Characteristic Equations

1. $\ddot{x}+\omega^2 x=0$ (harmonic oscillator)

$x=\cos\omega t$
$\dot{x}=-\omega\sin\omega t \Rightarrow -\omega^2\cos\omega t-\omega^2\cos\omega t=0$
$\ddot{x}=-\omega^2\cos\omega t$

$x=\sin\omega t$
$\dot{x}=\omega\cos\omega t \Rightarrow -\omega^2\sin\omega t+\omega^2\sin\omega t=0$
$\ddot{x}=-\omega^2\sin\omega t$

2.
$x=A\cos(\omega t-\phi)$
$\dot{x}=-A\omega\sin(\omega t-\phi)$
$\ddot{x}=-A\omega^2\cos(\omega t-\phi)$
$\Rightarrow -A\omega^2\cos(\omega t-\phi)+\omega^2 A\cos(\omega t-\phi)=0$

3. $x(t)=A\cos(\omega t-\phi)$
$x(0)=A\cos(-\phi)=0 \Rightarrow \phi=\frac{\pi}{2}+k\pi \quad k\in\mathbb{Z}$
or $A=0$

Uniqueness theorem seen thus far applies to first order eq.

4. Given $x(0)=x_0$, $\dot{x}(0)=\dot{x}_0$, solve $\ddot{x}+\omega^2 x=0$

$x(t)=c_1\cos(\omega t)+c_2\sin(\omega t) \Rightarrow x(0)=c_1=x_0$

$x'(t)=-\omega c_1\sin\omega t+\omega c_2\cos\omega t \Rightarrow \dot{x}(0)=\omega c_2=\dot{x}_0 \Rightarrow c_2=\frac{\dot{x}_0}{\omega}$

$x(t)=x_0\cos\omega t+\frac{\dot{x}_0}{\omega}\sin\omega t$

one solution (one pair of c_1, c_2)

5. r a constant.

e^{rt} sol'n to $\ddot{x}+kx=0 \Rightarrow r=\pm\sqrt{-k}$

6. $\ddot{x}-a^2x=0$ ↗ a spring with positive spring constant

$x_1(0)=1$ ↘ starts stretched

$\dot{x}_1(0)=0$ ↘ but not moving

$\Rightarrow r=\pm\sqrt{-(-a^2)}=\pm a$

$\Rightarrow x(t)=c_1e^{at}+c_2e^{-at}$
$x(0)=c_1+c_2=1 \Rightarrow c_1=\frac{1}{2},\ c_2=\frac{1}{2}$
$\dot{x}(t)=ac_1e^{at}-ac_2e^{-at}$
$\dot{x}(0)=ac_1-ac_2=0 \Rightarrow c_1=c_2$

$x(t)=\frac{e^{at}+e^{-at}}{2}$ (explosive)

$x_2(0)=0$ ↗ starts at equil.

$\dot{x}_2(0)=1$ ↗ but becoming stretched

$c_1+c_2=0 \Rightarrow c_1=-c_2$
$ac_1-ac_2=1 \Rightarrow a(c_1-c_2)=1$
$a(c_1-(-c_1))=1 \Rightarrow 2ac_1=1 \Rightarrow c_1=\frac{1}{2a}$
$c_2=-\frac{1}{2a}$

$x(t)=\frac{e^{at}-e^{-at}}{2a}$ ↘ does not exist if $a=0$

* $\ddot{x}+kx=0$
$x(t)=e^{rt}$
$e^{rt}\cdot r^2+ke^{rt}=0$
$e^{rt}(r^2+k)=0$
$r^2=-k$
$r=\pm\sqrt{-k} \Rightarrow r=\pm\sqrt{k}\,i$

$=\cos(\sqrt{k}t)+i\sin(\sqrt{k}t)$

$k<0 \Rightarrow e^{t\sqrt{k}i}=e^{t\sqrt{|k|}}$
$e^{t(-\sqrt{k})i}=e^{t(-\sqrt{|k|})}$

$x(t)=c_1e^{t\sqrt{|k|}}+c_2e^{-t\sqrt{|k|}}$

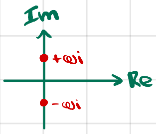
$k>0$ (hooke's law case)
$\Rightarrow e^{\sqrt{k}ti}$, apply Euler's Formula

$x(t)=c_1\cos(\sqrt{k}t)+c_2\sin(\sqrt{k}t)$
$=A\cos(\sqrt{k}t-\phi)$ ↘ sinusoidal general sol'n

Damped Oscillators

1. $\ddot{x}+\omega^2 x=0$ (simple harm. oscill.)

charact. eq. $r^2+\omega^2=0$



roots: $r=\pm\omega i$

exp. sol'ns: $e^{i\omega t}$, $e^{i(-\omega t)}$

$e^{i\omega t}=\cos\omega t+i\sin\omega t$
$e^{i(-\omega t)}=\cos(-\omega t)+i\sin(-\omega t)=\cos(\omega t)-i\sin(\omega t)$

general sol'n: $c_1\cos\omega t+c_2\sin\omega t$

2. $y(t)=e^{-t/2}\cos(3t)$ sol'n to $m\ddot{x}+b\dot{x}+kx=0$ $m,b,k\in\mathbb{R}$

a) This solution is real, and has the form of a basic real solution when roots are complex.

