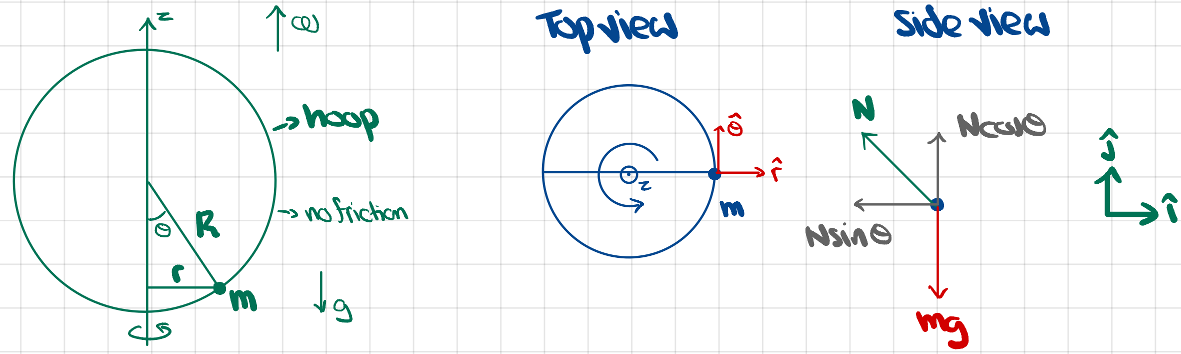
Problem Set 3

Problem 1 - Bead on Rotating Hoop



a) $\begin{cases} -N\sin\theta = m[-r\omega^2] \Rightarrow N\sin\theta = mR\sin\theta\,\omega^2 \Rightarrow \sin\theta(N - mR\omega^2) = 0 \\ N\cos\theta - mg = 0 \Rightarrow N = \dfrac{mg}{\cos\theta} \end{cases}$

$\Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

or $N = mR\omega^2 \Rightarrow \omega = \sqrt{\dfrac{N}{mR}} = \sqrt{\dfrac{g}{R\cos\theta}}$

$r = R\sin\theta$

we could also have divided one eq. by the other (assuming $\sin\theta \neq 0$)

$$\tan\theta = \frac{r\omega^2}{g} \Rightarrow \omega(r, g, \theta) = \sqrt{\frac{g\tan\theta}{r}} \qquad \theta \neq 0$$

interpretation: when we assume circular motion of the bead at fixed angle θ, the application of the 2nd law leads us to the relationships that need to hold for that motion to exist. First, the vertical component of N needs to offset mg. This determines what N is. As θ increases N needs to increase to keep vertical resultant force zero. But $\uparrow\theta$ and $\uparrow N$ increase $\vec{N}$'s horizontal component $N\sin\theta$. This component generates radial acceleration. At higher θ, radial acceleration has to be large because $N\sin\theta$ is large. Radial acceleration is a function of v, or in other terms, of $\theta'(t)$, and also of r: $a_r = r\theta'(t)^2$.

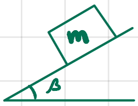
Note that we can't just increase θ without increasing r: we have the constraint $r = R\sin\theta$.
$\uparrow\theta \Rightarrow \uparrow r$.
$\uparrow\theta \Rightarrow \uparrow N$ (to keep $N\cos\theta = mg$), $\uparrow\sin\theta \Rightarrow \uparrow N\sin\theta$, which needs to equal ma_r
$a_r = r\omega^2$, we know $\uparrow r$ because $\uparrow\theta$, but we can't fully offset $\uparrow N\sin\theta$ because $\uparrow r$ comes fully from $R \cdot \uparrow\sin\theta$. Therefore $\uparrow\omega$ must also happen

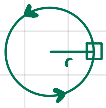
we can thus write $\omega(g, R, \theta) = [g/R\cos\theta]^{1/2}$ to know for each θ (and thus r) the required ω (and thus a_r) to have uniform circular motion.

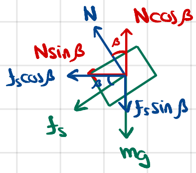
Note that at a specific θ, if $\omega < [g/R\cos\theta]^{1/2}$, N will not be large enough to power the uniform circular motion at this θ. The bead slides down to the θ that restores $\omega = [g/R\cos\theta]^{1/2}$.

b) At any θ, $\omega \leq \sqrt{g/r}$ means N can't sustain uniform circular motion at that θ, and the bead slides down to restore $\omega = [g/R\cos\theta]^{1/2}$ but fails to do so. θ reaches 0 but $\theta = 0$ is also a solution, because now there are only vertical forces $N = mg$, $N\sin\theta = 0$, $a_r = 0$.

