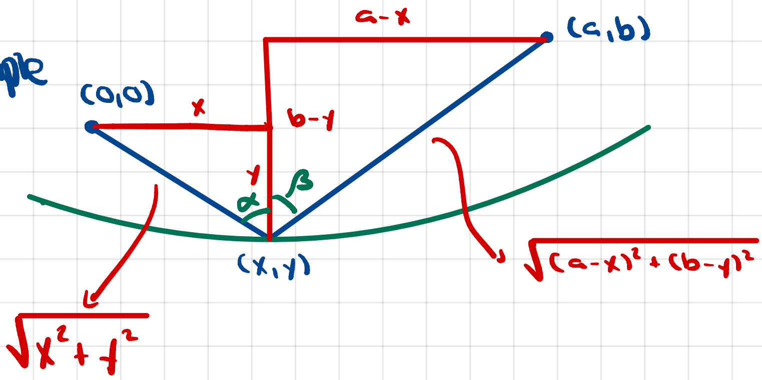
$V = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 \cdot h = \frac{1}{3}\pi\left(\frac{2}{5}\right)^2 \cdot h^3$

$\frac{dV}{dt} = \frac{1}{3}\pi \cdot \frac{4}{25} \cdot 3h^2 h' = \frac{4\pi}{25} h^2 h'$

$2 = \frac{4\pi}{\cancel{25}} \cdot \cancel{5^2} h'$

$h' = \frac{1}{2\pi}$ ft/min

Example



this is the constraint

$\sqrt{x^2+y^2} + \sqrt{(a-x)^2+(b-y)^2} = L$ → constant

$y = y(x)$ defined implicitly $\qquad y' = 0$ (critical point)

Diff. Implicitly

$$\frac{2}{2}\frac{x + \cancel{y y'}}{(x^2+y^2)^{1/2}} - \frac{(a-x) + \cancel{(b-y)y'}}{\sqrt{(a-x)^2+(b-y)^2}} = 0$$

we're looking for (x,y) where $y' = 0$

$$\frac{x}{\sqrt{x^2+y^2}} = \frac{a-x}{\sqrt{(a-x)^2+(b-y)^2}}$$

$\sin\alpha = \sin\beta \Rightarrow \alpha = \beta$

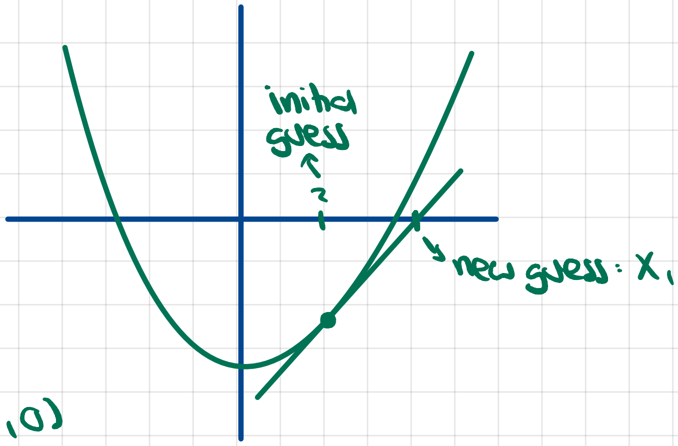
Newton's Method

Example solve $x^2 = 5$

$f(x) = x^2 - 5$, solve $f(x) = 0$

start w/ initial guess

$x_0 = 2$



tangent line

$y - y_0 = m(x - x_0)$

x_1 is the x-intercept: $(x_1, 0)$

$-y_0 = m(x_1 - x_0)$

$$x_1 = x_0 - \frac{y_0}{m} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Newton's Method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

In our example:

$f'(x) = 2x$

$x_0 = 2$

$$x_1 = x_0 - \frac{x_0^2 - 5}{2x_0} = \frac{2x_0^2 - x_0^2}{2x_0} + \frac{5}{2x_0}$$

$$x_1 = \frac{x_0}{2} + \frac{5}{2x_0}$$

Plugging in numbers

$x_1 = 9/4$

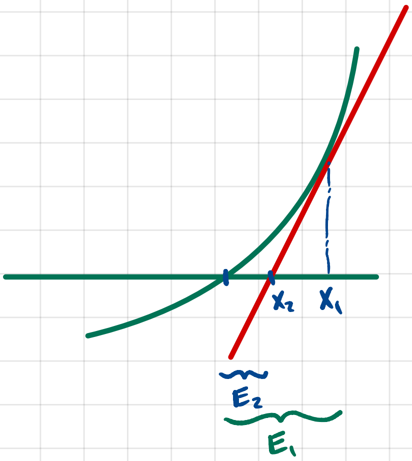
$x_2 = 161/72$

$(\dots)$

n	$\sqrt{5} - x_n$
0	$2 \cdot 10^{-1}$
1	10^{-2}
2	$4 \cdot 10^{-5}$
3	$4 \cdot 10^{-10}$

Lecture 14 - Newton's Method (CONT'D)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



error analysis

$E_1 = |x - x_1|$
$E_2 = |x - x_2|$

proportional to

$E_2 \sim E_1^2$

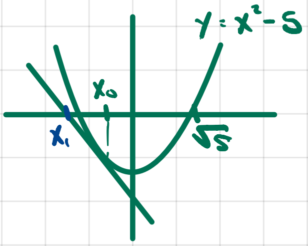
e.g.

