Previously

$df = f'(x)dx$ used to approximate $\Delta f = f(x+\Delta x) - f(x)$

now we want to approximate $\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y)$

Def:

$$df = f_x(x,y)\Delta x + f_y(x,y)\Delta y$$

approxim.:

linear function of $\Delta x, \Delta y$

$$\Delta f \approx df \Rightarrow f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y$$

Gradient Vector

$$\nabla f(\vec{x}) = \langle D_1 f(\vec{x}), D_2 f(\vec{x}), \dots, D_n f(\vec{x}) \rangle = \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \rangle$$

where

$$\vec{x} = \langle x_1, \dots, x_n \rangle$$

with vector notation we can more succinctly describe linear approx. of functions of many variables

$$\vec{h} = \langle h_1, \dots, h_n \rangle$$

$$f(\vec{x}+\vec{h}) \approx f(\vec{x}) + \nabla f(\vec{x}) \cdot \vec{h} \qquad \text{ie } df = \nabla f(\vec{x}) \cdot \vec{h}$$

Theorem

f is continuously differentiable in neighborhood of $\vec{a}$

$f(\vec{x})$ has continuous first order partial derivatives in a region that contains the neighborhood $|\vec{x}-\vec{a}| < r$.

$\vec{a}+\vec{h}$ lies in this neighborhood $\Rightarrow f(\vec{a}+\vec{h}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot \vec{h} + \underbrace{\epsilon(\vec{h}) \cdot \vec{h}}$

error in the lin. approx.

where $\epsilon(\vec{h}) = \langle \epsilon_1(\vec{h}), \dots, \epsilon_n(\vec{h}) \rangle$ approaches $\vec{0}$ as $\vec{h} \to 0$

note

$$\frac{\epsilon(\vec{h}) \cdot \vec{h}}{|\vec{h}|} = \underbrace{\epsilon_1(\vec{h})}_{\to 0} \cdot \underbrace{\frac{h_1}{|\vec{h}|}}_{\leq 1} + \dots + \epsilon_n(\vec{h}) \cdot \frac{\vec{h}_n}{|\vec{h}|} \longrightarrow 0 \text{ as } \vec{h} \to \vec{0}$$

divide by $|\vec{h}|$

$$\Rightarrow \lim_{\vec{h} \to 0} \frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - \nabla f(\vec{a})\vec{h}}{|\vec{h}|} = \lim_{\vec{h} \to 0} \epsilon(\vec{h}) = 0$$

Note

real-valued function $f(\vec{x})$ is differentiable at point a if $\exists\ \vec{c} = \langle c_1, \dots, c_n \rangle$ such that

$$\lim_{\vec{h} \to 0} \frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - \vec{c}\vec{h}}{|\vec{h}|} = 0$$

ie f diff at $\vec{a}$ if $\exists$ linear function $\vec{c} \cdot \vec{h} = c_1h_1 + \dots + c_nh_n$ that approximates the increment $f(\vec{a}+\vec{h}) - f(\vec{a})$ so closely that the error is small even in comparison with $|\vec{h}|$

That linear function of $\vec{h}$ is $\nabla f(\vec{a}) \cdot \vec{h}$, and $\nabla f(\vec{a})$ can be shown to be the only $\vec{c}$ giving such an function.

→ If $\nabla f(\vec{a})$ exists and its components are continuous near $\vec{a}$ then f is continuously differentiable at $\vec{a}$. But this means $\nabla f(\vec{a}) \cdot \vec{h}$ is the linear function mentioned above and the limit above exists

→ f differentiable