1A-1

a) $y = x^2 - 2x - 1$

$y = x^2 - 2x + 1 - 2$

$y + 2 = (x-1)^2$

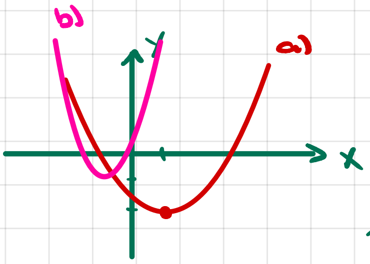
parabola w/ vertex at $(1,-2)$

Translate 2 units up, 1 unit to left: $x \to x+1$, $y \to y-2$

$u = x - 1 \Rightarrow x = u + 1$

$v = y + 2 \Rightarrow y = v - 2$

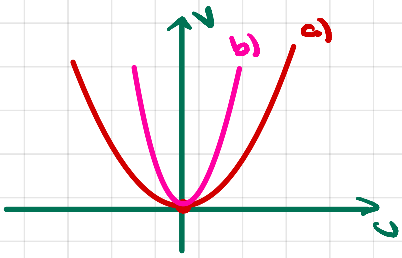
$\Rightarrow v = u^2$



b) $y = 3x^2 + 6x + 2$

$= 3(x^2 + 2x + 1) - 1$

$y + 1 = 3(x+1)^2$



parabola w/ vertex $(-1,-1)$

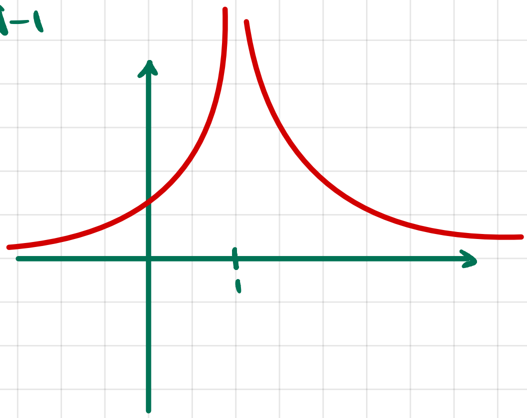
$u = x + 1 \Rightarrow x = u - 1$

$v = y + 1 \Rightarrow y = v - 1$

$\Rightarrow v = 3u^2$

1A-2

b) $y = \dfrac{2}{(x-1)^2}$

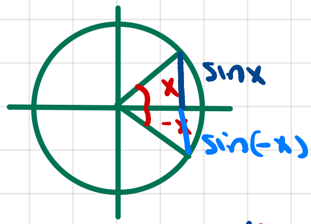


$f(x) = \sin(x)$

$f(-x) = \sin(-x)$

$= -\sin(x)$

$\Rightarrow$ odd



1A-3

even function: $f(-x) = f(x)$

odd function: $f(-x) = -f(x)$

a) $f(x) = \dfrac{x^3 + 3x}{1 - x^4}$

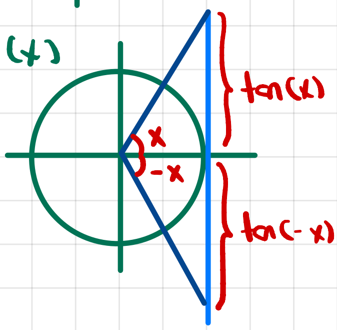
$f(-x) = \dfrac{-x^3 - 3x}{1 - x^4} = -\left[\dfrac{x^3 + 3x}{1 - x^4}\right] = -f(x)$