$\Rightarrow b^2-4mk<0$

b) $e^{-\frac{t}{2}}(\cos 3t+i\sin 3t)=e^{-\frac{t}{2}}e^{i3t}=e^{t(-\frac{1}{2}+3i)}=e^{(\frac{-b}{2m}+i\omega_d)t}$, $e^{t(-\frac{1}{2}\pm 3i)}$

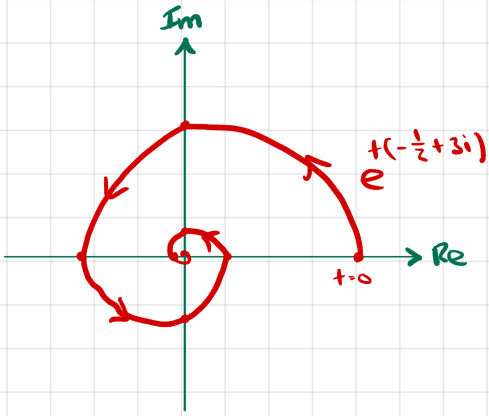
$\frac{-b}{2m}=-\frac{1}{2}\Rightarrow b=m$

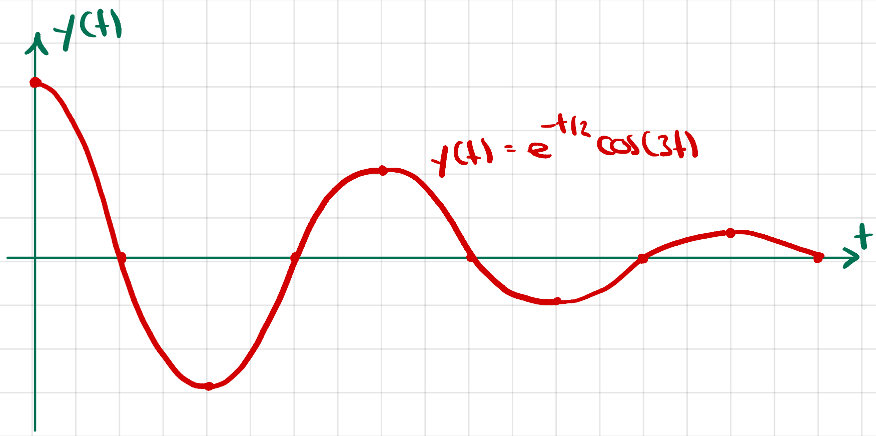
$3=\omega_d=\frac{\sqrt{b^2-4mk}}{2m}\Rightarrow 6m=(m^2-4mk)^{1/2}\Rightarrow 36m^2=m^2-4mk\Rightarrow 35m^2+4mk=0$

$\Rightarrow m(35m+4k)=0\Rightarrow k=-\frac{35}{4}m$ $\quad b=m=-\frac{4}{35}k$

$mr^2+br+k=0$

c)





d) $y(t)=e^{-\frac{t}{2}}(c_1\cos 3t+c_2\sin 3t)$

3. $\omega > 0$

3. Let $\omega > 0$. A damped sinusoid $x(t) = Ae^{-at}\cos(\omega t)$ has "pseudo-period" $2\pi/\omega$. The pseudo-period, and hence ω, can be measured from the graph: it is twice the distance between successive zeros of $x(t)$, which is always the same. Now what is the spacing between successive maxima of $x(t)$? Is it always the same, or does it differ from one successive pair of maxima to the next?

$x(t) = Ae^{-at}\cos(\omega t)$

pseudo-period $\frac{2\pi}{\omega}$

For any $t = t_1$ s.t. $x(t_1) = 0$ we have

$x(t_1) = Ae^{-at_1}\cos(\omega t_1) = 0 \Rightarrow \omega t_1 = \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z} \Rightarrow t_1(k) = \frac{\pi + 2\pi k}{2\omega}$

$t_1(k+1) - t_1(k) = \frac{\pi}{\omega}$

$x(t)$ reaches a local max when $x'(t) = 0$ and $x''(t) < 0$

$x'(t) = -a\cancel{A}\cancel{e^{-at}}\cos\omega t - \cancel{A}\cancel{e^{-at}}\omega\sin\omega t \Rightarrow -a\cos\omega t = \omega\sin\omega t \Rightarrow \frac{\sin\omega t}{\cos\omega t} = \tan\omega t = -\frac{a}{\omega}$

$\Rightarrow t(k) = \frac{\tan^{-1}(-a/\omega) + k\pi}{\omega} \quad k \in \mathbb{Z}$

$t(k+1) - t(k) = \frac{\tan^{-1}(-a/\omega)}{\omega} + \frac{(k+1)\pi}{\omega} - \frac{\tan^{-1}(-a/\omega)}{\omega} - \frac{k\pi}{\omega} = \frac{\pi}{\omega}$

spacing between successive critical points, ie between a max and a min & vice versa.

