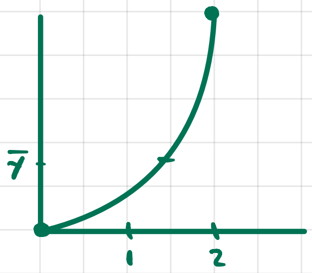
5.6

1 $f(x) = x^4 \quad [0,2]$



$\int_0^2 x^4 dx = \frac{32}{15}$. since f is continuous on $[0,2]$

the avg value theorem says $\bar{y} = f(\bar{x})$ exists, such that $\bar{x} \in [0,2]$ and

$$\bar{y} = \frac{1}{2-0} \int_0^2 x^4 dx = \frac{1}{2}\left[\frac{x^5}{5}\right]_0^2 = \frac{32}{25} \cdot \frac{1}{2} = \frac{16}{25}$$

limit of Riemann sum, given x, area between a and x

$$F(x) = \int_a^x t^4 dt \Rightarrow F'(x) = t^4$$

$$\frac{d}{dx}\left[\int_0^x t^4 dt\right] = x^4$$

rate of increase of area is derivative of the integral, ie the value of function whose area we want to compute.

2 $g(x) = \sqrt{x} \quad [1,4]$

$$\int_1^4 \sqrt{x}\, dx = \frac{2}{3} x^{3/2} \Big|_1^4 = \frac{2}{3}\left[\sqrt{26} - 1\right] = \frac{14}{3}$$

g cont. on $[1,4]$. $\bar{y} = \frac{1}{4-1} \cdot \frac{14}{3} = \frac{14}{9}$

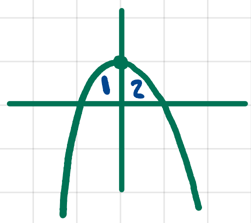
13 $$\int_{-1}^3 dx = x\Big|_{-1}^3 = 3-(-1) = 4$$

17 $$\int_1^3 \frac{3t-5}{t^4} dt = 3\int_1^3 t^{-3} dt - 5\int_1^3 t^{-4} dt = 3 \cdot \frac{1}{-2} t^{-2}\Big|_1^3 - 5 \frac{1}{-3} t^{-3}\Big|_1^3 = -\frac{3}{2}\left(\frac{1}{9} - 1\right) + \frac{5}{3}\left(\frac{1}{27} - 1\right)$$

$$= -\frac{3}{2} \cdot \left(-\frac{8}{9}\right) + \frac{5}{3}\left(-\frac{26}{27}\right) = \frac{24}{18} - \frac{130}{81} = \frac{108 - 130}{81} = -\frac{22}{81}$$

27 $$\int_4^8 \frac{1}{x} dx = \ln x \Big|_4^8 = \ln 8 - \ln 4 = \ln 2$$

29 $f(x) = \begin{cases} 1-x^4 & x \leq 0 \\ 1-x^3 & x \geq 0 \end{cases}$

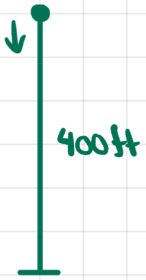


① $$\int_{-1}^0 (1-x^4) dx = \left(x - \frac{x^5}{5}\right)\Big|_{-1}^0$$

$$= -\left[-1 - \frac{-1}{5}\right] = 1 - 1/5 = 4/5$$

② $$\int_0^1 (1-x^3) dx = \left(x - \frac{x^4}{4}\right)\Big|_0^1 = 1 - 1/4 = 3/4$$

33



$a(t) = g \; m/s^2$

$v(t) = \int a(t)\,dt = gt + C = g \cdot t + v_0 = gt$

$s(t) = \int v(t)\,dt = \frac{gt^2}{2} + s_0 = \frac{gt^2}{2}$ ↘ $v_0 = 0,\ s_0 = 0$

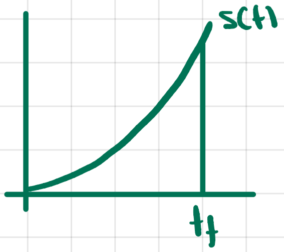
↘ area under $v(t)$

rate of change of area, $s'(t) = v(t)$

$400 = gt_f^2/2 \Rightarrow t_f^2 = \frac{800}{g} \Rightarrow t_f = \sqrt{800/g}$

