Basic DEs

Oryx population

→ natural growth rate: k year^{-1}

→ harvesting rate: a oryxes · year^{-1}

1. $\dot{p} = kp - a$ oryxes · year^{-1}

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2. $a = 0 \Rightarrow \dot{p} = kp \Rightarrow \frac{1}{kp} dp = dt \Rightarrow \frac{1}{k} \ln|kp| = t + C_1 \Rightarrow |kp| = e^{kt + C_1} \Rightarrow kp = \pm e^{C_1} e^{kt}$

$kp \neq 0$

$\Rightarrow p(t) = Ce^{kt} \quad C \neq 0, \quad p(0) = C \Rightarrow p(t) = p_0 e^{kt}$

* note $p(t) = 0$ is a lost solution.
  $C \neq 0$. If C could be zero we could accidentally have recovered the lost solution.
  It is still accurate, however to include $C = 0$ and obtain the general solution $p(t) = Ce^{kt}$, $C \in \mathbb{R}$.

Doubling Time: $2p_0 = p_0 e^{kt} \Rightarrow e^{kt} = 2 \Rightarrow kt = \ln 2 \Rightarrow t = \frac{\ln 2}{k} \approx 0.69/k$

note: $p(t + \frac{\ln 2}{k}) = p_0 e^{k \ln 2 / k} = p_0 \cdot 2$ ie the population doubles after $\frac{\ln 2}{k}$ starting from any point in time.

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3. $\frac{dp}{dt} = kp(t) - a \Rightarrow \frac{1}{kp - a} dp = dt \Rightarrow \frac{1}{k} \ln|kp - a| = t + C_1 \Rightarrow \ln|kp - a| = kt + kC_1$

$kp - a \neq 0 \Rightarrow p \neq a/k$

$\Rightarrow |kp - a| = e^{kC_1} e^{kt} = C_2 e^{kt} \Rightarrow kp - a = \pm C_2 e^{kt} \Rightarrow kp = C_3 e^{kt} + a \Rightarrow p(t) = Ce^{kt} + \frac{a}{k}, \quad C \neq 0$

Solution when $k = 0$

$\frac{dp}{dt} = -a \Rightarrow p(t) = -at + C = -at + p_0$

4. check solutions

$p(t) = Ce^{kt} + \frac{a}{k} \qquad p'(t) = kCe^{kt} \qquad kp(t) - a = k \cdot (Ce^{kt} + \frac{a}{k}) - a = kCe^{kt} = p'(t)$

$p(t) = a/k \qquad p'(t) = 0 \qquad kp(t) - a = ka/k - a = 0 = p'(t)$

$p(t) = -at + p_0 \qquad p'(t) = -a \qquad \cancel{k}^{0}p(t) - a = -a$

General Solution

$p(t) = Ce^{kt} + \frac{a}{k}, C \in \mathbb{R}$

$p(t) = -at + p_0$

5. $C = 0 \Rightarrow p(t) = \frac{a}{k}, \; k \neq 0$

→ natural growth rate: k year^{-1}

→ harvesting rate: a oryxes · year^{-1}

$\dot{p} = kp - a$ oryxes · year^{-1}

Example:

$a = 10$ oryx/year harvested

$k = 0.25$ / year

if we harvest the same number of oryx per year than the number born each year, the population stays at the same value.

$$a = k \cdot \bar{p} \Rightarrow \bar{p} = \frac{a}{k}. \text{ In our ex. } \bar{p} = \frac{10}{0.25} = 40$$

Note the other solution for $k \neq 0$ has this a/k term as well: $p(t) = Ce^{kt} + \frac{a}{k}$

$p(0) = C + a/k \Rightarrow C = p_0 - a/k$

$$p(t) = \left(p_0 - \frac{a}{k}\right)e^{kt} + \frac{a}{k}$$

in our numerical example: $p(t) = (p_0 - 40)e^{.25t} + 40$

if $p_0 \leq a/k$ the population decreases exponentially.

$p_0 k \leq a$ means the harvesting of oryx is larger (in oryx/year) than births of oryx, so the population decreases each year.

6. $p(t) = a/k$ is unstable because any change in a or k or both that alters a/k leads to either explosive growth or decay.

