Example 1

$f(x,y) = \frac{1}{180}[7400 - 4x - 9y - 0.03xy]$ °celsius

$P(200,200)$

$\vec{v} = \langle 3,4 \rangle \quad |\vec{v}| = 5 \quad \hat{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

$D_{\hat{u}} f(x,y) = \nabla f(x,y) \cdot \hat{u} = (\dots)$

$D_{\hat{u}} f(200,200) = -0.1$ °C/km

Example 3

Now we add speed to Ex1. $x = x(t)$ $y = y(t)$, t time $v = \frac{ds}{dt} = 5$ km/min

$\frac{dw}{dt} = \frac{dw}{ds} \cdot \frac{ds}{dt}$ → rate of change of distance w/ respect to time

deriv. w/ respect to distance in direction $\hat{u}$

Example 4

$w = f(x,y,z) = \frac{1}{180}[7400 - 4x - 9y - 0.03xy] - 2z$ °C

$P(200,200,5) \quad \vec{A} = \langle 3,4,-12 \rangle \quad |\vec{A}| = \sqrt{9+16+144} = \sqrt{169} = 13$

$\hat{u} = \frac{\langle 3,4,-12 \rangle}{13}$

x,y,z are a function of t. We don't know what $x(t), y(t), z(t)$ are i.e. the trajectory represented by $\vec{r}(t) = \langle x,y,z \rangle$ → km but we do know $\vec{r}\,'(t) = \vec{v}(t)$ at a particular instant t^*. $|\vec{v}(t^*)| = 3$ km/min → km/min in the direction of $\hat{u}$ ⇒ $\vec{v} = 3\hat{u}$.

$\frac{df}{dt} = f_x x_t + f_y y_t + f_z z_t = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = \nabla f(\vec{r}(t)) \cdot \vec{v}(t) = \nabla f(\vec{r}(t)) |\vec{v}(t)| \hat{u}$
$= |\vec{v}(t)| \cdot D_{\hat{u}} f(\vec{r}(t))$

↳ °C/min

$\nabla f = \langle \frac{1}{180}(-4 - 0.03y), \frac{1}{180}(-9 - 0.03x), -2 \rangle$

$\nabla f(200,200,5) = \langle -1/18, -15/180, -2 \rangle = \langle -1/18, -1/12, -2 \rangle$

$\frac{df}{dt}(\vec{r}(t^*)) = 3 \cdot \langle -1/18, -1/12, -2 \rangle \cdot \frac{\langle 3,4,-12 \rangle}{13} = -\frac{3 \cdot 3}{18 \cdot 13} - \frac{3 \cdot 3}{12 \cdot 13} + \frac{2 \cdot 3 \cdot 12}{13}$

$= 5.44$ °C/min

13 $f(x,y) = \sin x \cos y \quad P(\pi/3, -2\pi/3) \quad v = \langle 4,-3\rangle$

$\nabla f(x,y) = \langle \cos x \cos y, -\sin x \sin y\rangle$

$|\vec{v}| = \sqrt{16+9} = 5$

$\hat{u} = \langle 4/5, -3/5\rangle$

$D_u f = \frac{4}{5}\cos x \cos y + \frac{3}{5}\sin x \sin y$

$D_u f(\pi/3 - 2\pi/3) = \frac{4}{5}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right) + \frac{3}{5}\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = -\frac{4}{20} + \frac{3\cdot 3}{20} = \frac{1}{4}$

tangent plane at P

$f(\pi/3, -2\pi/3) = z = \frac{\sqrt{3}}{2}\cdot\left(-\frac{1}{2}\right) = -\frac{\sqrt{3}}{4}$

$z + \frac{\sqrt{3}}{4} = \left(\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\right)(x - \pi/3) + \left(-\frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{2}\right)\right)\cdot(y + 2\pi/3)$

If $y = -2\pi/3$ we slice f with that plane and get a curve function of x, a line passing through P with slope $f_x(\vec{P})$

25 $f(x,y,z) = 3x^2 + y^2 + 4z^2 \quad \vec{P}(1,5,-2)$

$\nabla f = \langle 6x, 2y, 8z\rangle$

$\nabla f(\vec{P}) = \langle 6, 10, -16\rangle$

$\nabla f(\vec{P})\hat{u} = |\nabla f(\vec{P})|\cdot\cos\theta$

$\cos\theta = 1 \Rightarrow \max \nabla f(\vec{P})\hat{u} \Rightarrow \hat{u}$ same direction as $\nabla f(\vec{P})$

$\rightarrow \nabla f(\vec{P})$ is vector pointing in direction of max derivative of f

