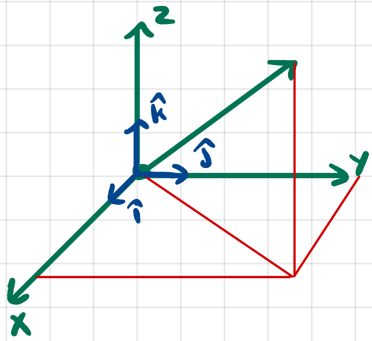
Lecture 1 - Vectors, Dot Product

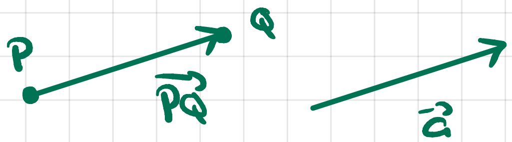
Vectors ↗ direction
↘ magnitude/length



$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \langle a_1, a_2, a_3 \rangle$

Length $|\vec{A}|$ (a scalar)

Direction $dir(\vec{A})$



same vectors

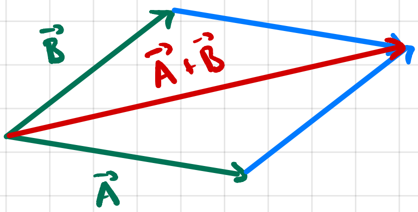
Length of $\langle 3,2,1 \rangle = \sqrt{9+4+1} = \sqrt{14}$

In general, $\vec{A} = \langle a_1, a_2, a_3 \rangle$

$\Rightarrow |\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

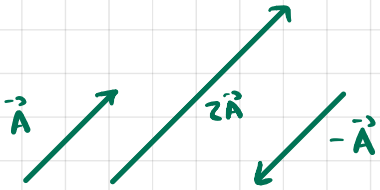
Vector Addition $\vec{A} + \vec{B}$
↙
geometric and computational objects

Geometric



Numeric: $\vec{A} + \vec{B} = \langle a_1+b_1, a_2+b_2, a_3+b_3 \rangle$

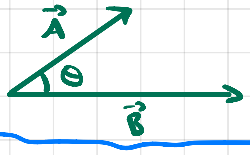
Multiplication by Scalar



Dot Product

Def: $\vec{A} \cdot \vec{B} = \sum a_i b_i$ ← a scalar

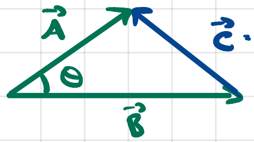
Geometrically, $\vec{A} \cdot \vec{B} = |A| \cdot |B| \cdot \cos(\theta)$ ← theorem



What does this mean?

1) $\vec{A} \cdot \vec{A} = |\vec{A}|^2 = a_1^2 + a_2^2 + \dots + a_n^2$

2)



$\vec{C} = \vec{A} - \vec{B}$

Law of cosines → Fact about sides of triangle

$|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|\cos\theta$

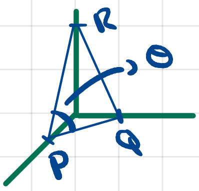
$|\vec{C}|^2 = \vec{C} \cdot \vec{C} = (\vec{A} - \vec{B})(\vec{A} - \vec{B})$

$= \vec{A}\cdot\vec{A} - \vec{A}\cdot\vec{B} - \vec{B}\cdot\vec{A} + \vec{B}\cdot\vec{B} = |A|^2 + |B|^2 - 2\vec{A}\cdot\vec{B}$

Applications

1) computing lengths and angles

$P = (1,0,0)$ $Q = (0,1,0)$ $R = (0,0,2)$



$\vec{PQ} \cdot \vec{PR} = |\vec{PQ}| \cdot |\vec{PR}| \cdot \cos\theta$

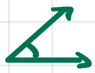
$\cos\theta = \dfrac{\vec{PQ} \cdot \vec{PR}}{|\vec{PQ}||\vec{PR}|}$

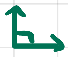
By simple inspection, $\vec{PQ} = \langle -1, 1, 0 \rangle$, $\vec{PR} = \langle -1, 0, 2 \rangle$

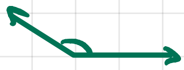
$|\vec{PQ}| = \sqrt{2}$, $|\vec{PR}| = \sqrt{5}$

$\Rightarrow \cos\theta = \dfrac{1 + 1 \cdot 0 + 0 \cdot 2}{\sqrt{10}} = \dfrac{1}{\sqrt{10}} \Rightarrow \theta = \cos^{-1}(1/\sqrt{10})$

→ sign of dot product is determined by sign of $\cos\theta$

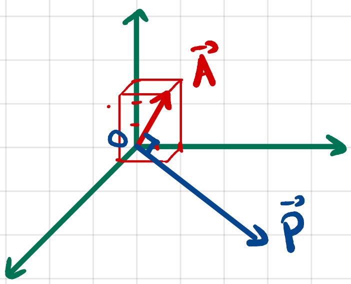
$\theta < 90^\circ \Rightarrow A \cdot B > 0$ 

$\theta = 90^\circ \Rightarrow A \cdot B = 0$ 

$\theta > 90^\circ \Rightarrow A \cdot B < 0$ 

2) Detect orthogonality

ex: $x + 2y + 3z = 0$ is eq. of plane



$\vec{OP} = \langle x, y, z \rangle$

$\vec{A} = \langle 1, 2, 3 \rangle$

All vectors $\vec{OP}$ perpendicular to $\vec{A}$

$\vec{OP} \cdot \vec{A} = 0 \Rightarrow x + 2y + 3z = 0$

$\vec{A}$ and $\vec{OP}$ are perpendicular.

→ remember: $\vec{A} \cdot \vec{B} = 0 \Leftrightarrow \cos\theta = 0$

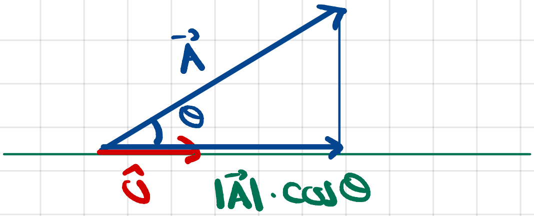
$\theta = 90^\circ$

$\vec{A} \perp \vec{B}$

Lecture 2

Applications of Dot Product (cont'd)

3) Components of $\vec{A}$ along direction $\hat{u}$ (unit vector)

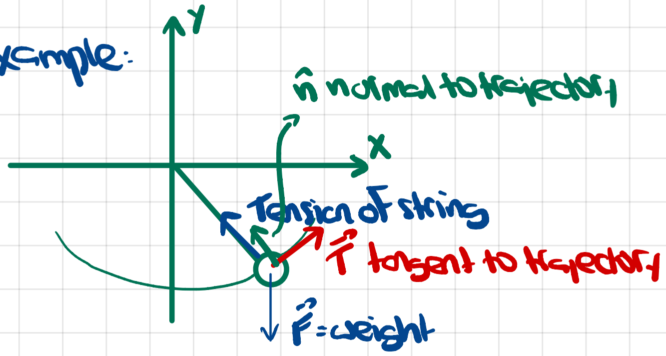


$|\vec{A}| \cdot \cos\theta$

$= |\vec{A}||\hat{u}|\cos\theta = \vec{A}\cdot\hat{u}$

($|\hat{u}| = 1$)

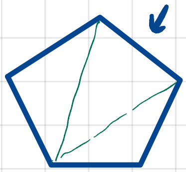
Example:



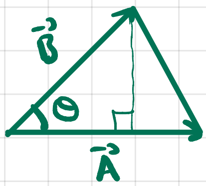
component of $\vec{F}$ along $\vec{T}$ causes pendulum to swing

" " " " $\hat{n}$ respons. for tension of string

Area? can we compute area using vectors?



easier: area of triangle



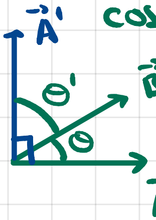
Area: $\frac{1}{2}|\vec{A}||\vec{B}|\sin\theta$

we need $\sin\theta$; we could find $\cos\theta$ then solve $\cos^2\theta + \sin^2\theta = 1$, but there is an easier way.

→ we want to find another angle θ' such that $\cos(\theta') = \sin\theta$, so the area equation becomes a dot product.

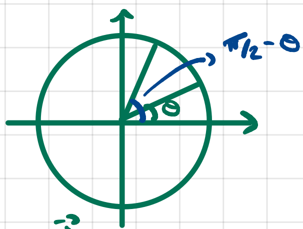
If we rotate $\vec{A}$ by 90° we have $\vec{A}'$ that forms θ' with $\vec{B}$. θ' and θ are complementary, so

$\cos(90-\theta) = \sin(\theta) = \cos(\theta')$.



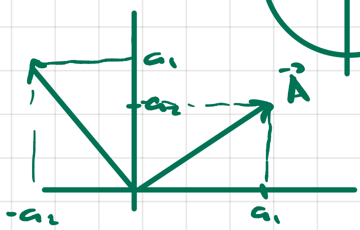
$\vec{A}'$ is $\vec{A}$ rotated 90°

$\theta' = \frac{\pi}{2} - \theta \Rightarrow \cos(\theta') = \sin(\theta)$



Now we have Area $= \frac{1}{2}|\vec{A}'|\cdot|\vec{B}|\cdot\cos(\theta') = \frac{1}{2}\vec{A}'\cdot\vec{B}$

But what does it mean to rotate counterclockwise by 90°?

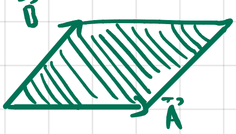


$\vec{A}' = \langle -a_2, a_1\rangle$

$\vec{A}'\cdot\vec{B} = \langle -a_2, a_1\rangle\langle b_1, b_2\rangle = a_1 b_2 - a_2 b_1$

$= \det(\vec{A}, \vec{B}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ determinant of vectors $\vec{A}$ and $\vec{B}$

measures area of parallelogram



(note neg. area if $\theta < 0$)

Note

$\pm$ area(▱) $= |\vec{A}| \cdot |\vec{B}| \sin\theta = \vec{A} \cdot \vec{B} = \det(\vec{A}, \vec{B})$

$\pm$ area(Δ) $= \frac{1}{2} |\vec{A}||\vec{B}| \sin\theta = \frac{1}{2} \vec{A} \cdot \vec{B} = \frac{1}{2} \det(\vec{A}, \vec{B})$

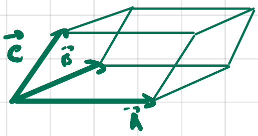
Determinant in space

$\rightarrow$ 3 vectors $\vec{A}, \vec{B}, \vec{C}$

$$\det(\vec{A}, \vec{B}, \vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

box with parallel sides

Theorem geometrically, $\det(\vec{A}, \vec{B}, \vec{C}) = \pm$ volume of parallelepiped



