15.1 Vector Fields

Vector field defined on a region T in space: vector-valued function $\vec{F}$:

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle = P\hat{i} + Q\hat{j} + R\hat{k}$$

Gradient Vector Field $\quad \vec{\nabla} f = \langle f_x, f_y, f_z \rangle = \text{grad } f$

Vector Differential operator ∇: operation that, when applied to the scalar function f, yields its gradient vector field $\vec{\nabla} f$

Divergence of a vector field: $\text{div } \vec{F} = \nabla \cdot \vec{F} = P_x + Q_y + R_z$

Curl of Vector Field

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

Note

vector diff. operator applied to scalar function

$\rightarrow \vec{\nabla}(af + bg) = \langle af_x + bg_x, af_y + bg_y \rangle = a\vec{\nabla} f + b\vec{\nabla} g$

$\rightarrow \vec{\nabla}(fg) = \langle f_x g + g_x f, f_y g + g_y f \rangle = g\vec{\nabla} f + f\vec{\nabla} g$

$\rightarrow \vec{\nabla} \cdot (a\vec{F} + b\vec{G}) = \langle \partial/\partial x, \partial/\partial y \rangle \cdot \langle aP_F + bP_G, aQ_F + bQ_G \rangle$

dot product of vector diff operator and a vector field (sum of vector fields)

$$= aP_{F_x} + bP_{G_x} + aQ_{F_y} + bQ_{G_y}$$

$$= a\vec{\nabla} \cdot \vec{F} + b\vec{\nabla} \cdot \vec{G}$$

$\rightarrow \vec{\nabla} \cdot (f\vec{G}) = \langle \partial/\partial x, \partial/\partial y \rangle \langle fP, fQ \rangle = f_x P + f P_x + f_y Q + f Q_y = f\vec{\nabla} \cdot G + G \cdot \nabla f$

$\rightarrow \nabla \times (a\vec{F} + b\vec{G}) = \text{curl}(a\vec{F} + b\vec{G}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & 0 \\ aP_F + bP_G & aQ_F + bQ_G & 0 \end{vmatrix}$

$\vec{F} = \langle P_F, Q_F \rangle \quad \vec{G} = \langle P_G, Q_G \rangle$

$$= \hat{k}\left[aQ_{F_x} + bQ_{G_x} - aP_{F_y} - bP_{G_y}\right] = \hat{k}\left[a(Q_{F_x} - P_{F_y}) + b(Q_{G_x} - P_{G_y})\right]$$

$$= a\vec{\nabla} \times \vec{F} + b\vec{\nabla} \times \vec{G} = a(\text{curl } \vec{F}) + b(\text{curl } \vec{G})$$

Also, $\text{curl}(f\vec{G}) = \nabla f \times G$