PSet - Integrating Factors - Part I

Problem 1 $\frac{dy}{dx} + y = 2 \qquad y(0)=0$

$u(x) = e^{\int dx} = e^x$

$(e^x y)' = 2e^x \Rightarrow e^x y = \int 2e^x dx + C = 2e^x + C$

$y(x) = 2 + Ce^{-x}$

$y(0) = 2 + C = 0 \Rightarrow C = -2 \qquad \Rightarrow y(x) = 2 - 2e^{-x}$

Problem 2 $xy' - y = x \qquad y(1) = 7$

$y' - \frac{1}{x} y = 1 \qquad u(x) = e^{-\int x^{-1}dx} = e^{-\ln|x|} = |x|^{-1}$

$|x|^{-1}y' - |x|^{-1} \cdot \frac{1}{x} y = |x|^{-1}$

$(|x|^{-1} y)' = |x|^{-1}$

$|x|^{-1} y = \int |x|^{-1} dx + C = \begin{cases} \ln(x) + C & x \geq 0 \\ -\ln(-x) + C & x < 0 \end{cases}$

$\Rightarrow y(x) = \begin{cases} x\ln(x) + Cx & x \geq 0 \\ x\ln(-x) - Cx & x < 0 \end{cases}$

$y(1) = 7 = C$

Problem 3 $y' = 1 + x + y + xy \qquad y(0) = 0$

$y' = y(1+x) + 1 + x = (1+x)(1+y)$

$\frac{1}{1+y} dy = (1+x)dx \qquad y \neq -1$

$\Rightarrow \ln|1+y| = x + \frac{x^2}{2} + C_1$

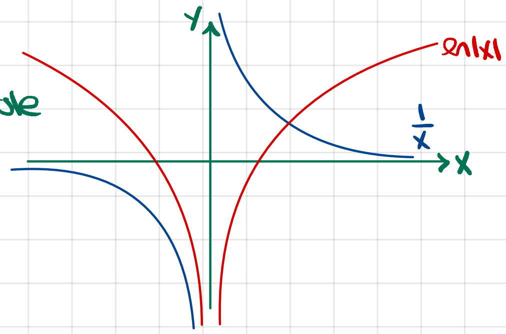
we will find the solution for $y > -1$ because our initial value is $y(0) = 0 > -1$.

$\ln(1+y) = x + \frac{x^2}{2} + C_1 \Rightarrow 1 + y = e^{C_1} e^{x + \frac{x^2}{2}} \Rightarrow y = 1 - Ce^{x+\frac{x^2}{2}}$

$y(0) = 0 = 1 - C \Rightarrow C = 1$

$y(x) = 1 - e^{x + \frac{x^2}{2}}$

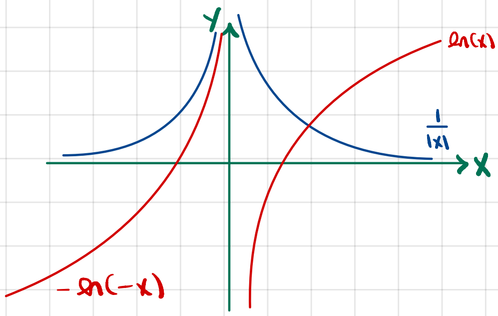
* Note



$* \; f(x) = |x|^{-1} = \begin{cases} x^{-1} & x \geq 0 \\ -x^{-1} & x < 0 \end{cases}$

$f'(x) = \begin{cases} -x^{-2} < 0 & x \geq 0 \\ x^{-2} > 0 & x < 0 \end{cases}$

$= \begin{cases} \frac{-1}{x|x|} < 0 & x \geq 0 \\ \frac{-1}{x|x|} > 0 & x < 0 \end{cases}$

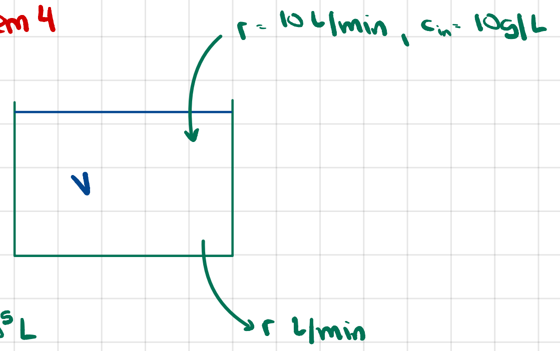


$\int f(x)dx = \begin{cases} \int x^{-1}dx = \ln x \\ -\int x^{-1}dx = -\ln(|x|) = -\ln(-x) \end{cases}$

