$\to \frac{dy}{dx} = 2y+1 \Rightarrow \frac{1}{2y+1} = dx$, $2y+1 \neq 0 \Rightarrow y \neq -\frac{1}{2} \Rightarrow \frac{1}{2}\ln|2y+1| = x + C_1 \Rightarrow |2y+1| = e^{2x+2C_1} = e^{2C_1}e^{2x} = C_2e^{2x}$

$C_2 = e^{2C_1} \neq 0 \Rightarrow 2y+1 = \pm C_2 e^{2x} \Rightarrow y = \frac{\pm C_2}{2}e^{2x} - \frac{1}{2} \Rightarrow y(x) = Ce^{2x} - \frac{1}{2}$

$y(x) = \frac{1}{2}$ is a lost solution.

---

lost solution $y(x) = 1$

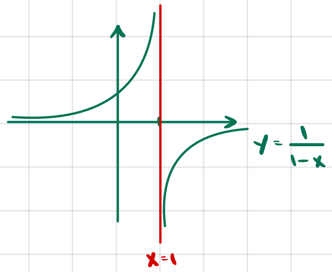
$\to \frac{dy}{dx} + xy = x \Rightarrow \frac{dy}{dx} = x(1-y) \Rightarrow \frac{1}{1-y}dy = x\,dx \Rightarrow -\ln|1-y| = \frac{x^2}{2} + C_1 \Rightarrow \ln|1-y| = -\frac{x^2}{2} + C_2$

$\Rightarrow |1-y| = e^{-\frac{x^2}{2}+C_2} = e^{C_2}e^{-x^2/2} = C_3e^{-x^2/2}$

$\Rightarrow 1-y = \pm C_3 e^{-x^2/2} \Rightarrow y = 1 \mp C_3 e^{-x^2/2} \Rightarrow y(x) = 1 + Ce^{-\frac{x^2}{2}}$

$y' = y^2$, $y(0) = 1 \xrightarrow{y \neq 0} y^{-2}dy = dx \Rightarrow -y^{-1} = x + C_1 \Rightarrow \frac{-1}{x+C_1} = y$, $y(0) = \frac{-1}{C_1} = 1 \Rightarrow C_1 = -1$

$\Rightarrow y(x) = \frac{-1}{x-1} = \frac{1}{1-x}$



"y blows up at x=1"

we can split the sol'n into two:

$y(x) = \frac{1}{1-x}$ $\quad x \in (-\infty, 1)$

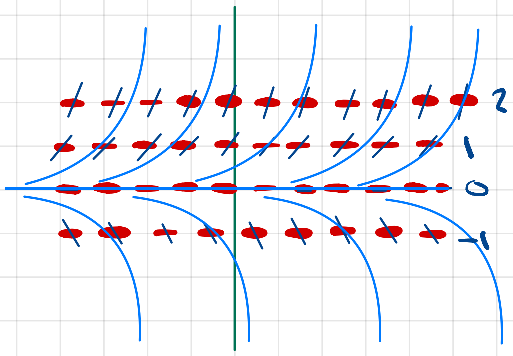
$y(x) = \frac{1}{1-x}$ $\quad x \in (1, +\infty)$

and the first one is the sol'n to this IVP.

$y' = y$

isoclines $y' = y = 0$

3 behaviors: exponential growth, exp. decay, constant $y = 0$.

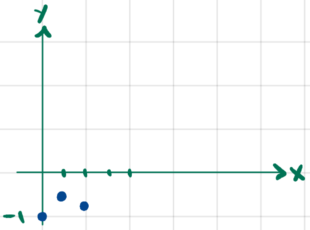


$U(x) = -\sqrt{x-1}$

$L(x) = -\sqrt{x}$

$D(x) = U(x) - L(x) = -\sqrt{x-1} + \sqrt{x} = \sqrt{x}\left(1 - \sqrt{\frac{x-1}{x}}\right) = \sqrt{x}\left(1 - \sqrt{1 - \frac{1}{x}}\right) \Rightarrow \lim_{x \to \infty} D(x) = 0$

Euler Method $y' = y^2 - x$ $\quad y(0) = -1$ $\quad$ estimate $y(1)$, Euler's Method, $h = 0.5$

