Theorem Differentiation Formulas

1 $D_t[\vec{u}(t)+\vec{v}(t)] = \vec{u}'(t)+\vec{v}'(t)$

2 $D_t[c\vec{u}(t)] = c\vec{u}'(t)$

3 $D_t[h(t)\vec{u}(t)] = h'(t)\vec{u}(t)+h(t)\vec{u}'(t)$

4 $D_t[\vec{u}(t)\vec{v}(t)] = \vec{u}'(t)\vec{v}(t)+\vec{u}(t)\vec{v}'(t)$

5 $D_t[\vec{u}(t)\times\vec{v}(t)] = \vec{u}'(t)\times\vec{v}(t)+\vec{u}(t)\times\vec{v}'(t)$

Proof

1 $\vec{u}(t)=\langle f_1(t), g_1(t)\rangle \qquad v(t)=\langle f_2(t), g_2(t)\rangle$

$D_t[\vec{u}(t)+\vec{v}(t)] = D_t[\langle f_1+f_2, g_1+g_2\rangle]$

$= \langle D_t(f_1+f_2), D_t(g_1+g_2)\rangle$

$= \langle f_1'+f_2', g_1'+g_2'\rangle = \langle f_1', g_1'\rangle+\langle f_2', g_2'\rangle = \vec{u}'(t)+v'(t)$

2 $D_t[cu(t)] = D_t[\langle cf_1, cg_1\rangle] = \langle cf_1', cg_1'\rangle = c\cdot\vec{u}'(t)$

3 $D_t[h(t)\vec{u}(t)] = D_t[h(t)\langle f(t), g(t)\rangle] = D_t[\langle f\cdot h, g\cdot h\rangle]$

$= \langle f'h+fh', g'h+gh'\rangle = \langle f'h, g'h\rangle+\langle fh', gh'\rangle = h\vec{u}'+\vec{u}h'$

5 $\vec{u}=\langle f_1, g_1, h_1\rangle \quad \vec{v}=\langle f_2, g_2, h_2\rangle$

$\vec{u}\times\vec{v} = \langle g_1h_2-h_1g_2, h_1f_2-f_1h_2, f_1g_2-g_1f_2\rangle$

$D_t[\vec{u}\times\vec{v}] = \langle g_1'h_2+g_1h_2'-h_1'g_2-h_1g_2', h_1'f_2+h_1f_2'-f_1'h_2-f_1h_2', f_1'g_2+f_1g_2'-g_1'f_2-g_1f_2'\rangle$

$= \langle g_1'h_2-h_1'g_2, h_1'f_2-f_1'h_2, f_1'g_2-g_1'f_2\rangle$

$\quad +\langle g_1h_2'-h_1g_2', h_1f_2'-f_1h_2', f_1g_2'-g_1f_2'\rangle$

$= \vec{u}'(t)\times\vec{v}(t)+\vec{u}(t)\times\vec{v}'(t)$

Ex 6 $r(t) = \langle t, t^2 \rangle$

$v(t) = r'(t) = \langle 1, 2t \rangle$

$a(t) = r''(t) = v'(t) = \langle 0, 2 \rangle$

$t = 2 \Rightarrow v(2) = \langle 1, 4 \rangle \qquad |v(2)| = \sqrt{1+8} = 3$

$a(2) = \langle 0, 2 \rangle \qquad |a(2)| = 2$

1 $x = t$

$y = \sin 5t$

$z = \cos 5t \Rightarrow y^2 + z^2 = 1, \forall x$

5 $\vec{r}(t) = \langle 3, -2 \rangle \quad t = 1 \qquad \vec{r}'(t) = \langle 0, 0 \rangle = \vec{r}''(t)$

9 $\vec{r}(t) = \langle 3\cos(2\pi t), 3\sin(2\pi t) \rangle \quad t = 3/4$

$\vec{r}'(t) = \langle -6\pi \sin(2\pi t), 6\pi \cos(2\pi t) \rangle \qquad \vec{r}'(3/4) = 6\pi \langle -\sin(\frac{3\pi}{2}), \cos(\frac{3\pi}{2}) \rangle$

$\vec{r}''(t) = \langle -12\pi^2 \cos(2\pi t), -12\pi^2 \sin(2\pi t) \rangle \qquad \vec{r}''(3/4) = -12\pi(\cos(3\pi/4), \sin(3\pi/4))$

