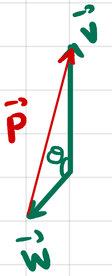
PSet 1

1A-6



$|\vec{v}| = 200$

$|\vec{w}| = 50$

$\vec{v} = \vec{p} + \vec{w}$

$\theta = \frac{3\pi}{4}$

$\vec{w} = -w\hat{\imath} - w\hat{\jmath} \Rightarrow w^2 + w^2 = 2500$

$\Rightarrow w = \pm\sqrt{1250}$

$\Rightarrow \vec{w} = \langle -25\sqrt{2}, -25\sqrt{2} \rangle$

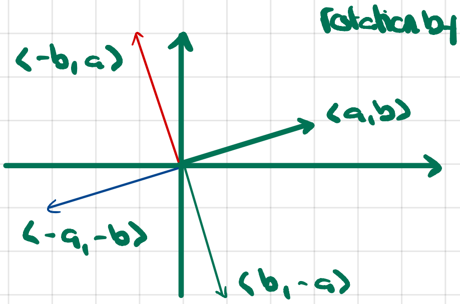
$\vec{v} = \langle 0, 200 \rangle = \langle p_1, p_2 \rangle + \langle -25\sqrt{2}, -25\sqrt{2} \rangle$

$p_1 - 25\sqrt{2} = 0 \Rightarrow p_1 = 25\sqrt{2}$

$p_2 - 25\sqrt{2} = 200 \Rightarrow p_2 = 200 + 25\sqrt{2}$

1A-7

$A = a\hat{\imath} + b\hat{\jmath}$

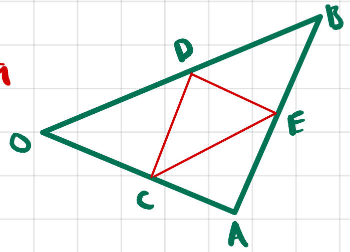


rotation by 90° clockwise

$A' = \langle b, -a \rangle$

$A'' = \langle -b, a \rangle$

1A-9



show $\vec{CD} = \frac{1}{2}\vec{AB}$, $\vec{DE} = \frac{1}{2}\vec{OA}$, $\vec{CE} = \frac{1}{2}\vec{OB}$

$\vec{CD} = \vec{OD} - \vec{OC}$

$\vec{OD} = \frac{1}{2}\vec{OB}$, $\vec{OC} = \frac{1}{2}\vec{OA}$ $\Rightarrow \vec{CD} = \frac{1}{2}(\vec{OB} - \vec{OA})$

$\vec{AB} = \vec{OB} - \vec{OA} \Rightarrow \boxed{\vec{CD} = \frac{1}{2}\vec{AB}}$

$\vec{DE} = \vec{OE} - \vec{OD}$

$\vec{OE} = \vec{OA} + \vec{AE}$

$\vec{AE} = \frac{1}{2}\vec{AB} = \frac{1}{2}(\vec{OB} - \vec{OA})$

$\Rightarrow \vec{OE} = \vec{OA} + \frac{1}{2}\vec{OB} - \frac{1}{2}\vec{OA}$

$= \frac{1}{2}(\vec{OA} + \vec{OB})$

$\Rightarrow \vec{DE} = \frac{1}{2}(\vec{OA} + \vec{OB}) - \frac{1}{2}\vec{OB} \Rightarrow \boxed{\vec{DE} = \frac{1}{2}\vec{OA}}$

$\vec{CE} = \vec{OE} - \vec{OC} = \frac{1}{2}(\vec{OA} + \vec{OB}) - \frac{1}{2}\vec{OA}$

$\Rightarrow \boxed{\vec{CE} = \frac{1}{2}\vec{OB}}$

IA-1

a) $\hat{\imath} + \hat{\jmath} + \hat{k} = \vec{v}$

$|\vec{v}| = \sqrt{1+1+1} = \sqrt{3}$ = magnitude

$dir(\vec{v}) = \dfrac{\vec{v}}{|\vec{v}|} = \langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} \rangle$

b) $\vec{v} = \langle 2, -1, 2 \rangle$

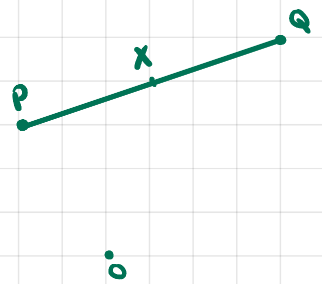
$|\vec{v}| = \sqrt{4+1+4} = 3$

$dir(\vec{v}) = \dfrac{\vec{v}}{|\vec{v}|} = \langle 2/3, -1/3, 2/3 \rangle$

c) $\vec{v} = \langle 3, -6, -2 \rangle$

$|\vec{v}| = \sqrt{9+36+4} = 7$

$dir(\vec{v}) = \vec{v}/7$



IA-4

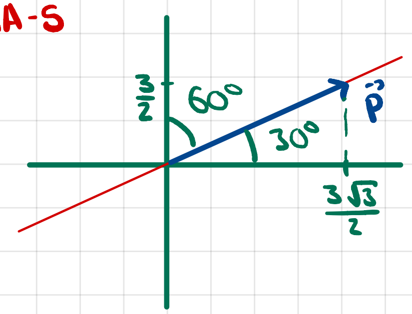
$\vec{OX} = \vec{OP} + \vec{PX}$

$\vec{PX} = \frac{1}{2}\vec{PQ}$

$\vec{PQ} = \vec{OQ} - \vec{OP}$ $\Rightarrow \vec{PX} = \frac{1}{2}(\vec{OQ} - \vec{OP})$

