Types of Functions

- single-variable real-valued $f: \mathbb{R} \to \mathbb{R}$, e.g. $f(x) = x^2$
- multi-variable real-valued $f: \mathbb{R}^n \to \mathbb{R}$, e.g. $f(x,y) = x^2 + y^2$
- single-variable vector-valued $f: \mathbb{R} \to \mathbb{R}^n$ e.g. $f(t) = \langle x(t), y(t) \rangle$
- multi-variable vector-valued $f: \mathbb{R}^n \to \mathbb{R}^m$ e.g. $f(x,y) = \langle x+y, x-y \rangle$

points → points
" → vectors ← most often
vectors → "
" → points

$$F(x_1, \dots, x_n) = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \cdots \\ f_m(x_1, \dots, x_n) \end{bmatrix}$$

component fns of F

Limits

Ex: $G: \mathbb{R}^2 \to \mathbb{R}^3$, $G(x,y) = \langle P(x,y), Q(x,y), R(x,y) \rangle$

$$\lim_{(x,y)\to(a,b)} G(x,y) = \left\langle \lim_{(x,y)\to(a,b)} P, \lim_{(x,y)\to(a,b)} Q, \lim_{(x,y)\to(a,b)} R \right\rangle$$ provided all three limits of component fns exist.

Derivatives

Ex: $G: \mathbb{R}^2 \to \mathbb{R}^3$, $G(x,y) = \langle P(x,y), Q(x,y), R(x,y) \rangle$

Derivative of G: $DG = DG(x,y) = DG(p)$ $p \in \mathbb{R}^2$ $= G'(p) = \begin{bmatrix} \partial P/\partial x & \partial P/\partial y \\ \partial Q/\partial x & \partial Q/\partial y \\ \partial R/\partial x & \partial R/\partial y \end{bmatrix}$

Jacobian matrix

* not every multiv. vector-valued fn has a derivative, because the real-valued component fns may not have all partial derivatives

e.g. $F(x,y) = \langle |y|, 3x^2y \rangle$ does not have $P_y(0,0)$.

* Jacobian Determinant

$n = m$, $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ ⇒ DF is a square matrix and we can take its determinant $\det D\vec{F}$, the Jacobian determinant of $\vec{F}$.

In the case of a transformation of variables

$T(u,v) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g(u,v) \\ h(u,v) \end{bmatrix}$ we have $\det(D\vec{F})$.

Sometimes instead of $x = g(u,v)$ we don't name the function and write $x = x(u,v)$. In this case

we'd have $\begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$ instead of "det", or we'd write $\dfrac{\partial(x,y)}{\partial(u,v)}$.