PSet - characteristic Eq.

opposite of damping

1. $y''-3y'+2y=0$

$y=e^{rt} \Rightarrow r^2-3r+2=0 \qquad \Delta=9-8=1$

$r=\frac{3\pm1}{2}$ → 2, 1

$y(t)=c_1e^{2t}+c_2e^{t}$

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2. $y''+2y'-3y=0 \qquad y(0)=1,\ y'(0)=-1$

spring stretched

moving towards equilibrium level

$r^2+2r-3=0 \qquad \Delta=4+12=16$

$r=\frac{-2\pm4}{2}$ → 1, -3

$y(t)=c_1e^{t}+c_2e^{-3t} \qquad y'(t)=c_1e^{t}-3c_2e^{-3t}$

$y(0)=c_1+c_2=1$

$y'(0)=c_1-3c_2=-1 \Rightarrow 1-c_2-3c_2=-1 \Rightarrow 4c_2=2$

$\Rightarrow c_2=\frac{1}{2},\ c_1=\frac{1}{2}$

$y(t)=\frac{1}{2}e^{t}+\frac{1}{2}e^{-3t}$

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3. $y(x)=c_1+c_2e^{-5x}$

$y(x)=e^{rx}=1 \Rightarrow r=0$

$e^{rx}=e^{-5x} \Rightarrow r=-5$

charact. eq.: $r(r+5)=r^2+5r=e^{rt}r^2+e^{rt}\cdot 5r=0$

$y''+5y'=0$

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4. $y=c_1e^{5x}+c_2e^{-5x}$

$r=5,\ r=-5 \Rightarrow (r-5)(r+5)=r^2-25$

$=e^{rt}r^2-25e^{rt}=0 \Rightarrow y''-25y=0$

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5. $y=c_1+c_2x$

$e^{rx}\cdot x=x$

$e^{rx}=1 \Rightarrow r=0 \Rightarrow e^{rx}\cdot r^2=0 \Rightarrow y''=0$

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6. $y''-4y=0$

$e^{rt}(r^2-4)=e^{rt}(r+2)(r-2)=0$ → $r=2$, $r=-2$

$y(t)=c_1e^{2x}+c_2e^{-2x}$

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* $mx''=-kx-cx'$

Newton, Hooke, Dashpot damping

$x''+\frac{c}{m}x'+\frac{k}{m}x=0$

$x''+Ax'+Bx=0$

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7. $2y''-3y'=0$

$2r^2-3r=r(2r-3)=0$ → $r=0$, $r=3/2$

$y(t)=c_1+c_2e^{\frac{3t}{2}}$

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8. $4y''-12y'+9y=0$

$e^{rt}(4r^2-12r+9)=0$

$\Delta=144-4\cdot4\cdot9=0 \qquad y=\frac{12}{8}=\frac{3}{2}$

$y_1(t)=e^{\frac{3t}{2}} \qquad y_2(t)=te^{\frac{3t}{2}}$

$y(t)=c_1e^{\frac{3t}{2}}+c_2te^{\frac{3t}{2}}$

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9. $y^{(4)}-8y''+16y=0$

$e^{rt}(r^4-8r^2+16)=0$

$m=r^2$

$m^2-8m+16=0 \qquad \Delta=64-4\cdot16=0$

$m=\frac{8}{2}=4 \Rightarrow r^2=4 \Rightarrow r=\pm2$

$e^{rt}(x+2)^2(x-2)^2=0$

$e^{2x}, e^{-2x}, xe^{2x}, xe^{-2x}$

$y(x)=c_1e^{2x}+c_2e^{-2x}+c_3xe^{2x}+c_4xe^{-2x}$

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10. $y''+2y'+2y=0$

$e^{rt}(r^2+2r+2)=0 \qquad \Delta=4-4\cdot2=-4$

$r=\frac{-2\pm2i}{2}=-1\pm i$

$e^{-t+ti}=e^{-t}(\cos t+i\sin t)$ is sol'n. So are $e^{-t}\cos t, e^{-t}\sin t$

$y(t)=c_1e^{-t}\cos t+c_2e^{-t}\sin t=e^{-t}A\cos(t-\phi)$

(underdamping)

