4.2 Increments, Differentials, Linear Approxim.

$\Delta y = f(x+\Delta x) - f(x)$
↳ increment

↗ differential, hypothetical change in y along tangent
$dy = f'(x)\Delta x$

for fixed x, dy is linear function of increment Δx
⇒ dy is linear approxim. to Δy

we can approximate $f(x+\Delta x)$:

lin. approx. formula

$$f(x+\Delta x) = f(x) + \Delta y \approx f(x) + dy = f(x) + f'(x)\Delta x$$

equivalently,

$$\Delta y \approx f'(x)\Delta x = dy$$

Absolute error in measured or approximated value = Actual Value − Approximate Value

$$\text{Relative error} = \frac{\text{absolute error}}{\text{actual value}}$$

Error in Lin Approxim.

we compare $f(x)$ to its linear approx. $L(x)$ near point $x=a$

$y = f(x)$

Actual Function Value

$\Delta x = x - a$

$f(a+\Delta x) = f(x) = f(a) + \Delta y$

⇒ Error in linear approx.

$$f(x) - L(x) = \cancel{f(a)} + \Delta y - \cancel{f(a)} - dy$$
$$= \Delta y - dy$$

Linear Approx.

$L(x) = f(a) + f'(a)\Delta x = f(a) + dy$

But, $\Delta y - dy = f(a+\Delta x) - f(a) - f'(a)\Delta x$

$$\epsilon(\Delta x) = \frac{\Delta y - dy}{\Delta x} = \frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)$$

$$\lim_{\Delta x \to 0} \epsilon(\Delta x) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a) = 0$$

⇒ $\Delta y - dy = \epsilon(\Delta x)\cdot\Delta x$, the error in the lin. approx., is the product of two quantities that $\to 0$ as $\Delta x \to 0$

we can rewrite the above as

↗ lin approx. + error term, as $\Delta x \to 0$ the error $\to 0$ ⇒ lin approx. gets better

$$f(a+\Delta x) = f(a) + f'(a)\Delta x + \epsilon(\Delta x)\Delta x$$

Example 5

$y = f(x) = x^3$

$\Delta x = x - a$

$\Delta y = f(x) - f(a) = f(a+\Delta x) - f(a) = (a+\Delta x)^3 - a^3 = 3a^2\Delta x + 3a(\Delta x)^2 + (\Delta x)^3$

$dy = f'(a)\Delta x = 3a^2\Delta x$

error in lin. approx. $= \Delta y - dy = 3a(\Delta x)^2 + (\Delta x)^3$

For $a = 1$, and $\Delta x = 0.1$, $\Delta y - dy = 0.031$

Note

$dy = f'(a)\Delta x$ can be viewed as a function of Δx, for given x.

$\Delta y = f(a+\Delta x) - f(a)$ can also be viewed as function of Δx.

$\Rightarrow$ given an x, we know Δy, the actual change; we know dy, the linear approx. change. therefore we know the error and we can vary Δx to see the corresponding error in the approxim.

Differentials

$f(x+\Delta x) \approx f(x) + f'(x)\Delta x$ is often written $f(x+dx) \approx f(x) + f'(x)dx$

dx is an independent variable called the differential of x, x is fixed.

Thus, the differentials of x and y: $dx = \Delta x$ $\quad dy = f'(x)\Delta x = f'(x)dx$

$\Rightarrow \dfrac{dy}{dx} = f'(x)$

This is differential notation.

$z = f(u)$

$dz = f'(u)du$

$z = f(u) = u^n \Rightarrow d(u^n) = nu^{n-1}du$

$z = f(u) = \sin u \Rightarrow d(\sin u) = \cos u \, du$

$\Rightarrow$ we can write differentiation rules in differential form without having to identify the independent variable

I think this means that in, say
$u = f(t), v = g(t), w = u + v$

$\dfrac{dw}{dt} = \dfrac{d(u+v)}{dt} = \dfrac{du}{dt} + \dfrac{dv}{dt}$, in differential notation

we can actually cancel the dt terms in the denom.

$d(u+v) = du + dv$

Applying differential notation to compositions, we obtain the chain rule

$$\begin{aligned} z &= f(u) \Rightarrow dz = f'(u)\,du \\ u &= g(x) \Rightarrow du = g'(x)\,dx \end{aligned} \Rightarrow dz = f'(g(x))g'(x)\,dx \Rightarrow \frac{dz}{dx} = f'(g(x))g'(x)$$