Differentiable function of 1 variable: $f'(x)$ exists at each point in domain

Differentiable at x_0: $f'(x_0)$ exists, ie $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists

<=> f has non-vertical tangent line at $(x_0, f(x_0))$

<=> f locally linear at x_0, ie can be approximated by linear function

Differentiable at x_0 -> continuous at x_0

Continuously Differentiable Fn: differentiable $f(x)$ and $f'(x)$ continuous

-> class C^1 functions

Class C^n Function: first n derivatives exist and are continuous

In higher dimensions: $f: \mathbb{R}^m \to \mathbb{R}^n$

differentiable at x_0: $\exists$ linear map $J: \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{\vec{h} \to \vec{0}} \frac{\| \vec{f}(\vec{x}_0 + \vec{h}) - \vec{f}(\vec{x}_0) - \vec{J}(\vec{h}) \|_{\mathbb{R}^m}}{\| \vec{h} \|_{\mathbb{R}^n}} = \vec{0}$$

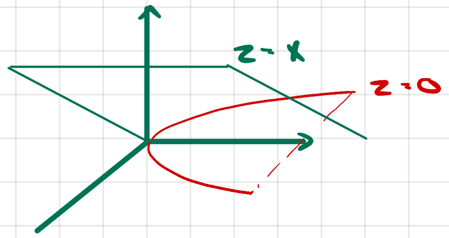
e.g. $f(x,y) = \langle g(x,y), h(x,y) \rangle : \mathbb{R}^2 \to \mathbb{R}^2$

$$f'(x,y) = \lim_{(h_1,h_2) \to (0,0)} \frac{\| f(x+h_1, y+h_2) - f(x,y) \|}{\| \langle h_1, h_2 \rangle \|}$$

$$= \lim_{(h_1,h_2) \to (0,0)} \frac{\| \langle g(x+h_1, y+h_2) - g(x,y), h(x+h_1, y+h_2) - h(x,y) \|}{\| \langle h_1, h_2 \rangle \|}$$

Ex:

$$f(x,y) = \begin{cases} x & y \neq x^2 \\ 0 & y = x^2 \end{cases}$$



At $(0,0)$

$$D_{\hat{u}} f = \lim_{h \to 0} \frac{f(\vec{x} + h\hat{u}) - f(\vec{x})}{h}$$

$$D_{\hat{u}} f(0,0) = \lim_{h \to 0} \frac{f(h\hat{u}) - f(0,0)}{h}$$

if $\hat{u} = \langle u_1, u_2 \rangle$ and $u_2 \neq u_1^2$ then $D_{\hat{u}} f(0,0) = \dfrac{hu_1 - 0}{h} = u_1$

if $u_2 = u_1^2$ then $D_{\hat{u}} f(0,0) = \dfrac{0 - 0}{h} = 0$

$\Rightarrow$ All directional derivatives exist.