PSet - Exponential Input - Part I

1. $y^{(3)} + y'' - y' + 2y = 2\cos x$

3rd order, linear, sinusoidal input

$z^{(3)} + z'' - z' + 2z = 2e^{ix}$ 3rd order, linear, exponential input

charact. eq: $p(r) = r^3 + r^2 - r + 2 = 0$

is i a root? $p(i) = i^3 + i^2 - i + 2 = -i - 1 - i + 2 = 1 - 2i$

$p'(r) = 3r^2 + 2r + 1$ $p'(i) = -3 + 2i + 1$

$\Rightarrow z_p(t) = \frac{2}{p(i)} e^{ix} = \frac{2e^{ix}}{1-2i} = \frac{(1+2i)}{5} \cdot 2(\cos x + i\sin x)$

$\Rightarrow y_p(t) = \frac{2}{5}\cos x - \frac{4}{5}\sin x$

roots: $-2, \frac{1}{2}, \pm\frac{\sqrt{3}}{2}i$ double root

sol'ns: $e^{-2t}, e^{\frac{t}{2}}, e^{\frac{\sqrt{3}}{2}it}$

$e^{\frac{\sqrt{3}}{2}i} = \cos\left(\frac{\sqrt{3}t}{2}\right) + i\sin\left(\frac{\sqrt{3}t}{2}\right)$

$\Rightarrow \cos(\sqrt{3}t/2), \sin(\sqrt{3}t/2)$ are real sol'ns.

$y(t) = \frac{2}{5}\cos t - \frac{4}{5}\sin x + c_1 e^{-2t} + c_2 e^{\frac{t}{2}} + c_3\cos(\sqrt{3}t/2) + c_4\sin(\sqrt{3}t/2)$

(wrong!)

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2. $y'' - 2y' + 4y = e^x\cos x$

1. Complexify

$e^x\cos x = Re(e^x e^{ix})$

$z'' - 2z' + 4z = e^{x+ix} = e^{x(1+i)}$

2. Exponential Input. Find roots of char. eq to determine which exp. response formula to use.

$p(r) = r^2 - 2r + 4 = 0 \Rightarrow r = \frac{2 \pm (4-16)^{1/2}}{2} = \frac{2 \pm (-12)^{1/2}}{2} = 1 \pm \sqrt{3}i$

$p(x) = p(1+i) = \cancel{1} + \cancel{2i} \cancel{-1} - 2 - \cancel{2i} + 4 = 2$

$z_p(x) = \frac{1}{2}e^x \cdot e^{ix} = \frac{e^x}{2}(\cos x + i\sin x)$

$y_p = Re(z_p(x)) = \frac{e^x\cos x}{2}$

3. $y'' - 6y' + 9y = e^{3x}$

$p(r) = r^2 - 6r + 9 = 0 \qquad r = \frac{6 \pm (36-36)}{2} = 3$

$p(r) = (r-3)^2$

$p(\alpha) = p(3) = 0$

$p'(r) = 2r - 6 = 0 \Rightarrow r = 3$

$p''(r) = 2 \neq 0$

$\Rightarrow y_p = \frac{x^2}{p''(3)} e^{3x} = \frac{e^{3x} x^2}{2}$

$y_h = c_1 e^{3t} + c_2 t e^{3t}$

$\Rightarrow y(t) = c_1 e^{3t} + C_2 t e^{3t} + \frac{e^{3t} t^2}{2}$

4. $x^{(3)} - x = e^{2t}$

$p(r) = r^3 - 1 = 0 \Rightarrow r = \sqrt[3]{1}$

$r = \rho e^{i\phi} = (e^{i2\pi k})^{1/3} = e^{\frac{i2\pi k}{3}}$

$\Rightarrow \rho = 1 \quad \phi = \frac{2\pi}{3}k \quad k = 0, 1, 2$

$* \; 1 = r\cos\theta + r\sin\theta$

$\Rightarrow r\cos\theta = 1 \Rightarrow r = 1$

$r\sin\theta = 0 \Rightarrow \theta = 2\pi k$

$\Rightarrow$ roots

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$e^{\frac{2\pi i}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$e^{\frac{4\pi i}{3}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$p(\alpha) = p(2) = 8 - 1 = 7$

$x_p(t) = \frac{1}{7} e^{2t}$

homog. sol'ns

e^t

$e^{-\frac{t}{2}} e^{\frac{\sqrt{3}ti}{2}} = e^{-\frac{t}{2}}(\cos(\sqrt{3}t/2) + i\sin(\sqrt{3}t/2))$

$e^{-\frac{t}{2}} e^{-\frac{\sqrt{3}ti}{2}} = e^{-\frac{t}{2}}(\cos(\sqrt{3}t/2) - i\sin(\sqrt{3}t/2))$

$\Rightarrow \quad e^{-\frac{t}{2}}\cos(\sqrt{3}t/2), \; e^{-\frac{t}{2}}\sin(\sqrt{3}t/2), \; -e^{-\frac{t}{2}}\sin(\sqrt{3}t/2)$ are sol'ns

$y(t) = c_1 e^t + c_2 e^{-\frac{t}{2}}\cos(\sqrt{3}t/2) + c_3 e^{-\frac{t}{2}}\sin(\sqrt{3}t/2)$

5. $y'' - 4y = \frac{1}{2}(e^{2t} + e^{-2t})$

$p(r) = r^2 - 4$
$p'(r) = 2r$

Use superposition principle.

