PSet 3

1E-2, 1E-3, 1E-5 done in PSet 2

1E-4 line through (0,1,2) and (2,0,3), intersection with $x+4y+z=4$

strategy: find parametric eq. for line
insert into plane eq., find t

$P_0 = (0,1,2) \quad P_1 = (2,0,3)$

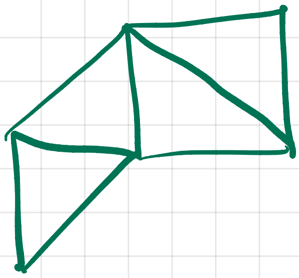
$\vec{P_0P_1} = \langle 2,-1,1\rangle$

$\langle x,y,z\rangle = \langle 0,1,2\rangle + t\langle 2,-1,1\rangle$

$x(t) = 2t$

$y(t) = 1-t \quad \Rightarrow \quad 2t+4(1-t)+2+t=4$
$2t+4-4t+2+t=4$
$-t+6=4 \Rightarrow t=2$

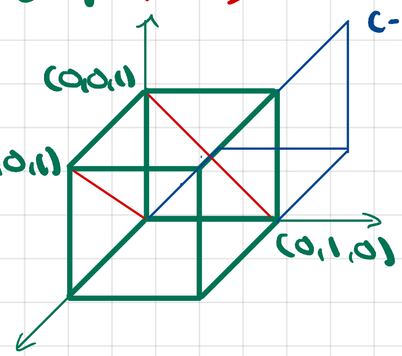
$z(t) = 2+t$



$\langle x(2), y(2), z(2)\rangle = \langle 4,-1,4\rangle$

check plane eq: $\cancel{4}+4(\cancel{-1})+4=4$

1E-7



method
1 find a vector parallel to each line
2 take cross product to obtain vector $\vec{v}$ pointing towards shortest path
3 find intersection of $\vec{v}$ parallel line and the line not passing through origin

1
line 1, through origin: $\langle x(t), y(t), z(t)\rangle = \vec{r}_1(t) = t\langle 1,0,1\rangle$

line 2: $\vec{r}_2(n) = \langle 0,0,1\rangle + n\langle 0,1,-1\rangle = \langle 0,n,1-n\rangle$

2
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} = \langle -1, -(-1), 1\rangle = \langle -1,1,1\rangle$$

shortest path line is parallel to $\langle -1,1,1\rangle$

$\vec{r}_3(m) = \vec{r}_{3_0} + m\langle -1,1,1\rangle = \langle t,0,t\rangle + \langle -m,m,m\rangle = \langle 0,n,1-n\rangle$

$t-m=0 \Rightarrow t=m$
$m=n$
$t+m = 1-n \Rightarrow \quad m+m=1-m \Rightarrow 3m=1 \Rightarrow m=1/3, t=1/3, n=1/3$

check

$\vec{r}_1(1/3) = \langle 1/3, 0, 1/3\rangle = \vec{r}_{3_0}$

$\vec{r}_3(1/3) = \langle 1/3, 0, 1/3\rangle + \langle -1/3, 1/3, 1/3\rangle$

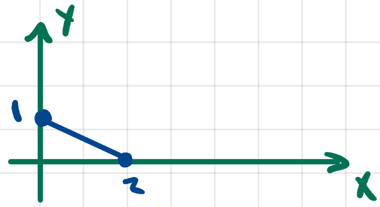
$= \langle 0, 1/3, 2/3\rangle$, intersection point

$r_2(1/3) = \langle 0, 1/3, 2/3\rangle$

II-3

a) $\vec{r} = \langle 2\cos^2 t, \sin^2 t \rangle$

t	x	y
0	2	0
$\pi/2$	0	1
π	2	0
$3\pi/2$	0	1



$x(t) = 2\cos^2(t) = 2 - 2\sin^2(t)$
$y(t) = \sin^2(t)$

$$\frac{x(t)}{2} = 1 - \sin^2(t) \Rightarrow \frac{x(t)}{2} + y(t) = 1 \Rightarrow y = 1 - \frac{x}{2}$$

b) $\vec{r} = \langle \cos 2t, \cos t \rangle$

$\cos^2 t = \frac{1}{2}(1 + \cos 2t)$

$2\cos^2 t = 1 + \cos 2t \Rightarrow \cos 2t = 2\cos^2 t - 1$

$x(t) = \cos(2t)$

$= 2\cos^2 t - 1$

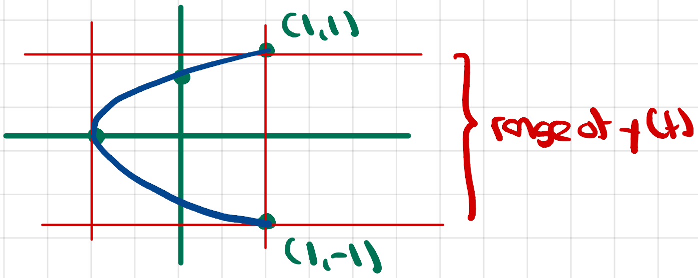
t	x	y
0	1	1
$\pi/4$	0	$\sqrt{2}/2$
$\pi/2$	-1	0
π	1	-1
$3\pi/2$	-1	0
2π	1	1

$2 \cdot y(t)^2 = 2\cos^2 t$

$\Rightarrow 2y(t)^2 - x(t) = \cancel{2\cos^2 t} - \cancel{2\cos^2 t} + 1$

$2y(t)^2 - x(t) = 1$

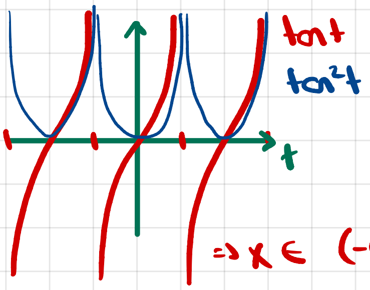
Position vector travels from $(1,1)$ at $t=0$ to $(1,-1)$ at $t=\pi$ and back to $(1,1)$ at $t = 2\pi$.



