

## Chapter 21 - Rigid Body Dynamics

### System of particles

→ if system is treated as point particle of mass  $m_T$  at CM moving at  $\vec{v}_{cm}$  we have

$$\vec{F}_{ext} = \frac{d\vec{p}_{S,TS}}{dt} = \frac{d}{dt}(m_T \vec{v}_{cm})$$

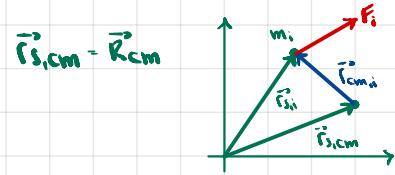
### (Translational Eq. of Motion)

### For system of particles

$$\vec{T}_S^{ext} = \sum_i^n \vec{r}_{Si} \times \vec{F}_i$$

+ assumption that all internal torques cancel in pairs

→ S origin of frame O



$$\text{sub in } \vec{r}_{Si} = \vec{r}_{cm} + \vec{r}_{cm,ii}$$

$$\vec{T}_S^{ext} = \sum_i (\vec{r}_{Si} \times \vec{r}_{cm,ii}) \times \vec{F}_i$$

$$= \sum_i \vec{r}_{Si} \times \vec{F}_i + \sum_i \vec{r}_{cm,ii} \times \vec{F}_i$$

$$= \vec{r}_{Si} \times \vec{F}_{ext} + \sum_i \vec{r}_{cm,ii} \times \vec{F}_i$$

$\rightarrow$   $\vec{T}_S^{ext} = \vec{T}_{S,cm}^{ext} + \vec{T}_{cm}^{ext}$   $\rightarrow$  external torque about CM in CM frame  
 $\rightarrow$  external torque about S with all forces acting on CM

External torque about S can be decomposed into two parts

$$\vec{T}_S^{ext} = \vec{T}_{S,cm}^{ext} + \vec{T}_{cm}^{ext}$$

This result is linked to another result defined previously.

$$\vec{L}_{S,TS} = \vec{L}_{S,cm} + \vec{L}_{cm}$$

$$= \vec{L}_S^{\text{orbital}} + \vec{L}_S^{\text{spin}}$$

$$= \vec{r}_{S,cm} \times \vec{p}_{S,TS} + \sum_i^n \vec{r}_{cm,ii} \times m_i \vec{v}_{cm,ii}$$

We also know that

$$\vec{T}_S^{ext} = \frac{d\vec{L}_{S,TS}}{dt}$$

$$\rightarrow \vec{T}_S^{ext} + \vec{T}_{cm}^{ext} = \frac{d\vec{L}_S^{\text{orbital}}}{dt} + \frac{d\vec{L}_S^{\text{spin}}}{dt}$$

$$\frac{d\vec{L}_S^{\text{orbital}}}{dt} = \frac{d}{dt} \vec{r}_{S,cm} \times \vec{p}_{S,TS}$$

$$+ \vec{r}_{S,cm} \times \frac{d\vec{p}_{S,TS}}{dt}$$

$$= \vec{v}_{S,cm} \times m \vec{a}_{S,cm}$$

$$+ \vec{r}_{S,cm} \times m \vec{a}_{cm} \cdot \vec{a}_{cm}$$

$$= \vec{r}_{S,cm} \times \vec{F}_{ext} = \vec{T}_{S,cm}^{ext}$$

The time derivative of orbital angular momentum about S is the external torque about S if the system were a point particle at CM.

$$\frac{d\vec{L}_S^{\text{spin}}}{dt} = \sum_i \vec{v}_{cm,ii} \times m \vec{a}_{cm,ii}$$

$$+ \sum_i \vec{r}_{cm,ii} \times m_i \vec{a}_{cm,ii}$$

$$= \vec{T}_{cm}^{ext}$$

### 21.5 Work-Energy Theorem

We've previously defined a result for a system of particles:

$$K = \frac{\sum m_i v_{Si}^2}{2} = K_{S,cm} + K_{cm}$$

Considering a rigid body as system of particles

$$K = K_{trans} + K_{rot}$$

Here is another definition.

$$W_{trans} = \int \vec{F}_{ext} \cdot d\vec{r} = \int \frac{dm \vec{v}_{cm}}{dt} \cdot d\vec{R}_{cm}$$

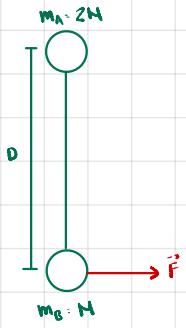
$$= m \int \frac{d\vec{v}_{cm}}{dt} \cdot \vec{v}_{cm} dt$$

$$= \frac{m}{2} \int d(\vec{v}_{cm} \cdot \vec{v}_{cm})$$

$$= \frac{m(v_{cm,f}^2 - v_{cm,i}^2)}{2}$$

$$= \Delta K_{trans}$$

### Example 21.1 - Angular Impulse



Rigid body system

we know:

-> motion of rigid body can be decomposed into translation of CM about initial frame and rotation about CM.

-> more specifically, for infinitesimal dm; on the rigid body

$$d\vec{r}_i = d\vec{r}_{\text{cm}} + \vec{r}_{\text{cm},i}$$

where  $d\vec{r}_{\text{cm}}$  is translation of the CM and  $\vec{r}_{\text{cm},i}$  is a rotational displacement.

