# 14.8 - Surface Area

1 plane: $z = x + 3y$

elliptical cylinder: $\frac{x^2}{4} + \frac{y^2}{9} = 1$

we want to integrate over the area of a piece of the surface $z = x + 3y$.

The intersection of the plane and cylinder tells us the region R in xy-plane over which we integrate.

we have the transformation

$$(u,v) \to \langle u, v, u+3v\rangle = \langle x(u,v), y(u,v), z(u,v)\rangle = \vec{r}(u,v)$$

parametric surface

The Riemann sum for surface area involves summing small areas that approximate small pieces of the actual surface. Such small areas are a cross product of two vectors that are tangent to the surface at a certain point in each element of the partition being used.

$$a(S) = \sum \Delta S_i \approx \sum \Delta P_i = \sum |r_u(u,v)\Delta u \times r_v(u,v)\Delta v| = \sum |\vec{N}(u_i,v_i)|\Delta u \Delta v$$

↳ area of a parallelogram approximating the actual curvilinear figure on the surface

In the limit, we have that surface area $a(S)$ is an integral $\iint_R |\vec{N}(u,v)|\,du\,dv$

In this problem:

The surface is $z = f(x,y) = x + 3y$ which we have parametrized as above. The cylinder simply gives us information on the integration bounds, which is an ellipse in the xy-plane.

$$\iint_R \underbrace{(1 + z_x^2 + z_y^2)^{1/2}}\,dx\,dy = \int_{-3}^{3}\int_{-2\sqrt{1-y^2/9}}^{2\sqrt{1-y^2/9}} \sqrt{11}\,dx\,dy = 6\pi\sqrt{11}$$

$$|r_u \times r_v| = \text{abs}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{vmatrix} = |\langle -1, -3, 1\rangle| = (1+9+1)^{1/2} = \sqrt{11}$$

$r_u = \langle 1, 0, 1\rangle$

$r_v = \langle 0, 1, 3\rangle$

**3** paraboloid $z=9-x^2-y^2$ above $z=5$

$(x,y) \to (x,y,9-x^2-y^2)$

$\mathbb{R}^2 \to \mathbb{R}^3$, transformation

region of integration: $9-x^2-y^2=5 \Rightarrow x^2+y^2=4$

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (1+4x^2+4y^2)^{1/2}\,dx\,dy = \frac{17\sqrt{17}\,\pi}{6} - \frac{\pi}{6}$$

**5** $z=x+y^2 \quad 0\le x\le 1 \quad 0\le y\le 2$

Maple

Textbook

$$\int_0^2\int_0^1 (1+1+4y^2)^{1/2}\,dx\,dy = 3\sqrt{2}+\frac{\operatorname{arcsinh}(2\sqrt{2})}{2} = 3\sqrt{2}+\frac{1}{2}\ln(3+2\sqrt{2}) \approx 1.7627$$

**7** $2x+3y+z=6$, first octant

$x=y=0 \Rightarrow z=6$
$y=z=0 \Rightarrow x=3$
$x=z=0 \Rightarrow y=1$

$z=6-2x-3y$

$z_x=-2 \quad z_y=-3$

$$\int_0^1\int_0^3 (1+4+9)^{1/2}\,dx\,dy = 3\sqrt{14}$$

15 cylinder $x^2+z^2=a^2$ within cylinder $r^2=x^2+y^2=a^2$

## rectangular

1) $x^2+z^2=a^2$ $\Rightarrow x^2=a^2-z^2 \Rightarrow a^2-z^2+y^2=a^2 \Rightarrow z^2=y^2$

2) $x^2+y^2=a^2$

3) $\Rightarrow y^2=z^2$

1), 2) $\Rightarrow$ 3)

parametrized intersection: $f(t)=(\pm\sqrt{4-t^2},\, t,\, \pm t)$

## spherical

$x=\rho\sin\phi\cos\theta$

$y=\rho\sin\phi\sin\theta$

$z=\rho\cos\phi$

1) $\rho^2\sin^2\phi\cos^2\theta+\rho^2\cos^2\phi=a^2$

2) $\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta=a^2 \Rightarrow \rho^2\sin^2\phi=a^2$

