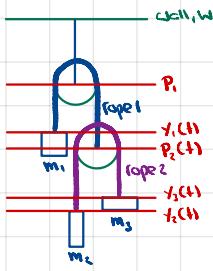


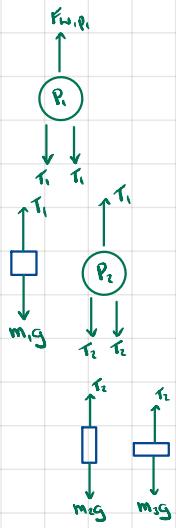
Example 8.9



$$l_1 = y_1(t) - P_1 + \pi R + P_2(t) - P_1 = \text{constant} \Rightarrow 0 = a_1 + a_{P_1} \Rightarrow a_1 = -a_{P_1}$$

$$l_2 = y_2(t) - P_2 + \pi R + y_3(t) - P_2(t) = \text{constant} \Rightarrow a_2 - a_{P_2} + a_3 - a_{P_2} = 0 \Rightarrow a_2 + a_3 = 2a_{P_2} = -2a_1$$

$$\Rightarrow a_2 + a_3 = -2a_1$$



$$m_1 g - T_1 = m_1 a_1$$

$$2T_2 - T_1 = 0 \Rightarrow T_1 = 2T_2$$

$$m_2 g - T_2 = m_2 a_2$$

$$m_3 g - T_3 = m_3 a_3$$

$$a_2 + a_3 = -2a_1$$

$$g(m_2 + m_3) - T_1 = (m_2 + m_3)a_{P_2}$$

$$g(m_2 + m_3) - T_1 = (m_2 + m_3)a_{P_2}$$

$$a_2 + a_3 = -2a_1$$

$$1. T_1 = m_1(g - a_1)$$

$$2. T_1 = 2T_2$$

$$3. T_2 = m_2(g - a_2)$$

$$4. T_2 = m_3(g - a_3)$$

unknowns: $T_1, T_2, a_1, a_2, a_3, a_{P_2}$

$$5. T_1 = (m_2 + m_3)(g - a_{P_2})$$

$$6. a_2 + a_3 = -2a_1$$

$$7. a_1 = -a_{P_2}$$

Solving the system of equations

\rightarrow 1, 5, and 7. are 3 eq. in 3 unknowns T_1, a_1, a_{P_2} .

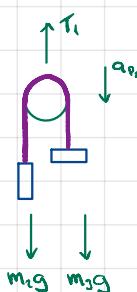
\rightarrow Then, 2. gives T_2

3. gives a_2

4. gives a_3

We did not need 6.

\rightarrow If we only use 2nd law eq. for individual objects (i.e. we drop eq. 5), we have 6 eq and 6 unknowns, so we could still solve that but it'd be more complicated.



using the first path of solving the eq.

$$(m_2 + m_3)(g + a_1) = m_1(g - a_1) \Rightarrow g(m_2 + m_3) + a_1(m_2 + m_3) = m_1g - m_1a_1 \Rightarrow a_1(m_2 + m_3 + m_1) = g(m_1 - m_2 - m_3)$$

$$\Rightarrow a_1 = \frac{g(m_1 - m_2 - m_3)}{m_1 + m_2 + m_3}, a_{P_2} = -a_1, T_1 = m_1g - m_1a_1 = \frac{m_1g(m_1 + m_2 + m_3) - m_1g(m_1 - m_2 - m_3)}{m_1 + m_2 + m_3}$$

$$= \frac{m_1^2 g + m_1 m_2 g + m_1 m_3 g - m_1^2 g + m_1 m_2 g + m_1 m_3 g}{m_1 + m_2 + m_3} = \frac{2g(m_1 m_2 + m_1 m_3)}{m_1 + m_2 + m_3}$$

