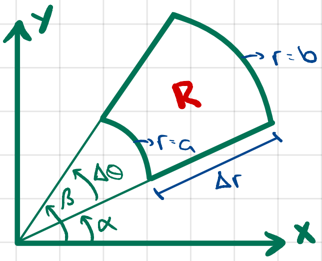
polar rectangle: region described in polar coordinates by

$$a \le r \le b \qquad \alpha \le \theta \le \beta$$



Area of circular sector with radius r, central angle θ = $\frac{1}{2}r^2\theta$

$$\text{Area}_R = \frac{1}{2}b^2(\beta-\alpha) - \frac{1}{2}a^2(\beta-\alpha) = \frac{1}{2}(\beta-\alpha)(b^2-a^2)$$

$$= \frac{1}{2}(\beta-\alpha)(b+a)(b-a) = \left[\frac{1}{2}(b+a)\right](b-a)(\beta-\alpha)$$

$$= \bar{r}\,\Delta r\,\Delta\theta$$

average radius

Assume $z = f(x,y)$, we want $\iint_R f(x,y)\,dA$.

→ start with a **polar partition** P of region R into $k = mn$ polar rectangles $R_1, R_2, \dots R_k$

$a = r_0 < r_1 < \dots < r_m = b$

$\alpha = \theta_0 < \theta_1 < \dots < \theta_n = \beta$

* **norm $|P|$**: max length of diagonals of the polar subrectangles

For each R_i we have a particular point inside (r_i^*, θ_i^*) where r_i^* is average radius of R_i.

The corresponding rectangular coordinates: $x_i^* = r_i^*\cos\theta_i^*$, $y_i^* = r_i^*\sin\theta_i^*$.

$\sum_{i=1}^{k} f(x_i^*, y_i^*)\Delta A_i$ is a Riemann sum for f associated with the polar partition above.

As computed above, the area ΔA_i of a polar rectangle is $r_i^*\Delta r\Delta\theta$

$$\Rightarrow \sum_{i=1}^{k} f(x_i^*, y_i^*)\, r_i^*\,\Delta r\,\Delta\theta = \sum_{i=1}^{k} f(r_i^*\cos\theta_i, r_i^*\sin\theta_i)\, r_i^*\,\Delta r\,\Delta\theta$$

$$= \sum_{i=1}^{k} g(r_i^*, \theta_i^*)\,\Delta r\,\Delta\theta$$

These sums are Riemann sums. As $|P| \to 0$ we get, by definition, $\iint_R f(x,y)\,dA$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$