Time to extinction if $p_0 = \frac{a}{2k}$

$$p(t) = \left(\frac{a}{2k} - \frac{a}{k}\right)e^{kt} + \frac{a}{k} = 0 \Rightarrow -\frac{a}{2k}e^{kt} = -\frac{a}{k} \Rightarrow e^{kt} = 2 \Rightarrow kt = \ln 2 \Rightarrow t_e = \frac{\ln 2}{k}$$

General time to extinction

$$p(t) = 0 = (p_0 - a/k)e^{kt} + a/k \Rightarrow e^{kt} = \frac{-a}{k(p_0 - a/k)} \Rightarrow t_e = \frac{1}{k}\ln\left[\frac{a}{a - kp_0}\right]$$

Recap

$$\frac{dp}{dt} = kp(t) - a$$

Case 1 $a = 0, k = 0 \Rightarrow p'(t) = 0 \Rightarrow p(t) = p_0$ no natural growth, no harvesting, no change

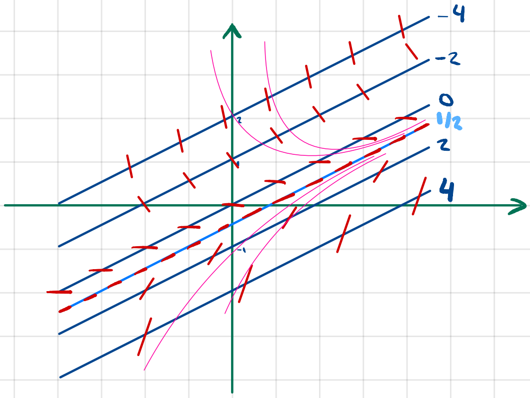
Case 2 $a = 0, k \neq 0 \Rightarrow p'(t) = kp(t) \Rightarrow p(t) = p_0 e^{kt}$ natural growth, no harvesting, exponential growth/decay depending on sign of k.

Case 3 $a \neq 0, k = 0 \Rightarrow p'(t) = -a \Rightarrow p(t) = p_0 - at$ only harvesting $\Rightarrow$ linear decrease from initial population

Case 4 $a \neq 0, k \neq 0 \Rightarrow p'(t) = kp(t) - a \Rightarrow p(t) = (p_0 - a/k)e^{kt} + \frac{a}{k}$ natural growth and harvesting, behavior of $p(t)$ depends on relative magnitudes of p_0 and a/k.

Geometric Methods

$\frac{dy}{dx} = x - 2y$



1. $x - 2y = m \Rightarrow y = \frac{x}{2} - \frac{m}{2}$

2. $y = mx + b \Rightarrow y' = m$

$x - 2(mx+b) = m$

$\Rightarrow x - 2mx - 2b = m$

$x(1-2m) - 2b = m$

$\Rightarrow 1 - 2m = 0 \Rightarrow m = \frac{1}{2}$

$-2b = m \Rightarrow b = -\frac{m}{2} = -\frac{1}{4}$

$\Rightarrow y = \frac{x}{2} - \frac{1}{4}$

3. $\frac{dy}{dx} = F(x,y)$

$y = mx+b$ solution $\Rightarrow F(x, mx+b) = m \Rightarrow$ the solution is an isocline.

The slope is constant on a straight line. As a solution, the slope at each point must be the slope of the slope field at that point ⇒ the slope field has the same slope at each point on the line ⇒ the line is an isocline.

4. Isoclines are straight lines. $y = \frac{x}{2} - \frac{1}{4}$ is a separatrix. Any isocline above it is an upper fence, and any isocline below it is a lower fence. Together any two isoclines, one above and one below the separatrix, form a funnel.

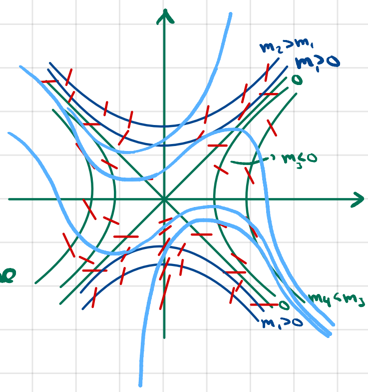
5. The isocline $y' = x - 2y = 0$ represents critical points of the solutions to $y' = x - 2y$.
From the Existence and Uniqueness Theorem we know that for any point (a,b) on the nullcline, ie where $y' = a - 2b = 0$, there is exactly one solution that passes through (a,b). After passing the nullcline, an upper fence, the solution does not cross the nullcline again. Each solution has one critical point.

The nullcline is $y = \frac{x}{2}$. The separatrix isocline is $y = \frac{x}{2} - \frac{1}{4}$. Any solution w/ $y(0) = y_0 > -\frac{1}{4}$ crosses the nullcline, ie any solution lying above the separatrix has a critical point, and it is a minimum. Also, $y'' = 1 - 2y'$. When $y' = 0$ then $y'' = 1 > 0$ ⇒ critical points are local minima, and since there is only one critical point per solution, it is also a global minimum.