$\Rightarrow$ odd

b) $f(x) = \sin^2 x$

$f(-x) = \sin^2(-x) = \sin^2(x)$

$\Rightarrow$ even



c) $f(x) = \dfrac{\tan x}{1 + x^2}$

$f(-x) = \dfrac{\tan(-x)}{1 + x^2}$

$= -\tan(x) / 1 + x^2 \Rightarrow$ odd

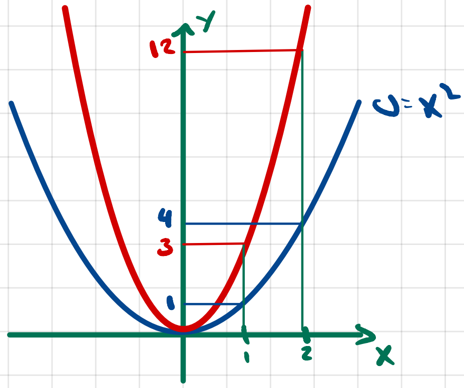
IA-3 e) $J_0(x^2)$

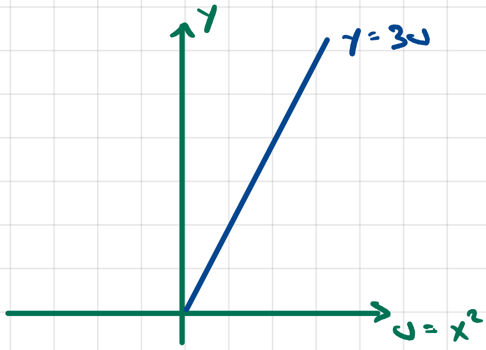
example:

$f(x) = 3x^2$

$u = x^2 \Rightarrow 0 \leq u < +\infty$

$f(x^2) = f(u) = 3u$





The image of $u(x) = x^2$ is $[0, +\infty)$, which is the domain of $f(x^2)$.

For a generic $J_0(x)$, if we define $u(x) = x^2$ we reach the same conclusion about the domain of $J_0(x^2)$: it is $[0, +\infty) \Rightarrow$ neither odd nor even function.

IA-6

b) $f(x) = \sin x - \cos x$

$A\sin(x+c) = A\sin x\cos c + A\sin c\cos x$

$$\begin{matrix} A\cos c = 1 \\ A\sin c = -1 \end{matrix} \Rightarrow \begin{matrix} A^2\cos^2 c = 1 \\ A^2\sin^2 c = 1 \end{matrix} \Rightarrow A^2 = 2 \Rightarrow A = \pm\sqrt{2}$$

$$\Rightarrow \begin{matrix} \cos c = 1/\sqrt{2} = \sqrt{2}/2 \Rightarrow c = \pi/4 \text{ or } 7\pi/4 \\ \sin c = -1/\sqrt{2} = -\sqrt{2}/2 \Rightarrow c = 5\pi/4 \text{ or } 7\pi/4 \end{matrix} \Rightarrow c = 7\pi/4$$

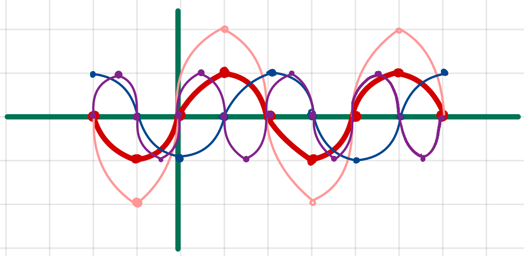
$f(x) = \sqrt{2}\sin\left(x + \frac{7\pi}{4}\right)$

OR

$$\begin{matrix} \cos c = -\sqrt{2}/2 \\ \sin c = \sqrt{2}/2 \end{matrix} \Rightarrow \begin{matrix} c = \frac{3\pi}{4} \text{ or } \frac{5\pi}{4} \\ c = \pi/4 \text{ or } 3\pi/4 \end{matrix} \Rightarrow c = \frac{3\pi}{4}$$

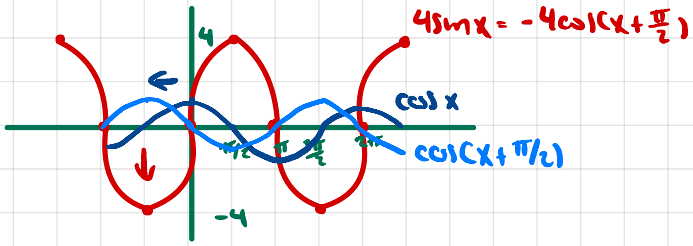
IA-7

b)



sinx
2sinx double amplitude
$\sin(x - \pi/2)$ shifted right (phase angle $\pi/2$)
$\sin(2x)$ period cut in half
↳ period is $\frac{2\pi}{b}$ in $\sin(bx)$

$-4\cos(x + \pi/2)$
amplitude = 4
phase angle = $-\pi/2$ (shift left)
period no alteration = 2π



$\cos(x) = \sin(x + \pi/2)$
$\cos(x + \pi/2) = \sin(x + \pi) = -\sin(x)$

$\Rightarrow -4\cos(x + \pi/2) = 4\sin(x)$

1B-1

400ft

a) $s(t) = 16t^2 \quad s(0) = 0 \quad s(2) = 64 \qquad v_{AVG}(t_f, t_i) = \dfrac{s(t_f) - s(t_i)}{t_f - t_i}$

$$v_{AVG} = \frac{\Delta s}{\Delta t} = \frac{64}{2} = 32 \text{ ft/s}$$

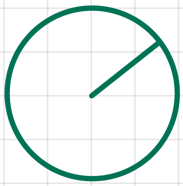
b) $16t_f^2 = 400 \Rightarrow t_f^2 = 25 \Rightarrow t_f = 5 \Rightarrow v_{AVG}(5,3) = \dfrac{400 - 144}{2} = 128 \text{ ft/s}$