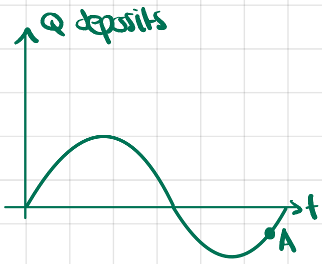


n	x_n	y_n	A_n	hA_n
0	0	-1	1	0.5
1	0.5	-0.5	-0.25	-0.125
2	1	-.625		

$x_{n+1} = x_n + h$

$y_{n+1} = y_n + hA_n$

$A_n = y'(x_n)$



$Q(A) < 0$, $Q'(A) > 0$

negative balance, positive deposits being made

$\dot{x}(t) = k(T_{ext}(t) - x(t))$

k large means larger rates of internal temp change for any difference in temp with outside cooler world.

$\Rightarrow$ less insulation.

units

$x(t) = {}^\circ C$

$\dot{x}(t) = {}^\circ C/\text{hour}$

$T_{ext} = {}^\circ C$

$kT_{ext} = {}^\circ C/h \Rightarrow k = h^{-1}$

$\dot{x} + 2x = 4$

$\dot{x} + p(t)x = 0 \Rightarrow \frac{1}{x}dx = -p(t)dt \Rightarrow \ln|x| = -\int p(t)dt + C_1 \Rightarrow |x| = e^{-\int p(t)dt + C_1}$

$\dot{x} + p(t)x = q(t)$

$\Rightarrow x(t) = Ce^{-\int p(t)dt} = Cx_h(t)$

$u(t) = e^{\int p(t)dt}$

$x(t) = \frac{1}{u(t)}\left[\int u(t)q(t)dt + C\right] = x_p(t) + Cx_h(t)$

For this problem

$x_h(t) = Ce^{-\int 2dt} = e^{-2t}$

$u(t) = \frac{1}{x_h(t)} = e^{2t}$

$\Rightarrow x_p(t) = e^{-2t}\left(\int e^{2t} \cdot 4dt\right) = e^{-2t} \cdot 2e^{2t} = 2$

$\Rightarrow x(t) = 2 + Ce^{-2t}$

$\dot{x} + p(t)x = q(t)$

$x(t) = x_p(t) + c x_h(t)$ is a solution.

x_p sol. to $\dot{x} + p(t)x = q(t)$

$\Rightarrow$ by superpos. principle, $c_1 x_p + c_2 x_h$ sol. to $\dot{x} + p(t)x = q(t)$

x_h sol. to $\dot{x} + p(t)x = 0$

$\dot{x} + kx = t$

$u(t) = e^{\int k\,dt} = e^{kt}$

$x(t) = e^{-kt}\left(\int te^{kt}dt + c\right) = e^{-kt}\left(\frac{e^{kt}(tk-1)}{k^2} + c\right) = \frac{tk-1}{k^2} + ce^{-kt}$

$\int te^{kt}dt = \frac{te^{kt}}{k} - \frac{1}{k}\int e^{kt}dt = \frac{te^{kt}}{k} - \frac{e^{kt}}{k^2} = \frac{e^{kt}(tk-1)}{k^2}$

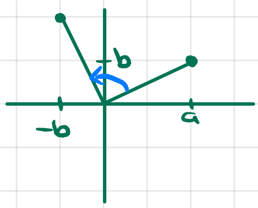
$u = t \quad du = dt$

$dv = e^{kt} \quad v = \frac{1}{k}e^{kt}$

$x(t) = \frac{tk-1}{k^2} + ce^{-kt}$

particular solution means choosing a C.

$i(a+bi) = ai - b$ 90° counterclockwise rotation



$z = a+bi$

$\bar{z} = -z$

$\bar{z} = a - bi = -a - bi \Rightarrow a = -a \Rightarrow a = 0$

$\Rightarrow$ if $z = \bar{z}$ then z is purely imaginary

$(1+i)^4$

$z = 1+i \quad \Rightarrow |z| = \sqrt{2}$

$\Rightarrow z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = r(\cos\theta + i\sin\theta)$

