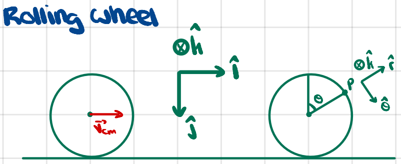
20.2 Constrained Motion: Translation and Rotation

Rolling wheel



$\vec v_{g,p} = \vec v_{g,cm} + \vec v_{cm,p}$

$\vec v_{g,cm} = v_{g,cm}\hat i$

$\vec r_{g,cm}(t) = (x_{g,cm,0} + v_{g,cm}t)\hat i$

$\vec\omega_{cm} = \omega_{cm}\hat k$

In CM frame P has uniform circular motion

$\vec v_{cm,p} = R\omega_{cm}\hat\theta$

We want to know $\vec v_{g,p}$, but $\vec v_{cm,p}$ is in polar, $\vec v_{cm,p}$ is in Cartesian.

Note the unusual choice for measuring θ and $\hat r, \hat\theta$.

$\Rightarrow \hat r = \sin\theta\,\hat i - \cos\theta\,\hat j$

$\hat\theta = \cos\theta\,\hat i + \sin\theta\,\hat j$

$\vec v_{cm,p} = R\omega_{cm}(\cos\theta\,\hat i + \sin\theta\,\hat j)$

$\Rightarrow \vec v_{g,p} = v_{g,cm}\hat i + R\omega_{cm}(\cos\theta\,\hat i + \sin\theta\,\hat j)$

$\vec v_{g,p} = \hat i(v_{g,cm} + R\omega_{cm}\cos\theta) + \hat j R\omega_{cm}\sin\theta$

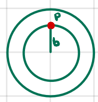
$\theta = \pi \Rightarrow$ p is contact point w/ ground

$v_{g,p}(\theta=\pi) = \hat i(v_{g,cm} - R\omega_{cm})$

if the wheel is not slipping, $v_{g,cm} = R\omega_{cm}$

$\Rightarrow v_p(\theta=\pi) = \vec 0$

Example 20.1 Bicycle wheel Rolling w/o Slipping



a) $\vec r_{g,p} = \vec r_{g,cm} + \vec r_{cm,p}$

$= (x_{g,cm,0} + v_{g,cm}t)\hat i + b(\sin\theta\,\hat i - \cos\theta\,\hat j)$

$= \hat i(x_{g,cm,0} + v_{g,cm}t + b\sin\theta) - b\cos\theta\,\hat j$

$\vec v_{g,p} = \hat i(v_{g,cm} + b\theta'(t)\cos\theta) + b\theta'(t)\sin\theta\,\hat j$

as expected given formula derived for $\vec v_{g,p}$ in general.

$\vec a_{g,p} = \hat i(b\theta''(t)\cos\theta - b\theta'^2(t)\sin\theta)$

$+ \hat j(b\theta''(t)\sin\theta + b\theta'^2(t)\cos\theta)$

in the CM frame

$\vec r_{cm,p} = b\hat r$

$\vec v_{cm,p} = b\omega\hat\theta$

$\vec a_{cm,p} = -b\omega^2\hat r$

ie uniform circular motion

Let's obtain these expressions from our results for the ground frame.

$\vec r_{cm,p} = \vec r_{g,p} - \vec r_{g,cm}$

$= \hat i(x_{g,cm,0} + v_{g,cm}t + b\sin\theta) - b\cos\theta\,\hat j$

$- (x_{g,cm,0} + v_{g,cm}t)\hat i$

$= \hat i\,b\sin\theta - \hat j\,b\cos\theta = b\hat r$

$\vec v_{cm,p} = \vec v_{g,p} - \vec v_{g,cm}$

$= \hat i(v_{g,cm} + b\omega_{cm}\cos\theta) + \hat j\,b\omega_{cm}\sin\theta$

$- v_{g,cm}\hat i$

$= \hat i(b\omega_{cm}\cos\theta) + \hat j\,b\omega_{cm}\sin\theta$

$= b\omega_{cm}\hat\theta$

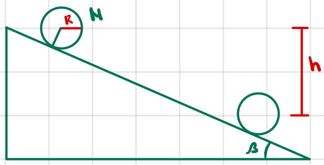
$\vec a_{cm,p} = b\omega_{cm}\cdot\frac{d\hat\theta}{dt}$