$\Rightarrow \vec{OX} = \frac{1}{2}(\vec{OQ} + \vec{OP})$

IA-5



$|\vec{p}| = 3 = \sqrt{p_1^2 + p_2^2}$

$\vec{p} \cdot \hat{\imath} = |\vec{p}| \cdot |\hat{\imath}| \cdot \cos 30° = 3 \cdot 1 \cdot \dfrac{\sqrt{3}}{2} = \dfrac{3\sqrt{3}}{2} = p_1$

$\Rightarrow 9 = \dfrac{9 \cdot 3}{4} + p_2^2 \Rightarrow p_2^2 = \dfrac{36-27}{4} = \dfrac{9}{4} \Rightarrow p_2 = \pm \dfrac{3}{2}$

$\vec{p} = p_1\hat{\imath} + p_2\hat{\jmath}$

Just to check: $\vec{p} \cdot \hat{\jmath} = p_2 = |\vec{p}||\hat{\jmath}| \cos 60° = 3 \cdot 1/2$

1A-8

a) $\vec{A} = \langle a, b, c \rangle$

$\vec{A} \cdot \hat{\imath} = \langle a, b, c \rangle \cdot \langle 1, 0, 0 \rangle = a = |\vec{A}| \cdot 1 \cdot \cos\alpha = |\vec{A}|\cos\alpha$

$\vec{A} \cdot \hat{\jmath} = |\vec{A}|\cos\beta$

$\vec{A}\,\hat{k} = |\vec{A}|\cos\gamma$

$$\operatorname{dir}(\vec{A}) = \frac{\vec{A}}{|\vec{A}|} = \frac{\cancel{|\vec{A}|}\langle \cos\alpha, \cos\beta, \cos\gamma \rangle}{\cancel{|\vec{A}|}}$$

b) $\cos\alpha = \dfrac{a}{|\vec{A}|} \quad \cos\beta = \dfrac{b}{|\vec{A}|} \quad \cos\gamma = \dfrac{c}{|\vec{A}|}$

$\vec{v} = \langle -1, 2, 2 \rangle \quad |\vec{v}| = \sqrt{1+4+4} = 3 \quad \operatorname{dir}(\vec{v}) = \langle {-1}/3, 2/3, 2/3 \rangle$

$\cos\alpha = -1/3 \quad \cos\beta = 2/3 \quad \cos\gamma = 2/3$

c) Given $A = \langle a, b, c \rangle$, then $\operatorname{dir}(\vec{A}) = \left\langle \dfrac{a}{|\vec{A}|}, \dfrac{b}{|\vec{A}|}, \dfrac{c}{|\vec{A}|} \right\rangle = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$

But

1) $\Rightarrow$

t, u, v are direction cosines of $\vec{A}$ $\Rightarrow$ Since $|\operatorname{dir}(\vec{A})| = \left|\dfrac{A}{|\vec{A}|}\right| = \left[\dfrac{a^2+b^2+c^2}{|\vec{A}|^2}\right]^{1/2} = \dfrac{(a^2+b^2+c^2)^{1/2}}{|\vec{A}|} = 1$

and $|\operatorname{dir}(\vec{A})| = |\langle \cos\alpha, \cos\beta, \cos\gamma \rangle|$

Then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow t^2 + u^2 + v^2 = 1$

2) $\Leftarrow$

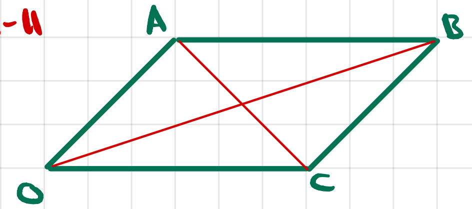
$t^2 + u^2 + v^2 = 1$ $\Rightarrow$ note that $|t| \le 1, |u| \le 1, |v| \le 1$

if we assign t, u, v as $t = \cos\alpha$, $u = \cos\beta$, $v = \cos\gamma$

then $\langle t, u, v \rangle = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$ is the direction vector

for some vector $\vec{A} = c \cdot \langle t, u, v \rangle$, $c = |\vec{A}|$

1A-11



$\vec{OX} = \frac{1}{2}\vec{OB}$ = pos. vector midpoint $\vec{OB}$ diagonal

$\vec{OY} = \frac{1}{2}(\vec{OA} + \vec{OC})$ $\Rightarrow \vec{OY} = \frac{1}{2}\vec{OB}$

But $\vec{OB} = \vec{OA} + \vec{OC}$

1B-2 $\vec{v} = \langle c, 2, -1\rangle$ $\quad \vec{w} = \langle 1, -1, 2\rangle$

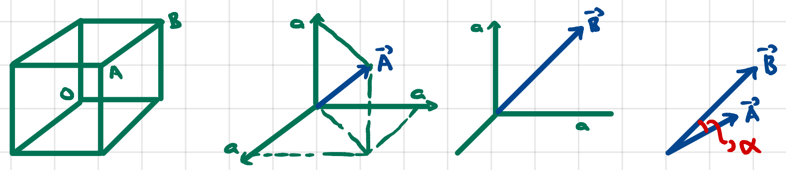
a) $\vec{v}\cdot\vec{w} = 0 : c - 2 - 2 = 0 \Rightarrow c = 4$

b) $\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\theta$

θ acute angle $\Rightarrow \cos\theta > 0 \Rightarrow \vec{v}\cdot\vec{w} > 0 \Rightarrow c - 4 > 0 \Rightarrow c > 4$

1B-3

a)