$\Rightarrow \rho^2\sin^2\phi(1-\cos^2\theta)=\rho^2\cos^2\phi$

$\Rightarrow \rho^2\sin^2\phi\sin^2\theta=\rho^2\cos^2\phi \Rightarrow \sin^2\phi\sin^2\theta=\cos^2\phi$

$\Rightarrow \tan^2\phi=1/\sin^2\theta$

From 2)

$$\sin^2\phi=\frac{a^2}{\rho^2} \Rightarrow \sin\phi=\pm\sqrt{\frac{a^2}{\rho^2}} \Rightarrow \phi=\sin^{-1}\left(\pm\frac{a}{\rho}\right)=\sin^{-1}\left(\frac{a}{\rho}\right)$$

$$\cos^2\phi=1-\frac{a^2}{\rho^2}=\frac{\rho^2-a^2}{a^2}$$

insert into 1)

$$\cancel{\rho^2}\cdot\frac{a^2}{\cancel{\rho^2}}\cos^2\theta+\cancel{\rho^2}\,\frac{(\rho^2-a^2)}{\cancel{\rho^2}}=a^2 \Rightarrow a^2\cos^2\theta+\rho^2-a^2=a^2 \Rightarrow \rho^2=2a^2-a^2\cos^2\theta$$

$$\Rightarrow \rho=\pm\sqrt{2a^2-a^2\cos^2\theta}$$

parametrized intersection: $\left(\pm\sqrt{2a^2-a^2\cos^2\theta},\, \theta,\, \sin^{-1}\left(\pm\frac{a}{\pm\sqrt{2a^2-a^2\cos^2\theta}}\right)\right)$

$\rho>0$ $\nearrow$ $+\frac{a}{\rho}$ $+$ $+\ s^{-1}$

$\searrow$ $+\frac{a}{-\rho}$ $+\ -$ $+\ -s^{-1}$

$\rho<0$ $+\frac{a}{-\rho}$ $-\frac{a}{\rho}$ $-$ $-=$ $--s^{-1}$

$-\frac{a}{-\rho}$ $\frac{a}{\rho}$ $-$ $+=-+$ $=-+s^{-1}$

## Alternative parametrization

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$

1) $\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \cos^2\phi = a^2$

2) $\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta = a^2 \Rightarrow \rho^2 \sin^2\phi = a^2$

$$\rho^2 = \frac{a^2}{\sin^2\phi} \Rightarrow \rho = \pm \frac{a}{\sin\phi}$$

$$\cancel{\frac{a^2}{\sin^2\phi}}\, \cancel{\sin^2\phi} \cos^2\theta + \frac{\cancel{a^2}}{\sin^2\phi} \cos^2\phi = \cancel{a^2}$$

$$\cos^2\theta + \frac{1}{\tan^2\phi} = 1 \Rightarrow \cos^2\theta = 1 - \frac{1}{\tan^2\phi} \Rightarrow \cos\theta = \pm\left(1 - \frac{1}{\tan^2\phi}\right)^{1/2}$$

$$\theta = \cos^{-1}\left[\pm\left(1 - \frac{1}{\tan^2\phi}\right)^{1/2}\right]$$

## Cylindrical

$x^2 + y^2 = a^2 \Rightarrow r^2 = a^2$
$x^2 + z^2 = a^2 \Rightarrow r^2\cos^2\theta + z^2 = a^2 \Rightarrow a^2\cos^2\theta + z^2 = a^2 \Rightarrow z^2 = a^2(1 - \cos^2\theta)$

