Practice Exam 2

P1 $f(x,y) = xy - x^4$

a) $\nabla f = \langle y - 4x^3, x \rangle$

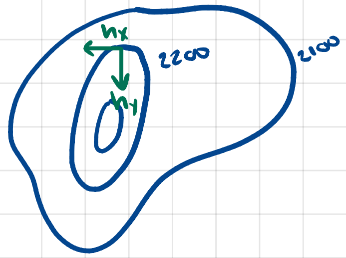
$\nabla f(1,1) = \langle 1-4, 1\rangle = \langle -3, 1\rangle$

b) $\Delta w \approx -3\Delta x + \Delta y$

P2 a) $\hat{u} = \langle 1,1 \rangle / \sqrt{2}$

$d_{\hat{u}} h(\vec{P}) \approx \frac{100}{500} = \frac{1}{5}$

b)



$\frac{\partial h}{\partial y}(\vec{Q}) \approx \frac{100}{250} = \frac{2}{5}$

P3 $x^3 y + z^2 = 3$ tangent plane at $(-1,1,2)$

Strategy: find ∇ at $(-1,1,2)$, $\perp$ to tangent plane there

$\nabla f(x,y,z) = \langle 3x^2 y, x^3, 2z \rangle$ $\qquad \nabla f(-1,1,2) = \langle 3, -1, 4 \rangle$

$\langle x+1, y-1, z-2 \rangle \langle 3, -1, 4 \rangle = 0$

$3x + 3 - y + 1 + 4z - 8 = 0$

$3x - y + 4z = 4$

P4 $V(x,y,z) = x \cdot y \cdot z$ $\quad$ min V s.t. $g = x^2 + y^2 + z = 1$

a) $V(x,y) = xy(1 - x^2 - y^2) = xy - x^3 y - xy^3$

$\min_{x,y} V(x,y) \to$ find critical points

$\nabla V(x,y) = \langle y - 3x^2 y - y^3, x - x^3 - 3xy^2 \rangle = 0$

b) $y(1 - 3x^2 - y^2) = 0$

$x(1 - 3y^2 - x^2) = 0$

$\to y = 0, x = 0$

$\to y = 0, \ 1 - x^2 = 0 \Rightarrow x = \pm 1$

$\to x = 0, \ 1 - y^2 = 0 \Rightarrow y = \pm 1$

$\to y^2 = 1 - 3x^2$

$1 - 3(1 - 3x^2) - x^2 = 0$

$1 - 3 + 9x^2 - x^2 = 0$

$-2 + 8x^2 = 0$

$8x^2 = 2 \Rightarrow x^2 = 1/4 \Rightarrow x = \pm 1/2$

$x = \frac{1}{2} \Rightarrow y^2 = 1 - 3 \cdot \frac{1}{4} = \frac{1}{4} \Rightarrow y = \pm 1/2$

$\Rightarrow$ critical points: $(0,0), (-1,0), (1,0), (0,-1), (0,1)$
$(1/2, 1/2), (1/2, -1/2), (-1/2, 1/2), (-1/2, -1/2)$

first quadrant critical points: $(1/2, 1/2)$

c) $V = xy - x^3y - xy^3$ $\quad \nabla V(x,y) = \langle y - 3x^2y - y^3, x - x^3 - 3xy^2 \rangle$

$V_{xx} = -6xy \quad V_{yy} = -6xy \quad V_{xy} = 1 - 3x^2 - 3y^2$

$V_{xx}V_{yy} - V_{xy} = 36x^2y^2 - 1 + 3x^2 + 3y^2$

At $(1/2, 1/2)$: $36 \cdot \frac{1}{4} \cdot \frac{1}{4} - 1 + 3 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} = \frac{36 - 16 + 12 + 12}{16} = \frac{44}{16} = \frac{11}{4} > 0$, and

