Exam II

1

a) $f(x)$ $x=a$ quadratic approx.

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

b) $\ln(1.2)$

$f(x) = \ln x \Rightarrow f(x) \approx \ln 1 + 1\cdot(x-1) - \frac{1}{2}(x-1)^2$

$f'(x) = 1/x$

$f''(x) = -1/x^2$ $\qquad \approx x - 1 - \frac{(x-1)^2}{2}$

$f'(1) = 1$

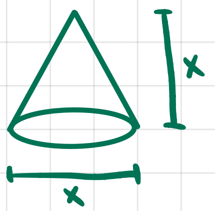
$f''(1) = -1$

$$\ln(1.2) \approx 0.2 - \frac{0.2^2}{2} \approx 0.2 - \frac{0.04}{2} = 0.2 - 0.02$$

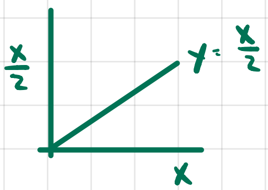
≈ 0.18

2

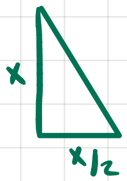
$30\ ft^3/min$



Volume of cone



$$\int_0^x \pi\left[\frac{t}{2}\right]^2 dt = \frac{\pi}{4}\int_0^x t^2\,dt = \frac{\pi}{4}\cdot\frac{t^3}{3}\Big|_0^x = \frac{\pi x^3}{12}$$



$$V(x) = \frac{\pi x^3}{12}\ ft^3$$

$$V(t) = \frac{\pi x(t)^3}{12} \Rightarrow 12V(t) = \pi x(t)^3$$

$$12V'(t) = \pi\cdot 3x(t)^2\cdot x'(t)$$

At t_0, $V'(t_0) = 30$, $x(t_0) = 10$, $V(t_0) = \frac{1000\pi}{12}$

$$12V'(t_0) = 3\pi x(t_0)^2 x'(t_0) \Rightarrow x'(t_0) = \frac{12V'(t_0)}{3\pi x^2(t_0)} = \frac{4V'(t_0)}{\pi x^2(t_0)} = \frac{4\cdot 30}{\pi\cdot 100} = \frac{12}{10\pi}\ ft/min$$

3 $f(x) = x - 3x^{1/3}, \quad x \in \mathbb{R}$

roots

$F(x) = 0 \Rightarrow x\left(1 - \frac{3}{x^{2/3}}\right) = 0 \nearrow \boxed{x = 0}$

$\searrow \sqrt[3]{x^2} = 3 \Rightarrow x = \pm\sqrt{27} = \boxed{\pm 3\sqrt{3}}$

slope

$f'(x) = 1 - \cancel{3} \cdot \frac{1}{\cancel{3}} x^{-2/3} = 1 - \frac{1}{\sqrt[3]{x^2}} \Rightarrow$ critical points: $\frac{1}{\sqrt[3]{x^2}} = 1 \Rightarrow \sqrt[3]{x^2} = 1$

$\Rightarrow \boxed{x = \pm 1}$

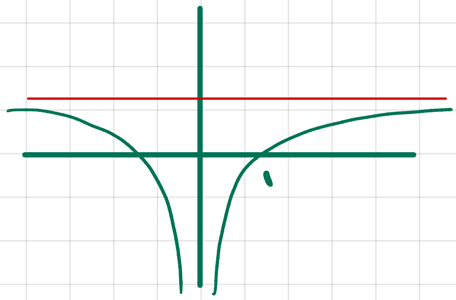
$$\boxed{\begin{array}{l} f(1) = 1 - 3 = -2 \\ f(-1) = -1 - 3(-1) = +2 \end{array}}$$

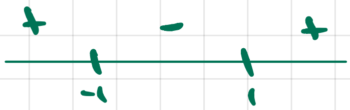
near zero

$\lim_{x \to 0^+} F'(x) = -\frac{1}{\sqrt[3]{0^+}} = -(+\infty) = -\infty$

$\lim_{x \to 0^-} f'(x) = -\frac{1}{\sqrt[3]{(0^-)^2}} = -\frac{1}{0^+} = -\infty$

