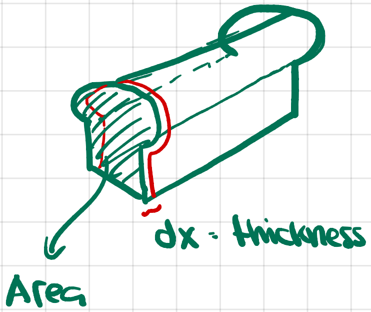
Lecture 22 - Volumes by Slicing

Intuitive Intro: slice of bread



$\Delta V \approx A \cdot \Delta x$

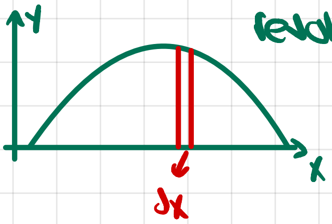
$dV = A(x)dx$

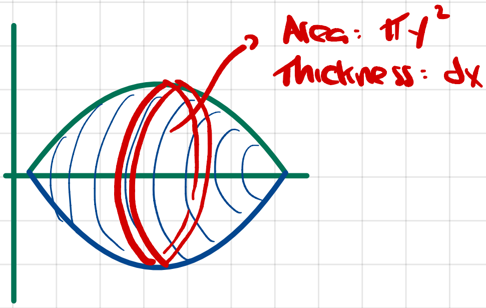
$V = \int A(x)dx \approx \sum A_i dx$

Solids of Revolution

method 1: method of disks

revolve around x-axis

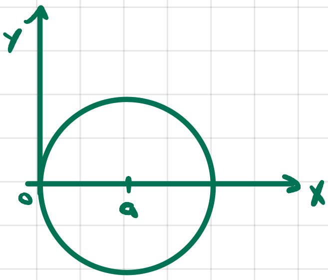




$dV = \pi y^2 dx$ volume of disks

Example:

volume of ball of radius a

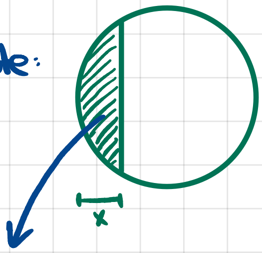


$(x-a)^2 + y^2 = a^2$

$y^2 = a^2 - (x-a)^2 = \cancel{a^2} - (x^2 - 2ax + \cancel{a^2}) = 2ax - x^2$

$V = \int_0^{2a} \pi(2ax - x^2)dx = \pi\left(ax^2 - \frac{x^3}{3}\right)\Big|_0^{2a} = \pi\left(4a^3 - \frac{8a^3}{3}\right) = \frac{4\pi a^3}{3}$

Note:

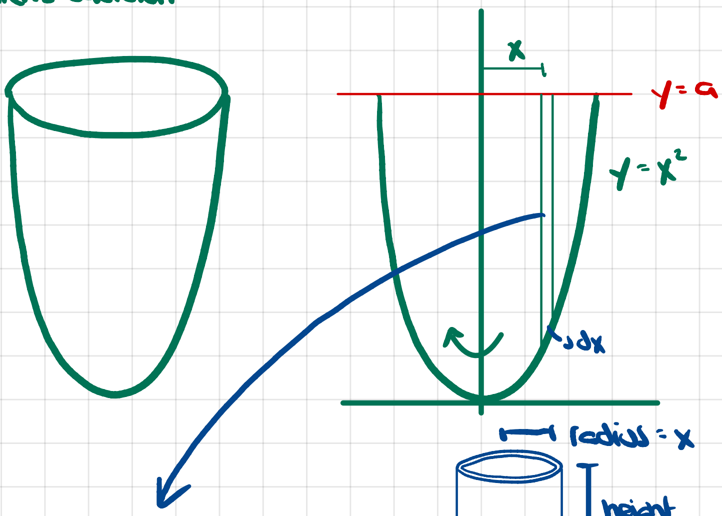


$V(x)$ = volume of portion of width x of ball

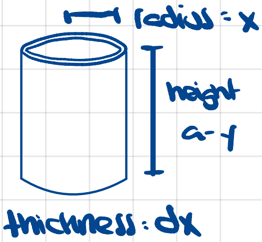
$= \pi(ax^2 - x^3/3)$

Method 2: Method of Shells

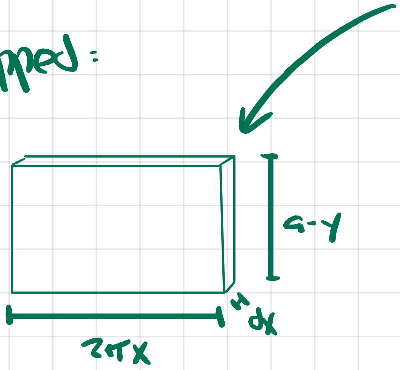
Witch's Cauldron



revolve around y axis, take a thin shell:



unwrapped:



$$dV = (2\pi x)(a-y)\,dx = 2\pi x(a-x^2)\,dx = 2\pi(ax-x^3)\,dx$$

$$V = \int_0^{\sqrt{a}} 2\pi(ax-x^3)\,dx = 2\pi\left(a\frac{x^2}{2}-\frac{x^4}{4}\right)\Big|_0^{\sqrt{a}} = 2\pi\left(\frac{a^2}{2}-\frac{a^2}{4}\right) = 2\pi\frac{a^2}{2} = \pi a^2$$

Beware of Units

$a = 100\text{ cm}$

$$V = \frac{\pi}{2}\cdot 100^2\ \text{cm}^3 = \frac{\pi}{2}\cdot 10\ (1000\ \text{cm}^3)$$

Both answers are correct

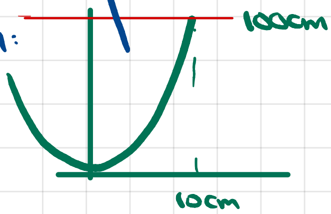
$a = 1\text{m}$

$$V = \frac{\pi}{2}\cdot 1^2\ \text{m}^3 = \frac{\pi}{2}\cdot(100\text{ cm})^3 = \frac{\pi}{2}\cdot 10^6\ \text{cm}^3 = \frac{\pi}{2}\cdot 1000\cdot(1000\ \text{cm}^3)$$

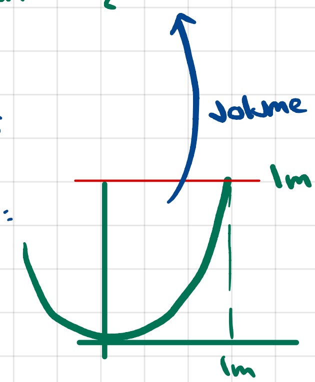
volume

$y = x^2$ can be interpreted in two ways

in cm:



in m:

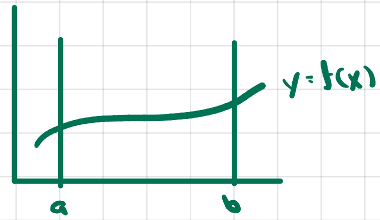


Lecture 23

Average Value $\dfrac{y_1+\cdots+y_n}{n} \longrightarrow \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$ ← continuous average

$a = x_0 < x_1 < \cdots < x_n = b$

$y_1 = f(x_1),\ y_2 = f(x_2),\ \ldots,\ y_n = f(x_n)$



$\Delta x = \dfrac{b-a}{n}$

Riemann sum $(y_1+\cdots+y_n)\Delta x \xrightarrow[\Delta x \to 0]{} \displaystyle\int_a^b f(x)\,dx$

Divide by $b-a$

$$\frac{y_1+\cdots+y_n}{b-a}\cdot\Delta x$$

$$\frac{\Delta x}{b-a} \to 0 \quad (n\to\infty)$$

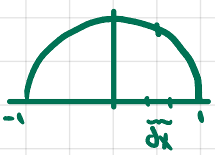
Example:

$f(x) = c$

$$\frac{1}{b-a}\int_a^b c\,dx = c$$

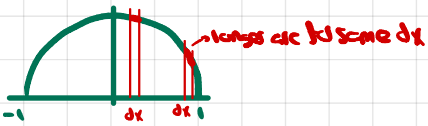
$\mathrm{Avg}(c) = c$ explains $\dfrac{1}{b-a}$ (and $\dfrac{1}{n}$)

Example: Average height of point on semi-circle



$$\frac{1}{1-(-1)}\underbrace{\int_{-1}^{1}\sqrt{1-x^2}\,dx}_{\text{area of unit semi-circle}} = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4} \to \text{avg height}$$

Example: Average height with respect to arc length θ.

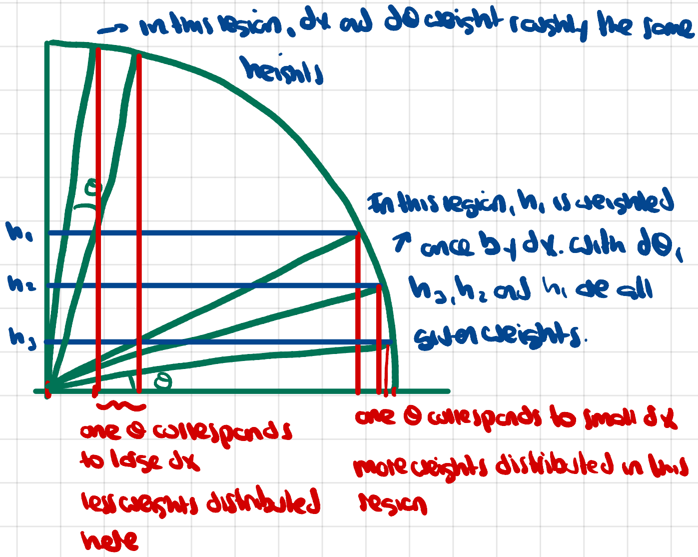


$0 \le \theta \le \pi$

$y = \sin\theta$

$$\frac{1}{\pi}\int_0^\pi \sin\theta\,d\theta = \frac{1}{\pi}(-\cos\theta)\Big|_0^\pi = -\frac{1}{\pi}(\overset{-1}{\cos\pi} - \overset{1}{\cos 0}) = \frac{2}{\pi}$$