$\Rightarrow \theta = \pi/4, \; r = \sqrt{2}$

$\Rightarrow z = \sqrt{2}e^{\frac{\pi i}{4}}$

$(1+i)^4 = z^4 = 2^2 e^{\pi i} = 4(-1) = -4$

$z = a + bi = r(\cos\theta + i\sin\theta)$

$a = r\cos\theta$

$b = r\sin\theta$

$e^z = e^{a+bi} = e^a e^{bi} = e^a(\cos b + i\sin b)$

$e^{zt} = e^{at} e^{ibt} = e^{at}(\cos bt + i\sin bt)$

$a = 0, b \neq 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \pi/2$, $z = bi = ir\sin\frac{\pi}{2}$

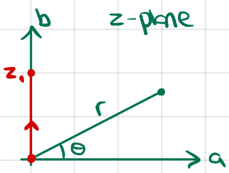
$z_1(t) = bti = itr\sin\frac{\pi}{2} = itr$

$\Rightarrow$ $z_1 = ibt$ means all the purely imaginary numbers, which lie on the y-axis.

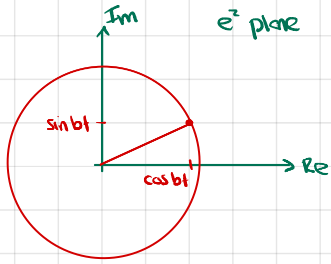
$\Rightarrow$ e^{z_1} is another number, distinct from z_1.

$\Rightarrow$ $e^{z_1 t} = e^{ibt} = \cos bt + i\sin bt$

*Note: $a + bi$ represents a point (a, b) in Cartesian coordinates. $(a+bi)t$ represents $\langle at, bt\rangle = t\langle a, b\rangle$, a parametrized line through $(0,0)$ and (a,b).



$\langle 0, bt\rangle$



$\langle \cos bt, \sin bt\rangle$

e^z is a complex number. $e^z = e^a(\cos b + i\sin b)$, ie represents point $(e^a\cos b, e^a\sin b)$

e.g. $e^{a+i2\pi}$ represents $(e^a, 0)$

e^{zt} represents $(e^{at}\cos(bt), e^{at}\sin(bt)) = e^{at}(\cos bt, \sin bt)$.

$(\cos bt, \sin bt)$ is a parametrized circle of radius 1.

we know because at each point distance to origin is $(\cos^2 bt + \sin^2 bt)^{1/2} = 1$

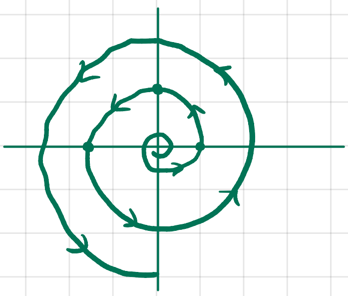
For e^{zt} the radius is e^{at}: $|z_1| = [e^{2at}(\cos^2 bt + \sin^2 bt)]^{1/2} = e^{at}$, which increases from 1 to ∞.

$z = e^{(a+bi)t} = e^{at} \cdot e^{ibt} = e^{at}(\cos bt + i\sin bt)$

$|z| = e^{at}$

$\arg(z) = bt$

$e^{(1+2\pi i)t} = e^t(\cos 2\pi t + i\sin 2\pi t)$, represents points $e^t(\cos(2\pi t), \sin(2\pi t))$, a spiral centered at the origin.



$A\cos(\omega t - \phi)$

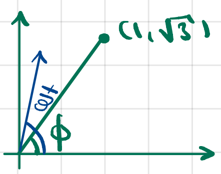
$P = \frac{2\pi}{\omega} = 8 \Rightarrow \omega = \frac{\pi}{4}$ rad/s $\Rightarrow \frac{\pi}{4}$ cycles in 2π time ie $t = 0$ $\cos(0)$, $t = 2\pi$ $\cos(\frac{\pi^2}{2})$, $\frac{\pi^2/2}{2\pi} = \frac{\pi}{4}$ cycles

$A = 2$

$\Rightarrow$ $2\cos(\frac{\pi}{4}t + \frac{\pi}{4})$

time lag $= \tau = -1 = \frac{\phi}{\omega} \Rightarrow \phi = -1 \cdot \pi/4$

$\cos\omega t + \sqrt{3}\sin\omega t = \sqrt{(1+3)}\cos(\omega t-\phi) = 2\cos(\omega t-\pi/3)$



$\phi = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

$(1,\sqrt{3})$ represents a complex number $1+i\sqrt{3}$.

