Pset 1 - Part I

Problem 1

$\frac{dy}{dx} = 2 - y \quad y(0) = 0$

$\frac{1}{2-y}\, dy = dx \Rightarrow -\ln|2-y| = x + C_1 \,,\; y \neq 2$

$|2-y| = e^{-x} e^{-C_1} \Rightarrow 2 - y = \pm e^{-C_1} e^{-x} \Rightarrow 2 - y = C_2 e^{-x} \Rightarrow y = 2 + Ce^{-x}$

$y(0) = 2 + C \Rightarrow C = y_0 - 2 \Rightarrow y(x) = 2 + (y_0 - 2)e^{-x},$
which includes $y(x) = y_0 = 2$

$y_0 = 0 \Rightarrow y(x) = 2 - 2e^{-x}$

Problem 2

$\frac{dy}{dx} = \frac{(y-1)^2}{(x+1)^2} \Rightarrow \frac{1}{(y-1)^2}\, dy = \frac{1}{(x+1)^2}\, dx$

$y \neq 1$

$\Rightarrow -(y-1)^{-1} = -(x+1)^{-1} + C_1$

$1/(y-1) = 1/(x+1) + C_2$

$$y - 1 = \frac{1}{\frac{1}{x+1} + C_2} = \frac{x+1}{C_2(x+1)+1}$$

$$y = \frac{x+1}{C_2(x+1)+1} + 1$$

$$y(0) = \frac{1}{2C_2+1} + 1 = \frac{2C_2+2}{2C_2+1} = \frac{2(C_2+1)}{2C_2+1} = y_0$$

$\Rightarrow 2C_2 y_0 + y_0 = 2C_2 + 2$

$2C_2(1 - y_0) = y_0 - 2 \Rightarrow C_2 = \frac{y_0 - 2}{2(1-y_0)}$

$$y(x) = \frac{x+1}{\frac{(x+1)(y_0-2)}{2(1-y_0)} + 1} + 1 = \frac{y_0(x-3) - 2(x-1)}{y_0(x-1) - 2x}$$

includes lost solution $y = y_0 = 1$

Problem 3

$\frac{dp}{dt} = k\sqrt{p}$

$p^{-\frac{1}{2}}\, dp = k\,dt \Rightarrow 2p^{1/2} = kt + C_1 \Rightarrow p = (\tfrac{1}{2}kt + C_2)^2$

$p(0) = C_2^2 \Rightarrow C_2 = \sqrt{p_0}$

$p(t) = (\tfrac{1}{2}kt + \sqrt{p_0})^2$

Problem 4

$\frac{dv}{dt} = kv^2$

$v \neq 0$

$v^{-2}\, dv = k\,dt \Rightarrow -v^{-1} = kt + C_1 \Rightarrow v = -\frac{1}{kt + C_1}$

$v(0) = -\frac{1}{C_1} \Rightarrow C_1 = -\frac{1}{v_0} \qquad v(t) = -\frac{1}{kt - \frac{1}{v_0}}$

includes $v = v_0 = 0$

Problem 5

identify relevant parameters

popul. size: S

people who have heard rumor: N

people who haven't: $N_n = S - N$

assumptions

$N'(t) = f(S - N), \; f(0) = 0$

→ we don't know exactly what f looks like, but we can linearize it.

$f(z) \approx f(0) + f'(0)z = kz$

using the linear approx. to f to model $N'(t)$ we have

$$\frac{dN}{dt} = k(S - N(t)) = \frac{k}{S}\left(1 - \frac{N}{S}\right)$$

the larger k is the faster the rumor spreads for any given level of N.

$$\frac{1}{1 - N/S}\, dN = \frac{k}{S}\, dt \Rightarrow -S\ln|1 - N/S| = \frac{k}{S}t + C_1$$

$$\Rightarrow \ln|1 - N/S| = -\frac{k}{S^2}t + C_1 \Rightarrow |1 - N/S| = e^{-\frac{k}{S^2}t + C_1}$$

$$1 - N/S = \pm e^{\frac{C_1}{S^2}} e^{-kt} = C_2 e^{-kt} \Rightarrow \frac{N}{S} = 1 - C_2 e^{-kt}$$

$$\Rightarrow N(t) = S(1 - C_2 e^{-kt}) = S + Ce^{-kt}$$

$N(0) = S + C \Rightarrow C = N_0 - S \Rightarrow N(t) = S + (N_0 - S)e^{-kt}$

Because $N_0 \leq S$, $N(t) \leq S$. In particular $N_0 - S \leq 0$.

