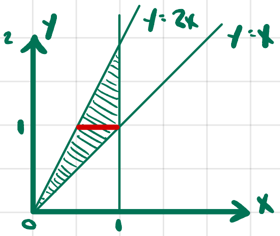
18.02 - Practice Exam 3b

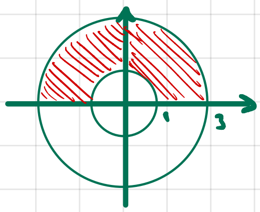
1

a)



b) $\int_0^1\int_x^{2x} dy\,dx = \int_0^1\int_{y/2}^{y} dx\,dy + \int_1^2\int_{y/2}^{1} dx\,dy$

2 a) $1 < x^2+y^2 < 9$ $(y \geq 0)$ $\quad \delta(x,y) = \frac{y}{x^2+y^2}$



mass $= \iint_R \delta(x,y)\,dA = \int_0^\pi\int_1^3 \frac{r\sin\theta}{r^2}\, r\,dr\,d\theta = \int_0^\pi\int_1^3 \sin\theta\,dr\,d\theta = \int_0^\pi 2\sin\theta\,d\theta$

$= 2(-\cos\theta)\Big|_0^\pi = -2(\cos\pi - \cos 0) = -2(-1-1) = 4$

b) $\bar{x} = 0$ by symmetry: δ and R are symmetric about y-axis

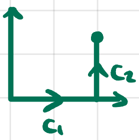
$\bar{x} = \int_0^\pi\int_1^3 \frac{r\sin\theta}{r^2}\cdot r\cos\theta\, r\,dr\,d\theta = \int_0^\pi\int_1^3 r\sin\theta\cos\theta\,dr\,d\theta = \int_0^\pi \frac{1}{2}(9-1)\sin\theta\cos\theta\,d\theta = 4\cdot\frac{\sin^2\theta}{2}\Big|_0^\pi = 0$

3 $\vec{F} = \langle 3x^2-6y^2, -12xy+4y\rangle$

a) $\vec{F}$ conservative $\iff$ $\vec{F} = \vec{\nabla} f$ $\iff$ $N_y = N_y$ $\Rightarrow$ curl $\vec{F} = 0$ $\iff$ path independent

$M_y = -12y$ $\quad N_x = -12y$ $\Rightarrow$ curl $\vec{F} = 0 \Rightarrow \vec{F}$ conservative

b) method of line integral



$\int_C \vec{F}\cdot d\vec{r} = \int_{\vec{r}(a)}^{\vec{r}(b)} \vec{F}\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$, $\vec{r}(b) = \langle x, y\rangle$, $\vec{r}(a) = \langle 0,0\rangle$

$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = f(x,y) + c \Rightarrow f(x,y) = \int_C \vec{F}\cdot d\vec{r} + c$

$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} (3x^2-6y^2)\,dx + (-12xy+4y)\cdot 0 = \int_0^{x_1} 3x^2\,dx = x_1^3$

$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^{y_1} (-12x_1y+4y)\,dy = -12x_1\frac{y_1^2}{2} + 4\frac{y_1^2}{2} = -6y_1^2x_1 + 2y_1^2$

$f(x,y) = x^3 + 2y^2 - 6xy^2$

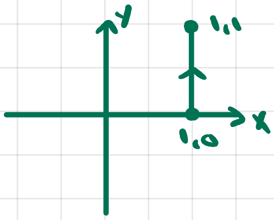
method of antiderivative

$\int M_x\,dx = x^3 - 6xy^2 + g(y)$

$\frac{d}{dy}\left(\int M_x\,dx\right) = -12xy + g'(y) \Rightarrow g'(y) = 4y \Rightarrow g(y) = 2y^2$

