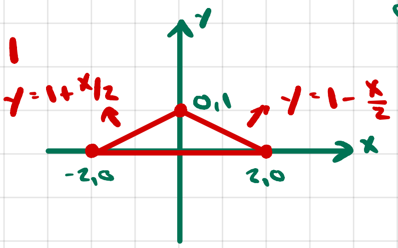
Practice Exam 3

uniform density $\delta(x,y)=1$

1

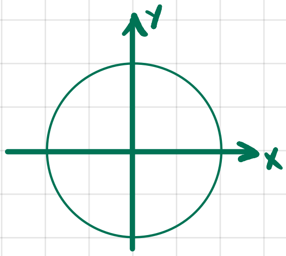


a) mass $= \iint_R 1\cdot dx\,dy = \text{Area}(R) = 4\cdot 1/2 = 2$

$$\bar{y} = \frac{\iint_R y\cdot dm}{m} = \frac{\int_0^1 \int_{2(y-1)}^{2(1-y)} y\,dx\,dy}{2}$$

b) $\bar{x} = 0$ by symmetry

2



$\delta(x,y) = x$

$$I_0 = \iint_R (x^2+y^2)x\,dA = \iint_R x^2\cdot x\,dA + \iint y^2 x\,dA = I_x + I_y$$

$$= \int_0^{2\pi}\int_0^1 r^2\cdot|r\cos\theta|\,r\,dx\,dy = 4\int_0^{\pi/2}\int_0^1 r^4\cos\theta\,dr\,d\theta$$

$$= \frac{4}{5}\int_0^{\pi/2}\cos\theta\,d\theta = \frac{4}{5}\sin\theta\Big|_0^{\pi/2} = \frac{4}{5}$$

3 $\vec{F} = \langle ax^2y + y^3 + 1,\ 2x^3 + bxy^2 + 2\rangle = \langle M, N\rangle$

a) $\vec{F} = \vec{\nabla} f \iff F$ conservative $\iff M_y = N_x$

$M_y = ax^2 + 3y^2 \qquad N_x = 6x^2 + by^2$

$M_y = N_x \Rightarrow x^2(a-6) = y^2(b-3)$

which holds for all x, y when $a=6, b=3$

b) $\vec{F} = \langle 6x^2y + y^3 + 1,\ 2x^3 + 3xy^2 + 2\rangle$

Antiderivative Method

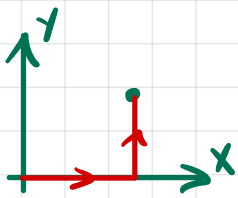
$\int N\,dx = 2x^3y + xy^3 + x + g(y)$

$\frac{d}{dy}\left(\int M\,dx\right) = 2x^3 + 3xy^2 + g'(y)$

$\Rightarrow g'(y) = 2 \Rightarrow g(y) = 2y$

$\Rightarrow f(x,y) = 2x^3y + xy^3 + x + 2y$

line integral method



$\int_C \vec{F}\cdot d\vec{r} = \int_A^B \vec{\nabla} f\,d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)),\ \vec{B} = \vec{r}(b),\ \vec{A} = \vec{r}(a)$

$\vec{B} = \langle x_1, y_1\rangle,\ \vec{A} = \langle 0,0\rangle \Rightarrow f(x_1,y_1) = \int_C \vec{F}\cdot d\vec{r} + f(0,0)$

$\int_C = \int_{C_1} + \int_{C_2} = \int_0^{x_1} 1\,dx + \int_0^{y_1}(2x_1^3 + 3x_1y^2 + 2)\,dy = x_1 + 2x_1^3y_1 + x_1y_1^3 + 2y_1$

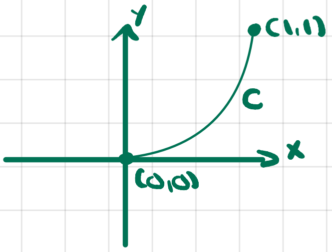
$\Rightarrow f(x,y) = 2x^3y + xy^3 + x + 2y$

c) $C: x = e^t\cos t,\ y = e^t\sin t \quad 0 \le t \le \pi$ $\qquad f(x,y) = 2x^3y + xy^3 + x + 2y$