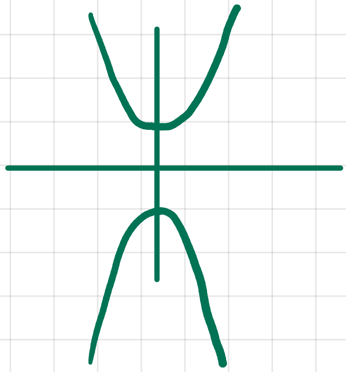
d) $\vec{r}(t) = \langle \tan t, \sec t \rangle = \left\langle \frac{\sin t}{\cos t}, \frac{1}{\cos t} \right\rangle = \frac{1}{\cos t}\langle \sin t, 1 \rangle$

$y^2(t) = 1 + x^2(t)$

$\sec^2 t = 1 + \tan^2 t$

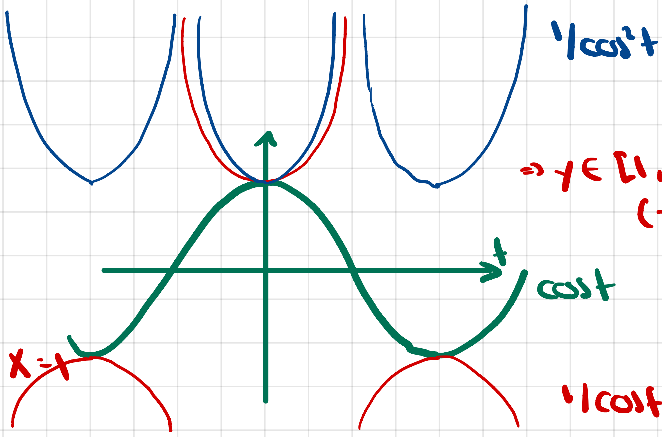


$\Rightarrow x \in (-\infty, \infty)$



$y^2 = 1 + x^2 = y = \pm\sqrt{1+x^2}$

t	x	y
0	0	1
$\pi/4$	1	$\sqrt{2}$
$\pi/2$	∞	∞
$3\pi/4$	-1	$-\sqrt{2}$
$-\pi/4$	-1	$\sqrt{2}$
$-\pi/2$	$-\infty$	∞



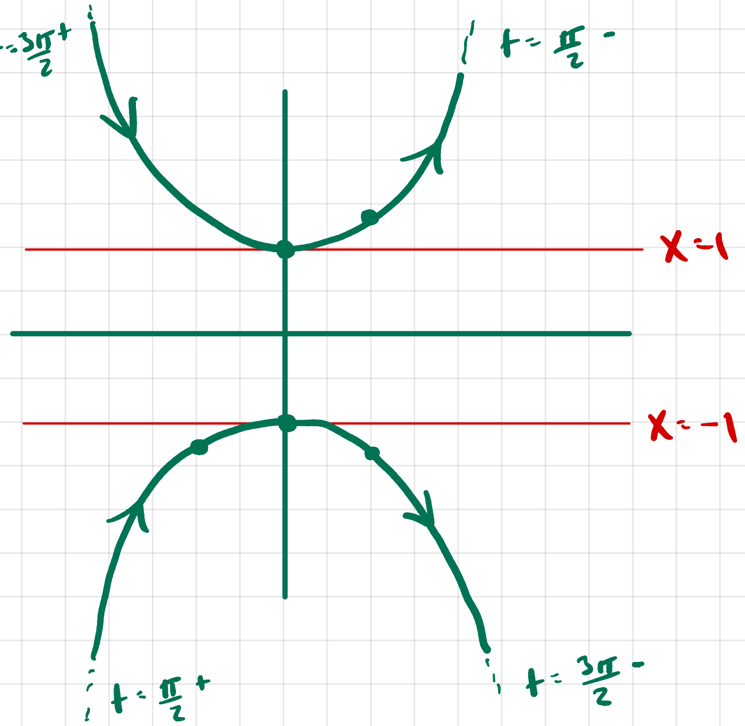
$\Rightarrow y \in [1, \infty) \cup (-\infty, -1]$

$1/\cos t = \sec t$

$\vec{r}(t) = \langle \tan t, \sec t \rangle$

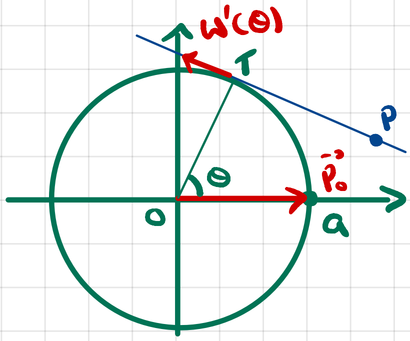
$y^2(t) = 1 + x^2(t)$

$\sec^2 t = 1 + \tan^2 t$



t	x	y
0	0	1
$\pi/4$	1	$\sqrt{2}$
$\pi/2^-$	∞	∞
$\pi/2^+$	$-\infty$	$-\infty$
$3\pi/4$	-1	$-\sqrt{2}$
π	0	-1
$5\pi/4$	1	$-\sqrt{2}$
$6\pi/4^-$	∞	$-\infty$
$6\pi/4^+$	$-\infty$	∞

II-S



$\vec{OP} = \vec{OT} + \vec{TP}$

$|\vec{TP}| = a\theta$

$\vec{OT} = \vec{w}(\theta) = \langle a\cos\theta, a\sin\theta \rangle$

$w'(\theta) = \langle -a\sin\theta, a\cos\theta \rangle$ = tangent vector

$$\vec{TP} = -\frac{w'(\theta)}{|w'(\theta)|} \cdot |\vec{P}(\theta)| = -\frac{\langle -a\sin\theta, a\cos\theta \rangle}{a} \cdot a\theta = \langle a\theta\sin\theta, -a\theta\cos\theta \rangle$$

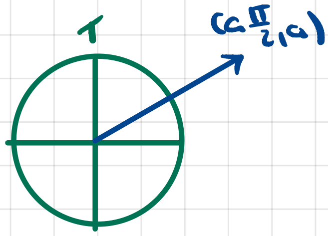
$$\vec{OP} = \vec{P}(\theta) = \langle a\cos\theta + a\theta\sin\theta, a\sin\theta - a\theta\cos\theta \rangle$$

$$= a \cdot \langle \cos\theta + \theta\sin\theta, \sin\theta - \theta\cos\theta \rangle$$