$\frac{d\hat\theta}{dt} = \omega_{cm}(-\sin\theta\,\hat i + \cos\theta\,\hat j)$

$= -\omega_{cm}\hat r$

$\Rightarrow \vec a_{cm,p} = -b\omega_{cm}^2\hat r$

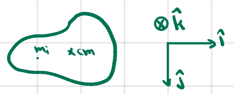
Example 20.2



M rolls w/o slipping

The ball starts w/ zero linear and angular momentum. To roll, a torque must act to change the angular momentum.

Gravity doesn't provide torque. We know that torque due to gravity on a rigid body is equivalent to a torque applied to a point particle at the center of mass.

* 

$\vec\tau_{s,i} = \vec r_{s,i}\times(m_i g\hat j)$

$= (\vec r_{s,i}\cdot m_i)\times g\hat j$

$\sum\vec\tau_{s,i} = \sum(\vec r_{s,i}\cdot m_i)\times g\hat j$

$= m\cdot\vec r_{s,cm}\times g\hat j$

$= \vec r_{s,cm}\times mg\hat j$

$= \vec r_{s,cm}\times\vec F_g$

$= \vec\tau_{s,g}^{\,particle}$

About the CM gravity applies no torque.

Friction at the contact point provides torque relative to CM.

$\vec\tau_{cm,f} = R\hat j\times(-f_s\hat i)$

$= Rf_s\hat k$, the net torque about CM axis

$= I_{cm}\alpha\hat k$

$\Rightarrow \frac{MR^2}{2}\alpha = Rf_s$

$\alpha = \frac{2f_s}{MR}$

Using 2nd law we can find the acceleration of the CM. Recall we can do this because the wheel is a system of particles

$\vec p_{sys} = \sum m_i\vec v_i = m_{sys}\cdot\vec v_{cm}$

$\frac{d\vec p_{sys}}{dt} = \vec F = m_{sys}\cdot\vec a_{cm}$

$\hat i$: $mg\sin\beta - f_s = M\cdot a_{cm}$

$\hat j$: $mg\cos\beta - N = 0$

The torque equation told us how much angular acceleration there is as a function of friction.

2nd law tells us what friction is as a function of acceleration of center of mass

If the ball rolls w/o slipping then

$a_{cm} = R\alpha$

Therefore we have

$mg\sin\beta - f_s = M\cdot a_{cm}$

$\alpha = \frac{2f_s}{MR}$

$a_{cm} = R\alpha$

$f_s = \mu N$

$N = mg\cos\beta$

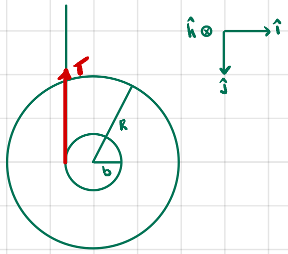
5 equations

unknowns: N, f_s, α, a_{cm}

$a_{cm} = \frac{2\mu mg\cos\beta}{M}$

$\alpha = \frac{2\mu mg\cos\beta}{MR}$

Example 20.3 Falling Yo-yo



net torque eq. relates forces to angular acceleration

$$bT = \frac{mR^2}{2}\alpha \Rightarrow \alpha = \frac{2bT}{mR^2}$$

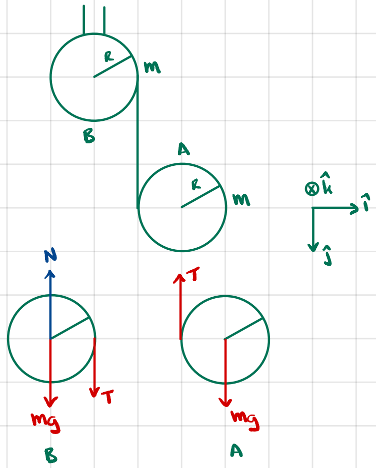
2nd law relates forces to linear acceleration

