Example 3 $\int_0^4 (x^3-2x)dx = \lim_{n\to\infty}\sum_{i=1}^n (x_i^3-2x_i)\Delta x$

$\Delta x = \frac{4}{n}$ $\quad x_i = \frac{4}{n}i$

$$\Rightarrow \lim_{n\to\infty}\sum_{i=1}^n \left[\left(\frac{4i}{n}\right)^3 - 2\left(\frac{4i}{n}\right)\right]\frac{4}{n}$$

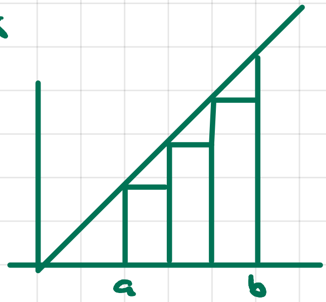
$$= \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^{\infty}\left[\frac{64i^3}{n^3} - \frac{8i}{n}\right] = \lim_{n\to\infty}\left[\frac{256}{n^4}\sum i^3 - \frac{32}{n^2}\sum i\right] = \lim_{n\to\infty}\left[\frac{256}{n^4}\cdot\frac{n^2(n+1)^2}{4} - \frac{32}{n^2}\frac{n(n+1)}{2}\right]$$

$$= \lim\left[\frac{64(n+1)^2}{n^2} - \frac{16(n+1)}{n}\right] = \lim\left[64\left(1+\frac{1}{n}\right)^2 - 16\left(1+\frac{1}{n}\right)\right]$$

$$= 64-16 = 48$$

Example 4

$\int_a^b x\,dx$



$$= \lim \sum_{i=1}^n f(x_i)\Delta x$$

$\Delta x = \frac{b-a}{n}$ $\quad x_i = a+\Delta x\cdot i$

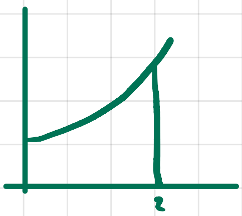
$$= \lim \sum (a+i\Delta x)\cdot\Delta x = \lim\left[\sum a\Delta x + (\Delta x)^2\sum i \right. = \lim\left[a\Delta x\sum 1 + \Delta x^2 \sum i\right]$$

$$= \lim_{n\to\infty}\left[a\Delta x\cdot n + \Delta x^2\cdot\frac{n(n+1)}{2}\right] = \lim\left[a(b-a) + \frac{(b-a)^2(n+1)}{2n}\right]$$

$$= a(b-a) + (b-a)^2 \lim \frac{1}{2}\left(1+\frac{1}{n}\right) = a(b-a)+(b-a)^2\cdot\frac{1}{2} = (b-a)\left(a+\frac{b-a}{2}\right)$$

$$= (b-a)\left(\frac{a+b}{2}\right) = \frac{b^2-a^2}{2} = \frac{b^2}{2}-\frac{a^2}{2}$$

Example 5 $\int_0^2 e^x\,dx$



$\Delta x = \frac{2-0}{n} = \frac{2}{n}$

$x_i = 0+\frac{2}{n}i = \frac{2}{n}i$

$$\lim_{n\to\infty}\sum_{i=1}^n e^{\frac{2}{n}i}\cdot\frac{2}{n} = \lim \frac{2}{n}\sum\left(e^i\right)^{2/n}$$

5.4 Problems

1 $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} (2x_i - 1)\Delta x$ over $[1,3]$ $\quad \Delta x = x_i - x_{i-1} = \frac{b-a}{n} \Rightarrow x_i = a + \frac{(b-a)i}{n}$

$$= \lim \sum \left(\frac{b-a}{n}\right) \cdot \left[2\cdot\left(a + \frac{(b-a)i}{n}\right) - 1\right]$$

$$= \lim \left[\left(\frac{b-a}{n}\right) \cdot 2a \sum 1 + \left(\frac{b-a}{n}\right)^2 \cdot 2 \sum i - \left(\frac{b-a}{n}\right) \sum 1\right]$$

$$= \lim \left[\frac{(b-a)}{\cancel{n}} \cdot 2a\cancel{n} + \frac{(b-a)^2}{n^{\cancel{2}}} \cdot \frac{\cancel{2}\cancel{n}(n+1)}{\cancel{2}} - \frac{(b-a)}{\cancel{n}} \cdot \cancel{n}\right]$$

$$= \lim \left[2a(b-a) + \frac{n+1}{n}(b-a)^2 - (b-a)\right]$$

$$= (b-a)(2a-1) + (b-a)^2 \cdot \lim \left(1 + \frac{1}{n}\right)$$

$$= (b-a)(2a-1+b-a) = (b-a)(a+b-1) = \cancel{ab} + b^2 - b - \cancel{ab} - a^2 + a$$

$$= b^2 - b - (a^2 - a)$$

$$\int_a^b (2x-1)\,dx = (x^2 - x)\Big|_a^b = (b^2 - b) - (a^2 - a)$$

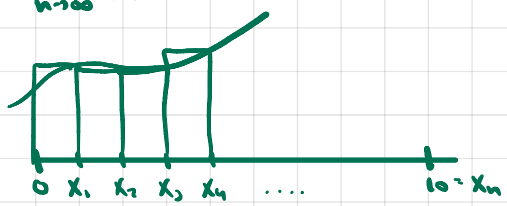
2 $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} (2 - 3x_{i-1})\Delta x$ over $[-3,2]$

