PSet 12

6D-1

a) $\vec{F} = \langle y, z, -x \rangle \qquad C: \langle t, t^2, t^3 \rangle \quad t \in [0,1]$

$dx = dt \quad dy = 2t\,dt \quad dz = 3t^2\,dt$

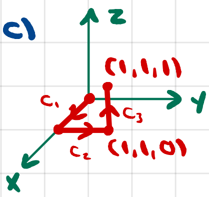
$$\int_C \vec{F}\cdot d\vec{r} = \int_C y\,dx + z\,dy - x\,dz = \int_0^1 (t^2 dt + t^3 \cdot 2t\,dt - t\cdot 3t^2 dt) = \int_0^1 (t^2 + 2t^4 - 3t^3)\,dt$$

$$= \frac{t^3}{3} + \frac{2t^5}{5} - \frac{3t^4}{4}\Big|_0^1 = \frac{1}{3} + \frac{2}{5} - \frac{3}{4} = \frac{20+24-45}{60} = -\frac{1}{60}$$

b) $C: \langle t, t, t \rangle \quad t \in [0,1] \qquad dx = dy = dz = dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C t\,dt + t\,dt - t\,dt = \int_0^1 t\,dt = \frac{t^2}{2} = \frac{1}{2}$$

c)



$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} 0\,dx + \int_{C_2} 0\,dy + \int_{C_3} -x\,dz = \int_0^1 -1\,dt = -1$$

$C_3: \langle 1, 1, t \rangle \quad t \in [0,1]$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & -x \end{vmatrix} = \langle 0-1, -(-1-0), 0-1 \rangle = \langle -1, 1, -1 \rangle \neq \vec{0} \Rightarrow \vec{F} \text{ not conservative}$$

d) $\vec{F} = \langle zx, zy, x \rangle \qquad C: \langle \cos t, \sin t, t \rangle, \ (1,0,0) \text{ to } (1,0,2\pi) \Rightarrow 0 \le t \le 2\pi$

$dx = -\sin t\,dt \quad dy = \cos t\,dt \quad dz = dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C zx\,dx + zy\,dy + x\,dz = \int_0^{2\pi} -t\cos t \sin t\,dt + t\sin t\cos t\,dt + \cos t\,dt$$

$$= \int_0^{2\pi} \cos t\,dt = \sin t\Big|_0^{2\pi} = 0$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ zx & zy & x \end{vmatrix} = \langle 0-y, -(1-x), 0-0 \rangle = \langle -y, x-1, 0 \rangle$$

6D-2 $\vec{F} = \langle x, y, z\rangle$ C any curve on sphere radius a centered origin

$x^2+y^2+z^2 = a^2$ on any point on C

$\vec{r}(t) = \langle x(t), y(t), [a^2 - x^2(t) - y^2(t)]^{1/2}\rangle$

$\vec{v}(t) = \langle x', y', \frac{1}{2[a^2-x^2-y^2]^{1/2}}(-2xx'-2yy')\rangle = \langle x', y', \frac{-(xx'+yy')}{\sqrt{a^2-x^2-y^2}}\rangle$

$\hat{T} = \frac{\vec{v}(t)}{|\vec{v}(t)|} \Rightarrow \vec{F}\cdot\hat{T} = \frac{\vec{F}\cdot\vec{v}(t)}{|\vec{v}(t)|} = \frac{xx'+yy'-(xx'+yy')}{|\vec{v}(t)|} = 0$

$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\hat{T}\,ds = 0$

Geometrically, $\vec{F}$ points radially outwards, $\hat{T}$ is tangent to sphere, $\vec{F}\perp\hat{T}$ everywhere on sphere.

6D-4

a) $f(x,y,z) = x^2+y^2+z^2$

$\vec{F} = \nabla f = \langle 2x, 2y, 2z\rangle$

b) $C: \langle \cos t, \sin t, t\rangle \quad t\in[0, 2\pi n]$

work done by $\vec{F}$ moving unit point mass along C

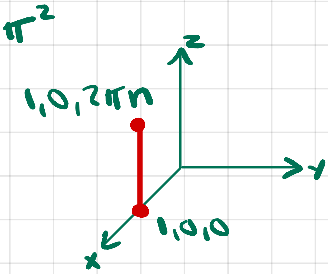
i) direct calculation

$\int_C \vec{F}\cdot d\vec{r} = \int_C 2x\,dx + 2y\,dy + 2z\,dz = \int_0^{2n\pi} \cancel{-2\cos t\sin t\,dt} + \cancel{2\sin t\cos t\,dt} + 2t\,dt$

$x = \cos t \quad dx = -\sin t\,dt \qquad = t^2\Big|_0^{2n\pi} = 4n^2\pi^2$

$y = \sin t \quad dy = \cos t\,dt$

$z = t \quad dz = dt$



ii) $\vec{F} = \nabla f \Rightarrow \int_C \vec{F}\cdot d\vec{r}$ path independent

$\int_C \vec{F}\cdot d\vec{r} = \int_C 2z\,dz = \int_0^{2\pi n} 2t\,dt = 4n^2\pi^2$

iii) $\int_C \vec{F}\cdot d\vec{r} = f(1,0,2\pi n) - f(1,0,0) = (1+4\pi^2n^2) - 1 = 4\pi^2 n^2$

6D-5 $\vec{F}=\nabla f$ $\quad f=\sin(xyz)$ $\quad \nabla f=\cos(xyz)\langle yz,xz,xy\rangle$

max value of $\int_C \vec{F}\cdot d\vec{r}$ over all possible C?