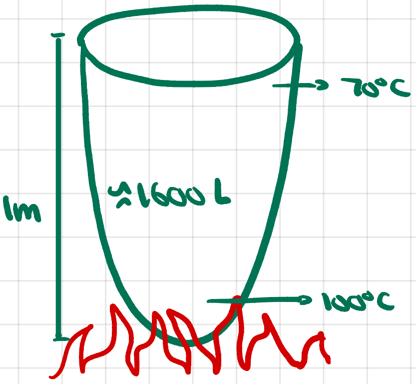
Note $\dfrac{2}{\pi} < \dfrac{\pi}{4}$

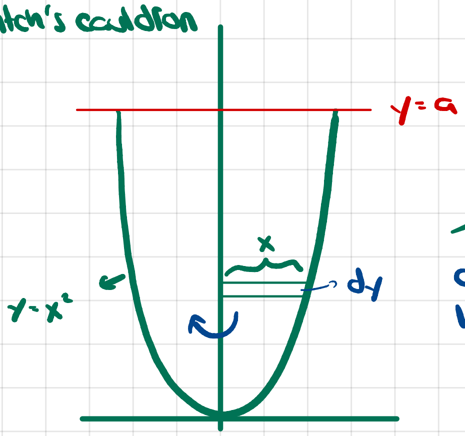


Weighted Average

$$\frac{\displaystyle\int_a^b f(x)\,w(x)\,dx}{\displaystyle\int_a^b w(x)\,dx}$$

Witch's Cauldron: boil water in witch's cauldron





$T = 100 - 30y$
constant temperature on same y level.

cal ft·lb
= heat = work

outside: 0°C = initial temp. cauldron, energy = volume · temperature
kWh
erg

How much heat do they need?

$$40\pi\ \cancel{°C}\cdot\cancel{m^3}\cdot\frac{1\ \text{cal}}{\cancel{°C}\cdot\cancel{cm^3}}\cdot\left(\frac{100\ \cancel{cm}}{\cancel{m}}\right)^3$$

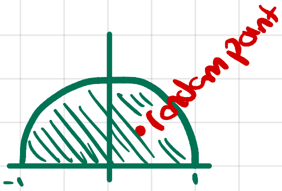
$$\int_0^1 T(\pi x^2)\,dy = \int_0^1 (100-30y)\pi y\,dy = (50\pi y^2 - 10\pi y^3)\Big|_0^1 = 40\pi\ °C\cdot m^3 = 40\pi\cdot 10^6\ \text{cal}$$
$$= 40\pi\cdot 10^3\ \text{kcal}$$
$$\approx 125\cdot 10^3\ \text{kcal}$$

Average Final Temperature

$$\frac{\int_0^1 T\,\pi y\,dy}{\int_0^1 \pi y\,dy} = \frac{40\pi}{\pi/2} = 80°$$ compare with $\frac{T_{max} + T_{min}}{2} = 85°$

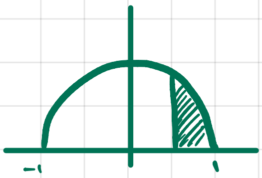
Probability

Pick a "point" at random in region $0 < y < 1 - x^2$



What is the probability that $x > 1/2$? ie, $P(x > 1/2)$

Probability = $\frac{\text{Part}}{\text{Whole}}$



$$\frac{\int_{1/2}^1 (1-x^2)\,dx}{\int_{-1}^1 (1-x^2)\,dx} = \frac{(x - \frac{x^3}{3})\Big|_{1/2}^1}{(x - x^3/3)\Big|_{-1}^1} = \frac{\left[1 - \frac{1}{3} - \left(\frac{1}{2} - \frac{1}{24}\right)\right]}{\left[1 - 1/3 - \left(-1 - \frac{-1^3}{3}\right)\right]}$$

weightings

$$= \frac{\frac{2}{3} - \frac{11}{24}}{\frac{2}{3} + \frac{2}{3}} = \frac{\frac{5}{24}}{\frac{4}{3}} = \frac{5}{\cancel{24}_{8}}\cdot\frac{\cancel{3}}{4} = \frac{5}{32}$$

General Formula

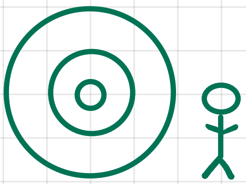
$a \le x_1 < x_2 \le b$ $\quad P(x_1 < x < x_2)$

$$\frac{\int_{x_1}^{x_2} w(x)\,dx}{\int_a^b w(x)\,dx}$$

Lecture 24

Dartboard example

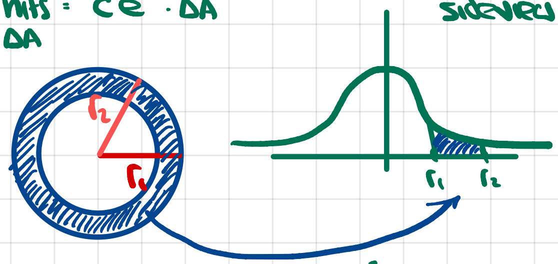
What is probability that the person gets hit by a dart?



Assumptions. bell curve (normal)

$\#$ hits in ΔA $= ce^{-r^2} \cdot \Delta A$

sideview of e^{-r^2}



The pdf e^{-r^2} sits atop the disks.

$r_1 < r = x < r_2$

shells $\displaystyle\int_{r_1}^{r_2} (2\pi r) e^{-r^2} dr = -\pi e^{-r^2} \Big|_{r_1}^{r_2} = \pi (e^{-r_1^2} - e^{-r_2^2})$

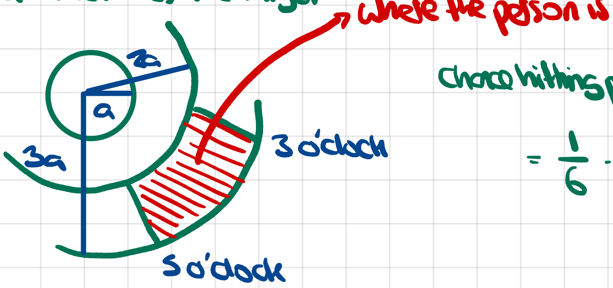
Part $= c\pi(e^{-r_1^2} - e^{-r_2^2})$

$\Rightarrow$ Probability $(r_1 < r < r_2) = \dfrac{\text{Part}}{\text{Whole}} = e^{-r_1^2} - e^{-r_2^2}$

Whole $= 0 \le r < +\infty$

$= c\pi(e^{-0^2} - e^{-\infty^2}) = c\pi$

a = radius of the target



Chance hitting person is $\frac{1}{6} P(2a < r < 3a)$

$= \frac{1}{6} \cdot \frac{1}{16} \approx \frac{1}{100} = 1\%$

$P(0 < r < a) = e^{-0^2} - e^{-a^2} = 1 - e^{-a^2}$

Assumption. $1 - e^{-a^2} = \frac{1}{2}$ $\Rightarrow$ thrower hit target $(r < a)$ half the time

$P(2a < r < 3a) = e^{(-2a)^2} - e^{(-3a)^2} = e^{(-a^2)^4} - e^{(-a^2)^9} = (\frac{1}{2})^4 - (\frac{1}{2})^9 \approx \frac{1}{16}$

Weight in this problem

$w(r) = 2\pi c r e^{-r^2}$

isn't the weight just $2\pi c r$?
the weight increases with r as the bands have larger areas.

Numerical Integration

→ many integrals don't have formulas : you have to compute them with a machine

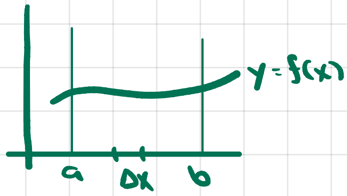
→ types

1. Riemann Sums (inefficient)

2. Trapezoidal Rule (inefficient)

3. Simpson's Rule

① Riemann Sums



$\Delta x_i = x_i - x_{i-1}$

$a = x_0 < x_1 < \cdots < x_n$

$y_0 = f(x_0) \ldots y_n = f(x_n)$

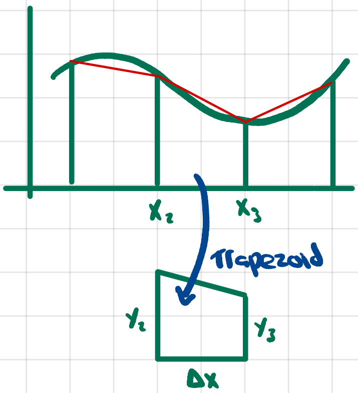
Goal : to "average" or add y's to get approximation to $\int_a^b f(x)dx$

$R = (y_0 + y_1 + \cdots + y_{n-1})\Delta x$ (left hand)

or

$(y_1 + \cdots + y_n)\Delta x$ (right hand)

② Trapezoidal Rule



Area = $\Delta x \cdot \frac{(y_2 + y_3)}{2}$

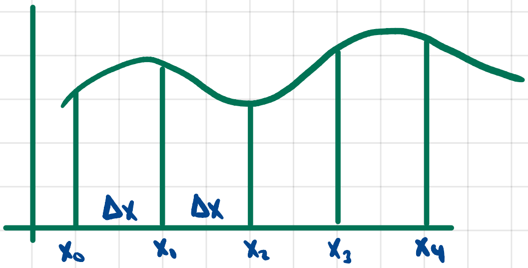
base × avg height

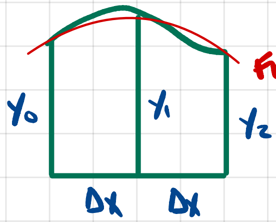
Add trapezoids to get formula

$$\Delta x \left(\frac{y_0 + y_1}{2} + \frac{y_1 + y_2}{2} + \cdots + \frac{y_{n-1} + y_n}{2} \right)$$

$$= \Delta x \left(\frac{y_0}{2} + y_1 + \cdots + y_{n-1} + \frac{y_n}{2} \right) = \frac{\text{Left Rieman Sum} + \text{Right Rieman Sum}}{2} = \text{Avg of L and R Riemann Sums}$$

③ Simpson's Rule





Area under parabola = base × avg height

computation on homework

$$2\Delta x\left(\frac{y_0+4y_1+y_2}{6}\right)$$

Total area = $\frac{2\Delta x}{6}\left[(y_0+4y_1+y_2)+(y_2+4y_3+y_4)+(y_{n-2}+4y_{n-1}+y_n)\right]$

pattern

```
1 4 1
    1 4 1
        1 4 1
-------------
1 4 2 4 2 4 1
```

$$\boxed{\text{Simpson's rule} = \frac{\Delta x}{3}\left[y_0+4y_1+2y_2+4y_3+2y_4+\cdots+2y_{n-2}+4y_{n-1}+y_n\right]}$$

Lecture 25

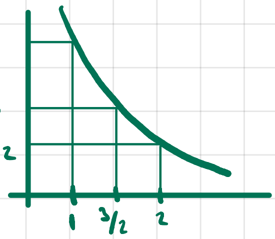
Numerical Integration (cont'd)

$$\int_1^2 \frac{dx}{x} = \ln x \Big|_1^2 = \ln 2 - \ln 1 = \ln 2 \approx 0.693147$$

Two intervals $y_0 = 1$

$y_1 = 2/3$

$y_2 = 1/2$



$a = 1$

$b = 2$

$b - a = 1$

$n = 2$

$\Delta x = \frac{b-a}{n} = \frac{1}{2}$

Trapezoidal rule: $\Delta x\left(\frac{1}{2}y_0 + y_1 + \frac{1}{2}y_2\right) = \frac{1}{2}\left[\frac{1}{2}\cdot 1 + \frac{2}{3} + \frac{1}{2}\cdot\frac{1}{2}\right] \approx 0.708$

Simpson's Rule: $\frac{\Delta x}{3}\left[y_0 + 4y_1 + y_2\right] = \frac{1}{6}\left[1 + 4\cdot\frac{2}{3} + \frac{1}{2}\right] \approx 0.69444$

$|\text{Simpson's} - \text{Exact}| \approx (\Delta x)^4$ e.g. $\Delta x \approx 1/10 \Rightarrow (\Delta x)^4 \approx 10^{-4}$

→ Simpson's Rule is derived using the exact answer for all degree 2 polynomials. Also exact for cubics.