E_1	E_2	E_3	E_4
10^{-1}	10^{-2}	10^{-4}	10^{-8}

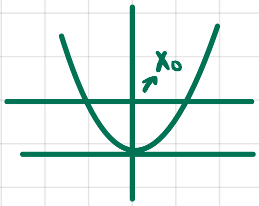
digits of accuracy doubles at each step

Newton's method works very well if $|f'|$ not small, $|f''|$ not too big, x_0 starts near target

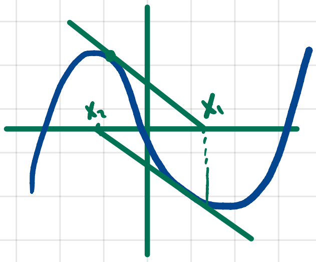
ways the method can fail



x_0 too far away -> wrong root



$f'(x_0)$ is a disaster



$x_1 \rightleftarrows x_2$

Mean Value Theorem

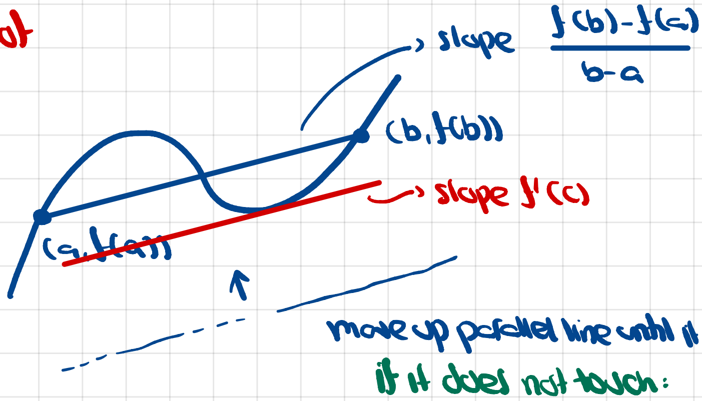
"If you go from Boston to LA in 6 hours (3000mi) then at some point you are going at the average speed (3000/6 = 500mi/h)."

Theorem (MVT)

$$(*) \quad \frac{f(b)-f(a)}{b-a} = f'(c) \qquad \text{for some } c,\ a<c<b$$

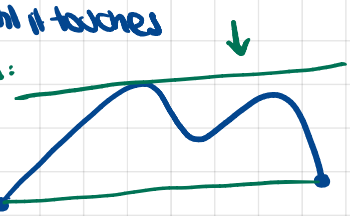
provided f differ. in (a,b), continuous in $[a,b]$

Proof



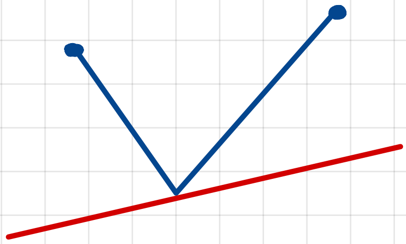
move up parallel line until it touches

if it does not touch:



then bring down parallel line from above

Note



one bad point can ruin the above proof

$\Rightarrow$ we need $f'(x)$ to exist $\forall x$ in (a,b)

Applications to Graphing

1 $f'>0 \Rightarrow f$ increasing

2 $f'<0 \Rightarrow f$ decreasing

3 $f'=0 \Rightarrow f$ constant

Proof

Rewrite $(*)$

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$f(b)-f(a) = f'(c)(b-a)$$

$$f(b) = f(a) + f'(c)(b-a)$$

$a<b \Rightarrow b-a>0$

1) $f'(c)>0 \Rightarrow f(b)>f(a) \Rightarrow$ increas.

2) $f'(c)<0 \Rightarrow f(b)<f(a) \Rightarrow$ decreas.

3) $f'(c)=0 \Rightarrow f(b)=f(a) \Rightarrow$ constant

Inequalities

1. $e^x > 1+x \quad (x>0)$

proof: $f(x) = e^x - (1+x)$

starts $f(0) = e^0 - (1+0) = 0$
and $f'(x) = e^x - 1 > 0$ for $x>0$

$\therefore f(x) > f(0), \quad x>0$

This means $e^x - (1+x) > 0 \iff e^x > 1+x$

2. $e^x > 1+x+\frac{x^2}{2}$

$g(x) = e^x - (1+x+x^2/2)$

$g(0) = 0$

$g'(x) = e^x - (1+x) > 0$ for $x>0$ because of 1

$\Rightarrow g$ is increasing $\Rightarrow g(x) > g(0) \Rightarrow e^x > 1+x+x^2/2$

Lecture 15

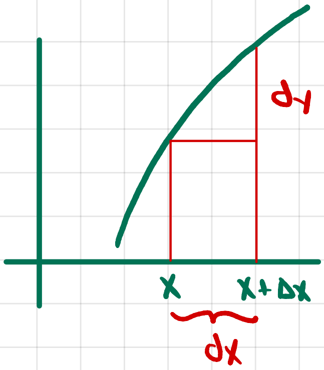
Differentials

$y = f(x)$

Leibniz interpretation of derivative as ratio of infinitesimal

By def, $dy = f'(x)\,dx$, aka df $\iff$ $\frac{dy}{dx} = f'(x)$

$dx = \Delta x$



use in linear approx.