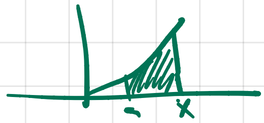
$\bar{s} = s(\bar{t}) = \frac{1}{t_f - 0} \cdot \int_0^{t_f} s(t)\,dt = \frac{1}{t_f} \cdot \left[\frac{gt^3}{3}\right]_0^{t_f} = \frac{1}{t_f} \cdot \frac{gt_f^3}{3} = \frac{g \cdot t_f^2}{3} = \frac{g \cdot 800}{3g} = \frac{800}{3}$ = avg height

$\bar{v} = v(\bar{t}) = \frac{1}{t_f - 0} \int_0^{t_f} v(t)\,dt = \frac{1}{t_f} \cdot s(t)\Big|_0^{t_f} = \frac{1}{t_f} s(t_f) = \frac{1}{\cancel{t_f}} \cdot \frac{g}{2} \cdot t_f^{\cancel{2}} = \frac{g}{2}\sqrt{\frac{800}{g}} = \frac{\sqrt{800g}}{2}$



Note

$f(t) = \int_a^x f(t)\,dx$



$g(x) = \int v(x)\,dt$

$s(t) = \frac{gt^2}{2}$

$\int_0^t s(t)\,dt = \frac{gt^3}{3}$

$\bar{s} = \frac{1}{t} \cdot gt^3/3 = \frac{gt^2}{3}$

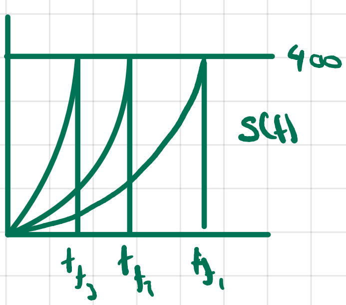
↗ stronger gravity, faster time to reach ground

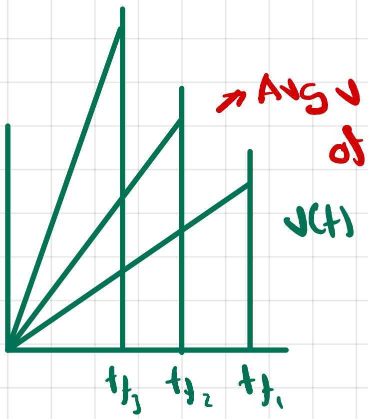
But, $s(t_f) = 400 \Rightarrow t_f = \sqrt{800/g}$

$\bar{s} = \frac{800}{3}$, independent of g

$\bar{v} = \sqrt{800g}/3$ depends on g

$\Rightarrow$ ↑g means v reaches a higher level ⇒ avg higher





↗ Avg v goes ↑ when t_f goes ↓ (because of ↑g)

↑g means ↑ avg v, so destination is reached earlier but the end value of s is always the same.

$t_f = \sqrt{\frac{800}{g}}$ $\quad \bar{s} = \frac{\text{Area}}{t_f}$ $\quad$ Area $= \frac{gt_f^3}{3} = \frac{t_f^3\sqrt{g}}{3}$

So we see area is proportional to $\sqrt{g}$, t_f to $1/\sqrt{g}$, so the ratio is constant.

34 $P(t) = 100 + 10t + 0.02t^2$ $[0,10]$

$$\bar{P} = \frac{1}{10-0}\int_0^{10}(100+10t+0.02t^2)\,dt = \frac{1}{10}\left[100t + 5t^2 + \frac{2}{300}t^3\right]_0^{10} = \frac{1}{10}\left[1000 + 500 + \frac{1000}{150}\right]$$

$$= 150 + \frac{2}{3} = \frac{452}{3}$$

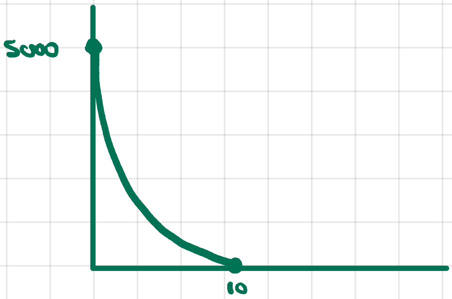
35 5000 L

10 min to drain

$V(t) = 50(10-t)^2$ liters, amount of water in tank

$$\int_0^{10} 50(10-t)^2\,dt = -\frac{50}{3}(10-t)^3\Big|_0^{10} = -\left[-\frac{50}{3}\cdot 10^3\right] = \frac{50\cdot 10^3}{3}$$

$$\bar{V} = V(\bar{t}) = \frac{1}{10}\cdot\frac{50000}{3} = \frac{5000}{3}$$



$V' = 100(10-t)(-1) = -100(10-t) = 100t - 1000$

$V'' = 100$

36 $T(t) = 80 + 10\sin\left(\frac{\pi}{12}(t-10)\right)$ = temp t hours past midnight

Avg between 12 and 18?