Problem 6

$\frac{dA(t)}{dt} = kA(t) \Rightarrow A(t) = Ce^{kt} = A_0 e^{kt}$

we have a process of decay $\Rightarrow k < 0$

half life $\frac{1}{2}A_0 = A_0 e^{kt} \Rightarrow e^{kt} = \frac{1}{2} \Rightarrow kt = \ln(1/2) = -\ln 2$

$\Rightarrow t_{1/2} = \frac{-\ln 2}{k}$

we know that $t_{1/2} = 5$ hours $\Rightarrow \frac{-\ln 2}{k} = 5 \Rightarrow k = \frac{-\ln 2}{5}$

$A(t) = A_0 e^{-\frac{\ln 2}{5}t}$

Also, $A(1) = 50 \cdot 60 = 3000$ (at least) $\Rightarrow$

$3000 = A_0 e^{-\frac{\ln 2}{5}} \Rightarrow A_0 = 3000\, e^{\frac{\ln 2}{5}} \Rightarrow A_0 = 3446$ mg

Problem 7

relevant parameters, variables

$x(t)$: distance travelled, m

$r(t)$: rate of snow clearance m^3/h

$d(t)$: m^3 snow/m road = m^2

w: plow/road width m

$h(t)$: snow height at time t at $x(t)$

→ we can imagine that the speed of the truck is determined simply by multiplying the inverse of the amount of snow per unit length of road, ie (m^3 snow/m road)$^{-1}$ by rate of snow clearance. Equivalently, divide rate of snow clearance by amount of snow per length road, giving us number of meters travelled per hour.

$$\frac{r}{d}\Delta t = \Delta x \qquad \text{e.g. } r = 600\,m^3/h,\ d = 2\,m^2/m \Rightarrow \frac{600}{2}\cdot 1h = 300\,m$$

$$\Rightarrow \frac{dx}{dt} = \frac{r(t)}{d(t)}$$

→ if we know how wide the plow is, we can have $d(t) = (\text{width } m)\cdot(1\text{ m length})\cdot h\ m\ /(1\text{ m road}) = h$, ie if we know the width then $S(t) = w\,h(t)$.

$$\Rightarrow \frac{dx}{dt} = \frac{r}{w\,h(x(t))}$$

→ how does snow height vary with time? assumption: linearly: $h(t) = h_0 + k_s t$

$$\Rightarrow \frac{dx}{dt} = \frac{r}{w(h_0+k_s t)} \Rightarrow \frac{r}{w}(h_0+k_s t)^{-1}dt = dx \Rightarrow x(t) = \frac{r}{w}\cdot\frac{1}{k_s}\ln|h_0+k_s t| + C = c_1\ln(h_0+k_s t)+c_2$$

$$x(0) = \frac{r}{wk_s}\ln(h_0)+C \Rightarrow C = x_0 - \frac{r\ln(h_0)}{wk_s} \Rightarrow x(t) = \frac{r}{w}\cdot\frac{1}{k_s}\ln(h_0+k_s t) + \left[x_0 - \frac{r\ln(h_0)}{wk_s}\right]$$

$$x(t_1+1) - x(t_1) = 2000 = \frac{r}{w}\cdot\frac{1}{k}\ln(h_0+k(t_1+1)) - \frac{r}{w}\cdot\frac{1}{k}\ln(h_0+kt_1) = \frac{r}{wk}\ln\left[\frac{h_0+k(t_1+1)}{h_0+kt_1}\right]$$

$$x(t_1+3) - x(t_1+1) = 2000 = \frac{r}{wk}\ln\left[\frac{h_0+k(t_1+3)}{h_0+k(t_1+1)}\right]$$

$$\Rightarrow \ln\left[\frac{h_0+k(t_1+3)}{h_0+k(t_1+1)}\right] = \ln\left[\frac{h_0+k(t_1+1)}{h_0+kt_1}\right] \Rightarrow \frac{h_0+k(t_1+3)}{h_0+k(t_1+1)} = \frac{h_0+k(t_1+1)}{h_0+kt_1}$$

