15.5 Surface Integrals

→ is to surfaces what a line integral is to curves in the plane

intuitive setup

curved, thin metal sheet shaped like surface S, with variable density $f(x,y,z)$

we want to define $\iint_S f(x,y,z)\,dS$ to be total mass of the sheet

if $f(x,y,z) \equiv 1$ then we obtain surface area

derivation

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

smooth parametric surface, (u,v) in region D in uv-plane

assumptions: continuous partials of components of $\vec{r}$

$\vec{r}_u, \vec{r}_v$ nonzero, nonparallel in D

In deriving an integral for surface area (14.8) we had

$$A \approx \sum \Delta S_i = \sum |r_u(u_i,v_i) \times r_v(u_i,v_i)|\,\Delta u \Delta v = \sum |\vec{N}(u_i,v_i)|\,\Delta u \Delta v$$

area of a parallelogram

$$m \approx \sum f(\vec{r}(u_i,v_i))\,\Delta S_i = \sum f(\vec{r}(u_i,v_i))\,|\vec{N}(u_i,v_i)|\,\Delta u \Delta v$$

Riemann sum for surface integral of f over surface S

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,|\vec{N}(u,v)|\,du\,dv$$

expanding the cross product determinant

$$\iint_D f(x(u,v), y(u,v), z(u,v)) \left[\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(u,v)}\right)^2 \right]^{1/2} du\,dv$$