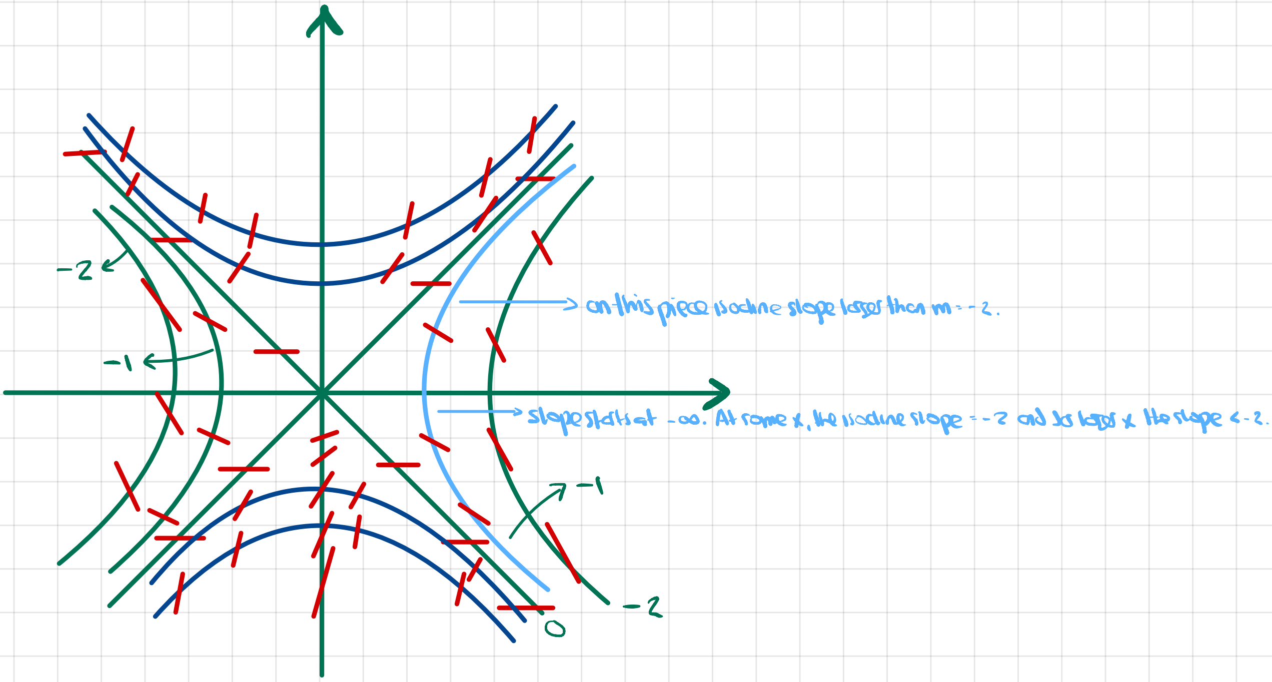
6. $\frac{dy}{dx} = y^2 - x^2$ isoclines: $y = \pm\sqrt{m + x^2}$

7. isocline with $m = 1$ seems to be a separatrix.

8. Any isocline with $m < -1$ is an upper fence in a funnel with lower fence $m = -1$, for x larger than some value. Solutions trapped in this funnel approach



Similarly any isocline w/ $m < -1$ is a lower fence in a funnel with upper fence $m = -1$ for x smaller than a certain value. Solutions trapped in this funnel approach $y = (-1 + x^2)^{1/2}$ as $x \to -\infty$.



$y^2 = x^2 - m$

−2-isocline: $y^2 = x^2 - 2 \Rightarrow 2yy' = 2x \Rightarrow y' = \frac{x}{y}$

$\frac{x}{y} > -2$ with $y^2 = x^2 - 2 \Rightarrow x > -2y,\ x^2 > 4y^2 \Rightarrow x^2 - 2 > 4y^2 - 2$

$\Rightarrow y^2 > 4y^2 - 2 \Rightarrow 3y^2 < 2 \Rightarrow y^2 < 2/3 \Rightarrow -(2/3)^{1/2} < y < (2/3)^{1/2}$

