Pset 6

2I-1 $V(x,y,z) = xyz$

max V s.t. $f(x,y,z) = c$

a) $f(x,y,z) = x+2y+3z = 18$ $\qquad \nabla f = \lambda \nabla V \Rightarrow$ $yz = \lambda$, $xz = 2\lambda$, $xy = 3\lambda$, $x+2y+3z=18$

$\nabla f = \langle 1,2,3\rangle$ $\quad \nabla V = \langle yz, xz, xy\rangle$

$yz = \frac{xz}{2} \Rightarrow 2y = x \Rightarrow y = \frac{x}{2}$

$yz = \frac{xy}{3} \Rightarrow 3z = x \Rightarrow z = \frac{x}{3}$

$x + 2\frac{x}{2} + 3\frac{x}{3} = 18 \Rightarrow 3x = 18 \Rightarrow x = 6, y = 3, z = 2 \Rightarrow (6,3,2)$ candidate

$V(6,3,2) = 6\cdot 3\cdot 2 = 36$ $\qquad$ There are no points where the constraint gradient ∇f is $\vec{0}$.

The boundaries of the problem involve one of x,y,z being 0, but volume is zero in those cases.

$\Rightarrow (6,3,2)$ is max point.

b) $f(x,y) = x^2+2y^2+4z^2-12 = 0$

$\nabla f = \langle 2x, 4y, 8z\rangle$

$\nabla V = \langle yz, xz, xy\rangle$

$\Rightarrow \nabla V = \lambda \nabla f \Rightarrow$ $yz = \lambda \cdot 2x$, $xz = \lambda\cdot 4y$, $xy = \lambda \cdot 8z$, $x^2+2y^2+4z^2=12$

$4y^2 = 2x^2$, $x^2 = 2y^2$, $y^2 = x^2/2$

$8z^2 = 2x^2$, $x^2 = 4z^2$, $z^2 = x^2/4$

$\frac{yz}{2x} = \frac{xz}{4y} = \frac{xy}{8z}$

$x^2+x^2+x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$ $\qquad x = 2 \Rightarrow y^2 = 2 \Rightarrow y = \pm\sqrt{2}$, $z^2 = 1 \Rightarrow z = \pm 1$ $\qquad x = -2 \Rightarrow y = \pm\sqrt{2}$, $z = \pm 1$

min/max candidates

$(2,\sqrt{2},1), (2,\sqrt{2},-1), (2,-\sqrt{2},1), (2,-\sqrt{2},-1)$

$(-2,\sqrt{2},1), (-2,\sqrt{2},-1), (-2,-\sqrt{2},1), (-2,-\sqrt{2},-1)$

only candidate in 1st octant is $(2,\sqrt{2},1)$, $V(2,\sqrt{2},1) = 2\sqrt{2}$ $\qquad$ lemma / are

2I-2 min $x^2+y^2+z^2$ s.t. $g(x,y,z)=x^3y^2z-6\sqrt{3}=0$

$\nabla g = \langle 3x^2y^2z,\ 2x^3yz,\ x^3y^2 \rangle$

$\nabla f = \langle 2x, 2y, 2z \rangle$

$\nabla f = \lambda \nabla g \Rightarrow$

$$2x = \lambda 3x^2y^2z$$
$$2y = \lambda \cdot 2x^3yz$$
$$2z = \lambda \cdot x^3y^2$$
$$x^3y^2z = 6\sqrt{3}$$

$$\frac{2}{3xy^2z} = \frac{2}{2x^3z} = \frac{2z}{x^3y^2}$$

$\Rightarrow 2x^3z = 3xy^2z \Rightarrow 2x^2 = 3y^2$

$\Rightarrow 3xy^2z\cdot z = x^3y^2 \Rightarrow 3xz^2 = x^3 \Rightarrow 3z^2 = x^2$

$\Rightarrow 2x^3z^2 = x^3y^2 \Rightarrow 2z^2 = y^2$

so, $x^3y^2z = 6\sqrt{3}$

$y^2 = 2z^2$

$x = \pm\sqrt{3}z$

$\Rightarrow \pm\sqrt{27}\,z^3\cdot 2z^2\cdot z = \pm 6\sqrt{3}\,z^6 = 6\sqrt{3} \Rightarrow \pm z^6 = 1$