$\rightarrow$ Geometrically, $\langle\cos\omega t, \sin\omega t\rangle \cdot \langle 1,\sqrt{3}\rangle = 1\cdot|\langle 1,\sqrt{3}\rangle|\cos(\omega t-\phi)$

$\cos\omega t + \sqrt{3}\sin\omega t = 2\cos(\omega t-\phi)$

we know ϕ because this is the polar angle of $(1,\sqrt{3})$.

Interpretation:

$\cos(\omega t)$: sinusoid with angular freq. ω, amplitude 1, no phase shift, period $\frac{2\pi}{\omega}$

$\sqrt{3}\sin(\omega t)$: " " " " " " $\sqrt{3}$ " " " " "

we sum them and get another sinusoid: $4\cos(\omega t-\pi/3)$ with amplitude 4, ang. freq. stays the same, phase shift $\frac{\pi}{3}$ (ie, shifted $\pi/3$ radians to right), time lag $\tau = \omega/(\pi/3)$ secs.

How did we obtain the parameters of the new sinusoid, A and ϕ?

A is the magnitude of the complex number repres. by $(1,\sqrt{3})$, ϕ its argument.

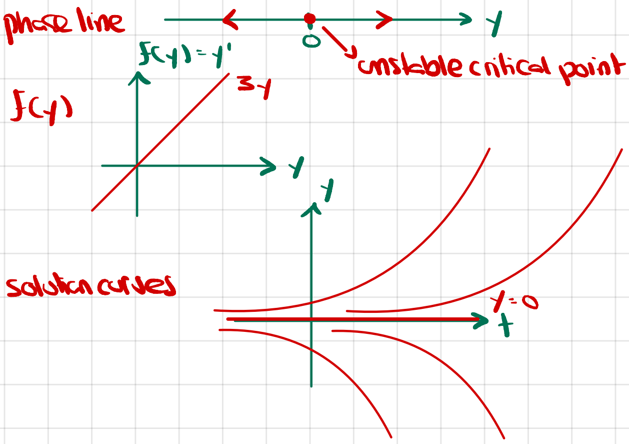
$\dot{y} = 3y$

critical points of DE: $y' = 3y = 0 \Rightarrow y = 0$

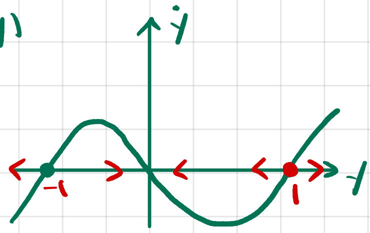
★ $\frac{1}{y}dy = 3dt \Rightarrow \ln|y| = 3t + c_1$

$|y| = e^{c_1}e^{3t}$

$y = \pm Ce^{3t}$



$\dot{y} = f(y)$



$y = 1$ is an unstable critical point

Autonomous DE: $\dot{y} = f(y)$

Assume there is a solution $y(t)$ such that $y'(t^*) = 0$.

$\Rightarrow y'(t^*) = f(y(t^*)) = 0 \Rightarrow f(y^*) = 0$, ie $y = y^*$ is a nullcline, but also a solution

because $y(t) = y^* \Rightarrow y'(t) = 0 \Rightarrow \dot{y} = 0 = f(y) = 0$

$\Rightarrow y$ does not have a local max at t^*.

Inflection Points

*a constant solution means

$y(t) = y_0$

$\dot{y} = 0$

$f(y) = 0$

$$\frac{dy}{dt} = f(y(t)) \qquad \frac{d^2y}{dt^2} = f'(y(t)) \cdot y'(t)$$

inflection point: $y'' = f' \cdot y' = 0$

Assume $y(t)$ nonconstant. y has inflection point at $t^* \Leftrightarrow f'(y(t^*)) = 0$ AND/OR $y'(t^*) = 0$

$\ddot{x} + 4x = 0$

$x_1(t) = \cos(2t)$

$x_1' = -2\sin(2t)$ $\qquad \Rightarrow -4\cos 2t + 4\cos 2t = 0$

$x_1'' = -4\cos(2t)$

Also, $x_2(t) = \sin 2t$ is a sol'n.

$\Rightarrow x(t) = a\cos(2t) + b\sin(2t)$ sol'n.