$s(3) = 16 \cdot 9 = 144$

c) $v'(t) = 32t \Rightarrow v'(5) = 160 \text{ ft/s}$

1C-1

a)



$A(r) = \pi r^2$

$$A'(r) = \lim_{\Delta r \to 0} \frac{A(r+\Delta r) - A(r)}{\Delta r} = \lim \frac{\pi(r+\Delta r)^2 - \pi r^2}{\Delta r} = \lim \frac{\pi(\cancel{r^2} + 2r\Delta r + \Delta r^2 - \cancel{r^2})}{\Delta r}$$

$$= \lim_{\Delta r \to 0} (2\pi r + \pi \Delta r) = 2\pi r$$

1C-3

a) $f(x) = \dfrac{1}{2x+1}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{2(x+h)+1} - \frac{1}{2x+1}}{h} = \lim_{h \to 0} \frac{\cancel{2x} + \cancel{1} - \cancel{2x} - 2h - \cancel{1}}{h(2x+1)(2(x+h)+1)}$$

$$\lim_{h \to 0} \frac{-2}{(2x+1)(2(x+h)+1)} = \frac{-2}{(2x+1)^2}$$

b) $f(x) = 2x^2 + 5x + 4$

$$f'(x) = \lim_{h \to 0} \frac{2(x+h)^2 + 5(x+h) + 4 - 2x^2 - 5x - 4}{h} = \lim \frac{\cancel{2x^2} + 4xh + 2h^2 + \cancel{5x} + 5h + \cancel{4} - \cancel{2x^2} - \cancel{5x} - \cancel{4}}{h}$$

$$= \lim \frac{4xh + 2h^2 + 5h}{h} = \lim_{h \to 0} 4x + 2h + 5 = 4x + 5$$

e) a) $f'(x) = \frac{-2}{(2x+1)^2} < 0 \ \forall x \in \mathbb{R} - \{-\frac{1}{2}\}$, $f'(x)$ undefined at $x = -\frac{1}{2}$

$\Rightarrow$ no points x where $f'(x)$ equals $+1$ or 0.

$f'(x) = -1 \Rightarrow -2/(2x+1)^2 = -1 \Rightarrow 4x^2 + 4x + 1 = 2 \Rightarrow 4x^2 + 4x - 1 = 0$