$\vec{OA} = \langle a, a, a\rangle$ $\quad \vec{OB} = \langle 0, a, a\rangle$

$|\vec{OA}| = a\sqrt{3}$ $\quad |\vec{OB}| = a\sqrt{2}$

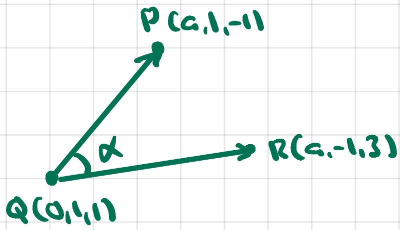
$\vec{OA}\cdot\vec{OB} = a^2 + a^2 = a\sqrt{3}\cdot a\sqrt{2}\cdot\cos\alpha$

$2\cancel{a^2} = \cancel{a^2}\sqrt{6}\cos\alpha \Rightarrow \cos\alpha = \frac{2}{\sqrt{2\cdot 3}} = \sqrt{\frac{2}{3}}$ $\Rightarrow \alpha = \cos^{-1}\left(\sqrt{2/3}\right)$

b) $\vec{OA}\cdot\hat{\imath} = a = a\sqrt{3}\cdot 1\cdot\cos\beta \Rightarrow \cos\beta = \frac{\sqrt{3}}{3} \Rightarrow \beta = \cos^{-1}\left(\sqrt{3}/3\right)$

1B-4

a)



$\vec{QP} = \vec{OP} - \vec{OQ} = \langle a, 0, -2\rangle$

$\vec{QR} = \vec{OR} - \vec{OQ} = \langle a, -2, 2\rangle$ $\quad \Rightarrow \vec{QP}\cdot\vec{QR} = a^2 - 4$

$|\vec{QP}| = \sqrt{a^2 + 4}$

$|\vec{QR}| = \sqrt{a^2 + 4 + 4} = \sqrt{a^2 + 8}$

$$\cos\alpha = \frac{\vec{QP}\cdot\vec{QR}}{|\vec{QP}||\vec{QR}|} = \frac{a^2 - 4}{\sqrt{(a^2+4)(a^2+8)}}$$

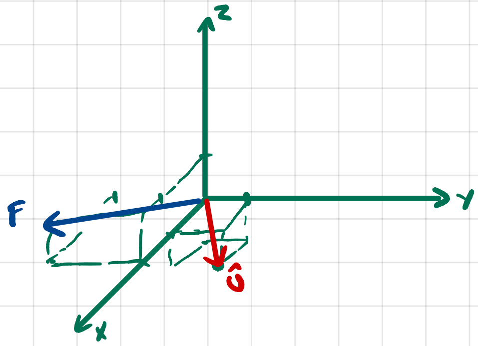
α right angle $\Rightarrow a^2 - 4 = 0 \Rightarrow a = \pm 2$

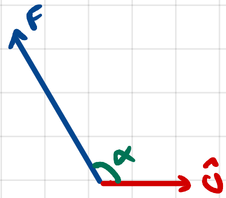
b) α acute $\Rightarrow \cos\alpha > 0 \Rightarrow a^2 - 4 > 0 \Rightarrow a^2 > 4 \Rightarrow |a| > 2$

1B-5

a) $\vec{F} = \langle 2,-2,1\rangle$

$\hat{u} = \dfrac{\langle 1,1,-1\rangle}{\sqrt{3}}$





α angle between $\vec{F}$ and $\hat{u}$.

$\text{comp}_{\hat{u}}\vec{F} = |\vec{F}|\cdot\cos\alpha = \dfrac{|\hat{u}||\vec{F}|\cos\alpha}{|\hat{u}|} = \dfrac{\vec{F}\cdot\hat{u}}{1} = \vec{F}\cdot\hat{u}$

$= \cancel{\dfrac{2}{\sqrt{3}}} - \cancel{\dfrac{2}{\sqrt{3}}} - \dfrac{1}{\sqrt{3}}$

$\vec{F}_u = (\text{comp}_{\hat{u}}\vec{F})\,\hat{u} = -\dfrac{1}{\sqrt{3}}\hat{u} = \langle -1/3, -1/3, 1/3\rangle$

* Find $F_{\perp u} = \vec{v}$ $\qquad \vec{v} = \langle v_1, v_2, v_3\rangle \qquad \vec{v} + \vec{F}_u = \vec{F}$

$\vec{v} = \vec{F} - \vec{F}_u = \langle 2,-2,1\rangle - \langle -1/3, -1/3, 1/3\rangle = \langle 2+1/3, -2+1/3, 1-1/3\rangle = \langle 7/3, -5/3, 2/3\rangle$

$\vec{v}\cdot\vec{u} = \dfrac{7}{3}\cdot\dfrac{1}{\sqrt{3}} - \dfrac{5}{3}\cdot\dfrac{1}{\sqrt{3}} + \dfrac{2}{3}\cdot\left(-\dfrac{1}{\sqrt{3}}\right) = \dfrac{7-5-2}{3\sqrt{3}} = 0$

b) $\hat{u} = \langle 3,2,-6\rangle$

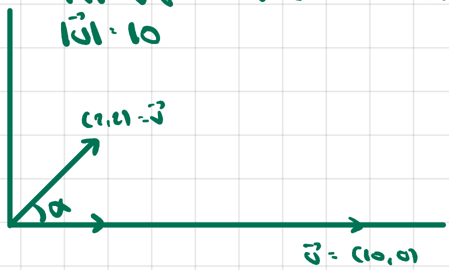
$|\hat{u}| = \sqrt{9+4+36} = 7$

$\text{comp}_u\vec{F} = |\vec{F}|\cos\alpha = \dfrac{|\vec{u}||\vec{F}|\cos\alpha}{|\vec{u}|} = \dfrac{\vec{F}\cdot\vec{u}}{|\vec{u}|} = \dfrac{6-4-6}{7} = -4/7$

