15.4 Green's Theorem

overview

→ relates a line integral around a simple closed plane curve to an ordinary double integral over the plane region R with boundary C

→ "simple closed plane curve"

if C parametrized by $r : [a,b] \to \mathbb{R}^2$ then

closed if $\vec{r}(a) = \vec{r}(b)$, simple if has no other self intersections

setup

C is piecewise smooth : consists of finite number of parametric arcs w/ continuous nonzero velocity vectors

⇒ C has unit tangent $\hat{T}$ everywhere except at finite number of corner points

→ positive, or counterclockwise direction along C : direction determined by a parametrization r(t) of C such that the region R remains on the left as $\vec{r}(t)$ traces the boundary curve C

→ C is positively oriented if it has such a parametrization

→ $\oint_C P\,dx + Q\,dy$ denotes line integral around C in positive direction

green's theorem

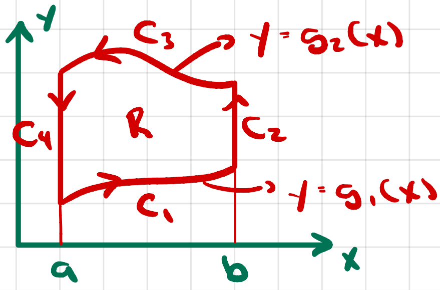
C positively oriented piecewise smooth simple closed curve that bounds region R in the plane.

P(x,y) and Q(x,y) have continuous first-order partial derivatives on R

$$\Rightarrow \oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\oint_C \vec{F} \cdot d\vec{r}$$

Proof of Green's Theorem when R is vertically simple



Vertically simple means R is described: $a \le x \le b,\ g_1(x) \le y \le g_2(x)$

$C = C_1 \cup C_2 \cup C_3 \cup C_4$, parametrized so as to be positively oriented

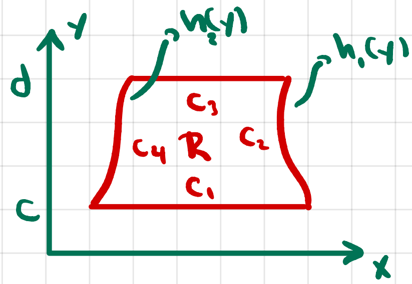
$$\oint_C P\,dx = \int_{C_1} P\,dx + \cancelto{0}{\int_{C_2} P\,dx} + \int_{C_3} P\,dx + \cancelto{0}{\int_{C_4} P\,dx}$$

x constant $\Rightarrow dx = 0$

$$= \int_a^b P(x, g_1(x))\,dx + \int_b^a P(x, g_2(x))\,dx = \int_a^b P(x, g_1(x))\,dx - \int_a^b P(x, g_2(x))\,dx$$

$$= -\int_a^b \left[P(x, g_2(x)) - P(x, g_1(x)) \right] dx = -\int_a^b \int_{g_1(x)}^{g_2(x)} \partial P/\partial y \; dy\,dx$$

$$= -\iint_R \partial P/\partial y \; dA$$



$$\oint_C Q\,dy = \cancelto{0}{\int_{C_1} Q\,dy} + \int_{C_2} Q\,dy + \cancelto{0}{\int_{C_3} Q\,dy} + \int_{C_4} Q\,dy$$

$$= \int_c^d Q(h_1(y), y))\,dy + \int_d^c Q(h_2(y), y))\,dy$$

$$= \int_c^d \left[Q(h_1(y), y) - Q(h_2(y), y) \right] dy = \int_c^d \int_{h_2(y)}^{h_1(y)} \partial Q/\partial x \; dx\,dy = \iint_R \partial Q/\partial x \, dA$$

$\Rightarrow$ Given a general region R, if we can subdivide it into regions that are each both vertically and horizontally simple, then the line integral on the positively oriented curve around the region (let's call the region R_i) is $\oint_{R_i} P\,dx + Q\,dy = \iint_{R_i} Q_x\,dA - \iint_{R_i} P_y\,dA = \iint_R (Q_x - P_y)\,dA$

Corollary to Green's Theorem

Let A be area of region R bounded by posit. oriented piecewise smooth simple closed curve C