11. $y''-2y'+5y=0 \qquad y(0)=1,\ y'(0)=-1$

$e^{rt}(r^2-2r+5)=0 \qquad \Delta=4-20=-16$

$r=\frac{2\pm4i}{2}=1\pm2i$

$e^{t+2ti}=e^t(\cos 2t+i\sin 2t)$

$y(t)=c_1e^t\cos 2t+c_2e^t\sin 2t=e^tA\cos(2t-\phi)$

$y(0)=c_1=1$

$y'(t)=c_1e^t\cos 2t-2c_1e^t\sin 2t+c_2e^t\sin 2t+2c_2e^t\cos 2t$

$y'(0)=c_1+2c_2=-1 \Rightarrow 2c_2=-2 \Rightarrow c_2=-1$

$y(t)=e^t(\cos 2t-\sin 2t)=e^t\sqrt{2}\cos(2t+\pi/4)$

12. $y''-4y'+4y=0$

$r^2-4r+4=0 \qquad \Delta=16-4\cdot4=0$

$r=\frac{4}{2}=2$

$y(t)=c_1e^{2t}+c_2te^{2t}$

$y'(t)=2c_1e^{2t}+c_2e^{2t}+c_2t\cdot2e^{2t}$

$y(0)=c_1=1$

$y'(0)=2c_1+c_2=1 \Rightarrow c_2=1-2=-1$

$y(t)=e^{2t}-te^{2t}$

13. $y''+6y'+9y=0$

$r^2+6r+9=0 \qquad \Delta=36-4\cdot9=0$

$r=\frac{-6}{2}=-3$

$y(t)=c_1e^{-3t}+c_2te^{-3t}$

14. $y''-4y'+3y=0 \qquad y(0)=7 \qquad y'(0)=11$

$r^2-4r+3=0 \qquad \Delta=16-4\cdot3=4$

$r=\frac{4\pm2}{2}$ $\nearrow 3$, $\searrow 1$

$y(t)=c_1e^{3t}+c_2e^{t} \qquad y'(t)=3c_1e^{3t}+c_2e^t$

$y(0)=c_1+c_2=7$

$y'(0)=3c_1+c_2=11$

$3c_1+7-c_1=11 \Rightarrow 2c_1=4 \Rightarrow c_1=2$

$\Rightarrow c_2=7-2=5$

$y(t)=2e^{3t}+5e^t$

15. $y''-6y'+25y=0 \qquad y(0)=3 \qquad y'(0)=1$

$r^2-6r+25=0 \qquad \Delta=36-4\cdot25=-64$

$r=\frac{6\pm8i}{2}=3\pm4i$

$y(t)=e^{3t}e^{4ti}=e^{3t}(\cos 4t+i\sin 4t)$

$y(t)=e^{3t}(c_1\cos 4t+c_2\sin 4t)$

$y'(t)=3y(t)+e^{3t}(-4c_1\sin 4t+4c_2\cos 4t)$

$y(0)=3=c_1$

$y'(0)=1=3\cdot3+4c_2 \Rightarrow c_2=\frac{1-9}{4}=-2$

$y(t)=e^{3t}(3\cos 4t-2\sin 4t)$

$=e^{3t}\sqrt{13}\cos(4t-\tan^{-1}(-2/3))$

16. $y'' + 2y' + cy = 0$

a) $r^2 + 2r + c = 0 \qquad \Delta = 4 - 4c$

$$r = \frac{-2 \pm \sqrt{4-4c}}{2} = -1 \pm \sqrt{1-c}$$

two real roots: $4 - 4c > 0 \Rightarrow c < 1$

$$y(t) = C_1 e^{t(-1+\sqrt{1-c})} + C_2 e^{t(-1-\sqrt{1-c})}$$

repeated real root: $c = 1$

$$r = \frac{-2}{2} = -1$$

$y(t) = c_1 e^{-t} + c_2 t e^{-t}$ stable

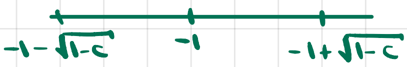
complex roots: $c > 1$

$r = -1 \pm \sqrt{1-c} = -1 \pm bi \quad b > 0$

$$y(t) = e^{-t} e^{bti} = e^{-t}(\cos bt + i \sin bt) = e^{-t}(c_1 \cos bt + c_2 \sin bt)$$

$= e^{-t} A \cos(bt - \phi) \quad A > 0$ stable (damped sinusoid)

b) $c < 1$



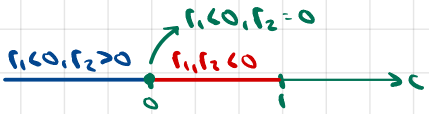
$-1 + \sqrt{1-c} < 0 \Rightarrow \sqrt{1-c} < 1 \Rightarrow 1 - c < 1 \Rightarrow c > 0$

$c \in (0,1) \Rightarrow$ roots both negative

$c = 0 \Rightarrow$ one negative, one zero

$c < 0 \Rightarrow$ one neg., one positive

c)



d) $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$c \in (0,1) \Rightarrow y(t \to \infty) = 0$

$c < 0 \Rightarrow y(t \to \infty) = \pm\infty$

$c = 0 \Rightarrow y(t \to \infty) = c_1$