Pset 10

5A-1

a) $\int_0^2\int_{-1}^1\int_0^1 (x+y+z)\,dx\,dy\,dz = \int_0^2\int_{-1}^1 \left(\frac12+y+z\right)dy\,dz = \int_0^2 \left[\frac12(1+1)+\frac12(1-1)+z(1+1)\right]dz$

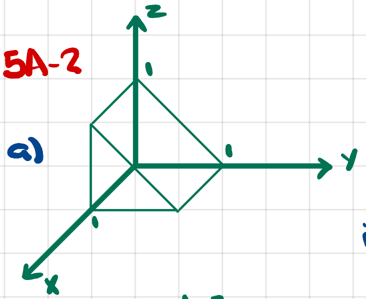
$= \int_0^2 (1+2z)\,dz = 2+4=6$

b) $\int_0^2\int_0^{\sqrt{y}}\int_0^{xy} 2xy^2z\,dz\,dx\,dy = \int_0^2\int_0^{\sqrt{y}} xy^2(xy)^2\,dx\,dy = \int_0^2 y^4 \frac{x^4}{4}\Big|_0^{\sqrt{y}} dy = \int_0^2 y^4\cdot y^2\cdot\frac14\,dy = \frac14\cdot\frac17 y^7\Big|_0^2$

$=\frac{1}{28}\cdot 128 = \frac{64}{14}=\frac{32}{7}$

5A-2

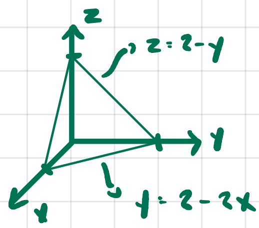
a)



i) $\int_0^1\int_0^1\int_0^{1-y} dz\,dy\,dx$

ii) $\int_0^1\int_0^{1-y}\int_0^1 dx\,dz\,dy$

iii) $\int_0^1\int_0^1\int_0^{1-z} dy\,dx\,dz$



b) $\int\int\int_0 dz\,dy\,dx$

i) find eq of plane passing through $(1,0,0)$, $(0,2,0)$, $(0,0,2)$

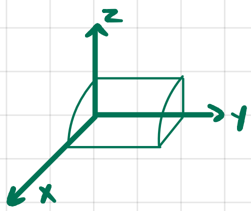
$\vec{v}_1 = \langle 0,2,-2\rangle$ $\vec{v}_2=\langle 1,0,-2\rangle$

$\vec{v}_1\times\vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & -2 \\ 1 & 0 & -2 \end{vmatrix} = \hat{i}(-4)-\hat{j}(2)+\hat{k}(-2) = \langle -4,-2,-2\rangle = \vec{n}$

plane: $\langle -4,-2,-2\rangle\langle x-1,y,z\rangle=0 \Rightarrow 4-4x-2y-2z=0 \Rightarrow z=2-2x-y$

$\int_0^1\int_0^{2-2x}\int_0^{2-2x-y} dz\,dy\,dx = \frac23$

c) $\int_0^{\pi/2}\int_0^1\int_0^2 r\,dy\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 2r\,dr\,d\theta = \int_0^{\pi/2} d\theta = \pi/2$

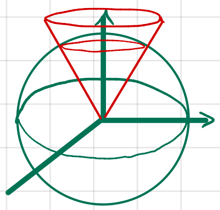


d) $\int_{-1}^1 \int_{-(1-x^2)^{1/2}}^{(1-x^2)^{1/2}} \int_{x^2+y^2}^{(2-x^2-y^2)^{1/2}} dz\,dy\,dx$

$a=\sqrt{2}$

$\Rightarrow x^2+y^2+z^2=a^2=2$

$\Rightarrow z=\pm(2-x^2-y^2)^{1/2}$

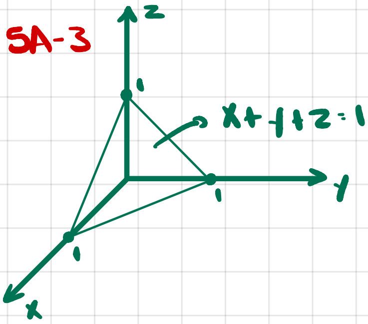


$=\int_0^{2\pi}\int_0^1\int_r^{(2-r^2)^{1/2}} r\,dz\,dr\,d\theta$

$x^2+y^2+x^2+y^2=2$

$x^2+y^2=1$ (intersection)

