Recap

1. Damped Harmonic Oscillator $x''+bx'+kx=F_{ext}(t)$

2. Simple Harmonic Oscillator $x''+kx=F_{ext}(t)$

1. a) homog. case

$x(t)=Ae^{rt} \Rightarrow p(r)=r^2+br+k=0$

$$r=\frac{-b\pm\sqrt{b^2-4k}}{2}$$

real roots: $x_h(t)=c_1e^{r_1t}+c_2e^{r_2t}$

repeated root r_1:

$x(t)=v(t)e^{r_1t} \Rightarrow (D^2+bD+k)v(t)e^{r_1t}=0$

$=D^2(ve^{r_1t})+bD(ve^{r_1t})+k(ve^{r_1t})$

$=v''e^{r_1t}+vr_1^2e^{r_1t}+bv'e^{r_1t}+bvr_1e^{r_1t}+kve^{r_1t}$

$=e^{r_1t}(v''+bv'+kv+vr_1^2+bvr_1)$

$=e^{r_1t}p(D)v+ve^{r_1t}\cancelto{0}{p(r_1)}=0$

$\Rightarrow p(D)v=0$

$\Rightarrow v''+bv'+k=0$

$v''=0,\ v'=-k/b \Rightarrow v(t)=\frac{-kt}{b}+c$

$b^2=4k$

$k=b^2/4$

$v=\frac{-b^2t}{4b}=\frac{-bt}{4}$

* $p(D)=(D-r_1)^2=D^2-2Dr_1+r_1^2$ *note $r_1^2=\frac{(-2r_1)^2}{4}$

$p(D)x=p(D)v(t)e^{r_1t}$

$(D-r_1)ve^{r_1t}=v'e^{r_1t}+vr_1e^{r_1t}-vr_1e^{r_1t}=v'e^{r_1t}$

$(D-r_1)^2ve^{r_1t}=v''e^{r_1t}$

$p(D)x=0 \Rightarrow v''=0 \Rightarrow v(t)=at+b$

e.g. $v(t)=t \Rightarrow te^{r_1t},\ e^{r_1t}$ two l.i. sol'ns

How do we know they are l.i.?

$c_1te^{r_1t}+c_2e^{r_1t}=0$ ↙ true because superpos.

$c_1(e^{r_1t}+tr_1e^{r_1t})+c_2r_1e^{r_1t}=0$ ↙ simple differen.

l.i. $\Leftrightarrow$ only sol'n (c_1,c_2) is $(0,0)$

$\Leftrightarrow \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}\neq 0$ $=tr_1e^{2r_1t}-(e^{r_1t}+tr_1e^{r_1t})e^{r_1t}$ $=\cancel{tr_1e^{2r_1t}}-e^{2r_1t}-\cancel{tr_1e^{2r_1t}}\neq 0$

$$\begin{vmatrix} te^{r_1t} & e^{r_1t} \\ e^{r_1t}+tr_1e^{r_1t} & r_1e^{r_1t} \end{vmatrix}$$

$x_h(t)=c_1te^{r_1t}+c_2e^{r_1t}$

complex roots

$x_1(t)=e^{-\frac{b}{2}+\frac{\sqrt{4k-b^2}}{2}i}$

$=e^{\frac{-b}{2}}\left(\cos\left(\frac{\sqrt{4k-b^2}}{2}\right)+i\sin\left(\frac{\sqrt{4k-b^2}}{2}\right)\right)$

$x_2(t)=e^{\frac{-b}{2}}\cos\frac{\sqrt{4k-b^2}}{2}$

$x_3(t)=e^{\frac{-b}{2}}\sin\frac{\sqrt{4k-b^2}}{2}$

$x_h(t)=e^{\frac{-b}{2}}\left(c_1\cos\frac{\sqrt{4k-b^2}}{2}+c_2\sin\frac{\sqrt{4k-b^2}}{2}\right)$

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1. b) non-homog. case, exponential input

$p(D)x=e^{\alpha x}$

$p(\alpha)\neq 0 \Rightarrow x_p(t)=\frac{e^{\alpha x}}{p(\alpha)}$

$p(\alpha)=0,\ p'(\alpha)\neq 0 \Rightarrow x_p(t)=\frac{te^{\alpha x}}{p'(\alpha)}$

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2. b) non-homog., input degree n polynomial

2. $x'' + \omega_0^2 x = F_0 \cos \omega t$

Homog. Case $x'' + \omega_0^2 x = 0$

$p(r) = r^2 + \omega_0^2 = 0 \Rightarrow r = \pm \omega_0 i$

$x_1 = e^{\omega_0 i t} = \cos \omega_0 t + i \sin \omega_0 t$

natural freq.

