PSet - Linear vs Nonlinear

Problem 1 $\dot{y} = (1-y)y - a$ logistic eq. w/ harvesting

$a = 3/16 = 0.1875$

y_0: stable critical point

$u = y - y_0 \Rightarrow y = u + y_0 \quad y' = u'$

a) $y_{crit} = \frac{1 \pm \sqrt{1-4a}}{2}$

$a = 3/16 \Rightarrow y_{crit} = 0.75$ stable $\Rightarrow y_0 = 0.75$

$y_{crit} = 0.25$ unstable

Let's change variables:

$u' = (1 - u - y_0)(u + y_0) - a$

$= u + y_0 - u^2 - uy_0 - uy_0 - y_0^2 - a$

$\Rightarrow \frac{du}{dt} = -u^2 + u(1 - 2y_0) + \underbrace{y_0 - y_0^2 - a}_{0} = g(u)$

Note that y_0 is critical point of $\dot{y} = (1-y)y - a \iff y_0 - y_0^2 - a = 0$

$\Rightarrow \dot{u} = -u^2 + u(1 - 2y_0)$

critical points: $g(u) = 0 \Rightarrow u(1 - 2y_0 - u) = 0$

$u_{crit} = \begin{cases} 0 \\ 1 - 2y_0 \end{cases}$

$y_0 = 0.75, a = 3/16 \Rightarrow u_{crit} = \begin{cases} 0 \\ -0.5 \end{cases}$

$\dot{u} = -u^2 + u(1 - 1.5)$

$\Rightarrow \dot{u} = -u^2 - u/2$

b) For $u = y - y_0$ small, ie y near y_0 we can discard u^2 term

g is now linear in u

$\frac{du}{dt} = -\cancel{u^2} + u(1 - 2y_0) = g(u)$

note we assume this is an approximation that is made for small u.

$u = 0 \Rightarrow \dot{u} = 0$ (linearized eq. near $u = 0$)

in general, $\dot{u} - ku = 0 \quad k = 1 - 2y_0$

$u \neq 0$ implicit

using int. factor e^{-kt}: $ue^{-kt} = C \Rightarrow u = Ce^{kt}$

sep. variables: $u^{-1}du = kdt \Rightarrow \ln|u| = kt + C \Rightarrow u = \pm e^{c}e^{kt}$

$u \neq 0$ implicit

$y_0 = 3/4 \Rightarrow \dot{u} = -\frac{1}{2}u \Rightarrow u(t) = Ce^{-\frac{1}{2}t}$

linearized

c) $y(10) - y_0 = b$, estimate $y(11)$, $y(12)$

$u(10) = y(10) - y_0 = b = Ce^{10k} \quad k = 1 - 2y_0$

$C = be^{-10k}$

$\Rightarrow u(t) = be^{-10k}e^{kt} = be^{kt - 10k} = be^{k(t-10)}$

$u(11) = be^{k}$

$u(12) = be^{2k}$

$y_0 = 3/4 \Rightarrow k = -0.5$

$\Rightarrow u(11) \approx 0.6065b$

$u(12) \approx 0.3678b$

$u = y - y_0 \Rightarrow y = y_0 + u$

$y(11) \approx y_0 + u(11) = \frac{3}{4} + 0.6065b$

approximate because we linearized $\dot{u}(t)$.

$u(t) = be^{-10k}e^{kt}$ is the general solution for the

d) $\dot{x} + p(t)x = q(t)$ autonomous

$\dot{x} = q(t) - p(t)x = f(x,t)$

$f_t = \dot{q} - \dot{p}x - p\dot{x} = 0$

$\Rightarrow \dot{x} = \frac{\dot{q}}{p} - \frac{\dot{p}}{p}x = q(t) - p(t)x$

$\underline{\dot{q}} - \dot{p}$

$q(t) = \frac{\dot{q}}{p}$

$p = \frac{\dot{p}}{p}$

$\dot{p} = p^2$

$qp = \dot{q}$