Cross Product of 2 vectors in 3d space

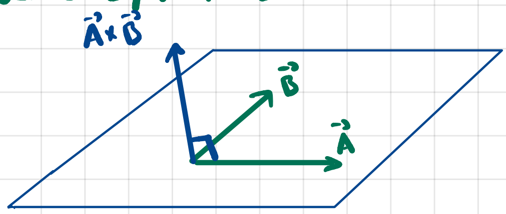
Def: $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\imath} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

vector

Theorem

$|\vec{A} \times \vec{B}| =$ area of parallelogram in space formed by $\vec{A}$ and $\vec{B}$

always positive



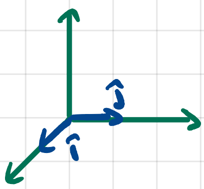
dir($\vec{A} \times \vec{B}$) is $\perp$ to plane of $\vec{A}$ and $\vec{B}$

using right-hand rule

right hand parallel to $\vec{A}$, curl fingers parallel to $\vec{B}$

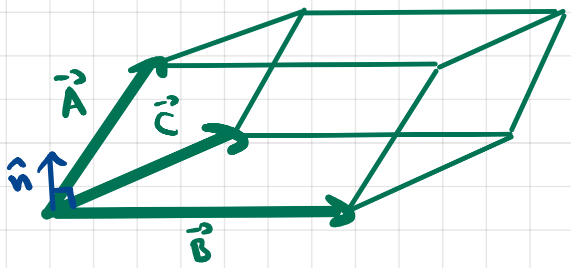
$\Rightarrow$ thumb points where $\vec{A} \times \vec{B}$ points

Ex:



$$\hat{\imath} \times \hat{\jmath} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \hat{\imath} \cdot 0 - \hat{\jmath} \cdot 0 + \hat{k} \cdot 1 = \hat{k}$$

Another look at volumes



volume = base area · height

base area = $|\vec{B} \times \vec{C}|$

height = $\vec{A} \cdot \hat{n}$ ($\vec{A}$ component in dir. of $\hat{n}$)

$$\hat{n} = \frac{\vec{B} \times \vec{C}}{|\vec{B} \times \vec{C}|}$$

$$\Rightarrow \text{volume} = \cancel{|\vec{B} \times \vec{C}|} \cdot \frac{(\vec{A} \cdot (\vec{B} \times \vec{C}))}{\cancel{|\vec{B} \times \vec{C}|}}$$

$$= \vec{A} \cdot (\vec{B} \times \vec{C})$$

we can check that this new expression equals the other formula we have for volume:

$$V = \det(\vec{A}, \vec{B}, \vec{C}) = \vec{A} \cdot (\vec{B} \times \vec{C})$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \langle a_1, a_2, a_3 \rangle \cdot \begin{vmatrix} \hat{i} & \hat{j} & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

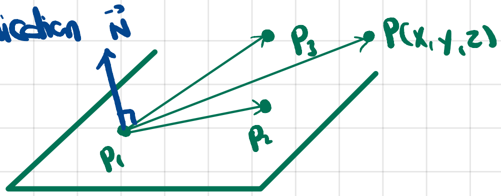
Lecture 3

cross product (cont'd)

$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

$\vec{A} \times \vec{A} = \vec{0}$

Application



Eq. of plane $P_1P_2P_3$ = condition on P

$\det(\vec{P_1P_2}, \vec{P_1P_3}, \vec{P_1P}) = 0$ $\Rightarrow$ volume parallelepiped is zero: $\vec{P_1P} \cdot (\vec{P_1P_3} \times \vec{P_1P_2}) = 0$

Alternatively,

P in the plane $\Leftrightarrow \vec{P_1P} \perp \vec{N}$

Triple Product

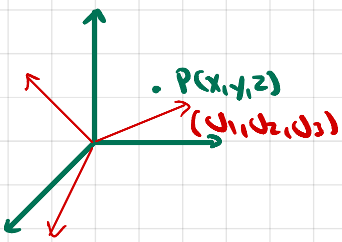
$\Leftrightarrow \vec{P_1P} \cdot \vec{N} = 0 = \vec{P_1P} \cdot (\vec{P_1P_2} \times \vec{P_1P_3}) = \det(\vec{P_1P_2}, \vec{P_1P_3}, \vec{P_1P})$

$\vec{N} = \vec{P_1P_2} \times \vec{P_1P_3}$

Matrices

often we have linear relations between variables

Ex: change of coordinate systems



$u_1 = 2x_1 + 3x_2 + 3x_3$
$u_2 = 2x_1 + 4x_2 + 5x_3$
$u_3 = x_1 + x_2 + 2x_3$

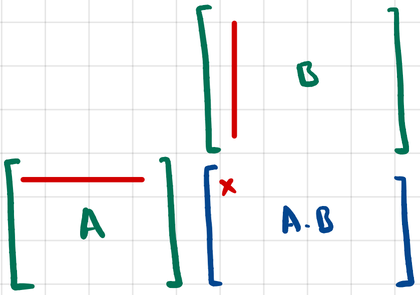
expressed using matrices:

$$\begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$A \cdot X = U$

dot product between rows of A and columns of X

Helpful way to set up matrix multiplication:



width A = Height B

What AB represents:

Do transformation B, then A.

$$\underbrace{(AB)}_{\text{Apply A to B}}X = A\underbrace{(BX)}_{\text{apply B to X}}$$

Note: $AB \neq BA$

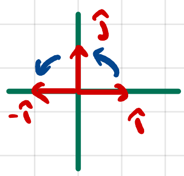
Identity Matrix I $\quad IX = X \quad I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

in general, $I_{n\times n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & & & 1 \end{bmatrix}$

Ex: In the plane, rotation by 90° counterclockwise

$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$R \cdot \hat{\imath} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \hat{\jmath}$



$R \cdot \hat{\jmath} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\hat{\imath}$

$R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_{2\times2}$

Inverse Matrix

$AN = NA = I$, $N = A^{-1}$

need: square matrix A, $n \times n$

solution to linear system $AX = B$ is $X = A^{-1}B$

$AX = B \Rightarrow A^{-1}AX = A^{-1}B \Rightarrow X = A^{-1}B$

"adjoint"

Formula for inverting $A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$

steps:

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix}$$

1) minors

$$\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} \quad \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} \quad \begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix}$$

$$\begin{bmatrix} 3 & -1 & -2 \\ 3 & 1 & -1 \\ 3 & 4 & 2 \end{bmatrix}$$

2) cofactors: flip signs in checkerboard

leave alone ↑ flip sign

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$\begin{bmatrix} 3 & 1 & -2 \\ -3 & 1 & 1 \\ 3 & -4 & 2 \end{bmatrix}$$

3) transpose

$$\begin{bmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{bmatrix} = \operatorname{adj}(A)$$

4) divide by $\det(A) = 3$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{bmatrix}$$

Lecture 4

Equations of Planes

Recall:

An eq. of plane is of form $ax+by+cz=d$

ex: plane through origin, normal vector $\vec{N}=\langle 1,5,10\rangle$

$\vec{OP}\cdot\vec{N}=0 \Rightarrow x+5y+10z=0$

ex: plane through $P_0(2,1,-1)$, $\vec{N}=\langle 1,5,10\rangle$

$\vec{P_0P}=\langle x-2, y-1, z+1\rangle$

$\vec{P_0P}\cdot\vec{N}=0 \Rightarrow (x-2)+5(y-1)+10(z+1)=0$

$\Rightarrow x+5y+10z=-3$

coeff. are normal vector coord.

In eq. $ax+by+cz=d$, $\langle a,b,c\rangle$ = normal vector $\vec{N}$

Can get $\vec{N}$ by cross product of two vectors in the plane

ex: $\vec{v}=\langle 1,2,-1\rangle$

plane $x+y+3z=5 \Rightarrow \langle 1,1,3\rangle=\vec{N}$

$\vec{v}\cdot\vec{N}=0 \Rightarrow \vec{v}$ perpendicular to $\vec{N}$, parallel to plane

3x3 linear system

$$\left.\begin{aligned} x \quad\;\; + z &= 1 \\ x + y \qquad &= 2 \end{aligned}\right\}$$ two planes intersect in a line

$x+2y+3z=3 \rightarrow$ third plane

The line $P_1 \cap P_2$ intersects P_3 in a point.

Solution to $AX=B$ is $X=A^{-1}B$

unless ...

→ unless 3rd plane parallel to line where P_1 and P_2 intersect

→ if $P_1 \cap P_2$ contained in P_3 ⇒ infinitely many solutions (any point on the line)

→ if $P_1 \cap P_2$ parallel to P_3 and not contained in it ⇒ no solutions

Recall $A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{adj}(A)$

A is invertible ⇔ $\det(A) \neq 0$

Start with homogeneous case: $AX = 0$

Trivial solution

e.g. $x + z = 0$
$x + y = 0$
$x + 2y + 3z = 0$

There's always an obvious solution $(0,0,0)$

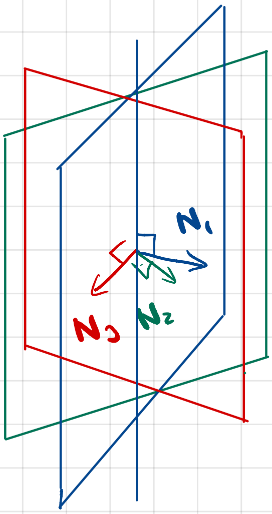
origin is solution because the three planes pass through origin.

if $\det(A) \neq 0$ $AX = 0$ ⇔ $X = A^{-1}0 = 0$, no other solution

equation coeff. in A are normal vector coord.

if $\det(A) = 0$ ⇔ $\det(\vec{N}_1, \vec{N}_2, \vec{N}_3) = 0$

⇔ $\vec{N}_1, \vec{N}_2, \vec{N}_3$ are coplanar (parallelipiped has volume 0)



line through O, perpend. to $\vec{N}_1, \vec{N}_2, \vec{N}_3$ is parallel to all three planes

and which is contained in them ⇒ ∞ solutions

Ex: $\vec{N}_1 \times \vec{N}_2 \perp \vec{N}_1, \vec{N}_2, \vec{N}_3$

non-trivial solution

General Case $AX = B$

$\det A \neq 0$ ⇒ unique solution $X = A^{-1}B$

$\det(A) = 0$ ⇒ either no solutions or ∞ solutions

Lecture 5

Equations of lines

→ line : intersection 2 lines

→ another way : line : trajectory of moving point : "parametric equation"

Ex: line through $Q_0 = (-1,2,2)$, $Q_1 = (1,3,-1)$

$Q(t)$ = moving point, constant speed on the line

what is $Q(t)$, position at time t?

$\overrightarrow{Q_0Q(t)} = t\,\overrightarrow{Q_0Q_1} = t\langle 2,1,-3\rangle$

$Q(t) = \langle x(t), y(t), z(t)\rangle$

$x(t) - x_0 = x(t) + 1 = 2t \Rightarrow x(t) = -1+2t$

$y(t) - 2 = t$

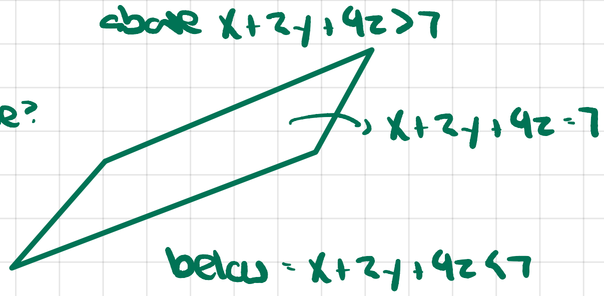
$z(t) - 2 = -3t$

$\Rightarrow Q(t) = Q_0 + t\,\overrightarrow{Q_0Q_1} = \langle -1,2,2\rangle + t\langle 2,1,-3\rangle$

Application

intersection with a plane?

consider $x+2y+4z=7$



where does line through Q_0, Q_1 intersect plane?

