Final Exam

1

a) $f(x) = x^3 e^x \Rightarrow f'(x) = 3x^2 e^x + x^3 e^x$

b) $f(x) = \sin(2x) \Rightarrow f'(x) = 2\cos(2x)$

$f''(x) = -4\sin(2x)$

$f'''(x) = -8\cos(2x)$

$f^{(4)} = -8 \cdot (-\sin 2x) \cdot 2$

$= 16\sin(2x)$

$f^{(5)} = 16\cos(2x) \cdot 2 = 32\cos(2x)$

$f^{(6)} = 32(-\sin 2x) \cdot 2 = -64\sin(2x)$

$f^{(7)} = -64\cos(2x) \cdot 2 = -128\cos 2x$

2

a) $y = 3x^2 - 5x + 2 \quad x = 2$

$y' = 6x - 5 \qquad y'(2) = 12 - 5 = 7$

$y(2) = 3 \cdot 4 - 5 \cdot 2 + 2 = 12 - 10 + 2 = 4$

$7 = \dfrac{4-y}{2-x} \Rightarrow 14 - 7x = 4 - y \Rightarrow y = 7x + 4 - 14 \Rightarrow y = 7x - 10$

b) $xy^3 + x^3 y = 4$

$y^3 + x \cdot 3y^2 y' + 3x^2 y + x^3 y' = 0$

horizontal tangent $\Rightarrow y' = 0$

$y^3 + 3x^2 y = 0 \Rightarrow y(y^2 + 3x^2) = 0 \nearrow y = 0$

$\searrow y^2 = -3x^2$

$y = 0$ is not on the curve $y(x)$.

$y^2 = -3x^2$ is only true if $y = x = 0$, but this point is not on $y(x)$.

Hence there are no points with horz. tangent (ie $y'(x) = 0$).

3

a) $\frac{d}{dx}\left(\frac{x}{x+1}\right)$

$f(x) = \frac{x}{x+1}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{x+h}{x+h+1} - \frac{x}{x+1}}{h} = \lim_{h\to 0} \frac{(x+h)(x+1) - x(x+h+1)}{h(x+h+1)(x+1)}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + \cancel{x} + \cancel{hx} + \cancel{h} - \cancel{x^2} - \cancel{hx} - \cancel{x}}{\cancel{h}(x+h+1)(x+1)} = \lim_{h\to 0} \frac{1}{(x+h+1)(x+1)} = \frac{1}{(x+1)^2}$$

b) $\lim_{x\to\sqrt{3}} \frac{\tan^{-1}(x) - \pi/3}{x-\sqrt{3}} = \frac{0}{0} = \lim_{x\to\sqrt{3}} \frac{1/(1+x^2)}{1} = \frac{1}{4}$

$f(x) = \tan^{-1}(x) - \pi/3$ $\qquad$ $g(x) = x - \sqrt{3}$

$f'(x) = 1+x^2$ $\qquad$ $g'(x) = 1$

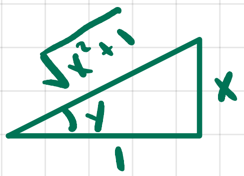
f and g continuous and diff. near $x=\sqrt{3}$, $g(x) \neq 0$ near $\sqrt{3}$.

