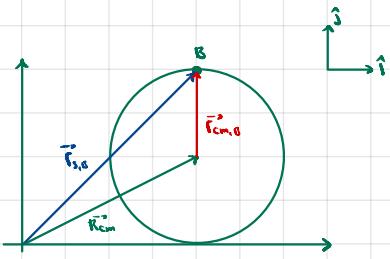


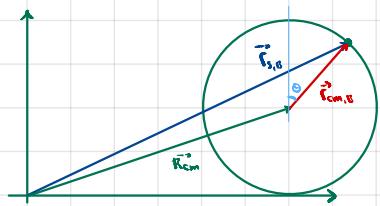
PSet 12

Problem 1 - Bicycle Wheel



$$\vec{r}_o = x_o \hat{i} + z_o \hat{j}$$

$$\vec{v}_{cm} = v_{cm} \hat{i} \Rightarrow \vec{r}_{cm} = (x_o + v_{cm}t) \hat{i} + z_o \hat{j}$$



$$\vec{r}_{s,B} = \vec{R}_{cm} + \vec{r}_{cm,B}$$

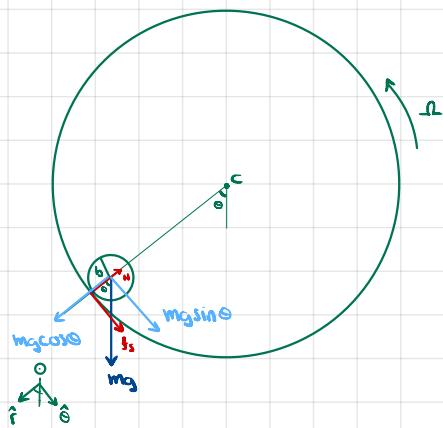
$$\vec{r}_{cm,B} = R \hat{i} + R(\sin\theta \hat{i} + \cos\theta \hat{j})$$

$$\text{no slipping} \Rightarrow v_{cm} = R\omega \Rightarrow \omega = \frac{v_{cm}}{R} \Rightarrow \theta(t) = \frac{v_{cm}t}{R}$$

$$\circ \vec{r}_{s,B} = (x_o + v_{cm}t) \hat{i} + R \hat{j} + R \sin\theta \hat{i} + R \cos\theta \hat{j}$$

$$\circ \vec{r}_{s,B} = \hat{i} \left(x_o + v_{cm}t + R \sin\left(\frac{v_{cm}t}{R}\right) \right) + \left(R + R \cos\left(\frac{v_{cm}t}{R}\right) \right) \hat{j}$$

Problem 2 - Rolling Without Slipping



First, suppose no gravity, no friction, ring at rest

$$N = -mR\omega^2 \Rightarrow N = mR\omega^2$$

no torques about CM $\Rightarrow \Delta L = 0$

$$\vec{F}_{c,sys} = (R-b)\hat{r} \times m\vec{v}_{cm}\hat{\theta} + I_{cm}\vec{\omega}_{cm}\hat{h}$$

$$= (m(R-b)\vec{v}_{cm} + I_{cm}\vec{\omega}_{cm})\hat{h}$$

$$\vec{v}_{cm} = (R-b)\cdot\vec{\omega}_c$$

$$\vec{F}_{c,sys} = (m(R-b)^2\vec{\omega}_c + I_{cm}\vec{\omega}_{cm})\hat{h}$$

The ball rotates with $\vec{\omega}_{cm}\hat{h}$ about CM, and $\vec{\omega}_c\hat{h}$ about the ring center.

\vec{v}_{cm} has constant magnitude.

Angular momentum is constant.

The ball will roll around the ring forever.

Now suppose the ring has angular velocity $-b\hat{h}$.

Since there is no friction, nothing changes.

Add gravity.

$$mg\cos\theta \cdot N = -mR\omega^2$$

$$mg\sin\theta \cdot m \cdot a_T = -mR\omega_c = \ddot{x}_c = -\frac{gsin\theta}{r}$$

$$\vec{F}_{c,sys} = \vec{r}\hat{r} \times mg\sin\theta\hat{\theta} + I_{cm}\vec{\omega}_{cm}\hat{h}$$

$$= rmgsin\theta\hat{h}$$

$$\vec{F}_{c,sys} = \vec{r}\hat{r} \times m\vec{v}_{cm,cm} - I_{cm}\vec{\omega}_{cm}\hat{h}$$

$$= \vec{r}\hat{r} \times m(-r\vec{\omega}_c\hat{\theta}) - I_{cm}\vec{\omega}_{cm}\hat{h}$$

$$= (-mr^2\vec{\omega}_c - I_{cm}\vec{\omega}_{cm})\hat{h}$$

$$\frac{d\vec{F}_{c,sys}}{dt} = -mr^2\vec{\alpha}_h$$

$$rmgsin\theta = -mr^2\ddot{x}_c \Rightarrow \ddot{x}_c = -\frac{gsin\theta}{r}$$

For small θ ,

$$\ddot{\theta} + \frac{g\theta}{r} = 0 \quad \text{second order, linear homog. ODE}$$

$$\text{guess } \vec{\theta}(t) = e^{int}$$

$$\dot{\theta} = me^{int}$$

$$\ddot{\theta} = m^2e^{int}$$

$$\Rightarrow e^{it}(m^2 + \frac{g}{r}) = 0$$

$$m^2 = -\frac{g}{r}$$

$$m = \pm \sqrt{\frac{g}{r}} i$$

$$\text{one soln is } \vec{\theta}(t) = e^{\frac{g}{r}it}$$

use Euler's formula

$$e^{\frac{g}{r}it} = \cos(\frac{g}{r}t) + i\sin(\frac{g}{r}t)$$

$$\theta_1(t) = \cos(\frac{g}{r}t)$$

$$\theta_2(t) = \sin(\frac{g}{r}t)$$

are real solns.

