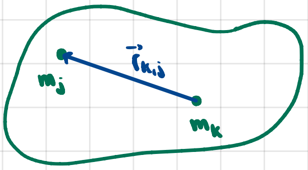
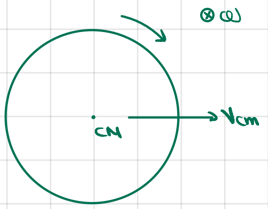
28.1 Rigid Bodies

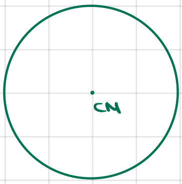


$|\vec{r}_{k,j}| \equiv r_{k,j} =$ constant for all j,k

ie, distance between any two points is fixed

28.2 Introduction to Rotation and Translation





ground frame

→ rotation + translation

$\vec{F}^{total}_{ext} = m_t \vec{a}_{cm}$

cm reference

→ pure rotation around cm

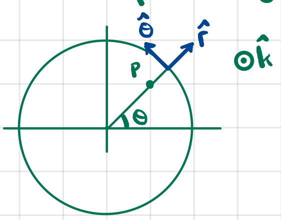
→ cm at rest, $\vec{\tau}_{cm} = I_{cm} \cdot \vec{\alpha}$ (torque)

→ $K^{rot} = \frac{1}{2} I_{cm} \omega^2$ = rotational energy

28.3 Review of Angular Velocity and Accel

angular acceleration α is given

→ choose point on rigid body, introduce coord. system



$\vec{\alpha}(t) = \theta''(t)\hat{k}$

find $\vec{\omega}(t) = \theta'(t)\hat{k}$ and $\theta(t)$

notation: $\alpha(z) = \frac{d\omega_z}{dt} = \theta''(t)$, $\omega_z = \theta'(t)$

$\omega_z(t) - \omega_z(0) = \int_0^t \alpha_z(t)dt \Rightarrow \omega_z(t) = \omega_z(0) + \int_0^t \alpha_z(t)dt$

$\theta(t) - \theta(0) = \int_0^t \omega_z(t)dt \Rightarrow \theta(t) = \theta(0) + \int_0^t \omega_z(t)dt$

---

* $\vec{\omega} = \theta'(t)\hat{k} = \omega_z \hat{k}$ = angular velocity vector, rad·s^{-1}

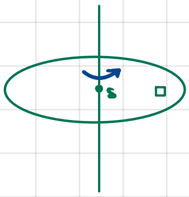
angular speed $= \omega = \theta'(t) = |\vec{\omega}|$

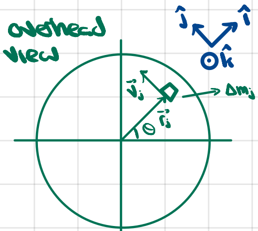
$\vec{v} = \vec{\omega} \times \vec{r} = \theta'(t)\hat{k} \times r\hat{r} = r\theta'(t)\hat{\theta}$

angular acceleration $= \frac{d\vec{\omega}}{dt} = \theta''(t)\hat{k} = \alpha_z \hat{k}$

$\alpha = |\vec{\alpha}| = |\theta''(t)|$

29.1 Kinetic Energy of Rotation





cut the rigid body up into many small Δm_j, each with position $\vec{r}_j$, velocity $\vec{v}_j = \vec{r} \times \vec{\omega} = r_j \theta'(t)\hat{\theta} = r_j \omega_z \hat{\theta}$, mass Δm_j.

Each small piece has rotational kinetic energy

$K_j^{rot} = \frac{1}{2}\Delta m_j v_j^2 = \frac{1}{2}\Delta m_j r_j^2 \omega_z^2$

$K^{rot} = \frac{1}{2}\omega_z^2 \sum_{j=1}^{n} \Delta m_j r_j^2$

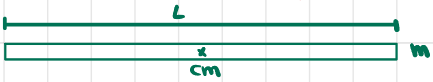
Definition: $I_S = \lim_{\substack{\Delta m_j \to 0 \\ n \to \infty}} \sum_{j=1}^{n} \Delta m_j r_j^2 = \int dm\, r_{S,dm}^2$

moment of inertia about axis passing through point S, ⊥ to plane of rotation

$K^{rot} = \frac{1}{2} I_S \omega_z^2$

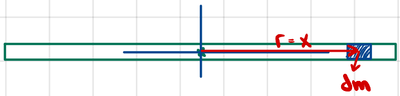
contrast this to $K^{translational} = \frac{1}{2} m v_{cm}^2$

