Problem Set 8

4A-1

a) $\vec{F} = \langle a, b \rangle$ = constant vectors at each point in vector field

$$\frac{d\vec{F}}{dx} = \frac{d\vec{F}}{dy} = \langle 0, 0 \rangle \text{, a continuous function}$$

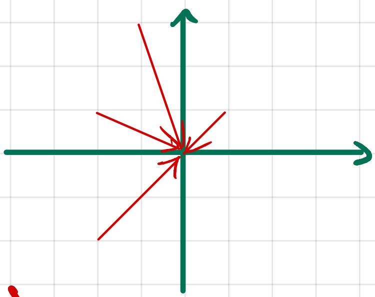
$\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\lim_{(x,y)\to(x_0,y_0)} \vec{F}(x,y) = \langle a,b \rangle = \vec{F}(x_0,y_0) \Rightarrow \vec{F} \text{ continuous.}$$

$$\lim_{(x,y)\to(x_0,y_0)} \vec{F}'(x,y) = \langle 0,0 \rangle = \vec{F}'(x_0,y_0) \Rightarrow \vec{F}'(x,y) \text{ is continuous.}$$

$\Rightarrow \vec{F}$ cont. differentiable in $\mathbb{R}^2$.

b) $\vec{F}(x,y) = \langle -x, -y \rangle = -(x,y)$



$F : \mathbb{R}^2 \to \mathbb{R}^2$

$$D\vec{F} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

continuous partial derivatives everywhere

$\Rightarrow$ continuously differentiable vector field

* Derivative of $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ using limits

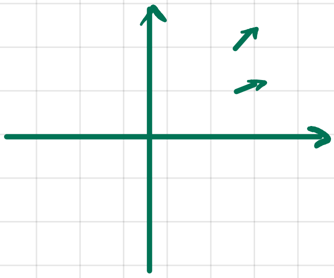
$\vec{p} = \langle x, y \rangle$ $\quad \vec{h} = \langle h_1, h_2 \rangle$

$$\lim_{\vec{h}\to\vec{0}} \frac{\vec{F}(\vec{p}+\vec{h}) - F(\vec{p}) - DF(\vec{p})\vec{h}}{|\vec{h}|} = \lim_{\vec{h}\to\vec{0}} \frac{\begin{bmatrix} -x+h_1 \\ -y+h_2 \end{bmatrix} - \begin{bmatrix} -x \\ -y \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}}{\sqrt{h_1^2+h_2^2}}$$

$$= \lim_{\vec{h}\to\vec{0}} \frac{\begin{bmatrix} h_1+h_1 \\ h_2+h_2 \end{bmatrix}}{\sqrt{h_1^2+h_2^2}} = \lim_{\vec{h}\to\vec{0}} \begin{bmatrix} \dfrac{2h_1}{\sqrt{h_1^2+h_2^2}} \\ \dfrac{2h_2}{\sqrt{h_1^2+h_2^2}} \end{bmatrix} = \lim_{\vec{h}\to\vec{0}} \begin{bmatrix} \dfrac{2}{\sqrt{1+(h_2/h_1)^2}} \\ \dfrac{2}{\sqrt{1+(h_1/h_2)^2}} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

differentiable for any point (x,y)

c) $\vec{F} = \langle \frac{x}{r}, \frac{y}{r} \rangle = \langle g(x,y), h(x,y) \rangle$



unit vectors pointing outwards radially

$$g(x,y) = \frac{x}{\sqrt{x^2+y^2}}$$

$$g_x = \frac{\sqrt{x^2+y^2} - x \cdot \frac{2x}{2\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{x^2+y^2-x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^{3/2}}$$

near $(0,0)$ $\lim\limits_{\vec{p} \to \vec{0}} \frac{y^2}{(x^2+y^2)^{3/2}}$ does not exist. We can check by using two different paths to $(0,0)$ and verifying two different results for the limit. With $y=0$, $\lim\limits_{(x,y) \to (0,0)} = 0$, with $x=0$ we have $\lim y^2/y^3 = \lim 1/y = \infty$

$\Rightarrow \vec{F}$ is not continuously differentiable at $(0,0)$.

Note the other partial derivatives:

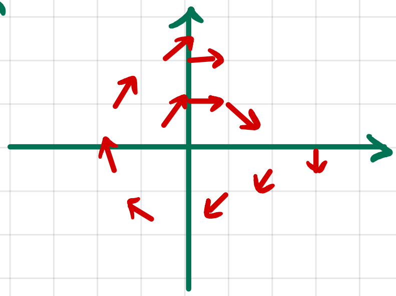
$$g_y = \frac{\frac{-x \cdot 2y}{2\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{-xy}{(x^2+y^2)^{3/2}} \qquad h_x = \frac{x^2}{(x^2+y^2)^{3/2}} \qquad h_y = \frac{-xy}{(x^2+y^2)^{3/2}}$$

g_x, g_y, h_x, h_y continuous everywhere except $(0,0)$

$\Rightarrow \vec{F}$ cont. diff. everywhere except $(0,0)$.

d) $\vec{F}(x,y) = \langle y/r, -x/r \rangle = \langle g(x,y), h(x,y) \rangle = \langle \frac{y}{\sqrt{x^2+y^2}}, \frac{-x}{\sqrt{x^2+y^2}} \rangle$

unit vectors tangent to the circle centered at origin passing through (x,y), clockwise direction.



alternatively, unit position vector rotated 90° clockwise.

$g_x = \frac{x^2}{\sqrt{x^2+y^2}}$ $\quad g_y = \frac{-xy}{\sqrt{x^2+y^2}}$

$h_x = \frac{xy}{\sqrt{x^2+y^2}}$ $\quad h_y = \frac{-y^2}{\sqrt{x^2+y^2}}$

=> cont. diff $\vec{F}$ except at $(0,0)$

4A-2

a) $w(x,y) = ax+by \qquad \vec{\nabla} w(x,y) = \langle a, b \rangle$

b) $w(x,y) = \ln(\sqrt{x^2+y^2}) \qquad \vec{\nabla} w = \langle \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}x, \frac{y}{x^2+y^2} \rangle$

$$= \langle \frac{x}{r^2}, \frac{y}{r^2} \rangle$$

c) $w(x,y) = f(r)$

$w_x = \frac{df}{dr} \cdot \frac{dr}{dx} = f'(r) \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = f'(r) \cdot \frac{x}{\sqrt{x^2+y^2}}$

