## 14.9

we want to evaluate $\iint_R F(x,y)\,dx\,dy$

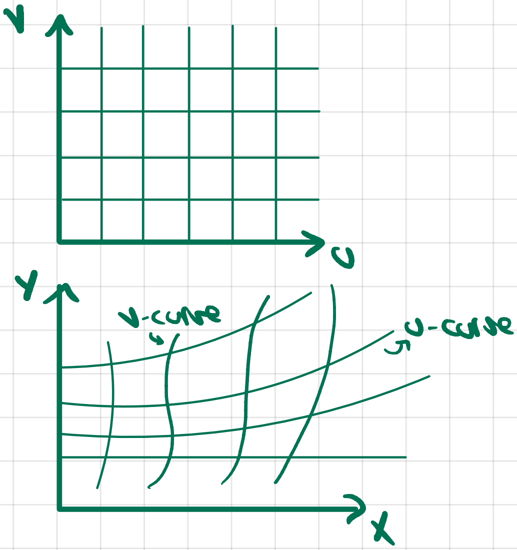
continuously diff. transformation T from uv-plane to xy-plane: $(u,v) \to (x,y) = T(u,v)$

image of (u,v) under T

⇒ T one-to-one ⇒ no two different (u,v) have same (x,y) image

⇒ can solve for u and v : $u = h(x,y)$  $v = k(x,y)$  of $T^{-1}$

**u-curves of T**: keep v constant, vary u (horizontal line in uv plane), obtain images in xy-plane



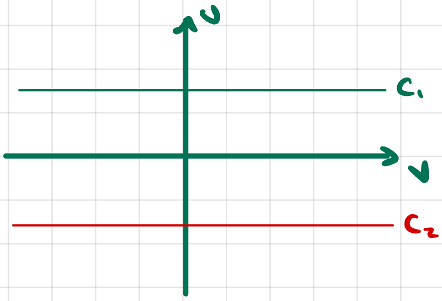

**v-curves analogous**

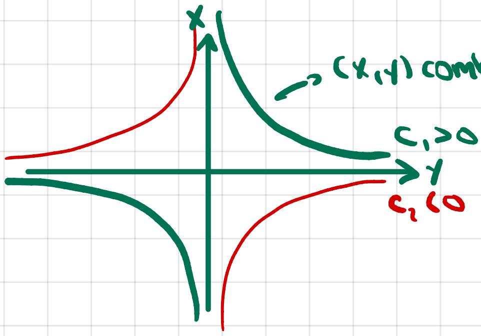
These have equations $k(x,y) = c_1$  u-curve

$h(x,y) = c_2$  v-curve

example:

$T^{-1}(x,y) = \langle xy, x^2 - y^2 \rangle = \langle u,v \rangle$



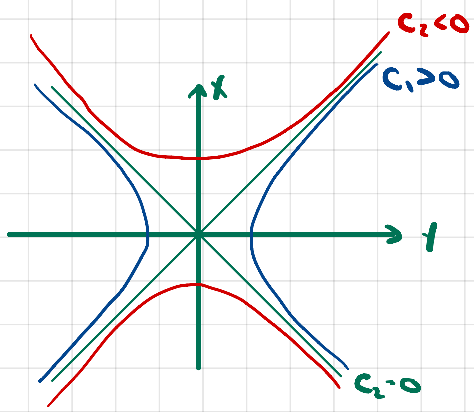




(x,y) combinations that produce $u = c_1$. each point on a v-curve corresponds to a different value for v.

$u = c = xy$ ⇒ v-curve

$y = \frac{c}{x}$

$v = c = x^2 - y^2$ ⇒ u-curve



# Change of Variables in Double Integrals

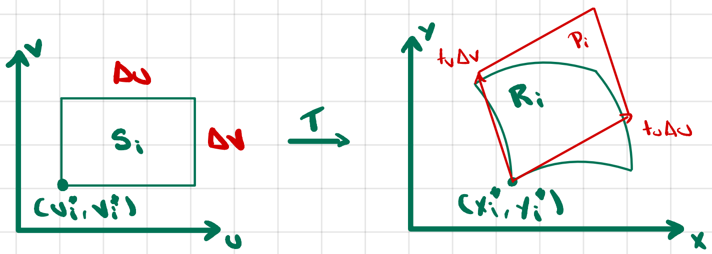
→ corresponding to transform. $T$: $(x,y) = T(u,v) = (f(u,v), g(u,v))$

## Setup

→ region $R$ in $xy$-plane is image under $T$ of region $S$ in $uv$-plane

→ $F(x,y)$ cont. on $R$

→ $\{S_1, S_2, \dots, S_n\}$ inner partition of $S$ into rectangles each $\Delta u$ by $\Delta v$; images $\{R_1, R_2, \dots, R_n\}$ are an inner partition of $R$




$(x_i^*, y_i^*) = (f(u_i^*, v_i^*), g(u_i^*, v_i^*))$

note we have parametric equations for $x$ and $y$ in parameters $u$ and $v$.
$\langle f(u,v), g(u,v)\rangle$ is a position vector.

suppose we keep $v$ constant and vary $u$ starting at $(u_i^*, v_i^*)$

In $xy$-space the velocity vector on the $u$-curve is $\vec{t}_u = \langle f_u(u_i^*, v_i^*), g_u(u_i^*, v_i^*)\rangle$

Similarly, velocity on the $v$-curve is $\vec{t}_v = \langle f_v(u_i^*, v_i^*), g_v(u_i^*, v_i^*)\rangle$

We use $\vec{t}_u \Delta u$ and $\vec{t}_v \Delta v$ to form a parallelogram that we use to approximate the area $\Delta A_i$ of $R_i$

$$\Delta A_i \approx a(P_i) = |(\vec{t}_u \Delta u) \times (\vec{t}_v \Delta v)| = |\vec{t}_u \times \vec{t}_v| \Delta u \Delta v$$

$$\vec{t}_u \times \vec{t}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} \hat{k}$$

↳ Jacobian of transformation $T: \mathbb{R}^2_{uv} \to \mathbb{R}^2_{xy}$

$$J_T(u,v) = \begin{vmatrix} x_u(u,v) & x_v(u,v) \\ y_u(u,v) & y_v(u,v) \end{vmatrix}$$

→ $\Delta A_i \approx |J_T(u_i^*, v_i^*)| \Delta u \Delta v$

$$\iint_R F(x,y)\,dx\,dy \approx \sum_{i=1}^{n} F(x_i^*, y_i^*)\,\Delta A_i$$

$$\approx \sum_{i=1}^{m} F(f(u_i^*, v_i^*), g(u_i^*, v_i^*))\,|J_T(u_i^*, v_i^*)|\,\Delta u \Delta v$$

$$\approx \iint_S F(f(u,v), g(u,v))\,|J_T(u,v)|\,du\,dv$$