Ex: $(64.1)^{1/3} = ?$

$y = x^{1/3}$ $\quad dy = \frac{1}{3}x^{-2/3}dx$

$y(64) = 4$

$dy = \frac{1}{3}(64)^{-2/3}dx = \frac{1}{3}\cdot\frac{1}{\sqrt[3]{2^{12}}} = \frac{1}{48}dx$

$dy = dx/48$

$dx = 0.1 \Rightarrow dy = 1/480$

$64.1^{1/3} \approx y + dy = 4 + 1/480 \approx 4.002$

compare to previous notation (review of lin. approx)

$f(x) = f(a) + f'(a)(x-a)$

$a = 64$
$f(x) = x^{1/3}$
$f(a) = f(64) = 4$
$f'(a) = \frac{1}{3}64^{-2/3} = 1/48$

$\Rightarrow x^{1/3} \approx 4 + \frac{1}{48}(x-a)$

$64.1^{1/3} \approx 4 + \frac{1}{48}(0.1) \approx 4.002$

Antiderivatives

integral sign

$G(x) = \int g(x)\,dx$

antiderivative of g (aka, indefinite integral of g)

we don't get a single answer, ambiguous up to a constant

1. $\int \sin x\,dx = -\cos x + C$

$G(x) = -\cos x + C$

$d(x^{a+1}) = (a+1)x^a\,dx$

2. $\int x^a dx = x^{a+1}/a+1 + C$

$a \neq -1$

3. $\int \frac{dx}{x} = \ln|x| + C$

$x > 0$ ✓

check $x < 0$: $\frac{d \ln|x|}{dx} = \frac{d \ln(-x)}{dx} = \frac{-1}{-x} = \frac{1}{x}$

4. $\int \sec^2 x \, dx = \tan x + C$

5. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

6. $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

Uniqueness of Antiderivatives up to a constant

theorem: If $F' = G'$ then $F(x) = G(x) + C$

Trickier Examples

$$\int x^3(x^4+2)^5 dx = \frac{(x^4+2)^6}{6 \cdot 4} + C = \frac{(x^4+2)^6}{24} + C$$

method of substitution

→ tailor-made for differential notation

$u = x^4 + 2$
$du = 4x^3 dx \Rightarrow x^3 dx = \frac{du}{4}$

$$\int x^3(x^4+2)^5 dx = \int u^5 \cdot \frac{1}{4} du = \frac{1}{4} \cdot \frac{1}{6} u^6 + C$$
$$= \frac{1}{24}(x^4+2)^6 + C$$

→ $\int \frac{x}{\sqrt{1+x^2}} dx = \int \frac{1}{2} \cdot \frac{1}{\sqrt{u}} du = \frac{1}{2} \int u^{-1/2} du = \frac{1}{2} u^{1/2} \cdot 2 + C = (1+x^2)^{1/2} + C$

$u = 1 + x^2$
$du = 2x \, dx$

Better way: advanced guessing: $(1+x^2)^{1/2}$

→ $\int e^{6x} dx$

guess e^{6x} $\quad (e^{6x})' = 6e^{6x} \Rightarrow \frac{1}{6} e^{6x} + C = \int e^{6x} dx$

$\int xe^{-x^2}dx = -\frac{1}{2}e^{-x^2} + C_1$

guess: e^{-x^2} $\quad (e^{-x^2})' = e^{-x^2}(-2x)$

$\int \sin x \cos x \, dx = \frac{1}{2}\sin^2 x + C$

guess $\sin^2 x$ $\quad (\sin^2 x)' = 2\sin x \cos x$

But also, $\frac{d}{dx}\cos^2 x = 2\cos x(-\sin x)$ $\Rightarrow$ another answer is $\int \sin x \cos x \, dx = -\frac{1}{2}\cos^2(x) + C_2$

$\frac{1}{2}\sin^2 x + C_1 - \left(-\frac{1}{2}\cos^2 x + C_2\right) = \frac{\sin^2 x + \cos^2 x}{2} + C_1 - C_2 = \underbrace{\frac{1}{2} + C_1 - C_2}$

the difference between the functions is constant.

$\boxed{\Rightarrow C_1 - C_2 = -\frac{1}{2}}$?

$\int \frac{dx}{x \ln x} = \int u^{-1} du = \ln|u| + C = \ln|\ln x| + C$

$u = \ln x$
$du = \frac{dx}{x}$

Lecture 16

Differential Equations

Ex: $\frac{dy}{dx} = f(x)$

solution: $y = \int f(x)dx$

For now, we assume we know how to solve this. We have one method for solving integrals (finding antiderivative): substitution. it has a version: advanced guessing.

