Ex1 centroid of $z = \sqrt{a^2-x^2-y^2}$ $\quad x^2+y^2 \le a^2 \quad \delta = 1$

$\vec{r}(x,y) = \langle x, y, \sqrt{a^2-x^2-y^2} \rangle$

$r_x = \langle 1, 0, \frac{-x}{\sqrt{a^2-x^2-y^2}} \rangle \qquad r_y = \langle 0, 1, \frac{y}{\sqrt{a^2-x^2-y^2}} \rangle$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{-x}{\sqrt{a^2-x^2-y^2}} \\ 0 & 1 & \frac{y}{\sqrt{a^2-x^2-y^2}} \end{vmatrix} = \left\langle \frac{x}{\sqrt{a^2-x^2-y^2}}, \frac{-y}{\sqrt{a^2-x^2-y^2}}, 1 \right\rangle$$

$$|\vec{r}_x \times \vec{r}_y| = [1 + z_x^2 + z_y^2]^{1/2} = \left[1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}\right]^{1/2} = \frac{1}{z}(x^2+y^2+z^2)^{1/2} = \frac{a}{z}$$

surface area $= \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} 1 \cdot \frac{a}{z} \, dx \, dy = 2\pi a^2$

centroid $\quad \bar{z} = \dfrac{\int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} z \cdot 1 \cdot \frac{a}{z} \, dx \, dy}{2\pi a^2} = \dfrac{a \cdot \pi a^2}{2\pi a^2} = \dfrac{a}{2}$

$\bar{x} = \bar{y} = 0$ by symmetry of the hemispherical surface.

Ex 2 $x^2+y^2+z^2=a^2$, spherical surface $\delta=k$

$r(\phi,\theta)=\langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi\rangle$

$|r_\phi \times r_\theta| = a^2\sin\phi \Rightarrow dS = a^2\sin\phi\, d\phi\, d\theta$

surface area $= \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi\, d\phi\, d\theta = a^2\int_0^{2\pi}[-\cos\pi+\cos 0]\,d\theta = a^2\int_0^{2\pi}(1+1)\,d\theta = 2a^2\cdot 2\pi = 4a^2\pi$

mass $= \int_0^{2\pi}\int_0^{\pi} k\, a^2\sin\phi\, d\phi\, d\theta = k\cdot 4a^2\pi$

$$\bar{z} = \frac{\int_0^{2\pi}\int_0^{\pi} a\cos\phi\, a^2\sin\phi\, d\phi\, d\theta}{m} = 0$$

$I_z = \int_0^{2\pi}\int_0^{\pi} a^2\sin^2\phi \cdot k\, a^2\sin\phi\, d\phi\, d\theta = a^4 k\int_0^{2\pi}\int_0^{\pi}\sin^3\phi\, d\phi\, d\theta = \frac{a^4 k\cdot 8\pi}{3}$

$x^2+y^2 = a^2-z^2 = a^2-a^2\cos^2\phi = a^2\sin^2\phi$