PSet 3

2A-1 $y = \sqrt{a+bx}$ $\quad a>0$

$$f(x) \approx f(0) + f'(0)(x-0) = \sqrt{a} + \frac{b}{2\sqrt{a}} \cdot x$$

$$f(0) = \sqrt{a}$$

$$f' = \frac{b}{2\sqrt{a+bx}}$$

$$f'(0) = \frac{b}{2\sqrt{a}}$$

Alternatively, $f(x) = \sqrt{a} \cdot \sqrt{1+\frac{bx}{a}} \approx \sqrt{a}\left(1+\frac{bx}{a}\right) = \sqrt{a} + \frac{bx}{2\sqrt{a}}$

But $\sqrt{1+x} \approx 1 + \frac{1}{2\sqrt{1+0}} \cdot x = 1 + \frac{x}{2}$

$$\sqrt{1+kx} \approx 1 + \frac{k}{2\sqrt{1+0}} \cdot x = 1 + \frac{kx}{2}$$

2A-2 $f(x) = \frac{1}{a+bx}$ $\quad a \neq 0$

$$f = \frac{1}{a(1+\frac{b}{a}x)} = \frac{1}{a} \cdot \left(1+\frac{b}{a}x\right)^{-1} \approx \frac{1}{a} \cdot \left(1 - \frac{b}{a}x\right) = \frac{(a-b)x}{a^2}$$

2A-3 $f(x) = \frac{(1+x)^{3/2}}{(1+2x)} = (1+x)^{3/2}(1+2x)^{-1} \approx \left(1+\frac{3x}{2}\right)(1-2x) = 1 - 2x + \frac{3}{2}x - 3x^2$

$= 1 - \frac{1}{2}x - \cancel{3x^2}$ throw away

$(1+x)^{3/2} \approx 1 + \frac{3}{2}x$

These are linearizations at $x=0$, and we consider small $x = \Delta x$. x^2 is negligible in the approx. for small Δx.

OR

$$f' = \frac{\frac{3}{2}(1+x)^{1/2}(1+2x) - (1+x)^{3/2} \cdot 2}{(1+2x)^2}$$

$$f'(0) = \left(\frac{3}{2} \cdot 1 - 1 \cdot 2\right)\Big|_1 = \frac{3}{2} - 2 = \frac{3}{2} - \frac{4}{2} = -\frac{1}{2}$$

$$f(0) = \frac{1}{1} = 1$$

$$f(x) \approx 1 - \frac{1}{2} \cdot x$$

$$* \ \frac{d}{dx}\sec x = \frac{d}{dx}\cos^{-1}x = -\cos^{-2}x(-\sin x) = \frac{\sin x}{\cos^2 x}$$

2A-6 quadratic approx to $\tan\theta$, $\theta \approx 0$

Altern.: $\tan\theta = \frac{\sin\theta}{\cos\theta} \approx \theta$

$\sin\theta \approx \theta$

$\cos\theta \approx 1$

$$\tan\theta \approx \tan\theta_0 + \tan'\theta_0 \Delta\theta + \frac{1}{2}\tan''\theta_0 \Delta\theta$$

$$\theta_0 = 0 \Rightarrow \tan\theta_0 = 0$$

$$\tan'\theta = \sec^2\theta \Rightarrow \tan'\theta_0 = \sec^2 0 = 1$$

$$\tan''\theta = 2\sec\theta \cdot \tan\theta \cdot \sec\theta \Rightarrow \tan''\theta_0 = 2 \cdot 1 \cdot 0 \cdot 1 = 0$$

$$\tan\theta \approx 0 + 1 \cdot \theta = \theta$$

2A-11

$pV^k = C \Rightarrow p(V) = CV^{-k}$

$V = V_0 + \Delta V$

$p(\Delta V) = C(V_0 + \Delta V)^{-k} = C(V_0(1 + \Delta V/V_0)^{-k} = CV_0^{-k}(1 + \Delta V/V_0)^{-k}$

we have a term $(1 + \Delta V/V_0)^{-k}$ that we can approximate with a quadratic function of ΔV.