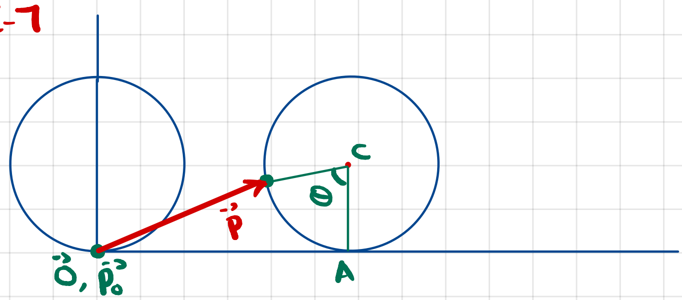
check if $\vec{P}(\theta)$ makes sense

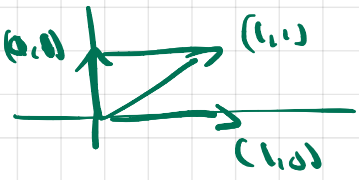
$\vec{P}(0) = a\langle 1, 0 \rangle$

$\vec{P}(\pi/2) = a\langle 0 + \frac{\pi}{2}, 1 - \frac{\pi}{2} \cdot 0 \rangle = a\langle \frac{\pi}{2}, 1 \rangle = \langle a\frac{\pi}{2}, a \rangle$

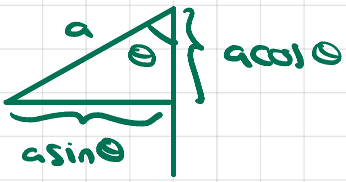


1I-7





$\vec{OP} = \vec{OA} + \vec{AC} + \vec{CP}$



$\vec{OA} = \langle a\theta, 0 \rangle$

$\vec{AC} = \langle 0, a \rangle$

$\vec{CP} = \langle -a\sin\theta, -a\cos\theta \rangle$

$\Rightarrow \vec{OP} = \vec{P}(\theta) = \langle a\theta - a\sin\theta, a - a\cos\theta \rangle$

check: does result make sense?

$\vec{P}(0) = \langle 0, 0 \rangle$

$P(\frac{\pi}{2}) = \langle a\frac{\pi}{2} - a, a - a\cdot 0 \rangle = \langle a(\frac{\pi}{2} - 1), a \rangle$

1J-1

* arc length along curve with position vector $\vec{r}(t)$, from $\vec{r}(a)$ to $\vec{r}(b)$ is defined

$$\int_a^b (x'(t)^2 + y'(t)^2 + z'(t)^2)^{1/2} dt$$

$$= \int_a^b v(t)\,dt \quad \text{where } v(t) = |\vec{v}(t)|$$

* arclength function

$$s(t) = \int_a^t v(t)\,dt$$

a) $\vec{r}(t) = \langle e^t, e^{-t} \rangle$

$\vec{v}(t) = \vec{r}'(t) = \langle e^t, -e^{-t} \rangle$

speed $= v(t) = |\vec{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2}$

$v(t) = \sqrt{e^{2t} + e^{-2t}} = \frac{ds}{dt}$ = time rate of change of arc length

unit tangent vector $= \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\langle e^t, -e^{-t} \rangle}{\sqrt{e^{2t} + e^{-2t}}}$

$\vec{a}(t) = \vec{v}'(t) = \langle e^t, e^{-t} \rangle$

b) $\vec{r}(t) = \langle t^2, t^3 \rangle$ $\quad \vec{v}(t) = \langle 2t, 3t^2 \rangle$ $\quad \vec{a}(t) = \langle 2, 6t \rangle$

$v(t) = \frac{ds(t)}{dt} = |\vec{v}(t)| = \sqrt{4t^2 + 9t^4}$

unit tangent velocity $= \frac{\langle 2t, 3t^2 \rangle}{\sqrt{4t^2 + 9t^4}}$

c) $\vec{r}(t) = \langle 1-2t^2, t^2, -2+2t^2 \rangle$

$\vec{v}(t) = \langle -4t, 2t, 4t \rangle$

$a(t) = \langle -4, 2, 4 \rangle$

$|\vec{v}(t)| = v(t) = \sqrt{16t^2 + 4t^2 + 16t^2} = 6t$

unit tangent velocity $= \dfrac{\langle -4t, 2t, 4t \rangle}{6t}$

1J-2 $\quad OP = \left\langle \dfrac{1}{1+t^2}, \dfrac{t}{1+t^2} \right\rangle = \vec{r}(t)$

a) $\vec{v}(t) = \left\langle \dfrac{-2t}{(1+t^2)^2}, \dfrac{1+t^2 - t\cdot 2t}{(1+t^2)^2} \right\rangle = \left\langle \dfrac{-2t}{(1+t^2)^2}, \dfrac{1-t^2}{(1+t^2)^2} \right\rangle$

$$s(t) = \int_0^t \left[\frac{4t^2}{(1+t^2)^4} + \frac{(1-t^2)^2}{(1+t^2)^4} \right]^{1/2} dt$$

$$\frac{ds}{dt} = v(t) = \frac{4t^2 + 1 - 2t^2 + t^4}{(1+t^2)^4} = \frac{t^4 + 2t^2 + 1}{(1+t^2)^4} = \frac{(1+t^2)^2}{(1+t^2)^4} = \frac{1}{(1+t^2)^2}$$

$$\vec{T} = \frac{\vec{v}(t)}{v(t)} = \left\langle \frac{-2t}{t^4+2t^2+1} \cdot (1+t^2)^2, \frac{(1-t^2)(1+t^2)^2}{t^4+2t^2+1} \right\rangle$$

b) $v(t) = (1+t^2)^{-2}$

$v'(t) = -2(1+t^2)^{-3} \cdot (2t) = \dfrac{-4t}{(1+t^2)^3}$

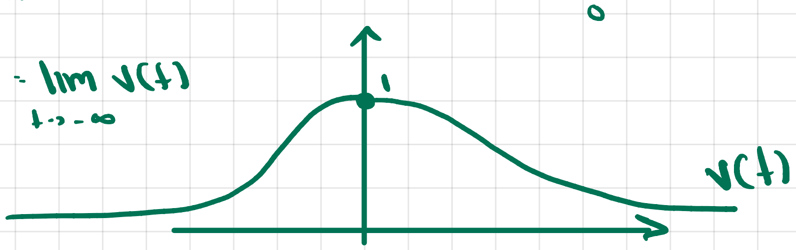
$v'(t) = 0 \Rightarrow -4t = 0 \Rightarrow t = 0$

