19.7 Angular Momentum and Torque for Fixed Axis Rotation

we know that $\vec{\tau}_S^{ext} = I_S\vec{\alpha}$

this is a special case of $\vec{\tau}_S^{ext} = \frac{d}{dt}\vec{L}_S^{sys}$

setup

→ rigid body, rotating around fixed axis passing through point S

→ fixed axis is the z-axis

→ angular velocity : $\vec{\omega} = \theta'(t)\hat{k} = \omega_z\hat{k}$

→ " acceleration : $\vec{\alpha} = \theta''(t)\hat{k} = \alpha_z\hat{k}$

Divide the rigid body

Δm_j moving in circle of radius $r_{S,j}^{\perp}$

$\vec{r}_{S,j}$ position of Δm_j

$\vec{p}_j$ is tangent to the circle

$\vec{L}_{S,j} = \vec{r}_{S,j} \times \vec{p}_j$

$\vec{r}_{S,j} = \vec{r}_{S,j}^{\parallel} + \vec{r}_{S,j}^{\perp}$

$\vec{v}_j = r_{S,j}^{\perp}\omega_z\hat{\theta}$

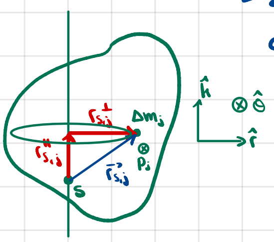
$\Rightarrow \vec{p}_j = \Delta m_j\vec{v}_j = \Delta m_j r_{S,j}^{\perp}\omega_z\hat{\theta}$

$\Rightarrow \vec{L}_{S,j} = \vec{r}_{S,j} \times \vec{p}_j$

$= (r_{S,j}^{\perp}\hat{r} + r_{S,j}^{\parallel}\hat{k}) \times \Delta m_j r_{S,j}^{\perp}\omega_z\hat{\theta}$

$= \Delta m_j {r_{S,j}^{\perp}}^2\omega_z\hat{k} + \Delta m_j r_{S,j}^{\parallel} r_{S,j}^{\perp}\omega_z(-\hat{r})$

→ z-component of L about S



$(L_{S,j})_z = \Delta m_j (r_{S,j}^{\perp})^2\omega_z$

$L_{S,z}^{sys} = \sum(\vec{L}_{S,j})_z = \sum \Delta m_j (r_{S,j}^{\perp})^2\omega_z$

For a continuous mass distribution

$L_{S,z}^{sys} = \int_{body} dm\,(r_{dm})^2\omega_z$

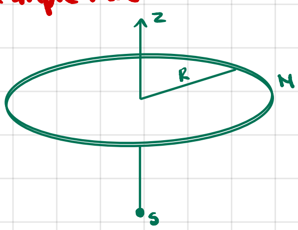
$= I_S\omega_z$

$\Rightarrow \vec{\tau}_S^{ext} = \frac{d}{dt}\vec{L}_S^{sys}$

For z-direction

$\tau_{S,z}^{ext} = \frac{d\vec{L}_{S,z}^{sys}}{dt} = \frac{d}{dt}(I_S\omega_z) = I_S\alpha_z$

Example 19.6



Apply formula : $L_{S,z} = I_S\omega_z$

$= MR^2\omega_z$

Alternatively

The ring is made up of pairs of particle masses. Their angular momentum about S is

$2\Delta m_i R^2\omega\hat{k}$

$L_S = \lim_{n\to\infty}\sum_{i=1}^{n} 2\Delta m_i R^2\omega$

$= \int_{ring} dm \cdot 2R^2\omega$

$dm = \frac{R\,d\theta}{2\pi R}M$

$= \int_0^{\pi} \frac{M}{\cancel{2}\pi}\cdot\cancel{2}R^2\omega\,d\theta = \frac{MR^2\omega}{\pi}\cdot\pi = MR^2\omega$