1.6 Substit. Methods and Exact Eq.

→ DEs neither separable nor linear

→ substit. methods: DE $\longrightarrow$ DE we know how to solve

$\frac{dy}{dx} = f(x,y)$

$v = \alpha(x,y)$

solve for $y = \beta(x,v)$

apply chain rule: $\frac{dy}{dx} = \beta_x + \beta_v \frac{dv}{dx}$

sub in rhs for $\frac{dy}{dx}$ in original DE, solve for $\frac{dv}{dx}$

$\Rightarrow \frac{dv}{dx} = g(x,v)$ which we hope is separable or linear

$v = v(x)$ sol'n $\Rightarrow y = \beta(x, v(x))$ sol'n of orig. DE.

Ex: $\frac{dy}{dx} = (x+y+3)^2$, nonlinear, first-order

$v = x+y+3 \Rightarrow y = v-x-3 \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

$\Rightarrow \frac{dv}{dx} - 1 = v^2 \Rightarrow \frac{dv}{dx} = 1+v^2$, a separable eq., non-linear

$\frac{1}{1+v^2} dv = dx \Rightarrow \tan^{-1} v = x + C \Rightarrow v = \tan(x+C)$

$y = \tan(x+C) - x - 3 \qquad x + C \in (-\pi/2, \pi/2) \Rightarrow -\frac{\pi}{2} - C < x < \frac{\pi}{2} - C$

General Result: any DE of form $\frac{dy}{dx} = F(ax+by+c)$ can be transformed into separable eq. by substit. $v = ax+by+c$.

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we now look at some standard substit. methods known to work

Homogeneous Eq.

$\frac{dy}{dx} = F(\frac{y}{x})$ homog. first-order DE

$v = \frac{y}{x} \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = \frac{dv}{dx}x + v$

subst. $\Rightarrow x\frac{dv}{dx} = F(v) - v$ separable

General Result

$P(x,y)y' = Q(x,y)$, polyn. coeffs. P and Q, is homog. if the terms in the polyn. all have same total degree K.

Ex: $2xy\frac{dy}{dx} = 4x^2 + 3y^2$

$P(x,y) = 2xy$

$Q(x,y) = 4x^2 + 3y^2$

Total degree is 2.

$\frac{dy}{dx} = \frac{4x^2}{2xy} + \frac{3y^2}{2xy} = 2\frac{x}{y} + \frac{3}{2}\frac{y}{x}$

$v = y/x \Rightarrow y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{2}{v} + \frac{3}{2}v$

$x\frac{dv}{dx} = \frac{2}{v} + \frac{v}{2} = \frac{4+v^2}{2v}$

$\frac{2v}{4+v^2} dv = \frac{1}{x} dx$

$\ln(4+v^2) = \ln|x| + C$

$4 + v^2 = |x|e^C$

$\frac{y^2}{x^2} = |x| \cdot C_1 - 4$

$y^2 = x^2|x|C_1 - 4x^2$

For a given C_1, $y(x)$ domain is

$x^2|x|C_1 - 4x^2 \geq 0$

$x^2|x|C_1 \geq 4x^2$

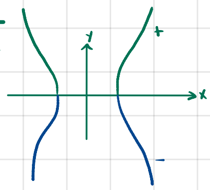
$|x|C_1 \geq 4$

$|x| \geq 4/C_1$ $\quad C_1 > 0$

$x > 0 \Rightarrow x \geq 4/C_1$

$x < 0 \Rightarrow -x \geq 4/C_1 \Rightarrow x \leq -4/C_1$

$y(x) = \pm\sqrt{x^2|x|C_1 - 4x^2}$



if $C_1 < 0$ then

$|x| \leq \frac{4}{C_1}$

$x > 0 \Rightarrow x \leq \frac{4}{C_1}$ not possible

$x < 0 \Rightarrow -x \leq \frac{4}{C_1} \Rightarrow x \geq -\frac{4}{C_1} > 0$

not possible

Bernoulli Equations

$\frac{dy}{dx} + P(x)y = Q(x)y^n$

$n=0$ or $n=1 \Rightarrow$ linear eq.

$n \neq 0,1 \Rightarrow$ substit. $v = y^{1-n}$

$y = v^{\frac{1}{1-n}}$ $\quad dy/dx = \frac{v^{\frac{n}{1-n}}}{1-n}\frac{dv}{dx} + P(x)v^{\frac{1}{1-n}} = Q(x)v^{\frac{n}{1-n}}$

$\Rightarrow \frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$

Ex: $2xyy' = 4x^2 + 3y^2$

$dy/dx = \frac{2x}{y} + \frac{3}{2}\frac{y}{x} \Rightarrow dy/dx - \frac{3}{2x}y = 2xy^{-1}$ (Bernoulli Eq.)

