Superposition

y_1, y_2 sol'ns of linear, homog. equation ⇒ any linear combin. of y_1, y_2 is also a solution

note: the coeff. don't have to be constant. Linearity is the key.

now consider

$$mx'' + bx' + kx = F_{ext}(t) \quad \text{and} \quad mx'' + bx' + kx = 0$$

x_p sol'n to inhomog. eq.
x_h sol'n to homog. eq. ⇒ $x_p + x_h$ sol'n to inhomog. eq.

Exponential Input Ae^{at}, A, a complex constants

$mx'' + bx' + kx = Be^{at}$ B, a constants

guess $x(t) = Ae^{at}$

$mAa^2e^{at} + bAae^{at} + kAe^{at} = Be^{at}$

$A(ma^2 + ba + k) = B \Rightarrow A = \dfrac{B}{ma^2 + ba + k} = \dfrac{B}{p(a)}$, $p(r) = mr^2 + br + k$ the characteristic polynomial

$\Rightarrow x_p(t) = \dfrac{B}{p(a)}e^{at}$ $p(a) \neq 0$

this is a single particular solution

Example $x'' + 8x' + 7x = 9\cos(2t)$

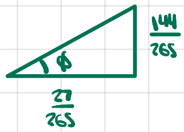
sinusoidal input

$\cos 2t = Re(e^{i2t})$

$z'' + 8z' + 7z = 9e^{i2t}$ $a = 2i,\ B = 9$

$$z_p = \frac{9}{a^2 + 8a + 7}e^{i2t} = \frac{9}{-4 + 16i + 7}e^{i2t} = \frac{9}{3 + 16i}e^{i2t} = \frac{9(3 - 16i)}{265}e^{i2t} = \frac{(27 - 144i)(\cos 2t + i\sin 2t)}{265}$$

$$x_p(t) = \frac{27}{265}\cos 2t + \frac{144}{265}\sin 2t = \frac{9\sqrt{265}}{265}\cos(2t - \phi)$$



$\tan\phi = \dfrac{144}{27} \Rightarrow \phi = \tan^{-1}\dfrac{16}{3}$

$x_h(t) = C_1e^{-7t} + C_2e^{-t}$

$$\Rightarrow x(t) = \frac{9\sqrt{265}}{265}\cos(2t - \phi) + C_1e^{-7t} + C_2e^{-t}$$

Interpretation: Sinusoidal input with amplitude 9. Output amplitude is $9 \cdot \underbrace{\dfrac{1}{|p(a)|}}_{\text{gain of the system}}$

output amplitude = input amplitude × gain

Simple Harmonic Oscillator

$mx'' + kx = F_{ext}(t)$

$\omega_n = \sqrt{k/m} \Rightarrow mx'' + m\frac{k}{m}x = F_{ext}(t) = m(x'' + \omega_n^2 x)$

n stands for "natural"

$\rightarrow F_{ext} = 0 \Rightarrow m(x'' + \omega_n^2 x) = 0 \quad p(r) = m(r^2 + \omega_n^2) = 0 \Rightarrow r = \pm \omega_n i$

$\Rightarrow x_h(t) = e^{\omega_n t i} = \cos(\omega_n t) + i\sin(\omega_n t)$

or $x_h(t) = c_1 \cos \omega_n t + c_2 \sin \omega_n t$, a sinusoidal output; ie, even w/o an external force the system oscillates with a natural frequency ω_n

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$F_{ext} = B\cos(\omega t) \Rightarrow m(x'' + \omega_n^2 x) = B\cos(\omega t)$

$m(z'' + \omega_n^2 z) = Be^{i\omega t}$

As seen previously, if we guess $z_p(t) = Ae^{\omega i t}$ then $A = \frac{B}{p(\omega i)} = \frac{B}{m((\omega i)^2 + \omega_n^2)} = \frac{B}{m(\omega_n^2 - \omega^2)}$

so $z_p(t) = \frac{B}{m(\omega_n^2 - \omega^2)} e^{\omega i t}$

Taking the real part, $x_p(t) = \frac{B}{m(\omega_n^2 - \omega^2)} \cos(\omega t)$

$x(t) = \frac{B}{m(\omega_n^2 - \omega^2)} \cos(\omega t) + c_1 \cos(\omega_n t) + c_2 \sin(\omega_n t)$

The gain is $\frac{1}{|p(a)|} = \frac{1}{m(\omega_n^2 - \omega^2)} = \frac{1}{|p(\omega i)|}$

we can view gain as a function of ω. As $\omega \to \omega_n$ we get pure resonance. $p(a) = p(i\omega) = 0$ in this case, the solution we derived above isn't correct. In its derivation we used $p(a) \neq 0$.

