18.03SC Unit 1 Practice Exam

1.

a) T: chip temperature

T_e: environment temperature

t: minutes

First order, linear, constant coeff.

$\Delta T = k(T_e - T(t))\Delta t \Rightarrow \frac{dT}{dt} = k(T - T_e)$ $k<0$

b) $\dot{T} - kT = -kT_e$

$u(t) = e^{\int -k\,dt} = e^{-kt}$

$\dot{T}e^{-kt} - ke^{-kt}T = -ke^{-kt}T_e$

$Te^{-kt} = -kT_e \int e^{-kt}dt + C$

$= -kT_e \cdot \frac{1}{-k}e^{-kt} + C = T_e e^{-kt} + C \Rightarrow T(t) = T_e + Ce^{kt}$

c) $T_e = 20°C$, $T(0) = 70°C$, $T(10) = 60$

$T(t) = 20 + Ce^{kt}$

$T(0) = 20 + C = 70 \Rightarrow C = 50$

$T(10) = 20 + 50e^{10k} = 60 \Rightarrow 50e^{10k} = 40 \Rightarrow e^{10k} = 4/5$

$\Rightarrow 10k = \ln 4 - \ln 5 \Rightarrow k = \frac{\ln 4 - \ln 5}{10}$

$T(t) = 20 + 50e^{\ln 0.8/10}$

2. $y' = y^2 - x^2$ $y(2) = 0$ estimate $y(2.2)$, Euler's Method, step size 0.1

Euler's Method: $y_{n+1} = y_n + y_n' h$

$x_{n+1} = x_n + h$

n	x_n	y_n	y_n'	hy_n'
0	2	0	-4	-0.4
1	2.1	-0.4	-4.25	-0.425
2	2.2	-.825		

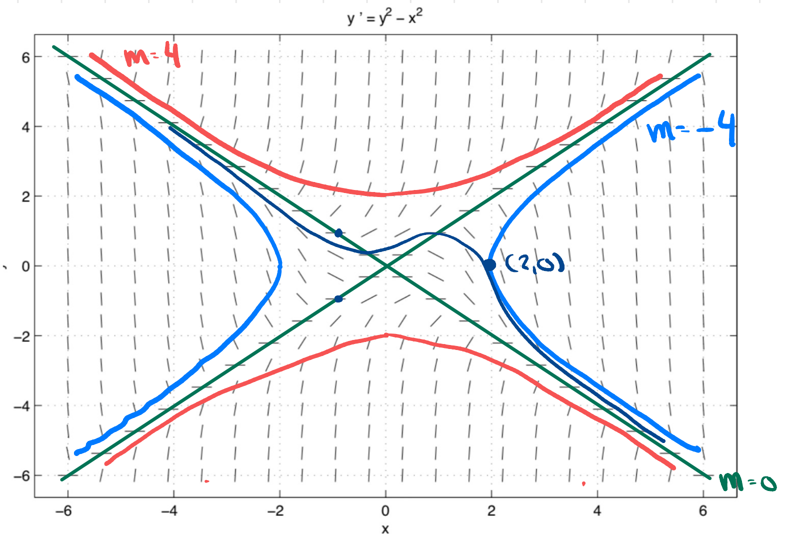
3. $y' = y^2 - x^2$ nonlinear, non-autonomous

a) isoclines $y^2 - x^2 = m \Rightarrow y = \pm\sqrt{m + x^2}$

$m = 0 \Rightarrow y = \pm x$

$m = -4 \Rightarrow y = \pm\sqrt{x^2 - 4}$ $x \geq 2, x \leq -2$

$m = 4 \Rightarrow y = \pm\sqrt{x^2 + 4}$



b) see direction field right.

c) estimate $y(100)$ given $y(2) = 0$

$u(x) = y(x) - (-x) = y(x) + x \Rightarrow y(x) = u(x) - x$

$u(2) = y(2) + 2 = 2$

$y'(x) = u'(x) - 1 = (u - x)^2 - x^2 = u^2 - 2ux + \cancel{x^2} - \cancel{x^2}$

$\Rightarrow u' = u^2 - 2ux + 1$

For u small (ie y near $-x$): $u' = -2ux + 1$

$\Rightarrow u' + 2xu = 1$

$u'e^{x^2} + 2xe^{x^2}u = e^{x^2}$

$ue^{x^2} = \int e^{x^2}dx + C$

$u(x) = e^{-x^2}\int e^{x^2}dx + C$ (dead end?)