$x_h(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t = A\cos(\omega_0 t - \phi)$

note: $c_1 e^{\omega_0 t i} + c_2 e^{-\omega_0 t i}$

$c_1 \omega_0 i e^{\omega_0 t i} - c_2 \omega_0 i e^{\omega_0 t i}$

$$W(x_1, x_2) = \begin{vmatrix} e^{\omega_0 t i} & e^{-\omega_0 t i} \\ \omega_0 i e^{\omega_0 t i} & -\omega_0 i e^{-\omega_0 t i} \end{vmatrix}$$

$= -\omega_0 i e^0 - \omega_0 i e^0 = -2\omega_0 i$

so the complex sol'ns are l.i.
However

$e^{\omega_0 t i} = \cos \omega_0 t + i \sin \omega_0 t$

$e^{-\omega_0 t i} = \cos(-\omega_0 t) + i \sin(-\omega_0 t)$

$= \cos(\omega_0 t) - i \sin(\omega_0 t)$

The real sol'ns they provide are the same.

2. b) non-homog., sinusoidal input

$y'' + \omega_0^2 y = B \cos \omega t$

$z'' + \omega_0^2 z = B e^{i\omega t}$, $p(i\omega) = -\omega^2 + \omega_0$

$z_p = \frac{B e^{i\omega t}}{\omega_0^2 - \omega^2}$ if $\omega \neq \omega_0$

$y_p = \text{Re}\left(\frac{B(\cos \omega t + i \sin \omega t)}{\omega_0^2 - \omega^2}\right) \Rightarrow y_p = \frac{B \cos \omega t}{\omega_0^2 - \omega^2}$

if $\omega = \omega_0$ we have resonance and $p'(r) = 2r$, $p'(i\omega) = 2i\omega$

$z_p = \frac{t B e^{i\omega t}}{2\omega i}$

$y_p = \text{Re}\left(\frac{tB(\cos \omega t + i \sin \omega t)}{2\omega i}\right)$

$= \text{Re}\left(\frac{-2\cancel{\omega} i \cdot tB(\cos \omega t + i \sin \omega t)}{4\omega^{\cancel{2}}}\right)$

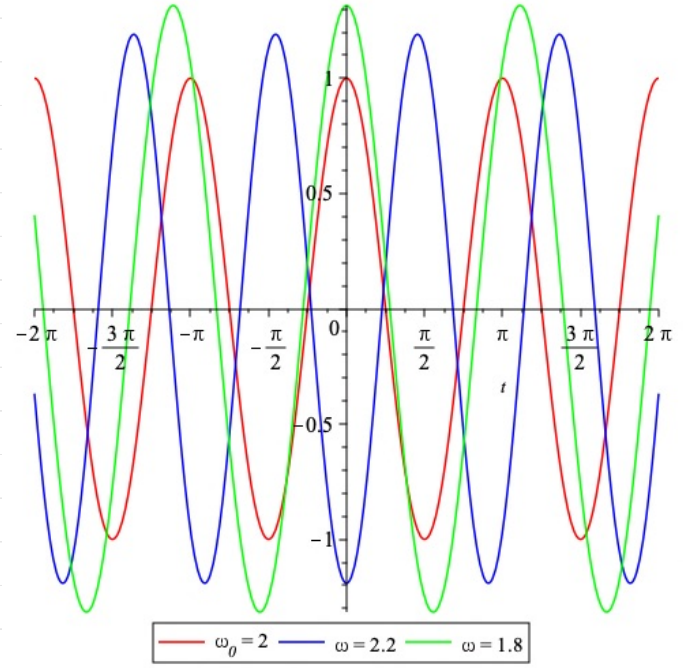
$y_p = \frac{tB \sin \omega t}{2\omega}$

we have the non-resonance case a) $y_p = \frac{B \cos \omega t}{\omega_0^2 - \omega^2}$

and the resonance case b) $y_p = \frac{tB \sin \omega t}{2\omega}$

a) $y_h' = -\omega_0 c_1 \sin \omega_0 t + \omega_0 c_2 \cos \omega_0 t$

$y_h(0) = c_1$ $\quad y_h'(0) = \omega_0 c_2$ $\quad y(t) = y_p \Rightarrow y_h = 0$



$y_h \neq 0 \Rightarrow y(t)$ is something like $\frac{B \cos \omega t}{\omega_0^2 - \omega^2} - \frac{\cos \omega_0 t}{\omega_0^2 - \omega^2}$

$c_1 = -(\omega_0^2 - \omega^2)^{-1}$

$c_2 = 0$

Take $B = 1$ for example.
Use $\cos B - \cos A = 2 \sin\left(\frac{A-B}{2}\right) \sin\left(\frac{A+B}{2}\right)$

$$\frac{\cos \omega t - \cos \omega_0 t}{\omega_0^2 - \omega^2} = \underbrace{\frac{2}{\omega_0^2 - \omega^2} \sin\left(\frac{t(\omega_0 - \omega)}{2}\right)}_{\text{amplitude}} \underbrace{\sin\left(\frac{t(\omega_0 + \omega)}{2}\right)}_{\text{sinusoid}}$$

$\omega \to \omega_0 \Rightarrow \downarrow$ period