$\vec{F}$ is conservative $\Rightarrow$ Fundamental Theorem says $\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

where C is parametrized by $\vec{r}(x,y,z)$

The max that this difference can be is $1-(-1)=2$, which occurs when C has endpoints

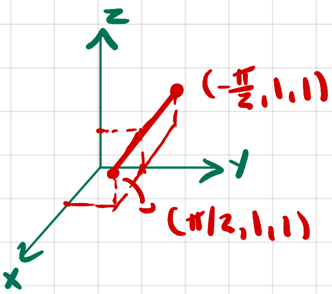
(x_0,y_0,z_0), $\sin(x_0y_0z_0)=-1$ $\qquad x_0y_0z_0=-\frac{\pi}{2}+n2\pi$

$\Rightarrow$

(x,y,z), $\sin(xyz)=1$ $\qquad xyz=\frac{\pi}{2}+n2\pi$

one example is:

C the path from $(-\frac{\pi}{2},1,1)$ to $(\frac{\pi}{2},1,1)$



6D-6 $\vec{F}=\nabla f$ $\quad f=\dfrac{1}{x+y+z+1}$ $\qquad C: \langle at,bt,ct\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = \lim_{t\to\infty}\int_0^t \nabla f\cdot d\vec{r} = \lim_{t\to\infty} f(\vec{r}(t)) - f(r(0)) = \lim_{t\to\infty}\frac{1}{at+bt+ct+1} - 1 = -1$$

6E-1

recap: given a function $f(x,y,z)$, its total differential is $f_x dx + f_y dy + f_z dz$.

given a differential $\langle h(x,y,z), g(x,y,z), t(x,y,z)\rangle\cdot\langle dx,dy,dz\rangle$ if it is the total differential of some fn f then the differential is said to be exact.

a) $x^2dx + y^2dy + z^2dz$

Take $\vec{F}=\langle x^2,y^2,z^2\rangle$. the differential is exact if $\vec{F}=\nabla f$ for some fn f.

The criterion to check this is curl $\vec{F}=0$

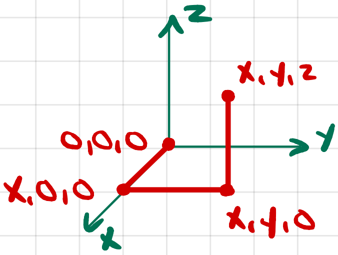
$$\text{curl }\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & y^2 & z^2 \end{vmatrix} = \langle 0-0, -(0-0), 0-0\rangle = \vec{0} \Rightarrow \text{exact}$$

To find f, there are two methods.

$$\int_C \nabla f\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) \Rightarrow f(r(\vec{b})) = f(x,y,z) = \int_C \nabla f\cdot d\vec{r} + C$$

$\Rightarrow$ we choose any path to a point (x,y,z) and the line integral gives us f plus a constant.

Let's take a = origin = $\langle 0,0,0\rangle$, $b = \langle x,y,z\rangle$



$$\int_C \vec{F}\cdot d\vec{r} = \int_0^x x^2\,dx + \int_0^y y^2\,dy + \int_0^z z^2\,dz + C$$
$$= \frac{1}{3}\left[x^3+y^3+z^3\right] + C$$

method 2: antiderivatives

$\vec{F} = \nabla f = \langle x^2, y^2, z^2\rangle$

$\Rightarrow f_x = x^2 \Rightarrow f(x,y,z) = \frac{x^3}{3} + g(y,z)$

$f_y = g_y(y,z) = y^2 \Rightarrow g(y,z) = \frac{y^3}{3} + h(z)$

$\Rightarrow f(x,y,z) = \frac{x^3}{3} + \frac{y^3}{3} + h(z)$

$\Rightarrow f_z = z^2 = h'(z) \Rightarrow h(z) = \frac{z^3}{3}$

$\Rightarrow f(x,y,z) = \frac{1}{3}\left[x^3+y^3+z^3\right] + C$

b) $\vec{F} = \langle y^2 z, 2xyz, xy^2\rangle = \langle P,Q,R\rangle$

$P_y = 2yz \qquad Q_x = 2yz$
$P_z = y^2 \qquad R_x = y^2 \qquad \Rightarrow \text{curl}\,\vec{F} = \vec{0} \Rightarrow \vec{F} = \nabla f$
$Q_z = 2xy \qquad R_y = 2xy$

Find f, line integral method

$$\int_C \nabla f\cdot d\vec{r} = \int_{C_1} \cancelto{0}{y^2 z}\,dx + \int_{C_2} \cancelto{0}{2xyz}\,dy + \int_{C_3} xy^2\,dz = xy^2\int_0^z dz = xy^2 z$$

c) $\vec{F} = \langle y(6x^2+z), x(2x^2+z), xy\rangle$

$P_y = 6x^2+z \qquad Q_x = 2\cdot 3x^2 + z = 6x^2+z$
$P_z = y \qquad R_x = y \qquad \Rightarrow \vec{F} = \nabla f$
$Q_z = x \qquad R_y = x$

$$\int_C \nabla f\cdot d\vec{r} = \int_{C_1} \cancelto{0}{y}(6x^2+\cancelto{0}{z})\,dx + \int_{C_2}(2x^3+\cancelto{0}{xz})\,dy + \int_{C_3} xy\,dz = \int_0^y 2x^3\,dy + \int_0^z xy\,dz$$
$$= 2yx^3 + xyz$$

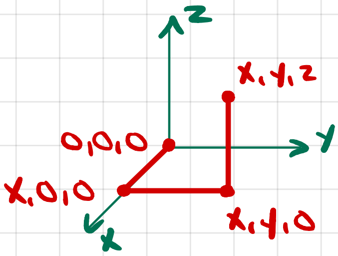
6E-2 $\vec{F} = \langle x^2y, yz, xyz^2 \rangle$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & yz & xyz^2 \end{vmatrix} = \langle xz^2 - y, -(yz^2 - 0), 0 - x^2 \rangle = \langle xz^2 - y, -yz^2, -x^2 \rangle$$