$$\Rightarrow \cancel{k^2t^2} + \cancel{2h_0kt} + 3kt^2 + \cancel{h_0^2} + 3h_0k$$

$$= \cancel{k^2t^2} + \cancel{2h_0kt} \qquad + \cancel{h_0^2} + 2h_0k \quad + 2k^2t + k^2$$

$$\Rightarrow 3kt^2 + h_0k = 2k^2t + k^2$$

$$3k\cdot t^2 - 2k^2\cdot t + (h_0k - k^2) = 0$$

$$\Delta = 4k^4 - 4\cdot 3k(h_0k - k^2) = 4k^4 - 12k^2h_0 + 12k^3$$

$$t = \frac{2k^2 \pm (4k^4 - 12k^2h_0 + 12k^3)^{1/2}}{6k}$$

(To be continued)

Problem 7 official solution

k_1: height/hour of snowfall
k_2: m^3/h snow clearance

snow height: $k_1 t$

$$\underbrace{\bar{w}}_{\text{width}} \cdot \underbrace{\Delta x}_{\text{length}} \cdot \underbrace{k_1 t}_{\text{height}} \approx \underbrace{k_2 \Delta t}_{\text{snow cleared in } \Delta t}$$

$$m \quad m \cdot m \qquad \frac{m^3}{h} \cdot h$$

$$\frac{\Delta x}{\Delta t} \approx \frac{k_2}{k_1 \bar{w} t} = \frac{k}{t}$$

$$\Rightarrow x(t) = k\ln(t) + C$$

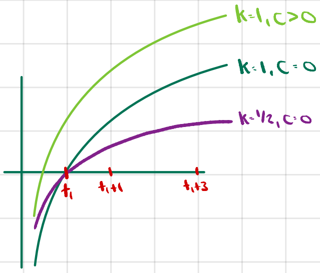
$$x(t_1+1) - x(t_1) = 2000 = k\ln\left((t_1+1)/t_1\right)$$
$$x(t_1+3) - x(t_1+1) = 2000 = k\ln\left((t_1+3)/(t_1+1)\right)$$

$$\ln\frac{t_1+1}{t_1} = \ln\frac{t_1+3}{t_1+1} \Rightarrow t_1 = 1$$

What we're saying here is that there is a particular t_1 for which the difference in x between t_1 and one time unit later is 2000, and same for between t_1+1 and t_1+3.

But actually the absolute magnitudes don't matter. What matters is that from t_1 to t_1+1 there was Δx and from t_1+1 to t_1+3, ie double the Δt, we got the same Δx change.

$x(t) = k\ln(t) + C$ are a family of logarithmic curves



Our result when we tried to find a t_1 that fit our equations was $t_1 = 1$, no matter the parameters k and C. The behavior we observed of Δx being the same between $t=1$ and $t=2$ as between $t=2$ and $t=4$ is consistent w/ any curve in the family $k\ln t + x_1$.

Because we have actual observations of $x_2 - x_1$ and $x_4 - x_2$, we can find the parameter k.

$$x_2 - x_1 = k\ln 2 + x_1 - x_1 = 2000 \Rightarrow k = \frac{2000}{\ln 2} \Rightarrow x(t) = \frac{2000}{\ln 2}\ln t + x_1$$

we can check $x_4 - x_2 = k\ln 4 + x_1 - x_2 = 2000 \Rightarrow 2k\ln 2 + \cancel{x_1} - k\ln 2 - \cancel{x_1} = 2000$

$\Rightarrow k = 2000/\ln 2$

x_1 can be any value. Changing x_1 just shifts the solution up or down but the relative changes Δx between times stay the same.