Solution using only individual object FBD diagrams

$$m_1 g - T_1 = m_1 a_1$$

$$a_1 = g - \frac{T_1}{m_1}$$

$$2T_2 - T_1 = 0 \Rightarrow T_1 = 2T_2 \Rightarrow T_1 = 2T_2$$

$$a_2 = g - \frac{T_2}{m_2}$$

$$m_3 g - T_3 = m_3 a_3$$

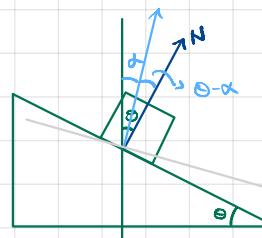
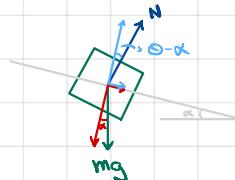
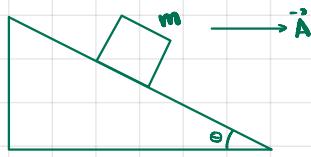
$$a_3 = g - \frac{T_3}{m_3}$$

$$\text{sub } a_1, a_2, a_3 \text{ into constraint eq. } g - \frac{T_2}{m_2} + g - \frac{T_2}{m_3} + 2g - \frac{4T_2}{m_1} = 0$$

$$\Rightarrow \frac{-T_2 m_1 m_3 - T_2 m_1 m_2 - 4T_2 m_2 m_3}{m_1 m_2 m_3} = -4g \Rightarrow T_2 = \frac{4g m_1 m_2 m_3}{m_1 m_3 + m_1 m_2 + 4m_2 m_3}$$

Example 8.10 Accelerating Wedge

Note: this was my attempt at a solution. It is correct I believe, but the textbook solution on the next page is much simpler, more straightforward, and more general.



→ The textbook solution obtains equations for x and y components of acceleration by applying the 2nd law in those directions.

→ Here we assume that because accelerations are constant along both x and y directions, the resultant acceleration will be along = line passing through the initial point of the block and forming an angle $0 \leq \alpha \leq \theta$ with the horizontal direction.

For this to occur, namely acceleration along pitch = line, we need:

$$N\cos(\theta-\alpha) = mgs\cos\alpha$$

$$N\sin(\theta-\alpha) + mgs\sin\alpha = ma_x$$

$$\Rightarrow N = \frac{mgs\cos\alpha}{\cos(\theta-\alpha)}$$

Here we are determining: given slope of angle θ and that acceleration of the block occurs at an angle α , what is the required normal force on

Given such an N , we have

$$a_x(\alpha, \theta, g, m) = \frac{N}{m} \sin(\theta-\alpha) + g\sin\alpha = g\cos\alpha \frac{\sin(\theta-\alpha)}{\cos(\theta-\alpha)} + g\sin\alpha = \frac{g\cos\alpha\sin(\theta-\alpha) + g\sin\alpha\cos(\theta-\alpha)}{\cos(\theta-\alpha)}$$

Given m and g , a slope θ , a normal force that together with gravitational force is a contributor to acceleration at an angle α , the magnitude of the acceleration is a_x .

As we derive on the next page we have a constraint eq.:

$$a_{b,y} = \tan\theta(A_{u,x} - a_{b,x}) \quad (\text{constraint eq.}) \Rightarrow a_{b,y} = \tan\theta(A_{u,x} - a_x\cos\alpha) \Rightarrow a_x\sin\alpha = \tan\theta(A_{u,x} - a_x\cos\alpha) \Rightarrow A_{u,x} = \frac{a_x[\cos\alpha + \tan\theta]}{\tan\theta}$$

recap:

$$a_x(\alpha, \theta, g) = \frac{g\cos\alpha\sin(\theta-\alpha) + g\sin\alpha\cos(\theta-\alpha)}{\cos(\theta-\alpha)}$$

$$A_{u,x}(\alpha, \theta) = \frac{a_x[\cos\alpha + \tan\theta]}{\tan\theta}$$

examples:

$$\alpha = 0, \theta = \frac{\pi}{4}, \text{ ie no vertical acceleration.}$$

$$a_x(0, \pi/4, g, m) = g$$

$$A_{u,x}(0, \pi/4) = a_x = g$$

Alternative solution

$$a_{b,x}(g, g, \pi/4) = g$$

$$a_{b,y}(g, g, \pi/4) = 0$$

* Alternative solution
(next page)

$$a_{b,x}(g, A_{u,x}, \theta) = \frac{g + A_{u,x}\tan\theta}{\cot\theta + \tan\theta}$$

$$a_{b,y}(g, A_{u,x}, \theta) = \frac{A_{u,x} - g\tan\theta}{\cot\theta + \tan\theta}$$

now let's go the other way around

Method 1

$$A_{u,x} = 5, \theta = \pi/3 \Rightarrow a_{b,x}(9.8, 5, \pi/3) = 7.9938$$

$$a_{b,y}(9.8, 5, \pi/3) = -5.1855$$

check the resultant acceleration's angle with horizontal.