$V_{xx}(1/2, 1/2) = -6 \cdot \frac{1}{2} \cdot \frac{1}{2} = -\frac{3}{2} < 0$

$\Delta > 0, V_{xx} < 0 \Rightarrow (1/2, 1/2)$ is local max

d) $V(1/2, 1/2) = \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{8} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{8} = \frac{4 - 1 - 1}{16} = \frac{2}{16} = \frac{1}{8}$

$\lim_{x \to 0} V(x,y) = \lim_{y \to 0} V(x,y) = 0$

$V = xy(1 - x^2 - y^2) \quad \lim_{x,y \to \infty} V = \infty \cdot \infty (-\infty) = -\infty$

$\Rightarrow$ first quadrant max at $(1/2, 1/2)$.

P5 $V(x,y,z) = x \cdot y \cdot z \quad$ min V s.t. $g = x^2 + y^2 + z = 1$

a) $\nabla V = \langle yz, xz, xy \rangle \quad \nabla g = \langle 2x, 2y, 1 \rangle$

$\nabla V = \lambda \nabla g \Rightarrow$
$yz = \lambda \cdot 2x \quad \lambda = yz/2x$
$xz = \lambda \cdot 2y \quad \lambda = xz/2y$
$xy = \lambda$
$x^2 + y^2 + z = 1$

b) $xy = \frac{xz}{2y} \Rightarrow y = \frac{z}{2y} \Rightarrow 2y^2 = z$

$xy = \frac{yz}{2x} \Rightarrow x = \frac{z}{2x} \Rightarrow 2x^2 = z$

$\Rightarrow \frac{z}{2} + \frac{z}{2} + z = 1 \Rightarrow 2z = 1 \Rightarrow z = \frac{1}{2} \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2} \Rightarrow y = \pm\frac{1}{2}$

candidate points $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$

$V(1/2, 1/2, 1/2) = 1/8$

P6 $w = f(u,v) \quad u = xy \quad v = x/y$

$$\frac{\partial w}{\partial x} = f_u u_x + f_v v_x = f_u y + \frac{f_v}{y}$$

$$\frac{\partial w}{\partial y} = f_u u_y + f_v v_y = f_u x + f_v\left(-\frac{x}{y^2}\right)$$

P7 $x^2y + xz^2 = 5 \qquad \left(\frac{\partial w}{\partial z}\right)_y$

$w = x^3y$

method 1

$$g(x,y,z) = x^2y + xz^2 = 5$$

$$dg = 0 = (2xy+z^2)\,dx + x^2 \overset{0}{\cancel{dy}} + 2xz\,dz \Rightarrow dx = \frac{2xz}{2xy+z^2}\,dz \Rightarrow \left(\frac{\partial x}{\partial z}\right)_y = \frac{-2xz}{2xy+z^2}$$

$$dw = 3x^2y\,dx + x^3 \overset{0}{\cancel{dy}} + 0\cdot dz = -3x^2y \cdot \frac{2xz}{(2xy+z^2)}\,dz$$

$$\left(\frac{\partial w}{\partial z}\right)_y = \frac{-6x^3yz}{2xy+z^2}$$

At $(1,1,2)$, $(\partial w/\partial z)_y = \dfrac{-6\cdot1\cdot1\cdot2}{2\cdot1\cdot1+4} = \dfrac{-12}{6} = -2$

method 2

$$\left(\frac{\partial w}{\partial z}\right)_y = 3x^2y\left(\frac{\partial x}{\partial z}\right)_y + x^3\overset{0,\ y\text{ kept constant}}{\cancel{\left(\frac{\partial y}{\partial z}\right)_y}}$$

using differentials we've seen $\left(\dfrac{\partial x}{\partial z}\right) = \dfrac{-2xz}{2xy+z^2}$

$$\Rightarrow \left(\frac{\partial w}{\partial z}\right)_y = \frac{-3x^2y\cdot 2xz}{2xy+z^2}$$