Ex 2: $\underbrace{\left(\frac{d}{dx} + x\right)}_{\text{annihilation operator (quantum mechanics)}} y = 0$

$$\frac{dy}{dx} = -xy$$

$$\frac{dy}{y} = -x\,dx$$ equation set up in terms of differentials

$$\int \frac{dy}{y} = \int -x\,dx$$

$\ln y = -x^2/2 + C$ (we're considering the $y > 0$ case here)

$$y = e^{-\frac{x^2}{2}+C} \Rightarrow y = Ae^{-\frac{x^2}{2}}$$

solution: $y = ae^{-x^2/2}$, a any constant ($<0, >0, =0$)

Separation of Variables

works when we have eq. in form $\frac{dy}{dx} = f(x)\cdot g(y)$

$$\frac{dy}{g(y)} = f(x)dx$$

$H(y) = \int \frac{dy}{g(y)}$

$\Rightarrow$ $H(y) = F(x) + C$ (implicit $y(x)$) $\Rightarrow$ $y = H^{-1}(F(x)+C)$

$F(x) = \int f(x)dx$

Remarks

$\rightarrow$ could have written $\ln|y| = -\frac{x^2}{2} + C$ $(y \neq 0)$, in Ex 2 above

$\Rightarrow |y| = A = e^{-x^2/2} \Rightarrow y = \pm Ae^{-x^2/2}$

→ $y=0$ is also a solution, but it was left out so far.

but not surprising because we divided $\frac{dy}{dx} = -xy$ by y in trying to find a solution,

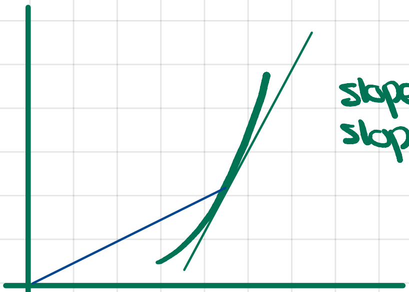
so $y=0$ as solution is missed

→ could have written $\ln y + C_1 = -\frac{x^2}{2} + C_2$, but we could just combine the constants

$$\ln y = -\frac{x^2}{2} + C_2 - C_1 = -\frac{x^2}{2} + C$$

Ex 1 via separation

$$\frac{dy}{dx} = f(x) \Rightarrow dy = f(x)\,dx \Rightarrow \int dy = \int f(x)\,dx \Rightarrow y = \int f(x)\,dx$$



slope of tangent twice the slope of ray from origin

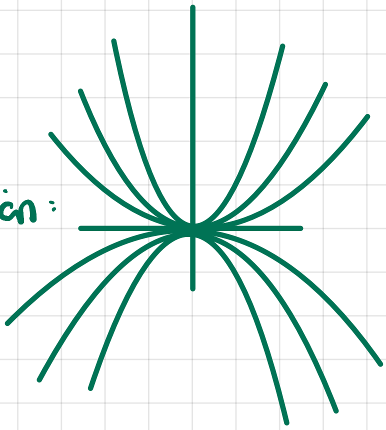
Example 3

$$\frac{dy}{dx} = 2\frac{y}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x} \quad \text{(separation step)}$$

$$\ln|y| = 2\ln x + C$$

$$|y| = e^{2\ln x + C} = e^C x^2 = Ax^2$$ → solution:



$$y = \pm Ax^2$$

check that solution satisfies initial diff eq:

$$\frac{dy}{dx} = 2Ax = \frac{2Ax^2}{x} = \frac{2y}{x}$$

Warning: notice $\frac{dy}{dx} = 2\frac{y}{x}$ is not defined at $x=0$

Example 4 .

Find curves perpendicular to parabolas from ex 3

$$\frac{dy}{dx} = \frac{-1}{2(y/x)} = \frac{-x}{2y}$$

$$2y\,dy = -x\,dx$$

$$y^2 = -\frac{x^2}{2} + C$$

clearly only $C>0$ work here

solutions: $\frac{x^2}{2} + y^2 = C$

implicit solution

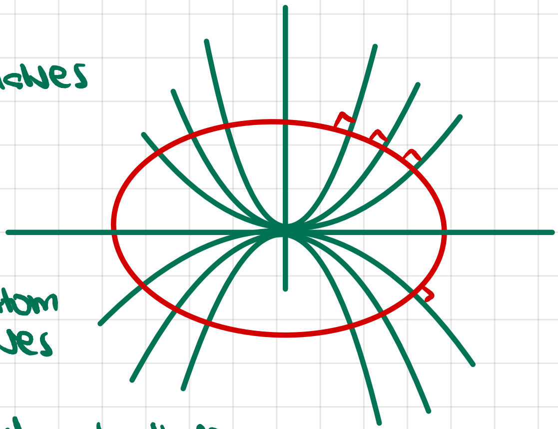
explicit solutions:

top halves

$$y = \sqrt{a^2 - x^2/2}$$

$$y = -\sqrt{a^2 - x^2/2}$$

bottom halves



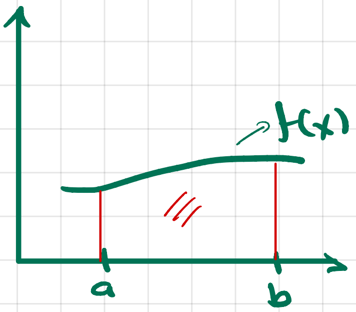
at $y=0$, vertical slopes, so solutions stop there

Lecture 18 - Intro to Integration

Geometric Interpretation

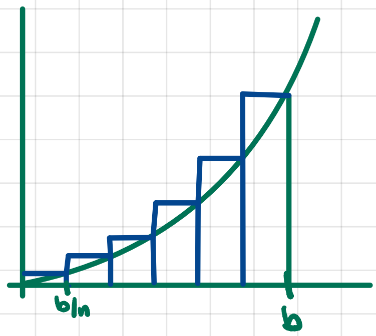
Area under a curve $= \int_a^b f(x)\,dx$