→ Derivation of $\frac{d}{dx}\tan^{-1}(x)$

$y = \tan^{-1}(x) \Rightarrow \tan(y) = x$

$\sec^2(y)\, y' = 1$

$y' = \frac{1}{\sec^2 y} = \cos^2 y = 1+x^2$



$\cos y = 1/\sqrt{1+x^2}$

$\cos^2(y) = 1/(1+x^2)$

4 $y = \frac{x}{x^2+1}$

max
min
increasing/decreas.
inflection
symmetries
vert. horiz. asymptotes

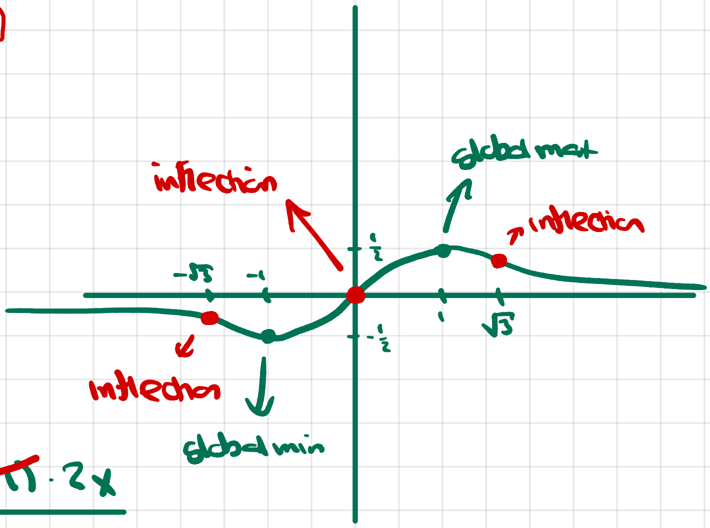
Roots $\boxed{y=0 \Rightarrow x=0}$

Slope $y' = \frac{x^2+1-x\cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

$y'=0 \Rightarrow 1-x^2=0 \Rightarrow x^2=1 \Rightarrow \boxed{x=\pm 1}$

	-1		$+1$	
$\ominus$		$\oplus$		$\ominus$ f'

$f(1) = \frac{1}{2}$ $\quad f(-1) = \frac{-1}{2}$



concavity

$$y'' = \frac{-2x(x^2+1)^{\cancel{2}\,1} - (1-x^2)2\cancel{(x^2+1)}\cdot 2x}{(x^2+1)^{\cancel{4}\,3}}$$

$= [-2x^3-2x-4x(1-x^2)] / (x^2+1)^3$

$= [-2x^3-2x-4x+4x^3] / (x^2+1)^3$

$= (2x^3-6x) / (x^2+1)^3$

$= 2x(x^2-3) / (x^2+1)^3$

$y''=0 \Rightarrow 2x(x^2-3)=0 \nearrow x=0$

$\searrow x^2=3 \Rightarrow x = \pm\sqrt{3}$

		$-\sqrt{3}$		0		$\sqrt{3}$	
x^2-3	$+$		$-$		$-$		$+$
$2x$	$-$		$-$		$+$		$+$
f''	$\ominus$		$\oplus$		$\ominus$		$\oplus$

Asymptotes

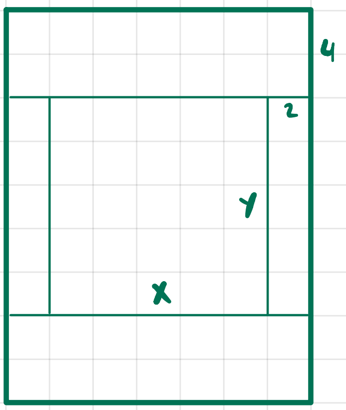
$\lim_{x\to\infty} f(x) = \lim \frac{x}{x^2+1} = \frac{\infty}{\infty} = \lim \frac{1}{2x} = 0$

$\lim_{x\to-\infty} f(x) = \frac{-\infty}{\infty} = 0$

5 50 in^2 printed type

4 in margin top, bottom

2 in margin sides



$xy = 50 \quad y = 50/x$

$A = (4+x)(8+y) = 32 + 4y + 8x + xy$

$= 32 + 200/x + 8x + 50$

$= 82 + \frac{200}{x} + \frac{8x^2}{x}$

$= \frac{8x^2 + 82x + 200}{x}$

$A'(x) = \frac{(16x+82)x - (8x^2+82x+200)}{x^2}$

$= (16x^2 + \cancel{82x} - 8x^2 - \cancel{82x} - 200)/x^2$

$= [8x^2 - 200]/x^2$

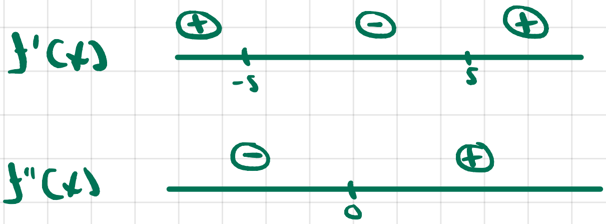
$A''(x) = [16x \cdot x^2 - 2x(8x^2 - 200)]/x^4$