$2(2/3)^{1/2} > -2y > -2(2/3)^{1/2}$

$x > -2y > 2(2/3)^{1/2}$

Numerical Methods

1. $\frac{dy}{dx} = y^2 - x^2 \qquad y(0) = -1 \qquad$ Estimate $y(1.5)$

Euler's Method

$h = 0.5$

Before calculating the Euler method iterations

$y'' = 2yy' - 2x$

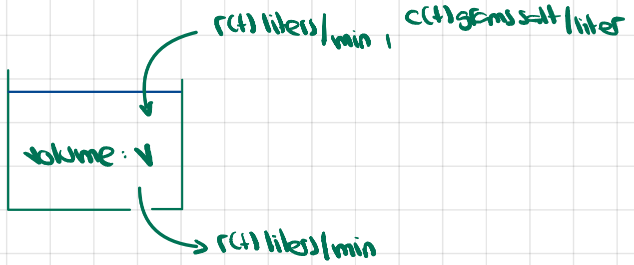
At $y(0) = -1$, $y''(0) = 2(-1)\cdot 1 - 0 = -2$

The solution passing through $(0,-1)$ is concave. Euler will be too high.

n	x_n	y_n	m_n	$m_n h$
0	0	−1	1	0.5
1	0.5	−0.5	0	0
2	1	−0.5	−.75	−.375
3	1.5	−.875		

Linear Models – Mixing Problem

1.



$x(t)$: amount of salt in tank, grams

$$-\frac{x(t)}{V}\,\frac{g}{L}\cdot r(t)\,\frac{L}{min}\cdot \Delta t \text{ min} + r(t)\,\frac{L}{min}\cdot c(t)\,\frac{g}{L}\cdot \Delta t \text{ min} = \Delta x \text{ g}$$

$$\frac{\Delta x}{\Delta t} = r(t)\cdot c(t) - \frac{x(t)\,r(t)}{V}$$

$$\frac{dx}{dt} + \frac{r(t)}{V}x(t) = r(t)c(t)$$

2. $\dot{x} + \frac{r}{V}x = rc \Rightarrow e^{\frac{r}{V}t}\dot{x} + e^{\frac{r}{V}t}\frac{r}{V}x = e^{\frac{r}{V}t}rc \Rightarrow \left(e^{\frac{r}{V}t}x\right)' = e^{\frac{r}{V}t}rc \Rightarrow e^{\frac{r}{V}t}x = \frac{V}{\cancel{r}}e^{\frac{r}{V}t}\cancel{r}c + C_1 = Vce^{\frac{r}{V}t} + C_1$

$r = 2\,L/min$

$V = 1\,L$

$x(0) = 0$

$u(x) = e^{\int \frac{r}{V}dt} = e^{\frac{r}{V}t}$

$\Rightarrow x(t) = Vc + C_1e^{-\frac{r}{V}t}$

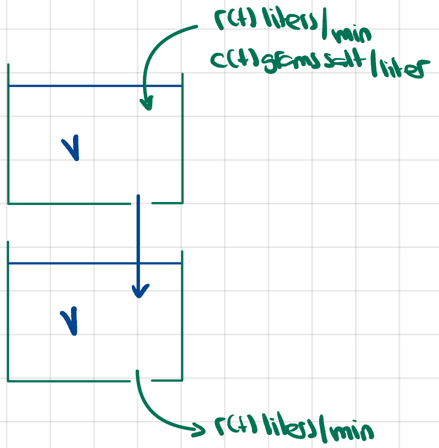
$x(0) = Vc + C_1 = 0 \Rightarrow C_1 = -Vc \quad \Rightarrow x(t) = Vc - Vce^{-\frac{r}{V}t} = Vc\left(1 - e^{-\frac{r}{V}t}\right)$

interpretation: Initially there is a volume V of water with no salt. A saline solution with concentration c is added at rate r. The same amount of the current volume V is removed. The concentration of V rises because the amount of salt increases as $V\cdot c\left(1-e^{-\frac{r}{V}t}\right)$ and V is constant. In particular, $x(t\to\infty) = V\cdot c = 1L\cdot c\ g/L = c\ g$

The tank contains half of the limiting value $V\cdot c$ when: $x(t) = \frac{\cancel{Vc}}{2} = \cancel{Vc}\left(1-e^{-\frac{r}{V}t}\right) \Rightarrow 1 = 2 - 2e^{-\frac{r}{V}t} \Rightarrow 2e^{-\frac{r}{V}t} = 1 \Rightarrow -\frac{r}{V}t = \cancel{\ln 1} - \ln 2$

$\Rightarrow t = \frac{V\ln(2)}{r} = \frac{\ln(2)}{2} \approx 0.35$ min

3.