Definite Integral



To compute the area

1. Divide into "rectangles"
2. Add up areas
3. Take limit as rectangles get thin



Example 1 $f(x) = x^2$, $a = 0$, b arbitrary

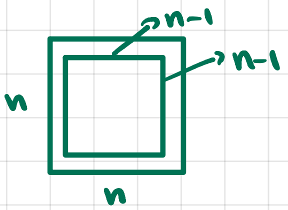
base length $= b/n$

Sum of areas of ▭'s

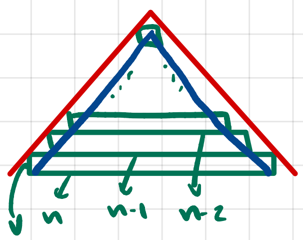
$(b/n)(b/n)^2 + (b/n)(2b/n)^2 + \dots + (b/n)\cdot(nb/n)^2$

$(b/n)^3(1 + 2^2 + 3^2 + \dots + n^2) = b^3 \dfrac{(1+2^2+\dots+n^2)}{n^3}$

x	$f(x)$
b/n	$(b/n)^2$
$2(b/n)$	$(2b/n)^2$
$(\dots)$	



"top view"
Volume $= 1^2 + 2^2 + \dots + n^2$



Vol inside $= \frac{1}{3}$ base $\cdot$ height

$= \frac{1}{3} n^2 \cdot n < 1^2 + 2^2 + \dots + n^2$

Vol outside $= \frac{1}{3}(n+1)^2(n+1) > 1^2 + 2^2 + \dots + n^2$

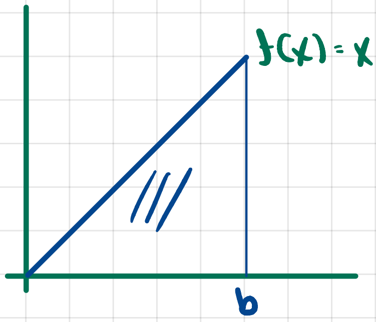
Divide by n^3: $\dfrac{1}{3} < \dfrac{1^2 + \dots + n^2}{n^3} < \dfrac{1}{3}\dfrac{(n+1)^3}{n^3} = \underbrace{\dfrac{1}{3}\left(1 + \dfrac{1}{n}\right)^3}_{\xrightarrow{n \to \infty} \frac{1}{3}}$

$\Rightarrow$ Total Area under $x^2 = b^3 \dfrac{(1+2^2+\dots+n^2)}{n^3}$

$= b^3/3$

with better notation: $\displaystyle\sum_{i=1}^{n} \frac{b}{n}\left(i\frac{b}{n}\right)^2 = \frac{b^3}{n^3}\sum i^2$

Example 3



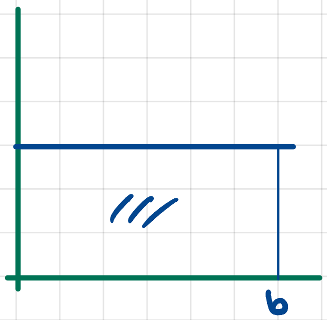
$A = \frac{1}{2} b \cdot b = \frac{b^2}{2}$

Pattern

$f(x)$	$\int_0^b f(x)\,dx$
x^2	$b^3/3$
x	$b^2/2$
1	$b = b^1/1$

Example 4

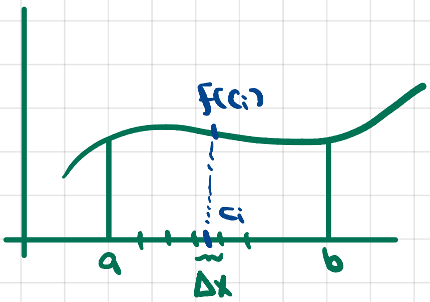
$f(x)=1$



$A = b \cdot 1 = b$

Notation Riemann Sums

General Procedure for definite integrals



n pieces $\Rightarrow \Delta x = (b-a)/n$

Pick any height of f in each interval

$$\sum_{i=1}^{n} \underbrace{f(c_i)}_{h_i} \underbrace{\Delta x}_{\text{base}} \xrightarrow[\Delta x \to 0]{} \int_a^b f(x)\,dx$$

Riemann sum

Integrals as Cumulative Sums

t time in years

$F(t)$ in \$/year, borrowing rate

$\Delta t = 1/365$ years

In day 45 ($t = 45/365$) ↗ how much was borrowed, in \$, on day 45

$f(45/365) \cdot \Delta t = f(45/365) \, 1/365$

$\sum_{i=1}^{365} f(i/365) \Delta t$ = total amount borrowed in year $\longrightarrow \int_0^1 f(t)\,dt$

Interest rate

borrow P, after time t you owe Pe^{rt}, r = interest rate

$$\sum_{i=1}^{365} \left(f\left(\frac{i}{365}\right)\Delta t\right) e^{r(1-i/365)} \longrightarrow \int_0^1 e^{r(1-t)} f(t)\,dt$$ = how much you owe end of year

$\tau = 1 - i/365$ = Amount of time left in the year

Lecture 19

Fundamental Theorem of Calculus (FTC1)

