X-axis as location of weightless lever arm supporting various point masses; origin: fulcrum

→ particle masses $m_0, m_1, \dots, m_n$ at locations $x_0, x_1, \dots, x_n$

law of the lever: the masses will balance if $\sum_{i=1}^{n} m_i x_i = 0$

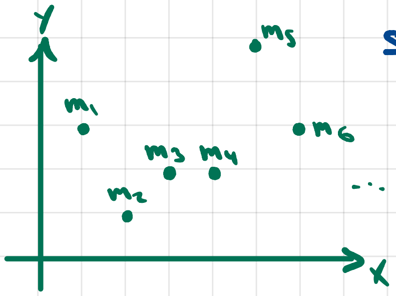
consider a single particle with mass $\sum m_i$

→ according to the law of the lever, $\sum m_i x_i - (\sum m_i)\bar{x} = 0$ ie the $\sum m_i$ mass placed at $-\bar{x}$ balances the other masses

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

$\sum m_i x_i$ } moment of the system of masses about the origin

$\bar{x}$ ↙ center of mass



System of n particles

in two dimensions:

moment about the y axis $= M_y = \sum m_i x_i$

" " " x " $M_x = \sum m_i y_i$

center of mass of the system $= (\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_y}{m} = \frac{\sum m_i x_i}{\sum m_i} \qquad \bar{y} = \frac{M_x}{m} = \frac{\sum m_i y_i}{\sum m_i}$$

Laminas and Thin Plates

→ ↑ number of particles, their masses decreasing in proportion ⇒ aggregate resembles plane region of varying density

→ lamina: thin plate, occupying bounded plane region R

consider density ρ to be constant $\rho = 1$ for convenience

centroid of plane region R: $(\bar{x}, \bar{y})$

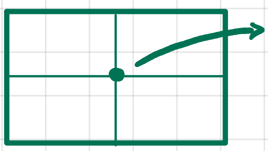
Def: $M_y(R)$, $M_x(R)$: moments of R about coordinate axes when $\rho = 1$

Def of physical principle of Additivity of Moments

R union of two non-overlapping regions S and T → $M_y(R) = M_y(S) + M_y(T)$

$M_x(R) = M_x(S) + M_x(T)$

consider the case of a rectangle, where is its centroid?



centroid because of symmetry principle → plane region R symmetric w/ respect to line L ⇒ centroid of R, considered lamina of constant density, lies on L

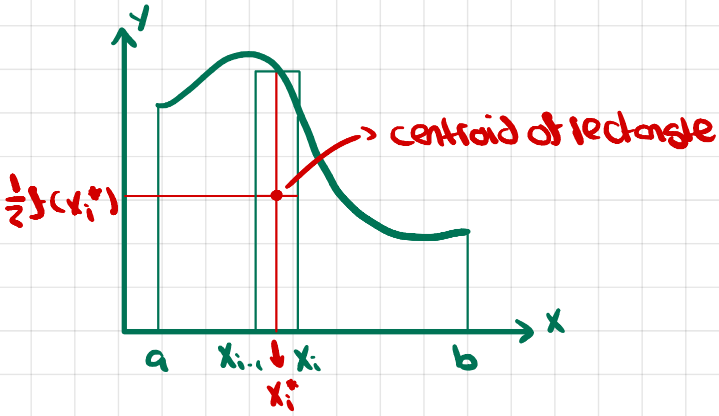
Def: moments of rectangle R with area A and centroid $(\bar{x}, \bar{y})$ are

$$M_y(R) = A\bar{x} \qquad M_x(R) = A\bar{y}$$

Goal: find centroid $(\bar{x}, \bar{y})$ of plane region

→ take the region between a function f, continuous and nonnegative on $[a,b]$, and the x-axis, for $a \le x \le b$.

→ partition $[a,b]$



→ i^{th} rectangle has centroid $(x_i^*, \frac{1}{2}f(x_i^*))$

area $f(x_i^*)\Delta x$, and moments

$$M_y = f(x_i^*)\Delta x \cdot x_i^* \qquad M_x = \frac{1}{2}f(x_i^*)^2\Delta x$$

→ P_n = union of all rectangles $i = 1, \dots, n$

each rectangle is non-overlapping w/ the others, so we can sum their moments

$$M_x(P_n) = \sum_{i=1}^{n} \frac{1}{2} f(x_i^*)^2 \Delta x \qquad M_y(P_n) = \sum_{i=1}^{n} f(x_i^*)\Delta x\, x_i^*$$

Def:

$$M_x(R) = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \frac{1}{2} f(x_i^*)^2 \Delta x = \int_a^b \frac{1}{2} f(x)^2\,dx$$

$$M_y(R) = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x\, x_i^* = \int_a^b x f(x)\,dx$$

$$\bar{x} = \frac{M_y(R)}{A} \qquad \bar{y} = \frac{M_x(R)}{A} \qquad A = \int_a^b f(x)\,dx$$