$= A\cos(2t - \phi)$

Period $= \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$

$\ddot{x} + \dot{x} + x = 0$

$y(t) = e^{rt} \Rightarrow r^2 + r + 1 = 0 \qquad \Delta = 1 - 4 = -3 \qquad r = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$

$y(t) = e^{-\frac{t}{2} \pm \frac{t\sqrt{3}}{2}i}$ complex solutions

Each complex solution gives us two real solutions

$e^{-\frac{t}{2}} e^{\frac{t\sqrt{3}}{2}i} = e^{-\frac{t}{2}}(\cos(\sqrt{3}t/2) + i\sin(\sqrt{3}t/2))$

$e^{-\frac{t}{2}}\cos(\sqrt{3}t/2)$ and $e^{-\frac{t}{2}}\sin(\sqrt{3}t/2)$ are both solutions.

By superposition, $y(t) = c_1 e^{-\frac{t}{2}}(\cos(\sqrt{3}t/2)) + c_2 e^{-\frac{t}{2}}\sin(\sqrt{3}t/2)$

$= e^{-\frac{t}{2}} A\cos(\sqrt{3}t/2 - \phi)$

$m\ddot{x} + b\dot{x} + kx = 0$ has charact. roots $a \pm ib$

$e^{t(a \pm ib)}$ are two complex sol'ns.

$e^{at + ibt} = e^{at}(\cos bt + i \sin bt)$

$x(t) = e^{at}(c_1 \cos bt + c_2 \sin bt) = e^{at} A \cos(bt - \phi)$

$\ddot{x} + 4\dot{x} + 4x = 0$

$r^2 + 4r + 4 = 0 \quad \Delta = 16 - 16 = 0 \quad r = \frac{-4}{2} = -2$

$x_1(t) = e^{-2t}$

$x_2(t) = e^{-2t} t$

$x_2' = -2te^{-2t} + e^{-2t}$

$x_2'' = 4te^{-2t} - 2e^{-2t} - 2e^{-2t}$

$4te^{-2t} - 2e^{-2t} - 2e^{-2t} - 8te^{-2t} + 4e^{-2t} + 4te^{-2t} = 0$

$\Rightarrow e^{-2t}(4t - 2 - 2 - 8t + 4 + 4t) = 0$

---

roots of homog. const. coeffs. lin. eq.: $3, 4, 4, 4, 5 \pm 2i, 5 \pm 2i$

$x(t) = c_1 e^{3t} + c_2 e^{4t} + c_3 e^{4t} t + c_4 e^{4t} t^2 + c_5 e^{5t} \cos 2t + c_6 t e^{5t} \cos 2t + c_7 e^{5t} \sin 2t + c_8 t e^{5t} \sin 2t$

$x_1(t) = e^{5t + 2ti} = e^{5t}(\cos 2t + i \sin 2t)$

$\left.\begin{array}{l} e^{5t} \cos 2t \\ e^{5t} \sin 2t \end{array}\right\}$ also solutions

$x_2(t) = e^{5t - 2ti} = e^{5t}(\cos(-2t) + i \sin(-2t))$

$= e^{5t}(\cos(2t) - i \sin(2t))$

$e^{5t} \cos 2t$

$-e^{5t} \sin 2t$

$\Rightarrow$ the linear eq. is of order eight

$*$ the two complex solutions each give two real sol'ns, but of the four only two are l.i.

$\ddot{x} + 8x + 15x = e^{-5t}$ linear non-homog. second order ODE w/ expon. input

The exponent $-5t$ tells us a lot about the particular sol'n. Because we use optimism method, our guess is Ae^{-5t}, which leads to a characteristic equation $p(\alpha) = 0$, where p is a second degree polynomial operator $p(\alpha) = \alpha^2 + 8\alpha + 15$, $p'(\alpha) = 2\alpha + 8$

$p(-5) = 25 - 40 + 15 = 0.$

$p'(-5) = 2(-5) + 8 = -2.$ $\Rightarrow -5$ is a simple root

Using the generalized exp. response formula, $x_p(t) = \dfrac{e^{-5t} t}{-2}$

$\ddot{x}+2\dot{x}+2x=e^{-t}\cos t$

$e^{-t}\cos t$ is real part of $e^{-t}\cdot e^{ti}=e^{t(-1+i)}$

$\ddot{z}+2\dot{z}+2z=e^{t(-1+i)}=e^{\alpha t},\ \alpha=-1+i$

$p(r)=r^2+2r+2$

$p(-1+i)=\cancel{1}-\cancel{1}-2i-\cancel{2}+2i+\cancel{2}=0$

$p'(r)=2r+2 \quad p'(-1+i)=2(-1+i+1)=2i$

$\Rightarrow z_p(t)=\dfrac{te^{t(-1+i)}}{2i}=\dfrac{-2ite^{-t}}{4}(\cos t+i\sin t)$

