Practice Final Exam

Problem 1

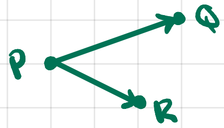
$P(1,1,-1) \qquad Q(1,2,0) \qquad R(-2,2,2)$

a) $\vec{PQ} = \langle 0,1,1\rangle$

$\vec{PR} = \langle -3,1,3\rangle$

$$\Rightarrow \vec{PQ}\times\vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ -3 & 1 & 3 \end{vmatrix} = \langle 3-1, -[0-(-3)], 0-(-3) \rangle = \langle 2,-3,3\rangle$$

b) plane through P, Q, R



$\langle x-1, y-1, z+1\rangle \langle 2,-3,3\rangle = 0$

vectors starting at $\vec{P}$ orthogonal to $\vec{PQ}\times\vec{PR}$

$2x-2-3y+3+3z+3=0 \Rightarrow 2x-3y+3z=-4$

Problem 2

$$A = \begin{bmatrix} 1 & 0 & c \\ 2 & c & 1 \\ 1 & -1 & 2 \end{bmatrix} \qquad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

a) $Ax = 0$
$$\begin{bmatrix} 1 & 0 & c \\ 2 & c & 1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$\Rightarrow$ intersection of three planes that pass through origin

either one (trivial solution $\vec{x} = \vec{0}$) or infinite solutions

$\det(A) \neq 0 \Rightarrow 1$ solution

$\det(A) = 0 \Rightarrow \infty$ solutions

$\det(A) = +1(2c-(-1)) + c(-2-c) = 0 \Rightarrow \cancel{2c}+1-\cancel{2c}-c^2 = 0 \Rightarrow c^2 = 1 \Rightarrow c = \pm 1$

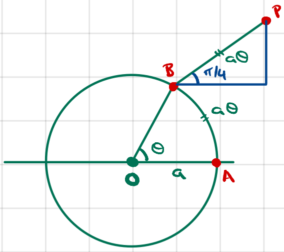
$\Rightarrow c = \pm 1 \Rightarrow \det(A) = 0 \Rightarrow Ax = 0$ has ∞ solutions

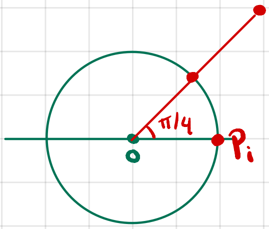
b) $c = 2 \Rightarrow A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix} \Rightarrow \det(A) = 1 - c^2 = 1-4 = -3 \neq 0$

$$A^{-1} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & x & \cdot \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & x \\ \cdot & \cdot & \cdot \end{bmatrix}^T \rightarrow \frac{(-)(1)\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}}{-3} = \frac{-1}{-3}(-1) = -\frac{1}{3}$$

Problem 3

$\vec{OP} = \vec{OB} + \vec{BP}$





$\vec{OB} = \langle a\cos\theta, a\sin\theta \rangle$

$\vec{BP} = \langle a\theta \frac{\sqrt{2}}{2}, a\theta \frac{\sqrt{2}}{2} \rangle$

$\Rightarrow \vec{OP} = \langle a(\cos\theta + \frac{\sqrt{2}\theta}{2}), a(\sin\theta + \frac{\sqrt{2}\theta}{2}) \rangle$

$x(\theta) = a(\cos\theta + \frac{\sqrt{2}\theta}{2}) = a\cos\theta + a\frac{\sqrt{2}}{2}\theta$

$y(\theta) = a(\sin\theta + \frac{\sqrt{2}\theta}{2}) = a\sin\theta + a\frac{\sqrt{2}}{2}\theta$

θ	$a\frac{\sqrt{2}}{2}\theta$
0	0
$\pi/4$	$a\pi\sqrt{2}/8$

$x'(\theta) = -a\sin\theta + \frac{a\sqrt{2}}{2} = 0 \Rightarrow \sin\theta = \frac{\sqrt{2}}{2}$

