PSet - Damped oscillators - Part II

1. $m\ddot{x}+b\dot{x}+kx=0$

$m=0.50$

$b=1.5$

$k=0.625$

a) $.5\ddot{x}+1.5\dot{x}+0.625x=0$

$.5r^2+1.5r+.625=0$

$$r=\frac{-1.5\pm\sqrt{1.5^2-4\cdot.5\cdot.625}}{2\cdot.5}\begin{cases}-0.5\\-2.5\end{cases}$$

b) $x(t)=c_1e^{-0.5t}+c_2e^{-2.5t}$

$x'(t)=-0.5c_1e^{-0.5t}-2.5c_2e^{-2.5t}$

$x(0)=c_1+c_2=x_0 \quad \Rightarrow c_1=x_0-c_2$

$x'(0)=-\frac{c_1}{2}-2.5c_2=\dot{x}_0$

$c_2-x_0-5c_2=2\dot{x}_0 \Rightarrow 4c_2=-2\dot{x}_0-x_0$

$$\Rightarrow c_2=\frac{-2\dot{x}_0-x_0}{4}$$

$$c_1=x_0+\frac{2\dot{x}_0+x_0}{4}=\frac{2\dot{x}_0+5x_0}{4}$$

$$\Rightarrow x(t)=\frac{2\dot{x}_0+5x_0}{4}e^{-0.5t}-\frac{2\dot{x}_0+x_0}{4}e^{-2.5t}$$

c) $c_1=0 \Rightarrow 2\dot{x}_0+5x_0=0 \Rightarrow x(t)=-\frac{2\dot{x}_0+x_0}{4}e^{-2.5t}$

$c_2=0 \Rightarrow 2\dot{x}_0+x_0=0 \Rightarrow x(t)=\frac{2\dot{x}_0+5x_0}{4}e^{-0.5t}$

d) $x(t)=0 \Rightarrow (2\dot{x}_0+5x_0)e^{-0.5t}=(2\dot{x}_0+x_0)e^{-2.5t}$

$$e^{2t}=\frac{2\dot{x}_0+x_0}{2\dot{x}_0+5x_0}>0 \Rightarrow t=\frac{1}{2}\ln\left[\frac{2\dot{x}_0+x_0}{2\dot{x}_0+5x_0}\right]>0$$

$$\Rightarrow \frac{2\dot{x}_0+x_0}{2\dot{x}_0+5x_0}>1$$

Case 1: $2\dot{x}_0+x_0>0,\ 2\dot{x}_0+5x_0>0$

$\Rightarrow 2\dot{x}_0+x_0>2\dot{x}_0+5x_0 \Rightarrow 4x_0<0 \Rightarrow x_0<0$

$\dot{x}_0>-\frac{x_0}{2},\quad \dot{x}_0>-\frac{5x_0}{2} \Rightarrow \dot{x}_0>-\frac{5x_0}{2}$

Case 2: $2\dot{x}_0+x_0<0,\ 2\dot{x}_0+5x_0<0$

$\Rightarrow 2\dot{x}_0+x_0<2\dot{x}_0+5x_0 \Rightarrow 4x_0>0 \Rightarrow x_0>0$

$\dot{x}_0<-\frac{x_0}{2},\quad \dot{x}_0<-\frac{5}{2}x_0 \Rightarrow \dot{x}_0<-\frac{5}{2}x_0$

Therefore, $x(t)=0$ with $t>0$ when

$x_0>0,\ \dot{x}_0<-\frac{5}{2}x_0$

$x_0<0,\ \dot{x}_0>-\frac{5}{2}x_0$

$$x(t) = \frac{2\dot{x}_0 + 5x_0}{4} e^{-0.5t} - \frac{2\dot{x}_0 + x_0}{4} e^{-2.5t}$$

consider $x(t) = c_1 e^{-0.5t} + c_2 e^{-2.5t} = 0 \Rightarrow c_1 e^{-0.5t} = -c_2 e^{-2.5t} \Rightarrow e^{2t} = -\frac{c_2}{c_1}$

e^{2t} is always positive. c_2 and c_1 must have opposite signs.

$2t = \ln\left|-\frac{c_2}{c_1}\right| \Rightarrow t = \frac{1}{2}\ln\left|-\frac{c_2}{c_1}\right|$

$t > 0 \Rightarrow \ln\left|-\frac{c_2}{c_1}\right| > 0 \Rightarrow \left|-\frac{c_2}{c_1}\right| > 1$

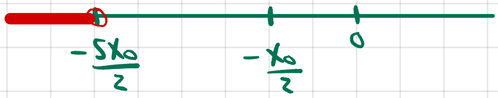
we can drop the absolute value because $-\frac{c_2}{c_1} > 0$ when $x(t) = 0$

$\Rightarrow -\frac{c_2}{c_1} > 1 \Rightarrow \frac{c_2}{c_1} < -1$.