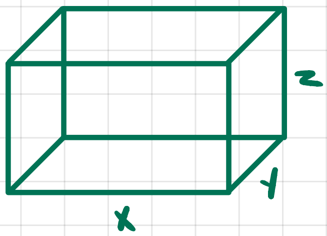
$\Rightarrow z^6 = \pm 1 \Rightarrow$ x must be $+\sqrt{3}z$ otherwise it does not satisfy the constraint

$\Rightarrow z = \pm 1 \Rightarrow x = \pm\sqrt{3} \Rightarrow y = \pm\sqrt{2}z = \pm\sqrt{2}$

candidates in 1st octant

$(\sqrt{3}, \sqrt{2}, 1)$

2I-3



$T(x,y,z) = 2\cdot 1\cdot yz + 2\cdot 2\cdot xz + 1\cdot 3\cdot xy = 2yz + 4xz + 3xy$

min T s.t. $V(x,y,z) = xyz = 1$

$\nabla V = \langle yz, xz, xy \rangle$ $\quad \nabla T = \langle 4z+3y,\ 2z+3x,\ 2y+4x \rangle$

$\nabla T = \lambda \nabla V \Rightarrow$

$$4z+3y = \lambda yz$$
$$2z+3x = \lambda xz$$
$$2y+4x = \lambda xy$$
$$xyz = 1$$

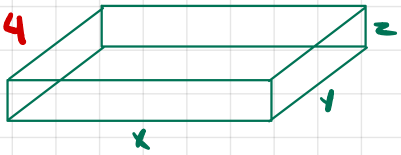
$\Rightarrow \lambda xyz = 4xz + 3xy = 2yz + 3xy = 2yz + 4xz$

$\Rightarrow 4xz = 2yz \Rightarrow 2x = y$

$\Rightarrow 3xy = 4xz \Rightarrow 3y = 4z$

$\Rightarrow \frac{y}{2}\cdot y\cdot \frac{3}{4}y = 1 \Rightarrow \frac{3}{8}y^3 = 1 \Rightarrow y^3 = \frac{8}{3} \Rightarrow y = \frac{2\sqrt{3}}{3} \Rightarrow x = \frac{\sqrt{3}}{3} \Rightarrow z = \frac{\sqrt{3}}{2}$

2I-4



$C(x,y,z) = 2\cdot 2\cdot yz + 1\cdot 2xz + 1\cdot 1\cdot xy + 1\cdot 4\cdot xz$
$= 4yz + 2xz + xy + 4xz$
$= 4yz + 6xz + xy$

a) $\max V(x,y,z) = xyz \quad \text{s.t.} \quad C(x,y,z) = 72$

$\langle 6z+y,\ 4z+x,\ 4y+6x\rangle = \lambda\langle yz, xz, xy\rangle$

$6z+y = \lambda yz \qquad \lambda xyz = 6xz + xy \Rightarrow 6xz = 4yz \Rightarrow 6x = 4y \Rightarrow 3x = 2y$
$4z+x = \lambda xz \qquad\quad = 4yz + xy \Rightarrow xy = 6xz \Rightarrow y = 6z$
$4y+6x = \lambda xy \qquad = 4yz + 6xz$

$\Rightarrow C(y) = 4y\cdot\frac{y}{6} + \cancel{6}\cdot\frac{2}{3}y\cdot\frac{y}{\cancel{6}} + \frac{2}{3}y\,y = y^2\left[\frac{4}{6}+\frac{2}{3}+\frac{2}{3}\right] = y^2\cdot 2 = 72$
$\Rightarrow y^2 = 36 \Rightarrow y = 6 \Rightarrow x = \frac{2}{3}\cdot 6 = 4 \Rightarrow z = 1 \qquad (4,6,1)$

$V(4,6,1) = 24$

b) $\min C(x,y,z) \quad \text{s.t.} \quad V(x,y,z) = 24$

$\langle 6z+y,\ 4z+x,\ 4y+6x\rangle = \lambda\langle yz, xz, xy\rangle$, same conditions as in a) so

