# 15.7 Stokes' Theorem

recall

Green's: $\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA = \iint_R \underset{\text{curl}}{\text{curl}}\,\vec{F}\,dA$

→ we are in 2D: C is a closed curve bounding region R in a plane, $\vec{F}$ is 2D vector field

→ if we are in 3D, we can write $\oint_C \vec{F}\cdot d\vec{r} = \oint \vec{F}\cdot\hat{T}\,ds = \iint_R (\text{curl}\,\vec{F})\cdot\hat{k}\,dA$

because $\text{curl}\,\vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle \Rightarrow \text{curl}\,\vec{F}\cdot\hat{k} = Q_x - P_y$

Stokes' Theorem generalizes $\oint_C \vec{F}\cdot\hat{T}\,ds = \iint_R \text{curl}\,\vec{F}\cdot\hat{k}\,dA$ when we replace the plane region R with an oriented bounded surface S in 3D space with boundary C consisting of one or more simple closed curves in space.

- → oriented surface: piecewise-smooth surface S plus a chosen unit normal vector field $\hat{n}$ that is continuous on each smooth piece of S
  - → positive orientation of boundary C of oriented surface: unit tangent vector $\hat{T}$ such that $\hat{n}\times\hat{T}$ always points into S

## Stokes' Theorem

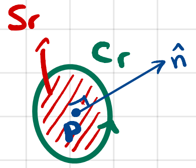
S oriented, bounded, piecewise smooth surface in space with positively oriented boundary C and unit normal vector field $\hat{n}$

$\hat{T}$ positively-oriented unit vector field tangent to C $\Rightarrow \oint_C \vec{F}\cdot\hat{T}\,ds = \iint_S (\text{curl}\,\vec{F})\cdot\hat{n}\,dS$

$\vec{F}$ cont. diff. in a space region containing S

## Physical Interpretation

→ Stokes' Theorem yields physical interpretation of curl $\vec{F}$




$S_r$ : circular disk, radius r

$a(S_r) = \pi r^2$

$C_r$ : positively oriented boundary circle

Apply MVT to $\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$

$\Rightarrow \underbrace{\oint_{C_r} \vec{F}\,d\vec{r}}_{\text{Stokes}} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS \underbrace{=}_{\text{MVT}} \text{Area}(S_r)\cdot\nabla\times F(\bar{P})\cdot\hat{n}$

For some $\bar{P}$ in $S_r$

* Mean Value Theorem for Def. Integrals
$f : [a,b] \to \mathbb{R}$ cont. fn

$\Rightarrow \exists\, c$ in $[a,b]$

$f(c) = \dfrac{\int_a^b f(x)\,dx}{b-a}$ = mean value of f on [a,b]

$(b-a)f(c) = \int_a^b f(x)\,dx$

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$$\Rightarrow \oint_{C_r} \vec F \cdot d\vec r \underset{\text{Stokes}}{=} \iint_S (\nabla\times\vec F)\cdot\hat n\, dS \underset{\text{MVT}}{=} \text{Area}(S_r)\cdot \nabla\times\vec F(\bar P)\cdot\hat n$$

$\text{Area}(S_r) = \pi r^2$

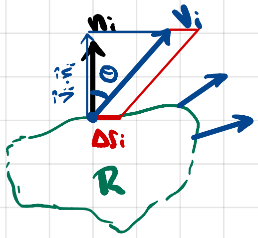
note that as $\bar P \to P$, $\nabla\times\vec F(\bar P) \to \nabla\times\vec F(P)$

$$\Rightarrow \lim_{r\to 0} [\nabla\times\vec F(\bar P)]\cdot\hat n = [\nabla\times F(P)]\cdot\hat n$$

$$\boxed{\Rightarrow [\nabla\times F(P)]\cdot\hat n = \lim_{r\to 0}\frac{1}{\pi r^2}\oint_{C_r}\vec F\cdot\hat T\,ds}$$

Suppose $\vec F = \delta\vec V$ $\quad (|\vec V_i|\cos\theta)\Delta s_i = \underbrace{\vec V_i\cdot\vec n_i}_{\text{height}}\underbrace{\Delta s_i}_{\text{base}}$ = Area portion of fluid leaving R $\quad \frac{m}{s}\times m = m^2/s$

* recall




Total mass of fluid leaving R per unit time

$$\approx \sum_{i=1}^{n} \delta_i \vec V_i\cdot\vec n_i\,\Delta s_i = \sum_{i=1}^{n} \vec F_i\cdot\vec n_i\,\Delta s_i, \quad \vec F = \delta\vec V$$

If instead of $\vec n_i$ we use $\hat T_i$ we have $\sum \delta_i \vec V_i\cdot\hat T_i\,\Delta s_i$

$\hat n_i$ replaced by $\hat T_i$

we get rate of flow of fluid mass around curve C, which is denoted

$\Gamma(C) = \oint_C \vec F\cdot\hat T\,ds$ = circulation of $\vec F$ around C

But $[\nabla\times\vec F(P)]\cdot\hat n \approx \dfrac{\Gamma(C_r)}{\pi r^2}$ if $C_r$ is a circle of very small radius r centered at P, $\perp$ to $\hat n$

curl $\vec F$ at center P gives approx. rate of flow of fluid mass around $C_r$, divided by area$(C_r)$

If $\nabla\times\vec F(P)\neq 0$ then circulation is max when we choose $\hat n$ in same direction as curl $\vec F$.

$\Rightarrow$ line through P determined by $\nabla\times\vec F(P)$ is the axis about which the fluid near P is revolving the most rapidly

If $\nabla\times\vec F(P) = 0$ everywhere then the fluid flow and vector field $\vec F$ are said to be irrotational.

$\Rightarrow$ vector field $\vec F$ defined on simply connected region D is irrotational

$\iff$ $\vec F$ conservative $\iff$ $\int_C \vec F\cdot\hat T\,ds$ path independent in D