Lecture 1 Geometric View of $y' = f(x,y)$

→ First-order ODE $\quad y' = f(x,y) \quad$ e.g. $y' = \frac{y}{x}$ $\qquad y' = x - y^2$ → not solvable

$y' = y - x^2$ → solvable

Geometric View of ODEs

Analytic		Geometric
$y' = f(x,y)$	⟷	direction field
$y_1(x)$ a solution	⟷	integral curve

integral curve has direction of field at all points of curve

→ $y_1(x)$ soln to $y' = f(x,y)$ ⟺ graph of $y_1(x)$ is an integral curve to dir. field assoc. with $y' = f(x,y)$

↙ means: ie $y_1'(x) = f(x, y_1(x))$

↙ means: slope of $y_1(x)$ = slope of direction field, $f(x, y_1(x))$

Drawing Direction Fields

Computer	Human
1. Pick (x,y) equally spaced	1. Pick slope c
2. Find $f(x,y)$	2. $f(x,y) = c$, plot equation (isocline)
3. Draw ⟋ slope	3.  isocline

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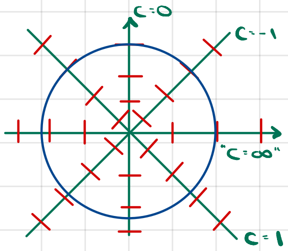
Example 1 $\quad y' = \frac{-x}{y} = c \Rightarrow y = \frac{-x}{c}$

integral curves are circles $\qquad y\,dy = -x\,dx$

$y^2/2 = -x^2/2 + C \Rightarrow x^2 + y^2 = C$

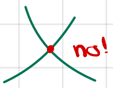


* it is hard to know what the domain of a typical solution is just by looking at the differential eq.
In this case, $y = \pm\sqrt{c - x^2}$, the domain of one of these is $[-c, +c]$.

→ Two important general principles

→ Two integral curves can't cross

can't have two slopes at a point

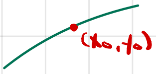


→ Two integral curves can't touch (can't be tangent)

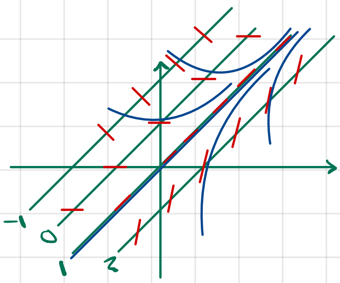
existence + uniqueness: through a point (x_0, y_0), $y' = f(x,y)$ has one and only one solution

→ hypotheses: $f(x,y)$ continuous near (x_0, y_0) (guarantees existence)

$f_y(x,y)$ " " " (" uniqueness)



Example 2 $\quad y' = 1 + x - y = c \Rightarrow y = x + (1 - c)$

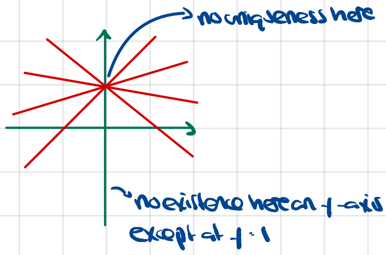


Example 3 $\quad xy' = y - 1 \Rightarrow (y-1)^{-1}dy = x^{-1}dx \Rightarrow \ln|y-1| = \ln|x| + C \Rightarrow |y-1| = C|x|$

$\Rightarrow y - 1 = Cx \Rightarrow y = 1 - Cx$

What's wrong? Need to write the diff eq. in standard form $\frac{dy}{dx} = \frac{1-y}{x}$ ↙ not continuous at $x = 0$

⇒ existence/uniqueness theorem does not apply

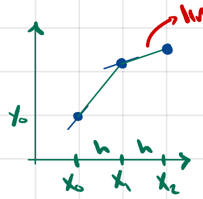


Lecture 2 - Numerical Solutions

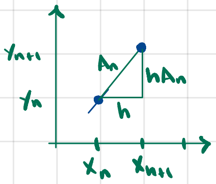
$y' = f(x, y)$
$y(x_0) = y_0$

Euler's Method

line segments called Euler struts



h : step size



$y_{n+1} - y_n = hA_n$
$\Rightarrow y_{n+1} = y_n + hA_n$

Euler Equations

$x_{n+1} = x_n + h$
$y_{n+1} = y_n + hA_n$
$A_n = f(x_n, y_n)$

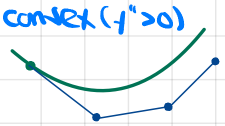
Example $y' = x^2 - y^2 \quad y(0) = 1 \quad h = 0.1$

n	x_n	y_n	A_n	hA_n
0	0	1	-1	-0.1
1	0.1	0.9	-0.8	-0.08
2	0.2	0.82		

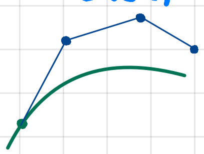
$\Rightarrow y(0.2) = 0.82 \leftarrow$ is this too high/low?

hypothetical cases for solution curve

convex ($y'' > 0$)



concave ($y'' < 0$)

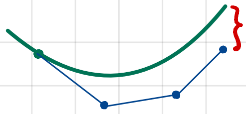


$y' = x^2 - y^2 \Rightarrow y'' = 2x - 2yy'$

At $(0,1)$,
$y(0) = 1$
$y'(0) = -1$
$y''(0) = 2 \cdot 0 - 2 \cdot 1 \cdot (-1) = 2 > 0$

The solution is convex at the starting point $(0,1)$
$\Rightarrow$ Euler prediction too low

Better method : smaller steps



} error depends on step size

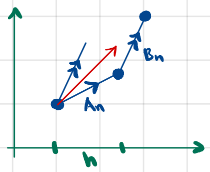
$e \sim c_1 h$

Euler is a first-order method
(h occurs to the first power)

$\rightarrow$ halve the step size, halve the error, approximately.

Still better method : Heun's Method / Improved / Modified Euler Method / RK2

$\rightarrow$ find better value for slope



$x_{n+1} = x_n + h$
$y_{n+1} = y_n + h\left(\frac{A_n + B_n}{2}\right)$
$\tilde{y}_{n+1} = y_n + hA_n$
$B_n = f(x_n, \tilde{y}_{n+1})$

second-order method
$e \sim c_2 h^2$

RK4 Runge-Kutta 4th-order

$\rightarrow$ standard method
$\rightarrow$ accurate

$$\frac{A_n + 2B_n + 2C_n + D_n}{6}$$

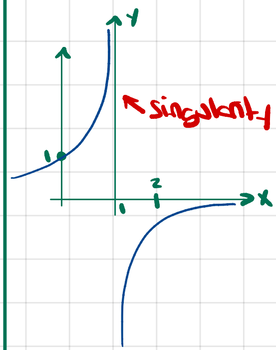
Pitfalls

1. Find in PSet.

2. $y' = y^2 \Rightarrow y = \frac{1}{c - x}$, $y(0) = 1$, $y(2)$?

singularity at $x = c$, which we know with the analytical solution but not from the original diff eq.

Tell a computer that $y(0) = 1$, slope field $y' = y^2$. It will apply a numerical method starting at $y(0) = 1$.
We know from the analytical solution that it looks as below:



The computer can't find $y(2)$.

Lecture 3 - First Order Linear

$a(x)y' + b(x)y = c(x)$ linear in y' and y

$c = 0 \to$ homogeneous

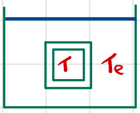
standard linear form: $y' + p(x)y = q(x)$

Examples of Models

- temperature concentration (conduction model)
- mixing
- decay
- bank account
- some motion problems

Examples

Conduction Model



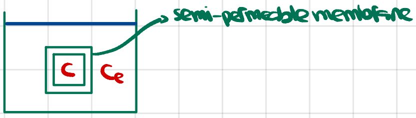
Newton cooling law

$$\frac{dT}{dt} = \underbrace{k}_{\text{conductivity}}(T_e - T) \qquad k > 0$$

$T(0) = T_0$

The first order diff eq. models the conduction of heat.

