PSet 6

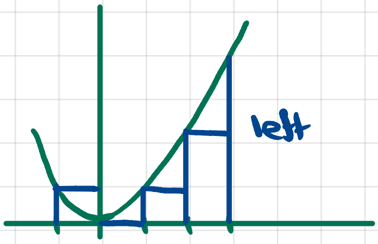
3B-2

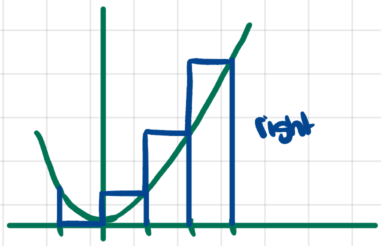
a) $3-5+7-9+11-13 = \sum_{i=1}^{6}(2i+1)\cdot(-1)^{i+1}$

b) $1+\frac{1}{4}+\frac{1}{9}+\dots+\frac{1}{n^2} = \sum_{i=1}^{n}\frac{1}{i^2}$

3B-3

b) $\int_{-1}^{3} x^2\,dx$

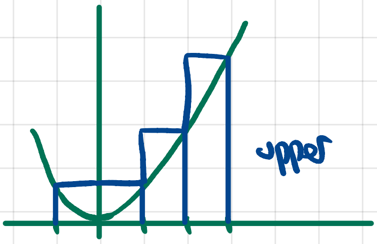




Right Riemann Sums:

$\Delta x_i = \frac{4}{n}$, $x_i = -1+i\Delta x_i = -1+\frac{4}{n}i$, $f(x_i) = (-1+\frac{4i}{n})^2$

$\sum_{i=1}^{4} x_i^2\cdot\Delta x_i = \sum_{i=1}^{4}(-1+\frac{4i}{4})^2\cdot\frac{4}{4} = \sum_{i=1}^{4}(-1+i)^2 = 0^2+1^2+2^2+3^2 = 14$



Left Sums:

$x_i = -1+(i-1)\cdot\frac{4}{4} = -1+i-1 = i-2$

$\sum_{i=1}^{4}(i-2)^2 = (-1)^2+0^2+1^2+2^2 = 6$

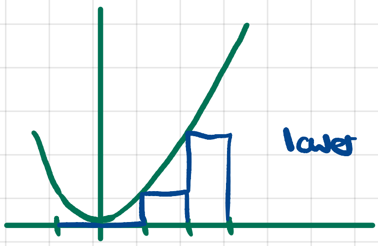
Upper Sums

$x_i^* = f^{-1}(\max \text{ of } f \text{ in } [x_{i-1}, x_i]) = \begin{cases} -1+\Delta x\cdot i & \text{if } i>1 \\ -1+\Delta x(i-1) & \text{if } i=1 \end{cases}$

$[-1+\frac{4}{4}(1-1)]^2\cdot\frac{4}{4} + \sum_{i=2}^{4}(-1+\frac{4}{4}\cdot i)^2\cdot\frac{4}{4}$

$= (-1)^2+\sum_{i=2}^{4}(-1+i)^2 = (-1)^2+(1)^2+(2)^2+(3)^2 = 15$

Lower Sums



$\sum_{i=1}^{4} f(x_i^*)\Delta x \qquad f(x_i^*) = \begin{cases} -1+\Delta x\cdot i & i=1 \\ -1+\Delta x(i-1) & i>1 \end{cases}$

$= (-1+1)^2+(-1+1)^2+(-1+2)^2+(-1+3)^2 = 0+0+1+2^2 = 5$

3B-4

a) $\int_0^b x^2 dx$

lower sum: $\sum \left[\frac{b}{n}(i-1)\right]^2 \cdot \frac{b}{n} = \left(\frac{b}{n}\right)^3 \cdot \sum (i-1)^2$

upper sum: $\sum \left(\frac{b}{n} i\right)^2 \cdot \frac{b}{n} = \left(\frac{b}{n}\right)^3 \sum i^2$

$\Rightarrow$ upper − lower $= \left(\frac{b}{n}\right)^3 \cancel{\sum} i^2 - \left(\frac{b}{n}\right)^3 \left(\cancel{\sum} i^2 - 2\sum i + \sum 1\right)$

$= \left(\frac{b}{n}\right)^3 \left(\cancel{2}\,\frac{n(n+1)}{\cancel{2}} - n\right) = \left(\frac{b}{n}\right)^3 \left(n^2 + \cancel{n} - \cancel{n}\right)$

$= \frac{b^3}{n} \longrightarrow 0$ as $n \to \infty$

$\sum_{i=1}^{n} f(x_i^*) \Delta x$

$\Delta x = \frac{b}{n}$

$x_i^* = \begin{cases} \text{lower sum: } \frac{b}{n}(i-1) \\ \text{upper sum: } \frac{b}{n} i \end{cases}$