$+ \quad \frac{\pi}{4} \quad - \quad \frac{3\pi}{4} \quad + \quad x'$

$x''(\theta) = -a\cos\theta \qquad - \quad \frac{\pi}{2} \quad + \quad x''$

Problem 4

$\vec{r}(t) = \langle 3\cos t, 5\sin t, 4\cos t \rangle$

a) $\vec{v}(t) = \langle -3\sin t, 5\cos t, -4\sin t \rangle$

speed $= |\vec{v}(t)| = (9\sin^2 t + 25\cos^2 t + 16\sin^2 t)^{1/2} = 5$

b) $A: (a,b,c)$ where trajectory crosses yz-plane

$x(t) = 0$ on yz-plane

$\Rightarrow 3\cos t = 0 \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2} \quad \Rightarrow A: \langle 0,5,0 \rangle, \langle 0,-5,0 \rangle$

Problem 5

$\omega = x^2y - xy^3 \qquad P = (2,1)$

a) $\left.\frac{d\omega}{ds}\right|_{\hat{u}}(P) \qquad \vec{v} = \langle 3,4 \rangle \quad \hat{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 3,4 \rangle}{5}$

$= \nabla\omega(P)\cdot\hat{u} = \langle 2xy - y^3, x^2 - 3xy^2 \rangle_P \cdot \hat{u} = \langle 3, -2 \rangle \cdot \frac{\langle 3,4 \rangle}{5} = \frac{9-8}{5} = \frac{1}{5}$

b) $\frac{1}{5}\cdot 0.01 = \frac{1}{500} = \frac{2}{1000} = 0.002$

Problem 6

a) tangent plane at $(1,1,1)$ to surface $x^2+2y^2+2z^2=5$

$f(x,y,z) = x^2+2y^2+2z^2-5 = 0 \qquad \nabla f = \langle 2x, 4y, 4z \rangle, \ \nabla f(1,1,1) = \langle 2,4,4 \rangle$

Gradient vector $\langle 2,4,4 \rangle$ is $\perp$ to any vector on the plane

$\Rightarrow \langle x-1, y-1, z-1 \rangle \langle 2,4,4 \rangle = 0 \Rightarrow 2x+4y+4z-2-4-4 = 0 \Rightarrow 2x+4y+4z = 10$

$\Rightarrow x+2y+2z = 5$

b) $\langle 1,2,2 \rangle = \vec{n}$ to plane $x+2y+2z=5$

$\vec{n}\cdot\hat{k} = |\vec{n}|\cos\theta \Rightarrow \cos\theta = \frac{2}{3} \qquad \theta = \cos^{-1}(2/3)$

Problem 7 point on $2x+y-z=6$ closest to origin

(Distance to origin)^2 = $D(x,y,z) = x^2+y^2+z^2$

min $x^2+y^2+z^2$ s.t. $g(x,y,z) = 2x+y-z-6=0$

$\nabla D = \langle 2x, 2y, 2z \rangle$

$\nabla g = \langle 2, 1, -1 \rangle$

$$\begin{cases} \nabla D = \lambda \nabla g \\ g(x,y,z)=0 \end{cases} \Rightarrow \begin{array}{ll} 2x = 2\lambda & \Rightarrow x = \lambda \\ 2y = \lambda & y = x/2 \\ 2z = -\lambda & z = -x/2 \\ 2x+y-z-6=0 & \end{array} \Rightarrow \begin{array}{l} 2x + \frac{x}{2} + \frac{x}{2} - 6 = 0 \Rightarrow 3x = 6 \Rightarrow x = 2 \\ y = 1,\ z = -1 \end{array}$$

Check:

$(2,1,-1)$ and boundary. Plane defined in $\mathbb{R}^3$. Boundary is as $x,y,z \to \infty$

But $\lim\limits_{x \text{ or } y \text{ or } z \to \infty} D = \infty$.

$D(2,1,-1) = 4+1+1 = 6$

min distance is $\sqrt{6}$ at $(x,y,z) = (2,1,-1)$

Problem 8

$\omega = f(x,y,z) \qquad g(x,y,z) = 3$

At $P(0,0,0)$, $\nabla f = \langle 1,1,2 \rangle$, $\nabla g = \langle 2,-1,-1 \rangle$

a) $\left(\frac{\partial z}{\partial x}\right)_y$ rate of chg of z when we vary x and keep y constant.

differentials

$dg = 0 \Rightarrow g_x dx + g_y dy + g_z dz = 0$

At P $\Rightarrow 2dx - dy - dz = 0$

$dy = 0 \Rightarrow 2dx = dz \Rightarrow \frac{dz}{dx} = 2$

chain rule

$\frac{dg}{dx} = g_x + g_y \cdot 0 + g_z \cdot \left(\frac{\partial z}{\partial x}\right)_y$

$= 2 - \left(\frac{\partial z}{\partial x}\right)_y = 0 \Rightarrow \left(\frac{\partial z}{\partial x}\right)_y = 2$

b) $\left(\frac{\partial \omega}{\partial x}\right)_y$ rate of chg of ω as we vary x, y kept constant

differentials

$d\omega = f_x dx + f_y \cancel{dy}^{\,0} + f_z dz = dx + 2dz$

dx and dz are related implicitly because of $g(x,y,z) = 3$
we need dz in terms of dx.

