12.6 Curvature and Acceleration

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = \langle x, y, z \rangle$

The arc length s along the curve with position vector $\vec{r}(t)$, from $\vec{r}(a)$ to $\vec{r}(b)$ is by definition:

$$s = \int_a^b \underbrace{\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}}_{|v(t)| = v(t)} \, dt = \int_a^b v(t)\,dt$$

arc length function $s(t) = \int_a^t v(\tau)\,d\tau$

$$\Rightarrow \frac{ds}{dt} = v(t)$$

"speed of moving point is the time rate of change of its arc-length function."

curvature of Plane Curves

"rate of change of direction"

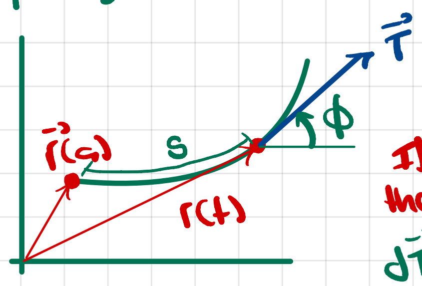
$\Rightarrow$ "rate at which $\vec{v}$ is turning"

direction is determined by velocity vector $\qquad \vec{v}(t) = \vec{r}'(t) \neq \vec{0}$

$\vec{r}(t) = \langle x(t), y(t) \rangle \quad a \leq t \leq b \quad$ position vector of differentiable plane curve that is smooth

$\Rightarrow$ unit tangent vector at point $\vec{r}(t)$ is $\vec{T}(t) = \dfrac{\vec{v}(t)}{|\vec{v}(t)|}$

Let ϕ = angle of inclination of $\vec{T}$ (counterclockwise from x-axis)



$\vec{T} = \langle \cos\phi, \sin\phi \rangle \quad (\langle |\vec{T}|\cos\phi, |\vec{T}|\sin\phi \rangle)$

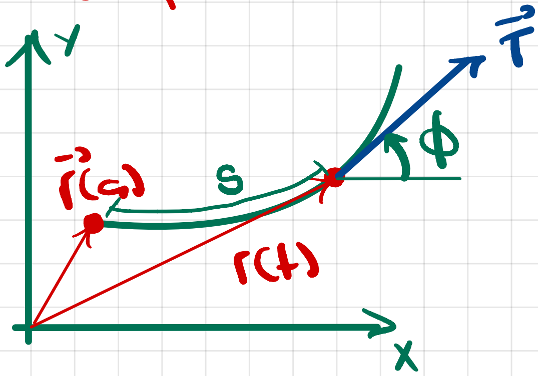
If we express $\vec{T}$ as function of arc-length, and take derivative, we get the rate at which $\vec{T}$ is turning:

$$\frac{d\vec{T}}{ds} = \frac{dT}{d\phi} \cdot \frac{d\phi}{ds} = \langle -\sin\phi, \cos\phi \rangle \frac{d\phi}{ds}$$

$$\left|\frac{dT}{ds}\right| = \sqrt{\sin^2\phi + \cos^2\phi}\, \left|\frac{d\phi}{ds}\right| = \left|\frac{d\phi}{ds}\right|$$

$$\kappa = \left|\frac{d\phi}{ds}\right| = \text{curvature}$$

Given a smooth parametric curve $x = x(t)$, $y = y(t)$ how do we compute its curvature?



$$\phi = \tan^{-1}\left(\frac{dy}{dx}\right) = \tan^{-1}\left(\frac{y'}{x'}\right), \quad x'(t) \neq 0$$

$$\frac{d\phi}{dt} = \frac{y''x' - y'x''}{(x')^2} \div \left[1 + \left(\frac{y'}{x'}\right)^2\right]$$

$$= \frac{x'y'' - x''y'}{(x')^2 + (y')^2}$$

$$\kappa = \left|\frac{d\phi}{ds}\right| = \left|\frac{d\phi}{dt} \cdot \frac{dt}{ds}\right| = \frac{1}{v}\left|\frac{d\phi}{dt}\right| \quad \text{where } v = \frac{ds}{dt} = \sqrt{x'^2 + y'^2} > 0$$

$$= \frac{x'y'' - x''y'}{[x'^2 + y'^2]^{3/2}}$$

Note

$$\vec{T} = \langle \cos\phi, \sin\phi \rangle$$

$$\frac{d\vec{T}}{ds} = \langle -\sin\phi, \cos\phi \rangle \frac{d\phi}{ds} \Rightarrow \vec{T} \cdot \frac{d\vec{T}}{ds} = 0$$

If $\frac{d\vec{T}}{ds} \neq \vec{0}$ then its unit vector is called principal unit normal vector, $\vec{N}$

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\phi}{ds}\right| \Rightarrow \frac{d\vec{T}}{ds} = \kappa \cdot \vec{N}$$

Curvature of Space Curves

→ moving particle in space
position vector, $\vec{r}(t)$, twice diff.