5A-3



$$\text{mass} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} 1\,dz\,dy\,dx = \int_0^1\int_0^{1-x} (1-x-y)\,dy\,dx = \int_0^1 \left(y - xy - \frac{y^2}{2}\right)\Big|_0^{1-x} dx$$

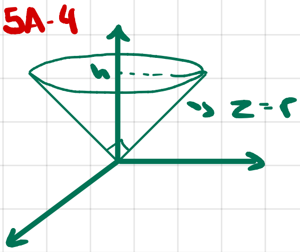
$$= \int_0^1 \left(1-x-x+x^2-\frac{1}{2}(1-2x+x^2)\right)dx = \frac{1}{2}\int_0^1 2-4x+2x^2-1+2x-x^2\,dx$$

$$= \frac{1}{2}\int_0^1 [1+x^2-2x]\,dx = \frac{1}{2}\left[x+\frac{1}{3}x^3-x^2\right]\Big|_0^1 = \frac{1}{2}\left(1+\frac{1}{3}-1\right) = \frac{1}{6}$$

$$\bar{x} = \frac{\int_0^1\int_0^{1-x}\int_0^{1-x-y} x\,dz\,dy\,dx}{m} = \frac{1/24}{1/6} = 1/4$$

By symmetry, $\bar{x} = \bar{y} = \bar{z} = 1/4$

5A-4



$\delta(x,y,z) = \sqrt{x^2+y^2} \Rightarrow \delta(r,\theta) = r$

$$m = \int_0^{2\pi}\int_0^h\int_r^h r\cdot r\,dz\,dr\,d\theta = \frac{h^4\pi}{6}$$

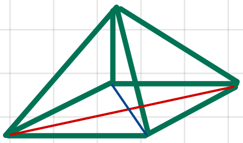
top cone rim: $r = h = z$

$\bar{x} = \bar{y} = 0$ by symmetry

$$\bar{z} = \frac{\int_0^{2\pi}\int_0^h\int_r^h z\,r\cdot r\,dz\,dr\,d\theta}{m} = \frac{\frac{2h^5\pi}{15}}{\frac{h^4\pi}{6}} = \frac{4h}{5}$$

5A-5

height $= 2$ $\delta = 1$



plane 1: $\langle 1,0,h\rangle \times \langle 0,h,1\rangle = \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & h \\ 0 & h & 1 \end{vmatrix} = \langle -h^2, -1, h\rangle$

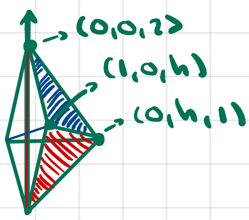
$-h^2x - y + hz = 0 \Rightarrow z = \frac{y}{h} + hx$

plane 2: $\langle 1,0,h-2\rangle \times \langle 0,h,-1\rangle = \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & h-2 \\ 0 & h & -1 \end{vmatrix} = \langle -h(h-2), -(-1), h\rangle$

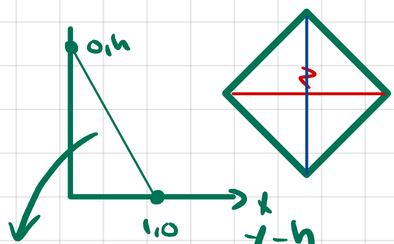
$= \langle 2h-h^2, 1, h\rangle$

$(2h-h^2)(x) + 1\cdot y + h(z-2) = 0$

$hz = 2h - y - x(2h-h^2) \Rightarrow z = 2 - \frac{y}{h} - x(2-h)$



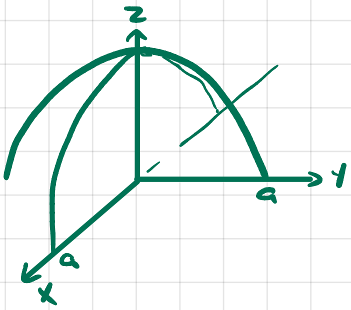
moment of inertia: $2\cdot\int_0^h\int_0^{1-\frac{y}{h}}\int_{\frac{y}{h}+hx}^{2-\frac{y}{h}-x(2-h)} (x^2+y^2)\cdot 1\,dz\,dx\,dy = 2\cdot\frac{1}{3}$



