Example 2 Bank Account

$x(t)$ dollars in bank account
can deposit, withdraw, earn interest
interest rate r year^{-1}

bank pays interest end of month on start of month balance

$\Rightarrow x(t+\Delta t) = x(t) + r\,x(t)\,\Delta t$ + [deposits − withdrawals between t and $t+\Delta t$]

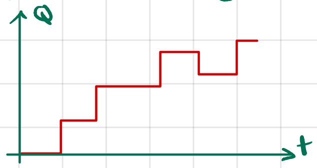
$\Delta t = \frac{1}{12}$

withdrawal = negative deposit

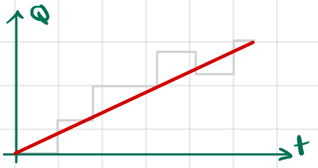
nowadays, interest is computed daily or continuously

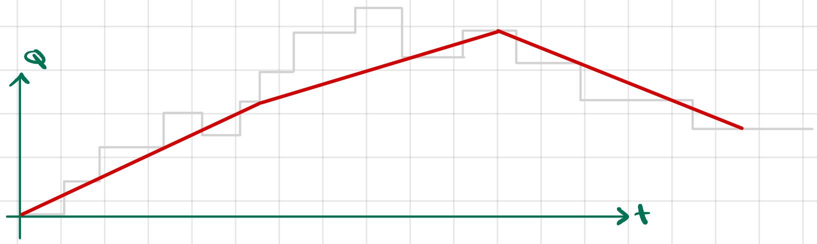
Let $Q(t)$ be total deposits (including withdrawals/negative deposits)

In reality, $Q(t)$ might look like



But considering start and end values of Q between two times, we can model the deposits as happening continuously at a constant rate. Q would then look like





$Q'(t) = q(t)$ is the rate of change of deposits at t.

$x(t+\Delta t) \approx x(t) + r\,x(t)\,\Delta t + q(t)\,\Delta t$

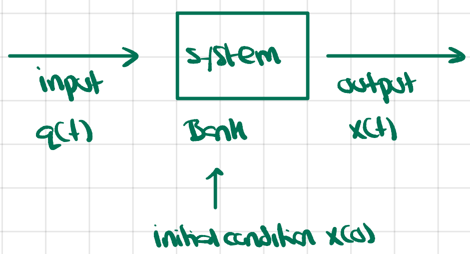
$\Rightarrow \frac{\Delta x}{\Delta t} \approx rx + q \quad \Rightarrow \frac{dx}{dt} = rx + q$

what comes into the bank; represents outside influence on the system; input signal

$\underbrace{\dot{x} - rx}_{} = q$ (standard form)

what happens at the bank.
represents the system (response), output signal

signal: in general, a function of t



the system responds to the input signal and yields a function $x(t)$

solve diff eq $\Leftrightarrow$ find response of system to input signal $q(t)$

most important property of first order linear equations: superposition principle

$\underbrace{\dot{y} + p(t)y}_{\text{system}} = \underbrace{q(t)}_{\text{input}}$

Actually, is "the defining characteristic of linear equations of any order"

notation: $q \rightsquigarrow y$ input q leads to output y

$q_1(t), q_2(t)$ signals

c_1, c_2 constants

$\Rightarrow c_1q_1(t) + c_2q_2(t)$ is a superposition of q_1 and q_2

aka linear combination

Superposition Principle

$$\left.\begin{aligned} q_1 &\rightsquigarrow y_1 \\ q_2 &\rightsquigarrow y_2 \end{aligned}\right. \Rightarrow c_1q_1 + c_2q_2 \rightsquigarrow c_1y_1 + c_2y_2$$

Example

DE	a solution
$\dot{x} + 2x = 1$	$x(t) = \frac{1}{2}$
$\dot{x} + 2x = e^{-2t}$	$x(t) = te^{-2t}$
$\dot{x} + 2x = 0$	$x(t) = e^{-2t}$
$\dot{x} + 2x = 1 + e^{-2t}$	$x(t) = \frac{1}{2} + te^{-2t}$
$\dot{x} + 2x = 1$	$x(t) = ce^{-2t} + \frac{1}{2}$
$\dot{x} + 2x = 1 + e^{-2t}$	$x(t) = \frac{1}{2} + te^{-2t} + Ce^{-2t}$