Pset 5

2C-1

a) $w = \ln(xyz)$

$$w_x = \frac{yz}{xyz} = \frac{1}{x} \qquad w_y = \frac{1}{y} \qquad w_z = \frac{1}{z}$$

$$dw = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}$$

b) $w = x^3y^3z \qquad w_x = 3x^2y^3z \quad w_y = 3x^3y^2z \quad w_z = x^3y^3$

$$dw = 3x^2y^3z\,dx + 3x^3y^2z\,dy + x^3y^3\,dz$$

c) $z = \dfrac{x-y}{x+y} \qquad z_x = \dfrac{\cancel{x}+y-(\cancel{x}-y)}{(x+y)^2} = \dfrac{2y}{(x+y)^2} \qquad z_y = \dfrac{-(x+\cancel{y})-(x-\cancel{y})}{(x+y)^2} = \dfrac{-2x}{(x+y)^2}$

$$dz = \frac{2y\,dx - 2x\,dy}{(x+y)^2}$$

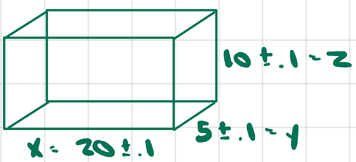
d) $w = \sin^{-1}\dfrac{u}{t} \qquad \dfrac{d}{dx}\sin^{-1}(x) = \dfrac{1}{\sqrt{1-x^2}}$

$$w_u = \frac{1}{(1-u^2/t^2)^{1/2}} \cdot \frac{1}{t} \qquad\qquad w_t = \frac{1}{(1-u^2/t^2)^{1/2}} \cdot \left(\frac{-u}{t^2}\right)$$

$$dw = \frac{t\,du - u\,dt}{t^2\sqrt{1-u^2/t^2}} = \frac{t\,du - u\,dt}{t\sqrt{t^2-u^2}}$$

$$t^2\sqrt{1-u^2/t^2} = \sqrt{t^4-t^2u^2} = \sqrt{t^2(t^2-u^2)} = t\sqrt{t^2-u^2}$$

2C-2



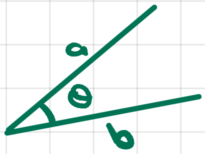
$V(x,y,z) = xyz \qquad\qquad V(20,5,10) = 1000$

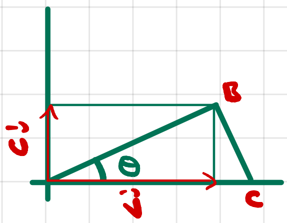
$V_x = yz \qquad V_x(20,5,10) = 50$
$V_y = xz \qquad V_y(20,5,10) = 200$
$V_z = xy \qquad V_z(20,5,10) = 100$

$$dV = 50dx + 200dy + 100dz = 350\cdot(\pm 0.1) = \pm 35$$

$$\Delta V \approx dV \le \pm 35 \Rightarrow |\Delta V| \le 35 \qquad V = 1000 \pm 35$$

2C-3

a) 



$\vec{u}+\vec{v}=\vec{OB}$
$\langle 0, a\sin\theta\rangle + \langle a\cos\theta, 0\rangle = \vec{OB}$
$= \langle a\cos\theta, a\sin\theta\rangle$

$\vec{OC} = \langle b, 0\rangle$

$A(a,b,\theta) = \frac{1}{2}\left|\vec{OB}\times\vec{OC}\right|$

$\vec{OB}\times\vec{OC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a\cos\theta & a\sin\theta & 0 \\ b & 0 & 0 \end{vmatrix} = \hat{\imath}(\cdot 0) - \hat{\jmath}(0) + \hat{k}(-ab\sin\theta)$

$|\vec{OB}|\times|\vec{OC}| = ab\sin\theta$

$\Rightarrow A(a,b,\theta) = \frac{ab\sin\theta}{2}$

$A_a = \frac{b\sin\theta}{2}$ $\quad A_b = \frac{a\sin\theta}{2}$ $\quad A_\theta = \frac{ab\cos\theta}{2}$

$\Rightarrow dA = \frac{1}{2}\left[b\sin\theta\, da + a\sin\theta\, db + ab\cos\theta\, d\theta \right]$

b) $a=1,\ b=2,\ \theta=\pi/6$ $\qquad A(1,2,\pi/6) = \frac{2}{2}\cdot\frac{1}{2} = \frac{1}{2}$

$dA = \frac{1}{2}\left[da + \frac{1}{2}db + \sqrt{3}\,d\theta \right]$

$\Rightarrow$ A most sensitive to θ, least sensitive to b.

c) $a = 1 \pm 0.02,\ b = 2 \pm 0.02,\ \theta = \pi/6 \pm 0.02$

$dA = 0.02 + 0.01 + \sqrt{3}\cdot 0.02 = 0.03 + \sqrt{3}\cdot 0.02$

$\Delta A \approx dA \quad \Rightarrow A = \frac{1}{2} \pm (0.03 + \sqrt{3}\cdot 0.02)$

