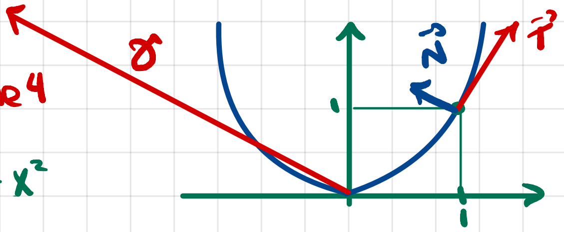
Example 4

$y = x^2$



$x(t) = t \quad y(t) = t^2$ $\qquad x' = 1 \quad y' = 2t$

$\qquad\qquad\qquad\qquad\qquad\quad\;\; x'' = 0 \quad y'' = 2$

$\vec{r}(t) = \langle t, t^2 \rangle$

$r'(t) = \langle 1, 2t \rangle \qquad |r'(t)| = \sqrt{1+4t^2}$

* radius of curvature at $t=1$ $= \dfrac{1}{k(1)} = \dfrac{[1+4]^{3/2}}{2} = \dfrac{5\sqrt{5}}{2}$

$r'(1) = \langle 1, 2 \rangle$

$\vec{T}(1) = \dfrac{\langle 1, 2 \rangle}{\sqrt{5}}$ = unit tangent vector

curvature = $k(t) = \dfrac{x'y'' - x''y'}{[x'^2 + y'^2]^{3/2}} = \dfrac{2-0}{[1^2 + 4t^2]^{3/2}} = \dfrac{2}{[1+4t^2]^{3/2}}$

curvature is a scalar. It is the length of $\frac{dT}{ds}$, ie rate of change of tangent velocity relative to arc length.

The curvature in this ex. starts at $k(0) = 2$, and $\lim_{t\to\infty} k(t) = 0$

At any point $\vec{r}(t)$ the radius of curvature is $1/k = [1+4t^2]^{3/2}/2$ which increases from $1/2$ to ∞.

To find $\vec{N}$

$$\begin{cases} \vec{N}(t) \cdot \vec{v}(t) = 0 \Rightarrow n_1 + n_2 \cdot 2t = 0 \\ n_1^2 + n_2^2 = 1 \end{cases}$$

$n_1 = -2t n_2 \qquad 4t^2 n_2^2 + 5n_2^2 = 1$

$n_2^2(4t^2+5) = 1$

$n_2 = \pm \sqrt{1/(4t^2+5)}$

At $t = 1 \Rightarrow n_1 = -2n_2, \quad 4n_2^2 + n_2^2 = 5n_2^2 = 1 \Rightarrow n_2 = \pm 1/\sqrt{5}$

$n_1 = \mp 2/\sqrt{5}$

So there are two vectors to choose from but we know $\vec{N}$ should point towards the center of the circle of curvature.

From the graph we see $\vec{N} = \langle -2/\sqrt{5}, 1/\sqrt{5} \rangle$

$\vec{\gamma}$ = center of curvature $= \vec{r}(t) + \rho \cdot \vec{N} = \langle t, t^2 \rangle + \dfrac{[1+4t^2]^{3/2}}{2} \cdot \vec{N}(t)$

At $t=1$ $\vec{\gamma}(1) = \langle 1, 1 \rangle + \dfrac{\sqrt{125}}{2} \langle -2/\sqrt{5}, 1/\sqrt{5} \rangle = \langle -4, 7/2$

Example 6

$x(t) = \frac{3}{2}t^2$ $\quad y(t) = \frac{4}{3}t^3$ $\quad t = 1$ $\quad \vec{a}$ decomposition in directions $\vec{T}$ and $\vec{N}$

strategy:

get $\vec{v}(t), v(t), v'(t)$ $\Rightarrow$ $\vec{T}$ comp. of $\vec{a}$ is $v'(t)$

get $\vec{T} = \vec{v}(t)/v(t)$

get dT/dt

calculate curvature $K = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{v(t)}\left|\frac{dT}{dt}\right|$

$\vec{N}$ comp. of $\vec{a}$ is $K(t)v(t)^2$

calculation

$\vec{v}(t) = \langle 3t, 4t^2 \rangle$

$v(t) = \sqrt{9t^2 + 16t^4} = t\sqrt{9 + 16t^2}$ $\Rightarrow$ $v'(t) = \frac{1}{2}\frac{(18t + 64t^3)}{\sqrt{9t^2 + 16t^4}} = \frac{9t + 32t^3}{\sqrt{9t^2 + 16t^4}}$

$\Rightarrow \vec{a}_T(t) = \frac{9t + 32t^3}{\sqrt{9t^2 + 16t^4}}$, $\quad a_T(1) = \frac{9 + 32}{5} = \frac{41}{5}$

$\vec{T} = \frac{\langle 3t, 4t^2 \rangle}{\sqrt{9t^2 + 16t^4}}$

$\vec{T}'(1) = \frac{12}{25}$ (maple) $\quad$ $K(1) = \frac{1}{v(1)}|T'(1)|$

$\Rightarrow$

$\vec{a}_N = v(1)^2 \cdot K = v(1)\,|T'(1)| = 5 \cdot \frac{12}{25} = \frac{12}{5}$

$v(1) = 5$