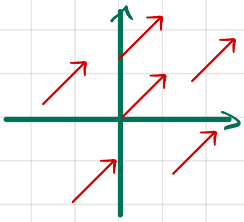
Lecture 19 Vector Fields

$\vec{F} = M\hat{i} + N\hat{j}$, M and N are functions of x and y

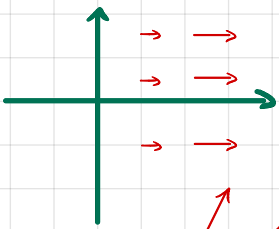
At each point (x,y), $\vec{F}$ a vector that depends on (x,y).

Ex:
- → velocity in fluid $\vec{v}$
- → force field $\vec{F}$

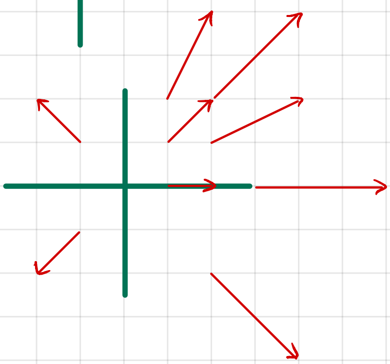
Ex: $\vec{F} = 2\hat{i} + \hat{j}$



Ex: $\vec{F} = x\hat{i}$

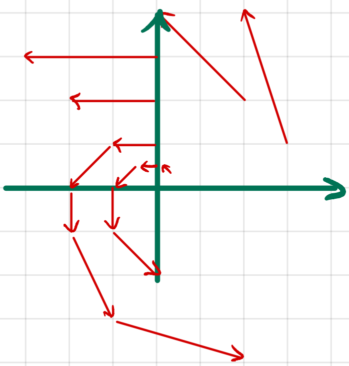


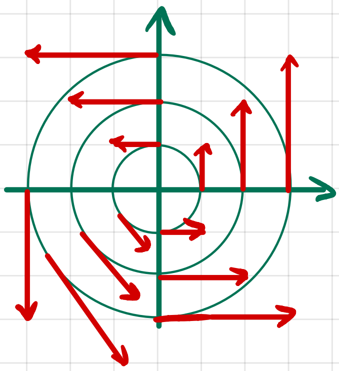
Ex: $\vec{F} = x\hat{i} + y\hat{j}$



Ex: $\vec{F} = -y\hat{i} + x\hat{j}$

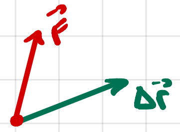
Velocity field for uniform rotation at unit angular velocity

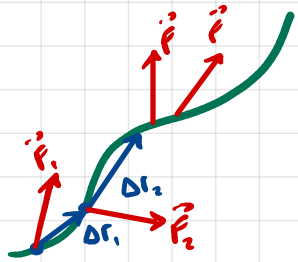




Work Done By Vector Field - Line Integrals

$W = \text{force} \cdot \text{distance} = \vec{F} \cdot \Delta\vec{r}$





Along the trajectory C the work adds up to

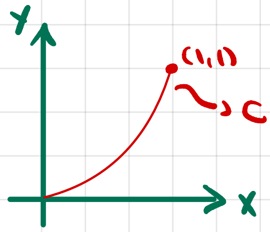
$$W = \int_C \vec{F}\, d\vec{r} = \int_{t_1}^{t_2} \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt$$

$$\lim_{\Delta r_i \to 0} \underbrace{\sum_i \vec{F} \Delta\vec{r}_i}_{\sum \vec{F}\left(\frac{\Delta\vec{r}}{\Delta t} \cdot \Delta t\right)}$$

$$\frac{d\vec{r}}{dt} = \vec{v}$$

Ex: $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$

C: $x(t) = t$, $y(t) = t^2$



$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt = \int_0^1 \langle -t^2, t\rangle \langle 1, 2t\rangle\, dt = \int_0^1 t^2\, dt = \frac{1}{3}$$

$\vec{F} = \langle -y, x\rangle = \langle -t^2, t\rangle$

not real vector, just notation

Another way: $\vec{F} = \langle M, N\rangle$ $\quad d\vec{r} = \langle dx, dy\rangle$

$\Rightarrow \vec{F} \cdot d\vec{r} = M\,dx + N\,dy$

$$\int_C \vec{F}\, d\vec{r} = \int_C (M\,dx + N\,dy)$$

Method to evaluate: express x, y in terms of a single variable and substitute

$$\int_C \vec{F}\, d\vec{r} = \int_C (-y\,dx + x\,dy) = \int_C (-t^2 dt + t\,2t\,dt) = \int_0^1 2t^2\, dt = \frac{1}{3}$$

$x = t$ $\quad y = t^2$

$dx = dt$ $\quad dy = 2t\,dt$

Note: $\int \vec{F}\, d\vec{r}$ depends on trajectory C but not on the parametrization.

e.g. could do $x = \sin\theta$, $y = \sin^2\theta$ $\quad 0 < \theta < \frac{\pi}{2}$, a harder way to get the same thing.

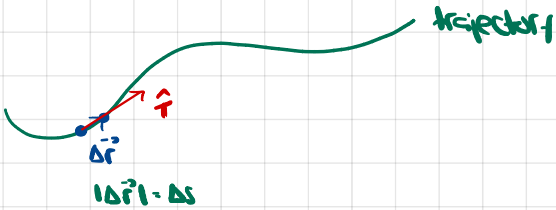
(NOT PRACTICAL)

Geometric Approach

$d\vec{r} = \langle dx, dy \rangle = \hat{T}\, ds$

direction of unit tangent vector

arc length



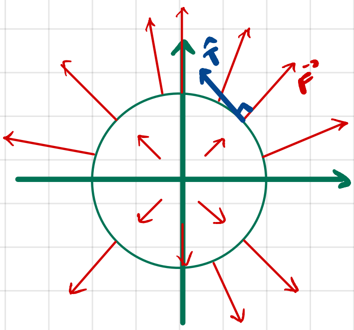
* note $\frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle = \hat{T} \frac{ds}{dt}$

So $\int_C \vec{F}\, d\vec{r} = \int_C M\,dx + N\,dy = \int_C \vec{F}\cdot\hat{T}\, ds$

Ex: C: circle of radius a at origin, counterclockwise

$\vec{F} = x\hat{\imath} + y\hat{\jmath}$

$\vec{F}\cdot\hat{T} = 0 \Rightarrow \int_C \vec{F}\cdot\hat{T}\, ds = 0$

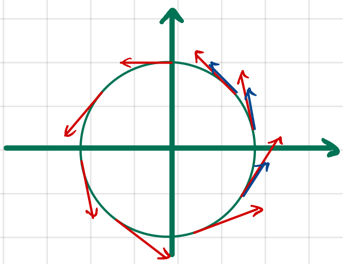


Ex: same C, $F = -y\hat{\imath} + x\hat{\jmath}$

$\vec{F} \parallel \hat{T} \Rightarrow \vec{F}\cdot\hat{T} = |\vec{F}| = a$

$\int_C \vec{F}\cdot\hat{T}\, ds = a\int_C ds = a \cdot \text{length}(C)$

$= a \cdot 2\pi a = 2\pi a^2$



Alternatively,

$$\int_C -y\,dx + x\,dy = \int_C (a^2\sin^2\theta\, d\theta + a^2\cos^2\theta\, d\theta) = \int_0^{2\pi} a^2\, d\theta = 2\pi a^2$$

$x = a\cos\theta$
$y = a\sin\theta$ $\quad \theta \in [0, \pi/2]$

$dx = -a\sin\theta$
$dy = a\cos\theta$

Lecture 20

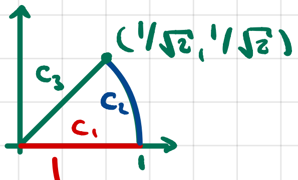
Recall from last lecture: Line Integrals for work

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\hat{T}\,ds = \int_C M\,dx + N\,dy, \quad \vec{F} = \langle M, N\rangle$$

Ex 1: $\vec{F} = \langle y, x\rangle$

$C = C_1 + C_2 + C_3$ enclosing sector of unit disk $0 \le \theta \le \pi/4$

need $\int_{C_i} y\,dx + x\,dy$



1) x-axis, C_1

$(0,0)$ to $(0,1)$ $\quad y = 0 \Rightarrow dy = 0$ $\quad$ $\vec{F} \parallel \hat{j}$, $\vec{F}\cdot\hat{T} = 0$

$$\int_{C_1} 0\,dx + x\,0 = 0$$

2) portion of unit circle, C_2

$x = \cos\theta \qquad dx = -\sin\theta\,d\theta$

$y = \sin\theta \qquad dy = \cos\theta\,d\theta$

$$\int_{C_2} y\,dx + x\,dy = \int_0^{\pi/4} \sin\theta(-\sin\theta\,d\theta) + \cos\theta\cdot\cos\theta\,d\theta = \int_0^{\pi/4}(\cos^2\theta - \sin^2\theta)\,d\theta = \int_0^{\pi/4}\cos 2\theta\,d\theta = \frac{1}{2}\sin 2\theta\Big|_0^{\pi/4}$$

$$= \frac{1}{2}$$

3) could do: parametrize line $x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}t \quad y = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}t \quad 0 \le t \le 1$

Better: $x = t$, $y = t$, t from 0 to $1/\sqrt{2}$ gives us $-C_3$ (C_3 backwards)

$$\int_{-C_3} = -\int_{C_3}$$

$$\int_{1/\sqrt{2}}^{0} x\,dx + x\,dy = \int_{1/\sqrt{2}}^{0} 2x\,dx = x^2\Big|_{1/\sqrt{2}}^{0} = -\frac{1}{2}$$

Total work $= 0 + \frac{1}{2} - \frac{1}{2} = 0$

Special case

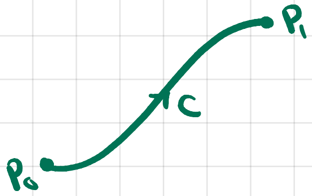
Say $\vec{F} = \nabla f$ gradient field

$f(x,y)$ is called potential

then we can simplify evaluation of $\int_C \vec{F} \, d\vec{r}$

Fundamental theorem of calculus for line integrals

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(P_1) - f(P_0)$$



$$\int_C f_x \, dx + f_y \, dy = \int_C df = f(P_1) - f(P_0)$$

Proof

$$\int_C \vec{\nabla} f \cdot d\vec{r} = \int_C f_x \, dx + f_y \, dy = \int_C f_x x'(t) \, dt + f_y y'(t) \, dt = \int_{t_0}^{t_1} \frac{df}{dt} \, dt = \Big[f(x(t), y(t)) \Big]_{t_0}^{t_1}$$
$$= f(P_1) - f(P_0)$$

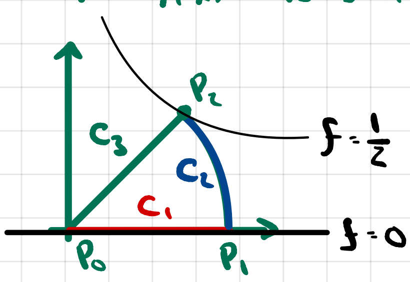
$C: x = x(t), \; y = y(t)$

$dx = x'(t)\,dt, \; dy = y'(t)\,dt$

$t_0 \le t \le t_1$

Ex 2: Ex 1 again, simpler solution method

$\vec{F} = \langle y, x \rangle = \nabla f \quad f(x,y) = xy$



$$\int_{C_1} \vec{\nabla} f \cdot d\vec{r} = f(P_1) - f(P_0) = 0$$

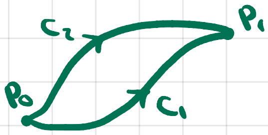
$$\int_{C_2} \vec{\nabla} f \cdot d\vec{r} = f(P_2) - f(P_1) = \frac{1}{2}$$

$$\int_{C_3} \vec{\nabla} f \cdot d\vec{r} = f(P_0) - f(P_2) = -\frac{1}{2}$$

* this method is nice but not all vector fields are gradient fields

Warning: everything today applies if $\vec{F}$ is a gradient field! Not true otherwise.

