6.4 Arclength, Surface of Revolution

Arclength

smooth arc: graph of smooth function defined on closed interval

smooth function: f on $[a,b]$ with f' continuous on $[a,b]$

no corner points on graph of f

our task is to approximate length s of smooth arc C. To do this, we:

inscribe in C polygonal arc, calculate length of polygonal arc

partition $[a,b]$ into n subintervals Δx

$P_i = (x_i, f(x_i))$ on the arc C

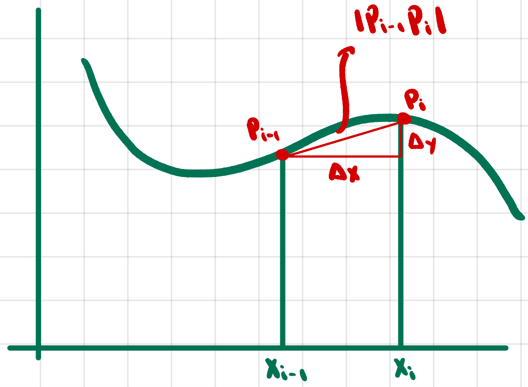
polygonal arc is the union of line segments $P_0P_1, P_1P_2, \ldots, P_{n-1}P_n$

$$s \approx \sum_{i=1}^{n} |P_{i-1}P_i|$$

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

Pythagoras

$$|P_{i-1}P_i| = [(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2]^{1/2}$$



MVT on $[x_{i-1}, x_i]$: there is x_i^* in $[x_{i-1}, x_i]$ s.t.

$$f'(x_i^*) = \frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})}$$

Hence,

$$|P_{i-1}P_i| = (x_i - x_{i-1})\left[1 + \left(\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})}\right)^2\right]^{1/2}$$

$$= \Delta x \sqrt{1 + f'(x_i^*)^2}$$

Riemann sum for $\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

$$\Rightarrow s = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \sqrt{1 + f'(x_i^*)^2}$$

Alternative derivation: take two points $P(x,y)$ and $Q(x+dx, y+dy)$ on the smooth arc C. ds is the length of the arc joining P and Q, which is approx. equal to line segment joining P and Q.

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (dy/dx)^2} \cdot dx = \sqrt{1 + (dx/dy)^2} \cdot dy$$

we then think of the entire arc s as a sum of small pieces ds.

$$s = \int_{*}^{**} ds$$

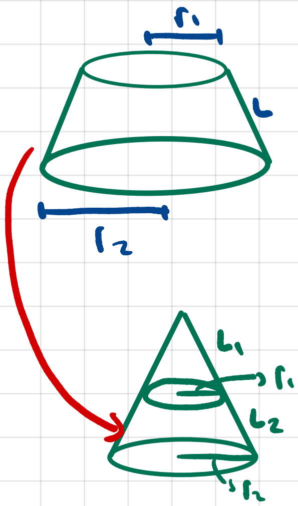
Surface of revolution: surface obtained revolving an arc of curve around an axis that lies in same plane as the arc

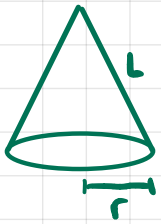
inscribe polygonal arc

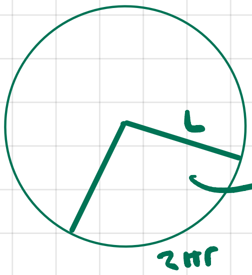
revolve the polygonal arc around axis generating an approximating surface

each revolved piece of the polygonal arc is a section of a cone, so we have a sum of sections of cones.

Let's derive a formula for the surface area of one such cone section:







$A = \frac{2\pi r}{2\pi L} \cdot \pi L^2 = \pi r L$

$A_{LC} =$ Area of lower section of cone $= \pi r_2(L_1+L_2) - \pi r_1 L_1 = \pi L_1(r_2-r_1) + \pi r_2 L_2$

note $\frac{r_1}{L_1} = \frac{r_2}{L_1+L_2} \Rightarrow r_1(L_1+L_2) = L_1 r_2 \Rightarrow r_1L_1 + r_1L_2 - L_1r_2 = 0$

$L_1(r_1-r_2) + r_1L_2 = 0$

$L_1(r_2-r_1) = r_1L_2$

so $A_{LC} = \pi r_1 L_2 + \pi r_2 L_2 = \pi L_2(r_1+r_2) = 2\pi L_2 \cdot \frac{(r_1+r_2)}{2}$

$A_{LC} = 2\pi L_2 \cdot \bar{r}$, $\bar{r} = \frac{r_1+r_2}{2}$

Recap of what we have so far:

$S = \lim_{n\to\infty} \Delta x \sqrt{1+f'(x_i^*)^2} = \int_a^b \sqrt{1+f'(x)^2}\, dx$ = arc length between a and b of $f(x)$.

$A_{LC} = 2\pi L \cdot \bar{r}$, $\bar{r} = \frac{r_1+r_2}{2}$ is surface area of cone section

Given $f(x)$, inscribe polygonal arc. Each piece of this polygonal arc, when revolved around x-axis generates a conic section. We need to find the analytical formula for the surface area of one such conical section, and then sum all the sections.