A sol'n in this case is $x_p(t) = \frac{B}{2m\omega} t\sin(\omega t)$

Amplitude grows in time.

Resonant Response Formula

$mx'' + bx' + kx = Be^{at}$ with characteristic polynomial p

$p(a) = 0,\ p'(a) \neq 0 \Rightarrow x_p(t) = \frac{B}{p'(a)} te^{at}$

Recap

→ how to find $x_p(t)$ satisfying $y''+Ay'+By=f(x)$

→ if we guess $x_p(t)=Ae^{at}$ then we have a few cases to consider (note $a\in\mathbb{C}$)

→ p is a linear operator. For the eq. above $p(r)=r^2+Ar+B$, $p(D)=D^2+AD+B$, and $p(D)y=f(x)$

→ if $p(a)=a^2+Aa+B\neq 0$ then $x_p(t)=\dfrac{e^{ax}}{p(a)}$. This is the exponential input theorem.

→ we can apply this result when the input is sinusoidal. The latter is the real part of a complex exponential. We find the z_p of the complex non-homog. ODE using the exponential input theorem and take the real part to get x_p.

→ we note that x_p is also sinusoidal and the amplitude is a multiple of the input sinusoid amplitude. The factor is $1/|p(a)|$, the gain.

→ we then consider the special case of the simple harmonic oscillator, with sinusoidal input. Nothing special here: we apply the theorem to get x_p. The amplitude is gain times input amplitude. The closer the frequency of input is to the natural frequency, the larger the amplitude.

we can't use this solution if the frequencies are the same. How does this case come about?

when solving the complexified ODE, the exponential input is $Be^{i\omega t}$

$$mz''+kz=Be^{i\omega t}$$

$$mz''+m\frac{k}{m}z=mz''+m\omega_n^2z=Be^{i\omega t}$$

$$m(z''+\omega_n^2z)=Be^{i\omega t}=p(D)z \qquad p(r)=m(r^2+\omega_n^2)$$

$$p(D)z=m(D^2+\omega_n^2)z$$

$$z=Ae^{\omega it}\Rightarrow p(D)z=m(Ae^{\omega it}(\omega i)^2+\omega_nAe^{\omega it})=mAe^{\omega it}(\omega_n-\omega^2)=Be^{i\omega t}$$

$$\Rightarrow A\underbrace{m(\omega_n^2-\omega^2)}_{p(\omega i)}=B$$

* $p'(r)=m(2r)$, $p'(i\omega)=2m\omega i\neq 0$ for $m,\omega>0$

$\Rightarrow$ all $z_p=\dfrac{B}{(\omega_n^2-\omega^2)m}$

if $p(\omega i)=0$ we don't have the exponential input theorem we've seen thus far.

How to proceed?

Therefore $\dfrac{ze^{i\omega t}}{p'(\omega i)}$ is a z_p when $p(\omega i)=0$ (but $p'(i\omega)\neq 0$), ie ωi is a single root of p, p the linear operator defining the ODE.

exponential shift rule: $p(D)e^{ax}u(x)=e^{ax}p(D+a)u(x)$

we use this to show that $\dfrac{e^{ax}x}{p'(a)}$ is a solution when $p(a)=0$ and a is a simple root, ie $p'(a)\neq 0$.

