PSet - Exponentials - Part I

Problem 1

a) $\dot{x} + 2x = e^{3t}$

guess $x_p(t) = Ae^{3t}$, $x_p'(t) = 3Ae^{3t}$

$3Ae^{3t} + 2Ae^{3t} = e^{3t} \Rightarrow Ae^{3t}(2+3) = e^{3t} \Rightarrow A = \frac{1}{5}$

$\Rightarrow x_p(t) = \frac{1}{5}e^{3t}$

$\Rightarrow x(t) = \frac{1}{5}e^{3t} + Ce^{-2t}$

$\dot{x} + 2x = 0 \Rightarrow x_h(t) = Ce^{-2t}$

b) $\dot{x} + 2x = e^{3it}$

guess $x_p(t) = Ae^{3it}$, $x_p'(t) = 3iAe^{3it}$

$\Rightarrow 3iAe^{3it} + 2Ae^{3it} = e^{3it} \Rightarrow Ae^{3it}(2+3i) = e^{3it} \Rightarrow A = \frac{1}{2+3i}$

$\Rightarrow x_p(t) = \frac{1}{2+3i}e^{3it}$

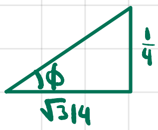
$\Rightarrow x(t) = \frac{1}{2+3i}e^{3it} + Ce^{-2t}$

$x_h(t) = Ce^{-2t}$

Part II

Problem 1

a) $\text{Re}\left(\frac{e^{3it}}{\sqrt{3}+i}\right) = \text{Re}\left(\frac{\sqrt{3}-i}{4}(\cos 3t + i\sin 3t)\right) = \frac{\sqrt{3}}{4}\cos(3t) + \frac{1}{4}\sin(3t) = \frac{1}{2}\cos(3t - \pi/6)$



$\phi = \tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}$ $\quad r = \left(\frac{3}{16} + \frac{1}{16}\right)^{1/2} = \frac{1}{2}$

* Alternatively, using polars:

$z = \frac{\sqrt{3}-i}{4} = \frac{\sqrt{3}}{4} - \frac{1}{4}i = \frac{1}{2}e^{-i\frac{\pi}{6}}$

$|z| = \left(\frac{3+1}{16}\right)^{1/2} = 1/2$

$\phi = \tan^{-1}\left(\frac{-1/4}{\sqrt{3}/4}\right) = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

$\Rightarrow \phi = -\pi/6$

$\Rightarrow \text{Re}\left(\frac{e^{3it}}{\sqrt{3}+i}\right) = \text{Re}\left(\frac{1}{2}e^{i(3t-\frac{\pi}{6})}\right) = \text{Re}\left(\frac{1}{2}\cos(3t-\frac{\pi}{6}) + i\frac{1}{2}\sin(3t-\frac{\pi}{6})\right) = \frac{1}{2}\cos(3t - \pi/6)$

b) $\dot{z} + 3z = e^{2it}$

guess $z_p(t) = Ae^{i2t}$, $z_p'(t) = Ae^{i2t} \cdot 2i$

$Ae^{i2t} \cdot 2i + 3Ae^{i2t} = e^{i2t}$

$Ae^{i2t}(2i+3) = e^{i2t} \Rightarrow A = \frac{1}{3+2i} = \frac{3-2i}{13} = \frac{3}{13} - \frac{2}{13}i$

$\Rightarrow z_p(t) = \left(\frac{3}{13} - i\frac{3}{13}\right)e^{i2t}$

$z(t) = z_p(t) + Ce^{-2t}$

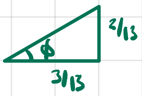
c) $\dot{x} + 3x = \cos(2t)$

$x(t)$ is the real part of the solution $\tilde{x}(t)$ to $\dot{\tilde{x}} + 3\tilde{x} = e^{2it}$

$\Rightarrow x(t) = \mathrm{Re}\left(\left(\frac{3}{13} - i\frac{3}{13}\right)e^{i2t}\right)$

$\left(\frac{3}{13} - i\frac{3}{13}\right)e^{i2t} = \left(\frac{3}{13} - i\frac{2}{13}\right)(\cos 2t + i\sin 2t)$

real part: $\frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t = \frac{\sqrt{13}}{13}\cos(2t - \tan^{-1}(2/3))$



$\phi = \tan^{-1}(2/3)$

$r = \left(\frac{4+9}{13^2}\right)^{1/2} = \frac{1}{\sqrt{13}}$