Pset 1

Problem 1 (same as Example 4.6)

$v_{c,0} = 12\ m\cdot s^{-1}$ $\quad v_c(t_2) = 0$ $\quad a_c(t) = \begin{cases} 0 & 0 \le t \le t_1 \\ b(t-t_1) & t_1 < t \le t_2 \end{cases}$ $\quad b = -6\ m\cdot s^{-3}$

a)

$$v_c(t) = \begin{cases} v_{c,0} & 0 \le t \le t_1 \\ v_{c,0} + \frac{b(t-t_1)^2}{2} & t_1 < t \le t_2 \end{cases}$$

$$x_c(t) = \begin{cases} v_{c,0}t + x_{c,0} & 0 \le t \le t_1 \\ v_{c,0}t + b(t-t_1)^3/6 + x_{c,0} & t_1 < t \le t_2 \end{cases}$$

inserting initial data

$$a_c(t) = \begin{cases} 0 & 0 \le t \le 1 \\ -6(t-1) & 1 < t \le t_2 \end{cases}$$

$$v_c(t) = \begin{cases} 12 & 0 \le t \le 1 \\ 12 - 3(t-1)^2 & 1 < t \le t_2 \end{cases}$$

$$x_c(t) = \begin{cases} 12t & 0 \le t \le 1 \\ 12t - (t-1)^3 & 1 < t \le t_2 \end{cases}$$

b) BIKE

$v_{b,0}$ constant speed throughout

$x_b(0) = -17$

$x_b(t_2) = x_c(t_2)$

$v_c(t_2) = 0$

$\Rightarrow$

$a_b(t) = 0$

$v_b(t) = v_{b,0}$

$x_b(t) = -17 + v_{b,0}t$

same position at $t = t_2$

$x_b(t_2) = x_c(t_2)$

$-17 + v_{b,0}t_2 = 12t_2 - (t_2 - 1)^3$

what is t_2?

$v_c(t_2) = 0 = 12 - 3(t_2^2 - 2t_2 + 1)$

$t_2^2 - 2t_2 + 1 = 4 \Rightarrow t_2^2 - 2t_2 - 3 = 0$

$\Delta = 4 - 4\cdot 1\cdot(-3) = 16$

$t_2 = \frac{2 \pm 4}{2} \nearrow 3 \quad \searrow -1$

$-17 + 3v_{b,0} = 36 - (2)^3$

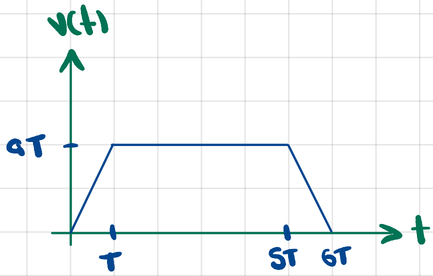
$\Rightarrow 3v_{b,0} = 28 + 17 = 45$

$\Rightarrow v_{b,0} = 15\ m\cdot s^{-1}$

Problem 2 - Elevator Trip

a)

Total interval of time: $[0, 6T]$



$$a(t) = \begin{cases} a & 0 \le t < T \\ 0 & T < t \le 5T \\ -a & 5T < t \le 6T \end{cases}$$

$$v(t) = \begin{cases} at & 0 \le t < T \\ aT + \int_T^t 0\,dt = aT & T < t \le 5T \\ aT + \int_{5T}^t (-a)\,dt = aT - (at - 5aT) = 6aT - at & 5T < t \le 6T \end{cases}$$

That is

$$v(t) = \begin{cases} at & 0 \le t < T \\ aT & T < t \le 5T \\ 6aT - at & 5T < t \le 6T \end{cases}$$

$$x(t) = \begin{cases} \int_0^t at\,dt = \frac{at^2}{2} \\ \frac{aT^2}{2} + \int_T^t aT\,dt = \frac{aT^2}{2} + aT(t-T) = aT\left(t - \frac{T}{2}\right) \\ aT\left(5T - \frac{T}{2}\right) + \int_{5T}^t (6aT - at)\,dt = \frac{9aT^2}{2} + 6aT(t-5T) - \frac{a(t^2 - 25T^2)}{2} \end{cases}$$

That is

$$x(t) = \begin{cases} \frac{at^2}{2} \\ aT\left(t - \frac{T}{2}\right) \\ \frac{9aT^2}{2} + 6aT(t-5T) - \frac{a(t^2-25T^2)}{2} \end{cases}$$