Problem 2 - Banked Turn







interpretation: if there is no friction force we have

$$\vec{F} = m\vec{a}$$

$$\hat{h} \quad N\cos\beta - mg = 0 \Rightarrow N = \frac{mg}{\cos\beta}$$

$$\hat{r} \quad -N\sin\beta = ma_r = -mr\theta'(t)^2$$

$$\Rightarrow \frac{\cancel{m}g\sin\beta}{\cos\beta} = \cancel{m}r\theta'(t)^2 \Rightarrow \theta'(t) = \sqrt{\frac{g}{r}\tan\beta} \Rightarrow v = \sqrt{rg\tan\beta}$$

I.e. at a particular β and r, the car traveling at constant $\omega = (g\tan\beta/r)^{1/2}$ doesn't experience friction in the direction of the incline. Of course there is friction in the direction of movement between tires and surface. To sustain constant speed, force must be applied by the tires on the surface to compensate for static friction. But we are not interested in that direction of movement here.

If we now increase $\theta'(t)$, a_r necessary for circular motion at this radius increases: $a_r = r\theta'(t)^2$. Radial force must increase. In the absence of friction the car would slide in a non-circular trajectory and eventually get back to a circular trajectory at a larger r and lower ω than the full increased ω.
But with friction coefficient >0, friction comes into play contributing to radial acceleration.

When that friction force is at its maximum magnitude we have

$$N\sin\beta + \mu_s N\cos\beta = mr\omega^2$$

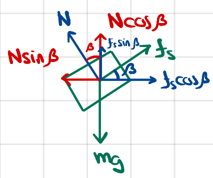
$$N\cos\beta - \mu_s N\sin\beta - mg = 0 \Rightarrow N = \frac{mg}{\cos\beta - \mu_s\sin\beta}$$

$$\Rightarrow \omega = [N(\sin\beta + \mu_s\cos\beta)/mr]^{1/2} = \left[\frac{g(\sin\beta + \mu_s\cos\beta)}{r(\cos\beta - \mu_s\sin\beta)}\right]^{1/2} \Rightarrow v = \left[\frac{gr(\sin\beta + \mu_s\cos\beta)}{(\cos\beta - \mu_s\sin\beta)}\right]^{1/2}$$

Note that $\dfrac{\sin\beta + \mu_s\cos\beta}{\cos\beta - \mu_s\sin\beta} > \dfrac{\sin\beta}{\cos\beta} = \tan\beta$

We're not done because it is also possible for us to decrease $\theta'(t)$ from the initial scenario of $\theta'(t) = (g\tan\theta/r)^{1/2}$. a_r required to be in circular motion at this radius decreases but $N\sin\theta$, the force generating a_r, is still the same. In the absence of friction, the car would slide on a new trajectory. With friction, the latter opposes the sliding at a max magnitude of $\mu_s N$.

$$\Rightarrow N\sin\beta - \mu_s N\cos\beta = mr\omega^2$$

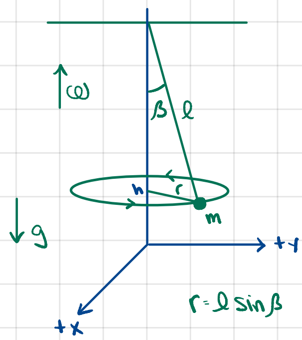


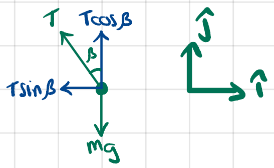
$$N\cos\beta + \mu_s N\sin\beta - mg = 0 \Rightarrow N = \frac{mg}{\cos\beta + \mu_s\sin\beta}$$

$$\Rightarrow \omega = \left[\frac{g(\sin\beta - \mu_s\cos\beta)}{r(\cos\beta + \mu_s\sin\beta)}\right]^{1/2} \Rightarrow v = \left[\frac{gr(\sin\beta - \mu_s\cos\beta)}{(\cos\beta + \mu_s\sin\beta)}\right]^{1/2}$$

$\Rightarrow$ For $\left[\dfrac{gr(\sin\beta - \mu_s\cos\beta)}{(\cos\beta + \mu_s\sin\beta)}\right]^{1/2} \le v \le \left[\dfrac{gr(\sin\beta + \mu_s\cos\beta)}{(\cos\beta - \mu_s\sin\beta)}\right]^{1/2}$ the car stays at the same radius r.