$(-1,2,2)$ $\quad -1+2\cdot 2+4\cdot 2 = 11$

$(1,3,-1)$ $\quad 1+2\cdot 3+4(-1) = 3$

$\Rightarrow Q_0$ and Q_1 are on different sides of $x+2y+4z=7$.

$(-1+2t) + 2(2+t) + 4(2-3t) = 7$

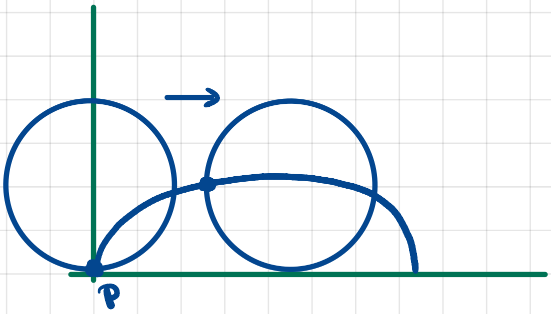
$\Rightarrow t = 1/2$

$Q(1/2) = \langle 0, 5/2, 1/2\rangle$

parametric eq. line, eq. of plane, plug one into the other, find intersection.

More generally, parametric eq. for arbitrary motion in the plane or in space.

Cycloid

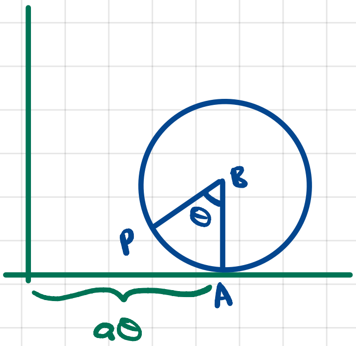


wheel of radius a rolling on floor = x-axis

P: point on rim of wheel, starts at 0

Question position $(x(t), y(t))$ of point P?

let t be θ, angle by which the wheel has rotated

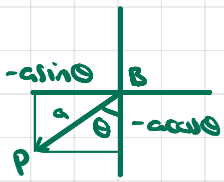


$\vec{OP} = \vec{OA} + \vec{AB} + \vec{BP}$

$\vec{OA} = \langle a\theta, 0 \rangle$ — amount by which wheel moved = arclength AP

$\vec{AB} = \langle 0, a \rangle$

$\vec{BP}$: $|\vec{BP}| = a$
angle θ w/ vertical



$\vec{BP} = \langle a\theta - a\sin\theta, a - a\cos\theta \rangle$

$= \langle x(\theta), y(\theta) \rangle$

Question: what happens near bottom point?

- Take length unit = radius a = 1

$x(\theta) = \theta - \sin\theta$
$y(\theta) = 1 - \cos\theta$

Taylor approx.: for small t $f(t) \approx f(0) + f'(0)t + f''(0)\frac{t^2}{2} + f'''(0)\frac{t^3}{6}$

For small θ, $\sin\theta \approx \theta - \frac{\theta^3}{6}$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$

$$\left.\begin{aligned} x(\theta) &\approx \theta - (\theta - \theta^3/6) = \theta^3/6 \\ y(\theta) &\approx 1 - (1 - \theta^2/2) = \theta^2/2 \end{aligned}\right\} \quad |x| \ll |y|, \quad \frac{y}{x} \approx \frac{3}{\theta} \longrightarrow \infty \text{ when } \theta \to 0$$

$\Rightarrow$ slope at origin is ∞

Lecture 6

position vector $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

↗ pos. of moving point

↘ vector whose components are coordinates of the point

Example: Cycloid (cont'd)

→ now we use radius 1, at unit speed ⇒ angle is the same thing as time
ie, speed = 1/unit of time = 1 radian/unit of time

$\vec{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$

e.g. if we have 2π rad rotation then 2π units of time have passed

Velocity vector $\vec{v} = \frac{d\vec{r}}{dt} = \langle dx/dt, dy/dt, dz/dt \rangle$

Speed (scalar) $|\vec{v}|$

Note: $\left|\frac{d\vec{r}}{dt}\right| \neq \frac{d|\vec{r}|}{dt}$

speed ?

Acceleration $\vec{a} = \frac{d\vec{v}}{dt}$

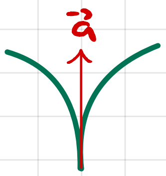
Ex: Cycloid (cont'd)

$\vec{v} = \langle 1 - \cos t, \sin t \rangle$

At $t = 0$, $\vec{v} = \langle 0, 0 \rangle$

$|\vec{v}| = \sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t}$

$\vec{a} = \langle \sin t, \cos t \rangle$



$t = 0 \Rightarrow \vec{a} = \langle 0, 1 \rangle$

Arc Length

s = distance travelled along trajectory

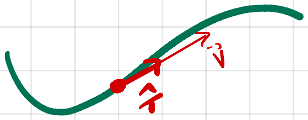
s versus t?

$\frac{ds}{dt}$ = speed = $|\vec{v}|$

Ex: (cycloid) length of one arch of cycloid is $\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$

Unit tangent vector $\hat{T}$

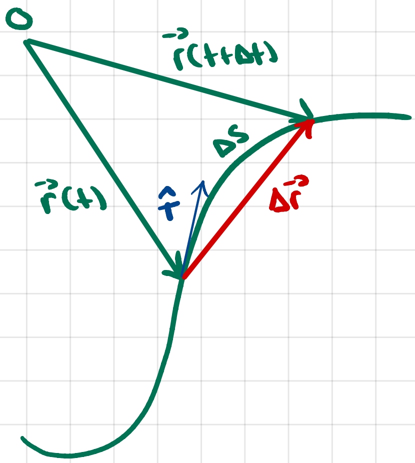
$$\hat{T} = \frac{\vec{v}}{|\vec{v}|} \Rightarrow \vec{v} = \hat{T}\,|\vec{v}|$$



$$\frac{d\vec{r}}{dt} = \vec{v} = \underbrace{\frac{d\vec{r}}{ds}}_{\hat{T}}\;\underbrace{\frac{ds}{dt}}_{|\vec{v}|,\ \text{speed}} = \hat{T}\frac{ds}{dt}$$

velocity has $\begin{cases} \text{direction tangent to trajectory, } \hat{T} \\ \text{length: speed } ds/dt \end{cases}$

How does r change when s changes, ie dr/ds?



In time Δt

$$\frac{\Delta s}{\Delta t} \approx \text{speed}$$

$$\Delta\vec{r} \approx \hat{T}\cdot\Delta s \Rightarrow \frac{\Delta r}{\Delta t} \approx \hat{T}\frac{\Delta s}{\Delta t}$$

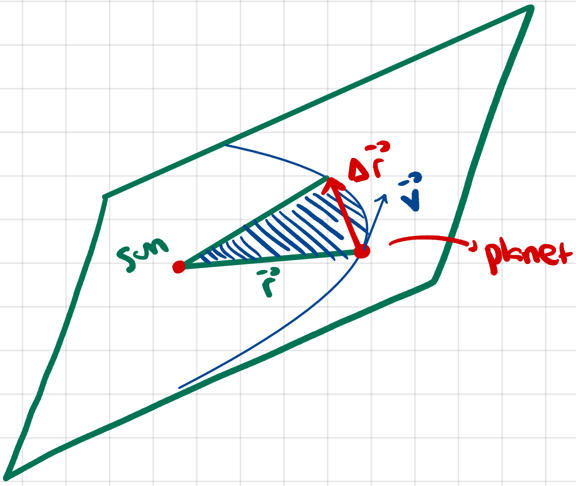
limit as $\Delta t \to 0$ gives $\frac{d\vec{r}}{dt} = \hat{T}\frac{ds}{dt}$

Example: Kepler's Second Law (1609)

Motion of planets is in a plane, and the area is swept out by the line from sun to planet at constant rate.

→ Newton later explained this using formula for gravitational attraction.

Kepler's Law in terms of vectors



1) Area swept in time Δt $\approx \frac{1}{2}|\vec{r}\times\Delta\vec{r}| \approx \frac{1}{2}|\vec{r}\times\vec{v}|\,\Delta t$

$\Delta r \approx \vec{v}\,\Delta t$

Law says area proportional to $\Delta t \Rightarrow |\vec{r}\times\vec{v}| = \text{constant}$

2) Plane of motion contains $\vec{r}$ and $\vec{v}$

$\Rightarrow \vec{r}\times\vec{v}$ normal to plane of motion

Kepler's 2nd Law $\Leftrightarrow \vec{r}\times\vec{v} = \text{constant vector}$

$$\Leftrightarrow \frac{d}{dt}(\vec{r}\times\vec{v}) = \vec{0} \Leftrightarrow \frac{d\vec{r}}{dt}\times\vec{v} + \vec{r}\times\frac{d\vec{v}}{dt} = 0 \Leftrightarrow \vec{v}\times\vec{v} + \vec{r}\times\vec{a} = 0$$

$$\Leftrightarrow \vec{r}\times\vec{a} = 0 \Leftrightarrow \vec{a} \parallel \vec{r} \Leftrightarrow \text{gravitational force} \parallel \vec{r}$$

Lecture 7

Topics: EVERYTHING!

1) vectors, dot product

→ find angles

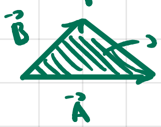
→ detect $\perp$

Practice 1A: Problem 1

2) cross product

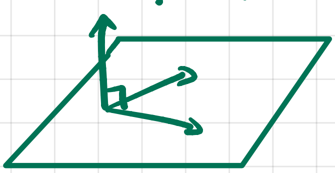
→ Determinants (2x2, 3x3)

→ Applications of cross product

→ Area of  → $\frac{1}{2}|\vec{A}\times\vec{B}|$

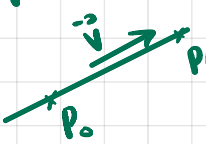
→ Finding normal vector to a plane

→ finding eq. of plane, $ax+by+cz=d$



Problem 5, practice 1A

→ equations of lines, find where line intersects plane



$P(t) = P_0 + t\vec{v}$

3) Matrices, Linear Systems

→ inverting matrices

$AX = B$

$\det A \neq 0 \Rightarrow X = A^{-1}B$

$\det A = 0 \Rightarrow$ 0 or ∞ solutions

In general we don't know how to tell if its 0 or ∞

4) Finding parametric equations

$\vec{r} = \vec{r}(t)$

→ velocity, acceleration

Differentiating vector identities

(Problems 2,4,6)

Problem 3

a)

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \qquad A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & x & y \\ -1 & -2 & 5 \\ 2 & 2 & -6 \end{bmatrix}$$

$$x = -\begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} = -(-2) = 2 \qquad y = +\begin{vmatrix} 3 & 2 \\ 0 & -1 \end{vmatrix} = -3$$

b) $AX = B$ $\quad X = A^{-1}B = \frac{1}{2}\begin{bmatrix} 1 & 2 & -3 \\ -1 & -2 & 5 \\ 2 & 2 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -6 \\ 8 \\ -8 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -4 \end{bmatrix}$

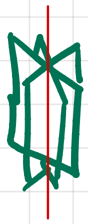
c) $M = \begin{bmatrix} 1 & 3 & c \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

M not invertible $\Leftrightarrow$ $\det M = 0$

$$\det M = 1(1) - 3(1) + c(2) = 1 - 3 + 2c = 2c - 2 = 0 \Rightarrow c = 1$$

$MX = 0$

$$\begin{aligned} x + 3y + z &= 0 \\ 2x \qquad - z &= 0 \\ x + y \qquad &= 0 \end{aligned}$$



$\vec{0}$ is a solution. So there are ∞ solutions.

