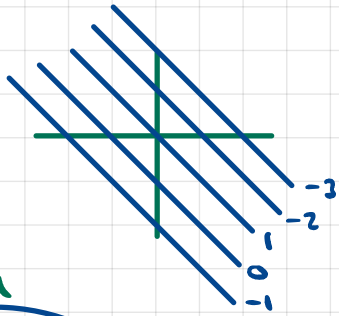
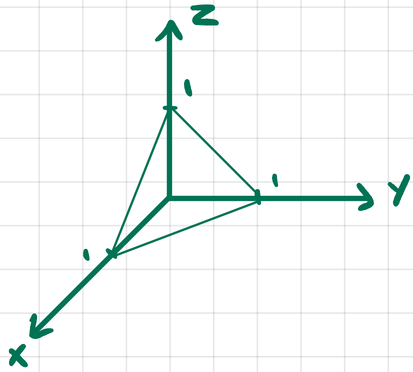
PSet 4

2A-1

a) $f(x,y) = 1-x-y$

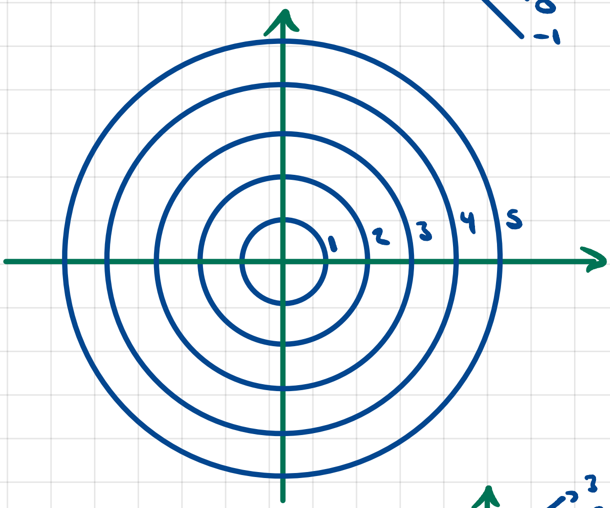
$1-x-y = C \Rightarrow y = -x-1-C = -x-(1+C)$

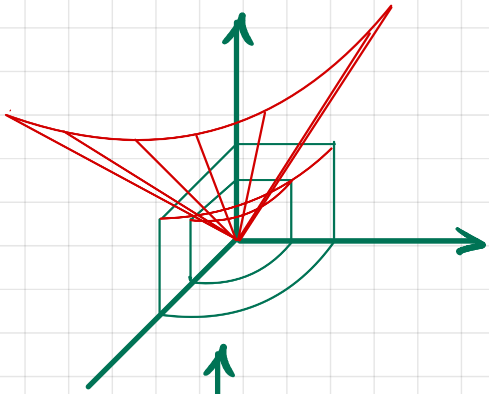


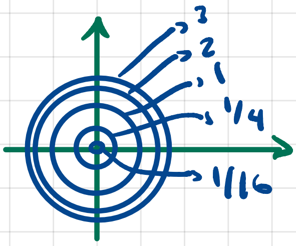


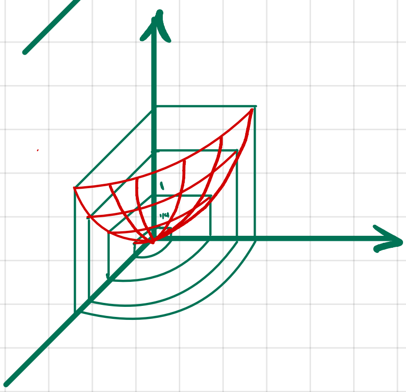
b) $f(x,y) = \sqrt{x^2+y^2}$

$x^2+y^2 = C^2$









c) $f(x,y) = x^2+y^2$

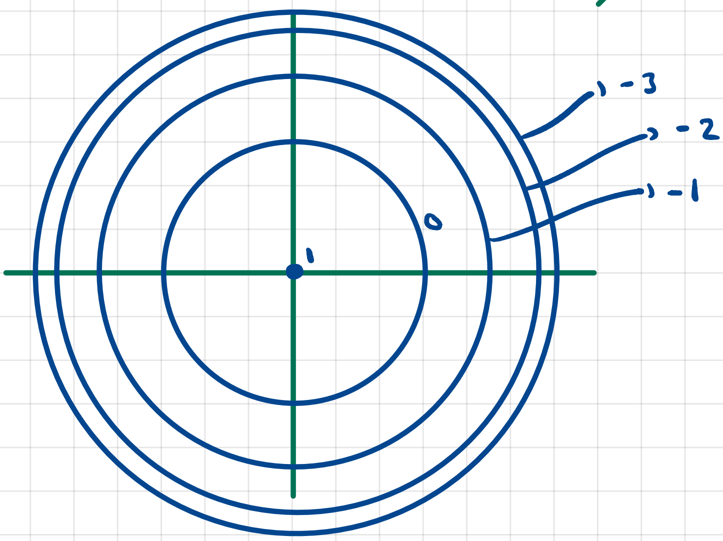
$x^2+y^2 = C \Rightarrow x^2+y^2 = (\sqrt{C})^2$

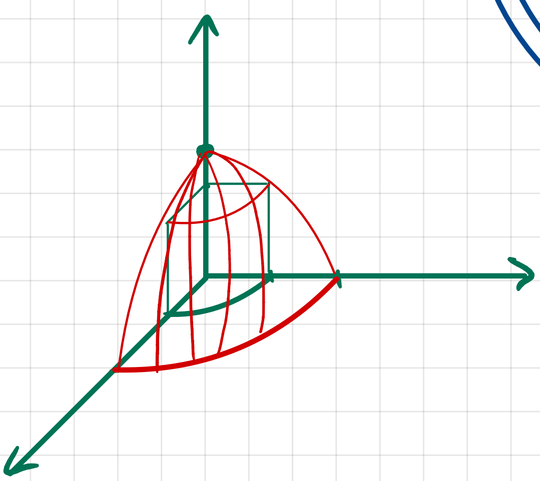
d) $f(x,y) = 1-x^2-y^2$

$1-x^2-y^2 = C \Rightarrow x^2+y^2 = 1-C$

$r^2 = 1-C \Rightarrow r = \sqrt{1-C}$

$\Rightarrow 1-C \geq 0 \Rightarrow C \leq 1$





2A-2

a) $w = x^3y - 3xy^2 + 2y^2$ $\quad w_x = 3x^2y - 3y^2$ $\quad w_y = x^3 - 6xy + 4y$

b) $z = \frac{x}{y}$ $\quad z_x = 1/y$ $\quad z_y = -x/y^2$

c) $f = \sin(3x+2y)$ $\quad f_x = \cos(3x+2y)\cdot 3$ $\quad f_y = \cos(3x+2y)\cdot 2$

d) $f = e^{x^2y}$ $\quad f_x = 2xy\,e^{x^2y}$ $\quad f_y = x^2e^{x^2y}$

e) $f = x\ln(2x+y)$ $\quad f_x = \ln(2x+y) + \frac{2x}{2x+y}$ $\quad f_y = \frac{x}{2x+y}$

f) $f = x^2z - 2yz^3$ $\quad f_x = 2xz$ $\quad f_y = -2z^3$ $\quad f_z = x^2 - 6yz^2$

2A-3

a) $f(x,y) = x^my^n$ $\quad m,n \in \mathbb{Z}^+$

$f_x = mx^{m-1}y^n$ $\qquad f_y = x^m ny^{n-1}$

$f_{xy} = mx^{m-1}ny^{n-1}$ $\qquad f_{yx} = mx^{m-1}ny^{n-1}$

b) $f(x,y) = \frac{x}{x+y} = x(x+y)^{-1}$

quotient rule

$$f_x = \frac{x+y-x}{(x+y)^2} = \frac{y}{(x+y)^2} = y(x+y)^{-2} \qquad f_y = \frac{-x}{(x+y)^2}$$

product rule

$$f_x = (x+y)^{-1} + x(-1)(x+y)^{-2} = \frac{x+y-x}{(x+y)^2} = \frac{y}{(x+y)^2}$$

$$f_y = x(-1)(x+y)^{-2} = \frac{-x}{(x+y)^2} = -x(x+y)^{-2}$$

quotient

$$f_{xy} = \frac{(x+y)^2 - y\cdot 2(x+y)}{(x+y)^4} = \frac{x+y-2y}{(x+y)^3} = \frac{x-y}{(x+y)^3}$$

$$f_{yx} = \frac{-(x+y)^2 - (-x)\cdot 2(x+y)}{(x+y)^4} = \frac{-x-y+2x}{(x+y)^3} = \frac{x-y}{(x+y)^3}$$

product

$$f_{xy} = (x+y)^{-2} + y(-2)(x+y)^{-3} = \frac{x+y-2y}{(x+y)^3} = \frac{x-y}{(x+y)^3}\ \checkmark$$

$$f_{yx} = -(x+y)^{-2} - x(-2)(x+y)^{-3} = \frac{-x-y+2x}{(x+y)^3} = \frac{x-y}{(x+y)^3}\ \checkmark$$