Problem 3





a) $T\cos\beta - mg = 0 \Rightarrow T = mg/\cos\beta$

$T\sin\beta = mr\omega^2 = ml\sin\beta\,\omega^2$

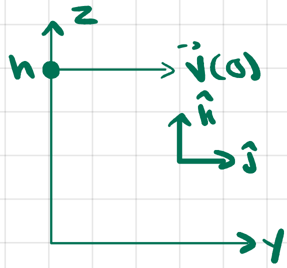
$$\Rightarrow \frac{\cancel{m}g\sin\beta}{\cos\beta} = \cancel{m}l\sin\beta\,\omega^2$$

$$\Rightarrow \sin\beta\left[l\omega^2 - g/\cos\beta\right] = 0 \quad \begin{cases} \sin\beta = 0 \\ \omega = \sqrt{\dfrac{g}{l\cos\beta}} \end{cases}$$

b) $\theta(t) = \omega t$

$$\vec{v}(t) = r\theta'(t)\hat{\theta} = \frac{l\sin\beta \cdot g^{1/2}}{l^{1/2}\cos\beta^{1/2}}\left(-\sin\theta(t)\hat{i} + \cos\theta(t)\hat{j}\right)$$

$$\vec{v}(0) = \sin\beta\sqrt{\frac{lg}{\cos\beta}}\,\hat{j} \qquad v_0 = \sin\beta\sqrt{\frac{lg}{\cos\beta}}$$



From point of release, motion is now on a plane zy.

$$\vec{a}(t) = \langle 0, -g\rangle \qquad \vec{v}(t) = \langle v_0, -gt\rangle \qquad \vec{r}(t) = \langle v_0 t, h - gt^2/2\rangle$$

Ball hits ground when $z = 0 \Rightarrow h - gt^2/2 = 0 \Rightarrow t = \sqrt{\dfrac{2h}{g}}$

$$d = y\left(\sqrt{\frac{2h}{g}}\right) = v_0(2h/g)^{1/2} = \sin\beta\left[\frac{l\cancel{g}\cdot 2h}{\cancel{g}\cos\beta}\right]^{1/2} = \sin\beta\sqrt{\frac{2lh}{\cos\beta}}$$

Note Keeping β constant, if you increase l the new circular motion has larger radius. Tension stays the same, so to have the same a_r as before, ω must go down. $a_r = r\omega^2$ means if you quadruple r, you only reduce ω by half.

Speed, however, is distance per time not angle per time. In the aforementioned situation, $\uparrow l$, $\uparrow r$, $\downarrow \omega$. r increases linearly with l but ω decreases at a rate of $1/\sqrt{l}$. So, $\uparrow v$ when $\uparrow l$, though it increases at a rate of $\sqrt{l}$.

All of this translates to expected relationships between l and h and distance d: longer l means higher v_0 and so more distance travelled, for a given h.

Lets consider the relationship between d and β.

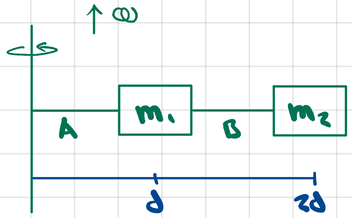
If you increase β with l fixed, ω increases. Because l, β, and r are dependent on each other, when we $\uparrow\beta$ and keep l fixed, r must increase. The forces are now different. $T\sin\beta$ increases, as does T itself. The increased radial force required generates a higher radial acceleration $a_r = r\omega^2$. Both r and ω increase.

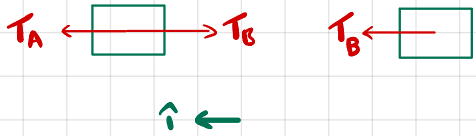
An interesting question is: given a fixed height H off the ground where the rope is attached, what is the farthest d and how is it obtained?

$$l\cos\beta + h = H \Rightarrow h = H - l\cos\beta, \quad d = \sin\beta\left[\frac{2l(H - l\cos\beta)}{\cos\beta}\right]^{1/2}. \quad d \to \infty \text{ as } \beta \to \frac{\pi}{2},$$

no matter what l is. However, since $\sin\beta/\sqrt{\cos\beta}$ and $H - l\cos\beta$ are monotonically increasing in β, for any given β, $l = H$ maximizes d.

