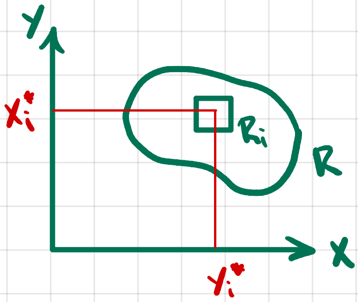
14.5 Applications of Double Integrals

Setting: lamina with variable density occupying a bounded region R in the xy-plane

density function: $\delta(x,y)$, density at (x,y) in units of mass per unit area



$P = \{R_1, \dots, R_n\}$ is an inner partition of R.

mass in R_i $\approx \delta(x_i^*, y_i^*)\Delta A_i$, ΔA_i = area(R_i)

$m \approx \sum_{i=1}^{n} \delta(x_i^*, y_i^*)\Delta A_i$, a Riemann sum.

mass of the lamina is defined as $\lim_{|P|\to 0} \sum_{i=1}^{n} \delta(x_i^*, y_i^*)\Delta A_i = \iint_R \delta(x,y)\, dA = \iint_R dm$

coordinates of centroid (center of mass):

$$\bar{x} = \frac{1}{m}\iint_R x\,\delta(x,y)\, dA \qquad \bar{y} = \frac{1}{m}\iint_R y\,\delta(x,y)\, dA$$

$$= \frac{1}{m}\iint_R x\, dm \qquad = \frac{1}{m}\iint_R y\, dm$$

$\bar{x}, \bar{y}$: average values of x and y w/ respect to mass in region R

First theorem of Pappus

Plane region R is revolved around an axis in its plane generating a solid of revolution w/ volume V, and assume the axis does not intersect R $\Rightarrow$ $V = A \cdot d$

area of R

distance traveled by the centroid of R

Centroid of Plane curves

Definitions

plane curve C, constant density $\delta \equiv 1$ $\Rightarrow$ $\bar{x} = \frac{1}{s}\int_C x\, ds \qquad \bar{y} = \frac{1}{s}\int_C y\, ds$

s = arc length of C

$$ds = \sqrt{1 + y'(x)^2}\, dx \quad \text{or} \quad \sqrt{1 + x'(y)^2}\, dy$$

$$= \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Second theorem of Pappus

curve C revolved around an axis in its plane. Axis does not intersect C. $\Rightarrow$ $A = s \cdot d$

arc length

distance traveled by centroid

Moments of Inertia

R a plane lamina

L a straight line that may or may not be in the xy plane

$\Rightarrow$ moment of inertia I of R around the axis L is defined

$$\iint_R p^2 \, dm$$

perpendicular distance to L
$p = p(x, y)$

- if L is the z-axis then $p = \sqrt{x^2 + y^2} = r$

$\Rightarrow I_0 = \iint_R r^2 \delta(x,y) \, dA = \iint_R (x^2 + y^2) \delta(x,y) \, dA = \iint_R (x^2 + y^2) \, dm$

polar moment of inertia of lamina R

But $\iint_R x^2 \, dm = \iint p_x^2 \, dm = I_x$ = polar moment around x-axis

$\iint_R y^2 \, dm = I_y$

$\Rightarrow I_0 = I_x + I_y$