$-\frac{h}{1}x = y-h \Rightarrow x = \frac{y-h}{-h} = 1-\frac{y}{h}$

5A-6 $\delta(x,y,z)=1$

hemisphere: $x^2+y^2+z^2=a^2,\ z\geq 0$



$\Rightarrow r^2+z^2=a^2,\ z>0$

$\Rightarrow z=(a^2-r^2)^{1/2}$

$$\int_0^{2\pi}\int_0^{a}\int_0^{(a^2-r^2)^{1/2}} r^2\cdot r\cdot 1\, dz\,dr\,d\theta$$

$$\int_0^{2\pi}\underbrace{\int_0^{a} r^3(a^2-r^2)^{1/2}\,dr}\,d\theta = \int_0^{2\pi} -\frac{2}{15}\left[-\left((a^2)^{5/2}\right)\right]d\theta = \frac{2}{15}\int_0^{2\pi} a^5\,d\theta = \frac{2\cdot a^5\cdot 2\pi}{15} = \frac{4\pi a^5}{15}$$

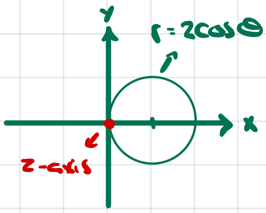
$$\int r^2\, r(a^2-r^2)^{1/2}dr = \frac{-r^2(a^2-r^2)^{3/2}}{3} + \frac{2}{3}\underbrace{\int r(a^2-r^2)^{3/2}dr}_{-\frac{1}{5}(a^2-r^2)^{5/2}}$$

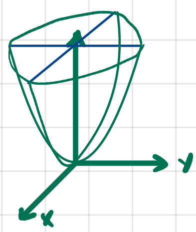
$u=r^2 \quad du=2r\,dr$

$dv=(a^2-r^2)^{1/2}\, r\,dr$

$v=\dfrac{-2(a^2-r^2)^{3/2}}{3\cdot 2} = \dfrac{-(a^2-r^2)^{3/2}}{3}$

5A-7





$z=2x$

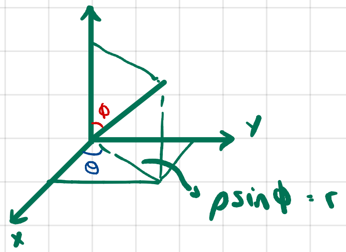
$z=x^2+y^2$

Intersection: $2x=x^2+y^2 \Rightarrow x^2-2x+1+y^2=1 \Rightarrow (x-1)^2+y^2=1$

$\hookrightarrow 2r\cos\theta=r^2 \Rightarrow r(r-2\cos\theta)=0 \Rightarrow r=2\cos\theta$

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\int_{r^2}^{2r\cos\theta} r^2\cdot r\cdot 1\,dz\,dr\,d\theta = \frac{2\pi}{3}$$

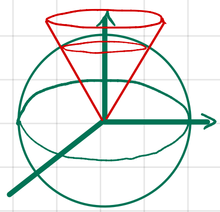
SB-1 $\int \int \int_0 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$



a)

cone: $z^2 = x^2 + y^2 \Rightarrow \rho^2\cos^2\phi = r^2 = \rho^2\sin^2\phi \Rightarrow \cos^2\phi = \sin^2\phi \Rightarrow \phi = \frac{\pi}{4}$

sphere: $\rho = a$



intersection: $\rho = a$ and $\phi = \pi/4$

$y = r\sin\theta = \rho\sin\phi\sin\theta$

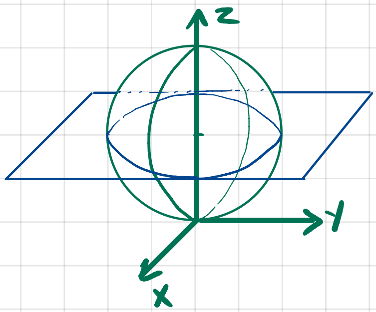
$x = r\cos\theta = \rho\sin\phi\cos\theta$

