# Unit 4 - Applications of Integration

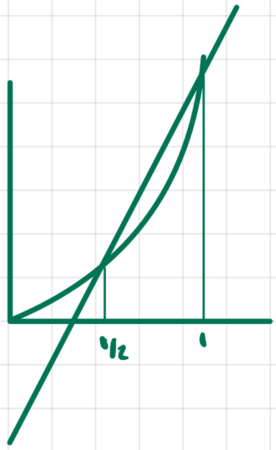
## 4A-1

a) $y=2x^2 \quad y=3x-1$

$2x^2=3x-1 \rightarrow 2x^2-3x+1=0$

$\Delta=9-4\cdot2\cdot1$

$x=\frac{3\pm1}{4} < \begin{matrix}1\\1/2\end{matrix}$



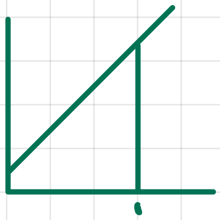

$\int_{1/2}^{1}(3x-1-2x^2)\,dx$

$=\frac{3x^2}{2}-x-\frac{2}{3}x^3\Big|_{1/2}^{1}=\frac{3}{2}-1-\frac{2}{3}-\left(\frac{3}{8}-\frac{1}{2}-\frac{2}{3}\cdot\frac{1}{8}\right)$

$=\frac{1}{2}-\frac{2}{3}-\left(\frac{3}{8}-\frac{4}{8}-\frac{2}{24}\right)=-\frac{1}{6}-\left(\frac{-3}{24}-\frac{2}{24}\right)=-\frac{1}{6}+\frac{5}{24}=\frac{1}{24}$

## 4F-1

a) $y=5x+2 \quad x\in[0,1] \qquad y'=5$




$\int_0^1\sqrt{1+25}\,dx=\sqrt{26}$

## 4F-2

$y=\frac{e^x+e^{-x}}{2} \quad x\in[0,b]$

$y'=\frac{1}{2}[e^x-e^{-x}]$

$\int_0^b\left[1+\left(\frac{e^x-e^{-x}}{2}\right)^2\right]^{1/2}dx=\int_0^b\left[\left(\frac{e^x+e^{-x}}{2}\right)^2\right]^{1/2}dx=\int_0^b\frac{e^x+e^{-x}}{2}\,dx=\frac{1}{2}[e^x-e^{-x}]\Big|_0^b$

$=\frac{1}{2}\left[(e^b-e^{-b})-(e^0-e^0)\right]=\frac{1}{2}(e^b-e^{-b})$

## 4F-3

$y=x^2 \quad x\in[0,b] \quad y'=2x$

$\int_0^b\sqrt{1+(2x)^2}\,dx$

4F-4

$x = t^2 \quad y = t^3 \quad t \in [0,2]$

$x' = 2t \quad y' = 3t^2$

$ds = \sqrt{4t^2+9t^4}\,dt = t\sqrt{4+9t^2}\,dt$

$\int_0^2 t\sqrt{4+9t^2}\,dt = \int \frac{1}{18} u^{1/2}\,du = \frac{1}{18}\frac{u^{3/2}}{\frac{3}{2}} = \frac{2}{3\cdot 18} u^{3/2}$

$\Rightarrow u = 4+9t^2$

$du = 18t\,dt$

$= \frac{1}{27}(4+9t^2)^{3/2}\Big|_0^2$

$= \frac{1}{27}\left[(4+36)^{3/2} - 4^{3/2}\right]$

$= \frac{1}{27}\left(40^{3/2} - 8\right)$

* $x = \cos t$
  $y = \sin t$ $\quad t \in [0, 4\pi]$

$\int y\,dx = \int_0^{2\pi} \sin t \cdot (-\sin t)\,dt$

$= -\int_0^{2\pi} \frac{1}{2}(1-\cos(2t))\,dt$

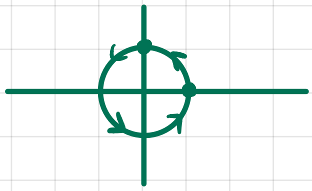
$= -\frac{1}{2}\left[t - \frac{\sin(2t)}{2}\right]\Big|_0^{2\pi}$

$= -\frac{1}{2}\left[(2\pi - 0) - (0-0)\right]$

$= -\frac{1}{2}\cdot 2\pi = -\pi$

should be π, need to integrate from 2π to 0

| t | x | y |
|---|---|---|
| 0 | 1 | 0 |
| | 0 | 1 |



$\int_{4\pi}^{0} y\,dx = -\frac{1}{2}\left[t - \frac{\sin(2t)}{2}\right]\Big|_{4\pi}^{0}$

$= -\frac{1}{2}\left[(0-0) - \left(4\pi - \frac{\sin(8\pi)}{2}\right)\right]$

$= -\frac{1}{2}(-4\pi + 0) = \frac{4\pi}{2} = 2\pi$

double the area of the circle because we went around the circle twice with t.

