PSet 7

3A-1

a) $\int_0^2\int_{-1}^{1}(6x^2+2y)\,dy\,dx = \int_0^2 12x^2\,dx = 12\frac{x^3}{3}\Big|_0^2 = 4x^3\Big|_0^2 = 4\cdot 8 = 32$

$\int_{-1}^{1}(6x^2+2y)\,dy = \left[6x^2y+y^2\right]_{-1}^{1} = 6x^2+1-(-6x^2+1) = 12x^2$

b) $\int_0^{\pi/2}\int_0^{\pi}(u\sin t + t\cos u)\,dt\,du = \int_0^{\pi/2}\left(2u+\frac{\pi^2}{2}\cos u\right)du = \left(u^2+\frac{\pi^2}{2}\sin u\right)\Big|_0^{\pi/2} = \frac{\pi^2}{4}+\frac{\pi^2}{2}-(0+0) = \frac{3\pi^2}{4}$

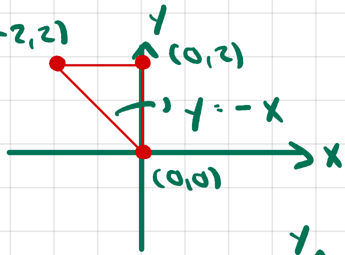
$\int_0^{\pi}(u\sin t + t\cos u)\,dt = \left[-u\cos t + \cos u\cdot\frac{t^2}{2}\right]_0^{\pi} = u+\frac{\pi^2}{2}\cos u - (-u+0) = 2u+\frac{\pi^2}{2}\cos u$

c) $\int_0^1\int_{\sqrt{x}}^{x^2} 2x^2y\,dy\,dx = \int_0^1 (x^6-x^3)\,dx = \left(\frac{x^7}{7}-\frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{7}-\frac{1}{4} = \frac{4-7}{28} = -\frac{3}{28}$

$\int_{\sqrt{x}}^{x^2} 2x^2y\,dy = x^2y^2\Big|_{\sqrt{x}}^{x^2} = x^2(x^4-x)$

d) $\int_0^1\int_0^{u}\sqrt{u^2+4}\,dv\,du = \int_0^1 u\sqrt{u^2+4}\,du = \frac{(u^2+4)^{3/2}}{(3/2)\cdot 2}\Big|_0^1 = \frac{(u^2+4)^{3/2}}{3}\Big|_0^1 = \frac{5^{3/2}-4^{3/2}}{3} = \frac{5\sqrt{5}-8}{3}$

$\int_0^{u}\sqrt{u^2+4}\,dv = u\sqrt{u^2+4}$



3A-2

ai $\int_{-2}^{0}\int_{-x}^{2} dy\,dx = \int_{-2}^{0}(2+x)\,dx = \left(2x+\frac{x^2}{2}\right)\Big|_{-2}^{0}$

$= 0-(-4+2) = 2$

aii $\int_0^2\int_{-y}^{0} dx\,dy = \int_0^2 y\,dy = \frac{y^2}{2}\Big|_0^2 = 2$

$y = 2x-x^2$

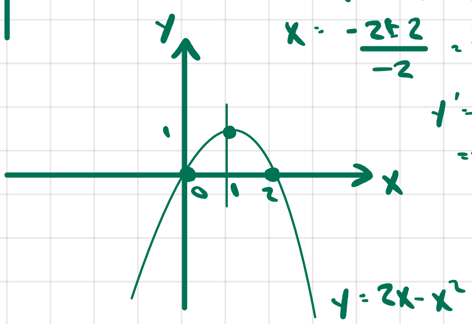
$\Delta = 4-4(-1)\cdot 0 = 4$

$x = \frac{-2\pm 2}{-2} \nearrow 0 \searrow 2$

$y' = 2-2x = 0$

$\Rightarrow x = 1$

$y(1) = 1$



bi $\int_0^2\int_0^{2x-x^2} dy\,dx = \int_0^2 (2x-x^2)\,dx$

$= \left(x^2-\frac{x^3}{3}\right)\Big|_0^2 = \left(4-\frac{8}{3}\right)-0 = 4/3$

$x^2-2x+y = 0$

$\Delta = 4-4y$

$x = \frac{2\pm\sqrt{4-4y}}{2} = 1\pm\sqrt{1-y}$

bii $\int_0^1\int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} dx\,dy = \int_0^1 2\sqrt{1-y}\,dy = -\frac{2(1-y)^{3/2}}{3/2}\Big|_0^1 = -2(0)-\left(-2\cdot 1\cdot\frac{2}{3}\right) = 4/3$

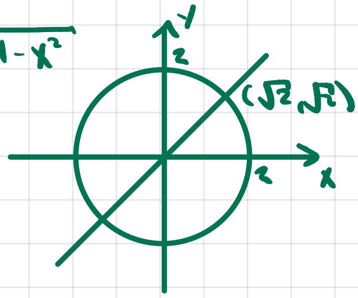
ci $\displaystyle\int_0^{\sqrt{2}}\int_0^{x} dy\,dx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} dy\,dx$

$\displaystyle= \int_0^{\sqrt{2}} x\,dx + \int_{\sqrt{2}}^{2}\sqrt{4-x^2}\,dx$

$x^2+y^2=4 \Rightarrow y=\pm\sqrt{4-x^2}$
$y=x$

$\Rightarrow 2x^2=4 \Rightarrow x^2=2$

$\Rightarrow x=\pm\sqrt{2},\ y=\pm\sqrt{2}$



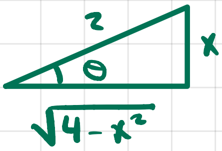
$\displaystyle\int\sqrt{4-x^2}\,dx = \int\sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta = \int 2\sqrt{1-\sin^2\theta}\ 2\cos\theta\,d\theta$

$x=2\sin\theta \quad dx=2\cos\theta\,d\theta$

$\displaystyle= 4\int\cos^2\theta\,d\theta = 4\cdot\frac{1}{2}\int(1+\cos 2\theta)\,d\theta = 2\left(\theta+\frac{\sin(2\theta)}{2}\right) = 2\theta+\sin 2\theta = 2\theta+2\sin\theta\cos\theta$

$\theta=\sin^{-1}(x/2)$

$\displaystyle= 2\cdot\sin^{-1}(x/2) + 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} = 2\sin^{-1}(x/2)+\frac{x\sqrt{4-x^2}}{2}$



$\displaystyle\Rightarrow \int_0^{\sqrt{2}} x\,dx + \int_{\sqrt{2}}^{2}\sqrt{4-x^2}\,dx = \frac{x^2}{2}\Big|_0^{\sqrt{2}} + \left[2\sin^{-1}(x/2)+\frac{x\sqrt{4-x^2}}{2}\right]_{\sqrt{2}}^{2}$

$\displaystyle= 1+\left[2\sin^{-1}(1)+0-\left(2\sin^{-1}(\sqrt{2}/2)+\frac{\sqrt{2}\sqrt{4-2}}{2}\right)\right.$

$\displaystyle= \cancel{1}+2\cdot\frac{\pi}{2}-2\cdot\frac{\pi}{4}\cancel{-1} = \pi-\frac{\pi}{2} = \pi/2$

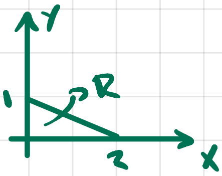
cii $\displaystyle\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} dx\,dy = \int_0^{\sqrt{2}}\left(\sqrt{4-y^2}-y\right)dy = \left[2\sin^{-1}(y/2)+\frac{y\sqrt{4-y^2}}{2}\right]_0^{\sqrt{2}} - \frac{y^2}{2}\Big|_0^{\sqrt{2}}$

$\displaystyle= 2\cdot\sin^{-1}(\sqrt{2}/2)+\frac{\sqrt{2}\sqrt{2}}{2}-\left[2\sin^{-1}(0)+0\right]-1$

$\displaystyle= 2\cdot\frac{\pi}{4}+1-1 = \pi/2$

3A-3

a) $\iint_R x\,dA$



$2y+x=2 \Rightarrow 2y=2-x$

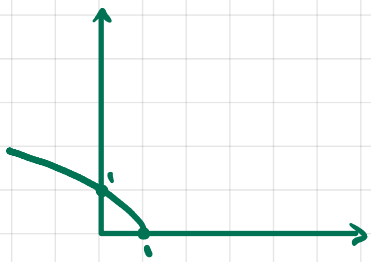
$\Rightarrow y=1-\frac{x}{2} \Rightarrow x=2-2y$

$$= \int_0^1\int_0^{2-2y} x\,dx\,dy = \frac{1}{2}\int_0^1(4-8y+4y^2)\,dy = 2\int_0^1(1-2y+y^2)\,dy = 2\left[y-y^2+\frac{y^3}{3}\right]_0^1 = 2\left[1-1+\frac{1}{3}\right]$$

$$= \frac{2}{3}$$

$$\int_0^{2-2y} x\,dx = \frac{x^2}{2}\Big|_0^{2-2y} = \frac{4-8y+4y^2}{2}$$

b)