$\vec{r}_{\text{cm},i}$  is constant,  $\vec{r}_{\text{cm},i} \cdot \vec{v}_{\text{cm},i} = 0$

$$\vec{\omega} = \vec{r}_{\text{cm},i} \times \vec{v}_{\text{cm},i}$$

Also

$$\vec{L}_{\text{ext}} = \vec{L}_{\text{sys}} + \vec{L}_{\text{cm}}$$

$$\vec{L}_{\text{sys}} = \vec{L}_{\text{cm}} + \vec{L}_{\text{cm}}$$

-> we also have 2nd law equation for the system

$$\vec{F}_{\text{ext}} \cdot m_T \cdot \vec{a}_{\text{cm}}$$

->  $\vec{F}$  applies an impulse on the system, changing its linear momentum.

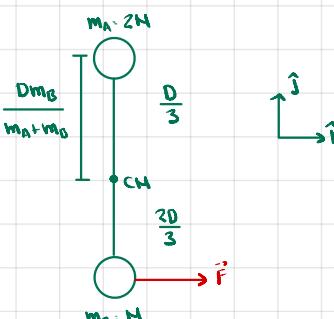
It also applies an angular impulse, changing angular momentum about the CM.

Find CM position:

$$\frac{m_A (-r_A \hat{j}) + m_B (-D - r_A) \hat{j}}{m_A + m_B} = 0$$

$$\Rightarrow m_A r_A - D m_B + m_B r_A = 0$$

$$\Rightarrow r_A = \frac{D m_B}{m_A + m_B} = \frac{D \cdot N}{3N} = \frac{D}{3}$$



$$\vec{F}_{\text{ext}} \cdot \vec{F} = m_T \cdot \vec{a}_{\text{cm}}$$

$$\int_{t_i}^t \vec{F} dt = m_T (\vec{v}_{\text{cm},f} - \vec{v}_{\text{cm},i})$$

$$\vec{F} \Delta t = m_T \vec{v}_{\text{cm},f}$$

$$\vec{v}_{\text{cm},f} = \frac{\vec{F} \Delta t}{m_A + m_B}$$

$$\vec{L}_{\text{cm},\text{sys}} = \vec{L}_{\text{cm},\text{cm}} + \vec{L}_{\text{cm}}$$

$$= \vec{r}_A \hat{j} \times m_A (\omega \hat{h} \times \vec{r}_A \hat{j})$$

$$+ \vec{r}_B \hat{j} \times m_B (\omega \hat{h} \times \vec{r}_B \hat{j})$$

$$= r_A \hat{j} \times m_A (r_A \omega (-\hat{i}) + r_B \hat{j} \times m_B r_B \omega (-\hat{i}))$$

$$= m_A r_A^2 \omega \hat{h} + m_B r_B^2 \omega \hat{h}$$

$$= I_{\text{cm},A} \vec{\omega} + I_{\text{cm},B} \vec{\omega}$$

$$= \left( \frac{m_A D^2 m_B \omega}{(m_A + m_B)^2} + \frac{m_B D^2 m_A \omega}{(m_A + m_B)^2} \right) \hat{h}$$

$$= \left( \frac{2m_D^2 N^2 \omega}{9N^2} + \frac{4N^2 m_A \omega}{9N^2} \right) \hat{h}$$

$$\Rightarrow \vec{L}_{\text{cm},\text{sys}} = \frac{2D^2 \omega N}{3} \hat{h}$$

$$\vec{L}_{\text{cm},\text{sys}} = \vec{L}_{\text{cm},\text{cm}} + \vec{L}_{\text{cm},\text{sys}}$$

$$= r_B \hat{j} \times \vec{F} - r_B F \hat{k}$$

$$= \frac{2DF}{3} \hat{k}$$

$$\int_{t_i}^t \frac{2DF}{3} \hat{k} dt = \frac{2DF \Delta t}{3}$$

$$\frac{1/2 \vec{F} \Delta t}{\Delta t} = \frac{1/2 D \omega N}{3}$$

$$\omega = \frac{F \Delta t}{DN}$$

### Interpretation

In this calculation we used the CM frame.

Torque about CM in CM frame is  $2DF \hat{k}/3$ , which acted for  $\Delta t$  applying an angular impulse  $\frac{2DF \Delta t \hat{k}}{3}$  increasing ang. momentum from 0 to  $2D^2 \omega N \hat{h}/3$ .

$$\Rightarrow \omega = F \Delta t / DN$$

If instead we do calculations from m\_B frame.

$$\vec{L}_{\text{B},\text{sys}} = \vec{L}_B^{\text{orb}} + \vec{L}_B^{\text{spin}}$$

$$= \frac{2D}{3} \hat{j} \times 3N \vec{v}_{\text{cm}}$$

$$+ 2N \cdot \frac{D^2}{9} \omega \hat{h} + N \cdot \frac{4D^2}{9} \omega \hat{h}$$

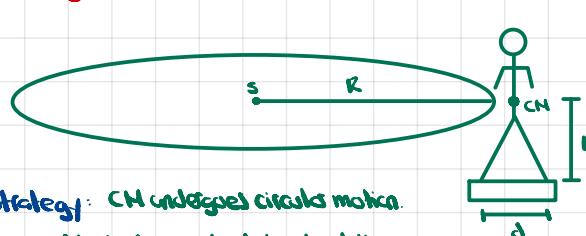
$$= 2D \hat{j} \times N \frac{F \Delta t}{3N} \hat{i} + \frac{2DN^2}{3} \omega \hat{h}$$

equal to 0 because there is no torque rel. to B so  $\Delta \vec{L}_B = \vec{0}$

$$\Rightarrow \frac{1/2 \vec{F} \Delta t}{\Delta t} = \frac{1/2 D^2 \omega}{\Delta t} \Rightarrow \omega = \frac{F \Delta t}{DN}$$

### Example 21.2 - Person on Railroad Car Moving in Circle

Strategy:



CM undergoes circular motion.