$p(D)z=p(D)\dfrac{e^{ax}x}{p'(a)}$, $p(D)$ has two roots a and b so $p(D)=(D-b)(D-a)$

$p'(D)=2D-a-b$. Now $p(D)e^{ax}\left(\dfrac{x}{p'(a)}\right)=e^{ax}p(D+a)\dfrac{x}{p'(a)}=e^{ax}(D+a-b)D\dfrac{x}{p'(a)}=e^{ax}\dfrac{(a-b)}{a-b}=e^{ax}$

Next we want to show that if $i\omega$ is a double root of p then $z_p = \dfrac{x^2 e^{\omega i x}}{p''(i\omega)}$

$i\omega$ double root $\Rightarrow$ $p(D) = (D - i\omega)^2$

$p'(D) = 2(D - i\omega)$

$p''(D) = 2$

Try the proposed sol'n $z_p = \dfrac{x^2 e^{\omega i x}}{p''(\omega i)}$

$p(D) z_p = p(D) e^{\omega i x} \dfrac{x^2}{p''(\omega i)}$

(expon. shift rule) $\Rightarrow$ $= e^{\omega i x} p(D + \omega i) \dfrac{x^2}{p''(\omega i)} = e^{\omega i x} D^2 \dfrac{x^2}{2} = e^{\omega i x}$

$p(D + \omega i) = (D + i\omega - i\omega)^2 = D^2$

Recap calculations

$y'' + Ay' + By = Ke^{\alpha x} \quad \alpha \in \mathbb{C}$

$p(D)y = e^{\alpha x} \qquad p(D) = D^2 + AD + B$

$y = Ce^{\alpha x} \Rightarrow Ce^{\alpha x}\alpha^2 + ACe^{\alpha x}\alpha + BCe^{\alpha x} = Ce^{\alpha x}(\alpha^2 + A\alpha + B) = Ce^{\alpha x}p(\alpha)$

ie $p(D)Ce^{\alpha x} = p(\alpha)Ce^{\alpha x} = Ke^{\alpha x} \Rightarrow C = \frac{K}{p(\alpha)}$ if $p(\alpha) \neq 0$

$p(\alpha) = \alpha^2 + A\alpha + B$

if $A = 0$ then $p(\alpha) = \alpha^2 + B = 0 \Rightarrow \alpha = \pm\sqrt{B}\,i$

Ex: $y'' + 4y = e^{2it} \qquad p(D) = D^2 + 4$

$y = Ae^{2it}$
$y' = 2iAe^{2it}$
$y'' = -4Ae^{2it}$

$\Rightarrow -4Ae^{2it} + 4Ae^{2it} = 0 \neq e^{2it}$
$= Ae^{2it}((2i)^2 + 4)$
$Ae^{2it}\,p(2i)$
$p(2i) = 0$

$y = Ate^{2it}$
$y' = A(e^{2it} + 2ite^{2it})$
$y'' = A(2ie^{2it} - 4te^{2it} + 2ie^{2it}) = A(4ie^{2it} - 4te^{2it}) = A4e^{2it}(i - t)$

$A4e^{i2t}(i-t) + A4te^{2it} = A4e^{2it}(i - t + t) = A4e^{2it}i = e^{2it} \Rightarrow A = \frac{1}{4i} = \frac{1}{p'(2i)}$

$p'(r) = 2r$
$p'(2i) = 4i$

$A \neq 0 \Rightarrow y'' + Ay' + By = Ke^{\alpha x} \qquad p(D)y = p(D)Ae^{\alpha x} = Ae^{\alpha x}p(\alpha) = Ke^{\alpha x}$

$p(\alpha) = \alpha^2 + A\alpha + B = 0 \Rightarrow \alpha = \frac{-A \pm (A^2 - 4B)^{1/2}}{2}$

$p'(\alpha) = 2\alpha + A$

i) Two real roots: $A^2 - 4B > 0 \Rightarrow p(\alpha) = 0,\ p'(\alpha) \neq 0$
ii) one (double) root $A^2 = 4B \Rightarrow \alpha = -A/2 \Rightarrow p(\alpha) = p'(\alpha) = 0$
iii) two complex roots $A^2 < 4B$

i) $A^2 - 4B > 0$, so there are two values α_1, α_2 for which $p(\alpha) = 0$.

$\alpha \neq \alpha_1, \alpha_2 \Rightarrow y_p = \frac{K}{p(\alpha)}e^{\alpha t}$

$\alpha = \alpha_1, \alpha_2 \Rightarrow y_p = \frac{Kt}{p'(\alpha)}e^{\alpha t} = \frac{Kt}{2\alpha + A}e^{\alpha t}$

ii) $A^2 - 4B = 0$, $\alpha = -A/2$. This would be the critically damped case in homog. eq.
$p(\alpha) = p'(\alpha) = 0$, $y_p = \frac{Kt^2}{p''(\alpha)}e^{\alpha t}$