$y^2=1-x \Rightarrow x=1-y^2$

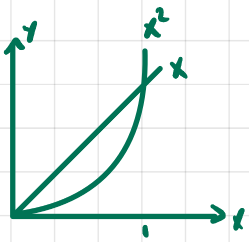
$y=\pm\sqrt{1-x}$

$$\int_0^1\int_0^{\sqrt{1-x}}(2x+y^2)\,dy\,dx = \int_0^1\int_0^{1-y^2}(2x+y^2)\,dx\,dy = \int_0^1\left[2\frac{x^2}{2}+y^2x\right]\Big|_0^{1-y^2}dy = \int_0^1\left[(1-y^2)^2+y^2(1-y^2)\right]dy$$

$$= \int_0^1\left[1-2y^2+\cancel{y^4}+y^2-\cancel{y^4}\right]dy = \int_0^1(1-y^2)\,dy = \left(y-\frac{y^3}{3}\right)\Big|_0^1 = 1-\frac{1}{3} = 2/3$$

3A-4

b)

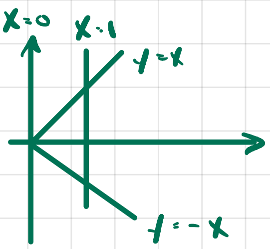


$$\int_0^1\int_{x^2}^{x} xy\,dy\,dx = \int_0^1 x\frac{y^2}{2}\Big|_{x^2}^{x}dx = \frac{1}{2}\int_0^1(x^3-x^5)\,dx = \frac{1}{2}\left[x^4/4 - x^6/6\right]_0^1$$

$$= \frac{1}{2}\left[\left(\frac{1}{4}-\frac{1}{6}\right)\right] = \frac{1}{2}\left[\frac{3-2}{12}\right] = \frac{1}{24}$$

$x^2=x \Rightarrow x(x-1)=0 \Rightarrow x=1,0$

c)



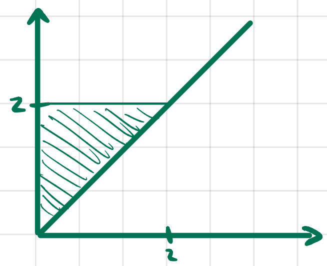
$$\int_0^1\int_{-x}^{x}(x^2-y^2)\,dy\,dx = \int_0^1\left(x^2y-\frac{y^3}{3}\right)\Big|_{-x}^{x}dx = \int_0^1\left(x^3-\frac{x^3}{3}-\left(-x^3+\frac{x^3}{3}\right)\right)dx$$

$$= \int_0^1\left(\frac{2x^3}{3}+\frac{2x^3}{3}\right)dx = \frac{4}{3}\int_0^1 x^3\,dx = \frac{4}{3}\cdot\frac{x^4}{4}\Big|_0^1 = \frac{1}{3}$$

$x^2-y^2=0 \Rightarrow x^2=y^2 \Rightarrow y=\pm x$

3A-5

a) $\int_0^2\int_x^2 e^{-y^2}\,dy\,dx = \int_0^2\int_0^y e^{-y^2}\,dx\,dy = \int_0^2 y e^{-y^2}\,dy = -\frac{e^{-y^2}}{2}\Big|_0^2 =$



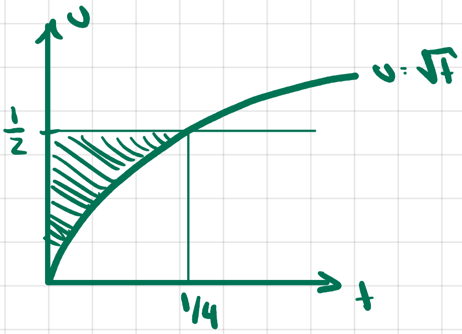
$= \frac{1}{2}\left[-e^{-4}-(-e^{0})\right] = \frac{1}{2}\left[-e^{-4}+1\right]$

$= \frac{1}{2}\left[1-e^{-4}\right]$

b) $\int_0^{1/4}\int_{\sqrt{t}}^{1/2} \frac{e^u}{u}\,du\,dt = \int_0^{1/2}\int_0^{u^2} \frac{e^u}{u}\,dt\,du = \int_0^{1/2} \frac{e^u}{u}\cdot u^2\,du = \int_0^{1/2} e^u\cdot u\,du = (e^u u - e^u)\Big|_0^{1/2}$

$= \frac{e^{1/2}}{2} - e^{1/2} - (0-1)$

$= -\frac{e^{1/2}}{2} + 1 = 1 - e^{1/2}/2$

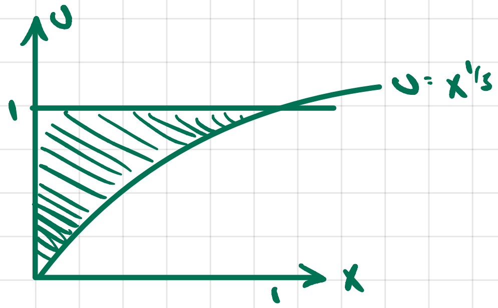


$m = u \quad dm = du$

$dn = e^u du \quad n = e^u$

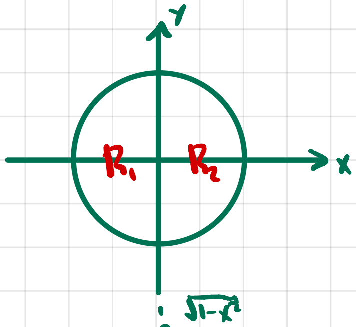
$\int e^u u\,du = e^u u - \int e^u\,du = e^u u - e^u$

c) $\int_0^1\int_{x^{1/3}}^1 \frac{1}{1+u^4}\,du\,dx = \int_0^1\int_0^{u^3} \frac{1}{1+u^4}\,dx\,du = \int_0^1 \frac{u^3}{1+u^4}\,du = \frac{\ln(1+u^4)}{4}\Big|_0^1$



$= \frac{(\ln 2 - \ln 1)}{4} = \frac{\ln 2}{4}$

3A-6



$\iint_R x\,dA = \iint_{R_1} x\,dA + \iint_{R_2} x\,dA = 0$ because for $f(x,y) = x$,

$f(x,y) = -f(-x,y)$

$\iint_R e^x\,dA = 2 \cdot$ integral over quadrants where $y>0$. $= 2\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} e^x\,dy\,dx$

$\iint_R x^2\,dA = 4\int_0^1\int_0^{\sqrt{1-x^2}} x^2\,dy\,dx$

$\iint_R x^2 y\,dA = 0$ because $f = x^2 y$

$\Rightarrow f(x^2,y) = -f(x^2,-y)$

$\iint_R (x^2+y)\,dA = \iint_R x^2\,dA$

$\iint_R xy\,dA \Rightarrow f(x,y) = xy,$

$f(-x,y) = -xy = -f(x,y)$

$f(-x,-y) = xy = f(x,y)$ $\Rightarrow \iint_R xy\,dA = 0$

$f(x,-y) = -xy = -f(x,y)$

3A-7 $f \le g$ on $R \Rightarrow \iint_R f\,dA \le \iint_R g\,dA$

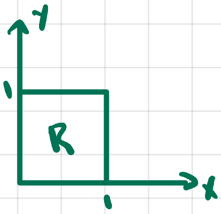
a) $\iint_R \frac{dA}{1+x^4+y^4}$

let $f(x,y) = \frac{1}{1+x^4+y^4}$ $\quad g(x,y) = 1$

$\forall x,y \in \mathbb{R} \quad f(x,y) \le g(x,y)$

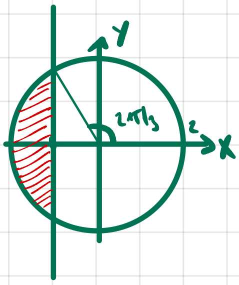
Area R $= \iint_R g\,dA \Rightarrow \iint_R f\,dA \le \iint_R g\,dA =$ Area R

b) $\iint_R \frac{x\,dA}{1+x^2+y^2} \le \iint_R \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_0^1 \ln(1+x^2) = \frac{1}{2}\ln 2 \approx 0.35$



3B-1

a)

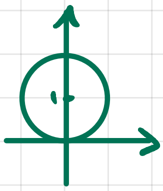


$$\int_{2\pi/3}^{4\pi/3} \int_{\frac{-1}{\cos\theta}}^{2} f(r,\theta)\, r\,dr\,d\theta$$

$x = -1 \Rightarrow r\cos\theta = -1 \Rightarrow r = -1/\cos\theta$

intersection: $r = 2 = -1/\cos\theta \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = 2\pi/3$

b)



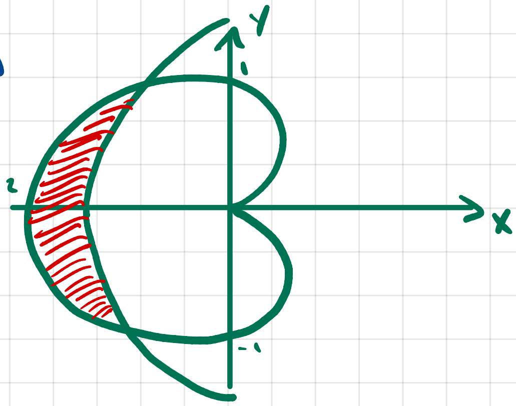
$$\int_0^{\pi} \int_0^{2\sin\theta} f(r,\theta)\, r\,dr\,d\theta$$