Case 1: $c_2 > 0, c_1 < 0 \Rightarrow c_2 > -c_1 > 0 \Rightarrow -2\dot{x}_0 - x_0 > -2\dot{x}_0 - 5x_0 > 0$

$\Rightarrow -2\dot{x}_0 - x_0 > 0 \Rightarrow 2\dot{x}_0 < -x_0 \Rightarrow \dot{x}_0 < -\frac{x_0}{2}$

$-2\dot{x}_0 - 5x_0 > 0 \Rightarrow 2\dot{x}_0 < -5x_0 \Rightarrow \dot{x}_0 < -\frac{5}{2}x_0$



$-x_0 > -5x_0 \Rightarrow -4x_0 < 0 \Rightarrow x_0 > 0$

Case 2: $c_2 < 0, c_1 > 0 \Rightarrow c_2 < -c_1 < 0 \Rightarrow -2\dot{x}_0 - x_0 < -2\dot{x}_0 - 5x_0 < 0$

$\Rightarrow \dot{x}_0 > -\frac{x_0}{2}, \ \dot{x}_0 > -\frac{5}{2}x_0, \ x_0 < 0$

$\ln\left|\frac{2x+0.25}{2x+1.25}\right| > 0 \Rightarrow \left|\frac{2x+0.25}{2x+1.25}\right| > 1$

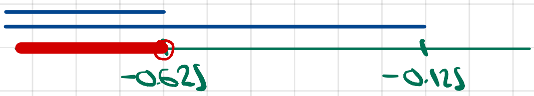
Case 1: $\frac{2x+0.25}{2x+1.25} > 0$

a) $2x+0.25 > 0, \ 2x+1.25 > 0 \Rightarrow \cancel{2x}+0.25 > \cancel{2x}+1.25 \Rightarrow 0.25 > 1.25$ not possible

b) $2x+0.25 < 0, \ 2x+1.25 < 0 \Rightarrow 2x+0.25 < 2x+1.25 \Rightarrow 1.25 > 0.25$

$2x < -0.25 \Rightarrow x < -0.125$

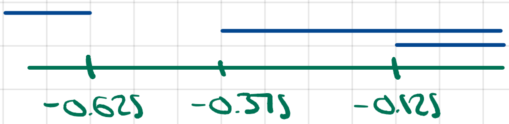
$2x < -1.25 \Rightarrow x < -0.625$



Case 2: $\frac{2x+0.25}{2x+1.25} < 0 \Rightarrow \frac{-2x-0.25}{2x+1.25} > 1$

a) $2x+0.25 > 0, \ 2x+1.25 < 0 \Rightarrow -2x-0.25 < 2x+1.25 \Rightarrow 4x > -1.5 \Rightarrow x > -0.375$

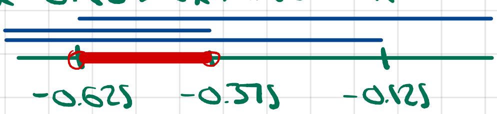
$x > -0.125, \ x < -0.625$



no solution

b) $2x+0.25 < 0, \ 2x+1.25 > 0 \Rightarrow -2x-0.25 > 2x+1.25 \Rightarrow x < -0.375$

$x < -0.125, \ x > -0.625$



2. $m\ddot{x}+b\dot{x}+kx=0$

$m=0.50$

$k=0.625$

a) critically damped case: repeated roots

$$0.5r^2+br+0.625=0$$

$$\Delta=b^2-4\cdot 0.5\cdot 0.625 \Rightarrow b^2=1.25 \Rightarrow b=\pm\sqrt{1.25}=\pm 1.1180$$

$$r=\pm 1.1180$$

$$x(t)=c_1e^{1.1180t}+c_2te^{1.1180t}$$

b) $b=0.25$ $\quad 0.5\ddot{x}+0.25\dot{x}+0.625x=0$

$$\Delta=\frac{1}{16}-4\cdot 0.5\cdot 0.625=\frac{1}{16}-1.25=\frac{1}{16}-\frac{5}{4}=\frac{1}{16}-\frac{20}{16}=\frac{-19}{16}$$

$$r=\frac{-0.25\pm\sqrt{19/16}\,i}{2\cdot 0.5}=-\frac{1}{4}\pm\frac{\sqrt{19}}{4}i$$