$$mg - T = m \cdot a$$

2 equations, 3 unknowns: T, a, α

no slipping $\Rightarrow a = b\alpha$

Example 20.4 – Unwinding Drum



The metal tape applies a torque on each drum.

$$RT = I_{cm}\alpha_B$$
$$RT = I_{cm}\alpha_A \Rightarrow \alpha_A = \alpha_B = \alpha$$

$$mg + T - N = 0$$
$$mg - T = m \cdot a_A$$

The tape has a fixed length. The vertical change in position of A equals the change in unrolled tape. Unrolling is either because of drum B or drum A

$$R\Delta\theta_B + R\Delta\theta_A = \Delta y_A$$

$$\frac{dy}{dt} = R\frac{d\theta_A}{dt} + R\frac{d\theta}{dt}$$

$$\Rightarrow \ddot{y} = R\ddot{\theta}_A + R\ddot{\theta}_B$$
$$= R(\alpha_A + \alpha_B)$$

20.3 Angular Momentum For System of Particles Undergoing Translation and Rotation

system of N particles

$$\vec{L}_S^{Total} = \sum L_{S,i} = \sum \vec{r}_{S,i} \times m_i \vec{v}_{S,i}$$

$$= \sum (\vec{r}_{S,cm} + \vec{r}_{cm,i}) \times m_i (\vec{v}_{S,cm} + \vec{v}_{cm,i})$$
$$= \sum \vec{r}_{S,cm} \times m_i \vec{v}_{S,cm}$$
$$+ \sum \vec{r}_{S,cm} \times m_i \vec{v}_{cm,i}$$
$$+ \sum \vec{r}_{cm,i} \times m_i \vec{v}_{S,cm}$$
$$+ \sum \vec{r}_{cm,i} \times m_i \vec{v}_{cm,i}$$

$$= \vec{r}_{S,cm} \times \left(\sum m_i\right) \vec{v}_{S,cm}$$
$$+ \vec{r}_{S,cm} \times \cancelto{0}{\sum m_i \vec{v}_{cm,i}}$$
$$+ \left(\cancelto{0}{\sum \vec{r}_{cm,i} \times m_i}\right) \vec{v}_{S,cm}$$
$$+ \sum \vec{r}_{cm,i} \times m_i \vec{v}_{cm,i}$$

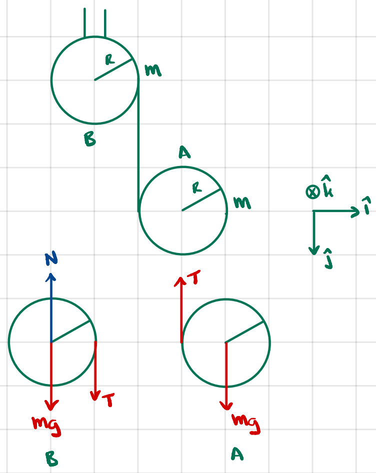
$$= \vec{r}_{S,cm} \times m_T \vec{v}_{S,cm} + \sum r_{cm,i} \times m_i \vec{v}_{cm,i}$$
$$= \underbrace{\vec{r}_{S,cm} \times \vec{p}_{sys}}_{\text{orbital angular momentum}} + \underbrace{\sum \vec{r}_{cm,i} \times m_i \vec{v}_{cm,i}}_{\text{spin angular momentum}}$$

orbital angular momentum

spin angular momentum: total ang. mom. of the system w/ respect to the CM in the CM frame