$\Delta = 16 + 16 = 32$

$$x = \frac{-4 \pm \sqrt{32}}{8} = \frac{-1 \pm \sqrt{2}}{2}$$

$f'(x) = -1$ at $x = \frac{-1 \pm \sqrt{2}}{2}$

b) $f'(x) = 4x + 5$

$f'(x) = 1 \Rightarrow 4x + 5 = 1 \Rightarrow 4x = -4 \Rightarrow x = -1$

$f'(x) = 0 \Rightarrow 4x = -5 \Rightarrow x = -5/4$

$f'(x) = -1 \Rightarrow 4x + 5 = -1 \Rightarrow 4x = -6 \Rightarrow x = -3/2$

1C-4

a) $f(x) = 1/(2x+1)$, $x = 1$ $\quad f(1) = 1/3$

$f'(x) = \frac{-2}{(2x+1)^2}$ $\quad f'(1) = -2/9$

point-slope eq. of tangent at $(1, 1/3)$: $y - 1/3 = -\frac{2}{9}(x-1)$

b) $f(x) = 2x^2 + 5x + 4$, $x = a$ $\quad f(a) = 2a^2 + 5a + 4$

$f'(x) = 4x + 5$ $\quad f'(a) = 4a + 5$

$y - (2a^2 + 5a + 4) = (4a + 5)(x - a)$

1C-5 $y = 1+(x-1)^2 = x^2-2x+2$

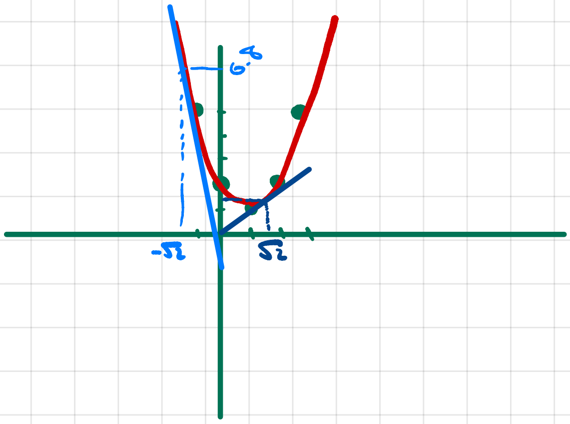
$f'(x) = 2x-2 = 0 \Rightarrow x=1$

$0 - y_0 = (2x_0-2)(0-x_0) = -2x_0^2+2x_0 \Rightarrow 2x_0^2 - 2x_0 = y_0$

$y_0 = x_0^2 - \cancel{2x_0} + 2 = 2x_0^2 - \cancel{2x_0} \Rightarrow x_0^2 = 2 \Rightarrow x_0 = \pm\sqrt{2}$

$x_0=\sqrt{2} \Rightarrow y_0 = 2\cdot 2 - 2\sqrt{2} = 4-2\sqrt{2} \approx 1.2$

$x_0=-\sqrt{2} \Rightarrow y_0 = 2\cdot 2 - 2\cdot(-\sqrt{2}) = 4+2\sqrt{2} \approx 6.8$



The two tangent lines going through o is in de

$y - \cancel{4} + \cancel{2\sqrt{2}} = 2(\sqrt{2}-1)(x-\sqrt{2})$

$= 2\sqrt{2}x - \cancel{4} - 2x + \cancel{2\sqrt{2}}$

$y = 2(\sqrt{2}-1)x$

$y - \cancel{4} - \cancel{2\sqrt{2}} = 2(-\sqrt{2}-1)(x+\sqrt{2})$

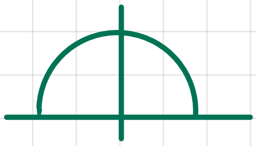
$= -2\sqrt{2}x - \cancel{4} - 2x - \cancel{2\sqrt{2}}$

$y = -2(\sqrt{2}+1)x$

1C-6

a) $x^2+y^2=R^2$

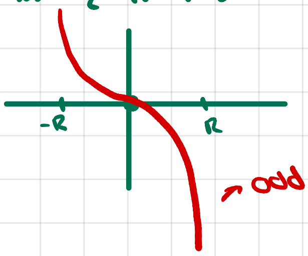
$y=\pm(R^2-x^2)^{1/2}$ $\quad R^2-x^2\ge 0 \Rightarrow x^2 \le R^2 \Rightarrow -R<x<R$



$f(x) = (R^2-x^2)^{1/2} \quad -R<x<R$

$f'(x) = \frac{1}{2}(R^2-x^2)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{R^2-x^2}}$

odd: $f'(-x) = \frac{x}{\sqrt{R^2-x^2}} = -f(x)$



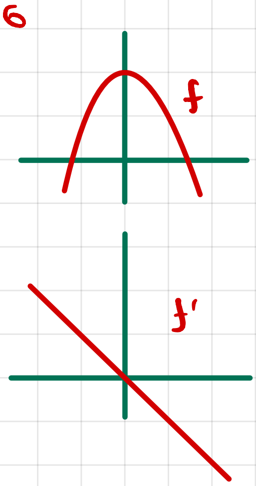
$f''(x) = \frac{-(R^2-x^2)^{1/2} - (-x)\cdot\cancel{\frac{1}{2}}(R^2-x^2)^{-\frac{1}{2}}(-\cancel{2}x)}{R^2-x^2} = -(R^2-x^2)^{-\frac{1}{2}} - x^2(R^2-x^2)^{-\frac{3}{2}}$

$= \frac{-(R^2-x^2)}{\sqrt{R^2-x^2}\,(R^2-x^2)} - \frac{x^2}{\sqrt{R^2-x^2}\,(R^2-x^2)} = \frac{-R^2}{(R^2-x^2)^{3/2}} < 0$