$x^2 + (y-1)^2 = 1 = r^2$

$r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + \cancel{1} = \cancel{1}$

$r^2 - 2r\sin\theta = 0 \quad r(r - 2\sin\theta) = 0 \Rightarrow r = 2\sin\theta$

c)

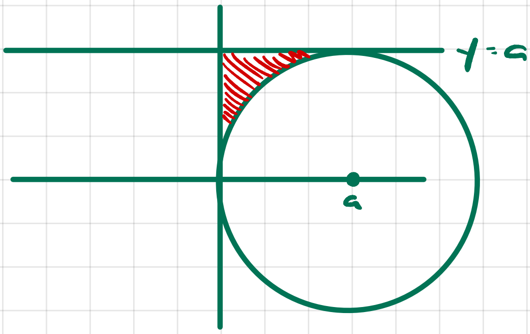


θ	$1-\cos\theta$
0	0
$\pi/2$	1
π	2
$3\pi/2$	1

intersection of $r = \frac{3}{2}$ and $r = 1-\cos\theta \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

$$\int_{2\pi/3}^{4\pi/3} \int_{3/2}^{1-\cos\theta} f(r,\theta)\, r\,dr\,d\theta$$

d)



$$\int_{\pi/4}^{\pi/2}\int_{2a\cos\theta}^{a/\sin\theta} f(r,\theta)\, r\, dr\, d\theta$$

$$(x-a)^2+y^2=a^2$$

$$y=a=r\sin\theta \Rightarrow r=\frac{a}{\sin\theta}$$

$$x^2-2ax+a^2+y^2=a^2$$

$$r^2-2ar\cos\theta=0$$

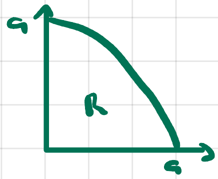
$$r(r-2a\cos\theta)=0 \Rightarrow r=2a\cos\theta$$

intersection

$$r=2a\cos\theta=\frac{a}{\sin\theta} \Rightarrow 2\sin\theta\cos\theta=1 \Rightarrow \sin(2\theta)=1 \quad 2\theta=\pi/2 \Rightarrow \theta=\pi/4$$

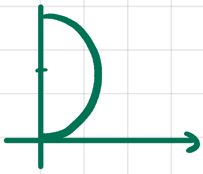
3B-2

b)



$$\iint_R \frac{1}{1+x^2+y^2}\,dx\,dy=\int_0^{\pi/2}\int_0^a \frac{r}{1+r^2}\,dr\,d\theta=\int_0^{\pi/2}\frac{1}{2}\ln(1+a^2)\,d\theta=\frac{\ln(1+a^2)}{2}\cdot\frac{\pi}{2}$$

d)



$$x^2+(y-1/2)^2=\frac{1}{4}$$

$$x^2+y^2-y+\frac{1}{4}=\frac{1}{4}$$

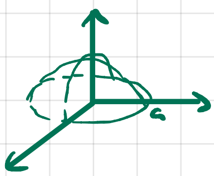
$$r^2-r\sin\theta=0 \Rightarrow r(r-\sin\theta)=0 \Rightarrow r=\sin\theta$$

$$\int_0^{\pi/2}\int_0^{\sin\theta}\frac{r}{\sqrt{1-r^2}}\,dr\,d\theta=-\int_0^{\pi/2}(1-r^2)^{1/2}\Big|_0^{\sin\theta}d\theta=-\int_0^{\pi/2}\left[(1-\sin^2\theta)^{1/2}-1\right]d\theta=-\int_0^{\pi/2}(\cos\theta-1)\,d\theta=-(\sin\theta-\theta)\Big|_0^{\pi/2}$$

$$-\left[\left(1-\frac{\pi}{2}\right)-(0-0)\right]=\pi/2-1$$

3B-3

a)



$$\int_0^{2\pi}\int_0^a\sqrt{a^2-r^2}\,r\,dr\,d\theta=\int_0^{2\pi}(a^2-r^2)^{3/2}\cdot\frac{2}{3}\left(-\frac{1}{2}\right)\Big|_0^a d\theta$$

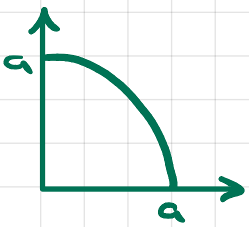
$$=-\frac{1}{3}\int_0^{2\pi}-a^3\,d\theta=\frac{1}{3}a^3\theta\Big|_0^{2\pi}=\frac{2\pi a^3}{3}$$

$$r=a \Rightarrow x^2+y^2+z^2=a^2$$

$$z=\sqrt{a^2-r^2\sin^2\theta-r^2\cos^2\theta}$$

$$=\sqrt{a^2-r^2}$$

b) $f(x,y) = xy$



$$\int_0^{\pi/2}\int_0^{a} r^2\sin\theta\cos\theta \, r\,dr\,d\theta = \frac{1}{4}\int_0^{\pi/2} \sin\theta\cos\theta \, a^4 \, d\theta$$

$$= \frac{a^4}{4}\,\frac{\sin^2\theta}{2}\Big|_0^{\pi/2} = \frac{a^4}{8}$$

$f(r,\theta) = r^2\sin\theta\cos\theta$

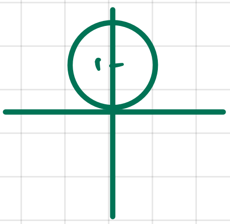
$\int \sin\theta\cos\theta\, d\theta =$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$= \int u\,du = \frac{u^2}{2} = \frac{\sin^2\theta}{2}$

c) $z = \sqrt{x^2+y^2}$



$$\int_0^{2\pi}\int_0^{2\sin\theta} r^2\,dr\,d\theta = \frac{1}{3}\int_0^{2\pi} 8\sin^3\theta\,d\theta = \frac{8}{3}\left[\frac{\cos^3\theta}{3} - \cos\theta\right]\Big|_0^{\pi}$$

$$= \frac{8}{3}\left[\left(-\frac{1}{3}+1\right) - \left(\frac{1}{3}-1\right)\right] = \frac{8}{3}\left[\frac{2}{3}+\frac{2}{3}\right] = \frac{32}{9}$$

$x^2+y^2-2y+1 = 1$

$r^2 - 2r\sin\theta = 0$

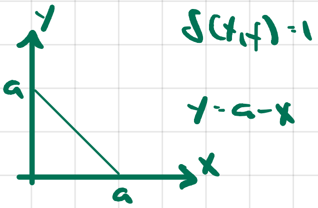
$r(r-2\sin\theta) = 0 \Rightarrow r = 2\sin\theta$

$$\int \sin^3\theta\,d\theta = \int \sin^2\theta\,\sin\theta\,d\theta = \int \sin\theta\,d\theta - \int \cos^2\theta\,\sin\theta\,d\theta$$

$$= -\cos\theta - \left[-\frac{\cos^3\theta}{3}\right] = \frac{\cos^3\theta}{3} - \cos\theta$$

3C-1

a)



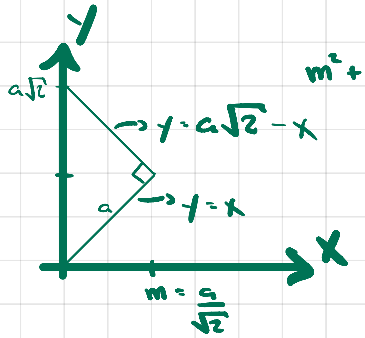
$$I=\int_0^a\int_0^{a-x} x^2\,dy\,dx=\int_0^a x^2(a-x)\,dx=\frac{a}{3}x^3-\frac{x^4}{4}\Big|_0^a=\frac{a^4}{3}-\frac{a^4}{4}=\frac{a^4}{12}$$

b)

$$I_0=\int_0^a\int_0^{a-x}(x^2+y^2)\,dy\,dx=\int_0^a\left[x^2(a-x)+\frac{1}{3}(a-x)^3\right]dx=\left[\frac{a}{3}x^3-\frac{x^4}{4}-\frac{(a-x)^4}{12}\right]\Big|_0^a$$

$$=\frac{a^4}{3}-\frac{a^4}{4}-\left[-\frac{a^4}{12}\right]=\frac{a^4}{12}+\frac{a^4}{12}=\frac{a^4}{6}$$

c)

