

PSet 1

Problem 1 (Same as Example 4.6)

$$v_{c,0} = 12 \text{ m}\cdot\text{s}^{-1} \quad v_c(t_1) = 0 \quad a_c(t) = \begin{cases} 0 & 0 \leq t \leq t_1 \\ b(t-t_1) & t_1 < t \leq t_2 \end{cases} \quad b = -6 \text{ m}\cdot\text{s}^{-2}$$

a)

$$v_c(t) = \begin{cases} v_{c,0} & 0 \leq t \leq t_1 \\ v_{c,0} + \frac{b(t-t_1)^2}{2} & t_1 < t \leq t_2 \end{cases}$$

$$x_c(t) = \begin{cases} v_{c,0}t + x_{c,0} & 0 \leq t \leq t_1 \\ v_{c,0}t + \frac{b(t-t_1)^3}{6} + x_{c,0} & t_1 < t \leq t_2 \end{cases}$$

inserting initial data

$$a_c(t) = \begin{cases} 0 & 0 \leq t \leq t_1 \\ -6(t-1) & t_1 < t \leq t_2 \end{cases}$$

$$v_c(t) = \begin{cases} 12 & 0 \leq t \leq t_1 \\ 12 - 3(t-1)^2 & t_1 < t \leq t_2 \end{cases}$$

$$x_c(t) = \begin{cases} 12t & 0 \leq t \leq t_1 \\ 12t - (t-1)^3 & t_1 < t \leq t_2 \end{cases}$$

b) BIKE

$v_{b,0}$ constant speed throughout

$$x_b(0) = -17$$

$$x_b(t_2) = x_c(t_2)$$

$$v_c(t_2) = 0$$

$$a_b(t) = 0$$

$$\Rightarrow v_b(t) = v_{b,0}$$

$$x_b(t) = -17 + v_{b,0}t$$

same position at $t = t_2$

$$x_b(t_2) = x_c(t_2)$$

$$-17 + v_{b,0}t_2 = 12t_2 - (t_2 - 1)^3$$

what is t_2 ?

$$v_c(t_2) = 0 = 12 - 3(t_2^2 - 2t_2 + 1)$$

$$t_2^2 - 2t_2 + 1 = 4 \Rightarrow t_2^2 - 2t_2 - 3 = 0$$

$$\Delta = 4 - 4 \cdot 1 \cdot (-3) = 16$$

$$t_2 = \frac{2 \pm 4}{2} \stackrel{?}{=} \begin{cases} 3 \\ -1 \end{cases}$$

$$-17 + 3v_{b,0} = 36 - (2)^3$$

$$\Rightarrow 3v_{b,0} = 28 + 17 = 45$$

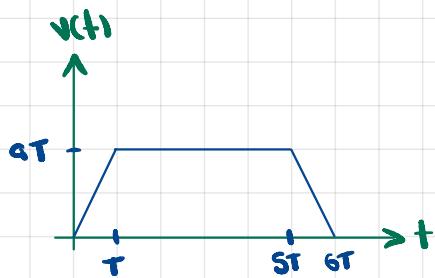
$$\Rightarrow v_{b,0} = 15 \text{ m}\cdot\text{s}^{-1}$$

Problem 2 - Elevator Trip

a)

Take interval of time: $[0, 6T]$

$$a(t) = \begin{cases} a & 0 \leq t < T \\ 0 & T \leq t \leq ST \\ -a & ST < t \leq 6T \end{cases}$$



$$v(t) = \begin{cases} at & 0 \leq t < T \\ aT + \int_0^t 0 dt = aT & T \leq t \leq ST \\ aT + \int_{ST}^t (-a) dt = aT - (at - aST) = 6aT - at & ST < t \leq 6T \end{cases}$$

That is

$$v(t) = \begin{cases} at & 0 \leq t < T \\ aT & T \leq t \leq ST \\ 6aT - at & ST < t \leq 6T \end{cases}$$

$$x(t) = \begin{cases} \int_0^t at dt = \frac{at^2}{2} & \\ \frac{aT^2}{2} + \int_T^t aT dt = \frac{aT^2}{2} + aT(t-T) = aT(t - \frac{T}{2}) & \\ aT(ST - \frac{T}{2}) + \int_{ST}^t (6aT - at) dt = \frac{9aT^2}{2} + 6aT(t-ST) - \frac{a(t^2 - 2ST^2)}{2} & \end{cases}$$