The easy answer here is that $y(100)$ is slightly lower than -100.

d) local extremum at $x=-1 \Rightarrow (-1,y)$ is critical point

0-isoclines are $y = \pm x \Rightarrow y(-1) = \pm(-1) = \pm 1 \Rightarrow$ the solution passes either through $(-1,-1)$ or $(-1,1)$.

Also, $y'' = 2yy' - 2x \Rightarrow y''(-1) = 2(-1)\cdot 0 - 2(-1) = 2$

the solution has a local minimum whether it passes through $(-1,-1)$ or $(-1,1)$.

4.

a) $t\dot{x} + 2x = t^2$

$\dot{x} + \frac{2}{t}x = t$

$u(t) = e^{\int 2t^{-1}dt} = e^{2\ln|t|} = |t|^2 = t^2$

$\dot{x}t^2 + t^2 \cdot \frac{2}{t}x = t^3$

$\dot{x}t^2 + 2tx = t^3$

$xt^2 = \frac{t^4}{4} + C \Rightarrow x(t) = \frac{t^2}{4} + Ct^{-2}$

b) $\dot{x} + 2x = \cos(2t)$ $\qquad z = x + iy$

$\dot{z} + 2z = e^{i2t} = \cos 2t + i\sin 2t$

guess $z_p = Ae^{i2t}$, $z' = 2iAe^{i2t}$

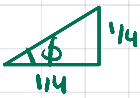
$2iAe^{i2t} + 2Ae^{i2t} = e^{i2t} \Rightarrow A = \frac{1}{2i+2}$

$A(2i+2) = 1$

$z_p(t) = \frac{1}{2+2i}e^{i2t} = \left(\frac{1}{4} - \frac{i}{4}\right)(\cos 2t + i\sin 2t) = \frac{1}{4}\cos 2t + \frac{1}{4}\sin 2t + \dots$

$e^{i2t} = \cos 2t + i\sin 2t$

$\frac{1}{2+2i} = \frac{2-2i}{8} = \frac{1}{4} - i\frac{1}{4}$



$\phi = \pi/4$

$r = (1/16 + 1/16)^{1/2} = 1/2\sqrt{2} = \frac{\sqrt{2}}{4}$

$\mathrm{Re}(z_p(t)) = \frac{\sqrt{2}}{4}\cos(2t - \pi/4)$

$\Rightarrow x(t) = \frac{\sqrt{2}}{4}\cos(2t - \pi/4) + Ce^{-2t}$

5. a) $\sqrt[3]{-8i} = z \Rightarrow \sqrt[3]{-i\,2^3} = 2\sqrt[3]{-i} = z$

$z = \rho e^{i\phi}$

$-i = e^{i\theta} = \cos\theta + i\sin\theta$

$\cos\theta = 0$

$\sin\theta = -1$ $\Rightarrow \theta = -\frac{\pi}{2} + 2\pi k$ $\Rightarrow -i = e^{-\frac{\pi}{2}i}$

$\rho e^{i\phi} = 2e^{i(-\frac{\pi}{2} + 2\pi k)\cdot\frac{1}{3}} = 2e^{i(-\frac{\pi}{6} + \frac{2\pi}{3}k)}$ $\quad k = 0,1,2$

$\rho = 2$

$\phi = -\frac{\pi}{6} + \frac{4\pi}{6}k$ $\quad k = 0,1,2$

Cube roots

$2e^{i(-\frac{\pi}{6} + \frac{2\pi}{3}k)}$

$= 2\cos(-\frac{\pi}{6} + \frac{2\pi}{3}k) + i\cdot 2\sin(-\frac{\pi}{6} + \frac{2\pi}{3}k)$ $\quad k = 0,1,2$

