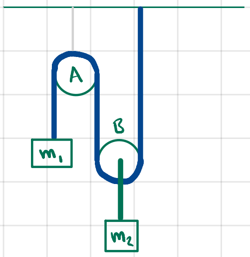
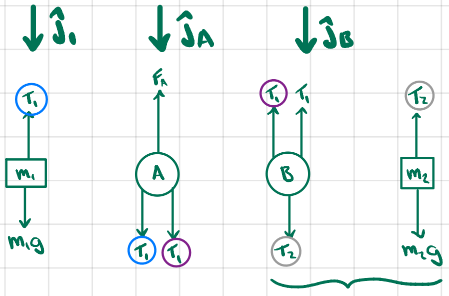
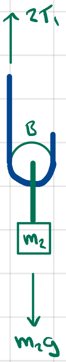
12.1 Pulley Problems





* Assumptions
$m_A = m_B = 0$
rope is slipping, massless $\Rightarrow$ T uniform



$m_1 g - T_1 = m_1 a_1$

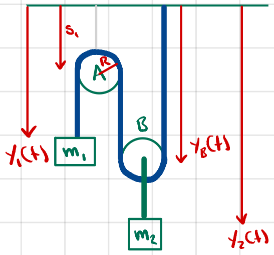
$m_2 g - 2T_1 = m_2 a_2$

Three unknowns, two equations
(T_1, a_1, a_2)

we need an extra equation, a constraint eq.

Let's look at three ways to obtain constraint equations.

1) Analytically



$\frac{d^2 y_1(t)}{dt^2} = a_1 \qquad \frac{d^2 y_2(t)}{dt^2} = a_2$

l_1: length of the string = constant

$= y_1(t) - s_1 + \pi R + (y_B(t) - s_1) + \pi R + y_B(t)$

$l_2 = \text{constant} = y_2(t) - y_B(t) \Rightarrow \frac{d^2 y_2(t)}{dt} - \frac{d^2 y_B(t)}{dt} = a_2(t) - a_B(t) = 0$

$\Rightarrow a_B(t) = a_2(t)$

$\frac{d^2 l_1}{dt} = 0 = \frac{d^2 y_1(t)}{dt} + 2\frac{d^2 y_B(t)}{dt} = 0 \Rightarrow a_1(t) = -2a_B(t) = -2a_2(t)$

we now have

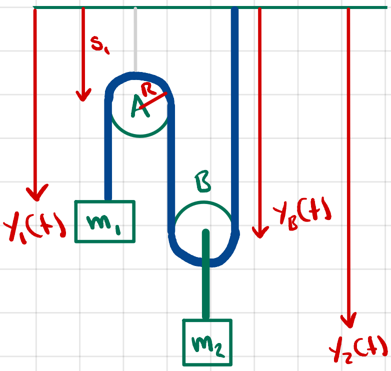
$a_1 = -2a_2$
$m_1 g - T_1 = m_1 a_1 \quad \Rightarrow T_1 = m_1(g + 2a_2)$
$m_2 g - 2T_1 = m_2 a_2 \quad \Rightarrow T_1 = m_2(g - a_2)$

$\Rightarrow m_1 g + 2m_1 a_2 = m_2 g - m_2 a_2 \Rightarrow a_2(2m_1 + m_2) = g(m_1 + m_2) \Rightarrow a_2 = \frac{g(m_1+m_2)}{3m_1 + m_2}$

$\Rightarrow a_1 = -\frac{2g(m_1+m_2)}{3m_1+m_2}$

$T_1 = m_1(g - a_1) = m_1\left[\frac{2m_1 g + g m_2 + 2m_1 g + 2m_2 g}{2m_1 + m_2}\right] = \frac{m_1(4m_1 g + 3m_2 g)}{2m_1 + m_2}$

2) Visual Argument



m_2 and B move together as a unit.

If we have a displacement $\Delta y_2 = \Delta y_B$, the rope on either side of B also gets displaced by Δy_2, so the total displacement of rope is $2\Delta y_2$.

m_1 is displaced Δy_1 due to the displacement of the rope

$\Rightarrow \Delta y_1 = -2\Delta y_2$.