$\Rightarrow \frac{2\pi}{\omega}$ distance between two successive maxima

$x''(t) = a^2Ae^{-at}\cos\omega t + aAe^{-at}\omega\sin\omega t + aAe^{-at}\omega\sin\omega t - Ae^{-at}\omega^2\cos\omega t$

$Ae^{-at}\left((a^2-\omega^2)\cos\omega t + (a\omega + a\omega)\sin\omega t\right) < 0$

$\Rightarrow \cos\omega t(a^2-\omega^2) < -\sin\omega t \cdot 2a\omega$

$-\frac{\pi}{2} < \omega t < \frac{\pi}{2}$

Here we consider two cases

1. $\cos\omega t > 0 \Rightarrow -\frac{\pi}{2} < \omega t < \frac{\pi}{2} \Rightarrow \frac{-\sin\omega t}{\cos\omega t} > \frac{a^2-\omega^2}{2a\omega} \Rightarrow -\tan\omega t > \frac{a^2-\omega^2}{2a\omega} \Rightarrow -\infty < \tan\omega t < \frac{\omega^2-a^2}{2a\omega}$

$\Rightarrow -\frac{\pi}{2\omega} < t < \frac{\tan^{-1}\left[\frac{\omega^2-a^2}{2a\omega}\right]}{\omega}$

2. $\cos\omega t < 0 \Rightarrow \frac{\pi}{2} < \omega t < \frac{3\pi}{2} \Rightarrow \frac{-\sin\omega t}{\cos\omega t} < \frac{a^2-\omega^2}{2a\omega} \Rightarrow -\tan\omega t < \frac{a^2-\omega^2}{2a\omega} \Rightarrow \tan\omega t > \frac{\omega^2-a^2}{2a\omega}$

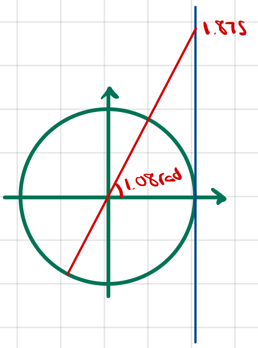
$\Rightarrow t > \frac{\tan^{-1}\left[\frac{\omega^2-a^2}{2a\omega}\right]}{\omega}$