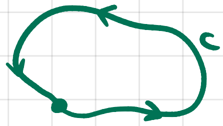
Consequences of fundamental theorem
(if $\vec{F}$ is gradient field then...)



→ Path independence

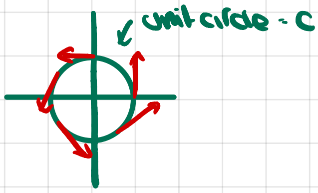
$\int_{C_1} \vec{\nabla} f \cdot d\vec{r} = \int_{C_2} \vec{\nabla} f \cdot d\vec{r}$ if C_1 and C_2 have same start and end points.

→ $\vec{F} = \vec{\nabla} f$ is conservative



C closed curve ⇒ $\int_C \vec{F} \cdot d\vec{r} = 0$

*** Remark**



$\vec{F} = \langle -y, x \rangle$

$\vec{F} \parallel \hat{T}$

$\vec{F} \cdot \hat{T} = |\vec{F}| = 1$

$\int \vec{F} \cdot \hat{T}\, ds = \int 1\, ds = 2\pi$

⇒ $\vec{F}$ is not conservative ⇒ $\vec{F}$ is not a gradient, not path independent

Physics if $\vec{F}$ is gradient of a potential $\vec{F} = \vec{\nabla} f$

→ Work of $\vec{F}$ = change in value of potential (e.g. gravitational/electric field vs gravitational/electrical potential)

→ conservativeness means no energy can be extracted from the field "for free"

→ total energy is conserved

Equivalent Properties

1) $\vec{F}$ is conservative : $\int_C \vec{F} \cdot d\vec{r} = 0$ along all closed curves C

2) $\int_C \vec{F} d\vec{r}$ is path independent

3) $\vec{F}$ is a gradient field $\vec{F} = \langle f_x, f_y \rangle$

*** Why?**

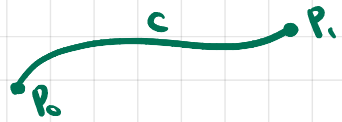
if 3) then 1), 2) by fundamental theorem

if 1) or 2), why is F a gradient? This will be how we find the potential.

4) $Mdx + Ndy$ is an exact differential df

Lecture 21 - Gradient Fields

Last time: $\vec{F} = \vec{\nabla} f$ then $\int_C \vec{F}\cdot d\vec{r} = f(P_1) - f(P_0)$



$\Rightarrow$ path independent line integral

$\Rightarrow$ the vector field is conservative: if C closed, $\oint_C \vec{F}\cdot d\vec{r} = 0$

Testing whether $\vec{F} = \langle M, N\rangle$ is gradient field.

$\boxed{\vec{F} = \vec{\nabla} f} \Rightarrow \begin{array}{l} M = f_x \\ N = f_y \end{array}$ then $f_{xy} = f_{yx} \Rightarrow \boxed{M_y = N_x}$

conversely:

if $\vec{F} = \langle M, N\rangle$ defined, diff. everywhere

and $M_y = N_x$

then $\vec{F}$ is a gradient field

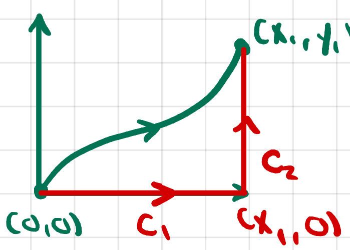
Ex: $\vec{F} = \langle -y, x\rangle$ $\quad$ $\begin{array}{l} M = -y \\ N = x \end{array}$ $\quad$ $\begin{array}{l} M_y = -1 \\ N_x = 1 \end{array}$ $\quad \Rightarrow M_y \neq N_x \Rightarrow \vec{F}$ not a gradient field

Ex: $\vec{F} = \langle 4x^2 + axy,\ 3y^2 + 4x^2\rangle$ $\quad M_y = ax \quad N_x = 8x$

$\Rightarrow$ if $a = 8$ then $\vec{F}$ is gradient field

Finding the potential, only if $M_y = N_x$

method 1: computing line integrals



$\int_C \vec{F}\cdot d\vec{r} = f(x_1, y_1) - f(0,0)$

$\Rightarrow f(x_1, y_1) = \int_C \vec{F}\cdot d\vec{r} + \underbrace{f(0,0)}_{\text{constant}}$

Ex: $\vec{F} = \langle 4x^2 + axy,\ 3y^2 + 4x^2\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (4x^2 + axy)\,dx + (3y^2 + 4x^2)\,dy = \int_0^{x_1} 4x^2\,dx + \int_0^{y_1} (3y^2 + 4x_1^2)\,dy$$

$$= \frac{4}{3}x_1^3 + [y_1^3 + 4x_1^2 y_1]$$

$$\Rightarrow f(x,y) = \frac{4}{3}x^3 + y^3 + 4x^2 y + C$$

method 2: antiderivatives

→ we want to solve $\begin{cases} f_x = 4x^2 + 8xy & (1) \\ f_y = 3y^2 + 4x^2 & (2) \end{cases}$

(1) ⇒ $f = \frac{4}{3}x^3 + 4x^2y + g(y)$

$f_y = 4x^2 + g'(y)$ match

⇒ $g'(y) = 3y^2$ ⇒ $g(y) = y^3 + C$

⇒ $f(x,y) = \frac{4}{3}x^3 + 4x^2y + y^3 + C$

Recap

$\vec{F} = \langle M, N \rangle$ is gradient field in a certain region of the plane ⇒ $N_x = M_y$ at every point
⇐ if $\vec{F}$ defined on entire plane (or in a simply connected region)

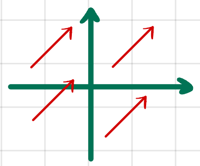
⟺

it is conservative if $\oint_C \vec{F} \cdot d\vec{r} = 0$ for closed C.

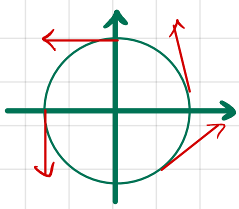
Def: $\text{curl}(\vec{F}) = N_x - M_y$

* $\vec{F}$ not a gradient field ⇒ $\text{curl}(\vec{F}) = 0$ (test for conservativeness)

→ For a velocity field, curl measures rotation component of motion

e.g. 

$\vec{F} = \langle a, b \rangle$
$N_x = M_y = 0$ ⇒ $\text{curl}\,\vec{F} = 0$, makes sense because no "swirling" motion.



$\vec{F} = \langle -y, x \rangle$

$M_y = -1 \quad N_x = 1 \Rightarrow \text{curl}\,\vec{F} = 1 - (-1) = 2$

→ "how much rotation is happening at any given point"

→ sign of curl tells you clockwise or counterclockwise

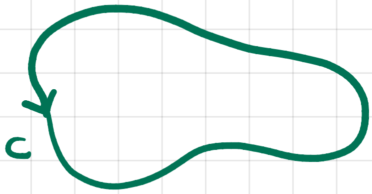
→ curl measures 2× angular velocity of rotation component of velocity field

→ curl of force field measures the torque exerted on a test object in the field.

→ $\frac{\text{torque}}{\text{moment of inertia}} = \frac{d}{dt}(\text{angular velocity})$, analogue of $\frac{\text{force}}{\text{mass}} = \frac{d}{dt}(\text{velocity})$

lecture 22

curl $\vec{F} = N_x - M_y$ $\quad\quad \vec{F} = \langle M, N \rangle$



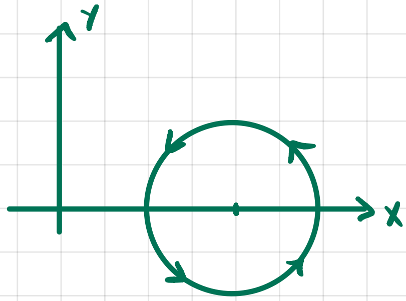
$\oint \vec{F} \cdot d\vec{r}$ ↗ direct calculation
↘ Green's theorem

Green's theorem

If C closed curve, enclosing a region R, counterclockwise, $\vec{F}$ vector field defined and differentiable in R $\Rightarrow$

$$\oint \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\, \vec{F}\, dA \quad \text{ie} \quad \oint M\,dx + N\,dy = \iint_R (N_x - M_y)\, dA$$

warning: only for closed curves!



Ex: Let C be circle, radius 1, centered (2,0)

$$\oint_C y e^{-x} dx + \left(\tfrac{1}{2}x^2 - e^{-x}\right) dy$$

$$= \iint_R x\, dA = \text{Area}(R) \cdot \bar{x} = \pi \cdot 2$$

$$\text{curl}\, \vec{F} = x + e^{-x} - (e^{-x}) = x$$

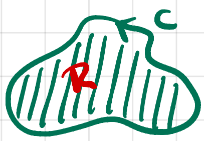
special case: curl $\vec{F} = 0 \Rightarrow \vec{F}$ conservative

Green's: $\oint \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\, \vec{F}\, dA = 0$ if curl $\vec{F} = 0$

This proves: curl $\vec{F} = 0$ everywhere in R $\Rightarrow \oint \vec{F} \cdot d\vec{r} = 0$

consequence: if $\vec{F}$ defined everywhere in the plane, and curl $\vec{F} = 0 \Rightarrow \vec{F}$ is conservative

proof:



$$\oint \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\, \vec{F}\, dA = 0$$

* cannot apply Green's theorem to vector field in Pset 8 Problem 2 when C encloses the origin!

$\vec{F} = -c\left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$, $\vec{F}$ not defined at (0,0) $\Rightarrow$ Green's theorem not applicable in regions enclosing R.

Proof of Green's Theorem: $\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA$

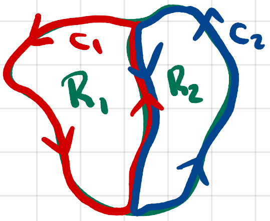
observe: $\int_C M\,dx = \iint_R -M_y\,dA$

easier to prove this statement first
special case $N=0$

A similar argument will show $\oint_C N\,dy = \iint_R N_x\,dA$

Summing, get Green's Theorem.

observe: we can decompose R into simpler regions.

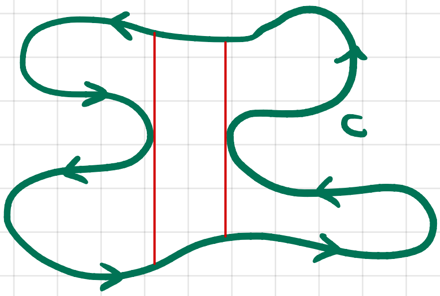


If we prove $\oint_{C_1} M\,dx = \iint_{R_1} -M_y\,dA$ and $\oint_{C_2} M\,dx = \iint_{R_2} -M_y\,dA$

then $\oint_C M\,dx = \oint_{C_1} + \oint_{C_2} = \iint_{R_1} + \iint_{R_2} = \iint_R -M_y\,dA$

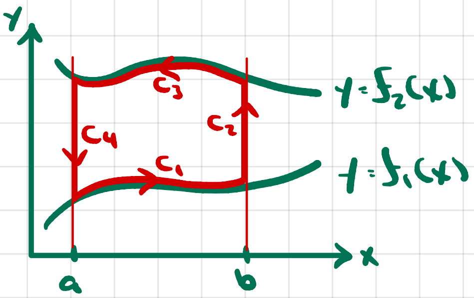
we go twice along boundary between R_1 and R_2 with opposite orientations.

imagine following region:



cut R into "vertically simple" regions

main step: prove $\oint_C M\,dx = \iint_R -M_y\,dA$ if R vertically simple, C boundary of R going counterclockwise



$\int_{C_1} M\,dx = \int_a^b M(x, f_1(x))\,dx$

$y=f_1(x)$, x from a to b

$\int_{C_2} M(x,y)\,dx = \int_{C_4} M(x,y) = 0$

$x=b$ $dx=0$ $\qquad$ $x=a$, $dx=0$

$\int_{C_3} M(x, f_2(x))\,dx = \int_b^a M(x, f_2(x))\,dx = -\int_a^b M(x, f_2(x))\,dx$