$\oplus \quad \ominus \quad v'(t) = a(t)$

0

$\lim_{t\to\infty} v(t) = 0 = \lim_{t\to-\infty} v(t)$

$v(0) = 1$

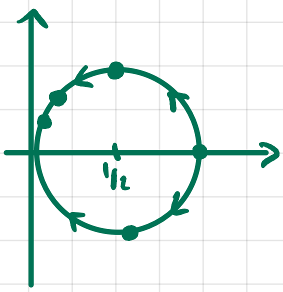


t	x	y
0	1	0
1	1/2	1/2
2	1/5	2/5
3	1/10	3/10

c) $x(t)^2 + y(t)^2 = \dfrac{1+t^2}{(1+t^2)^2} = \dfrac{1}{1+t^2} = x(t) \Rightarrow x^2 + y^2 = x$

$x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4} \Rightarrow (x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$

t	x	y
-1	1/2	-1/2
-2	1/5	-2/5



$\lim_{t\to\infty} x(t) = 0$

$\lim_{t\to\infty} y(t) = \lim \ ^1/_{2t} = 0$

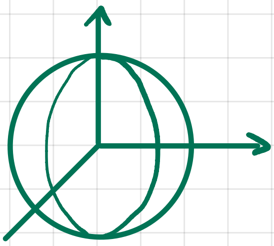
1J-3

$\vec{r}(t) = \langle r_1(t), r_2(t) \rangle$
$\vec{s}(t) = \langle s_1(t), s_2(t) \rangle$

$\vec{r} \cdot \vec{s} = r_1 s_1 + r_2 s_2$

$$\frac{d}{dt}\vec{r}\cdot\vec{s} = r_1' s_1 + r_1 s_1' + r_2' s_2 + r_2 s_2'$$
$$= (r_1' s_1 + r_2' s_2) + (r_1 s_1' + r_2 s_2')$$
$$= \langle r_1', r_2' \rangle \cdot \langle s_1, s_2 \rangle + \langle r_1, r_2 \rangle \langle s_1', s_2' \rangle$$
$$= \vec{r}\,'\vec{s} + \vec{r}\vec{s}\,'$$

1J-4



$\vec{OP} = \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

a) $|\vec{OP}| = k = \sqrt{x^2+y^2+z^2} = |\vec{r}(t)|$

$$\frac{2xx' + 2yy' + 2zz'}{2\sqrt{x^2+y^2+z^2}} = 0 \Rightarrow xx' + yy' + zz' = 0 \Rightarrow \vec{r}(t) r'(t) = 0$$

b) sphere: $|\vec{r}| = k \Rightarrow |\vec{r}|^2 = \vec{r}\cdot\vec{r} = k^2$

$$\frac{d}{dt}(\vec{r}\cdot\vec{r}) = \vec{r}\,'\vec{r} + \vec{r}\vec{r}\,' = 0 = 2\vec{r}\,'\vec{r} \Rightarrow \vec{r}\,'\vec{r} = 0 \Rightarrow \vec{v}\vec{r} = 0$$

c) $\vec{r}\cdot\vec{v} = 0$

$\vec{r}\cdot\vec{r} = |\vec{r}|^2$

$$\frac{d}{dt}(\vec{r}\cdot\vec{r}) = 2\vec{r}\,'\vec{r} = 2\vec{v}\vec{r} = 0$$

$$\Rightarrow \int \left[\frac{d}{dt}\vec{r}\cdot\vec{r}\right] dt = k$$

$\Rightarrow \vec{r}\cdot\vec{r} = k$
$|\vec{r}|^2 = k \Rightarrow |\vec{r}| = \sqrt{k}$

1J-5

a) $|\vec{v}(t)| = k = \vec{v}(t)\cdot\vec{v}(t)$

$\vec{v}\cdot\vec{v} = \vec{v}\vec{v}' + \vec{v}'\vec{v} = 0$

$2\vec{v}'\vec{v} = 0 \Rightarrow \vec{v}'\cdot\vec{v} = 0 \Rightarrow \vec{a}(t)\cdot\vec{v}(t) = 0$

b) $\vec{v}\cdot\vec{v}' = 0$

$|\vec{v}|^2 = \vec{v}\cdot\vec{v}$

$\frac{d}{dt}\vec{v}\cdot\vec{v} = 2\vec{v}\cdot\vec{v}' = 2\vec{v}\cdot\vec{a} = 0$

$\Rightarrow \vec{v}\cdot\vec{v} = |\vec{v}|^2 = k \Rightarrow |\vec{v}| = \sqrt{k}$

1J-6 $\vec{r}(t) = \langle a\cos t, a\sin t, bt\rangle$

a) $\vec{v}(t) = \langle -a\sin t, a\cos t, b\rangle$

$\vec{a}(t) = \langle -a\cos t, -a\sin t, 0\rangle$

$\vec{T} = \frac{\vec{v}(t)}{|v(t)|}$

$v(t) = |\vec{v}(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2 + b^2} = k$

$\frac{ds}{dt} = v(t)$

b) $\vec{a}(t) = -\langle a\cos t, a\sin t, 0\rangle$

$\Rightarrow \vec{a}(t)\vec{v}(t) = a^2\sin t\cos t - a^2\sin t\cos t = 0$

The point moves with constant speed, which means (by 1J-5) that $\vec{a}(t)\cdot\vec{v}(t) = 0$.

1J-9 $\vec{r}(t) = \langle 3\cos t, 5\sin t, 4\cos t\rangle$

a) $|\vec{r}(t)| = \sqrt{9\cos^2 t + 25\sin^2 t + 16\cos^2 t} = \sqrt{25} = 5$

b) $\vec{r}\,'(t) = \langle -3\sin t, 5\cos t, -4\sin t\rangle$

$$s(t) = \int_0^t \sqrt{9\sin^2 t + 25\cos^2 t + 16\sin^2 t}\, dt = \int_0^t v(t)\,dt = \int_0^t 5\,dt$$

$$\frac{ds}{dt} = v(t) = 5$$

c) $\vec{a}(t) = \vec{v}\,'(t) = \langle -3\cos t, -5\sin t, -4\cos t\rangle = -\vec{r}(t)$

d) $\vec{r}\cdot(\vec{v}\times\vec{a}) = \begin{vmatrix} 3\cos t & 5\sin t & 4\cos t \\ -3\sin t & 5\cos t & -4\sin t \\ -3\cos t & -5\sin t & -4\cos t \end{vmatrix}$

$= 3\cos t(-20\cos^2 t - 20\sin^2 t) - 5\sin t(\cancel{12\sin t\cos t - 12\sin t\cos t}) + 4\cos t(15\sin^2 t + 15\cos^2 t)$

$= -60\cos t + 60\cos t = 0$

$\Rightarrow$ $\vec{r}, \vec{v}, \vec{a}$ always on the same plane, but is it always the same plane?