$3x = 2y,\ z = y/6$

$\frac{2}{3}y\cdot y\cdot\frac{y}{6} = \frac{1}{9}y^3 = 24 \Rightarrow y^3 = 24\cdot 9 = 3\cdot 8\cdot 9 = 2^3\cdot 3^3 \Rightarrow y = 6 \Rightarrow x = 4 \Rightarrow z = 1$

2J-1 $w = x^2+y^2+z^2 \qquad z = x^2+y^2$

a) $dw = 2x\,dx + 2y\,dy + 2z\,\cancelto{0}{dz} \qquad \cancelto{0}{dz} = 2x\,dx + 2y\,dy \Rightarrow dx = -\frac{y}{x}dy$
$x\,dx = -y\,dy$
$= 2x\cdot\left(-\frac{y}{x}\right)dy + 2y\,dy$
$= -2y\,dy + 2y\,dy = 0 \quad .\ (\partial w/\partial x)_z = (\partial w/\partial y)_z = 0$

$2x\,dx = -2y\,dy + dz \Rightarrow dx = -\frac{y}{x}dy + \frac{1}{2x}dz$

$dw = 2x\left[-\frac{y}{x}dy + \frac{1}{2x}dz\right] + 2y\,dy + 2z\,dz$
$= -2y\,dy + dz + 2y\,dy + 2z\,dz = 3z\,dz$

$dw = 3z\,dz \Rightarrow (\partial w/\partial z)_x = 3z,\ (\partial w/\partial y)_z = 0$

b) $(\partial w/\partial z)_y = 3z$

2J-1 $w = x^2 + y^2 + z^2 \qquad z = x^2 + y^2$

method of substitution

we want $(\partial w/\partial y)_z$

how w changes when we change y, keeping z constant, but not x: x is dependent on y, z.

$x^2 = z - y^2 \Rightarrow w(y,z) = z - y^2 + y^2 + z^2 = z + z^2$

$dw = (1+2z)dz$

$\Rightarrow (\partial w/\partial y)_z = 0$

Also, $(\partial w/\partial z)_y = 1+2z$

method of differentials

$dw = 2x\,dx + 2y\,dy + 2z\,dz$

$\Rightarrow \nabla w = \langle 2x, 2y, 2z\rangle = \langle (\partial w/\partial x)_{y,z}, (\partial w/\partial y)_{x,z}, (\partial w/\partial z)_{y,x}\rangle$

these are the derivatives in directions $\hat{i}, \hat{j}, \hat{k}$ of $w = f(x,y,z)$

But $z = x^2 + y^2$, so we are not on the entirety of $f(x,y,z)$, we are on a part of it, a new surface.

$dz = 2x\,dx + 2y\,dy$. when we sub this into dw, we are obtaining an expression that says

"how w chgs when x and y change and z changes by an amount that is a function of the changes

in x and y. We aren't changing all three anymore so we aren't going in any direction we want.

$dw = 2x\,dx + 2y\,dy + 2z \cdot 2x\,dx + 2z \cdot 2y\,dy = (2x + 4zx)dx + (2y + 4yz)dy$

$$= \underbrace{(2x + 4x(x^2+y^2))}_{(\partial w/\partial x)_y}dx + \underbrace{(2y + 4y(x^2+y^2))}_{(\partial w/\partial y)_x}dy$$

$(\partial w/\partial y)_x = 2y(1 + 2(x^2+y^2))$

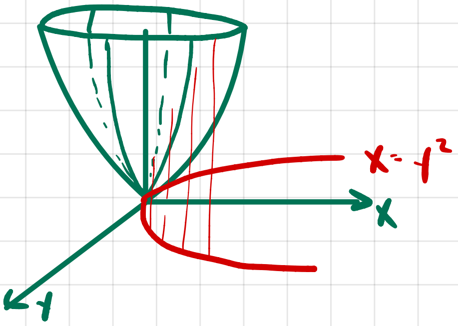
$dx = -\frac{y}{x}dy + \frac{1}{2x}dz \Rightarrow dw = 2x(-\frac{y}{x}dy + \frac{1}{2x}dz) + 2y\,dy + 2z\,dz = dz + 2z\,dz$

