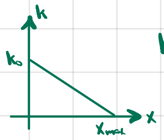
Exam 1

1. exponential growth $\dot{x} = kx \Rightarrow x(t) = x_0 e^{kt}$

a) $\frac{x(t+1)}{x(t)} = e \Rightarrow \frac{\cancel{x_0} e^{k(t+1)}}{\cancel{x_0} e^{kt}} = e^{kt+k-kt} = e^{k} \Rightarrow k = 1$

b) $\dot{x} = k(x)x \qquad k(x) = k_0\left(1 - \frac{x}{x_{max}}\right)$



$k_0 - \frac{k_0}{x_{max}} \cdot x = k_0\left(1 - \frac{x}{x_{max}}\right)$

$\dot{x} = k_0\left(1 - \frac{x}{x_{max}}\right)x \quad , \quad k_0 = 1, \; x_{max} = 1000$

$\dot{x} = \left(1 - \frac{x}{1000}\right)x$

c) $\dot{x} = \left(1 - \frac{x}{1000}\right)x - a$

crit. points no hunting: $x = 0, \; x = 1000$

crit. points hunting: $x - \frac{x^2}{1000} - a = 0$

$x^2 - 10^3 x + 10^3 a = 0$

$\Delta = 10^6 - 4 \cdot 10^3 \cdot a$

$x_c = \frac{10^3 \pm (10^6 - 4 \cdot 10^3 a)^{1/2}}{2}$

$\frac{x_c}{1000} = 0.75 \Rightarrow \frac{10^3 + (10^6 - 4 \cdot 10^3 a)^{1/2}}{2 \cdot 10^3} = 0.75$

$1500 = 1000 + (10^6 - 4 \cdot 10^3 a)^{1/2}$

$\Rightarrow 25 \cdot 10^4 = 10^6 - 4 \cdot 10^3 a \Rightarrow a = \frac{375}{2} = 187.5$

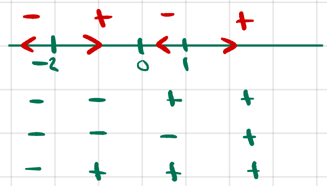
* Allen.

$\dot{x} = \left(1 - \frac{x}{1000}\right)x - a = 0 \Rightarrow a = \left(1 - \frac{x_c}{1000}\right)x_c$

set $x_c = 750 \Rightarrow a = 0.25 \cdot 750 = 187.5$

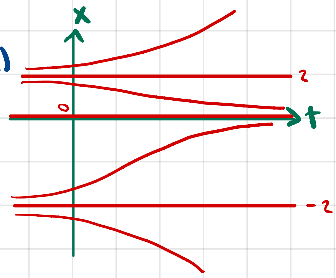
2. $\dot{x} = x(x-1)(x+2)$

a)



$x = -2$ unstable
$x = 0$ stable
$x = 1$ unstable

b)

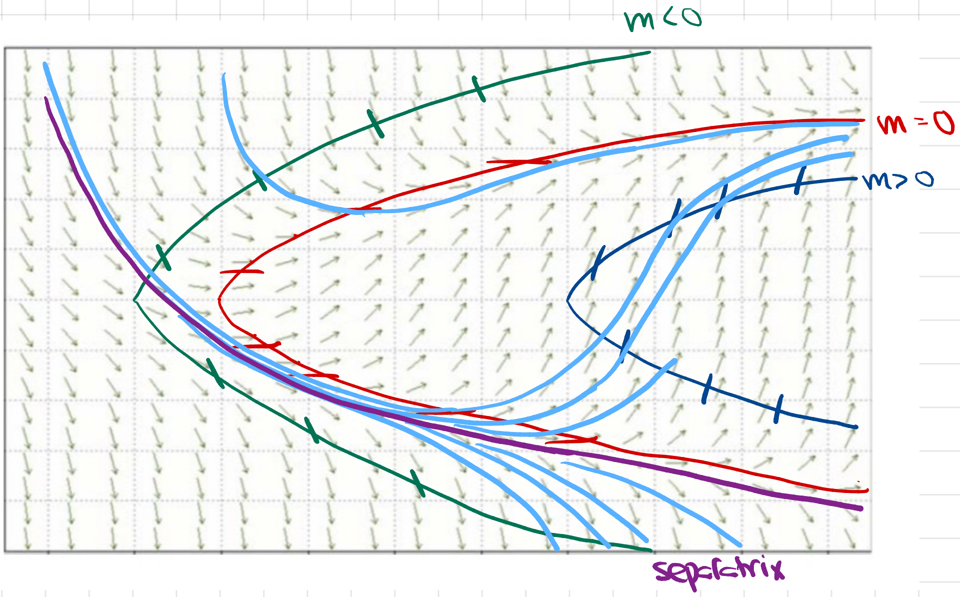


c) $y' = \frac{1}{4}(x - y^2)$

nullcline: $x = y^2$

d) $x - y^2 = 4m$
$x = 4m + y^2$

e) separatrix



f) True, the nullclines $y = \pm\sqrt{x}$ form a funnel. Any solution with a minimum crosses a nullcline into the funnel and stays in the funnel.