Was this problem contrived?

t_1 found because any logarithmic function has the proposed proportions at some t_1.

what if $x(t_1+1) - x(t_1) = \Delta x = k\ln\left((t_1+1)/t_1\right)$
$\qquad x(t_1+3) - x(t_1+1) = n\Delta x = k\ln\left((t_1+3)/(t_1+1)\right)$

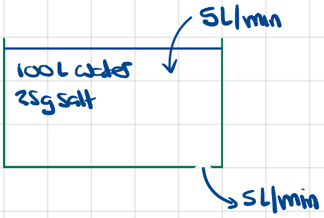
$$\Rightarrow \cancel{k}\ln\left(\frac{t_1+1}{t_1}\right) = \frac{\cancel{k}}{n}\ln\left(\frac{t_1+3}{t_1+1}\right) \Rightarrow n\ln\frac{t_1+1}{t_1} = \ln\frac{t_1+3}{t_1+1} \Rightarrow \left(\frac{t_1+1}{t_1}\right)^n = \frac{t_1+3}{t_1+1} \Rightarrow (t_1+1)^{n+1} = t_1(t_1+3)$$

For example: $n=2 \Rightarrow (t_1+1)^3 = t_1^2 + 3t_1 = t_1^3 + 3t_1^2 + \cancel{3t_1} + 1 = t_1^2 + \cancel{3t_1} \Rightarrow t_1^3 + 2t_1^2 + 1 = 0$ one real root -2.2057

$$x(-2.2057+1) - x(-2.2057) = k\ln\left(\frac{-1.2057}{-2.2057}\right) = 2000 \Rightarrow k = -3311.342$$

which k depends on the actual values of the Δx.

Problem 8



a)

$x(t)$: amount of salt in the water

$x(0) = 25$

$c(t)$: concentration salt g/L $= \frac{x(t)}{100}$

f_o : rate of outflow L/min

f_i : rate of inflow L/min

$V(t)$: total volume

$$-f_o \frac{L}{min} \cdot \frac{x(t)}{V(t)} \frac{g}{L} \cdot \Delta t \text{ min} = \Delta x \Rightarrow \frac{\Delta x}{\Delta t} = \frac{-f_o x(t)}{V(t)}$$

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = -f_o \frac{x(t)}{V(t)}$$

in this example, $f_o = 5$, $V(t) = 100$

$$\Rightarrow \frac{1}{x} dx = \frac{-1}{20} dt \Rightarrow \ln x = \frac{-t}{20} + C_1 \Rightarrow x = C e^{\frac{-t}{20}}$$

$$x(0) = C \Rightarrow x(t) = x_0 e^{-t/20}, \; x_0 = 25 \Rightarrow x(t) = 25 e^{-t/20}$$

b) $x(t) = 1 \Rightarrow e^{t/20} = 25 \Rightarrow \frac{t}{20} = \ln 25 \Rightarrow t = 20 \ln 25$

PSet I – Part II

Problem 1 oryx population

$k(t) = \frac{k_0}{(a+t)^2}$ $\quad t \geq 0,\ a, k_0 > 0$

$\frac{dp}{dt} = k(t)\,p(t)$

a) k is a natural growth rate; units are year^{-1}

since t is time (years), a is years because we must add same units.

then to check: $\frac{[\text{years}]}{[\text{years}^2]} = \text{years}^{-1}$

b) $\frac{dp}{dt} = \frac{p(t)k_0}{(a+t)^2}$

c) $\frac{1}{p}\,dp = \frac{k_0}{(a+t)^2}\,dt$ $\quad p \neq 0$

$\ln|p| = -k_0(a+t)^{-1} + C_1$

$|p| = e^{C_1} e^{-k_0/(a+t)}$

$p = \pm e^{C_1} e^{-k_0/(a+t)} \Rightarrow p(t) = Ce^{\frac{-k_0}{a+t}}$

$p(0) = Ce^{-\frac{k_0}{a}} \Rightarrow C = p_0 e^{\frac{k_0}{a}} \Rightarrow p(t) = p_0 e^{\left(\frac{k_0}{a+t} + \frac{k_0}{a}\right)}$

includes $p(t) = p_0 = 0$

d) $p'(t) = k(t)p(t) = \frac{k_0}{(a+t)^2} \cdot p_0 e^{\left(\frac{k_0}{a+t} + \frac{k_0}{a}\right)}$

$p'(t \to \infty) = 0 \cdot e^{k_0/a} = 0$

$p(t \to \infty) = p_0 e^{\frac{k_0}{a}} = C \Rightarrow p(t) = p_\infty e^{\frac{k_0}{a+t}}$