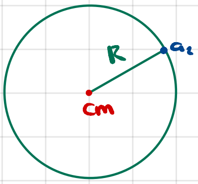
PSet 10 - Rotational Motion

Problem 1 - Disc and Washer

a)

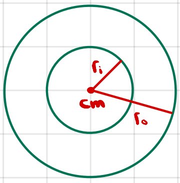


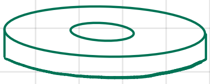
uniform disc ⇒ CM at center

i) $I_{cm} = \int_{disk} r^2 dm = \int_0^{2\pi}\int_0^{R} r^2 \cdot \frac{M}{\pi R^2} r\, dr\, d\theta = \frac{M}{\pi R^2}\int_0^{2\pi} \frac{R^4}{4} d\theta = \frac{MR^2}{4\pi}\cdot 2\pi = \frac{MR^2}{2}$

ii) $I_{a_2} = I_{cm} + M\cdot R^2 = \frac{3}{2} MR^2$

b) uniform washer



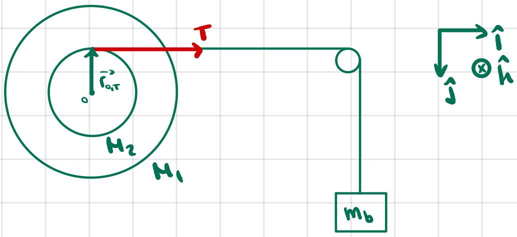


$I_{cm} = \int_{body} r^2 dm = \int_0^{2\pi}\int_{r_i}^{r_o}\int_0^{d} r^2\cdot\rho\cdot r\, dz\, dr\, d\theta = d\rho \int_0^{2\pi}\int_{r_i}^{r_o} r^3 dr\, d\theta = \frac{d\rho}{4}\int_0^{2\pi} (r_o - r_i)^4 d\theta = \frac{d\cdot\rho\cdot(r_o-r_i)^4\cdot 2\pi}{4}$

$= \frac{d\cdot\rho\cdot(r_o-r_i)^4\cdot\pi}{2} = 4\cdot 10^{-3}\cdot 7.8\cdot 10^{3}\cdot(31\cdot 10^{-3} - 13.5\cdot 10^{-3})^4\cdot\frac{\pi}{2} = 4.59\cdot 10^{-6}\ kg\cdot m^2$

$m\cdot\frac{kg}{m^3}\cdot m^4 = kg\cdot m^2$

Problem 2 - Compound Pulley



There is a torque applied by the tension force on the rope.

$$\vec{r}_{0,T} = -\frac{R}{2}\hat{j}$$

$$\vec{T} = T\hat{i}$$

$$\Rightarrow \vec{\tau}_{0,T} = \frac{R}{2}T\hat{k} = \frac{R^2(4M_1+M_2)}{8}\alpha_z\hat{k}$$

unknowns: α_z, T

$$I_1 = \frac{M_1R^2}{2}$$

$$I_2 = \frac{M_2R^2}{8}$$

$$\Rightarrow I_1 + I_2 = \frac{R^2(4M_1+M_2)}{8}$$

$$m_bg - T = m_b\cdot a$$

constraint: $\alpha_z\cdot\frac{R}{2} = a$

3 equations, 3 unknowns: α_z, T, a

$$\frac{RT}{2} = \frac{R^2(4M_1+M_2)}{8}\alpha_z$$

$$T = m_b(g-a)$$

$$\alpha_z = \frac{2a}{R}$$

$$\cancel{4}^{\,2}\cdot\cancel{R}\cdot m_b(g-a) = \cancel{R^2}(4M_1+M_2)\cdot\frac{\cancel{2}a}{\cancel{R}}$$

$$2m_bg - 2m_ba = (4M_1+M_2)a$$

$$a = \frac{2m_bg}{4M_1+M_2+2m_b}$$

$$\Rightarrow T = \frac{m_bg(4M_1+M_2)}{4M_1+M_2+2m_b}$$

1D Kinematics

$$y(t) = \frac{at^2}{2} = \frac{m_bgt^2}{4M_1+M_2+2m_b} = d$$

$$t = \pm\frac{\sqrt{m_bgd(4M_1+M_2+2m_b)}}{m_bg}$$

Extra: Energy Analysis

$$v = r\omega \Rightarrow \omega = \frac{v_1}{R}$$

$$\omega = \frac{v_2}{R/2}$$

$$E_i = 0$$

$$E_f = -m_bgh + \frac{I_1\omega^2}{2} + \frac{I_2\omega^2}{2} + \frac{m_bv^2}{2}$$