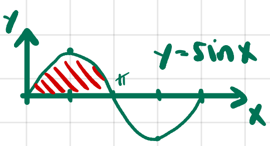


$$m^2+\left(\frac{a\sqrt{2}}{2}\right)^2=a^2 \Rightarrow m^2+\frac{a^2}{2}=a^2 \Rightarrow m^2=\frac{a^2}{2} \Rightarrow m=\frac{a}{\sqrt{2}}$$

$$I=\int_0^{a/\sqrt{2}}\int_x^{a\sqrt{2}-x} x^2\,dy\,dx=\int_0^{a/\sqrt{2}} x^2(a\sqrt{2}-x-x)\,dx=\int_0^{a/\sqrt{2}}(a\sqrt{2}x^2-2x^3)\,dx=a\sqrt{2}\cdot\frac{1}{3}x^3-\frac{x^4}{2}\Big|_0^{a/\sqrt{2}}=\frac{a\cancel{\sqrt{2}}}{3}\cdot\frac{a^3}{2\cancel{\sqrt{2}}}-\frac{a^4}{2\cdot 4}=\frac{a^4}{6}-\frac{a^4}{8}$$

$$=\frac{4a^4-3a^4}{24}=\frac{a^4}{24}$$

3C-2



a) $\delta(x,y)=1$

$$\text{mass}=\iint_R \delta(x,y)\,dA=\iint_R dm=\int_0^\pi\int_0^{\sin x} dy\,dx=\int_0^\pi \sin x\,dx=-\cos x\Big|_0^\pi=-\cos\pi-(-\cos 0)=-(-1)-(-1)=1+1$$

$$=2$$

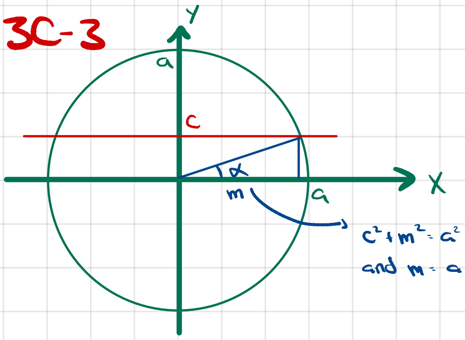
$\bar{x}=\pi/2$ by symmetry. $\bar{y}=\dfrac{\int_0^\pi\int_0^{\sin x} y\,dy\,dx}{m}=\dfrac{\int_0^\pi \frac{1}{2}\sin^2 x\,dx}{2}=\frac{1}{4}\cdot\frac{\pi}{2}=\pi/8$

b) $\delta=y$

$$m=\int_0^\pi\int_0^{\sin x} y\,dy\,dx=\frac{\pi}{4}$$

$$\bar{y}=\frac{\int_0^\pi\int_0^{\sin x} y^2\,dy\,dx}{m}=\frac{4}{\pi}\cdot\int_0^\pi \frac{1}{3}\sin^3 x\,dx=\frac{16}{9\pi}$$

3C-3



$x^2+y^2=a^2$

$y=c \Rightarrow x^2=a^2-c^2 \Rightarrow x=\pm\sqrt{a^2-c^2}$

$c^2+m^2=a^2 \Rightarrow m=\sqrt{a^2-c^2}$
and $m=a\cos\alpha$ $\Rightarrow a\cos\alpha=(a^2-c^2)^{1/2}$

a) $I=2\cdot\int_c^a\int_0^{\sqrt{a^2-y^2}} y\,dx\,dy = 2\int_c^a y\sqrt{a^2-y^2}\,dy = 2\cdot(a^2-y^2)^{3/2}\cdot\frac{2}{3}\cdot\left(-\frac{1}{2}\right)\Big|_c^a$

$= -\frac{2}{3}\left[-(a^2-c^2)^{3/2}\right] = \frac{2(a^2-c^2)^{3/2}}{3}$

b) $I=2\cdot\int_\alpha^{\pi/2}\int_{c/\sin\theta}^{a} r^2\sin\theta\,dr\,d\theta = 2\int_\alpha^{\pi/2}\sin\theta\,\frac{1}{3}r^3\Big|_{c/\sin\theta}^{a}d\theta = \frac{2}{3}\int_\alpha^{\pi/2}\sin\theta\left[a^3-\frac{c^3}{\sin^3\theta}\right]d\theta$

$y=c \Rightarrow r\sin\theta=c \Rightarrow r=\frac{c}{\sin\theta}$

$\frac{2a^3}{3}\int_\alpha^{\pi/2}\sin\theta\,d\theta - \frac{2c^3}{3}\int_\alpha^{\pi/2}\sin^{-2}\theta\,d\theta$

intersection

$r=\frac{c}{\sin\theta}$ $\quad r=a$

$\frac{c}{\sin\alpha}=a$

$\sin\alpha=\frac{c}{a}$

$\alpha=\sin^{-1}(c/a)$

$=\frac{2a^3}{3}\left(-\cancel{\cos\frac{\pi}{2}}^{\,0}+\cos\alpha\right)-\frac{2c^3}{3}\cdot\left(\cancel{\cot(\pi/2)}^{\,0}+\cot(\alpha)\right)$

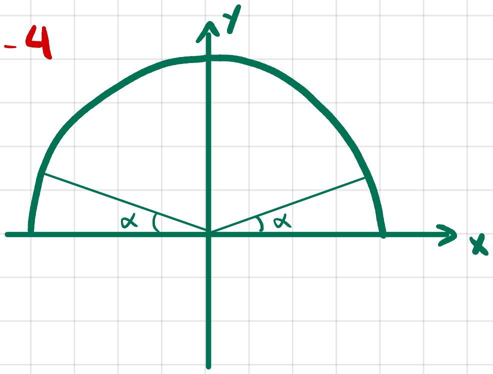
$=\frac{2a^3}{3}\cos\alpha-\frac{2c^3}{3}\cot(\alpha) = \frac{2a^3\cos\alpha}{3}-\frac{2c^3\cos\alpha}{3\sin\alpha}$

$=\frac{2a^3\cos\alpha}{3}-\frac{2c^{\cancel{3}\,2}\cos\alpha}{3}\cdot\frac{a}{\cancel{c}}$

$=\frac{2a}{3}\cos\alpha\left(a^2-c^2\right)=\frac{2}{3}(a^2-c^2)^{3/2}$

because $a\cos\alpha=\sqrt{a^2-c^2}$

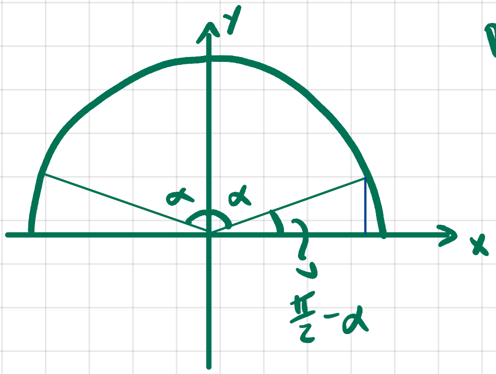
3C-4



$$\text{mass} = \int_{\alpha}^{\pi-\alpha}\int_0^a r\,dr\,d\theta = \int_{\alpha}^{\pi-\alpha} \tfrac{1}{2}a^2\,d\theta = \tfrac{1}{2}a^2(\pi-\alpha-\alpha) = \frac{a^2(\pi-2\alpha)}{2}$$

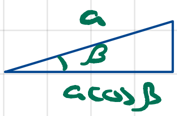
$$\bar{y} = \frac{\int_{\alpha}^{\pi-\alpha}\int_0^a r^2\sin\theta\,dr\,d\theta}{m} = \frac{\int_{\alpha}^{\pi-\alpha}\sin\theta\cdot\tfrac{1}{3}a^3\,d\theta}{m} = \frac{\tfrac{1}{3}a^3(-\cos\theta)\Big|_{\alpha}^{\pi-\alpha}}{m} = \frac{2a^{\cancel{3}}[-\cos(\pi-\alpha)+\cos(\alpha)]}{3\cancel{a^2}(\pi-2\alpha)}$$

$$= \frac{4a\cos\alpha}{3(\pi-2\alpha)}$$



previous α is now $\frac{\pi}{2}-\alpha$ so

$$\text{mass} = \frac{a^2(\pi-\cancel{2}\cdot(\frac{\pi}{\cancel{2}}-\alpha))}{2} = \frac{a^2}{2}(\pi-\pi+2\alpha) = \alpha a^2$$

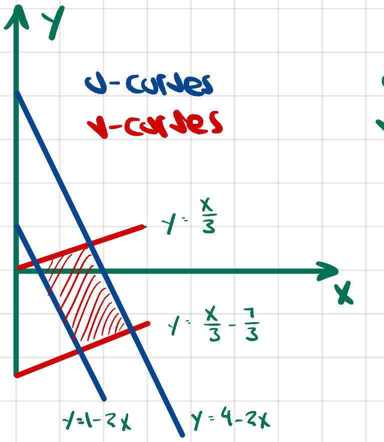


$$\bar{y} = \frac{2\cdot\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}}\int_0^a r^2\sin\theta\,dr\,d\theta}{m} = \frac{2\cdot\tfrac{1}{3}a^3(-\cos\theta)\Big|_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}}}{m} = \frac{2\cdot a^{\cancel{3}}}{3\alpha\cancel{a^2}}\left[-\cancelto{0}{\cos\tfrac{\pi}{2}}+\cos(\tfrac{\pi}{2}-\alpha)\right]$$

$$= \frac{2\cdot a}{3\alpha}\cos(\pi/2-\alpha) = \frac{2}{3}a\cdot\frac{\sin\alpha}{\alpha}$$

3D-1

$$\iint_R \frac{x-3y}{3x+y}\,dx\,dy = \iint_S \frac{u}{v}\cdot \overbrace{\frac{1}{7}}^{|J_T(u,v)|}\,du\,dv$$

