1.7 Population Models

$\beta(t)$: births / unit time · unit pop.
$\delta(t)$: deaths / " " " "

$\Delta P \approx \beta(t)\cdot P(t)\cdot \Delta t - \delta(t)\cdot P(t)\cdot \Delta t$

$\frac{\Delta P}{\Delta t} = P(t)(\beta(t) - \delta(t))$

$\dot{P} = P(t)(\beta(t) - \delta(t))$ general popul. eq.

Logistic Eq.

$\beta(P(t)) = \beta_0 - \beta_1 P \Rightarrow \dot{P} = (\beta_0 - \beta_1 P - \delta_0)P \Rightarrow \dot{P} = P(\beta_0 - \delta_0) - \beta_1 P^2$

$\Rightarrow \dot{P} = aP - bP^2 \;, a,b > 0 \qquad a = \beta_0 - \delta_0 \quad b = \beta_1$

$\Rightarrow \dot{P} = bP(\frac{a}{b} - P) \Rightarrow \dot{P} = kP(M-P) \qquad k = b \qquad M = \frac{a}{b} = \frac{\beta_0 - \delta_0}{\beta_1}$

Solution $\frac{1}{P(M-P)} dP = k dt$

partial fractions: $\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$

$A(M-P) + BP = 1 \Rightarrow P(B-A) + AM = 1 \Rightarrow B = A,\ AM = 1 \Rightarrow A = \frac{1}{M} = B$

$\left[\frac{1}{MP} + \frac{1}{M(M-P)}\right] dp = k dt \Rightarrow \frac{1}{M}\ln P - \frac{1}{M}\ln(M-P) = kt + C \Rightarrow \frac{1}{M}\ln\frac{P}{M-P} = kt + C \Rightarrow \left(\frac{P}{M-P}\right)^{\frac{1}{M}} = e^{kt+C}$

$\frac{P}{M-P} = e^{Mkt}e^{MC} \Rightarrow P(1 + e^{Mkt}e^{MC}) = Me^{Mkt}e^{MC} \Rightarrow P(t) = \frac{Me^{Mkt}e^{MC}}{1 + e^{Mkt}e^{MC}}$

$P(0) = \frac{Me^{MC}}{1+e^{MC}} = P_0 \Rightarrow Me^{MC} = P_0 + P_0 e^{MC} \Rightarrow e^{MC}(M - P_0) = P_0 \Rightarrow MC + \ln(M-P_0) = \ln P_0$

$\Rightarrow MC = \ln P_0 - \ln(M-P_0) \Rightarrow C = \frac{1}{M}\ln\frac{P_0}{M-P_0}$

$\Rightarrow e^{MC} = e^{\ln(P_0/M-P_0)} = \frac{P_0}{M-P_0}$

$\Rightarrow P(t) = \frac{Me^{Mkt}P_0/(M-P_0)}{1+e^{Mkt}P_0/(M-P_0)} = \frac{\frac{MP_0 \cdot \cancel{e^{Mkt}}}{\cancel{M-P_0}}}{\frac{\cancel{e^{Mkt}}}{\cancel{M-P_0}}(P_0 + (M-P_0)e^{-Mkt})} = \frac{MP_0}{P_0 + (M-P_0)e^{-Mkt}}$

$P_0 = M \Rightarrow P(t) = \frac{M^2}{M+0} = M$

$P_0 < M \Rightarrow \dot{P} > 0$ and $P(t) = \frac{MP_0}{P_0 + \text{pos. number}} < M$

$P_0 > M \Rightarrow \dot{P} < 0$ and $P(t) = \frac{MP_0}{P_0 + \text{neg. number}} > M$

$\lim_{t\to\infty} P(t) = M$

Exercise With Brazilian Population Numbers

$P(1800) = 3.64$ $\qquad$ $P(1850) = 7.25$

$P(1801) = 3.68$ $\qquad$ $P(1851) = 7.37$

Let's fit two models to this data

1. natural growth model

$\dot{P} = kP \Rightarrow P(t) = P_0 e^{kt}$

$P(50) = 3.64 e^{50k} = 7.25 \Rightarrow 50k = \ln 7.25/3.64 \Rightarrow k = 0.01378$

$\Rightarrow P(t) = 1.01387^t \cdot 3.64$

$\Rightarrow$ average pop. growth rate of 1.38% per year between 1800 and 1850.

$P(221) = 76.511$, massively underestimated.