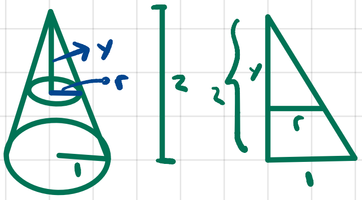
$$\int_{12}^{18}\left(80 + 10\sin\left(\frac{\pi}{12}(t-10)\right)\right)dt = \left[80t + 10\left(-\cos\left[\frac{\pi}{12}(t-10)\right]\cdot\frac{12}{\pi}\right)\right]_{12}^{18}$$

$$= 80(18-12) + \frac{120}{\pi}\left[-\cos\left(\frac{\pi}{12}\cdot 8\right) + \cos\left(\frac{\pi}{12}\cdot 2\right)\right]$$

$$= 480 + \frac{120}{\pi}\left(\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{2\pi}{3}\right)\right)$$

$$\bar{T} = T(\bar{t}) = \frac{1}{18-12}\left[480 + \frac{120}{\pi}\left(\cos\frac{\pi}{6} - \cos\frac{2\pi}{3}\right)\right]$$

39



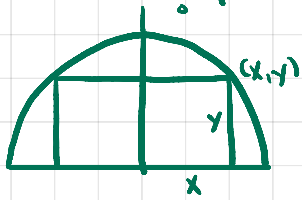
$\frac{r}{y} = \frac{1}{2} \Rightarrow r = \frac{y}{2}$

$A(y) = \pi r^2 = \pi \cdot \frac{y^2}{4}$

$\bar{A} = A(\bar{y}) = \frac{1}{2}\int_0^2 \frac{\pi y^2}{4} dy = \frac{1}{2}\frac{\pi y^3}{12}\Big|_0^2 = \frac{\pi}{24} \cdot 8 = \frac{\pi}{3}$

43

a)

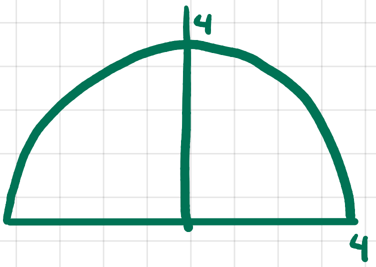


$y = \sqrt{16-x^2} \Rightarrow A(x) = 2 \cdot x\sqrt{16-x^2}$

b) $\bar{A} = A(\bar{x}) = \frac{1}{4}\int_0^4 2x\sqrt{16-x^2}\,dx = \frac{1}{2}(16-x^2)^{\frac{3}{2}} \cdot \cancel{\frac{2}{3}} \cdot (-\frac{1}{\cancel{2}})\Big|_0^4 = -\frac{1}{6}(16-x^2)^{3/2}\Big|_0^4 = \cancel{-\frac{1}{6}(16-16)^{3/2}} + \frac{1}{6}(16)^{3/2}$

$= \frac{1}{6} \cdot \sqrt{4^6} = \frac{4^3}{6} = \frac{64}{6} = \frac{32}{3}$ → avg area as x goes from 0 to 4

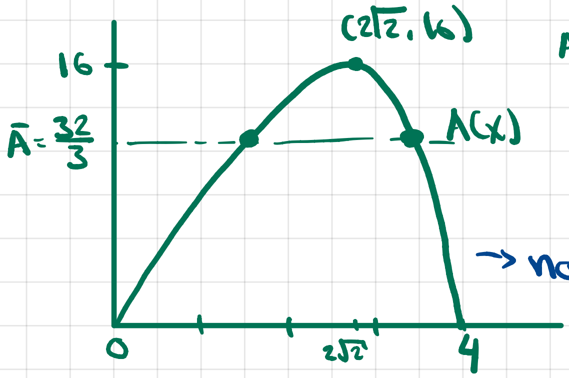
c)



A(x) is continuous in [0,4]

$A' = 2\sqrt{16-x^2} + \cancel{2}x \cdot \frac{1}{\cancel{2}}(16-x^2)^{-\frac{1}{2}} \cdot (-2x)$

$= 2\sqrt{16-x^2} - \frac{2x^2}{\sqrt{16-x^2}} = \frac{2(16-x^2) - 2x^2}{\sqrt{16-x^2}} = \frac{32-4x^2}{\sqrt{16-x^2}}$



$A' = 0 \Rightarrow 4x^2 = 32 \Rightarrow x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}$

x with max area.