$y=f_2(x)$, x from b to a

$\Rightarrow \oint_C M\,dx = \int_a^b M(x, f_1(x))\,dx - \int_a^b M(x, f_2(x))\,dx$

$\iint_R -M_y\,dA = -\iint_R \frac{\partial M}{\partial y}\,dA = -\int_a^b \int_{f_1(x)}^{f_2(x)} \partial M/\partial x\,dy\,dx = -\int_a^b [M(x, f_2(x)) - M(x, f_1(x))]\,dx$

$= \int_a^b M(x, f_1(x))\,dx - \int_a^b M(x, f_2(x))\,dx$

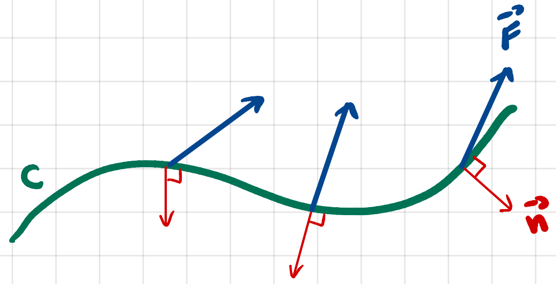
Lecture 23

Flux: another line integral

C plane curve

$\vec{F}$ vector field

Flux of $\vec{F}$ across C: $\int_C \vec{F}\cdot\vec{n}\,ds$ ← definition



break C into small pieces of length Δs, flux $= \lim_{\Delta s \to 0} \sum \vec{F}\cdot\hat{n}\,\Delta s$

recall work $= \int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\hat{T}\,ds$

→ summing tangential component of $\vec{F}$

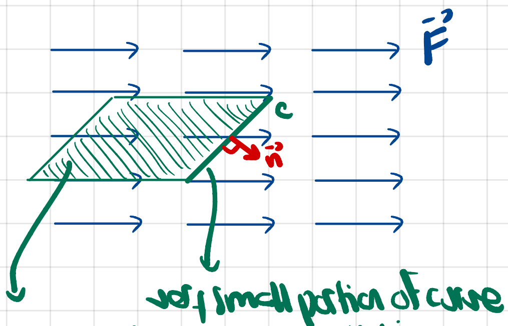
Flux sums the normal component of the vector field.

From point of view of computation, essentially the same as previous line integral we studied on for work

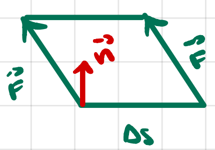
Interpretation

for $\vec{F}$ = velocity field

flux measures how much fluid passes through curve C per unit time.



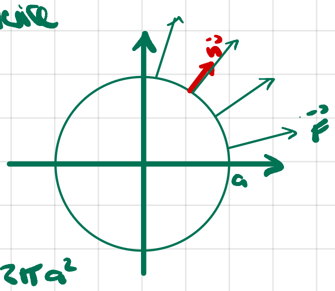
* flows from left-to-right: counted positively
" " right-to-left: " negatively



Area = base·height = $\Delta s\,(\vec{F}\cdot\hat{n})$

Ex 1: C: circle radius a, at origin, counterclockwise

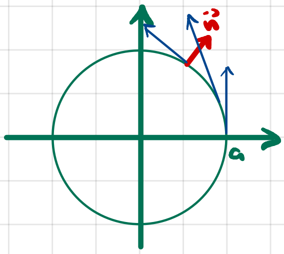
$\vec{F} = \langle x, y \rangle$



Along C, $\vec{F} \parallel \hat{n} \Rightarrow \vec{F}\cdot\hat{n} = |\vec{F}| = a$

Flux $= \int_C \vec{F}\cdot\hat{n}\,ds = \int_C a\,ds = a\,\text{length}(C) = 2\pi a^2$

Ex 1a $\vec{F} = \langle -y, x \rangle$



$\vec{F} \perp \hat{n} \Rightarrow \vec{F}\cdot\hat{n} = 0$

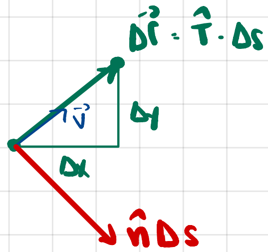
Flux $= 0$

Calculation using components

→ recall, $d\vec{r} = \hat{T}\,ds = \langle dx, dy \rangle$

$\hat{n}$ is $\hat{T}$ rotated 90° clockwise

$$\boxed{\hat{n}\,ds = \langle dy, -dx \rangle}$$



$\hat{T}\,ds = \Delta\vec{r} = \langle \Delta x, \Delta y \rangle$

$\hat{n}\,\Delta s = \langle \Delta y, -\Delta x \rangle$

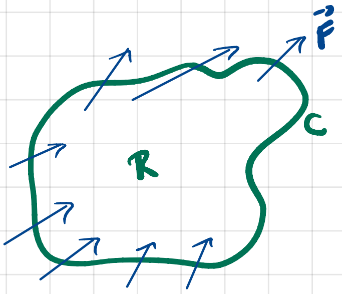
→ if $\vec{F} = \langle P, Q \rangle$ then $\int_C \vec{F}\cdot\hat{n}\,ds = \int_C \langle P, Q\rangle \cdot \langle dy, -dx \rangle = \int_C -Q\,dx + P\,dy$

Green's Theorem for Flux

If C encloses region R counterclockwise and $\vec{F} = \langle P, Q \rangle$ defined and differentiable in R then

$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \text{div}\,\vec{F}\,dA$$

$\text{div}\langle P, Q \rangle = P_x + Q_y$ → Green's Theorem in Normal Form (vs in tangential form)



Proof of $\oint -Q\,dx + P\,dy = \iint_{R} (P_x + Q_y)\,dA$

Let $M = -Q$ and $N = P$

$$\oint M\,dx + N\,dy = \iint_R \operatorname{curl} \vec{F}\, dA = \iint_R (N_x - M_y)\,dA = \iint_R (P_x + Q_y)\,dA$$

Ex 1 revisited $\vec{F} = \langle x, y \rangle$, C circle radius a, counterclockwise

$\operatorname{div} \vec{F} = 1 + 1 = 2$

$\text{Flux} = \iint_R 2\,dA = 2 \cdot \pi a^2$

Interpretation of $\operatorname{div} \vec{F}$

1) measures how much the flow is "expanding"

2) "source rate" = amount of fluid added to the system per unit time and area

Lecture 24 more about validity of Green's theorem

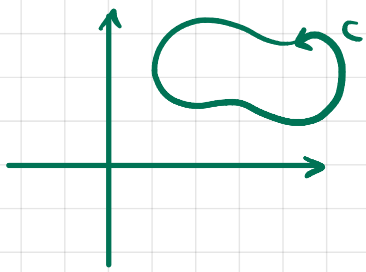
we've seen Green's Theorem

$$\oint \vec{F}\cdot\hat{T}\,ds = \iint_R \text{curl}\,\vec{F}\,dA$$

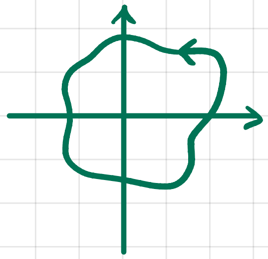
$$\oint \vec{F}\cdot\hat{n}\,ds = \iint_R \text{div}\,\vec{F}\,dA$$

only works if $\vec{F}$ and its derivatives are defined everywhere in R

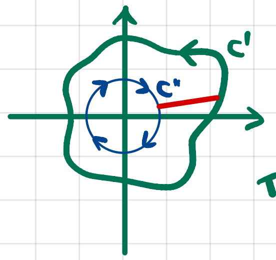
Ex: $\vec{F} = \dfrac{-y\hat{i}+x\hat{j}}{x^2+y^2}$ → domain not simply connected
→ not defined at origin
curl $\vec{F} = 0$ everywhere else



$\oint \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\,dA = 0$



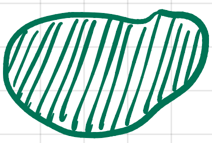
$\oint \vec{F}\cdot d\vec{r} = ?$ can't use Green directly



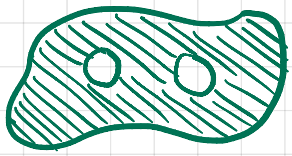
$\oint_{C'} \vec{F}\cdot d\vec{r} - \int_{C''} \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\cdot dA = 0$

Total line integral $= \oint_{C'} - \oint_{C''}$ + (red segments cancel out)

Def: a connected region in the plane is simply connected if the interior of any closed curve in R is also contained in R



simply connected



not simply connected

If domain where $\vec{F}$ is defined (and differentiable) is simply connected then we can always apply Green's theorem.

If curl $\vec{F} = 0$ and domain of $\vec{F}$ is simply connected then $\vec{F}$ is conservative and a gradient field.

Exam 3 Review

2 main objects $\iint_R f\,dA$ $\qquad \int_C \vec{F}\cdot\hat{T}\,ds,\ \int_C \vec{F}\cdot\hat{n}\,ds$

→ set up $\iint_R$ draw picture of R, take slices → iterated $\int$

remember area(R) = $\iint_R 1\,dA$

avg value $\bar{f}$ (in particular $\bar{x},\bar{y}$ = center of mass)

polar moment of inertia $I_0 = \iint_R (x^2+y^2)\,\delta\,dA$; I_x, I_y

mass

evaluating $\int$:

must know: usual $\int$
easy trigonometrics
substitution $\int_0^1 t\,dt/\sqrt{1+t^2}$ ← $u=1+t^2$

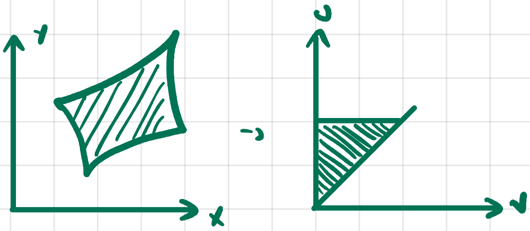
* not needed on exam: hard trig: $\int_0^{\pi/4} \cos^4\theta\,d\theta$
int. by parts

change of variables $u = u(x,y),\ v = v(x,y)$

1) $\frac{\partial(u,v)}{\partial(x,y)}$ $\qquad du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$

2) substitute x, y's in the integrand

3) setting up bounds



line integrals $\vec{F} = \langle M, N\rangle$

$\int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy$, $\int_C \vec{F}\cdot\hat{n}\,ds = \int_C -N\,dx + M\,dy$

→ evaluation: reduce to single parameter

$x = x(t),\ y = y(t)$, express $\int \underline{\quad t \quad}\,dt$

* if curl $\vec{F} = N_x - M_y = 0$ (and domain simply connected) then $\vec{F}$ is a gradient

$f_x = M$
$f_y = N$ $\iff \vec{F} = \vec{\nabla} f$

FTC $\int_C \vec{\nabla} f\cdot d\vec{r} = f(P_1) - f(P_0)$

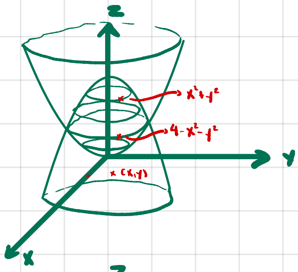
Green's theorem $\oint \vec{F}\cdot d\vec{r} = \iint_R \underbrace{\text{curl}\,\vec{F}}_{N_x - M_y}\,dA$, $\oint \vec{F}\cdot\hat{n}\,ds = \iint_R \underbrace{\text{div}\,\vec{F}}_{M_x + N_y}\,dA$

Lecture 25 - Triple Integrals

$\iiint_R f\,dV$

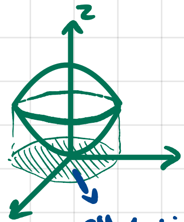
dV: volume element $= dx\,dy\,dz$

Ex 1: region between paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$



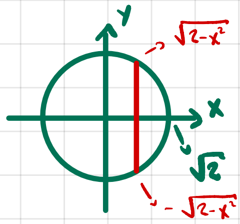
volume $\iiint_R 1\,dV = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dz$

Find shadow in x,y plane



$\rightarrow$ where $z_{bottom} < z_{top}$ $\Rightarrow$ $x^2 + y^2 < 4 - x^2 - y^2$

$\Rightarrow$ $x^2 + y^2 < 2$, disk of radius $\sqrt{2}$



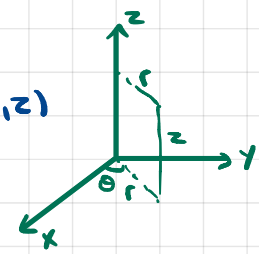
Better: use polar coord. instead of x,y

inner: $\int_{x^2+y^2}^{4-x^2-y^2} dz = z \Big|_{x^2+y^2}^{4-x^2-y^2}$

$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} 4 - 2x^2 - 2y^2 \, dy\,dz$: switch to polar coord.

