

## 1. Vectors and Matrices

## 1A. Vectors

**Definition.** A **direction** is just a unit vector. The **direction of  $\mathbf{A}$**  is defined by

$$\text{dir } \mathbf{A} = \frac{\mathbf{A}}{|\mathbf{A}|}, \quad (\mathbf{A} \neq \mathbf{0});$$

it is the unit vector lying along  $\mathbf{A}$  and pointed like  $\mathbf{A}$  (not like  $-\mathbf{A}$ ).

~~1A-1~~ Find the magnitude and direction (see the definition above) of the vectors

a)  $\mathbf{i} + \mathbf{j} + \mathbf{k}$     b)  $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$     c)  $3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$

1A-2 For what value(s) of  $c$  will  $\frac{1}{5}\mathbf{i} - \frac{1}{5}\mathbf{j} + c\mathbf{k}$  be a unit vector?

1A-3 a) If  $P = (1, 3, -1)$  and  $Q = (0, 1, 1)$ , find  $\mathbf{A} = PQ$ ,  $|\mathbf{A}|$ , and  $\text{dir } \mathbf{A}$ .

b) A vector  $\mathbf{A}$  has magnitude 6 and direction  $(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})/3$ . If its tail is at  $(-2, 0, 1)$ , where is its head?

~~1A-4~~ a) Let  $P$  and  $Q$  be two points in space, and  $X$  the midpoint of the line segment  $PQ$ . Let  $O$  be an arbitrary fixed point; show that as vectors,  $OX = \frac{1}{2}(OP + OQ)$ .

b) With the notation of part (a), assume that  $X$  divides the line segment  $PQ$  in the ratio  $r : s$ , where  $r + s = 1$ . Derive an expression for  $OX$  in terms of  $OP$  and  $OQ$ .

~~1A-5~~ What are the  $\mathbf{i}, \mathbf{j}$ -components of a plane vector  $\mathbf{A}$  of length 3, if it makes an angle of  $30^\circ$  with  $\mathbf{i}$  and  $60^\circ$  with  $\mathbf{j}$ . Is the second condition redundant?

~~1A-6~~ A small plane wishes to fly due north at 200 mph (as seen from the ground), in a wind blowing from the northeast at 50 mph. Tell with what vector velocity in the air it should travel (give the  $\mathbf{i}, \mathbf{j}$ -components).

~~1A-7~~ Let  $\mathbf{A} = a\mathbf{i} + b\mathbf{j}$  be a plane vector; find in terms of  $a$  and  $b$  the vectors  $\mathbf{A}'$  and  $\mathbf{A}''$  resulting from rotating  $\mathbf{A}$  by  $90^\circ$     a) clockwise    b) counterclockwise.

(Hint: make  $\mathbf{A}$  the diagonal of a rectangle with sides on the  $x$  and  $y$ -axes, and rotate the whole rectangle.)

c) Let  $\mathbf{i}' = (3\mathbf{i} + 4\mathbf{j})/5$ . Show that  $\mathbf{i}'$  is a unit vector, and use the first part of the exercise to find a vector  $\mathbf{j}'$  such that  $\mathbf{i}', \mathbf{j}'$  forms a right-handed coordinate system.

~~1A-8~~ The direction (see definition above) of a space vector is in engineering practice often given by its **direction cosines**. To describe these, let  $\mathbf{A} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$  be a space vector, represented as an origin vector, and let  $\alpha, \beta$ , and  $\gamma$  be the three angles ( $\leq \pi$ ) that  $\mathbf{A}$  makes respectively with  $\mathbf{i}, \mathbf{j}$ , and  $\mathbf{k}$ .

a) Show that  $\text{dir } \mathbf{A} = \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}$ . (The three coefficients are called the *direction cosines* of  $\mathbf{A}$ .)

b) Express the direction cosines of  $\mathbf{A}$  in terms of  $a, b, c$ ; find the direction cosines of the vector  $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ .

c) Prove that three numbers  $t, u, v$  are the direction cosines of a vector in space if and only if they satisfy  $t^2 + u^2 + v^2 = 1$ .

~~1A-9~~ Prove using vector methods (without components) that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length. (Call the two sides **A** and **B**.)

**1A-10** Prove using vector methods (without components) that the midpoints of the sides of a space quadrilateral form a parallelogram.

~~1A-11~~ Prove using vector methods (without components) that the diagonals of a parallelogram bisect each other. (One way: let  $X$  and  $Y$  be the midpoints of the two diagonals; show  $X = Y$ .)

**1A-12\*** Label the four vertices of a parallelogram in counterclockwise order as  $OPQR$ . Prove that the line segment from  $O$  to the midpoint of  $PQ$  intersects the diagonal  $PR$  in a point  $X$  that is  $1/3$  of the way from  $P$  to  $R$ .

