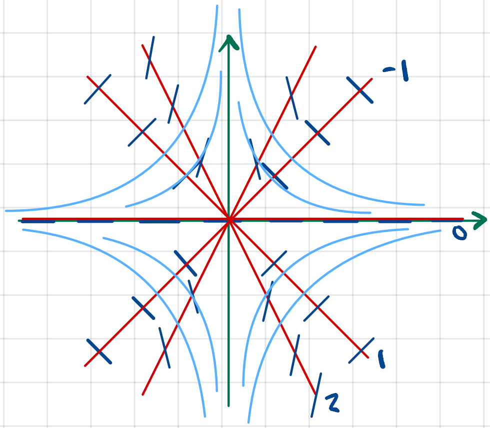
PSet - Geometric Methods - Part I

Problem 1 $y' = -\frac{y}{x}$

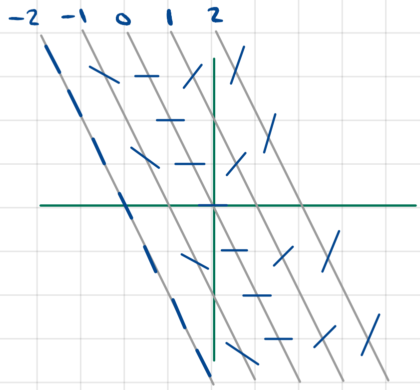
$\frac{1}{y}\,dy = -\frac{1}{x}\,dx \Rightarrow \ln|y| = -\ln|x| + C_1 \Rightarrow |y| = e^{C_1}e^{-\ln|x|} = e^{C_1}|x|^{-1} = C_2|x|^{-1}$

$\Rightarrow y = \pm\frac{C_2}{x}$



isoclines: $-\frac{y}{x} = m \Rightarrow y = -mx$

Problem 2 $y' = 2x + y$



isoclines: $2x + y = m \Rightarrow y = m - 2x$

If a solution is also an isocline, then the solution is a line $y = mx + b$.

$\Rightarrow m = 2x + mx + b$

$m = x(2+m) + b$

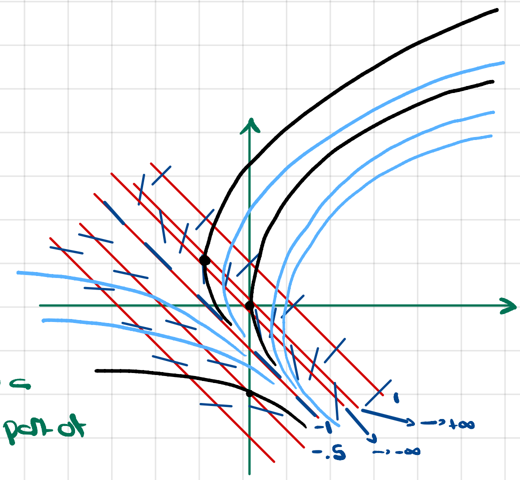
$\Rightarrow 2 + m = 0 \Rightarrow m = -2$

$b = m = -2$

$y = -2x - 2$

Problem 3 $y' = \frac{1}{x+y}$

isoclines: $\frac{1}{m} = x + y \Rightarrow y = -x + \frac{1}{m}$



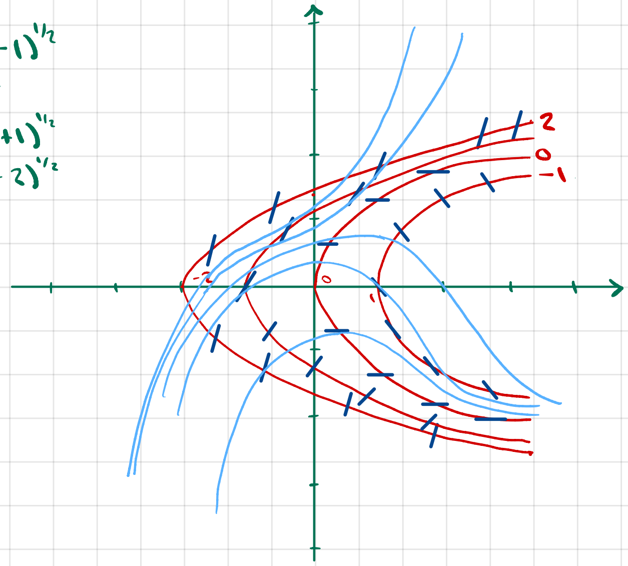
$m = -1 \Rightarrow y = -x - 1$, an isocline that is also a solution. Because each point in the slope field is part of exactly one solution, no solution can cross $y = -x - 1$.

