PSet 5

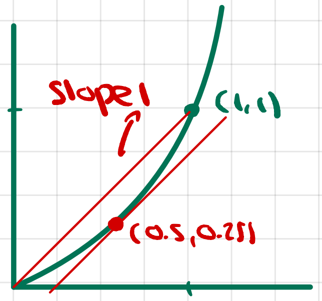
2G-1

a) x^2 on $[0,1]$

x^2 is continuous on $[0,1]$, diff. on $(0,1)$ $\Rightarrow \exists c \in [0,1] : f'(c) = \dfrac{f(1)-f(0)}{1-0} = 1$

$(x^2)' = 2x = 1 \Rightarrow x = 1/2$

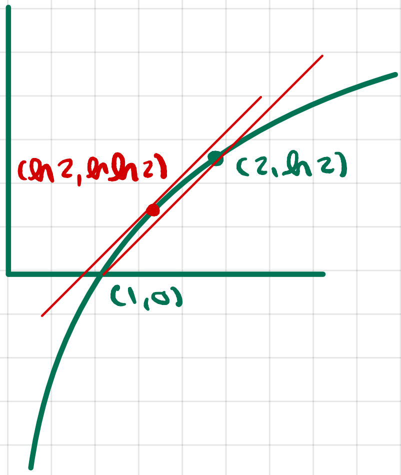
$c = 1/2$



b) $\ln x$ on $[1,2]$ $\qquad f'(c) = \dfrac{f(2)-f(1)}{2-1} = \ln 2 - \ln 1 = \ln 2$

$(\ln x)' = \dfrac{1}{x} = \ln 2 \Rightarrow x = 1/\ln 2$

$c = 1/\ln 2$



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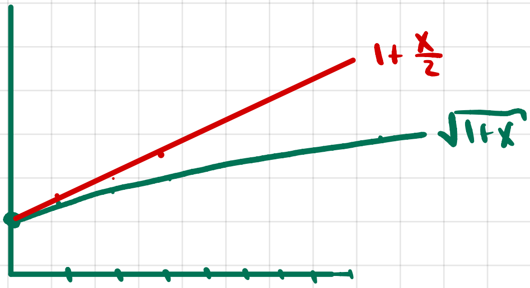
b) show $\sqrt{1+x} < 1 + \frac{x}{2}$ if $x > 0$



$f(x) = 1 + \frac{x}{2} - \sqrt{1+x}$

$f(0) = 1 + 0 - 1 = 0$

$f'(x) = \frac{1}{2} - \frac{1}{2}(1+x)^{-1/2} = \dfrac{\sqrt{1+x} - 1}{2\sqrt{1+x}} > 0$

$\Rightarrow f(x) = 1 + \frac{x}{2} - \sqrt{1+x} > 0 \quad \forall x > 0$

$\Rightarrow 1 + \frac{x}{2} > \sqrt{1+x}$

2G-5

a)

$f''(x)$ exists on I

$a<b<c \in I$

$f(a)=f(b)=f(c)=0$

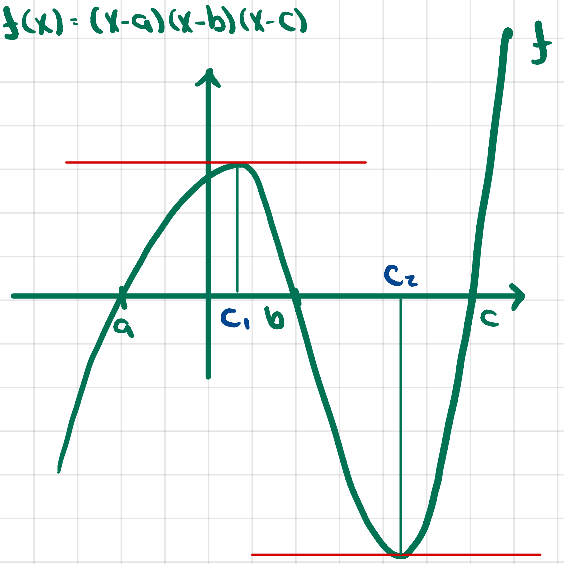
Show $\exists\, p$ on $[a,c]$ where $f''(p)=0$