$w_y = f'(r) \cdot \frac{y}{r}$

$\vec{\nabla} w = \langle f'(r) \frac{x}{r}, f'(r) \frac{y}{r} \rangle$

4A-3

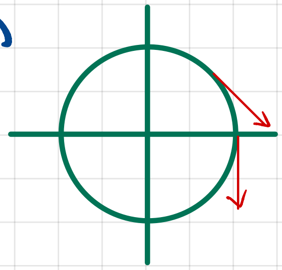
a) $\vec{v} = \langle 1,2 \rangle$ $\quad \vec{F}(x,y) = \langle 1,2 \rangle$

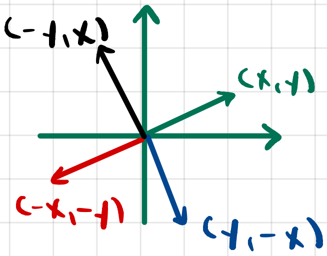
b) At (x,y) direction is $\langle -x,-y \rangle$

$\vec{v} = k\langle -x,-y \rangle$ $\quad \Rightarrow \vec{F}(x,y) = r\langle -x,-y \rangle$

$|\vec{v}| = k\sqrt{x^2+y^2} = r^2 = x^2+y^2 \Rightarrow k = \sqrt{x^2+y^2} = r$

c)





rotation by 90° clockwise T

$T(x,y) = \langle v,u \rangle = \langle y,-x \rangle$

$\Rightarrow$ At (x,y) the vector field vector is $\langle y,-x \rangle$

$|k\langle y,-x \rangle| = 1/r^2$

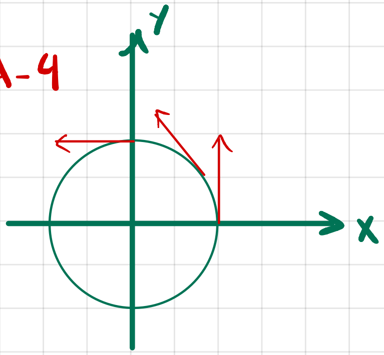
$k\sqrt{y^2+x^2} = \dfrac{1}{r^2} = kr \Rightarrow k = \dfrac{1}{r^3}$

$\Rightarrow \vec{F}(x,y) = \dfrac{\langle y,-x \rangle}{r^3}$

d) $\vec{F} = \langle 1,1 \rangle \cdot f(x,y) = \langle f(x,y), f(x,y) \rangle$

$|\vec{F}| = \sqrt{2}\, f(x,y)$

4A-4



rotation counterclockwise 90° $T(\langle x,y \rangle) = \langle -y,x \rangle$

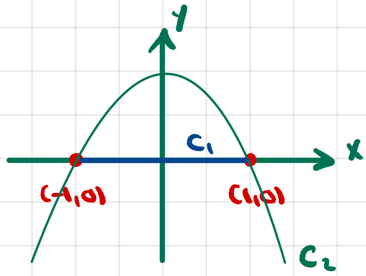
$\vec{F} = c\langle -y,x \rangle$

$|\vec{F}| = cr = \dfrac{k}{r} \Rightarrow c = \dfrac{k}{r^2}$

$\vec{F} = \dfrac{k}{r^2}\langle -y,x \rangle$

4B-1 $\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\hat{T}\,ds$

a) $\vec{F} = \langle x^2-y, 2x\rangle$



interpretation

→ we want the work done by the force field $\vec{F}$ on a particle moving from $(-1,0)$ to $(1,0)$ along two different paths C_1 and C_2.

$\vec{F}$ on a specific point in i^{th} subinterval of $t \in [a,b]$

$$\int_C \vec{F}\cdot\hat{T}\,ds = \lim_{\Delta t\to 0}\sum_{i=1}^{n} \vec{F}(x(t_i^*),y(t_i^*))\cdot\hat{T}(x(t_i^*),y(t_i^*))\cdot\Delta s_i = \lim_{\Delta t\to 0}\sum_{i=1}^{n}\Delta W_i$$

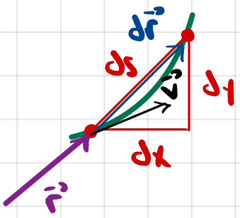
distance travelled

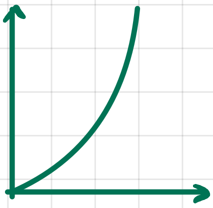
velocity vector

where we partitioned the range of parameter t that parametrizes C.

* what is $d\vec{r}$?

$$\hat{T}\,ds = \frac{\vec{v}}{\cancel{|\vec{v}|}}\cancel{|\vec{v}|}\,dt = \frac{d\vec{r}}{dt}\cdot dt = d\vec{r}$$





$\vec{r}(t) = \langle t, t^2\rangle$

$\vec{v}(t) = \langle 1, 2t\rangle$

$\hat{T}(t) = \frac{\langle 1, 2t\rangle}{\sqrt{1+4t^2}}$

$$ds = \sqrt{dx^2+dy^2} = \sqrt{(dx/dt)^2+(dy/dt)^2}\cdot dt = |\vec{v}|\,dt$$

when we think of ds, dx, dy as infinitesimals then the red ds is ∥ to $\vec{v}$ and it's length is distance traversed in dt.

$$d\vec{r} = \hat{T}\cdot ds = \frac{\langle 1,2t\rangle}{\sqrt{1+4t^2}}\cdot\sqrt{1+4t^2}\,dt = \langle dt, 2t\,dt\rangle$$

$d\vec{r}$ is the change in position vector. Its magnitude ds and direction $\hat{T}$.

$$|d\vec{r}| = \sqrt{dt^2+4t^2dt^2} = \sqrt{1+4t^2}\,dt = |\vec{v}(t)|\,dt = ds$$

solution $\vec{F} = \langle x^2-y, 2x\rangle$ $\quad C_1$: $\vec{r}(t) = \langle -1+2t, 0\rangle$

$\vec{v}(t) = \langle 2, 0\rangle$

$$\int_{C_1}\langle x^2-y, 2x\rangle\cdot\langle 2,0\rangle\,dt = \int_0^1 2(-1+2t)^2dt = \int_0^1 2(4t^2-4t+1)dt = 2\left[4\frac{t^3}{3}-2t^2+t\right]_0^1$$

$$= 2\left[\frac{4}{3}-2+1\right] = 2\,\frac{4-6+3}{3} = \frac{2}{3}$$

Alternatively,

$$\int\langle x^2-y, 2x\rangle\langle dx/dt, dy/dt\rangle\,dt = \int_{C_1}(x^2-y)\,dx + 2x\,dy$$