6E-3

ii) $\vec{F} = \langle 2xy + z, x^2, x \rangle$

a) $$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+z & x^2 & x \end{vmatrix} = \langle 0-0, -(1-1), 2x - 2x \rangle = \langle 0,0,0 \rangle$$



b) line integral method

$$\int_C \vec{F} \cdot d\vec{r} = \cancelto{0}{\int_{C_1} (2xy+z)\,dx} + \int_{C_2} x^2\,dy + \int_{C_3} x\,dz$$

$$= x^2y + xz$$

antiderivative method

$f_x = 2xy + z \Rightarrow f = x^2y + zx + g(y,z)$

$f_y = x^2 + g_y = x^2 \Rightarrow g(y,z) = h(z)$

$f = x^2y + zx + h(z) \Rightarrow f_z = x + h'(z) = x \Rightarrow h(z) = C$

$\Rightarrow f(x,y,z) = x^2y + zx + C$

6E-4 $\nabla f(x,y,z) = \nabla g(x,y,z)$

Given a curve C in space, $\int_C \nabla f \cdot d\vec{r} = \int_C \nabla g \cdot d\vec{r}$

If C is parametrized by $\vec{r}(t)$ for $a \le t \le t_f$ then

$f(\vec{r}(t_f)) - f(\vec{r}(a)) = g(\vec{r}(t_f)) - g(\vec{r}(a))$

$\Rightarrow f(\vec{r}(t_f)) = g(\vec{r}(t_f)) + C$

6E-5 $\vec{F} = \langle yz^2, xz^2 + ayz, bxyz + y^2 \rangle$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz^2 & xz^2 + ayz & bxyz + y^2 \end{vmatrix}$$

$$= \langle bxz + 2y - 2xz - ay, -(byz - 2yz), z^2 - z^2 \rangle$$

$$\cancel{bxz} + 2y - \cancel{2xz} - ay = 0 \Rightarrow y(2-a) = 0 \Rightarrow a = 2$$

$$byz - 2yz = 0 \Rightarrow yz(b-2) = 0 \Rightarrow b = 2$$

$$\Rightarrow \vec{F} = \langle yz^2, xz^2 + 2yz, 2xyz + y^2 \rangle$$

Find f, line integral method

$$\int_C \nabla f \cdot d\vec{r} = \int_{C_1} \cancelto{0}{yz^2}\, dx + \int_{C_2} (\cancelto{0}{xz^2} + 2\cancelto{0}{yz})\, dy + \int_{C_3} (2xyz + y^2)\, dz$$

$$= xyz^2 + y^2 z + C$$

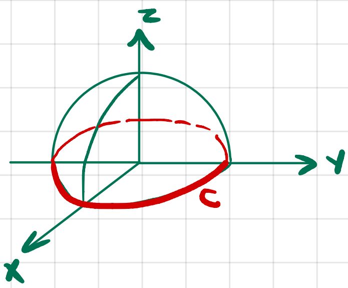
6F-1 S: upper hemisphere of sphere radius 1 centered origin

C: boundary of S

a) $\vec{F} = \langle x, y, z \rangle$

Stokes':

S oriented, bounded, piecewise smooth surface in space with positively oriented boundary C and unit normal vector field $\hat{n}$



$$\oint_C \vec{F} \cdot \hat{T}\, ds = \iint_S (\text{curl}\,\vec{F}) \cdot \hat{n}\, dS$$

$$\oint_C \vec{F} \cdot \hat{T}\, ds = \oint_C \vec{F} \cdot d\vec{r} = \oint_C x\,dx + y\,dy + z\,dz = \int -\cos t \sin t\, dt + \sin t \cos t\, dt = 0$$

C: $x^2 + y^2 = 1$, $x = \cos t$, $y = \sin t$, $dx = -\sin t\, dt$, $dy = \cos t\, dt$, $z = 0$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \langle 0-0, -(0-0), 0-0 \rangle = \vec{0}$$

$$\Rightarrow \iint_S (\text{curl}\,\vec{F}) \cdot \hat{n}\, dS = 0$$

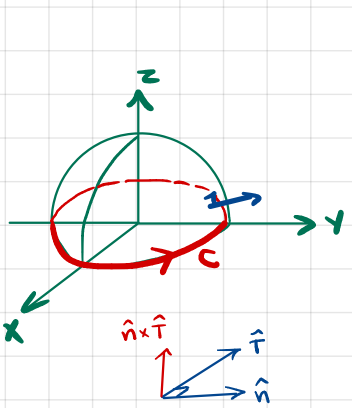
b) $\vec{F} = \langle y, z, x \rangle$

$\oint_C \vec{F}\cdot\hat{T}\,ds = \iint_S (\text{curl}\,\vec{F})\cdot\hat{n}\,dS$

$\oint_C \vec{F}\cdot\hat{T}\,ds = \oint_C y\,dx + \cancel{z}^{0}dy + x\,\cancel{dz}^{0} = \int_0^{2\pi} \sin t\,(-\sin t\,dt) = \int_0^{2\pi} -\sin^2 t\,dt$

$= -\int_0^{2\pi} (1-\cos(2t))\cdot\frac{1}{2}\,dt = -\frac{1}{2}\left[t - \frac{\sin(2t)}{2}\right]\Big|_0^{2\pi}$

$= -\frac{1}{2}\left[(2\pi - 0) - (0-0)\right] = -\frac{1}{2}\cdot 2\pi = -\pi$

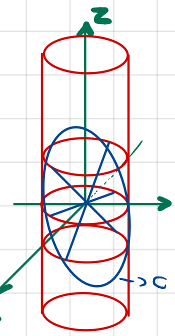


$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = \langle 0-1, -(1-0), 0-1 \rangle = \langle -1,-1,-1 \rangle$

$\iint_S \langle -1,-1,-1\rangle \frac{\langle x, y, z\rangle}{1}\,dS = \iint_S -(x+y+z)\,dS$

$= \int_0^{2\pi}\int_0^{\pi/2} -\left[\sin\phi\cos\theta + \sin\phi\sin\theta + \cos\phi\right]\cdot 1^2 \sin\phi\,d\phi\,d\theta = -\pi$

6F-2 $\vec{F} = \langle y, z, x \rangle \Rightarrow \text{curl}\,\vec{F} = \langle -1,-1,-1 \rangle$



cylinder: $x^2 + y^2 = 1$

plane: $x + y + z = 0$

$\oint_C \vec{F}\cdot\hat{T}\,ds = \iint_S (\text{curl}\,\vec{F})\cdot\hat{n}\,dS$ C an ellipse and S the elliptical disk bounded by C.