Part II

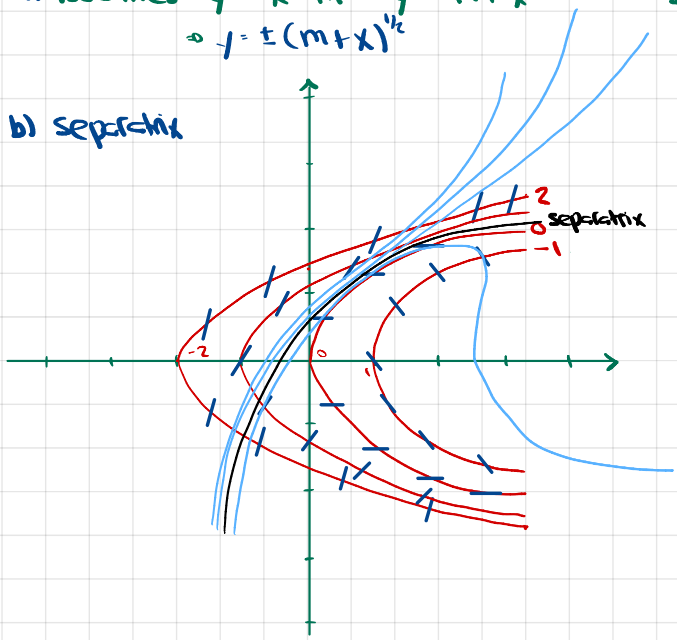
Problem 1 $\quad \dfrac{dy}{dx} = y^2 - x$

a) isoclines: $y^2 - x = m \Rightarrow y^2 = m + x$
$\Rightarrow y = \pm(m+x)^{1/2}$

m	
-1	$y = \pm(x-1)^{1/2}$
0	$y = \pm x^{1/2}$
1	$y = \pm(x+1)^{1/2}$
2	$y = \pm(x+2)^{1/2}$



b) separatrix



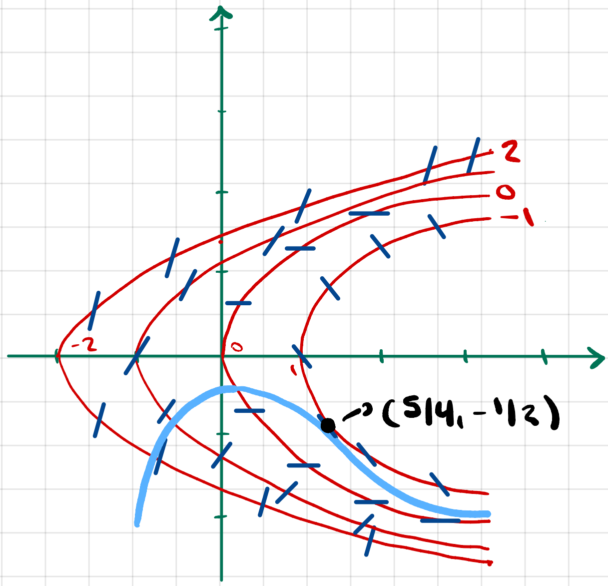
c) $y^2 = -1 + x$ (isocline)

at point (a,b) of tangency, $\dfrac{dy}{dx} = -1$

$y(x) = \pm(x-1)^{1/2}$ but the point must be on the negative portion.

$y'(x) = -\frac{1}{2}(x-1)^{-1/2} = -1 \Rightarrow (x-1)^{1/2} = \frac{1}{2} \Rightarrow x - 1 = \frac{1}{4} \Rightarrow x = \frac{5}{4}$

$\Rightarrow y = -\left(\frac{5}{4} - 1\right)^{1/2} = -\frac{1}{2} \qquad (a,b) = (5/4, -1/2)$



d) The isocline for $m = 0$ is $y = \pm\sqrt{x}$. The bottom half of the isocline is a lower fence: the slope of the isocline is smaller than the slope field slope at every point (on the bottom half isocline). Once $y(x)$ crosses this bottom nullcline, it can't cross back over it. The solution found in c) crosses the nullcline and so also does not cross back over it. Any solution with $y(a) < b$ lies below the c) solution because solutions can't cross. Therefore, any solution with $y(a) < b$ that crosses nullcline stays between the c solution and the nullcline.

All solutions with $y(a) < b$ lie below the solution from c). If such solutions cross the nullcline then for large x (ie x larger than the crossing point x) $y(x) > f(x)$.

di) At $x = a$, the -1 isocline slope equals the slope fields slope, and for $x > a$ has lesser slope, making it an upper fence.

dii) The bottom half of the nullcline is a lower fence starting at $x = 0$: no solution above the bottom half nullcline can cross it.

diii) Do all solutions with $y(a) < b$ cross the nullline?
→ every isocline below and to the left of the nullcline represents positive solution slope, ie the slope field is positive in such a region.
⇒ any solution with $y(a) < b$ is in such a region and is also bounded by the c) solution. Such solutions are therefore increasing; the nullcline is decreasing ⇒ all such solutions cross the nullcline.
→ we can also see that such solutions would have to stop increasing and start decreasing if they're to not cross the nullcline. But then they'd need to have a critical point, and the only location of critical points is on the nullcline.

e) $y'(c) = 0$, $y(c) = d$

(c, d) is on the nullcline, which is $y(x) = \pm\sqrt{x} \Rightarrow y(c) = \pm\sqrt{c} = d$

f) when a solution crosses the nullcline, its slope goes from positive to negative, so the crossing point is a local maximum.

$\frac{dy}{dx} = y^2 - x \Rightarrow y'' = 2yy' - 1$. At critical points, $y'' = -1 \Rightarrow$ critical points are local maxima