$y=0 \Rightarrow dy=0 \qquad \int_{-1}^{1}(x^2dx + 2x\cdot 0) = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3}-\left(-\frac{1}{3}\right) = \frac{2}{3}$

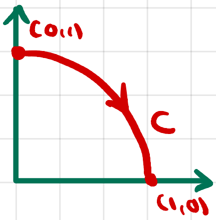
$C_2: \; y=1-x^2 \qquad \vec{r}(t)=\langle t, 1-t^2\rangle \quad t\in[-1,1]$

$$\int_{C_2}\vec{F}\,d\vec{r} = \int_{C_2}(x^2-y)\,dx + 2x\,dy = \int_{-1}^{1}\Big[[t^2-1+t^2]\,dt + 2t(-2t)\,dt\Big] = \int_{-1}^{1}(-1-2t^2)\,dt$$

$dx = dt \quad dy = -2t\,dt$

$$= -t-\frac{2}{3}t^3\Big|_{-1}^{1} = \left(-1-\frac{2}{3}\right)-\left(1+\frac{2}{3}\right) = -2-\frac{4}{3}$$

b) $\vec{F} = \langle xy, -x^2\rangle$



$y=\sqrt{1-x^2} \quad x\in[0,1]$

$\vec{r}(t) = \langle t, \sqrt{1-t^2}\rangle$

$r=1$

$x=\cos\theta \quad dx = -\sin\theta\,d\theta \qquad \vec{r}(t) = \langle\cos\theta, \sin\theta\rangle \quad \theta\in[0,\pi/2]$

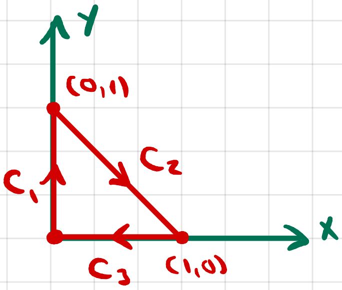
$y=\sin\theta \quad dy = \cos\theta\,d\theta$

$$\int_C \vec{F}\,d\vec{r} = \int_C xy\,dx - x^2\,dy = \int_0^{\pi/2}\cos\theta\sin\theta(-\sin\theta\,d\theta) + \cos^2\theta\cos\theta\,d\theta$$

$$= \int_{\pi/2}^{0}(-\cos\theta\sin^2\theta - \cos^3\theta)\,d\theta = \int_{\pi/2}^{0}\left[-\cos\theta + \cancel{\cos^3\theta} - \cancel{\cos^3\theta}\right]d\theta = -\int_{\pi/2}^{0}\cos\theta\,d\theta$$

$$= -\sin\theta\Big|_{\pi/2}^{0} = -(-1) = 1$$

c) $\vec{F} = \langle y, -x\rangle$



$$\int_C \vec{F}\,d\vec{r} = \int_{C_1}\vec{F}\,d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r} + \int_{C_3}\vec{F}\,d\vec{r}$$

$$\int_{C_1}\vec{F}\,d\vec{r} = \int_0^1 y\cdot 0 - 0\cdot dy = 0$$

C is piecewise smooth

$$\int_{C_2}\vec{F}\,d\vec{r} = \int_{C_2} y\,dx - x\,dy, \quad \begin{matrix} y = 1-x \\ dy = -dx \end{matrix} \Rightarrow \int_{C_2}(1-x)\,dx + x\,dx = \int_0^1 dx = 1$$

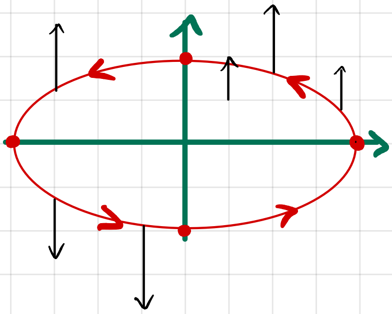
$$\int_{C_3}\vec{F}\,d\vec{r} = 0$$

$$\int_C \vec{F}\,d\vec{r} = 1$$

d) $\vec{F} = \langle y, 0 \rangle$

C: $x = 2\cos t$
$y = \sin t$

$dx = -2\sin t\, dt$



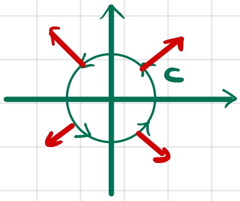
t	x	y
0	2	0
$\pi/2$	0	1
π	-2	0
$3\pi/2$	0	-1
2π	2	0

$$\int_C \vec{F}\, d\vec{r} = \int_0^{2\pi} \sin t(-2\sin t)dt = \int_0^{2\pi} -2\sin^2 t\, dt = -(t - \sin t\cos t)\Big|_0^{2\pi}$$

$$= -[2\pi - 0] - [-(0-0)] = -2\pi$$

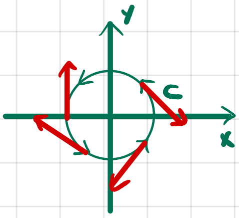
4B-2

a) $\vec{F} = \langle x, y \rangle$ radially outward vectors



$\int_C \vec{F}\, d\vec{r} = 0$

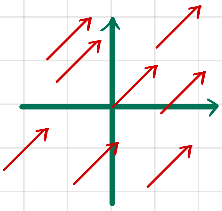
b) $\vec{F} = \langle y, -x \rangle$



$$\int_C \vec{F}\, d\vec{r} = \int_C y\,dx - x\,dy = \int_0^{2\pi} a\sin\theta \cdot (-a\sin\theta\, d\theta) - a\cos\theta\,(a\cos\theta\, d\theta)$$

$$= \int_0^{2\pi} (-a^2\sin^2\theta - a^2\cos^2\theta)d\theta = \int_0^{2\pi} -a^2\, d\theta = -a^2(2\pi)$$

4B-3 $\vec{F} = \langle 1, 1 \rangle$



a) $\int \vec{F} \cdot d\vec{r}$

$= \lim \sum_{i=1}^{n} \Delta W_i = \lim \sum \vec{F}(t_i) \cdot \vec{v}(t_i) \cdot \Delta s_i = \lim \sum \vec{F}(t_i) \cdot \hat{T} \cdot dt = \lim \sum |\vec{F}(t_i)| \cdot \cos\theta\, dt$

max when $\theta = 0 \Rightarrow \hat{T}$ and $\vec{F}$ have same direction: $\operatorname{dir}(\hat{T}) = \operatorname{dir}(\vec{F}) = \hat{u}$