$\vec{L}_S^{orbital} = \vec{r}_{S,cm} \times \vec{p}_{sys}$, angular momentum about S of point particle with mass of the system at center of mass.

$$\vec{L}_S^{total} = \vec{L}_S^{orbital} + \vec{L}_{cm}^{spin}$$



System: drum A + drum B

net torque about CM_B

$\vec{\tau}_{cm_{B,B,s}} = R\hat{i}\times T\hat{j} = RT\hat{k}$

$\vec{\tau}_{cm_{B,A,s}} = (2R\hat{i}+R\hat{j})\times mg\hat{j} + (R\hat{i}+R\hat{j})\times -T\hat{j}$

$= 2Rmg\hat{k} - RT\hat{k}$

$= (2Rmg - RT)\hat{k}$

$\vec{\tau}_{cm_{B,sys,s}} = 2Rmg\hat{k}$

Drum B

2nd law: $mg + T - N = m\cdot a_B = 0$

torque: $\vec{\tau}_{cm_{B,B,s}} = RT\hat{k} = I_{cm_{B,B}}\cdot\alpha_B\hat{k}$

$\Rightarrow RT = I_{cm_{B,B}}\cdot\alpha_B$

Drum A

2nd law $mg - T = m\cdot a_A$

torque $\vec{\tau}_{cm_{A,A,s}} = -R\hat{i}\times(-T\hat{j})$

$= RT\hat{k} = I_{cm_{A,A}}\cdot\alpha_A\hat{k}$

$\Rightarrow RT = I_{cm_{A,A}}\cdot\alpha_A$

we also have the equation: $a_A = R(\alpha_A+\alpha_B)$

Assume $I_{cm_{B,B}} = I_{cm_{A,A}} = I_{cm}$

$N = mg + T$

$RT = I_{cm}\cdot\alpha_B$

$RT = I_{cm}\cdot\alpha_A$

$T = m(g - a_A)$

$a_A = R(\alpha_A+\alpha_B)$

unknowns: $\alpha_A, \alpha_B, T, N, a_A$

$\Rightarrow \alpha_A = \alpha_B = \alpha = \frac{RT}{I_{cm}}$

$\Rightarrow T = m\left(g - \frac{2R^2T}{I_{cm}}\right)$

$T\left(1 + \frac{2R^2m}{I_{cm}}\right) = mg$

$$T = \frac{mg}{1+\frac{2R^2m}{I_{cm}}}$$

$$= \frac{mgI_{cm}}{I_{cm}+2R^2m}$$

$$\alpha = \frac{mgR}{I_{cm}+2R^2m}$$

$$a_A = \frac{2mgR^2}{I_{cm}+2R^2m}$$

$$N = \frac{mgI_{cm} + m^2g\cdot 2R^2 + mgI_{cm}}{I_{cm}+2R^2m}$$

$$= \frac{2mgI_{cm} + 2m^2gR^2}{I_{cm}+2R^2m} = \frac{2mg(I_{cm}+R^2m)}{I_{cm}+2R^2m}$$

If $I = \frac{mR^2}{2}$

$RT = \frac{mR^2}{2}\alpha$

↑m means ↑I, linearly, slope $\frac{R^2}{2}$

$T = \frac{mg}{5}$ increases linearly, slope $\frac{g}{5}$

$\alpha = \frac{2g}{5R}$ doesn't change

$$\frac{Rmg}{5} = \frac{mR^{\cancel{2}}}{\cancel{2}}\cdot\frac{\cancel{2}g}{5\cancel{R}} = \frac{mgR}{5}$$

$$\frac{Rg}{5} = \frac{R^2}{\cancel{2}}\cdot\frac{\cancel{2}g}{5R} = \frac{Rg}{5}$$