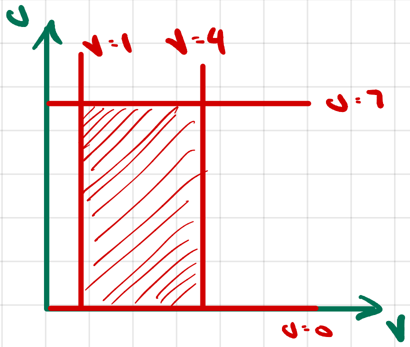


$u = f(x,y) = x-3y$

$v = g(x,y) = 3x+y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ 3 & 1 \end{vmatrix} = 1+6=7$$

$$\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)} = 1 \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{7}$$



$$= \frac{1}{7}\int_1^4\int_0^7 \frac{u}{v}\,du\,dv = \frac{1}{7}\int_1^4 \frac{1}{v}\,\frac{49}{2}\,dv = \frac{7}{2}\ln(v)\Big|_1^4 = \frac{7\ln 4}{2}$$

$$= 90/4 = 22.5$$

3D-2

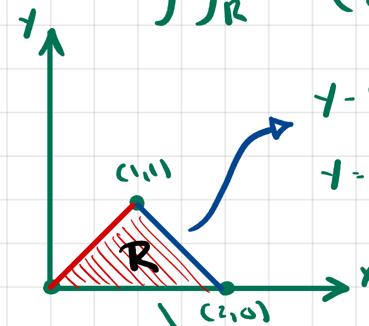
$$\iint_R \cos\left(\frac{x-y}{x+y}\right)dx\,dy$$

$u = x+y$

$v = x-y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1=-2$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$



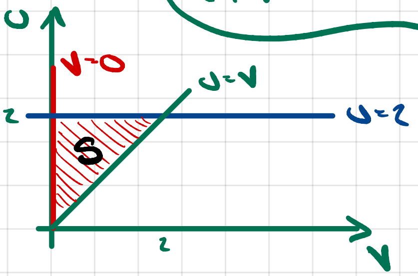
$y-1 = \frac{-1}{1}(x-1)$

$y = -x+2$

$\to y+x=2$

$u=2$

$y=0 \Rightarrow u=x$
$\phantom{y=0 \Rightarrow{}} v=x \Rightarrow u=v$



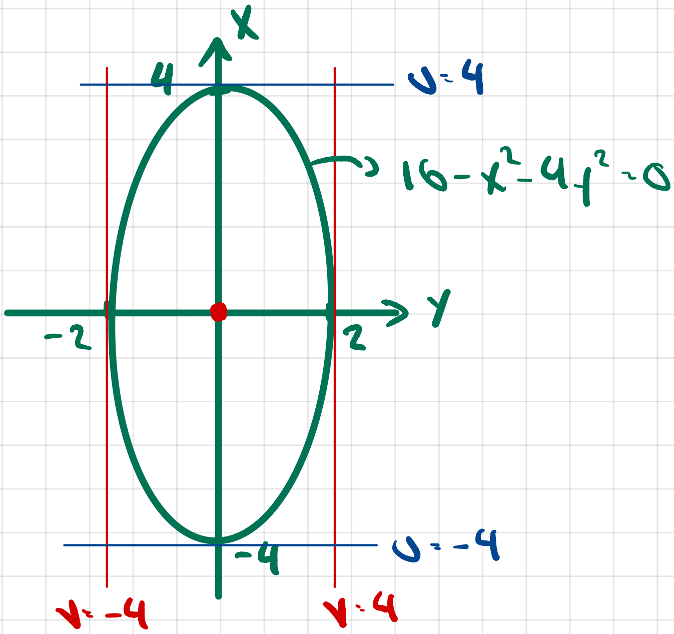
$$\iint_S \cos(v/u)\left(-\frac{1}{2}\right)du\,dv = -\frac{1}{2}\int_0^2\int_0^u \cos(v/u)\,dv\,du$$

$$= -\frac{1}{2}\int_0^2 u\sin(v/u)\Big|_0^u\,du = -\frac{1}{2}\int_0^2 u(\sin(1)-\sin 0)\,du = -\frac{1}{2}(\sin 1)\,\frac{u^2}{2}\Big|_0^2$$

$$= -\frac{1}{4}(\sin 1)\cdot 4 = -\sin 1$$

3D-3 $z = f(x,y) = 16 - x^2 - 4y^2$

$$u = x,\quad v = 2y \Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2$$

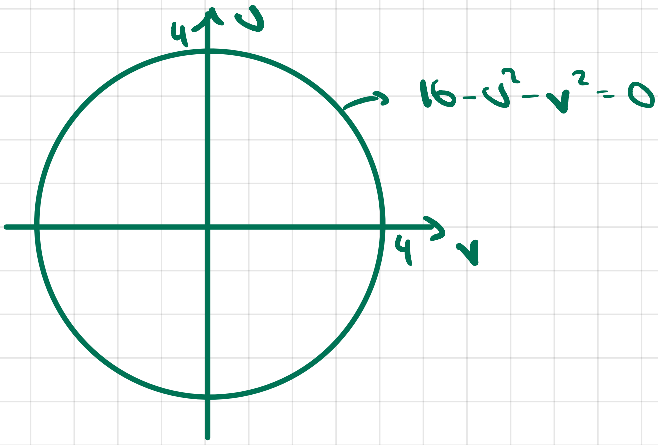


$$\iint_S (16 - u^2 - v^2) \cdot \frac{1}{2}\, du\, dv$$

$$= 4 \int_0^4 \int_0^{\sqrt{16-u^2}} (16 - u^2 - v^2)\frac{1}{2}\, dv\, du$$

$u = r\cos\theta \quad v = r\sin\theta$

$$\Rightarrow f(T_{r\theta}(T_{uv}(u,v))) = 16 - r^2 = 0 \rightarrow r = 4$$



$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$\iint_S (16 - u^2 - v^2) \cdot \frac{1}{2}\, du\, dv$$

$$= \iint_T (16 - r^2) \cdot \frac{1}{2} \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^4 \left(8r - \frac{r^3}{2}\right) dr\, d\theta = \int_0^{2\pi} \left(4r^2 - \frac{r^4}{8}\right)\Big|_0^4 d\theta$$

$$= \int_0^{2\pi} \left(64 - \frac{\overset{8}{\cancel{64}} \cdot 4}{\cancel{8}}\right) d\theta = 32\theta \Big|_0^{2\pi} = 64\pi$$

Alternatively, not ideal substitution:

$z = f(x,y) = 16 - x^2 - 4y^2$

$\iint_R f(x,y)\, dA$

$u = x^2$ $\qquad g(u,v) = \pm\sqrt{u} \quad h(u,v) = \frac{v}{2}$

$v = 2y$

$T(u,v) = (g(u,v), h(u,v))$

$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & 0 \\ 0 & 2 \end{vmatrix} = 4x$

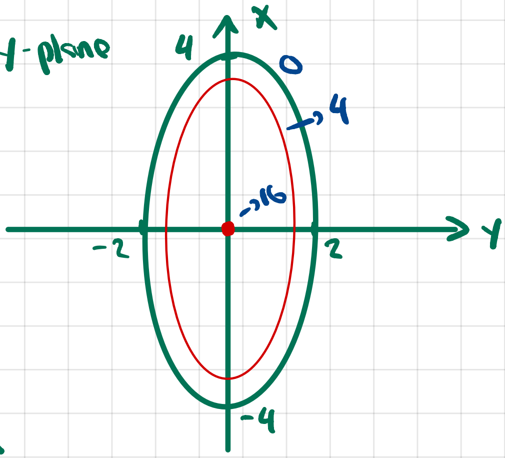
$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4x} = \frac{1}{4\sqrt{u}} = \left|\frac{1}{4\sqrt{u}}\right|$

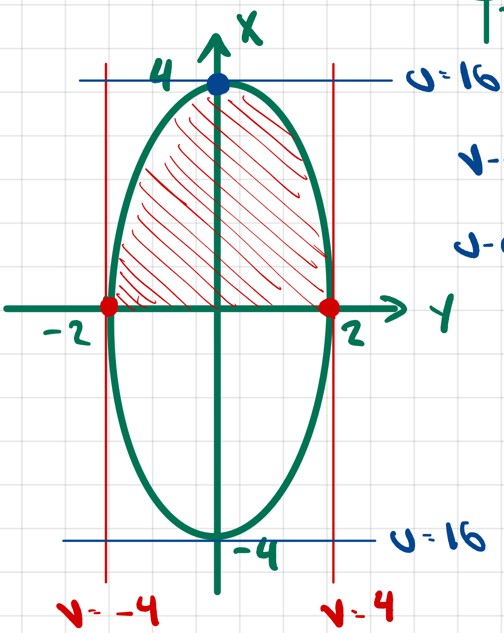
$\iint_S f(T(u,v)) \cdot \frac{1}{4\sqrt{u}}\, du\, dv$

$2 \cdot \frac{1}{4} \int_{-4}^{4} \int_{0}^{16-v^2} (16 - u - v^2) \cdot \frac{1}{\sqrt{u}}\, du\, dv$

