Practice Exam 4A

Problem 1

a) $F = \langle e^x yz, e^x z + 2yz, e^x y + y^2 + 1 \rangle$

$\vec{F}$ conservative, $\int_C \vec{F}\,d\vec{r}$ path independent $\iff \vec{F} = \nabla f \iff \text{curl}\,\vec{F} = \vec{0}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x yz & e^x z + 2yz & e^x y + y^2 + 1 \end{vmatrix}$$

$$= \langle e^x + 2y - e^x - 2y, -(e^x y - e^x y), e^x z - e^x z \rangle = \langle 0,0,0 \rangle$$

$*\ \vec{F} = \nabla f \Rightarrow \oint_C \vec{F}\,d\vec{r} = f(\vec{r}(a)) - f(\vec{r}(a)) = 0$

$\int_C \vec{F}\,d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

b) $\int_0^x e^x \cancel{y}^{0} z\,dx + \int_0^y (e^x \cancel{z}^{0} + 2y\cancel{z}^{0})\,dy + \int_0^z (e^x y + y^2 + 1)\,dz$

$= e^x yz + y^2 z + z = f(x,y,z)$

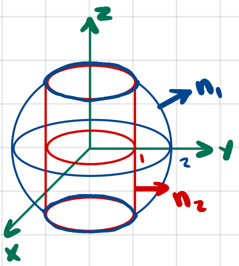
c) $\vec{G} = \langle y, x, y \rangle$

$$\text{curl}\,\vec{G} = \nabla \times \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x & y \end{vmatrix} = \langle 1, -(0-0), 1-1 \rangle = \langle 1,0,0 \rangle \neq \vec{0}$$

$\Rightarrow$ G is not conservative

Problem 2 S: part of spherical surface $x^2+y^2+z^2=4$ in $x^2+y^2>1$

a) $\vec{F}=\langle y,-x,z\rangle$ $\quad \operatorname{div}\vec{F}=0+0+1=1$



The boundary of R consists of an outer partially spherical portion and an internal cylindrical portion. This boundary is a surface S, a union of two surfaces.

$\hat{n}$ is a vector field. On the outer portion of S it points out from S and is normal to S, but on the cylindrical portion of S it points normally into R.

$$\iint_S \hat{F}\cdot\hat{n}\,dS = \iint_{S_1}\vec{F}\cdot\hat{n}\,dS - \iint_{S_2}\vec{F}\cdot\hat{n}\,dS$$

$$=\left[\int_0^{2\pi}\int_{\pi/6}^{5\pi/6}\int_0^{2} 1\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta + 2\cdot\pi\cdot1^2\cdot\sqrt{3}\cdot\frac{1}{3}\right]-\int_0^{2\pi}\int_{-\sqrt{3}}^{\sqrt{3}}\int_0^{1} r\,dr\,dz\,d\theta = 4\sqrt{3}\,\pi$$

We could've also reached this result with a single application of divergence theorem:

$$\int_0^{2\pi}\int_{\pi/6}^{5\pi/6}\int_{\sqrt{\frac{1}{\sin^2\phi}}}^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 4\sqrt{3}\,\pi$$

b) $\iint_{S_c}\hat{F}\cdot\hat{n}$ $\quad \hat{n}=\dfrac{\langle x,y,0\rangle}{(x^2+y^2)^{1/2}}$ $\quad \Rightarrow \vec{F}\cdot\hat{n}= xy-xy+0=0$

$\Rightarrow \text{Flux}_{S_c}=0$

c) Volume between S and C

region D between sphere and cylinder

$\rho=\sqrt{1+z^2}\,..\,2,\ \theta=0\,..\,2\pi,\ \phi=\frac{\pi}{6}\,..\,\frac{5\pi}{6}$

To find range for ϕ, need intersection:

$x^2+y^2=1 \Rightarrow 1+z^2=4 \Rightarrow z^2=3 \Rightarrow z=\pm\sqrt{3}$

$z=\rho\cos\phi \Rightarrow \cos\phi=\frac{z}{\rho}=\frac{\sqrt{3}}{2}\Rightarrow \phi=\frac{\pi}{6}$

To find range for ρ:

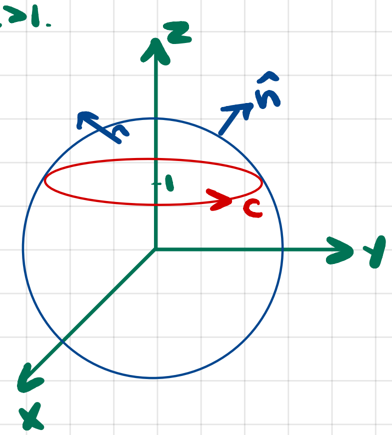
starts on cylinder: $\rho^2=1+z^2 \Rightarrow \rho=\sqrt{1+z^2} \Rightarrow \rho^2=1+\rho^2\cos^2\phi \Rightarrow \rho^2(1-\cos^2\phi)=1$