$$= -m_bgh + \frac{M_1\cancel{R^2}}{4}\cdot\frac{v_1^2}{\cancel{R^2}} + \frac{M_2\cancel{R^2}}{\cancel{16}\,4}\cdot\frac{v^2}{\cancel{R^2}}\cdot\cancel{4} + \frac{m_bv^2}{2}$$

$$\omega_1 = \omega_2 = \frac{v_1}{R} = \frac{v}{R/2} = \frac{2v}{R}$$

$$\Rightarrow v_1 = 2v$$

$$\Rightarrow -m_bgh + \frac{M_1}{4}\cdot 4v^2 + \frac{M_2v^2}{4} + \frac{m_bv^2}{2}$$

$$\Delta E = 0$$

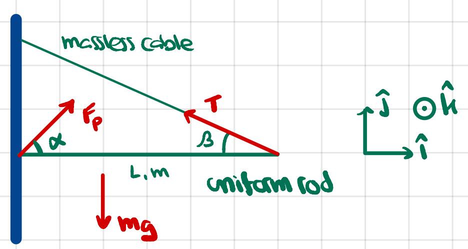
$$v^2\left(M_1 + \frac{M_2}{4} + \frac{m_b}{2}\right) = m_bgh$$

$$v = \sqrt{\frac{4m_bgh}{4M_1+M_2+2m_b}}$$

$$h(t) = \frac{m_bgt^2}{4M_1+M_2+2m_b}$$

$$v(t) = \sqrt{\frac{4m_b^2g^2t^2}{(4M_1+M_2+2m_b)^2}} = \frac{2m_bgt}{4M_1+M_2+2m_b}$$

Problem 3 - Suspended Rod



* Torque due to gravity

$$\sum \vec{r}_i \times m_i \vec{g} = \sum m_i \vec{r}_i \times \vec{g} = M_T \cdot \vec{r}_{cm} \times \vec{g}$$
$$= \vec{r}_{cm} \times M_T \vec{g} = \vec{r}_{cm} \times \vec{F}_{g_{Total}} = \vec{\tau}_{cm,g}$$

⇒ Torque on a rigid body due to gravity is the same as torque due to weight force of entire body acting at center of mass.

For rotation about the CM, $\vec{r}_{cm} = 0 \Rightarrow \vec{\tau}_{cm,g} = 0$

Static Equilibrium

1) $\sum \vec{F}_i = 0$ ⇒ CM at rest or moving w/ constant velocity
2) $\sum \vec{\tau}_{s,i} = 0$ ⇒ no rotational motion

1) $F_p \cos\alpha - T\cos\beta = 0$
$F_p \sin\alpha - mg + T\sin\beta = 0$

2) $T\sin\beta \cdot \frac{L}{2} - F_p \sin\alpha \cdot \frac{L}{2} = 0$ (lever law)

unknowns: α, F_p, T

$F_p \cos\alpha = T\cos\beta$
$F_p \sin\alpha = T\sin\beta$

⇒ $\tan\alpha = \tan\beta$ ⇒ $\alpha = \beta$ ⇒ $T = F_p$

$2T\sin\beta = mg$ ⇒ $T = F_p = \dfrac{mg}{2\sin\beta}$

a) $T = \dfrac{4 \cdot 9.8}{2 \cdot \frac{1}{2}} = 39.2\ N$

b) $\alpha = \beta = 30°$

c) $F_p = T = 39.2\,N$

Alternative Calculations

Torques about cable connection point:

$$F_p \sin\alpha\, \hat{j} \times (-L\hat{i}) + (-mg\,\hat{j})(-\tfrac{L}{2}\hat{i}) = 0$$

$$F_p \sin\alpha\, L - \frac{mgL}{2} = 0 \quad \text{(lever law)}$$

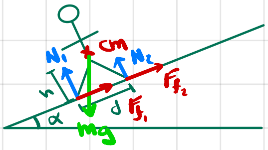
Equations for $\sum F_i = 0$ stay the same

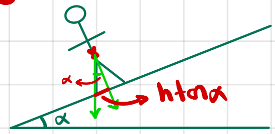
$F_p \cos\alpha - T\cos\beta = 0$
$F_p \sin\alpha - mg + T\sin\beta = 0$ same as before.