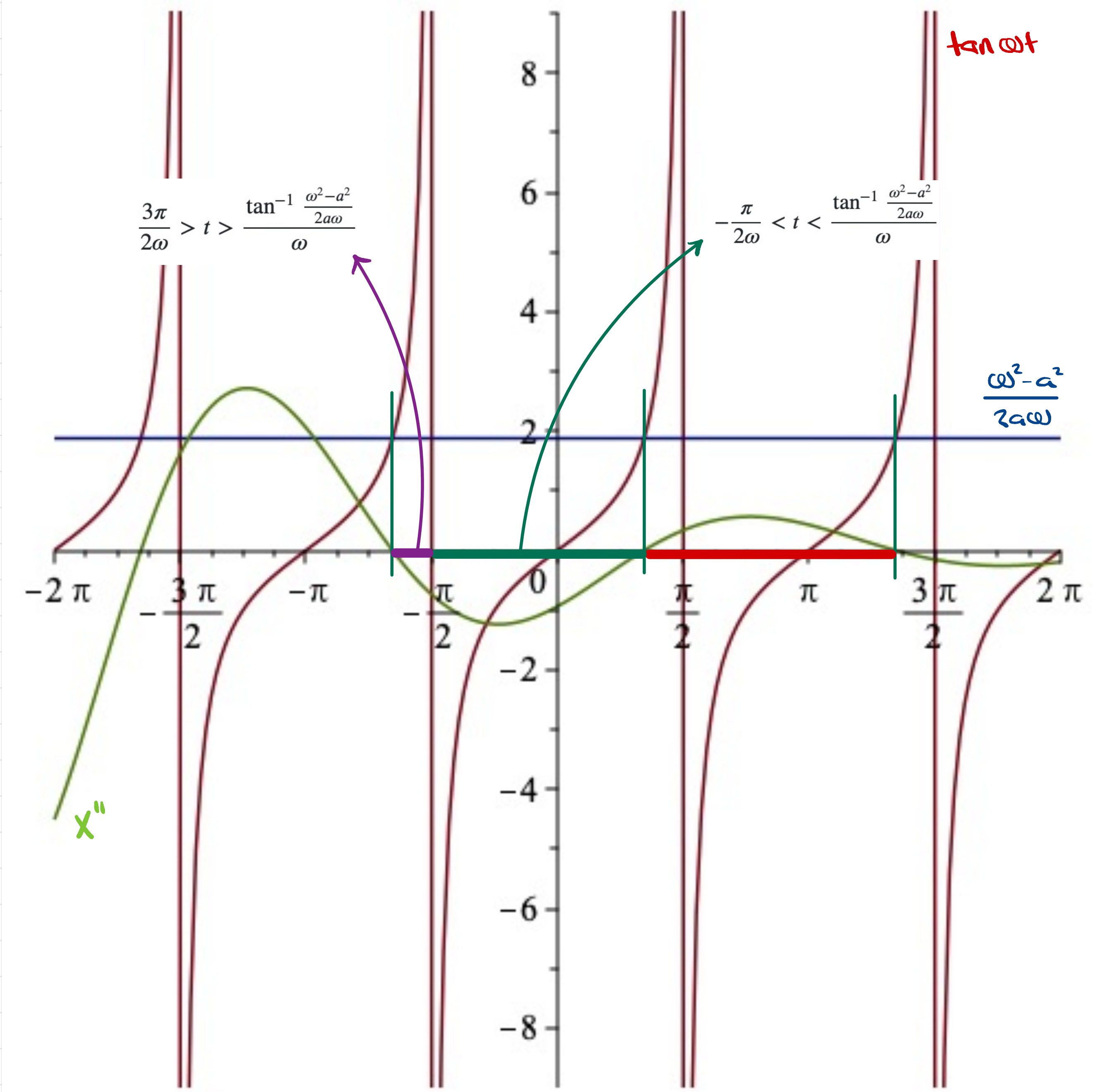
Ex: $x(t) = e^{-.25t}\cos(t)$

$a = 0.25$
$\omega = 1$

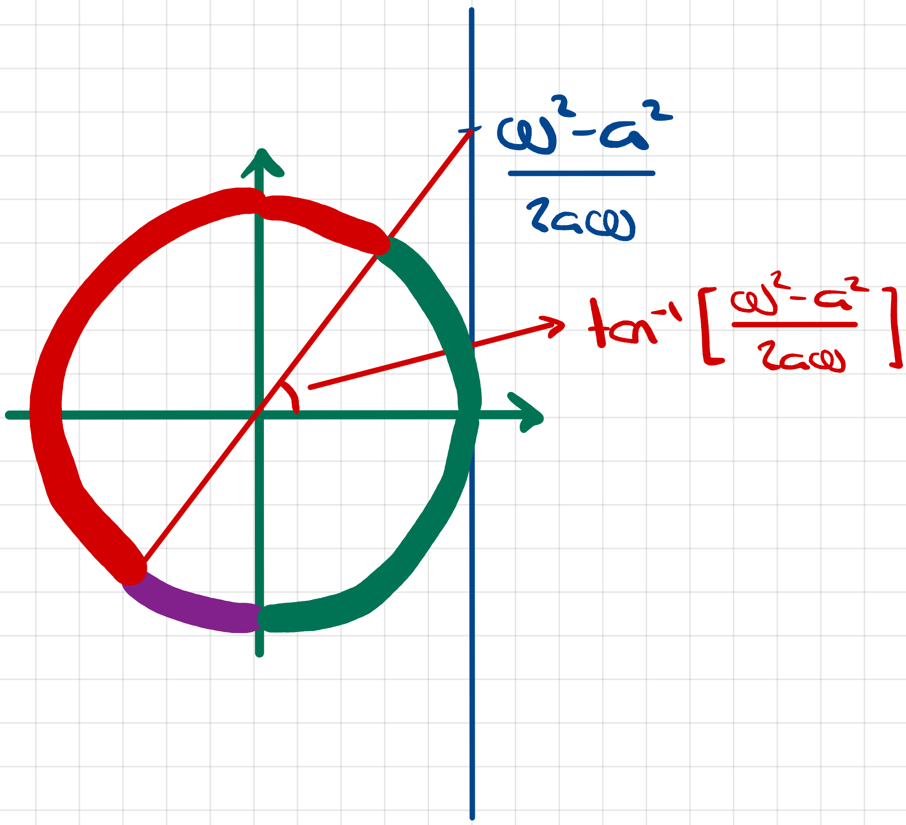
$\Rightarrow \tan(t) < \frac{1 - 0.25^2}{2 \cdot 0.25 \cdot 1} = 1.875$

$t < 1.08$





tan ωt
$\frac{3\pi}{2\omega} > t > \frac{\tan^{-1}\frac{\omega^2-a^2}{2a\omega}}{\omega}$
$-\frac{\pi}{2\omega} < t < \frac{\tan^{-1}\frac{\omega^2-a^2}{2a\omega}}{\omega}$
$\frac{\omega^2-a^2}{2a\omega}$
x''
8
6
4
2
0
−2
−4
−6
−8
−2 π
$-\frac{3\pi}{2}$
−π
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2 π