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b) The easy way to solve for a is to simply calculate the area under $v(t)$ between $t=0$ and $t=6T$.

$$\text{Area} = \frac{2 \cdot aT \cdot T}{2} + 4T \cdot aT = 5aT^2$$

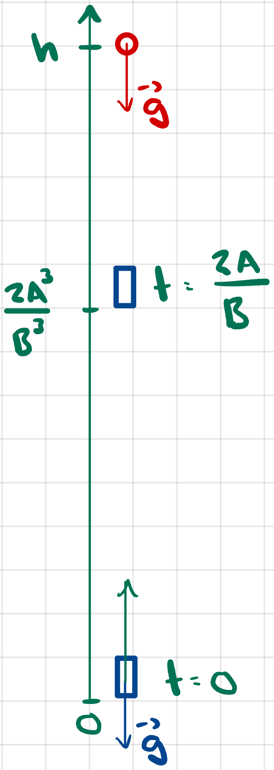
This area is actually the same as $\int_0^{6T} v(t)\,dt = x(6T) = h$. Thus, $5aT^2 = h \rightarrow a = \frac{h}{5T^2}$

Alternatively, calculate $x(6T)$ directly:

$$x(6T) = \frac{9aT^2}{2} + 6aT^2 - \frac{a(36T^2 - 25T^2)}{2} = \frac{21aT^2}{2} - \frac{11aT^2}{2} = 5aT^2$$

Thus, $5aT^2 = h$ and $a = \frac{h}{5T^2}$, as before.

Problem 3 - Rocket Launch



$a_r(t) = A - Bt \quad A, B > 0 \quad t > 0$

$v_r(t) = At - \frac{Bt^2}{2}$

$y_r(t) = \frac{At^2}{2} - \frac{Bt^3}{6}$

max height: $v_r(t) = 0 \Rightarrow t\left(A - \frac{Bt}{2}\right) \Rightarrow t = \frac{2A}{B}$

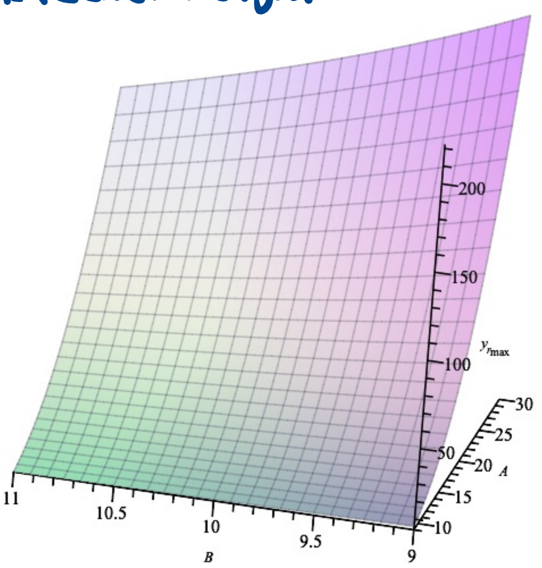
$y_r\left(\frac{2A}{B}\right) = \frac{A}{\cancel{2}} \cdot \frac{\cancel{4}^{2} A^2}{B^2} - \frac{B}{\cancel{6}_{3}} \cdot \frac{\cancel{8}^{4} A^3}{B^{\cancel{3}2}} = \frac{2A^3}{B^2} - \frac{4A^3}{3B^2} = \frac{2A^3}{3B^2}$

The stone falls for $\frac{2A}{B}$ seconds and $|\Delta y_s| = \frac{2A^3}{B^3} - h$

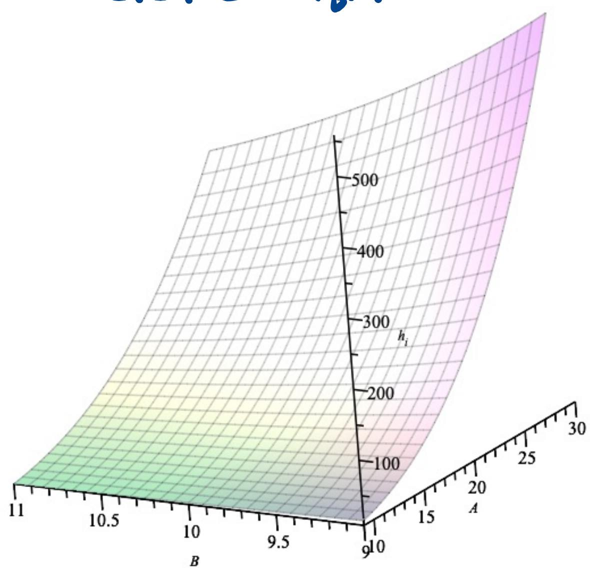
$a_s(t) = -gt \qquad v_s(t) = -\frac{gt^2}{2} \qquad y_s(t) = h - \frac{gt^3}{6}$