$T = \frac{mg}{5}$

$\alpha = \frac{2g}{5R}$

$a_A = \frac{4g}{5}$

$N = \frac{6mg}{5}$

$\alpha = \frac{2g}{5R}$

$\omega = \frac{2gt}{5R}$

$\theta = \frac{gt^2}{5R}$

$s = R\theta = \frac{gt^2}{5}$

$\vec{L}_{cm_{B,B}} = \frac{mR^2}{\cancel{2}}\cdot\frac{\cancel{2}gt}{5R}\hat{k} = \frac{Rgmt}{5}\hat{k}$

$\vec{L}_{cm_{B,A}} = \vec{L}_{cm_{B,cm_A}} + \vec{L}_{cm_{A,A}}$

$= (2R\hat{i}+R\hat{j})\times m\cdot\frac{4gt}{5}\hat{j} + \frac{mR^2}{\cancel{2}}\cdot\frac{\cancel{2}gt}{5R}\hat{k}$

$= \frac{8Rgmt}{5}\hat{k} + \frac{Rgmt}{5}\hat{k}$

$= \frac{9Rgmt}{5}\hat{k}$

$\vec{L}_{cm_B} = 2Rgmt\hat{k}$

$\frac{d\vec{L}_{cm_B}}{dt} = 2Rgm\hat{k} = \vec{\tau}_{cm_{B,sys}}$

Ex 20.5 Earth's Motion Around the Sun

$m_e = 5.97 \cdot 10^{24}\,kg$

$R_e = 6.38 \cdot 10^{6}\,m$

circular orbit, $r_{s,e} = 1.50 \cdot 10^{11}\,m$

$T_{orbit} = 365.25\,days$

$T_{spin} = 23h\,56min = 23 \cdot 3600 + 56 \cdot 60\,s$

axis of spin is tilted $23.5°$

Earth $\approx$ sphere $\Rightarrow I_{cm_e,e} = \frac{2 m_e R_e^2}{5}$

$\vec{L}_{s,e}^{\,orbital} = \vec{r}_{s,cm_e} \times m_e v_{cm_e} \hat{\theta}$

$= r_{s,e}\,\hat{r} \times m_e v_{cm_e} \hat{\theta}$

$= r_{s,e}\, m_e v_{cm_e} \hat{k}$

$= r_{s,e}^2\, m_e\, \omega_{orbit}\, \hat{k}$

$\omega_{orbit} = \frac{2\pi}{T_{orbit}} = \frac{2\pi}{365.25 \cdot 24 \cdot 3600}$

$= 1.991 \cdot 10^{-7}\,rad \cdot s^{-1}$

$\Rightarrow \vec{L}_{s,e}^{\,orbital} = (1.50 \cdot 10^{11})^2 \cdot 5.97 \cdot 10^{24} \cdot 1.991 \cdot 10^{-7}\,\hat{k}$

$= 2.68 \cdot 10^{40}\,\hat{k}$

$\vec{L}_{s,e}^{\,spin} = I_{cm_e,e} \times \vec{\omega}_{spin} = I_{cm_e,e} \times \omega_{spin}\,\hat{n}$

$= \frac{2 m_e R_e^2}{5} \cdot \frac{2\pi}{8.616 \cdot 10^{4}}\,\hat{n}$

$= 7.10 \cdot 10^{33}\,\hat{n}$

$\frac{L_{s,e}^{orbital}}{L_{s,e}^{spin}} = \frac{5\, r_{s,e}^2\, T_{spin}}{2\, R_e^2\, T_{orbital}} = 3.77 \cdot 10^{6}$