c) $f(x,y) = \cos(x^2+y)$

$f_x = -\sin(x^2+y)\cdot 2x$ $\qquad$ $f_y = -\sin(x^2+y)$

$f_{xy} = -\cos(x^2+y)\cdot 2x$ $\qquad$ $f_{yx} = -\cos(x^2+y)\cdot 2x$ ✓

2A-4 $f_x = axy + 3y^2$ $\qquad$ $f_y = x^2 + 6xy$

$f_{xy} = ax + 6y$ $\qquad$ $f_{yx} = 2x + 6y$

$ax + 6y = 2x + 6y \Rightarrow a = 2$

$\int f_x\,dx = 2yx^2/2 + 3y^2x = yx^2 + 3y^2x$ ✓

2A-5 $w_{xx} + w_{yy} = 0$

a) $w(x,y) = e^{ax}\sin(ay)$

$w_x = ae^{ax}\sin(ay)$ $\qquad$ $w_y = e^{ax}\cdot a\cos(ay)$

$w_{xx} = a^2e^{ax}\sin(ay)$ $\qquad$ $w_{yy} = -a^2e^{ax}\sin(ay)$ ✓

$\Rightarrow w_{xx} + w_{yy} = 0$

b) $w(x,y) = \ln(x^2+y^2)$

$w_x = \dfrac{2x}{x^2+y^2} = 2x(x^2+y^2)^{-1}$

$w_{xx} = 2(x^2+y^2)^{-1} + 2x(-1)\cdot 2x(x^2+y^2)^{-2}$

$= \dfrac{2x^2+2y^2-4x^2}{(x^2+y^2)^2} = \dfrac{2y^2-2x^2}{(x^2+y^2)^2}$ ✓

By symmetry,

$w_y = 2y(x^2+y^2)^{-1}$

$w_{yy} = \dfrac{2x^2-2y^2}{(x^2+y^2)^2}$ ✓

$\Rightarrow w_{xx} + w_{yy} = 0$

2B-1

a) $z = xy^2 \quad (1,1,1)$

$z = f(x,y)$

$f_x = y^2 \quad f_y = 2xy$

General plane equation

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

$$\langle A,B,C\rangle \langle x-x_0, y-y_0, z-z_0\rangle = 0$$

ie, any point (x,y,z) on the plane is such that the vector $\langle x-x_0, y-y_0, z-z_0\rangle$ is $\perp$ to a normal vector $\langle A,B,C\rangle$.

For a tangent plane at (x_0,y_0,z_0), $z_0 = f(x_0,y_0)$ we know that any line through the plane must be tangent to $f(x,y)$.

If $y=y_0$, then the points on the tangent plane still obey $A(x-x_0)+C(z-z_0)=0$

$\Rightarrow Cz - Cz_0 = A(x-x_0) \Rightarrow z-z_0 = a(x-x_0)$, a line. Since $y=y_0$ is fixed we are

on the y curve. The tangent to the y-curve of a function is f_y. So:

$\Rightarrow z-z_0 = a(x-x_0)$ is a line on the plane tangent at (x_0,y_0), specifically with $y=y_0$.

$\Rightarrow$ it is the line tangent to the y-curve

$\Rightarrow$ its slope, a, must be $f_y(x_0,y_0)$

$\Rightarrow$ analogously, $z-z_0 = b(y-y_0)$ is the line tangent to the x-curve $\Rightarrow b = f_x(x_0,y_0)$

So back to the tangent plane eq:

$$z-z_0 = a(x-x_0)+b(y-y_0) = f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(x-x_0)$$

$z = xy^2 \quad f_x = y^2 \quad f_y = 2xy$

For $(x_0,y_0,z_0) = (1,1,1)$, $z_0 = f(1,1) = 1 \quad f_x(1,1) = 1 \quad f_y(1,1) = 2$

$\Rightarrow z-1 = (x-1)+2(y-1)$ is the tangent plane. ✓

b) $w(x,y) = y^2/x \qquad P_0 = (1,2,4)$

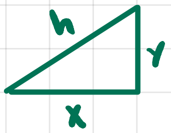
$w_x = -\dfrac{y^2}{x^2} \qquad w_y = \dfrac{2y}{x}$

tangent plane: $z - z_0 = -\left[\dfrac{y_0}{x_0}\right]^2 (x - x_0) + \left[\dfrac{2y_0}{x_0}\right](y - y_0)$

At $(1,2,4)$

$$z - 4 = -4(x-1) + 4(y-2) \checkmark$$

2B-3



$h^2 = x^2 + y^2$

$h(x,y) = \sqrt{x^2+y^2}$, a cone in space

$(x_0, y_0, z_0) = (3,4,5)$

$h_x = \dfrac{x}{\sqrt{x^2+y^2}} \qquad h_y = \dfrac{y}{\sqrt{x^2+y^2}}$

$z - f(x_0, y_0) = \dfrac{x_0}{\sqrt{x_0^2+y_0^2}}(x - x_0) + \dfrac{y_0}{\sqrt{x_0^2+y_0^2}}(y - y_0)$

$h_x(3,4) = \dfrac{3}{5} \quad h_y(3,4) = \dfrac{4}{5}$

$z - 5 = \dfrac{3}{5}(x-3) + \dfrac{4}{5}(y-4)$

$\Rightarrow h(3.01, 4.01) \approx 5 + \dfrac{3}{5}(0.01) + \dfrac{4}{5}(0.01) = 5 + 0.01 \cdot \dfrac{7}{5}$

$$\approx 5.014 \checkmark$$

2B-5

a) $f(x,y) = (x+y+2)^2$

Linearization

$f_x = 2(x+y+2) \qquad f_x(0,0) = f_y(0,0) = 4$

$f_y = 2(x+y+2) \qquad f_x(1,2) = f_y(1,2) = 10$

$f(x,y) \approx f(x_0,y_0) + 2(x_0+y_0+2)\Delta x + 2(x_0+y_0+2)\Delta y$

near $(0,0)$, $f(x,y) \approx 4 + 4x + 4y \checkmark$

near $(1,2)$, $f(x,y) \approx 25 + 10(x-1) + 10(y-2) \checkmark$

b) $f(x,y) = e^x \cos y$ $\quad f(0,0) = 1$ $\quad f(0,\pi/2) = 0$

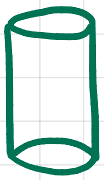
$f_x = e^x \cos y$ $\quad f_x(0,0) = 1$ $\quad f_x(0,\pi/2) = 0$

$f_y = -e^x \sin y$ $\quad f_y(0,0) = 0$ $\quad f_y(0,\pi/2) = -1$

near $(0,0)$ $\quad f(x,y) \approx 1 + x$ ✓

" $\quad (0,\pi/2)$ $\quad f(x,y) \approx -(y - \pi/2)$ ✓

2B-6 ✓



$V(r,h) = \pi r^2 h$ $\qquad V(2,3) = 12\pi$

we want linearization of V at $(2,3,12\pi)$
calculate $V(r+\Delta r, h+\Delta h) - V(2,3)$, check conditions for this difference $\leq .1$

$V_r = 2\pi r h$ $\qquad V_r(2,3) = 12\pi$

$V_h = \pi r^2$ $\qquad V_h(2,3) = 4\pi$

$V(2+\Delta r, 3+\Delta h) \approx 12\pi + 12\pi \Delta r + 4\pi \Delta h$

$|V(2+\Delta r, 3+\Delta h) - V(2,3)| = 12\pi \Delta r + 4\pi \Delta h \leq 0.1$

Assume $\Delta r = p \cdot 2$, $\Delta h = p \cdot 3$ $\quad p_r, p_h \geq 0$

$24\pi p + 12\pi p < 0.1$

$p < \frac{0.1}{36\pi} = \frac{1}{360\pi} \approx .08\%$ ie we measure radius and height with some percentage of error relative to what's being measured, the percentage must be $\leq 0.08\%$.

If instead we measure with the same instrument (for example), the error isn't proportional to what's being measured. $\Delta r = \Delta h = e \Rightarrow 16\pi e \leq 0.1 \Rightarrow e \leq 0.0019894 \approx 0.002$

In general $6\pi \Delta r + 4\pi \Delta h \leq 0.1$ must hold.

2B-9

a) $W(x,y) = x^2(y+1)$

$W_x = 2x(y+1) \qquad W_x(1,0) = 2$

$\Rightarrow$ more sensitive to Δx near $(1,0)$ ✓

$W_y = x^2 \qquad W_y(1,0) = 1$

b) $\Delta W(1,0) \approx 2\Delta x + \Delta y = 0 \Rightarrow \dfrac{\Delta y}{\Delta x} = -2$ ✓

2F-1

a) $xyz^2 = 1 \qquad D(x,y) = \sqrt{x^2+y^2+1/xy}$

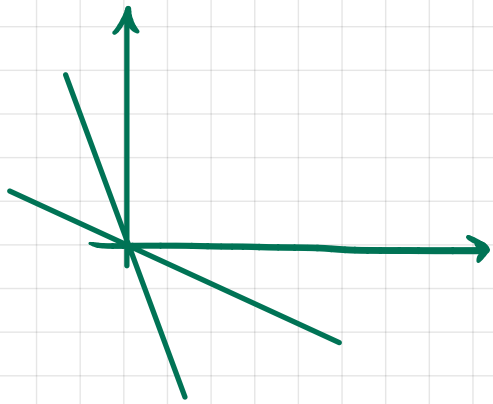
$z^2 = 1/xy \qquad D_2(x,y) = D^2 = x^2+y^2+1/xy$

$D_{2x} = 2x - \dfrac{1}{x^2y} = 0 \Rightarrow 2x^3y = 1$

$\Rightarrow \dfrac{x^2}{y^2} = 1 \Rightarrow x^2 = y^2$

$D_{2y} = 2y - \dfrac{1}{xy^2} = 0 \Rightarrow 2xy^3 = 1$

b) $x^2 - yz = 1 \Rightarrow x^2 = 1+yz$



$D_2(y,z) = 1+yz+y^2+z^2$

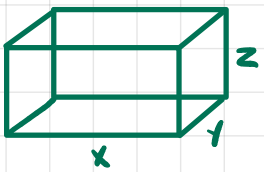
$D_{2y} = z+2y = 0$

$\Rightarrow 3y+3z = 0 \Rightarrow y = -z$

$D_{2z} = y+2z = 0$

$\Rightarrow z=0,\ y=0$

2F-2



$T(x,y,z) = 2yz+4xz+3xy$

$xyz = 1 \qquad z = 1/xy$

$T(x,y) = \dfrac{2}{x} + \dfrac{4}{y} + 3xy$

$T_x = \dfrac{-2}{x^2} + 3y = 0 \Rightarrow -2+3x^2y = 0 \Rightarrow y = \dfrac{2}{3x^2}$