$|\nabla f(\vec{P})|$ is max value of rate of chg of f at P

$= \sqrt{36+100+256} = \sqrt{392}$

31 $x^4 + xy + y^2 = 19 \quad P(2,-3)$

$\nabla f = \langle 4x^3 + y, x + 2y\rangle$

$\nabla f(\vec{P}) = \langle 32 - 3, 2 - 6\rangle = \langle 29, -4\rangle$

tangent line at $\vec{P}$ is $\langle x-2, y+3\rangle\langle 29,-4\rangle = 0$

$29(x-2) - 4(y+3) = 0$

35 $u = f(x,y) \quad v = g(x,y)$

$\nabla(au+bv) = \langle au_x + bv_x, au_y + bv_y \rangle = a\langle u_x, u_y\rangle + b\langle v_x, v_y\rangle = a\nabla u + b\nabla v$

36 $\nabla(uv) = \langle u_x v + u v_x, u_y v + u v_y \rangle = v\langle u_x, u_y\rangle + u\langle v_x, v_y\rangle = v\nabla u + u\nabla v$

38 $\nabla u^n \quad n \in \mathbb{Z}^+$

$\nabla u^n = \langle n u^{n-1} u_x, n u^{n-1} u_y \rangle = n u^{n-1} \nabla u$

39 $\nabla f = \langle f_x, f_y \rangle$

$\nabla f \cdot \hat{u} = |\nabla f| \cos\theta$

min when $\cos\theta = -1 \Rightarrow \theta = \pi$

41 $Ax^2+Bxy+Cy^2=D$ $P(x_0,y_0)$ tangent line at P?

2D curve consider $F(x,y)=Ax^2+Bxy+Cy^2-D$ $\Rightarrow F(x_0,y_0)=0$

$F_y=Bx+2Cy$ $F_y(x_0,y_0)=Bx_0+2Cy_0$

If $Bx_0+2Cy_0\neq 0$, we know that:

$F_x=2Ax+By$ $F_y=Bx+2Cy$, polynomials $\Rightarrow$ continuous everywhere

$\Rightarrow$ F contin. differentiable

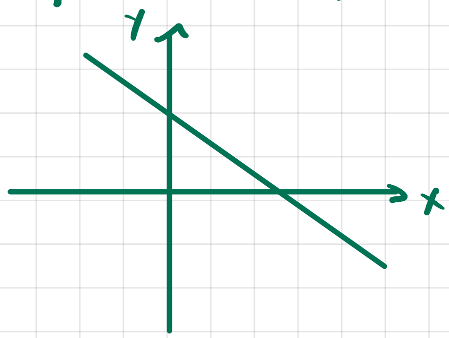
$\Rightarrow$ (implicit fn theorem) $\exists\ y=g(x)$ such that $g(x_0)=y_0$ and $F(x_0,g(x_0))=0$ for x near x_0

$\Rightarrow$ we can obtain $\frac{dy}{dx}$ by implicit diff.

$$2Ax+By+Bxy'+2Cyy'=0$$
$$y'(Bx+2Cy)=-2Ax-By \Rightarrow y'=\frac{-(2Ax+By)}{Bx+2Cy}$$

Note $\partial F/\partial x=2Ax+By$, $\partial F/\partial y=Bx+2Cy$ $\Rightarrow \frac{dy}{dx}=\frac{-\partial F/\partial x}{\partial F/\partial y}=\frac{-(2Ax+By)}{Bx+2Cy}$

$$(y-y_0)=\frac{-2Ax_0-By_0}{Bx_0+2Cy_0}(x-x_0)$$



point (x_0,y_0,D)

slope $\frac{-2Ax_0-By_0}{Bx_0+2Cy_0}$

$$\vec{v}=\left\langle 1,\frac{-2Ax_0-By_0}{Bx_0+2Cy_0},0\right\rangle$$

$$\langle x_0,y_0,D\rangle+t\left\langle 1,\frac{-2Ax_0-By_0}{Bx_0+2Cy_0},0\right\rangle=\left\langle x_0+t,\ y_0-t\left[\frac{2Ax_0\ By_0}{Bx_0+2Cy_0}\right],D\right\rangle$$

$f(x,y) = 3x^2 + 5xy + 2y^2$ $\quad\quad$ $f(1,2) = 3 + 10 + 8 = 21$

$f(x,y) = 21$

$\frac{\partial f}{\partial x} = 6x + 5y$ $\quad\quad$ $F_x(1,2) = 16$

$\frac{\partial f}{\partial y} = 5x + 4y$ $\quad\quad$ $F_y(1,2) = 5 + 8 = 13$

$\frac{dy}{dx} = \frac{-16}{13}$ $\quad\quad$ $\langle x,y,z\rangle = \langle 1,2,21\rangle + t\langle 1, -16/13, 0\rangle$

$= \langle 1+t, 2 - \frac{16t}{13}, 21\rangle$

tangent line at $(1,2,21)$ directly above the tangent line on the contour line $f(x,y) = 21$

45 $w = f(x,y,z) = 10 + xy + xz + yz$ $\quad\quad$ $Q(3,4,4)$

x, y, z in km
w in °C

hawk flies through $P(1,2,3)$ at 2 km/min towards $Q(3,4,4)$

its position $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ means

x, y, z are functions of t, time

we want the derivative of temperature w/ respect to time.