Find y_{1p} sol'n to $y'' - 4y = \frac{e^{2t}}{2}$

Find y_{2p} " " $y'' - 4y = \frac{e^{-2t}}{2}$

$\Rightarrow y_{1p} + y_{2p}$ sol'n to $y'' - 4y = \frac{1}{2}(e^{2t} + e^{-2t})$

$y_{1p}(t) = \frac{1/2}{4} te^{2t} = \frac{te^{2t}}{8}$

$y_{2p}(t) = \frac{1/2}{-4} te^{-2t} = \frac{te^{-2t}}{-8}$

$\Rightarrow y_p(t) = \frac{1/2}{4} te^{2t} + \frac{1/2}{(-4)} te^{-2t} = \frac{t}{8}(e^{2t} - e^{-2t})$

Part II

1.

a) $\dot{x} + 2x = e^{3t}\cos(4t)$

method 1 (complexify, ERF)

$\dot{z} + 2z = e^{3t}e^{4ti} = e^{t(3+4i)} = e^{\alpha t}$

$p(r) = r+2 = 0 \Rightarrow r = -2$

$p(3+4i) = 5+4i$

$z_p(t) = \frac{1}{5+4i}e^{t(3+4i)} = \frac{5-4i}{41}e^{3t}(\cos 4t + i\sin 4t)$

$x_p(t) = \frac{1}{41}e^{3t}\cdot 5\cos 4t + \frac{1}{41}e^{3t}(-4i)(i\sin 4t)$

$= \frac{5e^{3t}\cos 4t}{41} + \frac{4e^{3t}\sin 4t}{41}$

$= \frac{e^{3t}}{41}(5\cos 4t + 4\sin 4t)$

$= \frac{e^{3t}}{41}\sqrt{41}\cos(4t - \tan^{-1}(4/5))$

method 2: integrating factor

$u(t) = e^{\int 2dt} = e^{2t}$

$e^{2t}\dot{x} + 2e^{2t}x = e^{5t}\cos(4t)$

$xe^{2t} = \int e^{5t}\cos 4t\,dt + C = \frac{5e^{5t}\cos 4t}{41} + \frac{4e^{5t}\sin 4t}{41} + C$

$x(t) = \frac{5e^{3t}\cos 4t}{41} + \frac{4e^{3t}\sin 4t}{41} + C\cdot e^{-2t}$ (general sol'n)

b) $\ddot{x} + \dot{x} + 2x = \cos t$

$\ddot{z} + \dot{z} + 2z = e^{it}$

$p(r): r^2 + r + 2 = 0 \Rightarrow r = \frac{-1 \pm (1-8)^{1/2}}{2} = \frac{-1 \pm \sqrt{7}i}{2}$

$p(\alpha) = p(i) = i^2 + i + 2 = i+1 \neq 0$

$z_p(t) = \frac{1}{i+1}e^{it} = \frac{1-i}{2}(\cos t + i\sin t)$

$x_p(t) = \frac{\cos t}{2} + \frac{\sin t}{2} = \frac{\sqrt{2}}{2}\cos(t - \pi/4)$

c) $m\ddot{x} + b\dot{x} + kx = A\cos(\omega t)$

$m = b = 1$

$k = 2$

homog. case: $A = 0$

$\ddot{x} + \dot{x} + 2x = 0 \quad p(D)x = 0 \quad p(r) = r^2 + r + 2$

roots: $r = \frac{-1 \pm (1-8)^{1/2}}{2} = \frac{-1 \pm \sqrt{7}i}{2}$

homog. sol'ns

$e^{-\frac{t}{2}}e^{\pm t\frac{\sqrt{7}i}{2}}$

$e^{\frac{t\sqrt{7}i}{2}} = \cos(\sqrt{7}t/2) + i\sin(\sqrt{7}t/2)$

$\Rightarrow \cos(\sqrt{7}t/2), \sin(\sqrt{7}t/2)$ sol'ns.

$x_h(t) = e^{-\frac{t}{2}}(c_1\cos(\sqrt{7}t/2) + c_2\sin(\sqrt{7}t/2))$

$= e^{-\frac{t}{2}}A\cos(\sqrt{7}t/2 - \phi)$

$\omega = \frac{\sqrt{7}}{2}$

pseudo-period $= \frac{2\pi}{\frac{\sqrt{7}}{2}} = \frac{4\pi}{\sqrt{7}}$

d) $A = 1, \omega = 1$

$\ddot{x} + \dot{x} + 2x = \cos t$

$x(t) = \frac{\cos t}{2} + \frac{\sin t}{2} + e^{-\frac{t}{2}}(c_1\cos(\sqrt{7}t/2) + c_2\sin(\sqrt{7}t/2))$

$= \frac{\sqrt{2}}{2}\cos(t - \pi/4) + e^{-\frac{t}{2}}A\cos(\sqrt{7}t/2 - \phi)$

$x_p = \frac{\sqrt{2}}{2}\cos(t - \pi/4)$

amplitude $\frac{\sqrt{2}}{2} \sim 0.7$

$\dot{x}_p = -\frac{\sqrt{2}}{2}\sin(t - \pi/4)$

$\dot{x}_p(0) = -\frac{\sqrt{2}}{2}\cdot\left(-\frac{\sqrt{2}}{2}\right) = \frac{1}{2}$

e) $x_h = e^{-\frac{t}{2}}A\cos(\sqrt{7}t/2 - \phi)$

2. $\dot{x} + x = e^{-t}$

$p(r) = r + 1$ root: -1

$p'(r) = 1 \neq 0$

$\Rightarrow x_p = \frac{t}{1} e^{-t} = t e^{-t}$

$x_h = c_1 e^{-t}$

$x(t) = c_1 e^{-t} + t e^{-t}$