For $x \in [0,4]$, single max:

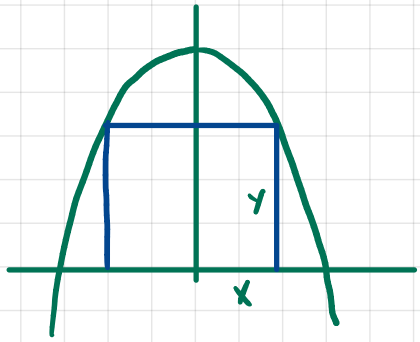
$x = 2\sqrt{2}$

→ not symmetric in 1st quadrant

$A(2\sqrt{2}) = 2 \cdot 2\sqrt{2}\sqrt{16-4 \cdot 2} = 2 \cdot \sqrt{8}\sqrt{8} = 16$

$A(x) = \frac{32}{3} \Rightarrow 2x\sqrt{16-x^2} = \frac{32}{3}$ ⇒ 2 solutions in [0,4] (maple)

44



$y = 16 - x^2$

$A(x) = 2 \cdot x(16 - x^2)$

$\bar{A} = \frac{1}{4} \cdot \int_0^4 2x(16-x^2)\,dx = \frac{1}{2}\int_0^4 (16x - x^3)\,dx = \frac{1}{2}\left[16x^2/2 - \frac{x^4}{4}\right]_0^4 = \frac{1}{2}\left[8 \cdot 4^2 - \frac{4^4}{4}\right] = 32$

$A(x) = 32 = 2x(16 - x^2) \Rightarrow 32x - 2x^3 - 32 = 0$

$\Rightarrow x = 1.078\ldots$

$x = 3.35\ldots$

45 $f(x) = \int_{-1}^{x} (t^2+1)^{17}\,dt$

f is the definite integral of $(t^2+1)^{17}$ from $t=-1$ to $t=x$.

From the FTC, $f'(x) = (x^2+1)^{17}$

f represents the area under $(t^2+1)^{17}$ from -1 to x.

$(x^2+1)^{17}$ is the slope of f at x, and gives the rate of change of its own area under curve from -1 to x.

changing -1 to s.th. else changes a constant in $f(x)$. f is still one of the infinite antiderivatives of $(t^2+1)^{17}$, so it's derivative is $(t^2+1)^{17}$.

46 $g(t) = \int_0^t \sqrt{x^2+25}\,dx$

$g'(t) = \sqrt{t^2+25}$

47 $h(z) = \int_2^z \sqrt[3]{u-1}\,du$

$h'(z) = \sqrt[3]{z-1}$

48) $A(x) = \int_1^x \frac{1}{t}\,dt$

$A'(x) = \frac{1}{x}$

49) $f(x) = \int_x^{10} (e^t - e^{-t})\,dt = -\int_{10}^{x} (e^t - e^{-t})\,dt$

$f'(x) = -(e^t - e^{-t})$

50 $G(x) = \int_2^x \frac{t}{t^2+1}\,dt \Rightarrow G'(x) = \frac{x}{x^2+1}$

51 $G(x) = \int_0^x \sqrt{t+4}\,dt \Rightarrow G'(x) = \sqrt{x+4}$

52 $G(x) = \int_0^x \sin^3 t\,dt \Rightarrow G'(x) = \sin^3 x$

53 $G(x) = \int_1^x \sqrt{t^3+1}\,dt \Rightarrow G'(x) = \sqrt{x^3+1}$

54 $f(x) = \int_0^{x^2} \sqrt{1+t^3}\,dt$

$u(x) = x^2$

$f(x) = g(u) = \int_0^u \sqrt{1+t^3}\,dt$

$f'(x) = D_x g(u) = g'(u) \cdot u'(x) = \sqrt{1+u^3} \cdot 2x = \sqrt{1+x^6} \cdot 2x$