From a), $dz = 2dx$

$\Rightarrow d\omega = dx + 2 \cdot 2dx = 5dx \Rightarrow \left(\frac{\partial \omega}{\partial x}\right)_y = 5$

chain rule

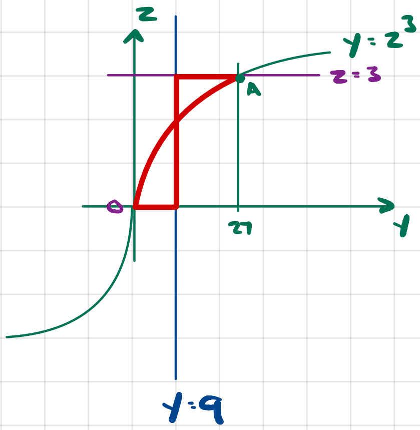
$\omega = f(x,y,z)$

$\left(\frac{d\omega}{dx}\right)_y = f_x + f_y \cdot 0 + f_z \left(\frac{dz}{dx}\right)_y = 1 + 2 \cdot 2 = 5$

Problem 9

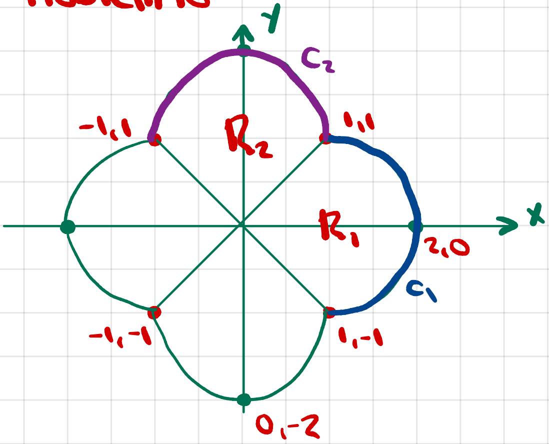
$$\int_0^3 \int_{z^3}^{9} x e^{-y^2}\, dy\, dz$$

$$\int_0^9 \int_0^{\sqrt[3]{y}} x e^{-y^2}\, dz\, dy + \int_9^{27} \int_{\sqrt[3]{y}}^{3} x e^{-y^2}\, dz\, dy$$



A: $y = z^3,\ z = 3 \Rightarrow y = 27$

Problem 10



$\delta(x,y) = 1$

$$I_{z_1} = \iint_{R_1} (x^2+y^2)\,\delta\, dx\, dy = \int_{-\pi/4}^{\pi/4} \int_0^{2\cos\theta} r^3\, dr\, d\theta$$

$$I_{z_2} = \iint_{R^2} (x^2+y^2)\,\delta\, dx\, dy = \int_{\frac{\pi}{4}}^{3\pi/4} \int_0^{2\sin\theta} r^3\, dr\, d\theta$$