# Types of Integral Computations

## Area under curve

$$A = \int_a^b y\,dx$$

## Volume by cross section, x-axis revol.

$$V = \sum \Delta V_i \approx \sum A(x_i)\Delta x$$

$$V = \int_a^b \underbrace{A(x)}_{\text{Area function}}\,dx = \int_a^b \pi y^2\,dx$$

## Vol. by Cylindrical Shells, y-axis rev.

$$V = \sum \Delta V_i \approx \sum 2\pi \bar{x}_i f(\bar{x}_i)\Delta x$$

$$V = \int_a^b 2\pi x\, y\,dx$$

## Arc length of curve

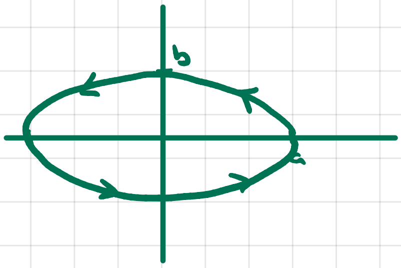
$$\int_0^s ds = \int_a^b \sqrt{1+(y'(x))^2}\,dx$$

## Area of Surface of Revolution, x-axis revol.

$$\int_{x=a}^{b} 2\pi y\,ds$$

4F-7

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , parameterized $x = a\cos t$
$y = b\sin t$



| $t$ | $x$ | $y$ |
| --- | --- | --- |
| $0$ | $a$ | $0$ |
| $\pi/2$ | $0$ | $b$ |
| $\pi$ | $-a$ | $0$ |
| $2\pi$ | $a$ | $0$ |

$x' = -a\sin t$

$y' = b\cos t$

$ds = \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt$

$$\int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt$$

4F-8 $x = e^t\cos t$ $y = e^t\sin t$ $t \in [0, 10]$

$dx = e^t\cos t - e^t\sin t\, dt$
$= e^t(\cos t - \sin t)dt$

$y' = e^t\sin t + e^t\cos t$
$= e^t(\sin t + \cos t)$

$ds = \left[e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2\right]^{1/2} dt$

$= e^t\left[\cos^2 t + \sin^2 t - \cancel{2\sin t\cos t} + \sin^2 t + \cos^2 t + \cancel{2\sin t\cos t}\right]^{1/2}$

$= e^t[2]^{1/2}$

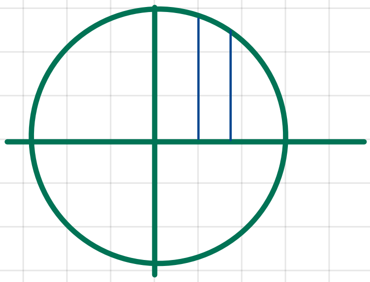
$$\int_0^{10} e^t\sqrt{2}\, dt = \sqrt{2}\, e^t\Big|_0^{10} = \sqrt{2}\,(e^{10} - 1)$$

## 4G-1

$x^2+y^2=R^2$

rev. around x-axis

$-R\le a<b\le R$



$2x+2yy'=0$

$ds=\sqrt{1+\frac{x^2}{R^2-x^2}}\,dx=\sqrt{\frac{R^2}{R^2-x^2}}\,dx$

$y'=-\frac{x}{y}=\frac{-x}{\sqrt{R^2-x^2}}$

$y=\pm\left[R^2-x^2\right]^{1/2}$

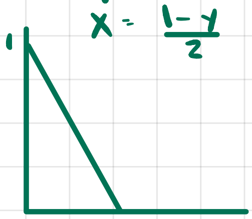
$$\int_a^b 2\pi\cdot y\cdot\sqrt{1+(y')^2}\,dx=2\pi\int_a^b\sqrt{R^2-x^2}\sqrt{\frac{R^2}{R^2-x^2}}\cdot dx=2\pi\int_a^b R\,dx=2\pi R(b-a)$$

