15.2 Line Integrals

Ex 3 $\int_C y\,dx + z\,dy + x\,dz$ is defined as $\int_C y\,dx + \int_C z\,dy + \int_C x\,dz$

C is $\langle x(t), y(t), z(t)\rangle = \langle t, t^2, t^3\rangle \quad 0 \le t \le 1$

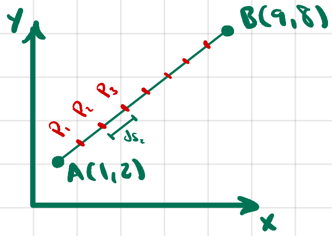
$$\int_C y\,dx = \lim_{\Delta t \to 0} \sum_{i=1}^{n} y(t_i^*)\Delta x_i = \int_0^1 t^2 \cdot 1\,dt = \frac{1}{3}$$

$$\int_C z\,dy = \lim_{\Delta t \to 0} \sum_{i=1}^{n} z(t_i^*)\Delta y_i = \int_0^1 t^3 \cdot 2t\,dt = \frac{2t^5}{5}\Big|_0^1 = \frac{2}{5}$$

$$\int_C x\,dz = \lim_{\Delta t \to 0} \sum_{i=1}^{n} x(t_i^*)\Delta z_i = \int_0^1 t \cdot 3t^2\,dt = \frac{3t^4}{4}\Big|_0^1 = \frac{3}{4}$$

$$\Rightarrow \int_C y\,dx + z\,dy + x\,dz = \frac{1}{3} + \frac{2}{5} + \frac{3}{4} = \frac{20+24+45}{60} = \frac{89}{60}$$

Ex 4



$x = 1 + 8t$
$y = 2 + 6t$ $\quad 0 \le t \le 1$

$f(x,y) = xy$

$$\int_C f(x,y)\,ds = \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(x(t_i^*), y(t_i^*))\,ds_i = \int_0^1 (1+8t)(2+6t)\sqrt{64+36}\,dt$$

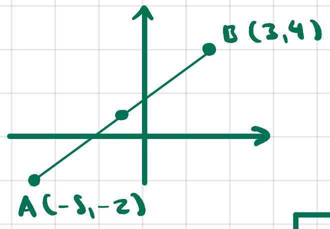
$$= \int_0^1 10(1+8t)(2+6t)\,dt = 290$$

$$\int_C f(x,y)\,dx = \lim_{\Delta t \to 0} \sum f(x(t_i^*), y(t_i^*))\Delta x_i = \int_0^1 (1+8t)(2+6t)8\,dt = 232$$

1 $f(x,y) = x^2 + y^2$

$\vec{r}(t) = \langle 4t-1, 3t+1 \rangle \quad t \in [-1,1]$

$$\int_C f(x,y)\,ds = \int_{-1}^{1} [(4t-1)^2 + (3t+1)^2]\, 5\,dt = \frac{310}{3}$$



$ds = \sqrt{16+9}\,dt = 5\,dt$

$= v\,dt$

t	$\vec{r}(t)$
-1	$-5,-2$
0	$-1,1$
1	$3,4$

*if $f(x,y) = 1$ the line integral Riemann sum is

$$\sum_{i=1}^{n} ds_i = \text{arc length}$$

$$\int_{-1}^{1} 5\,dt = 5(1-(-1)) = 10$$

From the graph, $\overline{AB} = \sqrt{(3-(-5))^2 + (4-(-2))^2}$

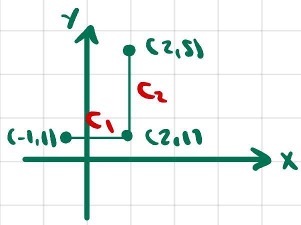
$= \sqrt{64+36} = 10$

if f is density then $\overline{AB}$ has mass $\frac{310}{3}$.

$$\int_C f(x,y)\,dx = \int_{-1}^{1} [(4t-1)^2 + (3t+1)^2] \cdot 4\,dt = \frac{248}{3}$$

$$\int_C f(x,y)\,dy = \int_{-1}^{1} [(4t-1)^2 + (3t+1)^2] \cdot 3\,dt = 62$$

9 $P(x,y) = x^2 y \qquad Q(x,y) = xy^3$



$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_{C_1+C_2} P\,dx + Q\,dy$$

C_1: either find parametrization $\quad x(0) = -1 \quad x(1) = 2 \Rightarrow x(t) = -1+3t$