$$-\frac{x_2(t)}{V}\cdot r(t)\cdot \Delta t + r(t)\cdot\frac{x(t)}{V}\cdot\Delta t = \Delta x_2$$

$$\Rightarrow \frac{\Delta x_2}{\Delta t} = \frac{rx}{V} - \frac{rx_2}{V} = \frac{r}{V}(x - x_2)$$

$$\Rightarrow \frac{dx_2}{dt} = \frac{r}{V}\left(x(t) - x_2(t)\right)$$

First Order Linear ODEs: Integrating Factors

1.

a) $y(t)$: ocean height

$x(t)$: bay height

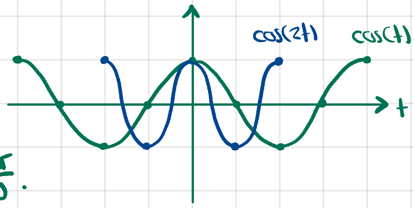
$$\Delta x = k(y(t)-x(t))\Delta t$$

$$\frac{\Delta x}{\Delta t} = k(y-x) \Rightarrow \frac{dx}{dt} = k(y(t)-x(t)) \Rightarrow \dot{x} + kx = ky$$

system: pond/channel/ocean

output: pond water height

input: ocean height



b) $y(t) = \cos(\omega t)$

The period of $\cos(\omega t)$ is $\frac{2\pi}{\omega}$.

If $T = 4\pi$ then $\frac{2\pi}{\omega} = 4\pi \Rightarrow \omega = \frac{1}{2}$

$$y(t) = \cos(t/2)$$

c) $\dot{x} + kx = k\cos(t/2)$

$$u(t) = e^{kt} \Rightarrow x(t) = e^{-kt}\left[\int k\cos(t/2)e^{kt}dt + c\right] = e^{-kt}\left[\frac{k(4ke^{kt}\cos(t/2) + 2e^{kt}\sin(t/2))}{1+4k^2} + c\right]$$

$$x(t) = \frac{4k^2\cos(t/2) + 2k\sin(t/2)}{1+4k^2} + ce^{-kt}$$

$$\int \cos(t/2)e^{kt}dt = \frac{e^{kt}}{k}\cos(t/2) + \int \frac{e^{kt}}{k}\cdot\frac{\sin(t/2)}{2}dt$$

$u = \cos(t/2)$ $\quad du = \frac{1}{2}(-\sin(t/2))dt$

$dv = e^{kt}dt$ $\quad v = \frac{e^{kt}}{k}$

$$\int \sin(t/2)e^{kt}dt = \frac{e^{kt}}{k}\sin(t/2) - \int \frac{e^{kt}}{k}\frac{\cos(t/2)}{2}dt$$

$u = \sin(t/2)$ $\quad du = \cos(t/2)\cdot\frac{1}{2}dt$

$dv = e^{kt}dt$ $\quad v = e^{kt}\cdot\frac{1}{k}$

$$\Rightarrow \int \cos(t/2)e^{kt}dt = \frac{e^{kt}}{k}\cos(t/2) + \frac{1}{2k}\left[\frac{e^{kt}}{k}\sin(t/2) - \frac{1}{2k}\int e^{kt}\cos(t/2)dt\right]$$

$$= \frac{e^{kt}}{k}\cos(t/2) + \frac{e^{kt}}{2k^2}\sin(t/2) - \frac{1}{4k^2}\int e^{kt}\cos(t/2)dt$$

$$\Rightarrow \left(1 + \frac{1}{4k^2}\right)\int \cos(t/2)e^{kt}dt = \frac{e^{kt}}{k}\cos(t/2) + \frac{e^{kt}}{2k^2}\sin(t/2)$$

$$\Rightarrow \int \cos(t/2)e^{kt}dt = \frac{4ke^{kt}\cos(t/2) + 2e^{kt}\sin(t/2)}{1+4k^2}$$

d) $x(t) = a\cos(\omega t) + b\sin(\omega t)$

$\dot{x} + kx = k\cos(t/2)$, sub in $x(t)$.

$$\dot{x}(t) = -a\omega\sin(\omega t) + b\omega\cos(\omega t)$$

$$-a\omega\sin(\omega t) + b\omega\cos(\omega t) + k(a\cos(\omega t) + b\sin(\omega t)) = k\cos(\omega t)$$

$$\sin(\omega t)(-a\omega + kb) + \cos(\omega t)(b\omega + ka) = k\cos(\omega t)$$