(Let  $\mathbf{A} = \overrightarrow{OP}$ , and  $\mathbf{B} = \overrightarrow{OR}$ ; express everything in terms of  $\mathbf{A}$  and  $\mathbf{B}$ .)

**1A-13\*** a) Take a triangle  $PQR$  in the plane; prove that as vectors  $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP} = \mathbf{0}$ .

b) Continuing part a), let  $\mathbf{A}$  be a vector the same length as  $\overrightarrow{PQ}$ , but perpendicular to it, and pointing outside the triangle. Using similar vectors  $\mathbf{B}$  and  $\mathbf{C}$  for the other two sides, prove that  $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$ . (This only takes one sentence, and no computation.)

**1A-14\*** Generalize parts a) and b) of the previous exercise to a closed polygon in the plane which doesn't cross itself (i.e., one whose interior is a single region); label its vertices  $P_1, P_2, \dots, P_n$  as you walk around it.

**1A-15\*** Let  $P_1, \dots, P_n$  be the vertices of a regular  $n$ -gon in the plane, and  $O$  its center; show without computation or coordinates that  $\overrightarrow{OP_1} + \overrightarrow{OP_2} + \dots + \overrightarrow{OP_n} = \mathbf{0}$ ,

a) if  $n$  is even;      b) if  $n$  is odd.

## 1B. Dot Product

**1B-1** Find the angle between the vectors

a)  $\mathbf{i} - \mathbf{k}$  and  $4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$       b)  $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$  and  $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ .

~~1B-2~~ Tell for what values of  $c$  the vectors  $c\mathbf{i} + 2\mathbf{j} - \mathbf{k}$  and  $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$  will

a) be orthogonal      b) form an acute angle

~~1B-3~~ Using vectors, find the angle between a longest diagonal  $PQ$  of a cube, and

a) a diagonal  $PR$  of one of its faces;      b) an edge  $PS$  of the cube.

(Choose a size and position for the cube that makes calculation easiest.)

~~1B-4~~ Three points in space are  $P : (a, 1, -1)$ ,  $Q : (0, 1, 1)$ ,  $R : (a, -1, 3)$ . For what value(s) of  $a$  will  $PQR$  be

a) a right angle      b) an acute angle ?

~~1B-5~~ Find the component of the force  $\mathbf{F} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$  in

a) the direction  $\frac{\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{3}}$       b) the direction of the vector  $3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$ .

**1B-6** Let  $O$  be the origin,  $c$  a given number, and  $\mathbf{u}$  a given direction (i.e., a unit vector). Describe geometrically the locus of all points  $P$  in space that satisfy the vector equation

$$OP \cdot \mathbf{u} = c|OP|.$$

In particular, tell for what value(s) of  $c$  the locus will be (Hint: divide through by  $|OP|$ ):

- a) a plane      b) a ray (i.e., a half-line)      c) empty

**1B-7** a) Verify that  $\mathbf{i}' = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}$  and  $\mathbf{j}' = \frac{-\mathbf{i} + \mathbf{j}}{\sqrt{2}}$  are perpendicular unit vectors that form a right-handed coordinate system

b) Express the vector  $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j}$  in the  $\mathbf{i}'\mathbf{j}'$ -system by using the dot product.

c) Do b) a different way, by solving for  $\mathbf{i}$  and  $\mathbf{j}$  in terms of  $\mathbf{i}'$  and  $\mathbf{j}'$  and then substituting into the expression for  $\mathbf{A}$ .

**1B-8** The vectors  $\mathbf{i}' = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}$ ,  $\mathbf{j}' = \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}$ , and  $\mathbf{k}' = \frac{\mathbf{i} + \mathbf{j} - 2\mathbf{k}}{\sqrt{6}}$  are three mutually perpendicular unit vectors that form a right-handed coordinate system.

- a) Verify this.      b) Express  $\mathbf{A} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$  in this system (cf. 1B-7b)

**1B-9** Let  $\mathbf{A}$  and  $\mathbf{B}$  be two plane vectors, neither one of which is a multiple of the other. Express  $\mathbf{B}$  as the sum of two vectors, one a multiple of  $\mathbf{A}$ , and the other perpendicular to  $\mathbf{A}$ ; give the answer in terms of  $\mathbf{A}$  and  $\mathbf{B}$ .

(Hint: let  $\mathbf{u} = \text{dir } \mathbf{A}$ ; what's the  $\mathbf{u}$ -component of  $\mathbf{B}$ ?)

~~**1B-10**~~ Prove using vector methods (without components) that the diagonals of a parallelogram have equal lengths if and only if it is a rectangle.

~~**1B-11**~~ Prove using vector methods (without components) that the diagonals of a parallelogram are perpendicular if and only if it is a rhombus, i.e., its four sides are equal.