$y(0) = 1 \quad y(1) = 1 \Rightarrow y(t) = 1$

$$\int_{C_1} P\,dx + Q\,dy = \int_0^1 (3t-1)^2 \cdot 1 \cdot 3\,dt + (3t-1) \cdot 1^3 \cdot 0\,dt = 3\int_0^1 (9t^2 - 6t + 1)\,dt = 3[3t^3 - 3t^2 + t]\Big|_0^1$$

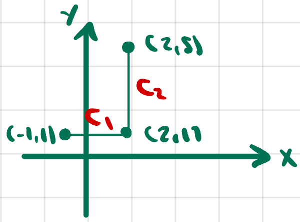
$$= 3[3 - 3 + 1] = 3$$

or compute the line integral directly in dx and dy differentials

Alternatively, $y \equiv 1 \Rightarrow dy = 0$

$$\int_{C_1} P\,dx + Q\,dy = \int_{x=-1}^{2} [1 \cdot x^2\,dx + x \cdot 0] = \int_{x=-1}^{2} x^2\,dx = \frac{1}{3}x^3\Big|_{-1}^{2} = \frac{1}{3}[8 - (-1)] = 3$$

C_2: $x \equiv 2 \Rightarrow dx = 0$



$$\int_{C_2} P\,dx + Q\,dy = \int_{y=1}^{5} 4y \cdot 0 + 2y^3\,dy = \int_1^5 2y^3\,dy$$

$$= \frac{2}{4} y^4 \Big|_1^5 = \frac{1}{2}(625-1) = 312$$

$$\Rightarrow \int_{C_1+C_2} P\,dx + Q\,dy = 312 + 3 = 315$$

11 $\vec{F}(x,y,z) = \langle z, x, -y \rangle = \langle P, Q, R \rangle$

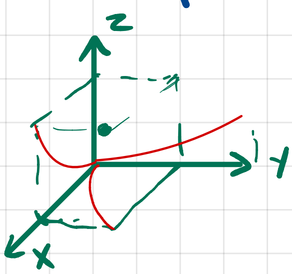
C: $\langle t, t^2, t^3 \rangle$ $t \in [0,1]$

we want $\int_C \vec{F} \cdot \vec{T}\,ds$ = work done by $\vec{F}$ in moving a particle from $\langle 0,0,0 \rangle$ to $\langle 1,1,1 \rangle$.

$$\int_C \vec{F} \cdot \vec{T}\,ds = \int_C P\,dx + Q\,dy + R\,dz = \int_C z\,dx + x\,dy - y\,dz = \int_0^1 t^3\,dt + t \cdot 2t\,dt - t^2 \cdot 3t^2\,dt$$

$$= \int_0^1 (t^3 + 2t^2 - 3t^5)\,dt = \frac{1}{4} + \frac{2}{3} - \frac{3}{6} = \frac{3+8-6}{12} = \frac{5}{12}$$

Geometric Interpretation



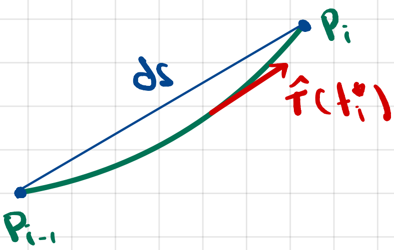
$r(t) = \langle t, t^2, t^3 \rangle$ a curve in space

particle moves along curve with $\vec{v} = \langle 1, 2t, 3t^2 \rangle$

$|\vec{v}|$ = speed = $\sqrt{1+4t^2+9t^2} = \sqrt{1+13t^2}$

$\Rightarrow$ speeds up quickly w/ time

On a small piece ds of the trajectory:



There is a $t_i^* \in [t_{i-1}, t_i]$ such that the speed at t_i^* is the average speed between t_{i-1} and t_i.

$$v(t_i^*) \cdot \Delta t = \int_C v(t)\,dt = \Delta s_i$$

$$\Delta W_i \approx \vec{F}(t_i^*) \cdot \hat{T}(t_i^*) \cdot \Delta s_i, \quad W \approx \sum_i \Delta W_i \Rightarrow W = \lim_{\Delta t \to 0} \sum_i \Delta W_i = \int_C F(t)\,T(t)\,ds$$

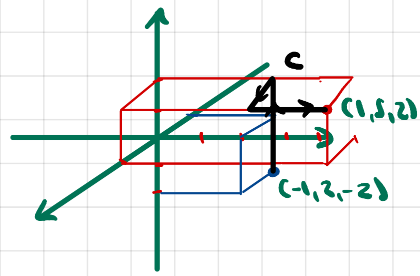
$$= \int \langle P, Q, R \rangle \langle x', y', z' \rangle \frac{1}{v} \cdot v\,dt = \int_C P\,dx + Q\,dy + R\,dz$$