→ Watch out $\frac{1}{x}$ for x near 0.

→ Mnemonic device: check $f(x) = 1$

Trapez. Rule $= \Delta x\left(\frac{y_0}{2} + y_1 + \cdots + y_{n-1} + \frac{y_n}{2}\right) = \Delta x\left((n-1)\cdot 1 + \frac{1}{2} + \frac{1}{2}\right) = \Delta x \cdot n = b - a$

Recall:

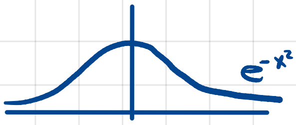
V = shell volume of 

rotated around axis:

circumf. ↗ thickness

$$\int_0^\infty 2\pi r e^{-r^2} dr = -\pi e^{-r^2} \Big|_0^\infty = \pi \Rightarrow V = \pi$$

height

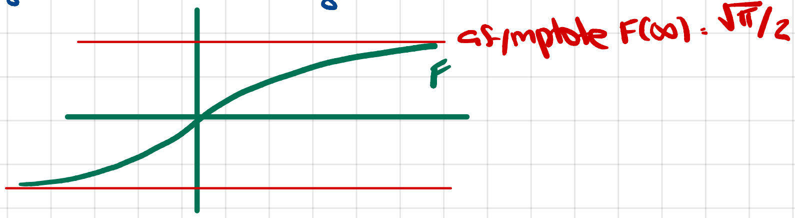
Now we want the value of: $Q = \int_{-\infty}^{\infty} e^{-t^2} dt$ = area under 

We will obtain it by relating it to the volume V of the solid whose volume we calculated using shells.

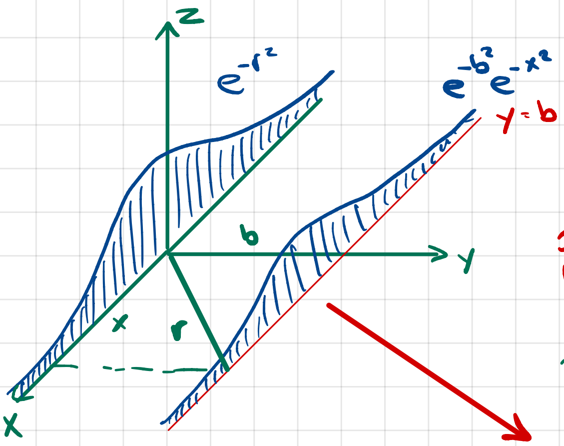
We will compute V by slices. The result will be $V = Q^2 \Rightarrow Q^2 = V = \pi \Rightarrow Q = \sqrt{\pi}$

We can then find the asymptotes of $F(x) = \int_0^x e^{-t^2} dt$, ie $F(\infty) = \int_0^\infty e^{-t^2} dt$, because

$Q = 2F(\infty) \Rightarrow F(\infty) = \sqrt{\pi}/2$



View of the slices in V:

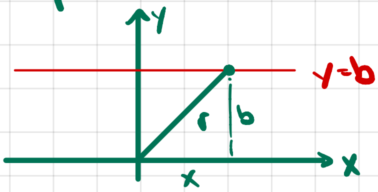


A(b) = area of slice y = b

If we find A(b), we can sum all slices (all possible values b) and obtain V of the solid:

$$V = \int_{-\infty}^{+\infty} A(y)\, dy$$

Top view:



To find A(b), fix y = b:

$r^2 = b^2 + x^2$, and on z axis has height e^{-r^2}

$e^{-r^2} = e^{-(b^2+x^2)} = e^{-b^2} \cdot e^{-x^2}$

⇒ each point on y = b has height $e^{-b^2} \cdot e^{-x^2}$

Therefore A(b) = area under some curve $e^{-b^2} e^{-x^2}$, for a constant e^{-b^2}

$$A(b) = \int_{-\infty}^{\infty} e^{-b^2} e^{-x^2} dx = e^{-b^2} \int_{-\infty}^{\infty} e^{-x^2} dx = e^{-b^2} \cdot Q$$

Now if we integrate the areas over all possible values of b (y = b, over $-\infty < b < \infty$), we are summing the infinitesimal volume slices:

$$V = \int_{-\infty}^{\infty} e^{-y^2} Q\, dy = Q \int_{-\infty}^{\infty} e^{-y^2} dy = Q^2$$

Exam 3 Review

1 Calculate definite integrals (FTC1, substitution)

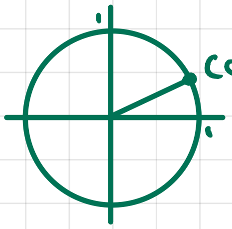
2 Numerical Approx.: Riemann, Trapez. Rule, Simpson's Rule

3 Areas / Volumes

4 Other cumulative sums (Avg Value, Probab., Work)

5 Sketch $F(x) = \int_a^x f(t)\,dt$

Lecture 27 - Trigonometric Integrals



$\sin^2\theta + \cos^2\theta = 1$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\sin(2\theta) = 2\sin\theta\cos\theta$

Half-angle formula

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$$

$$\Rightarrow \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

Similarly,

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

We already know:

$$d\sin x = \cos x\, dx \Rightarrow \int \cos x\, dx = \sin x + C$$

$$d\cos x = -\sin x\, dx \Rightarrow \int -\sin x\, dx = \cos x + C$$

① $\int \sin^n x \cdot \cos^m x\, dx$, $m, n \geq 0$

↳ this type of integral shows up a lot in science : important

easy case : at least one exponent is odd

ex1 $m=1$ $\quad \int \sin^n x \cdot \cos x\, dx = \int u^n du = \frac{u^{n+1}}{n+1} + C = \frac{(\sin x)^{n+1}}{n+1} + C$

substitution: $u = \sin x$

$du = \cos x\, dx$

here we are back in the same case of first example

ex2 $\int \sin^3 x \cos^2 x\, dx = \int (1-\cos^2 x)\sin x \cos^2 x\, dx = \int (\cos^2 x - \cos^4 x)\sin x\, dx$

use $\sin^2 x = 1 - \cos^2 x$ $\qquad u = \cos x,\ du = -\sin x\, dx$

$$= -\int (u^2 - u^4)\, du = -\frac{u^3}{3} + \frac{u^5}{5} + C = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$$

ex3 $\int \sin^3 x\, dx = \int (1-\cos^2 x)\sin x\, dx = \int (1-u^2)\, dx = u - \frac{u^3}{3} + C = \cos x - \frac{\cos^3 x}{3} + C$

$u = \cos x$

harder case: only even exponents

→ use half-angle formulas

ex1: $\int \cos^2 x\,dx = \int \frac{1+\cos(2x)}{2}\,dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C$

ex2: $\int \sin^2 x \cdot \cos^2 x\,dx = \int \frac{1-\cos(4x)}{8}\,dx = \frac{1}{8}x - \frac{\sin(4x)}{32} + C$

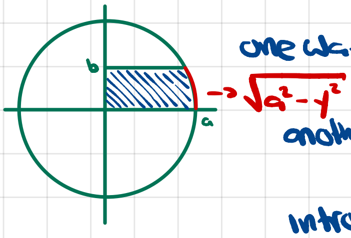
$\frac{(1-\cos 2x)(1+\cos 2x)}{4} = \frac{1-\cos^2(2x)}{4} = \frac{1}{4} - \frac{\cos^2(2x)}{4} = \frac{1}{4} - \left[\frac{1+\cos(4x)}{8}\right] = \frac{1-\cos(4x)}{8}$

same

Alternate method:

$\sin^2 x \cos^2 x = (\sin x \cos x)^2 = (\tfrac{1}{2}\sin(2x))^2 = \frac{\sin^2(2x)}{4} = \frac{1-\cos(4x)}{8}$

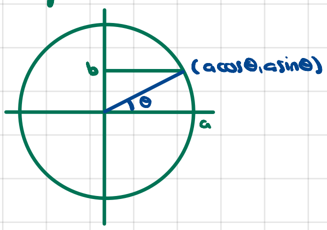
Application of Example of Trig. Substitution



one way: Area = $\int y\,dx$, but two different regions

another way: $\int_0^b x\,dy = \int_0^b \sqrt{a^2-y^2}\,dy$

introduce polar coordinates:



$y = a\sin\theta,\ dy = a\cos\theta\,d\theta$

$\sqrt{a^2-y^2} = \sqrt{a^2-a^2\sin^2\theta} = a\sqrt{1-\sin^2\theta} = a\cos\theta = x$

$\int\sqrt{a^2-y^2}\,dy = \int a\cos\theta \cdot a\cos\theta\,d\theta = \int a^2\cos^2\theta\,d\theta = a^2\int\cos^2\theta\,d\theta = a^2\left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right) + C$

$= a^2\left(\frac{\theta}{2} + \frac{2\cos\theta\sin\theta}{4}\right) + C = \frac{a^2\arcsin(y/a)}{2} + \frac{a\sin\theta \cdot a\cos\theta}{2} = \frac{a^2\arcsin(y/a) + y\sqrt{a^2-y^2}}{2} + C$

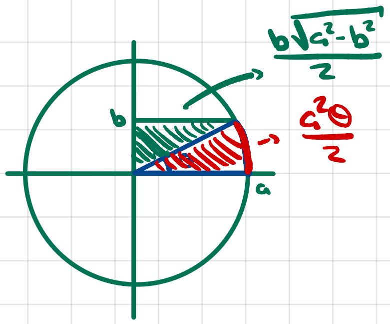
$\theta = \arcsin(y/a)$

ie, $\int\sqrt{a^2-y^2}\,dy = \frac{a^2\arcsin(y/a) + y\sqrt{a^2-y^2}}{2} + C$

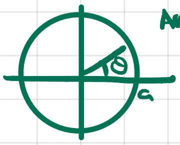
Area $= \int_0^b \sqrt{a^2-y^2}\,dy = \frac{a^2\arcsin(b/a) + b\sqrt{a^2-b^2}}{2} - \frac{a^2\cancel{\arcsin(0)}^{\,0} + 0}{2}$

note: $\arcsin(b/a) = \theta$

$= \frac{a^2\theta}{2} + \frac{b\sqrt{a^2-b^2}}{2}$



Formula for area of sector with angle θ:



Area = $\pi a^2 \cdot \frac{\theta}{2\pi} = \frac{a^2\theta}{2}$

Lecture 28

$\sec = \frac{1}{\cos}$ $\quad \csc = \frac{1}{\sin}$ $\quad \tan = \frac{\sin}{\cos}$ $\quad \cot = \frac{\cos}{\sin}$

$$\sec^2 x = \frac{1}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x$$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{-(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \tan x \cdot \sec x$$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx,\ u = \cos x,\ du = -\sin x\,dx,\ = -\int \frac{du}{u} = -\ln(u) + C = -\ln(\cos x) + C$$

$\int \sec x\,dx$

$$\frac{d}{dx}(\tan x + \sec x) = \sec^2 x + \tan x \sec x = \sec x(\sec x + \tan x)$$

$$u = \sec x + \tan x \Rightarrow u' = u \cdot \sec x \Rightarrow \sec x = \frac{u'}{u} = \frac{d}{dx}\ln(u) = \frac{d}{dx}\ln(\sec x + \tan x)$$

$\Rightarrow \int \sec x = \ln(\sec x + \tan x) + C$ logarithmic derivative

Example: $\int \sec^4 x\,dx = \int (1+\tan^2 x)\sec^2 x\,dx,\ u = \tan x,\ du = \sec^2 x\,dx,\ = \int (1+u^2)\,du = u + \frac{u^3}{3} + C$

$$= \tan x + \frac{\tan^3 x}{3} + C$$

Example: $\int \frac{dx}{x^2\sqrt{1+x^2}}$ we want to get rid of this square root

$x = \tan\theta \rightarrow$ trigon. substit.

$1 + x^2 = \sec^2\theta \Rightarrow \sqrt{1+x^2} = \sec\theta$

$dx = \sec^2\theta\,d\theta$

at this point, if we don't see a simplification, rewrite in terms of sin and cos

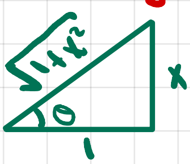
$$\int \frac{\sec^{\cancel{2}}\theta\,d\theta}{\tan^2\theta\,\cancel{\sec}\theta} = \int \frac{\sec\theta}{\tan^2\theta}d\theta = \int \frac{1}{\cancel{\cos}\theta}\cdot\frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}d\theta = \int \frac{\cos\theta}{\sin^2\theta}d\theta,\ u = \sin\theta,\ du = \cos\theta\,d\theta$$

need to undo the trigon. substit.

$$= \int \frac{du}{u^2} = -u^{-1} + C = -\sin^{-1}\theta + C = -\csc(\theta) + C$$

$$= -\frac{\sqrt{1+x^2}}{x} + C$$

* undoing trigon. substit.

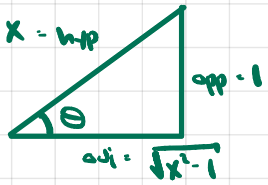


$\csc\theta = \frac{\sqrt{1+x^2}}{x}$

example: $\tan(\operatorname{arccsc} x)$

$\theta = \operatorname{arccsc}(x)$

$x = \csc(\theta) = \frac{1}{\sin\theta} = \frac{hyp}{opp}$



$\tan\theta = \tan(\operatorname{arccsc}(x)) = \frac{1}{\sqrt{x^2-1}}$

Summary of Trig Substitutions

If integral contains	Make substitution	To get
$\sqrt{a^2-x^2}$	$\begin{cases} x = a\cos\theta \\ x = a\sin\theta \end{cases}$ OR	$a\sin\theta$ $a\cos\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$a\sec\theta$
$\sqrt{x^2-a^2}$	$x = a \cdot \sec\theta$	$a^2\tan^2\theta$

Completing the square

Example: $\int \frac{dx}{\sqrt{x^2+4x}} = \int \frac{du}{\sqrt{u^2-4}}$, $u = 2\sec\theta, du = 2\sec\theta\tan\theta\, d\theta \Rightarrow \int \frac{1}{\cancel{2}\cancel{\tan}\theta} \cdot \cancel{2}\sec\theta \cancel{\tan}\theta\, d\theta$

rewrite quadratic as $(x+a)^2 + c$

$\sqrt{4(\sec^2\theta - 1)}$

$x^2+4x+4-4 = (x+2)^2 - 4$

$= \sqrt{4\tan^2\theta} = 2\tan\theta$

$u = x+2$, $du = dx$

$x^2+4x = u^2-4$

$= \int \sec\theta\, d\theta$

$= \ln(\sec\theta + \tan\theta) + C$

$= \ln\left(\frac{u}{2} + \sqrt{u^2-4}\right) + C$

$= \ln\left(\frac{x+2}{2} + \sqrt{(x+2)^2-4}\right. + C$

$= \ln\left(\frac{x+2}{2} + \sqrt{x^2+4x}\right) + C$

Lecture 29

$\frac{P(x)}{Q(x)}$ = rational function = ratio of two polynomials

Method of partial fractions: splits P/Q into "easier" pieces

Example: $\int \left(\frac{1}{x-1} + \frac{3}{x+2}\right) dx = \ln(x-1) + 3\ln(x+2) + C$

Difficulty:

$$\frac{1}{x-1} + \frac{3}{x+2} = \frac{x+2+3(x-1)}{(x-1)(x+2)} = \frac{4x-1}{x^2+x+2}$$ disguised, no longer clear how to integrate

Algebra problem: detect "easy" pieces

cover-up method

factor denominator

now solve for A and B

$$\frac{4x-1}{x^2+x+2} \overset{①}{=} \frac{4x-1}{(x-1)(x+2)} \overset{②}{=} \frac{A}{x-1} + \frac{B}{x+2}$$

solve for A by multiplying by $(x-1)$: $\frac{4x-1}{x+2} = A + \frac{B(x-1)}{x+2}$

Plug in $x=1$: $\frac{4-1}{1+2} = A = 1$

solve for B: multiply by $(x+2)$: $\frac{4x-1}{x-1} = \frac{A}{x-1}(x+2) + B$

Plug $x=-2$ $B = \frac{-8-1}{-3} = 3$

Steps

1 Factor Q (denom.)

2 "setup": target you are aiming for

3 "cover-up": solve for unknown coeff.

* cover-up works if Q(x) has distinct linear factors and degree(P) < degree(Q)

$$\frac{4x-1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$x=1$ cover up $x-1$	$x=-2$, cover up $(x+2)$
$\frac{4-1}{1+2} = A$	$\frac{4(-2)-1}{-2-1}$

Example 2 Q with repeated linear factors, $\deg(P) < \deg(Q)$

$$\frac{x^2+2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} \rightarrow$$

Analogy $\frac{7}{16} = \frac{0}{2^0} + \frac{0}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$

① step 1 done

cover-up works for coeff B and C, not A.

C: $\frac{(-2)^2+2}{(-2-1)^2} = C = \frac{6}{9} = \frac{2}{3}$

B: $\frac{1^2+2}{1+2} = B = 1$

actually a convenient number

For A, plug in your favorite number (if not yet used for other coeff.)

$x=0 \Rightarrow \frac{0^2+2}{-1^2 \cdot 2} = \frac{A}{-1} + \frac{1}{1} + \frac{2/3}{2}$

$1 = -A + 1 + 1/3 \Rightarrow A = 1/3$

Example 3 Q has a quadratic factor, $\deg(P) < \deg(Q)$

$$\frac{x^2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

cover-up for A: $\frac{1^2}{1^2+1} = A = \frac{1}{2}$

for B and C, clear denominator: $x^2 = \frac{1}{2}(x^2+1) + (Bx+C)(x-1)$

x^2 term: $1 = \frac{1}{2} + B \Rightarrow B = \frac{1}{2}$

x^0 term: $0 = \frac{1}{2} - C \Rightarrow C = \frac{1}{2}$

* alternatively, we could plug in numbers to find B, C

$$\int \frac{x^2}{(x-1)(x^2+1)}dx = \int \frac{1/2}{x-1}dx + \int \frac{(1/2)x}{x^2+1}dx + \int \frac{1/2}{x^2+1}$$

$$= \frac{1}{2}\ln|x-1| + \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + C$$

What if deg(P) > deg(Q)? → Analogy w/ numbers: improper fraction

Example: $\frac{x^3}{(x-1)(x+2)} = \frac{x^3}{x^2+x-2} =$

reverse factorization

long division

$$\begin{array}{r} x-1 \quad \text{→ quotient} \\ x^2+x-2 \,\overline{)\, x^3 \qquad\qquad} \\ \underline{x^3+x^2-2x} \\ -x^2-2x \\ \underline{-x^2-x+2} \\ -3x-2 \end{array}$$

→ remainder

$= x-1 + \frac{3x-2}{x^2+x-2}$

cover-up, deg(P) < deg(Q)

easy

Lecture 30

Partial Fractions (cont'd)

Always works, but maybe slowly

step 0: long division → $\frac{R(x)}{Q(x)} = \underbrace{p(x)}_{\text{quotient}} + \frac{\overbrace{R(x)}^{\text{remainder}}}{Q(x)}$

Note:

→ $\frac{\qquad}{(x+2)^4(x^2+2x+3)(x^2+4)^3}$

① Factorization is hard!

Factor Q has degree $4+2+3 = 12$

12 unknowns
12 equations ⇒ need machine

$$= \frac{A_1}{x+2} + \frac{A_2}{(x+2)^2} + \frac{A_3}{(x+2)^3} + \frac{A_4}{(x+2)^4} + \frac{B_0x+C_0}{x^2+2x+3} + \frac{B_1x+C_1}{x^2+4} + \frac{B_2x+C_2}{(x^2+4)^2} + \frac{B_3x+C_3}{(x^2+4)^3}$$

why 12 equations?

We know the coefficients on R(x).
We multiply the partial fractions by Q(x), and then equate the coefficients to obtain 12 eq. in 12 unknowns.