$\frac{\omega^2-a^2}{2a\omega}$
$\tan^{-1}\left[\frac{\omega^2-a^2}{2a\omega}\right]$

4. $x(t) = Ae^{-at}\cos(\omega t)$

maxima at t_0 and t_1

$$\frac{x(t_1)}{x(t_0)} = \frac{Ae^{-at_1}\cos(\omega t_1)}{Ae^{-at_0}\cos(\omega t_0)} = e^{-a(t_1-t_0)}\cdot\frac{\cos(\omega t_1)}{\cos(\omega t_0)} = e^{-a(t_1-t_0)}$$

maxima occur when $\omega t = k2\pi$ $k \in \mathbb{Z}$ $\Rightarrow \cos(\omega t) = 1$

If we know t_0 and $x(t_0)$ then we can determine a.

$$x(t_0) = Ae^{-at_0} \Rightarrow a = \frac{\ln(A/x(t_0))}{t_0}$$

Given t_1 we know $x(t_1) = x(t_0)e^{-a(t_1-t_0)}$

5. $\ddot{x} + b\dot{x} + x = 0$

$r^2 + br + 1 = 0$ $\quad\Delta = b^2 - 4 = 0 \Rightarrow b = \pm 2$ critical damping

$$r = \frac{-b}{2} = \pm 1$$

$x(t) = e^{\pm t}$

A general sol'n: $x(t) = c_1e^{t} + c_2e^{-t}$

$x'(t) = c_1e^{t} - c_2e^{-t}$

$x(0) = c_1 + c_2$
$x'(0) = c_1 - c_2$

$x(0) = 0$, $x'(0) = 1$ $\Rightarrow$ $c_1 = -c_2$, $2c_1 = 1 \Rightarrow c_1 = 1/2,\ c_2 = -1/2$

$x(0) = 1$, $x'(0) = 0$ $\Rightarrow$ $2c_1 = 1 \Rightarrow c_1 = 1/2,\ c_2 = 1/2$, $c_1 = c_2$

$$\Rightarrow \quad X_1(t) = \frac{e^{t} - e^{-t}}{2}$$
$$X_2(t) = \frac{e^{t} + e^{-t}}{2}$$
, normalized pair of solutions

Now we want sol'n with $x(0) = 2$ and $\dot{x}(0) = 3$.

$$x(t) = 2X_2(t) + 3X_1(t)$$

$$x(0) = 2X_2(0) + 3X_1(0) = 2$$

$$x'(t) = 2X_2'(t) + 3X_1'(t)$$

$$x'(0) = 2X_2'(0) + 3X_1'(0) = 3$$

Exponential and Sinusoidal Input Signals

$x(t) = A\sin 3t$
$x'(t) = 3A\cos 3t$
$x''(t) = -9A\sin 3t$

$\Rightarrow -9A\sin 3t + 4A\sin 3t = \sin 3t$
$-5A\sin 3t = \sin 3t$
$A = -\frac{1}{5}$

1. $\ddot{x} + 4x = \sin(3t)$
   simple harmonic oscillator

$\sin(3t) = \mathrm{Im}(e^{i3t})$

$x_p = \mathrm{Im}(z_p)$

$\ddot{z} + 4z = e^{i3t}$, a second order, linear ODE, const. coeff., exponential input.

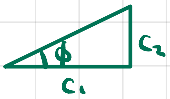
$p(D)z = e^{i3t}$ $\quad p(r) = r^2 + 4$ $\quad$ roots: $r^2 = -4 \Rightarrow r = \pm 2i$

$z_p = Ae^{3it} \Rightarrow z_p = \frac{1}{(3i)^2+4} e^{3it} = \frac{1}{-5} e^{3it} = -\frac{1}{5}(\cos 3t + i\sin 3t)$

$\Rightarrow x_p(t) = -\frac{1}{5}\sin 3t$

x_h: $e^{rt}(r^2+4) = 0 \Rightarrow e^{2it}$ is one sol'n.