2C-5

a) $\frac{1}{w} = \frac{1}{t} + \frac{1}{u} + \frac{1}{v}$

$-\frac{1}{w^2}dw = -\frac{dt}{t^2} - \frac{du}{u^2} - \frac{dv}{v^2}$

$dw = \left(\frac{1}{t} + \frac{1}{u} + \frac{1}{v}\right)^2\left(\frac{dt}{t^2} + \frac{du}{u^2} + \frac{dv}{v^2}\right)$

b) $u^2 + 2v^2 + 3w^2 = 10$

$\cancel{2}u\,du + \cancel{4}^{2}v\,dv + \cancel{6}^{3}w\,dw = 0$

$\Rightarrow dw = \frac{-u\,du - 2v\,dv}{3w}$

2E-1

a) $w = xyz \quad x = t \quad y = t^2 \quad z = t^3$

$$\frac{dw}{dt} = yz + xz \cdot 2t + xy \cdot 3t^2 = t^5 + t^4 \cdot 2t + t^3 \cdot 3t^2 = 6t^5$$

$w = t^6 \quad \frac{dw}{dt} = 6t^5$

b) $w = x^2 - y^2 \quad x = \cos t \quad y = \sin t$

$$\frac{dw}{dt} = 2x(-\sin t) - 2y\cos t = -2\sin t\cos t - 2\sin t\cos t = -4\sin t\cos t$$

$w = \cos^2 t - \sin^2 t$

$$\frac{dw}{dt} = -2\cos t\sin t - 2\sin t\cos t = -4\sin t\cos t$$

c) $w = \ln(u^2 + v^2) \quad u = 2\cos t \quad v = 2\sin t$

$$\frac{dw}{dt} = \frac{2u}{u^2+v^2} \cdot (-2\sin t) + \frac{2v}{u^2+v^2} \cdot 2\cos t$$

$$= \frac{-8\sin t\cos t + 8\sin t\cos t}{4} = 0$$

$w = \ln(4) \Rightarrow \frac{dw}{dt} = 0$

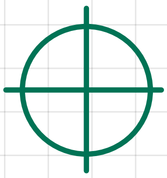
2E-2

b) $\vec{\nabla w} = \langle y, x \rangle \quad x = \cos t \quad y = \sin t$

$w = f(x(t), y(t)) \quad f_x = y \quad f_y = x$

$$\frac{dw}{dt} = y \cdot (-\sin t) + x\cos t = -\sin^2 t + \cos^2 t$$

$dw/dt = 0 \Rightarrow \sin^2 t = \cos^2 t \Rightarrow \sin t = \pm\cos t$



$\Rightarrow t = \pi/4 + k\,\pi/2 \quad k \in \mathbb{Z}$

c) $\nabla f = \langle 1, -1, 2 \rangle$ at $(1,1,1)$

$x = t \quad y = t^2 \quad z = t^3 \qquad \frac{df}{dt}$ at $t = 1$? $\qquad t = 1 \Rightarrow \langle x, y, z \rangle = \langle 1, 1, 1 \rangle$

$$\frac{df}{dt} = 1 - 2t + 6t^2$$

At $t = 1$ $\frac{df}{dt} = 1 - 2 + 6 = 5$

d) $\nabla f = \langle 3x^2y, x^3 + z, y \rangle \quad x = t \quad y = t^2 \quad z = t^3$

$$\frac{df}{dt} = 3x^2y + (x^3 + z)2t + y \cdot 3t^2$$

$$= 3t^4 + 2t^4 + 2t^4 + 3t^4 = 10t^4$$

2E-4 $w = f(x, y)$

$\nabla w = \langle 2, 3 \rangle$ at $(0, 1)$

$x = u^2 - v^2 \quad y = uv$

$$\frac{\partial w}{\partial u} = w_x \cdot x_u + w_y \cdot y_u = 2 \cdot 2 + 3 \cdot 1 = 7$$

$x_u = 2u \quad y_u = v \qquad x_v = -2v \quad y_v = u$

$w_x(1,1) = 2$

$w_y(1,1) = 3$

$x_u(1,1) = 2$

$y_u(1,1) = 1$

$x_v(1,1) = -2$

$y_v(1,1) = 1$

$$\frac{\partial w}{\partial v} = w_x x_v + w_y y_v = 2 \cdot (-2) + 3 \cdot 1 = -4 + 3 = -1$$

2E-5 $w = f(x,y)$ $x = r\cos\theta$ $y = r\sin\theta$

a) $w_\theta = w_x \cdot (-\sin\theta\, r) + w_y\, r\cos\theta$

$w_r = w_x \cos\theta + w_y \sin\theta$

$$w_\theta^2 = w_x^2 r^2\sin^2\theta + w_y^2 r^2\cos^2\theta - 2 w_x w_y r^2 \sin\theta\cos\theta$$

$$w_r^2 = w_x^2\cos^2\theta + w_y^2\sin^2\theta + 2w_x w_y \sin\theta\cos\theta$$

$$\frac{w_\theta^2}{r^2} = w_x^2 \sin^2\theta + w_y^2\cos^2\theta - 2w_x w_y \sin\theta\cos\theta$$

$$\Rightarrow w_r^2 + \frac{w_\theta^2}{r^2} = w_x^2(\sin^2\theta + \cos^2\theta) + w_y^2(\sin^2\theta + \cos^2\theta)$$
$$= w_x^2 + w_y^2$$

b) $\nabla w = \langle 2, -1 \rangle$ at $(1,1)$, what are w_r and w_θ at $r = \sqrt{2}, \theta = \pi/4$?