~~**1B-12**~~ Prove using vector methods (without components) that an angle inscribed in a semicircle is a right angle.

~~**1B-13**~~ Prove the trigonometric formula:  $\cos(\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2$ .

(Hint: consider two unit vectors making angles  $\theta_1$  and  $\theta_2$  with the positive  $x$ -axis.)

**1B-14** Prove the law of cosines:  $c^2 = a^2 + b^2 - 2ab \cos \theta$  by using the algebraic laws for the dot product and its geometric interpretation.

**1B-15\*** The **Cauchy-Schwarz inequality**

a) Prove from the geometric definition of the dot product the following inequality for vectors in the plane or space; under what circumstances does equality hold?

$$(*) \quad |\mathbf{A} \cdot \mathbf{B}| \leq |\mathbf{A}||\mathbf{B}|.$$

b) If the vectors are plane vectors, write out what this inequality says in terms of  $\mathbf{i}\mathbf{j}$ -components.

c) Give a different argument for the inequality (\*) as follows (this argument generalizes to  $n$ -dimensional space):

- i) for all values of  $t$ , we have  $(\mathbf{A} + t\mathbf{B}) \cdot (\mathbf{A} + t\mathbf{B}) \geq 0$  ;
- ii) use the algebraic laws of the dot product to write the expression in (i) as a quadratic polynomial in  $t$  ;
- iii) by (i) this polynomial has at most one zero; this implies by the quadratic formula that its coefficients must satisfy a certain inequality — what is it?

### 1C. Determinants

~~1C-1~~ Calculate the value of the determinants a)  $\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix}$  b)  $\begin{vmatrix} 3 & -4 \\ -1 & -2 \end{vmatrix}$

~~1C-2~~ Calculate  $\begin{vmatrix} -1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & -2 & -1 \end{vmatrix}$  using the Laplace expansion by the cofactors of:

- a) the first row b) the first column

~~1C-3~~ Find the area of the plane triangle whose vertices lie at

- a)  $(0, 0), (1, 2), (1, -1)$ ; b)  $(1, 2), (1, -1), (2, 3)$ .

~~1C-4~~ Show that  $\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$ .

(This type of determinant is called a **Vandermonde** determinant.)

~~1C-5~~ a) Show that the value of a  $2 \times 2$  determinant is unchanged if you add to the second row a scalar multiple of the first row.

- b) Same question, with “row” replaced by “column”.

~~1C-6~~ Use a Laplace expansion and Exercise 5a to show the value of a  $3 \times 3$  determinant is unchanged if you add to the second row a scalar multiple of the third row.

~~1C-7~~ Let  $(x_1, y_1)$  and  $(x_2, y_2)$  both range over all unit vectors.

Find the maximum value of the function  $f(x_1, x_2, y_1, y_2) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ .

~~1C-8\*~~ The base of a parallelepiped is a parallelogram whose edges are the vectors  $\mathbf{b}$  and  $\mathbf{c}$ , while its third edge is the vector  $\mathbf{a}$ . (All three vectors have their tail at the same vertex; one calls them “coterminal”.)

- a) Show that the volume of the parallelepiped  $\mathbf{abc}$  is  $\pm \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ .

- b) Show that  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) =$  the determinant whose rows are respectively the components of the vectors  $\mathbf{a}, \mathbf{b}, \mathbf{c}$ .

(These two parts prove the volume interpretation of a  $3 \times 3$  determinant.)

**1C-9** Use the formula in Exercise 1C-8 to calculate the volume of a tetrahedron having as vertices  $(0, 0, 0)$ ,  $(0, -1, 2)$ ,  $(0, 1, -1)$ ,  $(1, 2, 1)$ . (The volume of a tetrahedron is  $\frac{1}{3}(\text{base})(\text{height})$ .)

**1C-10** Show by using Exercise 8 that if three origin vectors lie in the same plane, the determinant having the three vectors as its three rows has the value zero.

### 1D. Cross Product

~~**1D-1**~~ Find  $\mathbf{A} \times \mathbf{B}$  if

a)  $\mathbf{A} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ ,  $\mathbf{B} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$       b)  $\mathbf{A} = 2\mathbf{i} - 3\mathbf{k}$ ,  $\mathbf{B} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ .

~~**1D-2**~~ Find the area of the triangle in space having its vertices at the points

$$P : (2, 0, 1), \quad Q : (3, 1, 0), \quad R : (-1, 1, -1).$$

~~**1D-3**~~ Two vectors  $\mathbf{i}'$  and  $\mathbf{j}'$  of a right-handed coordinate system are to have the directions respectively of the vectors  $\mathbf{A} = 2\mathbf{i} - \mathbf{j}$  and  $\mathbf{B} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ . Find all three vectors  $\mathbf{i}'$ ,  $\mathbf{j}'$ ,  $\mathbf{k}'$ .