If $F'(x) = f(x)$ then $\int_a^b f(x) = F(b) - F(a) = F(x)\Big|_a^b$

$F = \int_a^b f(x)\,dx$

Notation

$F(b) - F(a) = F(x)\Big|_a^b = F(x)\Big|_{x=a}^{x=b}$

Example 1

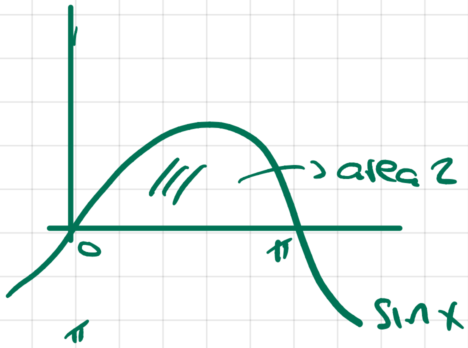
$F(x) = \frac{x^3}{3}$

$F'(x) = x^2$

By FTC1, $\int_a^b x^2 = \frac{b^3}{3} - \frac{a^3}{3}$

$\int_0^b x^2\,dx = \frac{b^3}{3}$

Example 2: Area under one hump of sin(x)



$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = -\cos\pi - (-\cos(0))$
$= 1 - (-1) = 2$

Example 3

$\int_0^1 x^{100}\,dx = \frac{x^{101}}{101}\Big|_0^1 = \frac{1}{101}$

Intuitive Interpretation

$x(t) =$ position

$x'(t) = \frac{dx}{dt} = v(t) =$ speed

$\int_a^b v(t)\,dt = \underbrace{x(b) - x(a)}_{\text{distance travelled}}$

Extend Integration to the case $f<0$ or $f>0$

Example
$$\int_0^{2\pi} \sin x \, dx = -\cos x \Big|_0^{2\pi} = 0$$

True Geometric Interpretation: area above x axis − area below x axis

Properties of Integrals
$$\int_a^b (f+g)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Lecture 20

FTC I: $F' = f \Rightarrow \int_a^b f(x)dx = F(b) - F(a)$

used to evaluate integrals.

Today, ↗ reversal of point of view

$F(b) - F(a) = \int_a^b f(x)dx$

use f to understand F.

info on F' ⇒ info on F

compare FTC I with MVT

Notation

$\Delta F = F(b) - F(a)$

$\Delta x = b - a$

$\Rightarrow \Delta F = \int_a^b f(x)dx$ (FTC I)

$\frac{\Delta F}{\Delta x} = \underbrace{\frac{1}{b-a}\int_a^b f(x)dx}_{\text{Avg of } f \text{ on } [a,b]}$

Inequalities

FTC I ⇒ $\Delta F = \text{Avg}(F') \cdot \Delta x \le \max(F')\Delta x$

MVT ⇒ $\Delta F = \underbrace{F'(c)}_{}\Delta x$

↳ vague, some c in (a,b), but we can say: $\min(F')\Delta x \le F'(c)\Delta x \le \max(F')\Delta x$

on exam, following problem solved w/ MVT:

$F'(x) = \frac{1}{1+x}$, $F(0) = 1$

MVT implies $A < F(4) < B$ for which A, B?

MVT: $F(4) - F(0) = F'(c)(4-0) = \frac{1}{1+c} \cdot 4$

↗ range: $\frac{1}{1+0} \cdot 4$ to $\frac{1}{1+4} \cdot 4$: 4 to $\frac{4}{5}$

conclusion: $F(4) - F(0)$ between $\frac{4}{5}$ and 4 ⇒ $F(4)$ between $\frac{9}{5}$ and 5

could be solved using FTC I:

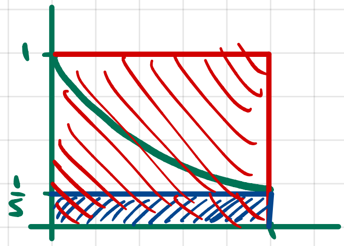
$F(4) - F(0) = \int_0^4 \frac{1}{1+x}dx = \ln(1+x)\Big|_0^4 = \ln 5 - \ln 1 = \ln 5 \Rightarrow f(4) = 1 + \ln 5$

But note that given $F(4)-F(0)=\int_0^4 \frac{dx}{1+x}$ is:

→ less than $\int_0^4 dx = 4$, as 1 is the max value of $F'(x)$ on $[0,4]$

→ greater than $\int_0^4 \frac{dx}{5} = \frac{4}{5}$, as $\frac{1}{5}$ is the min value of $F'(x)$ on $[0,4]$

$\Rightarrow \frac{4}{5} \le \int_0^4 F'(x)\,dx \le 4$



Two Riemann sums w/ one rectangle.

In, $\frac{F(4)-F(0)}{4-0} = F'(c)$, we don't know the slope, because we don't know what $F(4)$ is. Since we know $F'(x)$, we have an expression for $F'(c)$, so we can figure out a lower and an upper bound for the slope and solve for the bounds on $F(4)$.

With the FTC, $F(4)-F(0)$ is now linked to a specific way of finding F: $\int_0^4 F'(x)\,dx$.

We just antidifferentiate F', and get $F(4)-F(0)$ exactly. We can, alternatively, just find bounds on $F(4)$ by finding bounds on the integral as we did above.