$T_y = \dfrac{-4}{y^2} + 3x = 0 \Rightarrow -4+3xy^2 = 0 \Rightarrow \dfrac{12x}{9x^4} = 4 \Rightarrow 12 = 36x^3 \Rightarrow x^3 = \dfrac{1}{3} \Rightarrow x = 3^{-1/3}$

$\Rightarrow y = \dfrac{2}{3\cdot 3^{-2/3}} = \dfrac{2}{3^{1/3}} = 2\cdot 3^{-1/3}$

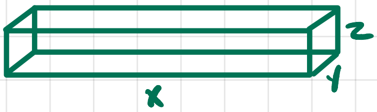
$\Rightarrow z = \left[2\cdot 3^{-1/3}\cdot 3^{-1/3}\right]^{-1} = \left[2\cdot 3^{-2/3}\right]^{-1} = \dfrac{3^{2/3}}{2} = \dfrac{3}{2}\cdot 3^{-1/3}$

$x = \dfrac{1}{\sqrt[3]{3}}$

$y = 2\cdot\dfrac{1}{\sqrt[3]{3}}$

$z = \dfrac{3}{2}\cdot\dfrac{1}{\sqrt[3]{3}}$

2F-5



$$C(x,y,z) = xy + xz + 2\cdot 2\cdot yz + 4xz$$

$$xyz = 2.5 \quad z = 2.5/xy$$

$$C(x,y) = xy + \frac{2.5}{y} + \frac{10}{x} + \frac{10}{y} = xy + \frac{10}{x} + \frac{12.5}{y}$$

$$C_x = y - \frac{10}{x^2} = 0 \Rightarrow y = \frac{10}{x^2}$$

$$C_y = x - \frac{12.5}{y^2} = 0$$

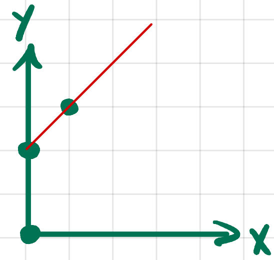
$$x - \frac{12.5x^4}{100} = 0 \Rightarrow x\left(1 - \frac{12.5x^3}{100}\right) = 0$$

$$\Rightarrow 12.5x^3 = 100 \Rightarrow x^3 = 8 \Rightarrow x = 2$$

$$\Rightarrow y = \frac{5}{2} \Rightarrow z = \frac{2.5}{2.5\cdot 2} = \frac{1}{2}$$

2G-1

a)



x_i	y_i
0	0
0	2
1	3

$$D(a,b) = \sum_{i=1}^{3} (y_i - (ax_i + b))^2$$

using formulas on right

$\Sigma x_i = 1$

$\Sigma x_i^2 = 1$

$\Sigma x_i y_i = 3$

$(\Sigma x_i)^2 = 1$

$\Sigma y_i = 5$

$$b = \frac{5\cdot 1 - 1\cdot 3}{3\cdot 1 - 1} = \frac{2}{2} = 1$$

$$a = \frac{3 - 1\cdot 1}{1} = 2$$

In general,

$$D(a,b) = \sum_{i=1}^{n} (y_i - (ax_i + b))^2$$

$$D_a = \Sigma 2(y_i - ax_i - b)(-x_i) = 0$$

$$\Rightarrow \Sigma (y_i - ax_i - b)x_i = 0$$

$$\Sigma y_i x_i = a\Sigma x_i^2 + b\Sigma x_i$$

$$a = \frac{\Sigma y_i x_i - b\Sigma x_i}{\Sigma x_i^2}$$

$$D_b = \Sigma 2(y_i - ax_i - b)(-1)$$

$$\Rightarrow \Sigma (y_i - ax_i - b) = 0$$

$$\Sigma y_i = a\Sigma x_i + bn$$

$$\Sigma y_i = \left[\frac{\Sigma y_i x_i - b\Sigma x_i}{\Sigma x_i^2}\right]\Sigma x_i + bn$$

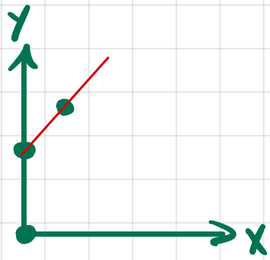
$$\Sigma y_i = \frac{\Sigma x_i \Sigma y_i x_i - b(\Sigma x_i)^2}{\Sigma x_i^2} + bn$$

$$\Sigma y_i \Sigma x_i^2 = \Sigma x_i \Sigma y_i x_i - b(\Sigma x_i)^2 + bn\Sigma x_i^2$$

$$\Sigma y_i \Sigma x_i^2 = \Sigma x_i \Sigma y_i x_i + b\left[n\Sigma x_i^2 - (\Sigma x_i)^2\right]$$

$$b\left[n\Sigma x_i^2 - (\Sigma x_i)^2\right] = \Sigma y_i \Sigma x_i^2 - \Sigma x_i \Sigma y_i x_i$$

$$b = \frac{\Sigma y_i \Sigma x_i^2 - \Sigma x_i \Sigma y_i x_i}{n\Sigma x_i^2 - (\Sigma x_i)^2}$$



x_i	y_i
0	0
0	2
1	3

$$D(a,b) = \sum_{i=1}^{3} (y_i - (ax_i + b))^2$$

From scratch

$$D(a,b) = b^2 + (2-b)^2 + (3-a-b)^2$$

$$D_a = 2(3-a-b)(-1) = -6+2a+2b = 0 \Rightarrow a+b=3 \Rightarrow a=3-b$$

$$D_b = 2b + 2(2-b)(-1) + 2(3-a-b)(-1)$$

$$= 2b - 4 + 2b - 6 + 2a + 2b = 0$$

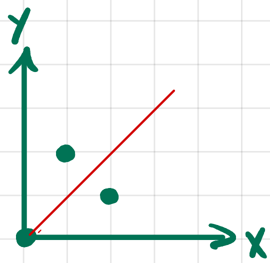
$$6b + 2a - 10 = 0$$

$$6b + 6 - 2b - 10 = 0$$

$$4b = 4$$

$$b = 1$$

$$a = 3-1 = 2$$



b)

x_i	y_i
0	0
1	2
2	1

$$D(a,b) = (-b)^2 + (2-a-b)^2 + (1-2a-b)^2$$

$$= b^2 + (2-a-b)^2 + (1-2a-b)^2$$

$$D_a = -2(2-a-b) - 4(1-2a-b) = 0$$

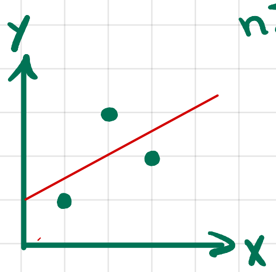
$$-4 + 2a + 2b + -4 + 8a + 4b \Rightarrow -8 + 10a + 6b = 0$$

$$D_b = 2b - 2(2-a-b) - 2(1-2a-b)$$

$$= 2b - 4 + 2a + 2b - 2 + 4a + 2b \Rightarrow 6b + 6a - 6 = 0 \Rightarrow a+b=1 \Rightarrow a = 1-b$$

$$-4 + 8 - 8b + 4b = 4 - 4b = 0 \Rightarrow b = 1 \Rightarrow a = 0$$

c) $b = \dfrac{\sum y_i \sum x_i^2 - \sum x_i \sum y_i x_i}{n\sum x_i^2 - (\sum x_i)^2}$ $\qquad a = \dfrac{\sum y_i x_i - b\sum x_i}{\sum x_i^2}$



x_i	y_i
1	1
2	3
3	2

$\sum y_i = 6$
$\sum x_i = 6$
$\sum x_i^2 = 1+4+9 = 14$
$\sum y_i x_i = 1+6+6 = 13$
$(\sum x_i)^2 = 36$

$$b = \frac{6\cdot 14 - 6\cdot 13}{3\cdot 14 - 36} = \frac{6}{6} = 1$$

$$y = \frac{x}{2} + 1$$

$$a = \frac{13 - 1\cdot 6}{14} = \frac{7}{14} = \frac{1}{2}$$

2G-4

$z = a + bx + cy$

(x_i, y_i, z_i) $\quad i = 1, \dots, n$

$$D(a,b,c) = \sum_{i=1}^{n} (z_i - a - bx_i - cy_i)^2$$

$$D_a = \sum 2(z_i - a - bx_i - cy_i)(-1) = 0 \Rightarrow \sum z_i - an - b\sum x_i - c\sum y_i = 0$$

$$D_b = \sum 2(z_i - a - bx_i - cy_i)(-x_i) = 0 \Rightarrow \sum z_i x_i - a\sum x_i - b\sum x_i^2 - c\sum y_i x_i = 0$$

$$D_c = \sum 2(z_i - a - bx_i - cy_i)(-y_i) = 0 \Rightarrow \sum z_i y_i - a\sum y_i - b\sum x_i y_i - c\sum y_i^2 = 0$$

2H-1

a) $f(x,y) = x^2 - xy - 2y^2 - 3x - 3y + 1$

$f_x = 2x - y - 3 = 0 \Rightarrow -8y - 6 - y - 3 = 0 \Rightarrow -9y - 9 = 0 \Rightarrow y = -1$