$x(\sqrt{2}, \pi/4) = \sqrt{2} \cdot \sqrt{2}/2 = 1$ $y(\sqrt{2}, \pi/4) = \sqrt{2}\sqrt{2}/2 = 1$

$$w_r(1,1) = w_x(x(\sqrt{2},\pi/4), y(\sqrt{2},\pi/4)) \cdot x_r(\sqrt{2},\pi/4)$$
$$= 2\cos\pi/4 - \sin\pi/4 \quad w_y(x(\sqrt{2},\pi/4), y(\sqrt{2},\pi/4)) \cdot y_r(\sqrt{2},\pi/4)$$
$$= \sqrt{2}/2$$

$w_\theta = -2r\sin\theta - r\cos\theta$ $w_\theta = -2 \cdot \sqrt{2} \cdot \frac{\sqrt{2}}{2} - \sqrt{2} \cdot \frac{\sqrt{2}}{2}$
$= -2 - 1 = -3$

Verifying:

$$2^2 + (-1)^2 = \left(\frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2} \cdot (-3)^2$$

$$4 + 1 = \frac{1}{2} + \frac{1}{2} \cdot 9 = \frac{10}{2} \Rightarrow 5 = 5$$

2E-7

$x = x(u,v)$
$y = y(u,v)$

$J = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$

$\nabla f(x,y) = \langle f_x, f_y \rangle$

$\nabla f(x(u,v), y(u,v)) = \langle f_u, f_v \rangle = \langle f_x x_u + f_y y_u, f_x x_v + f_y y_v \rangle$

$= [f_x \ f_y] \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \nabla f(x,y) \cdot J$

2E-8

a) $w = f(y/x)$ Show $x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} = 0$

$u = y/x$

$w = f(u)$

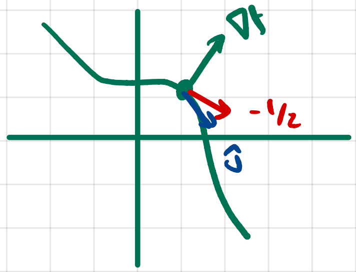
$w_x = w_u \cdot u_x = \frac{-w_u y}{x^2}$ $\Rightarrow$ $w_x x + w_y y = -\frac{w_u y}{x^{\cancel{2}}} \cdot \cancel{x} + \frac{w_u}{x} \cdot y$

$w_y = w_u \cdot u_y = \frac{w_u}{x}$ $= -\frac{w_u y}{x} + \frac{w_u y}{x} = 0$

2D-1

a) $f(x,y) = x^3 + 2y^3$ $P(1,1)$ $\vec{u} = \langle 1, -1 \rangle$ $|\vec{u}| = \sqrt{2}$

$\nabla f(x,y) = \langle 3x^2, 6y^2 \rangle$ $\nabla f(1,1) = \langle 3, 6 \rangle$



$D_{\hat{u}} f(x,y) = \langle 3x^2, 6y^2 \rangle \cdot \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$

$= \frac{3x^2 - 6y^2}{\sqrt{2}}$

$D_{\hat{u}} f(1,1) = -3/\sqrt{2}$

$x^3 + 2y^3 = 3$ level curve that (1,1) is on.

$3x^2 + 6y^2 y' = 0 \Rightarrow y' = -\frac{x^2}{2y^2} \Rightarrow y'(1,1) = -\frac{1}{2}$

b) $w = f(x,y,z) = \frac{xy}{z}$ $P(2,-1,1)$ $\vec{u} = \langle 1,2,-2\rangle$ $|\vec{u}| = \sqrt{1+4+4} = 3$

$\nabla f(x,y,z) = \langle \frac{y}{z}, \frac{x}{z}, \frac{-xy}{z^2}\rangle$ $\hat{u} = \frac{\langle 1,2,-2\rangle}{3}$

$\nabla f(2,-1,1) = \langle -1,2,2\rangle$

$D_{\hat{u}} f(-1,2,2) = \langle -1,2,2\rangle\langle 1/3, 2/3, -2/3\rangle = -\frac{1}{3} + \frac{4}{3} - \frac{4}{3} = -\frac{1}{3}$

c) $z = f(x,y) = x\sin y + y\cos x$ $P(0,\pi/2)$ $\vec{u} = \langle -3,4\rangle$ $|\vec{u}| = 5$

$\nabla f(x,y) = \langle \sin y - y\sin x, x\cos y + \cos x\rangle$ $\hat{u} = \frac{\langle -3,4\rangle}{5}$

$\nabla f(0,\pi/2) = \langle 1-0, 0+1\rangle = \langle 1,1\rangle$

$D_{\hat{u}} f(x,y) = \langle 1,1\rangle\langle -3/5, 4/5\rangle = -\frac{3+4}{5} = \frac{1}{5}$

e) $f(u,v,w) = (u+2v+3w)^2$ $P(1,-1,1)$ $\vec{u} = \langle -2,2,-1\rangle$ $|\vec{u}| = \sqrt{4+4+1} = 3$

$\nabla f = \langle 2(u+2v+3w), 2(u+2v+3w)\cdot 2, 2(u+2v+3w)\cdot 3\rangle$ $\hat{u} = \frac{\vec{u}}{3}$