$(1+cx)^{-k} \approx 1 - kcx + k(k+1)c^2 \cdot \frac{1}{2}x^2$

since ΔV is small, we can approx. $(1 + \Delta V/V_0)^{-k}$ near 0

$\approx 1 - \frac{k}{V_0}\Delta V + k(k+1) \cdot \frac{1}{V_0^2}\frac{\Delta V^2}{2}$

$\Rightarrow p(\Delta V) \approx \frac{C}{V_0^k}\left[1 - \frac{k}{V_0}\Delta V + k(k+1)\frac{\Delta V^2}{2V_0^2}\right]$

2A-12

a) $f = \frac{e^x}{1-x}$, quadratic approx. for $x \approx 0$

$e^x \approx 1 + x + \frac{x^2}{2}$

$f \approx \frac{1}{1-x} + \frac{x}{1-x} + \frac{x^2}{2(1-x)} = (1-x)^{-1} + x(1-x)^{-1} + x^2(1-x)^{-1}/2$

$(1-x)^{-1} \approx 1 - 1(1-0)^{-2}(-1)x - \cancel{2}(1-0)^{-3}(-1) \cdot \frac{x^2}{\cancel{2}}$
$= 1 + x + x^2$

$x(1-x)^{-1} \approx x(1 + x + x^2)$

$x^2(1-x) \approx x^2(1 + x + x^2)$

$\Rightarrow f \approx (1 + x + x^2) + x(1 + x + x^2) + x^2(1 + x + x^2)/2$

$= (1 + x + x^2)(1 + x + \frac{x^2}{2})$

Did not need to go through all this with:

$f = \frac{e^x}{1-x} = e^x(1-x)^{-1} \approx (1 + x + \frac{x^2}{2})(1 + x + x^2)$

2A-12

d) $f(x) = \ln(\cos x)$, quadratic approx near 0.

$\cos x \approx \cos x_0 - \sin x_0 \, \Delta x - \frac{1}{2}\cos(x_0)\,\Delta x^2$

near $x_0 = 0$,

$\approx 1 - x^2/2$

$f(x) \approx \ln(1 - x^2/2)$

$g(x) = \ln(1 - kx^2)$

$g'(x) = \frac{1}{1-kx^2}(-2kx) = \frac{-2kx}{(1-kx^2)} = -2kx(1-kx^2)^{-1}$

$g''(x) = -2k(1-kx^2)^{-1} - 2kx(-1)(1-kx^2)^{-2}(-2kx)$

$g'(0) = 0$

$g''(0) = -2k$

$\Rightarrow f(x) \approx \ln(1) + \frac{1}{2}\cdot\left(-2\cdot\frac{1}{2}\right)x^2$

$= -\frac{1}{2}x^2$

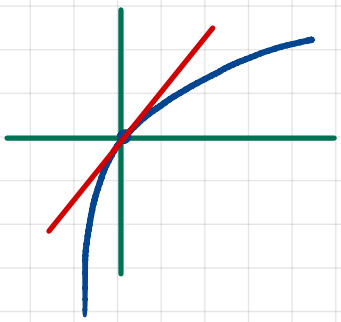
Alternatively

$f = \ln(\cos x) \approx \ln(1 + (-\frac{1}{2})x^2)$

$\ln(1+x) \approx x \Rightarrow \ln(1 + (-x^2/2)) \approx -x^2/2$

$\Rightarrow \ln(\cos x) \approx -x^2/2$

On the approx of $\ln(1+x)$:



We can use $\ln(1+x) \approx x$ in the middle of another function because that other function is also approximating near 0, so x is small.

2A-12

e) $f(x) = x \ln x$ quadratic approx. near $x_0 = 1$

$x = 1+h \Rightarrow F(h) = (1+h)\ln(1+h)$

if we approximate $f(h)$ near $h_0 = 0$, this is the same as approx $f(x)$ near $x_0 = 1+0 = 1$.

