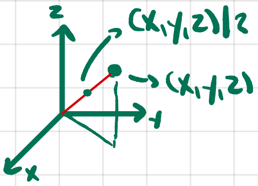
PSet 11

6A-1

a) $\vec{F} = \langle x, y, z\rangle/\rho$: unit vectors pointing radially outward from origin

b) $\vec{F} = \langle -x, 0, -z\rangle$: vectors pointing towards y-axis, larger magnitude the further away from y-axis they are

6A-2



$$\frac{-\langle x, y, z\rangle}{2}$$

6A-3 rotation, constant angular velocity

$\langle 0, -z, y\rangle k$ $\qquad k\sqrt{y^2+z^2} = \omega \Rightarrow k = \dfrac{\omega}{\sqrt{y^2+z^2}}$

6A-4 plane: $3x-4y+z=2 \Rightarrow z = 2-3x+4y$

vectors parallel to plane: $\langle x, y, 3x+4y\rangle$

however, we can have vectors of form $\langle P, Q, R\rangle$, $\langle P, Q, R\rangle\langle 3, -4, 1\rangle = 0$

$P = P(x,y,z)$, $Q = Q(x,y,z)$, $R = R(x,y,z)$ $\Rightarrow \langle P, Q, 2+4Q-3P\rangle$

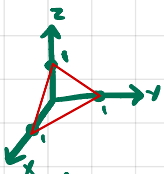
6B-1 $\vec{F} = \langle x, y, z\rangle$

Flux: $\iint_S \vec{F}\cdot\hat{n}\,dS = a\iint_S dS = a\cdot 4\pi a^2 = 4\pi a^3$

$\hat{n} = \dfrac{\langle x, y, z\rangle}{a} \Rightarrow \vec{F}\cdot\hat{n} = |\vec{F}|$, which is a on the sphere

6B-2 $F = \hat{k} = \langle 0,0,1\rangle$ flux through $x^2+y^2=1$

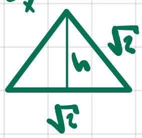
$\hat{n} = \langle x, y, 0\rangle \Rightarrow \vec{F}\cdot\hat{n} = 0$, flux $= 0$



6B-3 $\vec{F} = \hat{\imath} = \langle 1,0,0\rangle$ flux through $x+y+z=1$

$\hat{n} = \dfrac{\langle 1,1,1\rangle}{\sqrt{3}}$ $\qquad \vec{F}\cdot\hat{n} = \dfrac{\sqrt{3}}{3} \Rightarrow$ flux $=$ surface area $\cdot \dfrac{\sqrt{3}}{3}$

$= \dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{3}}{3} = \dfrac{1}{2}$



$h^2 + \dfrac{2}{4} = 2 \Rightarrow h^2 = \dfrac{3}{2} \Rightarrow h = \sqrt{3/2}$

Area $= \sqrt{\dfrac{3}{2}}\cdot\sqrt{2}\cdot\dfrac{1}{2} = \dfrac{\sqrt{3}}{2}$

6B-4 $\vec{F} = \langle 0, y, 0 \rangle$ S: half of sphere $x^2+y^2+z^2 = a^2$, $y \geq 0$

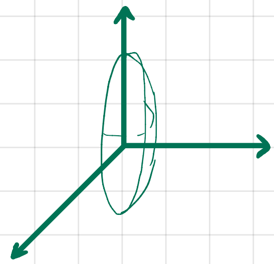
$x = a\cos\theta \quad z = a\sin\theta$

$a^2 - z^2 = a^2(1-\sin^2\theta) = a^2\cos^2\theta$

$a^2 - x^2 - z^2 = a^2 - r^2$

$y = [a^2-x^2-z^2]^{1/2} \qquad y_x = \dfrac{-x}{(a^2-x^2-z^2)^{1/2}} \qquad y_z = \dfrac{-z}{(a^2-x^2-z^2)^{1/2}}$