General soln:

$$\vec{\theta}(t) = c_1 \cos(\frac{g}{r}t) + c_2 \sin(\frac{g}{r}t)$$

→ the wheel behaves like a simple harmonic oscillator.

$$N = m(g\cos\theta + r\omega_c^2)$$

$$\omega_c = \dot{\theta}(t)$$

$$= -c_1 \frac{g}{r} \sin(\frac{g}{r}t) + c_2 \frac{g}{r} \cos(\frac{g}{r}t)$$

Now add friction.

$$(mg\sin\theta + f_s)\hat{\theta} = m \cdot \ddot{a}_{g,cm}$$

$$\ddot{a}_{g,cm} = \frac{mg\sin\theta + f_s}{m}\hat{\theta}$$

$$\vec{F}_{c,sys} = \vec{r}\hat{r} \times (mg\sin\theta + f_s)\hat{\theta}$$

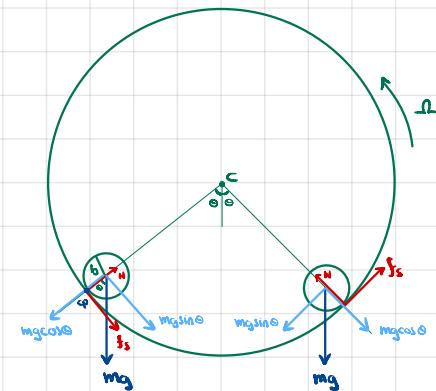
$$+ b\hat{r} \times f_s\hat{\theta}$$

$$= \hat{h} (rmgsin\theta + f_s + bf_s)$$

$$\vec{F}_{c,sys} = I_{cm}\vec{\omega}_{cm}\hat{h}$$

$$= bS_s\hat{h}$$

$$\Rightarrow \ddot{x}_{cm} = \frac{bf_s}{I_{cm}}$$



Though there is lots to explore here, the question simply asks for $\vec{v}_{g,cm}$ when the ball rolls without slipping and $\vec{\omega}_{cm} \neq 0$, ie clockwise rotation about CM.

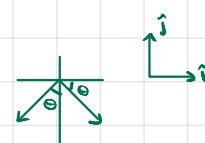
$$\text{no slipping} \Rightarrow \vec{v}_{g,cm} = -\vec{v}_{cm,cp}$$

$$= -(\omega_{cm}h \times b\hat{r}) = -b\omega_{cm}\hat{\theta}$$

$$\vec{v}_{g,cm} = \vec{v}_{g,ir} + \vec{v}_{r,cm}$$

$$= -\vec{R}\hat{h} \times R\hat{r} - b\omega_{cm}\hat{\theta}$$

$$= -(Rb - b\omega_{cm})\hat{\theta}$$



$$\hat{r} = -\sin\theta\hat{i} - \cos\theta\hat{j}$$

$$\hat{\theta} = \cos\theta\hat{i} - \sin\theta\hat{j}$$

$$\frac{dr}{dt} = -\cos\theta\hat{i} + \sin\theta\hat{j}$$

$$= \theta'(\cos\theta\hat{i} - \sin\theta\hat{j})$$

$$= \theta'\hat{r}$$

$$= \omega_c\hat{r}$$

$$\frac{d\theta}{dt} = -\sin\theta\hat{i} - \cos\theta\hat{j}$$

$$= \theta'(-\sin\theta\hat{i} - \cos\theta\hat{j})$$

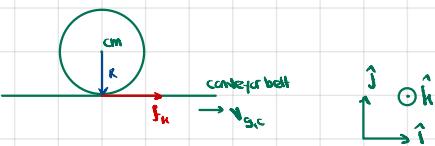
$$= \theta'\hat{r}$$

$$= \omega_c\hat{r}$$

$$\vec{r}_{c,cm} = \vec{r}\hat{r}, \quad r = R-b$$

$$\vec{v}_{c,cm} = -r\omega_c\hat{\theta}$$

$$\ddot{x}_{c,cm} = -r\dot{x}_c\hat{\theta} - r\omega_c^2\hat{r}$$



$$\vec{r}_{cm,sp} = R \cdot \vec{f}_h \hat{h} \cdot \vec{\alpha}_{cm} \times \vec{\omega}_{cm} \hat{h}$$

$$\Rightarrow \vec{\alpha}_{cm} = \frac{R \vec{f}_h}{I_{cm}}$$

$$\vec{J}_h \hat{h} = m \vec{a}_{cm} \hat{i} = \vec{a}_{g,cm} = \frac{\vec{f}_h}{m} \hat{i}$$

$$\vec{v}_{g,cm} = \frac{\vec{f}_h t}{m} \hat{i}$$

$$\vec{\omega}_{cm} = \frac{R \vec{f}_h t}{I_{cm}} \hat{h}$$

$$v_{g,cm} = v_{g,c} + v_{c,cm}$$

$$\Rightarrow \frac{\vec{f}_h t}{m} \hat{i} = v_{g,c} + v_{c,cm}$$

$$\vec{v}_{g,sp} = \vec{v}_{g,c} + \vec{v}_{c,sp}$$

$$\begin{aligned} \vec{v}_{c,sp} &= \vec{\omega} \times \vec{r}_{cp} \\ &= \frac{R \vec{f}_h t}{I_{cm}} \hat{h} \times R (-\hat{j}) \\ &= \frac{R^2 \vec{f}_h t}{I_{cm}} \hat{i} \end{aligned}$$

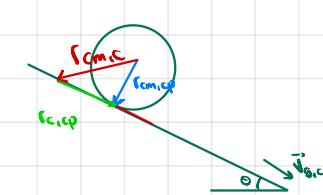
The conveyor belt moves with constant speed rel. to ground $\vec{v}_{g,c}$, ball starts at rest. Difference in speed \Rightarrow kinetic friction f_h accelerates ball and applies torque \Rightarrow rotation.

