Initial Definitions

differential eq. (DE): eq. expressing a relation between a function and its derivatives

ordinary DE: function in DE has single independent variable

order of DE: order of largest derivative in the DE

The Most Important DE $\dot{y} = ay$ "rate of change of y is proportional to y"

aka $\frac{dy}{dt} = ay(t)$ $y' = ay$ $\dot{y} - ay = 0$

solution: $y(t) = Ce^{at}$

models exponential growth $(a>0)$ or decay $(a<0)$

growth/decay constant

First Technique for Solving DEs

separable eq: algebra can be used to separate the two variables such that each is on one side of the eq.

$$y' = 2x(1-y)^2 \Rightarrow \frac{1}{(1-y)^2}dy = 2x\,dx \Rightarrow \frac{1}{1-y} = x^2 + C \Rightarrow y = 1 - \frac{1}{x^2+C}$$

This separation step is only valid if $y \neq 1$

this parametrized family of solutions does not include $y(x) = 1$.

$y(x) = 1$ is a lost solution

→ in general, for separable DE $y' = f(x)g(y)$, all roots of $g(y)$ give lost (constant) sol'ns.

Ex: $\frac{dy}{dt} = ky \Rightarrow y^{-1}dy = k\,dt$ $y \neq 0$ $\Rightarrow \ln|y| = kt + C_1 \Rightarrow |y| = e^{C_1}e^{kt} \Rightarrow y = \pm C_2 e^{kt}$, $C_2 = e^{C_1} \neq 0 \Rightarrow y = Ce^{kt}$

$C = \pm C_2 \neq 0$

$y(t) = 0$ is a lost sol'n.

Savings Account Model

→ $x(t)$ dollars in account at time t
→ accrues interest at rate r, in %/year
→ interest period Δt, e.g. $1/12$

$\Delta x = r x(t) \Delta t$ years^{-1} USD years = USD

$$\frac{\Delta x}{\Delta t} = r x(t)$$

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = x'(t) = r x(t), \text{ ie } \dot{x} = r x$$

note that we made no assumptions on r; it could be $r(t)$, $r(x,t)$, a constant, etc.
→ if $r = r(x)$ then the DE is nonlinear. We won't consider that case here.

→ Assume now that contributions are made to the account
q: rate of savings in USD/year
e.g. monthly payments $q\Delta t$, $\Delta t = 1/12$

$$\Delta x = r x(t) \Delta t + q \Delta t \Rightarrow \frac{\Delta x}{\Delta t} = r x(t) + q$$

$$\Rightarrow x'(t) = r x(t) + q \qquad \dot{x} = r x + q$$

in general, we have a first order linear ODE $\dot{x} - r(t) x = q(t)$

A Model For Linear Insulation

→ context: a cooler holds a beverage. How does the temp. inside change w/ time?

(1) identify relevant parameters time t, $x(t)$ inside temp, $y(t)$ outside temp

→ insulation affects the rate of change of inside temp.

→ we assume the rate of change depends on $x(t), y(t)$: $\dot{x} = F(x,y)$ 1st-order DE

→ what is the form of the relationship? we assume $\dot{x} = f(y-x)$ fn of one variable

→ $f(0) = 0$, ie no temp diff ⇒ no change of inside temp

→ tangent line approx to $f(y-z) = f(z)$

$$f(z) \approx f(0) + f'(0)(z-0) = kz$$

→ sub in the approximation of $f(y-x)$ into the diff. eq. $\dot{x} = k(y-x)$, ie we linearized the eq.

→ justify the use of the approx. by modeling the rate of chg of inside temp for small $y-x$.

→ rewrite as $\dot{x} + kx = ky$ (Newton's Law of Cooling)

coupling constant, large if bad insulation