PSet 16 - Frequency Response - Part I

1. $\ddot{x} + 6\dot{x} + 45x = 50\cos\omega t$

$\ddot{z} + 6\dot{z} + 45z = 50e^{i\omega t}$

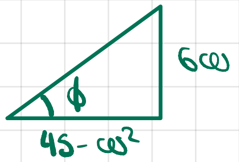
$p(r) = r^2 + 6r + 45$

$p(i\omega) = -\omega^2 + 6i\omega + 45$

response: $x_p(t) = \dfrac{50}{|p(i\omega)|} \cdot \cos(\omega t - \phi)$

$|p(i\omega)| = \sqrt{(45-\omega^2)^2 + 36\omega^2}$

$\phi = \mathrm{Arg}(p(i\omega)) = \tan^{-1}\left(\dfrac{6\omega}{45-\omega^2}\right)$



$$\text{gain}(\omega) = \frac{1}{\sqrt{(45-\omega^2)^2 + 36\omega^2}}$$

max gain when min $(45-\omega^2)^2 + 36\omega^2$

$2(45-\omega^2)(-2\omega) + 72\omega = 0$

$-4\omega(45-\omega^2) + 72\omega = -180\omega + 4\omega^3 + 72\omega = 0$

$\omega(4\omega^2 - 108) = 0$

$\Rightarrow \omega^2 = 27 \Rightarrow \omega = \sqrt{27}$

Part II

1. $\ddot{x} + \frac{1}{2}\dot{x} + 4x = \cos 2t$ $\qquad \ddot{z} + \frac{\dot{z}}{2} + 4z = e^{2it}$

$p(r) = r^2 + \frac{r}{2} + 4$

$p(2i) = -4 + i + 4 = i$

$|p(2i)| = 1$

a) gain $= 1$
phase $= \pi/2$

b) $z_p = \frac{1}{p(\omega i)} \cdot e^{i\omega t}$

complex gain $= \frac{1}{p(\omega i)} = \frac{1}{4 - \omega^2 + \frac{\omega i}{2}}$

$= \frac{4-\omega^2 - \omega i/2}{(4-\omega^2)^2 + \omega^2/4}$

$p(\omega i) = -\omega^2 + \frac{\omega i}{2} + 4$

gain $= \frac{1}{|p(i\omega)|} = \frac{1}{\sqrt{(4-\omega^2)^2 + \omega^2/4}}$

$\phi = Arg(p(i\omega))$

$\tan\phi = \frac{\omega/2}{4-\omega^2}$

2. a) max gain $\Rightarrow 2(4-\omega^2)(-2\omega) + \frac{\omega}{2} = 0$

$\Rightarrow -16\omega + 4\omega^3 + \frac{\omega}{2} = 0 \quad \Rightarrow 8\omega^3 - 31\omega = 0$

$\Rightarrow \omega(8\omega^2 - 31) = 0$

$\Rightarrow \omega_r = \sqrt{\frac{31}{8}} < 2$

$g(\omega_r) = 1.0079...$

b) $\phi = \pi/4 \Rightarrow \frac{\omega}{8-2\omega^2} = 1 \Rightarrow 2\omega^2 + \omega - 8 = 0$

$\Delta = 1 - 4\cdot 2\cdot(-8) = 65 \Rightarrow \omega = \frac{-1+\sqrt{65}}{4} = 1.76...$

c) Nyquist Plot

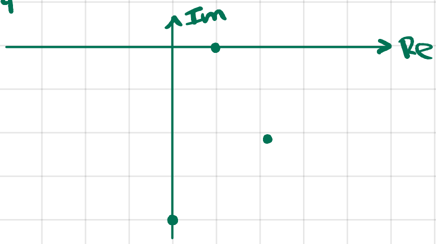
$\tilde{g}(\omega) = \frac{1}{p(\omega i)} = \frac{1}{4-\omega^2+\frac{\omega i}{2}} = \frac{4-\omega^2 - \frac{\omega}{2}i}{(4-\omega^2)^2 + (\frac{\omega}{2})^2}$

on complex plane $\left\langle \frac{4-\omega^2}{(4-\omega)^2+(\omega/2)^2}, \frac{-\omega/2}{(4-\omega)^2+(\omega/2)^2} \right\rangle$