Then, $A = \frac{1}{2} \oint_C -y\,dx + x\,dy = -\oint_C y\,dx = \oint_C x\,dy$

Proof

With $P(x,y) = -y$ and $Q(x,y) \equiv 0$ apply Green's Theorem:

$$\oint P\,dx + Q\,dy = \iint_R (0+1)\,dA = A$$

With $P(x,y) = 0$ and $Q(x) = x$ apply Green's:

$$\oint P\,dx + Q\,dy = \iint_R (1-0)\,dA = A$$

Divergence and Flux of a Vector Field

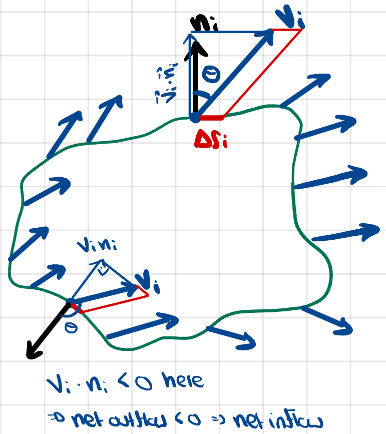
setting: steady flow of thin layer of fluid in the plane → $\vec{v}$ and δ depend only on x, y not t

$\vec{v}(x,y)$ velocity vector field

$\delta(x,y)$ fluid density

goal: compute rate at which the fluid flows out of region R bounded by positively oriented simple closed curve C

Δs_i is a short segment of C.



The area of the portion of fluid that flows out of R across Δs_i per unit time is approx the area of the parallelogram

$$(|\vec{v}_i|\cos\theta)\,\Delta s_i = \underbrace{\vec{v}_i \cdot \vec{n}_i}_{\text{height}}\,\underbrace{\Delta s_i}_{\text{base}} \qquad \frac{m}{s} \times m = m^2/s$$

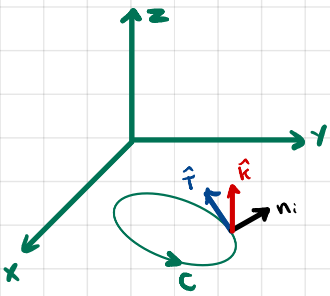
multiplied by $\delta(x_i^*, y_i^*)$ we obtain mass of fluid per unit time

Total mass of fluid leaving R per unit time $\approx \sum_{i=1}^{n} \delta_i \vec{v}_i \cdot \vec{n}_i \Delta s_i = \sum_{i=1}^{n} \vec{F}_i \cdot \vec{n}_i \Delta s_i$, $\vec{F} = \delta \vec{v}$

when we take a limit as $n \to \infty$ we obtain a line integral called the flux of vector field $\vec{F}$ across curve C

$$\Phi = \oint_C \vec{F} \cdot \vec{n}\, ds$$

$\vec{n}$ ↳ outer unit normal vector to C



$\vec{n} = \hat{T} \times \hat{k}$

$$\hat{T} = \frac{\langle x'(t), y'(t), 0\rangle}{v(t)} = \frac{\langle dx/dt, dy/dt, 0\rangle}{ds/dt} = \langle dx/ds, dy/ds, 0\rangle$$

$$\Rightarrow \vec{n} = \langle dx/ds, dy/ds, 0\rangle \times \hat{k} = \left(\frac{dx}{ds}\hat{\imath} + \frac{dy}{ds}\hat{\jmath}\right) \times \hat{k} = \frac{dx}{ds}(-\hat{\jmath}) + \frac{dy}{ds}\hat{\imath}$$

$$\Rightarrow \vec{n} = \langle dy/ds, -dx/ds, 0\rangle$$

substitute $\vec{n}$ into the flux integral

$$\Phi = \oint_C \langle M, N\rangle \langle dy/ds, -dx/ds\rangle\, ds = \oint_C -N\,dx + M\,dy$$

here we have closed C (piecewise smooth, simple) and $P(x,y) = -N$, $Q(x,y) = M$, so we can apply Green's Theorem:

$$\Phi = \iint_R \underbrace{(M_x + N_y)}_{}\, dA = \iint_R \nabla \cdot \vec{F}\, dA$$

Divergence of 2D vector field $\vec{F} = \langle M, N\rangle$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = M_x + N_y$$