3B-5

$$\lim_{n\to\infty} \frac{\sin(b/n) + \sin(2b/n) + \dots + \sin((n-1)b/n) + \sin(nb/n)}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{ib}{n}\right) \cdot \frac{1}{n} = \int_0^1 \sin(bx)\,dx = -\frac{\cos(bx)}{b}\Big|_0^1 = -\frac{\cos(b)}{b} - \left(-\frac{\cos 0}{b}\right) = \frac{-\cos b + 1}{b}$$

$\Delta x = \frac{1}{n}$ $\quad x_i = i \cdot \frac{1}{n}$ $\quad f(x_i) = \sin\left(i \cdot \frac{1}{n} \cdot b\right) = \sin(bx_i)$

3C-1

$y = \frac{1}{\sqrt{x-2}}$, $x \in [3,6]$

$$\int_3^6 (x-2)^{-\frac{1}{2}} dx = 2(x-2)^{1/2} \Big|_3^6 = 2(\sqrt{4} - \sqrt{1}) = 2$$

3C-2 a) $\int_0^2 \sqrt{3x+5}\,dx = \frac{1}{3}(3x+5)^{3/2} \cdot \frac{2}{3} \Big|_0^2 = \frac{2}{9}(3x+5)^{3/2} \Big|_0^2 = \frac{2}{9}\left[\sqrt{11^3} - \sqrt{5^3}\right]$

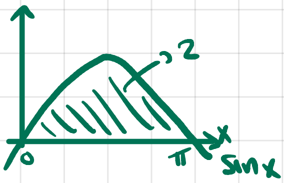
3C-3 a) $\int_1^2 \frac{x}{x^2+1} dx = \frac{1}{2}\ln(x^2+1) \Big|_1^2 = \frac{1}{2}[\ln 5 - \ln 2] = \frac{\ln 5/2}{2}$

$u = x^2+1$

$du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2} du$

$\frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C = \frac{1}{2}\ln(x^2+1) + C$

3C-5 a)



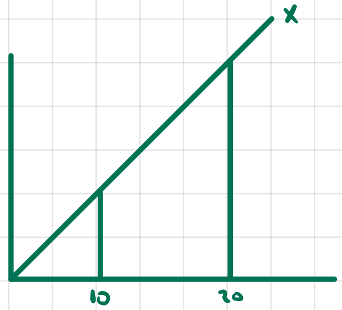
$\int_0^\pi \sin(x)\,dx = -\cos(x)\Big|_0^\pi = (-\cos \pi) - (-\cos 0) = -(-1) - (-1) = 1 + 1 = 2$

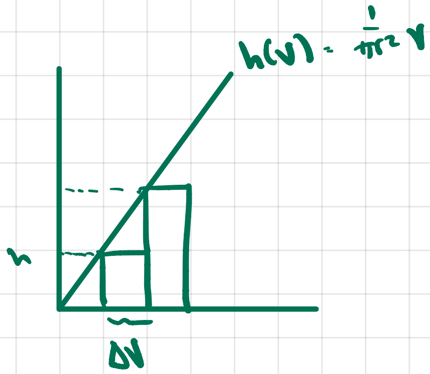
3E-6 b) $\int_0^\pi \sin^2 x\,dx$. We know $\int_0^\pi \sin x\,dx = 2$. Also, $x \in [0,\pi] \Rightarrow \sin x \in [0,1] \Rightarrow \sin^2 x \le \sin(x)\ \forall x \in [0,1]$

$\Rightarrow \int_0^\pi \sin^2 x\,dx < \int_0^\pi \sin x\,dx$

c) $\int_{10}^{20} (x^2+1)^{1/2}\,dx > 150$

$\int_{10}^{20} x\,dx = \frac{x^2}{2}\Big|_{10}^{20} = \frac{1}{2}(400-100)$
$= 150$

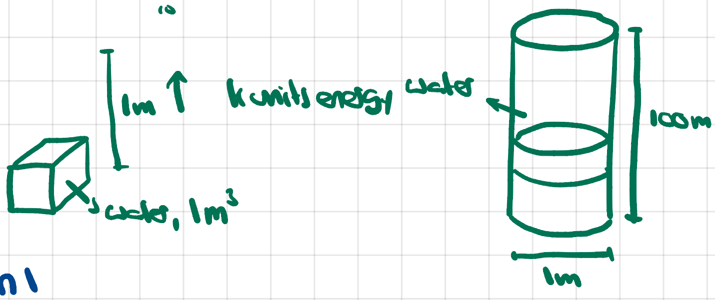




In $[10,20]$, $(x^2+1)^{1/2} > x$.