We know both $\vec{\alpha}_{cm}$ and $\vec{a}_{g,cm}$.

$$\Rightarrow \vec{a}_{cm} = \frac{R \vec{f}_h}{I_{cm}} \hat{h}$$

$$\vec{a}_{g,cm} = \frac{\vec{f}_h}{m} \hat{i}$$

$$\vec{v}_{g,cm} = \frac{\vec{f}_h t}{m} \hat{i}$$



$$f_{c,sp} = f_{c,cm} + f_{c,sp}$$

No slipping means that

$$\vec{v}_{c,cm} = R \hat{j} \times \vec{\omega}_{cm} \hat{h}$$

$$= -R \vec{\omega}_{cm} \hat{i}$$

we also have an expression for $\vec{v}_{c,cm}$

$$\vec{v}_{c,cm} = \left(\frac{\vec{f}_h}{m} t - \vec{v}_{g,c} \right) \hat{i}$$

Speed rel. to conveyor belt decreases in time until it reaches $-R \vec{\omega}_{cm}$. The contact point is at rest w/ the belt, there is no more friction.

$$\frac{\vec{f}_h t}{m} - \vec{v}_{g,c} = - \frac{R^2 \vec{f}_h t}{I_{cm}} \hat{i}$$

$$\Rightarrow t \left[\frac{\vec{f}_h}{m} + \frac{J_h R^2}{I_{cm}} \right] = \vec{v}_{g,c}$$

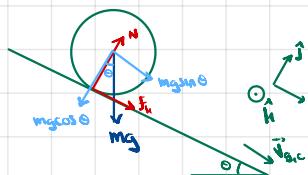
$$\Rightarrow t = \frac{\vec{v}_{g,c}}{\frac{\vec{f}_h}{m} + \frac{J_h R^2}{I_{cm}}}$$

$$\vec{v}_{cm,sp} = \vec{v}_{cm,c} + \vec{v}_{c,sp}$$

$$\begin{aligned} \vec{v}_{cm,sp} &= R \hat{j} \times \vec{\omega}_{cm} \hat{h} \\ &= -R \vec{\omega}_{cm} \hat{i} \end{aligned}$$

$$\vec{v}_{cm,c} = -\vec{v}_{cm,c} = R \vec{\omega}_{cm} \hat{i}$$

$\Rightarrow \vec{v}_{c,sp} \cdot \vec{\omega} \neq 0$ when not slipping.



$$\text{Torque as before } \vec{r}_{cm,sp} = R \cdot \vec{f}_h \hat{h} \cdot I_{cm} \vec{\omega}_{cm} \hat{h} \Rightarrow \vec{\alpha}_{cm} = \frac{R \vec{f}_h}{I_{cm}}$$

$$\text{2nd law: } (\vec{f}_h + mg \sin \theta) \hat{i} = m \vec{a}_{cm} \Rightarrow \vec{a}_{cm} = \frac{\vec{f}_h + mg \sin \theta}{m} \hat{i}$$

$$\vec{\omega}_{cm} = \frac{R \vec{f}_h}{I_{cm}} \hat{h}$$

$$\vec{v}_{c,cm} = \vec{v}_{c,sp} + \vec{v}_{c,cm}$$

$$= -\vec{v}_{cm,sp} = -(-R \hat{i}) \times$$

$$\vec{v}_{g,cm} = \frac{(\vec{f}_h + mg \sin \theta) t}{m} \hat{i}$$

$$\text{no slipping: } \vec{v}_{c,cm} = \vec{v}_{c,sp} + \vec{v}_{c,cm} = -\vec{v}_{cm,sp} = -[\vec{\omega}_{cm} \hat{h} \times (-R \hat{j})]$$

$$= -[-R \vec{\omega}_{cm} (-\hat{i})] = -R \vec{\omega}_{cm} \hat{i}$$

$$\vec{v}_{c,cm} = \vec{v}_{g,cm} - \vec{v}_{g,c} = \frac{(\vec{f}_h + mg \sin \theta) t}{m} \hat{i} - \vec{v}_{g,c} \hat{i}$$

$$\Rightarrow -\frac{R^2 \vec{f}_h t}{I_{cm}} = -\vec{v}_{g,c} + \frac{(\vec{f}_h + mg \sin \theta) t}{m}$$

$$\Rightarrow t = \frac{\vec{v}_{g,c}}{\frac{R^2 \vec{f}_h}{I_{cm}} + \frac{\vec{f}_h + mg \sin \theta}{m}} < \frac{\vec{v}_{g,c}}{\frac{\vec{f}_h}{m} + \frac{J_h R^2}{I_{cm}}}$$

The ball takes less time to reach no-slip velocity. However, $mg \sin \theta$ still present means the ball will continue to accelerate.

$$\vec{v}_{c,sp} = \vec{v}_{cm,sp} - \vec{v}_{cm,c} = \vec{v}_{cm,sp} + \vec{v}_{c,cm}$$

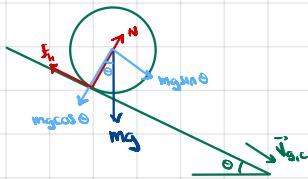
$$= -R \hat{j} \times \vec{\omega}_{cm} \hat{h} + \frac{(\vec{f}_h + mg \sin \theta) t}{m} \hat{i} - \vec{v}_{g,c} \hat{i}$$

$$= R \vec{\omega}_{cm} \hat{i} + \frac{(\vec{f}_h + mg \sin \theta) t}{m} \hat{i} - \vec{v}_{g,c} \hat{i}$$

$$= \hat{i} \left[-\vec{v}_{g,c} + \frac{R^2 \vec{f}_h t}{I_{cm}} + \frac{(\vec{f}_h + mg \sin \theta) t}{m} \right]$$