$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} dz\, r\,dr\,d\theta$

Cylindrical coordinates (r, θ, z)

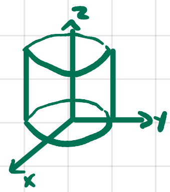


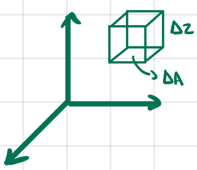
$x = r\cos\theta$

$y = r\sin\theta$

$\rightarrow$ $dx\,dy\,dz = r\,dr\,d\theta\,dz$

* Note: $r = a$, a cylinder





$\Delta V = \Delta A \cdot \Delta z$

$dV = dA \cdot dz$

Applications

$\rightarrow$ mass: density $\delta = \frac{\Delta m}{\Delta V}$ $\quad dm = \delta \cdot dV$

mass $= \iiint_R \delta\,dV$

$\rightarrow$ average value of $f(x,y,z)$ in R

$\bar{f} = \frac{1}{Vol(R)} \iiint_R f\,dV$

or, with density (weighted avg) $\frac{1}{mass(R)} \iiint_R f\,\delta\,dV$

→ center of mass $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \frac{1}{mass(R)} \iiint_R x \delta \, dV$$

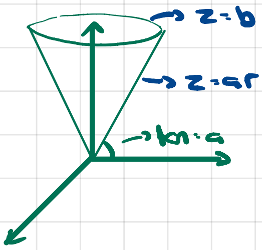
→ moment of inertia w/ respect to an axis

$$\iiint_R (\text{distance to axis})^2 \delta \, dV$$

$$I_z = \iiint_R r^2 \delta \, dV = \iiint_R (x^2+y^2) \delta \, dV$$

$$I_x = \iiint_R (y^2+z^2) \delta \, dV$$

$$I_y = \iiint_R (x^2+z^2) \delta \, dV$$

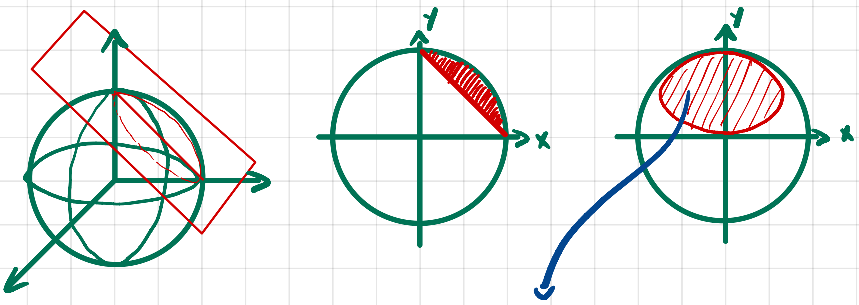


Ex 2: I_z of a solid cone between $z = ar$, $z = b$

$$I_z = \int_0^b \int_0^{2\pi} \int_0^{z/a} r^2 \, r \, dr \, d\theta \, dz = \frac{\pi b^5}{10a^4}$$

Ex 3 setup $\iiint$ for region $z > 1-y$ inside unit ball centered at origin

$$\int_0^1 \int_{-\sqrt{2y-2y^2}}^{\sqrt{2y-2y^2}} \int_{1-y}^{\sqrt{1-x^2-y^2}} dz \, dx \, dy$$

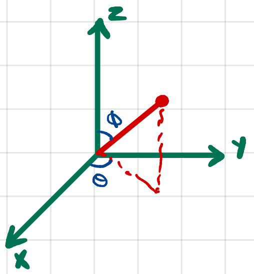


plane below sphere

$$1-y < \sqrt{1-x^2-y^2}$$

$$(1-y)^2 < 1-x^2-y^2$$

Lecture 26 - Spherical coordinates



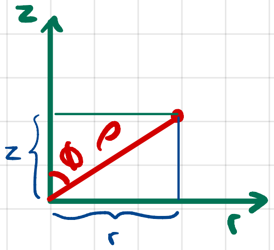
ρ = distance from origin = rho

$\varphi = \phi$ = angle from positive z-axis, 0 to π = phi

θ = same as before

On a sphere $\quad \phi \leftrightarrow$ latitude $\quad \theta \leftrightarrow$ longitude $\quad \rho = a$

Take a piece of a plane that starts on the z-axis



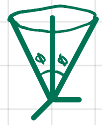
"it is as if we were doing polar coordinates in this plane rz"

$$z = \rho\cos\phi \qquad r = \rho\sin\phi \quad \Rightarrow \quad x = r\cos\theta = \rho\sin\phi\cos\theta \qquad y = r\sin\theta = \rho\sin\phi\sin\theta$$

$$\rho = \sqrt{r^2+z^2} \Rightarrow \rho = \sqrt{x^2+y^2+z^2}$$

Examples: $\rho = a \Rightarrow$ sphere, centered origin, radius a

$\phi = \pi/4 \Rightarrow$ cone



$\phi = \pi/2 \Rightarrow$ xy plane

Triple $\int$ in spherical coord.

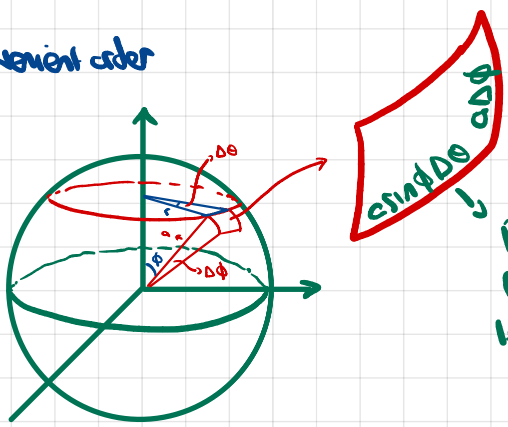
$dV = ???\ \ d\rho\, d\phi\, d\theta$
→ usually most convenient order

$\Delta V \approx \_\_\_\_\ \Delta\rho\,\Delta\phi\,\Delta\theta$

→ surface area on sphere of radius a

$$\Delta S \approx a\sin\phi\,\Delta\theta \cdot a\,\Delta\phi = a^2\sin\phi\,\Delta\phi\,\Delta\theta$$

$$dS = a^2\sin\phi\, d\phi\, d\theta$$

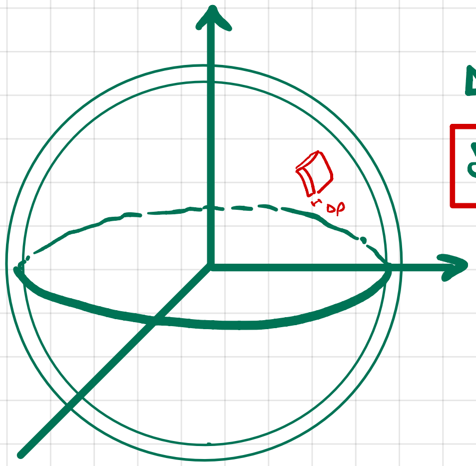


piece of circle of radius a
length = $a\Delta\phi$; it is on a meridian around the sphere

piece of circle of radius
$r = \rho\sin\phi = a\sin\phi$
length = $a\sin\phi\,\Delta\theta$

next we take two concentric spheres of radii ρ and $\rho+\Delta\rho$

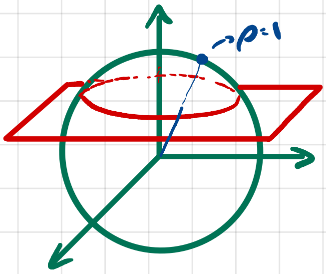
→ now we form a small rectangular volume between them



$\Delta V \approx \Delta\rho \cdot \Delta S = \rho^2 \sin\phi \, \Delta\rho \, \Delta\phi \, \Delta\theta$

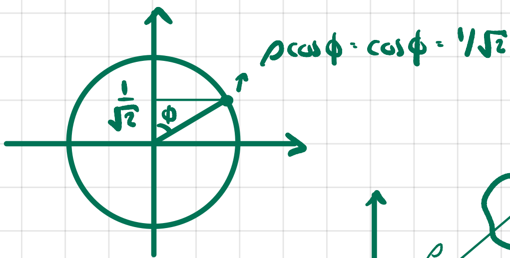
$$\boxed{dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

Ex Volume of portion of unit sphere above $1/\sqrt{2}$



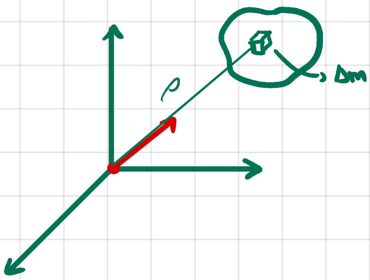
$$\int_0^{2\pi}\int_0^{\pi/4}\int_{\frac{1}{\sqrt{2}}\sec\phi}^{1} 1\,\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{2\pi}{3} - \frac{5\pi}{6\sqrt{2}}$$



plane: $z = \frac{1}{\sqrt{2}}$

$\rho\cos\phi = \frac{1}{\sqrt{2}} \Rightarrow \rho = \frac{1}{\cos\phi\sqrt{2}} = \frac{1}{\sqrt{2}}\sec\phi$



Applications

→ Gravitational force exerted by Δm at (x,y,z) on a mass m at origin

$|\vec{F}| = \frac{G \cdot \Delta M \cdot m}{\rho^2}$ $\quad$ $\text{dir}\,\vec{F} = \langle x,y,z\rangle/\rho$ $\quad$ $\vec{F} = |\vec{F}| \cdot \text{dir}\,\vec{F}$

Integrating $\quad \Delta M \approx \delta \Delta V \quad \vec{F} = \iiint \frac{Gm\langle x,y,z\rangle}{\rho^3} \delta \, dV$

Set up: place solid so z-axis is an axis of symmetry

Then $\vec{F} = \langle 0,0,F\rangle$

z component: $Gm\iiint \frac{z}{\rho^3}\delta \, dV$

Use spherical coordinates: $Gm\iiint \frac{\cancel{\rho}\cos\phi}{\cancel{\rho^3}} \underbrace{\delta \cancel{\rho^2} \sin\phi \, d\rho \, d\phi \, d\theta}_{dV}$

So, $\overset{\text{z component}}{F_z} = Gm\iiint \delta \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$

Ex: Newton's Theorem: grav. attraction of spherical planet with uniform density is equal to that of a point mass of same total mass at its center

Lecture 27 - Vector Fields in Space

→ At every point $\vec{F} = \langle P, Q, R \rangle$
↳ functions of x, y, z

Examples

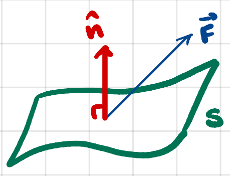
→ Force fields

gravitational attraction of solid mass at origin on mass at (x,y,z)

$\vec{F}$ directed towards origin, magnitude $\sim \frac{c}{\rho}$, $\vec{F} = \dfrac{-c\langle x,y,z\rangle}{\rho^3}$

→ Electric fields, magnetic fields → Velocity fields → Gradient fields

Flux



"orientation"

