second-order DE in unknown fn $y(x)$: $G(x, y, y', y'') = 0$

DE linear $\iff$ G linear in y and derivatives

nonhomog: $A(x)y'' + B(x)y' + C(x)y = F(x)$

we assume A, B, C, F contin. on some open interval I where we want to solve the DE, but they do not have to be linear

homog.: $A(x)y'' + B(x)y' + C(x)y = 0$

assume $A(x) \neq 0$ in I $\Rightarrow$ $y'' + p(x)y' + q(x)y = f(x)$
homog: $y'' + p(x)y' + q(x)y = 0$

Theorem (Principle of Superpos. for Homog. Eq.)

y_1, y_2 sol'ns to $y'' + p(x)y' + q(x)y = 0$ on I $\Rightarrow$ $c_1y_1 + c_2y_2$ sol'ns on I

Theorem (Existence and Uniqueness for Linear Eq.)

p, q, f cont. on open I containing a
b_0, b_1 two numbers
$\Rightarrow$ $y'' + p(x)y' + q(x)y = f(x)$ has exactly one sol'n on the entire I that satisfies $y(a) = b_0$, $y'(a) = b_1$

ie, every second-order IVP above has a unique sol'n on entire I

Note given $y'' + p(x)y' + q(x)y = 0$, we can always find two lin. indep. sol'ns:

$y_1(a) = 1$ $\quad y_1'(a) = 0$
$y_2(a) = 0$ $\quad y_2'(a) = 1$
$\Rightarrow$ $ky_2(a) \neq y_1(a)$ $\Rightarrow$ $y_2(a) \neq ky_1(a)$ $\Rightarrow$ y_1 and y_2 lin. indep.

we don't know what y_1 and y_2 look like, but we know they exist because of Exist./Uniq. theorem

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Theorem (Wronskian of Sol'ns)

y_1, y_2 sol'ns of $y'' + p(x)y' + q(x)y = 0$ on open I on which p, q contin.

$\Rightarrow$ y_1, y_2 lin dep $\Rightarrow$ $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \equiv 0$ on I

y_1, y_2 lin indep. $\Rightarrow$ $W(y_1, y_2) \neq 0$ on I

Theorem General Sol'ns of Homog. Eq.

y_1, y_2 sol'ns of $y'' + p(x)y' + q(x)y = 0$ on open I on which p, q contin.
Y any sol'n of same DE
$\Rightarrow$ $\exists c_1, c_2$ s.t. $Y(x) = c_1y_1(x) + c_2y_2(x)$ $\forall x$ in I

ie, if we find two l.i. sol'ns we can find all other sol'ns. $Y = c_1y_1 + c_2y_2$ is a general sol'n.

Proof

choose a in I

Consider the IVP $y'' + p(x)y' + y = 0$, $y(a) = Y(a)$, $y'(a) = Y'(a)$.

We know the sol'n is unique, and that it in fact exists.

Can we represent it as a linear comb. of the l.i. sol'ns y_1, y_2?

y_1, y_2 sol'ns $\Rightarrow$
$$c_1 y_1(a) + c_2 y_2(a) = Y(a)$$
$$c_1 y_1'(a) + c_2 y_2'(a) = Y'(a)$$

$$\Rightarrow \begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} Y(a) \\ Y'(a) \end{bmatrix}$$

(c_1, c_2) exists as sol'n if $W(y_1, y_2) \neq 0$. It is $\neq 0$ because y_1, y_2 are l.i. by assumption

$\Rightarrow$ For any IVP of this type we can find c_1, c_2 s.t. $c_1 y_1 + c_2 y_2$ is a sol'n, and by Exist./Uniq. it is the only sol'n.

Because any two l.i. sol'ns can be combined in a lin. comb. to obtain all other sol'ns, such a lin comb is a general sol'n. There are many general sol'ns (infinite) because we can find infinite pairs of l.i. sol'ns.

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homog. second-order linear DE $\quad ay'' + by' + cy = 0$

char. eq: $ar^2 + br + c = 0$

distinct real roots $\Rightarrow$ $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

repeated root $\Rightarrow$ $(r - r_1)^2 = r^2 - 2rr_1 + r_1^2$ is charact. eq.

$\Rightarrow y'' - 2r_1 y' + r_1^2 y = 0$

l.i. sol'ns: $e^{r_1 t}$, $te^{r_1 t}$