20.4 Kinetic Energy of System of Particles

$$K = \frac{\sum m_i v_{s,i}^2}{2} = \frac{\sum m_i (v_{s,cm} + v_{cm,i})^2}{2}$$

$$= \frac{\sum m_i v_{s,cm}^2 + \sum m_i v_{cm,i}^2 + \sum m_i \cdot 2 v_{s,cm} v_{cm,i}}{2}$$

$$= \frac{m_T \cdot v_{s,cm}^2}{2} + \frac{\sum m_i v_{cm,i}^2}{2} + v_{cm} \cancelto{0}{\sum m_i v_{cm,i}}$$

$$= K_{s,cm} + K_{cm}$$

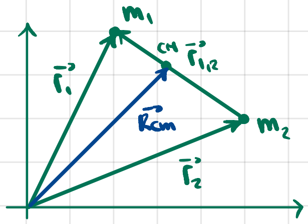
20.5 Rigid Body Undergoing Fixed Axis Rotation

$K_{sys} = K_{s,cm} + K_{cm} = K_{trans} + K_{rot}$

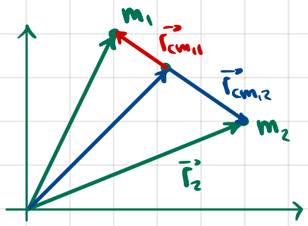
$= \frac{m_T v_{s,cm}^2}{2} + \frac{I_{cm}\,\omega_{cm}^2}{2}$

Appendix: Chasles' Theorem

"motion of any rigid body consists of a translation of the CM and rotation about the CM"



$$\vec{R}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$



$$\vec{r}_{cm,1} = \vec{r}_1 - \vec{R}_{cm} = \vec{r}_1 - \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$

$$= \frac{m_2(\vec{r}_1 - \vec{r}_2)}{m_1+m_2} = \frac{\mu}{m_1}\vec{r}_{1,2} \qquad \mu = \frac{m_1 m_2}{m_1+m_2}$$

$$\vec{r}_{1,2} = \vec{r}_1 - \vec{r}_2 = (\vec{r}_{cm,1} + \vec{R}_{cm}) - (\vec{r}_{cm,2} + \vec{R}_{cm})$$
$$= \vec{r}_{cm,1} - \vec{r}_{cm,2}$$
$$= \vec{r}_{cm,1,2}$$

relative position vectors are the same in any reference frame.

In CM frame $\frac{m_1\vec{r}_{cm,1} + m_2\vec{r}_{cm,2}}{m_1+m_2} = \vec{0}$

$$\Rightarrow m_1\vec{r}_{cm,1} = -m_2\vec{r}_{cm,2}$$

$$\Rightarrow m_1|\vec{r}_{cm,1}| = m_2|\vec{r}_{cm,2}|$$

$$d\vec{r}_{cm,1} = \frac{\mu}{m_1}d\vec{r}_{1,2}$$

Analogous calculations for $\vec{r}_{cm,2}$ yield

$$\vec{r}_{cm,2} = \vec{r}_2 - \vec{R}_{cm} = \vec{r}_2 - \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$

$$= \frac{m_1(\vec{r}_2 - \vec{r}_1)}{m_1+m_2} = \frac{\mu}{m_2}\vec{r}_{2,1}$$

$$= -\frac{\mu}{m_2}\vec{r}_{1,2}$$

$$d\vec{r}_{cm,2} = -\frac{\mu}{m_2}d\vec{r}_{1,2}$$

At this point we have positions of m_1 and m_2 from CM frame in terms of positions in initial frame. Same for position displacement vectors $d\vec{r}_{cm,i}$.

Arbitrary displacement of i^{th} particle

$$d\vec{r}_i = d\vec{r}_{cm,i} + d\vec{R}_{cm}$$

We have positions in CM in terms of initial frame, and the changes in the positions.

constraint: distance between particles is constant, ie rigid body

$$|\vec{r}_{1,2}|^2 = (\vec{r}_1 - \vec{r}_2)(\vec{r}_1 - \vec{r}_2)$$

But we're dealing with a specific kind of system, that of a rigid body.