$\iint_S (\text{curl}\,\vec{F})\cdot\hat{n}\,dS$

S: $z(x,y) = -x-y$, $x^2+y^2 \leq 1$

$\hat{n}$ pointing upwards from S $\Rightarrow \langle 1,1,1\rangle/\sqrt{3}$

$dS = \sqrt{1+1+1}\,dx\,dy = \sqrt{3}\,dx\,dy$

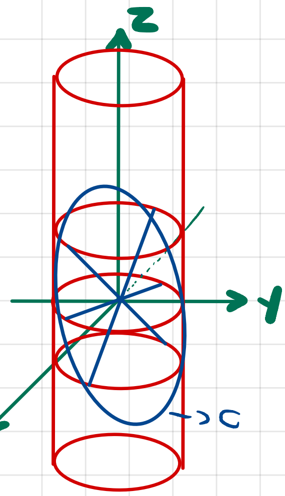
$\iint_S \langle -1,-1,-1\rangle\cdot\frac{\langle 1,1,1\rangle}{\sqrt{3}}\,dS = \iint_S -\frac{3}{\sqrt{3}}\cdot\sqrt{3}\,dx\,dy = -\int_0^{2\pi}\int_0^1 3r\,dr\,d\theta = -3\pi$

$\oint_C \vec{F}\cdot\hat{T}\,ds$ $\quad \vec{F} = \langle y, z, x\rangle \Rightarrow \text{curl}\,\vec{F} = \langle -1,-1,-1\rangle$

$C: \begin{cases} x^2+y^2=1 \\ x+y+z=0 \end{cases} \Rightarrow \begin{array}{l} x = \pm\sqrt{1-y^2} \\ z = \pm\sqrt{1-y^2} - y \end{array}$

$x = \cos\theta \quad y = \sin\theta \Rightarrow z = -(\sin\theta + \cos\theta)$

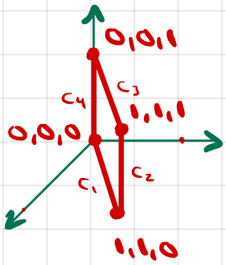
$dx = -\sin\theta\, d\theta \quad dy = \cos\theta\, d\theta \quad dz = -(\cos\theta - \sin\theta)d\theta = (\sin\theta - \cos\theta)d\theta$



$\oint_C \vec{F}\cdot d\vec{r} = \oint_C y\,dx + z\,dy + x\,dz$

$= \int_0^{2\pi} [-\sin^2\theta - \cancel{\sin\theta\cos\theta} - \cos^2\theta + \cancel{\sin\theta\cos\theta} - \cos^2\theta]\,d\theta$

$= \int_0^{2\pi} [-\sin^2\theta - 2\cos^2(\theta)]\,d\theta = -3\pi$



6F-3 $\vec{F} = \langle yz, xz, xy\rangle$

$\oint_C \vec{F}\cdot\hat{T}\,ds = \iint_S (\text{curl}\,\vec{F})\cdot\hat{n}\,dS$

$\oint_C \vec{F}\cdot\hat{T}\,ds = \int_{C_1} y\cancelto{0}{z}\,dx + x\cancelto{0}{z}\,dy + \int_{C_2} 1\cdot 1\,dz + \int_{C_3} yz\,dx + xz\,dy + xy\cancelto{0}{dz} + \int_{C_4} \cancel{0}\,dz$

$= \int_0^1 dz + \int_0^1 (1-t)(-dt) + (1-t)(-dt) = 1 - 2\int_0^1 (1-t)\,dt = 1 - 2 + 1 = 0$

$C_3: \langle 1-t, 1-t, 1\rangle \quad 0 \le t \le 1 \quad \Rightarrow \quad dx = -dt \quad dy = -dt \quad dz = 0$

$\iint_S (\text{curl}\,\vec{F})\cdot\hat{n}\,dS$

$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix} = \langle x-x, -(y-y), z-z\rangle = \vec{0}$

$\Rightarrow \vec{F}$ conservative, $\oint_C \vec{F}\cdot d\vec{r}$ path independent.

6F-5

S: surface formed by

cylinder: $x^2+y^2=a^2$, $0\le z\le h$

top circular disk

normal vector: up or out

$\vec{F}=\langle -y, x, x^2\rangle$

Flux of $\nabla\times\vec{F}$ through S

$\to \nabla\times\vec{F}=\text{curl}(\vec{F})$, a vector field.

a) Direct calculation

we want $\iint_S (\text{curl}\,\vec{F})\cdot\hat{n}\,dS = \iint_{sides}\text{curl}\,\vec{F}\cdot\hat{n}\,dS + \iint_{top}\text{curl}\,\vec{F}\cdot\hat{n}\,dS$

Flux of curl $\vec{F}$ through S

$\hat{n}_{top}=\langle 0,0,1\rangle$

$\hat{n}_{sides}=\langle x,y,0\rangle/(x^2+y^2)^{1/2}$

$$\text{curl}\,\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ -y & x & x^2\end{vmatrix}=\langle 0,-(2x-0),1-(-1)\rangle=\langle 0,-2x,2\rangle$$

$$\iint_{sides}\text{curl}\,\vec{F}\cdot\hat{n}\,dS=\iint_{sides}\langle 0,-2x,2\rangle\cdot\frac{\langle x,y,0\rangle}{r}\,dS=\iint_{sides}\frac{-2xy}{r}\,dS$$