$\Rightarrow f(x,y) = x^3 - 6xy^2 + 2y^2$

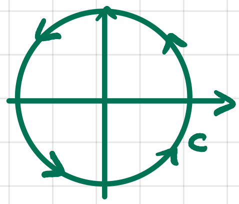
c) $C: x = 1 + y^3(1-y)^3 \quad 0 \leq y \leq 1$



$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(1,1) - f(0,1) = -3 - 1 = -4$

$f(x,y) = x^3 - 6xy^2 + 2y^2, \quad f(1,1) = 1 - 6 + 2 = -3$
$f(1,0) = 1$

4 $\vec{F} = \langle 5x + 3y, 1 + \cos y \rangle \Rightarrow \vec{F} \neq \nabla f \Rightarrow$ can't apply fundamental theorem



a) $W = \oint_C \vec{F} \cdot d\vec{r} = \oint_C (5x + 3y)\,dx + (1 + \cos y)\,dy$

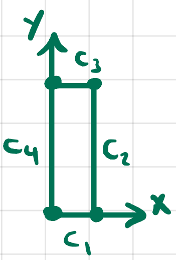
$x = \cos t,\ dx = -\sin t\,dt$
$y = \sin t,\ dy = \cos t\,dt$
$\Rightarrow W = \int_0^{2\pi} (5\cos t + 3\sin t)(-\sin t\,dt) + (1 + \cos(\sin t))\cos t\,dt$

b) $\vec{F} = \langle M, N \rangle$, C posit. oriented, smooth, simply closed curve bounding R, M, N have continuous 1st-order partial derivatives on R

$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = \iint_R (N_x - M_y)\,dA = \iint_R \text{curl}\,\vec{F}\,dA = -3 \iint_R dA = -3\pi$

$N_x = 0 \quad M_y = 3$

5



$\vec{F} = \langle xy + \sin x \cos y, -\cos x \sin y \rangle$

a) Flux $= \oint_C \vec{F} \cdot \hat{n}\,ds = \oint_C \langle P, Q \rangle \langle dy, -dx \rangle = \oint_C P\,dy - Q\,dx = \oint_C \vec{G}\,d\vec{r}$ for $\vec{G} = \langle -Q, P \rangle$

$\Rightarrow$ by Green's Theorem $= \iint_R \text{curl}\,\vec{G}\,dA = \iint_R \text{div}\,\vec{F}\,dA$

$*$ $\text{curl}\,\vec{G} = P_x + Q_y$

$\text{div}\,\vec{F} = P_x + Q_y$

$P_x = y + \cos x \cos y \quad Q_y = -\cos x \cos y$

$\Rightarrow \text{div}\,\vec{F} = y$

$\Rightarrow$ Flux $= \int_0^4 \int_0^1 y\,dx\,dy = \int_0^4 y\,dy = \frac{1}{2} \cdot 16 = 8$

b) $\int_{C_4} \vec{F} \cdot \hat{n}\,ds = \int_{C_4} \langle M, N \rangle \langle dy, -dx \rangle = \int_{C_4} \langle M, N \rangle \langle dy, 0 \rangle = \int_{C_4} 0\,dy = 0$

$\Rightarrow \int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$

6 $z = (2x-y)^2 + (x+y-1)^2$, $z = 4$

region R: $(2x-y)^2 + (x+y-1)^2 = 4$

$u = 2x - y$ $\quad v = x + y - 1$ $\to$ region S $u^2 + v^2 = 4$

$$\iint_R z(x,y)\, dA = \iint_S z(x(u,v), y(u,v)) \frac{\partial(x,y)}{\partial(u,v)}\, du\, dv = \iint_S (u^2+v^2)\frac{1}{3}\, du\, dv$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 2+1 = 3$$

$$= \int_0^{2\pi}\int_0^{2} r^2 \cdot \frac{1}{3} \cdot r\, dr\, d\theta = \frac{1}{3}\int_0^{2\pi} \frac{r^4}{4}\Big|_0^2 d\theta = \frac{4}{3}\int_0^{2\pi} d\theta = \frac{8\pi}{3}$$