Overview/Plan

→ Given sinusoidal input, how does system response change with input frequency changes?

In particular:

→ gain (freq. response) which freq. maximizes response (resonant freq. of the system)

* in undamped systems (ie simple harmonic oscillator), this freq. is called pure resonance, corresp. to infinite amplitude
  in damped " , its called practical resonance; finite amplitude

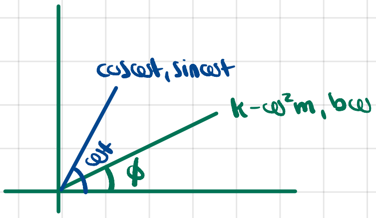
Damped Spring-Mass

$mx'' + bx' + kx = B\cos\omega t \qquad p(r) = mr^2 + br + k, \quad p(i\omega) = k - \omega^2 m + b\omega i$

$$mz'' + bz' + kz = Be^{i\omega t} \quad \Rightarrow \quad z_p = \frac{Be^{i\omega t}}{k - \omega^2 m + b\omega i} = \frac{(k-\omega^2 m - b\omega i)B(\cos\omega t + i\sin\omega t)}{(k-\omega^2 m)^2 + b^2\omega^2}$$

$$x_p = \frac{B(k-\omega^2 m)\cos\omega t + b\omega B\sin\omega t}{(k-\omega^2 m)^2 + b^2\omega^2} = \frac{B\left[(k-\omega^2 m)\cos\omega t + b\omega\sin\omega t\right]}{(k-\omega^2 m)^2 + b^2\omega^2}$$

$$= \frac{B\cos(\omega t - \phi)\sqrt{(k-\omega^2 m)^2 + b^2\omega^2}}{(k-\omega^2 m)^2 + b^2\omega^2}$$



$$\tan\phi = \frac{b\omega}{k-\omega^2 m}$$

$$= \frac{B\cos(\omega t - \phi)}{\sqrt{(k-\omega^2 m)^2 + b^2\omega^2}} = A\cos(\omega t - \phi)$$

$$\langle \cos\omega t, \sin\omega t\rangle \langle k-\omega^2 m, b\omega\rangle = \left[(k-\omega^2 m)^2 + (b\omega)^2\right]^{1/2}\cos(\omega t - \phi)$$

$$A = \frac{B}{\sqrt{(k-\omega^2 m)^2 + b^2\omega^2}}$$

complex gain: $\dfrac{\text{amplit. output}}{\text{amplit. input}}$ in complexified eq. $= \tilde{g}(\omega) = \dfrac{1}{k-\omega^2 m + b\omega i} = \dfrac{1}{p(i\omega)}$

gain $= g(\omega) = \dfrac{1}{\sqrt{(k-\omega^2 m)^2 + b^2\omega^2}} = \dfrac{1}{|p(i\omega)|}$ = amplitude response of the system

phase lag $= \phi = \phi(\omega) = \tan^{-1}\dfrac{b\omega}{k-\omega^2 m} = \mathrm{Arg}(p(i\omega))$ = phase response of the system

} frequency response

if the amplitude reaches a max ω_r we call it the practical resonance frequency

→ there may or may not be a max depending on damping intensity

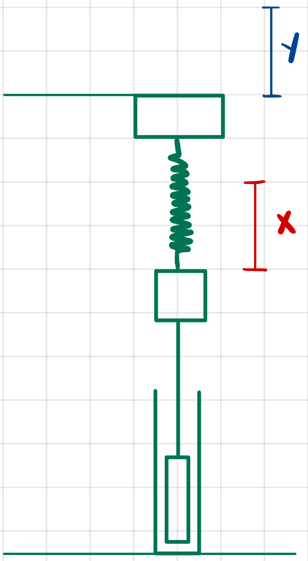
→ denom. $\sqrt{(k-\omega^2 m)^2 + b^2\omega^2}$ reaches a min

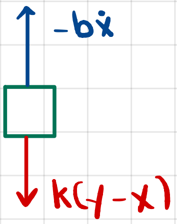
→ $f(\omega) = (k-\omega^2 m)^2 + b^2\omega^2 \Rightarrow f'(\omega) = 2(k-\omega^2 m)(-2\omega m) + 2\omega b^2 \Rightarrow 2\omega(b^2 - 2mk + 2m^2\omega^2)$

$\Rightarrow \omega = 0$ or $m^2\omega^2 = mk - b^2/2 \Rightarrow \omega = \sqrt{\dfrac{mk - b^2/2}{m^2}}$

$mk - b^2/2 > 0 \iff$ practical resonance

mechanical vibration system: driving through spring





2nd law: $\underset{\text{mass}}{m\ddot{x}} = \underset{\text{spring (driving force)}}{k(y-x)} - \overset{\text{dashpot}}{b\dot{x}}$

$m\ddot{x} + b\dot{x} + kx = ky$

y: input
x: response

suppose $y = B_1\cos(\omega t)$, then we expect $x_p = A\cos(\omega t - \phi)$

gain: $\dfrac{A}{B_1} = \dfrac{k}{|p(i\omega)|} = \dfrac{k}{\sqrt{(k-m\omega^2)^2 + b^2\omega^2}}$

phase lag: $\phi(\omega) = \tan^{-1}\left(\dfrac{b\omega}{k-m\omega^2}\right)$

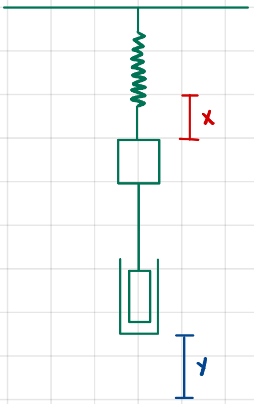
$mx'' + bx' + kx = B$

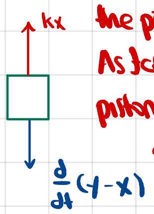
$x(t) = C \Rightarrow kC = B \Rightarrow C = \dfrac{B}{k}$