$y_s(2A/B) = \frac{2A^3}{B^3} = h - \frac{g}{\cancel{6}_{3}} \cdot \frac{\cancel{8}^{4} A^3}{B^3} \Rightarrow h = \frac{A^3}{B^3}\left[2 + \frac{4}{3}g\right]$

Plot of $y_r(2A/B) = \frac{2A^3}{2B^2} = y_{r_{max}}$
= max rocket height



Plot of $h = h(A, B) = \frac{A^3}{B^3}\left[2 + \frac{4}{3}g\right]$
= initial stone height



Problem 4 - Throw and Catch

Person

$\vec{a}_p(t) = \langle Bt, 0 \rangle, \ B > 0$

$\vec{v}_p(t) = \langle Bt^2/2, 0 \rangle$

$\vec{r}_p(t) = \langle Bt^3/6 + d, 0 \rangle$

Ball

$\vec{a}_b(t) = \langle 0, -g \rangle$

$\vec{v}_b(t) = \langle v_0 \cos\theta, v_0 \sin\theta - gt \rangle$

$\vec{r}_b(t) = \langle v_0 \cos\theta \, t, v_0 \sin\theta \, t - \frac{gt^2}{2} \rangle$

Catching the Ball

$\vec{r}_b(t) = \vec{r}_p(t)$

$$\begin{cases} t v_0 \cos\theta = Bt^3/6 + d \Rightarrow \\ t v_0 \sin\theta - gt^2/2 = 0 \end{cases} \Rightarrow t(v_0 \sin\theta - gt/2) = 0 \rightarrow t = \frac{2 v_0 \sin\theta}{g}$$

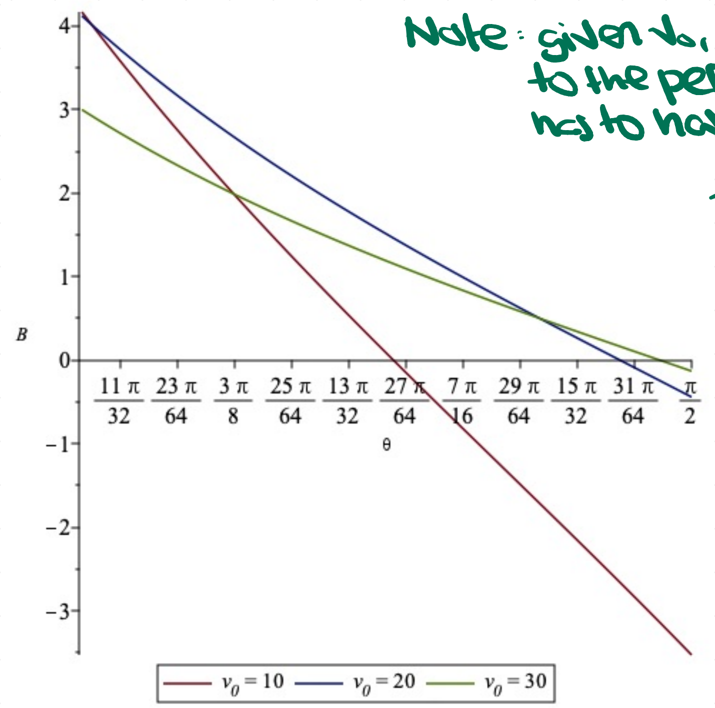
sub. into first eq.

$$\frac{2 v_0 \sin\theta}{g} \cdot v_0 \cos\theta = \frac{B}{\cancel{6}_3} \cdot \frac{\cancel{8}^4 v_0^3 \sin^3\theta}{g^3} + d$$

$$\Rightarrow \left[\frac{\sin(2\theta) v_0^2}{g} - d \right] \cdot \frac{3}{4} \frac{g^3}{v_0^3 \sin^3\theta} = B$$

$$\Rightarrow B(\theta, v_0, d, g) = \left[\sin(2\theta) v_0^2 - dg \right] \cdot \frac{3g^2}{4 v_0^3 \sin^3\theta}$$

Maple Plots



Note: given v_0, above a certain θ the ball does not make it to the person before falling below $y=0$. The person has to have $B<0$, ie run towards origin, to catch the ball

The larger v_0 is, the larger such a max angle is. Larger v_0 means larger x and y components of velocity $\therefore$ the ball travels farther all else equal.