3. a) $y' = x + y$ $\quad y(0) = 1$ $\quad$ step size $h = \frac{1}{2}$ $\quad y(3/2)$?

$x_{n+1} = x_n + h$
$y_{n+1} = y_n + h\,y_n'$
$y_n' = x_n + y_n$

n	x_n	y_n	y_n'	$h y_n'$
0	0	1	1	1/2
1	1/2	3/2	2	1
2	1	5/2	7/2	7/4
3	3/2	17/4		

b) $t\dot{x} + x = \cos t$ $\quad x(\pi) = 1$

$\dot{x} + \frac{1}{t}x = \frac{\cos t}{t}$

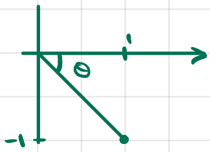
$e^{\int \frac{1}{t}dt} = e^{\ln t} = t$

$\dot{x}t + x = \cos t$
$xt = \int \cos t\, dt = \sin t + c$
$x(t) = \frac{\sin t}{t} + \frac{c}{t}$

$x(\pi) = 0 + \frac{c}{\pi} = 1 \Rightarrow c = \pi \Rightarrow x(t) = \frac{\sin t}{t} + \frac{\pi}{t}$

4. a) $\frac{1}{3+2i} = \frac{3-2i}{13}$

b) $1-i = \sqrt{2}e^{-\frac{\pi}{4}i}$



$\tan^{-1}\theta = -1$ $\quad r = (1+1)^{1/2} = \sqrt{2}$
$\Rightarrow \theta = -\frac{\pi}{4}$

c) $(1-i)^8 = (\sqrt{2}e^{-\frac{\pi}{4}i})^8 = 2^4 e^{-2\pi i} = 2^4(\cos(-2\pi) + i\sin(-2\pi))$
$= 2^4$

d) a, b $\quad b>0$, $a+bi$ cube root of -1

$z = \sqrt[3]{-1} \Rightarrow \rho e^{i\phi} = e^{i(\pi+k2\pi)/3} \Rightarrow \rho = 1 \quad \phi = \frac{\pi}{3} + \frac{2\pi}{3}k \quad k = 0,1,2$

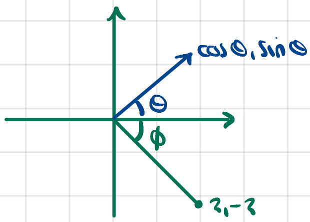
$-1 = \cos(\pi + k\cdot 2\pi) = e^{i(\pi+k2\pi)}$

cube roots: $e^{\frac{\pi}{3}i} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$
$e^{\pi i} = -1$
$e^{i5\pi/3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$a = \frac{1}{2},\ b = \frac{\sqrt{3}}{2}$

e) $e^{\ln 2 + i\pi} = e^{\ln 2}e^{i\pi} = 2e^{i\pi} = 2(\cos\pi + i\sin\pi) = 2(-1) = -2$
$a = -2,\ b = 0$

f) $f(t) = 2\cos(4t) - 2\sin(4t) = \sqrt{8}\cos(4t + \frac{\pi}{4})$



$\langle 2,-2\rangle\langle\cos\theta, \sin\theta\rangle$
$= \sqrt{8}\cdot 1\cdot\cos(\theta - \phi)$
$\phi = \tan^{-1}(-1) = -\frac{\pi}{4}$
$\theta = 4t$

5. a) $\dot{x} + 3x = e^{2t}$

$x_p = Ae^{2t} \Rightarrow 2Ae^{2t} + 3Ae^{2t} = e^{2t} \Rightarrow Ae^{2t}(5) = e^{2t} \Rightarrow A = 1/5$

$x_p(t) = \frac{e^{2t}}{5}$

b) $x(t) = \frac{e^{2t}}{5} + Ce^{-3t}$
$x(0) = \frac{1}{5} + C = 1 \Rightarrow C = 1 - \frac{1}{5} = \frac{4}{5}$
$\Rightarrow x(t) = \frac{e^{2t}}{5} + \frac{4}{5}e^{-3t}$

c) $\dot{z} + 3z = \cos 2t + i\sin 2t = e^{i2t}$

d) $z = Ae^{i2t} \Rightarrow 2i\cdot Ae^{i2t} + 3Ae^{i2t} = e^{i2t}$
$Ae^{i2t}(2i+3) = e^{i2t}$
$A = \frac{1}{3+2i} \Rightarrow z_p(t) = \frac{1}{3+2i}e^{i2t} = \frac{3-2i}{13}e^{i2t} = \left(\frac{3}{13} - \frac{2}{13}i\right)(\cos 2t + i\sin 2t)$

$\mathrm{Re}(z_p(t)) = \frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t = x_p(t)$