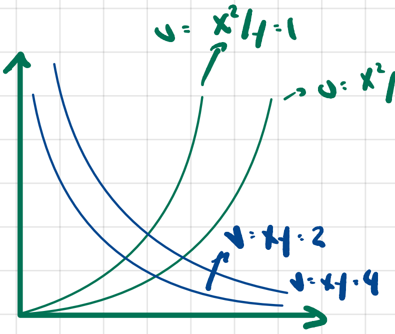
$\vec F$ conservative $\Rightarrow \int_C \vec F\cdot d\vec r = f(r(\pi)) - f(r(0)) = f(-e^\pi, 0) - f(1,0)$

$= -e^\pi - 1$

4 $\vec F = \langle yx^3, y^2\rangle$ $\qquad \int_C \vec F\cdot d\vec r = \int_C yx^3\,dx + y^2\,dy = \int_0^1 x^5\,dx + x^4\cdot 2x\,dx = \int_0^1 3x^5\,dx = \frac{x^6}{2}\Big|_0^1 = \frac12$



$y = x^2 \Rightarrow dy = 2x\,dx$



5 $y = x^2 \quad y = x^2/5 \quad xy = 2 \quad xy = 4$

a) $u = \frac{x^2}{y} \qquad v = xy \qquad \langle u,v\rangle = T(\langle x,y\rangle) = \langle x^2/y,\ xy\rangle$

$du\,dv = \frac{\partial(u,v)}{\partial(x,y)}\,dx\,dy \qquad = \begin{vmatrix} 2x/y & -x^2/y^2 \\ y & x \end{vmatrix} = \frac{2x^2}{y} + \frac{x^2}{y} = \frac{3x^2}{y}\,dx\,dy$

$\Rightarrow dx\,dy = \frac{y}{3x^2}\,du\,dv = \frac{1}{3u}\,du\,dv$

b) $\frac13\int_2^4\int_1^5 u^{-1}\,du\,dv = \frac13\int_2^4(\ln 5 - \ln 1)\,dv = \frac{\ln 2}{3}\int_2^4 dv = \frac23\ln 5$

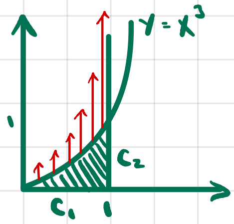
6) C simple closed curve counterclockwise around R.

a) $M = M(x,y) \qquad \oint_C M\,dx = \iint_R(-M_y)\,dA$

* if $\vec F = \langle M, 0\rangle$ then $\oint_C \vec F\cdot d\vec r = \oint_C M\,dx + 0\,dy = \oint M\,dx = \iint_R(-M_y)\,dA$ by Green's theorem

b) $\delta(x,y) = (x+y)^2 \qquad$ mass $= \iint_R \delta(x,y)\,dA \qquad \delta(x,y) = (x+y)^2 = -M_y \qquad M = -\frac{(x+y)^3}{3}$

7)



a) $\vec F = \langle 0, 1+y^2\rangle$

Flux $= \int_C \vec F\cdot\hat n\,ds = \int_C \overset{P}{(-1-y^2)}\,dx + \overset{Q}{0}\,dy = \iint_R (Q_x - P_y)\,dA = \iint_R 2y\,dA$

$= \int_0^1\int_0^{x^3} 2y\,dy\,dx = \int_0^1 y^2\Big|_0^{x^3}dx = \int_0^1 y^6\,dx = \frac17$

b) $\int_{C_1}\vec F\cdot\hat n\,ds = \int_{C_1} 0\,dy + (1+y^2)(-dx) = \int_0^1 -dx = -x\Big|_0^1 = -1$

$\int_{C_2}\vec F\cdot\hat n\,ds = \int_{C_2} 0\,dy + (1+y^2)(-0) = 0$

$\int_C \vec F\cdot\hat n\,ds = \frac17 = -1 + 0 + \int_{C_3}\vec F\cdot\hat n\,ds \Rightarrow \int_{C_3}\vec F\cdot\hat n\,ds = 8/7$