Diffusion Model



C: salt concentration inside

$$\frac{dc}{dt} = k_1(C_e - C) \qquad k_1 > 0$$

→ Both physical phenomena are modelled by same diff eq $\frac{dT}{dt} = kT = \overset{\text{general}}{kT_e}$, a fairly general diff eq.

k: usually constant, not necessarily

To solve $y' + p(x)y = q(x)$ need integrating factor: $u(x)$

$$\underbrace{uy' + puy}_{(uy)'} = qu$$

$(uy)' \to u' = pu \to \frac{du}{dx} = pu \to u^{-1}du = p\,dx$

$\to \ln(u) = \int p(x)dx \to u(x) = e^{\int p(x)dx}$ ← integrating factor

Method

1. Standard Linear Form
2. Calculate $e^{\int p\,dx}$ int. factor
3. Multiply both sides by int. factor
4. Integrate

Ex 1 $xy' - y = x^3$

$y' - \frac{1}{x}y = x^2$

$u(x) = e^{-\int x^{-1}dx} = e^{-\ln x} = x^{-1}$

$x^{-1}y' - x^{-2}y = x$

$\left(\frac{1}{x}y\right)' = x$

$\frac{1}{x}y = \frac{x^2}{2} + C \to y(x) = \frac{x^3}{2} + Cx$

Ex 2 $(1+\cos x)y' - \sin(x)y = 2x$

$y' - \frac{\sin x}{1+\cos x}y = \frac{2x}{1+\cos x}$

$u(x) = e^{-\int [\sin(x)/(1+\cos(x))]dx} = 1+\cos x$

$\int \frac{-\sin x}{1+\cos x}dx = \ln(1+\cos x)$

$\to y'(1+\cos x) - y\sin(x) = 2x$

$(y(1+\cos x))' = 2x \to y(1+\cos x) = x^2 + C$

$\to y(x) = \frac{x^2 + C}{1+\cos x}$

Linear with k constant

→ Temp. Model: $\frac{dT}{dt} + kT = kT_e$

$u(x) = e^{\int k\,dt} = e^{kt}$

$T'e^{kt} + e^{kt}kT = e^{kt}kT_e \to (Te^{kt}) = e^{kt}kT_e$

$\to Te^{kt} = T_e e^{kt} + C \to T(t) = \underbrace{T_e}_{\text{steady state}} + \underbrace{Ce^{-kt}}_{\text{transient}}$

Sometimes (engineering):

$Te^{kt} = \int_0^t e^{kt}kT_e\,dt = T_e e^{kt} - T_e$

$\to T(t) = T_e - T_e e^{-kt} = T_e(1 - e^{-kt})$

Lecture 4 - Substitutions

→ Scaling $x_1 = \frac{x}{a}$ $\quad y_1 = \frac{y}{b}$ $\quad$ a,b constants

→ reasons to do this

1. change units
2. make variables dimensionless (w/o units)
3. reduce number or simplify constants

ex: $\frac{dT}{dt} = k(M^4 - T^4)$ (Big temp. difference)

constant ext. temp $\quad$ int. temp

$T_1 = \frac{T}{M}$ dimensionless, sub in $T = MT_1$

$M\frac{dT_1}{dt} = k(M^4 - M^4T_1^4) = kM^4(1-T_1^4)$

⇒ $M\frac{dT_1}{dt} = k_1(1-T_1^4)$ $\quad (\text{time})^{-1}$

one less constant
simpler units
$k_1 = kM^4$, "lumping constants"

→ There are two kinds of substitutions

→ Direct: new vars some comb of old vars $\quad T_1 = T/M$

Inverse: old " " " new " $\quad T = T_1M$

* in calculus, we have a similar distinction

$\int x\sqrt{1-x^2}\,dx$, $u = 1-x^2$ (direct)

$\int \sqrt{1-x^2}\,dx$, $x = \sin u$ (inverse)

→ Direct sub in Bernoulli Eq.

Bernoulli Eq.

$y' = p(x)y + q(x)y^n$ $\quad (n \neq 0)$ * $n = 0,1$ ⇒ separable

$\frac{y'}{y^n} = p(x)y^{1-n} + q(x)$ $\qquad -1+n-1 \qquad n-2$

$v = y^{1-n}$ $\qquad v' = (1-n)y^{-n}\cdot y' = (1-n)\frac{y'}{y^n}$

⇒ $\frac{y'}{y^n} = \frac{v'}{1-n}$

⇒ $\frac{v'}{1-n} = p(x)v + q(x)$ (linear)

Ex: $y' = \frac{y}{x} - y^2$ ⇒ $\frac{y'}{y^2} = \frac{1}{x}\cdot\frac{1}{y} - 1$

$v(x) = \frac{1}{y(x)}$ $\qquad v'(x) = \frac{-1}{y^2}y' = \frac{-y'}{y^2}$ ⇒ $\frac{y'}{y^2} = -v'$

$-v' = \frac{1}{x}v - 1$ ⇒ $v' = -\frac{1}{x}v + 1$

⇒ $v' + \frac{1}{x}v = 1$ $\qquad u(x) = e^{\int x^{-1}dx} = e^{\ln x} = x$

$(xv)' = x$ ⇒ $xv = \frac{x^2}{2} + c$ ⇒ $v(x) = \frac{x}{2} + \frac{c}{x} = \frac{x^2+2c}{2x}$

$y(x) = \frac{1}{v(x)} = \frac{2x}{x^2+c_1}$

Homogeneous ODEs

→ invariant under zoom

→ increase the scale equally on both axes eg

e.g. $x = ax_1$, $y = ay_1$ ⇒ $dx = a\,dx_1$, $dy = a\,dy_1$ ⇒ $\frac{dy}{dx} = \frac{dy_1}{dx_1}$

→ $\frac{dy}{dx} = F(y/x) = \frac{dy_1}{dx_1} = F(y_1/x_1)$

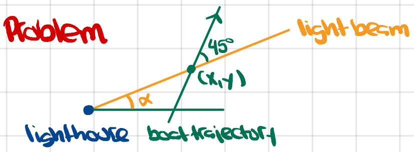
ex: $y' = \frac{x^2y}{x^3+y^3} = \frac{y/x}{1+(y/x)^3}$

ex: $xy' = \sqrt{x^2+y^2}$ ⇒ $y' = \sqrt{1+(y/x)^2}$

→ $y' = F(\frac{y}{x})$ $\qquad z = \frac{y}{x}$ ⇒ $y = zx$ ⇒ $y' = z'x + z$

$z'x + z = F(z)$ ⇒ $x\frac{dz}{dx} = F(z) - z$

Problem



lighthouse keeps shining light on boat, boat maintains 45° angle with beam. what is boat trajectory?

$y = y(x)$ unknown trajectory

$\tan\alpha = y/x$

$y' = \tan(\alpha + 45°) = \frac{\tan\alpha + \tan 45°}{1 - \tan\alpha\tan 45°} = \frac{y/x + 1}{1 - y/x} = \frac{y+x}{x-y}$ (homogeneous ODE)

$z = \frac{y}{x}$ $\quad y = zx$ $\quad y' = z'x + z$

⇒ $z'x + z = \frac{z+1}{1-z}$ ⇒ $x\frac{dz}{dx} = \frac{z+1-z(1+z)}{1-z} = \frac{1+z^2}{1-z}$ ⇒ $\frac{1-z}{1+z^2}dz = \frac{1}{x}dx$

⇒ $\tan^{-1}z - \frac{1}{2}\ln(1+z^2) = \ln x + C$

⇒ $\tan^{-1}z = \ln\sqrt{1+z^2} + \ln x + C$ ⇒ $\tan^{-1}(y/x) = \ln\sqrt{x^2+y^2} + C$

⇒ $\theta = \ln(r) + C$ ⇒ $r = C_1e^{\theta}$ (exponential spiral)

Lecture 5

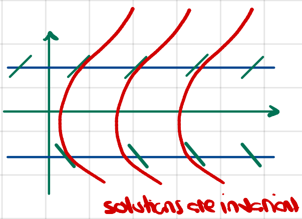
$\frac{dy}{dt} = f(y)$ → no indep. variable on RHS, ie autonomous

↘ common in practice, often difficult to integrate directly

→ could use separation of variables, but want to solve another problem: get qualitative info about solutions w/o solving the eq.