The rigid body does not rotate about the CM.

There is no angular acceleration of CM, so

there is no torque of CM rel. to curve center, and  $\Delta \vec{L}_{\text{cm,CM}} = \vec{0}$ .

There are torques about the CM, but the net  $\vec{L}_{\text{cm}}$  is zero because there is also no angular acceleration (or velocity).

$\vec{L}_{\text{cm}} = \vec{0}$  is one equation involving  $N_1, N_2$ .

2nd law in  $\hat{j}$  direction relates  $N_1, N_2$  to  $mg$ .

$$\vec{L}_{\text{cm}} = \left(\frac{d}{2}\hat{i} - L\hat{j}\right) \times (N_2\hat{j} - \mu N_1\hat{i}) + \left(-\frac{d}{2}\hat{i} - L\hat{j}\right) \times (N_1\hat{j} - \mu N_2\hat{i})$$

$$= \frac{dN_2}{2}\hat{k} - LN_2\hat{k} - \frac{dN_1}{2}\hat{k} - LN_1\hat{k}$$

$$= \hat{k}\left(\frac{d}{2}(N_2 - N_1) - LN(N_1 + N_2)\right) = \vec{0}$$

$$\frac{d}{2}(N_2 - N_1) = LN(N_1 + N_2)$$

$$N_1 + N_2 - mg = 0 \Rightarrow N_1 + N_2 = mg$$

$$\vec{L}_{\text{s,cm}} = R\hat{i} \times (-mg\hat{j} + N_1\hat{j} + N_2\hat{j}) = \vec{0}$$

$$\Rightarrow (-Rmg + RN_1 + RN_2)\hat{k} = 0$$

$$\Rightarrow N_1 + N_2 = mg$$

Friction at the feet causes radial acceleration.

$$-(f_1 + f_2)\hat{r} = -R\omega^2\hat{r}$$

$$\Rightarrow \mu(N_1 + N_2) = R\omega^2$$

$\vec{L}_{\text{cm,sfs}} = \vec{0}$  because the person is at rest in CM frame.

$$\vec{L}_{\text{s,sfs}} = \vec{L}_{\text{s,cm}} = I_{\text{s,cm}}\omega\hat{k} = mR^2\omega\hat{k}$$

$$\frac{d}{2}(N_2 - N_1) = LN(N_1 + N_2)$$

$$N_1 + N_2 = mg$$

$$N(N_1 + N_2) = R\omega^2$$

$$Nmg = R\omega^2$$

$$\omega = \sqrt{\frac{Nmg}{R}}$$

$$\frac{d}{2}(N_2 - mg + N_2) = LNmg$$

$$LN_2 - \frac{dmg}{2} = LNmg$$

$$N_2 = \frac{dmg + 2LNmg}{2d}$$

$$\Rightarrow N_2 = \frac{mg(d + 2LN)}{2d} = \frac{mg}{2} + \frac{mgLN}{d}$$

$$= \frac{mg}{2} + \frac{RL\omega^2}{d} = \frac{mg}{2} + \frac{L\omega^2}{R}$$

$$N_1 = \frac{2dmg - mgd - mg2LN}{2d}$$

$$\Rightarrow N_1 = \frac{mg(d - 2LN)}{2d} = \frac{mg}{2} - \frac{mgLN}{d}$$

## Textbook Solution

use cylindrical coordinates



use two dynamical equations of motion

$$\vec{F}_{ext} = \frac{d\vec{p}_{sys}}{dt} = \frac{d}{dt}(m_r \vec{v}_{cm}) = m \vec{a}_{cm}$$

$$\vec{\tau}_{cm,z} = I_{cm} \vec{\alpha}_{cm,z}$$

$$\vec{\alpha}_{cm,z} = \vec{0}$$

2nd law  $\vec{r} d\vec{r}$

$$-(f_1 + f_z) \hat{r} = -m R \omega^2 \hat{r} - m \frac{v^2}{R} \hat{r}$$

$$\Rightarrow f_1 + f_z = \frac{mv^2}{R}$$

$$2nd law \hat{h} \quad N_1 + N_2 = mg$$

rotational eq. of motion

$$\vec{\tau}_{cm,foot_1} = \vec{0}$$

$$= (-\frac{d}{2} \hat{r} - L \hat{k}) \times (-f_1 \hat{r} + N_1 \hat{h})$$

$$= -\frac{d}{2} N_1 (-\hat{o}) + L f_1 \hat{o}$$

$$= (L f_1 + \frac{dN_1}{2}) \hat{o}$$

$$\vec{\tau}_{cm,foot_2} = (\frac{d}{2} \hat{r} - L \hat{k}) \times (-f_2 \hat{r} + N_2 \hat{h})$$

