12.5 Curves and Motion in Space

→ point moving along 3d space, position: parametric equations $x = f(t), y = g(t), z = h(t)$, coordinates as function of time

→ parametric curve: (f, g, h), triple of coord. functions
↳ informally: trajectory

→ position vector: $\vec{r}(t) = \langle \underbrace{x(t)}_{}, y(t), z(t) \rangle = x\hat{i} + y\hat{j} + z\hat{k}$ (component)
↳ vector-valued function $t \to \vec{r}(t)$

We now look at how calculus of real-valued functions applies to vector-valued functions:

$$\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$, provided the latter three limits exist

$\vec{r} = \vec{r}(t)$ is continuous at a if $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$

i.e. $\vec{r}(t)$ continuous at a $\Leftrightarrow$ component functions cont. at a

$$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - r(t)}{\Delta t}$$ provided this limit exists

$r'(t)$ is a vector tangent to curve C with position vector $\vec{r}(t)$

Theorem

$r(t) = \langle f(t), g(t), h(t) \rangle$
f, g, h differentiable
$\Rightarrow$ $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$

i.e $\dfrac{d\vec{r}}{dt} = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt} \right\rangle$

Theorem Differentiation Formulas

$D_t[\vec{u}(t) + \vec{v}(t)] = \vec{u}\,'(t) + \vec{v}\,'(t)$

$D_t[c\vec{u}(t)] = c\vec{u}\,'(t)$

$D_t[h(t)\vec{u}(t)] = h'(t)\vec{u}(t) + h(t)\vec{u}\,'(t)$

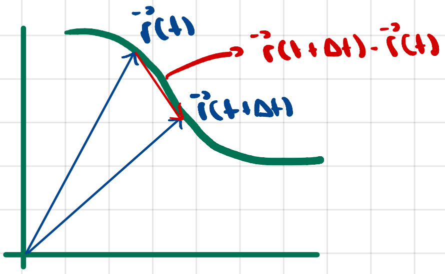
$D_t[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$

$D_t[\vec{u}(t) \times \vec{v}(t)] = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$

Velocity and Acceleration Vectors

recall

$$\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - r(t)}{\Delta t}$$ provided this limit exists



↳ the quotient is average velocity between t and $t+\Delta t$, giving both direction and magnitude

the limit yields a vector that represents instantaneous velocity

→ define $v(t) = r'(t) = \langle f', g', h' \rangle = \frac{dr}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$

↳ velocity vector

↳ differential notation

→ define $a(t) = r''(t) = \langle f'', g'', h'' \rangle$

↳ acceleration vector

* "speed", $v(t)$, is the magnitude of velocity vector
* scalar acceleration is magnitude of accel. vector

→ $v(t) = |\vec{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$

$a(t) = |\vec{a}(t)| = \sqrt{x''(t), y''(t), z''(t)}$

Integration of Vector-Valued Functions

$$\int_a^b \vec{r}(t)\,dt = \lim_{\Delta t \to 0} \sum_{i=1}^{n} \vec{r}(t_i^*)\,\Delta t$$

t_i^*: point in i^{th} subinterval

$[a,b]$ subdivided into n subintervals

$\vec{r}(t) = \langle f(t), g(t) \rangle$, continuous on $[a,b]$, take the limit defining integral componentwise:

$$\int_a^b \vec{r}(t)\,dt = \left\langle \lim \sum f(t_i^*)\Delta t,\ \lim \sum g(t_i^*)\Delta t \right\rangle$$

$$= \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt \right\rangle$$

Antiderivatives

$\vec{R}(t)$ antiderivative of $r(t)$ $\Rightarrow$ $\vec{R}'(t) = \vec{r}(t)$

$$\vec{R}(t) = \langle F(t), G(t) \rangle \Rightarrow \vec{R}'(t) = \langle f(t), g(t) \rangle = \vec{r}(t)$$

$$\Rightarrow \int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt \right\rangle = \langle F(b) - F(a),\ G(b) - G(a) \rangle$$

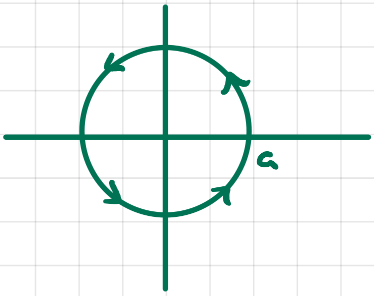
$$= \langle F(b), G(b) \rangle - \langle F(a), G(a) \rangle$$

$$= \vec{R}(b) - \vec{R}(a)$$

Example: Helical Motion

$$\vec{r}(t) = \underbrace{\langle a\cos(\omega t), a\sin(\omega t)}, bt\rangle$$

In xy-plane



once around circle every $\frac{2\pi}{\omega}$

$$\vec{v}(t) = \langle -a\omega\sin(\omega t), a\omega\cos(\omega t), b\rangle$$

$$\vec{a}(t) = \langle -a\omega^2\cos(\omega t), -a\omega^2\sin(\omega t), 0\rangle = -\omega^2\cdot\vec{r}(t) - \langle 0,0,bt\rangle$$

$$|\vec{v}(t)| = \sqrt{a^2\omega^2\sin^2(\omega t) + a^2\omega^2\cos^2(\omega t) + b^2}$$

$$= \sqrt{a^2\omega^2 + b^2} = \text{constant}$$

$\Rightarrow \vec{v} \perp$ any derivative vector

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \left\langle \frac{-a\omega}{|\vec{v}|}\sin(\omega t), \frac{a\omega}{|\vec{v}|}\cos(\omega t), \frac{b}{|\vec{v}|}\right\rangle$$

$$\frac{dT}{dt} = \left\langle \frac{-a\omega^2}{|\vec{v}|}\cos(\omega t), \frac{-a\omega^2}{|\vec{v}|}\sin(\omega t), 0\right\rangle$$

curvature $= \kappa = \frac{1}{v(t)}\cdot\left|\frac{dT}{dt}\right| = \frac{1}{|\vec{v}|}\sqrt{\left(\frac{-a\omega^2}{|\vec{v}|}\right)^2\left(\cos^2\omega t + \sin^2\omega t\right)}$

$$= \frac{a\omega^2}{|v|^2} = \frac{a\omega^2}{a^2\omega^2 + b}$$