$$dS = \left[1 + \frac{x^2+z^2}{a^2-x^2-z^2}\right]^{1/2} dx\,dz = \left[\frac{a^2}{a^2-x^2-z^2}\right]^{1/2} dx\,dz = \frac{a}{y}\,dx\,dz$$



$$\hat{n} = \frac{\langle x, y, z \rangle}{a}$$

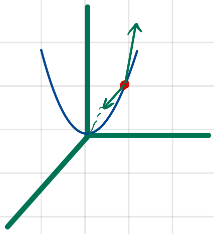
$$\vec{F} \cdot \hat{n} = \frac{y^2}{a}$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n}\, dS = \iint_S \frac{y^2}{a}\, dS = \int_0^{\pi}\int_0^{\pi} \frac{1}{a} a^2 \sin^2\phi \sin^2\theta \, a^2 \sin\phi \, d\phi\, d\theta$$

$$= \int_0^{\pi}\int_0^{\pi} a^3 \sin^3\phi \sin^2\theta \, d\phi \, d\theta = \frac{2a^3\pi}{3}$$

6B-6 $\vec{F} = \langle x, y, z \rangle$

$z = x^2+y^2$ beneath $z = 1$, $\hat{n}$ generally pointing upwards



$\langle x, y, x^2+y^2 \rangle = \vec{r}(x,y)$

$r_x = \langle 1, 0, 2x \rangle \qquad r_y = \langle 0, 1, 2y \rangle$

$$\hat{N} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = \langle -2x, -2y, 1 \rangle$$

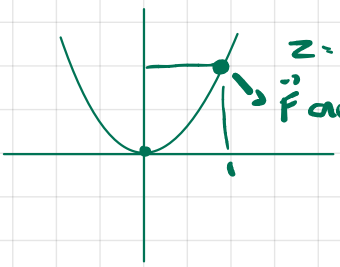
$$\hat{n} = \frac{\langle -2x, -2y, 1 \rangle}{(4x^2+4y^2+1)^{1/2}} \qquad dS = (1+4x^2+4y^2)^{1/2}\,dx\,dy$$

$$\vec{F} \cdot \hat{n} = \frac{-2x^2-2y^2+z}{(4x^2+4y^2+1)^{1/2}}$$

$$\text{Flux} = \iint_S \frac{-2x^2-2y^2+z}{\cancel{(4x^2+4y^2+1)^{1/2}}} \cancel{(1+4x^2+4y^2)^{1/2}}\,dx\,dy = \iint_S (-x^2-y^2)\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 (-r^2)\, r\, dr\, d\theta = -\int_0^{2\pi} \frac{1}{4}\, d\theta$$

$$= -\frac{1}{4} \cdot 2\pi = -\frac{\pi}{2}$$

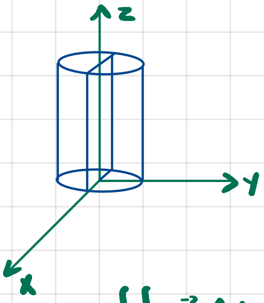


$z = x^2$, $z = x \Rightarrow x = x^2 \Rightarrow x(x-1) = 0$

$\vec{F}$ and $\hat{n}$ $\perp$

6B-8 $\vec{F} = \langle 0, y, 0 \rangle$

S: $x^2 + y^2 = a^2$ between $z=0$ and $z=h$, to right of xz plane



$\hat{n} = \dfrac{\langle x, y, 0 \rangle}{a}$

$\vec{F} \cdot \hat{n} = \dfrac{y^2}{a}$

$$\iint_S \vec{F} \cdot \hat{n}\, dS = \int_0^{\pi} \int_0^{h} \frac{y^2}{\cancel{a}}\, dz\, \cancel{a}\, d\theta = \int_0^{\pi} \int_0^{h} a^2 \sin^2\theta\, dz\, d\theta = a^2 \int_0^{\pi} \int_0^{h} \sin^2\theta\, dz\, d\theta = \frac{a^2 h \pi}{2}$$