$$= \frac{d}{2} N_2 (-\hat{o}) + L f_2 \hat{o}$$

$$= \hat{o} (L f_2 - \frac{N_2 d}{2})$$

$$\vec{\tau}_{cm,f_1} + \vec{\tau}_{cm,f_2} = \vec{0}$$

$$\Rightarrow L f_1 + \frac{N_1 d}{2} + L f_2 - \frac{N_2 d}{2} = 0$$

## Equations

$$\text{Torque: } L f_1 + \frac{N_1 d}{2} + L f_2 - \frac{N_2 d}{2} = 0$$

$$\text{2nd law: } f_1 + f_2 = \frac{mv^2}{R}$$

$$N_1 + N_2 = mg$$

unknowns:  $N_1, N_2, f_1, f_2$

$$L(f_1 + f_2) = \frac{d}{2} (N_2 - N_1)$$

$$\frac{Lmv^2}{R} = \frac{d}{2} (N_2 - mg + N_1)$$

$$dN_2 = \frac{Lmv^2}{R} + \frac{mgd}{2}$$

$$N_2 = \frac{mg}{2} + \frac{Lmv^2}{Rd}$$

$$N_1 = \frac{mg}{2} - \frac{Lmv^2}{Rd}$$

$$N_1 = 0 \Rightarrow \frac{Lmv_{max}^2}{Rd} = \frac{mg}{2}$$

$$\Rightarrow v_{max} = \sqrt{\frac{Rgd}{2L}}$$

$$\Rightarrow N_2 = \frac{mg}{2} + \frac{Lm}{Rd} \cdot \cancel{\frac{kgd}{2L}}$$

$$= mg$$

What happens if  $v > v_{max}$ ?

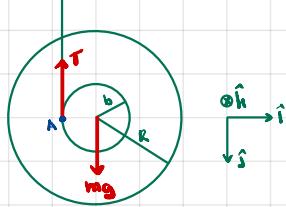
$N_1$  cannot be  $< 0$  so  $N_{2max} = mg$ .

The torque eq. doesn't balance out anymore.

$$L(f_1 + f_2) - \frac{mgd}{2} > 0$$

$$\Rightarrow \vec{\alpha}_{cm} > 0$$

### Example 21.3 Torsion, Rotation, and Translation



Tension applies a torque about CM. This changes angular momentum.

Additionally,  $mg$  applies torque about A changing angular momentum.

$mg - T$  accelerates the CM.

$$\vec{T}_{cm,sys} = \vec{T}_{cm,cm} + \vec{T}_{cm,sys}$$

$$= (-b\hat{i}) \times (-T\hat{j}) = bT\hat{h} = I_{cm}\alpha\hat{h}$$

$$\Rightarrow bT = \frac{mR^2}{2}\alpha$$

$$mg - T = m \cdot a \Rightarrow T = m(g - a)$$

$$a = b\alpha$$

Three equations,  $T, a, \alpha$  unknowns

$$T = m(g - b\alpha)$$

$$\Rightarrow bT(g - b\alpha) = \frac{mR^2\alpha}{2}$$

$$bg - b^2\alpha = \frac{R^2\alpha}{2}$$

$$\Rightarrow \alpha \left( \frac{R^2}{2} + b^2 \right) = bg$$

$$\alpha = \frac{2bg}{R^2 + 2b^2}$$

$$a = \frac{2b^2g}{R^2 + 2b^2}$$

$$T = m \left( \frac{g(R^2 + 2b^2 - 2b^2)}{R^2 + 2b^2} \right)$$

$$= \frac{mgR^2}{R^2 + 2b^2}$$

After a length  $l$  has been unwound

$$v = \frac{2b^2gt}{R^2 + 2b^2}$$

$$t = \frac{b^2g t^2}{R^2 + 2b^2}$$

$$t_e = \sqrt{\frac{l(R^2 + 2b^2)}{b^2g}}$$

$$v_e = 2 \sqrt{\frac{lb^2g}{R^2 + 2b^2}}$$

$$\omega = \frac{2}{R} \sqrt{\frac{l\sqrt{g}}{R^2 + 2b^2}} = \sqrt{\frac{4g\alpha}{R^2 + 2b^2}}$$

we could also have integrated  $\alpha$

$$\omega = \frac{2bgt}{R(R + 2b)}$$

and substituted in  $t_e$

Alternatively we solve using energy!