$\vec{v}(t)$ never zero

→ Def: $\vec{T}(t) = \dfrac{\vec{v}(t)}{|\vec{v}(t)|} = \dfrac{\vec{v}(t)}{v(t)} \Rightarrow \vec{v}(t) = v(t)\vec{T}(t)$

Before, we had $k = \left|\dfrac{d\phi}{ds}\right|$ but now there isn't a single angle determining $\vec{T}$'s direction.

→ different approach

$\vec{T}\cdot\vec{T} = 1$, diff. w/resp. to s

$\vec{T}\cdot\dfrac{d\vec{T}}{ds} = 0 \Rightarrow \vec{T}$ and $\dfrac{d\vec{T}}{ds}$ are $\perp$

*note that for any $\vec{v}$ if $|\vec{v}| = k$ then the derivative vector cannot have a component in the same direction as $\vec{v}$, because if it did, the length of the vector would change.

$\vec{v}\cdot\vec{v} = k \Rightarrow \vec{v}\dfrac{d\vec{v}}{dx} = 0 \quad \forall x$

define curvature $k = \left|\dfrac{d\vec{T}}{ds}\right|$, as before

but now we decompose $d\vec{T}/ds$ differently

$k = \left|\dfrac{d\vec{T}}{ds}\right| = \left|\dfrac{d\vec{T}}{dt}\cdot\dfrac{dt}{ds}\right| \Rightarrow k = \dfrac{1}{v(t)}\cdot\left|\dfrac{d\vec{T}}{dt}\right|$

for $k \neq 0$, define principal unit normal vector $\vec{N}$

$\vec{N} = \dfrac{d\vec{T}/ds}{|d\vec{T}/ds|} = \dfrac{1}{k}\cdot\dfrac{d\vec{T}}{ds} \Rightarrow \dfrac{d\vec{T}}{ds} = k\vec{N}$

Recap:

Define $k = \left|\dfrac{d\vec{T}}{ds}\right|$ obtain $k = \dfrac{1}{v(t)}\left|\dfrac{d\vec{T}}{dt}\right|$

Because $\vec{T}$ is of constant length 1 it is $\perp$ to any of its derivatives,

in particular $d\vec{T}/ds$. $\vec{N}$ is just the unit vector in the direction of $d\vec{T}/ds$,

and given the defn. of k we have $d\vec{T}/ds = k\vec{N}$

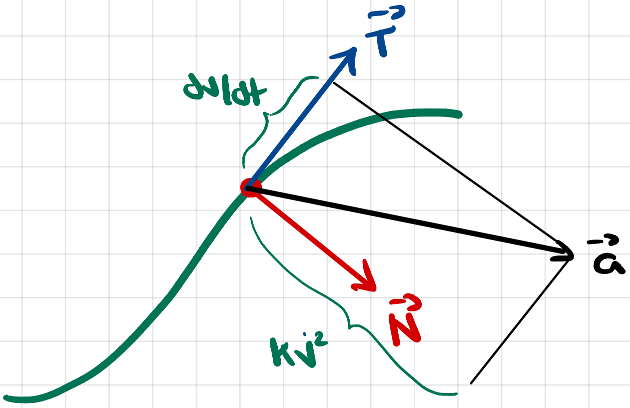
Normal and Tangential Components of Acceleration

$$\frac{d\vec{T}}{ds} = k\vec{N}$$

$$\vec{v} = v\vec{T}$$

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{dv(t)}{dt}\vec{T}(t) + v(t)\frac{dT(t)}{dt}$$

$$= \frac{dv(t)}{dt}\vec{T}(t) + v(t)\underbrace{\frac{dT(t)}{ds}}_{k\vec{N}}\overbrace{\frac{ds}{dt}}^{v(t)}$$



$\Rightarrow$

$$\vec{a}(t) = \frac{dv(t)}{dt}\vec{T}(t) + v(t)^2 k\vec{N}$$

rate of chg of speed — rate of change of direction of motion

ie $\vec{a} = \frac{dv}{dt}\vec{T} + kv^2\vec{N}$

$\vec{T}$ and $\vec{N}$ are $\perp$ so the expr. above is a decomposition of acceleration into components tangent and normal to the trajectory