6B-12 hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$

Avg height $= \bar{z}$

$$\text{mass} = \int_0^{2\pi} \int_0^{\pi/2} a^2 \sin\phi\, d\phi\, d\theta = 2\pi a^2$$

$$\bar{z} = \frac{\iint_S f(x,y,z)\, dS}{m} = \frac{\int_0^{2\pi} \int_0^{\pi/2} a\cos\phi\; a^2 \sin\phi\, d\phi\, d\theta}{m} = \frac{a^3 \int_0^{2\pi} \int_0^{\pi/2} \sin\phi \cos\phi\, d\phi\, d\theta}{m} = \frac{a^{\cancel{3}}\cancel{\pi}}{2\cancel{\pi}\cancel{a^2}} = \frac{a}{2}$$

$f(x,y,z) = z$

6C-1

a) $\vec{F} = \langle x^2 y, xy, xz \rangle$ $\quad$ $\text{div}\, \vec{F} = 2xy + x + x = 2x(1+y)$

6C-2 $\vec{F} = \rho^n \langle x, y, z \rangle$ $\quad$ $\rho = \rho(x,y,z)$

$$\frac{\partial}{\partial x}(\rho^n x) = n\rho^{n-1} \rho_x x + \rho^n = n\rho^{n-1} \frac{x}{\rho} x + \rho^n$$

$$\text{div}\, \vec{F} = n\rho^{n-2}(x^2 + y^2 + z^2) + 3\rho^n = 0 \Rightarrow n\rho^n + 3\rho^n = 0 \Rightarrow \rho^n (n+3) = 0 \rightarrow n = -3$$

6C-3 $\vec{F} = \langle x, y, z\rangle$ S: upper half sphere radius a, center origin, circular disk radius a

S is closed and piecewise smooth bounding region T

outer unit normal vector field $\Rightarrow \int_S \vec{F}\cdot\hat{n}\, dS = \iiint \nabla\cdot\vec{F}\, dV$

$= \langle x, y, z\rangle / \rho$ on half sphere

$\langle 0, 0, 1\rangle$ on bottom disk

F cont. diff on T

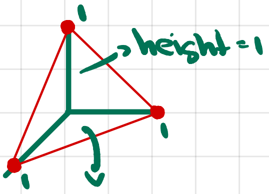
Verification

$$\vec{F}\cdot\hat{n} = \frac{\langle x,y,z\rangle\langle x,y,z\rangle}{a} = \frac{a^2}{a} = a \Rightarrow \iint_S a\, dS = \int_0^{2\pi}\int_0^{\pi/2} a\, a^2 \sin\phi\, d\phi\, d\theta = 2a^3\pi$$

$\vec{F}\cdot\hat{n} = \langle x, y, 0\rangle\langle 0,0,1\rangle = 0$ ie flux $= 0$ on bottom disk

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} 3\, d\rho\, d\phi\, d\theta = 3 \cdot \tfrac{1}{2}\text{volume(sphere)} = \tfrac{1}{2} 4\pi a^3 = 2\pi a^3$$

$\text{div}\,\vec{F} = 1 + 1 + 1 = 3$



6C-5 $\vec{F} = \langle x, z^2, y^2\rangle$

$$\iint_S \vec{F}\cdot\hat{n}\, dS = \iiint_V dV = \int_0^1\int_0^{1-z}\int_0^{1-y-z} dx\, dy\, dz = \frac{1}{6}$$

$\text{div}\,\vec{F} = 1 + 0 + 0 = 1$

base $= 1\cdot 1/2 = \frac{1}{2}$

$(x-1) + y + z = 0$

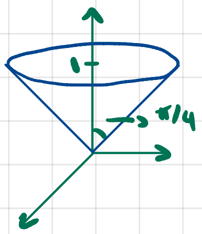
$x + y + z = 1$

$V = \frac{1}{3} b\cdot h = \frac{1}{3}\cdot\frac{1}{2}\cdot 1 = \frac{1}{6}$

6C-6 cone $z^2 = x^2 + y^2$ below, $z = 1$ above, flux?