13 $\vec{r}(t) = \langle t, 3e^t, 4e^t \rangle$

$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \langle 1, 3e^t, 4e^t \rangle$

$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \langle 0, 3e^t, 4e^t \rangle$

speed $= |\vec{v}(t)| = \sqrt{1^2 + 9e^{2t} + 16e^{2t}} = \sqrt{1 + 25e^{2t}}$

Ex 8 $\vec{r}(0) = \langle 2, 0 \rangle \qquad \vec{r}'(0) = \langle 1, -1 \rangle \qquad \vec{a}(t) = \langle 2, 6t \rangle \qquad \vec{r}(t), \vec{v}(t)$?

$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle \int 2\,dt, \int 6t\,dt \rangle = \langle 2t, 3t^2 \rangle + \langle C_1, C_2 \rangle$

$\vec{r}'(0) = \vec{v}(0) = \langle 0, 0 \rangle + \langle C_1, C_2 \rangle = \langle 1, -1 \rangle \Rightarrow \vec{v}(t) = \langle 2t + 1, 3t^2 - 1 \rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle t^2 + t, t^3 - t \rangle + \langle C_1, C_2 \rangle$

$\vec{r}(0) = \langle 0, 0 \rangle + \langle C_1, C_2 \rangle = \langle 2, 0 \rangle \Rightarrow \vec{r}(t) = \langle t^2 + t + 2, t^3 - t \rangle$

17 $\int_0^{\pi/4} \langle \sin t, 2\cos t\rangle\, dt = \langle -\cos t, 2\sin t\rangle \Big|_0^{\pi/4} = \langle -\frac{\sqrt{2}}{2}, \sqrt{2}\rangle - \langle -1, 0\rangle$

23 $\vec{u}(t) = \langle \cos t, \sin t\rangle \qquad \vec{v}(t) = \langle \sin t, -\cos t\rangle$

$D_t[\vec{u}(t)\cdot\vec{v}(t)] = \langle -\sin t, \cos t\rangle \cdot \langle \sin t, -\cos t\rangle + \langle \cos t, \sin t\rangle \langle \cos t, \sin t\rangle$

$= -\sin^2 t - \cos^2 t + \cos^2 t + \sin^2 t = 0$

27 $\vec{a}(t) = \langle 2, 0, -4\rangle \qquad \vec{v}(t) = \int \vec{a}(t)\,dt = \langle 2t + C_1, C_2, -4t + C_3\rangle$

$\vec{r}(0) = \langle 0, 0, 0\rangle \qquad \vec{v}(0) = \langle 0, 10, 0\rangle = \langle C_1, C_2, C_3\rangle$

$\vec{r}\,'(0) = \vec{v}(0) = \langle 0, 10, 0\rangle \qquad \vec{v}(t) = \langle 2t, 10, -4t\rangle$

$r(t)$? $\qquad \vec{r}(t) = \langle t^2 + C_1, 10t + C_2, -2t^2 + C_3\rangle$

$\vec{r}(0) = \langle 0, 0, 0\rangle = \langle C_1, C_2, C_3\rangle$

$\Rightarrow \vec{r}(t) = \langle t^2, 10t, -2t^2\rangle$

35 $x(t) = 3\cos(2t) \qquad y(t) = 3\sin(2t) \qquad z(t) = 8t$

$t = 7\pi/8$

$\vec{r}(t) = \langle 3\cos(2t), 3\sin(2t), 8t\rangle$

$\vec{v}(t) = \vec{r}\,'(t) = \langle -6\sin(2t), 6\cos(2t), 8\rangle$

$\vec{a}(t) = \langle -12\cos(2t), -12\sin(2t), 0\rangle$

$\vec{v}(7\pi/8) = \langle -6\sin(7\pi/4), 6\cos(7\pi/4), 8\rangle = \langle 3\sqrt{2}, 3\sqrt{2}, 8\rangle$

$\vec{a}(7\pi/8) = \langle -6\sqrt{2}, 6\sqrt{2}, 0\rangle$

$|\vec{v}(t)| = \sqrt{36\sin^2(2t) + 36\cos^2(2t) + 64} = \sqrt{36+64} = 10$