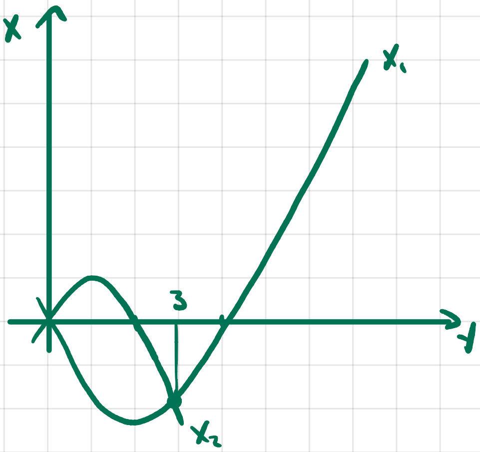
EXAM 3

1 $x_1 = y^2 - 4y$

$x_2 = 2y - y^2$



$$A = \int_0^3 \left[2y - y^2 - (y^2 - 4y)\right] dy$$

$$= \int_0^3 \left[2y - y^2 - y^2 + 4y\right] dy$$

$$= \int_0^3 \left[6y - 2y^2\right] dy = \left(\frac{6y^2}{2} - \frac{2y^3}{3}\right)\Big|_0^3 = \left(3y^2 - \frac{2}{3}y^3\right)\Big|_0^3 = 3\cdot 9 - \frac{2}{3}\cdot 27 = 27 - 18 = 9$$

$x_1(y) = y^2 - 4y = y(y-4)$ $\nearrow 0$ $\searrow 4$

$x_1' = 2y - 4 \Rightarrow y = 2$

$x_1(2) = 4 - 8 = -4$

$x_2(y) = 2y - y^2 = y(2-y)$ $\nearrow 0$ $\searrow 2$

$x_2' = 2 - 2y \Rightarrow y = 1$

$x_2(1) = 1$

$x_1 = x_2 \Rightarrow$

$y^2 - 4y = 2y - y^2$

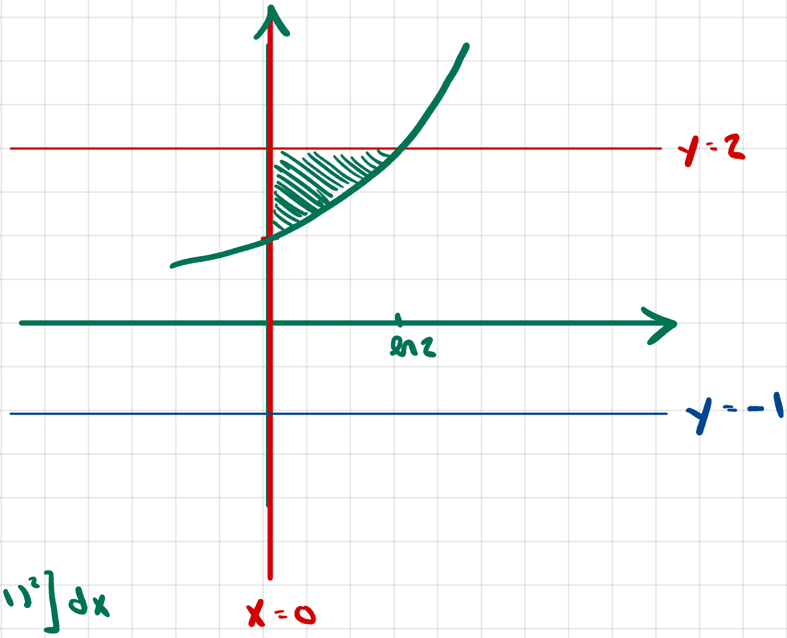
$2y^2 - 6y = 2y(y-3) = 0$

$y = 0,\ y = 3$

2 $y_1 = e^x$

$y_2 = 2$

$x = 0$



$e^x = 2 \Rightarrow x = \ln 2$

$$V = \int_0^{\ln 2} \left[\pi (y_2 + 1)^2 - \pi (y_1 + 1)^2 \right] dx$$

$$= \int_0^{\ln 2} \pi \left[9 - (e^x + 1)^2 \right] dx$$

3

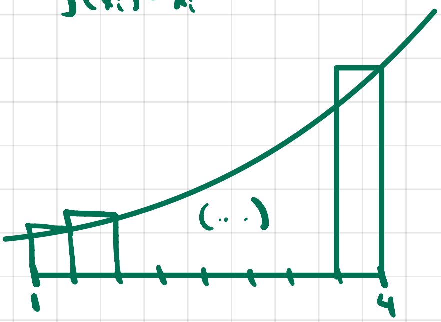
a) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1+i\cdot\frac{3}{n}\right)^2 \cdot \frac{3}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