That is

$$x(t) = \begin{cases} \frac{at^2}{2} \\ aT(t - \frac{T}{2}) \\ \frac{9aT^2}{2} + 6aT(t-ST) - \frac{a(t^2 - 2ST^2)}{2} \end{cases}$$

b) The easy way to solve for a is to simply calculate the area under $v(t)$ between $t=0$ and $t=6T$.

$$\text{Area} = \frac{2 \cdot aT \cdot T}{2} + 4T \cdot aT = 5aT^2$$

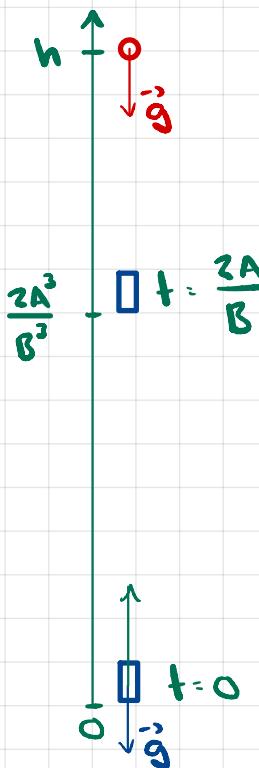
$$\text{This area is actually the same as } \int_0^{6T} v(t) dt = x(6T) = h. \text{ Thus, } 5aT^2 = h \rightarrow a = \frac{h}{5T^2}$$

Alternatively, calculate $x(6T)$ directly:

$$x(6T) = \frac{9aT^2}{2} + 6aT^2 - \frac{a(36T^2 - 2ST^2)}{2} = \frac{21aT^2}{2} - \frac{11aT^2}{2} = 5aT^2$$

$$\text{Thus, } 5aT^2 = h \text{ and } a = \frac{h}{5T^2}, \text{ as before.}$$

Problem 3 - Rocket Launch



$$a_r(t) = A - Bt \quad A, B > 0 \quad t > 0$$

$$v_r(t) = At - \frac{Bt^2}{2}$$

$$y_r(t) = \frac{At^2}{2} - \frac{Bt^3}{6}$$

$$\text{max height: } v_r(t) = 0 \Rightarrow t(A - \frac{Bt}{2}) = 0 \rightarrow t = \frac{2A}{B}$$

$$y_r\left(\frac{2A}{B}\right) = \frac{A}{2} \cdot \frac{4A^2}{B^2} - \frac{B}{3} \cdot \frac{8A^3}{B^3} = \frac{2A^3}{B^2} - \frac{4A^3}{3B^2} = \frac{2A^3}{3B^2}$$