$z = \rho\cos\phi$

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^{\pi/4} d\rho \, d\phi \, d\theta$$

b) $$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\infty} d\rho \, d\phi \, d\theta$$

c)



$$\int_0^{2\pi} \int_0^{\pi/4} \int_{1/\cos\phi}^{2\cos\phi} d\rho \, d\phi \, d\theta$$

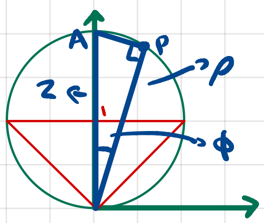
plane: $z = 1 \Rightarrow \rho\cos\phi = 1$

sphere: $x^2 + y^2 + (z-1)^2 = 1 \Rightarrow \rho^2\sin^2\phi + \rho^2\cos^2\phi - 2\rho\cos\phi + \cancel{1} = \cancel{1} \Rightarrow \rho^2 - 2\rho\cos\phi = 0$

$\Rightarrow \rho(\rho - 2\cos\phi) = 0 \Rightarrow \rho = 2\cos\phi$ or $\rho = 0$

intersection: $\frac{1}{\cos\phi} = 2\cos\phi \Rightarrow \cos^2\phi = \frac{1}{2} \Rightarrow \cos\phi = \pm\frac{\sqrt{2}}{2} \Rightarrow \phi = \frac{\pi}{4}$

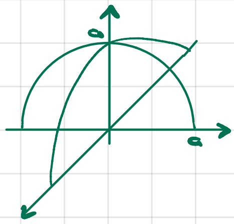
* Alternative way to obtain sphere equation in spherical coord.



$\cos\phi = \frac{\rho}{2} \Rightarrow \rho = 2\cos\phi$

5B-2

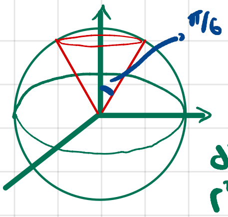
mass $= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} 1\,\rho^2 \sin\phi \,d\rho\, d\phi\, d\theta = \frac{2a^3\pi}{3}$



$$\bar{z} = \frac{\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} \rho^3 \cos\phi \sin\phi \,d\rho\, d\phi\, d\theta}{2a^3\pi/3} = \frac{\frac{a^4\pi}{4}}{\frac{2a^3\pi}{3}} = \frac{3a}{8}$$

$\bar{x} = \bar{y} = 0$ by symmetry

5B-3



$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^{a} \underbrace{\rho^2\sin^2\phi}_{\text{distance}^2}\, \underbrace{\rho\cos\phi}_{\delta}\, \underbrace{\rho^2 \sin\phi \,d\rho\, d\phi\, d\theta}_{dV} = \frac{a^6\pi}{192}$$

distance to central axis (z-axis) : r

$r^2 = x^2 + y^2 = \rho^2 \sin^2\phi$

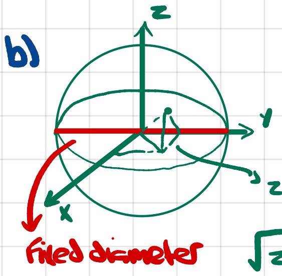
density : distance to xy plane $= z = \rho\cos\phi$

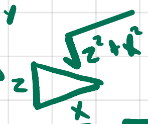
5B-4

a) mass $= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \sin\phi \,d\rho\, d\phi\, d\theta = \frac{4a^3\pi}{3}$

$$\text{Avg dist.} = \frac{\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho \cdot \rho^2 \sin\phi \,d\rho\, d\phi\, d\theta}{4a^3\pi/3} = \frac{a^4\pi}{4a^3\pi/3} = \frac{3}{4}a$$

b)





$$\frac{\int_0^{2\pi}\int_0^{\pi}\int_0^{a} (\rho^2\cos^2\phi + \rho^2\sin^2\phi\cos\theta)^{1/2} \rho^2 \sin\phi \,d\rho\, d\phi\, d\theta}{4a^3\pi/3} =$$

