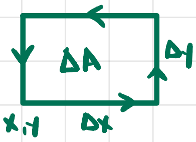
2D Divergence



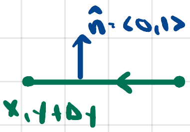
$\vec{F} = \langle M, N \rangle$

$\vec{F}$ cont. diff $\leadsto$ $\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$ is continuous $\Rightarrow$ approx. constant for small $\Delta x, \Delta y$

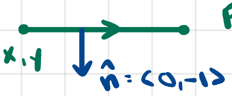
Apply Green's theorem to the small rectangle

$$\oint_C M\,dy - N\,dx = \iint_R (M_x + N_y)\,dA = \iint_R \operatorname{div} \vec{F}\,dA \approx (M_x + N_y)\Delta A$$

If we consider each rectangle side on its own:

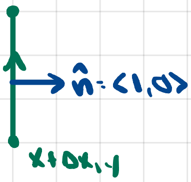


Flux across top $\approx \vec{F}(x, y+\Delta y) \cdot \langle 0,1 \rangle \Delta x = N(x, y+\Delta y)\Delta x$



Flux across bottom $\approx \vec{F}(x,y) \cdot \langle 0,-1 \rangle \Delta x = -N(x,y)\Delta x$

$\text{Flux}_{top} + \text{Flux}_{bottom} = (N(x, y+\Delta y) - N(x,y))\Delta x \approx (N_y \cdot \Delta y)\Delta x$



Flux across right side $\approx F(x+\Delta x, y) \cdot \langle 1,0 \rangle \Delta y = M(x+\Delta x, y)\Delta y$

Flux across left side $\approx F(x,y) \cdot \langle -1,0 \rangle \Delta y = -M(x,y)\Delta y$

$\text{Flux}_{left} + \text{Flux}_{right} = (M(x+\Delta x, y) - M(x,y))\Delta y \approx (M_x \Delta x)\Delta y$

$\text{Flux}_{top} + \text{Flux}_{bottom} + \text{Flux}_{left} + \text{Flux}_{right} \approx (M_x + N_y)\Delta x \Delta y$

Physical Interpretation

- total flux over rectangle positive $\Rightarrow$ net outflow $\Rightarrow$ source adding fluid to rectangle
  " " " " negative $\Rightarrow$ " inflow $\Rightarrow$ sink withdrawing fluid

Flux over sides of rectangle = source rate for rectangle $\approx (M_x + N_y)\Delta A$

source rate at $(x,y) = M_x + N_y = \operatorname{div} \vec{F}$

Source rate for $R = \iint_R \operatorname{div} \vec{F}\,dA$