Real part $=\dfrac{te^{-t}\sin t}{2}$

Note:

$x''+8x'+7x=2e^{-t}$

$p(r)=r^2+8r+7 \qquad r=\dfrac{-8\pm\sqrt{64-28}}{2}$ → -1, → -7

$p'(r)=2r+8$

$x_p(t)=Ate^{-t} \Rightarrow A=\dfrac{1}{p'(-1)}=\dfrac{1}{6}$

$\Rightarrow x_p(t)=\dfrac{2te^{-t}}{6}$

$=\dfrac{te^{-t}}{3}$

homog. sol'n

$x_h=e^{rt} \Rightarrow e^{rt}(r^2+8r+7)=0 \Rightarrow$ $r=-1$, $r=-7$

$x_h(t)=c_1e^{-t}+c_2e^{-7t}$

$\Rightarrow x(t)=\dfrac{te^{-t}}{3}+c_1e^{-t}+c_2e^{-7t}$

$y'=6x(y-1)^{2/3} \Rightarrow (y-1)^{-\frac{2}{3}}\dfrac{dy}{dx}=6x \Rightarrow (y-1)^{1/3}=x^2+C$ implicit sol'n $\Rightarrow y=1+(x^2+C)^3$

$y\neq 1$
lost sol'n

note $f(x,y)=6x(y-1)^{2/3}$, $f'(x,y)=6x\cdot\frac{2}{3}(y-1)^{-\frac{1}{3}}$, not defined for $y=1$.

$y(x)=1 \Rightarrow 6x(1-1)^{2/3}=y'(x)=0$

This sol'n really is lost in the deriv. of $y(x)=1+(x^2+C)^3$

IVPs with $y(a)=1$ do not have unique sol'ns.

$x'' + x' + 2x = \cos t$

$z'' + z' + 2z = e^{it}$

$p(r) = r^2 + r + 2$

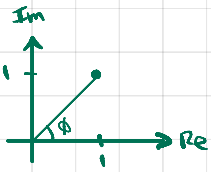
roots: $\dfrac{-1 \pm (1-8)^{1/2}}{2} = \dfrac{-1 \pm \sqrt{7}i}{2}$

$p(i) = i^2 + i + 2 = 1 + i \neq 0$

$\Rightarrow z_p = \dfrac{1}{1+i} e^{it} = \dfrac{(1-i)}{2}(\cos t + i \sin t)$

$x_p = \frac{1}{2}\cos t + \frac{1}{2}\sin t = \frac{\sqrt{2}}{2}\cos(t - \phi)$

$\phi = \tan^{-1} 1 = \frac{\pi}{4}$

note: $\mathrm{Arg}(p(i\omega)) = \mathrm{Arg}(p(i)) = \mathrm{Arg}(1+i) =$  $= \frac{\pi}{4}$

gain $= g = \dfrac{1}{|p(i)|} = \dfrac{\sqrt{2}}{2}$

phase lag $= \frac{\pi}{4}$

$\ddot{x} + b\dot{x} + 2x = \cos t$ $\qquad p(r) = r^2 + br + 2$

roots $r = \dfrac{-1 \pm (b^2 - 8)^{1/2}}{2}$

$p(i) = i^2 + bi + 2 = 1 + bi$

$z_p = \dfrac{1}{1+bi} e^{it} = \dfrac{(1-bi)}{1+b^2}(\cos t + i \sin t)$

$x_p = \dfrac{1}{1+b^2}\cos t + \dfrac{b}{1+b^2}\sin t = \dfrac{1}{(1+b^2)^{1/2}}\cos(t - \tan^{-1} b)$

$\dfrac{b}{1+b^2}$ $\qquad \phi = \tan^{-1} b$

$\dfrac{1}{1+b^2}$

Amplitude decreases w/ ↑b.

