10.4 Taylor Series

Ex 4 $f(x) = e^x \quad a = 0$

$f' = e^x$

$f'' = e^x$

$f^{(n)} = e^x$

$$P_n(x) = e^0 + e^0 x + \frac{e^0}{2}x^2 + \frac{e^0}{3!}x^3 + \cdots + \frac{e^0}{n!}x^n$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

1 $f(x) = e^{-x} \quad n = 5 \quad a = 0$

$f' = -e^{-x}$
$f'' = e^{-x}$
$f''' = -e^{-x}$

$f^{(k)} = (-1)^k e^{-x}$

$$f(x) = e^{-0} + \frac{-e^{-0}}{1!}x + \frac{e^{-0}}{2!}x^2 + \frac{-e^{-0}}{3!}x^3 + \frac{e^{-0}}{4!}x^4 + \frac{-e^{-0}}{5!} + \frac{e^{-z}}{6!}x^6$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{e^{-z}}{6!}x^6 \quad z \in (0, x)$$

Ex 3 $f(x) = \ln(x) \quad a = 1$

$f' = \frac{1}{x} \qquad f^{(k)} = (-1)^{k-1}\frac{(k-1)!}{x^k} \quad k \geq 1$

$f'' = \frac{-1}{x^2}$

$$P_3(x) = \ln(1) + \frac{1/1}{1!}(x-1) - \frac{1/1}{2!}(x-1)^2 + \frac{2/1}{3!}(x-1)^3 = x-1 - \frac{(x-1)^2}{2} + \frac{2(x-1)^3}{6}$$

$f''' = \frac{2}{x^3}$

$$P_3(1.1) = 0.1 - \frac{0.1^2}{2} + \frac{0.1^3}{3} \approx 0.0953$$

$f^{(4)} = \frac{-2 \cdot 3}{x^4}$

The remainder term in a third degree Taylor Formula is $\left(\frac{-3!}{z^4}\right)\Big/_{4!}(x-1)^4$

$x = 1.1 \Rightarrow \frac{-3!}{z^4 \cdot 4!} \cdot 0.1^4$ where $z \in (1, 1.1)$

$f^{(5)} = \frac{2 \cdot 3 \cdot 4}{x^5}$

11 $f(x) = e^x \quad a = 1 \quad n = 4$

$$f(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \frac{f^{(5)}(z)}{5!}(x-1)^5$$

$$= e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \frac{e}{24}(x-1)^4 + \frac{e^z}{5!}(x-1)^5$$

21 $f(x) = e^x$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(k+1)}(z)}{(k+1)!}(x-a)^{k+1}$$, z between a and x

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1} = \frac{e^z}{(n+1)!}(x-a)^{n+1}$$

$a = 0 \Rightarrow \frac{e^z x^{n+1}}{(n+1)!}$ z between 0 and x

$x < 0 \Rightarrow z < 0 \Rightarrow e^z < 1 \Rightarrow 0 < R_n(x) < \frac{|x|^{n+1}}{(n+1)!}$

so as $n \to \infty$ $0 < R_\infty(x) < 0$

$x > 0 \Rightarrow z > 0 \Rightarrow e^z < e^x \Rightarrow 0 < R_n(x) < \frac{e^x |x|^{n+1}}{(n+1)!}$, which also $\to 0$ as $n \to \infty$

so we can write that the Taylor series $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x.

Therefore, given $g(x) = e^{-x}$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

22 $f(x) = e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum 2^n \frac{x^n}{n!} = 1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{3!} + \cdots$

27 $f(x) = \sin(x^2)$

$$\sin(x) = 1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin(x^2) = 1 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

29 $f(x) = \ln(1+x) \quad a=0$

$f' = \frac{1}{1+x} \qquad f'(a) = 1$

$f'' = \frac{-1}{(1+x)^2} \qquad f''(a) = -1$

$f''' = \frac{2}{(1+x)^3} \qquad f'''(a) = 2$

$f^{(4)}(a) = (-1)^3\, 3! = -6$

$f^{(k)} = \frac{(k-1)!}{(1+x)^k} \cdot (-1)^{k-1} \qquad f^{(5)}(a) = (-1)^4\, 4! = 24$

$f^{(k)}(a) = (-1)^{k-1}(k-1)! \qquad k \geq 1$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot x^n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(n-1)!}{n!} \cdot x^n = \sum_{n=0}^{\infty} (-1)^{n-1} \cdot \frac{x^n}{n}$$

$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ = Taylor series of $\ln(1+x)$

Taylor Formula Remainder = $\frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} = \frac{n!}{(1+z)^{n+1}} \cdot (-1)^n \cdot \frac{x^{n+1}}{(n+1)!} = \frac{(-1)^n x^{n+1}}{(1+z)^{n+1}(n+1)}$

$x < |1| \Rightarrow \lim_{n\to\infty} R_n(x)$

$= \lim (-1)^n \frac{x^{n+1}}{n+1} = 0 \Rightarrow$ for $x < |1|$ the Taylor series converges to $f(x)$

40 $f(x) = \sqrt{1+x}\ , \ x \geq -1\ , \ a = 0$

$f' = \frac{1}{2\sqrt{1+x}} = \frac{(1+x)^{-1/2}}{2} \qquad f'' = -\frac{1}{2} \cdot \frac{1}{2}(1+x)^{-3/2} = \frac{-(1+x)^{-3/2}}{4}$