FTC2: If f is cont., and $G(x) = \int_a^x f(t)\,dt$, $a \le t \le x$ then $G'(x) = f(x)$

$G(x)$ solves the diff eq. $y' = f$, $y(a) = 0$

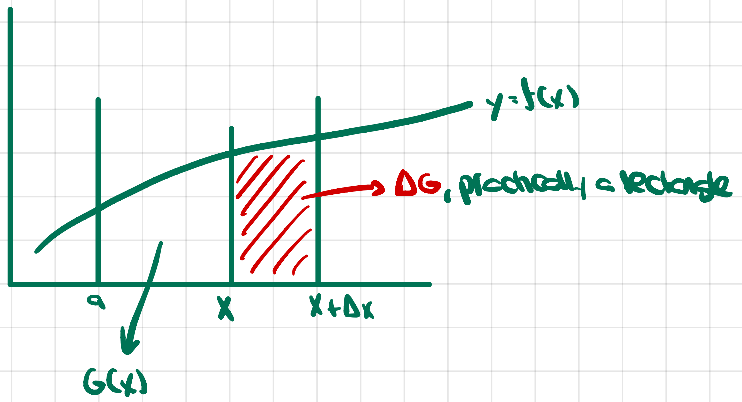
The theorem says we can always solve the diff eq. above.

example: $\frac{d}{dx}\int_1^x \frac{dt}{t^2} = \frac{1}{x^2}$

check: $\int_1^x t^{-2}\,dt = -t^{-1}\Big|_1^x = -x^{-1} - (-1)^{-1} = -x^{-1}+1$

$\frac{d}{dx}(-x^{-1}+1) = -1(-x^{-2}) = x^{-2}$

Proof of FTC2



$\Delta G \approx \Delta x \cdot f(x)$
base · height

$\lim_{\Delta x \to 0} \frac{\Delta G}{\Delta x} = f(x)$

Proof of FTC1

Start with $F' = f$. Assume f continuous.

Define $G(x) = \int_a^x f(t)\,dt$

FTC2 $\Rightarrow G'(x) = f(x)$

$F'(x) = G'(x) \Rightarrow F'(x) = G'(x) + C$
uses MVT

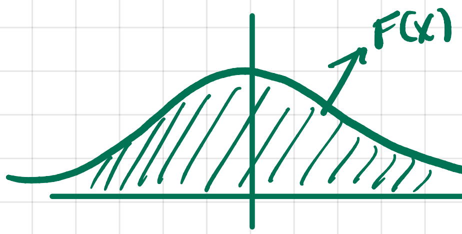
Hence $F(b) - F(a) = G(b) + \cancel{C} - (G(a) + \cancel{C}) = G(b) - \cancel{G(a)}^{\,0} = \int_a^b f(x)\,dx$

$L'(x) = \frac{1}{x}$, $L(1) = 0$

FTC → solution: $L(x) = \int_1^x \frac{dt}{t}$

"New" functions:

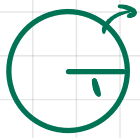
$y' = e^{-x^2}$ $y(0) = 0$ $\Rightarrow y(x) = \int_0^x e^{-t^2}\,dt$



e^{-x^2} = bell curve

F(x) cannot be expressed in terms of fn's we've seen before (log, exp, sin, etc)

Analogy:



→ Area = π "new number"
not the root of any algebraic eq. w/ rational coeff.

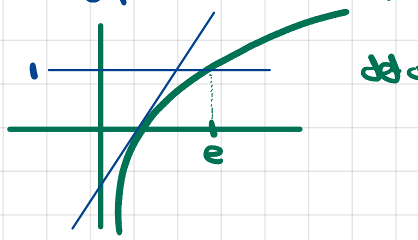
Lecture 21 - Applications to logarithms

FTC2 $\quad \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

$y' = \frac{1}{x} \qquad L(x) = \int_1^x \frac{dt}{t}$

Take as definition of logarithm

Sketch of graph

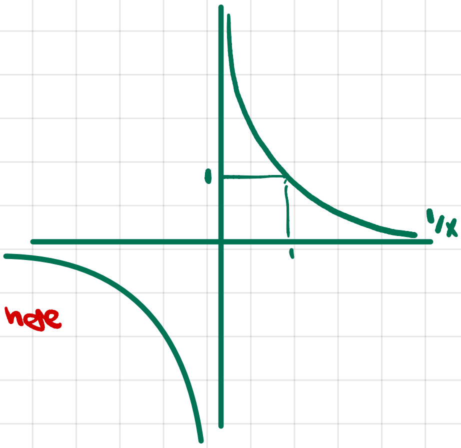


def of e: $L(e)=1$

$L'(x) = \frac{1}{x}$

$L(1) = \int_1^1 \frac{dt}{t} = 0, \quad L'(1) = 1$

$L''(x) = \frac{-1}{x^2} \qquad \Rightarrow$ concave down



why is $L(x) < 0$ for $x < 1$?