1C-6

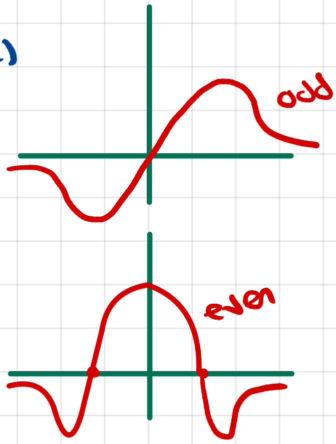
b)

f

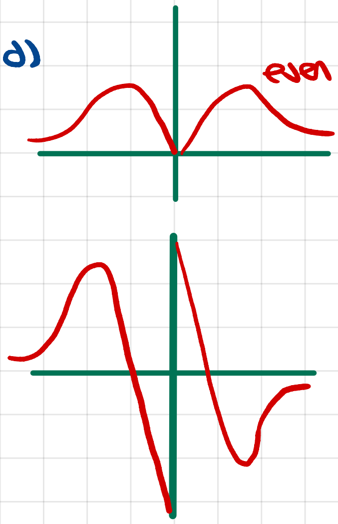


f'

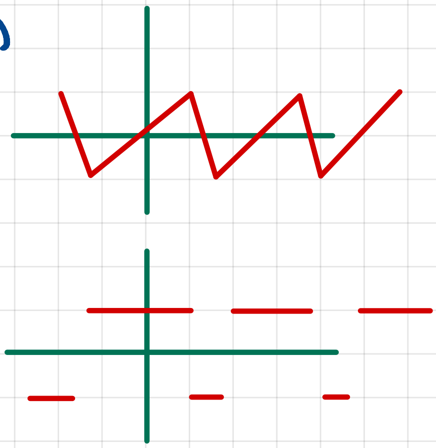
c)



d)



e)



1C-2

$$f(x) = (x-a)g(x)$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h-a)g(x+h) - (x-a)g(x)}{h}$$

$$= \lim \frac{(x-a)g(x+h) - (x-a)g(x) + hg(x+h)}{h}$$

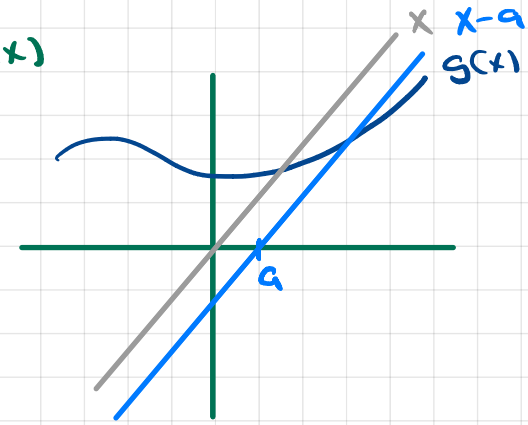
$$= \lim \left[\frac{(x-a)(g(x+h)-g(x))}{h} + \frac{hg(x+h)}{h} \right]$$

$$= (x-a)g'(x) + g(x)$$

$$f'(a) = g(a)$$

$$f(a) = 0$$

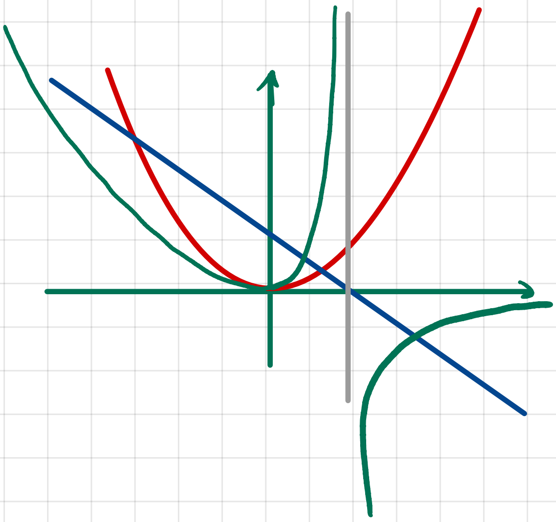
$$f(a+1) = g(a+1)$$



10-1

a) $\lim_{x\to 0} \frac{4}{x-1} = -4$

b) $\lim_{x\to 2} \frac{4x}{x+1} = \frac{8}{3}$



c) $\lim_{x\to -2} \frac{4x^2}{x+2} = +\infty$

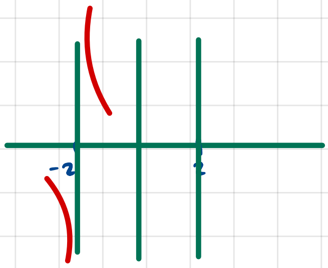
d) $\lim_{x\to 2^+} \frac{4x^2}{2-x} = -\infty$

f) $\lim_{x\to \infty} \frac{4x^2}{x-2} = +\infty$

g) $\lim_{x\to\infty} \left[\frac{4x^2}{x-2} - 4x\right] = \lim \frac{4x^2-4x^2+8x}{x-2} = \lim \frac{8x}{x-2} = 8 \cdot \lim \frac{1}{1-2/x} = 8$