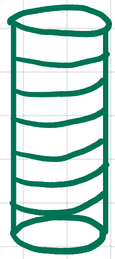
$\Rightarrow \int_{10}^{20} (x^2+1)^{1/2}\,dx > \int_{10}^{20} x\,dx = 150$

4J-1



Solution 1

Divide the cylinder into n parts



i^{th} part has volume: $\pi \cdot \left(\frac{1}{2}\right)^2 \cdot \Delta h_i$

Energy to pump water out: $k \cdot h_i^* \cdot \pi \left(\frac{1}{2}\right)^2 \Delta h_i = \frac{k h_i^* \pi}{4} \Delta h_i$

Energy to pump all water $\approx \sum_{i=1}^{n} \frac{k h_i^* \pi}{4} \Delta h_i$. We could choose a regular partition with $\Delta h_i = \frac{100}{n}$ and use right sums with $h_i^* = i \cdot \Delta h_i = i \cdot \frac{100}{n}$. This is a Riemann sum and $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{k h_i^* \pi}{4} = \int_0^{100} \frac{k h \pi}{4}\,dh$

$= \frac{k\pi}{4} \frac{h^2}{2}\Big|_0^{100} = \frac{10^4 k\pi}{8}$

Solution 2

volume to depth h: $\pi r^2 \cdot h$

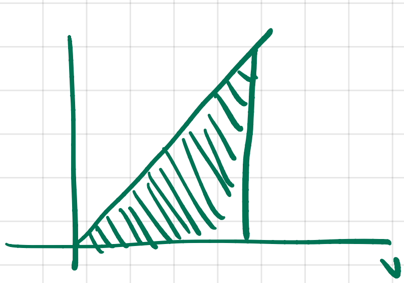
depth given volume: $h(v) = \frac{v}{\pi r^2}$, $v \in [0, \pi r^2 H]$

energy to take volume v up: $k \cdot h(v)$

The area under $kh(v)$ is the limit of sums of form $\Delta v \cdot kh(v)$, the energy to take volume Δv up a depth of $h(v)$. Intuitively, the limit of this sum should tell us the energy to pump all the water

$\int_0^{\pi r^2 H} \frac{kv}{\pi r^2}\,dv = \frac{k}{\pi r^2}\left[\frac{1}{2}v^2\right]_0^{\pi r^2 H} = \frac{k}{2\cancel{\pi r^2}} \cdot \pi^{\cancel{2}} r^{\cancel{4}\,2} H^2 = \frac{k\pi r^2 H^2}{2}$

with $r = 1/2$, $H = 100$, $= \frac{k\pi}{2} \cdot \frac{1}{4} \cdot 10^4 = \frac{10^4 k\pi}{8}$

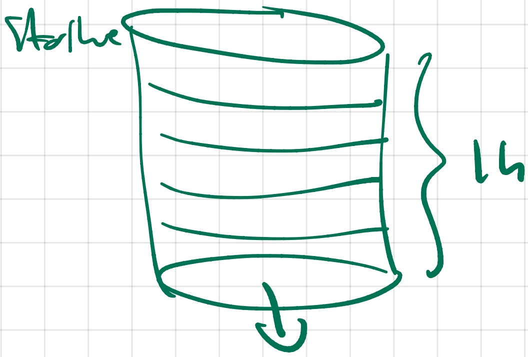


4J-2

$x = x_0 e^{-kt}$ grams at t minutes, exponential decline

1 gram material → r units radiation/minute

x_0 grams, 1 hour, how much radiation produced?



cut 1 h into n units Δt

grams material in i^{th} unit = $x_0 e^{-kt_i}$

radiation in i^{th} unit = $(x_0 e^{-kt_i}) \cdot r \cdot \Delta t$

sum of radiation $= \sum_{i=1}^{n} (x_0 e^{-kt_i} \cdot r \cdot \Delta t)$ $\quad t_i = \Delta t \cdot i, \; \Delta t = \frac{T}{n}$

$= \sum_{i=1}^{n} x_0 e^{-k\frac{T}{n}i} \cdot r \cdot \frac{T}{n}$ → Riemann sum

$\int_0^{60} x_0 e^{-kt} \cdot r \, dt$

$x_0 r \int_0^{60} e^{-kt} = x_0 r e^{-kt} \left(\frac{1}{-k}\right)\Big|_0^{60}$

$= \frac{x_0 r}{-k}\left(e^{-k60} - e^0\right) = -\frac{x_0 r}{k}\left(e^{-60k} - 1\right)$