29.2 Moment of Inertia of a Rod



$I_{cm} = \int_{body} dm\, r^2$

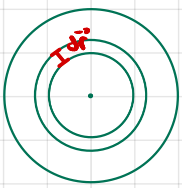
choose coord. system:



$dm = \frac{m}{L}dx$

$I_{cm} = \int_{-L/2}^{L/2} \frac{m}{L} dx \cdot x^2 = \frac{m}{L}\frac{1}{3}x^3\Big|_{-L/2}^{L/2} = \frac{1}{12}mL^2$

29.3 Moment of Inertia of a Disc



uniform thin disc, mass m, radius R

$\sigma = \frac{m}{\pi R^2}$

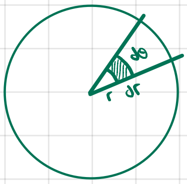
$dm = \sigma \cdot (\pi(r+dr)^2 - \pi r^2) = \frac{m}{\pi R^2}(\cancel{\pi r^2} + 2\pi r dr + \pi dr^2 - \cancel{\pi r^2})$

$= \frac{m}{\pi R^2}(2\pi r dr + \pi dr^2)$

limit $dr \to 0 \Rightarrow dm = \frac{m}{\pi R^2} \cdot 2\pi r dr$

$I_{cm} = \frac{m}{\pi R^2}\int 2\pi r dr\, r^2 = \frac{2m}{R^2}\int_0^R r^3 dr = \frac{2m}{R^2} \cdot \frac{R^4}{4} = \frac{mR^2}{2}$

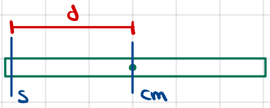
Alten. Calculation



$$\int_0^{2\pi}\int_0^R \underbrace{\frac{m}{\pi R^2}\cdot r\,dr\,d\theta}_{dm}\cdot r^2$$

$$= \frac{m}{\pi R^2}\int_0^{2\pi}\frac{R^4}{4}\,d\theta = \frac{mR^2}{4\pi}\cdot 2\pi = \frac{mR^2}{2}$$

29.4 Parallel Axis theorem

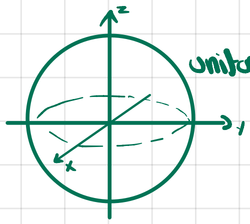


$$I_s = I_{cm} + md^2$$

Ex: $I_{cm} = \frac{mL^2}{12}$

$$I_s = \frac{mL^2}{12} + m\frac{L^2}{4} = \frac{mL^2}{3}$$

29.5 Moment of Inertia of a Sphere



uniform sphere, mass m, radius R

$$I_{cm,z} = \int_{sphere} dm\,(x^2+y^2)$$

$$I_{cm,x} = \int_{sphere} dm\,(y^2+z^2)$$

$$I_{cm,y} = \int_{sphere} dm\,(x^2+z^2)$$

By symmetry, $I_{cm,x} = I_{cm,y} = I_{cm,z} = I_{cm}$

$$3I_{cm} = I_{cm,x} + I_{cm,y} + I_{cm,z}$$
$$= \int_{sphere} dm\cdot 2\underbrace{(x^2+y^2+z^2)}_{r^2 \text{ of a sphere of thickness } dr}$$

$dm = \rho\, dV$

ρ: volume density $= \frac{m}{\frac{4}{3}\pi R^3}$

dV = volume of spherical shell, dr thick

$= 4\pi r^2 dr$

$$dm = \frac{3m}{\cancel{4\pi} R^3}\cdot \cancel{4\pi} r^2 dr = \frac{3mr^2}{R^3}$$

$$\Rightarrow 3I_{cm} = 2\int dm\, r^2 = 2\int_0^R \frac{3mr^2 dr}{R^3}\cdot r^2$$

$$\Rightarrow I_{cm} = \frac{2m}{R^3}\int_0^R r^4 dr = \frac{2mR^5}{5R^3} = \frac{2mR^2}{5}$$

