01 - Complex Numbers

Definitions

→ $i = \sqrt{-1}$, note $i^2 = -1$

→ complex number: expression of form $a + ib$, $a, b \in \mathbb{R}$

imaginary part

real part

$a = Re(a+ib)$ $\quad$ $b = Im(a+ib)$

→ equality: $a+ib = c+id \iff a = c, b = d$

→ addition: $(a+bi) + (c+di) = (a+c) + (b+d)i$

→ multiplication: $(a+ib)(c+id) = ac - bd + i(ad + bc)$

→ division: $\dfrac{a+ib}{c+id} = \dfrac{a+ib}{c+id}\,\dfrac{c-id}{c-id} = \dfrac{ac-bd+i(ad+bc)}{c^2+d^2}$

→ complex conjugate of $z = a + bi$ is $\bar{z} = a - bi$ $\Rightarrow z\bar{z} = a^2 + b^2 + i(\cancel{-ab+ab}) = a^2 + b^2$

→ size of complex number: absolute value or modulus

$|z| = |a+ib| = \sqrt{a^2+b^2}$ $\quad \Rightarrow z\bar{z} = |z|^2$

The Complex Plane

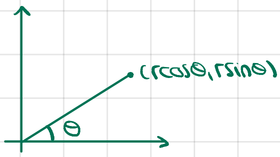
→ complex numbers are represented geometrically by points in the plane

→ $a+ib$ is (a,b) in Cartesian coordinates ; the plane thought of as representing complex numbers in this way is called the complex plane

in polar coord.

$x = r\cos\theta$

$y = r\sin\theta$ $\quad \Rightarrow x + iy = r(\cos\theta + i\sin\theta)$



$|x+iy| = \sqrt{x^2+y^2} = r$

θ = polar angle or argument of $x+iy$

$= \arg(x+iy)$

Examples

$-i = 0 - 1i = r(\cos 3\pi/2 + i\sin 3\pi/2) = -ri \Rightarrow r = -1$

$\Rightarrow -i = i\sin 3\pi/2$

$z = 1 + i$

$|z| = \sqrt{2} = r$ $\quad \Rightarrow z = \sqrt{2}(\cos\theta + i\sin\theta) \Rightarrow \sqrt{2}\cos\theta = \sqrt{2}\sin\theta = 1 \Rightarrow \theta = \pi/4$

$z = -1 + i\sqrt{3}$

$|z| = 2$ $\quad \Rightarrow z = 2\cos\theta + 2\sin\theta \cdot i \Rightarrow \cos\theta = -\frac{1}{2}$, $\sin\theta = \sqrt{3}/2$ $\Rightarrow \theta = \frac{2\pi}{3}$ $\Rightarrow z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

2. Euler's Formula

Definition $e^{i\theta} = \cos\theta + i\sin\theta$

exponential of an imaginary power

→ Given this definition, we can derive some results:

1. The polar repres. of $x+iy$ is $r(\cos\theta + i\sin\theta) = re^{i\theta}$

2. Given $r_1e^{i\theta_1} = r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2e^{i\theta_2} = r_2(\cos\theta_2 + i\sin\theta_2)$
we have that $r_1e^{i\theta_1} \cdot r_2e^{i\theta_2} = r_1r_2\left[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\right]$
$= r_1r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$
$= r_1r_2e^{i(\theta_1+\theta_2)}$

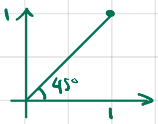
3. $\frac{1}{re^{i\theta}} \cdot re^{i\theta} = 1 \Rightarrow \frac{1}{re^{i\theta}} = (re^{i\theta})^{-1} = \frac{1}{r}e^{-i\theta}$

4. $\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)}\frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)} = \frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2))}{\cos^2\theta_2 + \sin^2\theta_2}$
$= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)) = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

5. $(x+iy)^n = (re^{i\theta})^n = r^n(e^{i\theta})^n = r^ne^{in\theta}$

$r=1 \Rightarrow (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$ (De Moivre Formula)

Example



a) $(1+i)^6$

$z = 1+i$
$|z| = \sqrt{2}$
$z = r(\cos\theta + i\sin\theta) = \sqrt{2}\cos\theta + i\sqrt{2}\sin\theta$