$$E_i = mgL$$

$$E_f = \frac{mv_f^2}{2} + \frac{mR^2}{2} \cdot \frac{v_f^2}{b^2} \cdot \frac{1}{2}$$

no nonconservative forces doing work

$$\Rightarrow DE = 0$$

$$v_f^2 + \frac{R^2 v_f^2}{2b^2} = 2gL$$

$$v_f^2 \left( \frac{2b^2 + R^2}{2b^2} \right) = 2gL$$

$$v_f = \sqrt{\frac{4b^2gL}{R^2 + 2b^2}}$$

$$\Rightarrow \omega_3 = \frac{v_f}{b} = \sqrt{\frac{4g\alpha}{R^2 + 2b^2}}$$

$$\int_0^{t_e} \left( mg - \frac{mgR^2}{R^2 + 2b^2} \right) dt$$

$$= \int_0^{t_e} \frac{mg \cdot 2b^2}{R^2 + 2b^2} dt$$

$$= \frac{mg \cdot 2b^2}{R^2 + 2b^2} \sqrt{\frac{b(R^2 + 2b^2)}{b^2 g}}$$

$$= \frac{4m^2 g b^2 l}{R^2 + 2b^2} \cdot \Delta P_1$$

$$P_{f3} = m \cdot z \cdot \sqrt{\frac{lb^2g}{R^2 + 2b^2}}$$

$$= \sqrt{\frac{4m^2 g b^2 l}{R^2 + 2b^2}}$$

$$\int_0^{t_e} b \frac{mgR^2}{R^2 + 2b^2} dt$$

$$= \frac{bmgR^2}{R^2 + 2b^2} \sqrt{\frac{l(R^2 + 2b^2)}{b^2 g}}$$

$$= \sqrt{\frac{m^2 g R^4 l}{R^2 + 2b^2}} \cdot \Delta L_{cm}$$

$$\vec{T}_{cm,sys} = \frac{mR^2}{2} \cdot \sqrt{\frac{4g\alpha}{R^2 + 2b^2}}$$

$$= \sqrt{\frac{m^2 g R^4 l}{R^2 + 2b^2}}$$

### Recap of steps

draw forces and possibly torque diagram

write equation for torque of system about a chosen point

$$\vec{T}_{s,sys} = \vec{T}_{s,cm} + \vec{T}_{cm}$$

write 2nd law equations for the CM.

$$\vec{F}_{ext} = M_T \cdot \vec{a}_{cm}$$

use constraint equations such as  $\vec{a}_{cm} = b\vec{x}$

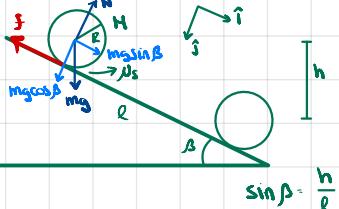
solve for unknown forces, acceleration and angular accel.

Mathematics can now be used to insert time into the analysis. We can now find velocity, position, angular velocity, angular displacement, and find the time it takes to reach specified values of these variables.

In the case of speed and displacement, we can use the energy theorem.

## Ex 21.4 - Cylinder Rolling Down Inclined Plane

Inclined Plane



### Solution 1

$$\vec{F}_{p,sys} = R\hat{j} \times (-\hat{s}\hat{i}) = R\hat{k}$$

$$\Rightarrow Rf = \frac{mR^2}{2}\alpha \quad \text{unknowns: } f, N, \alpha, a_{cm}$$

$$mg\cos\beta \cdot N$$

$$mg\sin\beta - f = m \cdot a_{cm}$$

$$a_{cm} = R\alpha$$

$$J = \frac{mR^2}{2} \cdot \frac{\alpha}{R} = \frac{m}{2}$$

$$mg\sin\beta = \frac{3}{2}a_{cm}$$

$$a_{cm} = \frac{2}{3}g\sin\beta$$

$$f = \frac{1}{3}mg\sin\beta$$

$$\alpha = \frac{2g\sin\beta}{3R}$$

$$v_{cm} = \frac{2g\sin\beta}{3}t$$

$$x_{cm} = \frac{g\sin\beta t^2}{3} = \frac{l}{2}$$

$$\Rightarrow t_c = \sqrt{\frac{3l}{g\sin\beta}}$$

$$v_{cm,2} = \frac{2g\sin\beta}{3} \sqrt{\frac{3l}{g\sin\beta}}$$

$$= \sqrt{\frac{4lg\sin\beta}{3}} = \sqrt{\frac{4gh}{3}}$$

$v_{cm}$  changes due to  $F_{gx} = f$ .

Rotation occurs due to  $T_{cm,f}$ .

The no-slip condition is  $v_{cm} = R\omega$ ,

which are assumed true because

circle  $a_{cm} = R\alpha$ .

We reached  $f = \frac{1}{3}mg\sin\beta$  for the equation to balance w/ no-slip.

Experimentally we have another eq.

$$F_{max} \cdot N = \mu mg\cos\beta$$

$$\Rightarrow \frac{W/\sin\beta}{3} \leq \mu W/\cos\beta$$

$$\Rightarrow \tan\beta \leq 3\mu, \text{ i.e. } \mu_s \geq \frac{\tan\beta}{3}$$

### Solution 2: Energy Methods

With by non-conservative friction equals change in mechanic energy.