Recall flux in 2D $= \int_C \vec{F} \cdot \hat{n}\, ds$

In 3D flux is measured through a surface ; surface integral

$\vec{F}$ vector field, S surface in space, $\hat{n}$ unit normal vector to S

Flux $= \iint_S \vec{F} \cdot \hat{n}\, dS$
↳ surface area element

* notation : $d\vec{S} = \hat{n}\, dS$
↳ often easier to compute

Examples

1) Flux of $\vec{F} = \langle x,y,z \rangle$ through sphere radius a centered origin

$\iint_S \vec{F} \cdot d\vec{S} = \iint_S a\, dS = a \iint_S dS = a \cdot \text{Area}(S) = a \cdot 4\pi a^2 = 4\pi a^3$

$\hat{n} = \frac{1}{a}\langle x,y,z \rangle$, $\vec{F} \cdot \hat{n} = |\vec{F}|$ since $\vec{F} \parallel \hat{n}$
$= a$

2) same sphere, $\vec{H} = z\hat{k}$

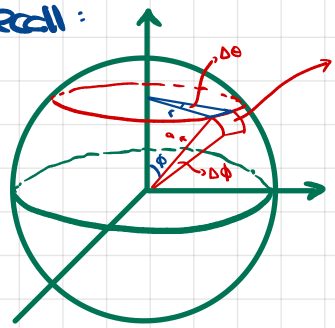
we still have $\hat{n} = \dfrac{\langle x,y,z\rangle}{a}$

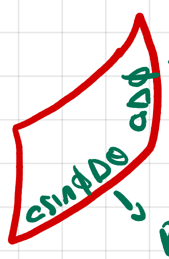
$\vec{H} \cdot \hat{n} = \langle 0,0,z \rangle \cdot \langle x,y,z \rangle / a = \dfrac{z^2}{a}$

Flux $= \iint_S \frac{z^2}{a} dS = \int_0^{2\pi}\int_0^{\pi} a\cos^2\phi \cdot a^2 \sin\phi\, d\phi\, d\theta = a^3 \cdot 2\pi \left[-\frac{1}{3}\cos^3\phi\right]_0^{\pi} = \frac{4}{3}\pi a^3$

what is dS in terms of $d\phi\, d\theta$?

recall :





piece of circle of radius a
length $= a\Delta\phi$; it is on a meridian around the sphere

piece of circle of radius
$r = \rho\sin\phi = a\sin\phi$
length $= a\sin\phi\, \Delta\theta$

$\Delta S \approx a\sin\phi\,\Delta\theta \cdot a\Delta\phi$ $\qquad dS = a^2 \sin\phi\, d\phi\, d\theta$

Conclusion

→ use geometry or need to set up $\iint_S \vec{F}\cdot\hat{n}\,dS$

0) S = horizontal plane z = a

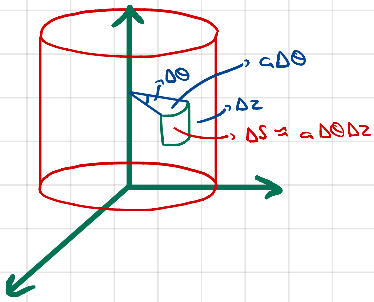
$\hat{n} = \pm\hat{k}$, $dS = dx\,dy$

→ vertical planes ∥ yz-plane, x = a

$\hat{n} = \pm\hat{\imath}$, $dS = dy\,dz$

1) sphere, radius a, center origin

$\hat{n} = \dfrac{\langle x, y, z\rangle}{a}$, $dS = a^2 \sin\phi\, d\phi\, d\theta$



2) cylinder of radius a, centered z-axis

$\hat{n} = \dfrac{\pm\langle x, y, 0\rangle}{a}$, $dS = a\,d\theta\,dz$

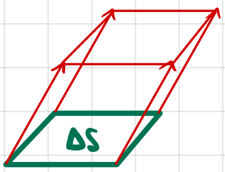
3) graph z = f(x, y)

$$\hat{n}\,dS = \pm \underbrace{\langle -f_x, -f_y, 1\rangle}_{\text{not } \hat{n}} \underbrace{dx\,dy}_{\text{not } dS}$$

To set up bounds on $\iint \dots dx\,dy$ look at shadow of S in xy-plane

* Geometric interpretation

→ if $\vec{F}$ is a velocity field then Flux = amount of matter through S per unit time



Lecture 28

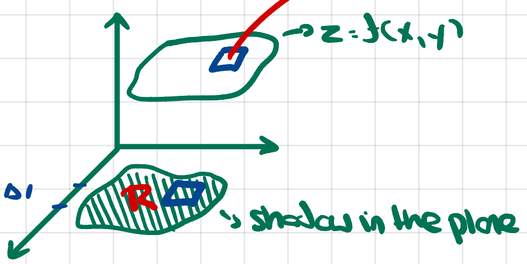
→ Flux of $\vec{F}$ through surface S is $\iint_S \vec{F}\cdot\hat{n}\,dS$

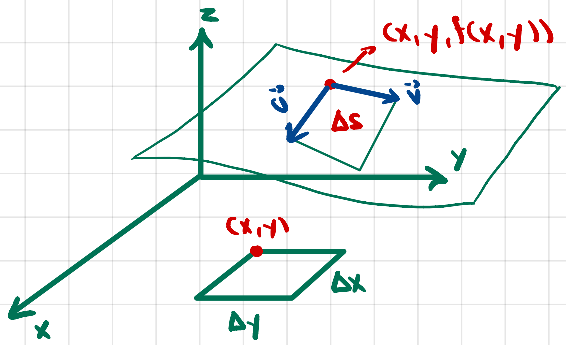
* if S is the graph of $z = f(x,y)$

$$\hat{n}\,dS = \pm\langle -f_x, -f_y, 1\rangle\,dx\,dy$$

setup

→ what is this area and what is its normal vector?





$\vec{u}\times\vec{v} = \Delta S\cdot\hat{n}$

↳ cross prod gives us the area of the small parallelogram and a normal vector which is what we need for $\hat{n}\,dS$

what are $\vec{u}$ and $\vec{v}$?

$\vec{u}$: from $(x,y,f(x,y))$ to $(x+\Delta x, y, \underbrace{f(x+\Delta x,y)}_{\approx f(x,y)+\Delta x f_x})$

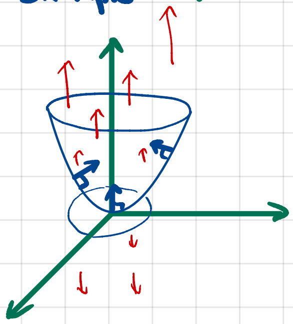
$\Rightarrow \vec{u} \approx \langle \Delta x, 0, f_x\Delta x\rangle = \langle 1,0,f_x\rangle\,\Delta x$

$\vec{v} \approx \langle 0, \Delta y, f_y\Delta y\rangle = \langle 0,1,f_y\rangle\,\Delta y$

$$\hat{n}\,\Delta S = \vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}\Delta x\,\Delta y = \langle -f_x, -f_y, 1\rangle\,\Delta x\,\Delta y$$

(with $\hat{n}$ upwards)

Example $\vec{F} = z\hat{k}$ through portion of $z = x^2+y^2$ above unit disk normal upwards.

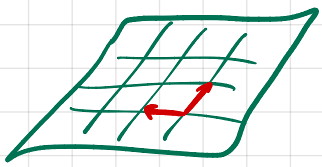


$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S \langle 0,0,z\rangle \langle -2x,-2y,1\rangle\,dx\,dy = \iint_S z\,dx\,dy$$

$$= \iint_S (x^2+y^2)\,dx\,dy = \int_0^{2\pi}\int_0^1 r^2\,r\,dr\,d\theta = \pi/2$$

Note generally: given parametric description of S

S: $x = x(u,v)$
$y = y(u,v)$
$z = z(u,v)$



sides $\frac{\partial \vec{r}}{\partial u}\Delta u$

$\frac{\partial \vec{r}}{\partial v}\Delta v$

$\vec{r} = \vec{r}(u,v) = \langle x,y,z\rangle$

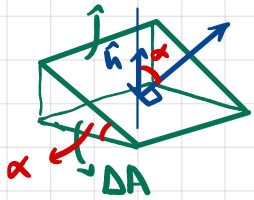
$\hat{n}\,dS = \underline{\quad ? \quad}\,du\,dv$ $\qquad \hat{n}\,\Delta S = \left(\frac{\partial \vec{r}}{\partial u}\Delta u\right)\times\left(\frac{\partial \vec{r}}{\partial v}\Delta v\right) = \vec{r}_u \times \vec{r}_v\,\Delta u\,\Delta v$

→ if we know a normal vector $\vec{N}$ (not necess. unit) to the surface.

examples: plane $ax+by+cz=d$ $\quad \vec{N} = \langle a,b,c\rangle$

S given by $g(x,y,z)=0$ $\quad \vec{N} = \nabla g$

ΔS = slanted plane



surface element: $\Delta A = \Delta S \cos\alpha$

$\cos\alpha = \frac{\vec{N}\cdot\hat{k}}{|\vec{N}|}$

$\Rightarrow \Delta S = \frac{1}{\cos\alpha}\Delta A = \frac{|\vec{N}|}{\vec{N}\cdot\hat{k}}\Delta A$

$$\hat{n}\,\Delta S = \frac{|\vec{N}|\cdot\hat{n}}{\vec{N}\cdot\hat{k}}\Delta A = \pm\frac{\vec{N}}{\vec{N}\cdot\hat{k}}\Delta A = \pm\frac{\vec{N}}{\vec{N}\cdot\hat{k}}\,dx\,dy$$

Example $\underbrace{z - f(x,y)}_{g(x,y,z)} = 0$ $\quad \vec{N} = \nabla g = \langle -f_x, -f_y, 1\rangle$ $\quad \frac{\vec{N}}{\vec{N}\cdot\hat{k}}\,dx\,dy = \langle -f_x,-f_y,1\rangle\,dx\,dy$

Divergence Theorem ("Gauss-Green Theorem")

→ 3d analogue of Green for flux

If S is closed surface enclosing region D, oriented with $\hat{n}$ outwards, and $\vec{F}$ defined and differ. everywhere in D, then $\oiint \vec{F} \cdot d\vec{S} = \iiint_D \operatorname{div} \vec{F} \, dV$

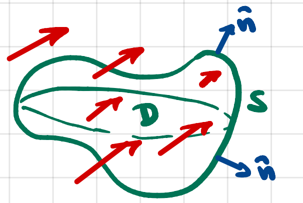
where $\operatorname{div}(\langle P, Q, R \rangle) = P_x + Q_y + R_z$

Ex: last time, flux of $z\hat{k}$ through sphere radius a was $\frac{4\pi a^3}{3}$

$$\iint_S z\hat{k} \cdot \hat{n} \, dS = \iiint \underbrace{\operatorname{div}(z\hat{k})}_{0+0+1} \, dV = \text{Volume (sphere)} = \frac{4\pi a^3}{3}$$

Lecture 29

Divergence Theorem



$$\oiint \vec{F} \cdot d\vec{S} = \iiint_D \operatorname{div} \vec{F}\, dV$$

$$\oiint_S \langle P, Q, R \rangle \cdot \hat{n}\, dS = \iiint_D (P_x + Q_y + R_z)\, dV$$

∇ notation

"del" $= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

$\nabla f = \langle f_x, f_y, f_z \rangle$

$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \langle P, Q, R \rangle = P_x + Q_y + R_z$

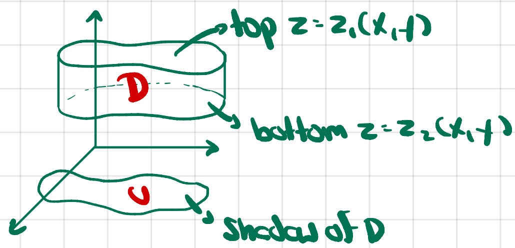
Physical Interpretation

$\operatorname{div} \vec{F}$ = "source rate" = amount of flux generated per unit volume

incompressible fluid flow: given mass occupies a fixed volume

$\iiint_D \operatorname{div} \vec{F}\, dV = \iint_S \vec{F} \cdot \hat{n}\, dS$ = amount of fluid leaving D per unit time

Proof of $\oiint \langle 0, 0, R \rangle \cdot \hat{n}\, dS = \iiint_D R_z\, dV$, then get general case by summing three such identities, one for each component.