$\vec{F} = \langle 0, 0, z\rangle$ $\text{div}\,\vec{F} = 1$

$$\iint_S \vec{F}\cdot\hat{n}\, dS = \iiint_V dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{1/\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{\pi}{3}$$



$x^2 + y^2 = 1$, $z = 1$, intersection cone/plane $\Rightarrow \rho^2\sin^2\phi = 1$, $\rho\cos\phi = 1$

cone: $\rho^2\cos^2\phi = \rho^2\sin^2\phi \Rightarrow \sin\phi = \pm\cos\phi \Rightarrow \phi = \pi/4$

6C-7

a) S: closed surface, portion of cylinder $x^2+y^2=1$ between $z=0, z=1$

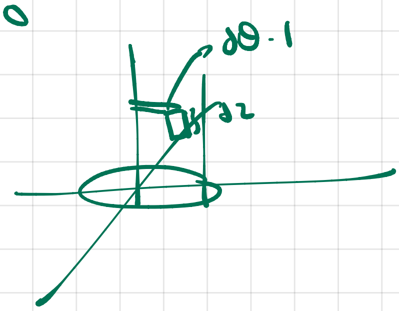
$\vec{F} = \langle x^2, xy, 0\rangle \quad \hat{n} = \langle x, y, 0\rangle / r \quad \vec{F}\cdot\hat{n} = x^3 + xy^3 = (r\cos\theta)^3 + r\cos\theta(r\sin\theta)^3$
$= \cos^3\theta + \cos\theta\sin^3\theta$

$dS = 1\cdot d\theta\, dz$

$$\text{Flux} = \iint_S \vec{F}\cdot\hat{n}\, dS = \int_0^{2\pi}\int_0^1 (x^3+xy^3)\,dz\,d\theta = \int_0^{2\pi}\int_0^1 (\cos^3\theta + \cos\theta\sin^3\theta)\,dz\,d\theta = 0$$

$\text{Flux}_{bottom} = 0 \qquad \text{Flux}_{top} = 0$

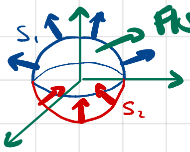
$\hat{n} = \langle 0,0,-1\rangle,\ \vec{F}\cdot\hat{n} = 0 \qquad \hat{n} = \langle 0,0,1\rangle\ \ \vec{F}\cdot\hat{n} = 0$



$\operatorname{div}\vec{F} = 2x + x + 0 = 3x$

$$\int_0^{2\pi}\int_0^1\int_0^1 3r\cos\theta\, r\,dz\,dr\,d\theta = 0$$

6C-8



$\text{Flux} = 0 = \iiint_V \operatorname{div} F\, dV$

$\operatorname{div}\vec{F} = 0$

a) S_1 and S_2 are hemispheres. They don't include the disk on the xy-plane, only the outer shell of the sphere that they form together.

$\int_{S_2} \vec{F}\cdot\hat{n}_1 dS = -\int_{S_2'} \vec{F}\cdot\hat{n}_2 dS$ where S_2' is S_2 but with normal vectors pointing outwards (ie S_2' has a different orientation of the normal vector.

$$\int_S \vec{F}\cdot\hat{n}\,dS = \iint_{S_1}\vec{F}\cdot\hat{n}\,dS + \iint_{S_2'}\vec{F}\cdot\hat{n}\,dS = \iiint_V \operatorname{div} F\, dV = 0$$

$$\Rightarrow \iint_{S_1} F\cdot\hat{n}\,dS = -\iint_{S_2'}\vec{F}\cdot\hat{n}\,dS = \iint_{S_2}\vec{F}\cdot\hat{n}\,dS$$

b) Given a closed surface S and $\operatorname{div}\vec{F} = 0$, we can split S into two non-overlapping surfaces S_1 and S_2.

$$\iint_S \vec{F}\,d\vec{S} = 0 = \iint_{S_1}\vec{F}\,d\vec{S} + \iint_{S_2}\vec{F}\,d\vec{S} \Rightarrow \iint_{S_1}\vec{F}\,d\vec{S} = -\iint_{S_2}\vec{F}\,d\vec{S} = \iint_{S_2'}\vec{F}\,d\vec{S}$$ where

S_2' is S_2 with reversed orientation.