$= (\cancel{16x^3} - \cancel{16x^3} + 400x)/x^4$

$= 400x/x^4 = 400/x^3$

$A'(x) = 0 \Rightarrow 8x^2 - 200 = 0$

$x^2 = 25 \Rightarrow x = \pm 5$



$x \in [0, \infty)$

At critical point, $x = 5$, $y = 10$, $A = 9 \cdot 18 = 729$, a local minimum

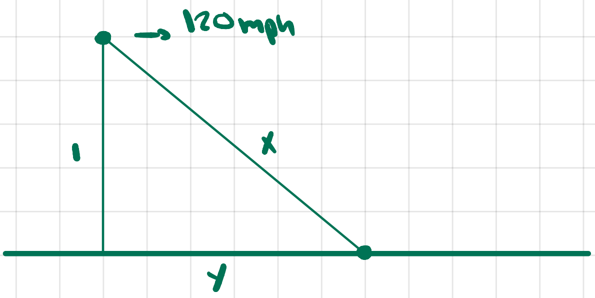
check boundaries of domain:

$x = 0 \Rightarrow y = +\infty \Rightarrow A = +\infty$

$x = \infty \Rightarrow y = 0 \Rightarrow A = +\infty$

$\Rightarrow x = 5$ is global minimum on $x \in [0, \infty)$.

6



At t^*, $x = 1.5$

$x' = -136$

$y^2 + 1 = (3/2)^2 \Rightarrow y^2 = \frac{9}{4} - 1 = \frac{5}{4} \Rightarrow y = \frac{\sqrt{5}}{2}$

$x^2 = 1 + y^2$

$2xx' = 2yy'$

$xx' = yy'$

$1.5 \cdot (-136) = \frac{\sqrt{5}}{2} \cdot y' \Rightarrow y' = \frac{3 \cdot (-136)}{\sqrt{5}} = \frac{-408}{\sqrt{5}} \approx -185.4$

$\frac{-408}{\sqrt{5}} = 120 - v \Rightarrow v = 120 - 185.4 = -65.4$ mph

7

a) $\lim_{n\to\infty} \sum_{i=1}^{n} \left[\sqrt{1+\frac{2i}{n}}\right]\frac{2}{n} = \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\sqrt{1+\frac{2i}{n}}$

$= \lim_{n\to\infty} \frac{2}{n}\left[\sqrt{1+\frac{2}{n}} + \sqrt{1+\frac{4}{n}} + \sqrt{1+\frac{6}{n}} + \cdots + \sqrt{1+\frac{2(n-1)}{n}} + \sqrt{3}\right]$

Looks like a Riemann sum.

$\Delta x = \frac{2}{n} = \frac{3-1}{n}$

$f(x_i) = \sqrt{1+i\Delta x}$

$x_i = 1+i\Delta x$

$\Rightarrow \int_1^3 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_1^3 = \frac{2\sqrt{x^3}}{3}\Big|_1^3 = \frac{2}{3}\left[\sqrt{27}-1\right] = 2\sqrt{3}-\frac{2}{3}$

b) $\lim_{h\to 0} \frac{1}{h}\int_2^{2+h} \sin(x^2)\,dx$

$= \sin(4)$

8

a) $\int_0^{\pi/4} \tan x\sec^2 x\,dx = \int u\,du = \frac{u^2}{2} = \frac{\tan^2 x}{2}\Big|_0^{\pi/4} = \frac{1}{2}\left(\tan\frac{\pi}{4} - \tan 0\right) = \frac{1}{2}$

$u = \tan x$

$du = \sec^2 x\,dx$

$\frac{1}{2}\int_1^2 x\,dx = \frac{1}{2}\frac{x^2}{2} = \frac{x^2}{4}$

b) $\int_1^2 x\ln x\,dx$ $= \frac{x^2\ln x}{2}\Big|_1^2 - \int_1^2 \frac{\cancel{x^2}}{2}\cdot\frac{1}{\cancel{x}}\,dx = \left[\frac{x^2\ln x}{2} - \frac{x^2}{4}\right]\Big|_1^2$

$u = \ln x \quad du = \frac{dx}{x}$

$dv = x\,dx \quad v = \frac{x^2}{2}$

$= \frac{4\ln 2}{2} - \left[\frac{4}{4} - \frac{1}{4}\right] = 2\ln 2 - \frac{3}{4}$

9

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot \cancel{3\cos\theta}\,d\theta}{\cancel{3\cos\theta}} = 9\int \sin^2\theta\,d\theta = \frac{9}{2}\int (1-\cos 2\theta)\,d\theta$$