$\Rightarrow \vec{v}(t) = h(t)\langle 1, 1 \rangle,\ h(t) > 0 \Rightarrow \vec{r}(t) = H(t)\langle 1, 1 \rangle \quad H(t) = \int h(t)dt$

b) $\cos\theta = -1 \Rightarrow \theta = \pi$, $\operatorname{dir}(\hat{T}) = -\operatorname{dir}(\vec{F})$

$\vec{v}(t) = h(t)\langle -1, -1 \rangle,\ h(t) > 0 \Rightarrow \vec{r}(t) = H(t)\langle -1, -1 \rangle \quad H(t) = \int h(t)dt$

c) $\cos\theta = 0 \Rightarrow \theta = \pi/2$

$\langle 1, 1 \rangle \cdot \vec{v}(t) = 0$, for $\vec{v}(t) = \langle v_1(t), v_2(t) \rangle$ this means $v_1(t) + v_2(t) = 0$

$\Rightarrow v_1(t) = -v_2(t) \quad \Rightarrow \vec{v}(t) = \langle v_1(t), -v_1(t) \rangle = h(t)\langle 1, -1 \rangle$

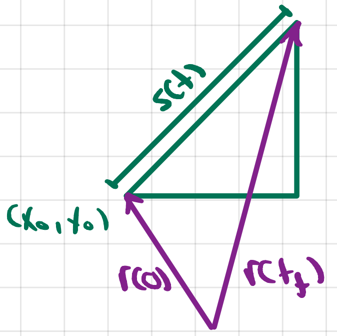
d) C has length 1, parallel to $\langle 1,1\rangle$, can be parametrized

$\vec{r}(t) = \langle r_x(t), r_y(t)\rangle \quad t\in[0,t_f]$ $\qquad \vec{v}(t) = h(t)\langle 1,1\rangle$

max $\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot\frac{\vec{v}}{v}\cdot v\,dt$

$= \int_C \vec{F}\cdot\vec{v}\,dt = \int_0^{t_f} 2h(t)\,dt = \frac{2}{\sqrt{2}}\int_0^{t_f}\sqrt{2}\,h(t)\,dt$

$= \frac{2}{\sqrt{2}}\, s(t)\Big|_0^{t_f} = \frac{2}{\sqrt{2}}(1-0) = \sqrt{2}$

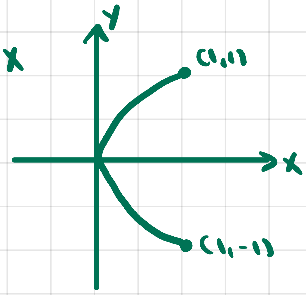


min $\int_C \vec{F}\cdot d\vec{r} = \int_{t_f}^{0} 2h(t)\,dt = \frac{2}{\sqrt{2}}(0-1) = -\sqrt{2}$

$|\vec{v}(t)| = \sqrt{2h(t)^2} = \sqrt{2}\,h(t)$

$s(t) = \int_0^t \sqrt{2}\,h(t)\,dt$

4C-1 $f(x,y) = x^3y + y^3$ $\qquad$ C: $y^2 = x$



a) $\vec{F} = \vec{\nabla} f = \langle 3x^2y,\, x^3+3y^2\rangle$

b) i) $\int_C \vec{F}\cdot d\vec{r} = \int_C 3x^2y\,dx + (x^3+3y^2)\,dy$

$\vec{r}(t) = \langle t^2, t\rangle \qquad dx = 2t\,dt \quad dy = dt$

$\int_C \vec{F}\,d\vec{r} = \int_{-1}^{1} 3t^5\cdot 2t\,dt + (t^6+3t^2)\,dt = \int_{-1}^{1}(7t^6+3t^2)\,dt = t^7+t^3\Big|_{-1}^{1} = (1+1)-(-1-1) = 4$

$\sqrt{-1^7} =$

*alternatively:

$\int_0^1 3x^2\sqrt{x} - \int_1^0 3x^2\sqrt{x}\,dx + \int_{-1}^{1}(y^6+3y^2)\,dy = \int_0^1 3x^{5/2}dx - \int_1^0 3x^{5/2}dx + \int_{-1}^{1}(y^6+3y^2)\,dy$

$= 3\cdot\frac{2}{7}x^{7/2}\Big|_0^1 - 3\cdot\frac{2}{7}x^{7/2}\Big|_1^0 + \left(\frac{y^7}{7}+y^3\right)\Big|_{-1}^{1} = \frac{6}{7} - \left[-\frac{6}{7}\right] + \left[\frac{1}{7}+1-\left(-\frac{1}{7}-1\right)\right]$

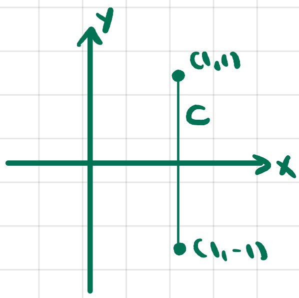
$= \frac{12}{7} + \frac{2}{7} + 2 = 4$

ii) Independence of Path Theorem $\int_C \vec{F}\,d\vec{r}$ of continuous vector field $\vec{F}$ is indep. of path in plane or space region D $<=>$ $\vec{F} = \vec{\nabla} f$ for some f defined on D

In this problem, $\vec{F} = \vec{\nabla} f$ by definition over $\mathbb{R}^2$, so specifically $\int_C \vec{F}\,d\vec{r}$ is path independent. Let's choose a simpler C:

$x = 1,\ dx = 0 \qquad \int_C 3y\cdot 0 + (1+3y^2)\,dy = \int_{-1}^{1}(1+3y^2)\,dy = (y+y^3)\Big|_{-1}^{1}$

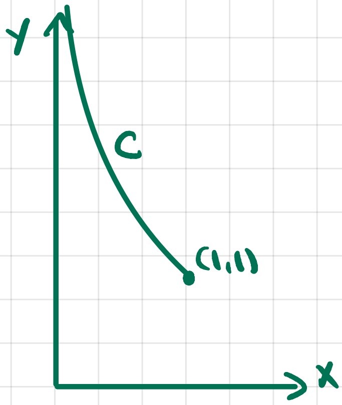
$= 1+1-(-1-1) = 4$



iii) Fundamental theorem for line integrals: $f:\mathbb{R}^2\to\mathbb{R}$ C is smooth, parametrized by $\vec{r}(t)=\langle t^2,t\rangle$, $t\in[-1,1]$. $\nabla f=\langle 3x^2y,\ x^3+3y^2\rangle$, a continuous vector-valued fn everywhere in domain.