$\vec{r}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\cos t & 5\sin t & 4\cos t \\ -3\sin t & 5\cos t & -4\sin t \end{vmatrix}$

$= \langle -20\sin^2 t - 20\cos^2 t, -(-12\sin t\cos t + 12\sin t\cos t), 15\cos^2 t + 15\sin^2 t\rangle$

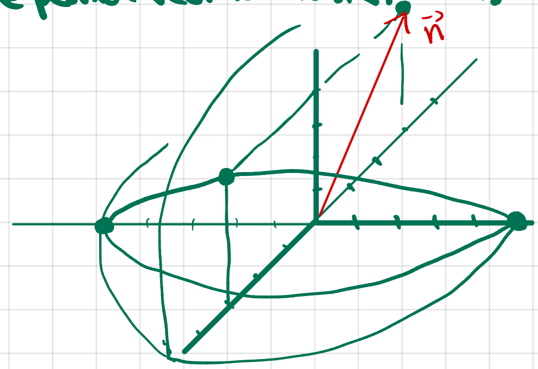
$= \langle -20, 0, 15\rangle = \vec{n}$

$\Rightarrow$ $\vec{r}\times\vec{v}$ gives a normal vector to the plane shared by $\vec{r}$ and $\vec{v}$. Since this normal vector is always the same (ie doesn't depend on t), the plane of motion is always the same. The plane passes through the origin because the position vectors all start at origin, and they are all on the same plane

e) $\vec{r}(0) = \langle 3, 0, 4\rangle$

$\vec{r}(\pi/2) = \langle 0, 5, 0\rangle$

$\vec{r}(-\pi/2) = \langle 0, -5, 0\rangle$



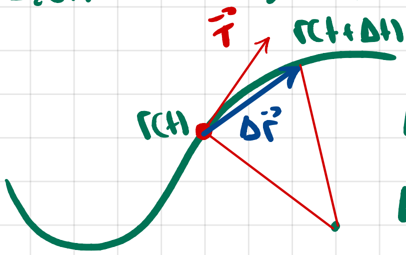
IK-2

$\vec{s}(t)$ vector function, prove $\frac{d\vec{s}}{dt} = \vec{0} \Rightarrow \vec{s}(t) = \vec{K}$

$$\frac{d}{dt}(\vec{s}(t)) = \left\langle \frac{d}{dt}s_1(t), \frac{d}{dt}s_2(t) \right\rangle = \langle 0, 0 \rangle$$

$\Rightarrow s_1'(t) = 0$
$\quad\; s_2'(t) = 0$ $\Rightarrow \int \left\langle \frac{d}{dt}s_1(t), \frac{d}{dt}s_2(t) \right\rangle dt = \int \langle 0, 0 \rangle dt = \langle k_1, k_2 \rangle$

IK-3



Area swept from $\vec{r}(t)$ to $\vec{r}(t+\Delta t)$ is approx. $\frac{1}{2}(\vec{r} \times \Delta\vec{r})$

$\Delta\vec{r} \approx \vec{v}(t)\Delta t \Rightarrow \text{Area} \approx \frac{1}{2}(\vec{r} \times \vec{v})\Delta t$

We start with the assumption that $\vec{a} \parallel \vec{r}$

$\Leftrightarrow \vec{r} \times \vec{a} = 0$ ie $\vec{r}$ and $\vec{a}$ are in the same plane through origin

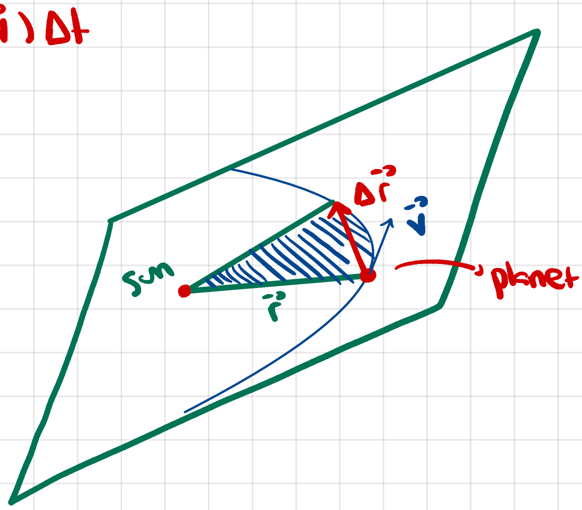
$$\Leftrightarrow \vec{r} \times \frac{d\vec{v}}{dt} = 0 \Leftrightarrow \vec{v} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} = 0 \Leftrightarrow \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} = 0 \Leftrightarrow \frac{d}{dt}(\vec{r} \times \vec{v}) = 0$$

Use result from IK-2: $\frac{d\vec{s}}{dt} = \vec{0} \Rightarrow \vec{s}(t) = \vec{K}$

$$\frac{d}{dt}(\vec{r} \times \vec{v}) = 0 \Rightarrow \vec{r} \times \vec{v} = \vec{K}$$

But the area swept by the moving point is $\approx \frac{1}{2}(\vec{r} \times \vec{v})\Delta t$