Partial Fractions still need to be integrated

e.g. $\int \frac{x}{(x^2+4)^3}dx$, easy

$\int \frac{dx}{(x^2+4)^3}$, trig. substitution $x = 2\tan u$

$\int \frac{dx}{x^2+2x+3}$, complete the square

Integration by Parts

$(uv)' = u'v + uv' \Rightarrow uv' = (uv)' - u'v \Rightarrow \int uv'\,dx = uv - \int u'v\,dx$

For definite integral:

$$\int_a^b uv'\,dx = uv\Big|_a^b - \int u'v\,dx$$

Ex: (Reduction Formula)

$\int \ln(x)^n dx \qquad u = \ln(x)^n \quad du = n\ln(x)^{n-1}\frac{1}{x}$
$\qquad\qquad\qquad dv = dx\,,\ v = x$

$= x\ln(x)^n - \int x n \ln(x)^{n-1} \cdot \frac{1}{x}\,dx = x\ln(x)^n - n\int \ln(x)^{n-1}dx$

If $F_n(x) = \int \ln(x)^n dx$ then $F_n(x) = x\ln(x)^n - nF_{n-1}(x)$

$n \to n-1 \to n-2 \to \cdots \to 1 \to 0$ (idea of a reduction formula)

$F_0(x) = \int dx = x$

$F_1(x) = \int \ln x\,dx = x\ln x - F_0(x) = x\ln x - x$

$F_2(x) = \int \ln(x)^2 dx = x\ln(x)^2 - 2F_1(x) = x\ln(x)^2 - 2[x\ln x - x]$

also works for cosx, sinx

Ex: $\int x^n e^x dx \qquad u = x^n \quad du = nx^{n-1}$
$\qquad\qquad\qquad dv = e^x dx \quad v = e^x$

$= x^n e^x - \int e^x n x^{n-1} dx$

reduction formula

$\Rightarrow G_n(x) = \int x^n e^x dx = x^n e^x - nG_{n-1}(x)$

$G_0(x) = \int e^x dx = e^x$

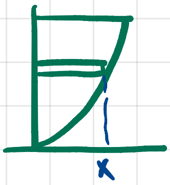
$G_1(x) = x^1e^x - 1e^x = e^x(x-1)$

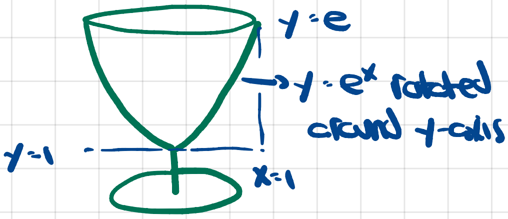
$G_2(x) = x^2e^x - 2e^x(x-1)$

Ex: (Application) Find volume of exponential wine glass

Two methods: horizontal / vertical slices

a) horizontal



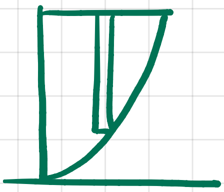


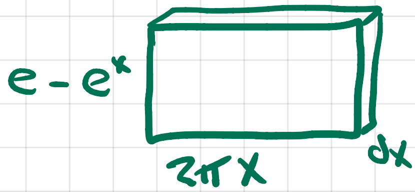
$$\int_1^e \pi x^2\,dy = \int_1^e \pi(\ln y)^2\,dy = \pi\int_1^e \ln(y)^2\,dy =$$

$$= \pi\left[y\ln y - 2(y\ln y - y)\right]\Big|_1^e = \pi\left[(e\ln e - 2\cancelto{0}{(e\ln e - e)}) - (1\cancelto{0}{\ln 1} - 2(1\cancelto{0}{\ln 1} - 1))\right]$$

$$= \pi[e-2]$$

b) vertical slicing (shells)





$$\int_0^1 2\pi x(e-e^x)\,dx$$

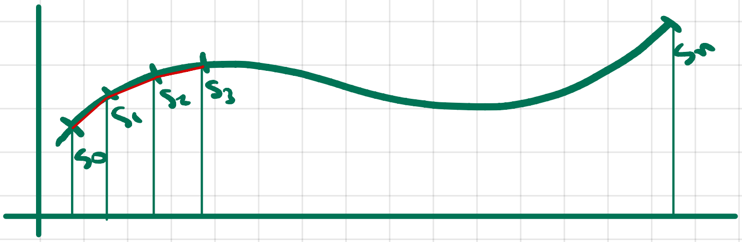
$$= 2\pi e\int_0^1 x\,dx - 2\pi\int_0^1 xe^x\,dx$$

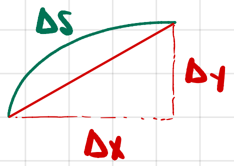
$$= 2\pi e\,\frac{x^2}{2}\Big|_0^1 - 2\pi e^x(x-1)\Big|_0^1 = \pi e - 2\pi\left[e^1(0) - e^0(-1)\right]$$

$$= \pi e - 2\pi(1) = \pi(e-2)$$

Lecture 31

Arclength





$\Delta s = s_i - s_{i-1}$

$\Delta s^2 \approx (\Delta x)^2 + (\Delta y)^2$

$(ds)^2 = (dx)^2 + (dy)^2$

$ds^2 = dx^2 + dy^2$

$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + (dy/dx)^2}$

Arclength $= s_n - s_0 = \int_a^b \sqrt{1 + (dy/dx)^2}\, dx$ $\left(= \text{informally, } \int ds\right)$

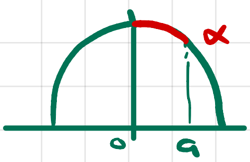
$= \int_a^b \sqrt{1 + f'(x)^2}\, dx$

Ex 1 $y = mx$ $\quad y' = m$ $\quad ds = \sqrt{1 + (y')^2}\, dx = \sqrt{1 + m^2}\, dx$

length $0 \le x \le 10$

$\int_0^{10} \sqrt{1+m^2}\, dx = 10\sqrt{1+m^2}$

Ex 2 $y = \sqrt{1-x^2}$ $\quad y' = \frac{-x}{\sqrt{1-x^2}}$

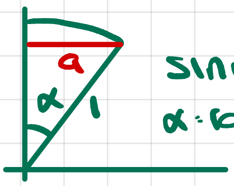


$ds = \sqrt{1 + \left(\frac{x^2}{1-x^2}\right)}\, dx = \sqrt{\frac{1}{1-x^2}}\, dx = \frac{dx}{\sqrt{1-x^2}}$

$\alpha = \int_0^a \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_0^a = \sin^{-1} a$

↖ Fundamental definition of $\sin x$, radians

Geometric Interpret.:



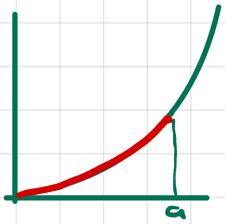
$\sin \alpha = a$

α = radians

Ex 3 length of parabola

$y = x^2 \quad y' = 2x$

$ds = \sqrt{1 + (2x)^2}$



long calculation

$$\int_0^a \sqrt{1+4x^2}\,dx = (\dots) = \left[\frac{1}{4}\ln\left(2x + \sqrt{1+4x^2}\right) + \frac{1}{2}x\sqrt{1+4x^2}\right]_0^a$$

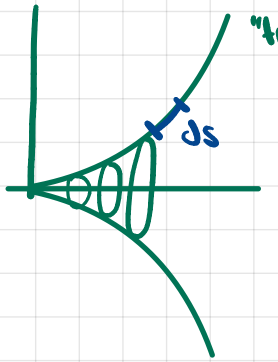
$x = \frac{1}{2}\tan u$

$dx = \frac{1}{2}\sec^2 u\,du$

Surface Area

Ex: Surface of Rotation

$y = x^2$ rotated around x-axis



circumference

we want surface area $dA = \int 2\pi y \cdot ds$

aka. S length

from previous ex

$$= \int_0^a 2\pi x^2 \sqrt{1+4x^2}\,dx = (\dots)$$

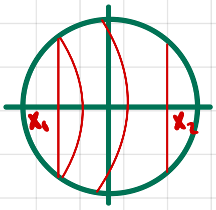
long calculation
trig substit. $x = \frac{\tan u}{2}$

Ex: Surface Area of Sphere

$y = \sqrt{a^2 - x^2}$

$y' = \frac{-x}{\sqrt{a^2 - x^2}}$

$\Rightarrow 1 + (y')^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2}$



$$\text{Area} = \int_{x_1}^{x_2} 2\pi y\,ds = \int_{x_1}^{x_2} 2\pi \cancel{\sqrt{a^2 - x^2}} \cdot \frac{a}{\cancel{\sqrt{a^2 - x^2}}}\,dx = \int_{x_1}^{x_2} 2\pi a\,dx = 2\pi a(x_2 - x_1)$$

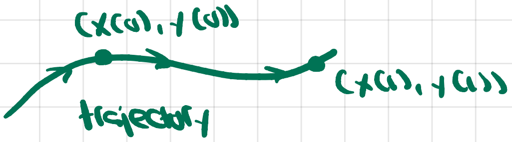
Special cases

hemisphere: $x_1 = 0,\ x_2 = a \Rightarrow \text{Area} = 2\pi a^2$

whole sphere: $x_1 = -a,\ x_2 = a \Rightarrow \text{Area} = 2\pi \cdot (a - (-a)) = 4\pi a$

Parametric Curves

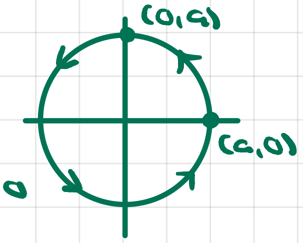
$x = x(t)$ $\quad$ t : parameter
$y = y(t)$



ex: $x = a\cos t$
$y = a\sin t$

$x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2 \Rightarrow$ circle

which direction are we going in on circle?



$t = 0 \Rightarrow (x,y) = (a\cos 0, a\sin 0) = (a,0)$

$t = \frac{\pi}{2} \Rightarrow (x,y) = (a\cos\frac{\pi}{2}, a\sin\frac{\pi}{2}) = (0,a)$

$\Rightarrow$ counterclockwise

Lecture 32 - Parametric Curves (cont'd)

Arclength

differential of arclength

$$ds^2 = dx^2 + dy^2 \Rightarrow ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$$

$$\frac{dx}{dt} = -a\sin t \qquad \frac{dy}{dt} = a\cos t$$

$$ds = \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt = \sqrt{a^2}\, dt = a\,dt$$

$$\frac{ds}{dt} = a$$ speed

new speed: $x = a\cos kt$, $y = a\sin kt \Rightarrow \frac{ds}{dt} = ak \quad (a>0, k>0)$

Notation

$$\Delta s^2 \approx \Delta x^2 + \Delta y^2$$

$$\left(\frac{\Delta s}{\Delta t}\right)^2 \approx \left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2$$

$$\frac{ds}{dt} = \sqrt{(dx/dt)^2 + (dy/dt)^2}$$

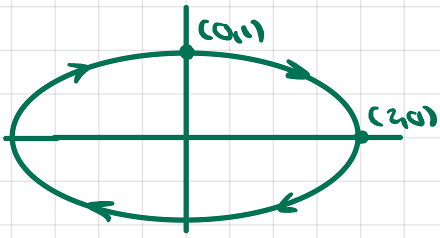
Note

use $\left(\frac{dx}{dt}\right)^2$

never $\cancel{\frac{dx^2}{dt^2}}$, which is different from $\frac{d^2x}{dt^2} = \left(\frac{d}{dt}\right)^2 x$

Ex: $x = 2\sin t$
$y = \cos t$

$\frac{1}{4}x^2 + y^2 = \sin^2 t + \cos^2 t = 1$ (ellipse)