To find other solutions

$\langle x, y, z\rangle \langle 1, 3, 1\rangle = 0$

$\langle x, y, z\rangle \langle 2, 0, -1\rangle = 0$

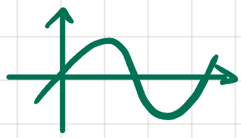
$\langle x, y, z\rangle \langle 1, 1, 0\rangle = 0$

To find $\langle x,y,z\rangle \perp \langle 1,3,1\rangle$ and $\langle 2,0,-1\rangle$ take the cross product.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 1 \\ 2 & 0 & -1 \end{vmatrix} = \langle -3, 3, -6\rangle$$

Lecture 8

Function of 1 variable $f(x) = \sin x$



" " 2 variables

given $(x,y) \to$ get number $f(x,y)$

Ex: $f(x,y) = x^2 + y^2$

$f(x,y) = \sqrt{y}$ only defined if $y \geq 0$

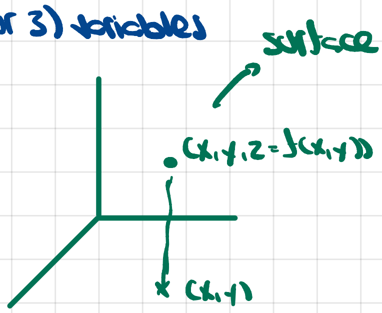
$f(x,y) = 1/(x+y)$ $\quad x+y \neq 0$

Ex: $f(x,y)$ = temp. at (x,y)

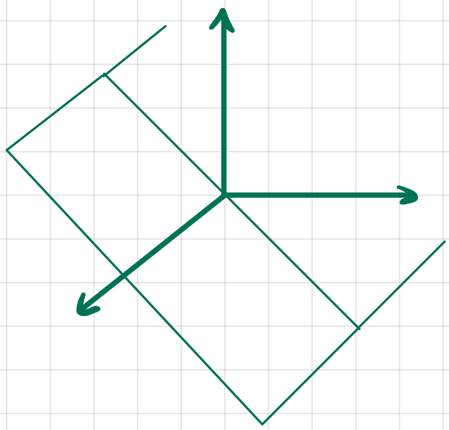
For simplicity focus mostly on 2 (or 3) variables

How to visualize $f = f(x,y)$

$\to$ graph: $z = f(x,y)$

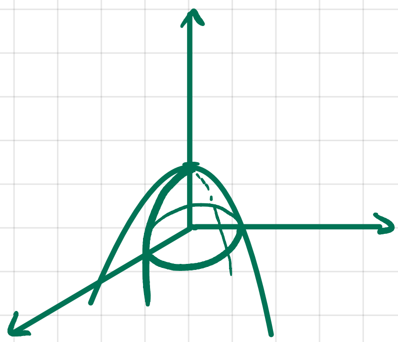


Ex: $f(x,y) = -y$



$z = -y$

Ex $f(x,y) = 1 - x^2 - y^2$

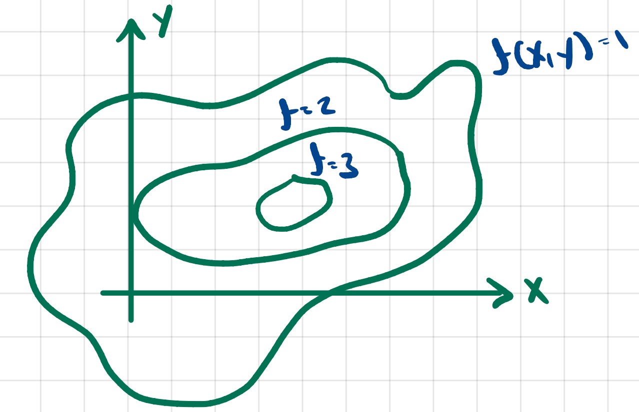


in yz plane, $x=0 \Rightarrow z = 1-y^2$

in xz plane, $y=0$, $z = 1-x^2$

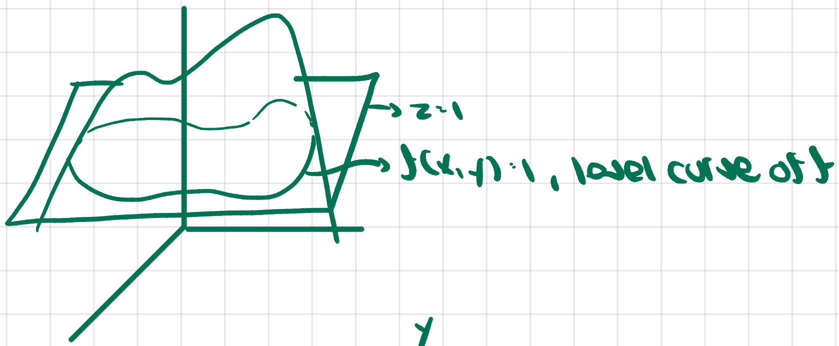
hit xy-plane? $z=0 \Rightarrow x^2+y^2=1$, a circle $r=1$

Contour Plot

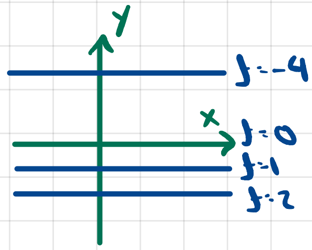


shows all points where $f(x,y)$ = some fixed constants, typically chosen at regular intervals

$\Leftrightarrow$ we slice the graph by horizontal planes $z = c$



Ex $f(x,y) = -y$



Ex $f(x,y) = 1 - x^2 - y^2$

$f = 0 \Rightarrow x^2 + y^2 = 1$

$f = m \Rightarrow x^2 + y^2 = 1 - m$ radius $= \sqrt{1-m}$

$f =$

radius increases slower than m so the concentric level curves get closer and closer for same increase in m

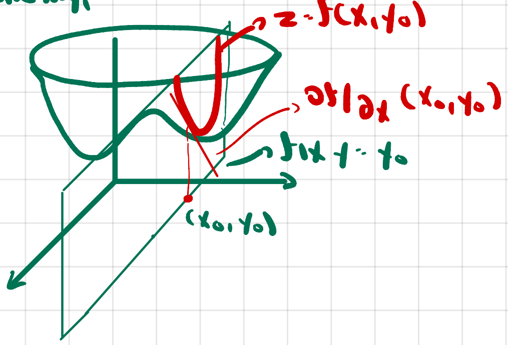
Partial Derivatives

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

"partial"

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0+\Delta y) - f(x_0, y_0)}{\Delta y}$$

Geometrically,



How to compute?

Treat x as variable, y as constant:

$$f(x,y) = x^3 y + y^2$$

$$\frac{\partial f}{\partial x} = 3x^2 y$$

$$\frac{\partial f}{\partial y} = x^3 + 2y$$

Lecture 9

Approxim. Formula

change $x \rightarrow x+\Delta x$
$y \rightarrow y+\Delta y$

$$\boxed{\Rightarrow \Delta z \approx f_x \Delta x + f_y \Delta y}$$

$z = f(x,y)$

How do we justify this formula? tangent plane to $z=f(x,y)$

→ we know f_x, f_y are slopes of two tangent lines

$\Rightarrow \dfrac{\partial f}{\partial x}(x_0,y_0) = a \quad \Rightarrow L_1 : \begin{cases} z = z_0 + a(x-x_0) \\ y = y_0 \end{cases}$

↳ line through tangent on intersection of f with $y=y_0$ plane

$\dfrac{\partial f}{\partial y}(x_0,y_0) = b \quad \Rightarrow L_2 : \begin{cases} z = z_0 + b(y-y_0) \\ x = x_0 \end{cases}$

L_1, L_2 both tangent to graph of $z=f(x,y)$.

Together they determine a plane $z = z_0 + a(x-x_0) + b(y-y_0)$

* Alternatively, find parametric eq. for the lines, take the cross product of the vectors to obtain plane normal vector, and obtain the same plane formula.

Approx formula says graph of f is close to its tangent plane.

Application of Partial Deriv.

Optimization Problems

→ Find min/max of $f(x,y)$

→ at local min or max, $f_x = 0$ and $f_y = 0$

$\Leftrightarrow$ tangent plane to the graph is horizontal

Def: (x_0,y_0) is a critical point of f if $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$

Ex: $f(x,y) = x^2 - 2xy + 3y^2 + 2x - 2y$

$$\left.\begin{aligned} f_x &= 2x - 2y + 2 = 0 \\ f_y &= -2x + 6y - 2 = 0 \end{aligned}\right\} \text{Sum: } 4y = 0 \Rightarrow y = 0 \Rightarrow x = -1$$

$(-1,0)$ is a critical point.

Possibilities for critical point

- → local min
- → local max
- → saddle

Ex: (cont'd)

complete square

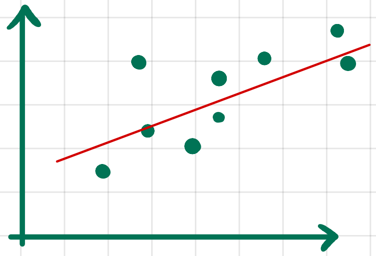
complete square again

$f(x,y) = (x-y)^2 + 2y^2 + 2x - 2y$

$= \underbrace{[(x-y)+1]^2}_{\geq 0} + \underbrace{2y^2}_{\geq 0} - 1$

$f(-1,0) = -1 \Rightarrow$ minimum

Least Squares Interpolation



Given experimental data $(x_1,y_1),(x_2,y_2),\dots(x_n,y_n)$

find the "best fit" line $y = ax+b$

Find "best" a and b

"Best" means minimizing total square deviation

Deviation for each data point: $y_i - (ax_i+b)$

Minimize $D(a,b) = \sum_{i=1}^{n} [y_i - (ax_i+b)]^2$

$\frac{\partial D}{\partial a} = \sum_{i=1}^{n} 2(y_i-(ax_i+b))(-x_i) = \sum_{i=1}^{n}(x_i^2a + x_ib - x_iy_i) \Rightarrow a\sum x_i^2 + b\sum x_i = \sum x_i y_i$

$\frac{\partial D}{\partial b} = \sum_{i=1}^{n} 2(y_i-(ax_i+b))(-1) = \sum_{i=1}^{n}(ax_i+b-y_i) \Rightarrow a\sum x_i + nb = \sum y_i$ → 2x2 linear system! solve for a and b.