→ Direction Field

→ isoclines: $f(y) = c \Rightarrow y(t) = f^{-1}(c) = \text{constant}$

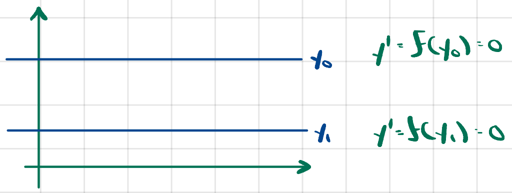


solutions are invariant under translation

→ Critical Points

1. Find critical points y_0, $f(y_0) = 0$

$y = y_0$ is an isocline and also a solution



→ other integral curves can't cross $y = y_0$ or $y = y_1$.

2. Graph $f(y) \gtrless 0$ so we know sign of $\frac{dy}{dx}$

Example

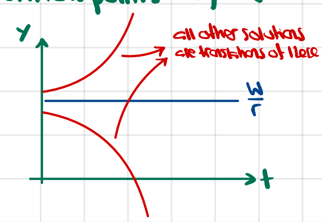
$y(t)$: money in account

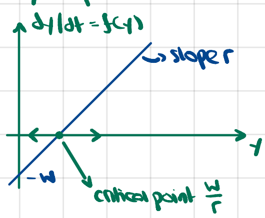
r: continuous interest rate

W = rate of embezzlement

$\dot{y} = ry - W$

critical points: $ry - W = 0 \Rightarrow y = \frac{W}{r}$





Example: Logistic Equation

Population behavior $y(t)$, $\frac{dy}{dt} = ky$

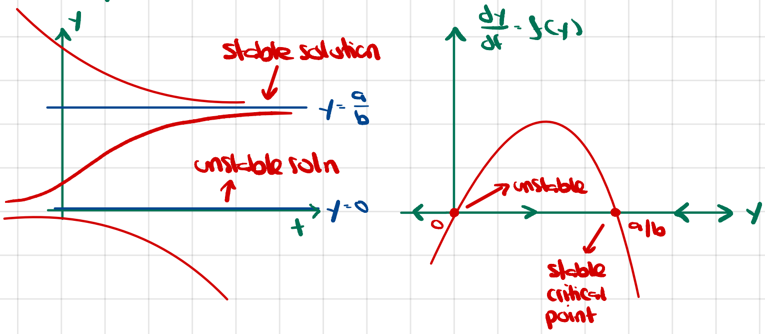
k constant: simple growth

logistic growth: k declines as y increases

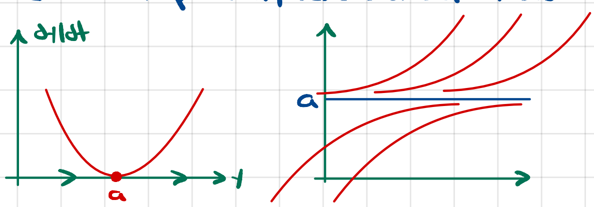
→ simplest choice: $k = a - by$

$\frac{dy}{dt} = ay - by^2$ can separate variables, partial fractions.

critical points: $y(a - by) = 0 \Rightarrow y = 0$ or $y = a/b$



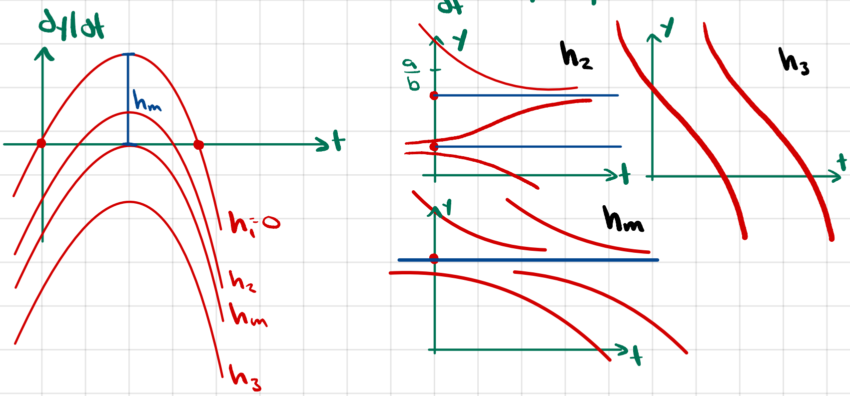
* there is a third possibility besides stable/unstable



stable from the left, unstable from right: semi-stable critical point

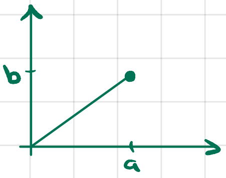
→ Logistic Equation w/ Harvesting

harvest at constant time rate $\Rightarrow \frac{dy}{dt} = ay - by^2 - h$



h_m: maximum rate of harvesting

Lecture 6 - Complex Numbers



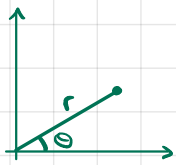
$z = a + bi$

$\bar{z} = a - bi$

$z\bar{z} = a^2 + b^2$

ex: $\frac{2+i}{1-3i} \cdot \frac{1+3i}{1+3i} = \frac{-1+7i}{10} = -\frac{1}{10} + \frac{7}{10} i$

Polar Repr'n



$a+bi = r\cos\theta + i\, r\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$\Rightarrow a+bi = re^{i\theta}$

r = modulus of α = $|\alpha|$

θ = argument of α = $\arg(\alpha)$

Advantage of Polar Form: good for multiplication

$r_1 e^{i\theta_1} \, r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$

Euler's Formula (actually a definition): $e^{i\theta} = \cos\theta + i\sin\theta$

Why this formula?

Exponential has following properties

1. Satisfies exponential law: $a^x \cdot a^y = a^{x+y}$
2. e^{at} sol'n to $\dot{y} = ay$, $y(0) = 1$
3. infinite series should work out

Is it the case that $e^{i\theta}$ defined as $\cos\theta + i\sin\theta$ has such properties?

1. $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$?
2. $\frac{d}{d\theta} e^{i\theta} = \theta e^{i\theta}$?
3. * we won't explore the infinite series aspect in 18.03

$\rightarrow e^{i\theta_1} \cdot e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$

$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)}$

complex-valued fn of a real variable

$\rightarrow e^{i\theta} = \cos\theta + i\sin\theta$

* $u(t) + iv(t)$ complex-val. fn of t

$D(u+iv) = Du + iDv$

$\rightarrow \frac{d}{d\theta} e^{it} = \frac{d}{dt}(\cos t + i\sin t) = -\sin t + i\cos t = i(\cos t + i\sin t) = ie^{it}$

$\rightarrow e^{i \cdot 0} = \cos 0 + i\sin 0 = 1$

Def: $e^{a+ib} = e^a e^{ib}$

→ we can calculate certain integrals by considering the integrand the real part of a number in the complex plane.

Ex: $\int e^{-x}\cos x \, dx$

$e^{-x}\underset{\text{real part of } e^{ix}}{\underbrace{\cos x}}$ = real part of e^{-x+ix}

$= \mathrm{Re}(e^{x(-1+i)})$

$\rightarrow \int e^{-x}\cos x\,dx = \mathrm{Re}\left(\int e^{x(-1+i)}dx\right)$

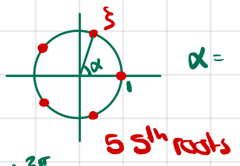
$= \mathrm{Re}\left[\frac{e^{x(-1+i)}}{-1+i}\right] = \mathrm{Re}\left[\frac{e^{-x}e^{ix}}{-1+i}\right]$

$= \mathrm{Re}\left[\frac{e^{-x}}{-1+i}(\cos x + i\sin x)\right]$

$= \mathrm{Re}\left[\frac{-1-i}{2} e^{-x}(\cos x + i\sin x)\right]$

$= \frac{e^{-x}(-\cos x + \sin x)}{2}$

$\rightarrow \sqrt[n]{1}$ n answers as complex numbers



$\alpha = \frac{2\pi}{5}$ rad

$\xi = e^{i\frac{2\pi}{5}}$ $(r=1)$

$\xi^5 = e^{i2\pi} = 1$ (since 2π and 0 are the same angle)

Lecture 7

Linear First order

$y'+p(t)y=q(t)$ (general form)

$y'+ky=q(t)$ (constant coefficients)

$y'+ky=kq_e(t)$ (special form)

$\Rightarrow \frac{1}{k}y'+y=q_e(t)$ (another form that is used)

useful form in models such as temperature conduction/diffusion, concentration, mixing

$y'+ky=q(t)$ $\underline{\underline{k>0}}$

uses $y(0)$

$$y(t)=\underbrace{e^{-kt}\int q(t)e^{kt}\,dt}_{\text{steady-state soln (long term)}}+\underbrace{Ce^{-kt}}_{\substack{0 \text{ as } t\to\infty \\ \text{transient}}}$$

input: $q(t)$
response: $y(t)$

physical input: $q_e(t)$

superposition of inputs
uses/is consequence of LINEARITY of ODE

$\rightarrow y'+ky=k\underset{\cos\omega t}{q_e(t)}$

ω: angular frequency: # complete oscillations in 2π

$\rightarrow$ Problem: $q_e=\cos\omega t$, find response

$\rightarrow$ complexify the problem (easier to integrate exponentials than trig)

$e^{i\omega t}=\cos\omega t+i\sin\omega t$

$\tilde{y}'+k\tilde{y}=ke^{i\omega t}$ $\tilde{y}=y_1+iy_2$, complex solution

Find $\tilde{y}$, then y_1 will solve the original real ODE.