3) Differentials

$$\ell_1 = y_1(t) + 2y_B(t) + 2\pi r - 2s_1$$

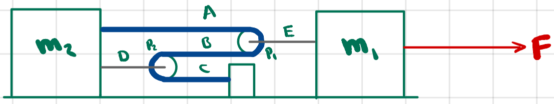
$$\ell_2 = y_2(t) - y_B(t)$$

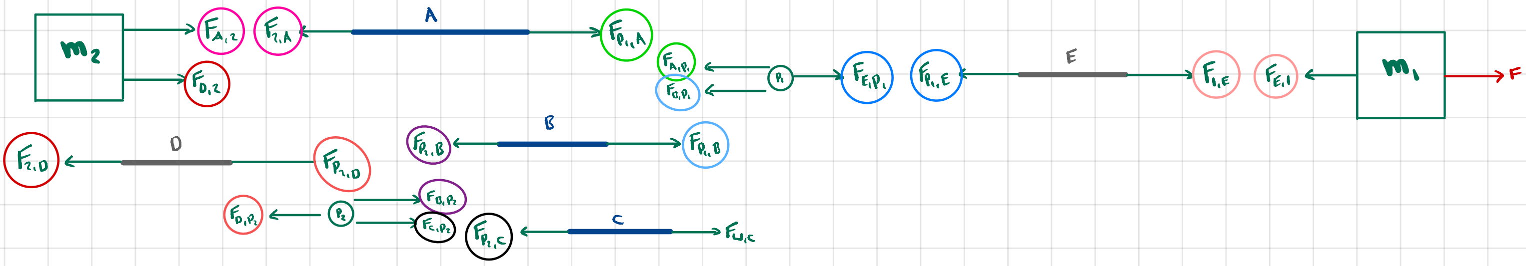
$$d\ell_2 = dy_2 - dy_B = 0 \Rightarrow dy_2 = dy_B$$

$$d\ell_1 = dy_1(t) + 2dy_B(t) = 0 \Rightarrow dy_1(t) = -2\,dy_2(t) \Rightarrow \frac{dy_1}{dt} = -2\frac{dy_2}{dt} \Rightarrow \frac{d^2y_1}{dt} = 2\frac{dy_2}{dt}$$

$$\Rightarrow a_1 = -2a_2$$

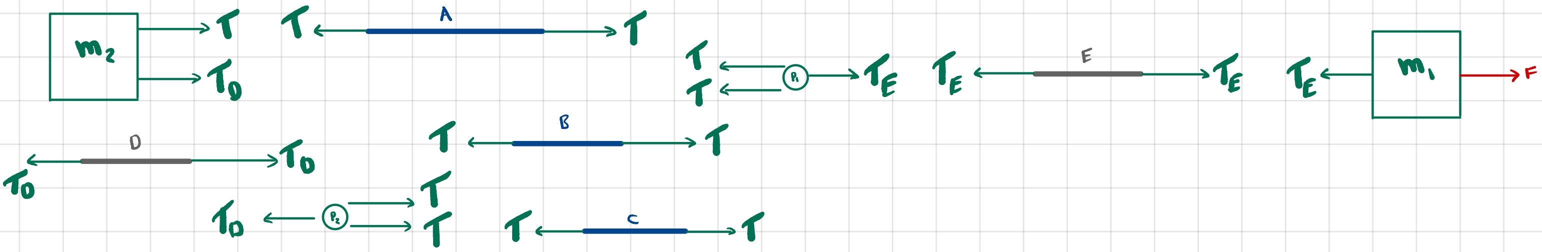
12.5 Worked Example - 2 Blocks, 2 Pulleys

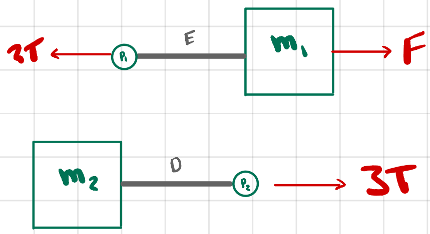




Note we assume the ropes are massless.

For ex, 3rd law: $F_{A,2} = F_{2,A}$, 2nd law: $F_{P_1,A} - F_{2,A} = 0 \Rightarrow F_{P_1,A} = F_{2,A}$