This equation is only valid up until no slipping happens. After this point we must start the analysis again because we once again have kinetic friction, but in the opposite direction.

$$\vec{\omega}_{cm} (t_{ns}) =$$



$$\vec{\omega}_{cm,sys} = -R\hat{j} \times (-S_h\hat{i}) - R\dot{S}_h\hat{h} = I_{cm}\vec{x}_{cm}\hat{h}$$

$$\Rightarrow \vec{x}_{cm} = \frac{-R\dot{S}_h}{I_{cm}} \hat{h}$$

$$\vec{\omega}_{cm} = (\omega_{ns} - \frac{R\dot{S}_h t}{I_{cm}})\hat{h}$$

$$(mg\sin\theta - S_h)\hat{i} - mg\sin\theta\hat{t} \Rightarrow \vec{a}_{g,cm} = \frac{mg\sin\theta - S_h}{m}\hat{i}$$

$$\vec{v}_{g,cm} = (v_{g,cm,ns} + \frac{(mg\sin\theta - S_h)t}{m})\hat{i}$$

$$\vec{v}_{c,cm} = \vec{v}_{g,cm} - \vec{v}_{gc} \quad \text{smaller than before}$$

$$= (v_{g,cm,ns} + \frac{(mg\sin\theta - S_h)t}{m})\hat{i} - v_{gc}\hat{i}$$

$$\vec{v}_{c,cp} = \vec{v}_{cm,cp} - \vec{v}_{mc,c} = \vec{v}_{cm,cp} + \vec{v}_{c,cm}$$

$$= [R\omega_{ns} - \frac{R^2\dot{S}_h}{I_{cm}}t + v_{g,cm,ns} + \frac{(mg\sin\theta - S_h)t}{m} - v_{gc}]\hat{i}$$

$$\vec{v}_{cm,cp} = (\omega_{ns} - \frac{R\dot{S}_h t}{I_{cm}})\hat{h} \times (-R\hat{j}) = (-R\omega_{ns} + \frac{R^2\dot{S}_h t}{I_{cm}})(-\hat{i})$$

$$= (R\omega_{ns} - \frac{R^2\dot{S}_h t}{I_{cm}})\hat{i}$$

$$\vec{a}_{c,cp} = \frac{R^2\dot{S}_h}{I_{cm}}\hat{i} + \frac{mg\sin\theta - S_h}{m}\hat{h}$$

$$= S_h\left(\frac{R^2}{I_{cm}} - \frac{1}{m}\right) + g\sin\theta$$

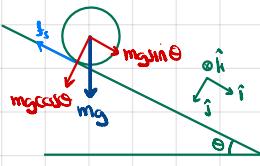
$$= S_h\left(\frac{s}{2m} - \frac{1}{m}\right) + g\sin\theta$$

$$= S_h \cdot \frac{3}{2m} + g\sin\theta > 0$$

$$\vec{v}_{g,cp} = \vec{v}_{gc} + \vec{v}_{c,cp}$$

$$= \vec{v}_{gc} + 0$$

$$= \vec{v}_{gc}$$



$$(mgsin\theta - f_s)\hat{i} = \vec{a}_{cm}$$

$$\vec{a}_{cm} = \frac{mgsin\theta - f_s}{m}\hat{i}$$

$$\vec{\tau}_{cm,s,p} = R\hat{j} \times (-f_s)\hat{i} = Rf_s\hat{i} = I_{cm}\alpha_{cm}\hat{i}$$

$$\Rightarrow \alpha_{cm} = \frac{Rf_s}{I_{cm}}\hat{i}$$

$$\text{no slipping} \Rightarrow \vec{v}_{c,cm} = -\vec{v}_{cm,p} = -\omega_{cm}\hat{i} \times R\hat{j}$$

$$= \omega_{cm}R\hat{i}$$

$$\Rightarrow \vec{\omega}_{cm} = \alpha_{cm}R\hat{i}$$

$$\Rightarrow \alpha_{cm}R = \frac{mgsin\theta - f_s}{m}$$

$$\frac{R^2f_s}{I_{cm}} = \frac{mgsin\theta - f_s}{m}$$

$$f_s \left(\frac{R^2}{I_{cm}} + \frac{1}{m} \right) = gsin\theta$$

$$f_s = \frac{gsin\theta}{\frac{R^2}{I_{cm}} + \frac{1}{m}}$$

$$I_{cm} = \frac{2\pi R^2}{5} \text{ (solid sphere)}$$

$$\Rightarrow f_s = \frac{gsin\theta}{\frac{5}{2m} + \frac{1}{m}} = \frac{gsin\theta}{\frac{7}{2m}}$$

$$= \frac{2mgsin\theta}{7}$$

$$f_{s,\max} = \mu mgcos\theta$$

$$f_s \leq f_{s,\max} \Rightarrow 2mgsin\theta \leq \mu mgcos\theta$$

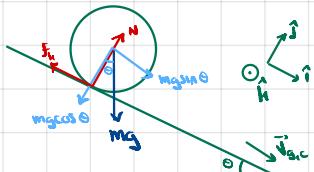
$$\Rightarrow tan\theta \leq \frac{1}{2}\mu$$

$$\vec{v}_{c,sp} = \vec{v}_{c,cm} + \vec{v}_{om,sp}$$

$$= \frac{mgsin\theta - f_s}{m}\hat{i} + \hat{i} - \frac{Rf_s t}{I_{cm}}\hat{i}$$

- (... sub in f_s)