$u(t)=e^{kt}$

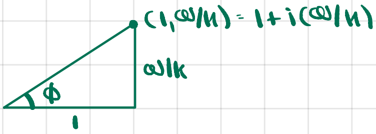
$(\tilde{y}e^{kt})'=e^{kt}ke^{i\omega t}=ke^{t(k+i\omega)}$

$\tilde{y}e^{kt}=\frac{k}{k+i\omega}e^{t(k+i\omega)}$

$\Rightarrow \tilde{y}(t)=\frac{1}{1+i(\omega/k)}e^{i\omega t}$ → polar, → cartesian

Take the real part
Two Methods
1. Go polar
2. Go cartesian

Polar Form of $\frac{1}{1+i(\omega/k)}$



Take $\alpha=1+i(\omega/k)$

$\Rightarrow |1/\alpha|=1/|\alpha|=1/(1+(\omega/k)^2)^{1/2}$

$\arg(1/\alpha)=-\arg(\alpha)=-\phi$

*recall α complex
consider $\alpha\cdot\frac{1}{\alpha}=$

$\Rightarrow \frac{1}{\alpha}\cdot\alpha=1$

Abs. values multiply
$\left|\frac{1}{\alpha}\right|\cdot|\alpha|=1$
$\Rightarrow \left|\frac{1}{\alpha}\right|=\frac{1}{|\alpha|}$

The arguments of α and $\frac{1}{\alpha}$ sum
$\arg(\alpha)+\arg(1/\alpha)=0$
$\Rightarrow \arg(\alpha)=-\arg(1/\alpha)$

$$\Rightarrow \frac{1}{1+i(\omega/k)}=\frac{1}{\sqrt{1+(\omega/k)^2}}e^{-i\phi}$$

$$\Rightarrow \tilde{y}(t)=\frac{1}{\sqrt{1+(\omega/k)^2}}e^{i(\omega t-\phi)}$$ (complex solution)

But $\tilde{y}=y_1+iy_2$

$$\Rightarrow y_1=\frac{1}{\sqrt{1+(\omega/k)^2}}\cos(\omega-\phi)$$ $\phi=\tan^{-1}(\omega/k)$ → phase lag

if k is conductivity, $\uparrow k\to\uparrow A$ because $\downarrow\omega/k$; $\downarrow\phi$

* Note: Given $y'+ky=k\cos\omega t$ we found

$y_1=\frac{1}{\sqrt{1+(\omega/k)^2}}\cos(\omega-\phi)$. Ie the response is also sinusoidal but with a phase lag.

Lecture 8 (continuation L7)

recall we had following diff eq. $y' + ky = k\cos\omega t$
To solve it, write the right term as a complex number in polar coord., solve the new diff eq. for a complex solution, take the real part and that solves the original diff eq.

This is the complex solution: $\tilde{y}(t) = \frac{1}{1+i(\omega/k)} e^{i\omega t}$

polar

cartesian

To obtain the real part, there are two methods: we either convert everything to polar coord. or cartesian coord.

→ last lecture we went polar: $\tilde{y}(t) = \frac{1}{\sqrt{1+(\omega/k)}} e^{i(\omega t - \phi)}$

⇒ real part: $\frac{1}{\sqrt{1+(\omega/k)}}\cos(\omega t - \phi)$

should be the same

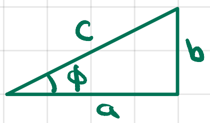
→ now, we go Cartesian:

$$\tilde{y}(t) = \frac{1 - i(\omega/k)}{1+(\omega/k)^2}(\cos\omega t + i\sin\omega t)$$

Real Part: $\frac{1}{1+(\omega/k)^2}\left(\cos\omega t + \frac{\omega}{k}\sin\omega t\right)$

To convert from the latter real part expression to the former, we will use:

$a\cos\theta + b\sin\theta = C\cos(\theta - \phi)$



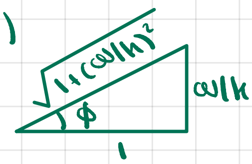
↳ sum of two sinusoidal functions of same frequency is another sinusoidal fn with that frequency

defn. $a\cos\omega t + b\sin\omega t = A\cos(\omega t - \phi)$

rectangular (Cartesian) form — amplitude-phase form

$$\frac{1}{1+(\omega/k)^2}\left(1+(\omega/k)^2\right)^{1/2}\cos(\omega t - \phi)$$

$$= \frac{1}{\sqrt{1+(\omega/k)}}\cos(\omega t - \phi)$$



→ Proof of the Formula $a\cos\theta + b\sin\theta = C\cos(\theta - \phi)$

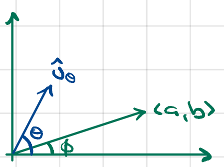
1. (high school proof)

$C\cos(\theta - \phi) = C[\cos\theta\cos(-\phi) - \sin\theta\sin(-\phi)]$

$= C[\cos\theta\cos\phi + \sin\theta\sin\phi]$

$= C\cos\phi\cos\theta + C\sin\phi\sin\theta = a\cos\theta + b\sin\theta$, $a = C\cos\phi$, $b = C\sin\phi$

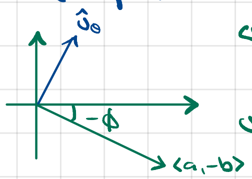
2. (18.03 proof)



$\langle a,b\rangle \cdot \langle\cos\theta, \sin\theta\rangle$

$= |\langle a,b\rangle| \cdot 1 \cdot \cos(\theta - \phi)$

3. (complex numbers proof)



write each vector as a complex number

$(a - bi)(\cos\theta + i\sin\theta)$

use polar representation

$= \sqrt{a^2+b^2}\,e^{-i\phi}\,e^{i\theta} = \sqrt{a^2+b^2}\,e^{i(\theta-\phi)}$

⇒ $(a - bi)(\cos\theta + i\sin\theta) = \sqrt{a^2+b^2}\,e^{i(\theta-\phi)}$

Take the real part on each side

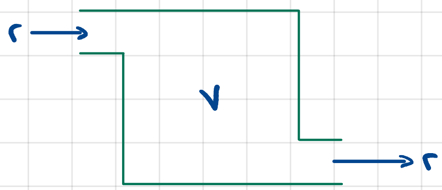
$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos(\theta - \phi)$

Basic Linear ODE

1. $y' + ky = kq_e(t)$ — most special form, controls temperature concentration model
2. $y' + ky = q(t)$
3. $y' + p(t)y = q(t)$

$k > 0$ in the cases so far.

Mixing Example



$x(t)$: amount of salt in tank at time t
C_e: concentration of incoming fluid
r: flow rate of fluid

$\frac{dx}{dt}$ = rate salt inflow − rate salt outflow

$= r \cdot C_e - r\frac{x(t)}{V}$

$$\frac{dx}{dt} + \frac{rx}{V} = rC_e$$

convert to concentration

$C(t) = \frac{x(t)}{V} \Rightarrow x(t) = C(t)V \Rightarrow \frac{dx}{dt} = V\frac{dC}{dt}$

$\Rightarrow V\frac{dC}{dt} + rC = rC_e \Rightarrow \frac{dC}{dt} + \frac{r}{V}C = \frac{r}{V}C_e$ (form 1.)