$e^{2it} = \cos 2t + i\sin 2t \Rightarrow x_h = c_1\cos 2t + c_2 \sin 2t = C\cos(2t - \phi)$



$C = (c_1^2 + c_2^2)^{1/2}$
$\phi = \tan^{-1}(c_2/c_1)$

$x(t) = -\frac{\sin 3t}{5} + C\cos(2t - \phi)$ $\quad \omega_n = 2$

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2. $\ddot{x} + 4x = \cos(\omega t)$

$x_p = A\cos(\omega t) \Rightarrow x_p' = -A\omega\sin\omega t$
$x_p'' = -A\omega^2\cos\omega t$

$-A\cos\omega t \cdot \omega^2 + 4A\cos\omega t = A\cos(\omega t)(4-\omega^2) = \cos\omega t$

$\Rightarrow A = \frac{1}{4-\omega^2} \Rightarrow x(t) = \frac{\cos\omega t}{4-\omega^2}$

$* \; p(r) = r^2+4$, $p'(r) = 2r$

$*$ Altern.: $\ddot{z}+4z = e^{i\omega t}$, roots $\pm 2i$, $z_p = e^{i\omega t} \cdot \frac{1}{(\omega i)^2+4} = \frac{e^{i\omega t}}{4-\omega^2}$

Steps to plot

plot input $\cos(\omega t)$

plot output $A\cos\omega t = \frac{\cos(\omega t)}{4-\omega^2}$

plot $A(\omega) = \frac{1}{4-\omega^2}$

resonance occurs when $\omega = 2$.

Note: $\ddot{x}+4x = 0 \Rightarrow \omega_n = 2$

$\omega^2 > 4 \Rightarrow$ output signal is of opposite signal to input.

3) $\dfrac{d^4x}{dt^4} - x = e^{-2t}$ $p(r) = r^4 - 1$ = charact. eq.

$p(r) = 0 \Rightarrow r^4 = 1 \Rightarrow r = \pm 1$, two double roots

$x_p = Ae^{-2t} \Rightarrow (-2)^4 Ae^{-2t} - Ae^{-2t} = Ae^{-2t}(16-1) = 15Ae^{-2t} = e^{-2t} \Rightarrow A = 1/15$

$$x_p(t) = \frac{e^{-2t}}{15}$$

4) $x^{(4)} - x = \cos 2t$

$x = A\cos 2t$
$x' = -2A\sin 2t$
$x'' = -4A\cos 2t$
$x''' = 8A\sin 2t$
$x^{(4)} = 16A\cos 2t$

$16A\cos 2t - A\cos 2t = 15A\cos 2t = \cos 2t$
$\Rightarrow A = 1/15$

$$x_p(t) = \frac{\cos 2t}{15}$$

5) $x^{(4)} - x = 0$

$e^{rt}(r^4 - 1) = 0$

$r^4 = 1 \Rightarrow r = \sqrt[4]{1}$

$\rho e^{i\theta} = (e^{i2\pi k})^{1/4} \quad k \in \mathbb{Z}$ $* \; e^{i2\pi k} = \cos(2\pi k) + i\sin(2\pi k) = 1$

$= e^{\frac{\pi}{2}ki} \quad k \in \mathbb{Z}$

$e^0 = 1 \Rightarrow c_1 e^t$
$e^{\frac{\pi}{2}i} = i \Rightarrow c_2 e^{it} = c_2(\cos t + i\sin t) \Rightarrow \cos t, \sin t$ are real solutions
$e^{\pi i} = -1 \Rightarrow c_3 e^{-t}$
$e^{\frac{3\pi}{2}i} = -i \Rightarrow c_4 e^{-it}$

$\Rightarrow x_h(t) = c_1 e^t + c_2 e^{-t} + c_3 \sin t + c_4 \cos t$

$$x(t) = c_1 e^t + c_2 e^{-t} + c_3 \sin t + c_4 \cos t + \frac{\cos 2t}{15}$$

Gain and Phase Lag

1. $p(D)x = Ae^{rt}$ → exponential input

$p(D)$ → linear operator w/ differentiation argument; x → function of t

$$x_p(t) = A\frac{e^{rt}}{p(r)}$$ → exponential response

$p(r)$ → char. polyn. w/ r arg, must be $\neq 0$ to use this formula for x_p

2. $m\ddot{x}+b\dot{x}+kx = ky$ $\quad m=1, b=3, k=4, y(t) = A\cos t$

$\ddot{x}+3\dot{x}+4x = 4A\cos t$ sinus. input, amplit. 4A

$\ddot{z}+3\dot{z}+4z = 4Ae^{it}$

$$p(r) = r^2+3r+4 = 0 \Rightarrow r = \frac{-3\pm(9-16)^{1/2}}{2} = \frac{-3\pm\sqrt{7}i}{2}$$

$p(i) = -1+3i+4 = 3+3i$

$$z_p = \frac{4A}{3+3i}e^{it} = \frac{(3-3i)}{18}4Ae^{it} = \frac{(3-3i)}{18}4A(\cos t + i\sin t) = \frac{2}{3}(1-i)A(\cos t + i\sin t)$$

$$x_p = \frac{2A}{3}\cos t + \frac{2A}{3}\sin t = \left(\frac{4A^2}{9}\cdot 2\right)^{1/2}\cos(t-\pi/4) = \frac{2\sqrt{2}}{3}A\cos(t-\pi/4)$$

$$H = \frac{3-3i}{18} = \frac{1-i}{6} = \frac{1}{6}\frac{2}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right) = \frac{1}{3\sqrt{2}}(\cos(-\pi/4)+i\sin(-\pi/4)) = \frac{e^{-\frac{\pi}{4}i}}{3\sqrt{2}}$$