$\Rightarrow kb = a\omega$

$k = b\omega + ka \Rightarrow k(1-a) = b\omega \Rightarrow b = \frac{k(1-a)}{\omega}$

$$\Rightarrow \frac{k^2(1-a)}{\omega} = a\omega \Rightarrow k^2 - k^2a = a\omega^2 \Rightarrow a(\omega^2 + k^2) = k^2$$

$$\Rightarrow a = \frac{k^2}{k^2+\omega^2}$$

$$b = \frac{k}{\omega}\left(\frac{\cancel{k^2} + \omega^2 - \cancel{k^2}}{k^2+\omega^2}\right) = \frac{k\omega^2}{\omega(k^2+\omega^2)}$$

$$\omega = \frac{1}{2} \Rightarrow a = \frac{k^2}{k^2+\frac{1}{4}} = \frac{4k^2}{1+4k^2}$$

2. $\frac{dy}{dx} = x - 2y$

$y' + 2y = x$

$u(x) = e^{2x}$

$e^{2x} y = \int e^{2x} x \, dx + C \Rightarrow y(x) = e^{-2x}\left[\frac{e^{2x}(2x-1)}{4} + C\right] \Rightarrow y(x) = \frac{2x-1}{4} + Ce^{-2x}$

$u = x \quad du = dx$

$dv = e^{2x} dx \quad v = \frac{e^{2x}}{2}$

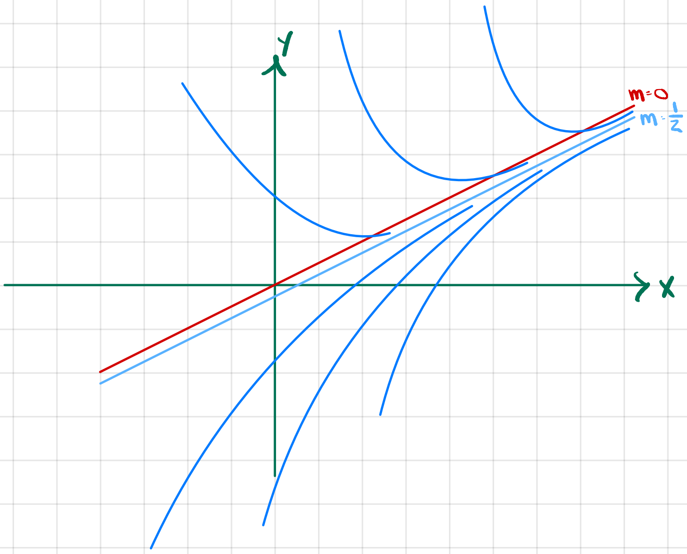
$\int e^{2x} x \, dx = \frac{1}{2} x e^{2x} - \int \frac{1}{2} e^{2x} dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} = \frac{e^{2x}(2x-1)}{4}$

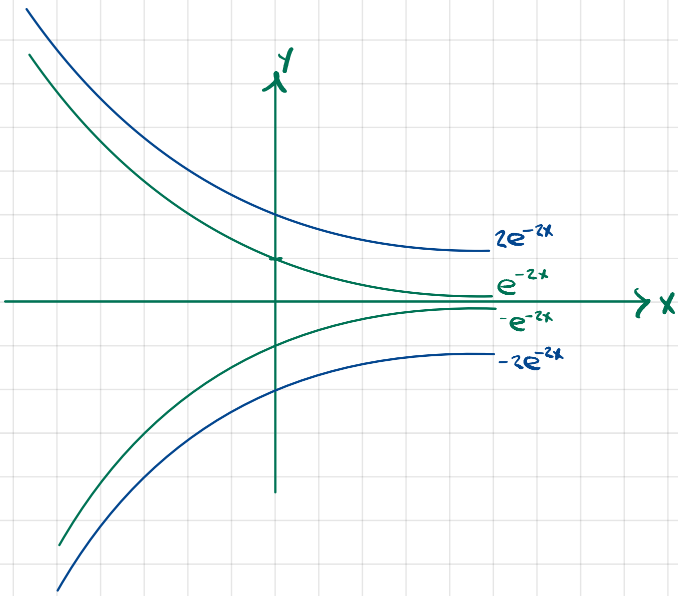
$\Rightarrow$ $C = 0$ is a straight line solution $y(x) = \frac{2x-1}{4}$

isoclines are lines $x - 2y = m \Rightarrow y = \frac{x-m}{2}$

$m = \frac{1}{2} \Rightarrow y = \frac{x}{2} - \frac{1}{4} = \frac{2x-1}{4}$, an isocline that is also a solution.