$dS = a\,d\theta\,dz$

$x=a\cos\theta$

$y=a\sin\theta$ $\Rightarrow xy=a^2\cos\theta\sin\theta$

$$=\int_0^{2\pi}\int_0^h -\frac{2a^2\cos\theta\sin\theta}{a}\cdot a\,dz\,d\theta=0$$

$$\iint_{top}\text{curl}\,\vec{F}\cdot\hat{n}\,dS=\iint_{top}\langle 0,-2x,2\rangle\langle 0,0,1\rangle dS=\int_0^{2\pi}\int_0^a 2r\,dr\,d\theta=2\pi a^2$$

$dS = r\,dr\,d\theta$

$\Rightarrow \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS = 2\pi a^2$

b) Using Stokes' Theorem

$\iint_S (\text{curl}\,\vec{F})\cdot\hat{n}\,dS = \oint_C \vec{F}\cdot\hat{T}\,ds$ where $C: x^2+y^2=a^2, z=0$

$$\oint_C \vec{F}\cdot\hat{T}\,ds=\oint_C -y\,dx+x\,dy+x^2\,dz=\int_0^{2\pi}-a\sin t(-a\sin t)dt+a\cos t\,a\cos t\,dt+0$$

$x=a\cos t$ $dx=-a\sin t\,dt$

$y=a\sin t$ $dy=a\cos t\,dt$

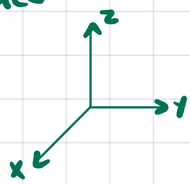
$z=0$ $dz=0$

$$=\int_0^{2\pi}(a^2\sin^2 t+a^2\cos^2 t)dt=\int_0^{2\pi}a^2\,dt=2\pi a^2$$

6G-1

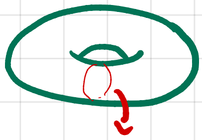
simply connected

First octant:



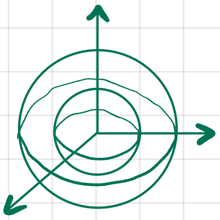
not simply connected

torus surface

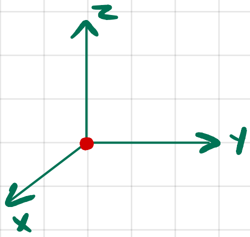


curve on surface does not bound a surface on the surface

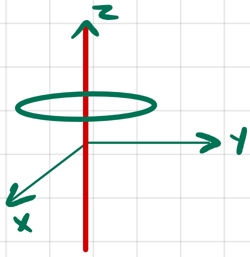
region between two concentric spheres



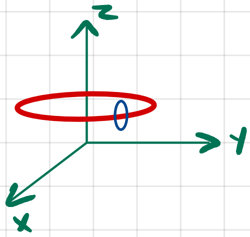
$\mathbb{R}^3$ - a point



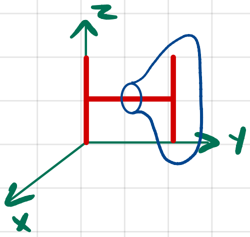
$\mathbb{R}^3$ - a line



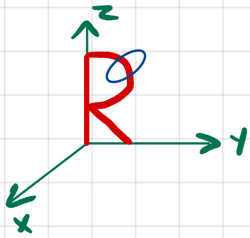
$\mathbb{R}^3$ - a circle



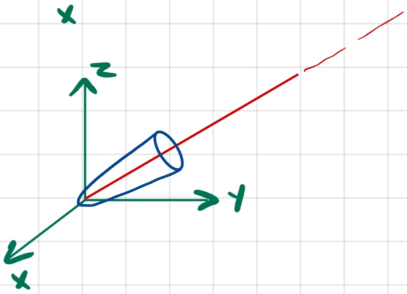
$\mathbb{R}^3$ - letter H



$\mathbb{R}^3$ - letter R



$\mathbb{R}^3$ - ray



6G-2 $\vec{F} = \rho^n \langle x, y, z \rangle$ show $F = \nabla f$ for any n

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \rho^n x & \rho^n y & \rho^n z \end{vmatrix} = \langle 0, 0, 0 \rangle$$

Find f

For $n=0$, $f_x = x$, $f_y = y$, $f_z = z \Rightarrow f = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{\rho^2}{2}$

$n=1$, $f_x = \rho x$, $f_y = \rho y$, $f_z = \rho z \Rightarrow f = \frac{\rho}{2}(x^2+y^2+z^2)$

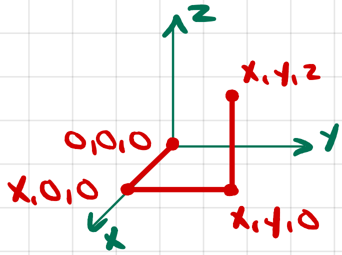
$f_x = (x^2+y^2+z^2)^{1/2} x \Rightarrow f = \frac{2}{3}(x^2+y^2+z^2)^{3/2} \cdot \frac{1}{2} = \frac{\rho^3}{3} \Rightarrow f(x,y,z) = \frac{\rho^3}{3}$

$n=2 \Rightarrow f_x = \rho^2 x = (x^2+y^2+z^2)x \Rightarrow f = (x^2+y^2+z^2)^2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{\rho^4}{4}$

$n=n \Rightarrow f_x = \rho^n x = (x^2+y^2+z^2)^{n/2} x \Rightarrow f = (x^2+y^2+z^2)^{\frac{n+2}{2}} \cdot \frac{2}{n+2} \cdot \frac{1}{2} = \frac{\rho^{n+2}}{n+2}$

if $n=-2$, $f_x = \rho^{-2} x = (x^2+y^2+z^2)^{-1} x \Rightarrow f = \ln(x^2+y^2+z^2) \cdot \frac{1}{2}$

