Stability criterion for second-order ODEs

$a_0 y'' + a_1 y' + a_2 y = r(t)$ stable $\Leftrightarrow$ all roots of $p(r)=0$ have negative real part

if $a_0 > 0$ this means

$p(r) = a_0 r^2 + a_1 r + a_2 = 0$

$$r = \frac{-a_1 \pm (a_1^2 - 4a_0 a_2)^{1/2}}{2a_0}$$

case 1: $a_1^2 - 4a_0 a_2 > 0$

$r < 0 \Rightarrow \quad -a_1 \pm (a_1^2 - 4a_0 a_2)^{1/2} < 0 \Rightarrow a_1 > + (a_1^2 - 4a_0 a_2)^{1/2} > 0 \quad \Rightarrow a_1 > 0$

$a_1^2 > 4a_0 a_2$ $(\dots)$? $a_1^2 > a_1^2 - 4a_0 a_2 \Rightarrow -4a_0 a_2 < 0 \Rightarrow a_2 > 0$

$a_2 < \frac{a_1^2}{4a_0}$

$\Rightarrow a_0, a_1, a_2 > 0$

case 2: $a_1^2 - 4a_0 a_2 = 0$

$r < 0 \Rightarrow \quad \frac{-a_1}{2a_0} < 0 \Rightarrow a_1 > 0$

$a_1^2 = 4a_0 a_2 \Rightarrow a_2 > 0$

$\Rightarrow a_0, a_1, a_2 > 0$

case 3: $a_1^2 - 4a_0 a_2 < 0$

$Re(r) < 0 \Rightarrow Re(-a_1 \pm i(4a_0 a_2 - a_1^2)^{1/2}) < 0 \Rightarrow \quad -a_1 < 0 \Rightarrow a_1 > 0$

$0 < a_1^2 < 4a_0 a_2 \quad \Rightarrow \quad 4a_0 a_2 > 0 \Rightarrow a_2 > 0$

$\Rightarrow a_0, a_1, a_2 > 0$

↘ coeff. criterion for stability of second order linear ODE with constant coeffs.

p(D) notation

linear constant coeff. DE $a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 x' + a_0 x = q(t)$ → $p(D)x = q(t)$

characteristic polynomial $p(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0$

homogenous case $a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 x' + a_0 x = 0$ → $p(D)x = 0$

sinusoidal input $p(D)x = B\cos(\omega t)$

complex replacement $p(D)z = Be^{i\omega t}$

exponential response formula $z_p = \dfrac{B}{p(i\omega)} e^{i\omega t}$

$x_p = Re(z_p) = \dfrac{B}{|p(i\omega)|} \cos(\omega t - \phi)$ $\phi = Arg(p(i\omega))$

periodic input $B\cos \omega t$

shifted by $Arg(p(i\omega))$

periodic output $x_p(t) = \dfrac{B}{|p(i\omega)|} \cos(\omega t - \phi)$

scaled by $g = \dfrac{1}{|p(i\omega)|}$

gain g

phase lag ϕ