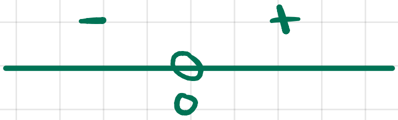
$\lim_{x \to -\infty} f'(x) = 1 - \frac{1}{\infty} = 1 = \lim_{x \to \infty} f'(x)$





concavity

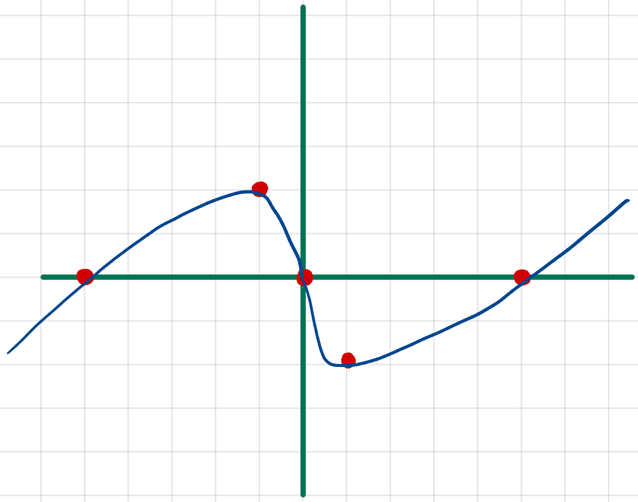
$f''(x) = -\left(-\frac{2}{3}\right) x^{-5/3} = \frac{2}{3\sqrt[3]{x^5}}$



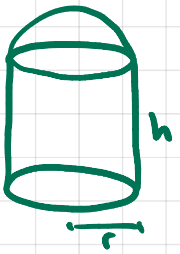
endpoints

$\lim_{x \to \infty} f(x) = \infty\left(1 - \frac{3}{\infty}\right) = \infty$

$\lim_{x \to -\infty} f(x) = -\infty\left(1 - \frac{3}{-\infty}\right) = -\infty$



4 Fixed Volume V



Volume $= \pi r^2 \cdot h + \frac{4}{3}\pi r^3 \cdot \frac{1}{2}$

$= \pi r^2 h + \frac{2}{3}\pi r^3$

metal used = total surface area

cylinder surfaces $= h \cdot 2\pi r + \pi r^2$

Dome surface $= 2\pi r^2$

Total surface $= A = 2\pi r^2 + \pi r^2 + 2\pi r h$

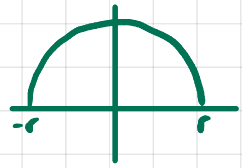
$= 3\pi r^2 + 2\pi r h$

$\pi r^2 h = V - \frac{2}{3}\pi r^3$

$h(r) = \frac{V}{\pi r^2} - \frac{2}{3}\frac{\cancel{\pi} r^{\cancel{3}}}{\cancel{\pi r^2}} = \frac{V}{\pi r^2} - \frac{2r}{3}$

$= \frac{3V - 2\pi r^3}{3\pi r^2}$

Dome Surface



$y = \sqrt{r^2 - x^2} \qquad y' = \frac{-2x}{2\sqrt{r^2-x^2}} = \frac{-x}{\sqrt{r^2-x^2}}$

$$\int_{-r}^{r} 2\pi \sqrt{r^2-x^2}\sqrt{1 + \left(-x/\sqrt{r^2-x^2}\right)^2}\, dx$$

$$= 2\pi \int_{-r}^{r} \sqrt{r^2 - \cancel{x^2} + \cancel{(r^2-x^2)}\frac{\cancel{x^2}}{\cancel{(r^2-x^2)}}}\, dx$$

$$= 2\pi \int_{-r}^{r} r\, dx = 2\pi r(r - (-r))$$

$$= 2\pi r \cdot 2r = 4\pi r^2$$

$$A(r) = 3\pi r^2 + 2\cancel{\pi}\cancel{r} \cdot \frac{[3V - 2\pi r^3]}{3\cancel{\pi} r^{\cancel{2}}} = 3\pi r^2 + \frac{6V - 4\pi r^3}{3r} = 3\pi r^2 + 2Vr^{-1} - \frac{4\pi r^2}{3}$$