Alten. Calculation

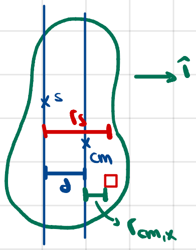
$$\text{mass density} = \frac{m}{\frac{4}{3}\pi R^3} = \frac{3m}{4\pi R^3}$$

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_0^R \underbrace{\rho^2\sin^2\varphi}_{\text{distance to z axis squared}}\Big(\underbrace{\frac{3m}{4\pi R^3}\cdot\rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi}_{dm}\Big)$$

$$= \frac{2mR^2}{5}$$

29.6 Derivation of Parallel Axis Theorem

Take a rigid body, consider two points, and parallel axes passing through them:



$$I_{cm} = \int_{body} dm\, r_{cm,x}^2$$

$$I_s = \int_{body} dm\, r_s^2$$

$$r_s = d + r_{cm,x} \Rightarrow r_s^2 = d^2 + r_{cm,x}^2 + 2d\,r_{cm,x}$$

$$\Rightarrow I_s = \int_{body} dm\, d^2 + \int_{body} dm\, r_{cm,x}^2 + 2d\int_{body} \cancel{dm\, r_{cm,x}} \rightarrow 0$$

$\vec{r}_{cm}$ is defined $\int dm\, \vec{r}_{cm} = 0 \Rightarrow \int dm\, r_{cm,x} = 0$

$$\Rightarrow I_s = d^2 m + I_{cm}$$

30.1 Introduction to Torque and Rotational Dynamics

$$\vec{F} = M\cdot\vec{a}_{cm}$$

Chasles' Theorem

Rotational equiv. to Newton's 2nd law

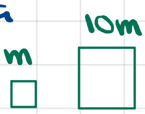
Rotational Eq. of Motion: $\vec{\tau} = I\vec{\alpha}$

$\vec{\tau}$ = torque

I = moment of inertia

$\vec{\alpha}$ = angular acceleration

Inertia



F = ma

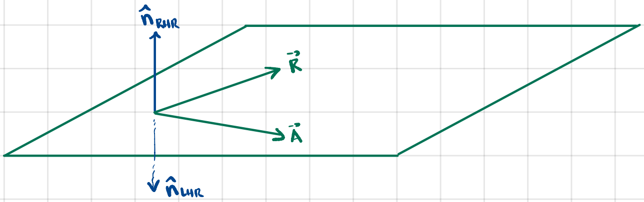
m represents inertia in translational motion

moment of inertia plays this role in rotational motion

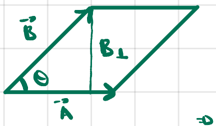
$I = \sum_{body} m_j r_j^2$ But in rot. motion, it's not just how much mass, but also how that mass is distributed.

30.2 Cross Product Review

Vector Product (cross product) $\vec{A} \times \vec{B} = \vec{C}$



$\vec{C} = |\vec{A}||\vec{B}| \sin\theta \, \hat{n}_{RHR}$ $\quad 0 \le \theta \le \pi$



Area $= B_\perp |\vec{A}|$
$= |\vec{A}||\vec{B}| \sin\theta$
$\Rightarrow |\vec{C}| =$ Area of ▱

30.3 Cross Product in Cartesian Coord.

define $\hat{k} = \hat{i} \times \hat{j}$ (right-handed coord. sys.)

(i, j, k) is a cyclic order, e.g. (j, i, k) anticyclic

$\hat{j} \times \hat{k} = \hat{i}$ $\quad (j, k, i)$
$\hat{k} \times \hat{i} = \hat{j}$ $\quad (k, i, j)$

$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

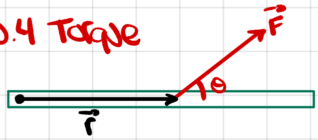
$\Rightarrow \hat{k} \times \hat{j} = -\hat{i}$ $\qquad \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$
$\hat{i} \times \hat{k} = -\hat{j}$
$\hat{j} \times \hat{i} = -\hat{k}$