$\Rightarrow 2e^{-\frac{\pi}{6}i} = \sqrt{3} - i$

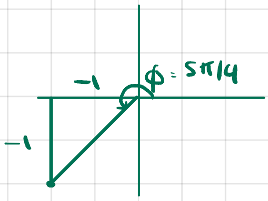
$2e^{\frac{\pi}{2}i} = 2i$

$2e^{\frac{7\pi}{6}i} = -\sqrt{3} - i$

$f(t) = -\cos(\frac{\pi}{2}t) - \sin(\frac{\pi}{2}t)$

b) $\langle -1,-1\rangle \langle \cos\frac{\pi}{2}t, \sin\frac{\pi}{2}t\rangle = \sqrt{2}\cos(\frac{\pi}{2}t - \phi)$

$\phi = \tan^{-1}(1) = \frac{5\pi}{4}$



$f(t) = \sqrt{2}\cos(\frac{\pi}{2}t - \frac{5\pi}{4})$

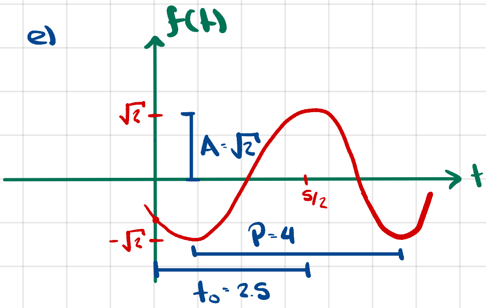
c) ω = angular frequency = cycles per 2π time = $\frac{\pi}{2}$

Period = time per cycle = $\frac{2\pi \text{ rad}}{\omega \text{ rad/time}} = \frac{2\pi}{\pi/2} = 4$

d) phase lag = $\frac{5\pi}{4}$

$\Rightarrow$ time lag = $\frac{\phi}{\omega} = \frac{5\pi/4}{\pi/2} = \frac{5\cancel{\pi}}{\cancel{4}_2} \cdot \frac{\cancel{2}}{\cancel{\pi}} = \frac{5}{2}$

e)



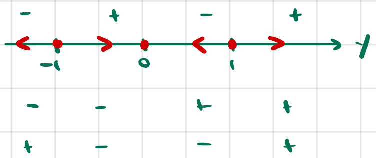
6. $\dot{y} = y^3 - y$ ↗ autonomous, nonlinear

a) critical points: $y(y^2-1) = 0$

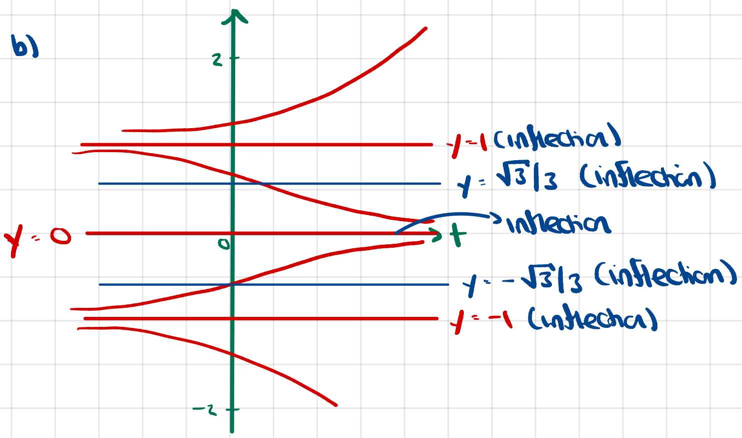
↗ $y = 0$ stable

↘ $y = 1$ unstable

$y = -1$ unstable



b)



c) inflection points

$\frac{d^2y}{dt^2} = 3y^2y' - y' = y'(3y^2 - 1) = 0$

$3y^2 = 1 \Rightarrow y^2 = \frac{1}{3} \Rightarrow y = \pm\frac{\sqrt{3}}{3}$

$y' = 0 \Rightarrow y = 0, y = 1, y = -1$