$$|H| = (1/36+1/36)^{1/2} = (1/18)^{1/2} = 1/3\sqrt{2}$$

$$\Rightarrow H = \frac{e^{-i\phi}}{|H|} \qquad \phi = \frac{\pi}{4}$$

input signal $4A\cos t$ real part of $4Ae^{it}$

complex sol'n $\frac{3-3i}{18}4Ae^{it} = H\cdot 4Ae^{it}$

complex gain H

$$x_p = Re(z_p) = Re(H\,y_{cx}) = Re\left(\frac{1}{3\sqrt{2}}\cdot e^{-\frac{\pi}{4}i}4Ae^{it}\right) = Re\left(\frac{2\sqrt{2}}{3}Ae^{i(t-\pi/4)}\right)$$

$$= \frac{2\sqrt{2}}{3}A\cos(t-\pi/4)$$

Note that the steady state vibration amplitude of the mass is $\frac{2\sqrt{2}}{3}A < 4A$, where $4A$ is the input amplitude.

3. $\ddot{x}+4x=\cos(2t)$

$\ddot{z}+4z=e^{2it}$

$p(r)=r^2+4=0 \Rightarrow r^2=-4 \Rightarrow r=\pm 2i$

$p(2i)=0$

$p'(r)=2r$

$p'(2i)=4i \qquad \Rightarrow z_p=\frac{1}{4i}te^{2it}=\frac{-4i}{16}t(\cos 2t+i\sin 2t)$

$x_p(t)=\frac{t\sin 2t}{4}$

Undetermined coeff.

1. $\ddot{x}-x=t^2+t+1$

$x_p=At^2+Bt+C$
$x_p'=2At+B$
$x_p''=2A$

$2A-At^2-Bt-C=-At^2-Bt+2A-C=t^2+t+1$

$\Rightarrow 1=-A \Rightarrow A=-1$
$1=-B \Rightarrow B=-1$
$1=2(-1)-C \Rightarrow C=-2-1=-3$

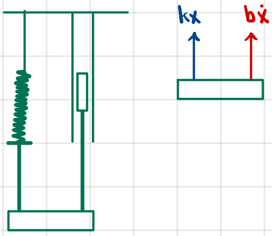
$x_p(t)=-t^2-t-3$

Frequency Response

$m\ddot{x}+b\dot{x}+kx = F_{ext} = A\cos\omega t$

$x_p = gA\cos(\omega t-\phi)$

$m=1,\ b=\frac{1}{4},\ k=2,\ A=1$



$m\ddot{x} = -kx - b\dot{x} + F_{ext}$

1. $\ddot{x} + \frac{1}{4}\dot{x} + 2x = \cos\omega t$

complexify $\ddot{z} + \frac{1}{4}\dot{z} + 2z = e^{i\omega t}$

$p(r) = r^2 + \frac{r}{4} + 2$

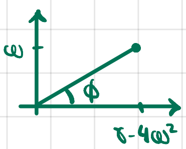
$p(i\omega) = -\omega^2 + \frac{i\omega}{4} + 2$

$$z_p = \frac{e^{i\omega t}}{2-\omega^2+\frac{i\omega}{4}} = \frac{4e^{i\omega t}}{8-4\omega^2+i\omega}$$

complex gain $= \tilde{g} = \frac{4}{8-4\omega^2+i\omega}$

2. $$z_p = \frac{(8-4\omega^2-i\omega)4(\cos\omega t + i\sin\omega t)}{(8-4\omega^2)^2+\omega^2}$$

$$x_p = Re(z_p) = \frac{4(8-4\omega^2)\cos\omega t + 4\omega\sin\omega t}{(8-4\omega^2)^2+\omega^2}$$



$$= \frac{4\cos(\omega t-\phi)}{\sqrt{(8-4\omega^2)^2+\omega^2}}$$

gain $= g = \frac{1}{|p(i\omega)|} = \frac{4}{\sqrt{(8-4\omega^2)^2+\omega^2}}$

$$g(1) = \frac{4}{(4^2+1)^{1/2}} = \frac{4}{\sqrt{17}} \approx 0.97$$

3. resonant angular freq = ω that maximizes the gain/amplit.

