PSet - Autonomous Eq. - Part I

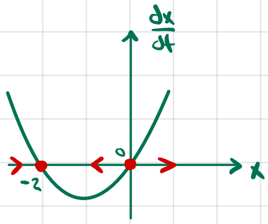
Problem 1

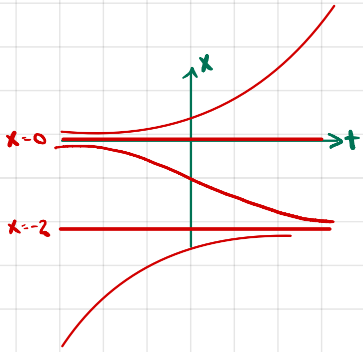
a) $x' = x^2 + 2x$

critical points: $\frac{dx}{dt} = x^2 + 2x = 0$

$x(x+2) = 0$ unstable

$x = 0,\ x = -2$ stable

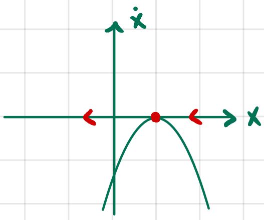


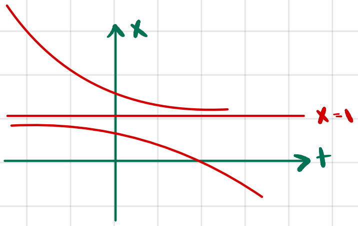


b) $x' = -(x-1)^2 = f(x)$

critical points: $(x-1)^2 = 0 \Rightarrow x = 1$

semi-stable





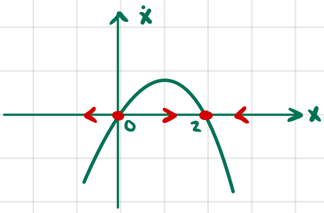
c) $x' = 2x - x^2$

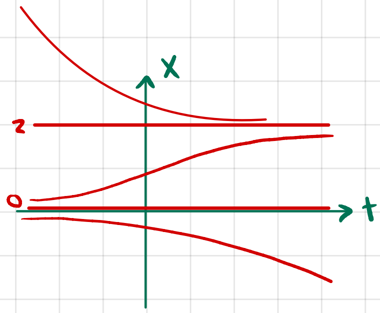
crit. points $2x - x^2 = 0$

$x(2-x) = 0$

$x = 0$ unstable

$x = 2$ stable

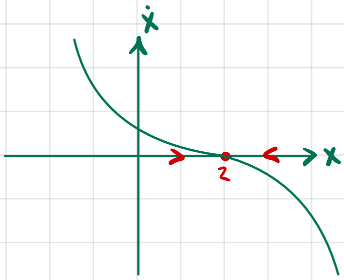


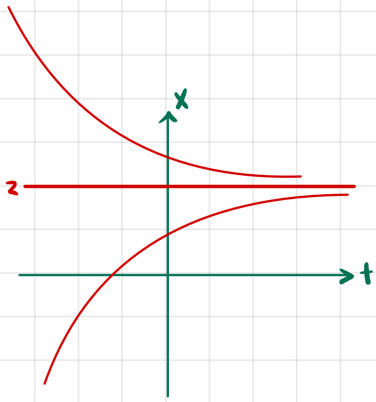


d) $x' = (2-x)^3$

crit points $(2-x)^3 = 0$

$x = 2$





Problem 2

a) $\dot{x}+2x=1$

i) $\dot{x}=1-2x \Rightarrow \frac{1}{1-2x}dx=dt \Rightarrow -\frac{1}{2}\ln|1-2x|=t+C \Rightarrow \ln|1-2x|=-2t+C \Rightarrow |1-2x|=e^{c}e^{-2t} \Rightarrow 1-2x=\pm e^{c}e^{-2t}$

$\Rightarrow 2x=1\pm e^{c}e^{-2t} \Rightarrow x(t)=\frac{1}{2}+Ce^{-2t}$

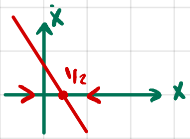
ii) $u(t)=e^{\int 2t}=e^{2t} \Rightarrow e^{2t}\dot{x}+2e^{2t}x=e^{2t} \Rightarrow (e^{2t}x)'=e^{2t} \Rightarrow e^{2t}x=\frac{1}{2}e^{2t}+C \Rightarrow x(t)=\frac{1}{2}+Ce^{-2t}$