$$\vec{a} = \langle 7.9938, -5.1855 \rangle \quad \cos\alpha = \frac{\vec{a} \cdot \hat{i}}{|\vec{a}|} \Rightarrow \alpha = 0.5754 \text{ rad}$$

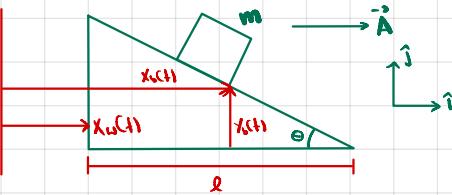
$$|\vec{a}| = 9.5284$$

$$a_x = \frac{a_{b,x}}{\cos\alpha} = \frac{a_{b,x}}{\cos\alpha} - \frac{a_{b,y}}{\sin\alpha} \Rightarrow \tan\alpha = \frac{a_{b,y}}{a_{b,x}} \Rightarrow \alpha = \tan^{-1} \frac{a_{b,y}}{a_{b,x}}$$

$$\Rightarrow \alpha = -0.5754 \text{ rad}$$

$$a_x = -9.5284$$

Example 8.10 - Textbook Solution



$$\tan \theta = \frac{y_b(t)}{l - (x_u(t) - x_0(t))} \Rightarrow y_b(t) = [l - (x_u(t) - x_0(t))] \tan \theta$$

Differentiate twice: $a_{b,x} = \tan \theta (A_{u,x} - a_{b,z})$ (constraint eq.)

$$N \sin \theta = m a_{b,x} \Rightarrow N = \frac{m a_{b,x}}{\sin \theta}$$

$$N \cos \theta - mg = m a_{b,y}$$

$$\frac{\sqrt{a_{b,x}^2 + a_{b,y}^2}}{\sin \theta} \cos \theta - mg = \sqrt{a_{b,x}^2 + a_{b,y}^2} \quad \text{sub in the constraint}$$

$$\frac{a_{b,x}}{\tan \theta} - g = \tan \theta (A_{u,x} - a_{b,z})$$

$$a_{b,x} \left[\frac{1}{\tan \theta} + \tan \theta \right] = g + A_{u,x} \tan \theta$$

$$\Rightarrow a_{b,x} = \frac{g + A_{u,x} \tan \theta}{\cot \theta + \tan \theta}$$

$$\Rightarrow a_{b,y} = \tan \theta \frac{A_{u,x} \cot \theta + A_{u,z} \tan \theta - g - A_{u,y} \tan \theta}{\cot \theta + \tan \theta}$$

$$a_{b,y} = \frac{A_{u,x} - g \tan \theta}{\cot \theta + \tan \theta}$$

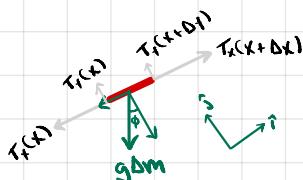
With $\theta = 45^\circ$ we have

$$a_{b,x} = \frac{g + A_{u,x}}{2} \quad a_{b,y} = \frac{A_{u,x} - g}{2}$$

Edge Cases

$$a_{b,y} = 0 \Rightarrow A_{u,x} = g \tan \theta \Rightarrow a_{b,x} = \frac{g(1 + \tan^2 \theta)}{\cot \theta + \tan \theta} = g \tan \theta$$

Example 8.3 - Tension in a Massive Rope

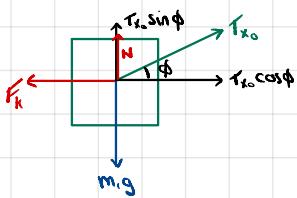


$$T_x(x+dx) - T_x(x) - g\Delta m \sin\phi = 0 \Rightarrow T_x(x+dx) - T_x(x) = \frac{g\Delta m \sin\phi}{d}$$

$$\Delta m = \frac{\Delta x}{d} m_2$$

$$\Rightarrow \frac{T_x(x+dx) - T_x(x)}{\Delta x} = \frac{g m_2 \sin\phi}{d} \Rightarrow \frac{dT_x(x)}{dx} = \frac{g m_2 \sin\phi}{d} \Rightarrow \int_{x=0}^{x(x)} dT_x \cdot \int \frac{g m_2 \sin\phi}{d} dx \Rightarrow T_x(x) - T_x(0) = \frac{g m_2 \sin\phi}{d} x$$