$\nabla f(1,-1,1) = \langle 4,8,12\rangle$

$D_{\hat{u}} f(1,-1,1) = \langle 4,8,12\rangle\langle -2/3, 2/3, -1/3\rangle$

$= -\frac{8}{3} + \frac{16}{3} - \frac{12}{3} = \frac{-4}{3}$

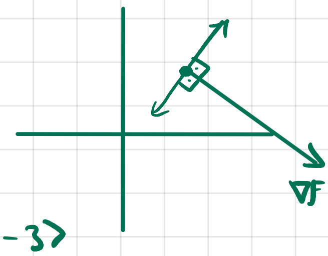
2D-2

a) $w = \ln(4x-3y)$ $P(1,1)$

i) $\hat{u} = \langle u_1, u_2\rangle$ $|\hat{u}| = 1$ $\frac{df}{ds} = D_{\hat{u}} f(x,y) = |\nabla f(x,y)|\cdot\cos\theta$

$\nabla f = \langle \frac{4}{4x-3y}, \frac{-3}{4x-3y}\rangle \Rightarrow \nabla f(1,1) = \langle 4,-3\rangle$ $|\nabla f(1,1)| = \sqrt{16+9} = 5$

$\Rightarrow \frac{df}{ds} = 5\cdot\cos\theta \Rightarrow \max D_{\hat{u}} f(1,1) = 5$, $\min D_{\hat{u}} f(1,1) = -5$



ii) max occurs in the direction of $\nabla f(1,1)$, ie $\frac{\langle 4,-3\rangle}{5}$

min " " " opposite direction of $\nabla f(1,1)$ ie $-\frac{\langle 4,-3\rangle}{5}$

iii) $\frac{df}{ds} = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \pi/2$ or $\theta = -\pi/2$

$\langle x,y\rangle\langle 4,-3\rangle = 0 \Rightarrow 4x-3y = 0 \Rightarrow \frac{x}{y} = \frac{3}{4}$

$\sqrt{x^2+y^2} = 1 \Rightarrow \sqrt{\frac{9}{16}y^2+y^2} = \sqrt{\frac{25y^2}{16}} = \frac{5}{4}|y| = 1 \Rightarrow |y| = \frac{4}{5} \Rightarrow y = \pm\frac{4}{5} \Rightarrow x = \pm\frac{3}{5}$

$\langle \frac{3}{5}, \frac{4}{5}\rangle, \langle -\frac{3}{5}, -\frac{4}{5}\rangle$ are directions where dw/ds is 0.

b) $w = f(x,y,z) = xy + yz + xz$ $\quad P(1,-1,2)$

i) $\nabla f = \langle y+z, x+z, y+x \rangle$

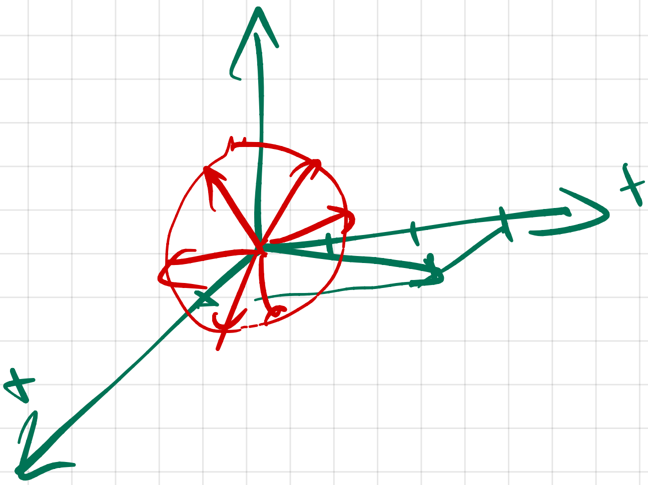
$\nabla f(1,-1) = \langle 1,3,0 \rangle$

max $\frac{df}{ds}\Big|_{\hat{u}}(1,-1,2) = |\nabla f(1,-1,2)| = \sqrt{1+9} = \sqrt{10}$

min $\frac{df}{ds}\Big|_{\hat{u}}(1,-1,2) = -\sqrt{10}$

ii) max dir $= \frac{\nabla f(1,-1,2)}{|\nabla f(1,-1)|} = \frac{\langle 1,3,0 \rangle}{\sqrt{10}}$

min dir $= \frac{-\langle 1,3,0 \rangle}{\sqrt{10}}$



iii) $\langle x,y,z \rangle \langle 1,3,0 \rangle = 0$

$x + 3y = 0$

$\frac{df}{ds} = D_{\hat{u}} f(x,y,z) = 0$ in the direction of any vector on the plane $x+3y=0$

in vector form, $\vec{u} = \langle -3y, y, z \rangle$ or $\langle -3,1,c \rangle$

the derivative in direction $\hat{u} = \frac{\langle -3,1,c \rangle}{\sqrt{10+c}}$ has $D_{\hat{u}} f = 0$

2D-3

$m - 12 =$

a) $xy^2z^3 = 12 \quad P(3,2,1)$

$\nabla f = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$

$\nabla f(3,2,1) = \langle 4 \cdot 1, 2 \cdot 3 \cdot 2 \cdot 1, 3 \cdot 3 \cdot 4 \cdot 1 \rangle = \langle 4,12,36 \rangle$

$m - 12 = 4(x-3) + 12(y-2) + 36(z-1)$

Altern. $\nabla f(3,2,1)$ is $\perp$ contour surface $\Rightarrow \langle f_x, f_y, f_z, -1 \rangle \perp$ tangent plane

$\langle x-3, y-2, z-1, m-12 \rangle \langle 1,3,9,-1/4 \rangle$

$(x-3) + 3(y-2) + 9(z-1) - \frac{1}{4}(m-12) = 0$ ← Equation of 4d plane tangent to $f(x,y,z) = m = xy^2z^3$ at $(3,2,1)$

The plane tangent to the contour surface is simply $\nabla f(3,2,1) \cdot \langle x-3, y-2, z-1 \rangle = 0$

$\Rightarrow 4(x-3) + 12(y-2) + 36(z-1) = 0 \Rightarrow x - 3 + 3y - 6 + 9z - 9 = 0 \; x + 3y + 9z = 18$

b) $x^2+4y^2+9z^2=14$ $\quad P(1,1,1)$

contour surface

View the desired contour surface of $w=f(x,y,z)=x^2+4y^2+9z^2$.
We want plane tangent to contour surface.

