# 10.3 Taylor series

→ so far, infinite series have constant terms, and if there is convergence, it is to a number

→ many functions have representations as infinite series with variable terms

$f(x) = \frac{1}{1-x}$ $|x|<1$ has infinite series representation $\frac{1}{1-x} = \sum_{i=0}^{n} x^n = 1 + x + x^2 + \dots$

$S_n = \sum_{i=1}^{n} x^i$ is now an $n^{th}$ degree polynomial approximating $f$.

$\frac{1}{1-x} \approx S_n$

Given $f(x)$, we want to approximate $f(x_0)$ with a polynomial $P(x)$ that is close to $f$ on an interval near $x_0$.

Simplest approx: linear approx. $f(x) \approx P_1(a) = f(a) + f'(a)(x-a)$

Notice $P_1(a) = f(a)$, $P_1'(a) = f'(a)$

In general, we can search for $P_n(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n$ such that

$$P_n(a) = f(a),\ P_n'(a) = f'(a), \dots, P_n^{(n)}(a) = f^{(n)}(a)$$

we have $n+1$ coefficients and $n+1$ equations.

It is easier if we define $P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n$

→ $P_n(a) = c_0 = f(a)$

$P_n'(x) = c_1 + 2c_2(x-a) + \dots \Rightarrow P_n'(a) = c_1 = f'(a)$

$P_n''(x) = 2c_2 + 3\cdot2\cdot c_3(x-a) + \dots \Rightarrow P_n''(a) = 2c_2 = f''(a) \Rightarrow c_2 = \frac{f''(a)}{2}$

$P_n'''(x) = 3\cdot2\cdot1\, c_3 + 4\cdot3\cdot2(x-a) \Rightarrow P_n'''(a) = 3!\,c_3 = f'''(a) \Rightarrow c_3 = \frac{f'''(a)}{3!}$

$(\dots)$

$P_n^{(n)}(x) = n!\,c_n(x-a) \Rightarrow P_n^{(n)}(a) = n!\,c_n = f^{(n)}(a) \Rightarrow c_n = \frac{f^{(n)}(a)}{n!}$

## Theorem: $n^{th}$ degree Taylor Polynomial

first $n$ derivatives of $f$ exist at $x=a$ ⇒ At $x=a$, $P_n(x)$ and all its deriv. agree w/ $f$ and its derivat.

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

→ $n^{th}$ degree Taylor Polyn. of $f$.

## Taylor's Formula

$R_n(x) = f(x) - P_n(x)$

$\swarrow$

$n^{th}$ degree remainder for $f(x)$ at $x=a$

## Theorem

$(n+1)$st derivative of $f$ exists on interval containing $a$ and $b$ $\Rightarrow$

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(z)}{(n+1)!}(b-a)^{n+1}$$

for some $z$ between $a$ and $b$

note:

$n=0 \Rightarrow f(b) = f(a) + f'(z)(b-a)$, the MVT

## $n^{th}$ degree Taylor formula with remainder at $x=a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

## Taylor Series

- If $f$ has derivatives of all orders, Taylor's formula can be written to any degree $n$ we want.
- Normally we don't know $z$.
- Sometimes we can show that $n \to \infty \Rightarrow R_n(x) \to 0$ for a particular $x$.

$f(x) = P_n(x) + R_n(x)$

$$\lim_{n\to\infty} [P_n(x) + R_n(x)] = \lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$\searrow$ Taylor Series of $f$ at $a$

Partial sums are successive Taylor polynomials for each $n$

- If $\lim R_n(x) = 0$ then the Taylor series converges to $f(x)$
- Examples: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e = \sum_{n=0}^{\infty} \frac{1}{n!}$

The number $\pi$

$$\frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \dots + (-1)^{k-1}x^{k-1} + \frac{(-1)^k x^k}{1+x}$$

To verify the above, multiply both sides by $1+x$

$$(1+x)\left(1 - x + x^2 - x^3 + \dots + (-1)^{k-1}x^{k-1} + \frac{(-1)^k x^k}{1+x}\right)$$

$$= 1 - \cancel{x} + \cancel{x^2} - \cancel{x^3} + \dots + \cancel{(-1)^{k-1}x^{k-1}} + \frac{(-1)^k x^k}{1+x}$$
$$+ \cancel{x} - \cancel{x^2} + \cancel{x^3} + \dots + \cancel{(-1)^{k-2}x^{k-1}} + (-1)^{k-1}x^k + \frac{(-1)^k x^{k+1}}{1+x}$$

$$= 1 + \frac{x^k(-1)^k}{\cancel{1+x}}\cancel{(1+x)} + (-1)^{k-1}x^k$$

$$= 1 + x^k\left((-1)^k + (-1)^{k-1}\right) = 1$$

Could it be done simpler?

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} x^k \cdot (-1)^k$$

$$(1+x)\sum_{k=0}^{\infty} x^k \cdot (-1)^k = \sum_{k=0}^{\infty} x^k(-1)^k + \sum_{k=0}^{\infty} x^{k+1}(-1)^k$$
$$= 1 + \underbrace{\sum_{k=1}^{\infty} x^k(-1)^k}_{\cancel{\sum_{k=0}^{\infty} x^{k+1}(-1)^{k+1}}} + \cancel{\sum_{k=0}^{\infty} x^{k+1}(-1)^k}$$

$$\Rightarrow (1+x)\cdot\frac{1}{1+x} = 1$$

substitute $t^2$ for $x$, $n+1$ for $k$: $\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 \dots + (-1)^n t^{2n} + \frac{(-1)^{n+1}(t^2)^{n+1}}{1+t^2}$

$D_t \tan^{-1} t = \frac{1}{1+t^2}$. Integrate from 0 to $x$:

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + R_{2n+1}$$

$$|R_{2n+1}| = \left|\int_0^x \frac{t^{2n+2}}{1+t^2}\,dx\right| \le \left|\int_0^x t^{2n+2}\,dx\right| = \frac{|x|^{2n+3}}{2n+3}$$

$\lim_{n\to\infty} R_n = 0$ because $\lim_{n\to\infty} \frac{|x|^{2n+3}}{2n+3} = 0$ for $|x| \le 1$

$\Rightarrow$ we have the Taylor series for $\tan^{-1}x$

$$\tan^{-1}x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}(-1)^n$$

For $x=1$, Leibniz's Series: $\sum \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$