$A = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$
$B = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$

$$\vec{A} \times \vec{B} = A_x B_y \hat{k} + A_x B_z (-\hat{j}) + A_y B_x (-\hat{k}) + A_y B_z \hat{i} + A_z B_x \hat{j} + A_z B_y (-\hat{i})$$

$$= \hat{k}(A_x B_y - A_y B_x) + \hat{i}(A_y B_z - A_z B_y) + \hat{j}(A_z B_x - A_x B_z)$$

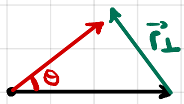
30.4 Torque



$\vec{\tau} = \vec{r} \times \vec{F} = |\vec{r}||\vec{F}| \sin\theta$

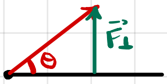
Let's analyze this expression in two ways

1) $\tau = |\vec{F}| \underbrace{(|\vec{r}| \sin\theta)}_{\text{moment arm}}$



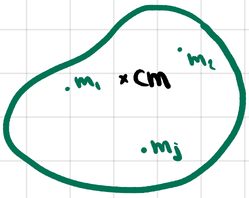
$|r_\perp| = |\vec{r}| \sin\theta$
(moment arm)

2) $\vec{\tau} = (|\vec{F}| \sin\theta) |\vec{r}|$



$|\vec{F}_\perp| = |\vec{F}| \sin\theta$

30.5 Torque From Gravity

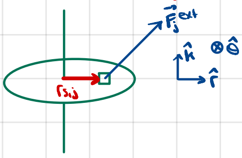


$$\vec{\tau} = \sum_{j=1}^{n} (\vec{r}_{s,j} \times m_j \vec{g})$$

$$= \left(\sum_{j=1}^{n} \vec{r}_{s,j} m_j \right) \times \vec{g}$$

$\Rightarrow \vec{\tau} = m_T \cdot \vec{r}_{s,cm} \times \vec{g}$

31.1 Relationship Between Torque and Angular Acceleration



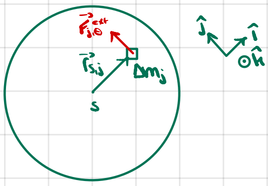
$$\vec{\tau}_s = \sum_{j=1}^{n} \vec{r}_{s,j} \times \vec{F}_j^{ext}$$

Goal: $\vec{\tau}_{s,z}$

$\vec{F}_j^{ext} = F_{j,r}^{ext} \hat{r} + F_{j,\theta}^{ext} \hat{\theta} + F_{j,z}^{ext} \hat{k}$

$\vec{r}_{s,j} \times \vec{F}_{j,z}^{ext} \hat{k}$ is not in the z (ie $\hat{k}$) direction. We ignore these terms.

$$\Rightarrow \vec{\tau}_{s,z} = \sum_{j=1}^{n} r_{s,j} \times (\vec{F}_{j,r}^{ext} \hat{r} + \vec{F}_{j,\theta}^{ext} \hat{\theta})$$



Note, however, that $\vec{r}_{s,j} \times F_{j,r}^{ext} \hat{r} = 0$

$$\Rightarrow \vec{\tau}_{s,z} = \sum_{j=1}^{n} \vec{r}_{s,j} \times \vec{F}_{j,\theta}^{ext} \hat{\theta} = \sum_{j=1}^{n} r_{s,j} F_{j,\theta} \hat{k}$$

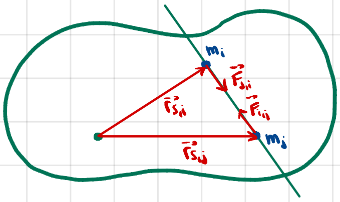
2nd Law: $F_{j,\theta} = \Delta m_j \cdot a_{j,\theta} = \Delta m_j \cdot (r_{s,j} \alpha_z)$

$$\Rightarrow \vec{\tau}_{s,z} = \left(\sum_{j=1}^{n} \Delta m_j r_{s,j}^2 \right) \alpha_z \hat{k}$$

$$\lim_{\Delta m_j \to 0} \sum \Delta m_j r_{s,j}^2 = \int_{body} dm \, r_{s,dm}^2$$

$\Rightarrow \vec{\tau}_{s,z} = I_s \vec{\alpha}$, $\vec{\alpha} = \alpha_z \hat{k}$