$2F_p \sin\alpha = mg$
$F_p \cos\alpha = T\cos\beta$ same as before
$mg = F_p \sin\alpha + T\sin\beta$

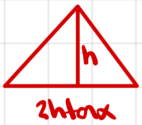
⇒ $2F_p\sin\alpha = F_p \sin\alpha + T\sin\beta$
$F_p \sin\alpha = T\sin\beta$ same eq. as before

Problem 4 - Person Standing on a Hill





The person is in static equil.



$$\sum \vec{F}_i = 0 \Rightarrow N_1 + N_2 - mg\cos\alpha = 0$$
$$mg\sin\alpha - f_1 - f_2 = 0$$

$$\sum \vec{\tau}_{cm,i} = 0 \Rightarrow \frac{d}{2} N_2 - \frac{d}{2} N_1 + hf_1 + hf_2 = 0$$

unknowns: N_1, N_2, f_1, f_2

eliminate an unknown if $f_1 = N_1 \mu_s$
$f_2 = N_2 \mu_s$

$N_1 + N_2 = mg\cos\alpha$
$\mu_s(N_1 + N_2) = mg\sin\alpha$

$2h\mu_s(N_1 + N_2) = d(N_1 - N_2)$

$\mu_s mg\cos\alpha = mg\sin\alpha \Rightarrow \tan\alpha = \mu_s$

$$2h = \frac{d(N_1 - N_2)}{mg\sin\alpha} \Rightarrow N_1 = N_2 + \frac{2hmg\sin\alpha}{d}$$

$$N_1 = mg\cos\alpha - N_1 + \frac{2hmg\sin\alpha}{d}$$

$$N_1 = \frac{mg\cos\alpha}{2} + \frac{hmg\sin\alpha}{d}$$

$$N_2 = mg\cos\alpha - N_1$$

$$= mg\cos\alpha - \frac{mg\cos\alpha}{2} - \frac{hmg\sin\alpha}{d}$$

$$= \frac{mg\cos\alpha}{2} - \frac{hmg\sin\alpha}{d} = 0$$

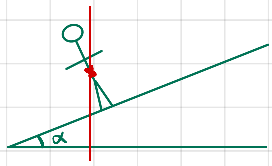
$$\Rightarrow d = \frac{2h\cancel{mg}\sin\alpha}{\cancel{mg}\cos\alpha} = 2h\tan\alpha$$

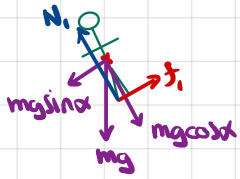
$N_2 = 0 \Rightarrow N_1 = mg\cos\alpha$

N_2 cannot be smaller than 0. For $d < 2h\tan\alpha$, the required N_2 to keep static equilibrium isn't possible.

torque: $\frac{d}{2} N_1 = h\mu_s N_1$ at d_{min}

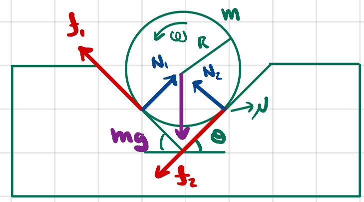
$h\tan\alpha = h\mu_s \Rightarrow \tan\alpha = \mu_s$

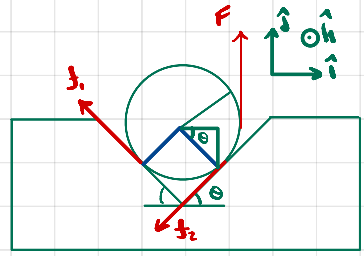


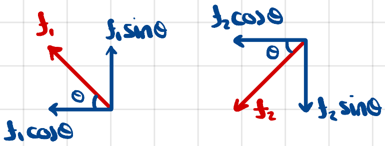


$$\tau_{cm} = -\frac{d}{2} N_1 + hf_1$$

Problem 5 - A Cylinder Rolling in a V-groove







Assume the cylinder is uniform, ie CM is in the middle.

$\vec{f}_1 = -f_1\cos\theta\,\hat{i} + f_1\sin\theta\,\hat{j}$

$\vec{f}_2 = -f_2\cos\theta\,\hat{i} - f_2\sin\theta\,\hat{j}$

$\vec{\tau} = (-R\sin\theta\,\hat{i} - R\cos\theta\,\hat{j}) \times (-f_1\cos\theta\,\hat{i} + f_1\sin\theta\,\hat{j})$
$+ (R\sin\theta\,\hat{i} - R\cos\theta\,\hat{j}) \times (-f_2\cos\theta\,\hat{i} - f_2\sin\theta\,\hat{j})$

$= \hat{h}(-Rf_1\sin^2\theta - Rf_1\cos^2\theta$
$- Rf_2\sin^2\theta - Rf_2\cos^2\theta)$