Flux "out" of S through S_1 equals flux "in" through S_2'.

6C-10 Flow field $\vec{F}$ incompressible : $\iint_S \vec{F} \, d\vec{S} = 0$ for all closed surfaces S.

Assume $\vec{F}$ cont. diff.

show: $\vec{F}$ flow field incompr. $\Leftrightarrow$ $\operatorname{div} \vec{F} = 0$

$\vec{F}$ incompr. $\Rightarrow \iint_S \vec{F} \, d\vec{S} = 0 \Rightarrow \iiint_T \operatorname{div} \vec{F} \, dV = 0$

$\Rightarrow \iiint_T (F_x + F_y + F_z) \, dV = \iiint_T F_x \, dV + \iiint_T F_y \, dV + \iiint_T F_z \, dV$

$\iiint_T F_x \, dV = \iint_S F \, dS$, by divergence theorem $\Rightarrow \iiint_T F_x \, dV = 0$

$\Rightarrow \operatorname{div} \vec{F} = 0$

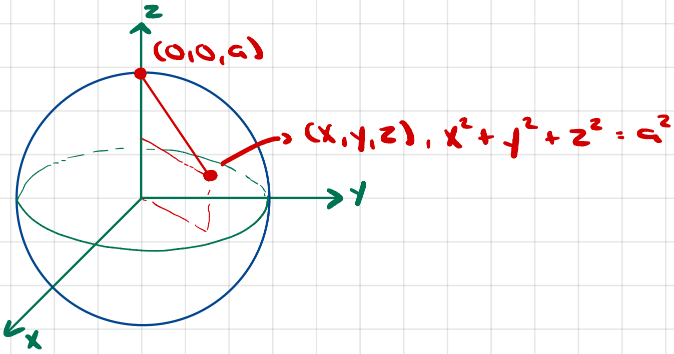
$\Rightarrow$ analogously $\iiint_T F_y \, dV = \iiint_T F_z \, dV = 0$

$\operatorname{div} \vec{F} = 0 \Rightarrow \iiint_V \operatorname{div} \vec{F} \, dV = 0 \Rightarrow \iint_S \vec{F} \, d\vec{S} = 0 \Rightarrow \vec{F}$ incompressible

6C-11 $\vec{F} = \langle x, y, z \rangle$ $\quad \operatorname{div} \vec{F} = 3$

$\iint_S \vec{F} \cdot \hat{n} \, dS = \iiint_T 3 \, dV = 3$ volume(T)

Problem 1



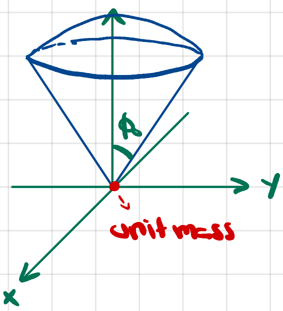
distance $= |\langle x, y, z-a \rangle| = (x^2+y^2+(a-z)^2)^{1/2} = f(x,y,z)$

$f(\rho,\phi,\theta) = [x^2+y^2+z^2+a^2-2az]^{1/2} = [a^2+a^2-2a\,a\cos\phi]^{1/2} = [2a^2(1-\cos\phi)]^{1/2}$

Avg distance $= \dfrac{1}{4\pi a^2} \iint_S (x^2+y^2+(a-z)^2)^{1/2}\, dS$

$= \dfrac{1}{4\pi a^2} \int_0^{2\pi}\int_0^{\pi} \sqrt{2}\, a \sqrt{1-\cos\phi}\; a^2 \sin\phi \, d\phi \, d\theta = \dfrac{1}{\cancel{4\pi a^2}} \cdot \dfrac{\overset{4}{\cancel{16}}\, a^{\cancel{3}}\, \cancel{\pi}}{3} = \dfrac{4a}{3}$

