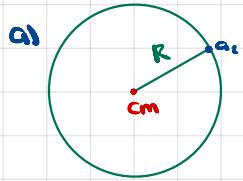


PSet 10 - Rotational Motion

Problem 1 - Disc and Washer

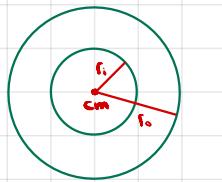


uniform disc \Rightarrow CM at center

$$\text{ii) } I_{\text{cm}} = \int_{\text{disc}} r^2 dm = \iint_{0}^{2\pi} r^2 \cdot \frac{M}{\pi R^2} r dr d\theta = \frac{M}{\pi R^2} \int_0^{2\pi} \frac{R^4}{4} d\theta = \frac{MR^2}{4\pi} \cdot 2\pi = \frac{MR^2}{2}$$

$$\text{iii) } I_{\omega_c} = I_{\text{cm}} + M \cdot R^2 = \frac{3}{2} MR^2$$

b) uniform washer

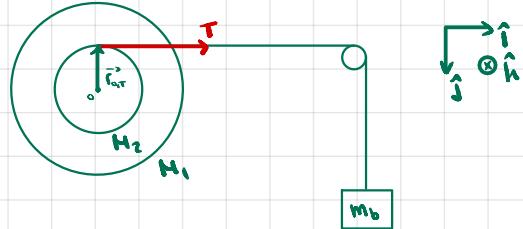


$$I_{\text{cm}} = \int_{\text{washer}} r^2 dm = \iint_{r_i}^{r_o} \int_0^{\theta} r^2 \cdot \rho \cdot r dz dr d\theta = \rho \int_{r_i}^{r_o} \int_0^{\theta} r^3 dr d\theta = \frac{\rho}{4} \int_{r_i}^{r_o} (r_o - r_i)^4 d\theta = \frac{\rho \cdot (r_o - r_i)^4 \cdot 2\pi}{4}$$

$$= \frac{\rho \cdot (r_o - r_i)^4 \cdot \pi}{2} = 4 \cdot 10^{-3} \cdot 7.8 \cdot 10^3 \cdot (31 \cdot 10^{-3} - 13.5 \cdot 10^{-3})^4 \cdot \frac{\pi}{2} = 4.59 \cdot 10^{-6} \text{ kg} \cdot \text{m}^2$$

$$\frac{m \cdot \text{kg}}{\text{m}^3} \cdot \text{m}^4 = \text{kg} \cdot \text{m}^2$$

Problem 2 - Compound Pulley



There is a torque applied by the tension force on the rope.

$$\vec{\tau}_{0,T} = -\frac{R}{2}\hat{j} \rightarrow \vec{\tau}_{0,T} \cdot \frac{R}{2}\hat{k} = \frac{R^2(4M_1+M_2)}{8} \alpha_z h$$

$$\vec{T} \cdot T\hat{i}$$

$$I_1 = \frac{M_1 R^2}{2} \Rightarrow I_1 + I_2 = \frac{R^2(4M_1+M_2)}{8}$$

$$I_2 = \frac{M_2 R^2}{8}$$

$$m_b g - T = m_b \cdot a$$

$$\text{constraint: } \alpha_z \cdot \frac{R}{2} = a$$



3 equations, 3 unknowns: α_z, T, a

$$\frac{RT}{2} = \frac{R^2(4M_1+M_2)}{8} \alpha_z$$

$$T = m_b(g-a)$$

$$\alpha_z = \frac{2a}{R}$$

$$\cancel{4} \cancel{R} \cdot m_b(g-a) = \cancel{R}(4M_1+M_2) \cdot \cancel{\frac{1}{R}a}$$

$$2m_b g - 2m_b a = (4M_1+M_2)a$$

$$a = \frac{2m_b g}{4M_1+M_2+2m_b}$$

$$\Rightarrow T = \frac{m_b g(4M_1+M_2)}{4M_1+M_2+2m_b}$$

1D Kinematics

$$y(t) = \frac{at^2}{2} = \frac{m_b g t^2}{4M_1+M_2+2m_b} = d$$

$$t = \pm \sqrt{\frac{m_b g d (4M_1+M_2+2m_b)}{m_b g}}$$

Ex 2: Energy Analysis

$$V = r\omega \Rightarrow \omega = \frac{V}{r}$$

$$E_i = 0$$

$$E_f = -m_b g h + \frac{I_1 \omega_1^2}{2} + \frac{I_2 \omega_2^2}{2} + \frac{m_b V^2}{2}$$