$\Rightarrow \theta = \frac{\pi}{4} \Rightarrow z = \sqrt{2}e^{\frac{\pi i}{4}}$

$(1+i)^6 = z^6 = (\sqrt{2})^6e^{\frac{3\pi i}{2}} = 8\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 8(0-i) = -8i$

b) $\frac{1+i\sqrt{3}}{\sqrt{3}+i} = \frac{2e^{\frac{\pi i}{3}}}{2e^{\pi i/6}} = e^{\frac{\pi}{6}i}$

$z_1 = 1+i\sqrt{3} \Rightarrow z_1 = 2\cos\theta + 2\sin\theta\, i$
$|z_1| = 2$
$\Rightarrow \cos\theta = \frac{1}{2} \quad \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3} \Rightarrow z_1 = 2e^{\frac{\pi i}{3}}$

$z_2 = \sqrt{3}+i \Rightarrow z_2 = 2(\cos\theta + i\sin\theta)$
$|z_2| = 2$
$\cos\theta = \frac{\sqrt{3}}{2} \quad \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \Rightarrow z_2 = 2e^{\frac{\pi i}{6}}$

Complex Exponentials

so far we've defined e^{ib}, exponential to pure imaginary power

now, we define

$$e^{a+ib} = e^a(\cos b + i\sin b) = e^a e^{ib}$$

Though $e^{a+ib} = e^a e^{ib}$ looks obvious and familiar from exp. w/ real numbers, we are dealing with complex numbers now and the eq. is not necessarily self-evidently true.

$$e^{a+ib+c+id} \overset{\text{by our new def}}{=} e^{a+c}e^{ib+id} \overset{\text{mult. rule for exp. of pure imag. numbers}}{=} e^a e^c e^{ib} e^{id} = e^a e^{ib} e^c e^{id} = e^{a+ib} e^{c+id}$$

$\rightarrow$ $e^{ix} = \cos x + i\sin x$ $\Rightarrow$ $\cos x = \mathrm{Re}(e^{ix})$

$\sin x = \mathrm{Im}(e^{ix})$

$$e^{-ix} = \cos(-x) + i\sin(-x)$$
$$= \cos(x) - i\sin(x)$$

$$e^{ix} + e^{-ix} = 2\cos x \Rightarrow \cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

$$e^{ix} - e^{-ix} = i \cdot 2\sin x \Rightarrow \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

Ex: $\cos^3(x) = \frac{1}{8}(e^{ix} + e^{-ix})^3 = \frac{1}{8}\left[e^{3ix} + 3e^{2ix}e^{-ix} + 3e^{ix}e^{-2ix} + e^{-3ix}\right] = \frac{1}{8}(e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix})$

$$= \frac{1}{4}\left[\frac{1}{2}(e^{3ix} + e^{-3ix}) + \frac{3}{2}(e^{ix} + e^{-ix})\right] = \frac{1}{4}(\cos 3x + 3\cos x)$$

complex-valued fn of a real variable x

$\rightarrow$ $u(x) + iv(x)$

$$D(u + iv) = Du + iDv$$

$$\int (u + iv)\,dx = \int u\,dx + i\int v\,dx$$

From this we can derive two results:

1. $D(e^{(a+ib)x}) = D(e^{ax}e^{ibx}) = D(e^{ax}(\cos bx + i\sin bx))$
$= D(e^{ax}\cos bx + e^{ax}\sin bx\, i)$
$= ae^{ax}\cos bx - be^{ax}\sin bx + i(ae^{ax}\sin bx + be^{ax}\cos bx)$
$= ae^{ax}(\cos bx + i\sin bx) + be^{ax}i\left(-\frac{\sin bx}{i} + \cos bx\right)$
$= ae^{ax}e^{ibx} + be^{ax}i(\cos bx + i\sin bx)$
$= ae^{(a+ib)x} + ibe^{ax}e^{ibx} = ae^{(a+ib)x} + ibe^{(a+ib)x} = (a+ib)e^{(a+ib)x}$

ie $D(e^{(a+ib)x}) = (a+ib)e^{(a+ib)x}$

2. $\int e^{(a+ib)x}\,dx = \int (e^{ax}\cos bx + e^{ax}\sin bx\, i)\,dx = (\ldots) = \frac{1}{a+bi}e^{(a+bi)x}$

Example: $\int e^x \cos 2x \, dx$

strategy: identify a complex number form for integrand. Integrate individual parts.