= 0



$$(mgsin\theta - f_s)\hat{i} = \vec{a}_{cm}$$

$$\vec{a}_{cm} = \frac{mgsin\theta - f_s}{m}\hat{i}$$

$$\vec{v}_{g,cm} = (v_{g,cm,0} + \frac{(mgsin\theta - f_s)t}{m})\hat{i}$$

$$\vec{v}_{om,sp} = (-R\hat{j}) \times (-f_s)\hat{i} = Rf_s\hat{i} = I_{cm}\alpha_{cm}\hat{i}$$

$$\Rightarrow \vec{\alpha}_{cm} = \frac{-Rf_s}{I_{cm}}\hat{i}$$

$$\vec{\omega}_{cm} = (\omega_{cm,0} - \frac{Rf_s t}{I_{cm}})\hat{k}$$

$$\text{no slipping} \Rightarrow \vec{v}_{c,cm} = -\vec{v}_{cm,p} = -\omega_{cm}\hat{i} \times R\hat{j}$$

$$= R\omega_{cm}(-\hat{i}) = -R\omega_{cm}\hat{i}$$

$$\Rightarrow \vec{v}_{c,cm} = -R(\omega_{cm,0} - \frac{Rf_s t}{I_{cm}})\hat{i}$$

Note difference vs the static concrete case: here we have $\vec{v}_{c,cm}$, not $\vec{v}_{g,cm}$. We can't yet directly relate this to $\vec{v}_{g,cm}$.

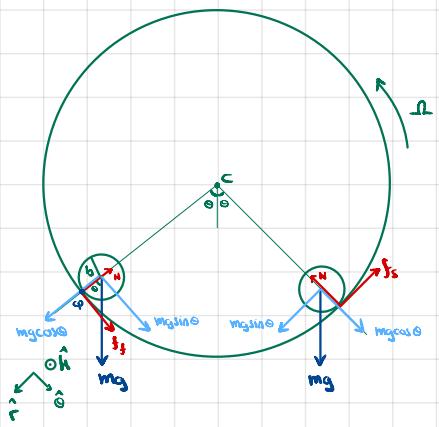
$$\vec{v}_{g,cm} = \vec{v}_{g,c} + \vec{v}_{c,cm} = v_{g,c}\hat{i} - R(\omega_{cm,0} - \frac{Rf_s t}{I_{cm}})\hat{i}$$

$$= \hat{i}[v_{g,c} - R(\omega_{cm,0} - \frac{Rf_s t}{I_{cm}})]$$

→ check $v_{g,cm}$ must be no slipping to occur

$$\Rightarrow v_{g,c} - R(\omega_{cm,0} - \frac{Rf_s t}{I_{cm}}) = v_{g,cm,0} + \frac{(mgsin\theta - f_s)t}{m}$$

→ $v_{g,cm}$ according to 2nd law law



$$\vec{r}_{c,sys} = r\hat{i} \times (mg\sin\theta + f_s)\hat{\theta} + b\beta_s\hat{k}$$

$$= r(mg\sin\theta + f_s)\hat{k} + b\beta_s\hat{k}$$

$$\vec{r}_{c,sys} = r\hat{i} \times m\vec{v}_{c,cm} + I_{cm}\omega_{cm}\hat{k}$$

$$= rm\vec{v}_{c,cm}\hat{k} + I_{cm}\omega_{cm}\hat{k} = I_c\omega\hat{k}$$

$$\frac{d\vec{r}_{c,sys}}{dt} = rm\alpha_{cm}\hat{k} + I_{cm}\alpha_{cm}\hat{k} = I_c\alpha\hat{k}$$

$$\Rightarrow I_{mac,cm}\hat{k} = / (mg\sin\theta + f_s)\hat{k}$$

$$\Rightarrow \alpha_{cm} = \frac{mg\sin\theta + f_s}{m}$$

$$(mg\sin\theta + f_s)\hat{\theta} - m\vec{a}_{g,cm} = \frac{mg\sin\theta + f_s}{m}\hat{\theta}$$

$$\vec{r}_{cm,sys} = b\hat{i} \times \beta_s\hat{\theta} - b\beta_s\hat{k}$$

$$\Rightarrow b\beta_s\hat{k} = \frac{dI_{cm,sys}}{dt} \cdot I_{cm}\alpha_{cm}\hat{k}$$

$$\Rightarrow \vec{r}_{cm} = \frac{b\beta_s}{I_{cm}}\hat{k}$$

$$(mg\cos\theta - N)\hat{i} - mr\omega^2\hat{r} \Rightarrow \vec{N} = -N\hat{i} = -m(r\omega^2 + g\cos\theta)$$

no slipping: $\vec{v}_{r,cm} = -\vec{v}_{cm,cp} = -\omega_{cm}\hat{k} \times b\hat{i}$

$$= -b\omega_{cm}\hat{\theta} \Rightarrow \vec{v}_{r,cm} = -b\omega_{cm}\hat{\theta}$$

$$\vec{v}_{g,cm} = \vec{v}_{g,r} + \vec{v}_{r,cm} = \vec{\omega}_r \times \vec{r}_{c,r} + \vec{v}_{r,cm}$$

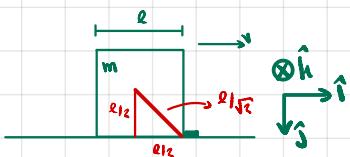
$$= -\Omega\hat{k} \times R\hat{i} + \vec{v}_{r,cm} = -\Omega R\hat{\theta} - b\omega_{cm}\hat{\theta}$$

$$\vec{v}_{g,cm} = \hat{\theta}(-\Omega R - b\omega_{cm})$$

From torque eq., $\vec{v}_{g,cm} = \frac{(mg\sin\theta + f_s)t}{m}\hat{\theta}$

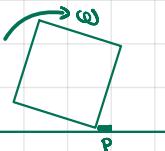
$$\vec{a}_{cm} = \underline{\underline{b\beta_s t}}$$