$$= -m_b g h + \frac{M_1 \cancel{R}}{4} \cdot \frac{V_1^2}{R} + \frac{M_2 \cancel{R}}{4} \cdot \frac{V_2^2}{R} + \frac{m_b V^2}{2}$$

$$\omega_1 = \omega_2 = \frac{V_1}{R} = \frac{V}{R/2} = \frac{2V}{R}$$

$$\Rightarrow V_1 = 2V$$

$$= -m_b g h + \frac{M_1}{4} \cdot 4V^2 + \frac{M_2 V^2}{4} + \frac{m_b V^2}{2}$$

$$\Delta E = 0$$

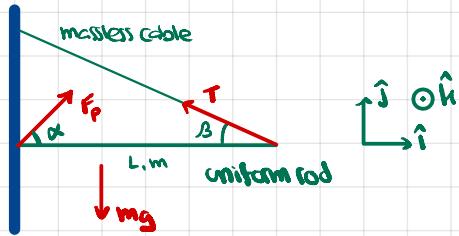
$$V^2 (M_1 + \frac{M_2}{4} + \frac{m_b}{2}) = m_b g h$$

$$V = \sqrt{\frac{4m_b g h}{4M_1+M_2+2m_b}}$$

$$h(t) = \frac{m_b g t^2}{4M_1+M_2+2m_b}$$

$$V(t) = \sqrt{\frac{4m_b^2 g^2 t^2}{(4M_1+M_2+2m_b)^2}} = \frac{2m_b g t}{4M_1+M_2+2m_b}$$

Problem 3 - Suspended Rod



* Torque due to gravity

$$\begin{aligned}\sum \vec{F}_i \times m_i \vec{g} &= \sum m_i \vec{r}_i \times \vec{g} = M_T \cdot \vec{r}_{cm} \times \vec{g} \\ &= \vec{r}_{cm} \times M_T \vec{g} = \vec{r}_{cm} \times \vec{F}_{\text{Total}} = \vec{I}_{cm,g}\end{aligned}$$

→ Torque on a rigid body due to gravity is the same as torque due to weight force of other body acting at center of mass.

For rotation about the CM, $\vec{I}_{cm} = 0 \Rightarrow \vec{I}_{cm,g} = 0$

static Equilibrium

- 1) $\sum \vec{F}_i = 0 \Rightarrow$ CM at rest or moving w/ constant velocity
- 2) $\sum \vec{\tau}_{s,i} = 0 \Rightarrow$ no rotational motion

$$\begin{aligned}1) \quad F_p \cos \alpha - T \cos \beta &= 0 \\ F_p \sin \alpha - mg + T \sin \beta &= 0\end{aligned}$$

$$2) \quad T \sin \beta \cdot \frac{L}{2} - F_p \sin \alpha \cdot \frac{L}{2} = 0 \quad (\text{lever law})$$

unknowns: α, F_p, T

$$\begin{aligned}F_p \cos \alpha &= T \cos \beta \\ F_p \sin \alpha &= T \sin \beta\end{aligned}$$

$$\Rightarrow \tan \alpha = \tan \beta \Rightarrow \alpha = \beta \Rightarrow T = F_p$$

$$2T \sin \beta = mg \Rightarrow T = F_p = \frac{mg}{2 \sin \beta}$$

$$\text{a) } T = \frac{4.98}{2 \cdot \frac{1}{2}} = 39.2 \text{ N}$$

$$\text{b) } \alpha = \beta = 30^\circ$$

$$\text{c) } F_p = T = 39.2 \text{ N}$$

Alternative Calculations

Torques about cable connection point.

$$F_p \sin \alpha \hat{j} \times (-L \hat{i}) + (-mg \hat{j}) (-\frac{L}{2} \hat{i}) = 0$$

$$F_p \sin \alpha L - \frac{mgl}{2} = 0 \quad (\text{lever law})$$

Equations for $\sum F_i = 0$ stay the same

$$F_p \cos \alpha - T \cos \beta = 0$$

$F_p \sin \alpha - mg + T \sin \beta = 0$ same as before.

$$2F_p \sin \alpha = mg$$

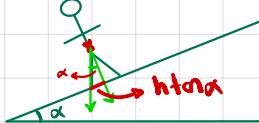
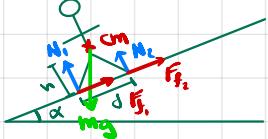
$$F_p \cos \alpha - T \cos \beta \text{ same as before}$$

$$mg = F_p \sin \alpha + T \sin \beta$$

$$\Rightarrow 2F_p \sin \alpha = F_p \sin \alpha + T \sin \beta$$

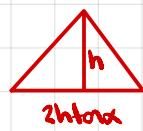
$$F_p \sin \alpha = T \sin \beta \text{ same eq. as before}$$