$t = 0 \quad x = 2\cdot 0 = 0 \quad y = 1 \quad (0,1)$

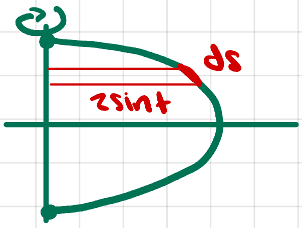
$t = \frac{\pi}{2} \quad x = 2\cdot 1 = 2 \quad y = 0 \quad (2,0)$

$$\frac{ds}{dt} = \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{(2\cos t)^2 + (-\sin t)^2}$$

$$\text{arclength} = \int_0^{2\pi} \sqrt{4\cos^2 t + \sin^2 t}\, dt$$

not elementary integral

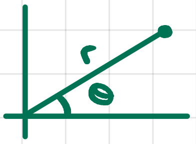
surface area
→ area of surface of ellipsoid formed by revolving prev. ex around y-axis



$$dA = \int 2\pi \cdot 2\sin t\, ds = \int_0^{\pi} 4\pi \sqrt{4\cos^2 t + \sin^2 t}\, dt$$

doable but long : $u = \cos t$ to start

Polar coordinates



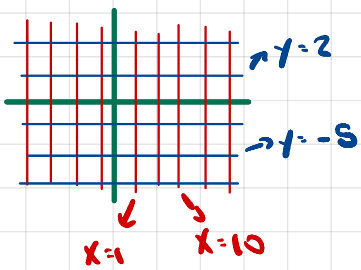
r = distance to origin

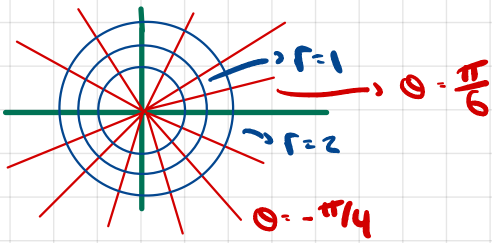
θ = angle of ray from origin with horizontal axis

$x = r\cos\theta$
$y = r\sin\theta$

$r = \sqrt{x^2 + y^2}$
$\theta = \tan^{-1}(y/x)$

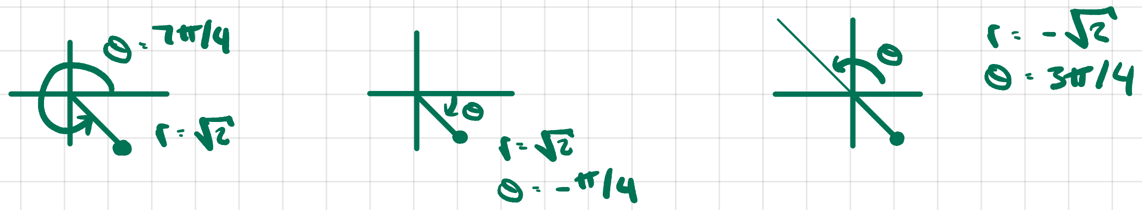
somewhat ambiguous, need to look at diagram sometimes (because signs)





Ex: $(x,y) = (1,-1)$

Repres. in polar coord.:



Ex2 $r = a$ (circle)

Ex3 $\theta = c$ (ray), implicitly assumes $0 \le r < \infty$

$-\infty < r < \infty$ gives whole line through origin

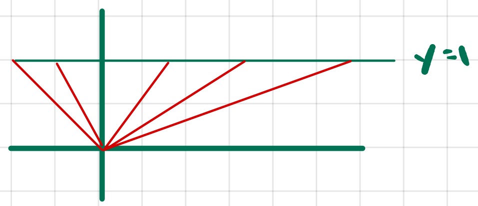
Typical conventions, used most of the time

$0 \le r < \infty$

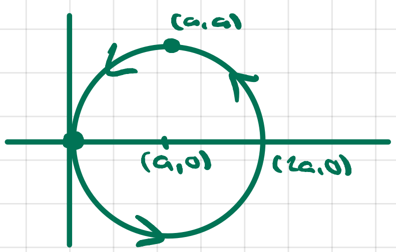
$-\pi < \theta \le \pi$ or $0 \le \theta \le 2\pi$

Ex4 $y = 1$

In polar coord: $y = r\sin\theta = 1 \Rightarrow r = \dfrac{1}{\sin\theta}$ $\quad 0 < \theta < \pi$



Ex 5 Off-center circle



$(x-a)^2 + y^2 = a^2$

$x^2 - 2ax + \cancel{a^2} + y^2 = \cancel{a^2}$

$r^2 - 2ax = r^2 - 2ar\cos\theta = 0$

$r(r - 2a\cos\theta) = 0$

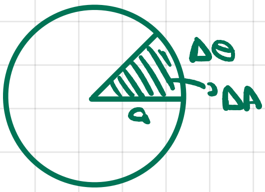
$r = 0$ or $r = 2a\cos\theta$ $\quad -\pi/2 \le \theta \le \frac{\pi}{2}$

θ	r	x	y	
$\pi/2$	0	0	$r=0$	$(0,0)$
$\pi/4$	$2a\frac{\sqrt{2}}{2} = a\sqrt{2}$	$\sqrt{2}a \cdot \frac{\sqrt{2}}{2} = a$	$\sqrt{2}a \cdot \frac{\sqrt{2}}{2} = a$	(a,a)
0	$2a$	$2a \cdot 1 = 2a$	$2a \cdot 0 = 0$	$(2a,0)$
$-\pi/2$	0	0	$-r = 0$	$(0,0)$

Lecture 33

Polar coordinates (cont'd)

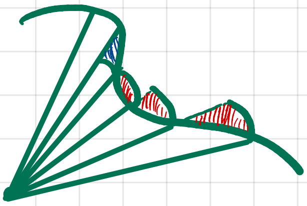
Area "Pie"

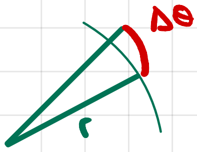


total $A = \pi a^2$

$\Delta A = \frac{\Delta\theta}{2\pi}\pi a^2 = \frac{1}{2}a^2\Delta\theta$

"Variable Pie"





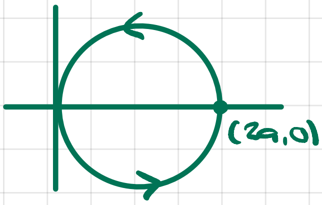
$\Delta A \approx \frac{1}{2}r^2\Delta\theta$

$A = \int \frac{1}{2}r^2 d\theta \quad (r = r(\theta))$

ex1: (area)

$r = 2a\cos\theta \Rightarrow (x-a)^2+y^2 = a^2$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$



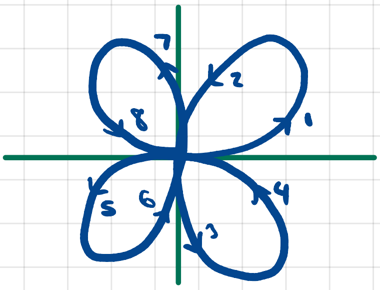
$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\cdot(2a\cos\theta)^2 d\theta = 2a^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\, d\theta = a^2\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)d\theta = a^2\left(\theta + \frac{\sin(2\theta)}{2}\right)\Big|_{-\pi/2}^{\pi/2}$$

$$= a^2(\theta + \sin\theta\cos\theta)\Big|_{-\pi/2}^{\pi/2} = a^2(\pi/2 + \pi/2) = a^2\pi$$

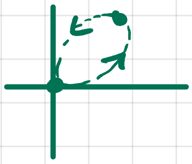
ex2: (drawing)

$r = \sin(2\theta)$

"4-leaf rose"

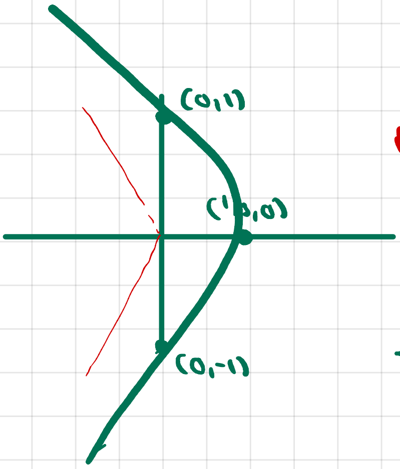


θ	r
0	0
$\pi/4$	1
$\pi/2$	0



ex3: $r = \frac{1}{1+2\cos\theta}$

θ	r	x	y
0	1/3		
$\pi/2$	1		
$-\pi/2$	1		



$2\cos\theta = -1 \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \pm\frac{2\pi}{3}$

rectangular eq.?

$r + 2r\cos\theta = 1 \Rightarrow r = 1-2x$

$r^2 = (1-2x)^2 \Rightarrow x^2+y^2 = 1-4x+4x^2$

$-3x^2+y^2+4x-1=0$ (hyperbola)

$r=0$ is the focus of the hyperbola

Kepler's Law $\quad \frac{dA}{dt} = \text{const}$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

hence, $r^2 \frac{d\theta}{dt}$ is constant

conservation of angular momentum

Exam 4 Review

1. Techniques of Integr.

   Trig substit.
   By Parts
   Partial Fractions

2. Parametric curves
   Arc length
   Area of Surface of Revol.

3. Polar coord. including area

Lecture 35

L'Hôpital's Rule

→ convenient way to calculate limits, including new ones

→ ex: $x \ln x \quad x \to 0^+$

$xe^{-x} \quad x \to \infty$

$\ln x / x \quad x \to \infty$

$f(x) = x^n - 1$

$f(1) = 0$

$f'(1) = \lim_{x \to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1} \frac{x^n - 1}{x-1}$

$= nx^{n-1}$

Ex1: $\lim_{x \to 1} \frac{x^{10}-1}{x^2-1} = \lim_{x \to 1} \frac{(x^{10}-1)/(x-1)}{(x^2-1)/(x-1)} = \frac{10x^9}{2x}\Big|_{x=1} = 5$

$\frac{0}{0}$, indeterminate form

We want to carry this out systematically

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x)/x-a}{g(x)/x-a} = \frac{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x \to a} \frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}$$

works if $g'(a) \neq 0$

$f(a) = g(a) = 0$

L'Hôpital's Rule (version 1)

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided $f(a) = g(a) = 0$ and the right-hand limit exists

subtle and important conditions here: $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ has to exist; note we are considering behavior at $x \neq a$, x near a

Ex2 $\lim_{x \to 0} \frac{\sin(5x)}{\sin(2x)} \overset{\text{L'Hôpital}}{=} \lim_{x \to 0} \frac{5\cos(5x)}{2\cos(2x)} = \frac{5}{2}$

Ex3 $\frac{\cos x - 1}{x^2} \underset{x \to 0}{\sim} \frac{-\sin x}{2x} \sim \frac{-\cos x}{2} = -\frac{1}{2}$

$x \to 0, \to \frac{0}{0}$

L'Hôpital

* both of these methods are valid

Linear Approxim.

