Solutions to Linear First Order ODEs

standard form, first order linear inhomogeneous ODE for unknown $x(t)$: $\dot{x} + p(t)x = q(t)$ $\quad q(t) \neq 0$

Associated homogeneous equation : $\dot{x} + p(t)x = \underbrace{0}_{\text{input signal} = 0 \text{ (null signal)}}$

Homogenous Eq. Solution

$$\frac{dx}{dt} = -p(t)x \Rightarrow \frac{1}{x}dx = -p(t)dt \Rightarrow \ln|x| = \int -p(t)dt + c_1 \Rightarrow |x| = e^{c_1}e^{-\int p(t)dt} \Rightarrow |x| = Ce^{-\int p(t)dt} \quad C>0$$

$$\Rightarrow x(t) = Ce^{-\int p(t)dt} \quad C \in \mathbb{R}$$

Each choice of C gives one particular solution. Each solution is a multiple of any one particular solution.

$$x(t) = C\,\underbrace{x_h(t)}_{\text{any arbitrary specific solution}}$$

→ To solve the inhomogeneous eq. we can use an integrating factor.

$\dot{x} + p(t)x = q(t)$

$u(t)\dot{x} + u(t)p(t)x = u(t)q(t)$

$$\frac{d}{dt}(ux) = \dot{u}x + u\dot{x} = u\dot{x} + upx \iff \dot{u} = up \Rightarrow \frac{1}{u}du = p\,dt \Rightarrow \ln|u| = \int p\,dt \Rightarrow u(t) = e^{\int p\,dt}$$

$$\Rightarrow (ux)' = \left(e^{\int p\,dt}x\right) = uq \Rightarrow ux = \int uq\,dt + C \Rightarrow x(t) = \frac{1}{u(t)}\left(\int u(t)q(t)\,dt + C\right), \quad u(t) = e^{\int p\,dt}$$

Note

$$u(t) = e^{\int p(t)dt}$$
$$x_h(t) = e^{-\int p(t)dt} \qquad \Rightarrow x_h(t) = \frac{1}{u(t)}$$

Superposition and Integrating Factors Solution

recall

$$\dot{x} + p(t)x(t) = q(t)$$

integrating factors solution

$$x(t) = \frac{1}{u(t)}\left[\int u(t)q(t)\,dt + c\right], \quad u(t) = e^{\int p(t)dt}$$

superposition principle

x_1 solution to $\dot{x} + p(t)x = q_1(t)$
x_2 " " " $= q_2(t)$
$\Rightarrow$ $\forall a,b$ constants $ax_1 + bx_2$ is solution to $\dot{x} + p(t)x = aq_1(t) + bq_2(t)$

$\Rightarrow$ written differently

$$q_1 \rightsquigarrow x_1, \quad q_2 \rightsquigarrow x_2 \Rightarrow a_1q_1 + a_2q_2 \rightsquigarrow ax_1 + bx_2$$

$\Rightarrow$ described in words

superposition of inputs leads to superposition of outputs

if we pick a constant $c = 0$ we obtain a specific (particular) solution to $\dot{x} + p(t)x = q(t)$

$$x_p = \frac{1}{u(t)}\int u(t)q(t)\,dt$$

But $x_h(t) = \frac{1}{u(t)}$, where $x_h(t)$ is an arbitrary solution to the homogeneous eq. $\dot{x} + p(t)x = 0$

$$\Rightarrow x(t) = \frac{1}{u(t)}\int u(t)q(t)\,dt + \frac{1}{u(t)}c = x_p(t) + c\,x_h(t)$$

$\Rightarrow$ recipe for solving the inhomogeneous eq:

1. obtain x_h by solving homog. eq.
2. find any solution to the inhomog. eq.
3. take a linear combin., coeff. are parameters.