~~**1D-4**~~ Verify that the cross product  $\times$  does not in general satisfy the associative law, by showing that for the particular vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , we have  $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} \neq \mathbf{i} \times (\mathbf{i} \times \mathbf{j})$ .

~~**1D-5**~~ What can you conclude about  $\mathbf{A}$  and  $\mathbf{B}$

a) if  $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}|$ ;      b) if  $|\mathbf{A} \times \mathbf{B}| = \mathbf{A} \cdot \mathbf{B}$ .

**1D-6** Take three faces of a unit cube having a common vertex  $P$ ; each face has a diagonal ending at  $P$ ; what is the volume of the parallelepiped having these three diagonals as coterminous edges?

~~**1D-7**~~ Find the volume of the tetrahedron having vertices at the four points

$$P : (1, 0, 1), \quad Q : (-1, 1, 2), \quad R : (0, 0, 2), \quad S : (3, 1, -1).$$

Hint: volume of tetrahedron  $= \frac{1}{6}(\text{volume of parallelepiped with same 3 coterminous edges})$

**1D-8** Prove that  $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ , by using the determinantal formula for the scalar triple product, and the algebraic laws of determinants in Notes D.

**1D-9** Show that the area of a triangle in the  $xy$ -plane having vertices at  $(x_i, y_i)$ , for

$$i = 1, 2, 3, \text{ is given by the determinant } \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}. \text{ Do this two ways:}$$

a) by relating the area of the triangle to the volume of a certain parallelepiped

b) by using the laws of determinants (p. L.1 of the notes) to relate this determinant to the  $2 \times 2$  determinant that would normally be used to calculate the area.

**1E. Equations of Lines and Planes**~~1E-1~~ Find the equations of the following planes:

- a) through  $(2, 0, -1)$  and perpendicular to  $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$
- b) through the origin,  $(1, 1, 0)$ , and  $(2, -1, 3)$
- c) through  $(1, 0, 1)$ ,  $(2, -1, 2)$ ,  $(-1, 3, 2)$
- d) through the points on the  $x$ ,  $y$  and  $z$ -axes where  $x = a$ ,  $y = b$ ,  $z = c$  respectively (give the equation in the form  $Ax + By + Cz = 1$  and remember it)
- e) through  $(1, 0, 1)$  and  $(0, 1, 1)$  and parallel to  $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

~~1E-2~~ Find the dihedral angle between the planes  $2x - y + z = 3$  and  $x + y + 2z = 1$ .~~1E-3~~ Find in parametric form the equations for

- a) the line through  $(1, 0, -1)$  and parallel to  $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$
- b) the line through  $(2, -1, -1)$  and perpendicular to the plane  $x - y + 2z = 3$
- c) all lines passing through  $(1, 1, 1)$  and lying in the plane  $x + 2y - z = 2$

~~1E-4~~ Where does the line through  $(0, 1, 2)$  and  $(2, 0, 3)$  intersect the plane  $x + 4y + z = 4$ ?~~1E-5~~ The line passing through  $(1, 1, -1)$  and perpendicular to the plane  $x + 2y - z = 3$  intersects the plane  $2x - y + z = 1$  at what point?~~1E-6~~ Show that the distance  $D$  from the origin to the plane  $ax + by + cz = d$  is given by the formula  $D = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$ .(Hint: Let  $\mathbf{n}$  be the unit normal to the plane. and  $P$  be a point on the plane; consider the component of  $OP$  in the direction  $\mathbf{n}$ .)~~1E-7\*~~ Formulate a general method for finding the distance between two skew (i.e., non-intersecting) lines in space, and carry it out for two non-intersecting lines lying along the diagonals of two adjacent faces of the unit cube (place it in the first octant, with one vertex at the origin).

(Hint: the shortest line segment joining the two skew lines will be perpendicular to both of them (if it weren't, it could be shortened).)

**1F. Matrix Algebra****1F-1\*** Let  $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 4 \end{pmatrix}$ ,  $B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 2 \end{pmatrix}$ ,  $C = \begin{pmatrix} 0 & 2 \\ -3 & 4 \\ 1 & 1 \end{pmatrix}$ . Compute

- a)  $B + C$ ,  $B - C$ ,  $2B - 3C$ .
- b)  $AB$ ,  $AC$ ,  $BA$ ,  $CA$ ,  $BC^T$ ,  $CB^T$
- c)  $A(B + C)$ ,  $AB + AC$ ;  $(B + C)A$ ,  $BA + CA$

**1F-2\*** Let  $A$  be an arbitrary  $m \times n$  matrix, and let  $I_k$  be the identity matrix of size  $k$ . Verify that  $I_m A = A$  and  $A I_n = A$ .**1F-3** Find all  $2 \times 2$  matrices  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  such that  $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ .

**1F-4\*** Show that matrix multiplication is not in general commutative by calculating for each pair below the matrix  $AB - BA$ :

$$\text{a) } A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{b) } A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & -1 \end{pmatrix}$$

~~**1F-5**~~ a) Let  $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ . Compute  $A^2, A^3$ . b) Find  $A^2, A^3, A^n$  if  $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ .

**1F-6\*** Let  $A, A', B, B'$  be  $2 \times 2$  matrices, and  $O$  the  $2 \times 2$  zero matrix. Express in terms of these five matrices the product of the  $4 \times 4$  matrices  $\begin{pmatrix} A & O \\ O & B \end{pmatrix} \begin{pmatrix} A' & O \\ O & B' \end{pmatrix}$ .

**1F-7\*** Let  $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ ,  $B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ . Show there are no values of  $a$  and  $b$  such that  $AB - BA = I_2$ .

~~**1F-8**~~ a) If  $A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ ,  $A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$ ,  $A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ , what is the  $3 \times 3$  matrix  $A$ ?

b)\* If  $A \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}$ ,  $A \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$ ,  $A \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$ , what is  $A$ ?

~~**1F-9**~~ A square  $n \times n$  matrix is called **orthogonal** if  $A \cdot A^T = I_n$ . Show that this condition is equivalent to saying that

- a) each row of  $A$  is a row vector of length 1,
- b) two different rows are orthogonal vectors.

**1F-10\*** Suppose  $A$  is a  $2 \times 2$  orthogonal matrix, whose first entry is  $a_{11} = \cos \theta$ . Fill in the rest of  $A$ . (There are four possibilities. Use Exercise 9.)

**1F-11\*** Show that if  $A + B$  and  $AB$  are defined, then

a)  $(A + B)^T = A^T + B^T$ , b)  $(AB)^T = B^T A^T$ .

### 1G. Solving Square Systems; Inverse Matrices

For each of the following, solve the equation  $A\mathbf{x} = \mathbf{b}$  by finding  $A^{-1}$ .

**1G-1\***  $A = \begin{pmatrix} 3 & 1 & -1 \\ -1 & 2 & 0 \\ -1 & -1 & -1 \end{pmatrix}$ ,  $\mathbf{b} = \begin{pmatrix} 8 \\ 3 \\ 0 \end{pmatrix}$ .

**1G-2\*** a)  $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$ ,  $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ ; b)  $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$ ,  $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ .

~~**1G-3**~~  $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix}$ ,  $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ . Solve  $A\mathbf{x} = \mathbf{b}$  by finding  $A^{-1}$ .

~~1G-4~~ Referring to Exercise 3 above, solve the system

$$x_1 - x_2 + x_3 = y_1, \quad x_2 + x_3 = y_2 \quad -x_1 - x_2 + 2x_3 = y_3$$

for the  $x_i$  as functions of the  $y_i$ .

~~1G-5~~ Show that  $(AB)^{-1} = B^{-1}A^{-1}$ , by using the definition of inverse matrix.

**1G-6\*** Another calculation of the inverse matrix.

If we know  $A^{-1}$ , we can solve the system  $A\mathbf{x} = \mathbf{y}$  for  $\mathbf{x}$  by writing  $\mathbf{x} = A^{-1}\mathbf{y}$ . But conversely, if we can solve by some other method (elimination, say) for  $\mathbf{x}$  in terms of  $\mathbf{y}$ , getting  $\mathbf{x} = B\mathbf{y}$ , then the matrix  $B = A^{-1}$ , and we will have found  $A^{-1}$ .

This is a good method if  $A$  is an upper or lower triangular matrix — one with only zeros respectively below or above the main diagonal. To illustrate:

$$\text{a) Let } A = \begin{pmatrix} -1 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}; \quad \text{find } A^{-1} \text{ by solving} \quad \begin{array}{l} -x_1 + x_2 + 3x_3 = y_1 \\ 2x_2 - x_3 = y_2 \\ x_3 = y_3 \end{array} \quad \text{for the } x_i$$

in terms of the  $y_i$  (start from the bottom and proceed upwards).

b) Calculate  $A^{-1}$  by the method given in the notes.

**1G-7\*** Consider the rotation matrix  $A_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  corresponding to rotation of the  $x$  and  $y$  axes through the angle  $\theta$ . Calculate  $A_\theta^{-1}$  by the adjoint matrix method, and explain why your answer looks the way it does.

**1G-8\*** a) Show:  $A$  is an orthogonal matrix (cf. Exercise 1F-9) if and only if  $A^{-1} = A^T$ .

b) Illustrate with the matrix of exercise 7 above.

c) Use (a) to show that if  $A$  and  $B$  are  $n \times n$  orthogonal matrices, so is  $AB$ .