The stone falls for $\frac{2A}{B}$ seconds and $|Dy_s| = h - \frac{2A^3}{B^2}$

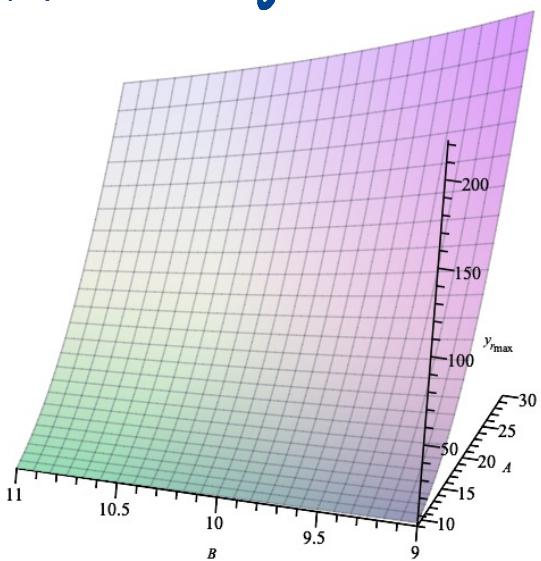
$$a_s(t) = -g$$

$$v_s(t) = -gt$$

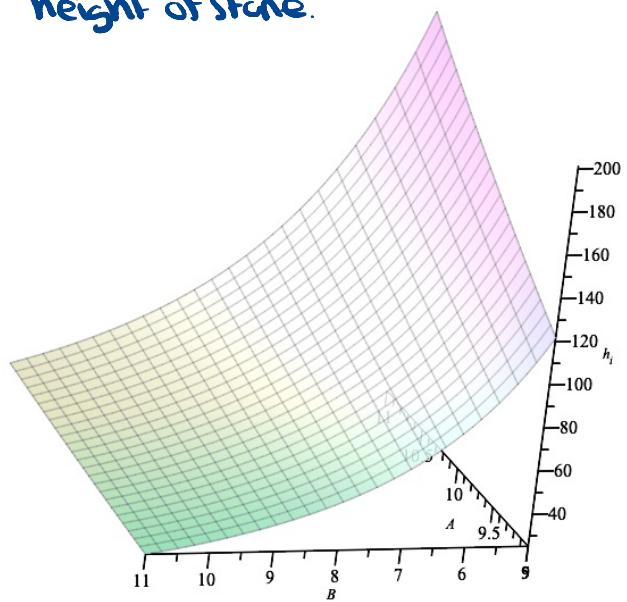
$$y_s(t) = h - \frac{gt^2}{2}$$

$$y_s\left(\frac{2A}{B}\right) = \frac{2A^3}{B^2} = h - \frac{g}{2} \frac{4A^2}{B^2} = h - \frac{2A^2g}{B^2} \rightarrow h = \frac{2A^2}{B^2} [A + g]$$

Plot of $y_r\left(\frac{2A}{B}\right) = \frac{2A^3}{2B^2} = y_{r_{\max}}$
= max rocket height



Plot of $h(A, B) = \frac{2A^2}{B^2}(A+g)$, initial height of stone.



Problem 4 - Throw and Catch

Person

$$\vec{a}_p(t) = \langle Bt, 0 \rangle, B > 0$$

$$\vec{v}_p(t) = \langle Bt^2/2, 0 \rangle$$

$$\vec{r}_p(t) = \langle Bt^3/6 + d, 0 \rangle$$

Ball



$$\vec{a}_b(t) = \langle 0, -g \rangle$$

$$\vec{v}_b(t) = \langle v_0 \cos \theta, v_0 \sin \theta - gt \rangle$$

$$\vec{r}_b(t) = \langle v_0 \cos \theta t, v_0 \sin \theta t - \frac{gt^2}{2} \rangle$$

Catching the Ball

$$\vec{r}_b(t) = \vec{r}_p(t)$$

$$\begin{cases} t v_0 \cos \theta = Bt^3/6 + d \\ t v_0 \sin \theta - gt^2/2 = 0 \end{cases} \Rightarrow t(v_0 \sin \theta - gt/2) = 0 \Rightarrow t = \frac{2v_0 \sin \theta}{g}$$

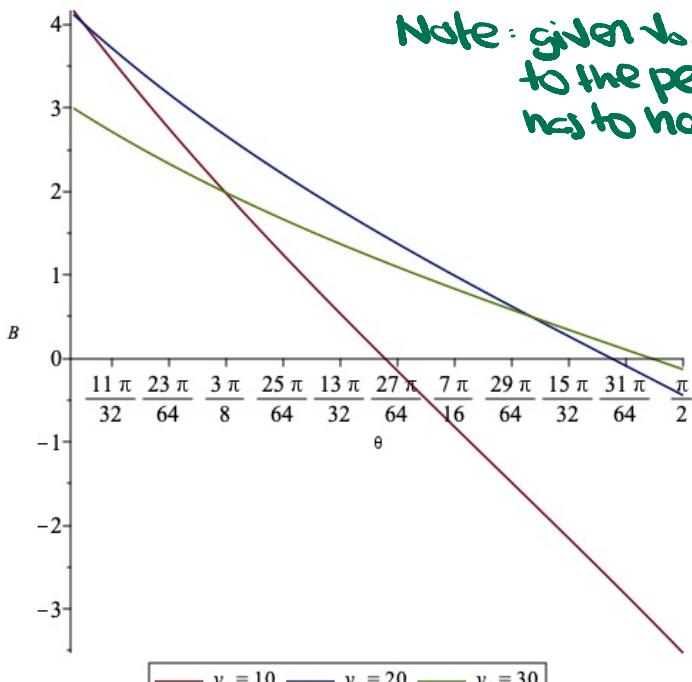
sub. into first eq.

$$\frac{3v_0 \sin \theta}{g} \cdot v_0 \cos \theta = \frac{B}{6} \cdot \frac{8v_0^3 \sin^3 \theta}{g^3} + d$$

$$\Rightarrow \left[\frac{\sin(2\theta) v_0^2}{g} - d \right] \cdot \frac{3}{4} \frac{g^3}{v_0^3 \sin^3 \theta} = B$$

$$\Rightarrow B(\theta, v_0, d, g) = [\sin(2\theta) v_0^2 - dg] \cdot \frac{3g^2}{4v_0^3 \sin^3 \theta}$$

Maple Plots



Note: given v_0 , above a certain θ the ball does not make it to the person before falling below $y=0$. The person has to have $B < 0$, i.e. run towards origin, to catch the ball

The larger v_0 is, the larger such an angle is. Larger v_0 means larger x and y components of velocity: the ball travels farther or else equal.

