Damped Harmonic Oscillator: $m\ddot{x}+b\dot{x}+kx=0$

damping constant

$b=0 \Rightarrow m\ddot{x}+kx=0 \quad \Rightarrow x(t)=c_1\cos(\omega t)+c_2\sin(\omega t) \quad \omega=\sqrt{k/m}$

(simple harmonic oscillator) $\quad = A\cos(\omega t-\phi)$

restrictions on coefficients: to have a physical model, $m>0,\ b\geq 0,\ k>0$

$\Rightarrow m\ddot{x}+b\dot{x}+kx=0$

$mr^2+br+k=0$

$$r=\frac{-b\pm\sqrt{b^2-4mk}}{2m}$$

$b^2-4mk<0 \Rightarrow$ underdamping, b small relative to m and k, complex roots

$b^2-4mk>0 \Rightarrow$ overdamping, real roots, both negative

$b^2=4mk$ critical damping b just between over and under damping

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underdamping

Let $\omega_d=\dfrac{\sqrt{b^2-4mk}}{2m} \quad \Rightarrow r=\dfrac{-b}{2m}\pm i\omega_d$

complex exponential sol'ns: $e^{\left(\frac{-b}{2m}\pm i\omega_d\right)t}$

basic real sol'ns: $e^{\frac{-bt}{2m}}\cos(\omega_d t),\ e^{\frac{-bt}{2m}}\sin(\omega_d t)$

general sol'n: $c_1e^{\frac{-bt}{2m}}\cos(\omega_d t)+c_2e^{\frac{-bt}{2m}}\sin(\omega_d t) = e^{\frac{-bt}{2m}}(c_1\cos(\omega_d t)+c_2\sin(\omega_d t))$

$= Ae^{\frac{-bt}{2m}}\cos(\omega_d t-\phi)$

damped angular (aka circular) frequency
aka pseudo-frequency of $x(t)$

$x(t)$ is not periodic; only periodic fns have frequency; $x(t)$ does oscillate, however, crossing $x=0$ twice each pseudo period.

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overdamping

$b^2-4mk>0 \quad \Rightarrow \quad b^2>b^2-4mk>0$ since $m,k>0 \quad \Rightarrow \quad \dfrac{-b\pm\sqrt{b^2-4mk}}{2m}<0 \quad \Rightarrow$ both real roots are negative

$b>\sqrt{b^2-4mk}$

exponential sol'ns: $e^{r_1t},\ e^{r_2t}$

general sol'n: $x(t)=c_1e^{r_1t}+c_2e^{r_2t}$

Critical Damping

$b^2 - 4mk = 0 \Rightarrow r = \frac{-b}{2m}$, real sol'n

basic sol'ns: $e^{\frac{-b}{2m}t}$, $te^{\frac{-b}{2m}t}$

general sol'n: $x(t) = e^{\frac{-b}{2m}t}(c_1 + c_2 t)$

like the overdamped case, there is no oscillation.

note that in the overdamped case the sol'ns had exponentials $e^{r_1 t}$, $e^{r_2 t}$ with $r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$

the larger of the two, $\frac{-b + \sqrt{b^2 - 4mk}}{2m}$, is larger than $\frac{-b}{2m}$.

The exponential w/ the largest exponent controls the rate at which x goes to zero.

$\Rightarrow$ the critically damped solution goes to zero the fastest out of all cases w/ real roots, ie the fastest return of the system to its equilibrium position.