$\Rightarrow \text{Area} \approx C\Delta t$



Kepler's Law in terms of vectors

1) $\text{Area} \approx \frac{1}{2}|\vec{r} \times \Delta\vec{r}| \approx \frac{1}{2}|\vec{r} \times \vec{v}|\Delta t$

swept in time Δt $\quad \Delta r \approx \vec{v}\Delta t$

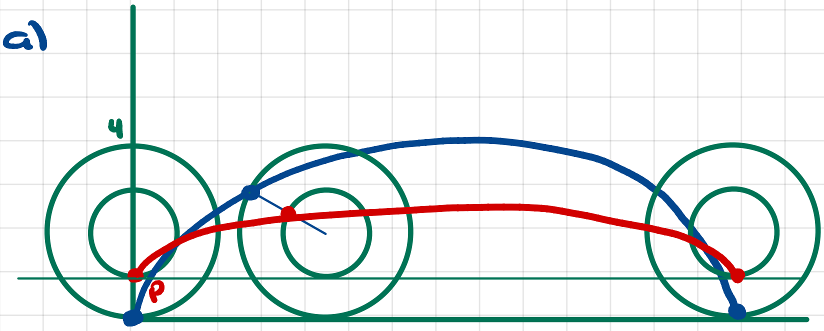
Law says area proportional to $\Delta t \Rightarrow |\vec{r} \times \vec{v}| = \text{constant}$

2) Plane of motion contains $\vec{r}$ and $\vec{v}$

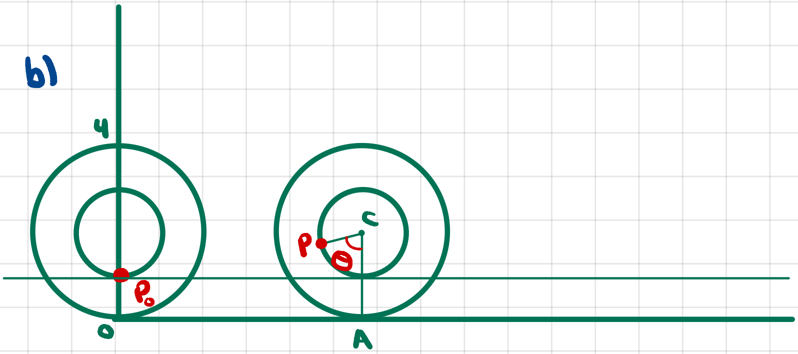
$\Rightarrow \vec{r} \times \vec{v}$ normal to plane of motion

Problem 1

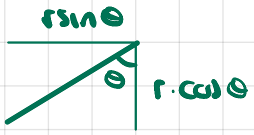
a)



b)



$\vec{OP} = \vec{OA} + \vec{AC} + \vec{CP}$



$\vec{OA} = \langle a\theta, 0 \rangle$

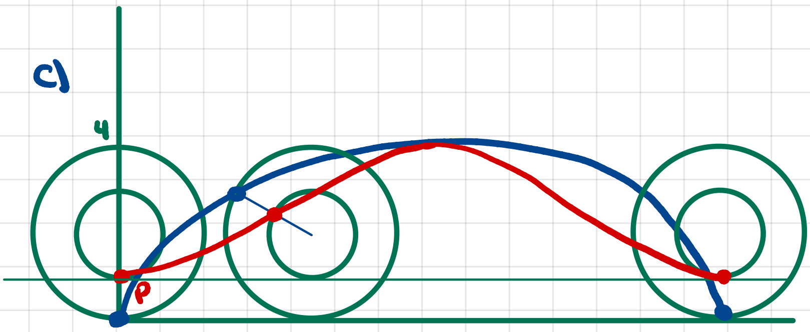
$\vec{AC} = \langle 0, a \rangle$

$\vec{CP} = \langle -r\sin\theta, -r\cos\theta \rangle$

$\vec{OP} = \langle a\theta - r\sin\theta, a - r\cos\theta \rangle$

For $a = 2, r = 1$: $\vec{OP} = \langle 2\theta - \sin\theta, 2 - \cos\theta \rangle$

c)



I thought the motion of P would be analogous to the motion on the outer rim of the wheel.

Problem 2

a) $\hat{u}, \hat{v}, \quad \hat{u} \perp \hat{v}$

$\vec{r}(t) = \cos t\,\hat{u} + \sin t\,\hat{v} = \langle u_1\cos t + v_1 \sin t, u_2\cos t + v_2\sin t, \dots \rangle$

$|\vec{r}(t)|^2 = \sqrt{(u_1\cos t + v_1\sin t)^2 + (u_2\cos t + v_2\sin t)^2 + (u_3\cos t + v_3\sin t)^2}$

$= \sqrt{\cos^2 t\underbrace{(u_1^2+u_2^2+u_3^2)}_{|\vec{u}|^2=1} + \sin^2 t\underbrace{(v_1^2+v_2^2+v_3^2)}_{|\vec{v}|^2=1} + \cancel{2\sin t\cos t}\underbrace{\cancel{(u_1v_1 + u_2v_2 + u_3v_3)}}_{=\vec{u}\cdot\vec{v}=0}}$

$= \sqrt{\cos^2 t + \sin^2 t} = 1$

$\Rightarrow |\vec{r}(t)| = 1$, which by itself could be a sphere, however $\vec{r}(t)$ is a lin. comb of $\hat{u}$ and $\hat{v}$. $u \times v$ is normal to the plane shared by $\hat{u}$ and $\hat{v}$ and

$\hat{u}\times\hat{v}\cdot\vec{r} = \hat{u}\times\hat{v}\cdot(\cos t\,\hat{u} + \sin t\,\hat{v}) = \hat{u}\times\hat{v}\cdot\hat{u}\cos t + \hat{u}\times\hat{v}\cdot\hat{u}\sin t = 0$

$\Rightarrow$ for all t, $\vec{r}(t)$ is always $\perp$ to the same normal vector $\hat{u}\times\hat{v}$ $\Rightarrow$ $\vec{r}(t)$ is always on the same plane. $\Rightarrow$ $\vec{r}(t)$ on a plane, $|\vec{r}| = 1 \Rightarrow \vec{r}(t)$ is a circle

b) $P: x + 2y + z = 0 \qquad \vec{n} = \langle 1,2,1\rangle \qquad \vec{r}(t)$ is $\perp$ to $\vec{n}$ (it is on P) and $|\vec{r}(t)| = 1$

we can use any two perpendicular vectors on P, make them unit vectors, and parametrize the circle as in a).

