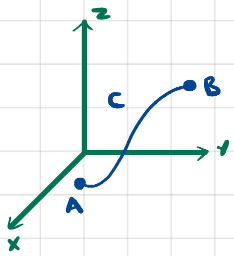
15.2 Line Integrals

→ we want to define integrals along curves in space or in a plane, aka "line integrals"

setup

smooth parametrized space curve C: $\langle x, y, z\rangle = \langle x(t), y(t), z(t)\rangle \quad a \le t \le b$

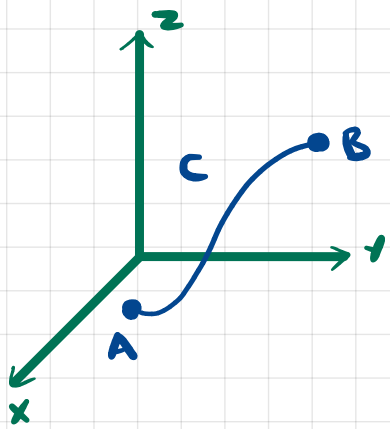


imagine a thin wire shaped like $\overline{AB} = C$

$f(x,y,z)$ = density of wire, units mass per unit length

we want to calculate the mass of the wire

approximation

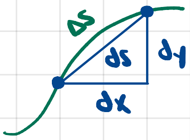


partition of $[a,b]$: $a = t_0 < t_1 < \cdots < t_n = b$

$P_i = (x(t_i), y(t_i), z(t_i)) \quad i = 0, 1, \ldots, n$

$\Rightarrow P_i$ are the subdivision points of C

recall:



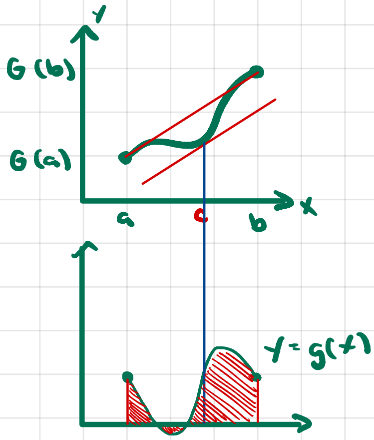
$$\Delta s \approx ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{(dx/dt)^2 + (dy/dt)^2} \cdot dt$$

if we have an entire arc, arc length is $\approx \sum_{i=1}^{n} ds$, arclength is $\lim_{n \to \infty} \sum_{i}^{n} ds = \int ds = \int \sqrt{x'(t)^2 + y'(t)^2}\, dt$

so, on C, each Δs_i = arc length between P_{i-1} and P_i = $\int_{t_{i-1}}^{t_i} [x'(t)^2 + y'(t)^2 + z'(t)^2]^{1/2} dt$

Average value theorem tells us:



$$\frac{\int_a^b g(x)\,dx}{b-a} = \frac{G(b) - G(a)}{b-a} = G'(c) = g(c) \quad \text{for } c \in (a,b)$$

$\Rightarrow$ Take $g(t) = [x'(t)^2 + y'(t)^2 + z'(t)^2]^{1/2}$ = speed at t

AVT says there is a $t_i^* \in (t_{i-1}, t_i)$ such that

$$g(t_i^*) = \frac{\int_{t_{i-1}}^{t_i} [x'(t)^2 + y'(t)^2 + z'(t)^2]^{1/2} dt}{\Delta t} = \frac{\Delta s_i}{\Delta t}$$

$\Rightarrow g(t_i^*)\Delta t = \Delta s_i$, ie there is a particular t at which the speed is the average speed between $t-1$ and t.

back to the problem of finding curve mass:

$$m \approx \sum_{i=1}^{n} \underbrace{f(x(t_i^*), y(t_i^*), z(t_i^*))}_{\text{density}} \underbrace{\Delta s_i}_{\text{arc length}}$$

limit as $\Delta t \to 0$ i.e. $n \to \infty$ is the mass.