Differentiate

$$0 = (\vec{r}_1' - \vec{r}_2')(\vec{r}_1 - \vec{r}_2) + (\vec{r}_1 - \vec{r}_2)(\vec{r}_1' - \vec{r}_2')$$

$$= 2(\vec{r}_1 - \vec{r}_2)(\vec{r}_1' - \vec{r}_2')$$

$$0 = 2(\vec{r}_1 - \vec{r}_2)(d\vec{r}_1 - d\vec{r}_2)$$

$$= 2\vec{r}_{1,2}\cdot d\vec{r}_{1,2}$$

(rigid body condition)

So we reach a condition reflecting that fact.

If $d\vec{r}_{1,2} = \vec{0}$ then the condition is satisfied.

$$\Rightarrow d\vec{r}_{cm,1} = d\vec{r}_{cm,2} = \vec{0}$$

$$\Rightarrow d\vec{r}_i = d\vec{R}_{cm}$$

(rigid body undergoes pure translation)

A system where the particles maintain relative position vector constant, not just the magnitude, is a rigid body in pure translation.

Suppose $d\vec{r}_{1,2} \neq \vec{0}$

rigid body condition rewritten

$$0 = d\vec{r}_{cm,1}\cdot\frac{m_1}{\mu\,\cancel{d\vec{r}_{1,2}}}\cdot 2\vec{r}_{1,2}\cdot\cancel{d\vec{r}_{1,2}}$$

$$= 2\frac{m_1}{\mu}\vec{r}_{1,2}\,d\vec{r}_{cm,1}$$

$$= 2\frac{m_1}{\mu}\vec{r}_{cm,1,2}\cdot d\vec{r}_{cm,1}$$

$$\Rightarrow 2\vec{r}_{cm,1,2}\cdot d\vec{r}_{cm,1}$$

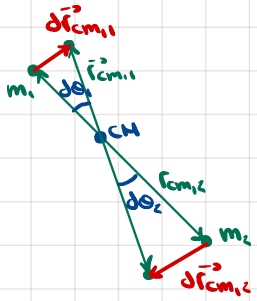
(rigid body condition in CM frame)

If however, the rel position vector changes direction, but not magnitude, we reach the conclusions that the infinitesimal displacement vectors $d\vec{r}_{cm,1}$ and $d\vec{r}_{cm,2}$ are $\perp$ to $\vec{r}_{cm,1,2}$, the vector from m_2 to m_1

$$\vec{r}_{cm,1,2} \perp d\vec{r}_{cm,1}$$

we can rewrite the rigid body condition differently and reach

$$\vec{r}_{cm,1,2} \perp d\vec{r}_{cm,2}$$



Furthermore, we conclude that m_1 and m_2 are rotating about the center of mass: change in position is always $\perp$ to position from center of mass and the angular displacement is the same for both masses.

$$d\theta_1 = \frac{|d\vec{r}_{cm,1}|}{|\vec{r}_{cm,1}|}$$

$$d\theta_2 = \frac{|d\vec{r}_{cm,2}|}{|\vec{r}_{cm,2}|}$$

But $d\vec{r}_{cm,1} = \frac{\mu}{m_1} \cdot \vec{r}_{1,2}$

$$d\vec{r}_{cm,2} = -\frac{\mu}{m_2} \vec{r}_{1,2}$$

$$\Rightarrow \quad d\theta_1 = \frac{\mu}{m_1} \frac{|\vec{r}_{1,2}|}{|\vec{r}_{cm,1}|}$$

$$d\theta_2 = \frac{\mu}{m_2} \frac{|\vec{r}_{1,2}|}{|\vec{r}_{cm,2}|}$$

we've also derived that

$$m_1 |\vec{r}_{cm,1}| = m_2 |\vec{r}_{cm,2}|$$

$$\Rightarrow \quad d\theta_1 = d\theta_2$$

$$d\vec{r}_i = \underbrace{dR_{cm}}_{\text{CM translation}} + \underbrace{d\vec{r}_{cm,i}}_{\text{rotation about CM}}$$