Using line integrals, since curl $\vec{F} = 0 \Rightarrow \vec{F}$ conservative



For $n \neq -2$:

$$\int_C \vec{F}\cdot d\vec{r} = f(x,y,z) - f(0,0,0)$$

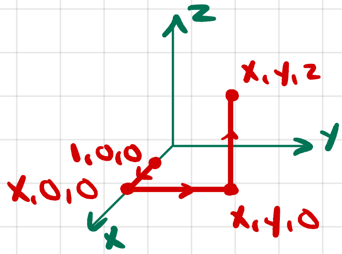
$$f(x,y,z) = \int_C \vec{F}\cdot d\vec{r} + C = \int_0^x \rho^n x\,dx + \int_0^y \rho^n y\,dy + \int_0^z \rho^n z\,dz + C \qquad \vec{F} = \rho^n \langle x,y,z\rangle$$

$$= \int_0^x x^n x\,dx + \int_0^y (x^2+y^2)^{\frac{n}{2}} y\,dy + \int_0^z (x^2+y^2+z^2)^{n/2} z\,dz$$

$$\int (x^2+y^2+z^2)^{\frac{n}{2}} x\,dx = (x^2+y^2+z^2)^{\frac{n+2}{2}}\cdot\frac{\cancel{2}}{n+2}\cdot\frac{1}{\cancel{2}} = \frac{(x^2+y^2+z^2)^{\frac{n+2}{2}}}{n+2}$$

$$\Rightarrow f(x,y,z) = \cancel{\frac{x^{n+2}}{n+2}} + \left[\cancel{\frac{(x^2+y^2)^{\frac{n+2}{2}}}{n+2}} - \cancel{\frac{x^{n+2}}{n+2}}\right] + \left[\frac{(x^2+y^2+z^2)^{\frac{n+2}{2}}}{n+2} - \cancel{\frac{(x^2+y^2)^{\frac{n+2}{2}}}{n+2}}\right]$$

$$= \frac{(x^2+y^2+z^2)^{\frac{n+2}{2}}}{n+2} = \frac{\rho^{n+2}}{n+2}$$



For $n = -2$

$$\int_C \vec{F}\cdot d\vec{r} = f(x,y,z) - f(0,0,0)$$

$$\Rightarrow f(x,y,z) = \int_C (x^2+y^2+z^2)^{-1} x\,dx + (x^2+y^2+z^2)^{-1} y\,dy + (x^2+y^2+z^2)^{-1} z\,dz$$

$$= \int_1^x \frac{x}{x^2}\,dx + \int_0^y \frac{y}{x^2+y^2}\,dy + \int_0^z \frac{z}{x^2+y^2+z^2}\,dz$$

$$= \left[\cancel{\frac{\ln(x^2)}{2}} - 0\right] + \frac{1}{2}\left(\cancel{\ln(x^2+y^2)} - \cancel{\ln(x^2)}\right) + \frac{1}{2}\left(\ln(x^2+y^2+z^2) - \cancel{\ln(x^2+y^2)}\right)$$

$$= \frac{1}{2}\left(\ln(x^2+y^2+z^2)\right)$$

6H-1 Prove $\nabla \cdot \nabla \times \vec{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$\nabla \times \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$\nabla \cdot \nabla \times \vec{F} = \cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}} = 0$$

To perform the calculation we needed $\vec{F}$ to have the mixed partials defined everywhere in the domain and continuity. For $P_{xy}(x,y,z) = P_{yx}(x,y,z)$ both P_{xy} and P_{yz} need to be continuous at (x,y,z).

6H-2 Show that

Any closed surface S
cont. diff. vector field $\vec{F}$ $\Rightarrow \iint_S \operatorname{curl} \vec{F} \, d\vec{S} = 0$

a) using divergence theorem

$\iint_S \nabla \times \vec{F} \cdot \hat{n} \, dS$ = Flux of curl $\vec{F}$ through S.

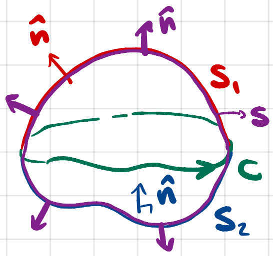
Divergence Theorem $\Rightarrow$ $= \iiint_D \operatorname{div}(\nabla \times \vec{F}) \, dV = \iiint_D 0 \, dV = 0$

$\operatorname{div}(\nabla \times \vec{F}) = \nabla \cdot \nabla \times \vec{F} = 0$ per 6H-1.

b) using Stokes Theorem

$\iint_S \operatorname{curl} \vec{F} \cdot \hat{n} \, dS =$

Let C be a closed curve on S



$$\iint_S \operatorname{curl} \vec{F} \cdot \hat{n} \, dS = \iint_{S_1} \operatorname{curl} \vec{F} \cdot \hat{n} \, dS - \iint_{S_2} \operatorname{curl} \vec{F} \cdot \hat{n} \, dS$$

$$= \oint_C \vec{F} \cdot d\vec{r} - \int_C \vec{F} \cdot d\vec{r} = 0$$

Problem 1

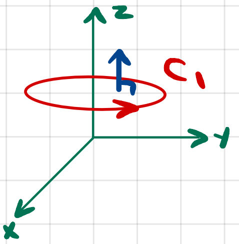
$\vec{F}(x,y,z) = \left\langle \frac{-z}{x^2+z^2},\ y,\ \frac{x}{x^2+z^2} \right\rangle \quad \forall (x,y,z) \in \mathbb{R}^3$ not on y-axis $(x^2+z^2>0)$

a) $\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{-z}{x^2+z^2} & y & \frac{x}{x^2+z^2} \end{vmatrix}$

$$= \left\langle 0-0,\ -\left[\underbrace{\frac{x^2+z^2-x\cdot 2x}{(x^2+z^2)^2}}_{\frac{z^2-x^2}{(x^2+z^2)^2}} - \underbrace{\frac{-(x^2+z^2)-(-z)\cdot 2z}{(x^2+z^2)^2}}_{\frac{-x^2-z^2+2z^2}{(x^2+z^2)^2} = \frac{z^2-x^2}{(x^2+z^2)^2}} \right],\ 0-0 \right\rangle$$

$= \langle 0,0,0 \rangle$

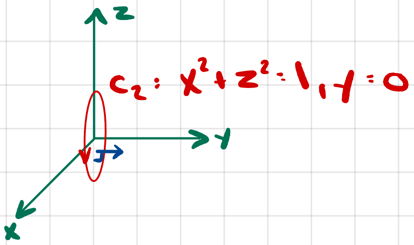
b) C_1: closed curve defined by $x^2+y^2=1$, $z=1$



$$\oint_{C_1} \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,dS = 0$$

Stokes Theorem

c)



The domain of $\vec{F}$ is not simply connected.

