Pset - Complex Numbers

Problem 1

a) $z = -1+i = re^{i\theta} = \sqrt{2}(\cos\theta + i\sin\theta) \Rightarrow \cos\theta = -1/\sqrt{2}$ $\Rightarrow \theta = 3\pi/4 + 2\pi k$

$|z| = \sqrt{2}$ $\qquad \sin\theta = 1/\sqrt{2}$

$-1+i = \sqrt{2}e^{(\frac{3\pi}{4}+2\pi k)i}$

b) $z = \sqrt{3} - i = 2e^{i\theta} \Rightarrow \cos\theta = \sqrt{3}/2$ $\Rightarrow \theta = -\pi/6 + 2\pi k$

$|z| = 2$ $\qquad \sin\theta = -1/2$

$z = \sqrt{3} - i = 2e^{(-\frac{\pi}{6}+2\pi k)i}$

Problem 2

$z = \dfrac{1-i}{1+i} = \dfrac{-2i}{2} = -i$

$1-i = \sqrt{2}e^{i\theta} \Rightarrow \cos\theta = 1/\sqrt{2}$ $\Rightarrow \theta = -\pi/4$

$\sin\theta = -1/\sqrt{2}$

$\Rightarrow 1-i = \sqrt{2}e^{-\frac{\pi i}{4}}$

$1+i = \sqrt{2}e^{i\theta} \Rightarrow \cos\theta = 1/\sqrt{2}$ $\Rightarrow \theta = \pi/4$

$\sin\theta = 1/\sqrt{2}$

$\Rightarrow 1+i = \sqrt{2}e^{\pi i/4}$

$\Rightarrow z = \dfrac{\sqrt{2}e^{-\frac{\pi}{4}i}}{\sqrt{2}e^{\frac{\pi}{4}i}} = e^{-\frac{\pi}{2}i} = \cos(-\pi/2) + i\sin(-\pi/2) = -i$

* $i = -\sqrt{1}$
$i^2 = -1$
$i^3 = -i$
$i^4 = -i^2 = 1$

Problem 3

a) $z = (1-i)^4$

using binomial thm: $z = 1 - 4\cdot1^3\cdot i + 6\cdot1^2 i^2 - 4\cdot1\cdot i^3 + i^4 = 1 - 4i - 6 + 4i + 1 = -4$

DeMoivre: $z = \left(\sqrt{2}e^{\frac{-\pi i}{4}}\right)^4 = 4\cdot e^{-\pi i} = 4(\cos(-\pi) + i\sin(-\pi)) = 4(-1) = -4$

b) $z = (1+i\sqrt{3})^3$

bin. thm: $z = 1^3 + 3\cdot1^2\cdot i\sqrt{3} + 3\cdot1\cdot i^2\cdot 3 + i^3\cdot 3^{3/2}$
$= 1 + 3\sqrt{3}i - 9 - 3\sqrt{3}i$
$= -8$

DeMoivre: $z_1 = 1+\sqrt{3}i \quad |z_1| = 2$ $\Rightarrow z = (2e^{\frac{\pi}{3}i})^3 = 8e^{\pi i} = 8(\cos\pi + \sin\pi\, i) = -8$

$z_1 = 2e^{i\theta} \quad \cos\theta = \frac{1}{2} \quad \theta = \pi/3$

$z_1 = 2e^{\pi i/3}$

Problem 4 Find sixth-roots of 1, $\sqrt[6]{1}$

$$z = \rho e^{i\phi} = (e^{i2\pi k})^{1/6} = e^{\frac{i\pi k}{3}} \Rightarrow \rho = 1,\ \phi = \frac{\pi k}{3}$$

$$1 = e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow \theta = 2\pi k \quad k \in \mathbb{Z}$$

$$\Rightarrow 1 = e^{i2\pi k}$$

$$\Rightarrow z = \sqrt[6]{1} = e^{\frac{\pi k}{3}i} = \begin{cases} 1 \\ e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i \\ e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ e^{\pi i} = -1 \\ e^{4\pi i/3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\ e^{5\pi i/3}\ \ \frac{1}{2} - \frac{\sqrt{3}}{2}i \end{cases}$$

Problem 5 $x^4 + 16 = 0$

$$x^4 = -16 \Rightarrow x = \sqrt[4]{-16} = 2\sqrt[4]{-1}$$

$$z = \rho e^{i\phi} = \sqrt[4]{-1} = (e^{i(\pi + 2\pi k)})^{1/4} = e^{\frac{i(\pi+2\pi k)}{4}} \Rightarrow \rho = 1,\ \phi = \frac{\pi}{4} + \frac{\pi k}{2}$$

$$-1 = e^{i\theta} = \cos\theta + i\sin\theta$$

$$\cos\theta = -1 \Rightarrow \theta = \pi + 2\pi k \quad k \in \mathbb{Z}$$

$$-1 = e^{i(\pi + 2\pi k)}$$

$$\Rightarrow z = e^{i(\frac{\pi}{4} + \frac{\pi k}{2})} \quad k = 0,1,2,3$$

$$x = 2e^{i(\frac{\pi}{4} + \frac{\pi k}{2})} \quad k = 0,1,2,3$$

$$= 2e^{i\frac{\pi}{4}} = 2\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \sqrt{2} + \sqrt{2}i$$

$$2e^{i\frac{3\pi}{4}}$$

$$2e^{\frac{5\pi}{4}i}$$

$$2e^{\frac{7\pi}{4}i}$$