**1G-9\*** a) Let  $A$  be a  $3 \times 3$  matrix such that  $|A| \neq 0$ . The notes construct a right-inverse  $A^{-1}$ , that is, a matrix such that  $A \cdot A^{-1} = I$ . Show that every such matrix  $A$  also has a left inverse  $B$  (i.e., a matrix such that  $BA = I$ .)

(Hint: Consider the equation  $A^T(A^T)^{-1} = I$ ; cf. Exercise 1F-11.)

b) Deduce that  $B = A^{-1}$  by a one-line argument.

(This shows that the right inverse  $A^{-1}$  is automatically the left inverse also. So if you want to check that two matrices are inverses, you only have to do the multiplication on one side — the product in the other order will automatically be  $I$  also.)

**1G-10\*** Let  $A$  and  $B$  be two  $n \times n$  matrices. Suppose that  $B = P^{-1}AP$  for some invertible  $n \times n$  matrix  $P$ . Show that  $B^n = P^{-1}A^nP$ . If  $B = I_n$ , what is  $A$ ?

**1G-11\*** Repeat Exercise 6a and 6b above, doing it this time for the general  $2 \times 2$  matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , assuming  $|A| \neq 0$ .



**1H. Theorems about Square Systems****1H-1** Use Cramer's rule to solve for  $x$  in the following:

$$\begin{array}{ll}
 3x - y + z = 1 & x - y + z = 0 \\
 \text{(a)} \quad -x + 2y + z = 2, & \text{(b)} \quad x - z = 1. \\
 x - y + z = -3 & -x + y + z = 2
 \end{array}$$

*(We did not cover Cramer's rule in this course.)***1H-2** Using Cramer's rule, give another proof that if  $A$  is an  $n \times n$  matrix whose determinant is non-zero, then the equations  $A\mathbf{x} = \mathbf{0}$  have only the trivial solution.*(We did not cover Cramer's rule in this course.)*

**1H-3** ~~a)~~ For what  $c$ -value(s) will  $\begin{matrix} x_1 - x_2 + x_3 = 0 \\ 2x_1 + x_2 + x_3 = 0 \\ -x_1 + cx_2 + 2x_3 = 0 \end{matrix}$  have a non-trivial solution?

~~b)~~ For what  $c$ -value(s) will  $\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = c \begin{pmatrix} x \\ y \end{pmatrix}$  have a non-trivial solution?  
 (Write it as a system of homogeneous equations.)

~~c)~~ For each value of  $c$  in part (a), find a non-trivial solution to the corresponding system. (Interpret the equations as asking for a vector orthogonal to three given vectors; find it by using the cross product.)

d)\* For each value of  $c$  in part (b), find a non-trivial solution to the corresponding system.

**1H-4\*** Find all solutions to the homogeneous system  $\begin{matrix} x - 2y + z = 0 \\ x + y - z = 0 \\ 3x - 3x + z = 0 \end{matrix}$  ;

use the method suggested in Exercise 3c above.

**1H-5** Suppose that for the system  $\begin{matrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{matrix}$  we have  $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$ . Assume that  $a_1 \neq 0$ . Show that the system is consistent (i.e., has solutions) if and only if  $c_2 = \frac{a_2}{a_1}c_1$ .

**1H-6\*** Suppose  $|A| = 0$ , and that  $\mathbf{x}_1$  is a particular solution of the system  $A\mathbf{x} = B$ . Show that any other solution  $\mathbf{x}_2$  of this system can be written as  $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{x}_0$ , where  $\mathbf{x}_0$  is a solution of the system  $A\mathbf{x} = \mathbf{0}$ .

**1H-7** Suppose we want to find a pure oscillation (sine wave) of frequency 1 passing through two given points. In other words, we want to choose constants  $a$  and  $b$  so that the function

$$f(x) = a \cos x + b \sin x$$

has prescribed values at two given  $x$ -values:  $f(x_1) = y_1$ ,  $f(x_2) = y_2$ .

a) Show this is possible in one and only one way, if we assume that  $x_2 \neq x_1 + n\pi$ , for every integer  $n$ .

- b) If  $x_2 = x_1 + n\pi$  for some integer  $n$ , when can  $a$  and  $b$  be found?

**1H-8\*** The method of partial fractions, if you do it by undetermined coefficients, leads to a system of linear equations. Consider the simplest case:

$$\frac{ax + b}{(x - r_1)(x - r_2)} = \frac{c}{x - r_1} + \frac{d}{x - r_2}, \quad (a, b, r_1, r_2 \text{ given; } c, d \text{ to be found});$$

what are the linear equations which determine the constants  $c$  and  $d$ ? Under what circumstances do they have a unique solution?