Phase lag increases w/ ↑b

$\left(\dfrac{1+b^2}{(1+b^2)^2} \right)^{1/2} = \dfrac{1}{(1+b^2)^{1/2}}$

$3x^{(4)} + 2x^{(3)} + x'' - x' + 4x = 2t^2+1$

polyn. input

$p(r) = 3r^4 + 2r^3 + r^2 - r + 4$

$p(0) = 4 \neq 0$ $\Rightarrow$ $p(D)x = 2t^2+1$ has exactly one sol'n and it is polyn.

input $q(t) = 2t^2+1$, degree 2 polyn.

$3x^{(4)} + 2x^{(3)} + x'' = 2t^2+1$

$p(r) = 3x^4 + 2x^3 + x^2$

$p(0) = 0$ $\Rightarrow$ special case

Try $x_p = At^4 + Bt^3 + Ct^2$, obtain polynomial sol'n.

$p(x) = x^2(3x^2+2x+1)$ $\quad \Delta = 4 - 4\cdot 3 = -8$

$x = \frac{-2 \pm 8i}{6} = \frac{-1 \pm 4i}{3}$

0 is a double root, so $e^{at} = C$ and $te^{at} = t$ are homog. sol'ns.

$\Rightarrow$ $c_1 t + c_2$ is sol'n.

$\Rightarrow$ there are infinite polynomial general sol'ns.

$D^3 e^{-t} \sin t$

Here we have the operator $p(D) = D^3$ applied to $e^{at}u(t) = e^{-t}\sin t$

By Exponential-Shift Rule $= e^{-t} p(D-1) \sin t = e^{-t}(2\cos t + 2\sin t)$

$p(D-1) = (D-1)^3 = D^3 - 3D^2 + 3D - 1$

$p(D-1)\sin t = -\cos t + 3\sin t + 3\cos t - \sin t$

$= 2\cos t + 2\sin t$

Altern. using substit. rule

complexify $e^{-t}\sin t$ $\Rightarrow$ $e^{-t}\sin t = Im(e^{-t}e^{it}) = Im(e^{t(-1+i)})$

$D^3 e^{-t}\sin t = (-1+i)^3 e^{(-1+i)t} = (-1+3i+3-i)e^{-t}(\cos t + i\sin t) = (2+2i)e^{-t}(\cos t + i\sin t)$

subst. rule

Take Imaginary part: $2e^{-t}\sin t + 2e^{-t}\cos t = 2e^{-t}(\sin t + \cos t)$

$x_p(t) = \sqrt{2}\sin(t/2 - \pi/4)$ sol'n to $2\ddot{x} + \dot{x} + x = \sin(t/2)$

Find sol'n to $2\ddot{x} + \dot{x} + x = \sin(t/2 - \pi/3)$ input signal has a phase lag

we have $p(D) = 2D^2 + D + 1$, a const. coeff operator
$p(D)x = \sin(t/2) = q(t)$

$\Rightarrow$ sol'n to $p(D)y = q(t - \pi/3)$ is just $x(t)$ shifted as well by $\frac{\pi}{3}$ units to the right, $x(t - \pi/3)$

ie $y_p = x_p(t - \pi/3) = \sqrt{2}\sin(t/2 - \pi/4 - \pi/3)$

$x\frac{dy}{dx} = y + \sqrt{x^2 - y^2}$ $\quad y(x_0) = 0 \quad x_0 > 0$

$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 - \frac{y^2}{x^2}}$ $\quad y = vx \quad \frac{dy}{dx} = \frac{dv}{dx}x + v$

$\frac{dv}{dx}x + \cancel{v} = \cancel{v} + \sqrt{1 - v^2}$ $\quad \frac{1}{\sqrt{1-v^2}}dv = \frac{1}{x}dx$

$\Rightarrow \sin^{-1} v = \ln x + C$

note we are searching for a sol'n defined near $x_0 > 0$, so $x > 0$.