$\nabla f(1,1,1)$ is $\perp$ contour surface at $\vec{P}$

$\nabla f=\langle 2x,8y,18z\rangle$

$\nabla f(1,1,1)=\langle 2,8,18\rangle$

$\langle 2,8,18\rangle\langle x-1,y-1,z-1\rangle=0$

$\langle 1,4,9\rangle\langle x-1,y-1,z-1\rangle=0$

$x+4y+9z=1+4+9=14$

2D-4 $T=\ln(x^2+y^2)$

a) $P(1,2)$ $\quad \nabla T=\left\langle \frac{2x}{x^2+y^2},\frac{2y}{x^2+y^2}\right\rangle$ $\quad \nabla T(1,2)=\langle 2/5,4/5\rangle$

Direction of $\nabla T(1,2)$ is of maximum increase in T.

b) $|\nabla T(1,2)|=\left(4/25+16/25\right)^{1/2}=\frac{2\sqrt{5}}{5}$ $\quad \hat{u}=\langle 2/5,4/5\rangle / 2\sqrt{5}/5$

$D_{\hat{u}}T(1,2)=\nabla T(1,2)\cdot\frac{\nabla T(1,2)}{|\nabla T(1,2)|}=\frac{|\nabla T(1,2)|^2}{|\nabla T(1,2)|}=|\nabla T(1,2)|=\frac{2\sqrt{5}}{5}$ °C/m

$dT=\left.\frac{dT}{ds}\right|_{\hat{u}}ds \Rightarrow 0.2=\frac{2\sqrt{5}}{5}ds \Rightarrow ds=\frac{1}{2\sqrt{5}}=\frac{\sqrt{5}}{10}$ m

c) $\vec{v}=\langle 1,1\rangle$ $\quad \hat{v}=\frac{\langle 1,1\rangle}{\sqrt{2}}$

$D_{\hat{v}}T(1,2)=\langle 2/5,4/5\rangle\langle 1/\sqrt{2},1/\sqrt{2}\rangle=\frac{2}{5\sqrt{2}}+\frac{4}{5\sqrt{2}}=\frac{6}{5\sqrt{2}}=\frac{6\sqrt{2}}{5\cdot 2}=\frac{3\sqrt{2}}{5}$

$dT=\frac{3\sqrt{2}}{5}\cdot ds \Rightarrow ds=\frac{5\cdot 0.12}{3\sqrt{2}}=\frac{0.20}{\sqrt{2}}=\frac{2}{\sqrt{2}}\cdot 10$ m ≈ 0.14m

d) $\langle 2/5,4/5\rangle\langle x,y\rangle=0$

$2x+4y=0 \Rightarrow y=-\frac{x}{2}$ $\quad \pm\frac{\langle 2,-1\rangle}{\sqrt{5}}$

2D-5 $T = x^2 + 2y^2 + 2z^2$

a) isotherm: $x^2 + 2y^2 + 2z^2 = C$, ellipsoids

b) $P(1,1,1)$

$\nabla T = \langle 2x, 4y, 4z \rangle$

$\nabla T(1,1,1) = \langle 2,4,4 \rangle$

$\hat{u} = \dfrac{\langle 2,4,4 \rangle}{6}$

most rapid decrease in direction $-\dfrac{\langle 2,4,4 \rangle}{6}$

c) $-1.2 = -6 \cdot ds \Rightarrow ds = 1/5$ m

d) $\vec{v} = \langle 1,-2,2 \rangle$ $|\vec{v}| = \sqrt{9} = 3$ $\hat{v} = \vec{v}/3$

$D_v T(1,1,1) = \langle 2,4,4 \rangle \cdot \frac{1}{3}\langle 1,-2,2 \rangle = \frac{1}{3}(2-8+8) = \frac{2}{3}$

$0.10 = \frac{2}{3} ds$ $ds = 0.15$ m

2D-8

$P(x,y,z) = 30 + (x+1)(y+2)e^z$

$P(0,0,0) = 30 + (1)(2)e^0 = 32$

$\nabla P(x,y,z) = \langle (y+2)e^z, (x+1)e^z, (x+1)(y+2)e^z \rangle$

$\nabla P(0,0,0) = \langle 2,1,2 \rangle$ $|\nabla P(0,0,0)| = 3$ $\hat{u} = \dfrac{\nabla P(0,0,0)}{3}$