Problem 4 - Person Skating on a Hill



The person is in static equilibrium.

$$\sum \vec{F}_i = 0 \Rightarrow N_1 + N_2 - mg\cos\alpha = 0 \\ mg\sin\alpha - f_1 - f_2 = 0$$



$$\sum \vec{F}_{cm,ii} = 0 \Rightarrow \frac{d}{2}N_2 - \frac{d}{2}N_1 + hf_1 + hf_2 = 0$$

unknowns: N_1, N_2, f_1, f_2

eliminate one unknown at $f_1 = N_1\mu_s$
 $f_2 = N_2\mu_s$

$$N_1 + N_2 = mg\cos\alpha$$

$$\mu_s(N_1 + N_2) = mg\sin\alpha$$

$$2h\mu_s(N_1 + N_2) = d(N_1 - N_2)$$

$$\mu_s mg\cos\alpha - mg\sin\alpha \Rightarrow \tan\alpha = \mu_s$$

$$zh = \frac{d(N_1 - N_2)}{mg\sin\alpha} \Rightarrow N_1 - N_2 = \frac{zhmg\sin\alpha}{d}$$

$$N_1 = mg\cos\alpha - N_2 + \frac{zhmg\sin\alpha}{d}$$

$$N_1 = \frac{mg\cos\alpha}{2} + \frac{hmg\sin\alpha}{d}$$

$$N_2 = mg\cos\alpha - N_1$$

$$= mg\cos\alpha - \frac{mg\cos\alpha}{2} - \frac{hmg\sin\alpha}{d}$$

$$= \frac{mg\cos\alpha}{2} - \frac{hmg\sin\alpha}{d} = 0$$

$$\Rightarrow d = \frac{2h\mu_s/\sin\alpha}{mg\cos\alpha} = 2htan\alpha$$

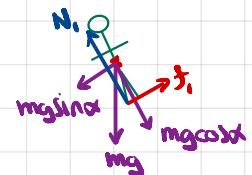
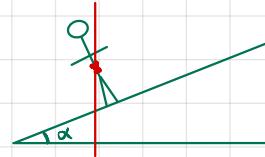
$$N_2 = 0 \Rightarrow N_1 = mg\cos\alpha$$

N_2 cannot be smaller than 0. For $d < 2htan\alpha$, the required N_2 to keep static equilibrium isn't possible.

$$\text{torque: } \frac{d}{2}N_1 = h\mu_s N_1$$

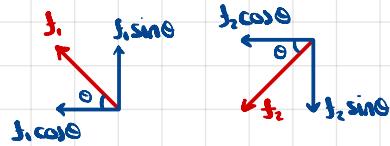
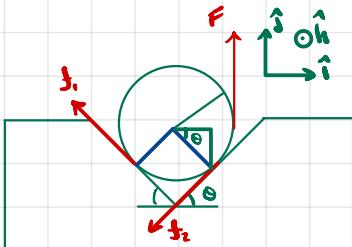
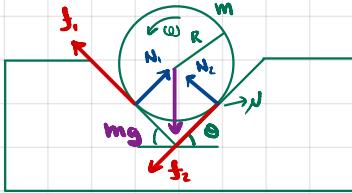
at d_{\min}

$$htan\alpha = h\mu_s \Rightarrow \tan\alpha = \mu_s$$



$$T_{cm} = -\frac{d}{2}N_1 + hf_1$$

Problem 5 - A Cylinder Rolling in a V-groove



Assume the cylinder is uniform, i.e CM is in the middle.

$$\vec{f}_1 = -f_1 \cos \theta \hat{i} + f_1 \sin \theta \hat{j}$$

$$\vec{f}_2 = -f_2 \cos \theta \hat{i} - f_2 \sin \theta \hat{j}$$

$$\vec{\tau} = (-R \sin \theta \hat{i} - R \cos \theta \hat{j}) \times (-f_1 \cos \theta \hat{i} + f_1 \sin \theta \hat{j}) \\ + (R \sin \theta \hat{i} - R \cos \theta \hat{j}) \times (-f_2 \cos \theta \hat{i} - f_2 \sin \theta \hat{j})$$

$$= \hat{h} (-R f_1 \sin^2 \theta - R f_1 \cos^2 \theta \\ - R f_2 \sin^2 \theta - R f_2 \cos^2 \theta)$$