$\sin^{-1}(v(x_0)) = \ln x_0 + C$

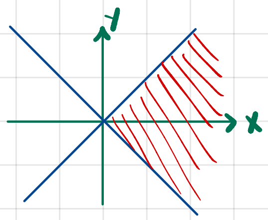
$v(x_0) = \frac{y(x_0)}{x_0} = 0 \Rightarrow 0 = \ln x_0 + C \Rightarrow C = -\ln(x_0)$

$\Rightarrow v = \frac{y}{x} = \sin(\ln x - \ln x_0) = \sin(\ln \frac{x}{x_0})$

$\Rightarrow y(x) = x\sin(\ln \frac{x}{x_0})$

Note:

$\rightarrow$ the original DE implies $x^2 - y^2 \geq 0 \Rightarrow x \geq \pm y$ $\nearrow x \geq y$ $\searrow x \geq -y$



$\rightarrow f(x,y) = y + (x^2 - y^2)^{1/2}$

$f_y = 1 - \frac{y}{(x^2-y^2)^{1/2}}$ undefined at $x^2 = y^2$, no unique sol'ns here.

in fact, $y = \pm x \Rightarrow y' = \pm 1 \Rightarrow \pm x = \pm x + 0$ ✓

also, $-x = x\sin(\ln x/x_0) \Rightarrow \sin(\ln x/x_0) = -1 \quad (x>0) \Rightarrow \ln(x/x_0) = -\frac{\pi}{2} + 2\pi k \quad k \in \mathbb{Z}$

$\Rightarrow \frac{x}{x_0} = e^{-\frac{\pi}{2} + 2\pi k} \Rightarrow x = x_0 e^{-\frac{\pi}{2} + 2\pi k}$

$\Delta P = b \cdot P \cdot \Delta t - dP\Delta t = P\Delta t(b-d)$

$\frac{\Delta P}{\Delta t} = (b-d)P \Rightarrow \dot{P} = (b-d)P$

Assume the logistic eq. is satisfied $\Rightarrow \delta(P) = b-d = A-BP$

$\dot{P} = AP - BP^2$

$0.75 = A \cdot 50 - B \cdot 50^2 \Rightarrow 1.5 = 100A - 2B \cdot 50^2$
$1 = A \cdot 100 - B \cdot 100^2$

Subtract: $0.5 = -5000B + 10000B = 5000B \Rightarrow B = 10^{-4}$
$1 = 100A - 10^{-4} \cdot 10^4$
$100A = 2 \Rightarrow A = 0.02 = 2 \cdot 10^{-2}$

$\Rightarrow \dot{P} = 2 \cdot 10^{-2}P - 10^{-4}P^2 = P(2 \cdot 10^{-2} - 10^{-4}P)$

$= 0 \Rightarrow P_{max} = \frac{2 \cdot 10^{-2}}{10^{-4}} = 2 \cdot 10^2 = 200$

$\frac{dp}{2p - 10^{-2}p^2} = \frac{1}{p(2-10^{-2}p)} = dt \cdot 10^{-2}$

$\frac{1}{p(2-10^{-2}p)} = \frac{Ap+B}{2-10^{-2}p} + \frac{C}{p}$

$C(2-10^{-2}P) + (Ap+B)p = 1$
$2C - 10^{-2}CP + Ap^2 + Bp = 1$
$Ap^2 + p(B - 10^{-2}C) + 2C = 1$

$A = 0 \quad 2C = 1 \Rightarrow C = \frac{1}{2} \qquad B - 10^{-2} \cdot \frac{1}{2} = 0 \Rightarrow B = \frac{1}{200}$

$\Rightarrow \int \left[\frac{1}{200}\frac{1}{2-10^{-2}p} + \frac{1}{2p}\right]dp = \int 10^{-2}dt$

$\frac{\ln p}{2} - \frac{1}{\cancel{200}_{2}} \cdot \cancel{100} \cdot \ln(200-p) = \frac{t}{100} + C$

$\frac{1}{2}\left(\ln\frac{p}{200-p}\right) = \frac{t}{100} + C$

$\frac{p}{200-p} = e^{\frac{t}{50}+2C}$

$p(1 + e^{\frac{t}{50}+2C}) = 200e^{\frac{t}{50}+2C}$

$p(t) = \frac{200e^{\frac{t}{50}+2C}}{1+e^{t/50+2C}}$

$p(0) = \frac{200e^{2C}}{1+e^{2C}} = 50 \Rightarrow e^{2C} \cdot 4 = 1 + e^{2C} \Rightarrow 3e^{2C} = 1 \Rightarrow 2C = \ln\frac{1}{3} = -\ln 3 \Rightarrow C = \frac{-\ln 3}{2}$

$p(t) = \frac{200e^{t/50 - \ln 3}}{1+e^{t/50 - \ln 3}}$

$p(115) = 153.75$