The motion of the rigid body is = CM translation and a rotation about the CM.

$$W_T = \int_{x_1}^{x_2} (mg\sin\beta - \frac{mg\sin\beta}{3}) dx \\ = \frac{2mg\sin\beta}{3}$$

$\rightarrow$  gravitational potential energy

$E_{kin} = mgh \rightarrow$  translational kinetic energy

$$E_{rot} = \frac{mV^2}{2} + \frac{m}{2} \cdot \frac{V^2}{R^2} \cdot \frac{1}{2} I \quad \text{rotational kinetic energy}$$

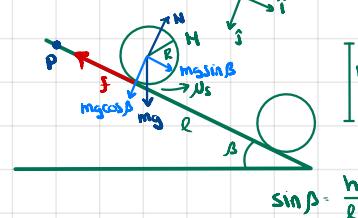
$W_f = 0$ , because the contact point is at rest relative to the surface.

$$\Delta E_m = W_f = 0$$

$$\frac{mV^2}{2} + \frac{mV^2}{4} - mgh = 0$$

$$\frac{3V^2}{4} = gh \Rightarrow V = \sqrt{\frac{4gh}{3}}$$

### Solution 3



Torque about P

$$\vec{F}_{p,mg} = \vec{F}_{p,cm,mg} + \vec{F}_{cm,mg}$$

$$= \vec{F}_{p,cm} \times \vec{F}_g$$

$$= (X_{p,cm}\hat{i} - Y_{p,cm}\hat{j}) \times mg(\sin\beta\hat{i} + \cos\beta\hat{j})$$

$$= (X_{p,cm}\cos\beta + R\sin\beta)mg\hat{h}$$

$$\vec{F}_{p,f} = \vec{F}_{p,cm,f} + \vec{F}_{cm,f}$$

$$= (X_{p,cm}\hat{i} - R\hat{j}) \times (-\hat{s}\hat{i}) + R\hat{j} \times (-\hat{s}\hat{i})$$

$$= -R\hat{f}\hat{h} + R\hat{f}\hat{h} = \vec{0}$$

$$\vec{F}_{p,N} = (X_{p,cm}\hat{i} - R\hat{j}) \times (-N\hat{j})$$

$$+ R\hat{j} \times N\hat{j}$$

$$= -N X_{p,cm}\hat{h}$$

$$\vec{F}_{p,sys} = \vec{F}_{p,N} + \vec{F}_{p,f} + \vec{F}_{p,mg}$$

$$= [(X_{p,cm}\cos\beta + R\sin\beta)mg - mg\cos\beta \cdot X_{p,cm}] \hat{h}$$

$$= Rmg\sin\beta \hat{h}$$

$$\vec{F}_{p,sys} = \vec{F}_{p,cm} + \vec{F}_{cm,sys}$$

$$= (X_{p,cm}\hat{i} - R\hat{j}) \times mV_{cm}\hat{i} + \frac{mR^2}{2} \cdot \omega \hat{h}$$

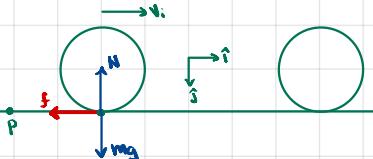
$$= (RmV_{cm} + \frac{mR^2\omega}{2}) \hat{h}$$

$$RmV_{cm} + \frac{mR^2\omega}{2} = Rmg\sin\beta$$

$$\frac{3}{2}a_{cm} = g\sin\beta$$

$$\Rightarrow a_{cm} = \frac{2g\sin\beta}{3}$$

### Example 21.5 - Bouncing Ball



If there is friction there is torque about CM and hence rotation and change in angular momentum.

Slipping means  $v_{cm} > R\omega$

Friction also generates negative acceleration. At some point  $v_{cm}$  equals  $R\omega$  and there is no slipping

$$\vec{\tau}_{p,i} = \vec{\tau}_{p,f} - \vec{\tau}_s$$

$$\vec{\tau}_{p,mg} = \vec{\tau}_{p,cm,mg} + \vec{\tau}_{cm,mg}$$

$$= (x_{p,cm}\hat{i} - R\hat{j}) \times mg\hat{j} + \vec{0}$$

$$= x_{p,cm}mg\hat{k}$$

$$\vec{\tau}_{p,N} = \vec{\tau}_{p,cm,N} + \vec{\tau}_{cm,N}$$

$$= (x_{p,cm}\hat{i} - R\hat{j}) \times (-N\hat{j}) + R\hat{j} \times (-N\hat{j})$$

$$= -Nx_{p,cm}\hat{k}$$

$$\Rightarrow \vec{\tau}_{p,s,f} = (x_{p,cm}mg - x_{p,cm}N)\hat{k}$$

2nd law:

$$N = mg$$

$$\Rightarrow \vec{\tau}_{p,s,f} = x_{p,cm}(mg - N)\hat{k} = \vec{0}$$

No torque about a fixed point along contact surface.

$$\Rightarrow \vec{\tau}_{p,i} = \vec{\tau}_{p,f}$$

$$\vec{\tau}_{p,i} = \vec{\tau}_{p,cm} + \vec{\tau}_{cm,i}$$

$$= (x_{p,cm}\hat{i} - R\hat{j}) \times m\vec{\omega}_{cm,i} + I_{cm}\vec{\alpha}_{cm,i}$$

$$= v_{cm,i}R\hat{k}$$

$$\vec{\tau}_{p,f} = (x_{p,cm}\hat{i} - R\hat{j}) \times m\vec{\omega}_{cm,f}\hat{i} + I_{cm}\vec{\alpha}_{cm,f}\hat{k}$$

$$v_{cm,f} = \omega_f \cdot R$$

$$I_{cm} = \frac{2mR^2}{5}$$

$$\Rightarrow \vec{\tau}_{p,f} = v_{cm,f} \cdot R\hat{k} + I_{cm}\vec{\alpha}_{cm,f}\hat{k}$$