nullcline: $x - 2y = 0 \Rightarrow y = \frac{x}{2}$



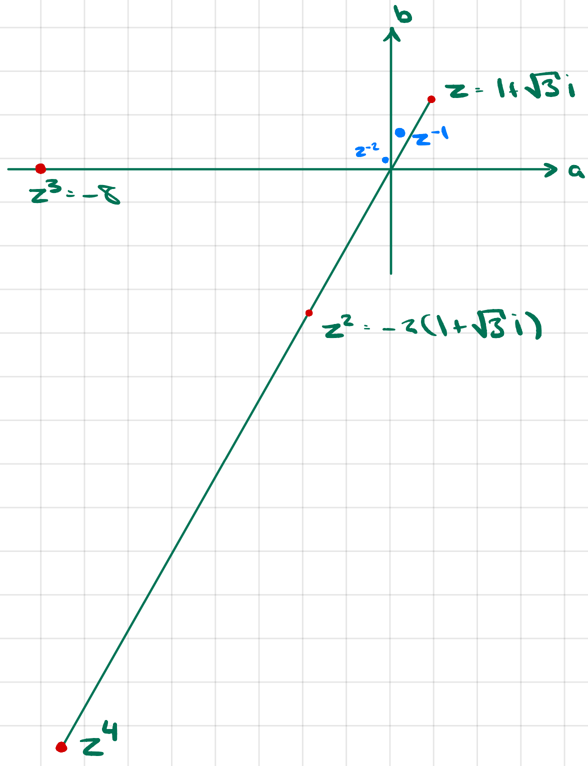


3. $x^2 \frac{dy}{dx} + 2xy = \sin(2x)$

$(x^2 y)' = \sin 2x \Rightarrow x^2 y = \int \sin 2x \, dx + C = -\frac{\cos 2x}{2} + C$

$\Rightarrow y(x) = \frac{-\cos(2x)}{2x^2} + \frac{C}{x^2}$

Complex Numbers



1. $z = 1+\sqrt{3}i$

$(1, \sqrt{3})$

$|z| = (1+3)^{1/2} = 2$

$a = |z|\cos\theta \Rightarrow 1 = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$b = |z|\sin\theta \Rightarrow \sqrt{3} = 2\sin\theta \Rightarrow \sin\theta = \sqrt{3}/2 \Rightarrow \theta = \pi/3$

$z^2 = (2e^{\frac{\pi}{3}i})^2 = 4e^{\frac{2}{3}\pi i} = 4(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$

$= 4(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) = -2 - 2\sqrt{3}i$

$z^n = 2^n e^{\frac{\pi n}{3}i} = 2^n(\cos\frac{\pi n}{3} + i\sin\frac{\pi n}{3})$

$z^3 = 8(\cos\pi + i\sin\pi) = -8$

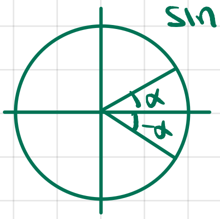
$z^4 = 16(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}) = 16(-\frac{1}{2}) + i\cdot 16(-\frac{\sqrt{3}}{2}) = -8 - i8\sqrt{3} = -8(1+\sqrt{3}i)$

$z^{-1} = \frac{1}{2}[\cos(-\pi/3) + i\sin(-\pi/3)] = \frac{1}{2}\cdot\frac{1}{2} + i(-\frac{\sqrt{3}}{2}) = \frac{1}{4} - \frac{\sqrt{3}}{2}i$

$z^{-2} = \frac{1}{4}[\cos(-\frac{2\pi}{3}) + \sin(-\frac{2\pi}{3})] = \frac{1}{4}(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}) = \frac{1}{4}(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) = -\frac{1}{8} - \frac{\sqrt{3}}{8}$

* note $\cos\alpha = \cos(-\alpha)$

$\sin\alpha = -\sin(-\alpha)$



2. $a+bi$ such that $e^{a+bi} = 1+i\sqrt{3}$

$2e^{\frac{\pi}{3}i} = 2(\cos\pi/3 + i\sin\pi/3) = 2(\frac{1}{2} + \frac{\sqrt{3}}{2}i) = 1+\sqrt{3}i$

$\Rightarrow 2e^{(\pi/3)i}$ works

$z = 1+\sqrt{3}i \Rightarrow |z| = 2$

$z = 2(\cos\theta + i\sin\theta) \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} + k2\pi$