→ if region D is vertically simple

Righthand side $\iiint_D R_z\, dV = \iint \int_{z_1(x,y)}^{z_2(x,y)} R_z\, dz\, dx\, dy$

$= \iint_U [R(x,y,z_2(x,y)) - R(x,y,z_1(x,y))]\, dx\, dy$

$\oiint_S \langle 0, 0, R \rangle \cdot \hat{n}\, dS = \iint_{top} + \iint_{bottom} + \iint_{sides}$

Top: graph $z = z_2(x,y)$ $\quad \hat{n}\, dS = \langle -z_{2_x}, -z_{2_y}, 1 \rangle\, dx\, dy$

$\Rightarrow \langle 0, 0, R \rangle \cdot \hat{n}\, dS = R\, dx\, dy$

$\iint_{top} \langle 0, 0, R \rangle \cdot \hat{n}\, dS = \iint_{top} R\, dx\, dy = \iint_U R(x, y, z_2(x,y))\, dx\, dy$

Bottom: graph $z = z_1(x,y)$ $\hat{n}\,dS = -\langle -z_{1,x}, -z_{1,y}, 1\rangle\,dx\,dy$

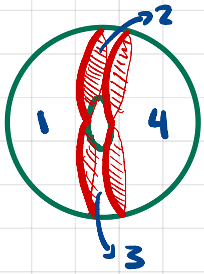
we need $\hat{n}$ pointing outwards from D

$$\iint_{bottom} \langle 0,0,R\rangle\cdot\hat{n}\,dS = \iint_{bottom} -R\,dx\,dy = \iint_{U} -R(x,y,z_1(x,y))\,dx\,dy$$

→ Sides are vertical ⇒ $\langle 0,0,R\rangle$ tangent to sides ⇒ Flux through sides = 0

So,
$$\iiint_D R_z\,dV = \oiint_{\substack{top + bottom\\ + sides}} \langle 0,0,R\rangle\cdot\hat{n}\,dS$$

If D not vertically simple, simply decompose it into simple pieces e.g.



4 pieces

Diffusion Equation = Heat Equation

governs motion of smoke in immobile air, dye in solution

u = concentration at given point = $u(x,y,z,t)$

$$\frac{\partial u}{\partial t} = k\underbrace{\nabla^2}_{\text{laplacian}} u = k\underbrace{\nabla\cdot\nabla u}_{\text{div}(\nabla u)} = k(u_{xx}+u_{yy}+u_{zz})$$

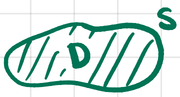
$\vec{F}$ = flow of smoke

1) Physics (+ common sense): smoke flows from high to low concentr. regions

→ $\vec{F}$ directed along $-\nabla u$

in fact $\vec{F} = -k\nabla u$

2) Relate $\vec{F}$ and $\partial u/\partial t$?



Flux out of D through S: $\oiint_S \vec{F}\cdot\hat{n}\,dS$ = amount of smoke through S per unit time $= -\frac{d}{dt}\underbrace{\left[\iiint_D u\,dV\right]}_{\text{chg in total smoke}}$

total amount smoke in D ↑ ($\iiint_D u\,dV$)

Note $-\frac{d}{dt}\left[\iiint_D u\,dV\right] = -\iiint_D \frac{\partial u}{\partial t}\,dV$

$$\frac{d}{dt}\sum_i u(x_i,y_i,z_i,t)\,\Delta V_i = \sum_i \frac{\partial}{\partial t} u(x_i,y_i,z_i,t)\,\Delta V$$

Then, Divergence theorem → $\oiint_S \vec{F}\cdot\hat{n}\,dS = \iiint_D \text{div}\,\vec{F}\,dV = -\iiint_D \frac{\partial u}{\partial t}\,dV$

$$\frac{1}{\text{vol}(D)}\iiint_D \text{div}\,\vec{F}\,dV = \frac{1}{\text{vol}(D)}\iiint -\frac{\partial u}{\partial t}\,dV \qquad \Rightarrow \text{div}\,\vec{F} = -\frac{\partial u}{\partial t}$$

Avg of div $\vec{F}$ in D = Avg of $-\partial u/\partial t$ in D, for any region D

Lecture 30

Diffusion equation (cont'd)

u = concentration of substance

$\vec{F}$ = flow of substance

1) Physics: $\vec{F} = -k\nabla u$

2) Diverg. theorem: $\operatorname{div}\vec{F} = -\partial u/\partial t$ $\Rightarrow$ $\dfrac{\partial u}{\partial t} = -\operatorname{div}\vec{F} = k\operatorname{div}(\nabla u) = k\nabla^2 u$

$\Rightarrow$ $\dfrac{\partial u}{\partial t} = k\nabla^2 u$ (diffusion eq.)

Line Integrals

→ vector field $\vec{F} = \langle P, Q, R\rangle$ repres. force

curve C in space, $d\vec{r} = \langle dx, dy, dz\rangle$

Work $= \int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$

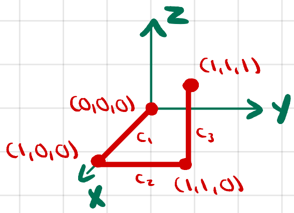
Evaluate by parametrizing C, express x, y, z, dx, dy, dz in terms of parameter

ex: $\vec{F} = \langle yz, xz, xy\rangle$ C: $x = t^3,\ y = t^2,\ z = t$ $0 \le t \le 1$

$dx = 3t^2\,dt$ $dy = 2t\,dt$ $dz = dt$

$\int_C \vec{F}\cdot d\vec{r} = \int_C yz\,dx + xz\,dy + xy\,dz = \int t^2 t\, 3t^2 + t^3 t\, 2t\,dt + t^3 t^2\,dt = (\ldots) = 1$

ex 2: same $\vec{F}$, C:



$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r} = 0$ ($z = 0$ and $dz = 0$)

$\int_{C_3} \vec{F}\cdot d\vec{r} = \int_0^1 1\,dz = 1$ ($x = y = 1$, $dy = dx = 0$)

In fact, $\vec{F}$ is conservative $\vec{F} = \nabla(xyz)$

Knowing this, Fund. Theorem Calculus for line integrals: $\int_C \nabla f\cdot dr = f(P_1) - f(P_0)$

$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(1,1,1) - f(0,0,0) = 1$

Test for gradient fields

$\vec{F} = \langle P, Q, R\rangle = \langle f_x, f_y, f_z\rangle$?

(defined in simply connected region)

If so, then $P_y = f_{xy} = f_{yx} = Q_x$
$P_z = f_{xz} = f_{zx} = R_x$
$Q_z = f_{yz} = f_{zy} = R_y$

$\Rightarrow$ criterion: $\vec{F} = \langle P, Q, R\rangle$ is gradient field $\Leftrightarrow$ $P_y = Q_x$, $P_z = R_x$, $Q_z = R_y$

In terms of differentials the criterion is

$$P\,dx + Q\,dy + R\,dz \text{ is exact} \Leftrightarrow \vec{F} = \langle P, Q, R\rangle \text{ is gradient field}$$

ex: for which a and b is $axy\,dx + (x^2 + z^3)dy + (byz^2 - 4z^3)dz$ exact?

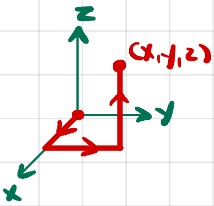
$axy\,dx + (x^3 + z^3)dy + (byz^2 - 4z^3)\,dz$ is a differential, and it is exact if it is the total differential of some function f. (total diff of $f(x,y)$ is $f_x dx + f_y dy$)

$P = axy$, $Q = x^2 + z^3$, $R = byz^2 - 4z^3$ $\Rightarrow$ $P_y = ax$, $P_z = 0$, $Q_z = 3z^2$; $Q_x = 2x$, $R_x = 0$, $R_y = bz^2$ $\Rightarrow$ $a = 2$, $b = 3$

Find potential?

method 1, line integrals

1) $f(x,y,z) = \int_C \vec{F} \cdot d\vec{r}$ + constant   C from $(0,0,0)$ to (x,y,z)



method 2, antiderivatives

$f_x = 2xy$ $f_y = x^2 + z^3$ $f_z = 3yz^2 - 4z^3$

$f = \int 2xy\,dx = x^2y + g(y,z)$

$f_y = x^2 + g_y(y,z) = x^2 + z^3 \Rightarrow g_y(y,z) = z^3 \Rightarrow g(y,z) = z^3y + h(z)$

$\Rightarrow f = x^2y + z^3y + h(z)$

$f_z = 3z^2y + h'(z) = 3z^2y - 4z^3 \Rightarrow h'(z) = -4z^3 \Rightarrow h(z) = -z^4 + C$

$\Rightarrow f(x,y,z) = x^2y + z^3y - z^4 + C$

Curl in 3D

if $\vec{F} = \langle P, Q, R \rangle$ then $\text{curl}\,\vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

→ if $\vec{F}$ defined in simply connected region,

$\vec{F}$ conservative $\Leftrightarrow$ $\text{curl}\,\vec{F} = 0$

* curl of vector field in space is a vector field

$\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$

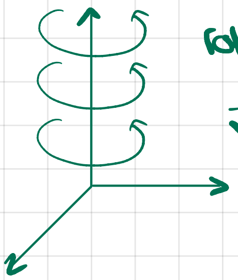
we've seen:

$\nabla f = \langle f_x, f_y, f_z \rangle$ and $\nabla \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z = \text{div}\,\vec{F}$

now we have:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \rangle = \text{curl}\,\vec{F}$$

Geometrically: curl measures rotation component in velocity field



rotation around z-axis at angular velocity ω

$\vec{v} = \langle -\omega y, \omega x, 0 \rangle$

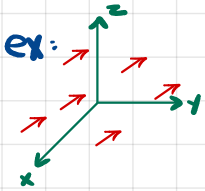
$\text{curl}\,\vec{v} = \langle 0, 0, 2\omega \rangle$

Lecture 31

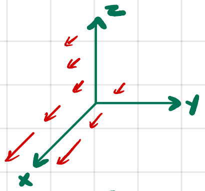
recall: $\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$

measures rotation part of a velocity field, direction = axis of rotation, magnitude = 2 · angular velocity

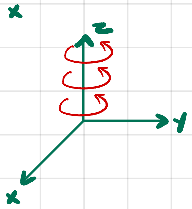
ex:



$\vec{F} = \langle a,b,c \rangle$
$\text{curl}\,\vec{F} = 0$



$\vec{F} = \langle x,0,0 \rangle$
$\text{curl}\,\vec{F} = 0$, but $\text{div}\,\vec{F} = 1$

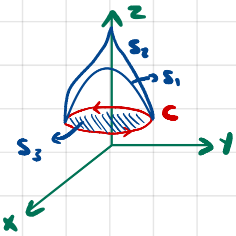


$\vec{F} = \langle -y,x,0 \rangle$
$\text{curl}\,\vec{F} = \langle 0,0,2 \rangle$

Stokes' Theorem

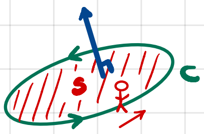
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \underbrace{(\nabla\times\vec{F})}_{\text{curl}\,\vec{F}}\cdot\hat{n}\,dS$$

C closed curve
S any surface bounded by C
$\vec{F}$ defined everywhere on S



← we can choose any of the surfaces

orientation: need orientations of C and S to be compatible

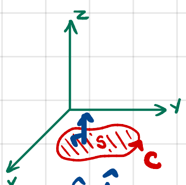


if I walk along C in the positive direction with S to my left then $\hat{n}$ is pointing up for me

alternatively, right hand rule: thumb along C positively, index tangent to S towards interior of S, middle finger points parallel to $\hat{n}$

comparing Stokes with Green

$\vec{F} = \langle P,Q,R \rangle$



$\hat{n} = \hat{k}$

S: portion of xy-plane bounded by a curve C (counterclockwise)

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy \overset{(z=0,\,dz=0)}{} \underset{\text{Stokes}}{=} \iint_S (\nabla\times\vec{F})\cdot\hat{k}\,dS = \iint_S (Q_x - P_y)\,dx\,dy$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \iint_S (Q_x - P_y)\,dx\,dy \quad \text{(Green's Thm)}$$

$(\nabla\times\vec{F})\cdot\hat{k}$ = z-component of curl = $Q_x - P_y$

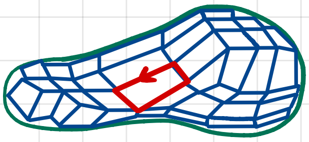
⇒ Green's Theorem is special case of Stokes in xy plane.