$$= \underbrace{0}_{(\partial w/\partial y)_z}dy + \underbrace{(1+2z)}_{(\partial w/\partial z)_y}dz$$

chain Rule w/ x and y indep., z dep., calculate $(\partial w/\partial y)_x$

$$\frac{\partial w}{\partial y} = 2x\cancelto{0}{\frac{\partial x}{\partial y}} + 2y\frac{\partial y}{\partial y} + 2z\frac{\partial z}{\partial y} = 2y + 2z \cdot 2y = 2y(1+2z) = 2y(1+2(x^2+y^2))$$

* $w(x,y) = x^2 + y^2 \quad x = y^2$



$w(y) = y^4 + y^2$

$\frac{\partial w}{\partial y} = 4y^3 + 2y$ → derivative of w when x is dependent variable. x is not held constant.

$dw = 2x\,dx + 2y\,dy \quad dx = 2y\,dy \Rightarrow dw = 4xy\,dy + 2y\,dy = (4xy + 2y)dy = (4y^3 + 2y)dy$

1J-2 $\quad dw = 2x\,dx + 2y\,dy + 2z\,dz \quad dz = 2x\,dx + 2y\,dy$

ai $\quad \frac{\partial w}{\partial y} = 2x\frac{\partial x}{\partial y} + 2y\frac{\partial y}{\partial y} + 2z\cancelto{0}{\frac{\partial z}{\partial y}}$

x dependent, z indep. kept fixed ⇒ $\partial z/\partial y = 0$

$x = h(y,z) \Rightarrow \cancelto{0}{\frac{\partial z}{\partial y}} = 2x\frac{\partial x}{\partial y} + 2y\frac{\partial y}{\partial y} \Rightarrow \frac{\partial x}{\partial y} = -\frac{y}{x}$

$\Rightarrow \left(\frac{\partial w}{\partial y}\right)_z = 2x(-y/x) + 2y = -2y + 2y = 0$

aii $\quad$ x dep., y kept constant ⇒ $1 = 2x\,\partial x/\partial z + 2y\,\cancelto{0}{\partial y/\partial z} \Rightarrow \partial x/\partial z = \frac{1}{2x}$

$\left(\frac{\partial w}{\partial z}\right)_y = 2x\frac{\partial x}{\partial z} + 2y\cancelto{0}{\frac{\partial y}{\partial z}} + 2z = 2x\cdot\frac{1}{2x} + 2z = 1 + 2z$

bi $\quad dw = 2x\,dx + 2y\,dy + 2z\,dz \qquad dz = 2x\,dx + 2y\,dy$

x dep. ⇒ sub in dx ⇒ $dx = -y/x \cdot dy + \frac{1}{2x}dz$

z indep. kept fixed ⇒ $dz = 0$

$\Rightarrow dw = -2y\,dy + dz + 2y\,dy + 2z\,dz = (1+2z)dz = 0$

$\Rightarrow \left(\frac{\partial w}{\partial y}\right)_z = 0$

bii $\quad (\partial w/\partial z)_y = (1+2z)$

2J-3 $w(x,y,z,t) = x^3y - z^2t \qquad xy = zt$

a) $(\partial w/\partial t)_{x,z}$ $\Rightarrow$ t, x, z indep. y dep.

$$\frac{\partial w}{\partial t} = (3x^2y)\frac{\partial x}{\partial t} + x^3\frac{\partial y}{\partial t} - 2zt\frac{\partial z}{\partial t} - z^2$$

x, z, t indep $\Rightarrow$ $\partial x/\partial t = \partial z/\partial t = 0$ $\Rightarrow$ $\left(\frac{\partial w}{\partial t}\right)_{x,z} = x^3\frac{\partial y}{\partial t} - z^2$

y dep $\Rightarrow$ $y = f(x,z,t) = \frac{zt}{x}$

$\Rightarrow \frac{\partial y}{\partial t} = \frac{z}{x} \Rightarrow (\partial w/\partial t)_{x,z} = x^3\cdot\frac{z}{x} - z^2 = x^2z - z^2 \qquad (\partial w/\partial t)_{x,z} = x^2z - z^2$

