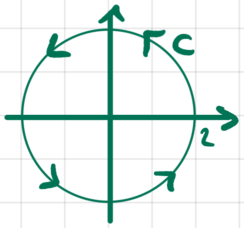
Ex1 $\oint (2y+\sqrt{9+x^3})\,dx + (5x+e^{\arctan y})\,dy$ C: posit. oriented $x^2+y^2=4$



Option 1: parametrize C, substitute x, y, dx, dy with expressions in t, dt, integrate

Option 2: Apply Green's Theorem: C is pos. oriented, smooth, simple, closed bounding a region R.

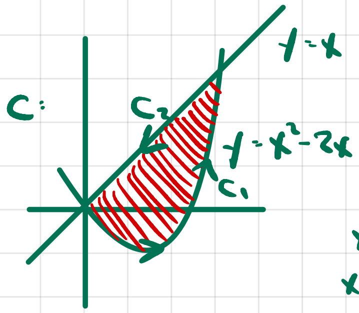
$P(x,y) = 2y+\sqrt{9+x^3}$ $P_y = 2$
$Q(x,y) = 5x+e^{\arctan y}$ $Q_x = 5$
} continuous partial derivatives everywhere

$\Rightarrow \oint P\,dx + Q\,dy = \iint_R \operatorname{curl} \vec{F}\, dA$ where $\vec{F} = \langle P, Q\rangle$

$= \iint_R Q_x - P_y\, dA = \iint_R 3\, dA = 3 \cdot \text{Area}(R) = 3\cdot\pi\cdot 4 = 12\pi$

If $\vec{F}$ is a force field $\oint \vec{F}\cdot d\vec{r}$ is the work done moving particle along C counterclockwise, 12π.

Ex2 $\oint 3xy\,dx + 2x^2\,dy$



option 1: direct calculation, two line integrals, for C_1 and C_2: $\int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$, need to parametrize C_1 and C_2

$x^2-2x = x \Rightarrow x^2-3x=0$
$x(x-3)=0 \to 0, 3$

option 2: $\oint P\,dx + Q\,dy = \iint_R \operatorname{curl}\vec{F}\,dA = \iint_R (4x-3x)\,dA = \int_0^3\int_{x^2-2x}^{x} x\,dy\,dx = \int_0^3 [x[x-x^2+2x]]\,dx$

$= \int_0^3 (3x^2-x^3)\,dx = \left(x^3 - \frac{x^4}{4}\right)\Big|_0^3 = 27 - \frac{81}{4} = \frac{27}{4}$

Ex 3 Area bounded by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

strategy: Find posit. oriented parametrization of C, curve bounding R

use vector field $\vec{F} = \langle -y(t), x(t)\rangle$

Calculate one of $\oint_C -y\,dx$, $\oint_C x\,dy$, $\frac{1}{2}\oint_C -y\,dx + x\,dy$: each gives the area of R

calculation:

C: $x = a\cos t$ $dx = -a\sin t$
$y = b\sin t$ $dy = b\cos t$

$\oint_C -y\,dx = \int_0^{2\pi} -b\sin t(-a\sin t\, dt) = \int_0^{2\pi} ab\sin^2 t\,dt = \frac{1}{2}ab\int_0^{2\pi}(1-\cos 2t)\,dt = \pi ab$

Ex 4 C : pos. oriented piecewise smooth simple closed curve enclosing $(0,0)$

$$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$P = \dfrac{-y}{x^2+y^2}$ $Q = \dfrac{x}{x^2+y^2}$ $\vec{F} = \langle P, Q \rangle$ $P_y = \dfrac{-x^2-y^2+y\cdot 2y}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

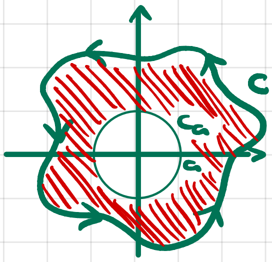
$\Rightarrow P_y - Q_x = 0$

curl $\vec{F} = 0$ except at $(0,0)$ where $\vec{F}$ is not defined $Q_x = \dfrac{x^2+y^2-x\cdot 2x}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$\Rightarrow$ Any R not enclosing $(0,0)$ has : $\int_C \vec{F}\cdot d\vec{r} = \iint_R 0\,dA = 0$

With R enclosing origin :

$\Rightarrow$ enclose $(0,0)$ with C_a



C_a : $x = a\cos t$
$y = a\sin t$ $0 \le t \le 2\pi$

Green's on region between C and C_a :

$$\oint_C P\,dx + Q\,dy - \oint_{C_a} P\,dx + Q\,dy = \iint_R 0\,dA$$