Directional Derivatives

Given $z = f(x,y)$, its first-order partial derivatives are:

$$f_x(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x} + h\hat{\imath}) - f(\vec{x})}{h} \qquad f_y(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x} + h\hat{\jmath}) - f(\vec{x})}{h}$$

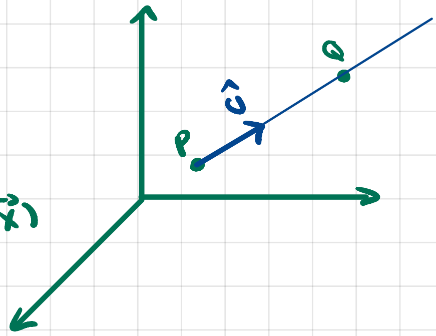
$\vec{x} = \langle x, y \rangle$, $\hat{\imath} = \langle 1, 0 \rangle$, $\hat{\jmath} = \langle 0, 1 \rangle$

If we replace $\hat{\imath}$ or $\hat{\jmath}$ with another unit vector $\hat{u}$ we obtain a directional derivative

$$D_{\hat{u}} f(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x} + h\hat{u}) - f(\vec{x})}{h}$$ provided the limit exists.

Note

$\vec{P} = \vec{x}$
$\vec{Q} = \vec{x} + h\hat{u}$



$\Delta w = f(Q) - f(P) = f(\vec{x} + h\hat{u}) - f(\vec{x})$

$\Delta s = |\vec{PQ}| = h$ = distance

$\dfrac{\Delta w}{\Delta s} = \dfrac{f(\vec{x} + h\hat{u}) - f(\vec{x})}{h}$ = average rate of change of f w/ respect to distance

If we take the limit as $h \to 0$ we obtain the instantaneous rate of change of w at P with respect to distance in the direction of $\hat{u}$ (ie dir. $\vec{P}$ to $\vec{Q}$)

$$\frac{dw}{ds} = \lim_{\Delta s \to 0} \frac{\Delta w}{\Delta s} = D_{\hat{u}} f(\vec{x})$$

Theorem

real-valued function f diff at $\vec{x}$
$\hat{u}$ a unit vector
$\Rightarrow$ $D_{\hat{u}} f(\vec{x})$ exists and $= \nabla f(\vec{x}) \cdot \hat{u}$

Chain Rule

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$
$f(x,y,z)$ differentiable function
$\Rightarrow$ $f(x(t), y(t), z(t))$ diff fn of t

velocity vector

$$D_t[f(\vec{r}(t))] = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt} = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$r(t)$ smooth parametric curve w/ nonzero velocity vector $\Rightarrow$ $\vec{v} = v \cdot \hat{u}$, $v = |\vec{v}|$

$$\Rightarrow D_t[f(\vec{r}(t))] = v \cdot \nabla f(\vec{r}(t)) \cdot \hat{u} = v D_{\hat{u}} f(\vec{r}(t)) = \frac{dw}{ds} \cdot \frac{ds}{dt}$$

where $w = f(\vec{r}(t))$

Gradient Vector as a Normal Vector

$F(x,y,z)=0$

implicit function theorem $\Rightarrow$ near any point P where $\frac{\partial F}{\partial z} \neq 0$, F defines z implicitly in terms of x and y, continuously diff.

$\Rightarrow$ near P, graph $F(x,y,z)=0$ coincides with the surface $z=f(x,y)$

Similarly, $\frac{\partial F}{\partial x} \neq 0$ near P $\Rightarrow$ $x=g(y,z)$, defined implicitly, coincides with F near P

Analogous result for $y=h(x,z)$ if $\frac{\partial F}{\partial y} \neq 0$ near P

$\Rightarrow \nabla F(P) \neq \vec{0}$, ie any of $F_x, F_y, F_z \neq 0$ near P, $\Rightarrow$ $F(x,y,z)$ looks like a surface near P

Theorem

$F(x,y,z)$ cont. diff

$P(x_0,y_0,z_0)$ point where $\nabla F(P_0) \neq \vec{0}$ $\Rightarrow \nabla F(P_0) \cdot r'(t_0)=0$

$\vec{r}(t)$ diff curve on this surface, $r(t_0)=\langle x_0,y_0,z_0\rangle$,

$r'(t_0) \neq 0$