$k = r/V$ is the basic parameter

fractional rate of inflow/outflow

→ suppose $C_e(t)$ is sinusoidal

e.g., polluted water flows from a factory into a lake, and the concentration of pollutant varies sinusoidally within a day.

How closely does $C(t)$ follow $C_e(t)$?

Because our diff. eq. for $C(t)$ is the same as the one for other physical problems we've already studied (e.g. temperature diffusion/conduction), we know that a large k (e.g. high conductivity, bad insulation) means $C(t)$ will track $C_e(t)$ closely.

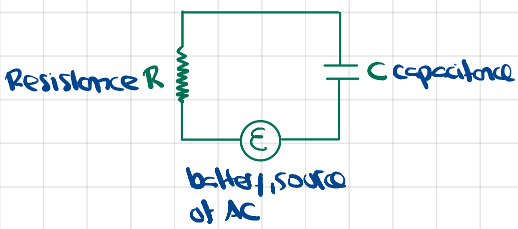
r large, ie large flow rate
V small
$\Rightarrow$ large k

Also, as we went through in lecture 7, if the specific form of $C_e(t)$ is sinusoidal then the response is sinusoidal with a phase lag $\phi = \tan^{-1}(\omega/k)$ and amplitude $A = \frac{1}{\sqrt{1+(\omega/k)^2}}$

$\uparrow k \Rightarrow \downarrow \phi$
$\downarrow A$

Examples of where we will need form 2.

q = charge on capacitance



$$\frac{dq}{dt} = i$$

$$R\frac{dq}{dt} + \frac{q}{C} = \mathcal{E}(t)$$

$$\dot{q} + \frac{q}{RC} = \frac{\mathcal{E}}{R}$$

Chain Decay

$$A \xrightarrow[\text{decay}]{} B \xrightarrow[\text{decay}]{}$$

$$\frac{dB}{dt} = k_1 A - k_2 B = k_1 A_0 e^{-k_1 t} - k_2 B$$

$$\dot{B} + k_2 B = k_1 A_0 e^{-k_1 t}$$

If $k<0$ none of the terminology transient, steady state, input, response applies.
→ techniques are the same, interpretations not.

Ex: $\frac{dy}{dt} - ay = q(t) \quad a>0 \quad$ ie $k = -a < 0$

$$e^{at}\int q(t)e^{-at}\,dt + ce^{at}$$

$\rightarrow \pm\infty$, not transient

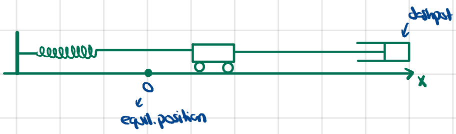
k typically negative in biology, economics, etc

Lecture 9 - Linear 2nd order ODE w/ constant coeff.

Standard form $y'' + Ay' + By = 0$ (homogeneous)

Assume general solution $y = c_1y_1 + c_2y_2$
where y_1, y_2 solutions

initial conditions are satisfied by choosing c_1, c_2



$m\underset{\text{Newton}}{x''} = -\underset{\text{Hooke}}{kx} - \underset{\text{Dashpot damping}}{cx'}$

$mx'' + cx' + kx = 0$

$x'' + \frac{c}{m}x' + \frac{k}{m}x = 0$ (typical model)

Find two solutions (independent)

Basic Method

$\Rightarrow$ Try $y = e^{rt}$ t = indep. variable

$r^2\cancel{e^{rt}} + Ar\cancel{e^{rt}} + B\cancel{e^{rt}} = 0$

$r^2 + Ar + B = 0$ (characteristic eq. of the system)

Case 1: roots $r_1 \neq r_2$ (real)

$y = c_1e^{r_1t} + c_2e^{r_2t}$ general solution

Ex $y'' + 4y' + 3y = 0$ $y(0) = 1$, $y'(0) = 0$

$r^2 + 4r + 3$

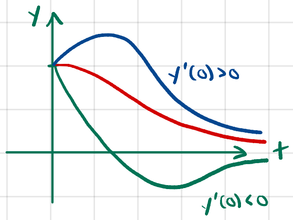
$(r+3)(r+1) = 0$

$\Rightarrow y = c_1e^{-3t} + c_2e^{-t}$

$y' = -3c_1e^{-3t} - c_2e^{-t}$

$1 = c_1 + c_2$
$0 = -3c_1 - c_2$ $\Rightarrow -2c_1 = 1 \Rightarrow c_1 = -\frac{1}{2}, c_2 = \frac{3}{2}$

$y = -\frac{1}{2}e^{-3t} + \frac{3}{2}e^{-t}$



Case 2: complex roots, $r = a + bi$

$\Rightarrow$ complex sol'n $y = e^{(a+bi)t}$

Theorem: if $u + iv$ complex sol'n to $y'' + Ay' + By = 0$ then u, v are real solutions.

Proof: $(u+iv)'' + A(u+iv)' + B(u+iv)$

$\underbrace{u'' + Au' + Bu}_{=0} + i\underbrace{(v'' + Av' + Bv)}_{=0} = 0$

$\Rightarrow u, v$ sol'ns

note that this proof relies on A, B being real constants.

$\Rightarrow$ case 2 solution

complex root $a+bi$ to characteristic eq.

$\Rightarrow y = e^{at+ibt}$ sol'n $= e^{at}\cos bt + ie^{at}\sin bt$, a complex sol'n

$\Rightarrow e^{at}\cos bt$, $e^{at}\sin bt$ real sol'ns

$\Rightarrow y(t) = e^{at}\underbrace{(c_1\cos bt + c_2\sin bt)}_{\text{sinusoidal oscillation}}$ general sol'n

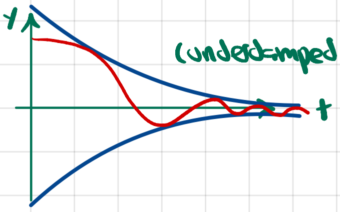
$= e^{at}\sqrt{c_1^2 + c_2^2}\cos(bt - \phi)$

Ex: $y'' + 4y' + 5y = 0$
$\uparrow$ stiffer spring

$r^2 + 4r + 5 = 0$

$r = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i$

$y = e^{t(-2+i)} = e^{-2t}(\cos t + i\sin t)$



(underdamped case)

$y(t) = e^{-2t}(c_1\cos t + c_2\sin t)$

$y(0) = 1$, $y'(0) = 0$

$\Rightarrow y = e^{-2t}(\cos t + 2\sin t) = e^{-2t}\cdot\sqrt{5}\cos(t - \phi)$ $\phi = \tan^{-1} 2$

Critically Damped Case: two equal roots to charact. eq.

$r = -a$

$(r+a)^2 = 0 \Rightarrow r^2 + 2ar + a^2 = 0 \Rightarrow y'' + 2ay' + a^2y = 0$

sol'n: $y = e^{-at}$ How to obtain a second sol'n? There are four ways.

know one sol'n $y_1 \Rightarrow$ another sol'n is $y_2 = y_1u$

$y = e^{-at}u$ $\times a^2$

$y' = -ae^{-at}u + e^{-at}u'$ $\times 2a$

$y'' = a^2e^{-at}u - ae^{-at}u' - ae^{-at}u' + e^{-at}u''$ $\times 1$

$= e^{-at}u'' \Rightarrow u'' = 0$

$u = c_1t + c_2$ $\Rightarrow y_2 = e^{-at}\cdot t$

Lecture 10 - oscillations

→ complex roots produce oscillations

$r = a \pm bi$ (complex conjugates)

→ $r = a + bi \Rightarrow y = e^{(a+ib)t} \xrightarrow{\text{Real}} e^{at}\cos bt = y_1(t)$, $e^{at}\sin bt = y_2(t)$ → real sol'ns

$\Rightarrow y(t) = c_1 y_1(t) + c_2 y_2(t)$

*we want real solutions!

$y(t) = c_1 e^{(a+bi)t} + c_2 e^{(a-bi)t}$ is a complex sol'n, including c_1, c_2

we want to be able to obtain the real part.