comparison with method of approximations

In ex2, $\sin(u) \approx u$ because $\sin(u) \approx \sin(0) + \cos(0)(u-0)$

$\approx u$

so $\frac{\sin 5x}{\sin 2x} \approx \frac{5x}{2x} = \frac{5}{2} \quad x \to 0$

In ex3, $\frac{\cos x - 1}{x^2} \approx \frac{(1 - x^2/2) - 1}{x^2} = \frac{-x^2/2}{x^2} = -\frac{1}{2}$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$ Prov. $f(a)=g(a)=0$, rh side exists

other cases allowed:

→ $a = \pm\infty$

we can handle $\frac{\infty}{\infty}$ case

→ $f(a), g(a) = \pm\infty$

→ rhs exists or could be $\pm\infty$

Ex 4 $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$

$0\cdot(-\infty)$ winner, $0/0$

Ex 5 $\lim_{x\to\infty} xe^{-px} = \lim_{x\to\infty}\frac{x}{e^{px}} = \lim_{x\to\infty}\frac{1}{pe^{px}} = 0$

$\infty\cdot 0$, $\frac{\infty}{\infty}$

⇒ x grows more slowly than e^{px}, $p>0$

Ex 5' $\lim_{x\to\infty}\frac{e^{px}}{x^{100}} = \lim\left[\frac{e^{px/100}}{x}\right]^{100} = \left[\lim\frac{e^{px/100}}{x}\right]^{100} = \left[\lim\frac{\frac{p}{100}e^{px}}{1}\right]^{100} = [\infty]^{100}$

$\frac{\infty}{\infty}$

⇒ e^{px} $p>0$ grows faster than any power of x

Ex 6 $\lim_{x\to\infty}\frac{\ln x}{x^{1/3}} = \lim\frac{1/x}{(1/3)x^{-2/3}} = \lim\left[3x^{-1/3}\right] = 0$

$\frac{\infty}{\infty}$

⇒ $\ln x$ grows more slowly than $x^{1/3}$ or any positive power of x

* Rule applies in cases $0/0$ or ∞/∞, a anything including $\pm\infty$

rhs must be either finite or $\pm\infty$

Another indeterminate form: 0^0

$$\lim_{x\to 0^+} x^x = \lim e^{x\ln x} = \lim e^{\ln x/(1/x)} = e^0 = 1$$

moving exponent so use base e

$$\frac{\sin x}{x^2} \underset{x\to 0}{\sim} \frac{\cos x}{2x} \sim \cancel{\frac{-\sin x}{2}} \xrightarrow[x\to 0]{} \cancel{0}$$

INCORRECT USE L'HOPITAL

lin. approx.:

Don't use L'Hôpital as a crutch!

$\sin x \sim x$ near 0

$$\frac{\sin x}{x^2} \approx \frac{x}{x^2} = \frac{1}{x} \xrightarrow[x\to 0^+]{} \infty$$

could apply L'Hôp five times
but better to:

$$\frac{x^5 - 2x^4 + 1}{x^4 + 2} \longrightarrow \frac{1 + 2/x + 1/x^5}{1/x + 2/x^5} \sim \frac{1}{1/x} \longrightarrow \infty$$

$\frac{x^5}{x^4}$ is the main term!

Lecture 36

Dealing with infinity (cont'd)

Recall: L'Hôpital's Rule

$\frac{\infty}{\infty}$ case

→ $f(x) \to \infty$
$g(x) \to \infty$
$\frac{f'(x)}{g'(x)} \to L$ as $x \to a$ $\Rightarrow$ $\frac{f(x)}{g(x)} \to L$ as $x \to a$

→ $\left.\begin{array}{l} a = \pm\infty \\ L = \pm\infty \end{array}\right\}$ are ok

Rates of Growth

$f(x) \ll g(x)$ means $\frac{f(x)}{g(x)} \to 0$ as $x \to \infty$

new notation

implicit assumption: $f, g > 0$

$\ln x \ll x^p \ll e^x \ll e^{x^2}$ $\quad p > 0$

All go to ∞ but at different rates

Rate of Decay

Take reciprocals

$$\frac{1}{\ln x} \gg \frac{1}{x^p} \gg \frac{1}{e^x} \gg \frac{1}{e^{x^2}}$$

Improper Integrals

Def: $\int_a^\infty f(x)dx = \lim_{N\to\infty} \int_a^N f(x)dx$

area is finite

The integral converges if the limit exists, diverges if not.

area is infinite

Ex 1: $\int_0^\infty e^{-kx}dx \quad k>0$

$$= \lim_{N\to\infty} \int_0^N e^{-kx}dx = \lim_{N\to\infty} \left[\frac{e^{-kx}}{-k} \right]_0^N = \lim_{N\to\infty} \left[-\frac{e^{-kN}}{k} - \left(-\frac{e^0}{k}\right)\right] = \lim_{N\to\infty} \left[\frac{1}{k} - \frac{1}{ke^{kN}} \right] = \frac{1}{k}$$

Shorthand

$$\int_0^\infty e^{-kx}dx = -\frac{1}{k}e^{-kx}\Big|_0^\infty = \cancel{-\frac{1}{ke^\infty}}^{\,0} - \left(-\frac{1}{k}\right) = 1/k$$

Physical Interpretation (Ex 1)

of particles on average that decay in a radioactive substance in time $0 \le t \le T$.

$\int_0^T Ae^{-kt}dt$ = # particles

$\int_0^\infty Ae^{-kt}dt$ = total # particles

Ex 2: $\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$

Key number in probability

Ex 3: $\int_1^\infty \frac{dx}{x} = \ln x \Big|_1^\infty = \infty - 0$

diverges

$$\int_1^\infty \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\Big|_1^\infty = \frac{\infty^{1-p}}{1-p} - \frac{1}{1-p}$$

$p<1 \Rightarrow \infty$, diverges

$p>1 \Rightarrow \frac{1}{p-1}$, converges

Limit Comparison

If $f(x) \sim g(x)$ as $x \to \infty$ then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge.

means $\frac{f(x)}{g(x)} \to 1$

Ex $\int_0^\infty \frac{dx}{\sqrt{x^2+10}} \sim \int_1^\infty \frac{dx}{x}$, which diverges (ignore $\int_0^1 \frac{dx}{\sqrt{x^2+10}}$)

$\sqrt{x^2+10} \sim \sqrt{x^2} = x$

note $\int_0^1 \frac{dx}{x}$ is ∞ for other reasons

Ex $\int_{10}^\infty \frac{dx}{\sqrt{x^3+3}} \sim \int_{10}^\infty \frac{dx}{x^{3/2}}$ convergent

$\frac{1}{\sqrt{x^3+3}} \sim \frac{1}{\sqrt{x^3}}$

Ex check convergence, not evaluation

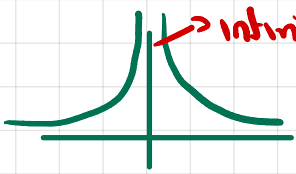
$$\int_{-\infty}^\infty e^{-x^2}\,dx = 2\int_0^\infty e^{-x^2}\,dx$$

$e^{-x^2} \le e^{-x}$ $\quad x \ge 1 \Rightarrow x^2 \ge x \Rightarrow -x^2 \le -x$

$$\int_{-\infty}^\infty e^{-x^2}\,dx \le 2\int_0^1 e^{-x^2}\,dx + 2\int_1^\infty e^{-x}\,dx$$

larger integrand $\Rightarrow$ converges

Improper Integral of 2nd Type



$\int_0^1 \frac{dx}{\sqrt{x}}$, $\int_0^1 \frac{dx}{x}$, $\int_0^1 \frac{dx}{x^2}$ are typical examples

consider $\int_{-1}^1 \frac{dx}{x^2} = -x^{-1}\Big|_{-1}^1 = -(1^{-1}) - (-(-1)^{-1}) = -1-1 = -2$ (WRONG)

Lecture 37

Improper Integrals, 2nd Type (singularity at finite place)

Def: $\int_0^1 f(x)dx = \lim_{a\to 0^+} \int_a^1 f(x)dx$

converges if limit exists, diverges if not

Ex: $\int_0^1 \frac{dx}{\sqrt{x}} = \int_0^1 x^{-1/2}dx = 2\cdot x^{1/2}\Big|_0^1 = 2-0=2$ convergent

Ex: $\int_0^1 \frac{dx}{x} = \ln x\Big|_0^1 = \ln 1 - \ln 0 = 0-(-\infty) = +\infty$ divergent

In general,

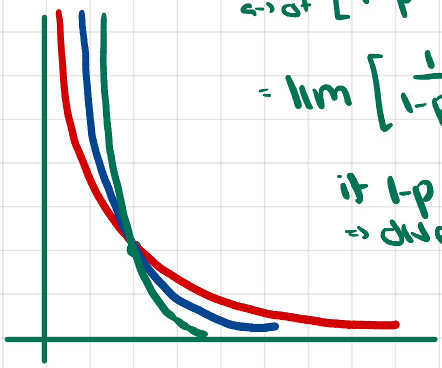
$\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}$ if $p\le 1$

diverges if $p>1$

$$\int_0^1 x^{-p}dx = \lim_{a\to 0^+}\int_a^1 x^{-p}dx = \lim_{a\to 0^+} \frac{x^{-p+1}}{-p+1}\Big|_a^1$$

$$= \lim_{a\to 0^+}\left[\frac{1}{1-p} - \left(\frac{a^{-p+1}}{-p+1}\right)\right]$$

$$= \lim\left[\frac{1}{1-p} + \frac{1}{1-p}a^{1-p}\right]$$



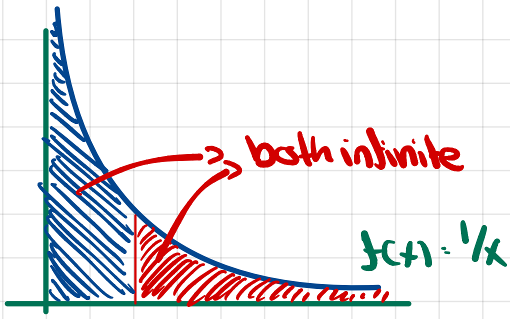
if $1-p<0$, there is a $+\infty$ term.
$\Rightarrow$ divergence. $p>1 \Rightarrow$ divergence

Contrast

divergent integrals

$\frac{1}{x^{1/2}} \ll \boxed{\frac{1}{x}} \ll \boxed{\frac{1}{x^2}}$ as $x\to 0^+$

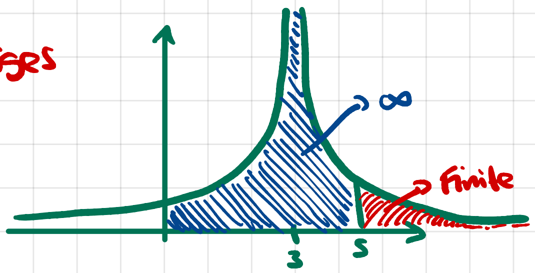
$\boxed{\frac{1}{x^{1/2}}} \gg \boxed{\frac{1}{x}} \gg \frac{1}{x^2}$ as $x\to\infty$