$D_{\hat{u}} P(0,0,0) = |\nabla P(0,0,0)| = 3$

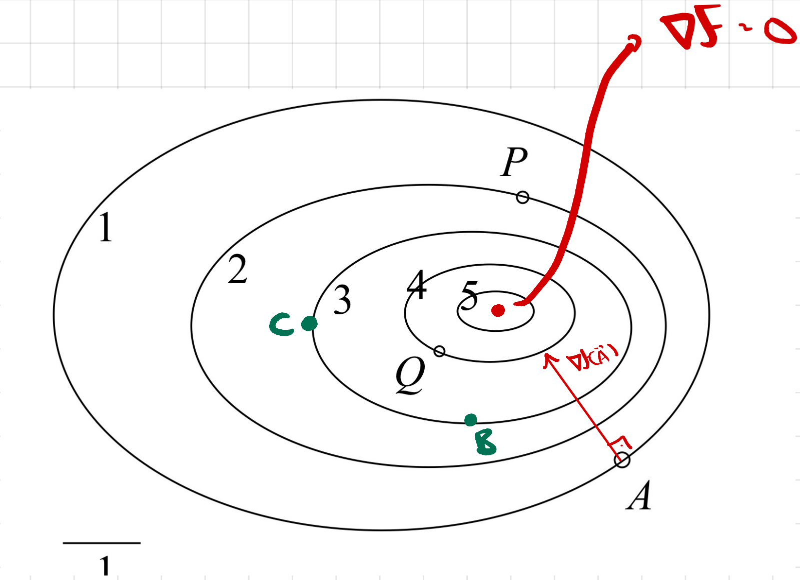
$-0.9 = D_{\hat{u}} P(0,0,0)\, ds \Rightarrow ds = \dfrac{-0.9}{3} \Rightarrow ds = -0.3$

we start at $(0,0,0)$ and move 0.3 units of distance in direction $\dfrac{-\langle 2,1,2 \rangle}{3}$

$= -\frac{3}{10}\langle 2,1,2 \rangle$ is the position vector of point closest to origin with P approx. equal to 31.1.

2D-9

a)



An estimate of the size of $\nabla f(\vec{A})$ is based on the distance between level curves 1 and 2 in the approximate direction of the gradient ($\perp$ to level curves)

8/14 distance unit between A and the next level curve, and f increases by 1

$\Rightarrow$ $1/(8/14) = \frac{14}{8}$ estimated slope = length of $\nabla f(\vec{A})$

b) $w(\vec{B}) = 3$ $\quad \frac{\partial w}{\partial x}(\vec{B}) = 0$

c) $w(\vec{C}) = 3$ $\quad \frac{\partial w}{\partial y}(\vec{C}) = 0$

d) $f_x \approx 1/(23/14) = \frac{14}{23}$

$f_y \approx 1/0.7 = \frac{10}{7}$

e) $\vec{v} = \langle 1, 1 \rangle$

$D_{\vec{v}} f(Q) \approx 1/0.5 = 2$

f) $\vec{v} = \langle 1, -1 \rangle$

$D_{\vec{v}} f(Q) = -1/1 = -1$

g) see level curves

Problem 1

$R(w,r) = k\frac{w}{r^4}$

a) $dR = \frac{k}{r^4}dw - \frac{4kw}{r^5}dr$

$\Delta r \approx dr$

b) $\frac{dR}{R} = \frac{\cancel{k}}{\cancel{r^4}} \cdot \frac{\cancel{r^4}}{\cancel{k}w}dw - \frac{4\cancel{kw}}{r^{\cancel{5}}} \frac{\cancel{r^4}}{\cancel{kw}}dr = \frac{dw}{w} - 4\frac{dr}{r}$

c) If $\frac{dw}{w} \approx \frac{dr}{r}$ then $\frac{dr}{dr}$ has larger absolute value impact.

To produce the largest relative change in R we should decrease $\frac{dr}{r}$ and increase $\frac{dw}{w}$.

$\frac{dR}{R}$ is a function of $\frac{dw}{w}$ and $\frac{dr}{r}$. The gradient is constant at $\langle 1, -4 \rangle$. This is the direction of max increase in $\frac{dR}{R}$ using linear approximation.

Problem 2

$f(x,y,z,t)$ smooth function

$\nabla f = \langle f_x, f_y, f_z \rangle$ gradient in space variables

$\vec{r} = \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ smooth curve

$\vec{v} = \vec{r}\,'(t)$

$\frac{Df}{Dt} = \frac{d}{dt} f(\vec{r}(t), t)$

Show $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f$

$\frac{d}{dt} f(\vec{r}(t), t) = \frac{d}{dt} f(x(t), y(t), z(t), t) = f_x x' + f_y y' + f_z z' + f_t = f_t + \nabla f \cdot r'$

Problem 3

$f = \rho(x,y,z,t)$ = fluid density

$\frac{D\rho}{Dt} = \frac{d}{dt}\rho(\vec{r}(t),t) = \nabla\rho\cdot\vec{v} + \frac{\partial\rho}{\partial t} = 0 \Rightarrow$ fluid incompressible

Physics of fluid motion

$\frac{D}{Dt}$ = rate of chg along moving path of some physical quantity (e.g. density) being transported by fluid currents

a) $\rho = \rho(t)$ show $\frac{D\rho}{Dt} = 0 \Leftrightarrow \rho(t)$ = constant