Let's use 1850 and 1900

$P(1900) = 18.1 \Rightarrow P(t) = 7.25 e^{kt} \Rightarrow P(50) = 18.1 = 7.25 e^{50t} \Rightarrow k = 0.018298$

$\Rightarrow P(t) = 1.01846^t \cdot 7.25$

$P(171) = 162.869$, still an underestimate.

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in general

$$P(t_1) = P_0 e^{kt_1} \Rightarrow e^{kt_1} = \frac{P_1}{P_0} \Rightarrow k = \frac{\ln \frac{P_1}{P_0}}{t_1} \Rightarrow P(t) = P_0 \cdot \left(\frac{P_1}{P_0}\right)^{\frac{t}{t_1}}$$

$$\Rightarrow P(1950) = 53.98 \Rightarrow P(t) = 18.1 \cdot \left(\frac{53.98}{18.1}\right)^{\frac{t}{50}} \Rightarrow P(121) = 254.74$$

$$* \; P(1850) = P_{1850} = c e^{1850k} \Rightarrow c = \frac{P_{1850}}{e^{1850k}}$$

$$P(t) = P_{1850} e^{k(t-1850)}$$

$$P(t_1) = P_1 = P_{1850} e^{k(t_1-1850)} \Rightarrow e^{k(t_1-1850)} = \frac{P_1}{P_{1850}} \Rightarrow k(t_1-1850) = \ln \frac{P_1}{P_{1850}} \Rightarrow k = \frac{1}{t_1-1850} \ln \frac{P_1}{P_{1850}}$$

$$\Rightarrow P(t) = P_{1850} e^{\frac{(t-1850)}{t_1-1850} \cdot \ln P_1/P_{1850}}$$

$$= P_{1850} \cdot \left(\frac{P_1}{P_{1850}}\right)^{\frac{t-1850}{t_1-1850}}$$

3. logistic equation

$\dot{P} = kP(M-P)$

$$P(t) = \frac{MP_0}{P_0 + e^{-Mkt}(M-P_0)}$$

$$P(t) = \frac{Me^{Mkt}e^{MC}}{1+e^{Mkt}e^{MC}}$$

$P(1850) = P_{1850} \Rightarrow P_{1850} + e^{Mk \cdot 1850} e^{MC} P_{1850} = Me^{1850Mk}e^{MC} \Rightarrow e^{MC}e^{1850Mk}(M - P_{1850}) = P_{1850}$

$$e^{MC} = \frac{P_{1850}}{e^{1850Mk}(M-P_{1850})} \Rightarrow C = \frac{1}{M}\left(\ln\left(e^{-1850Mk} \cdot \frac{P_{1850}}{M-P_{1850}}\right)\right) = \frac{1}{M}\left(-1850Mk + \ln\frac{P_{1850}}{M-P_{1850}}\right)$$

$$MC = -1850Mk + \ln\frac{P_{1850}}{M-P_{1850}}$$

$$e^{MC} = e^{-1850Mk} \cdot \frac{P_{1850}}{M-P_{1850}}$$

$$\Rightarrow P(t) = \frac{Me^{Mkt} \cdot e^{-1850Mk} \cdot \frac{P_{1850}}{M-P_{1850}}}{1+e^{Mkt} \cdot e^{-1850Mk} \cdot \frac{P_{1850}}{M-P_{1850}}} = \frac{Me^{Mk(t-1850)}\frac{P_{1850}}{M-P_{1850}}}{1+e^{Mk(t-1850)}\frac{P_{1850}}{M-P_{1850}}} = \frac{Me^{Mk(t-1850)}P_{1850}}{M-P_{1850}+e^{Mk(t-1850)}P_{1850}}$$

For any base year

$$P(t) = \frac{Me^{Mk(t-t_{base})} \cdot P_{base}}{M-P_{base} + Me^{Mk(t-t_{base})} \cdot P_{base}}$$

Let's use $(t_{base}, P_{base}) = (1800, 3.46)$ and then two data points $(1850, 7.25)$ and $(1900, 18.1)$ to solve for M and k.

$\Rightarrow M =$

$$\dot{P} = P(\beta(t) - \delta(t))$$

$$\beta(P) = \alpha P^2 + \beta P + \gamma$$

$$\dot{P} = P(\alpha P^2 + \beta P + \gamma - \delta_0)$$

$$\dot{P} = \alpha P^3 + \beta P^2 + (\gamma - \delta_0)P$$

$$\dot{P} = aP^3 + bP^2 + cP$$