55 $f(x) = \int_2^{3x} \sin(t^2)\,dt$

$u(x) = 3x,\ u'(x) = 3$

$g(u(x)) = \int_2^u \sin(t^2)\,dt \Rightarrow g'(u) = \sin(u^2)$

$f'(x) = \frac{dg}{dx} = g'(u) \cdot u'(x) = \sin(u^2) \cdot 3 = 3\sin(9x^2)$

56 $f(x) = \int_0^{\sin x} \sqrt{1-t^2}\,dt$

$u = \sin x$

$f'(x) = g(u(x)) = \int_0^u \sqrt{1-t^2}\,dt,\ g'(u) = \sqrt{1-u^2}$

$\frac{dg}{dx} = g'(u) \cdot u'(x) = \sqrt{1-\sin^2 x} \cdot \cos x = |\cos x| \cos x$

61 $\frac{dy}{dx} = \frac{1}{x} \quad y(1) = 0$

Let $y(x) = \int_1^x \frac{1}{x}\,dx$. Then, $y(1) = 0$ and by FTC, $y'(x) = \frac{1}{x}$.

So, $y(x) = \int_1^x 1/x\,dx$ is a solution.

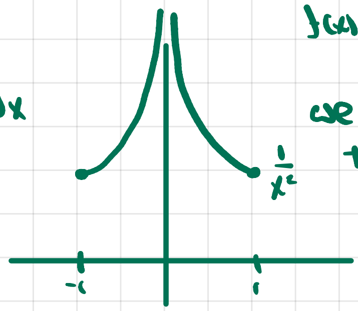
62 $\frac{dy}{dx} = \frac{1}{1+x^2} \quad y(1) = \frac{\pi}{4}$. $y(x) = \frac{\pi}{4} + \int_1^x \frac{1}{1+x^2}\,dx \Rightarrow y(1) = \frac{\pi}{4},\ y' = \frac{1}{1+x^2}$

63 $\frac{dy}{dx} = \sqrt{1+x^2}$ $y(5)=10$. $y(x) = 10 + \int_5^x \sqrt{1+x^2}\,dx \Rightarrow y(5)=10,\ y' = \sqrt{1+x^2}$

64 $\frac{dy}{dx} = \tan x$ $y(1)=2$. $y(x) = 2 + \int_1^x \tan x\,dx \Rightarrow y(1)=2,\ y' = \tan x$

65 $\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{1} = -2$

$\int_{-1}^{1} x^{-2}\,dx$



$f(x) = \frac{1}{x^2}$ is not integrable on $[-1,1]$ because it is not continuous

we cannot put a bound on the limit of a Riemann sum on this interval. the element of the Riemann sum for the subinterval containing $x=0$ can be made arbitrarily large by choosing $f(x_i^*)$ arbitrarily large, as $f(x_i^*)$ grows unbounded as $x_i \to 0$.

66 f differentiable

$\frac{f(b)-f(a)}{b-a}$ = avg rate of change on $[a,b]$

avg value of $f'(x) = \frac{1}{b-a}\int_a^b f'(x)\,dx = \frac{f(b)-f(a)}{b-a}$

$\int_0^2 2t + \int_2^t 8-2t$

$4 + (8t-t^2)\big|_2^t - (16-4)$

$4 + 8t - t^2 - 12$

$-8 + 8t - t^2$

67 $y = f(x) = \begin{cases} 2x & 0 \le x \le 2 \\ 8-2x & 2 \le x \le 6 \\ -4 & 6 \le x \le 8 \\ 2x-20 & 8 \le x \le 10 \end{cases}$

$g(x) = \int_0^x f(t)\,dt$

$g(x) = \begin{cases} x^2 & 0 \le x \le 2 \\ -x^2+8x-8 & 2 \le x \le 6 \end{cases}$

a) $[0,2] \Rightarrow g(x) = \int_0^x 2x\,dx = x^2$

$[2,6] \Rightarrow g(x) = \int_0^2 2x\,dx + \int_2^x (8-2x)\,dx = x^2\big|_0^2 + (8x-x^2)\big|_2^x = 4 + 8x - x^2 - (16-4) = -x^2+8x-8$

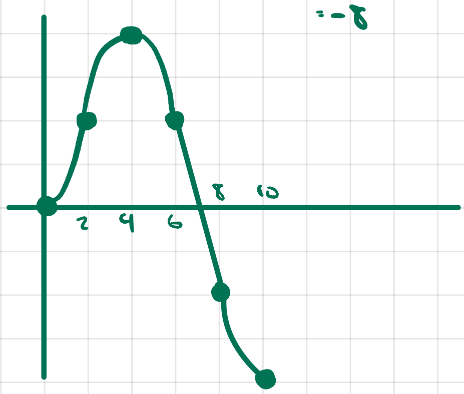
$[6,8] \Rightarrow g(x) = \int_0^6 f(x)\,dx + \int_6^x f(x)\,dx = g(6) + \int_6^x (-4)\,dx = (-36+48-8) - 4x\big|_6^x = 4 + (-4x) - (-24) = -4x+28$