31.2 Internal Torques Cancel in Pairs



$\vec{\tau}_S = \vec{r}_{S,i} \times \vec{F}_{j,i} + \vec{r}_{S,j} \times \vec{F}_{i,j}$

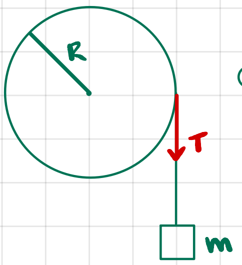
3rd law: $\vec{F}_{j,i} = -\vec{F}_{i,j}$

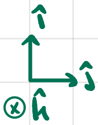
$\Rightarrow \vec{\tau}_S = (\vec{r}_{S,i} - \vec{r}_{S,j}) \times \vec{F}_{j,i}$

$F_{j,i} \parallel r_{S,i} - r_{S,j} \Rightarrow \vec{\tau}_S = 0$

$\Rightarrow$ we only need to address torque due to external forces

31.3 Worked Example: Moment of Inertia of a Disc From a Falling Mass





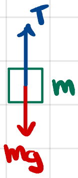
$\alpha = \alpha_z \hat{k}$ ⊗

$\vec{\tau}_{cm} = I_{cm} \cdot \vec{\alpha}$ ⊗

$\vec{\tau} = \vec{r} \times \vec{F} = I_S \cdot \vec{\alpha}$

$\vec{\tau}_{cm,T} = (R\hat{j}) \times (-T\hat{i}) = -RT\hat{k} = I_{cm} \cdot \alpha_z \hat{k}$

$\Rightarrow I_{cm} = \dfrac{-RT}{\alpha_z}$



What is T? 2nd law for gravity.

$mg - T = m \cdot a \Rightarrow T = m(g-a)$

The mass accelerates due to gravity. There is an offsetting tension force. The acceleration is the same as that experienced on the edge of the disc. The disc edge has circular motion. Therefore we can obtain angular acceleration from tangential acceleration. This relation is a constraint eq.

constraint condition $\quad a_T = R\alpha_z \Rightarrow \alpha_z = \dfrac{a_T}{R}$

At this point we've used the torque equations to set a formula for moment of inertia of the disc. It applies to any setup where a force T is pulling tangentially on the disc. With 2nd law we specified where T is coming from. Given the setup and physical reality of the rope, we have a constraint that tells us how the disc is accelerating under the effect of T.

Therefore, we can further refine our formula for I_{cm}

$$I_{cm} = \frac{-Rm(g-a)}{\frac{a}{R}} = \frac{-R^2 m(g-a)}{a}$$

$$= mR^2\left(\frac{g}{a} - 1\right)$$

The I_{cm} depends on various factors. We could observe a, or we could observe the mass falling a distance h in time t to know a.
This is 1D kinematics.

$h(t) = \dfrac{at^2}{2} \Rightarrow a = \dfrac{2h}{t^2}$

$\Rightarrow I_{cm} = mR^2\left(\dfrac{gt^2}{2h} - 1\right)$

Recap 31.3

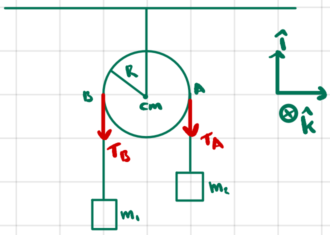
Torque eq: $-RT = I_{cm}\alpha_z$

2nd law eq: $T = m(g-a)$

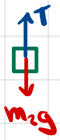
constraint eq: $\alpha_z = \dfrac{a}{R}$

unknowns: T, a, α_z, I_{cm}

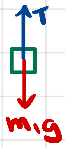
31.4 Worked Example - Atwood Machine



+ massless, frictionless pulley



$m_2 g - T = m_2 \cdot a$



$T - m_1 g = m_1 \cdot a$

$\Rightarrow$ $g(m_2 - m_1) = a(m_2 + m_1)$

$$a = \frac{m_2 - m_1}{m_2 + m_1} g$$

$m_2 > m_1$

point A

$\vec{\tau}_{cm,A} = \vec{r}_{cm,A} \times \vec{T}_A = (R\hat{j}) \times (-T\hat{i}) = -TR(-\hat{k}) = RT_A \hat{k}$