Ex: $y = \frac{1}{(x-3)^2}$

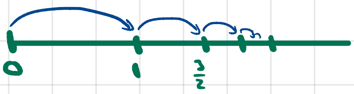
$$\int_0^\infty \frac{1}{(x-3)^2}dx = \underbrace{\int_0^5 \frac{1}{(x-3)^2}dx}_{\text{infinite}} + \underbrace{\int_5^\infty \frac{1}{(x-3)^2}dx}_{\text{finite}}$$

diverges



Infinite Series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=2$$



General Case: Geometric Series

$$1+a+a^2+a^3+\cdots=\frac{1}{1-a}, \quad |a|<1$$

Divergences

$a=1 \Rightarrow \quad 1+1+1+\cdots+\cdots = \cancel{\frac{1}{1-1}=\frac{1}{0}}=\infty$ diverges

$a=-1 \Rightarrow \quad 1-1+1-1+1+\cdots = \cancel{\frac{1}{1-(-1)}=\frac{1}{2}}$ diverges

$a=2 \Rightarrow \quad 1+2+4+8+\cdots = \cancel{\frac{1}{1-2}=-1}$ diverges

Notation

$S_N=\sum_{n=0}^{N} a_n$ partial sum

$S=\sum_{n=0}^{\infty} a_n=\lim_{n\to\infty} S_n$ ↗ exists ⇒ series converges. ↘ does not exist ⇒ series diverges

Ex: $\sum_{n=1}^{\infty}\frac{1}{n^2} \longleftrightarrow \int_1^{\infty}\frac{dx}{x^2}$ convergent $=1$

tricky to evaluate

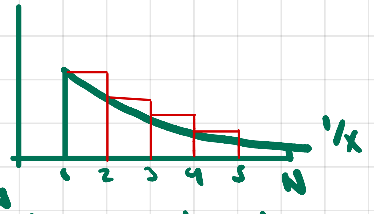
$=\pi^2/6$

Ex2: $\sum_{n=1}^{\infty}\frac{1}{n^3} \longleftrightarrow \int_1^{\infty}\frac{dx}{x^3}$ convergent

only recently proved to be irrational

Ex 3 $\sum_{n=1}^{\infty} \frac{1}{n} \leftrightarrow \int_1^{\infty} \frac{dx}{x}$ diverges

upper Riemann sum ($\Delta x = 1$)



$$\int_1^N \frac{dx}{x} < 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N-1} < S_N$$

$\ln x \big|_1^N = \ln N < S_N \Rightarrow N \to \infty \Rightarrow S_N \to \infty$

lower Riemann sum

$$\int_1^N \frac{dx}{x} > \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N} = S_N - 1$$

$\ln N > S_n - 1 \Rightarrow S_N < \ln N + 1$

$\Rightarrow \ln N < S_N < \ln N + 1$

Integral comparison

If $f(x)$ is decreasing, $f(x) > 0$, then $\left| \sum_{n=1}^{\infty} f(n) - \int_1^{\infty} f(x)\,dx \right| < f(1)$ and $\int_1^{\infty} f(x)\,dx$

and $\sum_{n=1}^{\infty} f(n)$ converge or diverge together.

Limit comparison

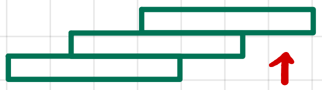
If $f(n) \sim g(n)$ and $g(n) > 0 \ \forall n$ then $\sum f(n)$ and $\sum g(n)$ both diverge or converge.

$\frac{f(n)}{g(n)} \to 1$ as $n \to \infty$

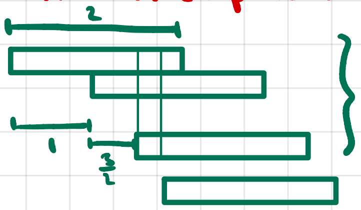
Ex: $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}} \leftrightarrow \sum \frac{1}{n}$ diverges

Ex: $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^5 - n}} \leftrightarrow \sum \frac{1}{\sqrt{n^5}} = \sum \frac{1}{n^{5/2}}$ converges

Lecture 38



Start with the top block



$$C_{n+1} = \frac{nC_n + 1\cdot(C_n+1)}{n+1}$$

$$= \frac{(n+1)C_n + 1}{n+1} \Rightarrow C_{n+1} = C_n + \frac{1}{n+1}$$

$C_0 = 1$

$C_1 = 1 + 1/2$

$C_2 = 1 + 1/2 + 1/3$

$C_3 = 1 + 1/2 + 1/3 + 1/4$

$C_n = \sum_{i=1}^{n} \frac{1}{i} = S_n$ $\qquad \ln n < S_n < \ln n + 1$

$n \to \infty \Rightarrow \ln n \to \infty \Rightarrow S_n \to \infty$

$\Rightarrow$ in principle you can stack an infinite number of blocks.

To get across the lab tables, length 13 blocks = 26 units (each block length 2)

$\ln N = 24 \Rightarrow N = e^{24}$

if block has height 3cm, $3\text{cm} \times e^{24} \approx 8 \cdot 10^8\,\text{m}$, twice the distance to the moon.

Power Series

$$1+x+x^2+x^3+\dots = \frac{1}{1-x} \quad |x|<1 \quad \text{(geometric series)}$$

$$1+x+x^2+\dots = S$$

multiply by

$$x+x^2+\dots+ = S\cdot x$$

subtract

$$1 = S - Sx = S(1-x) \Rightarrow S = \frac{1}{1-x}$$

The above reasoning is incomplete because requires that S exist, e.g. $x=1$

works when series are convergent, ie $|x|<1$

General Setup

$$a_0+a_1x+a_2x^2+a_3x^3+\dots = \sum_{n=0}^{\infty} a_n x^n$$ ← power series

$|x|<R$ (radius of convergence)

$-R<x<R$ where there is convergence.

$|x|>R \Rightarrow \sum a_n x^n$ diverges, $|a_n x^n| \not\to 0$

$|x|=R$ very delicate, borderline, not used by us

$|x|<R \Rightarrow |a_n x^n| \to 0$ exponentially

Comments

$$1+x+x^2+x^3+\dots = \frac{1}{1-x} \quad |x|<1$$

→ we can look at such eq. from both directions both are typically interesting.

→ series are flexible enough to represent all of the functions seen in this course so far

→ analogous to decimal expansions giving us a handle on all real numbers

Rules for convergent power series are just like polynomials

$f+g,\ f\cdot g,\ f(g(x)) = f\circ g,\ f/g,\ \frac{d}{dx}f,\ \int f\,dx$

we can do all these for power series

$$\frac{d}{dx}\left[a_0 + a_1x + a_2x^2 + \cdots\right] = a_1 + 2a_2x + 3a_3x^2 + \cdots$$

$$\int (a_0 + a_1x + \cdots)\,dx = C + a_0x + a_1\frac{x^2}{2} + \cdots$$

Taylor's Formula

$$f(x) = \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(0)}{n!}}_{a_n} x^n$$

$0! = 1$

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$$

$$f''(x) = 2\cdot1a_2 + 3\cdot2a_3x + 4\cdot3a_4x^2$$

$$f'''(x) = 3\cdot2\cdot1a_3 + 4\cdot3\cdot2a_4x + \cdots \qquad f'''(0) = 3\cdot2\cdot1a_3$$

in general, $f^{(n)}(0) = n!\,a_n \Rightarrow a_n = \dfrac{f^{(n)}(0)}{n!}$

Ex: $e^x = f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$

$f'(x) = f''(x) = \cdots = e^x$

$f^{(n)}(0) = 1$

$\Rightarrow e^1 = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

Ex: $f(x) = \sin(x) \approx x$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Lecture 39

Review of Power Series

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, generalized polynomials

ex: polynomial

$f(x) = a_0 + a_1 x + \cdots + a_n x^n$

one caution: there is R, $0 \leq R \leq \infty$ so that $|x| < R \Rightarrow f(x)$ converges, and $|x| > R \Rightarrow$ divergence of f

Radius of convergence

For $|x| < R$, $f(x)$ has all its derivatives.

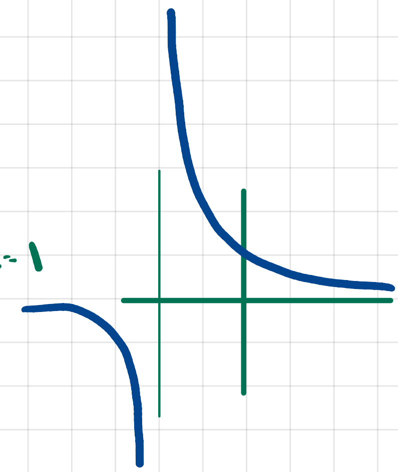
$$a_n = \frac{f^{(n)}(0)}{n!} \quad \text{(Taylor's Formula)}$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

Ex: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots +$ $\quad R = \infty$

Ex2: $\frac{1}{1+x} = f(x) = 1 - x + x^2 - x^3 + x^4$ $\quad R = 1$

↳ sum of infinite series $\sum (-x)^n$



Ex3 $f(x) = \sin(x)$

$f' = \cos x$
$f'' = -\sin x$
$f''' = -\cos x$
$f^{(4)} = \sin x$

$$f(x) = 0 + \cos(0)x - \frac{\sin 0}{2!}x^2 - \frac{\cos 0}{3!}x^3 + \frac{\sin 0}{4!}x^4$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

general term: $\frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1} \cdot \frac{x}{2} \cdot \frac{x}{3} \cdots \frac{x}{(2n+1)}$

fixed x

n grows

$\xrightarrow[n \to \infty]{} 0$ for any fixed x

$R = \infty$

New power series from old

Note even powers : odd × odd = even function

1 multiply

$$x\sin x = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \dots \qquad R = \infty$$

2 differentiate (maintains radius of convergence)

$$\frac{d}{dx}\sin x$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \dots \qquad (R = \infty)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \cos x$$

3 integrate

$$\int_0^x \frac{dt}{1+t} = \ln(1+x) \quad x > -1$$

R = 1

R = 1

$$= \int_0^x \left[1 - t + t^2 - t^3 + \dots\right] dt = \left[t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots\right]_0^x = \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right]$$

power series expansion of $\ln(1+x)$

4 substitute

$$e^{-t^2} = 1 + (-t^2) + \frac{(-t^2)^2}{2!} + \frac{(-t^2)^3}{3!} + \dots$$

$x = -t^2$ in e^x

$$= 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - \dots$$

error function - $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dx$ (so that $\lim_{x\to\infty} \text{erf}(x) = 1$)

$$= \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \dots\right]$$