min $(8-4\omega^2)^2+\omega^2$

$\Rightarrow 2(8-4\omega^2)(-8\omega) + 2\omega = 0$

$-16\cdot 8\omega + 16\cdot 4\omega^3 + 2\omega = 0$

$64\omega^3 - 126\omega = 0$

$2\omega(32\omega^2-63) = 0 \to \omega = 0,\ \omega^2 = 63/32$

$\omega = \sqrt{63/32} \approx 1.40$

4. $\tan\phi(\omega) = \frac{\omega}{8-4\omega^2} + k\pi \quad k\in\mathbb{Z}$

$\phi(\sqrt{63/32}) = \tan^{-1}(3\sqrt{14}) \approx 1.4819$ rad $\approx 84.90°$

$\phi = \frac{\pi}{2} \Rightarrow \omega = \sqrt{2}$

5. $\phi(\omega) = \phi_1 \Rightarrow \omega = (8-4\omega^2)\tan(\phi_1)$

$\Rightarrow 4\omega^2\tan\phi_1 + \omega - 8\tan\phi = 0$

$$\omega = \frac{-1\pm(1-4\cdot 8\cdot\tan\phi_1\cdot 4\tan\phi_1)^{1/2}}{8\tan\phi_1} = \frac{-1\pm(1-128\tan^2\phi)^{1/2}}{8\tan\phi_1}$$

$\phi_1 = \frac{\pi}{4} \Rightarrow \omega \approx 1.29$

$\phi_1 = \frac{3\pi}{4} \Rightarrow \tan\frac{3\pi}{4} = -1 = \frac{\omega}{8-4\omega^2}$

$$\omega = \frac{-1\pm\sqrt{129}}{8}$$

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6. $\ddot{x} + 3\dot{x} + 2x = te^{-t}$

$x(t) = v(t)e^{-t}$

$p(r) = r^2 + 3r + 2$

$p(D)x = te^{-t}$

$p(D)ve^{-t} = (D^2+3D+2)ve^{-t} = D(\dot{v}e^{-t} - ve^{-t}) + 3(\dot{v}e^{-t} - ve^{-t}) + 2ve^{-t}$

$= \ddot{v}e^{-t} - \dot{v}e^{-t} - \dot{v}e^{-t} + \cancel{ve^{-t}}\ \ +3\dot{v}e^{-t}\ \ \cancel{-3ve^{-t}}\ \ \cancel{+2ve^{-t}} = e^{-t}(\ddot{v}+\dot{v}) = te^{-t}$

$\Rightarrow \ddot{v} + \dot{v} = t$

$v(t) = at+b \Rightarrow a = t$ doesn't work

$v(t) = at^2 + bt$

$\dot{v} = 2at + b$

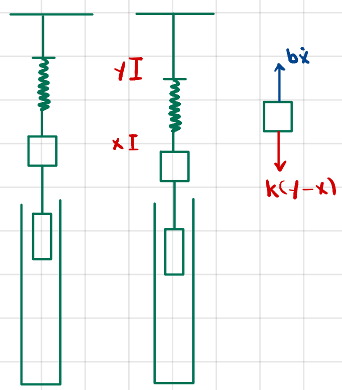
$\ddot{v} = 2a$

$\Rightarrow 2a + 2at + b = t(2a) + (2a+b) = t \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$

$2a + b = 0 \Rightarrow 1 + b = 0 \Rightarrow b = -1 \Rightarrow v(t) = \frac{t^2}{2} - t$

$x(t) = (\frac{t^2}{2} - t)e^{-t}$

7.



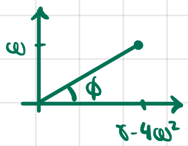
$m\ddot{x} = -k(y-x) + b\dot{x}$

Note we have

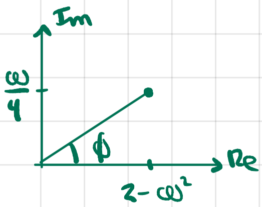
$$z_p = \frac{e^{i\omega t}}{2-\omega^2+\frac{i\omega}{4}} = \frac{4e^{i\omega t}}{8-4\omega^2+i\omega}$$

$$p(i\omega) = 2-\omega^2+\frac{i\omega}{4}$$

$$x_p = \mathrm{Re}(z_p) = \frac{4(8-4\omega^2)\cos\omega t + 4\omega\sin\omega t}{(8-4\omega^2)^2+\omega^2} = \frac{4\cos(\omega t-\phi)}{\sqrt{(8-4\omega^2)^2+\omega^2}}$$



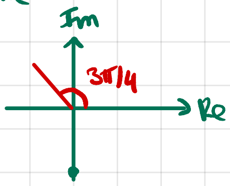
ϕ is $\mathrm{Arg}(p(i\omega))$



$\Rightarrow$ ϕ is $\frac{\pi}{2}$ when $\mathrm{Arg}(p(i\omega))$ is purely imaginary

$\Rightarrow \omega^2 = 2 \Rightarrow \omega = \sqrt{2}$

$\phi = \frac{3\pi}{4} \Rightarrow$



$$\frac{\omega/4}{\omega^2-2} = -1 \Rightarrow \omega = -4\omega^2+8 \Rightarrow 4\omega^2+\omega-8=0$$

$$\Delta = 1-4(4)(-8) = 129 \Rightarrow \omega = \frac{-1\pm\sqrt{129}}{8}$$