$\Rightarrow$ f continuously differentiable $\Rightarrow \int_C \nabla f\cdot d\vec{r} = f(\vec{r}(b))-f(\vec{r}(a))$

In this problem: $\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(1))-f(\vec{r}(-1)) = f(1,1)-f(1,-1) = 2-(-2)=4$



4C-2 $f(x,y)=xe^{xy}$

a) $\vec{F}=\nabla f=\langle e^{xy}+xe^{xy}\cdot y,\ xe^{xy}\cdot x\rangle$

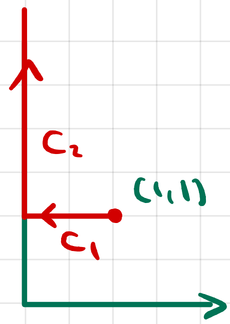
$=\langle e^{xy}(1+xy),\ x^2e^{xy}\rangle$

b) i) $x(t)=t$ $\quad dx=dt$ $\quad y(t)=1/t$ $\quad dy=-dt/t^2$

$\vec{\nabla} f(t)=\langle e(1+1),\ t^2\cdot e\rangle = \langle 2e,\ t^2e\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_1^0 2e\,dt + \cancel{t^2}e\left(\frac{-dt}{\cancel{t^2}}\right) = \int_1^0 (2e-e)\,dt = \int_1^0 e\,dt = -e$$

ii)



$$\int_{C_1}\vec{F}\,d\vec{r} = \int_1^0 (e^x(1+x))\,dx = (\cancel{e^x}+xe^x-\cancel{e^x})\Big|_1^0 = -e$$

$\int e^x x\,dx = e^x x-\int e^x\,dx = e^x x - e^x$

$dv=e^x dx$ $\quad v=e^x$

$u=x$ $\quad du=dx$

$$\int_{C_2}\vec{F}\,d\vec{r} = \int_0^\infty 0\,dy = 0$$

$$\Rightarrow \int_C \vec{F}\,d\vec{r} = -e+0=-e$$

4C-3 $f(x,y) = \sin x \cos y$

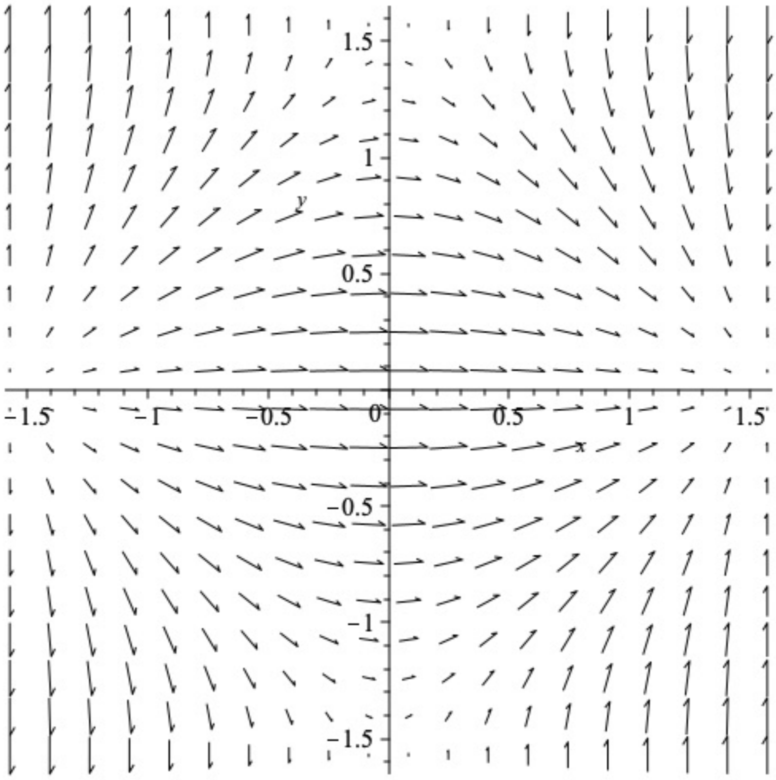
a) $\vec{F} = \nabla f = \langle \cos x \cos y, -\sin x \sin y \rangle$

b) vector field is $\vec{0}$ at regular intervals in $\mathbb{R}^2$.

$\langle x,y \rangle = \langle k_1 \cdot \pi, \frac{\pi}{2} + k_2 \pi \rangle$

or

$\langle x,y \rangle = \langle \frac{\pi}{2} + k_1 \pi, k_2 \pi \rangle$



$f(k_1 \pi, \frac{\pi}{2} + k_2 \pi)$

$= \sin(k_1 \pi) \cos(\pi/2 + k_2 \pi)$

$= 0 \cdot 0 = 0$

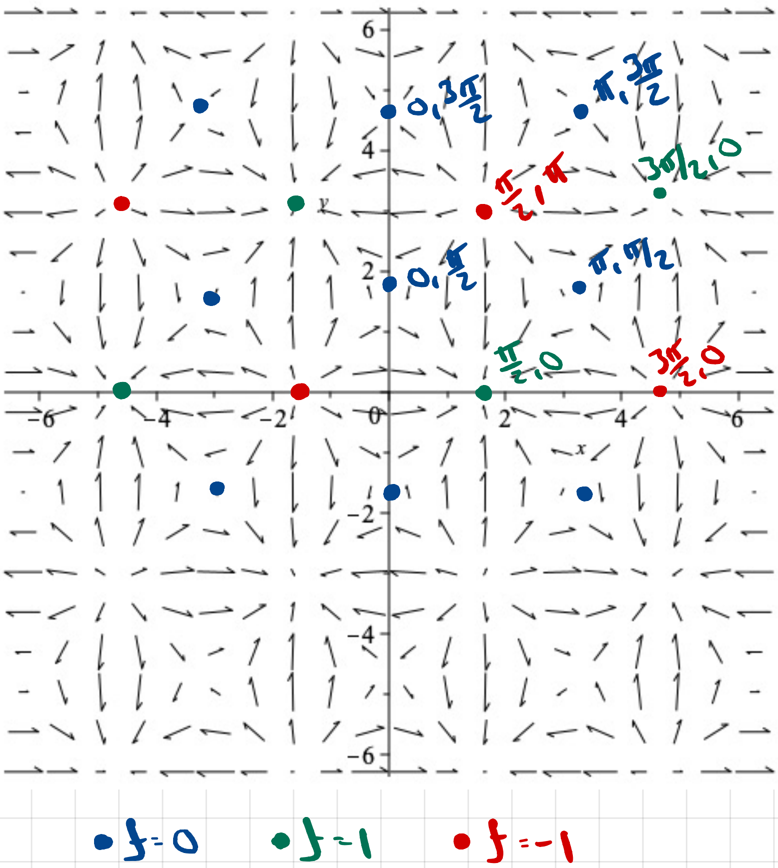
$f(\pi/2 + k_1 \pi, k_2 \pi)$

$= \sin(\pi/2 + k_1 \pi) \cos(k_2 \pi)$

$\pm 1 \cdot \pm 1 = \pm 1$

$f = +1 \Rightarrow \langle \frac{\pi}{2} + k_1 \cdot 2\pi, k_2 \cdot 2\pi \rangle$
or $\langle 3\pi/2 + k_1 \cdot 2\pi, \pi + k_2 \cdot 2\pi \rangle$