$\vec{\tau}_{cm,B} = \vec{r}_{cm,B} \vec{\alpha} = (-R\hat{j})(-T\hat{i}) = -RT_B \hat{k}$

$\vec{\tau}_{cm} = \vec{\tau}_{cm,A} + \vec{\tau}_{cm,B} = \hat{k} R(T_A - T_B)$

$\vec{\tau}_{cm} = I_{cm} \vec{\alpha} \Rightarrow R(T_A - T_B) = I_{cm} \alpha_z$

$I_{cm} = \dfrac{R(T_A - T_B)}{\alpha_z}$ 4 equations

$a = R\alpha_z$ unknowns: T_A, T_B, a, α_z

$T_A = m_2(g - a)$ It is assumed we know I_{cm}.

$T_B = m_1(a + g)$

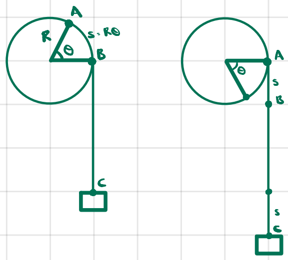
$$I_{cm} \cdot \frac{a}{R} = R\left(m_2(g-a) - m_1(g+a)\right)$$

$$I_{cm} \cdot a = R^2 \left[m_2 g - m_2 a - m_1 g - m_1 a \right]$$

$$a\left(I_{cm} + R^2(m_1 + m_2)\right) = R^2 g(m_2 - m_1)$$

$$a = \frac{R^2 g(m_2 - m_1)}{I_{cm} + R^2(m_1 + m_2)}$$

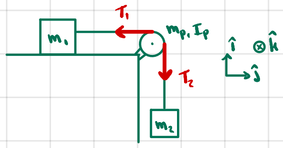
31.6 Non-slip condition



$$v_A = \frac{ds}{dt} = R\frac{d\theta}{dt} = v$$

$$\Rightarrow a = R\frac{d^2\theta}{dt} = R\alpha$$

31.7 Worked Example: Two Blocks and a Pulley Using Energy



clockwise rotation $\Rightarrow \hat{k}$ is $\otimes$

Find $v_{f,2}$

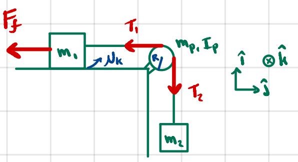
$$E_i = 0$$

$$E_f = -m_2gh + \frac{m_1v^2}{2} + \frac{m_2v^2}{2} + \frac{I_p\omega^2}{2}$$

$$= -m_2gh + \frac{v^2(m_1+m_2)}{2} + \frac{I_p}{2}\cdot\frac{v^2}{R^2}$$

$$\Delta E = 0 \Rightarrow v^2\left(m_1+m_2+\frac{I_p}{R^2}\right) = 2m_2gh$$

$$v = \sqrt{\frac{2m_2gh}{m_1+m_2+\frac{I_p}{R^2}}}$$



$m_1 > N_k m_2$

$$W_{F_f} = \Delta E_m$$

$$W_{F_f} = \int_0^x -N_k m_1 g\, dx = -N_k m_1 g x$$

Torque: $(R\hat{j})\times(-T_2\hat{i}) + (R\hat{i})\times(-T_1\hat{j}) = I_{cm}\alpha_z\hat{k}$

$\Rightarrow -RT_2(-\hat{k}) - RT_1\hat{k} = I_{cm}\alpha_z\hat{k}$

$\Rightarrow R(T_2 - T_1) = I_{cm}\alpha_z$

2nd law: $T_1 - F_f = m_1a \Rightarrow T_1 - N_k m_1 g = m_1 a \Rightarrow T_1 = m_1(a + N_k g)$

$m_2g - T_2 = m_2a \Rightarrow T_2 = m_2(g-a)$

constraint: $a = \alpha R$

unknowns: T_2, T_1, a, α_z

solution for a: $R(m_2g - m_2a - m_1a - m_1N_kg) = I_{cm}\cdot\frac{a}{R}$

$a(I_{cm} + R^2m_2 + R^2m_1) = R^2(m_2g - m_1N_kg)$

$$a = \frac{R^2g(m_2 - m_1N_k)}{I_{cm} + R^2(m_2+m_1)}$$