Definition: Line integral of function along a curve

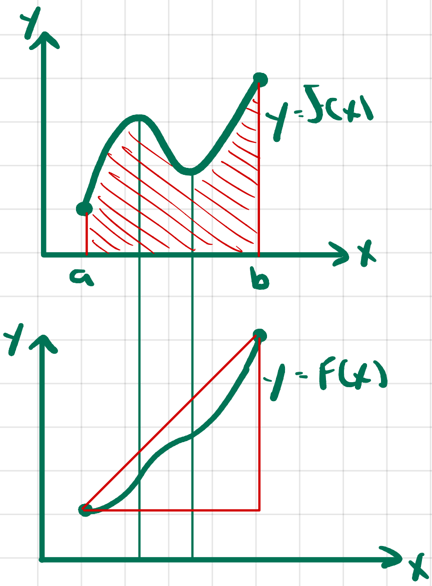
$$\int_C f(x,y,z)\,ds = \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(x(t_i^*), y(t_i^*), z(t_i^*))\Delta s_i$$ provided the limit exists

line integral of f w/ respect to arc length along C

assumptions: $f(x,y,z)$ defined at each point on $\underbrace{\text{smooth curve}}$ C

i.e. the component functions in its parametrization have continuous derivatives that are never simult. zero

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$$



$$\bar{y} = \frac{\int_a^b f(x)\,dx}{b-a} = \text{avg value of } f \text{ between } a \text{ and } b$$

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$$= \frac{F(b) - F(a)}{b-a} = F'(c) = f(c)$$

$$\text{avg value of } f = \frac{\int_a^b f(x)\,dx}{b-a} = \frac{\text{Area}}{\text{interval}}$$

= avg slope of antiderivative

$\Rightarrow$ There is $c \in (a,b)$ such that $F'(c)$ = avg slope of F = avg value of f

$\Rightarrow \exists\, c$ s.t. $f(c)$ = average of f in $[a,b]$

Line Integrals and Vector Fields

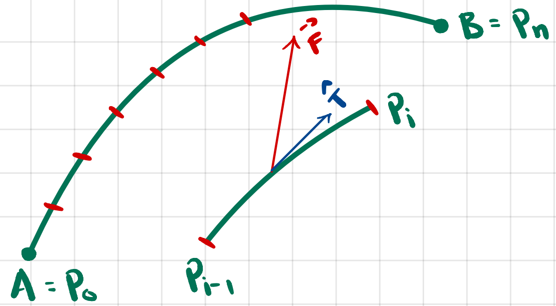
$\vec{F} = \langle P, Q, R \rangle$ = force field defined in region containing curve C from A to B

C parametrized as $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \quad t \in [a, b]$

$\vec{v}(t) = \langle x'(t), y'(t), z'(t) \rangle$

$v = |\vec{v}|$

$\hat{T} = \dfrac{\vec{v}}{v}$



A particle is moved along C from A to B by the force field $\vec{F}$. Note that at each point on C, the force is potentially different, ie $\vec{F}$ is function of x and y.

$\vec{F} \cdot \hat{T}$ is tangent to the trajectory on C.

we partition $[a,b]$ into n subintervals. To move the particle from the start to end of a subinterval, ie P_{i-1} to P_i, the work done is approx. $\Delta W_i \approx \vec{F}(x(t_i^*), y(t_i^*), z(t_i^*)) \cdot \hat{T}(t_i^*) \Delta s_i$

$\Rightarrow W = \sum_{i=1}^{n} \vec{F}(x(t_i^*), y(t_i^*), z(t_i^*)) \cdot \hat{T}(t_i^*) \Delta s_i$

by taking the limit as $\Delta t \to 0$ we have a line integral $W = \int_C \vec{F} \cdot \hat{T} \, ds$

integral w/ respect to arclength of the tangential component of the force

Another way to look at it

$dW = \vec{F} \cdot \vec{T} \, ds$ = infinitesimal work done by $\vec{F} \cdot \hat{T}$ in moving particle along ds

$\vec{r} = \langle x, y, z \rangle$

$d\vec{r} = \langle dx, dy, dz \rangle$

$\vec{T} ds = \dfrac{\vec{v}}{v} \cdot v \, dt = \vec{v} \, dt = \langle dx/dt, dy/dt, dz/dt \rangle dt = \langle dx, dy, dz \rangle$

$\Rightarrow \vec{T} ds = d\vec{r}$

$\Rightarrow W = \int_C \vec{F} d\vec{r}$

$W = \int_C \vec{F} \cdot \hat{T} ds = \int_a^b \langle P, Q, R \rangle \cdot \langle x', y', z' \rangle \cdot \frac{1}{v} \, v dt = \int_a^b (P x' + Q y' + R z') dt$

$= \int_C P dx + Q dy + R dz$

i.e. $\int_C \vec{F} \cdot \hat{T} ds = \int_C P dx + Q dy + R dz$