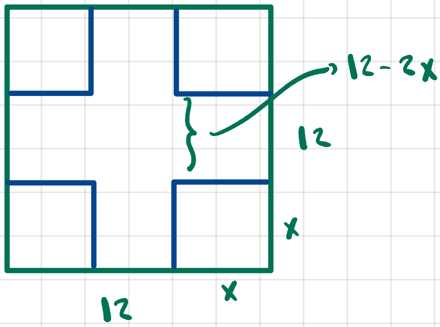
Pset 4

2C-1 $V = (12-2x)^2 \cdot x \quad x \in [0,6]$

$= (144 - 48x + 4x^2)x$

$= 4x^3 - 48x^2 + 144x$



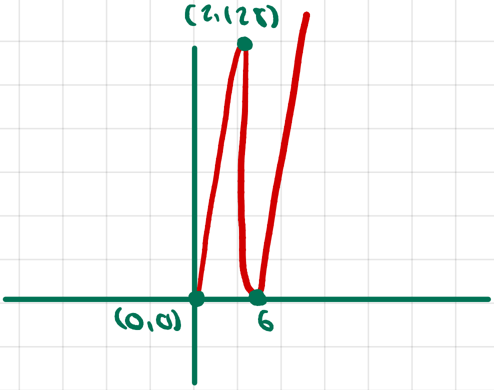
critical points

$V' = 12x^2 - 96x + 144 = 0$

$\Rightarrow x^2 - 8x + 12 = 0 \qquad \Delta = 64 - 48 = 16 \qquad x = \frac{8 \pm 4}{2} \nearrow 6 \searrow 2$

$V(2) = 8^2 \cdot 2 = 128$

$V(6) = 0$

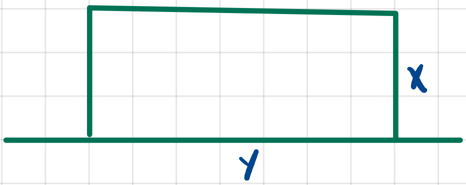


endpoints of domain

$V(0) = 0$

$V(6) = 0$

2C-2



$A = x \cdot y = 2 \cdot 10^4$ (constraint) $\Rightarrow y = \frac{2 \cdot 10^4}{x}$

$F = 2x + y \Rightarrow F(x) = 2x + \frac{2 \cdot 10^4}{x} = \frac{2x^2 + k}{x} \qquad k = 2 \cdot 10^4,\ x \in [0, \infty)$

$F'(x) = \frac{4x}{x} + (2x^2 + k)(-1)/x^2 = (4x^2 - 2x^2 - k)/x^2 = (2x^2 - k)/x^2 = 0$

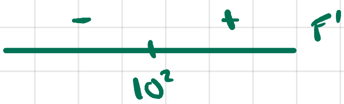
$\Rightarrow 2x^2 = 2 \cdot 10^4 \Rightarrow x^2 = 10^4 \Rightarrow x = 10^2 \Rightarrow y = 2 \cdot 10^2$

$F(10^2) = 2 \cdot 10^2 + 2 \cdot 10^2 = 400$

$x = 0 \Rightarrow y = \infty \Rightarrow F = \infty$

$x = \infty \Rightarrow y = 0 \Rightarrow F = \infty$

Note $F'(x) = \frac{2(x^2 - 10^4)}{x^2}$

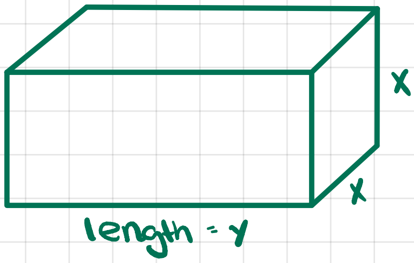


$\Rightarrow (10^2, 400)$ is the global min of $F(x)$.

ie $x < 10^2 \Rightarrow F(x) > F(10^2)$, and $F'(x) < 0$. $F(0) = +\infty \Rightarrow F$ decreases from $+\infty$ down to the min

$x > 10^2 \Rightarrow F(x) > F(10)^2$, $F(x) > 0 \Rightarrow F$ increases without bound as $F(\infty) = \infty$

2C-4



$y + 4x = 108$

$V = x^2 y = x^2(108-4x) = 108x^2 - 4x^3 \quad x \in [0,\infty)$

$V'(x) = 216x - 12x^2 = 0$

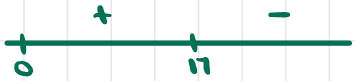
$x(216-12x) = 0 \Rightarrow x = 0$ or $x = 17$

