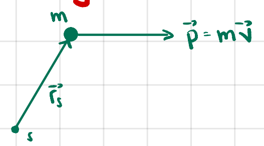
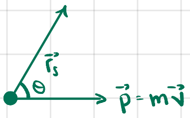
32.1 Angular Momentum For a Point Particle

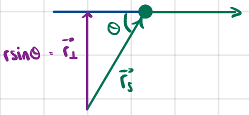


$\vec{L}_s \equiv \vec{r}_s \times \vec{p}$

angular momentum about point s

32.2 Calculating Angular Momentum



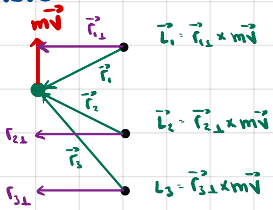


$\vec{L} = \vec{r} \times m\vec{v}$

$|\vec{L}| = rmv\sin\theta$

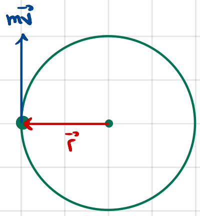
$= (r\sin\theta)mv$

note



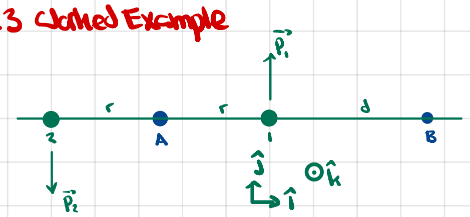
Angular momentum is the same for all three points.

circular motion



$\vec{L} = \vec{r} \times m\vec{v}$

32.3 Worked Example



$p_2 = -p_1 = p$

a) $\vec{L}_A = (-r\hat{i}) \times (-p\hat{j}) + (r\hat{i}) \times (p\hat{j})$

$= rp\hat{k} + rp\hat{k} = 2rp\hat{k}$

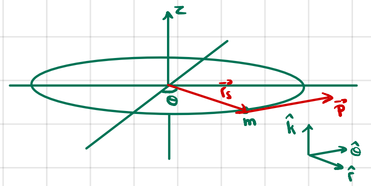
b) $\vec{L}_B = (-2r-d)\hat{i} \times (-p\hat{j}) + (-d)\hat{i} \times (p\hat{j})$

$= (2r+\cancel{d})p\hat{k} - \cancel{dp\hat{k}}$

$= 2rp\hat{k}$

c) $\vec{L}_{A_{sys}} = \vec{L}_{B_{sys}}$. whenever $\vec{p}^{sys} = 0$, then angular momentum is independent of the choice of point A.

32.4 Angular Momentum of Circular Motion



$\vec{\omega} = \omega_z\hat{k}, \ \omega_z > 0$

$\vec{L}_s = (R\hat{r}) \times (p\hat{\theta}) = Rp\hat{k}$

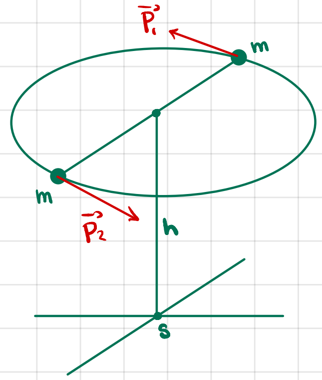
$p = mv_T = mR\omega_z$

$\Rightarrow \vec{L}_s = mR^2\omega\hat{k}$ ↗ $\vec{\omega}$

moment inertia of point particle about center

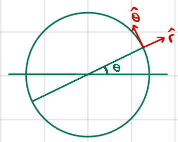
$\vec{L}_s = I_s\vec{\omega}$

33.1 Worked Example - Two Rotating Point Particles



$\vec{\omega} = \omega_z\hat{k}$

$\hat{r}_1 = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$



$\hat{r}_2 = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j} = -\hat{r}_1$

$\vec{L}_s = (R\hat{r}_1 + h\hat{k}) \times (p_1\hat{\theta}_1) + (R\hat{r}_2 + h\hat{k}) \times (p_2\hat{\theta}_2)$

$= (R\hat{r}_1 + h\hat{k}) \times (p_1\hat{\theta}_1) + (-R\hat{r}_1 + h\hat{k}) \times (-p_2\hat{\theta}_1)$

$= Rp_1\hat{k} + hp_1(-\hat{r}_1) + Rp_2\hat{k} - hp_2(-\hat{r}_1)$

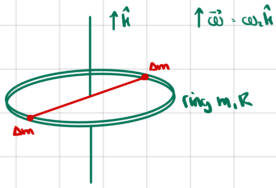
$= \hat{r}_1(-hp_1 + hp_2) + \hat{k}(Rp_1 + Rp_2)$