$f = -1 \Rightarrow \langle 3\pi/2 + k_1 2\pi, k_2 \cdot 2\pi \rangle$
$\langle \frac{\pi}{2} + k_1 2\pi, \pi + k_2 \cdot 2\pi \rangle$



$\vec{F}$ is a continuous gradient field in $\mathbb{R}^2$

$\Rightarrow \int_C \vec{F} d\vec{r}$ is path independent

$= f(x_f, y_f) - f(x_0, y_0)$, where

C starts at (x_0, y_0) ends at (x_f, y_f).

Max value occurs starting where

$f = -1$ and ending where $f = 1$.

$\Rightarrow \int_C \vec{F} d\vec{r} = 2$

For example, C being the curve from $(\pi/2, \pi)$ to $(3\pi/2)$

In general any path from $\langle \frac{\pi}{2} + k_1 \cdot 2\pi, \pi + k_2 \cdot 2\pi \rangle$ or $\langle \frac{3\pi}{2} + k_1 2\pi, k_2 \cdot 2\pi \rangle$

to $\langle \frac{\pi}{2} + k_1 \cdot 2\pi, k_2 \cdot 2\pi \rangle$ or $\langle 3\pi/2 + k_1 \cdot 2\pi, \pi + k_2 \cdot 2\pi \rangle$ has line integral $\int_C \vec{F} d\vec{r}$ equal to the max value of 2.

4C-5

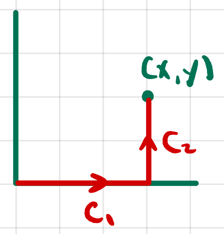
a) $\vec{F} = \langle y^2+2x, axy \rangle = \langle P(x,y), Q(x,y) \rangle$

If $\vec{F} = \nabla f$ then $P_y = Q_x$

$P_y = 2y \quad Q_x = ay \Rightarrow a = 2$

Two ways to find potential function f

1) line integral



$\int_{C_1+C_2} \vec{F}\, d\vec{r} = f(x,y) - f(0,0)$

$\int_{C_1} (y^2+2x)dx + 2xy\,dy$, on C_1 $y=0$ $dy=0$

$\Rightarrow \int_0^x 2x\,dx = x^2$

$\int_0^y 2xy\,dy = xy^2$

$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = x^2 + xy^2 = f(x,y) - \underbrace{f(0,0)}$ $\Rightarrow f(x,y) = x^2 + xy^2$

constant, we can set this to zero

∇f stays the same

2) antiderivatives

$P(x,y) = f_x \Rightarrow \int P(x,y)\,dx = f$

$\int (y^2+2x)dx = xy^2 + x^2 + g(y)$

$f_y = 2xy + g'(y) \Rightarrow g'(y) = 0$ when we compare with $Q(x,y) = f_y = 2xy$

$\Rightarrow f(x,y) = x^2 + xy^2$

b) $\vec{F} = \langle e^{x+y}(x+a), e^{x+y}x \rangle$

$P_y = e^{x+y}(x+a) \quad Q_x = e^{x+y}x + e^{x+y} \Rightarrow e^{x+y}(x+a) = e^{x+y}(x+1) \Rightarrow a = 1$

$f = \int e^{x+y}(x+a)\,dx = e^{x+y}\cdot a + \int e^{x+y}x\,dx = e^{x+y}a + e^{x+y}x - e^{x+y} = e^{x+y}(a+x-1) + g(y)$

$u = x \quad du = dx$

$dv = e^{x+y}dx$

$v = e^{x+y}$

$= e^{x+y}\cdot x - \int e^{x+y}dx$

$= e^{x+y}x - e^{x+y}$

$f_y = e^{x+y}(a+x-1) + g'(y)$

$= e^{x+y}x + [\cancelto{0}{e^{x+y}(a-1)} + g'(y)]$

a must be 1 here

$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C \Rightarrow f(x,y) = e^{x+y}x + C$

4C-6

a) Exact differential form: a scalar function $Q(x,y,z)$, and the expression

$$dQ \equiv (\partial Q/\partial x)_{yz}\,dx + (\partial Q/\partial y)_{xz}\,dy + (\partial Q/\partial z)_{xy}\,dz$$

Given a differential form $A(x,y,z)\,dx + B(x,y,z)\,dy + C(x,y,z)\,dz$

we check that it is exact by determining if $\langle A,B,C\rangle$ is the gradient of some f.

Note $\int_C A\,dx + B\,dy + C\,dz = \int_C \vec{F}\cdot d\vec{r}$ so if $\vec{F} = \langle A,B,C\rangle = \nabla f$ then we know

F is conservative.

$A(x,y) = y$ $\quad A_y = 1$

$\Rightarrow \langle A,B\rangle$ not a gradient field.

$B(x,y) = -x$ $\quad B_x = -1$

b) $y(2x+y)\,dx + x(2y+x)\,dy$

Is $\langle y(2x+y), x(2y+x)\rangle$ a gradient?

$\int A\,dx = \int (2xy+y^2)\,dx = x^2y + xy^2 + g(y)$ $\qquad \frac{d}{dy}(x^2y + xy^2 + g(y)) = x^2 + 2xy + g'(y)$

compare with $B = x^2 + 2xy \Rightarrow g'(y) = 0 \Rightarrow g(y) = C$

$\Rightarrow y(2x+y)\,dx + x(2y+x)\,dy$ is an exact differential

$\Rightarrow f(x,y) = x^2y + xy^2 + C$ is a potential function for a conservative vector field $\vec{F} = \nabla f$

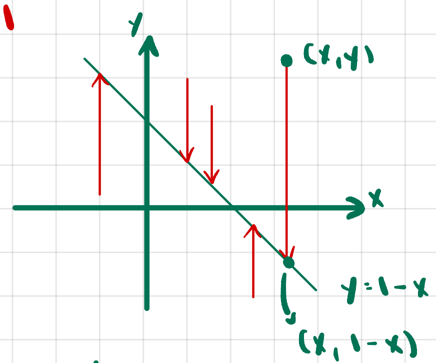
$= \langle y(2x+y), x(2y+x)\rangle$, and $y(2x+y)\,dx + x(2y+x)\,dy$ is an exact differential

$\Leftrightarrow \int_C \vec{F}\cdot d\vec{r} = \int_C y(2x+y)\,dx + x(2y+x)\,dy$ is path independent.