$V(0) = 0$

$x = 17 \Rightarrow y = 108 - 68 = 40$

$V(17) = 17^2 \cdot 40 = 289 \cdot 40 = 11560$

Note the sign of V':



$\Rightarrow (17, 11560)$ is global max

2C-10

$d_L = \sqrt{100^2 + x^2} \qquad d_W = \sqrt{100^2 + (a-x)^2}$

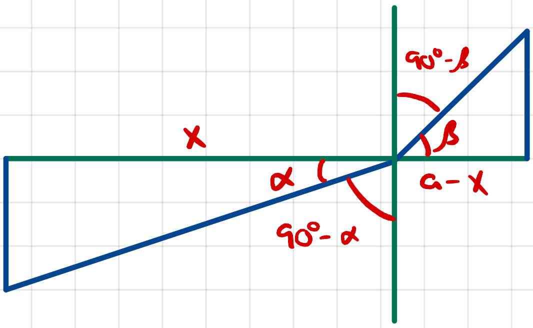
$$T = \frac{(100^2+x^2)^{1/2}}{5} + \frac{(100^2+(a-x)^2)^{1/2}}{2}$$

$$T' = \frac{1}{\cancel{2}\cdot 5}(100^2+x^2)^{-\frac{1}{2}}(\cancel{2}x) + \frac{1}{2}\cdot\frac{1}{\cancel{2}}(100^2+(a-x)^2)^{-1/2}\cdot\cancel{2}(a-x)(-1)$$

$$= \frac{x}{5\sqrt{100^2+x^2}} - \frac{a-x}{2\sqrt{100^2+(a-x)^2}} = 0 \Rightarrow \frac{1}{5}\,\frac{x}{\sqrt{100^2+x^2}} = \frac{1}{2}\,\frac{a-x}{\sqrt{100^2+(a-x)^2}}$$

But, $\cos\alpha = \dfrac{x}{\sqrt{100^2+x^2}}$, $\cos\beta = \dfrac{a-x}{\sqrt{100^2+(a-x)^2}}$ $\Rightarrow T' = \dfrac{\cos\alpha}{5} - \dfrac{\cos\beta}{2} \Rightarrow \dfrac{\cos\alpha}{\cos\beta} = \dfrac{5}{2}$

That, however is not the usual formulation of Snell's Law.



$\cos\alpha = \sin(90° - \alpha)$
$\cos\beta = \sin(90° - \beta)$

$$\Rightarrow \frac{\sin(90°-\alpha)}{\sin(90°-\beta)} = \frac{5}{2}$$

2C-13 y = passengers, x = price

a) $y(200) = 100$ $\quad y - 100 = -\frac{2}{5}(x-200) \Rightarrow y = 100 - \frac{2}{5}(x-200) = -\frac{2x}{5} + 180$

$\Rightarrow y > 0 \Rightarrow \frac{2}{5}x - 80 < 100 \Rightarrow \frac{2}{5}x < 180 \Rightarrow x < 450,\ x \in [0, 450]$

$R(x) = x \cdot y(x) = -\frac{2x^2}{5} + 180x$

$R'(x) = -\frac{4x}{5} + 180 = 0 \Rightarrow x = \frac{5 \cdot 180}{4} = 45 \cdot 5 = 225$

$R''(x) = -\frac{4}{5} < 0$ concave down $\Rightarrow x = 225$ is a max of R

b)

	Amount	Price	
production	x kWh/day	$10 - x \cdot 10^{-5}$ cents/kWh	$x \in [0, 8 \cdot 10^5]$
consumption	$10^5(10 - P/2)$ kWh/day	p cents/kWh	

$D(p)$ = demand = $10^5(10 - \frac{p}{2})$

$R(p)$ = revenue = $D(p) \cdot p$

$C(p)$ = cost = $D(p) \cdot (10 - D(p) \cdot 10^{-5})$

$Q(p)$ = profit = $D(p) \cdot p - D(p)(10 - D(p) \cdot 10^{-5}) = D \cdot p - 10D + D^2 \cdot 10^{-5}$

we assume utility sets price and produces amount demanded at that price.
because of production constraint, there is price constraint:

$D(p) \in [0, 8 \cdot 10^5] \Rightarrow D(p) = 0 = 10^6 - \frac{10^5 p}{2} \Rightarrow p = 2 \cdot 10 = 20$

$D'(p) = -\frac{10^5}{2}$ $\qquad D(p) = 8 \cdot 10^5 = 10^6 - \frac{10^5 p}{2} \Rightarrow p = \frac{2}{10^5} \cdot 2 \cdot 10^5 = 4$ $\qquad \Rightarrow p \in [4, 20]$