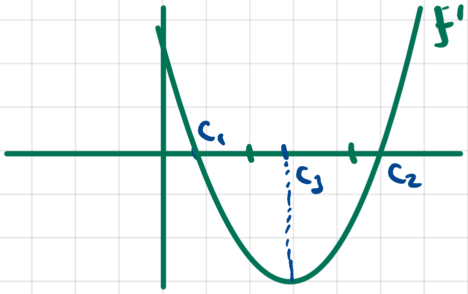
f'' exists $\Rightarrow f'$ is differentiable, continuous $\Rightarrow f$ differentiable, continuous

$f(a)=f(b)=0 \Rightarrow \exists c_1 \in (a,b) : f'(c_1)=0$
$f(b)=f(c)=0 \Rightarrow \exists c_2 \in (b,c) : f'(c_2)=0$

$f'(c_1)=f'(c_2)=0 \Rightarrow \exists c_3 \in (c_1,c_2) : f''(c_3)=0$

b) $f(x)=(x-a)(x-b)(x-c)$





2G-6

a) on $[a,b]$, $f'(x)>0 \Rightarrow f(x)$ increasing

$x \in [a,b] \Rightarrow a \le x \le b$, MVT on $[a,x] \Rightarrow f(x)-f(a)=f'(c)(x-a)$, $c \in (a,x)$. $f'(c)>0 \Rightarrow f(x)-f(a)>0$
$\Rightarrow f(x)$ increasing

b) $x \in [a,b]$, MVT on $[a,x] \Rightarrow f(x)-f(a)=0 \Rightarrow f(x)=f(a)\ \forall x \in [a,b] \Rightarrow f$ constant

3A-1

c) $d(x^{10}-8x+6) = (10x^9-8)\,dx$

d) $d(e^{3x}\sin x) = (e^{3x}\cdot 3\sin x + e^{3x}\cdot\cos x)\,dx$

e) $\sqrt{x}+\sqrt{y}=1$

$$\frac{dx}{2\sqrt{x}}+\frac{dy}{2\sqrt{y}}=0 \Rightarrow dy=-\frac{2\sqrt{y}}{2\sqrt{x}}dx=-\frac{\sqrt{y}}{\sqrt{x}}dx=\frac{\sqrt{x}-1}{\sqrt{x}}dx$$

3A-2

a) $\int(2x^4+3x^2+x+8)\,dx = \frac{2}{5}x^5+x^3+\frac{x^2}{2}+8x+C$

c) $\int\sqrt{8+9x}\,dx = \frac{2}{27}(8+9x)^{\frac{3}{2}}+C$

e) $\int\frac{x}{\sqrt{8-2x^2}}dx = -\frac{1}{2}(8-2x^2)^{1/2}+C$

g) $\int 7x^4e^{x^5}dx = \frac{7}{5}e^{x^5}+C$

i) $\int\frac{dx}{3x+2} = \frac{\ln(3x+2)}{3}+C$

k) $\int\frac{x}{x+5}dx \Rightarrow u=x+5,\ du=dx,\ x=u-5$

$$=\int\frac{u-5}{u}du=\int\left(1-\frac{5}{u}\right)du=u-5\ln u+C=x-5\ln(x+5)+C$$