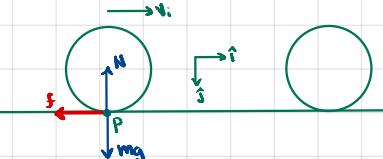
$$= (\omega_f R^2 m + \frac{2mR^2\omega_f}{5})\hat{k}$$

$$\vec{\tau}_{p,i} - \vec{\tau}_{p,f} = \cancel{N\vec{\alpha}_{cm,i}} - \frac{7\omega_f R^2}{5}\hat{k}$$

$$\Rightarrow \omega_f = \frac{5v_{cm,i}}{TR}$$

$$v_{cm,f} = \frac{5v_{cm,i}}{T}$$

can P be the moving contact point?



$$\vec{\tau}_{p,s,f} = \vec{\tau}_{p,cm} + \vec{\tau}_{cm,s,f}$$

$$\vec{\tau}_{p,s,f,N} = -R\hat{j} \times (-N\hat{j}) + R\hat{j} \times (-N\hat{j})$$

$$= \vec{0}$$

$$\vec{\tau}_{p,s,f,mg} = -R\hat{j} \times mg\hat{j} + R\hat{j} \times mg\hat{j}$$

$$= \vec{0}$$

$$\Rightarrow \vec{\tau}_{p,s,f} = \vec{0}$$

$$\vec{\tau}_{p,s,f,i} = \vec{\tau}_{p,cm,i} + \vec{\tau}_{cm,s,f,i}$$

$$= (-R\hat{j}) \times m\vec{\omega}_{cm,i}\hat{i} + I_{cm}\vec{\alpha}_{cm,i}\hat{k}$$

$$= Rm\vec{\omega}_{cm,i}\hat{k}$$

$$\vec{\tau}_{p,s,f,f} = -R\hat{j} \times m\vec{\omega}_{cm,f}\hat{i} + I_{cm}\vec{\alpha}_{cm,f}\hat{k}$$

$$= (Rm\vec{\omega}_{cm,f} + I_{cm}\vec{\alpha}_{cm,f})\hat{k}$$

$$\Delta \vec{\tau}_{p,s,f} = \vec{0}$$

$$\Rightarrow Rm\vec{\omega}_{cm,i} = Rm\vec{\omega}_{cm,f} + I_{cm}\vec{\alpha}_{cm,f}$$

In the final state, no slipping

$$\Rightarrow v_{cm,f} = R\omega_{cm,f}$$

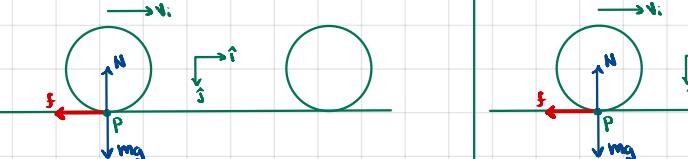
$$\text{Also, } I_{cm} = \frac{2mR^2}{5}$$

$$\Rightarrow \cancel{R\vec{\alpha}_{cm,i}} = Rm\vec{\omega}_{cm,f} + \frac{2mR^2\omega_f}{5}$$

$$= \frac{7\omega_f R^2}{5}$$

$$\Rightarrow v_{cm,f} = \frac{5}{7}v_{cm,i}$$

$$\omega_{cm,f} = \frac{5v_{cm,i}}{TR}$$



$$\vec{\tau}_{cm,s,f} = R\hat{j} \times (-Si) = Rf\hat{i} = I_{cm}\vec{\alpha}_{cm,f}$$

about CM system torque is now not zero anymore

$$-S\hat{i} = m a_{cm}\hat{i}$$

$a_{cm}\hat{i} = -\frac{f}{m}\hat{i}$  the ball is decelerating due to kinetic friction

while having accelerating angular speed, ie positive angular acceleration, due to torque applied by the same kinetic friction.  $\omega_{cm}$  is thus decreasing and  $\omega$  is increasing. At some

point the ball stops slipping, the contact point coincides with the ball is at rest rel. to surface. with  $R\omega$ . At that

there is no longer kinetic friction and static friction point there is no is zero.

$$v_{cm}(t_f) = v_{cm,i} - \frac{St_f}{m} = R\omega(t_f) = \frac{R^2 St_f}{I_{cm}}$$

$$t_f \left( \frac{R^2}{I_{cm}} + \frac{1}{m} \right) = v_{cm,i}$$

$$t_f = \frac{v_{cm,i}}{f} \cdot \frac{I_{cm}m}{R^2 m + I_{cm}}$$

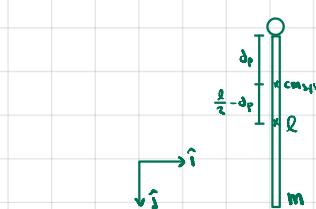
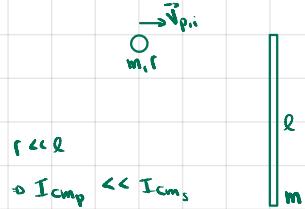
$$v_{cm,f} = v_{cm,i} - \frac{v_{cm,i} I_{cm}}{R^2 m + I_{cm}}$$

$$= \frac{v_{cm,i} R^2 m}{R^2 m + I_{cm}} = \frac{v_{cm,i} R^2 m}{\frac{5}{2} m R^2}$$