$\sqrt{z^2+x^2} = \sqrt{\rho^2\cos^2\phi + \rho^2\sin^2\phi\cos^2\theta}$

Alternatively, using z-axis as fixed diameter

$\sqrt{x^2+y^2} = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$

$= \sqrt{\rho^2\sin^2\phi} = \rho\sin\phi$

$$\Rightarrow \frac{\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho\sin\phi\, \rho^2 \sin\phi \,d\rho\, d\phi\, d\theta}{4a^3\pi/3} = \frac{a^4\pi^2/4}{4a^3\pi/3} = \frac{3a\pi}{16}$$

c) distance to plane xy : $z = \rho\cos\phi$

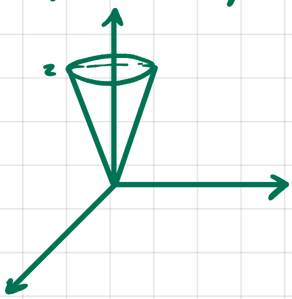
$$\frac{\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho\cos\phi \cdot \rho^2 \sin\phi \,d\rho\, d\phi\, d\theta = 0}{4a^3\pi/3}$$

5C-2 cone: $z^2 = 4(x^2+y^2)$ plane $z=2 \Rightarrow \rho\cos\phi = 2$

$\cancel{\rho}\cos^2\phi = 4r^2 = 4\cancel{\rho}^2\sin^2\phi$

$\Rightarrow \left(\frac{\sin\phi}{\cos\phi}\right)^2 = \frac{1}{4} \Rightarrow \tan\phi = \pm\frac{1}{2}$

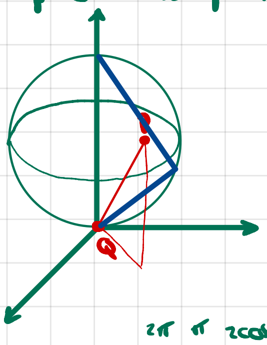
$\phi = \tan^{-1}(1/2)$



$$\int_0^{2\pi}\int_0^{\tan^{-1}(1/2)}\int_0^{2/\cos\phi} 1\cos\phi\sin\phi\, d\rho\, d\phi\, d\theta = 4\pi G\left(1-\frac{2\sqrt{5}}{5}\right)$$

5C-3 Solid sphere, radius 1, unit point mass on its surface at point Q

Sphere density at $P(x,y,z)$ is $|PQ|^{-1/2}$



$\vec{PQ} = \langle x,y,z\rangle$

$|\vec{PQ}| = \sqrt{x^2+y^2+z^2} = \rho$

sphere: $\cos\phi = \rho/2 \Rightarrow \rho = 2\cos\phi$

$$F_z = G\int_0^{2\pi}\int_0^{\pi}\int_0^{2\cos\phi} \rho\cdot\cos\phi\sin\phi\, d\rho\, d\phi\, d\theta = G\,\frac{8\sqrt{2}\,\pi}{5}$$

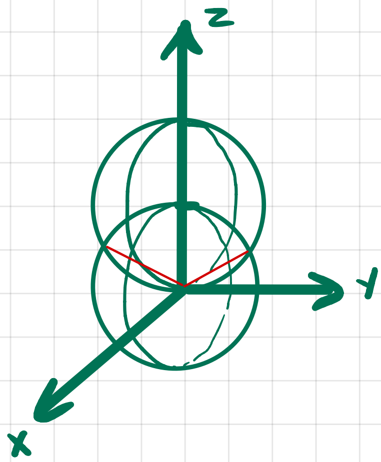
5C-4 sphere above: $x^2+y^2+z^2=1 \Rightarrow \rho^2=1$

" below: $x^2+y^2+z^2=2z \Rightarrow \rho^2 = 2\rho\cos\phi \Rightarrow \rho(\rho-2\cos\phi)=0 \Rightarrow \rho = 2\cos\phi$

$\delta = 1$ $x^2+y^2+(z-1)^2=1$

intersection: $\rho = 2\cos\phi = 1 \Rightarrow \cos\phi = 1/2 \Rightarrow \phi = \frac{\pi}{3}$

$$\int_0^{2\pi}\int_0^{\pi/3}\int_0^{1} \cos\phi\sin\phi\, d\rho\, d\phi\, d\theta + \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2\cos\phi} \cos\phi\sin\phi\, d\rho\, d\phi\, d\theta$$