$= \frac{1}{2} \cdot 128\pi = 64\pi$

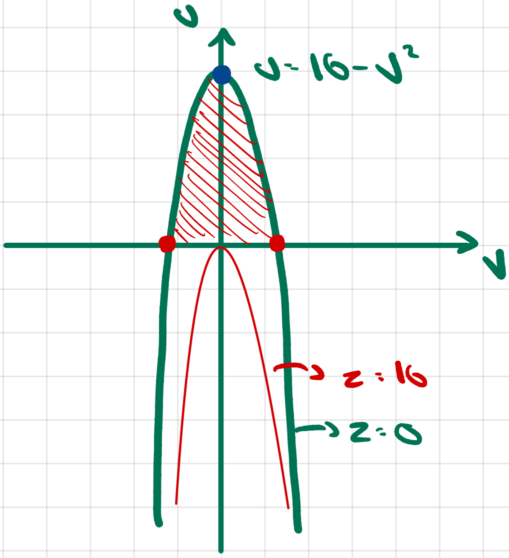
R region in xy-plane





v-curves: $x^2 = C_1 \Rightarrow x = \pm\sqrt{C_1}$

u-curves: $2y = C_2 \Rightarrow y = \frac{C_2}{2}$

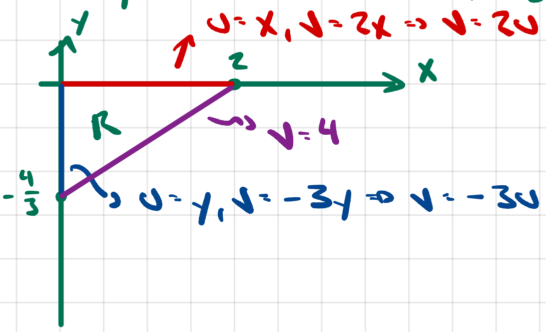


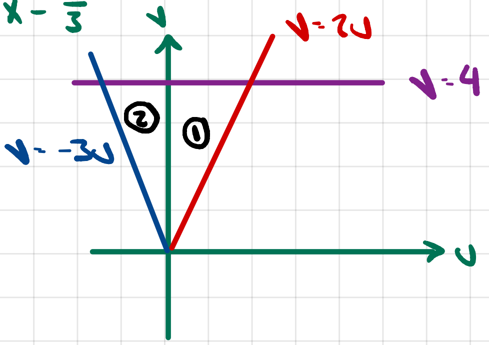
3D-4

$$\iint_R (2x-3y)^2(x+y)^2\,dx\,dy$$

$2x-3y=4 \Rightarrow 3y = 2x-4 \Rightarrow y = \frac{2}{3}x - \frac{4}{3}$

$\nearrow y=0 \Rightarrow 2x=4 \Rightarrow x=2$





chg of variables

$u = x+y$
$v = 2x-3y$

$f(T(u,v)) = v^2u^2$

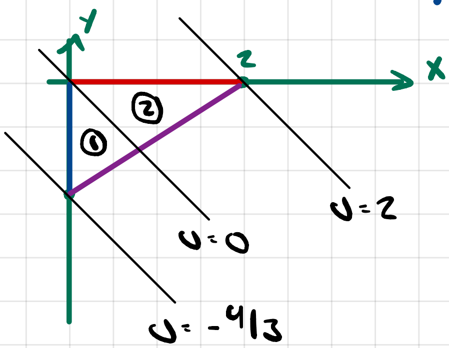
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -3-2 = -5$$

$$\iint_R f(x,y)\,dA = \iint_S u^2v^2 \cdot \frac{1}{5}\,dA'$$

Two ways of finding bounds for region S in uv-plane

1) u- and v-curves in xy-plane

2) graph of S in uv-plane

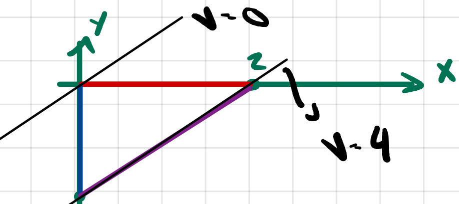


$\Rightarrow$ u from $-4/3$ to 0, v from $-3u$ to 4

+ u from 0 to 2, v from $2u$ to 4

= ① + ②

$$\int_{-4/3}^{0}\int_{-3u}^{4} u^2v^2 \cdot \frac{1}{5}\,dv\,du + \int_{0}^{2}\int_{2u}^{4} u^2v^2 \cdot \frac{1}{5}\,dv\,du = \frac{1792}{243}$$

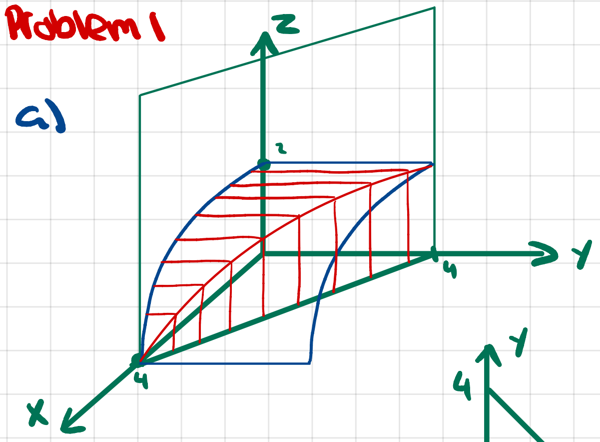


$\Rightarrow$ v from 0 to 4, u from $-\frac{v}{3}$ to $\frac{v}{2}$

$$\frac{1}{5}\int_{0}^{4}\int_{-v/3}^{v/2} u^2v^2\,du\,dv = \frac{1792}{243}$$

Problem 1

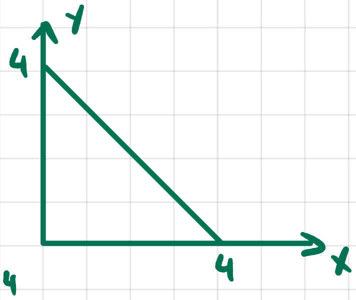
a)



plane: $x+y=4$

surface: $z=\sqrt{4-x}$

intersection: $\langle t, 4-t, \sqrt{4-t} \rangle$



b)

$$\int_0^4\int_0^{4-y}\sqrt{4-x}\,dx\,dy = \int_0^4 \frac{2}{3}(4-x)^{3/2}(-1)\Big|_0^{4-y}\,dy = -\frac{2}{3}\int_0^4 \left[y^{3/2}-2^3\right]dy$$

$$= \left[-\frac{2}{3}\cdot\frac{2}{5}y^{5/2}+\frac{2}{3}\cdot 8y\right]\Big|_0^4 = \left[-\frac{4}{15}\cdot 2^5+\frac{2\cdot 8\cdot 4}{3}\right]-\left[0\right]$$

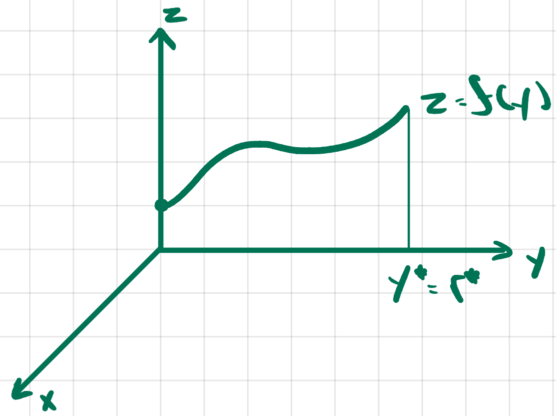
$$= \frac{-4\cdot 32}{15}+\frac{2\cdot 32}{3} = \frac{-4\cdot 32+10\cdot 32}{15} = \frac{\cancel{6}^{\,2}\cdot 32}{\cancel{15}_{\,5}} = \frac{64}{5}$$

Problem 2

$z = f(y)$ in yz-plane, rotate around z-axis $\rightarrow$ $z = f(r,\theta) = f(r)$

$h = f(0) > 0$

a)

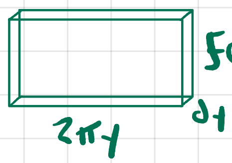


volume under $z = f(r)$

by double integral: $\displaystyle\int_0^{2\pi}\int_0^{r^*} f(r)\, r\, dr\, d\theta = \int_0^{r^*}\int_0^{2\pi} f(r)\, r\, d\theta\, dr = \int_0^{r^*} 2\pi r f(r)\, dr$

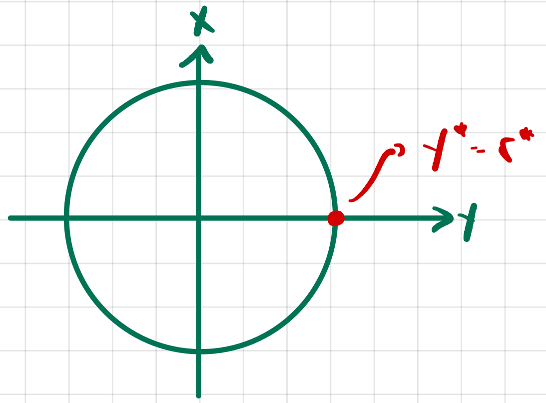
same integral

by shell volume:



$\displaystyle\int_0^{r^*} 2\pi y\, f(y)\, dy$

b)

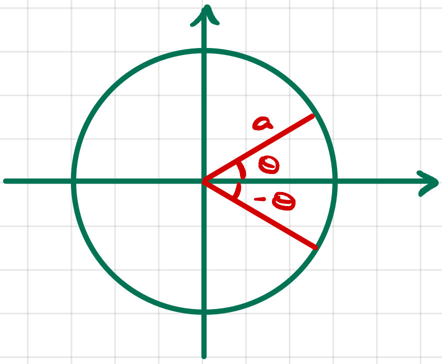


with double integrals, integration of $f(r)$ over a region R that is a circle of radius $y^* = r^*$.

with shells we integrate over a 1d range of 0 to $y^* = r^*$, adding volumes of cylindrical shells.