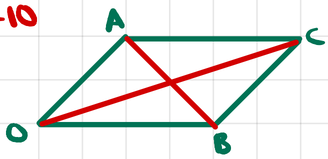
Just checking:

$|\vec{v}| = \sqrt{8} \qquad \vec{v}\cdot\vec{u} = 20 \qquad |\vec{v}|\cos\alpha = \dfrac{|\vec{v}||\vec{u}|\cos\alpha}{|\vec{u}|} = \dfrac{\vec{v}\cdot\vec{u}}{|\vec{u}|} = \dfrac{20}{10} = 2$

* $|\vec{u}| = 10$



1B-10



Prove $|\vec{AB}| = |\vec{OC}| \Leftrightarrow$ rectangle, ie $\vec{OA}\cdot\vec{OB} = 0$

1)

$\vec{AB} = \vec{OB} - \vec{OA}$

$\vec{OC} = \vec{OB} + \vec{OA}$

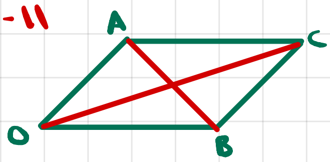
$\vec{AB}\cdot\vec{AB} = |AB|^2 \qquad \vec{OC}\cdot\vec{OC} = |OC|^2$

$$|\vec{AB}| = |\vec{OC}| \Leftrightarrow |\vec{AB}|^2 = |\vec{OC}|^2 \Leftrightarrow (\vec{OB} - \vec{OA})(\vec{OB} - \vec{OA}) = (\vec{OB} + \vec{OA})(\vec{OB} + \vec{OA})$$

$$\Leftrightarrow \cancel{\vec{OB}\cdot\vec{OB}} - 2\vec{OB}\cdot\vec{OA} + \cancel{\vec{OA}\cdot\vec{OA}} = \cancel{\vec{OB}\cdot\vec{OB}} + 2\vec{OB}\cdot\vec{OA} + \cancel{\vec{OA}\cdot\vec{OA}}$$

$$\Leftrightarrow 4\vec{OB}\cdot\vec{OA} = 0 \Leftrightarrow \vec{OB}\cdot\vec{OA} = 0$$

1B-11



Prove: $\vec{AB}\cdot\vec{OC} = 0 \Leftrightarrow |\vec{OA}| = |\vec{OB}|$

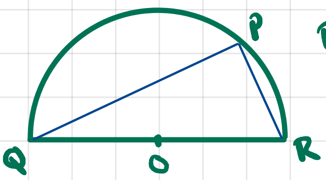
$\vec{AB} = \vec{OB} - \vec{OA} \qquad \vec{OC} = \vec{OA} + \vec{OB}$

$$\vec{AB}\cdot\vec{OC} = 0 \Leftrightarrow \cancel{\vec{OA}\cdot\vec{OB}} + \vec{OB}\cdot\vec{OB} - \vec{OA}\cdot\vec{OA} - \cancel{\vec{OA}\cdot\vec{OB}}$$

$$\Leftrightarrow |\vec{OB}|^2 - |\vec{OA}|^2 = 0$$

$$\Leftrightarrow |\vec{OB}|^2 = |\vec{OA}|^2 \Leftrightarrow |\vec{OB}| = |\vec{OA}|$$

1B-12



Prove $\vec{PR}\cdot\vec{QP} = 0$

$\vec{PR} = \vec{OR} - \vec{OP} \qquad |\vec{OP}| = |\vec{OR}| = |\vec{OQ}|$

$\vec{QP} = \vec{OP} - \vec{OQ}$

$$\vec{PR}\cdot\vec{QP} = \vec{OR}\cdot\vec{OP} - \underbrace{\cancel{\vec{OR}\cdot\vec{OQ}}}_{-1\cdot|OP|^2} - \cancel{|\vec{OP}|^2} + \vec{OP}\cdot\vec{OQ}$$

$$= \vec{OR}\cdot\vec{OP} + \vec{OP}\cdot\vec{OQ} = \vec{OP}\underbrace{(\vec{OR} + \vec{OQ})}_{0} = 0$$

Alternatively

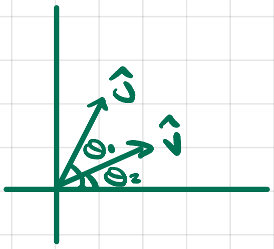
$\vec{QP} = \vec{QO} + \vec{OP} = \vec{OR} + \vec{OP}$, since $\vec{QO} = \vec{OR}$

$\vec{PR} = \vec{OR} - \vec{OP}$

$\vec{PR}\cdot\vec{QP} = |OR|^2 - |OP|^2 = 0$

1B-13

Prove $\cos(\theta_1-\theta_2)=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2$



$\hat{u}\cdot\hat{v}=|\hat{u}||\hat{v}|\cos(\theta_1-\theta_2)$
$=\cos(\theta_1-\theta_2)$

$\hat{u}=\langle\cos\theta_1,\sin\theta_1\rangle$

$\Rightarrow \hat{u}\cdot\hat{v}=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2$

$\hat{v}=\langle\cos\theta_2,\sin\theta_2\rangle$

$\Rightarrow \cos(\theta_1-\theta_2)=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2$

