The Instantaneous Motion of a Rigid Body - Dunham Jackson

instantaneous motion ~ velocities

reasoning independent of

- → forces which produce the motion
- → size and shape of body

each point in space has a defined vector velocity

- → ie velocity field
- → if rigidity condition fulfilled → rigid motion

motion: set of velocities

condition of rigidity: $\forall$ pair points P_1, P_2 → $\vec{v}_1$ and $\vec{v}_2$ have equal components along P_1P_2

- → ie distance P_1P_2 constant

two motions M', M'', resultant $M'+M''$

= motion in which vector field is created by vector sums at each point

⇒ resultant of two rigid motions is a rigid motion

For each pair P_1, P_2 the resultant in P_1P_2 direction is the sum of the same two vector components in P_1P_2 direction

Translation: motion in which all points have equal vector velocities

- → translation is a rigid motion

Rotation:

1. there is a straight line, the axis of rotation, all of whose points have zero velocity
2. points not on axis have velocity $\perp$ to plane passing through point and axis
3. points at equal distances from axis → same magnitude of velocity
4. points at different distances → speed proportional to distance to axis.
5. all velocities have same "sense" or "direction" of turning about axis

→ ratio of velocity to axis distance is constant

- → angular velocity of rotation

→ All essential characteristics identifying a rotation are represented by vector $\vec{\omega}$, vector velocity of rotation

→ O arbitrary point on axis, ρ position of P relative to O

⇒ velocity of P = $\vec{\omega} \times \vec{\rho}$

$\perp$ plane formed by $\vec{\rho}$ and $\vec{\omega}$ (2)

→ $\vec{\omega} \times \vec{\rho}$ defines set of vectors w/ characteristics of rotation

1) $\vec{\omega} \times \vec{0} = \vec{0}$

4) same $|\rho|$ → $|\vec{\omega} \times \vec{\rho}| = |\vec{\omega}||\rho| \sin\frac{\pi}{2} = |\vec{\omega}||\vec{\rho}|$

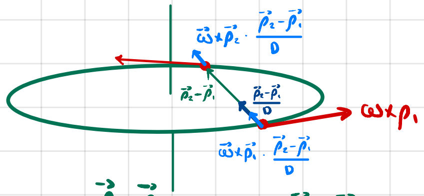
5) $\vec{\omega} \times \vec{\rho}$ all use right hand rule

→ consider P_1, P_2, ρ_1, ρ_2, and a given ω

$\vec{v}_1 = \vec{\omega} \times \vec{\rho}_1$

$\vec{v}_2 = \vec{\omega} \times \vec{\rho}_2$

$|P_1P_2| = D$ ⇒ $\vec{\omega} \times \rho_1 \cdot \frac{\vec{\rho}_2 - \vec{\rho}_1}{D}$ = component of $\vec{v}_1$ on P_1P_2



$$\vec{\omega} \times \vec{\rho}_1 \frac{\vec{\rho}_2 - \vec{\rho}_1}{D} - \vec{\omega} \times \vec{\rho}_2 \frac{\vec{\rho}_2 - \vec{\rho}_1}{D} = \vec{\omega} \times (\vec{\rho}_1 - \vec{\rho}_2) \cdot (\vec{\rho}_2 - \vec{\rho}_1) = 0$$

zero motion: a velocity vector field with $\vec{0}$ everywhere

General Rigid Motion

→ rigid motion w/ 3 non-collinear points have zero velocities

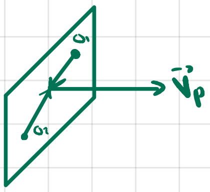
⇒ zero motion

points O_1, O_2, O_3 have zero velocities

P a point not in the same plane

rigidity condition ⇒ $\vec{v}_p$ has no component on O_1P, O_2P, O_3P

⇒ $\vec{v} \perp O_1P, O_2P, O_3P$



All the vectors perpendicular to $\vec{v}_p$ are on a plane through P. Not all three of O_1, O_2, O_3 share a plane w/ P so the rigidity condition can only be satisfied if $\vec{v}_p = \vec{0}$ (no components in any direction).

For a point Q in the plane of O_1, O_2, O_3, consider O_4 outside the plane. $\vec{v}_4$ must be zero. O_4 plus O_1 and O_2 are three non-collinear points w/ zero velocity ⇒ $\vec{v}_Q = \vec{0}$.

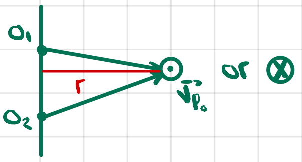
⇒ All points have zero velocity ⇒ zero motion

2) rigid motion, two distinct points have zero velocity $\Rightarrow$ rotation about the line of these points

O_1, O_2 zero velocity points
P_0 outside their line

$\vec{V}_{P_0} = \vec{0} \Rightarrow$ zero motion

$\vec{V}_{P_0} \neq \vec{0} \Rightarrow \vec{V}_{P_0} \perp \vec{O_1P_0}, \vec{O_2P_0}$



H denotes the rigid motion.