→ two ways

1. Multiply everything out, make Im part zero

2. Change i to −i, see if it stays the same

ex of 2. $y(t) = c_1 e^{(a+bi)t} + c_2 e^{(a-bi)t} \xrightarrow{i \to -i} \bar{c}_1 e^{(a-bi)t} + \bar{c}_2 e^{(a+bi)t}$

if $c_2 = \bar{c}_1$ and $c_1 = \bar{c}_2$ (this is only one condition actually) then $y(t)$ stayed the same w/ $i \to -i$

real solutions are $\underbrace{(c+id)}_{c_1} e^{(a+bi)t} + \underbrace{(c-id)}_{c_2 = \bar{c}_1} e^{(a-bi)t}$ (many scientists and engineers write the solutions this way)

how to change this to the old form (involving sines and cosines)?

method 1. expand everything out; the imag. part should go away because of the coeff. we chose.

nicer method: $e^{at}[c(e^{ibt} + e^{-ibt}) + id(e^{ibt} - e^{-ibt})]$

$= e^{at}[c \cdot 2\cos bt + id \cdot 2\sin bt]$

* $\cos a = \dfrac{e^{ia} + e^{-ia}}{2}$

$\sin a = \dfrac{e^{ia} - e^{-ia}}{2i}$

$mx'' + cx' + kx = 0$

$x'' + \frac{c}{m}x' + \frac{k}{m}x = 0$ standard form

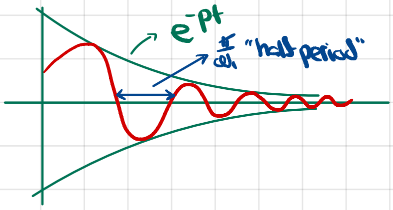
$y'' + 2py' + \omega_0^2 x = 0$ standard in engineering

oscillations

$r^2 + 2pr + \omega_0^2 = 0 \Rightarrow r = -p \pm \sqrt{p^2 - \omega_0^2}$

$p = 0$ → undamped, $y'' + \underbrace{\omega_0^2}_{\text{circular freq.}} y = 0$, $r = \pm i\omega_0$, $y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) = A\cos(\omega_0 t - \phi)$

damped case $\Rightarrow p^2 - \omega_0^2 < 0 \Rightarrow p < \omega_0$

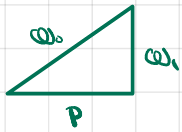


ω_1: pseudo-frequency, circular

$\uparrow c \Rightarrow \downarrow \omega_1$

$r = -p \pm \sqrt{p^2 - \omega_0^2} = -p \pm \sqrt{-(\omega_0^2 - p^2)} = -p \pm \sqrt{-\omega_1^2} = -p \pm \omega_1 i$

$\omega_1^2 = \omega_0^2 - p^2$



Sol'n: $e^{-pt}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t) = e^{-pt} A(\omega_1 t - \phi)$

$t_1 \qquad t_2 = t_1 + \frac{2\pi}{\omega_1}$

suppose sol'n crosses t axis for first time at $t = t_1$, what is t after one period?

$\omega_1 t_1 - \phi = \frac{\pi}{2} \Rightarrow y(t_1) = 0$

next time: $\omega_1 (t + \frac{2\pi}{\omega_1}) - \phi = \frac{\pi}{2} + 2\pi$

$\Rightarrow$ period $\frac{2\pi}{\omega_1}$

y crosses t axis every $\frac{\pi}{\omega_1} = \frac{\text{Period}}{2}$.

p depends only on ODE (c, m)

$\left.\begin{array}{l}\phi \\ A\end{array}\right\}$ depend on initial conditions

ω_1 depends only on ODE: $\omega_1^2 = \omega_0^2 - p^2$; ie depends on damping and spring constant

Lecture 11 - 2nd order linear homog. ODE

↳ linear in y'', y', y

$$y'' + p(x)y' + q(x)y = 0$$

→ sol'n method: find y_1, y_2 indep. sol'ns

independent: $y_2 \neq c y_1$, $y_1 \neq c' y_2$

* why do we write both conditions?
  if $y_1 = 0, y_2 \neq 0$ then $y_2 \neq c \cdot 0$ but $y_1 = 0 = 0 \cdot y_2$
  to exclude this case we need both $y_2 \neq c y_1$ and $y_1 \neq c' y_2$

All sol'ns are $y = c_1 y_1 + c_2 y_2$

Question 1: why are $c_1 y_1 + c_2 y_2$ sol'ns?

Question 2: why do they represent all the sol'ns?

Q1. answered by superposition principle.

superposition principle: y_1, y_2 sol'ns to linear homog. ODE
$\Rightarrow \underbrace{c_1 y_1 + c_2 y_2}_{\text{linear combination}}$ is a sol'n

---

→ $y'' + py' + qy = 0$

$D^2 y + pDy + qy = 0 \Rightarrow \underbrace{(D^2 + pD + q)}_{\text{Linear operator}} y = 0 \Rightarrow Ly = 0$

Linear operator
$L = D^2 + pD + q$

$u(x) \rightarrow \boxed{L} \rightarrow v(x)$

→ $L(u_1 + u_2) = L(u_1) + L(u_2)$ $\quad u_1, u_2$ fns

$L(cu) = cL(u)$, c const., u fn

Ex: D is linear: $D(u_1 + u_2) = (u_1 + u_2)' = u_1' + u_2'$
$D(cu) = (cu)' = cu'$

* proof that L is linear is in the homework

Proof of Superposition

ODE: $Ly = 0$

$L(c_1 y_1 + c_2 y_2) = L(c_1 y_1) + L(c_2 y_2) = c_1 L(y_1) + c_2 L(y_2)$
$= 0$

* key point: L is linear!

Answer 1: $c_1 y_1 + c_2 y_2$ is sol'n because the operator that gives us the ODE is linear.

→ Solving the IVP (fit init. values)

Theorem $\{c_1 y_1 + c_2 y_2\}$ is enough to satisfy any init. values.

Proof:

$y(x_0) = a$, $y'(x_0) = b$ init. values

$y = c_1 y_1 + c_2 y_2$
$y' = c_1 y_1' + c_2 y_2'$

plug in $x = x_0$ $\Rightarrow \begin{cases} c_1 y_1(a) + c_2 y_2(a) = a \\ c_1 y_1'(a) + c_2 y_2'(a) = b \end{cases}$ $\quad c_1, c_2$ unknown variables

solution exists if $\underbrace{W(y_1, y_2)}_{\text{Wronskian}} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}_{x_0} \neq 0$

Theorem: y_1, y_2 sol'ns to the ODE (lin., homog., 2nd order)

$\Rightarrow$ either $W(y_1, y_2) \equiv 0$ (for all x)

or $W(y_1, y_2)$ is never 0

$\{c_1 y_1 + c_2 y_2\} = \{c_1' u_1 + c_2' u_2\}$ $\quad u_1, u_2$ any other pair of indep. sol'ns

$u_1 = \bar{c}_1 y_1 + \bar{c}_2 y_2$
$u_2 = \bar{\bar{c}}_1 y_1 + \bar{\bar{c}}_2 y_2$

Finding Normalized Sol'ns: Y_1, Y_2
(at x_0, let's take $x_0 = 0$)

Y_1: $Y_1(0) = 1$, $Y_1'(0) = 0$ $\qquad$ Y_2: $Y_2(0) = 0$, $Y_2'(0) = 1$

Ex: $y'' + y = 0$

The standard sol'ns: $y_1 = \cos x$, $y_2 = \sin x$, and coincidentally: $Y_1 = y_1$, $Y_2 = y_2$

Ex: $y'' - y = 0$

$y_1 = e^x$, $y_2 = e^{-x}$ $\Rightarrow y = c_1 e^x + c_2 e^{-x}$
$y' = c_1 e^x - c_2 e^{-x}$

$y(0) = 1 = c_1 + c_2$
$y'(0) = 0 = c_1 - c_2$ $\Rightarrow c_1 = c_2 = \frac{1}{2} \Rightarrow Y_1 = \frac{e^x + e^{-x}}{2} = \cosh(x)$

Also $y(0) = 0$, $y'(0) = 1 \Rightarrow Y_2 = \frac{e^x - e^{-x}}{2} = \sinh(x)$

what's so good about normalized solutions?