$\frac{D\rho(t)}{Dt} = \frac{d\rho(t)}{dt} = 0 \Rightarrow \rho(t) = k$

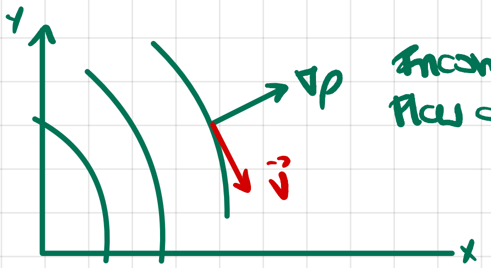
fluid: continuum of point masses

flow: particles moving along a path $r(t)$, with $\vec{v}$ tangent $\vec{v} = v(x,y,z,t)$

b) $\rho = \rho(x(t), y(t), z(t))$

$\frac{D}{Dt}\rho(t) = \frac{d}{dt}\rho(\vec{r}(t)) = \rho_x x' + \rho_y y' + \rho_z z' = \nabla\rho\cdot\vec{v} = 0$

In 2D,



Incompressible flow $\Rightarrow \vec{v}$ in direction of a level curve. Flow along level curve.

c) $\rho = (x(t), y(t))$

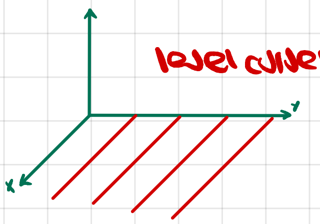
$\vec{v} = \vec{v}(x(t), y(t))$

stratified: depends only on $\vec{r}(t)$, not t, and incompressible: $\frac{d\rho}{dt} = \nabla\rho\cdot\vec{v} = 0$

steady: ρ and $\vec{v}$ don't depend explicitly on t

constant density along a path

$\Rightarrow$ path coincides with level curve

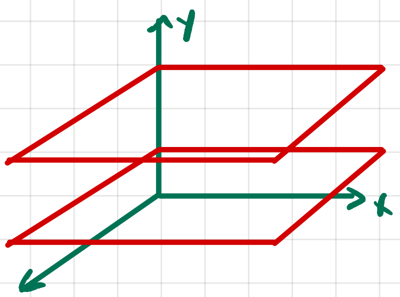


ρ is a function of y, constant for a given y.

an incompressible flow moves along a level curve, ie $\vec{r}(t) = \langle x(t), \bar{y}\rangle$, where $d\rho/dt = \cancel{\rho_x} x' + \rho_y \cancel{y'} = 0$

The paths (streamlines) are parallel to horizontal position x.

The velocity is constant for each position (x,y) in the path.



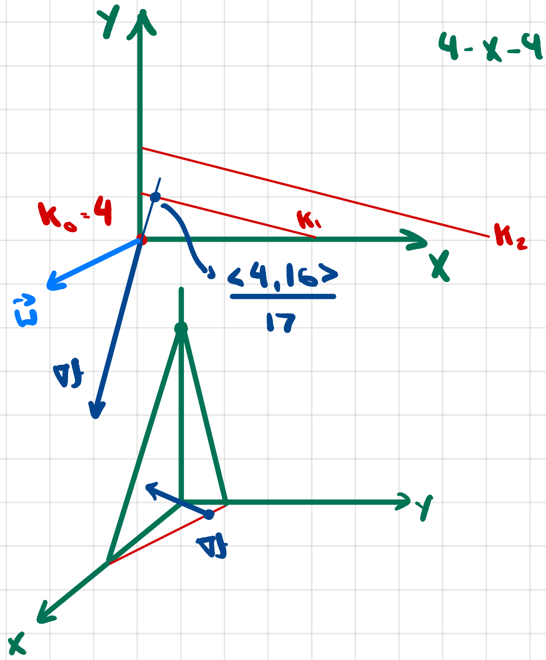
In 3D, ρ varies only with y so constant for any point on a plane $y = y^*$.

The flow is stratified, ie, collections of paths are on different planes. What makes each path steady is an additional constraint: v depends only on x, y, and z so each path, determined by $\vec{r}(t)$, is independent of time, ie, particles on a particular path follow the same trajectory over time.

Problem 4

$f(x,y) = 4 - x - 4y$

a) $4 - x - 4y = k \Rightarrow 4y = 4 - k - x \Rightarrow y = 1 - \frac{k}{4} - \frac{x}{4}$



$k_0 = 4 > k_1 = 0 > k_2$

b) $\nabla f = \langle -1, -4 \rangle$

c) line through origin in direction of ∇f

$t\langle -1,-4 \rangle = \langle -t, -4t \rangle$

level curve: $4 - x - 4y = 0 \Rightarrow 4 + t + 16t = 0 \Rightarrow 17t = -4 \Rightarrow t = \frac{-4}{17}$

point of intersection $\langle 4/17, 16/17 \rangle$

d) $\vec{w} = \langle -2, -1 \rangle$

$\hat{w} = \frac{\vec{w}}{\sqrt{5}}$

$D_{\hat{w}} f = \langle -1,-4 \rangle \langle -2,-1 \rangle \cdot \frac{1}{\sqrt{5}} = \frac{2+4}{\sqrt{5}} = \frac{6}{\sqrt{5}}$

