7.5 Rational Functions and Partial Fractions

→ methods to integrate any rational function in terms of elementary functions

$R(x) = \frac{P(x)}{Q(x)}$ → quotient of two polynomials

→ method of partial fractions: algebraic technique that decomposes $R(x)$ into a sum of terms:

$$R(x) = \frac{P(x)}{Q(x)} = \underbrace{p(x)}_{\text{polynomial}} + F_1(x) + \dots + F_k(x)$$ → partial fraction decomposition of $R(x)$

F_i is a fraction that can be integrated easily

In advanced algebra, there is a theorem that every rational function can be decomposed as above, with each F_i being a fraction either of form

$$\frac{A}{(ax+b)^n} \quad \text{or} \quad \frac{Bx+C}{(ax^2+bx+c)^n}, \quad A, B, C, a, b, c \text{ constants.}$$

partial fractions

ax^2+bx+c: irreducible quadratic polynomial: not a product of linear factors w/ real coeff.

ie, $ax^2+bx+c=0$ has no real roots

Finding the partial fractions decomposition

1 Find $p(x)$

→ if $\deg(P) < \deg(Q)$, then $p(x) \equiv 0$. In this case, $R(x)$ is said to be proper.

→ if $\deg(P) \geq \deg(Q)$, ie $R(x)$ not proper, then divide $Q(x)$ into $P(x)$. This yields a sum of $p(x)$ and proper rational function.

2 Factor denominator $Q(x)$ into product of linear factors and irreducible quadratic factors

→ possible in principle, may be difficult in practice

3 Write out the partial fractions based on the linear factors of $Q(x)$

$R(x) = \frac{P(x)}{Q(x)}$ proper rational function

linear factor $ax+b$ occurs n times in factorization of Q, ie $(ax+b)^n$ is highest power of $(ax+b)$ that divides evenly into $Q(x)$

multiplicity of factor $ax+b$ is n

⇒ partial fractions for $ax+b$ factor:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_n}{(ax+b)^n}$$

$A_1, \dots, A_n$ constants

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4 Write out partial fractions based on quadratic factors

$R(x) = \frac{P(x)}{Q(x)}$ proper rational function

multiplicity n

irreducible ax^2+bx+c occurs n times in Q's factorization

partial fractions for ax^2+bx+c factor

$$\Rightarrow \frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \dots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$