Y_1, Y_2 normalized at zero $\Rightarrow$ sol'n to IVP ODE + $\begin{cases} y(0)=a \\ y'(0)=b \end{cases}$ is $aY_1 + bY_2$

Existence and Uniqueness Theorem

$y''+py'+qy=0$

$\Rightarrow$ There is exactly one sol'n to $y(0)=A$, $y'(0)=B$

p, q cont. $\forall x$

---

we want all sol'ns to $y''+py'+qy=0$

claim: $\{c_1Y_1 + c_2Y_2\}$ are all the solutions

proof: given any sol'n $u(x)$ such that $u(0)=u_0$, $u'(0)=u_0'$, it is also true that

$u_0Y_1 + u_0'Y_2$ satisfies the same initial conditions as u.

By exist. and uniqueness theorem, $u(x) = u_0Y_1 + u_0'Y_2$

ie every solution is of form $c_1Y_1 + c_2Y_2$.

Lecture 12 - Inhomogeneous Equations

$y'' + p(x)y' + q(x)y = f(x)$ ↘ input, signal, driving (forcing) term

solution: $y(x)$, response, output

$y'' + p(x)y' + q(x)y = 0$ → assoc. homog. eq., reduced eq.

solution: $y(x) = c_1y_1(x) + c_2y_2(x)$ → y_c, y_h, complementary solution

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Example

$mx'' + bx' + kx = f(t)$ (spring-mass-dashpot)

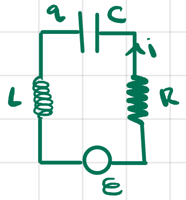
$mx'' = -kx - bx' + f(t)$ (Newton's 2nd Law)

↘ external force

$f(t) \neq 0 \Rightarrow$ forced system

$f(t) = 0 \Rightarrow$ passive system

Example



i: current

q: charge on capacitance

Sum of voltage drops = 0

$Li' + Ri + \frac{q}{C} = E(t)$

$\Rightarrow Li'' + Ri' + \frac{i}{C} = E'(t)$

$q' = i$

Theorem $Ly = f(x)$ (L linear operator)

solution: $y_p + y_c$, ie $y = y_p + c_1y_1 + c_2y_2$

y_p is a particular solution to $Ly = f(x)$

To prove

① $y_p + c_1y_1 + c_2y_2$ are sol'ns

$L(y_p + c_1y_1 + c_2y_2) = Ly_p + \underbrace{L(c_1y_1 + c_2y_2)}_{=0} = f(x) \Rightarrow Ly = f(x)$

② there are no other sol'ns

$u(x)$ sol'n $\Rightarrow L(u) = f(x)$

$L(y_p) = f(x)$

$L(u - y_p) = 0 \Rightarrow u - y_p = \tilde{c}_1y_1 + \tilde{c}_2y_2 \Rightarrow u = y_p + \tilde{c}_1y_1 + \tilde{c}_2y_2$

ie $u(x)$ is of form $y_p + y_c$ always.

$y' + ky = q(t)$

sol'n: $y = e^{-kt}\int q(t)e^{kt}dt + ce^{-kt}$

$= y_p + y_c$

$k > 0 \Rightarrow y = \underset{y_p}{\text{steady state}} + \underset{y_c \to 0}{\text{transient}}$

$y'' + Ay' + By = f(t)$

$y = y_p + \underbrace{c_1y_1 + c_2y_2}_{\text{use initial conditions}}$

under what conditions (A, B) does $c_1y_1 + c_2y_2 \to 0$ for all c_1, c_2?

→ if this is so, the ODE is called stable

$c_1y_1 + c_2y_2$ is the transient

y_p is the steady-state sol'n

---

char roots	sol'n	stability condition
$r_1 \neq r_2$	$c_1e^{r_1t} + c_2e^{r_2t}$	$r_1, r_2 < 0$
$r_1 = r_2$	$(c_1 + c_2t)e^{r_1t}$	$r_1 < 0$
$r = a \pm bi$	$e^{at}(c_1\cos(bt) + c_2\sin bt)$	$a < 0$

$\Rightarrow$ stable $\Leftrightarrow$ char. roots have negative real parts

Lecture 13

$y'' + Ay' + By = f(x)$

Find a particular sol'n y_p.

$\to$ important $f(x)$: e^{ax}, $\sin\omega x$, $\cos\omega x$, $e^{ax}\sin\omega x$, $e^{ax}\cos\omega x$

$\to$ All special cases of $e^{(a+i\omega)x} = e^{\alpha x}$

$(D^2 + AD + B)y = f(x)$
$p(D)y = f(x)$

$*$ $p(D)e^{\alpha x} = p(\alpha)e^{\alpha x}$

proof: $(D^2 + AD + B)e^{\alpha x} = D^2 e^{\alpha x} + ADe^{\alpha x} + Be^{\alpha x}$
$= \alpha^2 e^{\alpha x} + A\alpha e^{\alpha x} + Be^{\alpha x} = p(\alpha)e^{\alpha x}$

Exponential Input Theorem

$$y'' + Ay' + By = e^{\alpha x} \Rightarrow y_p = \frac{e^{\alpha x}}{p(\alpha)}$$

Proof: $p(D)y_p = p(D)\dfrac{e^{\alpha x}}{p(\alpha)} = \dfrac{p(\alpha)e^{\alpha x}}{p(\alpha)} = e^{\alpha x}$

$*$ what if $p(\alpha) = 0$? we assume $p(\alpha) \neq 0$ for now.

Example: $y'' - y' + 2y = 10e^{-x}\sin x$, Find particular sol'n.

$(D^2 - D + 2)\tilde{y} = 10e^{(-1+i)x}$ Im. part

$\tilde{y}_p = \dfrac{10e^{(-1+i)x}}{(-1+i)^2 - (-1+i) + 2}$, $\operatorname{Im}(\tilde{y}_p) = ?$

$= \dfrac{10e^{(-1+i)x}}{3 - 3i} = \dfrac{\overset{5}{\cancel{10}}}{3}\dfrac{(1+i)}{\cancel{2}}e^{-x}(\cos x + i\sin x)$

$\Rightarrow y_p = \operatorname{Im}(\tilde{y}_p) = \frac{5}{3}e^{-x}(\cos x + \sin x) = \frac{5}{3}e^{-x}\sqrt{2}\cos(x - \frac{\pi}{4})$

If $p(\alpha) = 0$

Let's rename α to a. Recall that $a \in \mathbb{C}$.

Exponential-Shift Rule $p(D)e^{ax}u(x) = e^{ax}p(D+a)u(x)$

Ex: $p(D) = D$

$De^{ax}u = e^{ax}Du + ae^{ax}u = e^{ax}(Du + au) = e^{ax}(D+a)u$

Ex: $p(D) = D^2$

$D^2 e^{ax}u = D(De^{ax}u) = D(e^{ax}(D+a)u)$

$= e^{ax}(D+a)(D+a)u = e^{ax}(D+a)^2 u$

$(D^2 + AD + B)y = e^{ax}$ $a \in \mathbb{C}$

if p is a simple root $p(x) = 0$

$y_p = \dfrac{xe^{ax}}{p'(a)}$

if a is a double root, $p(x) = p'(x) = 0$

$y_p = \dfrac{x^2 e^{ax}}{p''(a)}$

Proof (simple case)

$p(D) = (D-b)(D-a)$ $b \neq a$
$p'(D) = (D-a) + (D-b)$
$p'(a) = a - b$

$p(D)\dfrac{e^{ax}x}{p'(a)} \overset{\text{Exp. Shift Rule}}{=} e^{ax}p(D+a)\dfrac{x}{p'(a)}$

$p(D+a) = (D+a-b)(D+a-a) = (D+a-b)D$

$\Rightarrow e^{ax}(D+a-b)D\dfrac{x}{p'(a)} = e^{ax}\dfrac{(a-b)}{a-b} = e^{ax}$

Ex: $y'' - 3y' + 2y = e^x$

$p(D) = D^2 - 3D + 2$, $p(D)y = e^x$

1 is simple root of $D^2 - 3D + 2$

note $e^{\alpha x} = e^x \Rightarrow \alpha = 1$

$p(\alpha) = 1 - 3 + 2 = 0$

$\Rightarrow$ can't apply exp. input theorem

$p'(D) = 2D - 3$

$p'(\alpha) = -1 \neq 0$

$\Rightarrow y_p = \dfrac{xe^x}{-1}$

Lecture 14

$p(D)$ can be thought of in two ways:

i) formal polynomial in D
ii) linear operator on $y(t)$

we're interested in $f(t) = e^{\alpha t}$

we want y_p.