Problem 3

a) density $= \delta = 1$



$$\text{mass} = \int_{-\theta}^{\theta}\int_{0}^{a} r\,dr\,d\theta = \int_{-\theta}^{\theta} \frac{1}{2}a^2\,d\theta = \frac{1}{2}a^2(\theta-(-\theta)) = \frac{1}{2}a^2 \cdot 2\theta$$

$$= a^2\theta$$

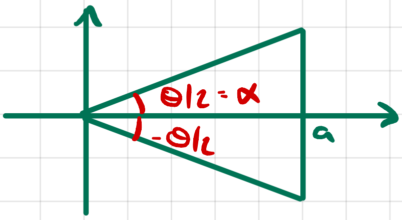
$$\bar{x} = \frac{\int_{-\theta}^{\theta}\int_{0}^{a} r^2\cos\theta\,dr\,d\theta}{m} = \frac{\int_{-\theta}^{\theta}\cos\theta \cdot \frac{1}{3}a^3\,d\theta}{m}$$

$\bar{y} = 0$ by symmetry.

$$= \frac{a^{\cancel{3}}}{3}\cdot\frac{1}{\cancel{a^2}\theta}\cdot(\sin\theta)\Big|_{-\theta}^{\theta} = \frac{a}{3\theta}(\sin\theta - \sin(-\theta))$$

$$= \frac{2a\sin\theta}{3\theta}$$

b)



$$\text{mass} = \int_{-\theta/2}^{\theta/2}\int_{0}^{a/\cos\theta} r\,dr\,d\theta = \int_{-\alpha}^{\alpha}\frac{1}{2}\left[\frac{a^2}{\cos^2\theta}\right]d\theta = \frac{a^2}{2}\int_{-\alpha}^{\alpha}\sec^2\theta\,d\theta$$

$$= \frac{a^2}{2}\tan\theta\Big|_{-\theta/2}^{\theta/2} = \frac{a^2}{2}\cdot 2\tan(\theta/2) = a^2\tan(\theta/2)$$

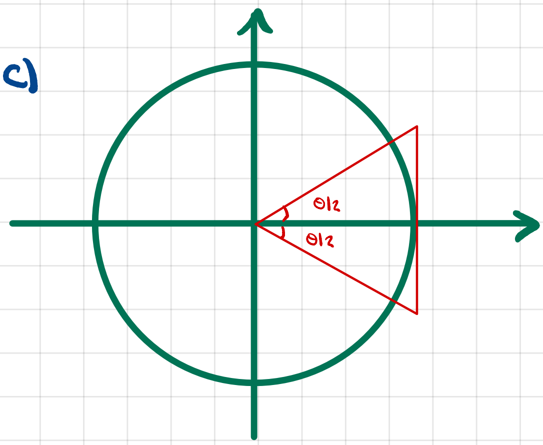
$x = a \Rightarrow r\cos\theta = a \Rightarrow r = \dfrac{a}{\cos\theta}$

$$\bar{x} = \frac{\int_{-\alpha}^{\alpha}\int_{0}^{a/\cos\theta} r^2\cos\theta\,dr\,d\theta}{m} = \frac{\frac{a^3}{3}\int_{-\alpha}^{\alpha}\cos^{-2}\theta\,d\theta}{m} = \frac{\frac{a^3}{3}2\cdot\cancel{\tan(\theta/2)}}{a^2\cancel{\tan(\theta/2)}} = \frac{2a}{3}$$

$\bar{y} = 0$, by symmetry

inner: $\cos\theta\cdot\frac{1}{3}\left[\frac{a^3}{\cos^3\theta}\right] = \frac{a^3}{3\cos^2\theta}$

c)



$$\bar{x}_{o} = \frac{2a}{3}\,\frac{\sin(\theta/2)}{(\theta/2)} < \frac{2a}{3} = \bar{x}_{\Delta}$$

$\Rightarrow$ centroid of circular sector closer to center than centroid of triangle.

Problem 4,5 Introduction

φ fluid flow map, smooth, one-to-one

(x,y,z) position of a point mass in the flow, time $t=0$

$(X,Y,Z)=\varphi(x,y,z,t)$ downstream position after elapsed time t

incompressible flow volume

For any bounded space region R in the flow $R_t=\varphi(R,t)=\{\varphi(x,y,z,t)\mid(x,y,z) \text{ in } R\}$,

region formed by the points from R which have been carried downstream by the flow

R_t can have different shapes for different t's, but the volume of R_t stays constant.

2D setup

$$(X(x,y,t),Y(x,y,t))=\varphi(x,y,t)$$

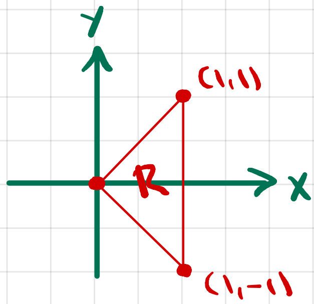
$$\vec{r}(t)=\langle X(t),Y(t)\rangle$$

$$\vec{v}(t)=\langle \partial X/\partial t,\partial Y/\partial t\rangle$$

A volume incompressible flow preserves areas: For any region R_t in xy-plane, ie $\varphi(R,t)$, representing the positions of point masses in region R at time t, R_t's area stays constant, though its shape can change.

$(x,y)\longrightarrow(X,Y)$ is the transformation $\varphi(x,y)$, given a t.

$$J(x,y,t)=\frac{\partial(X,Y)}{\partial(x,y)}$$



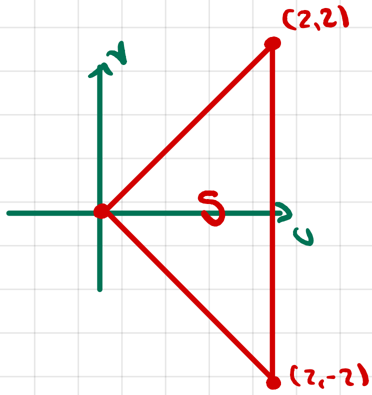
Problem 4

A)

i) $\varphi(x,y,t)=(x(1+t),y(1+t))$

$$J(x,y,t)=\frac{\partial(X,Y)}{\partial(x,y)}=\begin{vmatrix}1+t&0\\0&1+t\end{vmatrix}=(1+t)^2$$

Ex: $\varphi(x,y,1)=(2x,2y)$



$A_S=4\cdot 2/2=4$

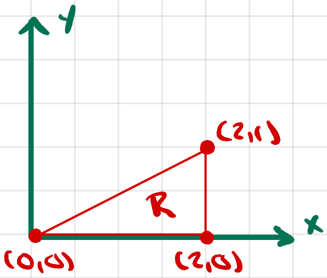
ii) $$A(R_t)=\iint_R(1+t)^2\,dA=\int_0^1\int_{-x}^{x}(1+t)^2\,dy\,dx$$

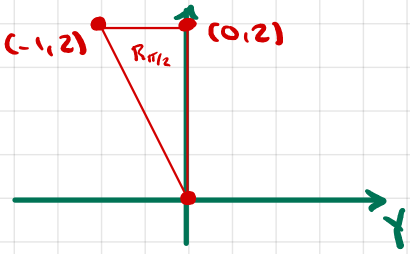
$$=(1+t)^2\int_0^1 2x\,dx=2(1+t)^2\frac{x^2}{2}\Big|_0^1=(1+t)^2\Rightarrow A(R_1)=4$$

B)