What is $T_x(0)$, i.e. tension in x direction at contact point with m_1?



$$\begin{cases} T_x0 \sin\phi + N - m_1 g = 0 \Rightarrow N = m_1 g - T_x0 \sin\phi \\ T_x0 \cos\phi - \mu_h N = m_1 a \end{cases}$$

unknowns: T_x0, N parameters: ϕ, μ_h, m_1, a

$$\Rightarrow T_x0 \cos\phi - \mu_h m_1 g + \mu_h T_x0 \sin\phi = m_1 a$$

$$T_x0 (\cos\phi + \mu_h \sin\phi) = m_1 (a + \mu_h g) \Rightarrow T_x0 = \frac{m_1 (a + \mu_h g)}{\cos\phi + \mu_h \sin\phi}$$

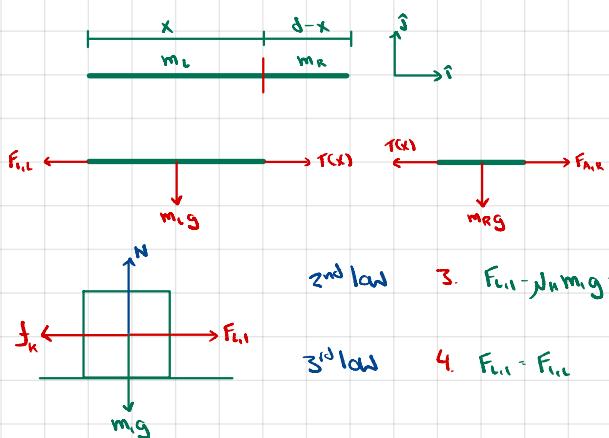
$$\Rightarrow T_x(x) = \frac{m_1 (a + \mu_h g)}{\cos\phi + \mu_h \sin\phi} + \frac{g m_2 \sin\phi}{d} x$$

There is, however, a component of $g\Delta m$ in the $-j$ direction. Because there is no acceleration in the $-j$ direction, there must be a component of tension force in that direction.

$$T_y(x+dx) + T_y(x) - g\Delta m \cos\phi = 0 \Rightarrow T_y(x) - T_y(0) = \frac{g m_2 \cos\phi}{d} x, T_{y0} = 0 \Rightarrow T_y(x) = \frac{g m_2 \cos\phi}{d} x$$

Textbook Solution

Assume ϕ very small: pulling and tension forces essentially act in horizontal direction.



$$4, 3, \text{ into } 2. \Rightarrow T(x) - \mu_h m_1 g - m_1 a_L = \frac{x}{d} m_2 a_L$$

But $a = a_L = a_R$ because rope and block move together.

$$\Rightarrow T(x) = \mu_h m_1 g + a(m_1 + \frac{x}{d} m_2)$$

Is the second solution a special case of the first?

First solution

$$\phi = 0 \quad T_x(x) = m_1 (a + \mu_h g) \Rightarrow T_{\text{horiz}}(x) = T_x(x) \cos\phi + T_y(x) \sin\phi = T_x(x)$$

$$T_y(x) = \frac{g m_2}{d} x$$

$$\phi = \pi/4 \quad T_x(x) = \frac{m_1 (a + \mu_h g)}{\sqrt{2} (1 + \mu_h)}$$

$$T_y(x) = \frac{g m_2 \sqrt{2} x}{2d} = \frac{m_1 (a + \mu_h g)}{1 + \mu_h} + \frac{g m_2 x}{2d}$$

Second solution

$$T_{\text{horiz}}(x) = \mu_h m_1 g + a(m_1 + \frac{x}{d} m_2)$$

$$T_{\text{vert}} = 0$$

not applicable



$$\tan\phi = \frac{\Delta y}{\Delta x}$$

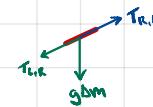
$$T_x(x + \Delta x, y + \Delta y) - T_x(x, y) = 0$$