$g(h) = \ln(1+h)$, $g(0) = 0$

$g'(h) = \frac{1}{1+h}$, $g'(0) = 1$

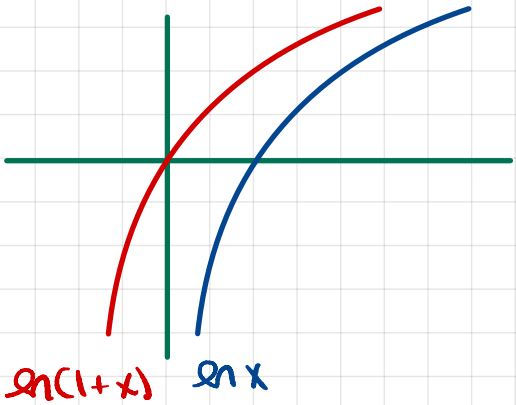
$g''(h) = \frac{-1}{(1+h)^2}$, $g''(0) = -1$

$\ln(1+h) \approx 0 + 1 \cdot h + \frac{1}{2}(-1)h^2 = h - \frac{h^2}{2}$

throw away

$\Rightarrow f(h) \approx (1+h)(h - \frac{h^2}{2}) = h - \frac{h^2}{2} + h^2 - \cancel{\frac{h^3}{2}} \approx h + \frac{h^2}{2}$

$f(x) \approx x - 1 + \frac{(x-1)^2}{2}$



$\ln x \to \ln(1+x)$ is a shift left of $\ln x$
The curve is still the same besides the shift.
We simply changed each coordinate linearly.

what if we hadn't used $x = 1+h$?

$f(x) = x \ln x$

$f' = \ln x + x \cdot 1/x = \ln x + 1$

$f'' = 1/x + 1/x + x(-1)/x^2 = \frac{2}{x} - \frac{x}{x^2} = \frac{2}{x} - \frac{1}{x} = \frac{1}{x}$

$f'(1) = \ln(1) + 1 = 1$

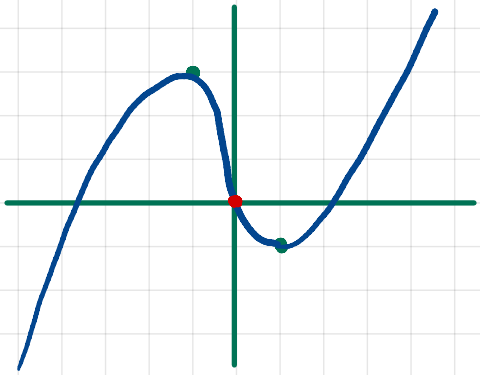
$f''(1) = 1$ $\quad f(1) = 1 \cdot \ln 1 = 0$

$f(x) \approx 0 + 1 \cdot (x-1) + \frac{1}{2} \cdot 1 (x-1)^2$

$= x - 1 + \frac{(x-1)^2}{2}$

2B-2

a) $y = x^3 - 3x + 1$



critical points

$y' = 3x^2 - 3 \qquad y' = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$y(1) = 1 - 3 + 1 = -1$

$y(-1) = -1 + 3 + 1 = 3$

y' sign

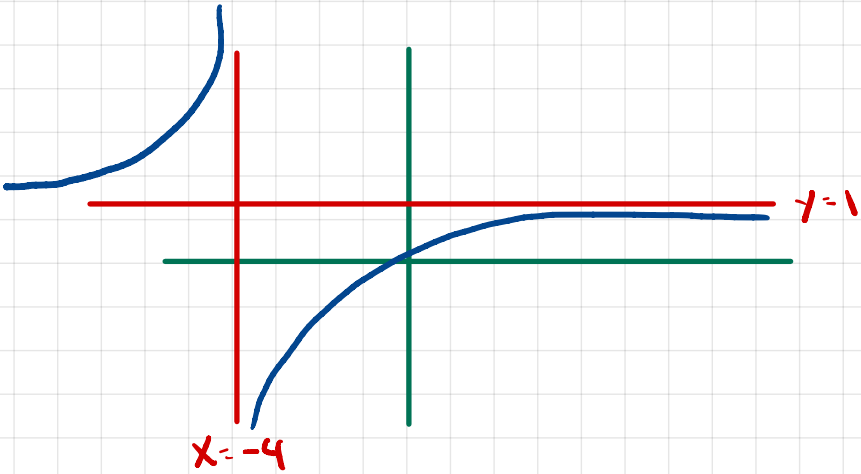
$+ \quad -1 \quad - \quad 1 \quad + \quad y'$