1C-1

a) $\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} = 1(-1) - 4(2) = -1 - 8 = -9$

b) $\begin{vmatrix} 3 & -4 \\ -1 & -2 \end{vmatrix} = -6 - (4) = -10$

1C-2

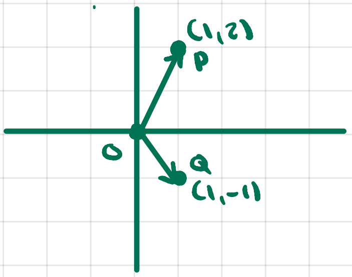
a) $\begin{vmatrix} -1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & -2 & -1 \end{vmatrix} = -1(-2-(-4)) + 4(-2-6)$

$= -1(2) + 4(-8) = -2 - 32 = -34$

b) $-1(2) - 1(-(-8)) + 3(-8) = -2 - 8 - 24 = -34$

1C-3

a)



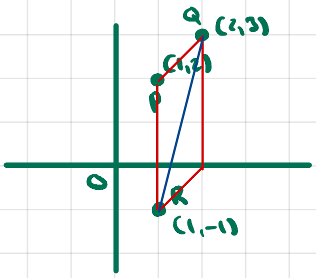
$\vec{OP} \times \vec{OQ} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 0 \\ 1 & -1 & 0 \end{vmatrix} = \hat{\imath} \cdot 0 - \hat{\jmath} \cdot 0 + \hat{k}(-1-2) = \langle 0, 0, -3 \rangle$

$\text{Area } OPQ = \dfrac{|\vec{OP} \times \vec{OQ}|}{2} = \dfrac{\sqrt{9}}{2} = \dfrac{3}{2}$

* note

$\vec{OQ} \times \vec{OP} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = \hat{k}(2+1) = \langle 0, 0, 3 \rangle$

b)



$\vec{PQ} = \vec{OQ} - \vec{OP} = \langle 1, 1 \rangle$

$\vec{PR} = \vec{OR} - \vec{OP} = \langle 0, -3 \rangle$

$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 0 \\ 0 & -3 & 0 \end{vmatrix} = \hat{k}(-3) = \langle 0, 0, -3 \rangle$

$\text{Area } PQR = \dfrac{|\vec{PQ} \times \vec{PR}|}{2} = \dfrac{3}{2}$

1C-5

a) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$

$\begin{vmatrix} a & b \\ c+ka & d+kb \end{vmatrix} = a(d + \cancel{kb}) - b(c + \cancel{ka})$

$= ad - bc$

1C-6 $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1+kc_1 & b_2+kc_2 & b_3+kc_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2+kc_2 & b_3+kc_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1+kc_1 & b_3+kc_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1+kc_1 & b_2+kc_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

1C-7 $f(x_1, x_2, y_1, y_2) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = |\vec{v}_1 \times \vec{v}_2|$, for $\vec{v}_1 = \langle x_1, y_1 \rangle$, $\vec{v}_2 = \langle x_2, y_2 \rangle$

↳ area of parallelogram, ie square

$|\vec{v}_1 \times \vec{v}_2| = |\vec{v}_1||\vec{v}_2| \sin\theta$ which is maxim. when $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$ ie $\vec{v}_1$ and $\vec{v}_2$ are

perpendiculars and $\vec{v}_1 \cdot \vec{v}_2 = 0 \Rightarrow x_1x_2 - y_1y_2 = 0.$

$$f(x_1, x_2, y_1, y_2) = |\langle x_1, y_1 \rangle \times \langle x_2, y_2 \rangle|$$

$$= |\langle x_1, y_1 \rangle| \, |\langle x_2, y_2 \rangle| \cdot \sin\theta$$

$$= \sin\theta$$

$$\Rightarrow \max f = 1$$

1D-1

a) $A = \langle 1,-2,1\rangle \quad B = \langle 2,-1,-1$

$$A\times B = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 1 \\ 2 & -1 & -1 \end{vmatrix} = (2+1)\hat{\imath} - (-1-2)\hat{\jmath} + (-1+4)\hat{k} = \langle 3,3,3\rangle$$

b) $$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & -3 \\ 1 & 1 & -1 \end{vmatrix} = (3)\hat{\imath} - (-2+3)\hat{\jmath} + (2)\hat{k} = \langle 3,-1,2\rangle$$

1D-2

$\vec{PQ} = \vec{OQ} - \vec{OP} = \langle 1,1,-1\rangle$

$\vec{PR} = \vec{OR} - \vec{OP} = \langle -3,1,-2\rangle$

Area $PQR = \frac{1}{2}\cdot|\vec{PQ}\times\vec{PR}| = \sqrt{42}/2$

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & k \\ 1 & 1 & -1 \\ -3 & 1 & -2 \end{vmatrix} = (-2+1)\hat{\imath} - (-2-3)\hat{\jmath} + (1+3)\hat{k} = \langle -1,5,4\rangle$$

$|\vec{PQ}\times\vec{PR}| = \sqrt{1+25+16} = \sqrt{42}$

1D-3

$\vec{A} = \langle 2,-1,0\rangle \quad \vec{B} = \langle 1,2,1\rangle$

$i' = \dfrac{\vec{A}}{|\vec{A}|} = \dfrac{\vec{A}}{\sqrt{5}} \qquad j' = \dfrac{\vec{B}}{|\vec{B}|} = \dfrac{\vec{B}}{\sqrt{6}}$

$|\vec{A}| = \sqrt{4+1} = \sqrt{5} \qquad |\vec{B}| = \sqrt{1+4+1} = \sqrt{6}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = (-1)\hat{\imath} - (2)\hat{\jmath} + (4+1)\hat{k} = \langle -1,-2,5\rangle$$