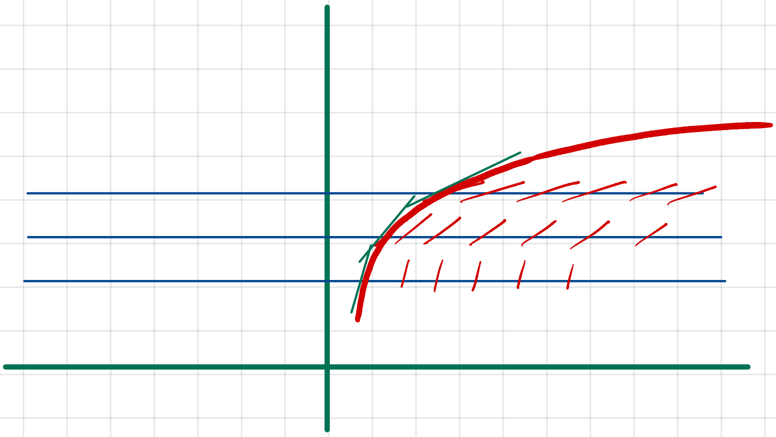
3A-3

a) $\int\sin(5x)\,dx = -\frac{\cos(5x)}{5}+C$

c) $\int\cos^2x\sin x\,dx = -\frac{1}{3}\cos^3x+C$

e) $\int\sec^2(x/5)\,dx = 5\tan(x/5)+C$

g) $\int\sec^9x\tan x\,dx = \frac{1}{9}\sec^9x+C$



3F-1

c) $\frac{dy}{dx}=\frac{3}{\sqrt{y}}$

$\sqrt{y}\,dy = 3dx$

$\int\sqrt{y}\,dy=\int 3dx \Rightarrow \frac{2}{3}y^{\frac{3}{2}}=3x+C$

$y^{3/2}=\frac{3}{2}(3x+C)=\left(\frac{9}{2}x+C\right)$

$y=\left(\frac{9}{2}x+C\right)^{\frac{2}{3}}$

Given an (x,y), $\frac{dy}{dx}$ is $3/\sqrt{y}$. Therefore, given a y^*, all points on $y=y^*$ have same slope. Remember y is family of curves.

$$y=\sqrt[3]{\left(\frac{9}{2}x+C\right)^2}>0$$

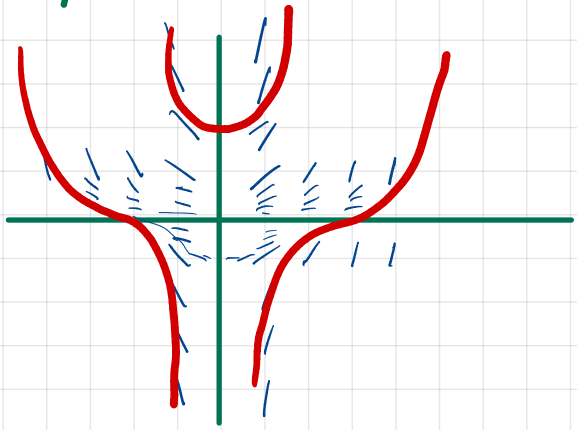
$$y'=\frac{2}{3}\left(\frac{9}{2}x+C\right)^{-1/3}\cdot\frac{9}{2}=\frac{3}{\left(\frac{9}{2}x+C\right)^{1/3}}=\frac{3}{\sqrt[3]{\frac{9}{2}x+C}}$$

$$=\frac{3}{\sqrt{\left(\frac{9}{2}x+C\right)^{2/3}}}=\frac{3}{\sqrt{y}}$$

d) $\frac{dy}{dx} = xy^2 \Rightarrow \frac{1}{y^2}dy = x\,dx$

$\int y^{-2}dy = \int x\,dx \Rightarrow -y^{-1} = \frac{x^2}{2} + C \Rightarrow \frac{1}{y} = -\frac{x^2}{2} + C$