b) $(\partial w/\partial z)_{x,y}$

$$\frac{\partial w}{\partial z} = 3x^2y\frac{\partial x}{\partial z} + x^3\frac{\partial y}{\partial z} - 2zt\frac{\partial z}{\partial z} - z^2\frac{\partial t}{\partial z}$$

x, y, z indep $\Rightarrow$ $\partial x/\partial z = \partial y/\partial z = 0$ $\Rightarrow$ $(\partial w/\partial z)_{x,y} = -2zt - z^2\frac{\partial t}{\partial z}$

t dep. $\Rightarrow$ $t = \frac{xy}{z}$ $\Rightarrow$ $\partial t/\partial z = -\frac{xy}{z^2}$ $\Rightarrow (\partial w/\partial z)_{x,y} = -2zt - z^2\left(-\frac{xy}{z^2}\right)$

$= -2\cancel{z}\frac{xy}{\cancel{z}} + xy = -xy \Rightarrow (\partial w/\partial z)_{x,y} = -xy$

2J-4

a) $dw = (3x^2y)dx + x^3dy - 2zt\,dz - z^2dt$ $\qquad (\partial w/\partial t)_{x,y,z}$

x, z kept constant $\Rightarrow$ $dx, dz = 0$ $\Rightarrow (dw)_{x,z} = x^3dy - z^2dt$

dy depends on dt because $y = f(x,z,t) \Rightarrow dy = \frac{z}{x}dt$

$\Rightarrow (dw)_{x,z} = x^3\frac{z}{x}dt - z^2dt = (x^2z - z^2)dt$

$\Rightarrow (\partial w/\partial t)_{x,z} = x^2z - z^2$

b) $dw = (3x^2y)dx + x^3dy - 2zt\,dz - z^2dt$

x, y constant $\Rightarrow dx, dy = 0 \Rightarrow (dw)_{x,y} = -2zt\,dz - z^2dt$

$t = f(x,y,z) = \frac{xy}{z} \Rightarrow dt = \frac{-xy}{z^2}dz$

$\Rightarrow (dw)_{x,y} = -2zt\,dz - \cancel{z^2}\cdot\frac{(-xy)}{\cancel{z^2}} = dz(-2zt + xy)$

$\Rightarrow (\partial w/\partial z)_{x,y} = -xy$

2J-5 $S = S(p,V,T)$ entropy of gas s.t. $pV = nRT$

a) $(\partial S/\partial p)_V$

$$\frac{\partial S}{\partial p} = S_p + S_V \frac{\partial V}{\partial p} + S_T \frac{\partial T}{\partial p}$$

p, V indep. $\Rightarrow \partial V/\partial p = 0 \Rightarrow (\partial S/\partial p)_V = S_p + S_T \frac{\partial T}{\partial p}$

T dep $\Rightarrow T = \frac{pV}{nR} \Rightarrow \frac{\partial T}{\partial p} = \frac{V}{nR}$

$\Rightarrow (\partial S/\partial p)_V = S_p + S_T \cdot \frac{V}{nR}$

b) $(\partial S/\partial T)_V$

$$\partial S/\partial T = S_p \frac{\partial p}{\partial T} + S_V \frac{\partial V}{\partial T} + S_T$$

V, T indep $\Rightarrow \partial V/\partial T = 0$

p dep $\Rightarrow p = p(V,T) = \frac{nRT}{V} \Rightarrow \partial p/\partial T = \frac{nR}{V}$