$n=-1$, sub $v = y^{1+1} = y^2 \Rightarrow y = v^{1/2} \Rightarrow dy/dx = \frac{1}{2v^{1/2}}\frac{dv}{dx}$

$\Rightarrow \frac{1}{2v^{1/2}}\frac{dv}{dx} - \frac{3}{2x}v^{1/2} = 2xv^{-1/2}$

$\Rightarrow \frac{dv}{dx} - \frac{3}{x}v = 4x$

$u(x) = e^{-3\int x^{-1}dx} = e^{-3\ln|x|} = |x|^{-3}$

$v|x|^{-3} = 4\int x|x|^{-3}dx + C$

$x>0 \Rightarrow vx^{-3} = 4\int x^{-2}dx + C = -4x^{-1} + C$

$v = -4x^2 + Cx^3 \Rightarrow y = \pm\sqrt{-4x^2 + Cx^3}$

$-4x^2 + Cx^3 \geq 0$
$4x^2 \leq Cx^3$
$4 \leq Cx$

$C>0 \Rightarrow x \geq \frac{4}{C}$
$C<0 \Rightarrow x \leq \frac{4}{C}$

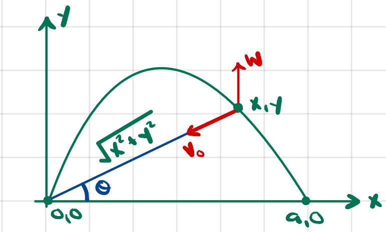
$x<0 \Rightarrow -vx^{-3} = -4\int x^{-2}dx + C = 4x^{-1} + C$

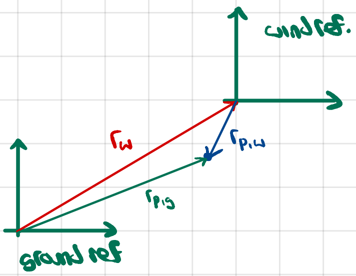
$v = -4x^2 - Cx^3 \Rightarrow y^2 = -4x^2 - Cx^3$

$-4x^2 - Cx^3 > 0 \Rightarrow 4x^2 < -Cx^3$
$4 < -Cx$
$C>0 \Rightarrow x < -\frac{4}{C} < 0$

$C<0 \Rightarrow x > -\frac{4}{C} > 0$ but $x<0$, contradiction

Plane Trajectory Problem





v_0 speed rel. to wind

$\vec{v}$ rel. to ground $= \langle v_x, v_y \rangle$

$$\frac{dx}{dt} = -v_0\cos\theta = \frac{-v_0 x}{\sqrt{x^2+y^2}}$$

$$\frac{dy}{dt} = w - v_0\sin\theta = w - \frac{v_0 y}{\sqrt{x^2+y^2}}$$

$$\frac{dy/dt}{dx/dt} = \frac{1}{v_0 x}\left(v_0 y - w\sqrt{x^2+y^2}\right) = \frac{dy}{dx}$$

$$k = \frac{w}{v_0} \Rightarrow \frac{dy}{dx} = \frac{y}{x} - k\sqrt{1+(y^2/x^2)}$$

$$y = vx \quad y' = v'x + \cancel{v} = \cancel{v} - k\sqrt{1+v^2}$$

$$\frac{1}{\sqrt{1+v^2}}\,dv = -\frac{k}{x}\,dx$$

$$\Rightarrow \ln\left(v+\sqrt{1+v^2}\right) = -k\ln x + C$$

initial condition $v(a) = \frac{y(a)}{a} = 0 \Rightarrow C = k\ln a$

$$\Rightarrow \ln\left(v+\sqrt{1+v^2}\right) = -k\ln x + k\ln a = k\left(\ln\frac{a}{x}\right)$$

$$v+\sqrt{1+v^2} = \left(\frac{a}{x}\right)^k \qquad (1)$$

note that $a^2-b^2 = (a+b)(a-b) \Rightarrow a-b = \frac{a^2-b^2}{a+b}$

$$\Rightarrow \sqrt{1+v^2} - v = \frac{1+\cancel{v^2}-\cancel{v^2}}{\sqrt{1+v^2}+v} = \left(\frac{x}{a}\right)^k \qquad (2)$$

subtract 2 from 1

$$2v = \left(\frac{a}{k}\right)^k - \left(\frac{a}{k}\right)^{-k}$$

$$v = \frac{1}{2}\left[\left(\frac{a}{x}\right)^k - \left(\frac{a}{x}\right)^{-k}\right]$$