15 $\vec{F}(x,y,z) = \langle yz^2, xz^2, 2xyz\rangle$

C_1: $x \equiv -1$ $dx = 0$
$y \equiv 2$ $dy = 0$

C_2: $z \equiv 2$, $y \equiv 2$

C_3: $x \equiv 1$, $z \equiv 2$



$$\int_C \vec{F}\cdot\hat{T}\,ds = \int_{C_1+C_2+C_3} \vec{F}\cdot\hat{T}\,ds$$

$$\int_{C_1} \vec{F}\cdot\hat{T}\,ds = \int_{C_1} P\cdot 0 + Q\cdot 0 + R\,dz$$

$$= \int_{-2}^{2} 2\cdot(-1)\cdot 2\cdot z\,dz = \int_{-2}^{2} -4z\,dz$$

$$= -\frac{4z^2}{2}\Big|_{-2}^{2} = -2(4-4) = 0$$

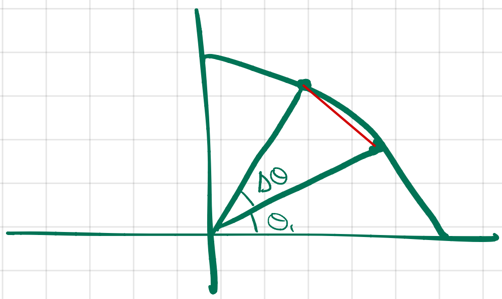
$$\int_{C_2} \vec{F}\cdot\hat{T}\,ds = \int_{-1}^{1} 2\cdot 2^2\,dx = 8(1-(-1)) = 16$$

$$\int_{C_3} \vec{F}\cdot\hat{T}\,ds = \int_{2}^{5} 1\cdot 2^2\,dy = 4(5-2) = 12$$

$$\Rightarrow \int_C \vec{F}\cdot\hat{T}\,ds = \int_{C_1+C_2+C_3} \vec{F}\cdot\hat{T}\,ds = 28$$

$ds = \sqrt{dx^2 + dy^2}$

$x = a\cos\theta$ $dx = a\sin\theta\,d\theta$
$y = a\sin\theta$ $dy = a\cos\theta\,d\theta$



17 $f(x,y,z) = 2x + 9xy$ $\quad C: \langle t, t^2, t^3\rangle \quad t\in[0,1]$

$$\int_C f(x,y,z)\,ds = \int_0^1 (2t+9t^3)\sqrt{1+4t^2+9t^4}\,dt = \frac{1}{4}\cdot\frac{2}{3}(1+4t^2+9t^4)^{3/2}\Big|_0^1$$

$$= \frac{1}{6}\left[(1+4+9)^{3/2} - 1\right] = \frac{14^{3/2}-1}{6}$$

19 wire of shape $x^2+y^2=a^2$, $a>0$, $y\geq 0$, $\delta = k$ $\qquad$ parametrization of C

$$m = \int_C dm = \int_C \delta\,ds = k\int_C ds = k\cdot 2\pi a/2 = k\pi a$$

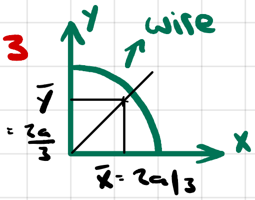
$$\bar{y} = \frac{\int_C y\,dm}{m} = \frac{\cancel{k}\int_C y\,ds}{\cancel{k}\pi a} = \frac{\int_0^\pi a^2\sin\theta\,d\theta}{\pi a} = \frac{-a^2\cos\theta\Big|_0^\pi}{\pi a} = \frac{-a^2(-1-1)}{\pi a} = \frac{2a^2}{\pi a} = \frac{2a}{\pi}$$

and $\bar{x} = 0$ by symmetry

$x = a\cos\theta$ $\quad dx = -a\sin\theta\,d\theta$
$y = a\sin\theta$ $\quad dy = a\cos\theta\,d\theta$

$$ds = \sqrt{dx^2+dy^2} = \sqrt{a^2(\sin^2\theta+\cos^2\theta)\,d\theta^2} = a\,d\theta$$