$Q'(p) = D'p + D - 10D' + 2DD' \cdot 10^{-5} = D'(p-10) + D + 2DD' 10^{-5}$

$= -\frac{10^5}{2}(p-10) + \cancel{10^5(10-\frac{p}{2})} + \cancel{2} \cdot \cancel{10^5(10-\frac{p}{2})}(-\frac{\cancel{10^5}}{\cancel{2}})\cancel{10^{-5}} = -\frac{10^5}{2}(p-10) = 0$

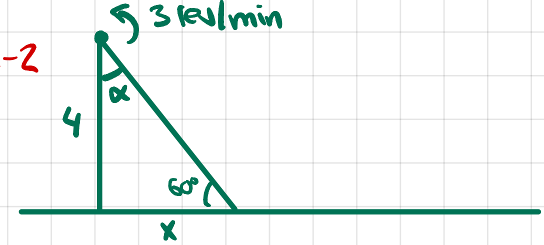
$\Rightarrow p = 10$ is critical point.

$D(10) = 5 \cdot 10^5$, $Q(10) = 5 \cdot 10^5 \cdot 10 - 5 \cdot 10^5(10 - 5 \cdot \cancel{10^5} \cdot \cancel{10^{-5}}) = 50 \cdot 10^5 - 25 \cdot 10^5 = 25 \cdot 10^5$

$Q''(p) = -\frac{10^5}{2} < 0 \Rightarrow$ concave down $\Rightarrow p = 10$ is global max

Note, $D(4) = 10^5 \cdot 8$, $Q(4) = 8 \cdot 10^5 \cdot 4 - 8 \cdot 10^5(10 - 8 \cdot \cancel{10^5} \cdot \cancel{10^{-5}}) = 32 \cdot 10^5 - 16 \cdot 10^5 = 16 \cdot 10^5$

2E-2



3rev/min = 6π rad/min

$$\frac{d\alpha}{dt} = 6\pi \text{ rad/min}$$

$$\tan\alpha = \frac{x \cancel{\text{miles}}}{4 \cancel{\text{miles}}}$$

$$x = 4\text{miles} \times \tan\alpha$$

$$x' = 4\sec^2\alpha \,\frac{\text{miles}}{\cancel{\text{rad}}} \cdot \frac{d\alpha}{dt}\,\frac{\cancel{\text{rad}}}{\text{minute}}$$

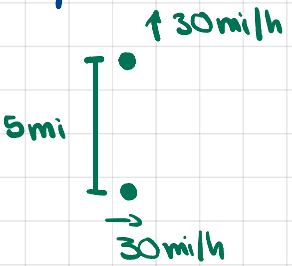
when shoreline angle is $\pi/3$, $\alpha = \frac{\pi}{6}$

$$x' = 4 \cdot \sec^2\frac{\pi}{6} \cdot 6\pi$$

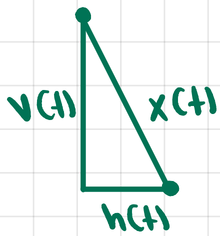
$$= 4 \cdot \frac{4}{\cancel{3}} \cdot \cancel{6}^{2}\pi = 32\pi \text{ miles/minute}$$

2E-3

initially:



1/3 h later:



$$\frac{30\text{mi}}{60\text{min}} = \frac{10\text{mi}}{20\text{min}}$$

$$\frac{30\text{mi}}{60\text{min}} \cdot 10\text{min} = 5\text{mi}$$

$v(0) = 5 \quad h(0) = 0$

$v(1/3) = 15 \quad h(1/3) = 10$

$v(t) = 5 + 30 \cdot t$

$h(t) = 30t$

we want $x'(1/3)$

$$x(t)^2 = v(t)^2 + h(t)^2 \Rightarrow x(t) = \sqrt{(5+30t)^2 + (30t)^2}$$

$$x'(t) = \frac{1}{2} \cdot \left((5+3t^2) + (30t)^2\right)^{1/2} \left(2(5+30t) \cdot 30 + 2 \cdot 30t \cdot 30\right.$$

$$= \frac{(300 + 1800t + 1800t)}{2 \cdot \sqrt{(5+30t)^2 + (30t)^2}}$$

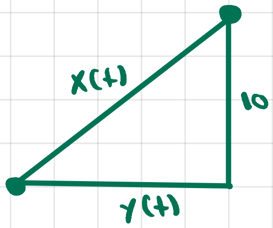
$$x'(1/3) = \frac{300 + 600 + 600}{2\sqrt{15^2 + 10^2}} = \frac{1500}{2\sqrt{325}} = \frac{750}{\sqrt{325}} = \frac{750}{5\sqrt{13}} = \frac{150}{\sqrt{13}} \text{ mi/h}$$

Alternative, easier way:

$$x^2 = v^2 + h^2 \Rightarrow xx' = vv' + hh' \Rightarrow x' = \frac{vv' + hh'}{x}$$

$$x'(1/3) = \frac{15 \cdot 30 + 10 \cdot 30}{\sqrt{325}} = \frac{750}{\sqrt{325}}$$

2E-5



$\frac{dx}{dt} = 4$

$x^2 = 10^2 + y^2$ what is $y'(t)$ when $y(t) = 20$?