$\varphi(x,y,t) = (x\cos t - y\sin t,\ x\sin t + y\cos t)$

$t = \pi/2 \quad \varphi(x,y,\pi/2) = (-y, x)$





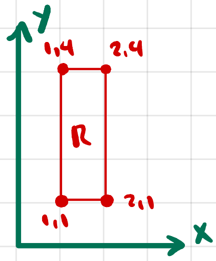
(x,y)	φ
(0,0)	(0,0)
(2,0)	(0,2)
(2,1)	(-1,2)

i) $\dfrac{\partial(X,Y)}{\partial(x,y)} = \begin{vmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$

ii) $A(R_t) = \iint_S dA = \iint_R |J(x,y,t)|\,dA = \int_0^2\int_0^{\frac{x}{2}} dy\,dx = \int_0^2 \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_0^2 = 1$

c) $\varphi(x,y,t) = \left(x(1+t),\ y\cdot\dfrac{1}{1+t}\right)$

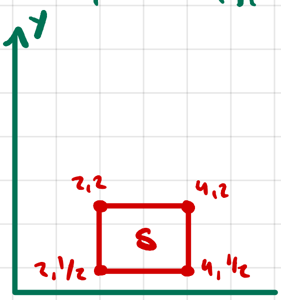
i) $J(x,y,t) = \dfrac{\partial(X,Y)}{\partial(x,y)} = \begin{vmatrix} 1+t & 0 \\ 0 & \frac{1}{1+t} \end{vmatrix} = 1$



ii) $\iint_S dA = \iint_R dA = \int_1^4\int_1^2 dx\,dy = 3$

For $t=1$, $\varphi(x,y,1) = (2x, \frac{y}{2})$

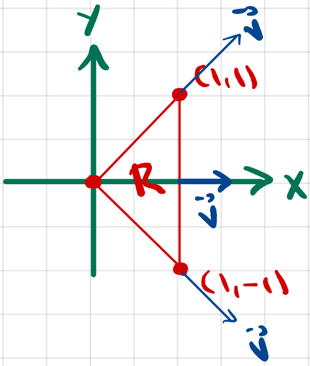
(x,y)	$\varphi(x,y,1)$
(1,1)	(2,1/2)
(2,1)	(4,1/2)
(1,4)	(2,2)
(2,4)	(4,2)

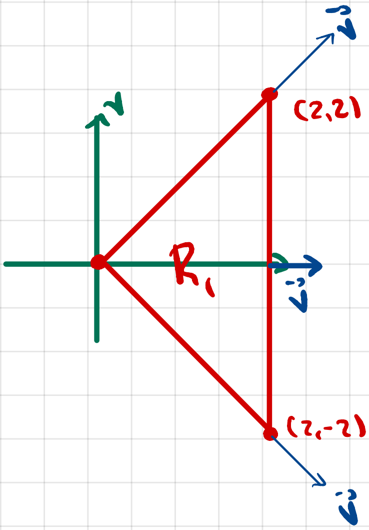


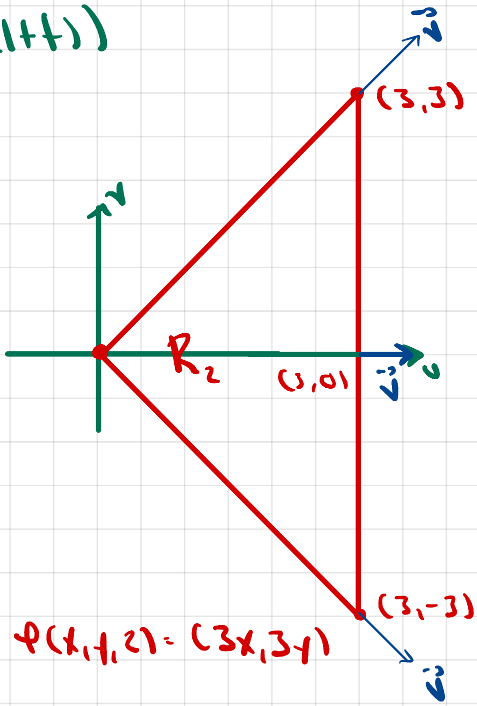
$A(R_1) = A(S) = 2\cdot\frac{3}{2} = 3$, which agrees with $A(R_1) = 3$

Problem 5

A) i), ii) $\varphi(x,y,t) = (x(1+t), y(1+t))$







$\varphi(x,y,1) = (2x,2y)$ $\varphi(x,y,2) = (3x,3y)$

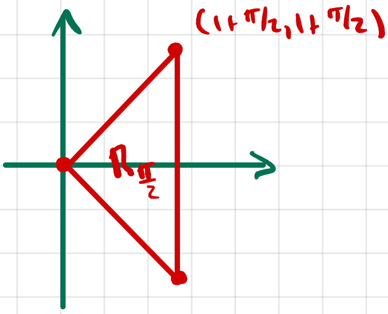
$A(R) = \frac{1\cdot 2}{2} = 1$ $A(R_1) = \frac{2\cdot 4}{2} = 4$ $A(R_2) = \frac{3\cdot 6}{2} = 9$

$\vec{r}(t) = (x_0(1+t), y_0(1+t))$

$\vec{v}(t) = (x_0, y_0)$

iii)

$t = \frac{\pi}{2} \Rightarrow \varphi(x,y,\pi/2) = (x(1+\pi/2), y(1+\pi/2))$



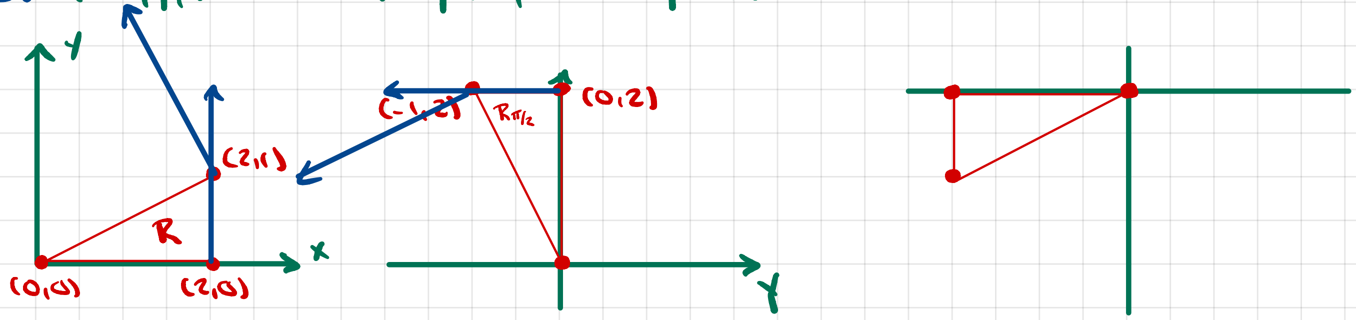
$A(R_{\pi/2}) = \frac{2(1+\pi/2)\cdot(1+\pi/2)}{2} = (1+\pi/2)^2$

$= (1+t)^2$

$t = 2 \Rightarrow A(R_2) = \frac{6\cdot 3}{2} = 9 = (1+2)^2 = (1+t)^2$

$t = 3 \Rightarrow A(R_3) = \frac{8\cdot 4}{2} = 16 = (1+3)^2 = (1+t)^2$

B) $\varphi(x,y,t) = (x\cos t - y\sin t,\ x\sin t + y\cos t)$



$t = \pi/2 \quad \varphi(x,y,\pi/2) = (-y, x)$ $\qquad$ $t = \pi \quad \varphi(x,y,\pi) = (-x,-y)$

$\vec{r}(t) = (x_0\cos t - y_0\sin t,\ x_0\sin t + y_0\cos t)$

$\vec{v}(t) = (-x_0\sin t - y_0\cos t,\ x_0\cos t - y_0\sin t)$

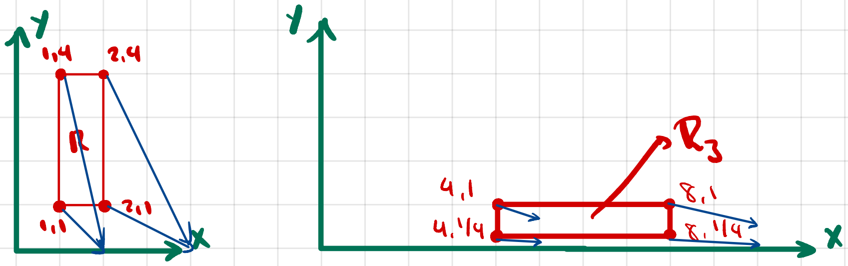
$\vec{v}(0) = (-y_0, x_0)$

$v(\pi/2) = (-x_0, -y_0)$

$A(R_{\pi/2}) = \frac{1\cdot 2}{2} = 1 = A(R_t)$

(x,y)	$\vec{v}(0)$	$\varphi(x,y,\pi/2)$	$v(\pi/2)$	$\varphi(x,y,\pi)$
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
(2,0)	(0,2)	(0,2)	(-2,0)	(-2,0)
(2,1)	(-1,2)	(-1,2)	(-2,-1)	(-2,-1)

C) $\varphi(x,y,t) = \left(x(1+t),\ y\cdot\frac{1}{1+t}\right)$



$\vec{r}(t) = (x_0(1+t),\ y_0\cdot {}^1/_{1+t})$

$\vec{v}(t) = \left(x_0,\ \frac{-y_0}{(1+t)^2}\right)$

$A(R_3) = \frac{3}{4}\cdot 4 = 3$

(x,y)	$\varphi(x,y,3) = (4x, y/4)$	$\vec{v}(3)$	$\vec{v}(0)$
1,1	4, 1/4	(1, −1/16)	(1,−1)
1,4	4, 1	(1, −1/4)	(1,−4)
2,1	8, 1/4	(2, −1/16)	(2,−1)
2,4	8, 1	(2, −1/4)	(2,−4)

iv) B and C are v-i : areas maintained under φ transformation.

v) A) point masses expand outward from origin along the line connecting its initial position ($t=0$) to origin

B) point masses move in circle around origin.