Problem 3 - Cubical Block Collision w/ Loss Ridge



no friction

inelastic, instantaneous collision



a) collision forces at point P apply no torque about P

$$\Rightarrow \vec{\tau}_{p,sfs, \text{collision}} = \frac{d\vec{\ell}_{p,sfs, \text{collision}}}{dt} = \vec{0}$$

$$\vec{\ell}_{p,sfs,A} = \left(-\frac{l}{2}\hat{i} - \frac{l}{2}\hat{j} \right) \times m\vec{v}_{x,i} \hat{i}$$

$$= \frac{l m v_{x,i}}{2} \hat{k}$$

$$\vec{\ell}_{p,sfs,B} = I_p \omega \hat{k}$$

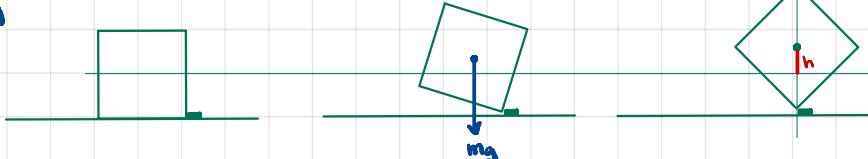
$$\Delta \vec{\ell} = \vec{0} \Rightarrow \omega = \frac{lm v_{x,i}}{2I_p}$$

$$I_p = I_{cm} + m \frac{l^2}{2} = \frac{ml^2}{6} + \frac{ml^2}{2}$$

$$= \frac{2ml^2}{3}$$

$$\Rightarrow \omega_i = \frac{3\cancel{m/l/v_{x,i}}}{4\cancel{m/l^2}} = \frac{3v_{x,i}}{4l}$$

b)



there is torque about P due to gravity.

$$E_{mi} = \frac{I_p \omega_i^2}{2}$$

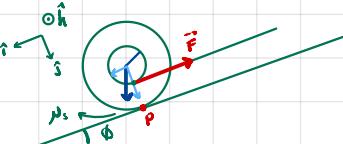
$$E_{mg} = mg \left(\frac{l}{\sqrt{2}} - \frac{l}{2} \right)$$

$$\frac{I_p \omega_i^2}{2} = mg \left(\frac{l}{\sqrt{2}} - \frac{l}{2} \right)$$

$$\Rightarrow \frac{1}{2} \cancel{I_p \omega_i^2} \cdot \cancel{\frac{3}{16} v_{x,i}^2} = mg \left(\frac{l}{\sqrt{2}} - \frac{l}{2} \right)$$

$$v_{x,i} = \sqrt{\frac{16}{3} g \left(\frac{l}{\sqrt{2}} - \frac{l}{2} \right)}$$

Problem 4 - Yo-Yo Rolling on Inclined Plane



First, consider case of no friction

$$\text{Torque: } \vec{\tau}_{\text{com,sys}} = \frac{R}{3} \hat{j} \times (-F\hat{i}) + R\hat{j} \times (-f_s)\hat{i} \\ \Rightarrow \vec{\alpha}_{cm} = \frac{RF}{3I_{cm}} \hat{k} \text{ angular accn due to torque}$$

$$\text{2nd law: } mgsin\theta - F \cdot m \cdot a_{cm} \Rightarrow a_{cm} = \frac{mgsin\theta - F}{m} \hat{i} \\ \text{accn of CM due to net force}$$

$$\text{no slipping: } \vec{v}_{cm} \cdot \vec{v}_{cm,p}, \text{ ie } \vec{v}_{cm} + \vec{\tau}_{cm} \cdot \vec{r}_p = 0$$

$$\Rightarrow \vec{v}_{cm} = -\omega_{cm} \hat{k} \times R\hat{j} = \omega_{cm} R \hat{i} \quad \begin{matrix} \text{velocity at each instant} \\ \text{compatible w/ no slipping} \end{matrix}$$

$$\vec{v}_{cm} = \frac{(mgsin\theta - F)t}{m} \hat{i} = \omega_{cm} R \hat{i} = \frac{R^2 t}{3I_{cm}} \hat{i}$$

$$\Rightarrow (mgsin\theta - F) \cdot 3I_{cm} = mR^2 F$$

$$F(mR^2 + 3I_{cm}) = mg \cdot 3I_{cm} sin\theta$$

$$F_{ns} = \frac{3mgsin\theta \cdot I_{cm}}{mR^2 + 3I_{cm}} \quad \begin{matrix} \text{Value of } F \text{ compatible w/ no} \\ \text{slipping.} \end{matrix}$$

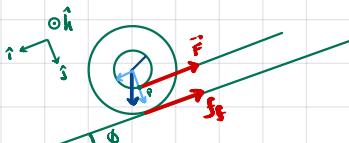
$$F > F_{ns} \Rightarrow \uparrow \alpha_{cm}, \downarrow a_{cm} \Rightarrow \text{too much torque, } |\vec{\tau}_{cm}| < |\omega_{cm} R| \\ \Rightarrow \text{slipping}$$

$$F < F_{ns} \Rightarrow \downarrow \alpha_{cm}, \uparrow a_{cm} \Rightarrow \text{too little torque, too little rotation}$$

\Rightarrow Skidding

Extreme case: $F=0 \Rightarrow$ no rotation only CM translation, extreme form of skidding.