$\ln|1+y| = \begin{cases} \ln(1+y) & y > -1 \\ \ln(-y-1) & y < -1 \end{cases}$

$\frac{d}{dy} \ln|1+y| = \begin{cases} 1/(1+y) & y > -1 \\ -1/[-(1+y)] = 1/(1+y) & y < -1 \end{cases}$

Problem 4



$V = 10^5$ L

a) $x(t)$: amount of pollutant

$$\Delta x = r \frac{L}{min} \cdot c_{in} \frac{g}{L} \cdot \Delta t \, min - r \frac{L}{min} \cdot \frac{x(t)}{V} \frac{g}{L} \cdot \Delta t \, min$$

$$\frac{\Delta x}{\Delta t} = rc - \frac{r x(t)}{V} = r\left(c_{in} - \frac{x(t)}{V}\right) \quad g/min$$

$$\frac{(\Delta x / V)}{\Delta t} = \frac{\Delta c}{\Delta t} = \frac{r}{V}(c_{in} - c(t)) \quad g/min^2$$

$$\frac{dc(t)}{dt} = \frac{r}{V}(c_{in} - c(t)) \quad g/min^2$$

b) $\dot{c}(t) = 10^{-4}(10 - c(t)) \Rightarrow \dot{c} + 10^{-4} c = 10^{-3}$

$$u(t) = e^{10^{-4} t} = e^{\frac{t}{10^4}}$$

$$c e^{\frac{t}{10^4}} = 10^{-3} \int e^{t \cdot 10^{-4}} dt + c_1 = \frac{10^{-3}}{10^{-4}} e^{t \cdot 10^{-4}} + c_1$$

$$c(t) = e^{-t \cdot 10^{-4}} \left[10 e^{t \cdot 10^{-4}} + c_1 \right] = 10 + c_1 e^{-t \cdot 10^{-4}}$$

$$c(0) = 10 + c_1 = 0 \Rightarrow c_1 = -10$$

$$c(t) = 10 - 10 e^{-t \cdot 10^{-4}} = 10(1 - e^{-t \cdot 10^{-4}})$$

c) $c(t) = 5 \Rightarrow \frac{1}{2} = 1 - e^{-t \cdot 10^{-4}} \Rightarrow \frac{1}{e^{t/10^4}} = \frac{1}{2}$

$$\Rightarrow e^{\frac{t}{10^4}} = 2 \Rightarrow t = 10^4 \cdot \ln 2 \ min$$

d) $c(t \to \infty) = 10$ g/L

Part II

Problem 1

interest rate I year^{-1}
deposits at rate $-q$ USD/year

a) $\Delta A = I \cdot A \cdot \Delta t - q \cdot \Delta t$

$\frac{\Delta A}{\Delta t} = IA - q \Rightarrow \frac{dA}{dt} = IA(t) - q$

$\dot{A} - IA = -q$

b) $u(x) = e^{-It}$

$(Ae^{-It})' = -qe^{-It} \Rightarrow Ae^{-It} = \frac{q}{I}e^{-It} + C$

$\Rightarrow A(t) = \frac{q}{I} + Ce^{It}$

c) $I = 0.05$ year^{-1}
$q = 12000$ USD $\cdot$ year^{-1}

$A(t) = \frac{12000}{0.05} + Ce^{0.05t} \Rightarrow A(t) = 240000 + Ce^{0.05t}$

$A(0) = 240000 + C \Rightarrow C = A_0 - 240000$

$A(t) = 240000 + (A_0 - 240000)e^{0.05t}$

$\dot{A}(t) = 0 \Rightarrow IA = q \Leftrightarrow A(t) = \frac{q}{I} = \frac{12000}{0.05} = 240000$

$A_0 = 240000 \Rightarrow A(t) = 240000$
Input is -12000, output is a constant $A(t) = 240000$.
$\dot{A} = 0 \Rightarrow IA = q \Rightarrow$ interest $=$ withdrawal

d) $A(20) = 0 = 240000 + A_0 e - 240000e \Rightarrow A_0 = \frac{240000(e-1)}{e} = 151708.93$

Problem 2

isotope of strontium, St
half life t_s

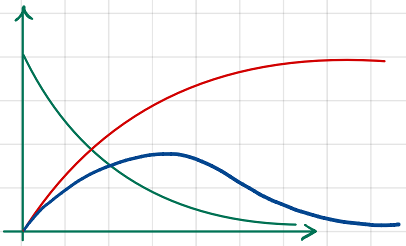
decays w/ equal probability into Midium or Endium

$$\begin{tikzcd} & Mi \arrow[d, "t_m"] \\ St \arrow[ur, "t_s"] \arrow[r, "t_s"'] & En \end{tikzcd} \qquad t_m \neq t_s$$

$$\left.\begin{array}{ll} x(t) & St \\ y(t) & Mi \\ z(t) & En \end{array}\right\} \text{Amounts}$$