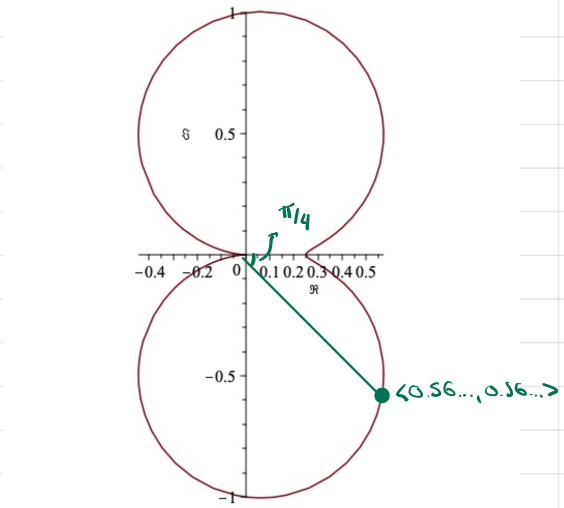
$\omega = 0 \Rightarrow \langle 1/4, 0 \rangle$

$\omega = 2 \Rightarrow \langle 0, -1 \rangle$

$\phi = \frac{\pi}{4} \Rightarrow \omega = 1.76... \Rightarrow \langle 0.56..., -0.56... \rangle$



$\left\langle \frac{4-\omega^2}{(4-\omega)^2+(\omega/2)^2}, \frac{-\omega/2}{(4-\omega)^2+(\omega/2)^2} \right\rangle =$



d) $p(r) = mr^2 + br + k$

$p(i\omega) = -m\omega^2 + b\omega i + k = (k-\omega^2) + b\omega i$

$\tilde{g}(\omega) = \frac{1}{k - m\omega^2 + b\omega i} = \frac{(k-\omega^2) - b\omega i}{(k-m\omega^2)^2 + b^2\omega^2}$

$\tilde{g}$ is purely imag. when $k - \omega^2 = 0$. This condition does not depend on b. The magnitude of the purely im. part does.

$\tilde{g}(2) = \frac{-2bi}{b^2 \cdot 4} = \frac{-i}{2b}$ $\qquad \uparrow b \Rightarrow \downarrow$ max real gain $= \frac{1}{2b}$

$|\tilde{g}(\omega)| = \frac{1}{|p(i\omega)|} = \frac{1}{\sqrt{(4-\omega^2)^2 + b^2\omega^2}}$

$\Rightarrow$ real gain depends on b

As we reduce k we are reducing the spring constant

Also,

$$mx'' + bx' + kx = 0$$

$$p(r) = mr^2 + br + k$$

$$e^{rt} p(r) = 0$$

$$\Delta = b^2 - 4mk$$

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$, assume $b^2 - 4mk < 0$

$$e^{\frac{-bt}{2m}} e^{\frac{\sqrt{4mk-b^2}}{2m} it}$$

call $\frac{\sqrt{4mk-b^2}}{2m} = \omega_0$

$$e^{\frac{\sqrt{4mk-b^2}}{2m} it} = e^{\omega_0 it} = \cos\omega_0 t + i\sin\omega_0 t$$

$$\Rightarrow e^{\frac{-bt}{2m}} \cos\omega_0 t, \ e^{\frac{-bt}{2m}} \sin\omega_0 t \text{ are sol'ns}$$

$$\Rightarrow x_h = c_1 e^{\frac{-bt}{2m}} \cos\omega_0 t + c_2 e^{\frac{-bt}{2m}} \sin\omega_0 t$$

$$= e^{\frac{-bt}{2m}} (c_1 \cos\omega_0 t + c_2 \sin\omega_0 t)$$

So w/o input, w/ complex roots, angular freq. (pseudo-freq.) is $\frac{\sqrt{4mk-b^2}}{2m}$. Reducing spring constant (or increasing damping) reduce this "natural" pseudo-freq.

$$\rightarrow \tilde{g}(\omega) = \frac{(k - m\omega^2) - b\omega i}{(k - m\omega^2)^2 + b^2\omega^2}$$ ⇒ purely imag (ie $\phi = \pi/2$) when

$$\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}}$$

$$\phi = \operatorname{Arg}(p(i\omega)) = \operatorname{Arg}(-m\omega^2 + bi\omega + k)$$

$$\Rightarrow \tan\phi = \frac{b\omega}{k - m\omega^2} \Rightarrow \uparrow b \rightarrow \uparrow \phi$$

$$\tilde{g}(\omega) = \frac{1}{k - m\omega^2 + b\omega i} = \frac{(k - m\omega^2) - b\omega i}{(k - m\omega^2)^2 + b^2\omega^2}$$

$$\phi = \frac{\pi}{2} \Rightarrow \frac{k - m\omega^2}{(k - m\omega)^2 + b^2\omega^2} = 0 \Rightarrow \omega = \sqrt{\frac{k}{m}}$$