3. Double Integrals

3A. Double Integrals in Rectangular Coordinates

3A-1 Evaluate each of the following iterated integrals:

a) $\displaystyle\int_0^2 \int_{-1}^1 (6x^2 + 2y)\,dy\,dx$ b) $\displaystyle\int_0^{\pi/2} \int_0^{\pi} (u \sin t + t \cos u)\,dt\,du$

c) $\displaystyle\int_0^1 \int_{\sqrt{x}}^{x^2} 2x^2 y\,dy\,dx$ d) $\displaystyle\int_0^1 \int_0^u \sqrt{u^2+4}\,dv\,du$

3A-2 Express each double integral over the given region R as an iterated integral, using the given order of integration. Use the method described in Notes I to supply the limits of integration. For some of them, it may be necessary to break the integral up into two parts. In each case, begin by sketching the region.

a) R is the triangle with vertices at the origin, $(0,2)$, and $(-2,2)$.

Express as an iterated integral: i) $\displaystyle\iint_R dy\,dx$ ii) $\displaystyle\iint_R dx\,dy$

b) R is the finite region between the parabola $y = 2x - x^2$ and the x-axis.

Express as an iterated integral: i) $\displaystyle\iint_R dy\,dx$ ii) $\displaystyle\iint_R dx\,dy$

c) R is the sector of the circle with center at the origin and radius 2 lying between the x-axis and the line $y = x$.

Express as an iterated integral: i) $\displaystyle\iint_R dy\,dx$ ii) $\displaystyle\iint_R dx\,dy$

d)* R is the finite region lying between the parabola $y^2 = x$ and the line through $(2,0)$ having slope 1.

Express as an iterated integral: i) $\displaystyle\iint_R dy\,dx$ ii) $\displaystyle\iint_R dx\,dy$

3A-3 Evaluate each of the following double integrals over the indicated region R. Choose whichever order of integration seems easier — given the integrand, and the shape of R.

a) $\displaystyle\iint_R x\,dA$; R is the finite region bounded by the axes and $2y + x = 2$

b) $\displaystyle\iint_R (2x + y^2)\,dA$; R is the finite region in the first quadrant bounded by the axes and $y^2 = 1 - x$; ($dx\,dy$ is easier).

c) $\displaystyle\iint_R y\,dA$; R is the triangle with vertices at $(\pm 1, 0)$, $(0,1)$.

3A-4 Find by double integration the volume of the following solids.

a) the solid lying under the graph of $z = \sin^2 x$ and over the region R bounded below by the x-axis and above by the central arch of the graph of $\cos x$

b) the solid lying over the finite region R in the first quadrant between the graphs of x and x^2, and underneath the graph of $z = xy$..

c) the finite solid lying underneath the graph of $x^2 - y^2$, above the xy-plane, and between the planes $x = 0$ and $x = 1$

3A-5 Evaluate each of the following iterated integrals, by changing the order of integration (begin by figuring out what the region R is, and sketching it).

a) $\displaystyle\int_0^2\int_x^2 e^{-y^2}\,dy\,dx$ b) $\displaystyle\int_0^{1/4}\int_{\sqrt{t}}^{1/2}\frac{e^u}{u}\,du\,dt$ c) $\displaystyle\int_0^1\int_{x^{1/3}}^1\frac{1}{1+u^4}\,du\,dx$

3A-6 Each integral below is over the disc consisting of the interior R of the unit circle, centered at the origin. For each integral, use the symmetries of R and the integrand

i) to identify its value as zero; or if its value is not zero,

ii) to find a double integral which is equivalent (i.e., has the same value), but which has a simpler integrand and/or is taken over the first quadrant (if possible), or over a half-disc. (Do not evaluate the integral.)

$$\iint_R x\,dA;\quad \iint_R e^x\,dA;\quad \iint_R x^2\,dA;\quad \iint_R x^2y\,dA;\quad \iint_R (x^2+y)dA;\quad \iint_R xy\,dA$$

3A-7 By using the inequality $f \le g$ on $R \Rightarrow \iint_R f\,dA \le \iint_R g\,dA$, show the following estimates are valid:

a) $\displaystyle\iint_R \frac{dA}{1+x^4+y^4} \le$ area of R

b) $\displaystyle\iint_R \frac{x\,dA}{1+x^2+y^2} < .35,$ R is the square $0 \le x, y \le 1$.

3B. Double Integrals in Polar Coordinates

In evaluating the integrals, the following definite integrals will be useful:

$$\int_0^{\pi/2} sin^n x\,dx = \int_0^{\pi/2} cos^n x\,dx = \begin{cases} \dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(n-1)}{2\cdot 4\cdot\cdots\cdot n}\,\dfrac{\pi}{2}, & \text{if } n \text{ is an even integer} \ge 2 \\ \dfrac{2\cdot 4\cdot\cdots\cdot(n-1)}{1\cdot 3\cdot\cdots\cdots n}, & \text{if } n \text{ is an odd integer} \ge 3. \end{cases}$$

For example: $\displaystyle\int_0^{\pi/2}\sin^2 x\,dx = \frac{\pi}{4}$, $\displaystyle\int_0^{\pi/2}\sin^3 x\,dx = \frac{2}{3}$, $\displaystyle\int_0^{\pi/2}\sin^4 x\,dx = \frac{3\pi}{16}$,

and the same holds if $\cos x$ is substituted for $\sin x$.

3B-1 Express each double integral over the given region R as an iterated integral in polar coordinates. Use the method described in Notes I to supply the limits of integration. For some of them, it may be necessary to break the integral up into two parts. In each case, begin by sketching the region.

a) The region lying inside the circle with center at the origin and radius 2. and to the left of the vertical line through $(-1, 0)$.

b)* The circle of radius 1, and center at $(0, 1)$.

c) The region inside the cardioid $r = 1 - \cos\theta$ and outside the circle of radius $3/2$ and center at the origin.

d) The finite region bounded by the y-axis, the line $y = a$, and a quarter of the circle of radius a and center at $(a, 0)$.