Problem 4 – Two Boxes Around a Shaft





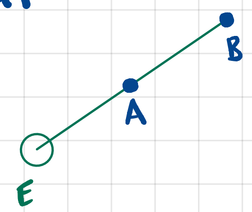
→ Assumptions: no gravity, massless strings

$$\begin{cases} T_A - T_B = m_1 d \omega^2 \\ T_B = m_2 2 d \omega^2 \end{cases}$$

$$\Rightarrow T_A = d\omega^2 (m_1 + 2m_2)$$

Problem 5 - Satellite

a)



when radius is increased, gravitational force drops proportionally to the square of r. radial accel. required to maintain orbit at the new r and same ω increases linearly in r. ω must decrease.

speed however is $v(r,\omega) = r\omega(r)$

$$\frac{Gm_1\cancel{m_2}}{r^2} = \cancel{m_2} r\omega^2$$

$$\omega(r) = \sqrt{\frac{Gm_1}{r^3}}$$

$$\frac{d\omega(r)}{dr} = \frac{d}{dr}\left((Gm_1)^{1/2} r^{-3/2}\right) = -\frac{3}{2} r^{-\frac{5}{2}} (Gm_1)^{1/2} = -\frac{3}{2}\,\frac{(Gm_1)^{1/2} r^{-3/2}}{r} = -\frac{3\omega(r)}{2r} < 0$$

$$\frac{dv(r)}{dr} = \omega(r) + r\frac{d\omega(r)}{dr} = \omega(r) - \cancel{r}\,\frac{3\omega(r)}{2\cancel{r}} = -\frac{\omega(r)}{2} < 0$$

$$\Rightarrow v(r_B) < v(r_A)$$

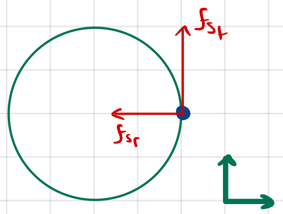
Alternatively

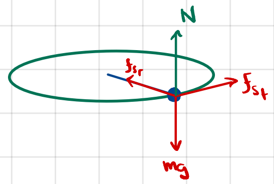
$$\frac{Gm_1\cancel{m_2}}{r^{\cancel{2}}} = \frac{\cancel{m_2}v^2}{\cancel{r}} \Rightarrow v(r) = \sqrt{\frac{Gm_1}{r}}, \quad \frac{dv(r)}{dr} = -\frac{1}{2}\sqrt{\frac{Gm_1}{r^3}} < 0$$

b) $T(\omega) = \dfrac{2\pi}{\omega} \Rightarrow T'(\omega) = \dfrac{-2\pi}{\omega^2} < 0$

$$T(\omega_B) < T(\omega_A)$$

Problem 6 - Coin on Rotating Dish





$\alpha(t) = \alpha$, constant angular accel.

$\omega(t) = \alpha t$

$\vec{r}(t) = r\hat{r}(t)$

$\vec{v}(t) = r\theta'(t)\hat{\theta}(t)$

$\vec{a}(t) = r\theta''(t)\hat{\theta}(t) - r\theta'(t)^2\hat{r}(t)$

$= r\alpha\hat{\theta}(t) - r\omega(t)^2\hat{r}(t)$

2nd law: $\hat{r}$: $f_{s_r} = -mr\omega(t)^2 = -mr\alpha^2 t^2$

$\hat{\theta}$: $f_{s_t} = mr\alpha$

a) $\vec{f}_{s_t} = \langle 0, mr\alpha \rangle$

$\vec{f}_{s_r} = \langle -mr\omega(t)^2, 0 \rangle$

$\vec{F}_s = \vec{f}_{s_t} + \vec{f}_{s_r} = -mr\omega(t)^2\hat{r} + mr\alpha\hat{\theta}$

$|\vec{F}_s| = \left[m^2r^2\omega(t)^4 + m^2r^2\alpha^2 \right]^{1/2} = \left[m^2r^2(\omega(t)^4 + \alpha^2) \right]^{1/2} = mr\sqrt{\omega(t)^4 + \alpha^2}$

For circular motion of the coin of mass m on the dish with angular speed $\omega(t)$ and angular acceleration α, at a radius of r, a force $\vec{F} = -mr\omega(t)^2\hat{r} + mr\alpha\hat{\theta}$ must act on the coin.

In this problem, static friction between coin and dish is the force causing the described motion. Since static frictional force has a maximum value, there is a limit to the combinations of parameters it can sustain.

Given m, r, and α there is a max ω that can be sustained before the coin slips, ie accelerates differently from the acceleration expected in a circular motion.

$$\mu_s N = \mu_s \cancel{m}g = \cancel{m}r\sqrt{\omega(t)^4 + \alpha^2} \Rightarrow \omega_{max} = \sqrt[4]{\frac{\mu_s^2 g^2}{r^2} - \alpha^2}$$

Alternatively, we could look at the time it takes to slide

$$\omega_{max} = \alpha t_{max} \Rightarrow t_{max} = \sqrt{\frac{\mu_s^2 g^2 - \alpha^2 r^2}{r^2\alpha^4}}$$