→ Least squares is more general

Ex:

→ Best exponential fit $y = ce^{ax}$ $\quad \ln y = \ln c + ax$

→ Best quadratic fit: $y = ax^2+bx+c$

$D(a,b,c) = \sum (y_i - (ax_i^2+bx_i+c)) \xrightarrow{\text{derivatives}}$ 3x3 linear system

Lecture 10

recall: critical points of $f(x,y)$: $f_x = 0$ and $f_y = 0$

how to decide between $\begin{cases} \text{local min} \\ \text{" max} \\ \text{saddle point} \end{cases}$

how do we find the global min or max

→ can occur either at a critical point or on boundary / infinity

Second Derivative Test

Ex: consider $w(x,y) = ax^2 + bxy + cy^2 \Rightarrow (0,0)$ is critical point

$w(x,y) = x^2 + 2xy + 3y^2 = (x+y)^2 + 2y^2$

In general, if $a \neq 0$, $w = a\left(x^2 + \frac{b}{a}xy\right) + cy^2 = a\left(x + \frac{b}{2a}y\right)^2 + \left(c - \frac{b^2}{4a}\right)y^2$

$= \frac{1}{4a}\left[4a^2\left(x + \frac{b}{2a}y\right)^2 + (4ac - b^2)y^2\right]$

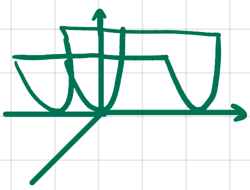
3 cases

1) $4ac - b^2 < 0$ $\quad \frac{1}{4a}\left[\underbrace{4a^2\left(x + \frac{b}{2a}y\right)^2}_{\geq 0} + \underbrace{(4ac - b^2)y^2}_{\leq 0}\right]$

→ saddle point

2) $4ac - b^2 = 0$ $\quad \frac{1}{4a}\left[4a^2\left(x + \frac{b}{2a}y\right)^2\right]$

ex: $w = x^2$



degenerate critical points (infinite critical points)

3) $4ac - b^2 > 0$ $\quad \frac{1}{4a}\left[\underbrace{4a^2\left(x + \frac{b}{2a}y\right)^2}_{\geq 0} + \underbrace{(4ac - b^2)y^2}_{\geq 0}\right]$

$a > 0 \Rightarrow (0,0)$ minimum

$a < 0 \Rightarrow (0,0)$ maximum

Note

$b^2 - 4ac$

rewrite $w(x,y)$ in a different way

$$w(x,y) = ax^2 + bxy + cy^2 = \underbrace{y^2}_{\geq 0}\left[\underbrace{a\left(\frac{x}{y}\right)^2 + b\frac{x}{y} + c}\right]$$

$b^2 - 4ac > 0 \Rightarrow \exists$ solutions, the expression takes on both + and − values
$\Rightarrow$ saddle point $(0,0)$

$b^2 - 4ac < 0 \Rightarrow$ no solution
$\Rightarrow$ expr. only + or only −
$\Rightarrow$ w always + or −
$\Rightarrow$ max or min at $(0,0)$

In general, look at second derivatives!

$\frac{\partial^2 f}{\partial x^2} = f_{xx}$ $\quad$ $\frac{\partial^2 f}{\partial y^2} = f_{yy}$ $\quad$ $f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$

Second Derivative Test

At critical point (x_0, y_0) of f, let $A = f_{xx}(x_0,y_0)$, $B = f_{xy}(x_0,y_0)$, $C = f_{yy}(x_0,y_0)$

$AC - B^2 > 0$ and $A > 0 \Rightarrow$ local minimum
$\qquad\qquad\qquad\quad A < 0 \Rightarrow$ local maximum

$AC - B^2 < 0 \Rightarrow$ saddle point

$AC - B^2 = 0 \Rightarrow$ can't conclude

Verify in special case of $w(x,y) = ax^2 + bxy + cy^2$

$w_x = 2ax + by$ $\quad$ $w_{xx} = 2a$ $\quad$ $w_{xy} = b$ $\quad$ $w_y = bx + 2cy$ $\quad$ $w_{yx} = b$ $\quad$ $w_{yy} = 2c$

$A = 2a$, $B = b$, $C = 2c$ $\Rightarrow$ $AC - B^2 = 4ac - b^2$

Quadratic Approxim.

$\Delta f \approx f_x \Delta x + f_y \Delta y$

At critical point $f_x = f_y = 0 \Rightarrow$ lin approx. no good, need more terms

$$\Delta f \approx f_x \Delta x + f_y \Delta y + \frac{1}{2} f_{xx} \Delta x^2 + f_{xy} \Delta x \Delta y + \frac{1}{2} f_{yy} \Delta y^2$$

$\Rightarrow$ the general case reduces to the quadratic case

$\Rightarrow$ in the degenerate case, what actually happens depends on higher order derivatives

Example: $f(x,y) = x + y + \dfrac{1}{xy}$ $\quad x, y > 0$

$f_x = 1 - \dfrac{1}{x^2 y} = 0 \Rightarrow x^2 y = 1$

$\Rightarrow x = y,\ y^3 = 1 \Rightarrow y = 1 \Rightarrow x = 1$ $\quad (1,1)$ critical point

$f_y = 1 - \dfrac{1}{x y^2} = 0 \Rightarrow x y^2 = 1$

$f_{xx} = \dfrac{2}{x^3 y} = A$ $\qquad AC - B^2 = \dfrac{4}{x^4 y^4} - \dfrac{1}{x^4 y^4} = \dfrac{3}{x^4 y^4}$

$f_{yy} = \dfrac{2}{x y^3} = C$ $\qquad$ At $(1,1)$ $= 3 > 0 \Rightarrow$ since $A(1,1) > 0$, local minimum

$f_{xy} = \dfrac{1}{x^2 y^2} = B$

maximum: $\underbrace{f \to \infty \text{ when } x \to \infty \text{ or } y \to \infty \text{ or } x, y \to 0}_{\text{boundaries}}$

Lecture 11 More tools to study functions

Differentials

-> recall implicit differentiation

$$y = f(x)$$

$$dy = f'(x)\,dx$$

ex: $y = \sin^{-1}(x)$

$$\sin y = x \Rightarrow dx = \cos y\, dy \;\Rightarrow\; \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

Total Differential

$$f(x,y,z)$$

$$df = f_x\,dx + f_y\,dy + f_z\,dz$$

-> important : df is not the same as Δf ← a number

not numbers, simply expressible in terms of other differentials

"differentials are strange objects"

-> can do:

1) encode how change in x, y, z affects f

2) placeholders for small variations $\Delta x, \Delta y, \Delta z$ to get approx. formula $\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z$

3) divide by something like dt to get an infinitesimal rate of change when $x = x(t), y = y(t), z = z(t)$.

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} \;\rightarrow \text{CHAIN RULE}$$

why is this valid?

1st attempt

$$df = f_x\,dx + f_y\,dy + f_z\,dz$$

$$dx = x'(t)\,dt \qquad dy = y'(t)\,dt \qquad dz = z'(t)\,dt$$

$$\Rightarrow df = f_x x'(t)\,dt + f_y y'(t)\,dt + f_z z'(t)\,dt$$

divide by dt => get chain rule

not completely convincing

Better way:

$$\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z$$

Divide by Δt

we are dividing by numbers here, not differentials

$$\frac{\Delta f}{\Delta t} \approx f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + f_z \frac{\Delta z}{\Delta t}$$

$\Delta t \to 0$

$$\frac{\Delta f}{\Delta t} \to \frac{df}{dt} \qquad \frac{\Delta x}{\Delta t} \to x'(t) \qquad (\ldots)$$

in the limit $\Delta t \to 0$

$$\Rightarrow \quad \frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$

Example: $w = x^2 y + z \qquad x = t \qquad y = e^t \qquad z = \sin t$

$$\frac{dw}{dt} = 2xy \cdot 1 + x^2 e^t + 1 \cdot \cos t$$

$$= 2te^t + t^2 e^t + \cos t$$

according to the chain rule

to double check, plug $x(t), y(t), z(t)$ into w

$$w(t) = t^2 e^t + \sin t$$

$$w'(t) = 2te^t + t^2 e^t + \cos t$$

Application: justify product and quotient rules

$f = uv \qquad u = u(t) \qquad v = v(t)$

$$\frac{d(uv)}{dt} = f_u \frac{du}{dt} + f_v \frac{dv}{dt} = v \frac{du}{dt} + u \frac{dv}{dt}$$

$$g = \frac{u}{v}$$

$$\frac{dg}{dt} = \frac{d}{dt}(u/v) = \frac{1}{v} \cdot \frac{du}{dt} + \frac{-u}{v^2} \frac{dv}{dt} = \frac{vu' - uv'}{v^2}$$

Chain Rule w/ More Variables

$f = f(x, y)$ where $x = x(u, v)$, $y = y(u, v)$

what are $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$ in terms of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, x_u, x_v, y_u, y_v$

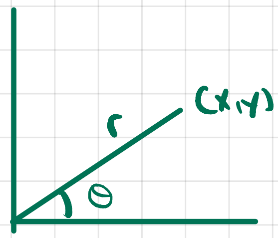
$$df = f_x\,dx + f_y\,dy$$
$$= f_x[x_u\,du + x_v\,dv] + f_y[y_u\,du + y_v\,dv]$$
$$= \underbrace{(f_x x_u + f_y y_u)}_{\partial f/\partial u}du + \underbrace{(f_x x_v + f_y y_v)}_{\partial f/\partial v}dv$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

no simplifications possible here

can't "cancel" partials

Example polar coord.



$x = r\cos\theta$

$y = r\sin\theta$

$f = f(x, y)$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta$$

Lecture 12

$\nabla \vec{w} = \langle w_x, w_y, w_z \rangle$ Gradient of w at some point (x,y,z)

$\frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$

$\frac{dw}{dt} = \nabla \vec{w} \cdot \frac{d\vec{r}}{dt}$

Ex: $w = a_1x + a_2y + a_3z$

$\nabla \vec{w} = \langle a_1, a_2, a_3 \rangle$

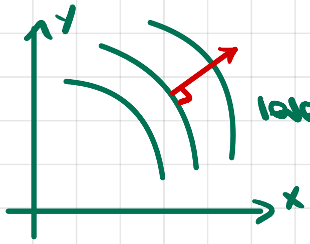
level surfaces

plane

$a_1x + a_2y + a_3z = c$

$\vec{n} = \langle a_1, a_2, a_3 \rangle = \nabla \vec{w}$

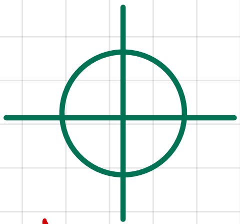
Theorem $\nabla w \perp$ level surface $w = \text{constant}$



level curves of $F(x,y)$

Ex: $w = x^2 + y^2$

$w = c$ is a circle



$\nabla \vec{w} = \langle 2x, 2y \rangle$

→ parallel to position vector, $\perp$ to level curve

Proof:

Take $\vec{r} = \vec{r}(t)$ that stays on the level surface $w = c$

$\vec{v} = \frac{d\vec{r}}{dt}$ is tangent to level $w = c$

By chain rule, $\frac{dw}{dt} = \nabla \vec{w} \cdot \frac{d\vec{r}}{dt} = \nabla \vec{w} \cdot \vec{v} = 0$ since $w(t) = c = \text{constant}$

$\Rightarrow \nabla \vec{w} \perp \vec{v}$ This is true for any motion on the level surface $w = c$

$\Rightarrow \vec{v}$ can be any vector tangent to the level surface

Given any vector $\vec{v}$ tangent to level, $\nabla \vec{w} \perp \vec{v}$

$\Rightarrow \nabla \vec{w} \perp$ tangent plane to the level

Ex: Find tangent plane to $x^2 + y^2 - z^2 = 4$ at $(2,1,1)$

$\nabla w = \langle 2x, 2y, -2z \rangle$ $\nabla w(2,1,1) = \langle 4, 2, -2 \rangle$ = normal vector to tangent plane

$4x + 2y - 2z = 8$

Another Way

$dw = 2x\,dx + 2y\,dy - 2z\,dz$

At $(2,1,1)$ $\quad dw = 4dx + 2dy - 2dz \qquad \Delta w \approx 4\Delta x + 2\Delta y - 2\Delta z$

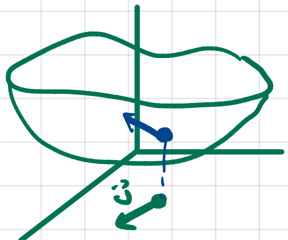
So, level : $\Delta w = 0$ $\quad$ is tangent plane $\quad 4\Delta x + 2\Delta y - 2\Delta z = 0 \quad 4(x-2) + 2(y-1) - 2(z-1) = 0$

Directional Derivatives

derivatives in direction $\hat{i}$ and $\hat{j}$

$w = w(x,y)$ $\qquad$ we know w_x and w_y

what if we move in direction of $\hat{u}$?



straight line trajectory

$\vec{r}(s),\ \dfrac{d\vec{r}}{ds} = \hat{u}$

s: arc length, distance along line

$\Rightarrow$ moving at unit speed along this line $\Rightarrow$ parametrizing by distance traveled along the curve

$*$ if $\hat{u} = \langle a,b \rangle$

$$\begin{cases} x(s) = x_0 + a \cdot s \\ y(s) = y_0 + bs \end{cases}$$

plug into w, take $\dfrac{dw}{ds}$

what is $\dfrac{d\vec{w}}{ds}$?