$A'(r) = 6\pi r - \frac{2V}{r^2} - \frac{8\pi r}{3} \qquad A'(r) = 0 \Rightarrow \frac{10\pi r}{3} - \frac{2V}{r^2} = 0 \Rightarrow \frac{10\pi r}{3} = \frac{2V}{r^2}$

$\Rightarrow 10\pi r^3 = 6V \Rightarrow r^3 = \frac{6V}{10\pi} = \frac{3V}{5\pi} \Rightarrow r = \sqrt[3]{\frac{3V}{5\pi}}$

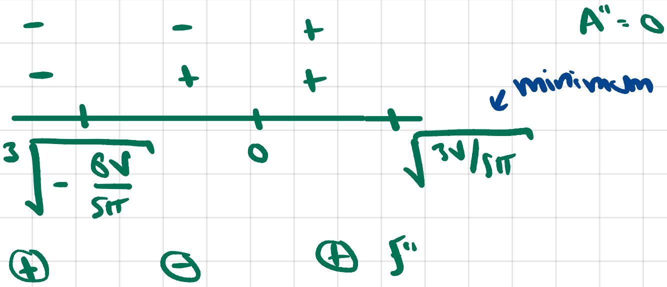
$h\left(\sqrt[3]{\frac{3V}{5\pi}}\right) = \frac{\cancel{3}\cdot\frac{5}{\cancel{3}}\pi r^3 - 2\pi r^3}{3\pi r^2} = \frac{\cancel{3}\cancel{\pi} r^3}{\cancel{3}\cancel{\pi} r^2} = r \qquad \Rightarrow V = \frac{5}{3}\pi r^3$

$\Rightarrow h^* = \sqrt[3]{3V/5\pi}$

$$A''(r) = 6\pi + \frac{4V}{r^3} - \frac{8\pi}{3} = \frac{18\pi r^3 + 12V - 8\pi r^3}{3r^3} = \frac{r^3 \cdot 10\pi + 12V}{3r^3}$$

$A'' = 0 \Rightarrow r^3 = -\frac{12V}{10\pi} = -\frac{6V}{5\pi}$

$r = \sqrt[3]{-\frac{6V}{5\pi}} < 0$



we must compare critical point with boundary solutions:

$V = \pi r^2 h + \frac{2}{3}\pi r^3 \qquad r \in [0,\infty),\ h \in [0,\infty)$

$A = 3\pi r^2 + 2\pi r h = 3\pi r^2 + 2Vr^{-1} - \frac{4\pi r^2}{3} = \frac{5\pi r^2}{3} + \frac{2V}{r}$

$h(r) = \frac{3V - 2\pi r^3}{3\pi r^2} \qquad r \in [0,\infty)$

- $r = h = \sqrt[3]{\frac{3V}{5\pi}} \Rightarrow V = \frac{5}{3}\pi r^3 \Rightarrow A = 3\pi \cdot \left[\frac{3V}{5\pi}\right]^{2/3} + 2\pi\left[\frac{3V}{5\pi}\right]^{2/3} = 5\pi\left[\frac{3V}{5\pi}\right]^{2/3}$

- $r = 0 \Rightarrow A = +\infty$ because $h(0) = \infty$ X

- $r = \infty \Rightarrow A = +\infty,\ h = -\infty$ X

- $h = 0 \Rightarrow V = \frac{2\pi r^3}{3} \Rightarrow A = \frac{5\pi r^2}{3} + \frac{4\pi r^3}{3r} = \frac{9\pi r^3}{3r} = 3\pi r^2 = 3\pi \cdot \left[\frac{3V}{2\pi}\right]^{2/3}$