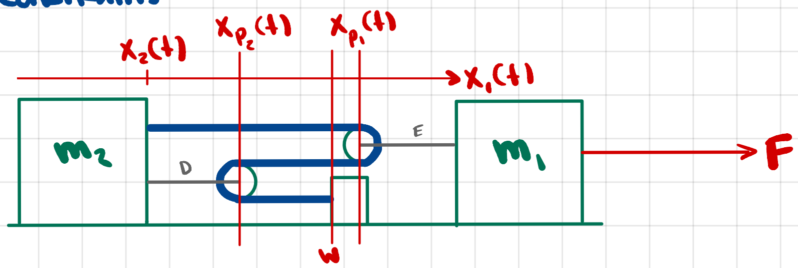


$F - 2T = m_1 a_1$

$3T = m_2 a_2$

2 eq. 3 unknowns T, a_1, a_2

Constraints



$\ell_D = \text{constant} = x_{P_2}(t) - x_2(t) \Rightarrow a_{P_2} = a_2$

$\ell_E = \text{constant} = x_1(t) - x_{P_1}(t) \Rightarrow a_1 = a_{P_1}$

$\ell = \text{constant} = x_{P_1}(t) - x_2(t) + \pi R + x_{P_1}(t) - x_{P_2}(t) + \pi R + w - x_{P_2}(t)$

$= 2x_{P_1}(t) - 2x_{P_2}(t) - x_2(t) + 2\pi R + w = 2x_1(t) + C_1 - 2x_2(t) + C_2 - x_2(t) + 2\pi R + w$

$= 2x_1(t) - 3x_2(t) + 2\pi R + w + C$

$\Rightarrow 2a_1 - 3a_2 = 0 \Rightarrow a_1 = \frac{3}{2}a_2$ ↗ extra equation ⇒ 3 eq. 3 unknowns

$F - 2T = m_1 a_1 \Rightarrow F - 2T = m_1 \cdot \frac{3a_2}{2}$

$3T = m_2 a_2 \Rightarrow T = \frac{m_2 a_2}{3}$

$a_1 = \frac{3}{2} a_2$

$\Rightarrow F = \frac{2}{3} m_2 a_2 + \frac{3}{2} m_1 a_2 \Rightarrow 6F = 4m_2 a_2 + 9m_1 a_2$

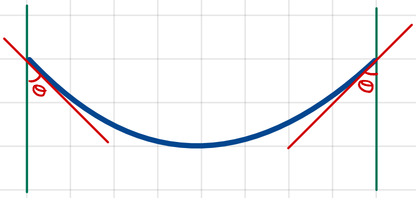
$\Rightarrow a_2(9m_1 + 4m_2) = 6F$

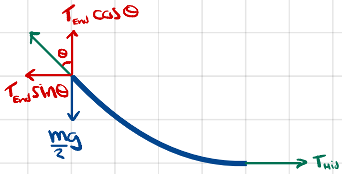
$$\Rightarrow a_2 = \frac{6F}{9m_1 + 4m_2}$$

$a_1 = \frac{3}{2} a_2$

$T = \frac{m_2 a_2}{3}$

13.1 Rope Hanging Between Trees

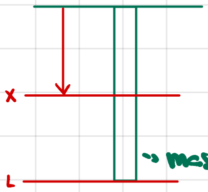




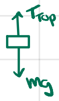
$$T_E\cos\theta - \frac{mg}{2} = 0 \Rightarrow T_E = \frac{mg}{2\cos\theta}$$

$$T_M - T_E\sin\theta = 0 \Rightarrow T_M = \frac{mg\tan\theta}{2}$$

13.2 Differential Analysis of a Massive Rope



→ massive rope hanging from the ceiling



little piece of rope at the top

$T_{top} = mg$

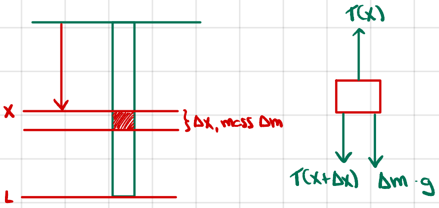
bottom of rope, $T=0$

At some position x what is the tension?

"cut" the rope at x & ask what tension force would be necessary to support the rope?

$$T(x) = \left(\frac{L-x}{L}\right)mg = (1 - x/L)mg$$

Alternative approach: differential analysis



$$T(x+\Delta x) + \Delta m g - T(x) = 0$$

$$T(x+\Delta x) - T(x) = -\Delta m g$$

* if massless rope ($\Delta m = 0$) then $T(x+\Delta x) = T(x)$

Note that $\Delta m = \frac{\Delta x}{L} m$

$$\Rightarrow T(x+\Delta x) - T(x) = -\frac{\Delta x\, m}{L} g \Rightarrow \frac{T(x+\Delta x) - T(x)}{\Delta x} = -\frac{mg}{L}$$

$$\lim_{\Delta x \to 0} \frac{T(x+\Delta x) - T(x)}{\Delta x} = \frac{dT(x)}{dx} = -\frac{mg}{L}$$

$$\Rightarrow dT = -\frac{mg}{L}dx \Rightarrow \int_{T(0)}^{T(x)} dT' = \frac{-mg}{L}\int_0^x dx' \Rightarrow T(x) - T(0) = \frac{-mgx}{L}$$

$T(0) = mg$

$$\Rightarrow T(x) = mg\left(1 - \frac{x}{L}\right)$$

* we reached same result but this approach is much more powerful. If the rope's mass were not uniformly distributed through the rope, it could still be modelled like this using a different expression for Δm.

