Vectors and Matrices

IA-2 $\vec{v} = \langle \frac{1}{5}, -\frac{1}{5}, c \rangle$ $\quad |\vec{v}| = 1 \Rightarrow \left(\frac{1}{25} + \frac{1}{25} + c^2\right)^{1/2} = 1 \Rightarrow c^2 = 1 - \frac{2}{25} \Rightarrow c^2 = \frac{23}{25}$

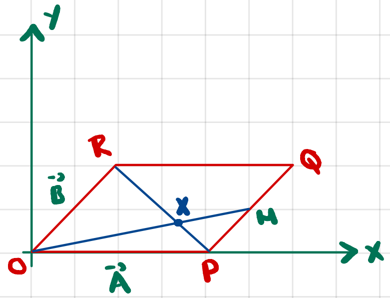
$$\Rightarrow c = \pm \frac{\sqrt{23}}{5}$$

IA-3

a) $P = (1,3,-1)$ $\quad Q = (0,1,1)$

$\vec{A} = \vec{PQ} = \langle -1,-2,2 \rangle$ $\quad |\vec{A}| = (1+4+4)^{1/2} = 3$ $\quad \text{dir}(\vec{A}) = \frac{\vec{A}}{|\vec{A}|} = \frac{\langle -1,-2,2 \rangle}{3}$

IA-12

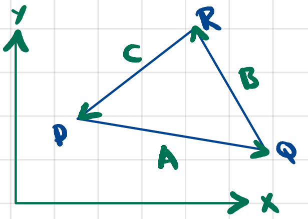


Prove $\vec{PX} = \frac{\vec{PR}}{3}$

$\vec{PR} = \vec{B} - \vec{A}$

IA-13

a)



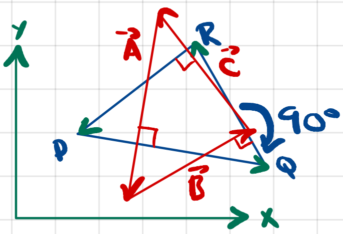
$\vec{A} = \vec{PQ}$
$\vec{B} = \vec{QR}$
$C = \vec{RP}$

$\vec{A} + \vec{B} + \vec{C}$

$\Rightarrow \vec{A} - \vec{C} - \vec{A} + \vec{C} = \vec{0}$

$\vec{B} = -\vec{C} - \vec{A}$

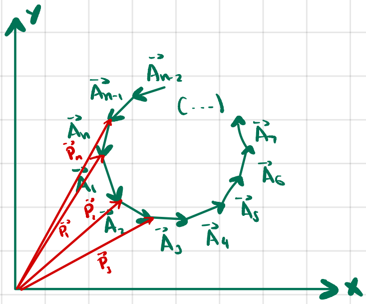
b)



rotate PQR 90° (cw or ccw). Each side of PQR is now ⊥ to the corresponding side of the rotated triangle, and vectors representing the new triangles still maintain previous relationships

$\Rightarrow \vec{A} + \vec{B} + \vec{C} = 0$

1A-14



closed polygon, vertices $P_0, \dots, P_n$ with position $\vec{P}_0, \dots, \vec{P}_n$

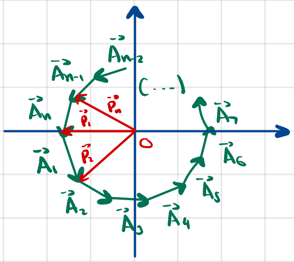
$$\vec{P}_1 + \vec{A}_1 = \vec{P}_2$$

$$\vec{P}_1 + \vec{A}_1 + \vec{A}_2 = \vec{P}_3$$

$$\vec{P}_1 + \vec{A}_1 + \vec{A}_2 + \vec{A}_3 = \vec{P}_4$$

$$\vec{P}_1 + \sum_{i=1}^{n} \vec{A}_i = \vec{P}_1 \Rightarrow \sum_{i=1}^{n} A_i = 0$$

1A-15



a) $\vec{P}_k + P_{k+(6\%(k+n/2))} = 0$

ie symmetric points relative to origin have opposite position vectors that cancel each other (sum = 0).