Now add friction



- 1) $\vec{F} = 0, \text{ no friction} \Rightarrow$ skidding, zero rotation
- 2) $\vec{F} = 0 \Rightarrow$ if f_s high enough, friction provides torque compatible with no slipping: $a_{cm} = \frac{R}{3} \alpha_{cm}$. Contact point is always at rest rel. to surface but there is a force from contact point onto surface.
- 3) $F > 0$, ie pulling on string \Rightarrow \uparrow torque, $\uparrow \alpha_{cm}$

$$\text{Torque: } \vec{\tau}_{\text{com,sys}} = \frac{R}{3} \hat{j} \times (-F\hat{i}) + R\hat{j} \times (-f_s)\hat{i}$$

$$= \frac{RF}{3} \hat{k} + Rf_s \hat{k} = I_{cm} \alpha_{cm} \hat{k} \Rightarrow \alpha_{cm} = \frac{R(\frac{F}{3} + f_s)}{I_{cm}} \hat{k}$$

$$\text{2nd law: } (mgsin\theta - F - f_s)\hat{i} = m \cdot a_{cm}\hat{i}$$

$$\vec{a}_{cm} = \frac{mgsin\theta - F - f_s}{m} \hat{i}$$

$$\text{no slipping: } \vec{v}_{cm} = -\omega_{cm} \hat{k} \times R\hat{j} = \omega_{cm} R \hat{i}$$

$$\Rightarrow \frac{R(\frac{F}{3} + f_s)\hat{k} \cdot R}{I_{cm}} = \frac{(mgsin\theta - F - f_s)\hat{i}}{m}$$

$$R^2 m (\frac{F}{3} + f_s) = I_{cm} (mgsin\theta - F - f_s)$$

$$F \left[\frac{R^2 m}{3} + I_{cm} \right] = I_{cm} (mgsin\theta - F - f_s) - R^2 m f_s$$

$$F = \frac{I_{cm} mgsin\theta - f_s (I_{cm} + R^2 m)}{\frac{R^2 m}{3} + I_{cm}}$$

If $F - mgsin\theta > 0$ then f_s will point in same direction as $mgsin\theta$, ie f_s will be negative.

If $f_{s,\max} = \mu mg \cos\theta$ is the largest magnitude of static friction then

$$F_{max} = \frac{I_{cm} mgsin\theta + f_{s,\max} (I_{cm} + R^2 m)}{\frac{R^2 m}{3} + I_{cm}}$$

$$= \frac{3I_{cm} mgsin\theta + 3\mu mg \cos\theta (I_{cm} + mR^2)}{mR^2 + 3I_{cm}}$$

$$> \frac{3mgsin\theta \cdot I_{cm}}{mR^2 + 3I_{cm}}$$

In the case of no friction

$$F - F_{\text{N}} = 0$$

$$\vec{a}_{\text{cm}} = \frac{mg \sin \theta \cdot R^2}{mR^2 + 3I_{\text{cm}}} \hat{i}$$

$$\vec{\alpha}_{\text{cm}} = \frac{mg \sin \theta \cdot R}{mR^2 + 3I_{\text{cm}}} \hat{h}$$

$$\text{For a solid sphere, } I_{\text{cm}} = \frac{2mR^2}{5}$$

$$\vec{a}_{\text{cm}} = \frac{5g \sin \theta}{11} \hat{i}$$

$$\vec{\alpha}_{\text{cm}} = \frac{5g \sin \theta}{11R} \hat{h}$$

$$\vec{F}_{\text{N}} = -\frac{6mg \sin \theta}{11} \hat{i}$$

In the case of friction and a solid sphere

$$\vec{a}_{\text{cm}} = \left(\frac{5g \sin \theta}{11} - \frac{10\mu g \cos \theta}{11} \right) \hat{i}$$

$$\vec{\alpha}_{\text{cm}} = \left(\frac{5g \sin \theta}{11R} - \frac{10\mu g \cos \theta}{11R} \right) \hat{h}$$

$$\vec{F}_{\text{max}} = -\left(\frac{6mg \sin \theta}{11} + \frac{21mg \cos \theta}{11} \right) \hat{i}$$

If $\theta = 45^\circ$ and $\mu = 1$ (rubber)

$$\vec{a}_{\text{cm}} = -\frac{5g}{11} \frac{\sqrt{2}}{2} \hat{i} < 0 \Rightarrow \text{cccl. up the slope}$$

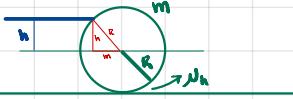
For the frictionless , $I_{\text{cm}} = \frac{mR^2}{2}$

$$\Rightarrow \vec{F}_{\text{max}} = -\frac{3mg(\sin \theta + 3\mu \cos \theta)}{5} \hat{i}$$

$$\vec{a}_{\text{cm}} = \left(\frac{2g \sin \theta}{5} - \frac{4\mu g \cos \theta}{5} \right) \hat{i}$$

$$\vec{\alpha}_{\text{cm}} = \left(\frac{2g \sin \theta}{5R} - \frac{4\mu g \cos \theta}{5R} \right) \hat{h}$$

Problem 5 - Billiards



The we stick applies an angular impulse $\int_{t_i}^{t_f} \vec{I}_{cm,sys} dt$, changing angular momentum. There is also impulse $\int_{t_i}^{t_f} \vec{F}_h dt$, changing momentum.

After the collision the ball has v_0 , ω_0 . The ball slides because $v_0 > R\omega_0$.



The ball accelerates at some point after the collision.

The only force acting in x-dir after collision is friction, so it must be in +x direction.

This is due to the contact point having negative velocity relative to the ground.

$$f_k = m \cdot a_{cm} \Rightarrow a_{cm} = \frac{f_k}{m} \quad \text{Friction accelerates the ball's CM}$$

$$v_{cm} = v_0 + a_{cm} t$$

we know final and initial speeds \Rightarrow we know $\frac{d}{dt} v_{cm} = a_{cm} t = \frac{2v_0}{T}$ impulse required

$$\int_{t_i}^{t_f} f_k dt = \frac{2v_0 m}{T}$$

$$\Rightarrow t_{fr} = \frac{2v_0 m}{7f_k} \quad \text{time of friction impulse}$$

f_k accelerates the ball but also reduces angular speed.