$$= \hat{h} (-R f_1 - R f_2) = -R(f_1 + f_2) \hat{h}$$

Add an external torque to obtain zero net torque

$$\vec{\tau} = -R(f_1 + f_2) \hat{h} + (R \hat{i}) \times (F \hat{j}) - I \cdot \vec{\alpha} = 0$$

$$\Rightarrow \hat{h} (-R(f_1 + f_2) + FR) = 0$$

$$\Rightarrow F = f_1 + f_2$$

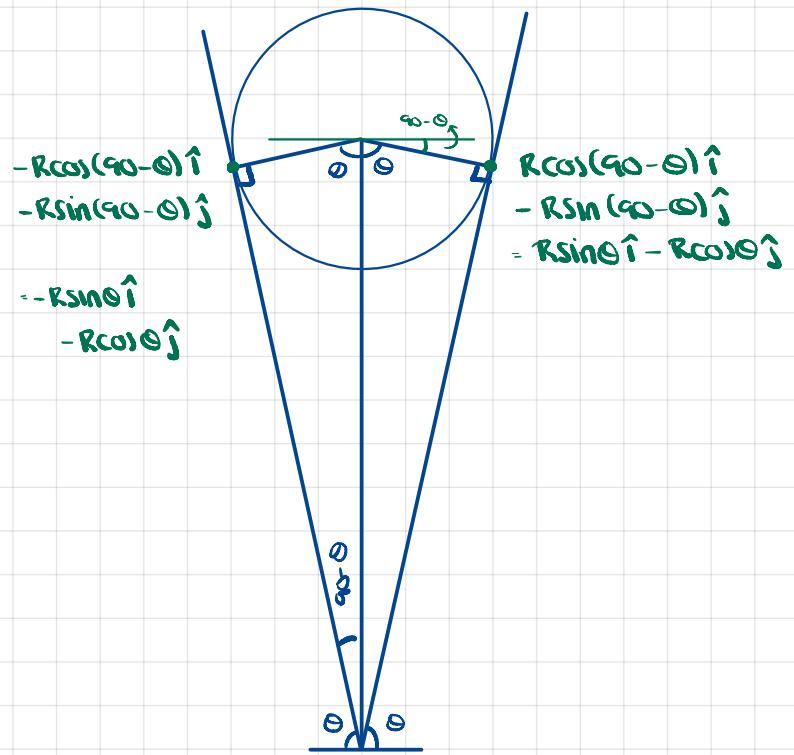
$$N_1 = N_2 = mg \cos \theta$$

$$\Rightarrow f_1 = f_2 = \mu mg \cos \theta$$

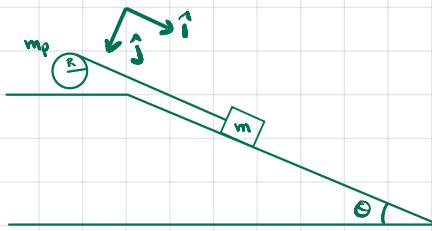
$$\Rightarrow F = 2\mu mg \cos \theta$$

$$\text{external torque} = FR = 2R\mu mg \cos \theta$$

$$\theta = 45^\circ \Rightarrow FR = \sqrt{2} R \mu mg$$



Problem 6 - A Massive Pulley and a Block on an Incline



Gravity pulls the block down the incline.



The rope applies a torque on the pulley, generating rotation.

$$\vec{\tau}_{cm} = (-R\hat{j}) \times (T\hat{i}) = \frac{m_p R^2}{2} \alpha_z$$

$$\Rightarrow RT \cdot \frac{m_p R^2}{2} \alpha_z$$

Angular acceleration of the pulley's rotation is related to the block's acceleration.

$$\alpha_z = \frac{a}{R}$$

3 equations, unknowns: T, a, α_z

$$mg \sin \theta - T = m \cdot a \Rightarrow T = m(g \sin \theta - a)$$

$$RT \cdot \frac{m_p R^2}{2} \alpha_z$$

$$\alpha_z = \frac{a}{R}$$

$$\Rightarrow 2T \cdot m(g \sin \theta - a) = m_p \cancel{R} \cdot \cancel{\frac{a}{R}}$$

$$a(m_p + 2m) = 2mg \sin \theta$$

$$a = \frac{2mg \sin \theta}{m_p + 2m}$$

10 Kinematics

$$x(t) = \frac{at^2}{2} = d \Rightarrow t = \sqrt{\frac{2d}{a}}$$

$$t_d = \sqrt{\frac{d(m_p + 2m)}{mg \sin \theta}}$$