$\Rightarrow \rho^2=\frac{1}{\sin^2\phi} \Rightarrow \rho=\sqrt{\frac{1}{\sin^2\phi}}$

goes to sphere: $\rho^2=x^2+y^2+z^2=4 \Rightarrow \rho=2$

d) $\iint_S\hat{F}\cdot\hat{n}\,dS=\iiint_D \operatorname{div}\vec{F}\,dV=\int_0^{2\pi}\int_{\pi/6}^{5\pi/6}\int_{\sqrt{\frac{1}{\sin^2\phi}}}^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 4\sqrt{3}\,\pi$

Problem 3 S: part of $x^2+y^2+z^2=2$, spherical surface lying in $z\geq 1$.



a) C: $z=1$, $x^2+y^2+1=2 \Rightarrow x^2+y^2=1,\ z=1$

$$I=\oint_C xz\,dx+y\,dy+y\,dz=\int_0^{2\pi}-\cos t\sin t\,dt+\sin t\cos t\,dt$$

$$=0$$

$x=\cos t \quad dx=-\sin t\,dt$

$y=\sin t \quad dy=\cos t\,dt$

$z=1 \quad dz=0$

b) $\vec{F}=\langle xz, y, y\rangle$

$$\text{curl}\,\vec{F}=\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ xz & y & y\end{vmatrix}=\langle 1,-(0-x),0\rangle=\langle 1,x,0\rangle$$

c) $$\underbrace{\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS}_{\text{Flux of curl }\vec{F}\text{ through }S}=\oint_C\vec{F}\cdot d\vec{r}=\oint_C xz\,dx+y\,dy+y\,dz=I=0$$

Problem 4

$\vec{F}=\langle 1,1,1\rangle \qquad \text{div}\,\vec{F}=0$

closed surface S: $x^4+y^4+z^4=1$

Flux of $\vec{F}$ across S: $\iint_S\vec{F}\cdot\hat{n}\,dS=\iiint_D \text{div}\,\vec{F}\,dV=0$

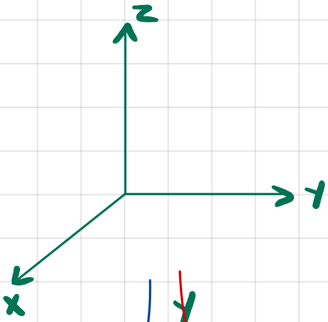
Problem 5

$S: \; z = (x^2+y^2+z^2)^2$

a) $z = (x^2+y^2+z^2)^2 \Rightarrow z \geq 0$

b) $z = \rho^4 \Rightarrow \rho\cos\phi = \rho^4 \Rightarrow \rho(\rho^3 - \cos\phi) = 0$ $\nearrow \rho = 0$ $\searrow \rho^3 = \cos\phi$

c) $$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\sqrt[3]{\cos\phi}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$



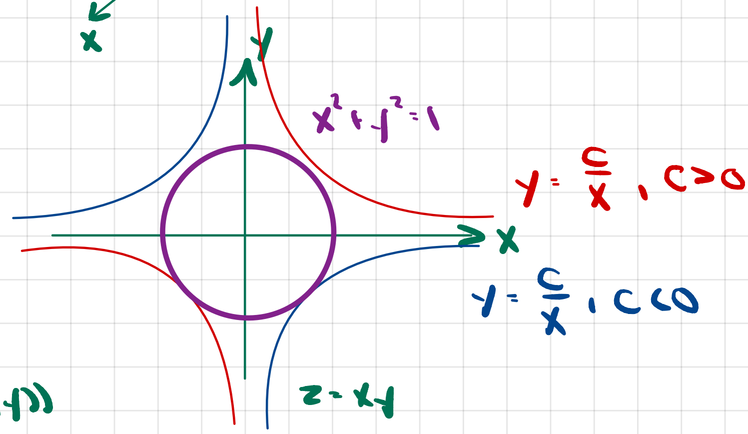
Problem 6

S: part of $z = xy$ where $x^2+y^2<1$

$\operatorname{div}\vec{F} = 0+0+1 = 1$

Flux of $\vec{F} = \langle y, x, z\rangle$ upward across S

$= \iint_S \vec{F}\cdot\hat{n}\, dS$



Finding $\hat{n}$

$\langle x, y, xy\rangle = \vec{r}(x(x,y), y(x,y), z(x,y))$

$r_x = \langle 1, 0, y\rangle \quad r_y = \langle 0, 1, x\rangle$

$$r_x \times r_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = \langle -y, -(x), 1\rangle = \langle -y, -x, 1\rangle$$

$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\, dS = \iint_S \langle y, x, z\rangle \langle -y, -x, 1\rangle \, dx\, dy$

$= \iint_S (-y^2 - x^2 + z)\, dx\, dy = \int_0^{2\pi}\int_0^{1} (-r^2 + \cos\theta\sin\theta)\, r\, dr\, d\theta = -\frac{\pi}{2}$

$z = xy$

$z_x = y \quad z_y = x$

$z_{xx} = 0 \quad z_{xy} = 1 \quad z_{yy} = 0$

$z_{xx}z_{yy} - z_{xy}^2 = 0 - 1^2 = -1$

$\Rightarrow$ Any critical point is a saddle