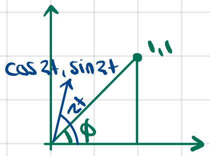
$\Rightarrow z = 2e^{i(\frac{\pi}{3}+2\pi k)}$, b is smallest when k=0 $\Rightarrow z = 2e^{\frac{\pi i}{3}}$

$z = 2e^{\frac{\pi i}{3}} = 1+i\sqrt{3} \Rightarrow z^n = 2^n e^{\frac{\pi n}{3}i} = 2^n(\cos\frac{\pi n}{3} + i\sin\frac{\pi n}{3})$

Sinusoids

1.

a) $\cos(2t) + \sin(2t)$

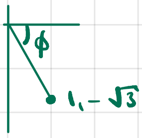


$\langle \cos 2t, \sin 2t \rangle \langle 1, 1 \rangle = 1 \cdot \sqrt{1+1} \cdot \cos(2t - \phi)$

where $\phi = \tan^{-1}(1) \Rightarrow \phi = \pi/4$.

$\Rightarrow \cos 2t + \sin 2t = \sqrt{2} \cos(2t - \pi/4)$

b) $\cos(\pi t) - \sqrt{3} \sin(\pi t) = 2\cos(\pi t + \pi/3)$



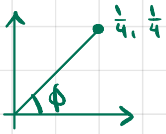
$\phi = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

c) $\mathrm{Re}\left[\frac{e^{it}}{2+2i}\right]$

$e^{it} = \cos t + i \sin t$

$\frac{e^{it}}{2+2i} = \frac{\cos t + i \sin t}{2+2i} \cdot \frac{2-2i}{2-2i} = \frac{2\cos t + 2\sin t + i(2\sin t - 2\cos t)}{8} = \frac{(\sin t + \cos t) + i(\sin t - \cos t)}{4}$

$\mathrm{Re}\left(\frac{e^{it}}{2+2i}\right) = \frac{1}{4}\sin t + \frac{1}{4}\cos t = \sqrt{1/8}\cos(t - \pi/4)$



$\phi = \tan^{-1} 1 = \pi/4$

Exponential Input

1. $\dot{x} + 2x = e^t$

guess $x_p(t) = We^t$, $x_p'(t) = We^t$

$We^t + 2We^t = e^t \Rightarrow e^t(3W) = e^t \Rightarrow W = \frac{1}{3}$

$x_p(t) = \frac{e^t}{3}$

$\dot{x} + 2x = 0 \Rightarrow \frac{dx}{dt} = -2x \Rightarrow x_h(t) = Ce^{-2t}$

$x(t) = \frac{e^t}{3} + Ce^{-2t}$

$\dot{z} + 2z = e^{i2t}$

guess $z_p(t) = Ae^{i2t}$, $z_p'(t) = Ae^{i2t} \cdot 2i$

$Ae^{i2t} \cdot 2i + 2Ae^{i2t} = e^{i2t}$

$Ae^{i2t}(2i + 2) = e^{i2t} \Rightarrow A = \frac{1}{2+2i} = \frac{2-2i}{8} = \frac{1}{4} - \frac{1}{4}i$

$\Rightarrow z_p(t) = \left(\frac{1}{4} - \frac{1}{4}i\right)e^{i2t}$

$z(t) = z_p(t) + Ce^{-2t}$

2. $\dot{x} + 2x = \cos(2t)$

$\cos(2t) = \mathrm{Re}(\cos 2t + \sin 2t\, i) = \mathrm{Re}(e^{i2t})$

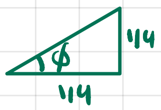
$\dot{\tilde{x}} + 2\tilde{x} = e^{i2t}$ $\qquad \tilde{x}(t) = x_1 + ix_2$

$\Rightarrow \tilde{x}(t) = \left(\frac{1}{4} - \frac{1}{4}i\right)e^{i2t}$

$= \left(\frac{1}{4} - \frac{1}{4}i\right)(\cos 2t + i\sin 2t)$

$= \frac{1}{4}\cos 2t + \frac{1}{4}\sin 2t + i\left(\frac{1}{4}\sin 2t - \frac{1}{4}\cos 2t\right)$

$\Rightarrow x_p(t) = \frac{1}{4}\cos 2t + \frac{1}{4}\sin 2t = (1/8)^{1/2}\cos(2t - \pi/4)$



$\phi = \pi/4$

Also, if $\dot{y} + 2y = \sin 2t$ then $y(t) = -\frac{1}{4}\cos 2t + \frac{1}{4}\sin 2t$

$= (1/8)^{1/2}\cos(2t + \pi/4)$