$y = \frac{-2}{x^2+C}$



3F-2

a) $\frac{dy}{dx} = 4xy \qquad y(1)=3,\ y(3)=?$

$\frac{1}{y}dy = 4x\,dx \Rightarrow \ln y = 2x^2 + C$

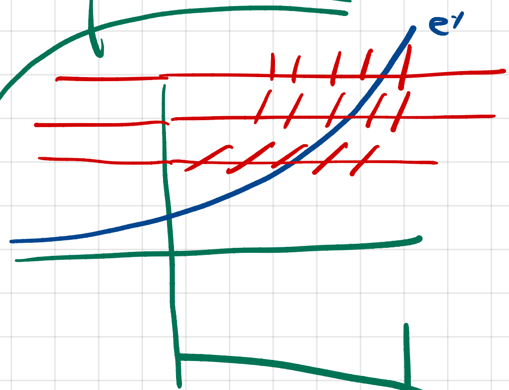
$y = e^{2x^2+C} = K\cdot e^{2x^2}$

$y(1) = Ke^2 = 3 \Rightarrow K = 3/e^2$

$y = 3e^{2x^2-2} = 3e^{2(x^2-1)}$

$y(3) = 3e^{16}$

e) $\frac{dy}{dx} = e^y \qquad y(3)=0$



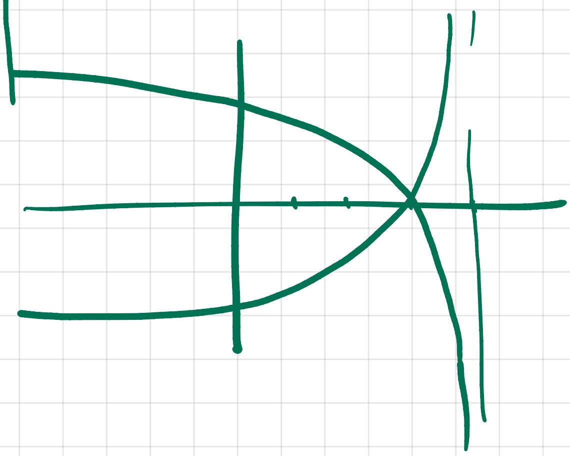
$e^{-y}dy = dx \Rightarrow -e^{-y} = x + C$

$e^y = \frac{-1}{x+C} = \frac{1}{-(x+C)}$

$y = -\ln(-x-C) \qquad -x-C>0 \Rightarrow x<-C$

$0 = -\ln(-3-C) \Rightarrow -3-C=1 \Rightarrow C=-4$

$\Rightarrow y = -\ln(4-x),\ x<4$



$e^y = \frac{-1}{x-4} = \frac{1}{4-x}$

$y = \ln 1 - \ln(4-x) = -\ln(4-x),\ x<4$

3F-4 $\frac{dT}{dt} = k(T_e - T)$

a) we want $\underbrace{T_e - T > 0 \Rightarrow dT/dt > 0}_{\text{external environment warmer} \to \text{body temperature increases}}$, so $k > 0$.

b) $\frac{1}{T_e - T}\,dT = k\,dt \Rightarrow -\ln|T_e - T| = kt + C \Rightarrow \ln|T_e - T| = -(kt + C) \Rightarrow |T_e - T| = Ae^{-kt}$

$\Rightarrow T > T_e \Rightarrow -(T_e - T) = Ae^{-kt} \Rightarrow T = T_e + Ae^{-kt}$

$T < T_e \quad " \quad T_e - T = Ae^{-kt} \Rightarrow T = T_e - Ae^{-kt}$

c) $\lim_{t\to\infty} T = \begin{cases} \lim_{t\to\infty} (T_e + Ae^{-kt}) = T_e \\ \lim_{t\to\infty} (T_e - Ae^{-kt}) = T_e \end{cases}$

d) $T = 680° \xrightarrow{8h} 200° \longrightarrow 50°$ ($?$)

$T_e = 40$

$|T - T_e| = Ae^{-kt}$

$640 = Ae^{-k\cdot 0} \Rightarrow A = 640$

$160 = 640e^{-k\cdot 8}$

$e^{8k} = 4 \Rightarrow 8k = \ln 4 \Rightarrow k = \frac{\ln 4}{8}$

$|T - T_e| = 640e^{-\frac{\ln 4}{8}t}$

$10 = 640e^{-\frac{\ln 4 t}{8}} \Rightarrow \left(e^{\frac{\ln 4}{8}}\right)^t = 64 \Rightarrow \frac{\cancel{\ln} 4 \cdot t}{8} = \ln 64 = \ln 4^3 = 3\cancel{\ln}4$

$\Rightarrow t = 8\cdot 3 = 24$

e) $1000° \xrightarrow{1h} 800°$ $\quad |T - T_e| = Ae^{-kt} \Rightarrow$ since $T > T_e$, $T - T_e = Ae^{-kt}$