$[8,10] \Rightarrow g(x) = \int_0^8 f(x)\,dx + \int_8^x f(x)\,dx = g(8) + \int_8^x (2x-20)\,dx = -32+28 + (x^2-20x)\big|_8^x = -4 + x^2 - 20x - (64-160)$

$= x^2 + 92 - 20x$

$\Rightarrow g(0)=0,\ g(2)=4,\ g(4) = -16+32-8 = 8,\ g(6) = -24+28 = 4,\ g(8) = -32+28 = -4,\ g(10) = 100+92-200$
$= -8$

b) increasing: $[0,4]$
decreasing: $[4,10]$



c) $g'(x) = f(x)$. f not diff. at $x = 2,6,8$. Endpoints at $x=0,10$

$g'(x)=0 \Rightarrow \begin{cases} 2x=0 \Rightarrow x=0 \\ x=4 \\ x=10 \end{cases}$ $\Rightarrow x=4$ critical point.

$g(0)=0$
$g(10)=-8$
$g(4)=8$ → global max
$g(2)=4$

69 $f(x) = x\sin x \quad [0, 4\pi]$

$$g(x) = \int_0^x f(t)dt = \int_0^x x\sin x\,dx$$

a)

critical points in $(0, 4\pi)$

$g'(x) = f(x) = 0 = x\sin x \Rightarrow x = 0$

$\sin(x) = 0 \Rightarrow x = \pi$, $x = 2\pi$, $x = 3\pi$, $x = 4\pi$

critical points: $\pi, 2\pi, 3\pi$

boundary points: $0, 4\pi$

From the graph: local maxima at $x = \pi, x = 3\pi$

" minima at $x = 2\pi$

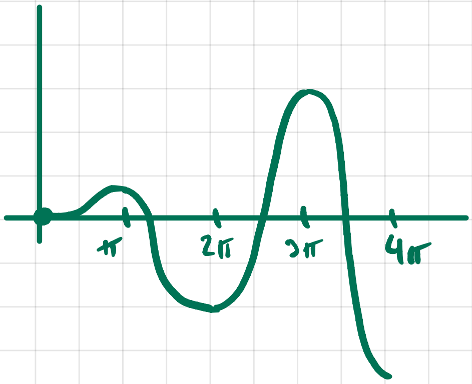
b) global max at $x = 3\pi$ because $g(3\pi) = \int_0^{3\pi} f(x)dx = \underbrace{\int_0^{\pi} f(x)dx}_{>0} + \underbrace{\int_0^{2\pi} f(x)dx}_{<0} + \underbrace{\int_{2\pi}^{3\pi} f(x)dx}_{>0}$

global min at $x = 4\pi$

$> g(\pi) > g(2\pi) > g(4\pi)$ simply from inspection of relative sizes of areas under the curve

c) x where $f''(x) = 0$, the critical points of $f(x)$, roughly $x = 2, x = 5, x = 8$

d)



$$g(x) = \int_0^x x\sin x\,dx = -\cos x \cdot x\Big|_0^x + \int_0^x \cos x\,dx = -x\cos x\Big|_0^x + \sin x\Big|_0^x = -x\cos x + \sin x$$

$u = x \quad du = dx$

$dv = \sin x\,dx$

$v = -\cos x$

$g(0) = 0$

$g(\pi) = -\pi \cdot (-1) = \pi$

$g(2\pi) = -2\pi(1) = -2\pi$

$g(3\pi) = -3\pi(-1) = 3\pi$

$g(4\pi) = -4\pi(1) = -4\pi$

70 $f(x) = \frac{\sin x}{x}$

$g(x) = \int_0^x f(x)\,dx$

a) $g'(x) = f(x) = \frac{\sin x}{x} = 0 \Rightarrow \sin x = 0 \Rightarrow x = \pi$
$x = 2\pi$
$x = 3\pi$

boundary points: $x = 0, x = 4\pi$

local maxima at: $x = \pi$, $x = 3\pi$

" minima at: $x = 2\pi$

b) global max: $x = \pi$

" min: $x = 0$

c) inflections at roughly $x = 4.5$, $x = 7.5$, $x = 11$

d)