$= h(p_2 - p_1)\hat{r}_1 + R(p_1 + p_2)\hat{k}$

$p_1 = mR\omega_z = p_2 \Rightarrow \vec{L}_s = R \cdot 2mR\omega\hat{k}$

$= 2R^2m\omega\hat{k}$

33.2 Angular Momentum of Symmetric Object

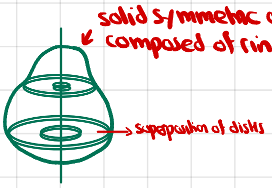


$\sum m^{pair} = \sum(2\Delta m) = m$

$\vec{L}_S^{pair} = 2\Delta m R^2\vec{\omega}$

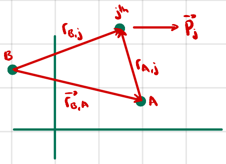
$\vec{L}_S^{ring} = \sum_{pairs} m^{pair} R^2\vec{\omega} = mR^2\vec{\omega} = I_{axis}\vec{\omega}$

solid symmetric object, composed of rings



$\vec{L}_S = I_{axis}\vec{\omega}$

33.4 If Momentum is Zero, Angular Momentum is independent of origin



A, B fixed: show $\vec{L}_A^{sys} = \vec{L}_B^{sys}$

$\vec{L}_A^{sys} = \sum \vec{r}_{A,j}\times\vec{p}_j$

$\vec{L}_B^{sys} = \sum \vec{r}_{B,j}\times\vec{p}_j$

$\vec{r}_{B,j} = \vec{r}_{B,A} + \vec{r}_{A,j}$

$\Rightarrow \vec{L}_B^{sys} = \sum\vec{r}_{B,A}\times\vec{p}_j + \sum\vec{r}_{A,j}\times\vec{p}_j$

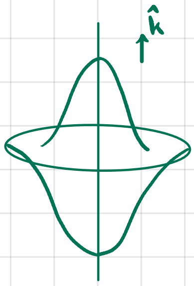
$= \vec{r}_{B,A}\times\sum\vec{p}_j + \vec{L}_A^{sys}$

$= \vec{r}_{B,A}\times\vec{p}^{sys} + \vec{L}_A^{sys}$

$\vec{p}^{sys} = 0 \Rightarrow \vec{L}_B^{sys} = \vec{L}_A^{sys}$

Note: Ref. Frame moving w/ CM has $\vec{p}^{sys} = 0$ by definition.

33.5 Kinetic Energy of Symmetric Object



$\vec{\omega} = \omega_z\hat{k}$

$\vec{L}_{axis} = I_{axis}\vec{\omega}$

$L_{axis} = |\vec{L}_{axis}| = I_{axis}\cdot\omega$

$$K_{axis}^{rot} = \frac{I_{axis}\omega^2}{2} = \frac{I_{axis}\cdot L_{axis}^2}{2\cdot I_{axis}^2} = \frac{L_{axis}^2}{2\cdot I_{axis}}$$

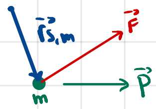
note that there is an analogous expression for linear motion kinetic energy

$\vec{p} = m\vec{v}$

$p = |\vec{p}| = mv$

$$K^{trans} = \frac{mv^2}{2} = \frac{m^2v^2}{2m} = \frac{p^2}{2m}$$

34.1 Torque Causes Ang. Mom. to change - Point Particle



$$\vec{F} = \frac{d\vec{p}}{dt}$$

$\vec{L}_S = \vec{r}_{S,m}\times\vec{p}$

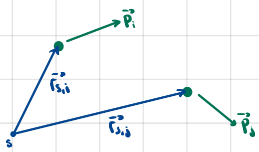
$$\frac{d\vec{L}_S}{dt} = \frac{d\vec{r}_{S,m}}{dt}\times\vec{p} + \vec{r}_{S,m}\times\frac{d\vec{p}}{dt}$$

$= \cancel{\vec{v}\times m\vec{v}}^{\,0} + \vec{r}_{S,m}\times\vec{F}$

$= \vec{\tau}_S$

$$\Rightarrow \vec{\tau}_S = \frac{d\vec{L}_S}{dt}$$

34.2 Torque Cause Ang. Mom. to Change System of Particles



$\vec{L}_S = \sum\vec{r}_{S,j}\times\vec{p}_j$

$$\frac{dL_S}{dt} = \sum\frac{d\vec{r}_{S,j}}{dt}\times\vec{p}_j + \sum\vec{r}_{S,j}\times\vec{F}_j$$