## 4G-3

$y=1-2x$

$x=\frac{1-y}{2}$

$ds=\left[1-(1/2)^2\right]^{1/2}=\left[3/4\right]^{1/2}$



$$\int_0^1 2\pi x\cdot\sqrt{3/4}\,dy=\int_0^1 \cancel{2}\pi\frac{(1-y)}{\cancel{2}}\cdot\frac{\sqrt{3}}{2}\,dy=\frac{\sqrt{3}\pi}{2}\int_0^1(1-y)\,dy=\frac{\pi\sqrt{3}}{2}\left(y-\frac{y^2}{2}\right)\Big|_0^1$$

$$=\frac{\pi\sqrt{3}}{2}\left[\left(1-\frac{1}{2}\right)-0\right]=\frac{\pi\sqrt{3}}{4}$$

4G-6 $x^{2/3}+y^{2/3}=a^{2/3}$ rev. x-axis, surface

$$\int_0^s 2\pi y\sqrt{1+y'}\,dx$$

$$y^{\frac{2}{3}}=a^{\frac{2}{3}}-x^{\frac{2}{3}} \Rightarrow y=\left[a^{2/3}-x^{2/3}\right]^{3/2}$$ → domain $[-s,s]$

$$y'=\frac{3}{2}\left[a^{2/3}-x^{2/3}\right]^{1/2}\cdot\frac{2}{3}x^{-1/3}=x^{-1/3}\left[a^{2/3}-x^{2/3}\right]^{1/2}$$

$$ds=\sqrt{1+x^{-\frac{2}{3}}\left[a^{2/3}-x^{2/3}\right]}\,dx$$

$$=\left[1+a^{2/3}x^{-2/3}-1\right]^{1/2}=\left[\frac{a}{x}\right]^{1/3}$$

$$2\cdot\int_0^s 2\pi\cdot\left(a^{2/3}-x^{2/3}\right)^{3/2}\cdot a^{1/3}x^{-1/3}dx$$

$$=4\pi a^{1/3}\int_0^s\left(a^{2/3}-x^{2/3}\right)^{3/2}\cdot x^{-1/3}=4\pi a^{1/3}\left(a^{2/3}-x^{2/3}\right)^{5/2}\cdot\frac{2}{5}\cdot\left(-\frac{3}{2}\right)\Big|_0^s$$

$$=\left(\frac{12\pi}{5}\right)a$$

$$x^{2/3}+y^{2/3}=a^{2/3}$$

$$\cancel{\frac{2}{3}}x^{-\frac{1}{3}}+\cancel{\frac{2}{3}}y^{-1/3}y'=0$$

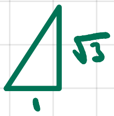
$$y'\cdot y^{-1/3}=-x^{-1/3}$$

$$y'=\left[\frac{y}{x}\right]^{1/3}$$

$$=\frac{\left[a^{2/3}-x^{2/3}\right]^{1/2}}{x^{1/3}}$$

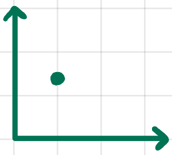
4H-1

a) $(0,3)$ $\quad (3, \pi/2)$




b) $(-2,0)$ $\quad (-2,0)$ or $(2,\pi)$

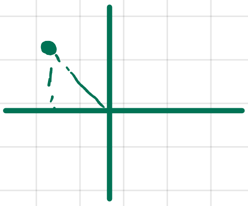
c) $(1,\sqrt{3})$



$r^2 = 1+3 \quad r = 2$

$\tan\theta = \sqrt{3} \quad \theta = \pi/3$

d) $(-2,2)$



$\tan(\theta - \pi) = \frac{2}{-2} = -1$

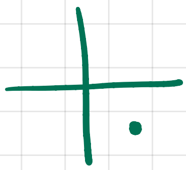
$\theta - \pi = -\pi/4$

$\theta = \frac{3\pi}{4}$

$(2\sqrt{2}, \frac{3\pi}{4})$

$(-2\sqrt{2}, -\pi/4)$

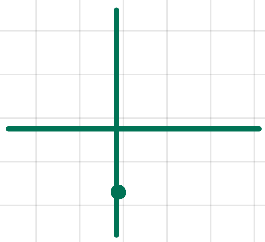
e) $(1,-1)$



$r = \sqrt{1+1} = \sqrt{2}$ $\quad (\sqrt{2}, -\pi/4)$

$\tan\theta = \frac{-1}{1} = -1 \quad \theta = -\pi/4$ $\quad (-\sqrt{2}, 3\pi/4)$

f) $(0,-2)$



$r = 2$ $\quad r = -2$

$\theta = -\pi/2$ $\quad \theta = \pi/2$

44-3

a) $r = \sec\theta \Rightarrow r\cos\theta = 1$

$x = r\cos\theta = 1$

b) $r = 2a\cos\theta$

$r^2 = r \cdot 2a\cos\theta = 2ax$

$x^2 + y^2 = 2ax$