$\Rightarrow (\partial S/\partial T)_V = S_p \cdot \frac{nR}{V} + S_T$

2J-6 $w(u,v) = u^3 - uv^2$ $\quad u = xy \quad v = u + x$

a) $(\partial w/\partial u)_x$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial u} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial u} = [3u^2 - v^2] - 2uv$$

x, u indep $\Rightarrow y, v$ dep. $\Rightarrow y(u,x) = u/x$, $v(u,x) = u + x$

$(\partial w/\partial u)_x = 3u^2 - v^2 - 2uv$

$\Rightarrow (\partial w/\partial u)_x = 3u^2 - v^2 - 2uv$

$(\partial w/\partial x)_u$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \cancelto{0}{\frac{\partial u}{\partial x}} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x} = -2uv \cdot 1$$

x, u indep $\Rightarrow y, v$ dep. $\Rightarrow y(u,x) = u/x$, $v(u,x) = u + x$

$\Rightarrow (\partial w/\partial x)_u = -2uv$

$w(u,v) = u^3 - uv^2 \quad u = xy \quad v = u + x$

b) $(\partial w/\partial u)_x$

$dw = [3u^2 - v^2]du - 2uv\,dv$

x, u indep $\Rightarrow y, v$ dep. $\Rightarrow y(u,x) = u/x \;,\; v(u,x) = u + x$

$dv = du + dx$

$\ast\; dw = [3u^2 - v^2]du - 2uv(du + dx)$

$dw = \underbrace{[3u^2 - v^2 - 2uv]}_{(\partial w/\partial u)_x}du \; \underbrace{-2uv}_{(\partial w/\partial x)_u}dx$

x constant $\Rightarrow dv = du$

$\Rightarrow dw = [3u^2 - v^2 - 2uv]du$

$\Rightarrow (\partial w/\partial u)_x = 3u^2 - v^2 - 2uv$

$(\partial w/\partial x)_u$

$dw = w_u \cdot \cancel{u_x}^{\,0} dx + w_v v_x dx = -2uv \cdot 1\, dx$

$(\partial w/\partial x)_u = -2uv$

2J-7 $\quad P(1,-1,1) \quad z = x^2 + y + 1$

$f(x,y,z)$ differentiable, $\nabla f(P) = \langle 2, 1, -3 \rangle$

$g(x,z) = f(x, y(x,z), z) \qquad \nabla g(1,1)$?

$\rightarrow y(x,z) = z - x^2 - 1 \;,\; y(1,1) = 1 - 1 - 1 = -1 \;,\; g(1,1) = f(1,-1,1) = f(P)$

we want $\nabla g = \langle (\partial f/\partial x)_z, (\partial f/\partial z)_x \rangle$ at $(1,1)$, so we need $(\partial f/\partial x)_z$ and $(\partial f/\partial z)_x$ at $(1,-1,1)$

$df = f_x dx + f_y dy + f_z dz \qquad dy = -2x\,dx + dz$

$\Rightarrow df = f_x dx + f_y(-2x\,dx + dz) + f_z dz = \underbrace{(f_x - 2x f_y)}_{(\partial f/\partial x)_z} dx + \underbrace{(f_y + f_z)}_{(\partial f/\partial z)_x} dz$

At $P(1,-1,1)$ we have $f_x = 2,\; f_y = 1,\; f_z = -3$

$\Rightarrow (\partial f/\partial x)_z = 2 - 2 \cdot 1 \cdot 1 = 0$

$(\partial f/\partial z)_x = 1 - 3 = -2$

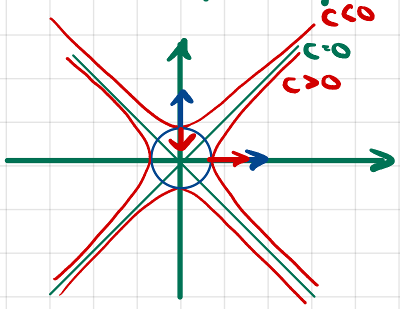
$\Rightarrow \nabla g(1,1) = \langle 0, -2 \rangle$

Problem 1 $f(x,y) = x^2 - y^2$ $\quad g(x,y) = x^2 + y^2$

a) $\nabla f = \langle 2x, -2y \rangle$ $\quad \nabla g = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g \Rightarrow 2x = \lambda 2x \Rightarrow \lambda = 1 \Rightarrow y = 0, x \in \mathbb{R}$

$\qquad\qquad -2y = \lambda 2y \Rightarrow \lambda = -1 \Rightarrow x = 0, y \in \mathbb{R}$