Problem 2



$\delta(x,y,z) = 1$

$\iiint_V G \dfrac{\cos\phi}{\rho^2}\, \delta \, dV$

$= G \int_0^{2\pi}\int_0^{\phi_0}\int_0^{a} \dfrac{\cos\phi}{\cancel{\rho^2}}\, \cancel{\rho^2} \sin\phi \, d\rho\, d\phi\, d\theta = G a \pi \sin^2(\phi_0)$

Problem 3 $a=\sqrt{2}$ $\phi_0=\pi/4$ $\vec{F}=\langle 0,0,z\rangle$

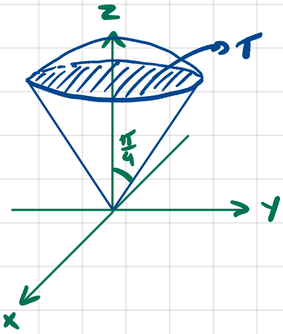
a)

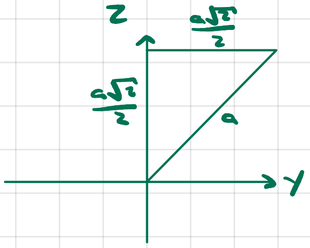
cone: $z=r$

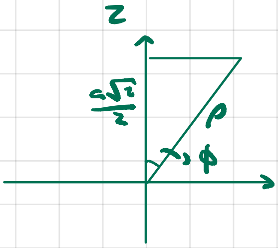
sphere: $x^2+y^2+z^2=r^2+z^2=a^2$

intersection: $2r^2=a^2 \Rightarrow r^2=a^2/2$

$$\Rightarrow r=\frac{a\sqrt{2}}{2},\quad z=r=\frac{a\sqrt{2}}{2}$$







$\cos\phi=\frac{a\sqrt{2}}{2\rho}$ $\Rightarrow \rho=\frac{a\sqrt{2}}{2\cos\phi}$

$\hat{n}=\langle 0,0,1\rangle$ $\vec{F}\cdot\hat{n}=z$

$$\text{Flux}=\iint_T \vec{F}\cdot\hat{n}\,dS=\int_0^{2\pi}\int_0^{\frac{a\sqrt{2}}{2}}\frac{a\sqrt{2}}{2}\,r\,dr\,d\theta=\frac{a^3\pi\sqrt{2}}{4}$$

with $a=\sqrt{2}$, Flux $=\pi$

b) U: conical bottom, S: spherical cap, Flux through U, S

U and S form a closed surface. $\operatorname{div}\vec{F}=1$

$$\text{Flux}_{cap}-\text{Flux}_{disk}=\text{Volume(spherical cap)} \Rightarrow \iint_S \vec{F}\cdot\hat{n}\,dS=\pi+\int_0^{2\pi}\int_0^{\pi/4}\int_{\frac{1}{\cos\phi}}^{\sqrt{2}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\pi+2\pi\left[\frac{2\sqrt{2}}{3}-\frac{5}{6}\right]=\frac{4\sqrt{2}\pi}{3}-\frac{2\pi}{3}$$

$$\text{Flux}_{\substack{cone\\bottom}}+\text{Flux}_{disk}=\text{Volume(cone)}$$

$$\Rightarrow \iint_U \vec{F}\cdot\hat{n}\,dS=\frac{1}{3}\pi\left(\frac{a\sqrt{2}}{2}\right)^2\cdot\frac{a\sqrt{2}}{2}-\pi=\frac{a^3\pi\sqrt{2}}{12}-\pi=\frac{\pi}{3}-\pi=-\frac{2\pi}{3}$$

$$\Rightarrow \iint_S \vec{F}\,d\vec{S}+\iint_U \vec{F}\,d\vec{S}=\frac{4\sqrt{2}\pi}{3}-\frac{4\pi}{3}$$

$$\text{Flux}=V(\text{spherical cap})+V(\text{cone})=V(\text{entire shape})=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\frac{4\sqrt{2}\pi}{3}-\frac{4\pi}{3}$$

c) $$\iint_U \vec{F}\cdot\hat{n}\,dS=\int_0^{2\pi}\int_0^{1} z\,r\,dz\,d\theta=\int_0^{2\pi}\int_0^{1} z^2\,dz\,d\theta=\int_0^{2\pi}\frac{1}{3}\,d\theta=\frac{2\pi}{3}$$