$\cos(2x)$ is real part of $\cos 2x + i\sin 2x = e^{i2x}$

$$\Rightarrow \int e^x \cos 2x \, dx = \mathrm{Re}\left(\int e^{x+i2x} dx\right) = \mathrm{Re}\left(\frac{1}{1+2i} e^{x+i2x}\right) = \mathrm{Re}\left(\frac{1-2i}{5} e^x(\cos 2x + i\sin 2x)\right)$$

$$= \mathrm{Re}\left(e^x\left(\frac{1}{5}\cos 2x - \frac{2}{5}\sin 2x + i\left(-\frac{2}{5}\cos 2x + \frac{1}{5}\sin 2x\right)\right)\right) = e^x\left(\frac{1}{5}\cos 2x - \frac{2}{5}\sin 2x\right)$$

n^{th} Roots of 1

problem: Find roots of $z^n = \alpha$.

special case: roots of $z^n = 1$

$(re^{i\theta})^n = 1 = e^{2\pi k i}$ ↗ $\cos(2\pi k) + i\sin(2\pi k) = 1$ $\quad k = 0, \pm 1, \pm 2, \dots$ $\quad \Rightarrow r^n e^{in\theta} = e^{i2\pi k}$

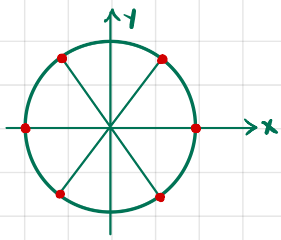
$\Rightarrow r^n = 1 \Rightarrow r = 1$ (r real, non-negative)

$n\theta = 2\pi k \Rightarrow \theta = \frac{2\pi k}{n}$ $\quad k = 0, 1, \dots, n-1$

↳ this is valid for all $k \in \mathbb{Z}$, but we need only consider $\{0, \dots, n-1\}$ because for the other k values the assoc. θ start to repeat. $\theta' = \frac{2\pi(k+an)}{n} = \theta + 2\pi a \Rightarrow e^{i\theta} = e^{i\theta'}$

$\Rightarrow$ n^{th} roots of 1 are the complex numbers $z = e^{i\theta}$ with $\theta = \frac{2\pi k}{n}$ $\quad k = 0, 1, \dots, n-1$

Ex: $z^6 = 1 \Rightarrow z = e^{\frac{\pi k i}{3}}$ $\quad k = 0, \dots, 5$



$\Rightarrow$ Notation: n^{th} roots of 1 are denoted $\zeta, \zeta^2, \dots, \zeta^{n-1}$, $\zeta = e^{\frac{2\pi i}{n}}$

$\Rightarrow$ General Case: n^{th} roots of w

$z^n = w$

$*\ w \in \mathbb{R}$

$w = w(\cos(2\pi k) + i\sin(2\pi k))$
$= we^{i2\pi k}$

$z^n = w = we^{i2\pi k}$

$\Rightarrow z = r^{\frac{1}{n}} e^{i\frac{2\pi k}{n}}$

write in polar form:

$w = re^{i\theta}$ $\quad \theta = \mathrm{Arg}(w)$ $\quad 0 \leq w \leq 2\pi$

$z = \rho e^{i\phi}$ $\quad \phi = \mathrm{Arg}(z)$

$\Rightarrow \rho^n e^{in\phi} = re^{i\theta} \Rightarrow \rho = r^{\frac{1}{n}}$

$n\phi = \theta + 2k\pi$ $\quad k \in \mathbb{Z} \Rightarrow \phi = \frac{\theta + 2k\pi}{n}$

$\Rightarrow z = r^{\frac{1}{n}} e^{\frac{\theta + 2\pi k}{n} i}$ $\quad k = 0, 1, \dots, n-1$