Why Stokes is true

* we know it for C, S in xy-plane

* also for C, S in any plane (using that work, flux, curl make sense independently of coordinate system!)

* given any S, decompose it into tiny, almost flat pieces

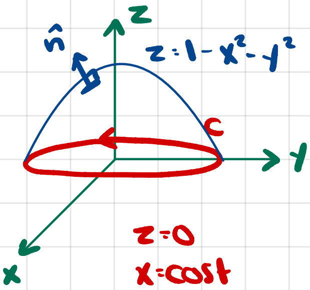


can apply Stokes to each small surface

all inner lines cancel out: sum of work around each little piece = work along C

sum of flux through each piece = flux through S

ex: work of $\vec{F} = \langle z, x, y \rangle$ around unit circle in xy-plane (counterclockwise)



$z = 0$
$x = \cos t$
$y = \sin t$

directly: $\oint_C z\,dx + x\,dy + y\,dz = \int_0^{2\pi} 0 + \cos t \cos t\,dt + \sin t \cdot 0$

$= \int_0^{2\pi} \cos^2 t\,dt = \pi$

* we are using paraboloid for sake of demonstration. Easier to choose flat disk.

stokes: $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\,dS = \iint_S \langle 1,1,1 \rangle \langle 2x, 2y, 1 \rangle\,dx\,dy = \iint (2x+2y+1)\,dx\,dy = (\dots) = \pi$

$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & y \end{vmatrix} = \langle 1-0, -(0-1), 1-0 \rangle = \langle 1,1,1 \rangle$

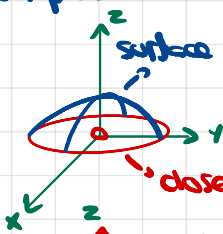
$\hat{n}\,dS = \langle -f_x, -f_y, 1 \rangle\,dx\,dy = \langle 2x, 2y, 1 \rangle\,dx\,dy$

Lecture 32

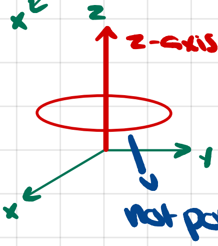
Stokes and Path Independence

Def: (Simply Connected Region) a region in space is simply connected if every closed loop inside it bounds a surface inside it.

examples



space w/ origin removed is simply connected



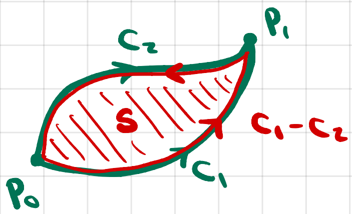
NOT simply connected

not possible to find surface whose boundary is this curve that doesn't include a piece of the z-axis (which isn't part of the region)

recall: if $\vec{F} = \nabla f$ is gradient field then curl $\vec{F} = 0$

theorem: if $\vec{F}$ defined in simply connected region and curl $\vec{F} = 0$ then $\vec{F}$ is a gradient, $\int_C \vec{F} \cdot d\vec{r}$ is path independent.

Proof



Assume curl $\vec{F} = 0$ we want to prove $\int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = 0$

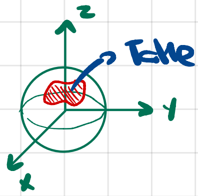
$\int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot d\vec{r}$, $C = C_1 - C_2$

C is closed curve, can apply Stokes but need a surface to apply it to

Given our assumption that $\vec{F}$ is defined in a simply connected region R, we know that any closed curve bounds a surface in R.

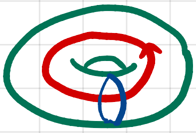
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \underbrace{(\text{curl}\,\vec{F})}_{0} \cdot d\vec{S} = 0$$

Remark Topology classifies surfaces in space



Take any curve on the surface, it bounds a surface on the sphere.

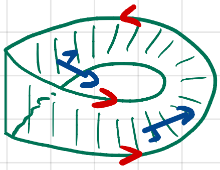
$\Rightarrow$ a sphere is simply connected



surface of a torus: R

2 independent loops that don't bound surfaces in R

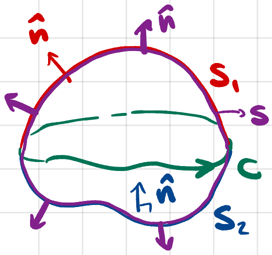
→ orientability



non-orientable surface

Flux cannot be defined

Stokes and "Surface Independence"



Stokes: $\oint_C \vec{F} \cdot d\vec{r} = \iint_{S_1} \text{curl} \vec{F} \cdot \hat{n} \, dS = \iint_{S_2} \text{curl} \vec{F} \cdot \hat{n} \, dS$

why are these the same?

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot \hat{n} \, dS - \iint_{S_2} (\nabla \times \vec{F}) \cdot \hat{n} \, dS = \iint_{S = S_1 - S_2} (\nabla \times \vec{F}) \cdot \hat{n} \, dS$$

S is S_1 with given orientation above plus S_2 with reversed orientation

By divergence theorem, since S is closed surface

$$\iint_{S = S_1 - S_2} (\nabla \times \vec{F}) \cdot \hat{n} \, dS = \iiint_D \text{div}(\nabla \times \vec{F}) \, dV$$

→ Can check: $\text{div}(\text{curl} \vec{F}) = \text{div}(\nabla \times \vec{F}) = 0$ always

$F = \langle P, Q, R \rangle \qquad \nabla \times \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

$$\text{div}(\nabla \times \vec{F}) = (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z$$

$$= \cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zx}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}} = 0$$

$\Rightarrow \nabla \cdot (\nabla \times \vec{F}) = 0$

* note: for "real" vectors $u \cdot (u \times v) = 0$; even though ∇ is not a "real" vector the same relation holds here for triple product.

Review

$$\iiint_R f\,dV$$

rect. coord.: $dV = dz\,dx\,dy$

cylind. coord: $dV = dz\,r\,dr\,d\theta$

spherical coord: $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

Applications

- mass
- avg value of f
- moment of inertia
- gravitational attraction on mass at origin

$$\iint_S \vec{F}\cdot\underbrace{\hat{n}\,dS}_{d\vec{S}}$$

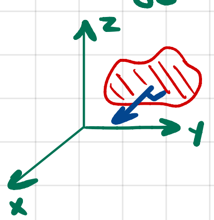
Formulas for $\hat{n}\,dS$

$\hat{n}\,dS = \ldots dx\,dy$ becomes $\iint \ldots dx\,dy$

→ horizontal plane

e.g. yz plane

$dS = dy\,dz \qquad \hat{n} = \pm\hat{\imath}$



→ spheres, cylinders centered at 0 on z-axis

$\hat{n}_{sph} = \dfrac{\langle x, y, z\rangle}{a} \qquad \hat{n}_{cyl} = \dfrac{\langle x, y, 0\rangle}{a}$

$dS = a^2 \sin\phi\, d\phi\, d\theta \qquad dS = a\,dz\,d\theta$

→ general cases

→ $z = z(x,y) \Rightarrow \hat{n} = \langle -z_x, -z_y, 1\rangle\, dx\,dy$

→ $\vec{N}$ given $\Rightarrow \hat{n}\,dS = \pm \dfrac{\vec{N}}{\vec{N}\cdot\vec{k}}\, dx\,dy$

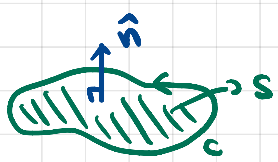
$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$

→ parametrize C, express in terms of a single variable

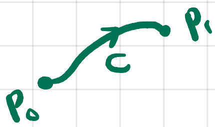
Relationships $\quad \iiint_R f\,dV \xleftrightarrow{\text{Divergence}} \iint_S \vec{F}\cdot\hat{n}\,dS \xleftrightarrow{\text{Stokes}} \int_C \vec{F}\cdot d\vec{r}$

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_D \operatorname{div}\vec{F}\,dV$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$$



$$\int_C \nabla f\cdot d\vec{r} = f(P_1) - f(P_0)$$



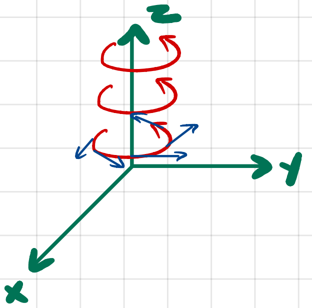
Lecture 33

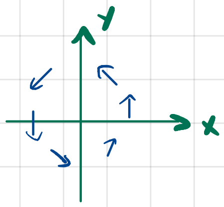
Curl recall $\vec{v}$ velocity field → curl $\vec{v}$ measures $2\times$ angular velocity vector (rotation part of the motion)

e.g. uniform rotation about z-axis

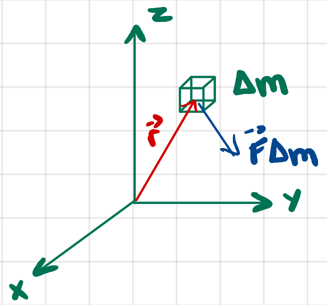
$\vec{v} = \omega\langle -y, x, 0\rangle$

$\nabla\times\vec{v} = \langle 0, 0, 2\omega\rangle$





For a force field



Torque $= \vec{\tau} = \vec{r}\times\underbrace{\vec{F}\Delta m}_{\text{force on mass }\Delta m}$

For translation motions $\dfrac{\text{Force}}{\text{Mass}} = \text{Acceleration} = \dfrac{d}{dt}(\text{velocity})$

curl curl curl

For rotation: $\dfrac{\text{Torque}}{\text{moment of inertia}} = \text{angular acceleration} = \dfrac{d}{dt}(\text{angular velocity})$

Consequence if $\vec{F}$ derives from a potential then curl $\vec{F} = 0$ → $\vec{F}$ does not generate any rotational motion

Electric / Magnetic Fields : Maxwell's Equations

$\vec{E}$: Electric field is vector field that tells you what kind of force will be exerted on a charged particle put in it

$\vec{B}$: Magnetic field

$\vec{F} = q\vec{E}$ $\qquad\qquad$ $\vec{F} = q\vec{v}\times\vec{B}$

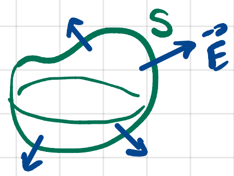
Div and Curl of $\vec{E}$

electric charge density

Gauss-Coulomb Law: $\nabla\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0}$ → partial diff. eq satisfied by $\vec{E}$

ε_0 → constant

Apply Divergence Theorem



S closed surface

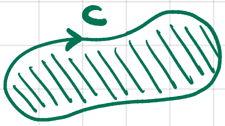
electric charge in D

Flux of $\vec{E}$ through S $= \oiint_S \vec{E}\cdot d\vec{S} = \iiint_D \operatorname{div}\vec{E}\, dV = \dfrac{1}{\varepsilon_0}\iiint_D \rho\, dV = \dfrac{Q}{\varepsilon_0}$

Faraday's Law tells you about $\nabla \times \vec{E}$

$$\text{curl}\,\vec{E} = \underbrace{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}_{\text{PDE}}$$

Apply Stokes:



$$\oint_C \vec{E}\cdot d\vec{r} = \iint_S (\nabla \times \vec{E})\cdot d\vec{S} = \iint_S \left(-\frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{S}$$