10-3

a) $f(x) = \frac{x-2}{x^2-4} = \frac{(x-2)}{(x+2)(x-2)} = \frac{1}{x+2}$



$\lim_{x\to 2} f(x) = \lim \frac{1}{x+2} = \frac{1}{4} \neq f(2)$ which is undefined

$\Rightarrow x=2$ is a removable discontinuity

$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{1}{x+2} = +\infty$

$\Rightarrow x=-2$ infinite discont.

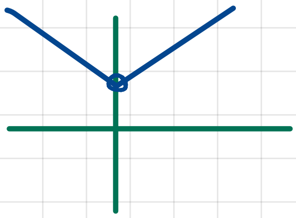
$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{1}{x+2} = -\infty$

b) $\frac{1}{\sin x}$ $\quad \sin x = 0$ for $x = i\cdot\pi$, $i = 0, 1, \ldots$

infinite number of infinite discont.

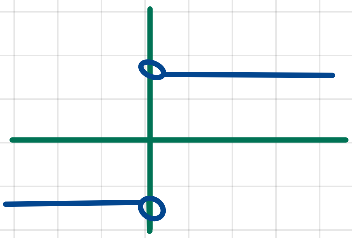
c) $\frac{x^4}{x^3} = f(x) = \frac{x \cdot x^3}{x^3}$

d) $f(x) = \begin{cases} x+a, & x>0 \\ a-x, & x<0 \end{cases}$



removable discont.

e) $f'(x) = \begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases}$

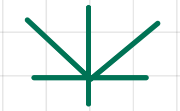


jump discont.

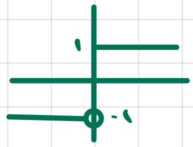
10-6

a)

$$f(x) = \begin{cases} x^2+4x+1, & x \geq 0 \\ ax+b, & x<0 \end{cases}$$



continuous, not differentiable at $x=0$



not cont. at $x=0$, not differentiable at $x=0$: $f'(0^+) \neq f'(0^-)$

$$f'(0^+) = \lim_{h \to 0^+} \frac{+1+h-1}{h} = 1$$

$$f'(0^-) = \lim_{h \to 0^-} \frac{-1+h-1}{h} = \lim \frac{-2}{h} + \lim 1 = \infty$$

$$f'(x) = \begin{cases} 2x+4 & x \geq 0 \\ a & x<0 \end{cases}$$

To be continuous at $x=0$ ⇒

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)$$

$$\lim_{x \to 0^+} f(x) = 1$$

$$\lim_{x \to 0^-} f(x) = b$$

$$f(0) = 1$$

$\Rightarrow b = 1 \Rightarrow f$ continuous at $x=0$

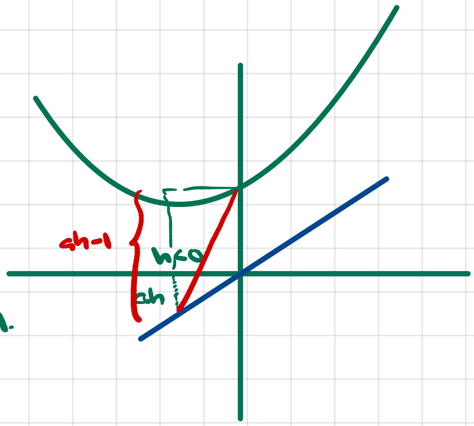
To be differentiable at $x=0$ ⇒

$$\underbrace{\lim_{h \to 0} \frac{f(0+h)-f(x)}{h}} = \lim \frac{f(0+h)-f(0)}{h}$$

$$f'(0^+) = 2 \cdot 0 + 4 \cdot 0 + 1 = 1$$

$$\lim_{x \to 0^-} \frac{a(0+h)+b-(0^2+4 \cdot 0+1)}{h} = \lim_{h \to 0^-} \frac{ah+b-1}{h} = a = f'(0^-)$$

$f'(0^+) = f'(0^-) \Rightarrow a = 1 \Rightarrow f$ differentiable



Note what it means to calculate $f'(0^-)$ in this type of case where f is not continuous: one of the one-sided limits has a weird geometric interpretation.