$= \left(\sqrt[n]{r}\, e^{\frac{i\theta}{n}}\right)\left(e^{\frac{2\pi i}{n}}\right)^k = z_0 \zeta^k$ $\quad k = 0, \dots, n-1$

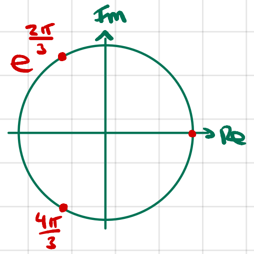
Examples

a) $\sqrt[3]{1}$ ie find the cube roots of 1

$z = \sqrt[3]{1} \Rightarrow re^{i\theta} = (e^{i2\pi k})^{1/3} \Rightarrow r = 1,\ \theta = \frac{2\pi k}{3} \quad k = 0,1,2$

$1 = (\cos(2\pi k) + i\sin(2\pi k)) \quad k \in \mathbb{Z}$

$= e^{i2\pi k}$



$\sqrt[3]{1} = 1, e^{\frac{2\pi}{3}i}, e^{\frac{4\pi}{3}i}$

ie $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

b) $\sqrt[4]{i}$, Find fourth roots of i

apply the formula $z = r^{\frac{1}{n}} e^{\frac{\theta + 2\pi k}{n} i} \quad k = 0,1,\dots,n-1$

$= (\sqrt[n]{r}\, e^{\frac{i\theta}{n}})(e^{\frac{2\pi i}{n}})^k = z_0 \zeta^k \quad k = 0,\dots,n-1$

write i in polar coord.: $r = 1,\ i = e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow \theta = \frac{\pi}{2}$

$i = e^{\frac{\pi i}{2}}$

$z_0 = \sqrt[4]{1}\, e^{\frac{\theta}{4} i}$

Four roots here

$\sqrt[4]{1} = (e^{\frac{2\pi i}{4}})^k \quad k = 0,\dots,3 \quad = 1, e^{\frac{\pi}{2}i}, e^{\pi i}, e^{\frac{3\pi}{2}i}$

$= 1, i, -1, -i$

$z_0 = e^{\frac{\pi}{8}i} \cdot \sqrt[4]{1}$

$\zeta = e^{\frac{\pi i}{2}} \qquad \zeta^k = e^{\frac{\pi k i}{2}} \quad k = 0,\dots,3$

$\Rightarrow \sqrt[4]{i}$ are

$k=0$: $e^{\frac{\pi}{8}i} \cdot 1 = e^{\frac{\pi}{8}i}$

$k=1$ $\quad e^{\frac{\pi}{8}i}\, i = i\cdot\cos\frac{\pi}{8} - \sin\frac{\pi}{8}$

$k=2$ $\quad e^{\frac{\pi}{8}i}(-1)$

$k=3$ $\quad e^{\frac{\pi}{8}i}(-i) = -i\cos\frac{\pi}{8} + \sin\frac{\pi}{8}$

Example: Solve $x^6 - 2x^3 + 2 = 0$

$y = x^3 \Rightarrow y^2 - 2y + 2 = 0$

$\Delta = 4 - 4\cdot1\cdot2 = -4 \qquad y = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$

$\Rightarrow x^3 = 1 \pm i$

$x = \sqrt[3]{1 \pm i}$ → Find cube roots of $1+i$ and $1-i$

$z = \sqrt[3]{1 \pm i}$

$1 \pm i = \sqrt{2}(\cos\theta \pm i\sin\theta) \Rightarrow \cos\theta = \frac{\sqrt{2}}{2} = \sin\theta$

$\Rightarrow \theta = \pm\frac{\pi}{4} + 2\pi k \quad k \in \mathbb{Z}$

$\rho e^{i\phi} = (\sqrt{2}\, e^{i(\pm\frac{\pi}{4} + 2\pi k)})^{1/3} = 2^{1/6} e^{i(\pm\frac{\pi}{12} + \frac{2}{3}\pi k)}$

$\rho = 2^{1/6}$

$\phi = \pm\frac{\pi}{12} + \frac{2}{3}\pi k \quad k = 0,1,2$

roots: $2^{1/6} e^{i(\pm\frac{\pi}{12} + \frac{2\pi}{3}k)} \quad k = 0,1,2$