$f_y = -x - 4y - 3 = 0 \Rightarrow x = -4y - 3 \Rightarrow x = -4(-1) - 3 = 1$

$(1,-1)$ critical point

$f_{xx} = 2 = A$ $\qquad AC - B^2 = -8 - 1 = -9 < 0 \Rightarrow (1,-1)$ is a saddle point
$f_{xy} = -1 = B$
$f_{yy} = -4 = C$

b) $f = 3x^2 + xy + y^2 - x - 2y + 4$

$f_x = 6x + y - 1 = 0 \Rightarrow y = 1 - 6x$

$f_y = x + 2y - 2 = 0 \Rightarrow x + 2 - 12x - 2 = 0 \Rightarrow x = 0 \Rightarrow y = 1$

$(0,1)$ critical point

$f_{xx} = 6$

$f_{xx} f_{yy} - f_{xy}^2 = 6 \cdot 2 - 1 = 11$ $\Rightarrow (0,1)$ is a local minimum

$f_{xy} = 1$

$f_{xx} > 0$

$f_{yy} = 2$

c) $f = 2x^4 + y^2 - xy + 1$

$f_x = 8x^3 - y = 0 \Rightarrow y = 8x^3$

$f_y = 2y - x = 0 \quad 16x^3 - x = 0 \Rightarrow x(16x^2 - 1) = 0$

$\nearrow x = 0,\ y = 0$

$\searrow x^2 = 1/16 \Rightarrow x = \pm \frac{1}{4}$

$x = 1/4 \Rightarrow y = 8/64 = 1/8$

$x = -1/4 \Rightarrow y = 8/(-64) = -1/8$

$(0,0)\ (1/4, 1/8),\ (-1/4, -1/8)$ critical points

$f_{xx} = 24x^2$

$f_{xy} = -1$

$f_{yy} = 2$

$(0,0) \quad \Delta = 0 \cdot 2 - 1 = -1$ saddle

$(1/4, 1/8) \quad \Delta = \frac{24}{16} \cdot 2 - 1 = \frac{48 - 16}{16} = 2 > 0,\ f_{xx}(1/4, 1/8) > 0 \Rightarrow$ local min.

$(-1/4, -1/8) \quad \Delta = \frac{24}{16} \cdot 2 - 1 = 2 > 0 \quad f_{xx}(-1/4, 1/8) > 0 \Rightarrow$ local min.

e) $f = (x^3+1)(y^3+1)$

$f_x = 3x^2(y^3+1) = 0 \Rightarrow x = 0$ or $y = -1$

$f_y = 3y^2(x^3+1) = 0 \Rightarrow y = 0$ or $x = -1$

$\left.\begin{matrix}(0,0)\\(-1,-1)\end{matrix}\right\}$ critical points

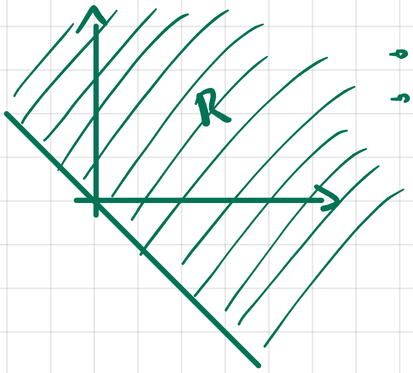
$f_{xx} = 6x(y^3+1)$

$f_{yy} = 6y(x^3+1)$

$f_{xy} = 3x^2 \cdot 3y^2 = 9x^2y^2$

	f_{xx}	f_{yy}	f_{xy}	Δ	
$(0,0)$	0	0	0	0	test fails
$(-1,-1)$	0	0	9	-9	saddle

2H-3 $f(x,y) = x^2 + y^2 + 2x + 4y - 1$



→ f increases without bound as $x \to \infty$ and as $y \to \infty$
→ no critical points in R
⇒ no max in R
⇒ min reached on boundary $y = -x$

$f(x,-x) = x^2 + x^2 + 2x - 4x - 1$
$2x^2 - 2x - 1 = g(x)$

$g'(x) = 4x - 2 = 0 \Rightarrow x = 1/2 \Rightarrow y = -1/2$

$f(1/2, -1/2) = \frac{1}{4} + \frac{1}{4} + 2 \cdot \frac{1}{2} - 4 \cdot \frac{1}{2} - 1$

$= \frac{1}{2} + 1 - 2 - 1 = -\frac{3}{2}$

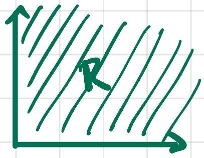
$f_x = 2x + 2 = 0 \Rightarrow x = -1$

$f_y = 2y + 4 = 0 \Rightarrow y = -2$

$(-1,-2)$ is outside of R

2H-4 $\quad f(x,y) = xy - x - y + 2$

a)



$f_x = y - 1 = 0 \Rightarrow y = 1$

$f_y = x - 1 = 0 \Rightarrow x = 1$

For fixed $y<1$, f decreases to $-\infty$ as $x \to \infty$
" " $x<1$, " " " " " $y \to \infty$

$\Rightarrow$ no minimum in R

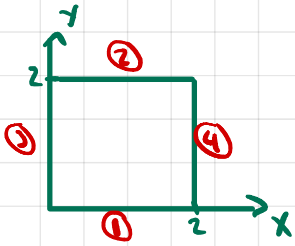
$f_{xx} = 0$
$f_{yy} = 0 \qquad f_{xx} f_{yy} - f_{xy} = 0 - 1 = -1 < 0$
$f_{xy} = 1 \qquad \Rightarrow (1,1)$ is saddle

For fixed $y > 1$, $f' > 0$ so $f \to \infty$ as $x \to \infty$

boundary $x=0$ $\quad f(0,y) = -y+2 = g(y) \Rightarrow g'(y) = -1 \neq 0 \Rightarrow$ decreasing, so max at $y=0$

boundary $y=0$ $\quad f(x,0) = -x+2 = h(x) \Rightarrow h'(x) = -1 \Rightarrow$ max at $h(0) = 2 \quad g(0) = 2$

b)



boundaries

1 $\; y = 0, \; 0 \le x \le 2 \Rightarrow (0,0)$ max value of 2, min value of 0 at $(2,0)$, $f(2,0) = 0$

2 $\; y = 2, \; 0 \le x \le 2 \Rightarrow 2x - x - 2 + 2 = x = g(x) \Rightarrow g'(x) = 1 \Rightarrow g$ increasing on 1

$(0,2)$ min, $(2,2)$ max on this boundary

$f(0,2) = -2+2 = 0$ min, $f(2,2) = 4-2-2+2 = 2$ max

3 $\; x = 0, \; 0 \le y \le 2 \Rightarrow f = -y + 2 = h(y) \Rightarrow h'(y) = -1 \Rightarrow$ on 3, max at $(0,0)$, min at $(0,2)$

4 $\; x = 2, \; 0 \le y \le 2 \Rightarrow f = 2y - 2 - y + 2 = y = h(y) \Rightarrow h'(y) = 1 = h$ increasing on 4 in y

$\Rightarrow (2,0)$ min, $(2,2)$ max

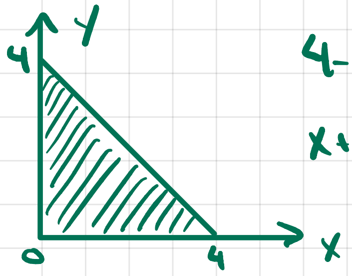
$\Rightarrow$ on boundaries: $\quad (0,0) \to 2$ $\quad$ maxima
$(2,0) \to 0$ $\Big\}$ minima
$(0,2) \to 0$
$(2,2) \to 2$

only critical point is a saddle point $\Rightarrow$ max/min on boundary

c) $f_{xx} = 0$
$f_{yy} = 0 \qquad f_{xx} f_{yy} - f_{xy} = 0 - 1 = -1 < 0$
$f_{xy} = 1 \qquad \Rightarrow (1,1)$ is saddle

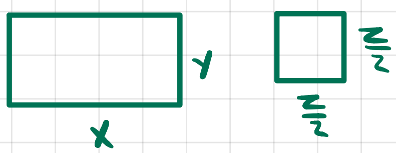
2H-6

a) $x+y+z=4 \Rightarrow z=4-x-y$



$4-x-y \geq 0$

$x+y \leq 4 \quad y \leq 4-x$



$A(x,y,z) = xy + \frac{z^2}{4}, \quad x,y,z \geq 0, \ x+y+z=4$

$A(x,y) = xy + \frac{(4-x-y)^2}{4} \quad y \leq 4-x, \ x,y \geq 0$

$A_x = y + \frac{2(4-x-y)(-1)}{4} = 0 \Rightarrow 2y-4+x+y = 3y+x-4=0$

$A_y = x + \frac{2}{4}(4-x-y)(-1) = 0 \Rightarrow 2x-4+x+y = 3x+y-4=0 \quad \Rightarrow x=4-3y$

$12-9y+y-4=0 \Rightarrow 8=8y \Rightarrow y=1 \Rightarrow x=4-3=1 \Rightarrow z=4-1-1=2$

$(1,1,2)$ critical point $\quad A(1,1,2) = 1+1=2$

boundaries

$x=0 \Rightarrow A(0,y) = g(y) = \frac{(4-y)^2}{4} \Rightarrow g'(y) = 2(4-y)(-1)/4 = \frac{y-4}{2}$

$g'(y) < 0$ in $(0,4) \Rightarrow (0,4)$ min, $(0,0)$ max

$g(0)=4, \ g(4)=0$

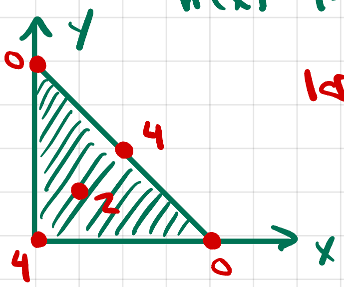
$y=0 \Rightarrow A(x,0) = h(x) = \frac{(4-x)^2}{4} \Rightarrow h'(x) = \frac{2(4-x)(-1)}{4} = \frac{x-4}{2}$

$\Rightarrow g'(x) < 0$ in $(0,4) \Rightarrow$ max at $(0,0)$, min $(4,0)$

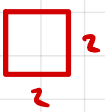
$g(0)=4, \ g(4)=0$

$y=4-x \Rightarrow A(x,4-x) = h(x) = x(4-x) + \frac{(\cancel{4}-\cancel{x}-\cancel{4}+\cancel{x})^2}{4} = 4x-x^2$

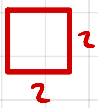
$h'(x) = 4-2x = 0 \Rightarrow x=2, y=2 \Rightarrow h(2) = 2\cdot 2 = 4$



largest total area = 4, at $(0,0,4)$ or $(2,2,0)$



and no rectangle, or of sides x,y



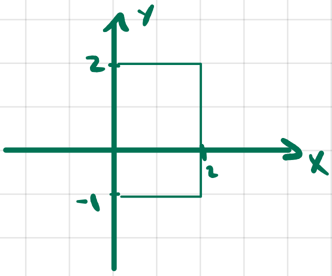
and no square of side $\frac{z}{2}$

b) check critical point $\quad A_{xx}=1 \quad A_{xy}=3 \quad A_{yy}=1$

$\Delta = 1\cdot 1 - 3^2 < 0 \Rightarrow (1,1,2)$ is a saddle point.