$df = y(2x+y)\,dx + x(2y+x)\,dy$

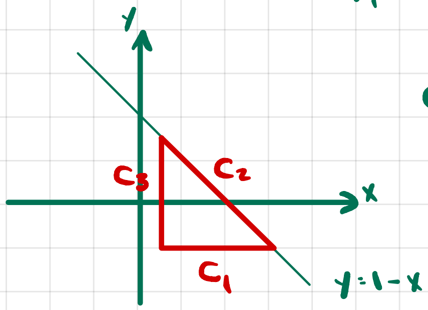
Problem 1

a)



$\vec F(x,y) = \langle x-x, 1-x-y\rangle = \langle 0, 1-x-y\rangle$

b)



C is a closed path.

$\vec F$ is conservative (ie $F = \nabla f$) $\iff \int_C \nabla f\, d\vec r = f(A) - f(A) = 0$, for all closed paths C.

So, if $\int_C \nabla f\, d\vec r \neq 0$ for C closed path then $\vec F$ not conservative ($\vec F \neq \nabla f$).

Also, $\vec F \neq \vec\nabla f \Rightarrow \int_C \nabla f\, dr$ is not $=0$ for all C.

$P_y = 0 \qquad Q_x = -1$

$\Rightarrow \vec F$ not conservative, $\vec F \neq \nabla f$

$\int_{C_1} \vec F\, d\vec r = 0$ because $\vec F \cdot \hat T = 0$ on C_1.

$\int_{C_3} \vec F \cdot d\vec r \neq 0$ because $\vec F \parallel \hat T \Rightarrow F \cdot \hat T \neq 0$ on C_2.

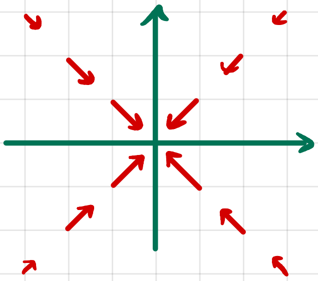
$\int_{C_1+C_2} \vec F\, d\vec r = \int_{C_2} \vec F \cdot d\vec r = 0$ because on C_2 $\vec F = 0$.

$\Rightarrow \int_C \vec F\, d\vec r = \cancelto{0}{\int_{C_1+C_2} \vec F\, d\vec r} + \int_{C_3} \vec F\, d\vec r \neq 0$

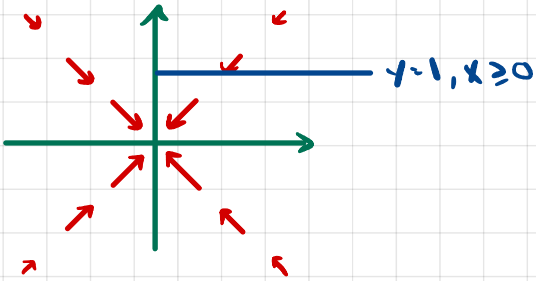
Note: starting at P_0, it is possible to get to $y=1-x$ by moving horizontally with zero work from $\vec F$. Moving along $y=1-x$ is also done with zero work from F. Move to position vertically above P_0 on $1-x$. Now, to get back to P_0 move down vertically with $W<0$ from $\vec F$.

Problem 2

$\vec{F} = -c\left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$



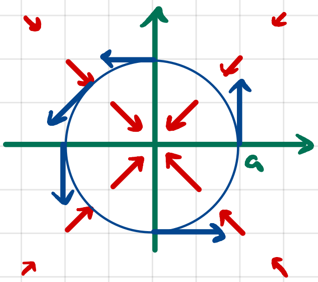
a)



$F = \left\langle \frac{-1}{x^2+y^2}, \frac{-1}{x^2+y^2} \right\rangle$ C: $y=1, x=t$

$$\int_C \vec{F}\cdot\hat{T}\,ds = \int_0^\infty \left(\frac{-t}{1+t^2}\right)dt = \lim_{n\to\infty}\int_0^n \frac{-t}{1+t^2}\,dt = \lim_{n\to\infty}\left[-\frac{\ln(1+t^2)}{2}\right]\Bigg|_0^n = \lim_{n\to\infty} -\frac{\ln(1+n^2)}{2} = -\infty$$

b)



$x^2+y^2=r$ $\vec{F} = \left\langle \frac{-1}{a}, \frac{-1}{a} \right\rangle$

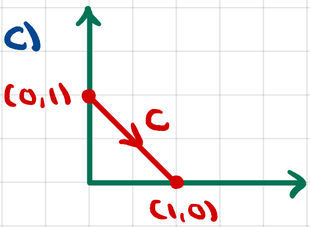
$x = a\cos\theta$ $dx = -a\sin\theta\,d\theta$

$y = a\sin\theta$ $dy = a\cos\theta\,d\theta$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \frac{-1}{\cancel{a}}(-\cancel{a}\sin\theta)\,d\theta - \frac{1}{\cancel{a}}\cancel{a}\cos\theta\,d\theta = \int_0^{2\pi}(\sin\theta - \cos\theta)\,d\theta$$

$$= -\cos\theta - \sin\theta\,\Big|_0^{2\pi} = (-\cos 2\pi - \sin 2\pi) - (-\cos 0 - \sin 0) = (-1-0) - (-1-0) = 0$$

c)



$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \frac{-t}{2t^2-2t+1}\,dt - \frac{(1-t)}{2t^2-2t+1}(-dt) = \int_0^1 \frac{-t+1-t}{2t^2-2t+1}\,dt = \int_0^1 \frac{-2t+1}{2t^2-2t+1}\,dt$$

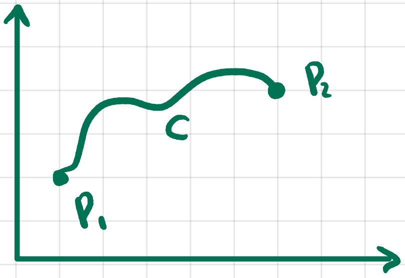
C: $y = 1-x$, parametrized $x=t$, $y=1-t$

$x^2+y^2 = t^2+1-2t+t^2 = 2t^2-2t+1$

Problem 3

a) $\vec{F} = -\nabla \ln r = -\ln\sqrt{x^2+y^2} = -\left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$

b)



$f(x,y) = \ln(\sqrt{x^2+y^2})$

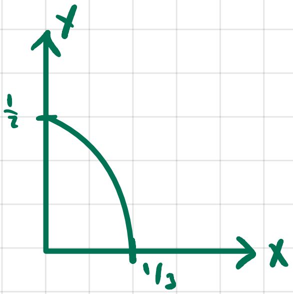
$\int_C \vec{F}\, d\vec{r} = f(P_2) - f(P_1)$ since $\vec{F}$ is path independent.