(If you are ambitious, try doing this also for three roots  $r_i$ ,  $i = 1, 2, 3$ . Evaluate the determinant by using column operations to get zeros in the top row.)

## 11. Vector Functions and Parametric Equations

**11-1** The point  $P$  moves with constant speed  $v$  in the direction of the constant vector  $a\mathbf{i} + b\mathbf{j}$ . If at time  $t = 0$  it is at  $(x_0, y_0)$ , what is its position vector function  $\mathbf{r}(t)$ ?

**11-2** A point moves *clockwise* with constant angular velocity  $\omega$  on the circle of radius  $a$  centered at the origin. What is its position vector function  $\mathbf{r}(t)$ , if at time  $t = 0$  it is at  
(a)  $(a, 0)$  (b)  $(0, a)$

**11-3** Describe the motions given by each of the following position vector functions, as  $t$  goes from  $-\infty$  to  $\infty$ . In each case, give the  $xy$ -equation of the curve along which  $P$  travels, and tell what part of the curve is actually traced out by  $P$ .

~~a)~~  $\mathbf{r} = 2\cos^2 t\mathbf{i} + \sin^2 t\mathbf{j}$     ~~b)~~  $\mathbf{r} = \cos 2t\mathbf{i} + \cos t\mathbf{j}$     c)  $\mathbf{r} = (t^2 + 1)\mathbf{i} + t^3\mathbf{j}$   
~~d)~~  $\mathbf{r} = \tan t\mathbf{i} + \sec t\mathbf{j}$

**11-4** A roll of plastic tape of outer radius  $a$  is held in a fixed position while the tape is being unwound counterclockwise. The end  $P$  of the unwound tape is always held so the unwound portion is perpendicular to the roll. Taking the center of the roll to be the origin  $O$ , and the end  $P$  to be initially at  $(a, 0)$ , write parametric equations for the motion of  $P$ .  
(Use vectors; express the position vector  $OP$  as a vector function of one variable.)

~~11-5~~ A string is wound clockwise around the circle of radius  $a$  centered at the origin  $O$ ; the initial position of the end  $P$  of the string is  $(a, 0)$ . Unwind the string, always pulling it taut (so it stays tangent to the circle). Write parametric equations for the motion of  $P$ .  
(Use vectors; express the position vector  $OP$  as a vector function of one variable.)

**11-6** A bow-and-arrow hunter walks toward the origin along the positive  $x$ -axis, with unit speed; at time 0 he is at  $x = 10$ . His arrow (of unit length) is aimed always toward a rabbit hopping with constant velocity  $\sqrt{5}$  in the first quadrant along the line  $y = 2x$ ; at time 0 it is at the origin.

- a) Write down the vector function  $\mathbf{A}(t)$  for the arrow at time  $t$ .  
b) The hunter shoots (and misses) when closest to the rabbit; when is that?

~~11-7~~ The cycloid is the curve traced out by a fixed point  $P$  on a circle of radius  $a$  which rolls along the  $x$ -axis in the positive direction, starting when  $P$  is at the origin  $O$ . Find the vector function  $OP$ ; use as variable the angle  $\theta$  through which the circle has rolled.  
(Hint: begin by expressing  $OP$  as the sum of three simpler vector functions.)

**1J. Differentiation of Vector Functions**

~~1J-1~~ 1. For each of the following vector functions of time, calculate the velocity, speed  $|ds/dt|$ , unit tangent vector (in the direction of velocity), and acceleration.

- a)  $e^t \mathbf{i} + e^{-t} \mathbf{j}$       b)  $t^2 \mathbf{i} + t^3 \mathbf{j}$       c)  $(1 - 2t^2) \mathbf{i} + t^2 \mathbf{j} + (-2 + 2t^2) \mathbf{k}$

~~1J-2~~ Let  $OP = \frac{1}{1+t^2} \mathbf{i} + \frac{t}{1+t^2} \mathbf{j}$  be the position vector for a motion.

- a) Calculate  $\mathbf{v}$ ,  $|ds/dt|$ , and  $\mathbf{T}$ .  
 b) At what point in the speed greatest? smallest?  
 c) Find the  $xy$ -equation of the curve along which the point  $P$  is moving, and describe it geometrically.

~~1J-3~~ Prove the rule for differentiating the scalar product of two plane vector functions:

$$\frac{d}{dt} \mathbf{r} \cdot \mathbf{s} = \frac{d\mathbf{r}}{dt} \cdot \mathbf{s} + \mathbf{r} \cdot \frac{d\mathbf{s}}{dt},$$

by calculating with components, letting  $\mathbf{r} = x_1 \mathbf{i} + y_1 \mathbf{j}$  and  $\mathbf{s} = x_2 \mathbf{i} + y_2 \mathbf{j}$ .