23



$\delta(x,y) = kxy$ $\qquad y = \sqrt{a^2 - x^2}$

$dy/dx = \dfrac{-x}{\sqrt{a^2-x^2}}$

$$m = \int_C \delta(x,y)\,ds = \int_0^a kxy\sqrt{1+(dy/dx)^2}\,dx = \int_0^a kxy\sqrt{1+x^2/a^2-x^2}\,dx = k\int_0^a xy\sqrt{\frac{a^2}{a^2-x^2}}\,dx$$

$$= ka\int_0^a \frac{x}{\cancel{\sqrt{a^2-x^2}}}\cdot\cancel{\sqrt{a^2-x^2}}\,dx = ka\,\frac{x^2}{2}\Big|_0^a = \frac{ka^3}{2}$$

by symmetry of region and δ, $\bar{x} = \bar{y}$

$$\bar{x} = \frac{\int_C x\cdot\delta(x,y)\cdot ds}{m} = \frac{\int_0^{\pi/2} a\cos\theta\cdot k\,a\cos\theta\cdot a\sin\theta\cdot a\,d\theta}{m} = \frac{a^4k\int_0^{\pi/2}\cos^2\theta\sin\theta\,d\theta}{m} = \frac{-a^4k\,\frac{\cos^3\theta}{3}\Big|_0^{\pi/2}}{m}$$

$x = a\cos\theta \quad dx = -a\sin\theta\,d\theta \quad ds = \sqrt{a^2 d\theta^2} = a\,d\theta$

$y = a\sin\theta \quad dy = a\cos\theta\,d\theta \Rightarrow$

$$= \frac{-\frac{a^4k}{3}(0-1)}{m} = \frac{\frac{a^{\cancel{4}}\cancel{k}}{3}}{\frac{\cancel{ka^3}}{2}} = \frac{2a}{3}$$

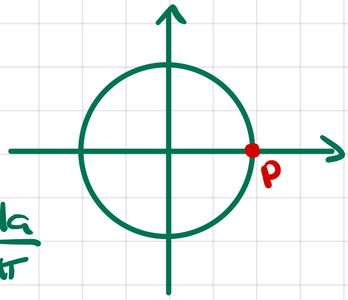
$$I_x = \int_C y^2\,dm = \int_0^{\pi/2} a^2\sin^2\theta\cdot ka^2\sin\theta\cos\theta\,a\,d\theta$$

$$= \int_0^{\pi/2} ka^5\sin^3\theta\cos\theta\,d\theta = \frac{ka^5\sin^4\theta}{4}\Big|_0^{\pi/2} = \frac{ka^5}{4}(1-0) = \frac{ka^5}{4} = \frac{ka^3}{2}\cdot\frac{a^2}{2} = \frac{1}{2}ma^2$$

by symmetry $I_y = I_x$.

$I_0 = I_x + I_y = ma^2$

27 $\bar{D} = \frac{1}{s}\int_C D(x,y)\,ds$ $\quad P(a,0) \quad D(x,y) = \sqrt{(a-x)^2 + y^2}$



$s = 2\pi a$

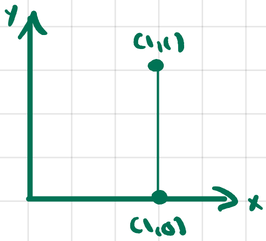
$$\bar{D} = \frac{1}{2\pi a}\int_C D(x,y)\,ds = \frac{\sqrt{2}}{2\pi a}\cdot a^2\int_0^{2\pi}(1-\cos\theta)^{1/2}\,d\theta = \frac{\cancel{\sqrt{2}}}{\cancel{2}\pi}a\cdot 4\cancel{\sqrt{2}} = \frac{4a}{\pi}$$

$x = a\cos\theta$
$y = a\sin\theta$ $\Rightarrow ds = a\,d\theta$

$$D(\theta) = \sqrt{(a-a\cos\theta)^2 + a^2\sin^2\theta} = \sqrt{a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta} = \sqrt{2a^2 - 2a^2\cos\theta}$$

$$= \sqrt{2a^2(1-\cos\theta)}$$

33 $\vec{F} = \dfrac{\langle kx, ky\rangle}{x^2+y^2}$



$C: \; x \equiv 1 \Rightarrow dx = 0$

$$W = \int_C \vec{F}\cdot\hat{T}\,ds = \int_C P\,dx + Q\,dy = \int_C \frac{k}{1+y^2}(0) + \frac{ky}{1+y^2}\,dy = \int_0^1 \frac{ky}{1+y^2}\,dy = \frac{k}{2}\ln(1+y^2)\Big|_0^1 = \frac{k\ln 2}{2}$$