$x = 0, y = 1 \Rightarrow z = -2 \qquad \langle 0,1,-2\rangle = \vec{u} \qquad |\vec{u}| = \sqrt{5} \qquad \hat{u} = \dfrac{\langle 0,1,-2\rangle}{\sqrt{5}}$

$\hat{u}\times\vec{n} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1/\sqrt{5} & -2/\sqrt{5} \\ 1 & 2 & 1 \end{vmatrix} = \hat{\imath}\left(\frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}}\right) - \hat{\jmath}\left(\frac{2}{\sqrt{5}}\right) + \hat{k}\left(-\frac{1}{\sqrt{5}}\right)$

$= \langle 5/\sqrt{5}, -2/\sqrt{5}, -1/\sqrt{5}\rangle = \left\langle \sqrt{5}, -\frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right\rangle$

$\hat{v} = \left\langle \frac{\sqrt{30}}{6}, -\frac{\sqrt{30}}{15}, -\frac{\sqrt{30}}{30}\right\rangle \Rightarrow \hat{u}\cdot\hat{v} = 0$

$\vec{r}(t) = \cos(t)\cdot\hat{u} + \sin(t)\cdot\hat{v}$

c) Maple plot:

```
with(VectorCalculus)
n := <1,2,1>
u := <0,1,-2>
uhat := u/Norm(u)
v := CrossProduct(uhat,n)
vhat := v/Norm(v)

r := cos(t)·uhat + sin(t)·vhat

plot3d(r, t=0..2π)
```

Problem 3

a) $x+2y+z=0$ is plane P

All lines passing through P are perpendicular to $\langle 1,2,1\rangle$.

For $\langle x,y,z\rangle$ on a line:

$\langle x,y,z\rangle = \langle 0,0,0\rangle + \langle a,b,c\rangle t$ where $\langle a,b,c\rangle \cdot \langle 1,2,1\rangle = 0$

$a+2b+c=0$

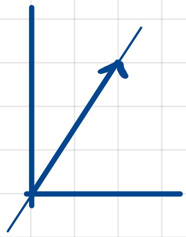
$\Rightarrow c = -a-2b$

$\langle x,y,z\rangle = \langle a,b,c\rangle t = \langle a,b,-a-2b\rangle t$

$\Rightarrow x(t) = at$
$y(t) = bt$
$z(t) = (-a+2b)t$

$\dfrac{x(t)}{y(t)} = \dfrac{a}{b} = \text{constant}$ $\Rightarrow$ we can scale

$*$ $\vec{r} = \langle 2t,3t\rangle = t\langle 2,3\rangle$



x and y maintain the same proportion. They are scaled up or down based on $\langle 2,3\rangle$. We can scale the vector up or down in size.

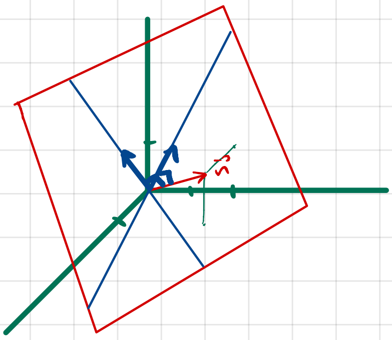
$\langle \frac{2}{3},1\rangle$ gives the same line

$\langle a,b,-a+2b\rangle$ such that the coeff. on x is the ratio between a and b. If the ratio is a parameter then all we need to specify is the ratio to get the other coord. based on their relationship with the param.

$\langle x,y,z\rangle = \langle \frac{a}{b}, 1, -\frac{a}{b}-2\rangle$

$= \langle \alpha, 1, -\alpha-2\rangle$

b)

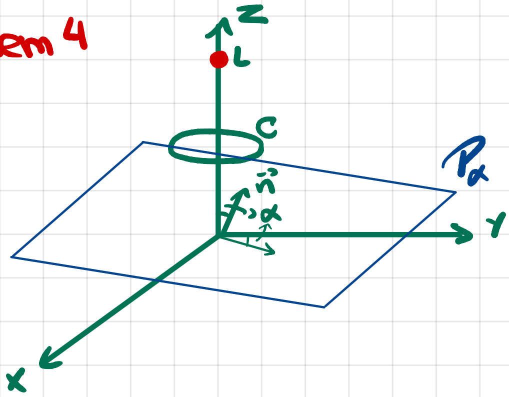


This family of lines contains lines that are perpendicular to $\langle 1,2,1\rangle$. Given one line we can get all the others by rotating the first line around the axis formed by the normal vector $\langle 1,2,1\rangle$.

We need one parameter only to identify one particular direction perpendicular to $\langle 1,2,1\rangle$, and that is α. The direction will then be that of the vector $\langle \alpha, 1, -\alpha-2\rangle$.

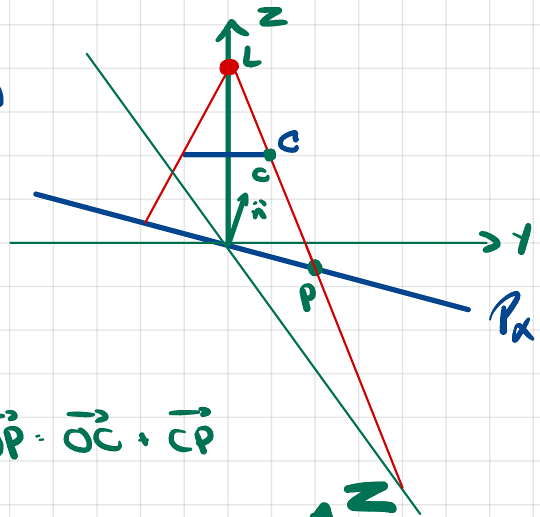
Problem 4

a)

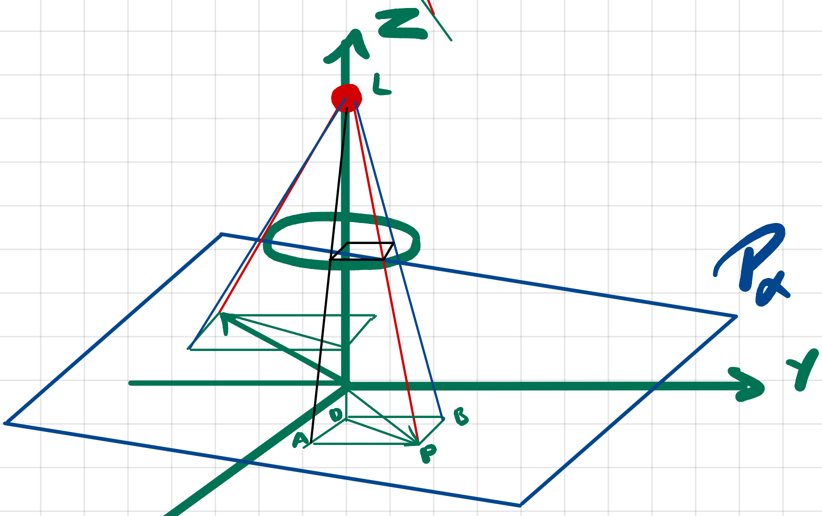


$\vec{C}(t) = \langle \cos t, \sin t, 2\rangle$

b)