$m\ddot{x} = k(1-x) - b\dot{x}$ $\qquad$ $\ddot{x} = 2(1-x) - \dot{x}$

$m\ddot{x} + b\dot{x} + kx = ky$ $\qquad$ $\ddot{x} + \dot{x} + 2x = 2$

mechanical vibration system: driving through the base of the dashpot. The force exerted by the dashpot is proportional to the speed of the piston moving through it. As force pulls down, it's as if piston accel. up, so force is downwards.





$m\ddot{x} = -kx + b\frac{d}{dt}(y-x)$ (2nd law)

$m\ddot{x} + b\dot{x} + kx = b\dot{y}$ ↗ system = input

system response x
" input y

$y = B_1 \cos\omega t$ → assump.: sinusoidal input
$\dot{y} = -B_1\omega\sin\omega t$

↗ $b\dot{y}$, notice y isn't here

$m\ddot{x} + b\dot{x} + kx = -b\omega B\sin\omega t$

↗ we know response is sinus.

$x_p = A\cos(\omega t - \phi)$

$g = \frac{A}{B_1}$

complexify. $\tilde{y} = B_1 e^{i\omega t}$, $\tilde{y}' = B_1\omega i e^{i\omega t}$. $y = \mathrm{Re}(\tilde{y})$
↘ complexify y, use the deriv.

$m\ddot{z} + b\dot{z} + kz = b\tilde{y}' = bB_1\omega i e^{i\omega t}$

$z_p = \frac{ibB_1\omega}{p(i\omega)} e^{i\omega t}$ ↗ complex sol'n
$p(r) = mr^2 + br + k$

complex gain

$\tilde{g}(\omega) = \frac{ibB_1\omega / p(i\omega)}{B_1} = \frac{ib\omega}{p(i\omega)}$ ↗

↘ $\tilde{y} = B_1 e^{i\omega t}$

$\Rightarrow z_p = B_1\tilde{g}(\omega)e^{i\omega t}$

↗ rewrite $\tilde{g} \in \mathbb{C}$ in expon. form

$\tilde{g} = |\tilde{g}|e^{-i\phi}$ $\phi = -\mathrm{Arg}(\tilde{g})$

$\Rightarrow z_p = B_1|\tilde{g}|e^{-i\phi}e^{i\omega t}$

↗ take real part of z_p

$\Rightarrow x_p = B_1|\tilde{g}|\cos(\omega t - \phi)$

$g = \frac{B_1|\tilde{g}|}{B_1} = |\tilde{g}|$ ↗ gain

↘ $y = B_1\cos(\omega t)$

$\phi = -\mathrm{Arg}(\tilde{g})$ → this comes from $\tilde{g}$

↗ calc. this to calc $|\tilde{g}|$

$p(i\omega) = m(i\omega)^2 + bi\omega + k = k - \omega^2 m + bi\omega$

$|\tilde{g}| = \frac{ib\omega}{k - m\omega^2 + bi\omega}$

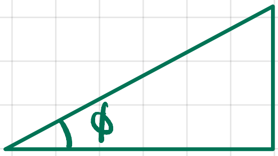
$\Rightarrow g = |\tilde{g}| = \frac{\omega b}{\sqrt{(k-m\omega^2)^2 + b^2\omega^2}}$

compute the phase lag $\phi = -\mathrm{Arg}(\tilde{g})$

$\tilde{g} = \frac{ib\omega}{k - \omega^2 m + bi\omega}$

$= \frac{ib\omega(k - \omega^2 m - ib\omega)}{(k-\omega^2 m)^2 - b^2\omega^2}$

$= \frac{ib\omega(k-\omega^2 m) + b^2\omega^2}{(k-\omega^2 m)^2 - b^2\omega^2}$



$\frac{b\omega(k-\omega^2 m)}{(k-\omega^2 m)^2 - b^2\omega^2}$

$\frac{b^2\omega^2}{(k-\omega^2 m)^2 - b^2\omega^2}$

$\mathrm{Arg}(\tilde{g}) = \tan^{-1}\frac{b\omega(k-\omega^2 m)}{b^2\omega^2} = \tan^{-1}\frac{k-\omega^2 m}{b\omega}$

$\phi = -\mathrm{Arg}(\tilde{g}) = \tan^{-1}\left[-\frac{k-\omega^2 m}{b\omega}\right]$

practical resonance freq.

we could diff g and find ω_r, or:

$\tilde{g} = \frac{ib\omega}{k - \omega^2 m + bi\omega} = \frac{1}{1 + (k - \omega^2 m)/i\omega b}$

$= \frac{1}{1 - i(k-\omega^2 m)/\omega b}$

$\Rightarrow g = |\tilde{g}| = \frac{1}{\sqrt{1 + \frac{(k-\omega^2 m)^2}{(\omega b)^2}}}$

this is largest when $k - \omega^2 m = 0$

$\Rightarrow \omega_r = \sqrt{\frac{k}{m}} = \omega_0$

$\Rightarrow$ resonant freq. = natural freq.