Remark $\nabla \cdot \vec{B} = 0$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

vector current density

Lecture 34 - Final Review

Unit 1

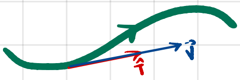
- vectors, dot product $\vec{A}\cdot\vec{B} = \sum a_i b_i = |\vec{A}||\vec{B}|\cos\theta$
- cross product $\vec{A}\times\vec{B}$
  - area in space (▱)
  - vector $\perp$ to $\vec{A}$ and $\vec{B}$
- equations of planes

  $ax+by+cz=d$ $\langle a,b,c\rangle$ normal vector
- equations of lines

  parametric $\langle x_0+at,\ y_0+bt,\ z_0+ct\rangle$

  $\langle x_0,y_0,z_0\rangle$ point on L, $\langle a,b,c\rangle$ parallel to line
- parametric equations of curves (decompose position vector into sum of simpler vectors)
- velocity $\vec{v}=\frac{d\vec{r}}{dt}$, speed $=|\vec{v}|$, $\vec{v}$ is tangent to trajectory

  $\vec{v}=\hat{T}\underbrace{\frac{ds}{dt}}_{\text{speed}}$



  acceleration $=\vec{a}=\frac{d\vec{v}}{dt}$
- matrices, determinants, linear systems

  3×3 linear system $\iff \underset{3\times3}{A}\,\underset{3\times1}{X}=\underset{3\times1}{B}$

  inverting (2×2 or 3×3) matrices

  $\underset{A}{\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}} \xrightarrow{} $ minors $[\ \ ]$ entries are 2×2 determinants formed by deleting 1 column and 1 row $\longrightarrow$ cofactors $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \longrightarrow$ transpose $\longrightarrow \frac{1}{\det(A)}$

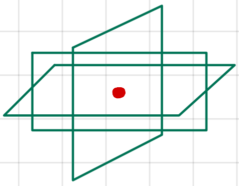
  * $\det A \neq 0 \iff$ matrix invertible

  Then $AX=B \iff X=A^{-1}B$

  otherwise $AX=B$ has either no solution or ∞ many solutions

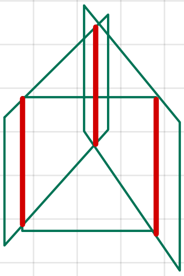
→ $AX = B$, intersection of three planes

$\det A \neq 0 \Rightarrow$



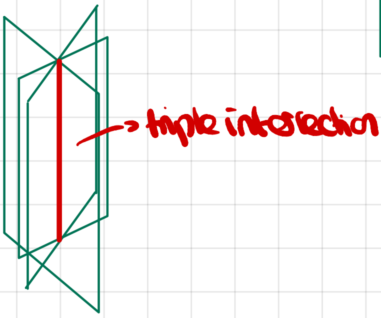
$\det = 0$, $B \neq 0$

→ all 3 planes are parallel to a same vector



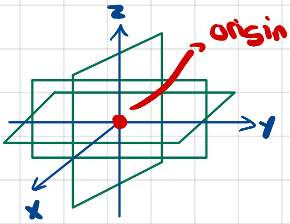
no triple intersection

→ infinite solutions

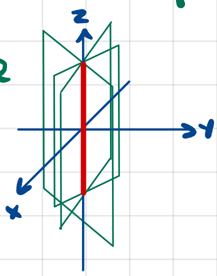


$\det = 0$, $B = 0$

→ All planes pass through origin $\Rightarrow \vec{0}$ is always a solution

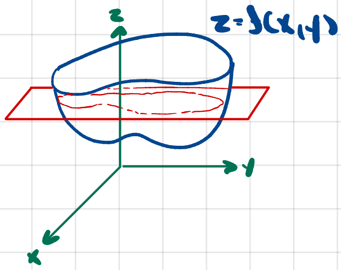


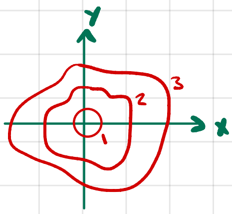
→ possible to fall in infinite solutions case



Unit 2

→ viewing $f(x,y)$: graph, contour plot





→ partial derivatives

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}$$

→ linear approximation

$$\Delta f \approx f_x \Delta x + f_y \Delta y$$

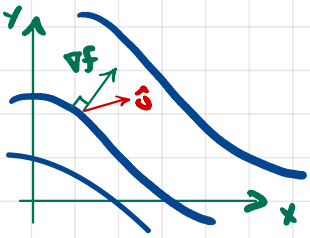
→ tangent plane to the graph of f

→ differentials, chain rules

$$df = f_x dx + f_y dy$$

If $\begin{array}{l} x = x(t) \\ y = y(t) \end{array} \Rightarrow \dfrac{df}{dt} = f_x \dfrac{dx}{dt} + f_y \dfrac{dy}{dt}$

→ gradient vector: $\nabla f = \langle f_x, f_y \rangle$



→ Directional Derivative $\frac{dF}{ds}\big|_{\hat{u}} = \nabla F \cdot \hat{u}$

→ max/min problems

critical points: $\nabla F = 0$ → second derivative test → local min / local max / saddle

also need to check values of f at boundaries of f

→ max/min f with non-independent variables

$g = c$ ⇒ lagrange multipliers, solve $\begin{cases} \nabla f = \lambda \nabla g \\ g = c \end{cases}$

* second derivative test does not apply here!

→ constrained partial derivatives: $g(x,y,z) = c$, $\frac{\partial F}{\partial x} = ?$

→ formal partial: $\frac{\partial f}{\partial x}$ x varies, y, z held constant

→ $\left(\partial F/\partial x\right)_y$ y held constant, z depends on x, y
x varies

→ $\left(\partial f/\partial x\right)_z$ y depends on x, z

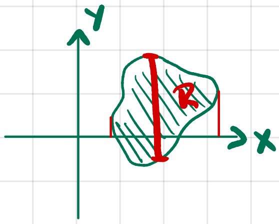
Lecture 35 - Final Review (cont'd)

Units 3 and 4

→ Double Integrals - setting up bounds

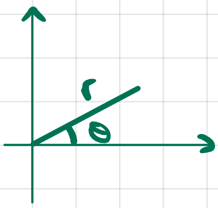
draw domain of integration

find bounds



$$\int_{x_{min}}^{x_{max}} \int_{y_{bottom}(x)}^{y_{top}(x)} f(x,y)\,dx\,dy$$

→ also in polar coordinates

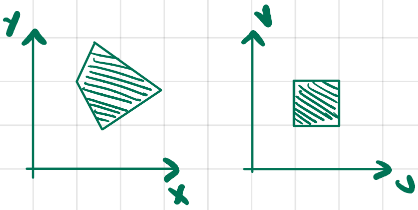


$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

→ changing to uv coordinates

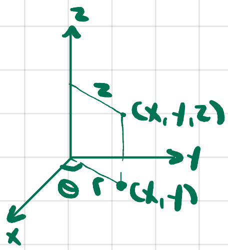


$$du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$$

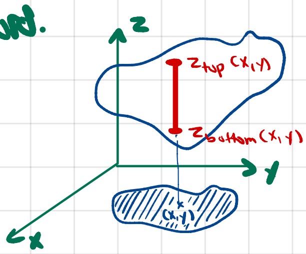
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

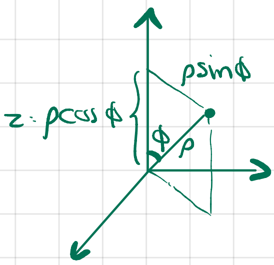
→ triple integrals

→ rectangular, cylindrical, spherical coord.



$dV = dz\,r\,d\theta$





$dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

→ Applications

→ area, volume, mass

$\iint 1\,dA \quad \iiint 1\,dV \quad \iint(\int)\,\delta\,\begin{matrix} dA \\ dV \end{matrix}$

→ average value of a function:

$$\bar{f} = \frac{1}{vol}\iiint_D f\,dV \quad \text{or (weighted):} \quad \frac{1}{mass}\iiint f\,\delta\,dV$$

→ center of mass $(\bar{x}, \bar{y}, \bar{z})$

→ moment of inertia $I_z = \iiint (x^2 + y^2)\,\delta\,dV$

→ gravitational attraction $\quad \vec{F} = Gm\iiint \frac{\delta\cos\phi}{\rho^2}\,dV$

→ Work and line integrals (in plane and space)

→ $\int_C \vec{F}\cdot d\vec{r}$ ⇒ $\int_C M\,dx + N\,dy$, express x and y in terms of single parameter

$F = \langle M, N \rangle$

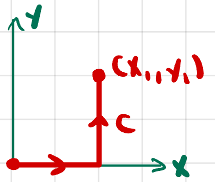
→ gradient fields and path independence

→ curl $\vec{F} = 0$ and F defined in simply connected region ⇒ $F = \nabla f$ for some f

in 2D, 1 condition $N_x = M_y$

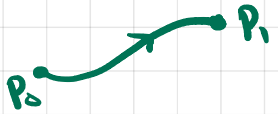
in 3D, 3 conditions

→ how to find the potential (2 methods)

① 

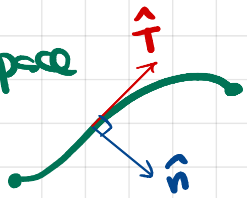
$\int_{C_1 + C_2} \vec{F}\cdot d\vec{r}$ gives $f(x_1, y_1)$

② start with $f_x = M$ $\xrightarrow[\int dx]{}$ $f = \underline{\qquad} + g(y)$
$g(y,z)$

→ once we have a potential, F.T.C 

$\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0)$

→ Flux in plane and space

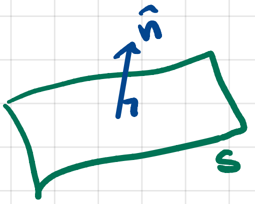
in the plane 

$\hat{n} = \hat{T}$ rotated 90° clockwise

$\hat{n}\,ds = \langle dy, -dx \rangle$

$\int_C \vec{F}\cdot\hat{n}\,ds = \int_C -Q\,dx + P\,dy$

$\vec{F} = \langle P, Q \rangle$

in space 

$\iint_S \vec{F}\cdot\hat{n}\,dS$

express $\hat{n}$ and dS geometrically or if S given by eqn $z = f(x,y)$ then

$\hat{n}\,dS = \langle -f_x, -f_y, 1 \rangle\,dx\,dy$

$\hat{n} = \pm \dfrac{\langle -f_x, -f_y, 1 \rangle}{(f_x^2 + f_y^2 + 1)^{1/2}}$ $dS = (1 + f_x^2 + f_y^2)\,dx\,dy$

or if we know some normal vector $\vec{N}$, $\hat{n}\,dS = \pm \dfrac{\vec{N}}{\vec{N}\cdot\hat{k}}\,dx\,dy$

→ slanted plane

→ $S: g(x,y,z) = 0$, $\nabla g = \vec{N}$

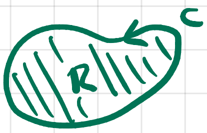
	2D	3D

Work

Green

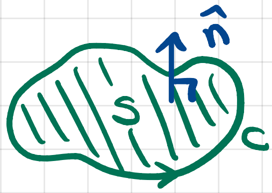
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\,dA$$

$$\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA$$



Stokes

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dA$$

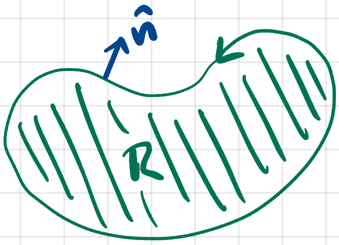


Flux

Green

$$\oint \vec{F}\cdot\hat{n}\,ds = \iint_R \text{div}(\vec{F})\,dA$$

$$\text{div}\langle M,N\rangle = M_x + N_y$$



Divergence

$$\oiint_S \vec{F}\cdot\hat{n}\,dS = \iiint_R \text{div}\,\vec{F}\,dV$$

$$\text{div}\langle P,Q,R\rangle = P_x + Q_y + R_z$$