Now let's get the surface of revolution:

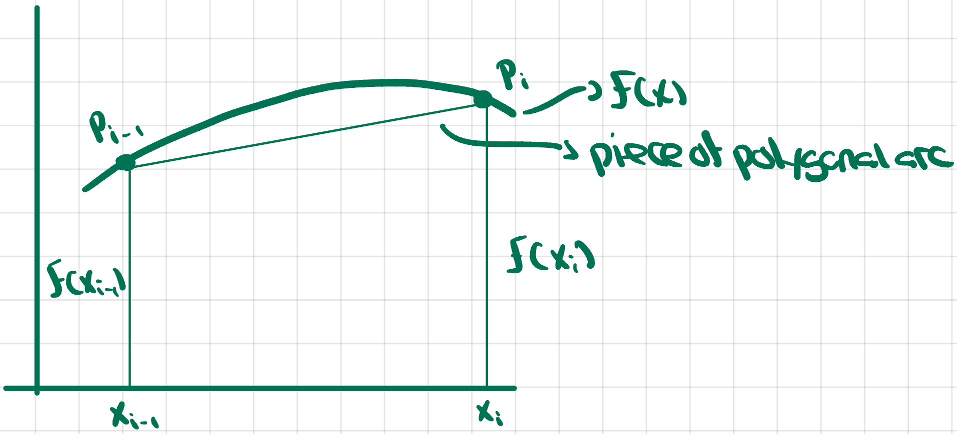
smooth arc $y = f(x)$, $a \le x \le b$, suppose $f(x) > 0$ on $[a,b]$

surface S, area A generated by revolving $f(x)$ around x-axis

approx. of A: divide $[a,b]$ into n subintervals Δx

$$L_i = |P_{i-1}P_i| = \sqrt{1+(f'(x_i^*))^2}\ \Delta x \quad , \ x_i^* \in [x_{i-1}, x_i]$$

revolve $P_{i-1}P_i$ around x-axis: this generates a conical section with slant height L_i and average radius $r_i = \frac{1}{2}[f(x_{i-1}) + f(x_i)]$



in the formula $A_{cc} = 2\pi r L \bar{r}$, L is the length of $P_{i-1}P_i$ which is $\Delta x\sqrt{1+f'(x_i^*)^2}$,

$\bar{r}$ is $\dfrac{f(x_i)+f(x_{i-1})}{2}$. $\bar{r}$ is a value between $f(x_i)$ and $f(x_{i-1})$, so by the intermediate value theorem, $\exists\, x_i^{**}$ s.t. $f(x_i^{**}) = \bar{r}$, with $x_i^{**} \in [x_{i-1}, x_i]$.

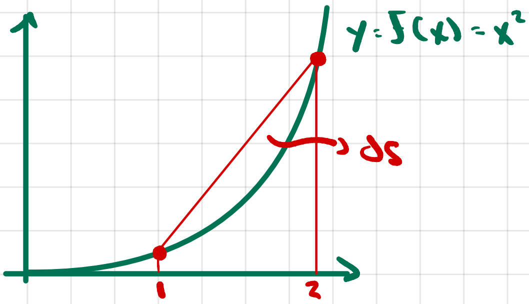
The area of the conical section is therefore $2\pi \bar{r}_i L_i = 2\pi f(x_i^{**}) \cdot \Delta x\sqrt{1+f'(x_i^*)^2}$

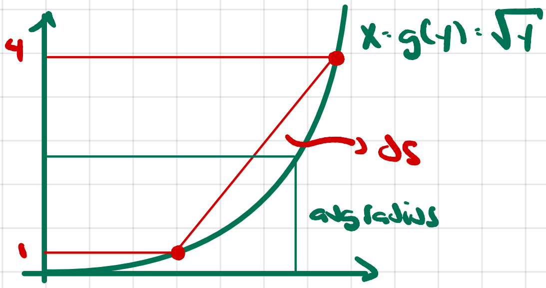
Total area $\approx \sum_{i=1}^{n} 2\pi f(x_i^{**}) \cdot \Delta x\sqrt{1+f'(x_i^*)^2}$

x_i^* and x_i^{**} are not necessarily the same point, but the sum above still approaches the integral $\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$ as $\Delta x \to 0$, and this integral is the definition of the area A of our surface.

To remember this formula it helps to think of it as $A = \int \underbrace{2\pi y\, \overset{\text{arc length}}{ds}}_{\text{conic section surface area}}$ (x-axis)

radius, the function





revolution around x-axis

$$\int 2\pi y\, ds = \int_1^2 2\pi x^2 \cdot \sqrt{1+(2x)^2}\, dx$$

$$= 2\pi \int x^2 \sqrt{1+4x^2}\, dx$$

$$\int 2\pi y\, ds = \int_1^4 2\pi \sqrt{y}\, \sqrt{1+(1/2\sqrt{y})^2}\, dy$$

$$= 2\pi \int_1^4 (y + 1/4)^{1/2}\, dy$$

$$= 2\pi \cdot \frac{2}{3} \cdot (y+1/4)^{3/2} \Big|_1^4$$

$$= (4\pi/3) \cdot \left[(17/4)^{3/2} - (5/4)^{3/2} \right]$$

$$= \frac{\cancel{4}\pi}{3} \left[\frac{17^{3/2} - 5^{3/2}}{\cancel{64}\,16} \right] = \frac{\pi}{48}\left(17^{3/2} - 5^{3/2}\right)$$