$|\vec{A}\times\vec{B}| = \sqrt{1+4+25} = \sqrt{30}$

$\hat{k} = \dfrac{\langle -1,-2,5\rangle}{\sqrt{30}}$

10-4 $\hat{i}\times\hat{i}=\vec{0}$ $\quad(\hat{i}\times\hat{i})\times\hat{j}=\vec{0}\times\hat{j}=\vec{0}$ $\quad$ i j k i j k i

$\hat{i}\times\hat{j}=\hat{k}$ $\quad\hat{i}\times(\hat{i}\times\hat{j})=\hat{i}\times\hat{k}=-\hat{j}$

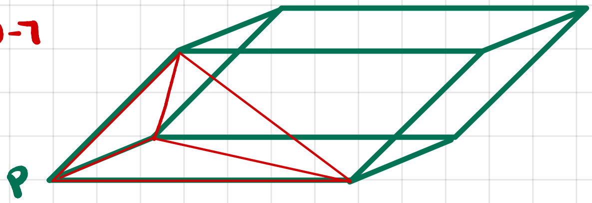
10-5

a) $|A\times B|=|A||B| \Rightarrow \sin\theta=1 \Rightarrow \theta=\pi/2$

b) $|A\times B|=A\cdot B \Rightarrow |A||B|\sin\theta=|A||B|\cos\theta \Rightarrow \sin\theta=\cos\theta$

$\theta=\pi/4$

10-7



P (1,0,1)
Q (-1,1,2)
R (0,0,2)
S (3,1,-1)

we can pick any vertex of the tetrahedron to be the shared vertex of three coterminous edges of a containing parallelepiped.

Let's choose P (1,0,1).

$|\vec{PQ}\times\vec{PR}|=$ Area parallelogram

$|\vec{PQ}\times\vec{PR}\cdot PS|=$ Volume parallelepiped $\Rightarrow$ Volume Tetrahedron $=\frac{1}{6}|\vec{PQ}\times\vec{PR}\cdot PS|$

$\vec{PQ}=\langle -2,1,1\rangle \qquad |\vec{PQ}|=\sqrt{4+1+1}=\sqrt{6}$

$\vec{PR}=\langle -1,0,1\rangle \qquad |\vec{PR}|=\sqrt{1+1}=\sqrt{2}$

$\vec{PS}=\langle 2,1,-2\rangle$

$\vec{PQ}\times\vec{PR}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\-2&1&1\\-1&0&1\end{vmatrix}=(1)\hat{i}-(-2+1)\hat{j}+(1)\hat{k}$

$=\langle 1,1,1\rangle$

$|\vec{PQ}\times\vec{PR}|=\sqrt{1+1+1}=\sqrt{3}=$ Area parallelogram

$\vec{PQ}\times\vec{PR}\cdot\vec{PS}=\langle 1,1,1\rangle\cdot\langle 2,1,-2\rangle$

$=2+1-2=1$

$|\vec{PQ}\times\vec{PR}\cdot PS|=1=$ Volume Parallelepiped

$V=\frac{1}{6}\cdot 1=\frac{1}{6}$

Alternatively,

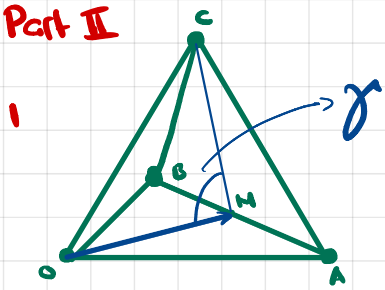
$$\begin{vmatrix}2&1&-2\\-2&1&1\\-1&0&1\end{vmatrix}$$

$=2\cdot 1-1(-2+1)-2(1)$

$=2+1-2=1=V$

Part II

1



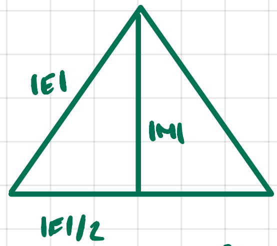
$\vec{OM} = \frac{1}{2}(\vec{OB}+\vec{OA})$

$\vec{MC} = \vec{OC} - \vec{OM} = \vec{OC} - \frac{1}{2}\vec{OB} - \frac{1}{2}\vec{OA}$

$\vec{OM}\cdot\vec{MC} = \frac{1}{2}(\vec{OB}+\vec{OA})(\vec{OC} - \frac{1}{2}\vec{OB} - \frac{1}{2}\vec{OA})$

$= \frac{1}{2}\Big[\vec{OB}\cdot\vec{OC} - \frac{1}{2}|\vec{OB}|^2 - \frac{1}{2}\vec{OB}\cdot\vec{OA} + \vec{OA}\cdot\vec{OC} - \frac{1}{2}\vec{OA}\cdot\vec{OB} - \frac{1}{2}|OA|^2$

$= \frac{1}{2}\Big[\vec{OB}\cdot\vec{OC} + \vec{OA}\cdot\vec{OC} - \vec{OA}\cdot\vec{OB} - \frac{1}{2}|\vec{OB}|^2 - \frac{1}{2}|\vec{OA}|^2\Big]$



$|OB| = |OC| = |OA| = |E|$

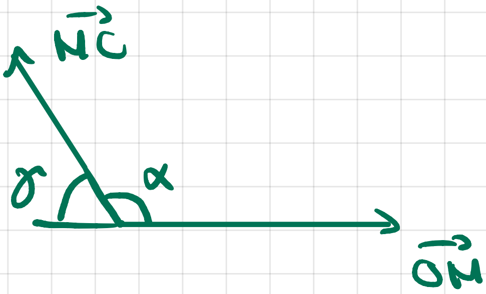
$\Rightarrow \frac{1}{2}\Big[|E|^2\cdot\cos 60^\circ - |E|^2\Big]$