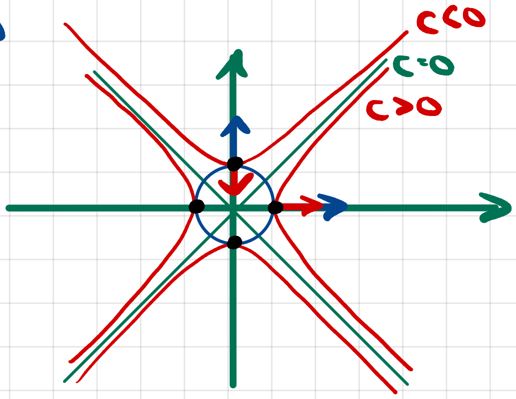
$f = c \Rightarrow y^2 = x^2 - c \Rightarrow y = \pm\sqrt{x^2 - c}$ $\quad x^2 \geq c \Rightarrow x \geq c, x \leq -c$

b) $g(x,y) = x^2 + y^2 = 3$

$y = 0 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

$x = 0 \Rightarrow y^2 = 3 \Rightarrow y = \pm\sqrt{3}$

c)



A constraint is a circle centred at the origin. It's gradient points outward from the circle because g is a paraboloid that grows outward relative to the origin.

For f, $c > 0$ means pairs of symmetric level curves that cross the x-axis at $\pm\sqrt{c}$. ∇f at these points points in the opposite direction of the origin, which coincides with the gradient of the constraint circle. These are the solutions with $\lambda = 1$

For $c < 0$, there is a pair of symmetric level curves that cross the y axis at $\pm\sqrt{c}$. This time, f's level curves have value that increases as c gets less negative so at $y = \pm\sqrt{c}$ ∇f points towards the origin, the opposite direction of ∇g at these points. $\lambda = -1$ for these solutions.

Problem 2

current I flows over resistor R $\Rightarrow$ energy loss: I^2R per second

a) Total energy loss: $I_1^2 R_1 + I_2^2 R_2$

$T(I_1, I_2) = I_1^2 R_1 + I_2^2 R_2$

$\min T(I_1, I_2)$ s.t. $I_1 + I_2 = I$

$T(I_1) = I_1^2 R_1 + (I - I_1)^2 R_2$

$T'(I_1) = 2I_1R_1 - 2(I - I_1)R_2 = 2I_1R_1 - 2IR_2 + 2I_1R_2 = 0$

$2I_1(R_1 + R_2) = 2IR_2$

$\Rightarrow I_1^* = \dfrac{IR_2}{R_1 + R_2} \Rightarrow I_2^* = \dfrac{IR_1 + IR_2 - IR_2}{R_1 + R_2} = \dfrac{IR_1}{R_1 + R_2}$

$T''(I_1) = 2R_1 + 2R_2 > 0 \Rightarrow$ min at I_1^*

b) $\min T(I_1, I_2, I_3) = I_1^2 R_1 + I_2^2 R_2 + I_3^2 R_3$ s.t. $g(I_1, I_2, I_3) = I_1 + I_2 + I_3 - I = 0$

$\nabla T = \langle 2I_1R_1, 2I_2R_2, 2I_3R_3 \rangle$

$\nabla g = \langle 1, 1, 1 \rangle$

$\nabla T = \lambda \nabla g \Rightarrow$
$$2I_1R_1 = \lambda$$
$$2I_2R_2 = \lambda \quad \Rightarrow I_1R_1 = I_2R_2 = I_3R_3$$
$$2I_3R_3 = \lambda$$
$$I_1 + I_2 + I_3 = I$$

$\Rightarrow I_1 + I_1\dfrac{R_1}{R_2} + I_1\dfrac{R_1}{R_3} = I \Rightarrow \dfrac{I_1(R_2R_3 + R_1R_3 + R_1R_2)}{R_2R_3} = I$

$I_1 = \dfrac{R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2}$ $\qquad I_2 = \dfrac{R_1R_3}{R_2R_3 + R_1R_3 + R_1R_2}$