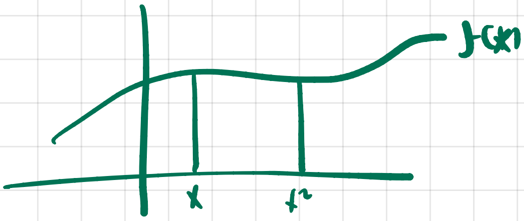
$\Delta x = \frac{3}{n}$ $\qquad = \lim \left[\left(1+\frac{3}{n}\right)^2 \cdot \Delta x + \dots + \left(1+n\cdot\frac{3}{n}\right)^2 \Delta x \right]$

$x_i = 1 + i\cdot\Delta x$

$= \int_1^4 x^2\,dx = \left.\frac{x^3}{3}\right|_1^4 = \frac{4^3}{3} - \frac{1}{3} = \frac{63}{3} = 21$

$f(x_i) = x_i^2$





b) $x\cdot\sin(\pi x) = \int_0^{x^2} f(t)\,dt$ $\qquad f(4)?$

$h(x) = \int_0^{x^2} f(t)\,dt$

$u(x) = x^2 \Rightarrow h(x) = g(u(x)) = \int_0^u F(t)\,dt$

$h'(x) = D_x[g(u(x))] = \frac{dg}{du}\frac{du}{dx} = F(u)\cdot 2x = f(x^2)\cdot 2x$

$\Rightarrow 1\cdot\sin(\pi x) + x\cos(\pi x)\pi = f(x^2)\cdot 2x$

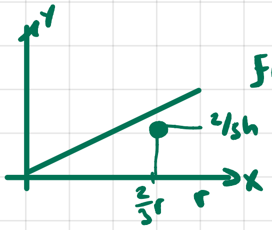
$F(x^2) = \frac{\sin(\pi x) + \pi x\cos(\pi x)}{2x} \Rightarrow f(2^2) = \frac{\sin 2\pi + 2\pi\cos 2\pi}{2\cdot 2}$

$= \frac{0+2\pi}{4} = \frac{\pi}{2}$

4

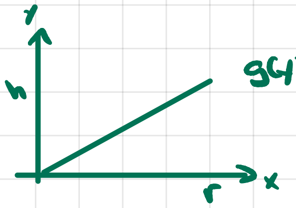
a)

center mass = $\left(\dfrac{\int x f(x)dx}{\int f(x)dx} , \dfrac{\int y g(y)dy}{\int g(y)dy} \right)$



$f(x) = \frac{h}{r}x$

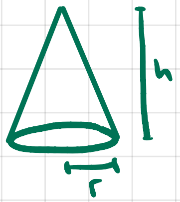
$$\frac{\frac{h}{r}\int_0^r x^2 dx}{\frac{h}{r}\int_0^r x\, dx} = \frac{\frac{r^3}{3}}{\frac{r^2}{2}} = \frac{r^3}{3}\cdot\frac{2}{r^2} = r\cdot\frac{2}{3}$$



$g(y) = r - \frac{r}{h}y$

$$\frac{\int_0^h y(r-\frac{r}{h}y)dy}{\int_0^h (r-\frac{r}{h}y)dy} = \frac{r\int_0^h (y-\frac{y^2}{h})dy}{r\int_0^h (1-\frac{y}{h})dy} = \frac{\left[\frac{y^2}{2} - \frac{y^3}{3h}\right]_0^h}{\left(y - \frac{y^2}{2h}\right)\Big|_0^h} = \frac{\frac{h^2}{2} - \frac{h^{\cancel{3}\,2}}{3\cancel{h}}}{h - \frac{h^{\cancel{2}}}{2\cancel{h}}} = \frac{\frac{3h^2-2h^2}{6}}{\frac{h}{2}} = \frac{h}{3}$$

b)