Two possible opposite rotations about O_1O_2,
$\omega = \frac{V_{P_0}}{r}$

Denote one rotation by R (assoc. w/ H), the other by $-R$.

resultant of H and $-R$ is $H-R$ is rigid motion with O_1, O_2, P_0 having zero velocities

$\Rightarrow$ zero motion

$\Rightarrow$ H identical to R.

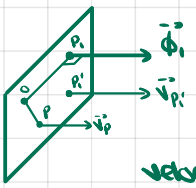
3) rigid motion, a point O has zero velocity
$\Rightarrow$ rotation about an axis through O

P_1 distinct from O, with vector velocity $\vec{\phi}_1$
O is on the plane $\perp$ to $\vec{\phi}_1$. The plane we call $\mathcal{P}_1$

P_1' any point on $\mathcal{P}_1$ outside OP_1

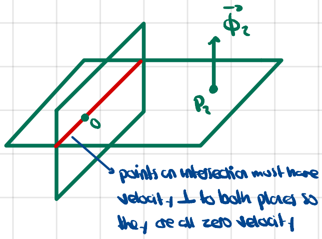
$\vec{V}_{P_1'} \perp OP_1', P_1P_1' \Rightarrow \vec{V}_{P_1'} \perp \mathcal{P}_1$

consider yet another point P in $\mathcal{P}_1$ outside of OP_1, and a point Q on OP_1. $\Rightarrow$ any point outside of OP has velocity $\perp$ to $\mathcal{P}_1$, including points on OP_1 like Q.



$\Rightarrow$ All points in $\mathcal{P}_1$ except O have velocity $\perp$ to $\mathcal{P}_1$.

Now consider P_2 outside of $\mathcal{P}_1$ with velocity $\vec{\phi}_2$



But given two zero velocity points (and less than three non-collinear) we are in case II, where we showed that the rigid motion is rotation about a specific axis.

4) Arbitrary point O $\Rightarrow$ most general rigid motion is resultant of rotation about axis thru O and a suitable translation

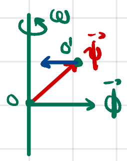
H given rigid motion
$\vec{\phi}$ vector vel. of O
T translation where all points have $\vec{\phi}$
$H - T = H + (-T) \Rightarrow$ O has vel. $\vec{0}$
$\Rightarrow$ 3) $\Rightarrow$ $H-T$ is rotation about axis thru O, call it R
$\Rightarrow$ H is resultant of R and T

5) Resultant of rotation and translation $\perp$ to axis of rotation is a rotation at equal $\vec{\omega}$ about parallel axis.

R rotation w/ $\vec{\omega}$
T translation w/ $\vec{\phi}$
$\vec{\omega} \cdot \vec{\phi} = 0$

$\vec{\psi} = \vec{\omega} \times \frac{\vec{\phi}}{\omega^2}$

O point on axis R
O' point s.t. $\vec{OO'}$ is $\vec{\psi}$



$\vec{V}_{O'} = \vec{\omega} \times \vec{\psi} = \vec{\omega} \times \left(\vec{\omega} \times \frac{\vec{\phi}}{\omega^2}\right)$

$= \cancel{\left(\vec{\omega} \cdot \frac{\vec{\phi}}{\omega^2}\right)}^{0} \vec{\omega} - \cancel{(\vec{\omega} \cdot \vec{\omega})} \frac{\vec{\phi}}{\cancel{\omega^2}}$

$= -\vec{\phi}$

* Triple Cross Product
$\vec{A} \times (\vec{B} \times \vec{C})$
$\vec{G} = \vec{B} \times \vec{C} \perp \mathcal{P}_{B,C}$
$\vec{F} = \vec{A} \times \vec{G} \perp \vec{G}$ ie $\perp \vec{B} \times \vec{C}$
$\Rightarrow \vec{F} = m\vec{B} + n\vec{C}$
$\Rightarrow \vec{F} \cdot \vec{A} = m\vec{A} \cdot \vec{B} + n\vec{A} \cdot \vec{C} = 0$
because $\vec{F} \perp \vec{A}$

$m = \lambda \vec{A} \cdot \vec{C}$
$n = -\lambda \vec{A} \cdot \vec{B}$ solves the eq., $\forall \lambda$

$\Rightarrow \vec{F} = \lambda(\vec{A} \cdot \vec{C})\vec{B} - \lambda(\vec{A} \cdot \vec{B})\vec{C} = \vec{A} \times (\vec{B} \times \vec{C})$

$\lambda = 1 \Rightarrow \vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$