$\Rightarrow$ all pairs sum to $\vec{0}$

b)

6 %	1	1
	2	2
	3	3
	4	1
	5	5
	6	6
	7	1
	8	
	9	
	10	

1B-1

a) $\vec{v}_1 = \langle 1,0,-1\rangle$ $\quad$ $\vec{v}_2 = \langle 4,4,-2\rangle$

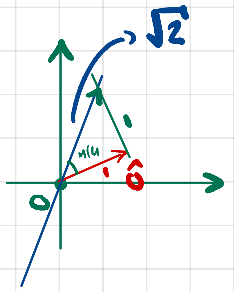
$\vec{v}_1\cdot\vec{v}_2 = 4+2 = |\vec{v}_1||\vec{v}_2|\cos\alpha$ $\qquad$ $|\vec{v}_1| = \sqrt{2}$ $\quad$ $|\vec{v}_2| = (16+16+4)^{1/2} = 6$

$$\cos\alpha = \frac{\vec{v}_1\cdot\vec{v}_2}{|\vec{v}_1||\vec{v}_2|} = \frac{6}{6\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \alpha = \pi/4$$

b) $\vec{v}_1 = \langle 1,1,2\rangle$ $\quad$ $\vec{v}_2 = \langle 2,-1,1\rangle$

$$\cos\alpha = \frac{2-1+2}{(6)^{1/2}\cdot(6)^{1/2}} = \frac{3}{6} = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}$$

1B-6



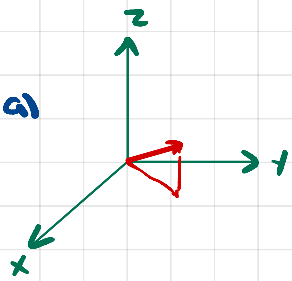
$\vec{OP}\cdot\hat{u} = c|\vec{OP}|$

projection of $\vec{OP}$ onto $\hat{u}$'s direction is a multiple of $\vec{OP}$'s length

$$\Rightarrow \frac{\vec{OP}}{|\vec{OP}|}\cdot\hat{u} = \cos\alpha = c \Rightarrow \alpha = \cos^{-1}c$$

ex: $c = \frac{\sqrt{2}}{2} \Rightarrow \alpha = \frac{\pi}{4}$

a)



$\alpha = \frac{\pi}{2} \Rightarrow c = 0 \Rightarrow$ for any vector $\vec{OP}$ on the plane that has normal vector $\hat{u}$, we have

$$\frac{\vec{OP}\cdot\hat{u}}{|\vec{OP}|} = 0$$

b) $\alpha = 0 \Rightarrow c = 1 \Rightarrow \frac{\vec{OP}\cdot\hat{u}}{|\vec{OP}|} = 1$ $\quad\Rightarrow \vec{OP}\cdot\hat{u} = |\vec{OP}|$, $\vec{OP}$ and $\hat{u}$ have same direction, namely $\hat{u}$.

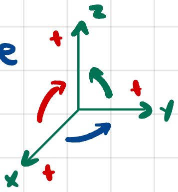
points $\vec{P}$ form a ray from origin in direction $\hat{u}$

$\alpha = \pi$ is similar: $\vec{OP}\cdot\hat{u} = -|\vec{OP}|$, $\vec{OP}$ is in direction $-\hat{u}$, and points P now form ray from origin in direction $-\hat{u}$. $c = -1$.

c) $c \notin [-1,1]$

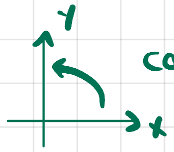
1B-7

note



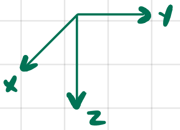
positive rotation about x, y, and z-axis are all counterclockwise

From the top (z-axis):

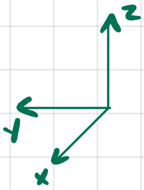


counterclockwise

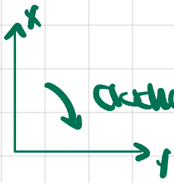
If we switch the orientation of the z-axis, we get a left-handed coordinate system:



$\rightarrow$



From positive z-axis:



clockwise

right handed: $\hat{\imath} = \langle x, y \rangle$, $\hat{\jmath} = \langle -y, x \rangle$

left-handed: $\hat{\imath} = \langle x, y \rangle$, $\hat{\jmath} = \langle y, -x \rangle$