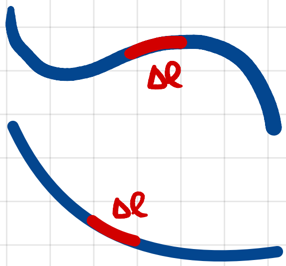
13.3 Differential Elements

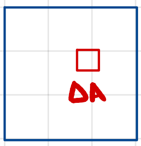
→ Frequently, we need to consider how the mass of an object is distributed throughout the object.

→ we define a small piece of the object and then consider the mass of that small piece

object in one dimension

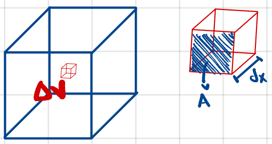
Δl, diff el of length, containing mass Δm





2D object

Area element



$\Delta V = A dx$

13.4 Density

We want to relate $\Delta l, \Delta A, \Delta V$ to Δm

1D, linear density = $\lambda = \Delta m / \Delta l$

2D: $\sigma = \Delta m / \Delta A$

3D: $\rho = \Delta m / \Delta V$

13.6 Summary of Differential Analysis

Steps when analyzing continuous mass distribution

1. Choose Δm, small mass element
2. Analyze forces acting on Δm
3. Take limit as $\Delta x \to 0$ ⇒ differential equation
4. Separate and integrate
5. Apply boundary condition

14.1 Intro to Resistive Forces

→ Drag forces in fluid (e.g. liquid, air)

depend on

(1) properties of the object (speed, size, shape)

(2) density, viscosity, compressibility of the fluid

Air drag

$$F_{drag} = \frac{1}{2} C_D \rho A v^2$$

drag coeff. (C_D), cross sectional area (A), object speed (v), resistive force (F_{drag}), density of air (ρ)

* drag coeff: different for different shapes

viscosity : η $\qquad |\vec{F}_{drag}| \sim v$, $\sim \eta$

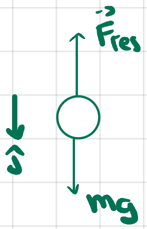
→ special case : sphere of radius R

$$\vec{F}_{drag} = -6\pi\eta R\vec{v} \quad \text{(Stokes' Law)}$$

14.2 Resistive Forces - Low Speed Case

$\vec{F}_{res} = -\alpha\vec{v}$

velocity dependent resistive force



e.g. marble falling in a vat of oil

units of α : $\frac{[F]}{[v]} = \frac{kg\,ms^{-2}}{ms^{-1}} = kg \cdot s^{-1}$

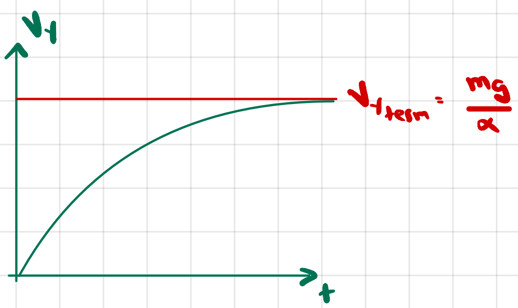
$\vec{F} = m\vec{a}$, $\hat{j}$ direction

$$mg - \alpha v_y = m\frac{dv_y}{dt} \Rightarrow \frac{dv_y}{dt} = g - \frac{\alpha}{m}v_y \quad \text{(linear 1}^{st}\text{ order diff eq.)}$$

$$\frac{1}{g - \alpha/m}dv_y = dt \Rightarrow \int_{v_y(0)}^{v_y(t)} \frac{dv_y}{-\frac{\alpha}{m}(v_y - \frac{mg}{\alpha})} = \int_{t'=0}^{t'=t} dt' \Rightarrow -\frac{m}{\alpha}\ln\left[\frac{v_y - mg/\alpha}{-mg/\alpha}\right] = t$$

$$\Rightarrow \ln\left[\frac{v_y - mg/\alpha}{-mg/\alpha}\right] = -\frac{\alpha}{m}t \Rightarrow \frac{v_y - mg/\alpha}{-mg/\alpha} = e^{-\frac{\alpha}{m}t}$$