$t = 0, T = 1000 \Rightarrow 1000 - T_e = A$

$t = 1, T = 800 \Rightarrow 800 - T_e = (1000 - T_e)e^{-k} \Rightarrow e^{-k} = \frac{800 - T_e}{1000 - T_e}$

$1000° \xrightarrow{2h} 700°$

$t = 2, T = 700 \Rightarrow 700 - T_e = (1000 - T_e)e^{-2k} = (1000 - T_e)(e^{-k})^2$

$\Rightarrow 700 - T_e = \cancel{(1000 - T_e)}\frac{(800 - T_e)^2}{(1000 - T_e)^{\cancel{2}}} \Rightarrow 1000\cdot 700 - 1000T_e - 700T_e + \cancel{T_e^2} = 800^2 - 1600T_e + \cancel{T_e^2}$

$\Rightarrow 100T_e = 700{,}000 - 640{,}000 = 60000 \Rightarrow T_e = 600$

f) $y(t) = T(t-t_0)$ Newton's Law of Cooling: $\frac{dT}{dt} = k(T_e - T)$

ie, we are shifting the function $T(t)$ to the right by t_0

$y'(t) = T'(t-t_0) = k(T_e - T(t-t_0)) = k(T_e - y(t))$

so $y(t)$ follows Newton's Law of cooling

Now we want to show that $y(t)$ is actually of the same form as $T(t)$ with specific k, T_e, T_0.

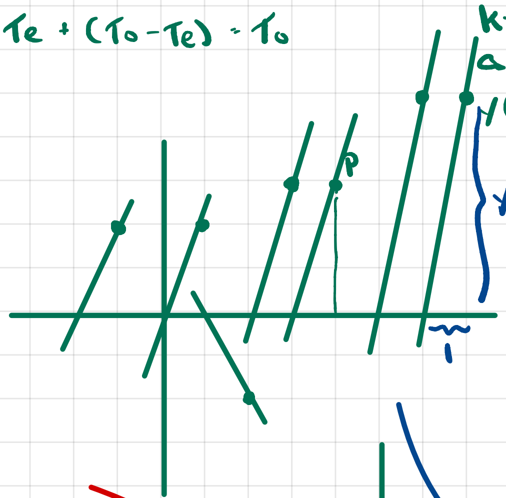
$y(t) = T(t-t_0) = T_e + (T_0 - T_e)e^{-k(t-t_0)} = a + (y(t_0) - a)e^{-c(t-t_0)}$