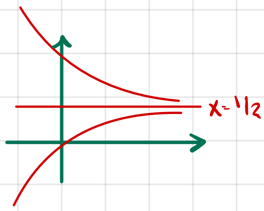
iii) guess $x_p(t)=A \Rightarrow x_p(t)=\frac{1}{2}$

$\Rightarrow x(t)=\frac{1}{2}+Ce^{-2t}$

$\dot{x}+2x=0 \Rightarrow x_h(t)=Ce^{-2t}$

b) $\dot{x}=1-2x=f(x)$





$f(x)=0 \Rightarrow x=\frac{1}{2}$ stable

c) Euler's Method for estimating $x(1)$ when $x(0)=0$. Three steps $\Rightarrow h$ = step size $=\frac{1}{3}$

$t_{n+1}=t_n+h$

$x_{n+1}=x_n+\dot{x}_n h$

$\dot{x}_n=1-2x_n$

n	t_n	x_n	$\dot{x}_n$	$\dot{x}_n h$
0	0	0	1	1/3
1	1/3	1/3	1/3	1/9
2	2/3	4/9	1/9	1/27
3	1	13/27		

Part II

Problem 1 Game preserve

→ case 1: no hunting ⇒ stable population of 1 kilo-oryx = 1000 oryx

→ case 2: hunting rate a

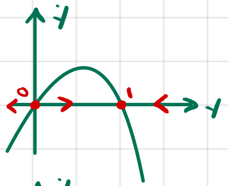
a) $\frac{dy}{dt} = (1-y)y - a$ $\quad$ * $y'(t) = ky - a$, $k(y) = a - by$

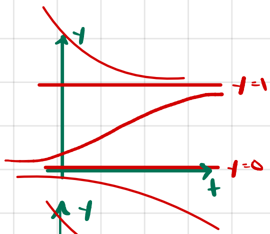
(logistic pop. model w/ harvesting) $\quad$ $k(0) = 1 \Rightarrow a = 1 \Rightarrow k(y) = 1 - y$

$k(1) = 1 \Rightarrow b = 1$

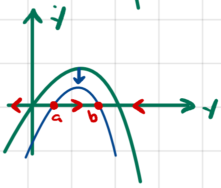
This model represents the population dynamics in the game preserve.
There is a natural growth rate that depends on the population level. As pop. increases the growth rate decreases, reaches zero at some level of y, and then becomes negative. In the absence of hunting ($a=0$), there is a stable equilibrium at some $y>0$.
Graphically, we have:

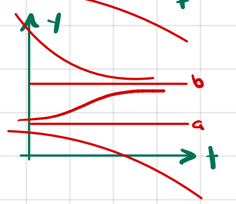
$\frac{dy}{dt} = (1-y)y$ $\quad$ crit. points: $y = 0, y = 1$





when we introduce hunting (a), $f(y)$ shifts down. There is now a level of population below which the popul. decreases exponentially to zero (extinction).



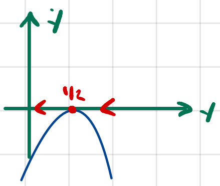


b) $\frac{dy}{dt} = (1-y)y - a$

critical points: $y - y^2 - a = 0 \Rightarrow y^2 - y + a = 0$ $\quad$ $\Delta = 1 - 4a = 0$ $\quad \Rightarrow y = \frac{1 \pm \sqrt{1-4a}}{2}$

→ $1 - 4a < 0 \Rightarrow a > 1/4$ then there are no critical points, $\dot{y} < 0$ and extinction occurs irrespective of y_0.

→ $1 - 4a = 0 \Rightarrow a = 1/4 \Rightarrow y_{crit} = 1/2$
There is a semi-stable equilibrium at $y = 1/2$.

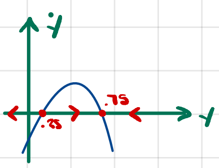


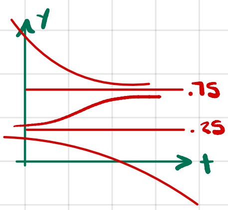
→ $1 - 4a > 0 \Rightarrow a < 1/4 \Rightarrow$ one stable critical point $y = \frac{1+\sqrt{1-4a}}{2}$, one unstable critical point at $y = \frac{1-\sqrt{1-4a}}{2}$.

c) $a = 0.1875 < 1/4$

⇒ stable popul. = 0.75 = 750 oryx
unstable popul. = 0.25, below which the trajectory is extinction.

d)





e) Bifurcation Diagram shows critical points in the ya-plane.

$y - y^2 - a = 0 \Rightarrow a = y - y^2$