~~1J-4~~ Suppose a point  $P$  moves on the surface of a sphere with center at the origin; let  $OP = \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}$ .

Show that the velocity vector  $\mathbf{v}$  is always perpendicular to  $\mathbf{r}$  two different ways:

- a) using the  $x, y, z$ -coordinates  
 b) without coordinates (use the formula in **1J-3**, which is valid also in space).  
 c) Prove the converse: if  $\mathbf{r}$  and  $\mathbf{v}$  are perpendicular, then the motion of  $P$  is on the surface of a sphere centered at the origin.

~~1J-5~~ a) Suppose a point moves with constant speed. Show that its velocity vector and acceleration vector are perpendicular. (Use the formula in **1J-3**.)

b) Show the converse: if the velocity and acceleration vectors are perpendicular, the point  $P$  moves with constant speed.

~~1J-6~~ For the helical motion  $\mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j} + bt \mathbf{k}$ ,

- a) calculate  $\mathbf{v}$ ,  $\mathbf{a}$ ,  $\mathbf{T}$ ,  $|ds/dt|$   
 b) show that  $\mathbf{v}$  and  $\mathbf{a}$  are perpendicular; explain using **1J-5**

**1J-7** a) Suppose you have a differentiable vector function  $\mathbf{r}(t)$ . How can you tell if the parameter  $t$  is the arclength  $s$  (measured from some point in the direction of increasing  $t$ ) without actually having to calculate  $s$  explicitly?

b) How should  $a$  be chosen so that  $t$  is the arclength if  $\mathbf{r}(t) = (x_0 + at) \mathbf{i} + (y_0 + at) \mathbf{j}$ ?

c) How should  $a$  and  $b$  be chosen so that  $t$  is the arclength in the helical motion described in Exercise **1J-6**?

**1J-8** a) Prove the formula  $\frac{d}{dt} u(t)\mathbf{r}(t) = \frac{du}{dt} \mathbf{r}(t) + u(t)\frac{d\mathbf{r}}{dt}$ .

(You may assume the vectors are in the plane; calculate with the components.)

b) Let  $\mathbf{r}(t) = e^t \cos t \mathbf{i} + e^t \sin t \mathbf{j}$ , the exponential spiral. Use part (a) to find the speed of this motion.

~~1J-9~~ A point  $P$  is moving in space, with position vector

$$\mathbf{r} = OP = 3 \cos t \mathbf{i} + 5 \sin t \mathbf{j} + 4 \cos t \mathbf{k}$$

- Show it moves on the surface of a sphere.
- Show its speed is constant.
- Show the acceleration is directed toward the origin.
- Show it moves in a plane through the origin.
- Describe the path of the point.

**1J-10** The **positive curvature**  $\kappa$  of the vector function  $\mathbf{r}(t)$  is defined by  $\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$ .

- Show that the helix of **1J-6** has constant curvature. (It is not necessary to calculate  $s$  explicitly; calculate  $d\mathbf{T}/dt$  instead and relate it to  $\kappa$  by using the chain rule.)
- What is this curvature if the helix is reduced to a circle in the  $xy$ -plane?

### 1K. Kepler's Second Law

**1K-1** (Same as 1J-3). Prove the product rule for differentiating the dot product of two plane vectors: do the calculation using an  $\mathbf{i} \mathbf{j}$ -coordinate system.  
(Let  $\mathbf{r}(t) = x_1(t) \mathbf{i} + y_1(t) \mathbf{j}$  and  $\mathbf{s}(t) = x_2(t) \mathbf{i} + y_2(t) \mathbf{j}$ .)

**1K-2** Let  $\mathbf{s}(t)$  be a vector function. Prove by using components that

$$\frac{d\mathbf{s}}{dt} = \mathbf{0} \quad \Rightarrow \quad \mathbf{s}(t) = \mathbf{K}, \quad \text{where } \mathbf{K} \text{ is a constant vector.}$$

~~1K-3~~ In our proof that Kepler's second law is equivalent to the force being central, used the following steps to show the second law implies a central force. Kepler's second law says the motion is in a plane and

$$(2) \quad 2 \frac{dA}{dt} = |\mathbf{r} \times \mathbf{v}| \text{ is constant.}$$

This implies  $\mathbf{r} \times \mathbf{v}$  is constant. So,

$$0 = \frac{d}{dt} (\mathbf{r} \times \mathbf{v}) = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} = \mathbf{r} \times \mathbf{a}.$$

This implies  $\mathbf{a}$  and  $\mathbf{r}$  are parallel, i.e. the force is central.

Reverse these steps to prove the converse: for motion under any type of central force, the path of motion will lie in a plane and area will be swept out by the radius vector at a constant rate. You will need the statement in exercise 1K-2.

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