Problem 5 - Vertical Collision

First can



$$a_1(t) = -g$$

$$v_1(t) = v_0 - gt$$

$$y_1(t) = v_0 t - \frac{gt^2}{2}$$

Second can

$$a_2(t) = \begin{cases} 0 & 0 \leq t \leq t_1 \\ -g & t > t_1 \end{cases}$$

$$v_2(t)$$

$$0 \leq t \leq t_1 \Rightarrow v_2(t) = 0$$

$$t > t_1 \Rightarrow v_2(t) = C - gt$$

$$v_2(t_1) = v_0 - C - gt_1 \Rightarrow C = v_0 + gt_1$$

$$\Rightarrow v_2(t) = \begin{cases} 0 & 0 \leq t \leq t_1 \\ v_0 + gt_1 - gt & t > t_1 \end{cases}$$

$$y_2(t)$$

$$0 \leq t \leq t_1 \Rightarrow y_2(t) = 0$$

$$t > t_1 \Rightarrow y_2(t) = C + (v_0 + gt_1)t - \frac{gt^2}{2}$$

$$y_2(t_1) = 0 = C + v_0 t_1 + gt_1^2 - \frac{gt_1^2}{2}$$

$$\Rightarrow C = v_0 t_1 + \frac{gt_1^2}{2} \Rightarrow C = -(v_0 t_1 + \frac{gt_1^2}{2})$$

$$\Rightarrow y_2(t)$$

$$= \begin{cases} 0 & 0 \leq t \leq t_1 \\ -(v_0 t_1 + \frac{gt_1^2}{2}) & t > t_1 \end{cases}$$

$$+ (v_0 + gt_1)t - \frac{gt^2}{2} \quad t > t_1$$

Collision at $y = y_c$

\Rightarrow calculate time t at which can 2 reaches y_c , given v_0, t_1 .

\Rightarrow given v_0, t_1 , calculate time for can 1 to drop to y_c .

\Rightarrow equate times to obtain v_0 as function of t_1 .

$$a) y_c = -(v_0 t_1 + \frac{gt_1^2}{2}) + (v_0 + gt_1)t - \frac{gt^2}{2}$$

$$\Rightarrow \frac{g}{2}t^2 - (v_0 + gt_1)t + (v_0 t_1 + \frac{gt_1^2}{2} + y_c) = 0$$

$$\Delta = (v_0 + gt_1)^2 - 4 \cdot \frac{g}{2} (v_0 t_1 + \frac{gt_1^2}{2} + y_c)$$

$$= v_0^2 + 2v_0 g t_1 + g^2 t_1^2 - 2v_0 g t_1 - g^2 t_1^2 - 2g y_c$$

$$= v_0^2 - 2g y_c \Rightarrow v_0^2 \geq 2g y_c$$

$$\Rightarrow t = \frac{v_0 + gt_1 \pm \sqrt{v_0^2 - 2g y_c}}{g}$$

\Rightarrow can 2 reaches y_c

$$v_0 t - \frac{gt^2}{2} = y_c \Rightarrow \frac{g}{2}t^2 - v_0 t + y_c = 0$$

$$\Delta = v_0^2 - 4 \cdot \frac{g}{2} \cdot y_c = v_0^2 - 2y_c g \Rightarrow v_0^2 \geq 2y_c g$$

$$t = \frac{v_0 \pm \sqrt{v_0^2 - 2y_c g}}{g} \Rightarrow \text{can 1 reaches } y_c$$

$$\frac{v_0 + \sqrt{v_0^2 - 2y_c g}}{g} = \frac{v_0 + gt_1 - \sqrt{v_0^2 - 2g y_c}}{g}$$

$$\Rightarrow 2\sqrt{v_0^2 - 2y_c g} = gt_1$$

$$4v_0^2 - 8y_c g = g^2 t_1^2$$

$$\Rightarrow v_0^2(t_1, g, t_1) = \frac{g^2 t_1^2 + 8y_c g}{4}$$

$$\Rightarrow v_0(t_1, g, t_1) = \sqrt{\frac{g^2 t_1^2 + 8y_c g}{4}}$$

$$b) v_0(5, 9.8, 4) = 31.95 \text{ m/s}$$

$$t_{\text{collision}} = 4.24 \text{ s}$$