$x(0)=1,\ y(0)=z(0)=0$

a)



b) $x(t_s)=0.5$

$\dot{x} = -\sigma x \quad \sigma>0 \quad \Rightarrow x(t) = x_0 e^{-\sigma t}$

$x(t_s) = \frac{x_0}{2} = x_0 e^{-\sigma t_s} \Rightarrow e^{-\sigma t_s} = \frac{1}{2} \Rightarrow e^{\sigma t_s} = 2 \Rightarrow \sigma t_s = \ln 2$

$\Rightarrow t_s = \frac{\ln 2}{\sigma}, \quad \sigma = \frac{\ln 2}{t_s}$

$\dot{y} = \frac{\sigma x}{2} - \nu y(t) = \frac{\ln 2}{2t_s} x - \nu y(t)$

$\dot{z} = \frac{\sigma x}{2} + \nu y(t) = \frac{\ln 2}{2t_s} x + \nu y(t)$

$\dot{x} + \dot{y} + \dot{z} = -\cancel{\frac{\ln 2}{t_s}} x + \cancel{\frac{\ln 2}{2t_s}} x - \cancel{\nu y(t)} + \cancel{\frac{\ln 2}{2t_s}} x + \cancel{\nu y(t)} = 0$

c) $x(t) = x_0 e^{-\sigma t}$

$\dot{y} + \nu y = \frac{\sigma}{2} x_0 e^{-\sigma t} \Rightarrow (e^{\nu t} y)' = \frac{\sigma}{2} x_0 e^{t(\nu-\sigma)}$

$\Rightarrow y e^{\nu t} = \frac{\sigma x_0}{2} \frac{e^{t(\nu-\sigma)}}{\nu-\sigma} + C \Rightarrow y(t) = \frac{\sigma x_0}{2(\nu-\sigma)} e^{-t\sigma} + Ce^{-t\nu}$

$u(t) = e^{\nu t}$

$y(0) = y_0 = \frac{\sigma x_0}{2(\nu-\sigma)} + C \Rightarrow C = y_0 - \frac{\sigma x_0}{2(\nu-\sigma)}$

$y(t_m) = \frac{y_0}{2} = \frac{\sigma x_0}{2(\nu-\sigma)} e^{-t_m\sigma} + \left[y_0 - \frac{\sigma x_0}{2(\nu-\sigma)}\right] e^{-t_m\nu}$

$y(0)=0 \Rightarrow y(t) = \left[\frac{\sigma x_0 e^{t(\nu-\sigma)}}{2(\nu-\sigma)} - \frac{\sigma x_0}{2(\nu-\sigma)}\right] e^{-\nu t}$

$\dot{z} = \frac{\sigma}{2} x_0 e^{-\sigma t} + \nu \left[\frac{\sigma x_0 e^{t(\nu-\sigma)}}{2(\nu-\sigma)} - \frac{\sigma x_0}{2(\nu-\sigma)}\right] e^{-\nu t}$

Solution z(t) in Maple workbook "Pset - VI - Integrating Factors"

d) $y'(t) = -\dfrac{t_s\left[\ln\left(\frac{\nu t_s}{2\ln 2} + \ln 2\right)\right]}{-\nu t_s + \ln 2}$

e) If $x(0)$ is increased and all other parameters (t_s, t_m) then the shapes of the curves $x(t), y(t), z(t)$ stay the same. For example, max of $y(t)$ occurs at the same time (though the level changes, ie it doubles).

f) $t\dot{x} + 2x = q(t)$

$x(t) = e^t$ solution

$te^t + 2e^t = q(t) = e^t(t+2)$

$x_p(t) = e^t$ is a particular solution

$t\dot{x} + 2x = 0 \Rightarrow \dot{x} = -\frac{2x}{t} \Rightarrow \frac{1}{x}dx = -\frac{2}{t}dt \Rightarrow \ln|x| = -2\ln|t| + c_1$

$|x| = e^{-2\ln|t|} e^{c_1} = |t|^{-2} e^{c_1} \Rightarrow x = \pm e^{c_1} t^{-2} \Rightarrow x_h(t) = Ct^{-2}$

By superposition principle we know that $x_p(t) + x_h(t) = e^t + Ct^{-2}$ is a solution to $t\dot{x} + 2x = e^t(t+2)$