15.3

recall: FTC: differentiation and integration are inverse processes for single-variable functions

FTC, part 2: $\int_a^b G'(t)\,dt = G(b) - G(a)$ if G' continuous on $[a,b]$

in vector calculus, there is a similar result: gradient vector differentiation and line integration are inverse processes for multivariable functions.

Theorem: Fundamental Theorem for Line Integrals

f function of two or three variables

C smooth curve parametrized by vector-valued $\vec{r}(t)$ $t \in [a,b]$

f cont. diff on C

$$\Rightarrow \int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Theorem: Independence of Path

$\int_C \vec{F} \cdot \hat{T}\,ds$ of continuous vector field $\vec{F}$ is independent of path in plane or space region D

$\Leftrightarrow$ $\vec{F} = \vec{\nabla} f$ for some f in D

Def: Conservative Fields and Potential Functions

$\exists$ scalar function f on region D such that $\vec{F} = \nabla f$ in D $\Rightarrow$ vector field $\vec{F}$ is conservative

f is called a potential function for $\vec{F}$

Theorem Conservative Fields and Potential Functions

vector field $\vec{F} = \langle P, Q \rangle$ continuously diff. in open rectangle in xy-plane

Then,

$\vec{F}$ conservative in R $\Leftrightarrow$ at each point in R, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

conservative force fields and conservation of energy

Given conservative force field $\vec{F}$, it is customary in Physics to write $\vec{F} = -\nabla V$, and to call $V(x,y,z)$ the potential energy at (x,y,z).

$$\Rightarrow W = \int_A^B \vec{F}\cdot\hat{T}\,ds = -\int_A^B \nabla V\cdot\hat{T}\,ds = -(V(B)-V(A)) = V(A)-V(B)$$

$\Rightarrow$ work done = decrease in potential energy

setup

particle mass m moves from A to B under influence of conservative force $\vec{F}$.

$\vec{r}(t)$ is position of the motion $t \in [a,b]$

Newton's Law: $\vec{F}(\vec{r}(t)) = m\vec{r}''(t) = m\vec{v}'(t)$

$$\int_A^B \vec{F}\cdot\hat{T}\,ds = \int_a^b m\vec{v}'(t)\cdot\vec{v}(t)\,dt = \int_a^b m\,D_t\left[\frac{1}{2}\vec{v}(t)\cdot\vec{v}(t)\right]dt = \frac{1}{2}m\,v(t)^2\Big|_a^b$$

$$= \frac{1}{2}m\left(v(b)^2 - v(a)^2\right)$$

we have:

$$W = \int_A^B \vec{F}\cdot\hat{T}\,ds = V(A)-V(B)$$

$$= \int_a^b \vec{F}\cdot\vec{v}(t)\,dt = \frac{1}{2}mv(b)^2 - \frac{1}{2}mv(a)^2 = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$$

$\Rightarrow V(A) + \frac{1}{2}mv_A^2 = V(B) + \frac{1}{2}mv_B^2$, the law of conservation of mechanical energy for a particle moving under the influence of a conservative force field

Total energy = kinetic energy + potential energy = constant

Recap

$\int_c \vec{F}\cdot d\vec{r} = \int_c \vec{F}\cdot\hat{T}\,ds$ ↗ obtained via Riemann sum perspective to calculating work

$\Delta W_i \approx \vec{F}(x(t_i^*),y(t_i^*),z(t_i^*))\cdot\hat{T}(t_i^*)\,\Delta s_i$

$W = \sum_{i=1}^{n}\vec{F}(x(t_i^*),y(t_i^*),z(t_i^*))\cdot\hat{T}(t_i^*)\,\Delta s_i$

$\lim_{\Delta t\to 0} W = \int_c \vec{F}\cdot\hat{T}\,ds$ → integral w/ respect to arclength of the tangential component of the force

obtained via differentials

$dW = \vec{F}\cdot\hat{T}\,ds$ = infinitesimal work done by $\vec{F}\cdot\hat{T}$ in moving particle along ds