Problem 4

$f(x,y,z) = \frac{1}{\rho} = \rho^{-1} \qquad \rho = (x^2+y^2+z^2)^{1/2}$

$\frac{d}{dx}\rho^{-1} = \frac{-1}{\rho^2}\cdot\frac{1}{2\rho}\cdot 2x = \frac{-x}{\rho^3}$

$\frac{d}{dx}\rho = \frac{x}{\rho}$

a) $\vec{F} = \nabla f = \left< \frac{-x}{\rho^3}, \frac{-y}{\rho^3}, \frac{-z}{\rho^3} \right>$

$\frac{d}{dx}(-x\rho^{-3}) = -\rho^{-3} - x(-3)\rho^{-4}\frac{x}{\rho} = \frac{3x^2}{\rho^5} - \frac{1}{\rho^3}$

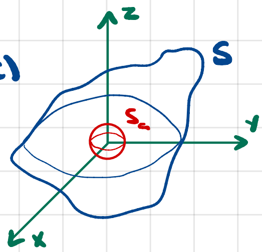
$\Rightarrow \operatorname{div}\vec{F} = \frac{3(x^2+y^2+z^2)}{\rho^5} - \frac{3}{\rho^3} = \frac{3\rho^2 - 3\rho^2}{\rho^5} = 0$

b) Note that $\vec{F}$ is not defined at origin. so $\vec{F}$ is not cont. diff in all points in the sphere.

We can't apply the Divergence theorem directly.

$\iint_S \vec{F}\,d\vec{S} = \int_0^{2\pi}\int_0^{\pi} -a^{-2}a^2 \sin\phi \, d\phi \, d\theta = -4\pi$

$\hat{n} = \frac{\langle x,y,z \rangle}{a} \qquad \vec{F}\cdot\hat{n} = \frac{-x^2-y^2-z^2}{a^4} = \frac{-a^2}{a^4} = -\frac{1}{a^2}$

c) 

$\iint_S \vec{F}\cdot d\vec{S} - \iint_{S_a} \vec{F}\cdot d\vec{S} = \iiint_R 0 \, dV = 0$

$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_{S_a} \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi} -\frac{1}{a^2}a^2 \sin\phi \, d\phi \, d\theta = -4\pi$

R: Volume between S and S_a

Problem 5

Laplacian of $f(x,y,z)$ defined

$$\nabla^2 f = f_{xx} + f_{yy} + f_{zz}$$

show $\iint_S |\nabla f| \, dS = \pm \iiint_G \nabla^2 f \, dV$

S isosurface of f, ie $f(x,y,z) = c$
G is interior of S

$\nabla f = \langle f_x, f_y, f_z \rangle$ $\quad |\nabla f| = (f_x^2 + f_y^2 + f_z^2)^{1/2}$

Because $f = c$ on S, ∇f is always normal to S at every point in S.

$\hat{n} = \pm \dfrac{\nabla f}{|\nabla f|}$ $\quad \nabla f \cdot \hat{n} = \pm \dfrac{|\nabla f|^2}{|\nabla f|} = \pm |\nabla f|$

$\Rightarrow \iint_S |\nabla f| \, dS = \iint_S \nabla f \cdot \hat{n} \, dS = \pm \iiint_G \nabla^2 f \, dV$

$\operatorname{div} \nabla f = f_{xx} + f_{yy} + f_{zz} = \nabla^2 f$