$= \sum\cancel{\vec{v}_j\times m_j\vec{v}_j}^{\,0} + \sum\vec{\tau}_j$

assume: internal torques cancel in pairs

$$\frac{d\vec{L}_S}{dt} = \vec{\tau}_{S_{total\,external}}$$

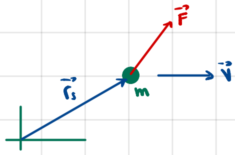
34.3 - Angular Impulse

$\vec{p} = m\vec{v}$

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\underbrace{\int_0^t\vec{F}\,dt}_{\text{Impulse}} = \int_{\vec{p}(0)}^{\vec{p}(t)} d\vec{p} = \vec{p}(t) - \vec{p}(0)$$

Ex: $\int\vec{F}\,dt = \vec{0} \rightarrow \Delta\vec{p} = 0$



$\vec{L}_S = \vec{r}_S\times\vec{p}$

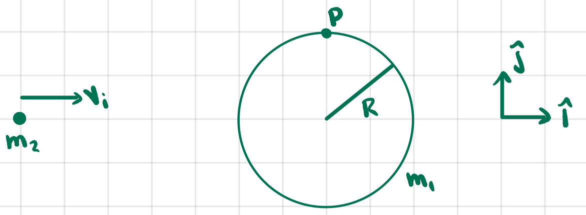
$$\vec{\tau}_S = \frac{d\vec{L}_S}{dt}$$

$$\int_0^t\vec{\tau}_S\,dt = \int_{\vec{L}_S(0)}^{\vec{L}_S(t)} d\vec{L}_S = \vec{L}_S(t) - \vec{L}_S(0)$$

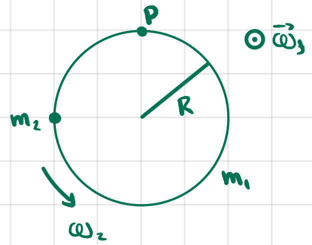
Ex: $\int_0^t\vec{\tau}_S\,dt = \vec{0} \Rightarrow \vec{L}_S(t) = \vec{L}_S(0)$

34.5 Worked Example – Particle Hits Pivoted Ring

A



B

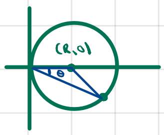


The collision is inelastic, so kinetic energy is not conserved.
The external forces are gravity and normal forces.
There is no motion in the direction of action of the external forces; they do no work. Momentum is conserved.

The collision involves internal forces and a pair of internal angular impulses.
Internal torques cancel each other.
Because there is no external torque, angular momentum is conserved.

Summary: linear and angular momentum constant, mechanical energy not conserved.

b) $I_{ring} = \int_0^{2\pi} r^2(\theta) \cdot \frac{m_1}{2\pi R} \cdot R\, d\theta$



$= 2m_1R^2$

$(x-R)^2 + y^2 = R^2$

$x^2 - 2xR + R^2 + y^2 = R^2$

$x^2 + y^2 = 2xR$

$r^2 = 2Rr\cos\theta$

$r(r - 2R\cos\theta) = 0$

$r = 2R\cos\theta$

$I_{m_2} = m_2 \left[\cancel{2}R\frac{\sqrt{2}}{\cancel{2}}\right]^2 = m_2 \cdot 2R^2$

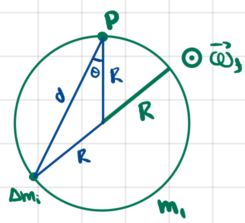
$I_P^{sys} = 2m_2R^2 + 2m_1R^2$

* Altern. calculation of $I_{P,ring}$

$I_{P,ring} = I_{cm,ring} + m_1R^2$

$= 2m_1R^2$

c) $\vec{L}^{sys}_{P,A} = (-R\hat{i} - R\hat{j}) \times m_2 v_i \hat{i}$



$= -Rm_2v_i(-\hat{k}) = Rm_2v_i\hat{k}$

$d = 2R\cos\theta$

$\vec{L}^{sys}_{P,B} = I_P \cdot \vec{\omega}$

$= \left(2m_2R^2 + 2m_1R^2\right) \cdot \omega\hat{k}$

d) $L^{sys}_{P,A} = L^{sys}_{P,B}$

$\cancel{R}m_2v_i = 2R^{\cancel{2}}(m_1 + m_2)\omega$

$\Rightarrow \omega = \frac{m_2 v_i}{2R(m_1 + m_2)}$