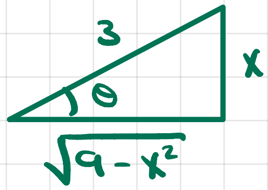
$x = 3\sin\theta$

$x^2 = 9\sin^2\theta$

$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)}$

$= \sqrt{9\cos^2\theta} = 3\cos\theta$

$dx = 3\cos\theta\,d\theta$



$\sin\theta = \frac{x}{3}$

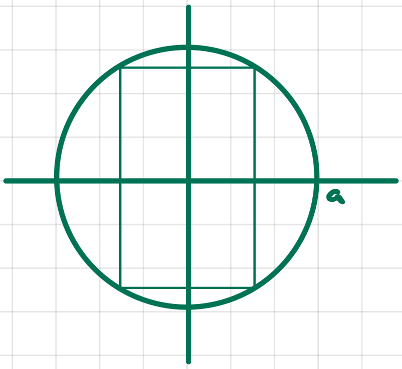
$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$$= \frac{9}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right]$$

$$= \frac{9}{2}\left[\theta - \sin\theta\cos\theta\right]$$

$$= \frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right] + C$$

10 $x^2+y^2=a^2$ $\qquad x=\frac{a}{2} \Rightarrow y^2=a^2-\frac{a^2}{2}=\frac{a^2}{2} \Rightarrow y=\pm\frac{a}{\sqrt{2}}$



$y=\pm\sqrt{a^2-x^2}$

$$\int_{a/2}^{a} 2\pi x\cdot\left[\sqrt{a^2-x^2}-\left(-\sqrt{a^2-x^2}\right)\right]dx = 2\pi\int_{a/2}^{a} x\cdot 2\sqrt{a^2-x^2}\,dx$$

$u=a^2-x^2$
$du=-2x\,dx$

$$\int x\cdot 2\sqrt{a^2-x^2}\,dx=\int -\frac{1}{2}\cdot 2\sqrt{u}\,du=-\int u^{1/2}du$$

$$=-\frac{2}{3}u^{3/2}$$

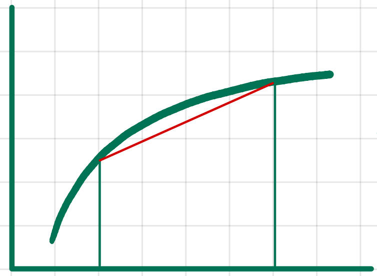
$$\Rightarrow 2\pi\left[-\frac{2}{3}(a^2-x^2)^{3/2}\right]\Bigg|_{a/2}^{a}=-\frac{4\pi}{3}(a^2-x^2)^{3/2}\Bigg|_{a/2}^{a}$$

$$=-\frac{4\pi}{3}\left[-\left(a^2-\frac{a^2}{4}\right)^{3/2}\right.$$

$$=\frac{4\pi}{3}\cdot\left[\frac{3}{4}a^2\right]^{3/2}=\frac{\cancel{4}\pi}{\cancel{3}}\cdot\frac{\cancel{3}\sqrt{3}\cdot a^3}{\cancel{8}\,2}$$

$$=\frac{\pi\sqrt{3}a^3}{2}$$

11 $\int_1^5 \frac{e^x}{x}\,dx$



$$\sum_{i=1}^{n} \Delta x \cdot \left(\frac{f(x_i)+f(x_{i-1})}{2}\right)$$

$$= \frac{\Delta x}{2}\left[f(x_0)+f(x_1)+f(x_1)+f(x_2) + \dots + f(x_n)\right]$$

$$= \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+\dots\right]$$

$\Delta x = \frac{5-1}{2} = 2$

$x_0 = 1$ $\quad f(x_0) = 2.7$

$x_1 = 3$ $\quad f(x_1) = 6.7$

$x_2 = 5$ $\quad f(x_2) = 29.7$

$$\begin{array}{r} {}^{1\ \ 1} \\ 2.7 \\ 13.4 \\ 29.7 \\ \hline 45.8 \end{array}$$

$$T_n = \frac{2}{2}\left[2.7 + 2\cdot 6.7 + 29.7\right] = 45.8$$

12

$\frac{dm}{dt}$ = rate of radioactive decay = $k\,m(t)$

$t=0 \Rightarrow m(0) = 100$ mg Radium

a) $\frac{dm}{dt} = k\,m(t)$

$\frac{1}{m(t)}\,dm = k\,dt$

$\ln m(t) = kt + C$

$m(t) = Ce^{kt}$

$m(0) = C = 100 \Rightarrow m(t) = 100e^{kt}$

$100e^{k \cdot 1600} = 50$

$e^{k1600} = \frac{1}{2} \Rightarrow 1600k = -\ln 2$

$k = \frac{-\ln 2}{1600}$

Alternat.