$\vec{v} = \vec{PQ} = \langle 2,2,1\rangle$ $\quad$ $|\vec{v}| = \sqrt{4+4+1} = 3$

$\hat{v} = \vec{v}/3$ is the direction the hawk flies in

The rate of change of w rel. to distance is $D_{\hat{v}} f(x,y,z) = \nabla f \cdot \hat{v}$

" " " " " " " " time is $\nabla f(x,y,z) \cdot \vec{v} = \nabla f(x,y,z) \cdot |\vec{v}| \cdot \hat{v}$

$\nabla f = \langle y+z, x+z, x+y\rangle$

At $(1,2,3)$ $\nabla f = \langle 5,4,3\rangle$

$|\vec{v}| = 2$ km/min

$\Rightarrow D_t f(1,2,3) = \langle 5,4,3\rangle \cdot \langle 2,2,1\rangle \cdot \frac{1}{3} \cdot 2$

$= (10 + 8 + 3) \cdot \frac{2}{3} = \frac{42}{3} = 14$ °C/min

47 $w = f(x,y,z) = 50 + xyz$

a) $P(3,4,1)$, $\frac{dw}{ds}$?

$\vec{v} = \langle 1,2,2 \rangle$

$|\vec{v}| = 3 \quad \hat{u} = \vec{v}/3$

$\nabla f = \langle yz, xz, xy \rangle$

At $(3,4,1)$ $\nabla f = \langle 4,3,12 \rangle$

$\frac{dw}{ds} = D_{\hat{u}} f = \nabla f \cdot \hat{u}$

$= \langle 4,3,12 \rangle \cdot \langle 1,2,2 \rangle \cdot \frac{1}{3}$

$= (4+6+24)/3 = 34/3$ °C/ft

b) The maximal directional deriv at $(3,4,1)$ is $|\nabla f(3,4,1)| = |\langle 4,3,12 \rangle| = \sqrt{16+9+144} = \sqrt{169} = 13$

in the direction $\frac{\nabla f(3,4,1)}{|\nabla f(3,4,1)|} = \frac{\langle 4,3,12 \rangle}{13}$

*note: $s = \vec{r}(t)$ = position

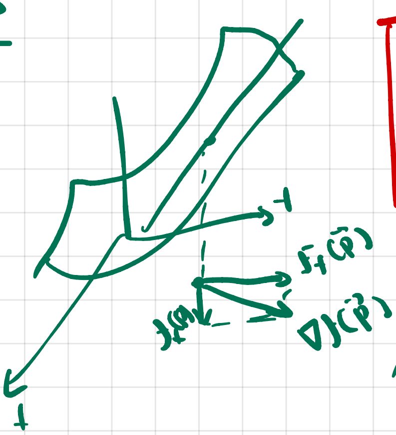
$\frac{dw}{ds}$ is rate of chg of w respect to position, ie change in position which is distance so dw/ds is $D_{\hat{u}} f(\vec{r}(t))$, ie the derivative in a certain direction.

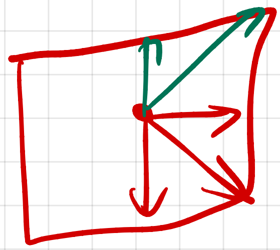
In 2D, $f(x) = y$

we can also consider $r(t) = x(t)$ = position.

$\frac{dy}{dr} = \frac{dy}{dx} = \frac{dy}{ds}$

$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{ds} \cdot v$





49 $z = f(x,y) = \frac{1}{10}(x^2 - xy + 2y^2)$ miles

a) plane tangent at $P(2,1,0.4)$

$\nabla f = \langle \frac{1}{10}(2x-y), \frac{1}{10}(-x+4y) \rangle$

$\nabla f(2,1) = \langle \frac{1}{10}(3), \frac{1}{10}(-2+4) \rangle = \langle \frac{3}{10}, \frac{1}{5} \rangle$

$z - 0.4 = \frac{3}{10}(x-2) + \frac{1}{5}(y-1)$

note $\langle \frac{3}{10}, \frac{1}{5}, -1 \rangle$ should be $\perp$ tangent plane

$\Rightarrow \frac{3}{10}x + \frac{1}{5}y - z = 0.4$

$\frac{3}{10} \cdot 2 + \frac{1}{5} - \frac{4}{10} = \frac{6+2-4}{10} = 0.4$

$\Rightarrow 3x + 2y - 10z = 4$