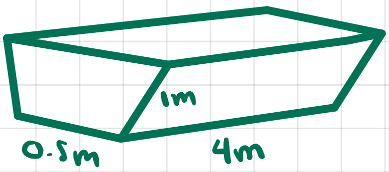
$xx' = yy'$

$y' = \frac{xx'}{y}$

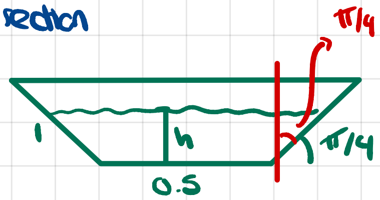
At t_*, $y(t_*) = 20$, $x(t_*) = \sqrt{500}$, $x' = 4$

$y' = \frac{4 \cdot \sqrt{500}}{20} = \frac{10\sqrt{5}}{20} = \sqrt{5}/2$ ft/s

2E-7



cross section



$V = 4 \cdot [\underbrace{0.5h}_{\text{central rectangle}} + \underbrace{2 \cdot h^2/2}_{\text{two side triangles}}]$ m^3

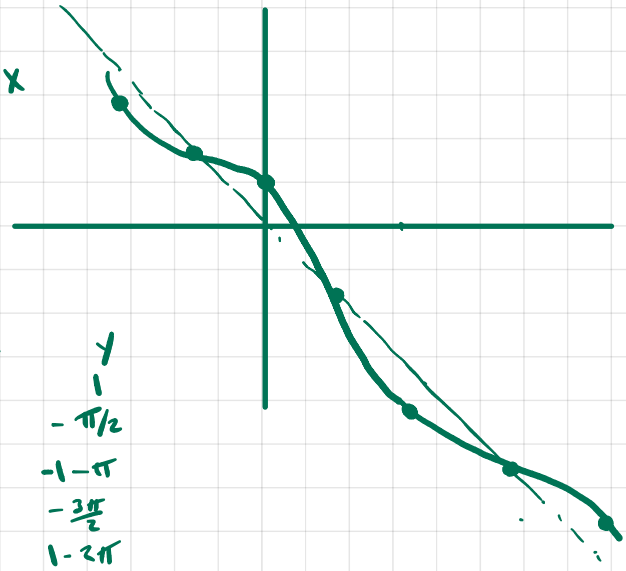
$V = 2h + 4h^2$, we know $V'(t) = 1$ m^3/s, we want $h'(t)$ when $h(t) = 0.5$

$V' = 2h' + 8hh' = h'(2 + 8h)$

$h' = \frac{V'}{2+8h} \Rightarrow h'(t_*) = \frac{1}{2+8 \cdot 0.5} = \frac{1}{6}$ m/s

2F-1

a) $y = \cos x - x$



x	$\cos x$	y
0	1	1
$\pi/2$	0	$-\pi/2$
π	-1	$-1-\pi$
$3\pi/2$	0	$-\frac{3\pi}{2}$
2π	1	$1-2\pi$
$-\pi/2$	0	$\frac{\pi}{2}$
$-\pi$	-1	$-1+\pi$

$x_n \sin x_n + x_n + \cos x_n - x_n$

b) $y = \cos x - x = 0$

$y' = -\sin x - 1 = 0$

$$x_{n+1} = x_n - \frac{\cos x_n - x_n}{-(\sin x_n + 1)} = x_n + \frac{\cos x_n - x_n}{\sin x_n + 1} = \frac{x_n \sin x_n + \cancel{x_n} + \cos x_n - \cancel{x_n}}{\sin x_n + 1}$$

$$= \frac{x_n \sin x_n + \cos x_n}{\sin x_n + 1}$$

$x_0 = \frac{\pi}{2} \Rightarrow x_1 = 0.785 \Rightarrow x_2 = 0.739 \Rightarrow x_3 = 0.739$

c) $x_{n+1} = \cos(x_n)$

Fixed point method takes 53 iterations to stabilize to 9 decimals.

Newton's method takes three iterations.