$$x_1(t)=e^{t\left(\frac{-1+\sqrt{19}i}{4}\right)}=e^{-\frac{t}{4}}e^{\frac{\sqrt{19}i}{4}}$$

$$e^{\sqrt{19}i/4}=\cos(\sqrt{19}t/4)+i\sin(\sqrt{19}t/4)$$

$$x(t)=e^{-\frac{t}{4}}(c_1\cos(\sqrt{19}t/4)+c_2\sin(\sqrt{19}t/4))$$

$$x(0)=c_1=0 \Rightarrow x(t)=e^{-\frac{t}{4}}c_2\sin(\sqrt{19}t/4)$$

$$x'(t)=-\frac{1}{4}e^{-\frac{t}{4}}c_2\sin(\sqrt{19}t/4)+e^{-\frac{t}{4}}\frac{c_2\sqrt{19}}{4}\cos(\sqrt{19}t/4)$$

$$\dot{x}(0)=\frac{c_2\sqrt{19}}{4}=1 \Rightarrow c_2=\frac{4}{\sqrt{19}}$$

$$x(t)=\frac{4}{\sqrt{19}}e^{-\frac{t}{4}}\sin(\sqrt{19}t/4)$$

c) $x(t)=0 \Rightarrow \frac{\sqrt{19}t}{4}=\pi k \quad k\in\mathbb{Z} \Rightarrow t=\frac{4\pi k}{\sqrt{19}} \quad k\in\mathbb{Z}$

$$t(k+1)-t(k)=\frac{4\pi}{\sqrt{19}}(k+1)-\frac{4\pi}{\sqrt{19}}k=\frac{4\pi}{\sqrt{19}}$$

$0.5\ddot{x}+0.25\dot{x}+0.625x=0$ represents a spring-mass-dashpot system. For example, a door with a spring and damper. $x(0)=0$ and $\dot{x}(0)=1$ represents a door that is being swung open at $t=0$ from position $x=0$. $x(t)$ is a damped oscillation for those parameters.

The oscillations have smaller and smaller amplitude, but the angular frequency is constant so the time between crossing the same position twice is always the same.

The pseudo-period is $2\pi\cdot 4/\sqrt{19}$.

$x(0)=x_0,\ \dot{x}(0)=\dot{x}_0$

$$x(t)=e^{-\frac{t}{4}}(x_0\cos(\sqrt{19}t/4)+c_2\sin(\sqrt{19}t/4))$$

$$\dot{x}(t)=-\frac{1}{4}x(t)+e^{-\frac{t}{4}}\left(-\frac{x_0\sqrt{19}}{4}\sin(\sqrt{19}t/4)+\frac{c_2\sqrt{19}}{4}\cos(\sqrt{19}t/4)\right)$$

$$\dot{x}(0)=-\frac{1}{4}x_0+\frac{c_2\sqrt{19}}{4}=\dot{x}_0$$

$$c_2=\frac{(\dot{x}_0+\frac{1}{4}x_0)\cdot 4}{\sqrt{19}}$$

$$x(t)=e^{-\frac{t}{4}}\left[x_0\cos(\sqrt{19}t/4)+\frac{4}{\sqrt{19}}\left(\dot{x}_0+\frac{x_0}{4}\right)\sin\left(\frac{\sqrt{19}t}{4}\right)\right]$$

$x(t)=0 \Rightarrow$

$$x_0\cos\left(\frac{\sqrt{19}t}{4}\right)=-\frac{\cancel{4}}{\sqrt{19}}\frac{(4\dot{x}_0+x_0)}{\cancel{4}}\sin\left(\frac{\sqrt{19}t}{4}\right)$$

$$\Rightarrow \sin\left(\frac{\sqrt{19}t}{4}\right)=-\frac{x_0\sqrt{19}}{4\dot{x}_0+x_0}\cos\left(\frac{\sqrt{19}t}{4}\right)$$

$$\tan\left(\frac{\sqrt{19}t}{4}\right)=-\frac{x_0\sqrt{19}}{4\dot{x}_0+x_0}$$

$$t=\frac{4}{\sqrt{19}}\tan^{-1}\left[-\frac{x_0\sqrt{19}}{4\dot{x}_0+x_0}\right]+\frac{4}{\sqrt{19}}k\pi$$

$$t(k+1)-t(k)=\frac{4}{\sqrt{19}}\pi$$

The distance between successive zeros of the gen. sol'n is always $\frac{4}{\sqrt{19}}\pi$ no matter the initial conditions $(\dot{x}_0, x_0)$.