① $L(1) = 0$ and L increasing.

not clear why $x<0$ not being considered here

② $L(x) = \int_1^x \frac{dt}{t} = -\int_x^1 \frac{dt}{dt} < 0$. For $x<1$, $\int_x^1 \frac{dt}{t}$ is positive as it is an area

Claim: $L(ab) = L(a) + L(b)$

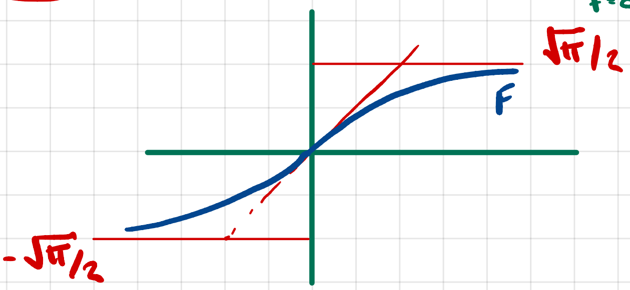
$$\int_1^{ab} \frac{dt}{t} = \int_1^a \frac{dt}{t} + \underbrace{\int_a^{ab} \frac{dt}{t}}_{} = L(a) + L(b)$$

$$\int_a^{ab} \frac{dt}{t} = \int_1^b \frac{a\,du}{au} = \int_1^b \frac{du}{u} = L(b)$$

$t = au \Rightarrow dt = a\,du \qquad t = a \Rightarrow u = 1$

$t = ab \Rightarrow u = b$

$F(x) = \int_0^x e^{-t^2}\,dt$

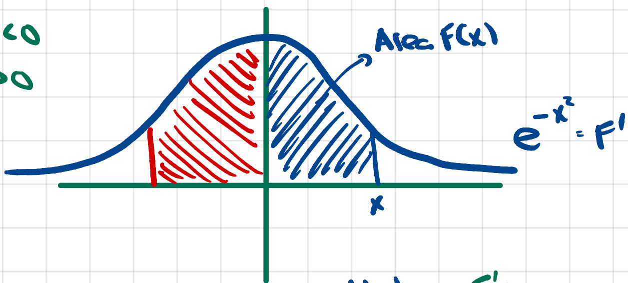


$F'(x) = e^{-x^2} > 0$

$F(0) = 0, \quad F'(0) = 1$

$F''(x) = -2xe^{-x^2} \begin{cases} > 0 \text{ for } x < 0 \\ < 0 \text{ for } x > 0 \end{cases}$

$F(-x) = -F(x) \quad \Rightarrow$ ODD



$\lim_{x\to\infty} F(x) = \sqrt{\pi}/2$

$$\boxed{\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}F(x)}$$

Note: F' is even: $F'(x) = F'(-x)$, and symmetric.

$F(-x) = \int_0^{-x} F'(x)\,dx = -\int_{-x}^0 F'(x)\,dx = -\int_0^x F'(x)\,dx = -F(x)$

$\Rightarrow F$ ODD

other examples of functions that cannot be expressed in terms of elementary functions:

$$C(x) = \int_0^x \cos(t^2)\,dt$$

$$S(x) = \int_0^x \sin(t^2)\,dt$$

→ Fresnel Integrals

$$H(x) = \int_0^x \frac{\sin(t)}{t}\,dt$$

$$Li(x) = \int_2^x \frac{dt}{\ln t}, \quad Li(x) \approx \#\text{primes} < x$$

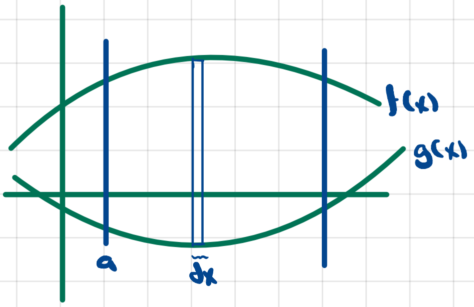
↑ Riemann Hypothesis

Areas Between Curves

$$\int_a^b [f(x) - g(x)]\,dx$$

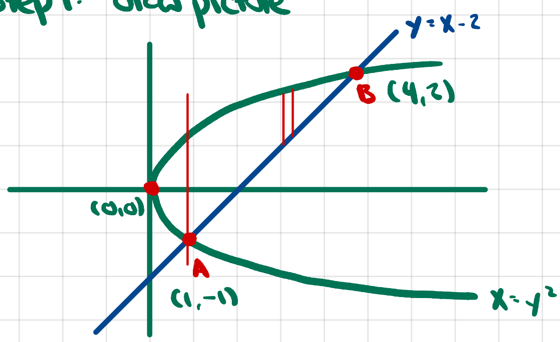
↑ identify limits → identify integrand ⇒ calculate integral

$[f(x)-g(x)]$: height, dx: base



Example: Find area between $x = y^2$ and $y = x-2$

step 1: draw picture



$x = y^2$ intersection with $y = x-2$

$y+2 = y^2$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

$\Rightarrow y = 2$ or $y = -1$

$y = -1 \Rightarrow x = 1$ A

$y = 2 \Rightarrow x = 4$ B

Method 1:

top: $y = \sqrt{x}$ bottom left: $y = -\sqrt{x}$ bottom right: $y = x-2$

$$\text{Area} = \int_0^1 [\sqrt{x} - (-\sqrt{x})]\,dx + \int_1^4 [\sqrt{x} - (x-2)]\,dx$$

Quicker way (Method 2): use horizontal slices

$$\int_{-1}^{2} [2+y-y^2]\,dy$$