$f''' = -\frac{1}{4}\left(-\frac{3}{2}\right)(1+x)^{-5/2} = \frac{3}{8}(1+x)^{-5/2} \qquad f^{(4)} = \frac{3}{8} \cdot \left(-\frac{5}{2}\right)(1+x)^{-7/2} = -\frac{15}{16}(1+x)^{-7/2}$

$f^{(5)} = \frac{7}{2} \cdot \frac{15}{16}(1+x)^{-9/2} = \frac{105}{32}(1+x)^{-9/2}$

n^{th} Maclaurin coeff = $\frac{(-1)^{n-1} \cdot (1 \cdot 3 \cdot 5 \cdots (2n-3))}{2^n \cdot n!}$

so, $f'(0) = \frac{1}{2}$

multiply and divide by $2 \cdot 4 \cdot 6 \cdots (2n-2) = 2^{n-1}(1 \cdot 2 \cdot 3 \cdots (n-1))$
$= 2^{n-1} \cdot (n-1)!$

$f''(0) = -\frac{1}{4}$

$f'''(0) = \frac{3}{8} = \frac{1 \cdot 3}{2^3} \qquad = \frac{(-1)^{n-1} \cdot (1 \cdot 3 \cdot 5 \cdots (2n-3))}{2^n \cdot n!} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2n-2)}{2^{n-1}(n-1)!}$

$f^{(4)}(0) = \frac{-15}{16} = \frac{-1 \cdot 3 \cdot 5}{2^4} \qquad = \frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}\, n!\,(n-1)!}$

$f^{(5)}(0) = \frac{105}{32} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5}$

(cont'd) $= \dfrac{(-1)^{n-1}(2n-2)!}{2^{2n-1}\,n!\,(n-1)!}$

So the Taylor Formula for $f(x) = \sqrt{1+x}$ has coefficients

x^0 : 1

x^1 $\dfrac{1!}{2 \cdot 1! \, 0!} = \dfrac{1}{2}$

x^2 $\dfrac{-2!}{2^3 \cdot 2! \, 1!} = -\dfrac{1}{8}$

x^3 $\dfrac{4!}{2^5 \, 3! \, 2!} = \dfrac{4}{2 \cdot 32} = \dfrac{1}{16}$

x^4 $\dfrac{-6!}{2^7 \, 4! \, 3!} = \dfrac{-30}{6 \cdot 128} = \dfrac{-5}{128}$

$$1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \dots + \underbrace{\underbrace{\frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot (2(n+1)-3)}{2^{n+1}} \cdot \frac{1}{(1+z)^{(1+2n)/2}}}_{f^{(n+1)}(z)} \cdot \frac{x^{n+1}}{(n+1)!}}_{R_n(x)}$$

z between 0 and x

41 $f(x) = \sin x \quad a = 0$

$f' = \cos x \qquad f'(0) = 1$
$f'' = -\sin x \qquad f''(0) = 0$
$f''' = -\cos x \qquad f'''(0) = -1$
$f^{(4)} = \sin x \qquad f^{(4)}(0) = 0$

$$\sin(x) = 0 + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^{2n-1}}{(2n-1)!} + (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} \cdot f^{(n+1)}(z)$$

$f^{(n+1)}(z)$ is always in $[-1,1]$ so $R_n(x) \to 0$ as $n \to \infty$

42 $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \cdots$

$$= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}(-1)^k$$

$$\frac{d}{dx}\sin x = \sum_{k=0}^{\infty} \frac{(2k+1)\,x^{2k}}{(2k+1)!}(-1)^k = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}(-1)^k = \cos(x)$$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}(-1)^k$$

$$\frac{d}{dx}\cos(x) = \sum_{k=1}^{\infty} \frac{2k\,x^{2k-1}}{(2k)!}(-1)^k = \sum_{k=1}^{\infty} \frac{x^{2k-1}}{(2k-1)!}(-1)^k = -\sin(x)$$

53 $\tan(A+B) = \dfrac{\tan(A) + \tan(B)}{1 - \tan A \tan B}$

$A = \tan^{-1} x$
$B = \tan^{-1} y$

$$\tan(A+B) = \frac{x+y}{1-xy}$$

$$A+B = \tan^{-1}\left[\frac{x+y}{1-xy}\right] \quad xy < 1$$

If we keep adding values like $\tan^{-1}x$ and can obtain an expression $\frac{x+y}{1-xy}$ equal to one, we'll have

$$\tan^{-1}(x) + \tan^{-1}(y) + \tan^{-1}(z) + \cdots = \tan^{-1}(1) = \frac{\pi}{4}$$

and the left side can be computed using the Taylor series for $\tan^{-1}$.

In particular,

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{7}{9}$$

$$\tan^{-1}\frac{7}{9} + \tan^{-1}\frac{1}{8} = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\approx \frac{1}{8} - \frac{1}{8^3 \cdot 3} + \frac{1}{8^5 \cdot 5} - \frac{1}{8^7 \cdot 7} + \frac{1}{8^9 \cdot 9}$$

$$\approx \frac{7}{9} - \frac{7^3}{9^3 \cdot 3} + \frac{7^5}{9^5 \cdot 5} - \frac{7^7}{9^7 \cdot 7} + \frac{7^9}{9^9 \cdot 9}$$