$I_3 = \dfrac{R_1R_2}{R_2R_3 + R_1R_3 + R_1R_2}$

Problem 3

$w = w(x,y) = \frac{1}{2}xy \qquad \tan\theta = \frac{y}{x} \qquad (\partial w/\partial x)_\theta \,,\, (\partial w/\partial \theta)_x$

a) w is dependent on x and y $\qquad x, \theta$ independent $\Rightarrow y$ dep. $\Rightarrow y(x,\theta) = x\tan\theta$

$w(x,\theta) = \frac{1}{2}x^2\tan\theta$

$$(\partial w/\partial x)_\theta = x\tan\theta \qquad (\partial w/\partial\theta)_x = \frac{x^2\sec^2\theta}{2}$$

b) $$\left(\frac{\partial w}{\partial x}\right)_\theta = \frac{\partial w}{\partial x}\cdot\left(\frac{\partial x}{\partial x}\right)_\theta + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial x}\right)_\theta = \frac{1}{2}\underbrace{y}_{x\tan\theta} + \frac{1}{2}x\tan\theta = x\tan\theta$$

$$(\partial w/\partial\theta)_x = \frac{\partial w}{\partial x}\cancelto{0}{\frac{\partial x}{\partial\theta}} + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial\theta}\right)_x = \frac{1}{2}x\cdot x\sec^2\theta = \frac{x^2\sec\theta}{2}$$

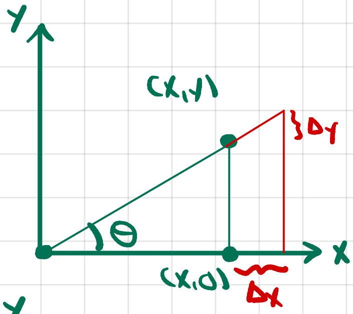
c) $dw = \frac{y}{2}dx + \frac{x}{2}dy \qquad dy = \tan\theta\, dx + x\sec^2\theta\, d\theta$

$$= \frac{y}{2}dx + \frac{x\tan\theta}{2}dx + \frac{x^2\sec^2\theta}{2}d\theta = \left[\frac{y + x\tan\theta}{2}\right]dx + \frac{x^2\sec^2\theta}{2}d\theta$$
$$= x\tan\theta\, dx + \frac{x^2\sec^2\theta}{2}d\theta$$

$\Rightarrow (\partial w/\partial x)_\theta = x\tan\theta$

$(\partial w/\partial\theta)_x = \frac{x^2\sec^2\theta}{2}$

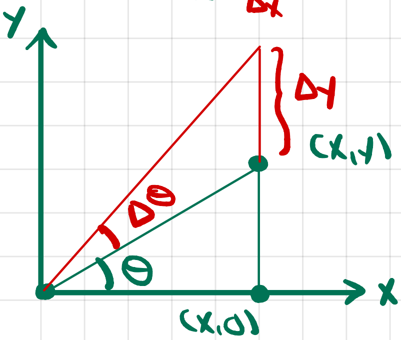
d)



$\Delta w = \Delta x\cdot y + \frac{\Delta x\cdot\Delta y}{2}$

$$\left(\frac{\Delta w}{\Delta x}\right)_\theta = y + \frac{\Delta y}{2} = x\tan\theta + \frac{\Delta x\cdot\tan\theta}{2}$$

$$\lim_{\Delta x\to 0}\left(\frac{\Delta w}{\Delta x}\right)_\theta = x\tan\theta$$



$\Delta w = \frac{1}{2}\Delta y\cdot x$

$\tan(\theta + \Delta\theta) = \frac{y + \Delta y}{x} \Rightarrow \Delta y = x\tan(\theta+\Delta\theta) - y$

$\Delta w = \frac{1}{2}\left[x^2\tan(\theta+\Delta\theta) - x^2\tan\theta\right]$

$$\Rightarrow \left(\frac{\Delta w}{\Delta\theta}\right)_x = \frac{x^2}{2}\left[\frac{\tan(\theta+\Delta\theta) - \tan\theta}{\Delta\theta}\right]$$

$$\lim_{\Delta\theta\to 0}(\Delta w/\Delta\theta)_x = \frac{x^2}{2}\cdot\frac{d}{d\theta}\tan\theta = \frac{x^2\sec^2\theta}{2}$$