$= \hat{h}(-Rf_1 - Rf_2) = -R(f_1+f_2)\hat{h}$

Add an external torque to obtain zero net torque

$\vec{\tau} = -R(f_1+f_2)\hat{h} + (R\hat{i})\times(F\hat{j}) = I\cdot 0 = 0$

$\Rightarrow \hat{h}(-R(f_1+f_2) + FR) = 0$

$\Rightarrow F = f_1 + f_2$

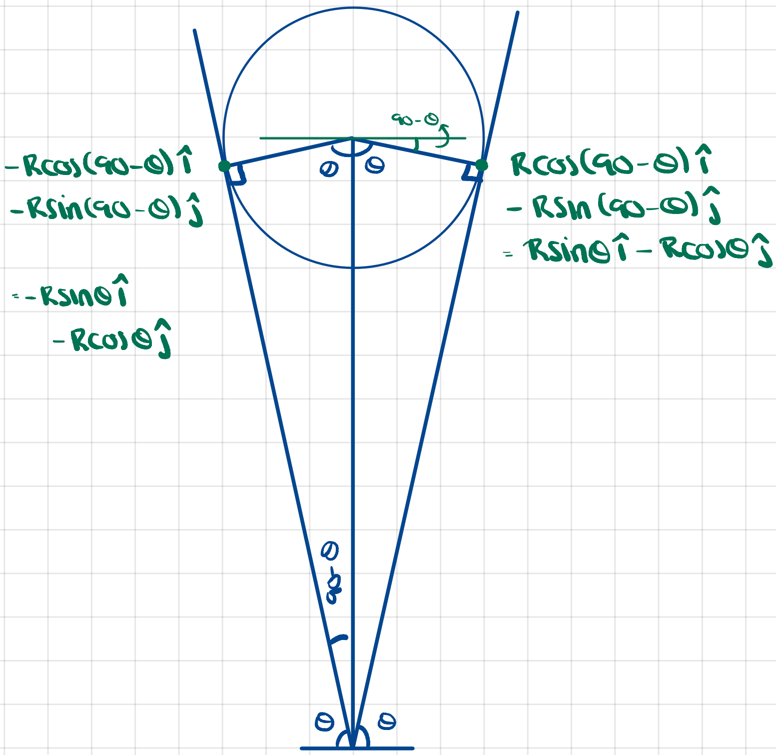
$N_1 = N_2 = mg\cos\theta$

$\Rightarrow f_1 = f_2 = \mu mg\cos\theta$

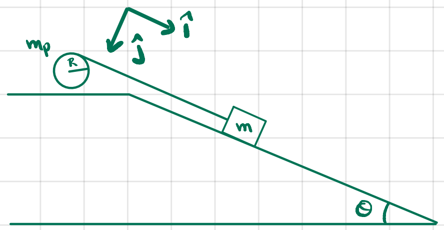
$\Rightarrow F = 2\mu mg\cos\theta$

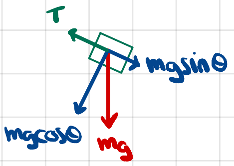
external torque $= FR = 2R\mu mg\cos\theta$

$\theta = 45^\circ \Rightarrow FR = \sqrt{2}R\mu mg$



Problem 6 - A Massive Pulley and a Block on an Incline





Gravity pulls the block down the incline.

$$mg\sin\theta - T = m \cdot a$$

The rope applies a torque on the pulley, generating rotation.

$$\vec{\tau}_{cm} = (-R\hat{j}) \times (T\hat{i}) = \frac{m_p R^2}{2} \cdot \alpha_z$$

$$\Rightarrow RT = \frac{m_p R^2}{2} \alpha_z$$

Angular acceleration of the pulley's rotation is related to the block's acceleration.

$$\alpha_z = \frac{a}{R}$$

3 equations, unknowns: T, a, α_z

$$mg\sin\theta - T = m \cdot a \Rightarrow T = m(g\sin\theta - a)$$

$$RT = \frac{m_p R^2}{2} \alpha_z$$

$$\alpha_z = \frac{a}{R}$$

$$\Rightarrow 2\cancel{R} \cdot m(g\sin\theta - a) = m_p \cancel{R} \cdot \frac{a}{\cancel{R}}$$

$$a(m_p + 2m) = 2mg\sin\theta$$

$$a = \frac{2mg\sin\theta}{m_p + 2m}$$

1D kinematics

$$x(t) = \frac{at^2}{2} = d \Rightarrow t = \sqrt{\frac{2d}{a}}$$

$$t_d = \sqrt{\frac{d(m_p + 2m)}{mg\sin\theta}}$$