This is $\left.\dfrac{dw}{ds}\right|_{\hat{u}}$ $\leftarrow$ directional deriv. in dir. $\hat{u}$

= slope of slice of graph by a vertical plane $\parallel$ to $\hat{u}$

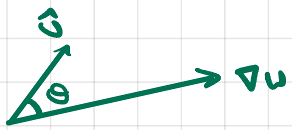
computing

$\dfrac{dw}{ds} = \nabla \vec{w} \cdot \dfrac{d\vec{r}}{ds} = \nabla \vec{w} \cdot \hat{u}$

$\nabla \vec{w} \cdot \hat{u}$: component of $\nabla \vec{w}$ in direction of $\hat{u}$

Ex: $\left.\dfrac{dw}{ds}\right|_{\hat{i}} = \nabla \vec{w} \cdot \hat{i} = \dfrac{\partial w}{\partial x}$

Geometry $\quad \left.\dfrac{dw}{ds}\right|_{\hat{u}} = \nabla w \cdot \hat{u} = |\nabla w|\underbrace{|\hat{u}|}_{1} \cdot \cos\theta$



$\Rightarrow$ maximum when $\cos\theta = 1 \Rightarrow \theta = 0 \Rightarrow \hat{u} = \text{dir}(\nabla w) \Rightarrow$ dir of $\nabla \vec{w}$ is dir of fastest increase of w

$\Rightarrow$ minimum when $\theta = 180°$

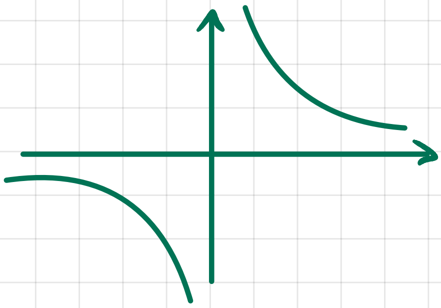
$\Rightarrow \left.\dfrac{dw}{ds}\right|_{\hat{u}} = 0 \Rightarrow \theta = 90° \Rightarrow \hat{u} \perp \nabla w \iff \hat{u}$ tangent to level

Lecture 13

Lagrange Multipliers

goal: min/max function $f(x,y,z)$ where x,y,z are not independent: $g(x,y,z)=c$

Ex: point closest to origin on hyperbola $xy=3$

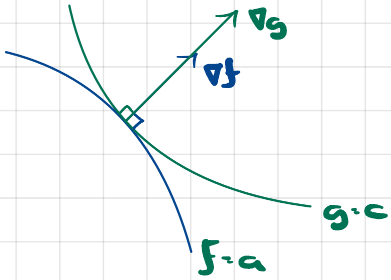


min $f(x,y)=\sqrt{x^2+y^2}$

better: min $f(x,y)=x^2+y^2$ subject to $xy=g(x,y)=3$

observe: at the minimum, the level curve of f is tangent to the hyperbola $g=3$

How to find (x,y) where level curves of f and g are tangent to each other?



$\nabla f \parallel \nabla g$ so $\nabla f=\lambda\nabla g$

$\to$ min/max with 2 variables x,y and constraint $g(x,y)=c$ becomes a system of eq. $\nabla f=\lambda\nabla g$

$$\begin{cases} f_x=\lambda g_x \\ f_y=\lambda g_x \\ g(x,y)=c \end{cases}$$

three eq., three vars

Ex: $f=x^2+y^2$, $g=xy$

$$\begin{aligned} f_x&=\lambda g_x &\Rightarrow\ 2x&=\lambda y &\Rightarrow\ 2x-\lambda y&=0 \\ f_y&=\lambda g_y &\Rightarrow\ 2y&=\lambda x &\Rightarrow\ \lambda x-2y&=0 \\ g&=3 & xy&=3 & xy&=3 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 2 & -\lambda \\ \lambda & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

* trivial solution does not satisfy constraint $xy=3$ ie $0\neq 3$

other solutions exist $\Leftrightarrow$ $\det(M)=0$

$\det M=-4+\lambda^2=0 \Leftrightarrow \lambda^2=4 \Leftrightarrow \lambda=\pm 2$

$\lambda=2 \Rightarrow x=y \Rightarrow x^2=3 \Rightarrow x=y=\pm\sqrt{3} \Rightarrow (\sqrt{3},\sqrt{3}),(-\sqrt{3},-\sqrt{3})$ are candidate solutions.

$\lambda=-2 \Rightarrow x=-y \Rightarrow -x^2=3$ no solutions

Why is this method valid?

At constrained min or max, in any direction along level $g=c$, the rate of change of f must be zero.

For any direction $\hat{u}$ tangent to $g=c$ we must have $\frac{df}{ds}\big|_{\hat{u}} = 0$.

$$\frac{df}{ds}\Big|_{\hat{u}} = \nabla f \cdot \hat{u}$$

$\Rightarrow$ any such $\hat{u} \perp \nabla f$

$\Rightarrow$ $\nabla f \perp$ level set of g

we know $\nabla g \perp$ level set of g $\Rightarrow$ $\nabla f \parallel \nabla g$

Warning: method doesn't tell whether solution is min or max!

$\to$ can't use second derivative test

To find min or max we compare values of f at various solutions to lagrange equations.

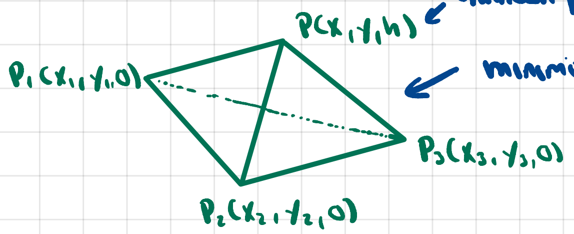
Advanced Example

Want to build a pyramid with given triangular base and given volume.

goal: min total surface area

$V = \frac{1}{3} A_b \cdot h$ ↗ fixed because V and A_b given

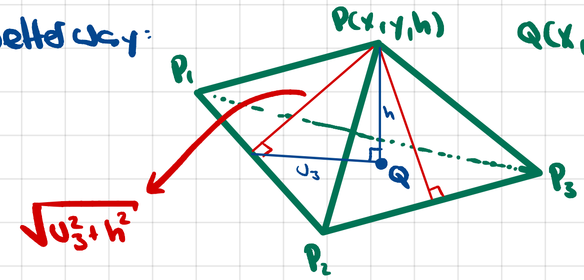
could do:



↙ minimize sum of three triangular sides.

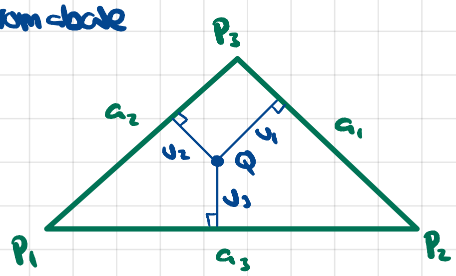
$\Rightarrow$ very complicated

better way:



$Q(x,y,0)$

from above

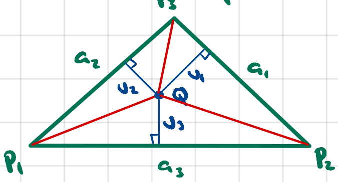


let u_1, u_2, u_3 = distance from Q to sides

heights of faces: $\sqrt{u_1^2+h^2}, \ldots$

side area $= \frac{1}{2}a_1\sqrt{u_1^2+h^2} + \frac{1}{2}a_2\sqrt{u_2^2+h^2} + \frac{1}{2}a_3\sqrt{u_3^2+h^2} = f(u_1,u_2,u_3)$ ↙ f

constraint obtained by dividing base into three triangles:



$A_{base} = \frac{1}{2}a_1u_1 + \frac{1}{2}a_2u_2 + \frac{1}{2}a_3u_3$ $\to g$

To recap the problem:

minimize $f(v_1,v_2,v_3) = \frac{1}{2}a_1\sqrt{v_1^2+h^2} + \frac{1}{2}a_2\sqrt{v_2^2+h^2} + \frac{1}{2}a_3\sqrt{v_3^2+h^2}$

subject to $g(v_1,v_2,v_3) = \frac{1}{2}a_1v_1 + \frac{1}{2}a_2v_2 + \frac{1}{2}a_3v_3 = A_{base}$

$$f_{v_i} = \frac{1}{2}a_i \frac{1}{\cancel{2}\sqrt{v_i^2+h^2}} \cdot \cancel{2}v_i = \frac{a_iv_i}{2\sqrt{v_i^2+h^2}}$$

$$g_{v_i} = \frac{1}{2}a_i$$

$$\nabla f = \lambda \nabla g \Rightarrow \frac{\cancel{a_i}v_i}{\cancel{2}\sqrt{v_i+h^2}} = \frac{\lambda\,\cancel{a_i}}{\cancel{2}} \Rightarrow \lambda = \frac{v_i}{\sqrt{v_i+h^2}}$$

$$\Rightarrow v_1 = v_2 = v_3$$

$\Rightarrow$ Q should be equidistant from the sides, Q is the incenter.

Lecture 14 Non-independent Variables

Ex $f(P,V,T)$ where $PV = nRT$

Ex $f(x,y,z)$ where $g(x,y,z) = c$

$\Rightarrow$ if $g(x,y,z) = c$ then $z = z(x,y)$.

How to find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$?