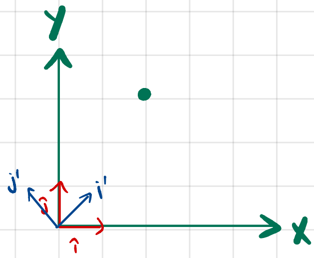
a) $i' = \dfrac{\langle 1,1 \rangle}{\sqrt{2}}$ $\quad j' = \dfrac{\langle -1,1 \rangle}{\sqrt{2}}$ $\quad i' \cdot j' = \dfrac{-1}{2} + \dfrac{1}{2} = 0$

we have $j' = \langle -i_y, i_x \rangle = i'$ rotated $90°$ cc $\Rightarrow$ right-handed coordinate system

b) $A = 2\hat{\imath} + 3\hat{\jmath}$ $\qquad A' = A\cdot\hat{\imath}\, i' + A\cdot\hat{\jmath}\, j'$

$\langle 2,3 \rangle \cdot \dfrac{\langle 1,1 \rangle}{\sqrt{2}} = \dfrac{5}{\sqrt{2}}$ $\qquad = \dfrac{\langle 5, 1 \rangle}{\sqrt{2}} = \dfrac{5}{\sqrt{2}} i' + \dfrac{1}{\sqrt{2}} j'$

$\langle 2,3 \rangle \cdot \dfrac{\langle -1,1 \rangle}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}$



c) $i' = \dfrac{1}{\sqrt{2}}\hat{\imath} + \dfrac{1}{\sqrt{2}}\hat{\jmath}$ $\qquad j' = \dfrac{-1}{\sqrt{2}}\hat{\imath} + \dfrac{1}{\sqrt{2}}\hat{\jmath} \Rightarrow \hat{\jmath} = \sqrt{2}\, j' + \hat{\imath}$

$i' = \dfrac{1}{\sqrt{2}}\hat{\imath} + j' + \dfrac{1}{\sqrt{2}}\hat{\imath} \Rightarrow \hat{\imath}\left(\dfrac{2}{\sqrt{2}}\right) = i' - j' \Rightarrow \hat{\imath} = \dfrac{\sqrt{2}}{2}(i' - j')$

$\Rightarrow \hat{\jmath} = \sqrt{2}\, j' + \dfrac{\sqrt{2}}{2}(i' - j') \Rightarrow \hat{\jmath} = \dfrac{\sqrt{2}}{2} i' + \left(\sqrt{2} - \dfrac{\sqrt{2}}{2}\right) j'$

$A' = \cancel{2}\dfrac{\sqrt{2}}{\cancel{2}}(i' - j') + 3\dfrac{\sqrt{2}}{2} i' + \dfrac{\sqrt{2}}{2} j'$

$= i'\left(\sqrt{2} + \dfrac{3}{2}\sqrt{2}\right) + j'\left(-\sqrt{2} + \dfrac{\sqrt{2}}{2}\right) = \dfrac{5}{2}\sqrt{2}\, i' - \dfrac{\sqrt{2}}{2} j'$

1B-8 $\hat{i}' = \frac{\langle 1,1,1\rangle}{\sqrt{3}}$ $\hat{j}' = \frac{\langle 1,-1,0\rangle}{\sqrt{2}}$ $\hat{k}' = \frac{\langle 1,1,-2\rangle}{\sqrt{6}}$

a) verify right handed coordinate system

The question is: relative to positive x, is positive y rotated ccw or cw?

If right-handed, then $i' \times j' = k'$, if lefthanded $i' \times j' = -k'$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{vmatrix} = \left\langle \frac{1}{\sqrt{6}}, -\left(-\frac{1}{\sqrt{6}}\right), -\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} \right\rangle = \left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right\rangle = k'$$

$\Rightarrow$ i', j', k' righthanded coordinate system

Also, $i' \cdot j' = \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} + 0 = 0$ $\quad i' \perp j'$

b) $\vec{A} = \langle 2,2,-1\rangle = 2\hat{i} + 2\hat{j} - \hat{k}$

To obtain coord. of A in i', j', k', we project $\vec{A}$ onto directions of i', j', k'.