Because the ball is initially sliding, $v_{cm} < R\omega_0$.

$$\vec{I}_{cm,sys} = -R\hat{j} \times f_k \hat{i} = R f_k \hat{h} = I_{cm} \alpha_{cm} \hat{h}$$

$$\alpha_{cm} = \frac{R f_k}{I_{cm}} \hat{h} \quad \text{angular acceleration caused by torque due to } f_k$$

$$\vec{\omega}_{cm} = (\omega_0 + \frac{R f_k t}{I_{cm}}) \hat{h} = \omega_{cm} \hat{h}$$

$$\omega_{cm,fin} = \omega_0 + \frac{R f_k}{I_{cm}} \cdot \frac{2v_0 m}{7f_k} = \omega_0 + \frac{2v_0 m}{7I_{cm}}$$

$$\text{no slipping} \quad \vec{v}_{cm} = -\vec{v}_{cm,rel}$$

$$= -\omega_{cm} \hat{h} \times (-R\hat{j}) = -R\omega_{cm} \hat{i}$$

$$\Rightarrow v_{cm,fin} = -R\omega_{cm,fin}$$

$$\frac{d}{dt} v_0 = -R(\omega_0 + \omega_{cm,fin}) = -(R\omega_0 + \frac{2R^2 m v_0}{7I_{cm}})$$

$$\Rightarrow \omega_0 = \frac{-2R^2 m v_0 - 9v_0 I_{cm}}{7I_{cm} \cdot R}$$

Initial angular impulse

$$\int_{t_i}^{t_f} \vec{I}_{cm,sys} dt = \vec{I}_{cm,sys,0} = I_{cm} \cdot \omega_0 \hat{h}$$

$$\vec{I}_{cm,sys} = (h\hat{j} - m\hat{i}) \times \vec{F}_h = -F_h h \hat{h}$$

$$\int_{t_i}^{t_f} -F_h h \hat{h} dt = -F_h t_i h \hat{h} = I_{cm} \omega_0 h \hat{h} \Rightarrow \omega_0 = \frac{-F_h t_i}{I_{cm}}$$

Initial Impulse

$$\int_{t_i}^{t_f} \vec{F}_h dt = \vec{P}_{sys,0} = m v_0$$

$$F_h t_i = m v_0$$

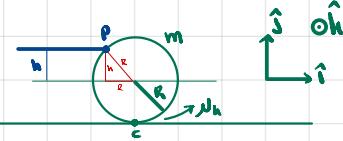
$$\Rightarrow \omega_0 = \frac{-h F_h t_i}{I_{cm}} = \frac{-h m v_0}{I_{cm}} = \frac{-Sh v_0}{2\pi R^2} = -\frac{Sh v_0}{2R^2}$$

$$\text{recall } \omega_0 = \frac{-2R^2 m v_0 - 9v_0 I_{cm}}{7I_{cm} \cdot R}$$

$$I_{cm} = \frac{2\pi m R^2}{5} \Rightarrow \omega_0 = -\frac{2v_0}{R}$$

$$-\frac{2v_0}{R} = -\frac{Sh v_0}{2R^2} \Rightarrow \frac{h}{R} = \frac{4}{5}$$

solution 2



$$\vec{L}_{\text{p,sys}} = \vec{0}$$

$$= \vec{L}_{\text{p,orbital}} + \vec{L}_{\text{p,spin}}$$

$$= (l\hat{i} - h\hat{j}) \times mV_0\hat{i} + I_{\text{cm}}\omega_0\hat{k}$$

$$= -hmV_0(-\hat{i}) + I_{\text{cm}}\omega_0\hat{k}$$

$$= \hat{k} (hmV_0 + I_{\text{cm}}\omega_0)$$

$$\Rightarrow mV_0h = -I_{\text{cm}}\omega_0$$

$$\vec{L}_{\text{c,sys},i} = R\hat{j} \times mV_0\hat{i} + I_{\text{cm}}\omega_0\hat{k}$$

$$= -RmV_0\hat{k} + I_{\text{cm}}\omega_0\hat{k}$$

$$= \hat{k} (-mV_0R - mV_0h)$$

$$= \hat{k} (-mV_0(R+h))$$

$$\vec{L}_{\text{c,sys},f} = R\hat{j} \times mV_f\hat{i} + I_{\text{cm}}\omega_f\hat{k}$$

$$V_f = -R\omega_f$$

$$\Rightarrow \vec{L}_{\text{c,sys},f} = -RmV_f\hat{k} - \frac{I_{\text{cm}}\omega_f}{R}\hat{k}$$

$$= \hat{k} \left(-RmV_f - \frac{2mR^2\omega_f}{s} \right)$$

$$= \hat{k} \left(\frac{-5RmV_f - 2Rm\omega_f}{s} \right)$$

$$= \hat{k} \left(\frac{-7mRV_f}{s} \right)$$

$$= \hat{k} \left(\frac{-1mR}{s} \cdot \frac{9V_0}{t} \right)$$

$$= \hat{k} \left(-\frac{9}{5}mRV_0 \right)$$

$$\vec{L}_{\text{c,sys},i} = \vec{L}_{\text{c,sys},f}$$

$$\Rightarrow -mV_0(R+h) = -\frac{9}{5}mRV_0$$

$$mV_0h = \frac{9}{5}mRV_0 - mRV_0$$

$$= \frac{4}{5}mRV_0 \Rightarrow \frac{h}{R} = \frac{4}{5}$$