Implicit Partial Differentiation

$z = g(x,y)$ but z not explicitly defined by a formula giving z in terms of x and y

z is instead defined implicitly by $F(x,y,z) = 0$

under certain hypotheses, the existence and differentiability of F are guaranteed.

Theorem

$F(x_1, \dots, x_n, z)$ continuously differentiable near $(\vec{a}, b) = (a_1, \dots, a_n, b)$ where $F(\vec{a}, b) = 0$ and

$D_z F(\vec{a}, b) \neq 0$

$\Rightarrow \exists$ continuously diff. fn $z = g(x_1, \dots, x_n)$ such that $g(\vec{a}) = b$ and $F(\vec{x}, g(\vec{x})) = 0$ for $\vec{x}$ near $\vec{a}$

Moreover, $g(\vec{x})$ is uniquely defined for $\vec{x}$ near $\vec{a}$.

$\Rightarrow$ we can take $F(x_1, \dots, x_n, z) = 0$ and calculate the partial derivatives of z by implicit diff.

$$\frac{\partial F}{\partial x_1} \cdot \cancel{\frac{\partial x_1}{\partial x_i}}^{\,0} + \dots + \frac{\partial F}{\partial x_i} \frac{\partial x_i}{\partial x_i} + \dots + \frac{\partial F}{\partial x_n} \cdot \cancel{\frac{\partial x_n}{\partial x_i}}^{\,0} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x_i} = 0$$

$\partial x_i / \partial x_j = 0$ unless $i = j$ $\Rightarrow \dfrac{\partial F}{\partial x_i} + \dfrac{\partial F}{\partial z} \cdot \dfrac{\partial z}{\partial x_i} = 0 \Rightarrow \dfrac{\partial z}{\partial x_i} = \dfrac{-\partial F / \partial x_i}{\partial F / \partial z} = \dfrac{-F_{x_i}}{F_z}$