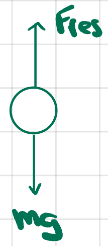
$$\Rightarrow v_y(t) = \frac{mg}{\alpha}\left(1 - e^{-\frac{\alpha}{m}t}\right)$$



Define $e^{-\frac{\alpha}{m}t} = e^{-t/\tau}$ $\qquad \tau = \frac{m}{\alpha} \Rightarrow [\tau] = \frac{kg}{kg\,s^{-1}} = s^{1}$

$v_{y\,term} \Leftrightarrow \frac{dv_y}{dt} = g - \frac{\alpha}{m}v_y = 0 \Leftrightarrow$ resistive force = gravitational force

14.3 Resistive Forces - High Speed Case



→ more complicated example of drag forces

→ model: $\vec{F}_{res} = -\beta v^2 \hat{j}$

$$[\beta] = \frac{kg\; m\; s^{-2}}{m^2\, s^{-2}} = kg \cdot m^{-1}$$

$\vec{F} = m\vec{a}$ in dir. $\hat{j}$

nonlinear 1st order diff eq.

$$mg - \beta v^2 = m\frac{dv}{dt} \Rightarrow \frac{dv}{dt} = g - \frac{\beta}{m}v^2 \Rightarrow \frac{dv}{g - \frac{\beta}{m}v^2} = dt \Rightarrow \frac{-dv}{g\left(1 - \frac{\beta v^2}{mg}\right)} = -dt \Rightarrow \frac{-dv}{1 - \frac{\beta v^2}{mg}} = -g\,dt$$

change of variables

$$u = \sqrt{\frac{\beta}{mg}}\, v \Rightarrow du = \sqrt{\frac{\beta}{mg}}\, dv$$

$u_0 = 0, v_0 = 0$

*Note: if the object starts at rest, initially it will be moving very slowly, so our resistive model proportional to v^2 doesn't apply. We're going to disregard this here. To be more accurate we'd need to model each stage of the motion differently

$$u(t) = [\beta/mg]^{1/2}\, v(t)$$

$$\Rightarrow -[mg/\beta]^{1/2} \int_0^{u(t)} \frac{du}{1-u^2} = -\int_0^t g\,dt$$

$$\Rightarrow \int_0^{u(t)} \frac{du}{1-u^2} = [\beta/mg]^{1/2} \int_0^t -g\,dt$$

$$= \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)\Bigg|_0^{u(t)} = -gt[\beta/mg] \Rightarrow -\ln\left(\frac{1+u}{1-u}\right)\Bigg|_0^{u(t)} = -2gt[\beta/mg]^{1/2}$$

$$= \ln\left[\frac{1 - [\beta/mg]^{1/2} v(t)}{1 + [\beta/mg]^{1/2} v(t)}\right] = -2gt[\beta/mg]^{1/2}$$

$$\frac{1 - [\beta/mg]^{1/2} v(t)}{1 + [\beta/mg]^{1/2} v(t)} = e^{-2gt[\beta/mg]^{1/2}}$$

Define $\tau = [mg/\beta] \cdot \frac{1}{2g}$ $\quad [\tau] = s$

$$\Rightarrow \frac{1 - \frac{v(t)}{2g\tau}}{1 + \frac{v(t)}{2g\tau}} = e^{-\frac{t}{\tau}} \Rightarrow 1 - \frac{v(t)}{2g\tau} = e^{-\frac{t}{\tau}} + \frac{e^{-t/\tau}\, v(t)}{2g\tau}$$

$$\Rightarrow \frac{v(t)}{2g\tau}\left[1 + e^{-t/\tau}\right] = 1 - e^{-t/\tau} \Rightarrow v(t) = \sqrt{\frac{mg}{\beta}}\, \frac{(1 - e^{-t/\tau})}{(1 + e^{-t/\tau})}$$

* $2g\tau = \sqrt{\frac{mg}{\beta}}$

$$\Rightarrow t \to \infty \quad e^{-t/\tau} \to 0 \quad v(t \to \infty) = v_{term} = \sqrt{\frac{mg}{\beta}} \Rightarrow \frac{dv}{dt} = 0$$