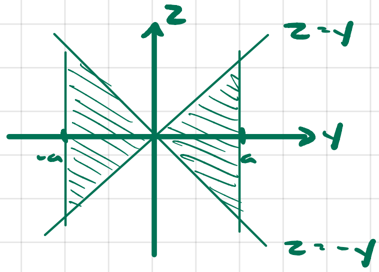
$\Rightarrow z^2 = a^2 \sin^2\theta \Rightarrow z = \pm a \sin\theta$

$(\pm a, \theta, \pm a\sin\theta)$

Equations for part of $x^2+y^2=a^2$ inside $x^2+z^2=a^2$

inside $x^2+z^2=a^2$ means $x^2+z^2\le a^2 \Rightarrow a^2-y^2+z^2\le a^2 \Rightarrow z^2\le y^2$

$$\begin{cases} z^2\le y^2 \\ x^2+y^2=a^2 \end{cases}$$




we can choose how to parametrize. One option is

$$x=\pm(a^2-y^2)^{1/2},\quad y=y,\quad z=z$$

$\vec{r}(y,z)=\langle g(y,z),y,z\rangle=(\pm\sqrt{a^2-y^2},y,z)$, which are two parametric surfaces.

We need the area of all four surfaces.

First surface

$$\vec{r}(y,z)=\langle\sqrt{a^2-y^2},y,z\rangle$$

$$g_y=\frac{-y}{\sqrt{a^2-y^2}}\qquad g_z=0$$

$$\text{Area}(S)=\int_0^a\int_{-y}^{y}\left(1+\frac{y^2}{a^2-y^2}\right)dz\,dy+\int_{-a}^{0}\int_{y}^{-y}\left(1+\frac{y^2}{a^2-y^2}\right)dz\,dy=2a^2+2a^2=4a^2$$

Second surface

$$\vec{r}(y,z)=\langle-\sqrt{a^2-y^2},y,z\rangle\qquad g_y=\frac{y}{\sqrt{a^2-y^2}}\qquad g_z=0$$

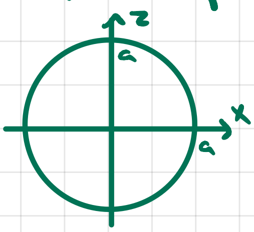
$$\text{Area}(S)=\int_0^a\int_{-y}^{y}\left(1+\frac{y^2}{a^2-y^2}\right)dz\,dy+\int_{-a}^{0}\int_{y}^{-y}\left(1+\frac{y^2}{a^2-y^2}\right)dz\,dy=4a^2$$

We could have chosen an alternative parametrization

$\vec{r}(x,z)=\langle x,\pm\sqrt{a^2-x^2},z\rangle$ where $y=h(x,z)=\pm\sqrt{a^2-x^2}$

We have two surfaces. For example, $r(x,z)=\langle x,\sqrt{a^2-x^2},z\rangle$

The relationship between x and z is: $z^2\le y^2=a^2-x^2 \Rightarrow x^2+z^2\le a^2$




$$\text{Area}=\int_{-2}^{2}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(1+\frac{x}{a^2-x^2}\right)^{1/2}dz\,dx$$

why can't we use $z=z(x,y)$?

From $z^2\le y^2 \Rightarrow z\le y,\ z\ge -y$ we don't have a function.

For each $y$, $z$ takes on infinite values in $[-y,y]$.

17 $y = f(x,z)$

$\vec{r}(x,z) = \langle x, f(x,z), z \rangle$

$\vec{r}_x = \langle 1, f_x, 0 \rangle \qquad \vec{r}_z = \langle 0, f_z, 1 \rangle$

$$\vec{r}_x \times \vec{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & f_x & 0 \\ 0 & f_z & 1 \end{vmatrix} = \langle f_x, -1, f_z \rangle$$

$|\vec{r}_x \times \vec{r}_z| = (1 + f_x^2 + f_z^2)^{1/2}$

$A(S) = \iint_R (1 + f_x^2 + f_z^2)^{1/2} \, dx\,dz$