Ex: $g = x^2 + yz + z^3 = 8$ at $(2,3,1)$

Take differential

$$2x\,dx + z\,dy + (y + 3z^2)\,dz = 0 \quad \swarrow dg = 0$$

at $(2,3,1)$ $4dx + dy + 6dz = 0$

if we view $z = z(x,y)$: $dz = -\frac{1}{6}(4dx + dy)$

$\partial z / \partial x = -\frac{4}{6} = -\frac{2}{3}$

$\searrow$ y constant, dy constant

$\partial z / \partial y = -1/6$

In general, $g(x,y,z) = c$ then $dg = g_x dx + g_y dy + g_z dz = 0$

Solve for dz: $dz = \frac{-g_x}{g_z} dx - \frac{g_y}{g_z} dy$

So $\frac{\partial z}{\partial x}$: set $dy = 0 \Rightarrow dz = -\frac{g_x}{g_z} dx \Rightarrow \frac{\partial z}{\partial x} = \frac{-g_x}{g_z}$

Say $f(x,y) = x + y$ $\quad \partial f / \partial x = 1$

change of variables $x = u$, $y = u + v$ $\Rightarrow f = x + y = 2u + v$

$\frac{\partial f}{\partial u} = 2$

this notation doesn't make clear what is kept constant

so, $x = u$ but $\partial f / \partial x \neq \partial f / \partial u$

chg $u = x$, keep y constant

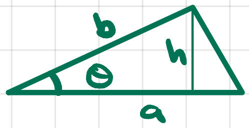
chg $u = x$, keep $v = y - x$ constant

Need clearer notation $\left(\frac{\partial f}{\partial x}\right)_y$ = keep y constant

$\left(\frac{\partial f}{\partial u}\right)_v$ = keep v constant

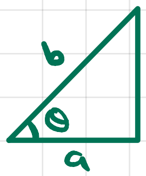
$\Rightarrow \left(\frac{\partial f}{\partial x}\right)_y \neq \left(\frac{\partial f}{\partial x}\right)_v = \left(\frac{\partial f}{\partial u}\right)_v$

Ex: area of triangle



$\underbrace{A = \frac{1}{2}ab\sin\theta}_{f(a,b,\theta)}$

Assume now it's a right triangle



$\Rightarrow \underbrace{a = b\cos\theta}_{\text{constraint}}$

what is rate of chg of A w/ respect to θ?

→ if a, b, θ independent: $\partial A/\partial\theta$, ie no constraint $(\partial A/\partial\theta)_{a,b} = \frac{1}{2}ab\cos\theta$
⇒ lose right-triangle property

→ we want the right triangle property

option 1: chg θ keep a constant ⇒ $b = b(a,\theta) = a/\cos\theta$ so we keep right angle
$(\partial A/\partial\theta)_a$

option 2: chg θ keep b constant ⇒ $a = a(b,\theta) = b\cos\theta$
$(\partial A/\partial\theta)_b$

These derivatives are definitions at this point, we don't know how to compute them.

compute $(\partial A/\partial\theta)_a$

method 0: not applicable in general, because can't always solve for b

→ solve for b in constraint, substitute

$a = b\cos\theta \Rightarrow b = \frac{a}{\cos\theta} = a\sec\theta$

$A(a,\theta) = \frac{1}{2}a^2\frac{\sin\theta}{\cos\theta} = \frac{1}{2}a^2\tan\theta$

$(\partial A/\partial\theta)_a = \frac{1}{2}a^2\sec^2\theta$

There are two systematic methods

method 1: differentials

→ keep a fixed $\Rightarrow da = 0$

figure out how db and $d\theta$ are related

→ constraint $a = b\cos\theta$

$$da = \cos\theta\, db - b\sin\theta\, d\theta$$

set $da = 0 \Rightarrow 0 = \cos\theta\, db - b\sin\theta\, d\theta \Rightarrow db = b\tan\theta\, d\theta$

→ $A = \frac{1}{2}ab\sin\theta \Rightarrow dA = \frac{1}{2}b\sin\theta\, \cancel{da}^{\,0} + \frac{1}{2}a\sin\theta\, db + \frac{1}{2}ab\cos\theta\, d\theta$

we will obtain dA as function solely of $d\theta$

$$dA = \frac{1}{2}a\sin\theta\,(b\tan\theta\, d\theta) + \frac{1}{2}ab\cos\theta\, d\theta = \frac{1}{2}ab(\sin\theta\tan\theta + \cos\theta)\,d\theta$$

$$= \frac{1}{2}ab\sec\theta\, d\theta$$

$$\Rightarrow \left(\partial A/\partial\theta\right)_a = \frac{1}{2}ab\sec\theta$$

Summary

→ write dA in terms of $da, db, d\theta$

→ $a = $ const $\Rightarrow$ set $da = 0$

→ diff constraint $\Rightarrow$ solve db in terms of $d\theta$

→ plug into dA, get answer

method 2: chain rule $(\partial/\partial\theta)_a$ in formula for A

$$\left(\partial A/\partial\theta\right)_a = A_\theta \cancel{\left(\partial\theta/\partial\theta\right)_a}^{\,1} + A_a \cancel{\left(\partial a/\partial\theta\right)_a}^{\,0} + A_b \underbrace{\left(\partial b/\partial\theta\right)_a}_{\text{use constraint}}$$

Lecture 15

Topics for Exam 2

- → functions of several variables
  - → contour plots
  - → partial derivatives
  - → gradient $\nabla f = \langle f_x, f_y, f_z \rangle$
    - → approximation $\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z = \nabla f \cdot \Delta \vec{r}$
      - = tangent plane approximation
      - * tangent plane to surface $f(x,y,z) = c$
      - normal vector = ∇f
- → min/max problems
  - → critical point ⇒ all partial deriv. = 0
  - → second derivative test for function of 2 variables
- → differentials
  - → $df = f_x\,dx + f_y\,dy + f_z\,dz$

    ↓ chain rule

    $x = x(u,v),\ y = y(u,v),\ z = z(u,v)$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u}$$

- → directional derivatives
- → non-independent variables $g(x,y,z) = c$
  - → min/max problems: Lagrange multipliers, min/max f s.t. constraint $g = c$

    $\nabla f = \lambda \nabla g,\ g = c$
- → constrained partial derivatives

  $f(x,y,z)$ where $g(x,y,z) = c$

  To find $(\partial f / \partial z)_y$ ↘ y const

  z varies

  $x = x(y,z)$

1) using differentials

$$df = f_x dx + f_y \cancel{dy}^{\,0} + f_z dz$$

y const $\Rightarrow dy = 0$

need dx in terms of dz

plug into df eq

$$dg = g_x dx + \cancel{g_y dy}^{\,0} + g_z dz = 0 \Rightarrow dx = \frac{-g_z}{g_x} dz$$

$$\Rightarrow df = f_x\left(-\frac{g_z}{g_x}\right)dz + f_z dz = \underbrace{\left[-f_x\frac{g_z}{g_x} + f_z\right]}_{(\partial f/\partial z)_y} dz$$

2) using chain rule

$$(\partial f/\partial z)_y = \frac{\partial f}{\partial x}\underbrace{\left(\frac{\partial x}{\partial z}\right)_y}_{\text{use constraint to find this}} + \frac{\partial f}{\partial y}\cancel{\left(\frac{\partial y}{\partial z}\right)_y}^{\,0} + \frac{\partial f}{\partial z}\underbrace{\left(\frac{\partial z}{\partial z}\right)_y}_{1}$$

$$(\partial g/\partial z)_y = 0 = \frac{\partial g}{\partial x}\left(\frac{\partial x}{\partial z}\right)_y + \frac{\partial g}{\partial y}\cancel{\left(\frac{\partial y}{\partial z}\right)_y}^{\,0} + \frac{\partial g}{\partial z}\overbrace{\left(\frac{\partial z}{\partial z}\right)_y}^{1}$$

$$0 = g_x(\partial x/\partial z)_y + g_z \Rightarrow (\partial x/\partial z)_y = -g_z/g_x$$

$$\Rightarrow (\partial f/\partial z)_y = \frac{\partial f}{\partial x}\left(-\frac{g_z}{g_x}\right) + \frac{\partial f}{\partial z}$$

Partial Differential Equations

→ equations involving the partial derivatives of an unknown function

ex: heat equation $\quad \frac{\partial f}{\partial t} = k\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right]$

↳ solved by $f(x,y,z,t)$ = temp at pos x,y,z time t

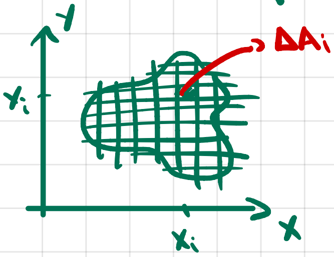
Lecture 16 Double Integrals

remember: $\int_a^b f(x)\,dx$ = area below graph of f over $[a,b]$

→ double integral = volume below graph $z = f(x,y)$ over a region R in xy-plane

$$\iint_R f(x,y)\,dA$$

→ cut R into small pieces of area ΔA



$$\sum_i f(x_i, y_i)\,\Delta A_i$$

Take limit as $\Delta A_i \to 0$, get $\iint$

To compute $\iint_R f(x,y)\,dA$ take slices

Let $S(x)$ = area of slice by plane $\parallel$ yz plane

$$\text{Volume} = \int_{x_{min}}^{x_{max}} S(x)\,dx$$

For given x, $S(x) = \int_{y_{min}(x)}^{y_{max}(x)} f(x,y)\,dy$

iterated integral

$$\iint_R f(x,y)\,dA = \int_{x_{min}}^{x_{max}} \int_{y_{min}(x)}^{y_{max}(x)} f(x,y)\,dy\,dx$$

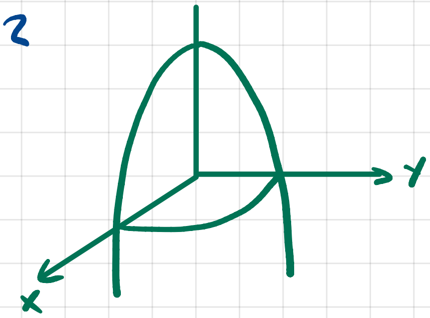
Example $z = 1 - x^2 - y^2 \quad 0 \le x \le 1,\ 0 \le y \le 1$

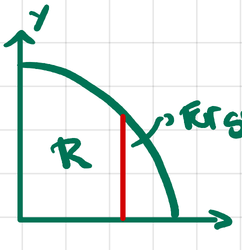
$$\int_0^1 \int_0^1 (1 - x^2 - y^2)\,dy\,dx = \int_0^1 \left[y - x^2 y - \frac{y^3}{3}\right]_0^1 dx = \int_0^1 \left[1 - x^2 - \frac{1}{3}\right] dx$$

$$= \int_0^1 \left[\frac{2}{3} - x^2\right] dx = \left(\frac{2}{3}x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