$$I_z = 4 \cdot \int_{-\pi/4}^{\pi/4} \int_0^{2\cos\theta} r^3\, dr\, d\theta$$

C_1: $(x-1)^2 + y^2 = 1$

$x^2 + y^2 - 2x = 0$

$r^2 - 2r\cos\theta = 0$

$r(r - 2\cos\theta) = 0$

$\Rightarrow r = 2\cos\theta$

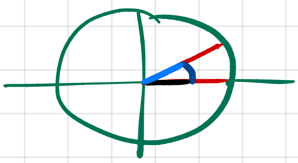
C_2: $x^2 + (y-1)^2 = 1$

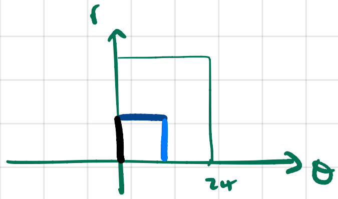
$x^2 + y^2 - 2y = 0$

$r^2 - 2r\sin\theta = 0$

$\Rightarrow r(r - 2\sin\theta) = 0$

$\Rightarrow r = 2\sin\theta$





$$\int_0^{r} \int_{-\sqrt{}}^{\sqrt{}} x^2 + y^2\, dx\, dy$$

$x = r\cos t$

$y = r\sin t$

$x^2 + y^2 = a^2$

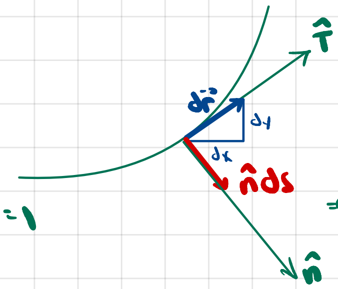
$r^2 = a^2$

$r = \pm a$

Problem 11

a) $\vec{F} = \langle P, Q\rangle$ $\quad \int_C \vec{F}\cdot\hat{n}\,ds = \int_C \langle P,Q\rangle\langle dy,-dx\rangle = \int_C -Q\,dx + P\,dy$

b) $\vec{F} = \langle ax, by\rangle$



$d\vec{r} = \hat{T}ds = \langle dx, dy\rangle$

$\Rightarrow \hat{n}ds = \langle dy, -dx\rangle$

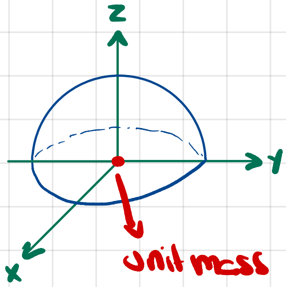
$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R dA \Rightarrow a+b=1$$

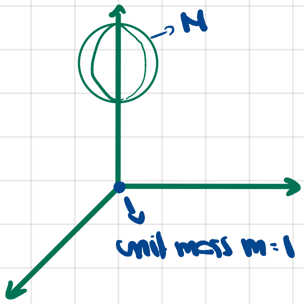
$$\oint \vec{F}\cdot\hat{n}\,ds \underset{\text{Green's}}{=} \iint_R \operatorname{div}\vec{F}\,dA = \iint_R (a+b)\,dA$$

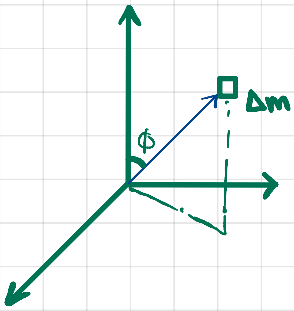
$\operatorname{div}\vec{F} = a+b$

Problem 12

$\delta = z$







we have a unit mass at origin $\Rightarrow m=1$

Δm exerts a gravitational force on the unit mass, in the same direction as the position vector of Δm. That direction is $\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$.

$$\vec{F} = \frac{GMm}{|\vec{R}|^2}\vec{r}$$

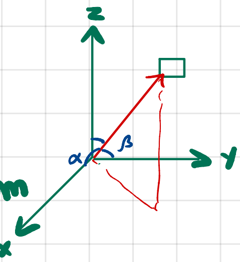
point mass, mass = M

$\vec{r} \to$ dir($\vec{R}$)

position vector from origin

Force exerted by point mass M on point mass m

$$\Delta\vec{F} = \frac{G\Delta m}{|R|^2}\cdot\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$$



To obtain the gravitational force of a whole object we must add the ΔF from Δm's forming the entire object.

ie

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{G\Delta m_i}{\rho^2}\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$$

$\Delta m_i = \delta\,\Delta V \Rightarrow \lim_{n\to\infty}\sum_i \frac{G\delta}{\rho^2}\Delta V\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$

$$\Rightarrow G\iiint\frac{\delta}{\rho^2}\langle\cos\alpha, \cos\beta, \cos\gamma\rangle\,dV$$

* Direction cosines

Given a vector $\vec{v}$ $\quad \vec{v}\cdot\hat{k} = |\vec{v}|\cos\gamma$

$\Rightarrow \cos\gamma = \frac{\vec{v}}{|\vec{v}|}\cdot\hat{k}$

Also, $\cos\alpha = \frac{\vec{v}}{|\vec{v}|}\cdot\hat{\imath}$ and $\cos\beta = \frac{\vec{v}}{|\vec{v}|}\cdot\hat{\jmath}$

$\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$ is the unit vector in the direction of $\vec{v}$.