$\vec{r} = \langle x,y,z\rangle$, remember we're moving along curve C

$d\vec{r} = \langle dx, dy, dz\rangle$

$\hat{T}ds = \frac{\vec{v}}{v}\cdot v\,dt = \vec{v}\,dt = \langle dx/dt, dy/dt, dz/dt\rangle dt = \langle dx, dy, dz\rangle$

$\Rightarrow \hat{T}ds = d\vec{r} \Rightarrow W = \int_c \vec{F}\,d\vec{r}$, an alternative notation for the same base concept of integrating tangential component of force w/ respect to ds

At this point we do some gymnastics w/ the differentials

$\vec{F} = \langle P, Q, R\rangle$, each component a fn of (x,y,z)

$\hat{T} = \frac{r'(t)}{v(t)} = \frac{\langle x'(t), y'(t), z'(t)\rangle}{v(t)} = \frac{\langle \partial x/\partial t, \partial y/\partial t, \partial z/\partial t\rangle}{v(t)}$

$ds = v(t)\cdot dt$

$W = \int_c \vec{F}\cdot\hat{T}\,ds = \int_a^b \langle P,Q,R\rangle\cdot\langle x',y',z'\rangle\cdot\frac{1}{v}\,v\,dt = \int_a^b (Px' + Qy' + Rz')dt$

$= \int_c P\,dx + Q\,dy + R\,dz$ i.e. $\int_c \vec{F}\cdot\hat{T}\,ds = \int_c P\,dx + Q\,dy + R\,dz$

$\vec{F}$ is the gradient of a fn f $\Rightarrow \int_c \vec{F}\cdot\hat{T}\,ds = \int_c \vec{\nabla} f\cdot\hat{T}\,ds = f(\vec{r}(b)) - f(\vec{r}(a))$, where

$C: r(t)\ t\in[a,b]$, ie the line integral is independent of path between a and b.

$\Rightarrow \vec{F}$ is a conservative vector field

$\int_c \vec{F}\cdot\hat{T}\,ds$ is path indep,
ie $\vec{F}$ is conservative $\Rightarrow \vec{F} = \vec{\nabla} f$ for some f

f is called the potential function of v.f. $\vec{F}$

$\int_C \vec{F} \cdot \hat{T} \, ds$ is path indep. $\iff P_y = Q_x$
ie $\vec{F}$ is conservative

* note $\vec{F} = \langle P(x,y), Q(x,y) \rangle$

and $\vec{F}$ conservative means $\vec{F}$ is $\vec{\nabla} f$

$\Rightarrow \vec{F} = \langle f_x, f_y \rangle = \langle P, Q \rangle$

$\Rightarrow P = f_x \,, \; Q = f_y$

$\Rightarrow P_y = f_{xy} = f_{yx} = Q_x$

$F(x) = x^2$

$\int x^2 dx = \frac{x^3}{3}$

$\int_0^2 x^2 dx = \frac{x^3}{3}\Big|_0^2 = (-E(x))\Big|_0^2 = E(0) - E(2)$

if $\frac{x^3}{3} = -E(x) \Rightarrow E(x) = -\frac{x^3}{3}$ then

$\int_0^2 x^2 dx =$

$x = 3t+2 \quad dx = 3dt$

$\int (3t+2)^2 \cdot 3\, dt = \frac{(3t+2)^3}{3} + C = \frac{27t^3 + 3\cdot 9t^2 \cdot 2 + 3\cdot 3t\cdot 4 + 8}{3} + C = 9t^3 + 27t^2 + 12t + \frac{8}{3} + C$

$\int 3(9t^2 + 12t + 4)dt = \int (27t^2 + 36t + 12)dt = 9t^3 + 18t^2 + 12t + C$

define $M(t) = 9t^3 + 18t^2 + 12t$

$\Rightarrow \int_a^b (3t^2+2)^2 \cdot 3\, dt = M(b) - M(a)$