  $\Rightarrow r = \sqrt[3]{\frac{3V}{2\pi}}$

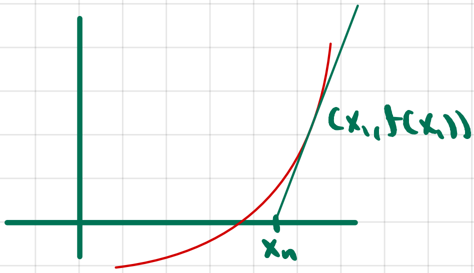
compare $\quad 3\pi\left({}^{3V}/_{2\pi}\right)^{2/3}$ and $5\pi\left({}^{3V}/_{5\pi}\right)^{2/3}$

cube: $\quad 27\pi^3 \cdot \frac{9V^2}{4\pi^2} \qquad 125\pi^3 \cdot \frac{9V^2}{25\pi^2}$

$V^2\pi \cdot \underbrace{\frac{27 \cdot 9}{4}}_{60.75} \quad > \quad V^2 \cdot \pi \cdot \underbrace{\frac{125 \cdot 9}{25}}_{45}$

$\Rightarrow r = h = \sqrt[3]{\frac{3V}{2\pi}}$ minimum dimensions.

5 $f(x) = x^3 - 3x + 7$ $\quad x_1 = 2$, Newton's method



$$0 - f(x_1) = f'(x_1)(x_n - x_1)$$

$$-f(x_1) = f'(x_1)x_n - f'(x_1)x_1$$

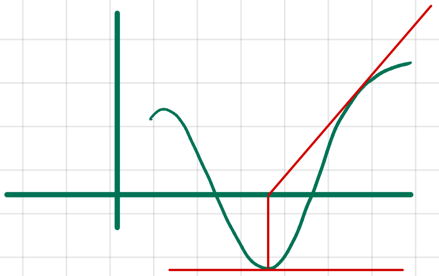
$$x_n = \frac{f'(x_1)x_1 - f(x_1)}{f'(x_1)} = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$f(x) = x^3 - 3x + 7$

$f'(x) = 3x^2 - 3$

$f(2) = 8 - 6 + 7 = 9$

$f'(2) = 9$



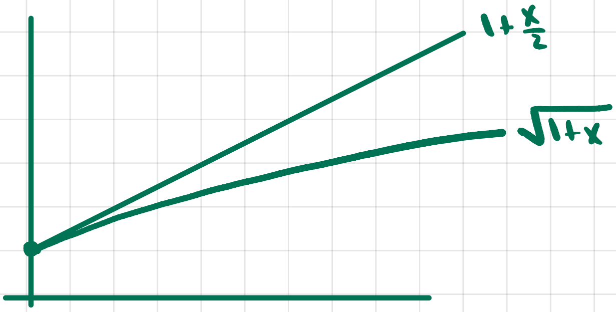
$x_{n_1} = 2 - 1 = 1$

$x_{n_2} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{5}{0}$

Newton's method does not work because of a situation analogous to that drawn above.

After one iteration, we obtain a value x_2, from which we want to obtain $x_3 = 1 - f(x_2)/f'(x_2)$. But $f'(x_2)$ is zero, so the expression for x_3 is undefined.

6 $\sqrt{1+x} < 1+\frac{x}{2}$ if $x>0$



$f(x) = 1+\frac{x}{2} - \sqrt{1+x} \quad x>0$

$f'(x) = \frac{1}{2} - \frac{1}{2\sqrt{1+x}} = \frac{1}{2}\left[1 - \frac{1}{\sqrt{1+x}}\right]$

For $x>0$, $\sqrt{1+x} > 1 \Rightarrow \frac{1}{\sqrt{1+x}} < 1 \Rightarrow 1 - \frac{1}{\sqrt{1+x}} > 0$

$\Rightarrow f'(x) > 0$

$f(0) = 1 + \frac{0}{2} - \sqrt{1+0} = 1-1=0$

$f(0)=0$ and $f'(x)>0$ for $x>0$

$\Rightarrow$ for any $x>0$, by the MVT $\frac{f(x)-f(0)}{x-0} = f'(c) \quad c \in [0,x]$.

we've shown $f'(c)>0$, so $\frac{f(x)}{x} > 0 \Rightarrow f(x)>0 \Rightarrow 1+\frac{x}{2} > \sqrt{1+x} \quad x>0$