2H-7

a) $f(x,y) = 2x^2 - 2xy + y^2 - 2x$



b) $f_{xx} = 4$ $f_{xy} = -2$ $f_{yy} = 2$

$\Rightarrow \Delta = 4 \cdot 2 - 4 = 4$

$\Delta > 0$, $f_{xx} > 0$ $\Rightarrow$ (1,1) is local min

$f_x = 4x - 2y - 2 = 0 \Rightarrow 2x = 2 \Rightarrow x = 1 \Rightarrow y = 1$

$f_y = -2x + 2y = 0 \Rightarrow x = y$

(1,1) critical point $f(1,1) = 2 - 2 + 1 - 2 = -1$

$f_{xx} = 4$ $f_{xy} = -2$ $f_{yy} = 2$ $\Rightarrow \Delta = 4 \cdot 2 - 4 = 4$

$\Delta > 0$, $f_{xx} > 0$ $\Rightarrow$ (1,1) is local min

boundaries

$y = -1 \Rightarrow f(x,-1) = 2x^2 + 2x + 1 - 2x = 2x^2 + 1 = g(x)$

$g'(x) = 4x = 0 \Rightarrow x = 0$

local min at $f(0,-1) = 1$, local max at $f(2,-1) = 9$, on this boundary

$x = 0 \Rightarrow f(0,y) = y^2 \Rightarrow$ min at $f(0,0) = 0$, max at $f(0,2) = 4$

$y = 2 \Rightarrow f(x,2) = 2x^2 - 4x + 4 - 2x = h(x) = 2x^2 - 6x + 4$

$h'(x) = 4x - 6 = 0 \Rightarrow x = 3/2$ $h'' = 4 \Rightarrow$ $(3/2, 2)$ is local min on boundary

$f(3/2,2) = 2 \cdot \frac{9}{4} - 2 \cdot \frac{3}{2} \cdot 2 + 4 - 3 = \frac{9}{2} - 6 + 4 - 3 = -\frac{1}{2}$

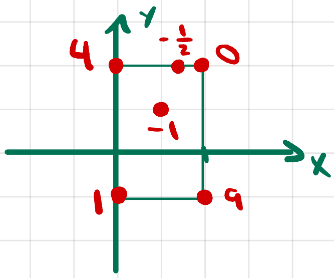
$f(0,2) = h(0) = 4$, $f(2,2) = h(2) = 8 - 12 + 4 = 0$

$x = 2 \Rightarrow f(2,y) = 8 - 4y + y^2 - 4 = g(y) = 4 - 4y + y^2$

$g' = -4 + 2y = 0 \Rightarrow y = 2$ $g'' = 2 \Rightarrow$ (2,2) min on this boundary

$f(2,2) = g(2) = 8 - 8 + 4 - 4 = 0$

$f(2,-1) = g(-1) = 8 + 4 + 1 - 4 = 9$



global max in R: $f(2,-1) = 9$

" min " " : $f(1,1) = -1$

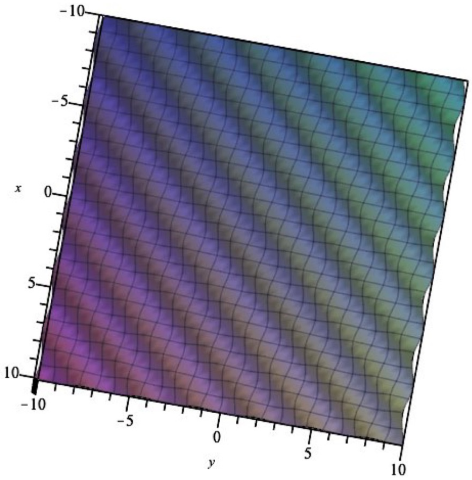
Problem 1 $\quad f(u) \quad F_c(x,t) = f(x-ct)$

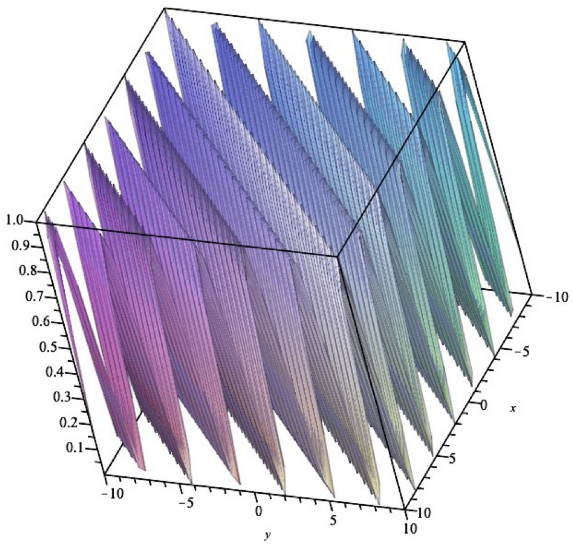
a) $f(u) = (\cos(u))^2$

$F(x,t) = f(x-ct) = \cos^2(x-ct)$

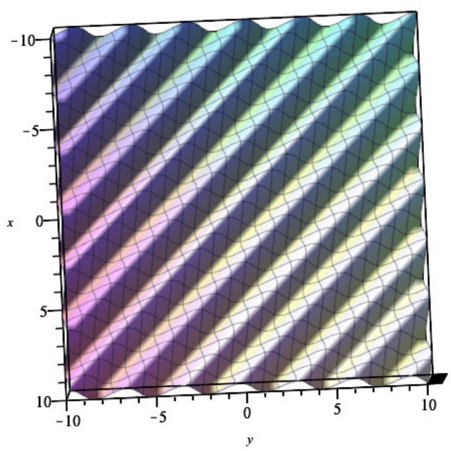
Plot in Maple

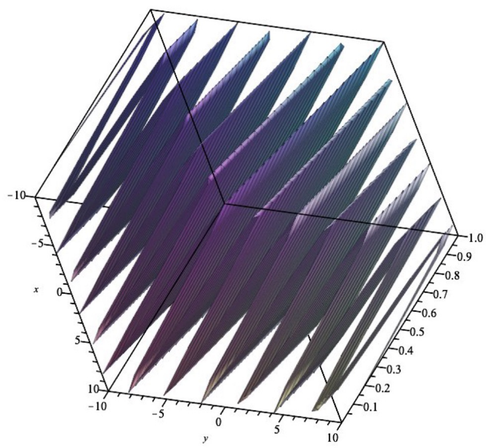
$c = -1$



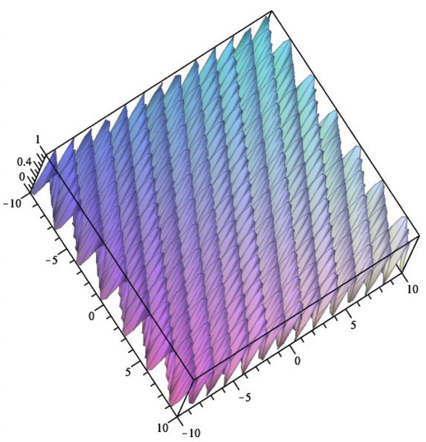


$c = 1$





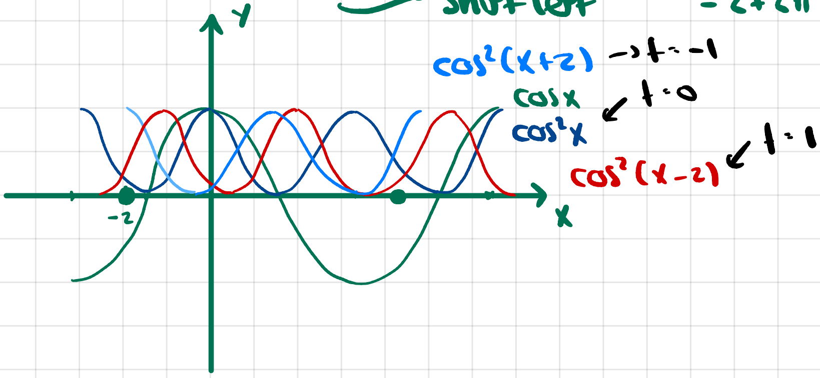
$c = 2$



$c = 2 \Rightarrow F(x,t) = f(x-2t) = \cos^2(x-2t)$

If we fix t we get $F_t(x) = \cos^2(x-2t)$

$t = -1$ $\quad F_t = \cos^2(x+2t)$ → shift left $\quad -2+2\pi$



t parameter shifts $\cos^2(x)$ left or right ($t<0 \Rightarrow$ left shift, $t>0 \Rightarrow$ right shift)

b) $F_t(x,t)$ describes vibration of the string, in the sense that each point on the string moves up and down from 0 to 1 and back again, completing one of these "round trips" every $\frac{\pi}{2}$ units of time.

