→ linear, constant coeff. ODE w/ exponential input signal

$\dot{x} + 2x = 4e^{3t}$

→ we could use an integrating factor
→ OR, we can essentially guess a particular solution: $y_p = Ae^{3t}$. This method is called method of optimism.

we find $A = 4/5 \Rightarrow x_p = \frac{4}{5}e^{3t}$

then we solve the homog. eq. to get $x_h = Ce^{-2t}$

Superposition $\Rightarrow x(t) = x_p(t) + x_h(t) = \frac{4}{5}e^{3t} + Ce^{-2t}$

→ linear, constant coeff., sinusoidal input

general case: $\dot{x} + kx = B\cos(\omega t)$

example: $\dot{x} + 2x = 2\cos(2t)$

introduce new variable y with own ODE: $\dot{y} + 2y = 2\sin(2t)$

combine x and y: $z = x + iy \quad \Rightarrow x = Re(z)$

$\dot{z} + 2z = 2\cos(2t) + i2\sin(2t) = 2e^{i2t}$

not sure how this passage works

if we find $z(t)$ then we get $x(t) = Re(z(t))$

The equation in $\dot{z}, z$ has exponential input → easy to solve by guessing $z_p(t) = Ae^{i2t}$

$(\dots) \Rightarrow z_p(t) = \dfrac{e^{2it}}{1+i}$

To obtain $x(t)$ we need the real part of z_p. z_p is a ratio of a complex number in Cartesian form and one in polar form. We must choose one form.

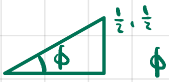
→ go polar: $\dfrac{e^{i2t}}{1+i} = \dfrac{1-i}{2}e^{i2t} = \dfrac{\sqrt{2}}{2}e^{i(-\pi/4)} \cdot e^{i2t} = \dfrac{\sqrt{2}}{2}e^{i(2t-\pi/4)} = \dfrac{\sqrt{2}}{2}\left(\cos\left(2t-\frac{\pi}{4}\right) + i\sin\left(2t-\frac{\pi}{4}\right)\right)$

Real Part: $\dfrac{\sqrt{2}}{2}\cos(2t-\pi/4)$

→ go cartesian $e^{i2t} = \cos 2t + i\sin 2t$

$\dfrac{(\cos 2t + i\sin 2t)(1-i)}{2} = \dfrac{\cos 2t + \sin 2t + i(\sin 2t - \cos 2t)}{2}$

Real part $= \frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t = \sqrt{\frac{1}{4}+\frac{1}{4}}\cos(2t-\pi/4) = \frac{\sqrt{2}}{2}\cos(2t-\pi/4)$



$\phi = \tan^{-1} 1$
$\phi = \pi/4$

$\Rightarrow x_p(t) = \frac{\sqrt{2}}{2}\cos(2t-\pi/4)$. We need $x_h(t)$, which is Ce^{-2t}

→ Let's consider the general case of linear, constant coeff. DE with sinusoidal input

$$\dot{x} + kx = B\cos(\omega t)$$

→ Following the steps laid out on previous page, we obtain

$$x_p(t) = \frac{B}{\sqrt{k^2+\omega^2}}\cos(\omega t - \phi) \qquad \phi = \tan^{-1}(\omega/k)$$

$$x_h(t) = Ce^{-kt}$$

$$\Rightarrow x(t) = x_p(t) + x_h(t)$$

Consider now multiplying the input also by k, just a constant

$$\dot{x} + kx = \underbrace{kB\cos(\omega t)}_{\text{input signal}}$$

input amplitude = B

input circular frequency = ω

$$\Rightarrow x_p(t) = \frac{kB}{\sqrt{k^2+\omega^2}}\cos(\omega t - \phi) = gB\cos(\omega t - \phi), \quad g = \frac{k}{\sqrt{k^2+\omega^2}} = \text{gain (amplitude response)}$$

phase lag $= \tan^{-1}(\omega/k)$

output amplitude = gB

$$\Rightarrow \frac{\text{output amplitude}}{\text{input amplitude}} = \frac{gB}{B} = g$$

k: coupling constant