3B-2 Evaluate by iteration the double integrals over the indicated regions. Use polar coordinates.

a) $\displaystyle\iint_R \frac{dA}{r}$; R is the region inside the first-quadrant loop of $r = \sin 2\theta$.

b) $\displaystyle\iint_R \frac{dx\,dy}{1+x^2+y^2}$; R is the first-quadrant portion of the interior of $x^2+y^2=a^2$

c) $\displaystyle\iint_R \tan^2\theta\,dA$; R is the triangle with vertices at $(0,0),\ (1,0),\ (1,1)$.

d) $\displaystyle\iint_R \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$; R is the right half-disk of radius $\frac{1}{2}$ centered at $(0,\frac{1}{2})$.

3B-3 Find the volumes of the following domains by integrating in polar coordinates:

a) a solid hemisphere of radius a (place it so its base lies over the circle $x^2+y^2=a^2$)

b) the domain under the graph of xy and over the quarter-disc region R of 3B-2b

c) the domain lying under the cone $z=\sqrt{x^2+y^2}$ and over the circle of radius one and center at $(0,1)$

d) the domain lying under the paraboloid $z = x^2+y^2$ and over the interior of the right-hand loop of $r^2 = \cos\theta$.

3B-4* Sometimes students wonder if you can do a double integral in polar coordinates iterating in the opposite order: $\iint_R d\theta\,dr$. Though this is uncommon, just to see if you can carry out in a new situation the basic procedure for putting in the limits, try supplying the limits for this integral over the region bounded above by the lines $x=1$ and $y=1$, and below by a quarter of the circle of radius 1 and center at the origin.

3C. Applications of Double Integration

If no coordinate system is specified for use, you can use either rectangular or polar coordinates, whichever is easier. In some of the problems, a good placement of the figure in the coordinate system simplifies the integration a lot.

3C-1 Let R be a right triangle, with legs both of length a, and density 1. Find the following ((b) and (c) can be deduced from (a) with no further calculation)
a) its moment of inertia about a leg;
b) its polar moment of inertia about the right-angle vertex;
c) its moment of inertia about the hypotenuse.

3C-2 Find the center of mass of the region inside one arch of $\sin x$, if: a) $\delta = 1$ b) $\delta = y$

3C-3 D is a diameter of a disc of radius a, and C is a chord parallel to D with distance c from it. C divides the disc into two segments; let R be the smaller one. Assuming $\delta = 1$, find the moment of R about D, giving the answer in simplest form, and using
(a) rectangular coordinates; (b) polar coordinates.

3C-4 Find the center of gravity of a sector of a circular disc of radius a, whose vertex angle is 2α. Take $\delta = 1$.

3C-5 Find the polar moment of inertia of one loop of the lemniscate $r^2 = a^2 \cos 2\theta$ about the origin. Take $\delta = 1$.

3D. Changing Variables in Multiple Integrals

~~3D-1~~ Evaluate $\displaystyle\iint_R \frac{x-3y}{2x+y}\, dx\, dy$, where R is the parallelogram bounded on the sides by $y = -2x+1$ and $y = -2x+4$, and above and below by $y = x/3$ and $y = (x-7)/3$. Use a change of variables $u = x - 3y$, $v = 2x + y$.

~~3D-2~~ Evaluate $\displaystyle\iint_R \cos\left(\frac{x-y}{x+y}\right) dx\, dy$ by making the change of variables $u = x + y$, $v = x - y$; take as the region R the triangle with vertices at the origin, (2,0) and (1,1).

~~3D-3~~ Find the volume underneath the surface $z = 16 - x^2 - 4y^2$ and over the xy-plane; simplify the integral by making the change of variable $u = x,\ v = 2y$.

~~3D-4~~ Evaluate $\displaystyle\iint_R (2x-3y)^2(x+y)^2\, dx\, dy$, where R is the triangle bounded by the positive x-axis, negative y-axis, and line $2x - 3y = 4$, by making a change of variable $u = x + y,\ \ v = 2x - 3y$.

3D-5 Set up an iterated integral for the polar moment of inertia of the finite "triangular" region R bounded by the lines $y = x$ and $y = 2x$, and a portion of the hyperbola $xy = 3$. Use a change of coordinates which makes the boundary curves grid curves in the new coordinate system.

3D-6* Verify that the Jacobian gives the right volume element in spherical coordinates. Recall spherical coordinates have

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

and the volume element is $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.

3D-7* Using the coordinate change $u = xy,\ v = y/x$, set up an iterated integral for the polar moment of inertia of the region bounded by the hyperbola $xy = 1$, the x-axis, and the two lines $x = 1$ and $x = 2$. Choose the order of integration which makes the limits simplest.

3D-8 For the change of coordinates in 3D-7, give the uv-equations of the following curves: a) $y = x^2$ b) $x^2 + y^2 = 1$.

3D-9* Prove the relation between Jacobians: $\displaystyle\frac{\partial(x,y)}{\partial(u,v)}\,\frac{\partial(u,v)}{\partial(x,y)} = 1$; use the chain rule for partial differentiation, and the rule for multiplying determinants: $|AB| = |A||B|$, where A and B are square matrices of the same size.

3D-10* Let $u = x + y$ and $v = x - y$; change $\displaystyle\int_0^1\int_0^x dy\, dx$ to an iterated integral in the order $\displaystyle\iint_R dv\, du$, and check your work by evaluating it. (You will have to break the region up into two pieces, using different limits of integration for the pieces.)