Given a mass that is symmetric about the z-axis, the integrals in δx and δy are 0, ie the gravitational force has no component in the directions $\hat{\imath}$ and $\hat{\jmath}$.

In direction $\hat{k}$, $F_z = \iiint_V \frac{G\delta}{\rho^2}\cos\gamma\,dV = G\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{1}\frac{z}{\cancel{\rho^2}}\cos\phi\,\cancel{\rho^2}\sin\phi\,d\rho\,d\phi\,d\theta$

$$= G\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{1}\rho\cos^2\phi\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= -\frac{G}{2\cdot 3}\int_0^{2\pi}\cos^3\phi\Big|_0^{\frac{\pi}{2}}\,d\theta = -\frac{G}{6}\int_0^{2\pi}(0-1)\,d\theta = \frac{G}{6}\cdot 2\pi = \frac{G\pi}{3}$$

Problem 14

a) $\vec{F} = \langle ay^2, 2y(x+z), by^2+z^2 \rangle$ $\quad$ $\vec{F}$ conservative $\iff \vec{F} = \nabla f \iff \int_C \vec{F}\cdot d\vec{r}$ path indep.

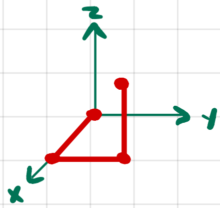
$\iff \text{curl}\,\vec{F} = 0$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ay^2 & 2y(x+z) & by^2+z^2 \end{vmatrix}$$

$$= \langle 2by - 2y, -(0-0), 2y - 2ay \rangle = \vec{0}$$

$y(b-1) = 0 \Rightarrow b = 1$

$y(1-a) = 0 \Rightarrow a = 1$



b) $\vec{F} = \langle y^2, 2y(x+z), y^2+z^2 \rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \cancel{y^2}^{\,0}\,dx + \int_{C_2} 2y(x+\cancel{z}^{\,0})\,dy + \int_{C_3} (y^2+z^2)\,dz$$

$$= \int_0^y 2yx\,dy + \int_0^z (y^2+z^2)\,dz = y^2x + y^2z + \frac{z^3}{3} = f(x,y,z)$$

c) surface S such that $\int_P^Q \vec{F}\cdot d\vec{r} = \int_P^Q \vec{F}\cdot\hat{T}\,ds = f(Q) - f(P) = 0 \Rightarrow f(Q) = f(P)$

$f(Q) = f(P)\ \forall P,Q \in \mathbb{R}^3 \Rightarrow$ we are on a level surface $f(x,y,z) = c$

$y^2x + y^2z + \frac{z^3}{3} = c$ is S

Problem 15

bottom $x^2+y^2=1$ $\quad$ upper: $z = 1-x^2-y^2$ $\quad z \geq 0$ $\quad \vec{F} = \langle x,y,z \rangle$

S is a closed surface: $\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \text{div}\,\vec{F}\,dV$ (divergence theorem)

$\text{div}\,\vec{F} = 3$ $$\int_0^{2\pi}\int_0^1\int_0^{1-r^2} 3\,dz\,r\,dr\,d\theta = 3\int_0^{2\pi}\int_0^1 (1-r^2)r\,dr\,d\theta = 3\int_0^{2\pi}\left(\frac{1}{2}-\frac{1}{4}\right)d\theta = \frac{3}{4}\cdot 2\pi = \frac{3\pi}{2}$$

Problem 16 bottom $x^2+y^2=1$ upper: $z=1-x^2-y^2$ $z\geq 0$ $\vec{F}=\langle x,y,z\rangle$

S is a closed surface: $\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \operatorname{div}\vec{F}\,dV$ (divergence theorem)

$\langle x,y,1-x^2-y^2\rangle = \vec{r}(x,y)$

$r_x=\langle 1,0,-2x\rangle$ $r_y=\langle 0,1,-2y\rangle$ $\vec{n}=r_x\times r_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x,2y,1\rangle$