$= \ln(\sqrt{x_2^2+y_2^2}) - \ln(\sqrt{x_1^2-y_1^2}) = \ln(r_2) - \ln(r_1) = \ln(r_2/r_1)$

Problem 4

$\vec{F} = \nabla(x^2y + 2xy^2) = \langle 2xy + 2y^2, x^2 + 4xy \rangle$

$C: 9x^2 + 4y^2 = 1$



a) $\int_C \vec{F}\cdot d\vec{r} = \int_C (2xy + 2y^2)\,dx + (x^2 + 4xy)\,dy$

b) $x = \frac{1}{3}\cos t \quad y = \frac{1}{2}\sin t \qquad dx = -\frac{\sin t}{3}dt \quad dy = \frac{\cos t}{2}dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi/2} \left(\frac{\sin t\cos t}{3} + \frac{\sin^2 t}{2} \right)\left(\frac{-\sin t}{3} \right)dt + \left(\frac{\cos^2 t}{9} + 4\cdot\frac{1}{3}\cdot\frac{1}{2}\cos t\sin t \right)\frac{\cos t}{2}dt$$

$$= \int_0^{\pi/2} \left[\frac{-\sin^2 t\cos t}{9} - \frac{\sin^3 t}{6} + \frac{\cos^3 t}{18} + \frac{1}{3}\cos^2 t\sin t \right] dt$$

c) $f(x,y) = x^2y + 2xy^2$

$\int_C \vec{F}\cdot d\vec{r} = \int_A^B \nabla f\cdot d\vec{r} = f(B) - f(A)$ where $B = (0, 1/2)$, $A = (1/3, 0)$

$f(B) = f(0, 1/2) = 0$

$f(A) = f(1/3, 0) = 0$

d) f is continuously diff. in $\mathbb{R}^2 \Rightarrow \int_C \nabla f\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$

ie $\vec{F} = \nabla f$ is path independent.

$\Rightarrow$ between $A = (1/3, 0)$ and $B = (0, 1/2)$, $\int_C \vec{F}\cdot d\vec{r} = 0$

Problem 5

$\vec{F} = \langle xy, x^3 \rangle$

a) How to show $\vec{F}$ not conservative?

We know that: $\vec{F}$ conservative $\iff \vec{F} = \vec{\nabla} f$

Using the $\Rightarrow$ portion, if $\vec{F}$ is not a gradient then $\vec{F}$ is not conservative.

Assume $\vec{F} = \nabla f = \langle f_x, f_y \rangle$

$\Rightarrow f_x = xy \qquad f_y = x^3$

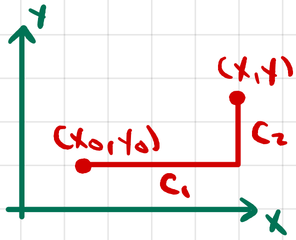
$f_{xy} = x \qquad f_{yx} = 3x^2$

But $f_{xy} \neq f_{yx} \Rightarrow F \neq \nabla f$ for some f. $\Rightarrow \vec{F}$ not conservative.

b) $\displaystyle\int_{(x_0,y_0)}^{(x,y)} \vec{F}\cdot d\vec{r} =$

$\displaystyle\int_0^{x_1} x y_0 \, dx = y_0 x_1^2 = 0$

$\displaystyle\int_0^{y_1} x_1^3 = x_1^3 y_1$



$\displaystyle\int_{C_1} \vec{F}\, d\vec{r} = \int_C xy\,dx = \int_{x_0}^{x_1} x y_0 \, dx = \frac{x^2}{2} y_0 \Big|_{x_0}^{x_1} = \frac{y_0}{2}(x_1^2 - x_0^2)$

$\displaystyle\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{y_0}^{y_1} x_1^3 \, dy = x_1^3 (y_1 - y_0)$

$\displaystyle\int_C \vec{F}\, d\vec{r} = \frac{y_0 x^2}{2} + x^3 y - \frac{y_0 x_0^2}{2} - x^3 y_0 = f(x,y)$

But $f_x = xy_0 + 3x^2 y - 3x^2 y_0 \neq xy$ $\Rightarrow \vec{F} \neq \nabla f$ as we already knew

$f_y = x^3$

Easier to see choosing a simpler path i.e. starting from $(x_0, y_0) = (0,0)$

$\Rightarrow f(x,y) = x^3 y$, $f_x = 3x^2 y \neq P(x,y)$, $f_y = x^3 = Q(x,y)$

The line integral results in a function that is not a potential for $\vec{F}$.

c) $\vec{F} = \langle xy, x^3 \rangle$

we know $\vec{F} \neq \nabla f$ for some $f: \mathbb{R}^2 \to \mathbb{R}$ because $P_y \neq Q_x$, ie curl $\vec{F} = Q_x - P_y \neq 0$

if we try to find potential function despite this using method 2:

$\int P\,dx = \int xy\,dx = \frac{x^2}{2}y + g(y)$

$\frac{d}{dy}\left[\int P\,dx\right] = \frac{x^2}{2} + g'(y)$, which should be equal to x^3

$\frac{x^2}{2} + g'(y) = x^3 \Rightarrow g'(y) = x^3 - \frac{x^2}{2}$, but this is a function of x, and we assumed it was not, contradiction. It means $g(y)$ is a function of x, which contradicts our initial integration of $P(x,y)$:

$\int g'(y)\,dy \Rightarrow g(y) = y(x^3 - x^2/2)$

$\Rightarrow f(x,y) = \frac{x^2}{y} + yx^3 - \frac{yx^2}{2}$

$f_x = \frac{2x}{y} + y \cdot 3x^2 - yx \neq xy = P(x,y)$