$y(t_0) = T(0) = T_e + (T_0 - T_e) = T_0$

$k = c$

$a = T_e$

$y(t_0) = T_0$



3F-8

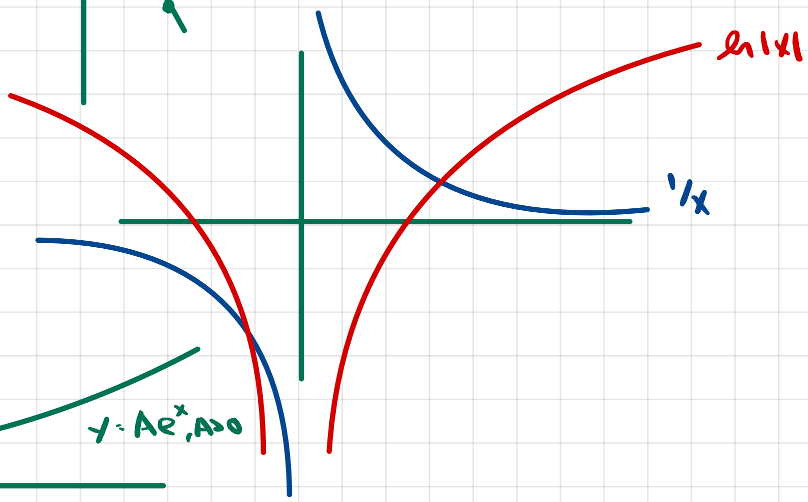
a) $\frac{dy}{dx} = y$

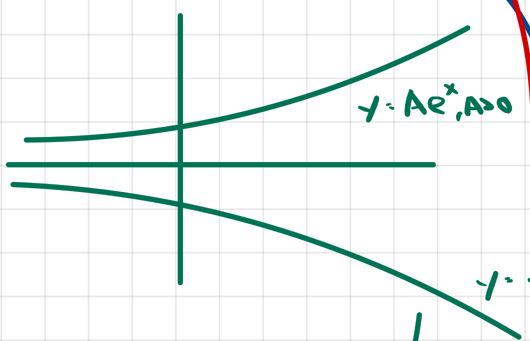
$\frac{1}{y}dy = dx$

$\ln|y| = x + C$

$|y| = Ae^x$

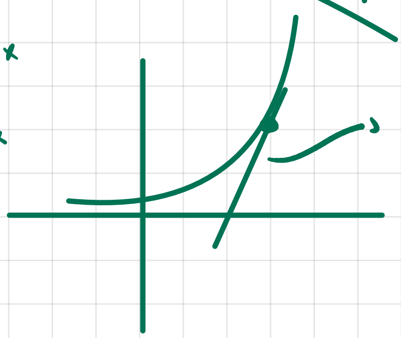
$y = \pm Ae^x$





ex: $y = 2e^x$

$y' = 2e^x$



$y(3) = 2e^3$

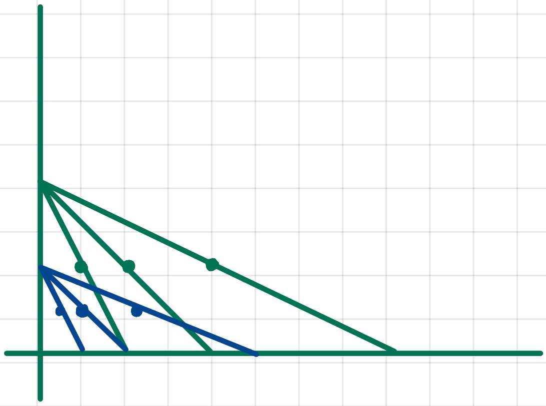
$y'(3) = 2e^3$

$y - 2e^3 = 2e^3(x-3)$

$-2e^3 = 2e^3(x-3) \Rightarrow x = -1+3 = 2$

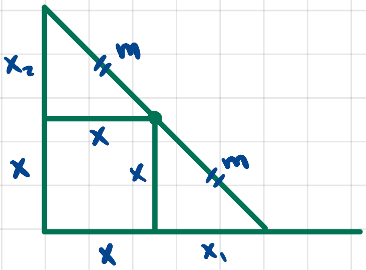
b)

$$\frac{dy}{dx} =$$



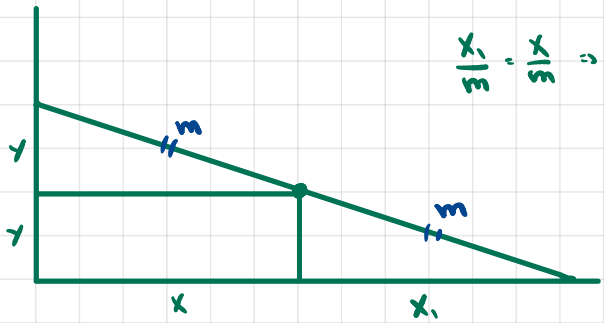
What is the slope at each point P?

For each P, the tangent's intersection with $y=0$ is $2y$:



$$\frac{x}{m} = \frac{x_1}{m} \Rightarrow x_1 = x$$

$$\frac{x}{m} = \frac{x_2}{m} \Rightarrow x_2 = x$$



$$\frac{x_1}{m} = \frac{x}{m} \Rightarrow x_1 = x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2y}{2x} = -\frac{y}{x} \Rightarrow -\frac{1}{y}\,dy = \frac{1}{x}\,dx \;,\; y, x > 0$$

$$\ln y = -\ln x + c$$

$$y = e^{-\ln x + c} = e^c (e^{\ln x})^{-1} = e^c x^{-1} = \frac{c}{x} \;,\; c > 0$$