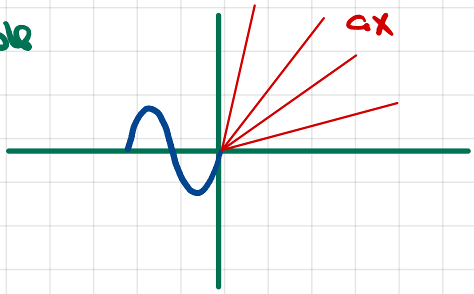
$$f(x) = \begin{cases} x^2+4x+1, & x \geq 0 \\ ax \quad, & x<0 \end{cases}$$

$$\lim_{h \to 0^-} \frac{a(0+h)-1}{h} = \lim \frac{ah-1}{h} = a - \frac{1}{h} = \infty$$

1D-8

a) continuous but not differentiable

$$f(x) = \begin{cases} ax+b & x>0 \\ \sin 2x & x\leq 0 \end{cases}$$



continuity:

$$\lim_{x\to 0^+} ax+b = \lim_{x\to 0^-} \sin 2x = f(0) = \sin 0 = 0$$

$\Rightarrow$ $b = 0$

differentiable

$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} \Big|_{x=0}$$

$$\lim_{h\to 0^+} \frac{a(0+h)-\sin 0}{h} = \lim \frac{ah}{h} = a$$

$$\lim_{h\to 0^-} \frac{\sin(2(0+h)) - \sin(0)}{h} = \lim \frac{\sin 2h - 0}{h} = \lim \frac{2\sin h\cos h}{h} = 2\cdot 1\cdot 1 = 2$$

$a \neq 2 \Rightarrow$ not differentiable

1E-1

a) $10x^9 + 15x^2 + 6x$

c) $\frac{1}{2}$

1E-2

a) $\frac{ax^2}{2} + bx + c$

c) $(x^3+1)^2 = x^6 + 2x^3 + 1 = f'$

$\Rightarrow f = \frac{x^7}{7} + \frac{2x^4}{4} + x + c$

1E-3

$y = x^3 + x^2 - x + 2$

$y' = 3x^2 + 2x - 1 = 0$

$\Delta = 4 - 4\cdot 3\cdot(-1) = 16 \quad x = \frac{-2\pm 4}{6}$ $\nearrow -1$ $\searrow 1/3$

$f(-1) = -1 + 1 + 1 + 2 = 3$

$f(1/3) = \frac{1}{27} + \frac{1}{9} - \frac{1}{3} + 2 = \frac{1+3-9+54}{27} = \frac{49}{27}$

$(-1, 3)$

$(1/3, 49/27)$

1E-4

b) f should be differentiable

$$f(x) = \begin{cases} ax^2+bx+4, & x \le 1 \\ 5x^5+3x^4+7x^2+8x+4, & x>1 \end{cases}$$

$f'(1^+) = 25\cdot1^4 + 12\cdot1^3 + 14\cdot1 + 8 = 59$

$f'(1^-) = 2a\cdot1 + b = 2a+b$

$\lim\limits_{x\to1^+} f(x) = 5+3+7+8+4 = 27$

$\lim\limits_{x\to1^-} f(x) = a+b+4$

$f(1) = a+b+4$

$a+b+4 = 27$ (for continuity)

$2a+b = 59$ (for diff)

$a + 59 - 2a + 4 = 27$

$a = 59+4-27 = 36$

$\Rightarrow b = 59-72 = -13$

1E-5

a) $f(x) = \dfrac{x}{1+x} \Rightarrow f'(x) = \dfrac{1}{1+x} + x(-1)(1+x)^{-2}$

$$= \frac{1+x-x}{(1+x)^2} = \frac{1}{(1+x)^2}$$

c) $f(x) = \dfrac{x+2}{x^2-1} \Rightarrow f'(x) = \dfrac{1}{x^2-1} + \dfrac{(x+2)(-1)\cdot 2x}{(x^2-1)^2}$

$$= \frac{(x^2-1) - 2x(x+2)}{(x^2-1)^2}$$

$$= \frac{x^2-1-2x^2-4x}{(x^2-1)^2} = \frac{-x^2-4x-1}{(x^2-1)^2}$$

1F-1

a) $f(x) = (x^2+2)^2 \Rightarrow f'(x) = 2(x^2+2)\cdot 2x = 4x(x^2+2)$

alternatively,

$2x(x^2+2) + (x^2+2)2x = 4x(x^2+2)$

b) $(x^2+2)^{1000}$ $\quad f'(x) = 1000(x^2+2)^{999}\cdot 2x$

$= 2000x(x^2+2)^{999}$

1F-2

$f(x) = x^{10}(x^2+1)^{10}$ $\quad f'(x) = 10x^9(x^2+1)^{10} + x^{10}\cdot 10(x^2+1)^9\cdot 2x$