Problem 2

a) $z_1 = f(x,y) = x^2 - y^2$

Find curve of intersection in parametric form

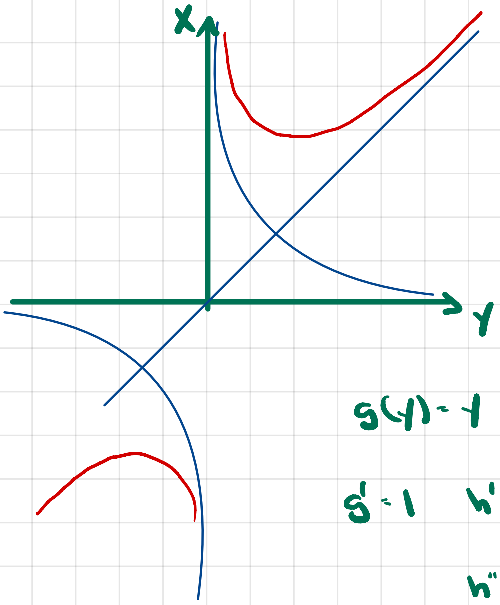
$z_2 = g(x,y) = 2 + (x-y)^2$

$z_1 = z_2 \Rightarrow \cancel{x^2} - y^2 = 2 + \cancel{x^2} - 2xy + y^2$

$2y^2 - 2xy + 2 = 0$

$y^2 - xy + 1 = 0$

$\Rightarrow x = \dfrac{y^2+1}{y} = y + \dfrac{1}{y}$



$g(y) = y \quad h(y) = \dfrac{1}{y}$

$g' = 1 \quad h' = \dfrac{-1}{y^2}$

$h'' = \dfrac{2}{y^3}$

$z_1 = f\left(y + \dfrac{1}{y}, y\right) = \cancel{y^2} + 2 + \dfrac{1}{y^2} - \cancel{y^2} = 2 + \dfrac{1}{y^2}$

$z_2 = g\left(y + \dfrac{1}{y}, y\right) = 2 + \left(\dfrac{1}{y}\right)^2 = 2 + \dfrac{1}{y^2}$

This curve of intersection can be parametrized:

$y(t) = t \qquad x(t) = t + \dfrac{1}{t} \qquad z(t) = 2 + \dfrac{1}{t^2} \qquad t \neq 0$

b) At $(2,1,3)$, let's obtain tangent planes, normal vectors, and angle between latter.

$f_x = 2x$
$f_y = -2y$

$z_1 - 3 = 4(x-2) - 2(y-1) \Rightarrow 4x - 8 - 2y + 2 + 3 - z_1 = 0$
$\Rightarrow 4x - 2y - z_1 = 3 \Rightarrow \vec{n}_1 = \langle 4, -2, -1 \rangle$

$g_x = 2(x-y)$
$g_y = -2(x-y)$

$z_2 - 3 = 2(x-2) - 2(y-1) \Rightarrow 2x - 4 - 2y + 2 + 3 - z_2 = 0$
$\Rightarrow 2x - 2y - z_2 = -1 \Rightarrow \vec{n}_2 = \langle 2, -2, -1 \rangle$

$|\vec{n}_1| = \sqrt{16+4+1} = \sqrt{21} \qquad |\vec{n}_2| = \sqrt{4+4+1} = 3$

$\vec{n}_1 \cdot \vec{n}_2 = 8 - 4 + 1 = 5 = \sqrt{21} \cdot 3 \cdot \cos\alpha \Rightarrow \cos\alpha = \dfrac{5 \cdot \sqrt{21}}{3 \cdot 21} = \dfrac{5\sqrt{21}}{63}$

$\Rightarrow \alpha = \arccos\left(\dfrac{5\sqrt{21}}{63}\right) \Rightarrow \alpha = 1.1986 \text{ rad} \approx 68.67°$

c) $\vec{r}(t) = \left\langle t + \dfrac{1}{t}, t, 2 + \dfrac{1}{t^2} \right\rangle \qquad \vec{r}(1) = \langle 2, 1, 3 \rangle$

$\vec{r}\,'(t) = \left\langle 1 - \dfrac{1}{t^2}, 1, -\dfrac{2}{t^3} \right\rangle \qquad \vec{r}\,'(1) = \langle 0, 1, -2 \rangle$

$\vec{r}\,'(1) \cdot \vec{n}_1 = \langle 0,1,-2 \rangle \langle 4,-2,-1 \rangle = -2 + 2 = 0$

$\vec{r}\,'(1) \cdot \vec{n}_2 = \langle 0,1,-2 \rangle \langle 2,-2,-1 \rangle = -2 + 2 = 0$

$\Rightarrow \vec{r}\,'(1)$ is on both tangent planes

Problem 3

$f(x,y) = \dfrac{4}{1+x^2+y^2}$, S the surface given by graph of f

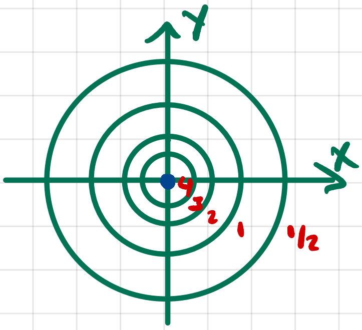
a) $\dfrac{4}{1+x^2+y^2} = c \Rightarrow x^2+y^2 = \dfrac{4}{c} - 1$

$c=3 \quad r^2 = 1/3 \quad r = \sqrt{3}/3 \approx 0.57$
$c=2 \quad r^2 = 1 \quad r = 1$
$c=1 \quad r^2 = 3 \quad r = \sqrt{3} \approx 1.7$
$c = 1/2 \quad r^2 = 7 \quad r = \sqrt{7} \approx 2.65$

$c = 3.25 \quad r^2 = \frac{3}{13} \quad r \approx 0.48$

$c = 3.5 \quad r^2 = \frac{8}{7} - 1 = \frac{1}{7} \quad r \approx .38$

$c = 3.75 \Rightarrow r \approx 0.258$



$f_x = \dfrac{-4 \cdot 2x}{(1+x^2+y^2)^2}$

$f_y = \dfrac{-4 \cdot 2y}{(1+x^2+y^2)^2}$

(0,0) is a critical point, tangent plane horizontal.

$$f_{xx} = \frac{-8(1+x^2+y^2)^2 + 8x \cdot 2(1+x^2+y^2) \cdot 2x}{(1+x^2+y^2)^4} = \frac{-8-8x^2-8y^2+32x^2}{(1+x^2+y^2)^3}$$

$$= \frac{24x^2 - 8y^2 - 8}{(1+x^2+y^2)^3}$$

$3x^2 - y^2 = 1 \quad y^2 = 3x^2 - 1$

$3y^2 - x^2 = 1 \quad 9x^2 - 3 - x^2 = 1$

$8x^2 = 4$

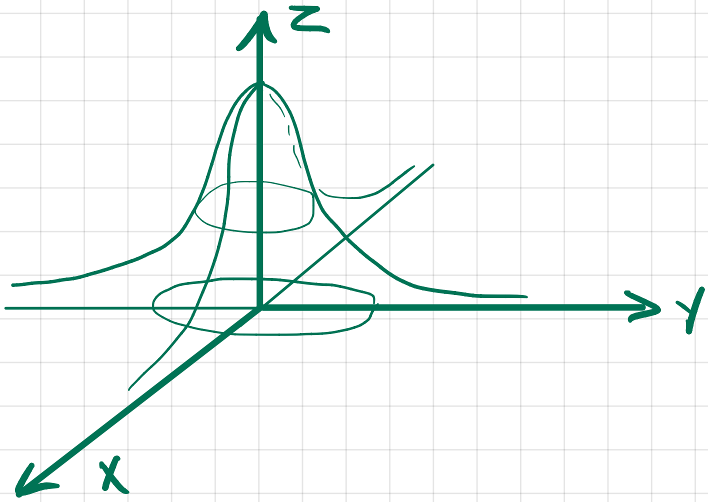
$x^2 = 1/2 \Rightarrow x = \pm 1/\sqrt{2}$

$$f_{yy} = \frac{24y^2 - 8x^2 - 8}{(1+x^2+y^2)^3}$$

$$f_{xy} = -4 \cdot 2x(-2)(1+x^2+y^2)^{-3} \cdot 2y = \frac{32xy}{(1+x^2+y^2)^3}$$

At (0,0) $\quad f_{xx}(0,0) = -8 \quad f_{yy} = -8 \quad f_{xy} = 0 \quad \Rightarrow \Delta = -8(-8) - 0 = 64$

$\Delta > 0$, $f_{xx} < 0 \Rightarrow$ (0,0) is a local max



b) C_2 curve in xy plane, projection on xy plane of curve C of surface S

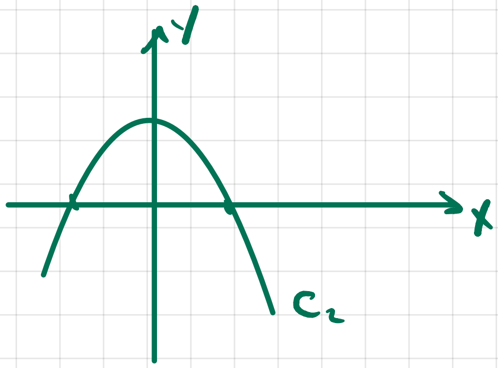
$\vec{r}_2(t) = \langle t, \frac{3}{2} - t^2 \rangle$