$|E|^2 = |M|^2 + \frac{|E|^2}{4} \Rightarrow |M|^2 = \frac{3}{4}|E|^2$

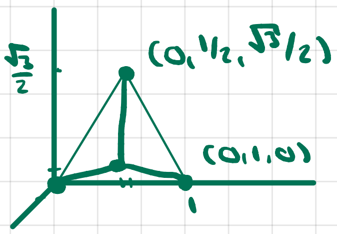
$= \frac{1}{2}\Big[|E|^2\cdot\frac{1}{2} - |E|^2\Big] = \frac{1}{2}\cdot\Big[-\frac{1}{2}|E|^2\Big] = -\frac{1}{4}|E|^2$

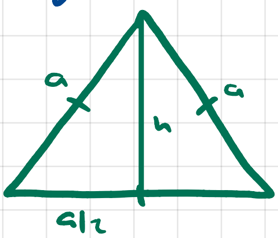
$\Rightarrow |M|^2\cdot\cos\alpha = -\frac{1}{4}|E|^2 \Rightarrow \cos\alpha = -\frac{1}{4}\frac{|E|^2}{|M|^2} = -\frac{1}{\cancel{4}}\cdot\cancel{|E|^2}\cdot\frac{\cancel{4}}{3\cancel{|E|^2}} = -\frac{1}{3}$

$\Rightarrow \alpha = \cos^{-1}(-1/3)$ $\quad \alpha = 109.47^\circ \quad \Rightarrow \gamma = 70.53^\circ$



Alternatively, there is an easier way:





$a^2 = h^2 + \frac{a^2}{4} \Rightarrow h^2 = \frac{3}{4}a^2 \Rightarrow h = \frac{\sqrt{3}}{2}a$

$P = \langle p_1, p_2, p_3 \rangle$

$$\begin{cases} p_1^2 + (p_2 - 1)^2 + p_3 = 1 \\ p_1^2 + (p_2 - 1/2)^2 + (p_3 - \frac{\sqrt{3}}{2})^2 = 1 \\ p_1^2 + p_2^2 + p_3^2 = 1 \end{cases}$$

$(p_2 - 1)^2 - p_2^2 = 0$

$\cancel{p_2^2} - 2p_2 + 1 - \cancel{p_2^2} = 0$

$p_2 = \frac{1}{2}$

$\Rightarrow p_1^2 + p_3^2 = \frac{3}{4}$

$p_1^2 + (p_3 - \sqrt{3}/2)^2 = 1$

$\rightarrow (p_3 - \sqrt{3}/2)^2 - p_3^2 = \frac{1}{4}$

$\cancel{p_3^2} - \sqrt{3}p_3 + \frac{3}{4} - \cancel{p_3^2} = \frac{1}{4} \Rightarrow \sqrt{3}p_3 = \frac{1}{2} \Rightarrow p_3 = \frac{1}{2\sqrt{3}}$

$\Rightarrow p_1^2 = \frac{3}{4} - \frac{1}{4\cdot 3} = \frac{9-1}{12} = \frac{2}{3} \Rightarrow p_1 = \frac{\sqrt{6}}{3}$

$\Rightarrow p = \langle \sqrt{6}/3, 1/2, \sqrt{3}/6 \rangle$

Two planes, two normal vectors.

$\langle 0, \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle \times \langle 0,1,0 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & 1 & 0 \end{vmatrix} = \hat{i}(-\frac{\sqrt{3}}{2})$

$= \langle -\sqrt{3}/2, 0, 0 \rangle$

$\langle \sqrt{6}/3, 1/2, \sqrt{3}/6 \rangle \times \langle 0,1,0 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sqrt{6}/3 & 1/2 & \sqrt{3}/6 \\ 0 & 1 & 0 \end{vmatrix}$

$= (-\sqrt{3}/6)\hat{i} + (\sqrt{6}/3)\hat{k}$

$= \langle -\sqrt{3}/6, 0, \sqrt{6}/3 \rangle$

we have two normal vectors, each perpendicular to one different face of the tetrahedron.

$$\vec{n}_1 = \langle -\sqrt{3}/2, 0, 0 \rangle$$

$$\vec{n}_2 = \langle -\sqrt{3}/6, 0, \sqrt{6}/3 \rangle$$

$$\cos\alpha = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = \frac{1/4}{\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

$$\vec{n}_1 \cdot \vec{n}_2 = \frac{3}{12} = \frac{1}{4}$$

$$\Rightarrow \alpha = \cos^{-1}(1/3)$$

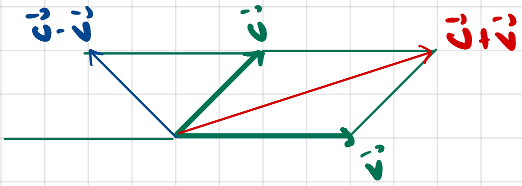
$$\Rightarrow \alpha = 70.53^\circ$$

$$|\vec{n}_1| = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$|\vec{n}_2| = \sqrt{\frac{3}{36} + \frac{6}{9}} = \sqrt{\frac{3+24}{36}} = \sqrt{\frac{27}{36}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$$

Problem 2

a) polarization identity: $\frac{1}{4}\left(|\vec{u}+\vec{v}|^2 - |\vec{u}-\vec{v}|^2\right) = \vec{u}\cdot\vec{v} \quad \forall\, \vec{v},\vec{u}$