$f'(x) = 10x^9(x^2+1)^9\left[x^2+1+x\cdot 2x\right] = 10x^9(x^2+1)^9(3x^2+1)$

1F-6

even: $f(x) = f(-x)$

$$f'(x) = f'(-x)(-1) \Rightarrow f' \text{ odd}$$

odd: $f(x) = -f(-x)$

$$f'(x) = -f'(-x)(-1) = f'(-x) \Rightarrow f' \text{ even}$$

1F-7

b) $$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = m_0(1 - v^2/c^2)^{-\frac{1}{2}}$$

$$\frac{dm}{dv} = m_0(-\frac{1}{\cancel{2}})(1 - v^2/c^2)^{-\frac{3}{2}}(-\frac{\cancel{2}}{c^2}v)$$

$$= \frac{m_0}{c^2}v(1 - v^2/c^2)^{-\frac{3}{2}}$$

c) $$F = \frac{mg}{(1+r^2)^{3/2}} = mg(1+r^2)^{-\frac{3}{2}}$$

$$\frac{dF}{dr} = mg(-\frac{3}{\cancel{2}})(1+r^2)^{-\frac{5}{2}} \cdot \cancel{2}r$$

$$= -3rmg(1+r^2)^{-\frac{5}{2}}$$

1G-1

b) $$f(x) = \frac{x}{x+5} = x(x+5)^{-1}$$

$$f'(x) = (x+5)^{-1} + x(-1)(x+5)^{-2} = \frac{x+5-x}{(x+5)^2} = \frac{5}{(x+5)^2}$$

$$f''(x) = 5(-2)(x+5)^{-3}$$

$$= -10(x+5)^{-3}$$

c) $$f(x) = \frac{-5}{x+5} = -5(x+5)^{-1}$$

$$f'(x) = -5(-1)(x+5)^{-2} = 5(x+5)^{-2}$$

$$f''(x) = -10(x+5)^{-3}$$

1G-5

a) $y = u(x)v(x)$

$$y' = u'v + uv'$$

$$y'' = u''v + u'v' + u'v' + uv'' = u''v + 2u'v' + uv''$$

$$y''' = u'''v + u''v' + 2u''v' + 2u'v'' + u'v'' + uv'''$$

$$= u'''v + 3u''v' + 3u'v'' + uv'''$$

1J-1

a) $y = \sin(5x^2)$ $\quad y' = \cos(5x^2)\cdot 10x$

b) $y = \sin^2 3x$ $\quad y' = 2\sin(3x)\cos(3x)\cdot 3$

e) $y = \dfrac{\sin x}{x}$ $\quad y' = \dfrac{\cos(x)}{x} - \dfrac{\sin x}{x^2} = \dfrac{x\cos x - \sin x}{x^2}$

k) $y = \tan^2 3x$ $\quad y' = 2\tan(3x)\cdot\sec^2(3x)\cdot 3$

$= \dfrac{6\cdot\sin(3x)}{\cos^3(3x)}$

m) $y_1 = \cos(2x)$

$y_2 = \cos^2 x - \sin^2 x$

$y_3 = 2\cos^2 x$

$y_1' = -\sin(2x)\cdot 2 = -2\sin(2x) = -2(2\sin x\cos x) = -4\sin x\cos x$

$y_2' = 2\cos x(-\sin x) - 2\sin x\cos x$
$= -4\sin x\cos x$

$y_3' = 2\cdot 2\cos x(-\sin x) = -4\sin x\cos x$

$y_1 = y_2$ because $\cos(x+y) = \cos x\cos y - \sin x\sin y$
$\Rightarrow \cos(2x) = \cos(x+x) = \cos^2 x - \sin^2 x$

y^3 is not the same function as the other two: $y_3(0) = 2,\ y_1(0) = y_2(0) = 1$

$y_2 = y_1 = \cos^2 x - \sin^2 x = \cos^2 x - (1-\cos^2 x) = 2\cos^2 x - 1$
therefore y_3 differs from the other two by a constant.

However the derivatives are the same at every point: if we graph y_3 it is exactly the same form but is shifted up one unit