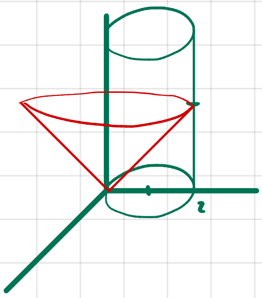
Problem 1

solid region:

→ inside cylinder: $x^2+(y-1)^2=1 \Rightarrow x^2+y^2-2y=0 \Rightarrow r^2=2y \Rightarrow r^2-2r\sin\theta=r(r-2\sin\theta)=0$

→ above $z=0$, plane

→ below cone: $z=\sqrt{x^2+y^2}$



a) $\displaystyle\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r} r\,dz\,dr\,d\theta = \frac{32}{9}$

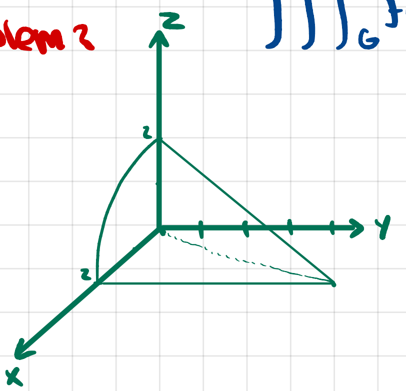
b) $\delta=1 \Rightarrow m = 32/9$

$$\bar{z} = \frac{\displaystyle\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r} z\,r\,dz\,dr\,d\theta}{32/9} = \frac{3\pi/4}{32/9} = \frac{27\pi}{128}$$

Problem 2

$$\iiint_G f(x,y,z)\,dV = \int_0^2\int_0^{2x}\int_0^{\sqrt{4-x^2}} f(x,y,z)\,dz\,dy\,dx$$

a)



$z^2+x^2=4 \Rightarrow z^2+\frac{y^2}{4}=4 \Rightarrow 4z^2+y^2=16$

$y=2x$

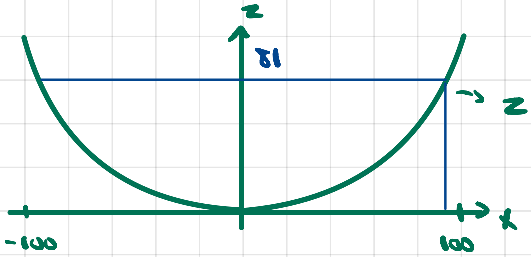
$\Rightarrow z^2=\frac{16-y^2}{4} \Rightarrow z=\pm\sqrt{4-y^2/4}$

b) $\displaystyle\int_0^2\int_0^{\sqrt{4-z^2}}\int_0^{2x} f(x,y,z)\,dy\,dx\,dz$

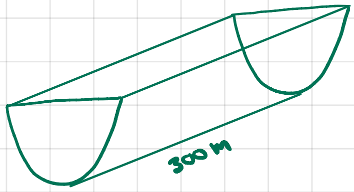
c) $\displaystyle\int_0^4\int_0^{\sqrt{4-y^2/4}}\int_{y/2}^{\sqrt{4-z^2}} f(x,y,z)\,dx\,dz\,dy$

d) Using Maple, all the integrals equal 16/3

Problem 3



$\rightarrow z = \frac{1}{100}x^4$

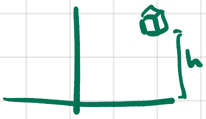


$$81 = \frac{x^4}{100} \Rightarrow x = (8100)^{1/4}$$

mass density of water = 10^3 kg/m^3

$$\cancel{m^3} \cdot \frac{kg}{\cancel{m^3}} \cdot \frac{m}{s^2} \cdot m = \frac{kg\, m^2}{s^2} = J$$

→ water is pumped into the vessel from height $z=0$



$$\underbrace{\Delta V \cdot \delta}_{dm} \cdot g \cdot h = dE$$

$$\sum_{i=1}^{n} m_i h_i g = \sum \delta g h_i \Delta V$$

$$\int_0^{300}\int_{-100}^{100}\int_{x^4/100}^{81} \delta g h \, dz\, dx\, dy \quad , \quad h(x,y,z) = z$$

$$9.8 \cdot 10^3 \int_0^{300}\int_{-\sqrt[4]{8100}}^{\sqrt[4]{8100}}\int_{x^4/100}^{81} z \, dz\, dx\, dy$$