y'', concavity

$y'' = 6x$

$y'' = 0 \Rightarrow x = 0$ inflection point

$- \quad 0 \quad + \quad y''$



Ends

$y(+\infty) = +\infty$

$y(-\infty) = -\infty$

e) $y = \dfrac{x}{x+4} = x(x+4)^{-1}$

critical points

$y' = (x+4)^{-1} + x(-1)(x+4)^{-2} = \dfrac{x+4-x}{(x+4)^2} = \dfrac{4}{(x+4)^2} > 0$

$\Rightarrow$ no critical points

discontinuities

$x = -4 \qquad \lim\limits_{x \to -4^+} f(x) = \dfrac{-4}{0^+} = -\infty \qquad \lim\limits_{x \to -4} f'(x) = +\infty$

$\lim\limits_{x \to -4^-} f(x) = \dfrac{-4}{0^-} = +\infty$

ends

$f = \dfrac{1}{1 + 4/x} \Rightarrow f(\pm\infty) = 1$

concavity

$y'' = -2(4)(x+4)^{-3} = \dfrac{-8}{(x+4)^3} \neq 0 \Rightarrow$ no inflection point

$+ \quad -4 \quad -$

$y''(-4^-) = \dfrac{-8}{0^-} = +\infty$

$y''(-4^+) = -8/0^+ = -\infty$

2B-2

h) $y = e^{-x^2} = \frac{1}{e^{x^2}}$

$y' = (-2x)e^{-x^2} = -2xe^{-x^2}$

$y'' = -2e^{-x^2} - 2x(-2x)e^{-x^2} = -2e^{-x^2} + 4x^2e^{-x^2}$

$= e^{-x^2}(4x^2-2)$

$y'' = 0 \Rightarrow 4x^2 = 2$

$x^2 = 1/2 \Rightarrow x = \pm 1/\sqrt{2}$

2B-4

f continuous in $[0,10]$

$f(4) = f(7) = f(9) = 0$

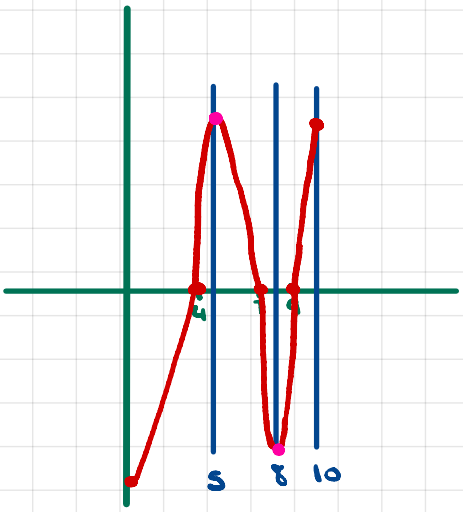
$f' > 0$ in $(0,5) \cup (8,10)$

$f' < 0$ in $(5,8)$

$\Rightarrow$ critical points:

$x = 5$

$x = 8$



To say s.th about the global max and min we need to know

$f(5), f(8), f(10), f(0)$.

$f(5)$ and $f(8)$ de local max and min, respect.

2B-6

cubic polynomial

local max $x = -1$

" min $x = 1$

easier way

$f'(-1) = f'(1) = 0 \Rightarrow f' = (x+1)(x-1) = x^2-1$

$\Rightarrow f = \frac{x^3}{3} - x$

complicated way:

$y = ax^3 + bx^2 + cx + d$

$y' = 3ax^2 + 2bx + c$

$y'' = 6ax + 2b$

$y'(-1) = 3a - 2b + c = 0$

$y''(-1) = -6a + 2b < 0 \Rightarrow a > b/3$

$y'(1) = 3a + 2b + c$

$y''(1) = 6a + 2b > 0 \Rightarrow a > -b/3$

$6a + 2c = 0 \Rightarrow 3a = -c$

$4b = 0 \Rightarrow b = 0$

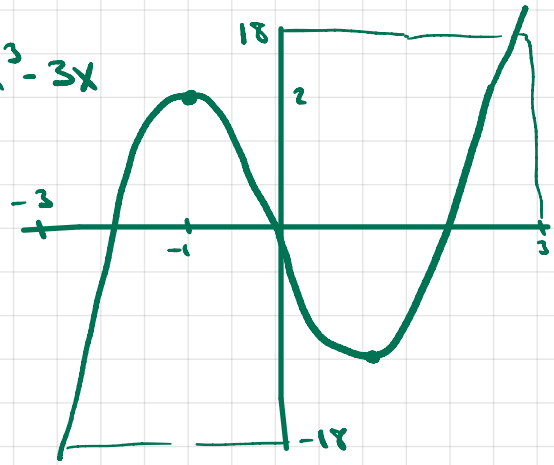
$\Rightarrow a > 0$

$\Rightarrow$ we have: $b = 0$, $a > 0$, $c = -3a$

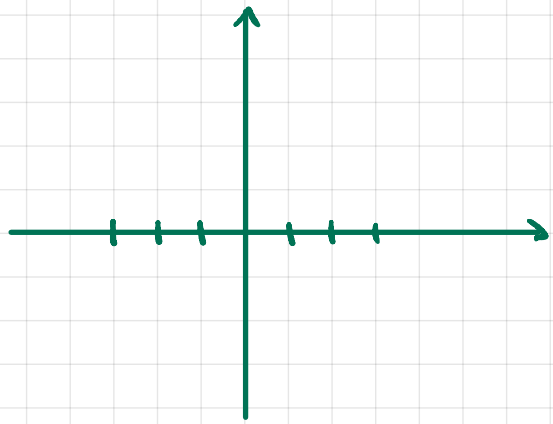
Use $a = 1 \Rightarrow c = -3$

$\Rightarrow y = x^3 - 3x$

b) $y = x^3 - 3x$



c) differentiable f in $[-3,3]$ has absolute max at $x=-1$, abs. min at $x=1$



$-\sin\left(\frac{\pi}{2}x\right)$ $\quad T = \frac{2\pi}{\pi/2} = \frac{4\pi}{\pi} = 4$

$f = \sin(kx) \cdot \frac{1}{x}$

$f'(x) = \cos(kx)k \cdot \frac{1}{x} + \sin(kx)(-1)/x^2 = \frac{\cos(kx)k}{x} - \frac{\sin(kx)}{x^2}$

$f'(-1) = 0 \Rightarrow \quad -\cos(-k)k - \sin(-k) = 0$

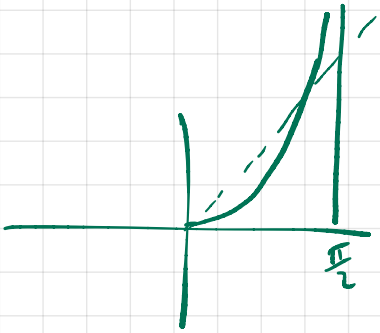
$-\cos(-k)k = \sin(-k)$

$k = -\frac{\sin(-k)}{\cos(-k)} \Rightarrow -k = \tan(-k)$

$f'(1) = \cos(k)k - \sin(k) = 0 \Rightarrow \frac{\sin k}{\cos k} = k \Rightarrow k = \tan(k)$

$k = \tan(k) \approx$

$\tan x \sim$



2B-7

a) $f(x)$ increasing $\Rightarrow x_1 > x_0 \Rightarrow f(x_1) \geq f(x_0)$
$f'(x_0)$ exists

$\Rightarrow x_1 - x_0 > 0 \Rightarrow f(x_1) - f(x_0) \geq 0 \Rightarrow \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} \geq 0 \Rightarrow \lim\limits_{x_1 \to x_0} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} \geq 0$

$\Rightarrow f'(x_0) \geq 0$

b) In part a), an increasing $f(x)$ means $x_1 > x_0 \Rightarrow f(x_1) \geq f(x_0)$. This means it is possible for $f(x_1) = f(x_0)$

$y = x^3$
$y' = 3x^2 > 0 \Rightarrow$ increasing $f(x)$

$y'(0) = 0$

Another example: $y = k$, $y' = 0$, $\forall x$