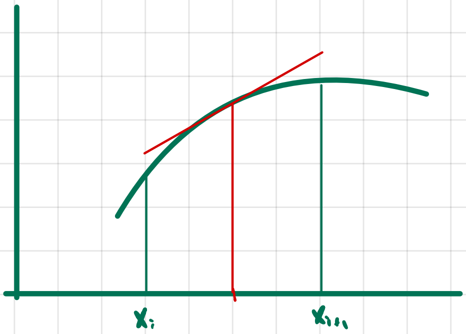


$A = \frac{rh}{2}$

distance travelled by rotated centroid $= 2\pi\cdot\left(\frac{r}{3}\right)$

$V = 2\pi\frac{r}{3}\cdot\frac{rh}{2} = \frac{1}{3}\pi r^2 h$

5 $\int_a^b f(x)\,dx$



$$T_n = \sum \Delta x_i \frac{f(x_i)+f(x_{i-1})}{2} = \frac{\Delta x}{2}\left[f(x_0)+f(x_1)+f(x_1)+f(x_2)+ \dots + f(x_{n-1})+f(x_n)\right]$$

$$= \sum_{i=1}^{n} \frac{\Delta x}{2}\left(f(x_0)+2f(x_1)+ \dots + 2f(x_{n-1})+f(x_n)\right)$$

$$M_n = \sum_{i=1}^{n} \Delta x_i\, f\left(\frac{x_i+x_{i-1}}{2}\right) = \Delta x\left[f\left(\frac{x_0+x_1}{2}\right)+f\left(\frac{x_1+x_2}{2}\right)+ \dots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right]$$

$$= \Delta x\left[f(x_{0.5})+f(x_{1.5})+ \dots + f(x_{n-0.5})\right]$$

$$\frac{1}{3}T_n + \frac{2}{3}M_n = \frac{\Delta x}{6}\left[f(x_0)+f(n)+\sum_{i=1}^{n-1} 2f(x_i)\right] + \frac{2}{3}\Delta x \sum_{i=1}^{n} f(x_{i-0.5})$$

$$= \frac{\Delta x}{6}\left[f(x_0)+4f(x_{0.5})+2f(x_1)+4f(x_{1.5})+ \dots + 4f(x_{n-0.5})+f(x_n)\right]$$

$$= \frac{(\Delta x/2)}{3}\left[f(x_0)+4f\left(x_0+1\cdot\frac{\Delta x}{2}\right)+2f\left(x_0+2\cdot\frac{\Delta x}{2}\right)+ \dots + f\left(x_0+2n\cdot\frac{\Delta x}{2}\right)\right]$$

we can redefine the intervals as $\Delta m = \frac{\Delta x}{2}$

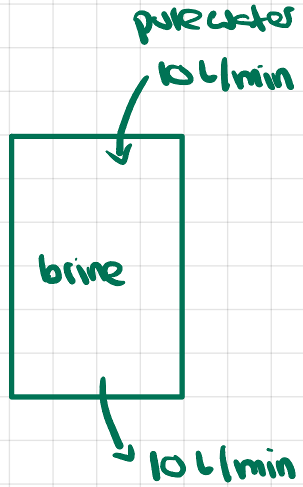
$$= \frac{\Delta m}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+ \dots + f(x_{2n})\right]$$

$$= S_{2n}$$

6 1000 L water

15 kg salt

constant volume



a) $\frac{dS(t)}{dt} = -S(t) \cdot \frac{10}{1000}$

$\frac{1}{S} dS = -0.01$

b) $\ln S = -0.01t + k$

$S(t) = \frac{e^k}{e^{0.01t}} = \frac{K}{e^{0.01t}}$

$S(0) = K = 15 \text{ kg}$

$S(t) = \frac{15}{e^{0.01t}}$

c) $\frac{1}{2} \cdot 15 = \frac{15}{e^{0.01t}} \Rightarrow e^{0.01t} = \frac{1}{2}$

$0.01t = \ln 2 \Rightarrow t = \frac{\ln 2}{0.01} = 100 \cdot \ln 2 = 69.3 \text{ min}$

* concentration

$\frac{dC(t)}{dt} = \frac{dS(t)}{dt} / 1000 = -\frac{S(t) \cdot 10}{1000 \cdot 1000}$

$= -C(t) \cdot 10/1000$

$\Rightarrow C(t) = \frac{K}{e^{0.01t}}$

$C(0) = K = 0.015$

$C(t) = 0.015 / e^{0.01t}$

$\frac{1}{2} 0.015 = 0.015 / e^{0.01t}$

$e^{0.01t} = 2 \Rightarrow t = 69.3 \text{ min}$