Every surface that has boundary C_2 crosses the y-axis and so has a point in it where $\vec{F}$ is not defined.

$\vec{F}$ is not continuously differentiable in the space region containing such a surface, and it would need to be for Stokes' to be applicable.

$\Rightarrow$ No, we cannot use Stokes'.

d) $\oint_{C_2} \vec{F}\,d\vec{r} = \oint_{C_2} \frac{-z}{x^2+z^2}\,dx + y\,dy + \frac{x}{x^2+z^2}\,dz$

$y=0 \quad dy=0$
$x=\cos t \quad dx=-\sin t\,dt$
$z=\sin t \quad dz=\cos t\,dt$

$$= \int_0^{2\pi} \sin^2 t\,dt + \cos^2 t\,dt = \int_0^{2\pi} dt = 2\pi$$

Problem 2

$$G(x,y,z) = \left\langle \frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2} \right\rangle$$ defined $\forall (x,y,z) \neq (0,0,0)$

a) $\operatorname{curl}\vec{G} = \nabla \times \vec{G} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{x}{x^2+y^2+z^2} & \frac{y}{x^2+y^2+z^2} & \frac{z}{x^2+y^2+z^2} \end{vmatrix}$

$$= \left\langle \frac{-y\cdot 2x}{\rho^4} - \frac{-x\cdot 2y}{\rho^4}, -\left[\frac{-z\cdot 2x}{\rho^4} - \frac{-x\cdot 2z}{\rho^4}\right], \frac{-y\cdot 2x}{\rho^4} - \frac{-x\cdot 2y}{\rho^4} \right\rangle = \langle 0,0,0\rangle, \quad (x,y,z) \neq (0,0,0)$$

b) $\oint_C \vec{F}\cdot d\vec{r}$ closed C not passing through origin

work by $\vec{F}$ on C $= \oint \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,dS = 0$, using Stokes'.

We can use Stokes because $\mathbb{R}^3$ - origin is simply connected. Given any closed curve in $\mathbb{R}^3$ - origin, we can find a surface w/ C as its boundary, and the assumptions to apply Stokes' will be valid. In particular, $\vec{F}$ is cont. diff in the space region containing S.

c) $\mathbb{R}^3$ - origin is simply connected. $\mathbb{R}^3$ - y-axis is not. When the latter is the domain of a vector field, $\vec{F}$ is not cont. diff. on every point of certain surfaces which have as boundary certain closed curves, namely, closed curves around the y-axis.
No such case exists for $\vec{F}$ defined on $\mathbb{R}^3$ - origin.

Problem 3

$\vec{F}(x,y,z,t) = \rho(x,y,z,t)\vec{v}(x,y,z,t) = \langle z\sin t, -z\cos t, -x\sin t + y\cos t \rangle$

non-steady flow

$\rho = 1$

$\nabla \cdot F = \operatorname{div} \vec{F}$ is defined using $\langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$ (only space variables)

$\Rightarrow$ for every t, Divergence theorem and Stokes' Theorem hold

a) show $\vec{F}$ satisfies $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{F} = 0$

Divergence at a point : source rate at that point = flux

change in density at a point

$\frac{\partial \rho}{\partial t} = 0$

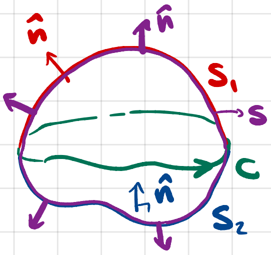
$\nabla \cdot \vec{F} = 0 + 0 + 0 = 0$

$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{F} = 0$

b) net outward flux through a simple closed surface?

$\text{Flux} = \iint_S \vec{F} \cdot \hat{n}\, dS = \iiint_D \nabla \cdot \vec{F}\, dV = \iiint_D 0\, dV = 0$

we can also see this by taking a closed curve on S, thus splitting S into two non-overlapping surfaces.
we apply Stokes' to each surface



$\text{Flux}_S = \text{Flux}_{S_1} + \text{Flux}_{S_2}$

$= \iint_{S_1} \vec{F} \cdot \hat{n}\, dS + \iint_{S_2} \vec{F} \cdot \hat{n}\, dS$

$= \oint_C \operatorname{curl} \vec{F} \cdot \hat{n}\, dS - \oint_C \operatorname{curl} \vec{F} \cdot \hat{n}\, dS = 0$

because of opposite orientation along C

Problem 4

$\vec{F}(x,y,z,t) = \rho(x,y,z,t)\vec{v}(x,y,z,t) = \langle z\sin t, -z\cos t, -x\sin t + y\cos t\rangle$

non-steady flow

$\rho = 1$

a) $\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z\sin t & -z\cos t & -x\sin t + y\cos t \end{vmatrix}$

$= \langle \cos t + \cos t, -(-\sin t - \sin t), 0 - 0\rangle = \langle 2\cos t, 2\sin t, 0\rangle$

b) $\text{curl}\,\vec{F}$ is a vector field. At any point (x,y,z), given t, $\text{curl}\,\vec{F}$ represents approximately the rate of flow of fluid mass around an axis that has direction $\text{curl}\,\vec{F}$, which happen to be the axis passing through (x,y,z) about which the fluid mass rotates most rapidly.