Recall: $f(x,y) = \dfrac{4}{1+x^2+y^2}$, S the surface given by graph of f

$$z(t) = \frac{4}{1+t^2+(\frac{3}{2}-t^2)^2}$$

$$\vec{r}(t) = \left\langle t, \frac{3}{2}-t^2, \frac{4}{1+t^2+(\frac{3}{2}-t^2)^2} \right\rangle$$

c)

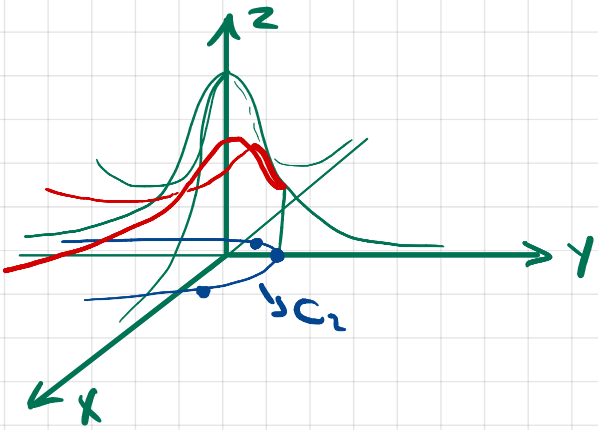


$x^2 + y = t^2 + \frac{3}{2} - t^2 = \frac{3}{2}$

$y = -x^2 + \frac{3}{2}$

$\Delta = -4(-1) \cdot \frac{3}{2} = 6$

$x = \dfrac{\pm\sqrt{6}}{-2}$



d) $$z(t) = \frac{4}{1+t^2+(\frac{3}{2}-t^2)^2}$$

$$z'(t) = \frac{-4}{(1+t^2+(\frac{3}{2}-t^2)^2)^2} \cdot \left(2t + 2(\tfrac{3}{2}-t^2)(-2t)\right)$$

$$= \frac{-4(2t-6t+4t^3)}{(1+t^2+(\frac{3}{2}-t^2)^2)^2} = \frac{-4(4t^3-4t)}{(1+t^2+(\frac{3}{2}-t^2)^2)^2} = \frac{-16t(t^2-1)}{(1+t^2+(\frac{3}{2}-t^2)^2)^2} = 0$$

$\Rightarrow t=0$ or $t=1$ or $t=-1$

$\vec{r}(1) = \langle 1, 1/2, 16/9 \rangle$

$\vec{r}(0) = \langle 0, 3/2, 16/13 \rangle$

Problem 4

$x,y,z \ge 0 \quad x^2+y^2+z^2=27 \quad f(x,y,z)=x^3+y^3+z^3$

$f(x,y)=x^3+y^3+(27-x^2-y^2)^{3/2}$

$f_x = 3x^2 + \frac{3}{\cancel{2}}(27-x^2-y^2)^{1/2}\cdot(-\cancel{2}x)$ $\quad f_y = 3y(y-\sqrt{27-x^2-y^2})$

$= 3x^2 - 3x(27-x^2-y^2)^{1/2} = 0$ $\quad \Rightarrow y=0$ OR $2y^2=27-x^2$

$3x(x-\sqrt{27-x^2-y^2})=0$

$\Rightarrow x=0$ OR

$x^2=27-x^2-y^2 \Rightarrow 2x^2=27-y^2$

critical points

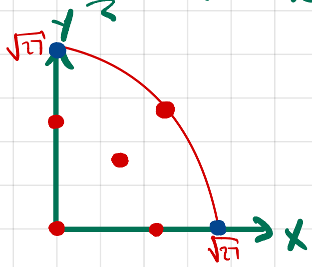
1) $x=0,\ y=0,\ z=\sqrt{27}$

2) $x=0,\ y=\sqrt{27/2},\ z^2+27/2=27 \Rightarrow z^2=27/2 \Rightarrow z=\sqrt{27/2}$

3) $y=0,\ x=\sqrt{27/2},\ z=\sqrt{27/2}$

4) $2y^2=27-\dfrac{(27-y^2)}{2} \Rightarrow 4y^2=54-27+y^2 \Rightarrow 3y^2=27$

$y^2=9 \Rightarrow y=3 \Rightarrow x^2=\dfrac{27-9}{2}=9 \Rightarrow x=3 \Rightarrow z^2=27-18 \Rightarrow z=3$



$(0,0,\sqrt{27})$

$(0,\sqrt{27/2},\sqrt{27/2})$

$(\sqrt{27/2},0,\sqrt{27/2})$

$(3,3,3)$

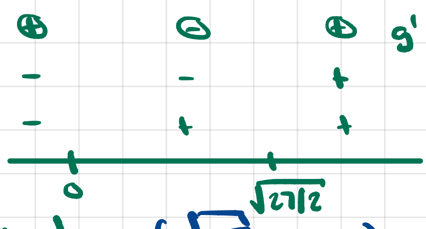
boundary points

$y=0$

$f(x,0)=g(x)=x^3+(27-x^2)^{3/2}$

$g'(x)=3x^2+\frac{3}{\cancel{2}}(27-x^2)^{1/2}\cdot(-\cancel{2}x)$

$=3x^2-3x(27-x^2)^{1/2}=3x(x-(27-x^2)^{1/2})$



$x=0$ or $x^2=27-x^2 \Rightarrow x^2=27/2 \Rightarrow x=\sqrt{27/2}$

$x=\sqrt{27} \Rightarrow (\sqrt{27},0,0)$ must be checked. $\quad (\sqrt{27},0,0)$

$x=0$

$f(0,y)=y^3+(27-y^3)^{3/2}=h(y)$ By symmetry, $h'(y)=0$ for $y=0$ and $y=\sqrt{27/2}$.

$y=\sqrt{27}$ must be checked as well $\quad (0,\sqrt{27},0)$

$x^2+y^2=27$

$\Rightarrow f(x,y)=x^3+y^3$

$g(x)=x^3+(27-x^2)^{3/2}$

$g'(x)=3x^2+\frac{3}{\cancel{2}}(27-x^2)^{1/2}(-\cancel{2}x)=3x^2-3x(27-x^2)^{1/2}=0$

$3x(x-(27-x^2)^{1/2})=0$

$x=0$ OR $x^2=27-x^2 \Rightarrow x^2=27/2$, $x=\sqrt{27/2}$

$\Rightarrow y=(27-27/2)^{1/2}=\sqrt{27/2}$

$(\sqrt{27/2},\sqrt{27/2},0)$

We have seven points to check $\quad f(x,y)=x^3+y^3+(27-x^2-y^2)^{3/2}$

$(0,0,\sqrt{27})$ $\quad f(0,0,\sqrt{27})=\sqrt{27^3}=\sqrt{3^9}=3^4\sqrt{3}=81\sqrt{3}$

$(0,\sqrt{27/2},\sqrt{27/2})$ $\quad f(0,\sqrt{27},0)=\sqrt{27^3}+\cancel{(27-27)}=81\sqrt{3}$

$(\sqrt{27/2},0,\sqrt{27/2})$ $\quad f(\sqrt{27},0,0)=81\sqrt{3}$ by symmetry

$(3,3,3)$ $\quad f(\sqrt{27/2},\sqrt{27/2},0)=2\sqrt{(27/2)^3}=\sqrt{\frac{27^3}{2^3}\cdot 2^2}$

$(\sqrt{27/2},\sqrt{27/2},0)$ $\quad =\sqrt{27^3/2}=\frac{81\sqrt{3}}{\sqrt{2}}$

$(\sqrt{27},0,0)$ $\quad =f(0,\sqrt{27/2},\sqrt{27/2})=f(\sqrt{27/2},0,\sqrt{27/2})$

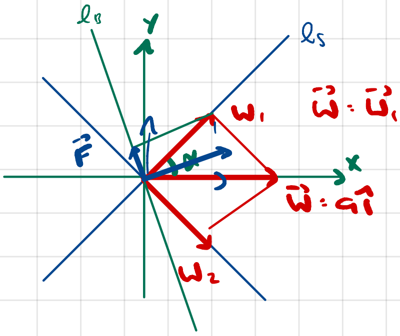
$(0,\sqrt{27},0)$ $\quad f(3,3,3)=27+27+27=81$

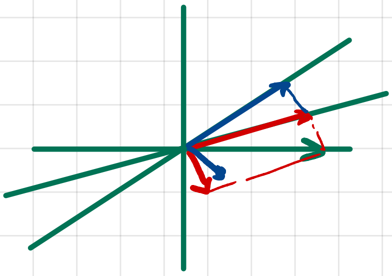
Max when two numbers are zero and the third is $\sqrt{27}$.

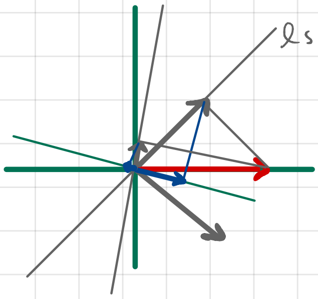
Min when $x=y=z=3$.

Max value is $81\sqrt{3}$, min value is 81.