$\vec{OP} = \vec{OC} + \vec{CP}$



$C(t) = \langle \cos t, \sin t, 2\rangle$

P_α $\quad my + z = 0 \quad m = \tan\alpha$

$\Rightarrow \frac{\sin\alpha}{\cos\alpha} y + z = 0$

$\Rightarrow \sin\alpha\, y + \cos\alpha\, z = 0$

normal vectors to P_α

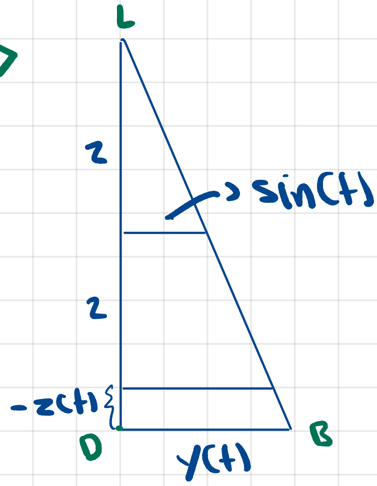
$\langle 0, \sin\alpha, \cos\alpha\rangle$ proportion $\frac{y}{z}$

$\langle 0, \tan\alpha, 1\rangle = \langle 0, m, 1\rangle = \vec{n}$

vectors on P_α

$z = -my$

$\langle x, y, -my\rangle$



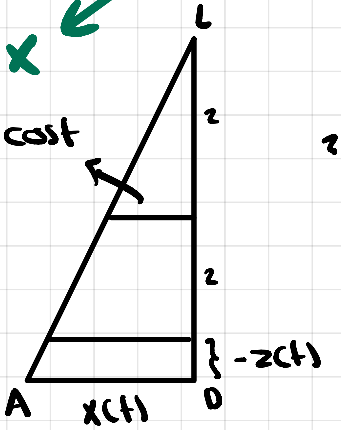
$\frac{2}{\sin t} = \frac{4 - z(t)}{y(t)}$, $z(t) = -my(t)$

$2y = 4\sin t + my\sin t$

$y(2 - m\sin t) = 4\sin t$

$\Rightarrow y(t) = \frac{4\sin t}{2 - m\sin t}$

$\Rightarrow z(t) = \frac{-4m\sin t}{2 - m\sin t}$



$\frac{4 + my(t)}{x(t)} = \frac{2}{\cos t}$

$2x = 4\cos t + \frac{m\cos t \cdot 4\sin t}{2 - m\sin t}$

$= \frac{8\cos t - 4m\sin t\cos t + 4m\sin t\cos t}{2 - m\sin t}$

$\Rightarrow x(t) = \frac{4\cos t}{2 - m\sin t}$

$\vec{r}(t) = \left\langle \frac{4\cos t}{2 - m\sin t}, \frac{4\sin t}{2 - m\sin t}, \frac{-4m\sin t}{2 - m\sin t} \right\rangle$

c) $\vec{r}_\alpha(t) = \left\langle \frac{4\cos t}{2-m\sin t}, \frac{4\sin t}{2-m\sin t}, \frac{-4m\sin t}{2-m\sin t} \right\rangle$

$\alpha = 0 \Rightarrow m = 0 \Rightarrow \vec{r}(t) = \langle 2\cos t, 2\sin t, 0 \rangle$

$\Rightarrow x^2 + y^2 = 4$, circle radius 2

$|\vec{r}_\alpha(t) - \vec{r}_0(t)|$

	$\vec{r}_\alpha(t)$	$\vec{r}_0(t)$	$\vec{r}_\alpha(t) - r_0(t)$	$\lvert\vec{r}_\alpha(t) - \vec{r}_0(t)\rvert$
$t=0$	$\langle \frac{4}{2} = 2, 0, 0 \rangle$ $\langle 2,0,0 \rangle$	$\langle 2,0,0 \rangle$	$\langle 0,0,0 \rangle$	0
$t = \pi/2$	$\langle 0, \frac{4}{2-m}, \frac{-4m}{2-m} \rangle$	$\langle 0,2,0 \rangle$	$\langle 0, \frac{4-(4-2m)}{2-m}, \frac{4m}{2-m} \rangle$ $\langle 0, \frac{2m}{2-m}, \frac{4m}{2-m} \rangle$	$\sqrt{\frac{4m^2 + 16m^2}{(2-m)^2}}$ $= \sqrt{\frac{20m^2}{(2-m)^2}}$
$t = \pi$	$\langle \frac{-4}{2}, 0, 0 \rangle$ $\langle -2,0,0 \rangle$	$\langle -2,0,0 \rangle$	$\langle 0,0,0 \rangle$	0
$t = \frac{3\pi}{2}$	$\langle 0, \frac{-4}{2+m}, \frac{4m}{2+m} \rangle$	$\langle 0,-2,0 \rangle$	$\langle 0, \frac{-4}{2+m} + 2, \frac{4m}{2+m} \rangle$ $\frac{-4+4+2m}{2+m}$ $= \frac{2m}{2+m}$ $\langle 0, \frac{2m}{2+m}, \frac{4m}{2+m} \rangle$	$\sqrt{\frac{4m^2 + 16m^2}{(2+m)^2}}$ $= \sqrt{\frac{20m^2}{(2+m)^2}}$

The largest distortion, as estimated using the distance $|\vec{r}_\alpha(t) - \vec{r}_0(t)|$, has largest value of $\sqrt{\frac{20m^2}{(2 \pm m)^2}}$. When $m > 0$ $(\alpha > 0)$, $\sqrt{20m^2/(2-m)^2}$ is largest and for $m < 0$ $(\alpha < 0)$ $\sqrt{20m^2/(2+m)^2}$ is largest. The part of the plane farthest from the light source is most distorted, ie the portion tilted below the xy plane.