$f(x_i) = 2 - 3x_{i-1}$

leftmost value of partition

$$= \int_{-3}^{2} (2-3x)\,dx = \left(2x - \frac{3}{2}x^2\right)\Big|_{-3}^{2} = (4-6) - \left(-6 - \frac{27}{2}\right) = -2 + \frac{39}{2} = \frac{35}{2}$$

3 $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} (x_i^2 + 4)\Delta x$ over $[0,10]$ $= \int_0^{10} (x^2+4)\,dx$



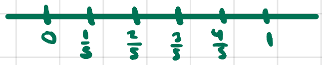
10 $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} e^{2x_i}\Delta x$ over $[0,1]$ $= \int_0^1 e^{2x}\,dx$

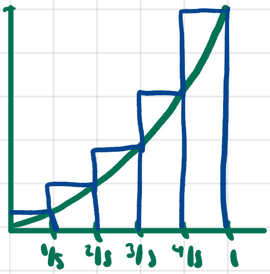
11 $f(x) = x^2$ on $[0,1]$, $n=5$

$$R = \sum_{i=1}^{5} x_i^2 \cdot \frac{1}{5} = \sum\left(\frac{1}{5}i\right)^2 \cdot \frac{1}{5} = \sum \frac{1}{25} i^2 \cdot \frac{1}{5} = \frac{1}{125}\sum_{i=1}^{5} i^2 = \frac{1}{125}(1+4+9+16+25) = \frac{55}{125} = \frac{11}{25}$$

$$\Delta x = \frac{1-0}{5}$$

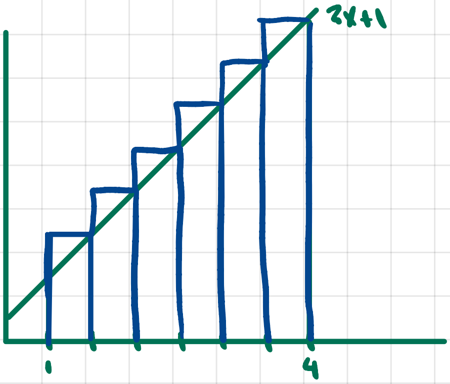
$x_i = 0 + \frac{1}{5} i$ $\quad f(x_i) = \left(\frac{1}{5}i\right)^2$





Exact Value: $\displaystyle\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

15 $f(x) = 2x+1$ on $[1,4]$, $n=6$



$$R = \sum_{i=1}^{6} f(x_i)\Delta x = \sum_{i=1}^{6}(3+i)\frac{1}{2} = \frac{3}{2}\cdot\sum_{i=1}^{6} 1 + \frac{1}{2}\sum_{i=1}^{6} i = \frac{3}{2}\cdot 6 + \frac{1}{2}(1+2+3+4+5+6) = 9 + 21 = 30$$

$$\Delta x = \frac{4-1}{6} = \frac{1}{2}$$

$x_i = 1 + i \cdot \frac{1}{2}$

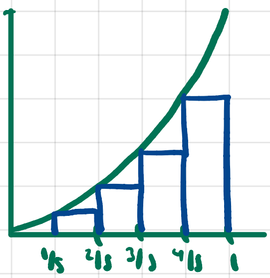
$f(x_i) = 2\left(1 + \frac{i}{2}\right) + 1 = 2 + i + 1 = 3 + i$

Exact Value: $\displaystyle\int_1^4 (2x+1)\,dx = (x^2+x)\Big|_1^4 = (16+4) - (1+1) = 18$

21 $f(x) = x^2$ on $[0,1]$, $n=5$

$\Delta x = \frac{1}{5}$

$x_i = 0 + \Delta x(i-1) = \frac{1}{5}(i-1)$



$$R = \sum_{i=1}^{5}\left[\frac{1}{5}(i-1)\right]^2 \cdot \frac{1}{5}$$

$$= \sum \frac{1}{25}(i-1)^2 \cdot \frac{1}{5} = \frac{1}{125}\cdot(0^2+1^2+2^2+3^2+4^2) = \frac{30}{125} = \frac{6}{25}$$

43 $\int_0^2 x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim \sum \frac{4i^2}{n^2}\cdot\frac{2}{n} = \lim \sum_{i=1}^{n} \frac{8}{n^3} i^2$

$(n^2+n)(2n+1) = (2n^3+n^2+2n^2+n)$
$= 2n^3+3n^2+n$

$\Delta x = \frac{2-0}{n} = \frac{2}{n}$

$= \lim_{n\to\infty} \frac{8}{n^3}\cdot\sum i^2 = \lim \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$

$x_i = 0 + i\Delta x = \frac{2}{n} i$

$= \lim \frac{8}{n^3} \frac{(2n^3+3n^2+n)}{6} = \lim_{n\to\infty}\left[\frac{16}{6} + \frac{24}{6n} + \frac{8}{6n^2}\right] = \frac{8}{3}$

$f(x_i) = \left(\frac{2i}{n}\right)^2$