PSet 14 – Linear Operators – Part I

1. $\ddot{x} + x = t^2 + \cos(2t-1)$ $\qquad p(D) = D^2+1$

Strategy: Find x_1 sol'n to $p(D)x = t^2$, and x_2 sol'n to $p(D)x = \cos(2t-1)$.

superposition $\Rightarrow$ $x_1 + x_2$ sol'n to $p(D)x = t^2 + \cos(2t-1)$

$\to$ $p(D)x = t^2$, polyn. input

$x_{1p} = At^2 + Bt + C$
$x'_{1p} = 2At + B$
$x''_{1p} = 2A$

$2A + At^2 + Bt + C = At^2 + Bt + (2A+C) = t^2$

$\Rightarrow$ $A = 1$
$B = 0$
$2 \cdot 1 + C = 0 \Rightarrow C = -2$
$x_{1p} = t^2 - 2$

$\to$ $p(D)x = \cos(2t-1)$, sinusoidal input

$p(D)z = e^{i(2t-1)} = e^{-i}e^{2it}$
$z = Ae^{2it}$
$\Rightarrow z' = 2iAe^{2it}$
$z'' = -4Ae^{2it}$

$\Rightarrow A\cancel{e^{2it}}(-4+1) = e^{-i}\cancel{e^{2it}} \Rightarrow A = \frac{e^{-i}}{-3}$

$\Rightarrow z_p(t) = -\frac{e^{-i}e^{2it}}{3} = -\frac{e^{i(2t-1)}}{3} = -\frac{1}{3}(\cos(2t-1) + i\sin(2t-1))$

$\Rightarrow x_{p_2}(t) = -\frac{1}{3}\cos(2t-1)$

$\Rightarrow x_p(t) = x_{1p} + x_{2p} = t^2 - 2 - \frac{1}{3}\cos(2t-1)$

2. $y'' + y' + y = 2xe^x$

$y_p = a_1xe^x + a_2e^x$

$y_p' = a_1e^x + a_1xe^x + a_2e^x$

$y_p'' = a_1e^x + a_1e^x + a_1xe^x + a_2e^x$

$e^x(2a_1 + a_2 + a_1 + a_2 + a_2) + xe^x(a_1 + a_1 + a_1) = 2xe^x$

$e^x(3a_1 + 3a_2) + xe^x\,3a_1 = 2xe^x$

$3a_1 = 2 \Rightarrow a_1 = 2/3$
$a_1 + a_2 = 0 \Rightarrow a_2 = -2/3$

$\Rightarrow y_p(t) = \frac{2}{3}xe^x - \frac{2}{3}e^x$

$y'' + y' + y = 0$

$p(r) = r^2 + r + 1 = 0$

$\Delta = 1 - 4 = -3$

$r = \frac{-1 \pm \sqrt{3}i}{2}$

complex homog. solution: $e^{-\frac{x}{2}}e^{\frac{\sqrt{3}x}{2}i}$
$= e^{-\frac{x}{2}}(\cos(\sqrt{3}x/2) + i\sin(\sqrt{3}x/2))$

$\Rightarrow e^{-\frac{x}{2}}\cos(\sqrt{3}x/2)$ and $e^{-\frac{x}{2}}\sin(\sqrt{3}x/2)$ are real sol'ns.

$\Rightarrow y_h(t) = e^{-\frac{x}{2}}(\cos(\sqrt{3}x/2) + \sin(\sqrt{3}x/2))$

$\Rightarrow y(t) = e^{-\frac{x}{2}}(\cos(\sqrt{3}x/2) + \sin(\sqrt{3}x/2)) + \frac{2}{3}xe^x - \frac{2}{3}e^x$

3. $y^{(4)} - 2y'' + y = xe^x$

$p(r) = r^4 - 2r^2 + 1 = (r^2-1)^2 = 0 \Rightarrow r^2 = 1 \Rightarrow r = \pm 1$, each a double root

$= (r+1)^2(r-1)^2$

$y_p = C_3x^2e^x + C_4x^3e^x$

$\Rightarrow e^x(8C_3 + 24C_4) + 24C_4xe^x = xe^x$

$\Rightarrow C_4 = 1/24 \;,\; C_3 = -1/8$

$\Rightarrow y_p(t) = -\dfrac{x^2e^x}{8} + \dfrac{x^3e^x}{24}$