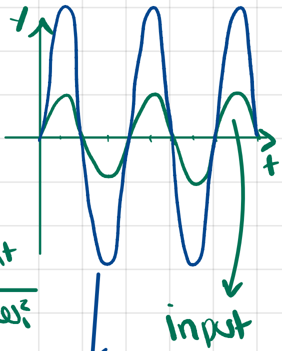
we derived $y_p = \dfrac{e^{\alpha t}}{p(\alpha)}$, $p(\alpha) \neq 0$

$= \dfrac{e^{\alpha t} t}{p'(\alpha)}$, $p'(\alpha) \neq 0$

Resonance

$y'' + \omega_0^2 y = \cos(\omega_1 t)$ $\quad \underbrace{\overset{\text{input freq}}{\omega_1}}_{} \neq \underset{\text{natural freq.}}{\omega_0}$

$(D^2 + \omega_0^2) y = \cos(\omega_1 t)$



complexify

$(D^2 + \omega_0^2)\tilde{y} = e^{i\omega_1 t}$

$\tilde{y}_p = \dfrac{e^{i\omega_1 t}}{(i\omega_1)^2 + \omega_0^2} = \dfrac{e^{i\omega_1 t}}{\omega_0^2 - \omega_1^2}$

$y_p = Re(\tilde{y}_p) = \dfrac{\cos \omega_1 t}{\omega_0^2 - \omega_1^2}$ → response

$\omega_1 \approx \omega_0 \Rightarrow$ large amplitude

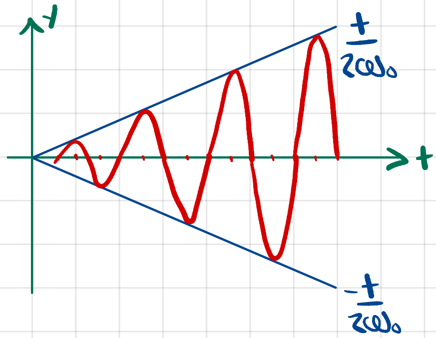
If $\omega_1 = \omega_0$ $\quad (D^2 + \omega_0^2) y = \cos(\omega_0 t)$

$\Rightarrow (D^2 + \omega_0^2)\tilde{y} = e^{i\omega_0 t}$

$i\omega_0$ is simple root of $D^2 + \omega_0^2$

$\Rightarrow \tilde{y}_p = \dfrac{t e^{i\omega_0 t}}{2i\omega_0}$

$y_p = Re(\tilde{y}_p) = \dfrac{t \sin(\omega_0 t)}{2\omega_0}$



consider again the case with $\omega_1 \neq \omega_0$

What do other particular sol'ns look like?

$y_p = \dfrac{\cos \omega_1 t}{\omega_0^2 - \omega_1^2}$ + anything that solves the homog. eq.

eg $\; - \dfrac{\cos \omega_0 t}{\omega_0^2 - \omega_1^2}$

Take the limit

$$\lim_{\omega_1 \to \omega_0} \frac{\cos \omega_1 t - \cos \omega_0 t}{\omega_0^2 - \omega_1^2}$$

use L'Hôpital

$$\lim_{\omega_1 \to \omega_0} \frac{-\sin(\omega_1 t) t}{-2\omega_1} = \frac{t \sin(\omega_0 t)}{2\omega_0}$$

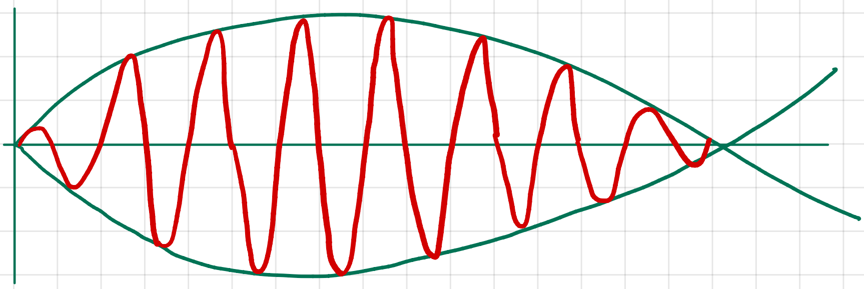
What's the geometric meaning of all this?

$\cos B - \cos A = 2 \sin\left(\frac{A-B}{2}\right) \sin\left(\frac{A+B}{2}\right)$

$B = \omega_1 t$
$A = \omega_0 t$

$$\Rightarrow \frac{\cos \omega_1 t - \cos \omega_0 t}{\omega_0^2 - \omega_1^2} = \underbrace{\frac{2}{\omega_0^2 - \omega_1^2} \sin\left(\frac{\overset{\text{very small freq.}}{(\omega_0 - \omega_1)} t}{2}\right)}_{\text{varying amplitude}} \underbrace{\sin\left(\frac{(\omega_0 + \omega_1)t}{2}\right)}_{\approx \sin(\omega_0 t)}$$

→ since $(\omega_0 - \omega_1)/2$ very small, $\sin\left(\frac{(\omega_0 - \omega_1)t}{2}\right)$ has a very large period.



The $\sin((\omega_0 + \omega_1)t/2)$ oscillation has time-varying amplit.
As $\omega_1 \to \omega_0$, the period increases to ∞.

Damped Resonance

→ note differences in notation

Book uses $mx'' + cx' + kx = f(t)$

$$x'' + 2px' + \omega_0^2 x = f_2(t) \qquad \omega_0 = \text{natural freq.}$$

visual $x'' + bx' + kx = f_3(t)$

ω_1 natural damped freq., "pseudo-freq."

ω_0 fixed by spring

$\uparrow p \Rightarrow \downarrow \omega_1$

$\uparrow$ damping $\Rightarrow \downarrow$ pseudo freq.

$\omega_1^2 = \omega_0^2 - p^2$ (from char. roots of damped eq.)

$y'' + 2py + \omega_0^2 = \cos \omega t$

problem: which input freq. gives max amplit. for response?

I give you p, ω_0

Answer: $\omega_r = \sqrt{\omega_0^2 - 2p^2}$

Lecture 15 - Fourier Series

up to now, we're solving $y'' + ay' + by = f(t)$ ← input

sol'n $y(t)$ response

input has been exponentials, $\sin t$, $\cos t$, ...

any $f(t)$ periodic (e.g. 2π) can be represented as sum

$$f(t) = c_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$$

↘ Fourier series

input	response
$a_n \sin(nt)$	$a_n y_n^{(s)}(t)$
$a_n \cos(nt)$	$a_n y_n^{(c)}(t)$
$f(t)$	$\sum (a_n y_n^{(c)} + a_n y_n^{(s)} + c_1)$

infinite series of fns
because of superposition
ok since ODE is linear

Program today: given $f(t)$, 2π as its period, find its Fourier Series

$u(t)$, $v(t)$ fns on $\mathbb{R}$, say 2π is a period,

orthogonal on $[-\pi, \pi]$ if $\int_{-\pi}^{\pi} u(t)v(t)\,dt = 0$

Theorem:

$$\begin{cases} \sin(nt) & n = 1, \dots, \infty \\ \cos(mt) & m = 0, \dots, \infty \end{cases}$$

any two distinct ones are orthogonal on $[-\pi, \pi]$

include $\int_{-\pi}^{\pi} \sin^2(nt)\,dt = \int_{-\pi}^{\pi} \cos^2(nt)\,dt = \pi$

① Trig. Identities
② Complex Exponentials
③ Use ODE ↘ this is the one that generalizes

Proof: $m \neq n$, $\sin(nt)$, $\cos(nt)$

satisfy: $u'' + n^2 u = 0$

u_n, u_m be any two of the fns

$$\int_{-\pi}^{\pi} u_n'' v_m\,dt = \underbrace{u_n' v_m \Big|_{-\pi}^{\pi}}_{=0} - \int_{-\pi}^{\pi} u_n' v_m'\,dt$$

integrate by parts

$$\int_{-\pi}^{\pi} u_n'' v_m\,dt = -n^2 \int_{-\pi}^{\pi} u_n v_m\,dt$$

use ODE

$$\int_{-\pi}^{\pi} v_m'' u_n\,dt = -m^2 \int_{-\pi}^{\pi} u_n v_m\,dt$$

therefore $\int_{-\pi}^{\pi} u_n v_m\,dt = 0$ if $m \neq n$