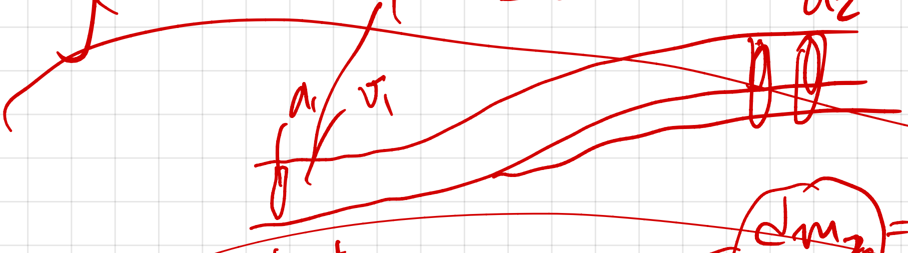
e) see sketches above

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v}\cdot\vec{\nabla}\rho = 0 \quad D$$

$$\boxed{\frac{d\rho}{dt} = 0} \Longrightarrow \rho(x(t), y(t), z(t), t) = cte?$$

$$\frac{d\rho}{dt} = 0$$

ρ_0 $\quad \rho =$



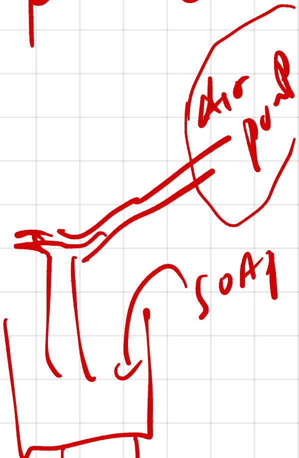
$dm_2 = \rho_2 a_2 v_2 dt$

$dm_1 = \rho_1 a_1 v_1 dt$

$\rho v^2 = cte$

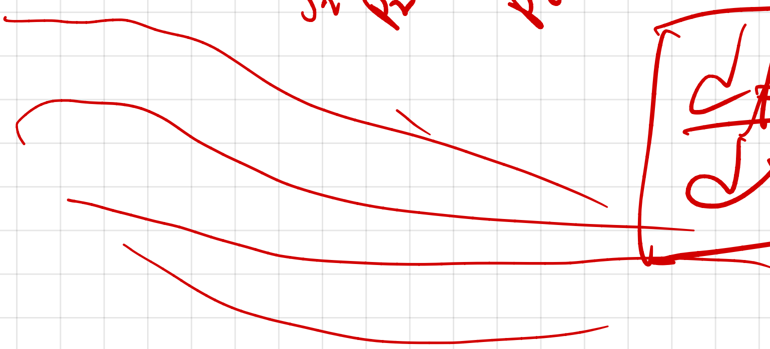
$\rho(x(t), y(t), \ldots)$

$a_1 v_1 = a_2 v_2$



p_1 tot

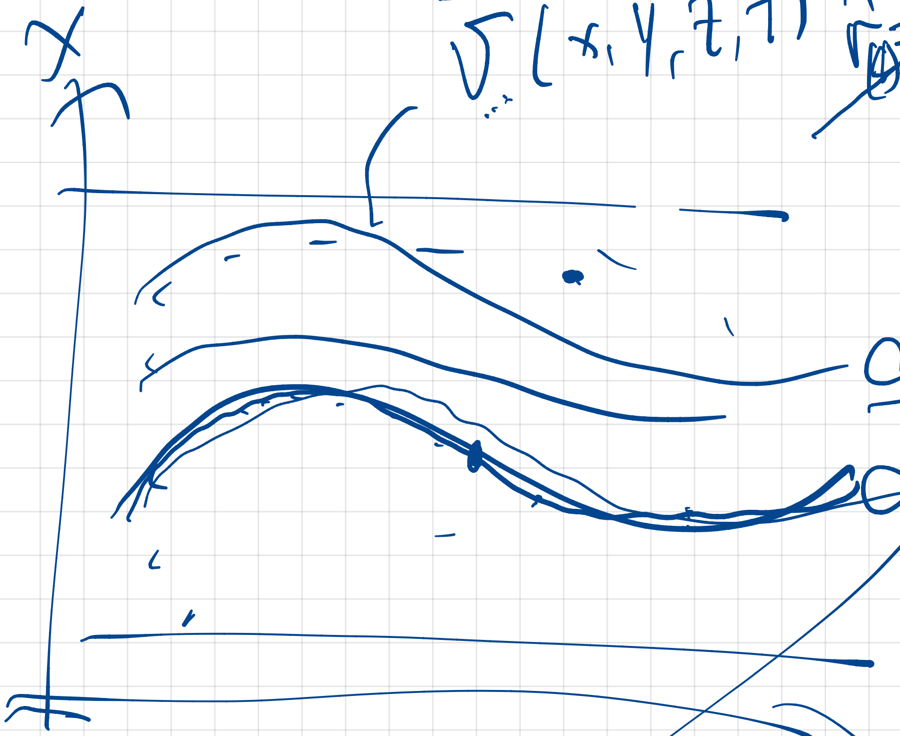
tube



$$\boxed{\frac{d\rho}{dt}} \Longrightarrow \boxed{\rho = cte}$$

$$\rho(x, y, z, t) = cte$$

$\vec{v}(x,y,z,t)$ $\quad \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$



$$\frac{d\rho}{dt} = \lim_{\Delta t \to 0} \frac{\rho(t+\Delta t) - \rho(t)}{\Delta t}$$

$t \quad t+\Delta t$

v_x

$\rho(x,y,z,t)$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \rho}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \rho}{\partial z}\frac{\partial z}{\partial t}$$

$v(x,y,z,t) \quad (x(t), y(t))$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{\nabla}\rho \cdot \vec{v}$$

$\vec{v}(t)$