$\hat{n}=\dfrac{\langle 2x,2y,1\rangle}{(4x^2+4y^2+1)^{1/2}}$

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S \langle x,y,z\rangle\cdot\frac{\langle 2x,2y,1\rangle}{\cancel{|\langle 2x,2y,1\rangle|}}\,\cancel{|\langle 2x,2y,1\rangle|}\,dx\,dy$$

$$=\iint_S (2x^2+2y^2+1-x^2-y^2)\,r\,dr\,d\theta$$

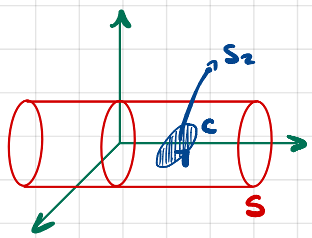
$$=\int_0^{2\pi}\int_0^1 (r^2+1)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r^3+r)\,dr\,d\theta = \int_0^{2\pi}\left(\frac{r^4}{4}+\frac{r^2}{2}\right)\Big|_0^1 d\theta = 2\pi\left(\frac{1}{4}+\frac{1}{2}\right)=2\pi\cdot\frac{3}{4}=\frac{3\pi}{2}$$

Problem 17 xz cylinder in 3-space: surface given by $f(x,z)=0$

$\vec{F}=\langle z^2,y^2,xz\rangle$ $\oint_C \vec{F}\cdot d\vec{r}$, simple closed curve C on cylinder

A closed curve on the yz-cylinder splits the cylinder into two surfaces that share the same boundary, the curve C.

$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\,dS = \iint_{S_1}\nabla\times\vec{F}\cdot\hat{n}\,dS + \iint_{S_2}\nabla\times\vec{F}\cdot\hat{n}\,dS = \oint_C \vec{F}\cdot d\vec{r} - \oint_C \vec{F}\cdot d\vec{r} = 0$$



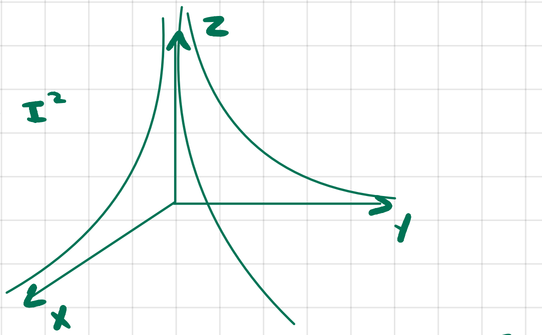
note that $\oint_C \vec{F}\cdot d\vec{r}$ is a line integral on C with counterclockwise orientation.

$\iint_{S_2}\nabla\times\vec{F}\cdot\hat{n}\,dS$ requires opposite orientation ie $=\oint_{-C}\vec{F}\cdot d\vec{r} = -\oint_C \vec{F}\cdot d\vec{r}$

Problem 18 $I = \int_0^\infty e^{-x^2}dx$

a) $\int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} \, dy \, dx$

$$= \lim_{n\to\infty} \int_0^n \left[\lim_{m\to\infty} \int_0^m e^{-y^2} dy \right] e^{-x^2} dx = \lim_{n\to\infty} I \int_0^n e^{-x^2} dx = I^2$$



x, y in $\mathbb{R}^{2+} \Rightarrow$ θ from $0 .. \pi/2$
r from $0 .. \infty$

b) $x = r\cos t$
$y = r\sin t$

$$\int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} \, dy \, dx = \int_0^{\pi/2} \int_0^\infty e^{-r^2} \, r \, dr \, d\theta$$

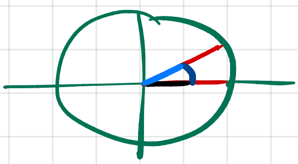
$$= \int_0^{\pi/2} -\frac{e^{-r^2}}{2} \Big|_0^\infty d\theta = \int_0 \left[\cancel{\frac{-1}{2e^\infty}}^{\,0} + \frac{1}{2} \right] d\theta = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

$$\Rightarrow I^2 = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}$$

* scribble about change of variables

$\langle x, y \rangle = \langle x(r,t), y(r,t) \rangle = \langle r\cos t, r\sin t \rangle$

$$\frac{\partial(x,y)}{\partial(r,t)} = \begin{vmatrix} x_r & x_t \\ y_r & y_t \end{vmatrix} = \begin{vmatrix} \cos t & -r\sin t \\ \sin t & r\cos t \end{vmatrix} = r\cos^2 t + r\sin^2 t = r$$



$$\int_0^r \int_{-\sqrt{}}^{+\sqrt{}} x^2 + y^2 \, dx \, dy$$

$x^2 + y^2 = a^2$
$r^2 = a^2$
$r = \pm a$

$x = r\cos t$
$y = r\sin t$