Problem 5







$\text{comp}_{\ell_s} \vec{\omega} = |\vec{\omega}| \cos\alpha = |a| \cos\alpha$

$\text{comp}_{\ell_\beta} \vec{\omega}_1 = |\vec{\omega}_1| \cdot \cos\beta = |\vec{\omega}| \cos\alpha \cos\beta = |a| \cos\alpha \cos\beta$

$\text{comp}_{\hat{i}} \vec{F} = |\vec{F}| \cdot \cos(\alpha+\beta) = |a| \cos\alpha \cos\beta \cos(\alpha+\beta)$

a) $\vec{\omega} = \langle 1,0,0 \rangle \Rightarrow a = 1$

$\text{comp}_{\hat{i}} \vec{F} = C_F(\alpha,\beta) = \cos\alpha \cos\beta \cos(\alpha+\beta)$

$C_{F_\alpha} = \cos\beta(-\sin\alpha \cos(\alpha+\beta) - \cos\alpha \sin(\alpha+\beta))$

$= -\cos\beta [\sin\alpha \cos(\alpha+\beta) + \cos\alpha \sin(\alpha+\beta)] = 0 \Rightarrow \cos(\alpha+\beta) = -\dfrac{\cos\alpha \sin(\alpha+\beta)}{\sin\alpha}$ (a)

$C_{F_\beta} = -\cos\alpha [\sin\beta \cos(\alpha+\beta) + \cos\beta \sin(\alpha+\beta)] = 0$

$\cos\beta = 0 \Rightarrow \beta = 0 \Rightarrow C_{F_\beta} = -\cos\alpha[\cos(\alpha)] \Rightarrow \cos\alpha = 0 = \alpha = 0$

$\cos\alpha = 0 \Rightarrow \alpha = 0 \Rightarrow C_{F_\alpha} = \cos\beta(-\cos\beta) \Rightarrow \cos\beta = 0 \Rightarrow \beta = 0$

or, substituting $\cos(\alpha+\beta)$ into C_{F_β}:

$$-\cos\alpha \left[-\frac{\sin\beta \cos\alpha \sin(\alpha+\beta)}{\sin\alpha} + \cos\beta \sin(\alpha+\beta) \right] = 0$$

$\cos\alpha = 0 \Rightarrow \cos\beta = 0$ as calculated previously

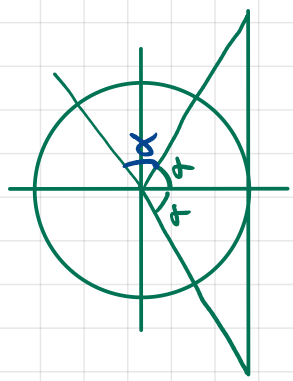
$\sin(\alpha+\beta)\left[\cos\beta - \cos\alpha \dfrac{\sin\beta}{\sin\alpha}\right] = 0 \Rightarrow \sin(\alpha+\beta) = 0$ OR $\cos\beta \cdot \sin\alpha = \cos\alpha \sin\beta$

$\Rightarrow \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{\sin\beta}{\cos\beta} \Rightarrow \tan\alpha = \tan\beta \Rightarrow \alpha = \beta$

$\Rightarrow$ in (a), $\tan(\alpha+\beta) = -\tan\alpha$

$\Rightarrow \tan 2\alpha = -\tan\alpha = -\tan\beta$

$\Rightarrow \alpha = \pi/3$ using geometry:

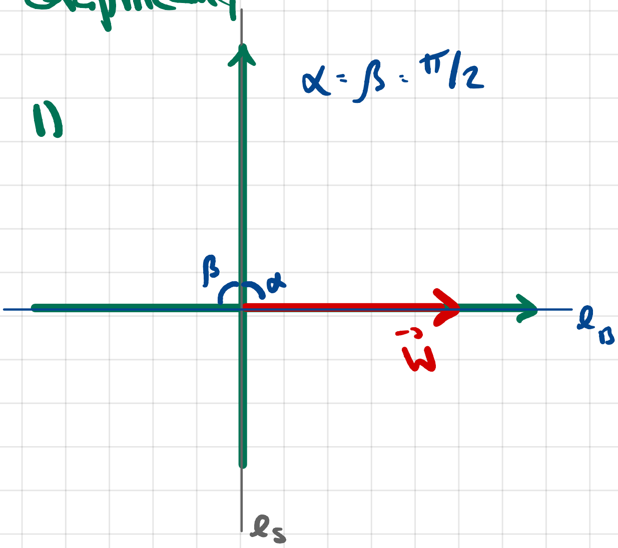


critical point conditions

1) $\cos\beta = \cos\alpha = 0 \Rightarrow \alpha = \beta = \pi/2$
2) $\sin(\alpha+\beta) = 0 \Rightarrow \alpha+\beta = 0$ or $\Rightarrow \alpha = \beta = 0$ or $\alpha = \beta = \pi/2$
3) $\alpha = \beta = 60°$

Graphically

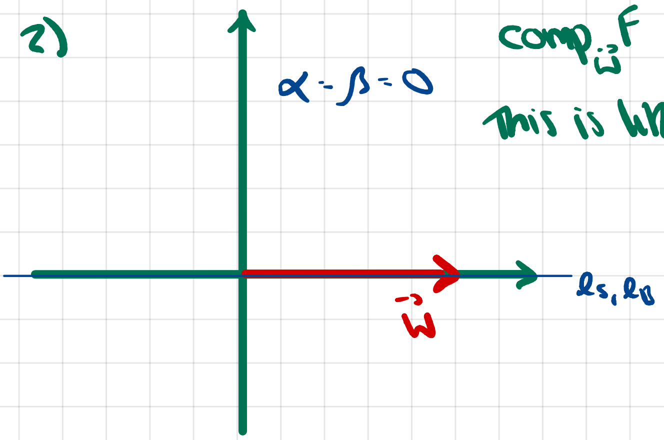
1)



$\alpha = \beta = \pi/2$

$\text{comp}_{\vec{w}}\vec{F} = 0$

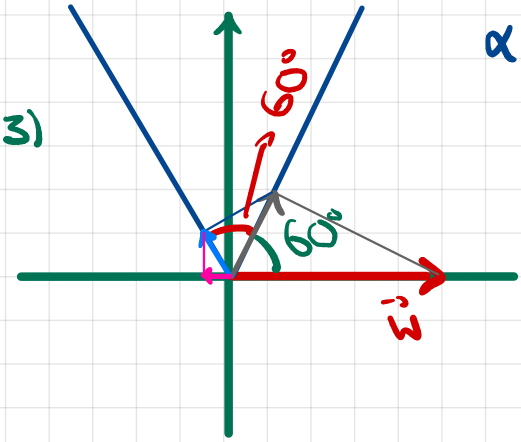
2)



$\alpha = \beta = 0$

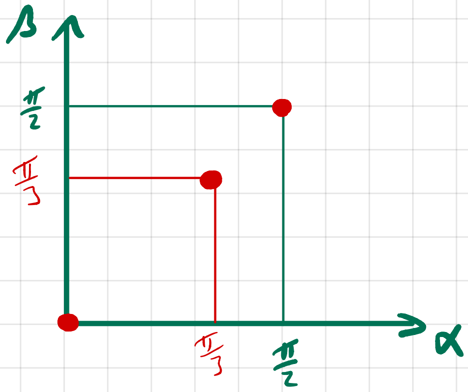
$\text{comp}_{\vec{w}}\vec{F} = 1 = |\vec{w}|$, all projections are on the same line. This is likely a max for $C_F(\alpha,\beta)$. Note we are looking for the minimum value of $C_F(\alpha,\beta)$, ie a negative projection onto $\vec{w}$.

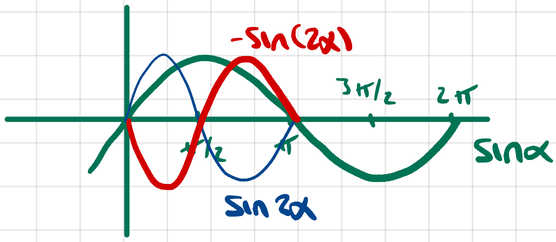
3)



$\alpha = \beta = 60°$

Note the domain of C_F $\quad C_F(\alpha,\beta) = \cos\alpha \cos\beta \cos(\alpha+\beta)$





$\beta = 0 \Rightarrow C_F(\alpha,0) = g(\alpha) = \cos^2\alpha$ $\qquad$ always <0 in $[0,\pi/2$

$g'(\alpha) = 2\cos\alpha(-\sin\alpha) = -2\cos\alpha\sin\alpha = -\sin(2\alpha) = 0 \Rightarrow 2\alpha = 0 \Rightarrow \alpha = 0$

$\alpha = 0 \Rightarrow C_F(0,\beta) = h(\beta) = \cos^2\beta \Rightarrow h'(\beta) = -\sin(2\beta)$, by symmetry.

$(0,0)$ is a single point on the boundary to consider for max/min.

$\lim\limits_{\alpha\to\pi/2} C_F(\alpha,\beta) = \lim\limits_{\beta\to\pi/2} C_F(\alpha,\beta) = 0$

Let us check the candidate points:

$C_F(0,0) = 1$

$C_F(\pi/2,\pi/2) = 0$

$C_F(\pi/3,\pi/3) = \frac{1}{2}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right) = -\frac{1}{8}$

$\alpha = \beta = \pi/3$ gives most negative $\hat{\imath}$ component of $\vec{u}_2$.

b) $1/8$

$$C_{F_\alpha} = -\cos\beta\left[\sin\alpha\cos(\alpha+\beta) + \cos\alpha\sin(\alpha+\beta)\right]$$

$$C_{F_\beta} = -\cos\alpha\left[\sin\beta\cos(\alpha+\beta) + \cos\beta\sin(\alpha+\beta)\right]$$

$$C_{F_{\alpha\alpha}} = -\cos\beta\left[\cos\alpha\cos(\alpha+\beta) + \sin\alpha(-\sin(\alpha+\beta)) + \sin\alpha\sin(\alpha+\beta) + \cos\alpha\cos(\alpha+\beta)\right]$$

$$= -\cos\beta\left[2\cos\alpha\cos(\alpha+\beta)\right] = -2\cos\alpha\cos\beta\cos(\alpha+\beta)$$

$$\Rightarrow C_{F_{\beta\beta}} = C_{F_{\alpha\alpha}}$$

$$C_{F_{\alpha\beta}} = \sin\beta\left[\sin\alpha\cos(\alpha+\beta) + \cos\alpha\sin(\alpha+\beta)\right] - \cos\beta\left[\sin\alpha(-\sin(\alpha+\beta)) + \cos\alpha\cos(\alpha+\beta)\right]$$