$m(t+1600) = \cancel{100}e^{k(t+1600)} = \frac{m(t)}{2} = \frac{\cancel{100}e^{kt}}{2}$

$e^{kt}e^{1600k} = \frac{e^{kt}}{2} \Rightarrow e^{1600k} = \frac{1}{2}$

$\Rightarrow 1600k = -\ln 2 \Rightarrow k = \frac{-\ln 2}{1600}$

$$m(t) = 100\,e^{-\frac{\ln 2}{1600}t} = 100\left(e^{-\ln 2}\right)^{t/1600} = 100 \cdot 0.5^{t/1600}$$

b) $m(1000) = 100e^{-\frac{1000}{1600}\ln 2} = \frac{100}{2^{10/16}} = 100 \cdot 0.65 = 65$ mg

13

$$x = C(t) = \int_0^t \cos(\pi u^2/2)\,du$$

$$y = S(t) = \int_0^t \sin(\pi u^2/2)\,du$$

Arc length, $t=0$ to $t=t_0$

In xy equations:

$$\int ds = \int \sqrt{1+f'(x)^2}\,dx$$

In parametric equations:

$x = C(t)$

$dx = C'(t)\,dt = \cos(\pi t^2/2)\,dt$

$y(t) = S(t)$

$y'(t) = S'(t) = \sin(\pi t^2/2)$

$$ds = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + (dy/dt)^2}\cdot dt$$

$$\int_0^{t_0} \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^{t_0} \left[\cos^2(\pi t^2/2) + \sin^2(\pi t^2/2)\right]^{1/2} dt = \int_0^{t_0} dt = t_0$$

14

a) $\ln(1+x)$ $a=0$ Taylor series

$f(x)=\ln(1+x)$ $\qquad f(0)=0$

$f'=\dfrac{1}{1+x}=(1+x)^{-1}$ $\qquad f'(0)=1 \quad 0!$

$f''=-1(1+x)^{-2}=\dfrac{-1}{(1+x)^2}$ $\qquad f''(0)=-1 \quad -1!$

$f'''=2(1+x)^{-3}=\dfrac{2}{(1+x)^3}$ $\qquad f'''(0)=2=2!$

$f^{(4)}=-3\cdot 2(1+x)^{-4}=\dfrac{-3\cdot 2}{(1+x)^4}$ $\qquad f^{(4)}(0)=-6=-3!$

$f^{(5)}=4\cdot 3\cdot 2(1+x)^{-5}$ $\qquad f^{(5)}(0)=4!$

$f^{(k)}=(-1)^{k-1}(k-1)!\,(1+x)^{-k}$ $\qquad f^{(k)}(0)=(-1)^{k-1}(k-1)!$

n^{th} coeff of Taylor series: $\dfrac{f^{(n)}(0)}{n!}x^n=\dfrac{(-1)^{n-1}(n-1)!}{n!}x^n=\dfrac{(-1)^{n-1}x^n}{n}$ $\quad n\geq 1$

Taylor series $= f(0)+\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$

$= 0+x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\cdots$

b) Taylor Formula $= f(0)+\displaystyle\sum_{k=1}^{n}\frac{(-1)^{n-1}x^n}{n}+\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$

$R_n(x)=\dfrac{(-1)^n n!\,(1+z)^{-(n+1)}}{(n+1)!}x^{n+1}=\dfrac{(-1)^n x^{n+1}}{(n+1)(1+z)^{n+1}}=(-1)^n\dfrac{1}{n+1}\cdot\left(\dfrac{x}{1+z}\right)^{n+1}$ $\quad$ z between 0 and x

c) $\ln(3/2)$

Taylor series $= f(0) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$

$= 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$\frac{3}{2} = 1 + x \Rightarrow x = \frac{1}{2}$

$\ln(3/2) = \ln(1 + 1/2) \approx \frac{1}{2} - \frac{(1/2)^2}{2} = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$

d) Taylor's Formula

$f(x) = P_2(x) + R_2(x)$ $\qquad f^{(k)} = (-1)^{k-1}(k-1)!\,(1+x)^{-k}$

$= x - \frac{x^2}{2} + \frac{2!}{(1+z)^3 \cdot 3!} x^3$

$R_2(x) = (-1)^2 \cdot 2!\,(1+z)^{-3} \cdot \frac{x^3}{3!}$ $\qquad \frac{x^3}{3(1+z)^3}$ z between 0 and x

$R_2(1/2) = \frac{1}{24(1+z)^3}$