$$= \frac{5}{7}v_{cm,i}$$

$$\omega_{cm,f} = \frac{5v_{cm,i}}{TR}$$

## Example 21.6 Rotation and Translation Object and Shell Collision



a) conservation

$$-\frac{mv_{pi}\hat{j}}{2m} + \left(\frac{l}{2} - d_p\right)\hat{i} = 0$$

$$\Rightarrow 2md_p = \frac{ml}{2}$$

$$d_p = \frac{l}{4}$$

b) linear velocity of system

no external forces  $\Rightarrow \Delta p = 0$   
 $\vec{p}_i = mv_{pi}\hat{i}$   
 $\vec{p}_f = 2m\vec{v}_{cm,f}$

$$\Rightarrow \vec{v}_{cm,f} = \frac{v_{pi}\hat{i}}{2}$$

c) mechanical energy

$$K_i = \frac{m v_{pi}^2}{2}$$

$$K_f = \frac{m v_{pi}^2}{2} + \frac{I_{cm,if} \omega^2}{2}$$

$$= \frac{m v_{pi}^2}{2}$$

$$+ \left(I_{cm} + \frac{ml^2}{8}\right) \left[\frac{2m}{8I_{cm} + ml^2}\right]^2 v_{pi}^2 \cdot \frac{1}{2}$$

$$= \frac{m v_{pi}^2}{2} \cdot \frac{6ml^2 + 8I_{cm}}{8ml^2 + 64I_{cm}} < \frac{m v_{pi}^2}{2}$$

The collision is totally inelastic.

Energy not conserved.

d) angular momentum

$$\vec{L}_{cm,sfi,i} = -\frac{l}{4}\hat{j} \times m v_{pi}\hat{i} = \frac{2mv_{pi}}{4}\hat{i}$$

$$\vec{L}_{cm,sfi,f} = \frac{2mv_{pi}}{4}\hat{i} + I_{cm} \omega \hat{i}$$

$$I_{cm} = I_{cm} + m \frac{l^2}{16}$$

$$v_{pi} = \frac{l}{4}\omega$$

$$\Rightarrow \vec{L}_{cm,sfi,f} = \left[\frac{2mv_{pi}}{16}\right] + \left(I_{cm} + \frac{ml^2}{16}\right)\omega \hat{i}$$

$$\Delta L = 0 \Rightarrow \frac{2mv_{pi}}{4} = \omega I_{cm} + \frac{ml^2}{8}\omega$$

$$\Rightarrow 2lmv_{pi} = 8I_{cm}\omega + ml^2\omega$$

$$\Rightarrow \omega = \frac{2lm}{8I_{cm} + ml^2} v_{pi}$$

$$v_{pi,f} = \frac{2lm}{16I_{cm} + 2ml^2}$$

e) The shell has  $\vec{v}_{g,cm,f} = \frac{v_{pi}\hat{i}}{2}$

The shell rotates w/  $\omega$ :  $\frac{2lm}{8I_{cm} + ml^2} v_{pi}$

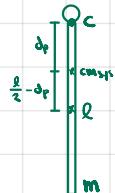
$$\vec{v}_{g,cm}(t) = \frac{v_{pi}\hat{i}}{2}$$

$$\Theta(t) = \frac{2lmv_{pi}t}{8I_{cm} + ml^2}$$

$$\Theta = 2\pi \Rightarrow t = \frac{\pi(8I_{cm} + ml^2)}{lmv_{pi}}$$

$$x_{g,cm}(t_f) = \frac{\pi(8I_{cm} + ml^2)}{2lm}$$

Repeat angular momentum calc about point C



$$\vec{L}_{c,sfi,i} = \vec{0}$$

$$\vec{L}_{c,sfi,f} = -\frac{l}{4}\hat{j} \times 2m\vec{v}_{c,cm,f} + \vec{L}_{cm,sfi,f}$$

$$= -\frac{l}{4}\hat{j} \times 2m\vec{v}_{c,cm,f} + \omega \left(I_{cm} + \frac{ml^2}{8}\right)\hat{i}$$

$$\vec{v}_{g,cm,f} = \vec{v}_{g,ci} + \vec{v}_{c,cm,f}$$

$$\vec{v}_{c,cm,f} = \frac{v_{pi}\hat{i}}{2} - \frac{l}{4}\omega_{cm}\hat{i}$$

$$\vec{L}_{c,sfi,f} = \frac{l}{4}\hat{j} \times 2m \left( \frac{v_{pi}}{2}\hat{i} - \frac{l}{4}\omega_{cm}\hat{i} \right)$$

$$+ \omega \left(I_{cm} + \frac{ml^2}{8}\right)\hat{i}$$

$$= -\frac{2mlv_{pi}}{8}\hat{i} + \frac{2ml^2\omega_{cm}}{16}\hat{i}$$

$$+ \omega \left(I_{cm} + \frac{ml^2}{8}\right)\hat{i}$$

$$- \frac{mlv_{pi}}{4}\hat{i} + \frac{ml^2\omega_{cm}}{8}\hat{i} + \omega_{cm}I_{cm} = 0$$

$$\omega(ml^2 + 8I_{cm}) = 2mlv_{pi}$$

$$\omega = \frac{2mlv_{pi}}{ml^2 + 8I_{cm}}$$