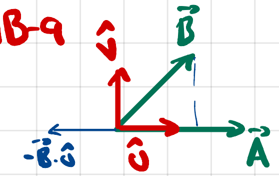
$\vec{A} \cdot i' = \langle 2,2,-1\rangle \langle 1,1,1\rangle / \sqrt{3} = (2+2-1)/\sqrt{3} = 3/\sqrt{3}$

$\vec{A} \cdot j' = \langle 2,2,-1\rangle \cdot \langle 1,-1,0\rangle / \sqrt{2} = (2-2+0)/\sqrt{2} = 0$

$\vec{A} \cdot \hat{k} = \langle 2,2,-1\rangle \langle 1,1,-2\rangle / \sqrt{6} = (2+2+2)/\sqrt{6} = 6/\sqrt{6}$

$\Rightarrow A' = \langle 3/\sqrt{3}, 0, 6/\sqrt{6}\rangle = \langle \sqrt{3}, 0, \sqrt{6}\rangle$

1B-9



$\hat{u} = \mathrm{dir}(\vec{A}) = \frac{\vec{A}}{|\vec{A}|} = \langle \cos\alpha, \cos\beta\rangle$

$\vec{B} \cdot \hat{u}$ is $\vec{B}$ coord in $\vec{A}$ direction

$\hat{v}$ $\quad \hat{v} \cdot \hat{u} = 0$ $\quad \vec{B} \cdot \vec{v}$ coord. in $\vec{v}$ direction

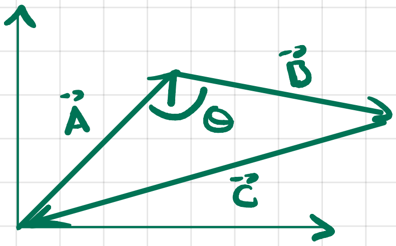
$\hat{v} = \langle -\cos\beta, \cos\alpha\rangle$

$\vec{B} = (\vec{B} \cdot \hat{u})\hat{u} + b_v \hat{v} \Rightarrow b_v \hat{v} = \vec{B} - (\vec{B} \cdot \hat{u})\hat{u}$

$\Rightarrow \vec{B} = (\vec{B} \cdot \hat{u})\hat{u} + (\vec{B} - (\vec{B} \cdot \hat{u})\hat{u})$

$$= \left[\vec{B} \cdot \frac{\vec{A}}{|\vec{A}|}\right] \frac{\vec{A}}{|\vec{A}|} + \left[\vec{B} - \left[\vec{B} \cdot \frac{\vec{A}}{|\vec{A}|}\right] \frac{\vec{A}}{|\vec{A}|}\right] = \frac{\vec{B} \cdot \vec{A}}{A \cdot A} \vec{A} + \left[\vec{B} - \frac{\vec{B} \cdot \vec{A}}{A \cdot A} \vec{A}\right]$$

1B-14 Prove law of cosines $c^2 = a^2 + b^2 - 2ab\cos\theta$



$$\vec{A} - \vec{B} = \vec{C}$$

$$(\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = |C|^2$$

$$|\vec{A}|^2 - \vec{A} \cdot \vec{B} - \vec{A} \cdot \vec{B} + |\vec{B}|^2 = |C|^2$$

$$\Rightarrow |A|^2 + |B|^2 - 2|A||B|\cos\theta = |C|^2$$

1C-4

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_2x_3^2 - x_2^2x_3) - (x_1x_3^2 - x_1^2x_3) + (x_1x_2^2 - x_1^2x_2)$$

$$(x_1 - x_2)(x_2 - x_3)(x_3 - x_1) = (x_1x_2 - x_1x_3 - x_2^2 + x_2x_3)(x_3 - x_1)$$

$$= \cancel{x_1x_2x_3} - x_1^2x_2 - x_1x_3^2 + x_1^2x_3 - x_2^2x_3 + x_2^2x_1 + x_2x_3^2 - \cancel{x_1x_2x_3}$$

$$= x_2x_3^2 - x_2^2x_3 - x_1x_3^2 + x_1^2x_3 + x_1x_2^2 - x_1^2x_2 = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix}$$