First order, autonomous DEs

self-governing: $\dot{x}$ dependent on x itself only, not time

$\dot{x} = f(x)$

in general non-linear

→ characteristics

→ separable, but can be hard to integrate
→ can be analyzed without solving
→ time invariant: $y(t)$ solution ⇒ $y(t-t_0)$ solution for any t_0

Simple Examples

→ $\dot{y} = ky$ (natural growth/decay)
→ $\dot{y} = I(y)\,y + q$ (bank account)
interest rate, savings

Logistic Population Model

$\dot{y} = k(y)\cdot y$

variable growth rate

model for population that takes into account the limits imposed by the environment. This is done through $k(y)$. We can think of the growth rate starting at k_0 when y is low, and decreasing as y increases. In its simplest form, $k(y)$ is a linear decreasing fn of y.

$k = k(y) = k_0\left(1 - \frac{y}{M}\right)$ ie, $y = M \Rightarrow k(M) = 0$
$y = 0 \Rightarrow k(0) = k_0$

With such a $k(y)$ we have the logistic population model

$\dot{y} = k_0\left(1 - \frac{y}{M}\right)y = f(y)$ (non-linear, autonomous)

The integrals involved in this DE can be solved using partial fractions. However, we'd like to develop ways of analyzing the solutions qualitatively w/o solving for them analytically.

Here are some steps we take:

1. look for constant solutions: $y = y_0,\ y' = f(y_0) = 0$

$\Rightarrow k_0\left(1 - \frac{y_0}{M}\right)y_0 = 0 \Rightarrow y_0 = 0$ or $y_0 = M$ (equilibrium sol'ns)

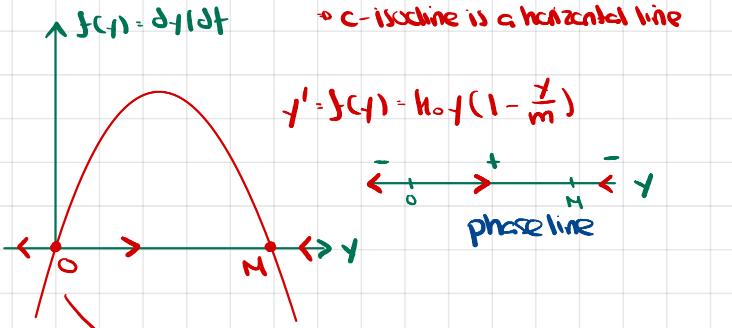
0 and M are critical points of the DE

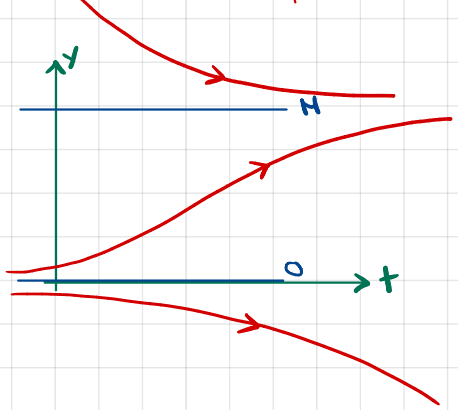
Note that 0 and M are the solutions lost in the separable equations method

2. Understand the non-constant solutions by studying the isoclines given the constant solutions.

$\frac{dy}{dt} = f(y) = c \Rightarrow k_0\left(1 - \frac{y}{M}\right)y = c \Rightarrow k_0 y - \frac{k_0 y^2}{M} = c$

$\Rightarrow y^2 - My + c = 0$ → the y values that solve this represent horizontal lines in ty-plane.
→ c-isocline is a horizontal line





→ note that in $y^2 - My + c = 0$ we have
$\Delta = M^2 - 4\cdot 1\cdot c \geq 0 \Rightarrow c \leq M^2/4$

M: carrying capacity of the environment

Time translate of a solution $y(t)$

$y(t-t_0)$ shifts $y(t)$ t_0 units to the right.