So far, we know that a general sol'n of an n^{th} order homog. lin. eq. is a lin comb. of n lin. indep. particular sol'ns.
To actually find solutions:

$L(D)y = a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0 \qquad a_0, \ldots, a_n$ real, $a_n \neq 0$

Guess $y(x) = e^{rx} \Rightarrow e^{rx}(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0) = 0$

Let's consider the case of repeated roots, e.g. a root r_0 of multiplicity 1 and r_1 of mult. $k = n-1 > 1$

char. eq: $(r - r_1)^k (r - r_0) = 0 \quad \Rightarrow \quad (r - r_1)^k = 0$

note the char. eq. is $L(r)$ so $L(r) = (r - r_1)^k (r - r_0)$

we know two sol'ns: $e^{r_0 t}, e^{r_1 t}$. we need $k+1$ l.i. sol'ns to build a general sol'n

Try $u(x) e^{r_1 x}$, $u(x)$ to be determined

$L(D)y = (D - r_0)(D - r_1)^k y$

$(D - r_1) u(x) e^{r_1 x} = Du e^{r_1 x} + u r_1 e^{r_1 x} - r_1 u e^{r_1 x} = Du e^{r_1 x}$

$(D - r_1)(Du e^{r_1 x}) = D^2 u e^{r_1 x} + Du r_1 e^{r_1 x} - Du r_1 e^{r_1 x} = D^2 u e^{r_1 x}$

$\Rightarrow (D - r_1)^k e^{r_1 x} = D^k u e^{r_1 x}$

$\Rightarrow u e^{r_1 x}$ is sol'n $\Leftrightarrow D^k u = 0 \Leftrightarrow u(x) = \sum_{i=1}^{k} c_i x^{i-1}$ a pol. fn. of degree at most $k-1$

$\Rightarrow y(x) = \left(\sum_{i=1}^{k} c_i x^{i-1}\right) e^{r_1 x}$

$\Rightarrow$ we have extra sol'ns (besides $e^{r_1 t}$) $x e^{r_1 t}, x^2 e^{r_1 t}, \ldots, x^{k-1} e^{r_1 t}$
so k sol'ns total