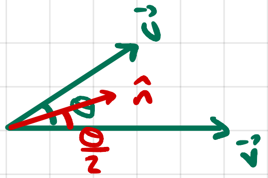


$\vec{u} = \frac{1}{2}\left[(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\right]$

$\Rightarrow \vec{u}\cdot\vec{v} = \frac{1}{4}\left[|\vec{u}+\vec{v}|^2 - |\vec{u}-\vec{v}|^2\right]$

$\vec{v} = \frac{1}{2}\left[(\vec{u}+\vec{v}) - (\vec{u}-\vec{v})\right]$

b)



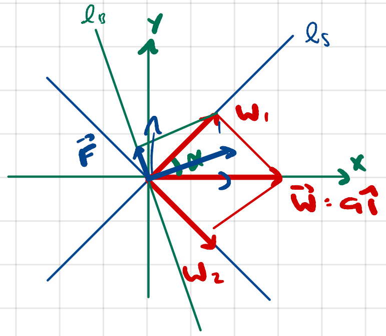
$\hat{n}\cdot\vec{u} = |\hat{n}||\vec{u}|\cos(\theta) = |\vec{u}|\cos\theta/2$

$\hat{n}\cdot\vec{v} = |\vec{v}|\cos\theta/2$

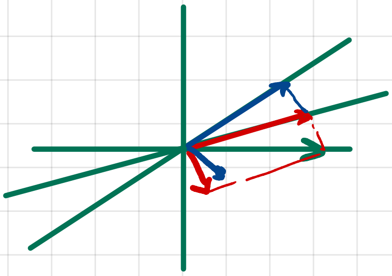
$\Rightarrow \hat{n}\cdot\hat{u} = \hat{n}\cdot\hat{v} = \cos\theta/2$

Problem 3

a)



$\vec\omega = \vec\omega_1 + \vec\omega_2$



b) $\text{comp}_{\ell_S}\vec\omega = |\vec\omega|\cos\alpha = |a|\cos\alpha$

$\text{comp}_{\ell_B}\vec\omega_1 = |\vec\omega_1|\cdot\cos\beta = |\vec\omega|\cos\alpha\cos\beta = |a|\cos\alpha\cos\beta$

$\text{comp}_{\vec\omega}\vec F = |\vec F|\cdot\cos(\alpha+\beta) = |a|\cos\alpha\cos\beta\cos(\alpha+\beta)$

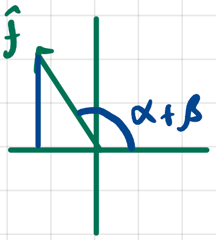
$\alpha=\beta=\pi/3 \Rightarrow \text{comp}_{\ell_S}\vec\omega = |a|\cdot\cos(\pi/3) = \frac{|a|}{2}$

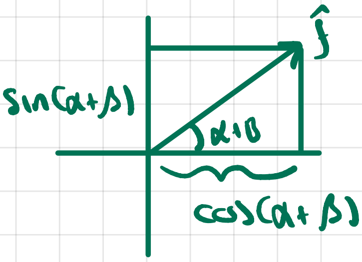
$\text{comp}_{\ell_B}\omega_1 = |a|\cos(\pi/3)\cdot\cos(\pi/3) = \frac{|a|}{4}$

$\text{comp}_{\vec\omega}\vec F = \frac{|a|}{4}\cdot\cos(2\pi/3) = \frac{|a|}{4}\cdot\left(-\frac{1}{2}\right) = -\frac{|a|}{8}$

For $\vec F$ to have a component in $-\hat\imath$, we need $\cos(\alpha+\beta)<0 \Rightarrow \alpha+\beta>0$

Note that if $\hat f$ is unit vector in direction of $\vec F$ is $\text{dir}(\vec F) = \frac{\vec F}{|\vec F|} = \langle \cos\alpha+\beta, \sin\alpha+\beta\rangle$





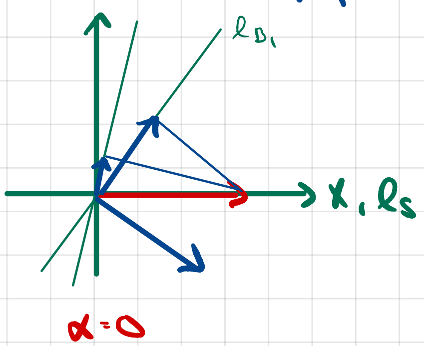
$\Rightarrow \vec F = |a|\cos\alpha\cos\beta\langle\cos\alpha+\beta, \sin\alpha+\beta\rangle$

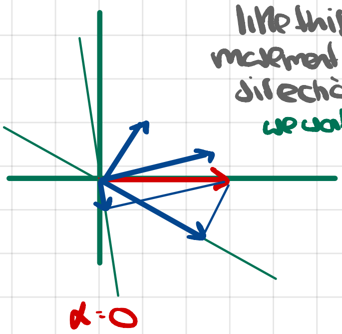
So to have a $-\hat\imath$ component,

$|a|\cos\alpha\cos\beta\cos(\alpha+\beta)<0$

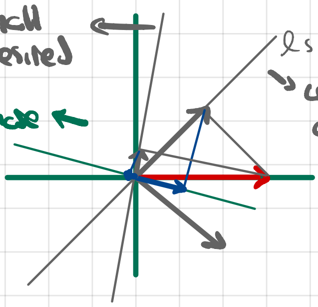
$\Rightarrow \alpha+\beta > \pi/2$ since $0<\alpha,\beta<\pi/2$

$\rightarrow \alpha=0$ (sail perpendicular to wind)





like this, small movement in desired direction we want to move



with sails like this we move in opposite direction than desired