$|\text{curl}\,\vec{F}| = (4\cos^2 t + 4\sin^2 t)^{1/2} = 2$

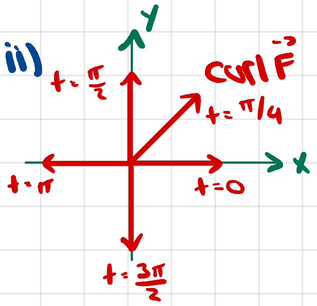
$= 2\omega_{max} \Rightarrow \omega_{max} = 1$ rad/unit time

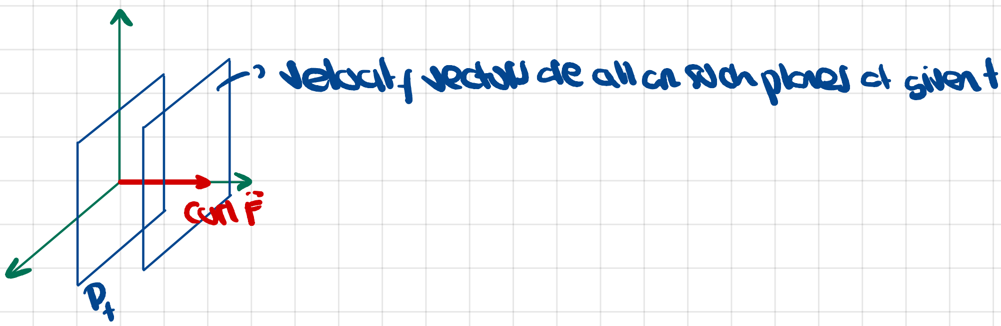
c) i) $n_t = \langle \cos t, \sin t, 0\rangle = \text{dir}(\text{curl}\,\vec{F})$

$\vec{F} = \vec{v}(x,y,z,t) \Rightarrow \vec{v} = \langle z\sin t, -z\cos t, -x\sin t + y\cos t\rangle$

$\vec{v}\cdot n_t = z\sin t\cos t - z\cos t\sin t = 0 \Rightarrow \vec{v} \perp n_t$, $\vec{v}$ at any (x,y,z) are on the same plane P_t.

ii)





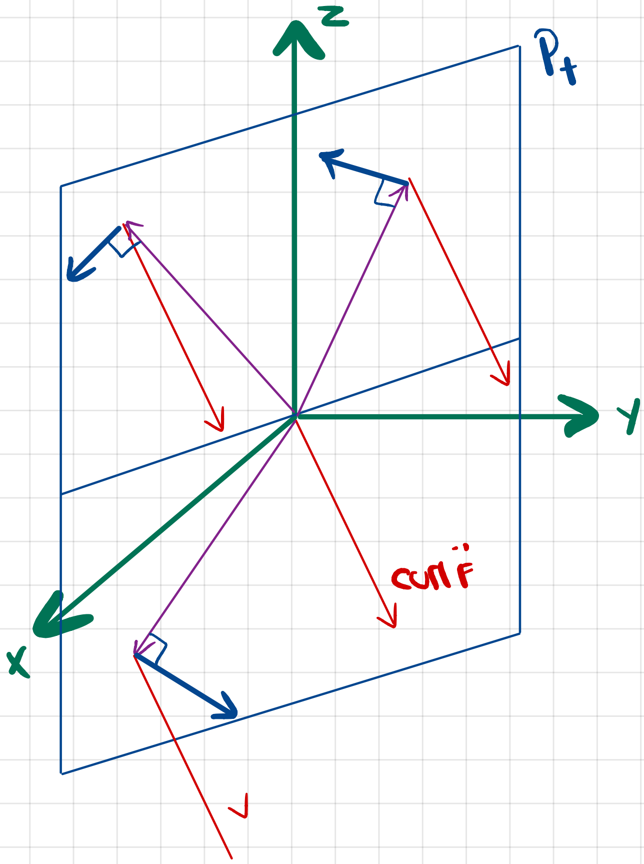
$\vec{r}(x,y,z)$ = radial vector = $\langle x,y,z\rangle$

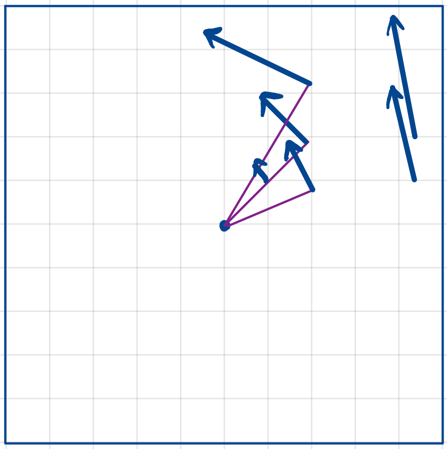
$\vec{r}\cdot\vec{v} = xz\cancel{\sin t} - yz\cancel{\cos t} - xz\cancel{\sin t} + yz\cancel{\cos t} = 0$

plane of greatest spin?

At time t, $\vec{v}$ are all on one plane P_t. Angular velocity at each point is constant, and P_t is the plane where the fluid has largest angular velocity.

d)





$\vec F = \vec V(x,y,z,t) \Rightarrow \vec V = \langle z\sin t, -z\cos t, -x\sin t + y\cos t\rangle$

$t = \frac{\pi}{4} \Rightarrow \vec V(x,y,z,t) = \vec F = \langle \frac{\sqrt{2}}{2} z, \frac{\sqrt{2}}{2}(-z), -x\frac{\sqrt{2}}{2} + y\frac{\sqrt{2}}{2} \rangle$

$= C\langle z, -z, y-x\rangle \quad C = \frac{\sqrt{2}}{2}$

For this particular choice of t, $y = -x \Rightarrow \vec V = \langle z, -z, 2y\rangle$

e) At time t, rotational flow on a plane P_t. P_t rotates with about the z-axis.