$$v = \frac{y}{x} \Rightarrow y(x) = \frac{1}{2}\left[a^k x^{1-k} - a^{-k}x^{1+k}\right]$$

$$\Rightarrow y(x) = \frac{a}{2}\left[\left(\frac{x}{a}\right)^{1-k} - \left(\frac{x}{a}\right)^{1+k}\right]$$

$$\vec{r}_{p,g} = \vec{r}_w + \vec{r}_{p,w}$$

$$\vec{v}_{p,g} = \vec{v}_w + \vec{v}_{p,w} = \langle 0, w\rangle + \langle -v_0\cos\theta, -v_0\sin\theta\rangle$$

$$\Rightarrow \langle \dot{x}, \dot{y}\rangle = \langle -v_0\cos\theta, w - v_0\sin\theta\rangle$$

$$y(x) = \frac{a}{2}\left[\left(\frac{x}{a}\right)^{1-k} - \left(\frac{x}{a}\right)^{1+k}\right]$$

recall $k = \frac{w}{v_0}$

$$k = 1 \Rightarrow y(x) = \frac{a}{2}\left[1 - \left(\frac{x}{a}\right)^2\right] \Rightarrow y(0) = \frac{a}{2}$$

$$k > 1 \Rightarrow y(x) = \frac{a}{2}\left[\left(\frac{a}{x}\right)^{k-1} - \left(\frac{x}{a}\right)^{1+k}\right]$$

$$\lim_{x\to 0} y(x) = \frac{a}{2}(\infty - 0) = +\infty$$

$$0 < k < 1 \Rightarrow \lim_{x\to 0} y(x) = \frac{a}{2}(0-0) = 0$$

max amount by which plane blown off course, ie max value of $y(x)$ for $0 \le x \le a$?

$$y'(x) = \frac{\cancel{a}}{2}\left[(1-k)\left(\frac{x}{a}\right)^{-k}\cdot\frac{1}{\cancel{a}} - (1+k)\left(\frac{x}{a}\right)^{k}\cdot\frac{1}{\cancel{a}}\right] = 0$$

$$\Rightarrow (1-k)\left(\frac{x}{a}\right)^{-k} = (1+k)\left(\frac{x}{a}\right)^{k}$$

$$\left(\frac{x}{a}\right)^{2k} = \frac{1-k}{1+k} \Rightarrow x = \left[\frac{1-k}{1+k}\right]^{\frac{1}{2k}}\cdot a$$

Exact Differential Equations

general sol'n $y(x)$ of first-order DE often implicitly defined $F(x, y(x)) = C$, C constant

differentiate to recover DE

$$F_x + F_y y' = 0$$

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{(differential form)}$$

ex: $y' = f(x,y) \Rightarrow f(x,y) - \frac{dy}{dx} = 0$, $M(x,y) = f(x,y)$, $N(x,y) = -1$

if we start with the eq. $M(x,y)\,dx + N(x,y)\,dy = 0$

then, if we can find F s.t. $F_x = M$ and $F_y = N$, it means $F(x,y) = C$ implicitly defines a general sol'n of the differential eq. we started with.

$\Rightarrow$ The differ. eq. is called an **exact DE**

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example $y^3\,dx + 3xy^2\,dy = 0$

$M(x,y) = y^3$ $\qquad$ $N(x,y) = 3xy^2$

$M_y = 3y^2$ $\qquad$ $N_x = 3y^2$ $\qquad$ $\Rightarrow M_y = N_x$, necess. condition for exactness met.

in fact $F(x,y) = xy^3$ is such that $F_x = M$ and $F_y = N$.

Thus, $F(x,y) = xy^3 = C$ is a general sol'n to the diff. eq.

if we divide the DE by y^2: $y\,dx + 3x\,dy = 0$ (non-exact)

$$y\,dx = -3x\,dy \Rightarrow -\frac{1}{3x}dx = \frac{1}{y}dy \Rightarrow \ln|y| = \frac{-\ln|3x|}{3} + C \Rightarrow |y| = e^C|3x|^{-\frac{1}{3}} \Rightarrow y = \pm C_1|3x|^{-\frac{1}{3}}$$

$$y = C_2|3x|^{-1/3}$$

$$x > 0 \Rightarrow y = C_2 3^{-1/3}x^{-1/3} \Rightarrow y = C_3x^{-1/3} \Rightarrow y^3x = C_4$$

$$x < 0 \Rightarrow y = C_2(-3^{-1/3})x^{-1/3} \Rightarrow y = C_3x^{-1/3} \Rightarrow y^3x = C_4$$ (same sol'n as the exact eq.)