* $dA = dy\,dx$ because $\Delta A_i = \Delta x_i\,\Delta y_i$

Example 2





→ For given x, range of y is $y=0$ to $\sqrt{1-x^2}$

$x^2+y^2=1$, $x,y\geq 0$

R: quarter disk

$$\iint_R (1-x^2-y^2)\,dA$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} (1-x^2-y^2)\,dy\,dx = \int_0^1 \frac{2}{3}(1-x^2)^{3/2}\,dx$$

$$= (\ldots) = \pi/8$$

$$\int_0^{\sqrt{1-x^2}} (1-x^2-y^2)\,dy$$

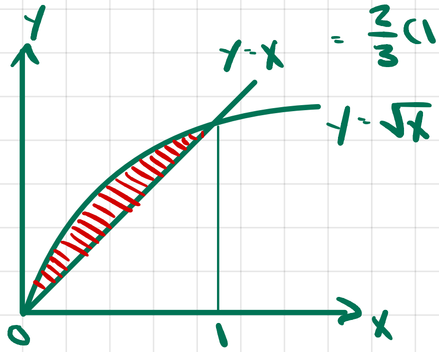
$$= \left[y(1-x^2) - \frac{y^3}{3}\right]_0^{\sqrt{1-x^2}}$$

$$= (1-x^2)^{3/2} - \frac{(1-x^2)^{3/2}}{3}$$

$$= \frac{2}{3}(1-x^2)^{3/2}$$

Exchanging order of Integration

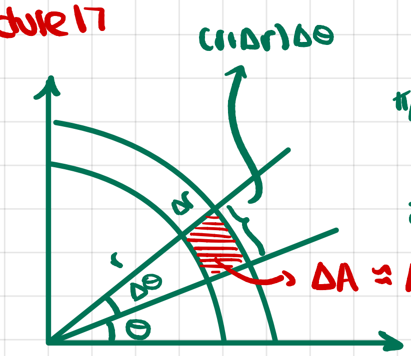
Ex: $$\int_0^1\int_x^{\sqrt{x}} \frac{e^y}{y}\,dy\,dx$$



$$= \int_0^1\int_{y^2}^{y} \frac{e^y}{y}\,dx\,dy$$

$$= \int_0^1 \frac{e^y}{y}(y-y^2)\,dy = \int_0^1 (e^y - e^y y)\,dy = (\ldots) = e-2$$

Lecture 17



$\Delta A \approx \Delta r \cdot (r+\Delta r)\Delta\theta$

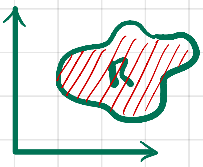
$$\int_0^{\pi/2}\int_0^1 f \cdot r\,dr\,d\theta$$

$dA = r\,dr\,d\theta$

ex: $f = 1 - x^2 - y^2 = 1-(x^2+y^2) = 1 - r^2$

$$\int_0^{\pi/2}\int_0^1 (1-r^2)\,r\,dr\,d\theta = (\dots) = \pi/8$$

Applications



1) Find area of region R: $\text{Area}_R = \iint_R 1\,dA$

mass of a flat object with density δ = mass per unit area

$\Delta m = \delta \Delta A$

$\text{Mass} = \iint_R \delta\,dA$

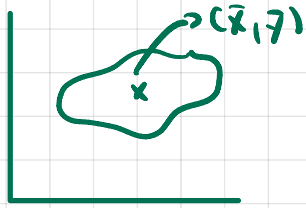
uniform average

2) Average value of f in R: Avg of $f = \bar{f} = \dfrac{\iint_R f\,dA}{\text{Area}(R)}$

Weighted Average of f with density δ: $\dfrac{1}{\text{Mass}(R)}\iint_R f\,\delta\,dA$

2a) Center of Mass of planar object w/ density δ:

$\bar{x} = \dfrac{1}{\text{mass}}\iint_R x\,\delta\,dA \qquad \bar{y} = \dfrac{1}{\text{mass}}\iint_R y\,\delta\,dA$

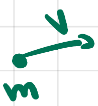


3) Moment of Inertia

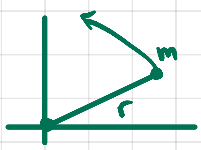
→ mass: how hard it is to impart translation motion to object

→ moment of inertia about an axis: how hard " " " rotation motion to object

idea: Kinetic energy



$\frac{1}{2}mv^2$



mass m, distance r, angular velocity ω

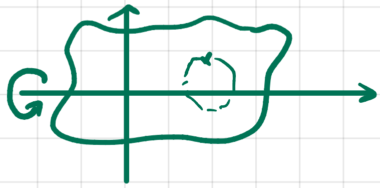
$v = r\omega \Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}\underbrace{mr^2}_{\text{moment of inertia}}\omega^2$

For a solid w/ density δ: $\Delta m \approx \delta \Delta A$, moment of inertia $\Delta m\, r^2 = r^2 \delta \Delta A$

$\Rightarrow$ moment of inertia about origin: $\iint_R \underbrace{r^2}_{x^2+y^2} \delta \, dA = I_0$

Rotational kinetic energy is $\frac{1}{2} I_0 \omega^2$

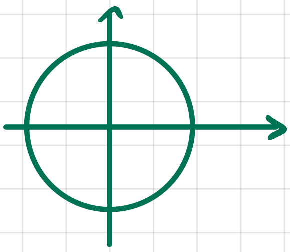
What about rotation about x-axis?



distance to x-axis is $|y|$

$I_x = \iint_R y^2 \delta \, dA$

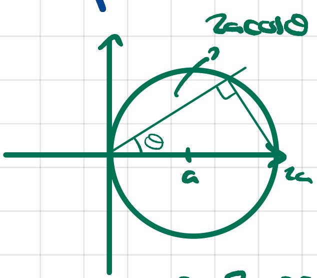
Ex:



disk, radius a, uniform density $\delta = 1$

$I_0 = \iint r^2 \, dA = \int_0^{2\pi}\int_0^{a} r^2 \, r \, dr \, d\theta$

About a point on the circumference:



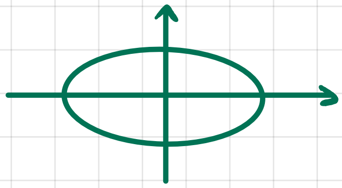
$I = \iint r^2 \, dA = \int_{-\pi/2}^{\pi/2}\int_0^{2a\cos\theta} r^2 \, r \, dr \, d\theta = (\dots) = \frac{3}{2}\pi a^4$

$r = 2a\cos\theta$

Lecture 18

Changing Variables in Double Integrals

Ex: area of ellipse w/ semiaxes a,b



$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

$$\iint dx\,dy \underset{?}{=} \iint_{u^2+v^2<1} ab\,du\,dv = ab\iint_{u^2+v^2<1} du\,dv = ab\cdot\text{Area unit disk} = ab\pi$$

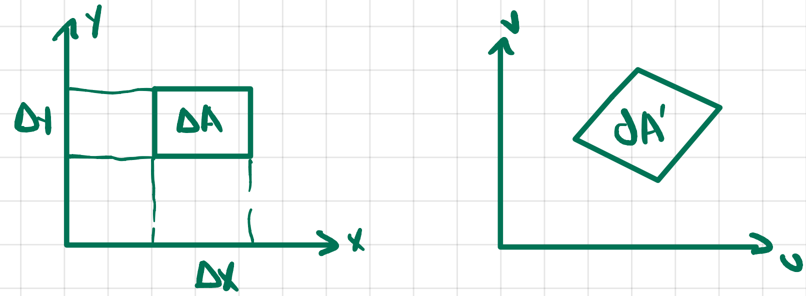
set $\frac{x}{a}=u$, $\frac{y}{b}=v$

$du=\frac{dx}{a}$ $\quad dv=\frac{dy}{b}$ $\quad\Rightarrow du\,dv=\frac{1}{ab}dx\,dy \Rightarrow dx\,dy = ab\,du\,dv$

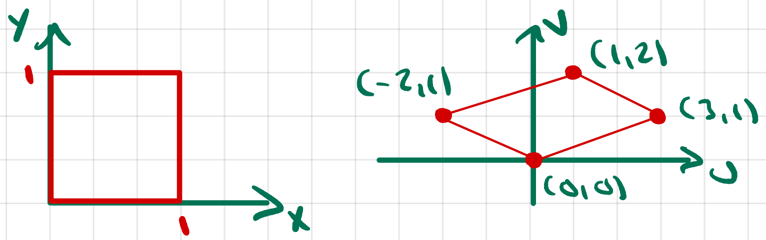
In general, need to find scaling factor ($dx\,dy$ vs $du\,dv$)

Ex: Say, $\left.\begin{array}{l} u=3x-2y \\ v=x+y \end{array}\right\}$ to simplify integrand or bounds

relation between $dA=dx\,dy$ and $dA'=du\,dv$



Area scaling factor here doesn't depend on choice of rectangle because linear chg of variables



$A=1$ $\qquad A'=\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix}=5$

For any other rectangle, area multiplied by 5

$\Rightarrow dA'=5\,dA \Rightarrow du\,dv = 5\,dx\,dy$

so $\iint \ldots dx\,dy = \iint \ldots \frac{1}{5}du\,dv$

General case

$u = u(x,y)$

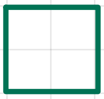
$v = v(x,y)$

$\Delta u \approx u_x \Delta x + u_y \Delta y$

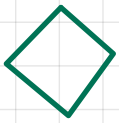
$\Delta v \approx v_x \Delta x + v_y \Delta y$

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \approx \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

small xy-rectangle

 Δy $\longleftrightarrow$ small uv-parallelogram

Δx



area = det

$\langle \Delta x, 0 \rangle \to \langle \Delta u, \Delta v \rangle \approx \langle u_x \Delta x, v_x \Delta x \rangle$

$\langle 0, \Delta y \rangle \to \langle u_y \Delta y, v_y \Delta y \rangle$

these are the sides of the parallelogram

their determinant

Jacobian: $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ determinant

Then $du\,dv = |J|\,dx\,dy = \left|\dfrac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$

absolute value

Ex: polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$dx\,dy = |r|\,dr\,d\theta = r\,dr\,d\theta$

Remark: $\dfrac{\partial(u,v)}{\partial(x,y)} \cdot \dfrac{\partial(x,y)}{\partial(u,v)} = 1$

Ex2 compute $\int_0^1\int_0^1 x^2\,dx\,dy$ by choosing $u=x,\ v=xy$

1) area element

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix}1 & 0\\ y & x\end{vmatrix} = x \qquad \text{so } du\,dv = x\,dx\,dy$$

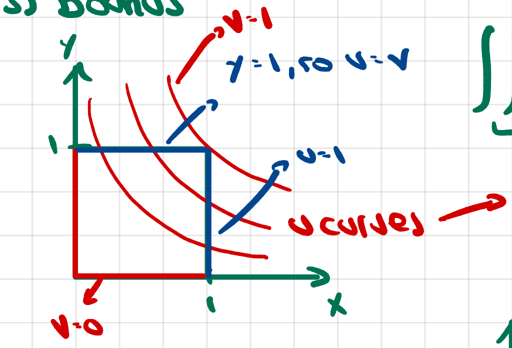
* x positive in our region $\Rightarrow |x| = x$

2) integrand in terms of u,v

$$x^2 y\,dx\,dy = x^2 y\,\frac{1}{x}\,du\,dv = xy\,du\,dv = v\,du\,dv$$

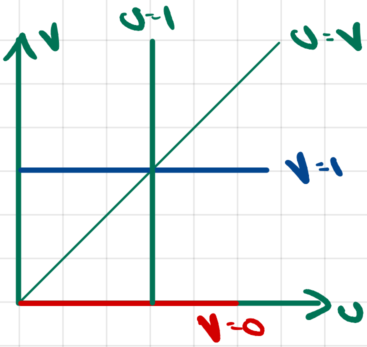
$$\iint_{???} v\,du\,dv \quad (\text{or } du\,dv)$$

3) bounds



$$\underbrace{\iint v\,du}_{}\,dv$$

u changes, v constant
$\Rightarrow xy$ = constant



$$\int_0^1\int_v^1 v\,du\,dv$$