Problem 5 - Vertical Collision

First can

$\vec{F} = -mg$

$a_1(t) = -g$

$v_1(t) = v_0 - gt$

$y_1(t) = v_0 t - \frac{gt^2}{2}$

Second can

$a_2(t) = \begin{cases} 0 & 0 \le t \le t_1 \\ -g & t_1 < t \end{cases}$

$v_2(t)$

$0 \le t \le t_1 \Rightarrow v_2(t) = 0$

$t > t_1 \Rightarrow v_2(t) = C - gt$

$v_2(t_1) = v_0 = C - gt_1 \Rightarrow C = v_0 + gt_1$

$\Rightarrow v_2(t) = \begin{cases} 0 & 0 \le t \le t_1 \\ v_0 + gt_1 - gt & t > t_1 \end{cases}$

$y_2(t)$

$0 \le t \le t_1 \Rightarrow y_2(t) = 0$

$t > t_1 \Rightarrow y_2(t) = C + (v_0 + gt_1)t - \frac{gt^2}{2}$

$y_2(t_1) = 0 = C + v_0 t_1 + gt_1^2 - gt_1^2/2$

$= C + v_0 t_1 + gt_1^2/2 \Rightarrow C = -(v_0 t_1 + gt_1^2/2)$

$\Rightarrow y_2(t)$

$= \begin{cases} 0 & 0 \le t \le t_1 \\ -(v_0 t_1 + gt_1^2/2) + (v_0 + gt_1)t - \frac{gt^2}{2} & t > t_1 \end{cases}$

Collision at $y = y_c$

→ calculate time at which can 2 reaches y_c, given v_0, t_1.

→ given v_0, t_1 calculate time for can 1 to drop to y_c.

→ equate times to obtain v_0 as function of t_1

a) $y_c = -(v_0 t_1 + gt_1^2/2) + (v_0 + gt_1)t - \frac{gt^2}{2}$

$\Rightarrow \frac{g}{2}t^2 - (v_0 + gt_1)t + (v_0 t_1 + \frac{gt_1^2}{2} + y_c)$

$\Delta = (v_0 + gt_1)^2 - \cancel{4}^{2} \cdot \frac{g}{\cancel{2}}(v_0 t_1 + \frac{gt_1^2}{2} + y_c)$

$= v_0^2 + \cancel{2v_0 g t_1} + \cancel{g^2 t_1^2} - \cancel{2v_0 g t_1} - \cancel{g^2 t_1^2} - 2gy_c$

$= v_0^2 - 2gy_c \Rightarrow v_0^2 \ge 2gy_c$

$\Rightarrow t = \frac{v_0 + gt_1 \pm [v_0^2 - 2gy_c]^{1/2}}{g}$ → can 2 reaches y_c

$v_0 t - \frac{gt^2}{2} = y_c \Rightarrow \frac{g}{2}t^2 - v_0 t + y_c = 0$

$\Delta = v_0^2 - \cancel{4}^{2} \cdot \frac{g}{\cancel{2}} \cdot y_c = v_0^2 - 2y_c g \Rightarrow v_0^2 \ge 2y_c g$

$t = \frac{v_0 \pm [v_0^2 - 2y_c g]^{1/2}}{g}$ → can 1 reaches y_c

$\frac{\cancel{v_0} + [v_0^2 - 2y_c g]^{1/2}}{\cancel{g}} = \frac{\cancel{v_0} + gt_1 - [v_0^2 - 2gy_c]^{1/2}}{\cancel{g}}$

$\Rightarrow 2[v_0^2 - 2y_c g]^{1/2} = gt_1$

$4v_0^2 - 8y_c g = g^2 t_1^2$

$\Rightarrow v_0^2(y_c, g, t_1) = \frac{g^2 t_1^2 + 8y_c g}{4}$

$\Rightarrow v_0(y_c, g, t_1) = \left[\frac{g^2 t_1^2 + 8y_c g}{4}\right]^{1/2}$

b) $v_0(5, 9.8, 4) = 21.95$ m/s

$t_{collision} = 4.24$ s