$$T_y(x + \Delta x, y + \Delta y) - T_y(x, y) - gdm = 0$$

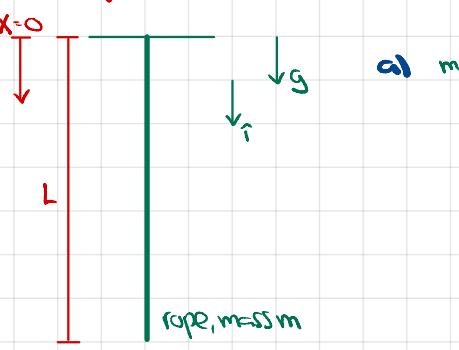
$$\Delta m = \frac{\sqrt{\Delta x^2 + \Delta y^2}}{\delta} m_2$$

$$\Rightarrow T_y(x + \Delta x, y + \Delta y) - T_y(x, y) = \frac{\sqrt{\Delta x^2 + \Delta y^2}}{\delta} m_2 g$$

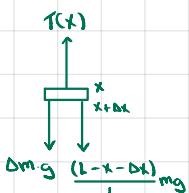
$$\lim_{\Delta y \rightarrow 0} \frac{T_y(x + \Delta x, y + \Delta y) - T_y(x, y)}{\sqrt{\Delta x^2 + \Delta y^2}} = \frac{m_2 g}{\delta} - \nabla T_y(x, y) \cdot \vec{v} = \frac{\partial T_y}{\partial x} \Delta x + \frac{\partial T_y}{\partial y} \Delta y, \quad \frac{\partial y}{\partial x} = \tan\phi$$



Example 8.4 Tension in a Suspended Rope



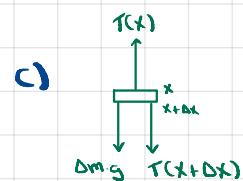
b) Let's consider a piece of rope located at position x , with mass Δm .



$$\Delta m \cdot g + \frac{L-x-\Delta x}{L} mg - T(x) = 0$$

$$\Delta m = \frac{\Delta x}{L} m \Rightarrow T(x) = \frac{\Delta x \cdot mg}{L} + \frac{L-x-\Delta x}{L} mg = \frac{mg}{L} (L-x)$$

$$T(x) = \frac{L-x}{L} mg$$



$$\frac{\Delta x}{L} mg + T(x+\Delta x) - T(x) = 0 \Rightarrow \lim_{\Delta x \rightarrow 0} \frac{T(x+\Delta x) - T(x)}{\Delta x} = -\frac{mg}{L} \Rightarrow \frac{dT(x)}{dx} = -\frac{mg}{L}$$

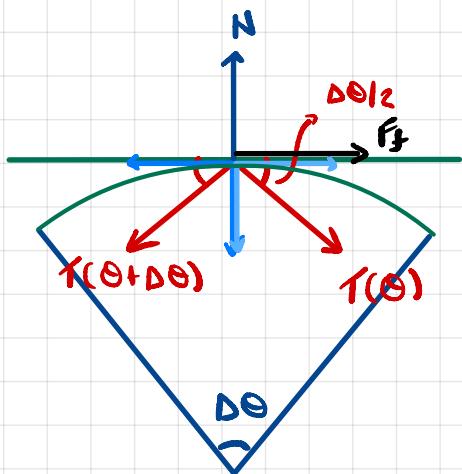
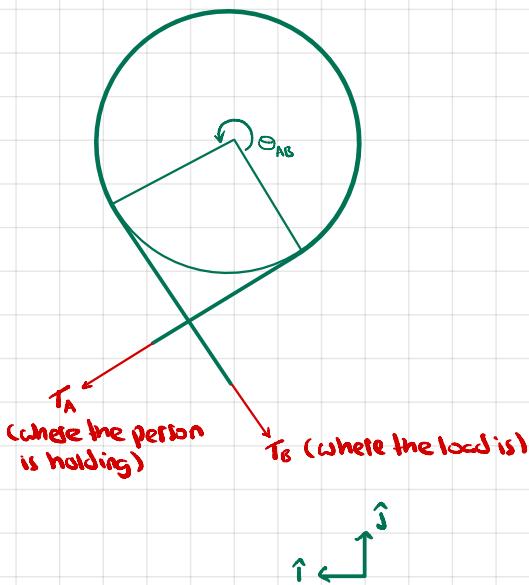
$$\Rightarrow \int_{T(0)}^{T(x)} dT = \int_0^x -\frac{mg}{L} dx \Rightarrow T(x) - T(0) = -\frac{mgx}{L}$$

$$\text{since } T(0) = mg, T(x) = mg(1 - x/L) = \frac{mg}{L} (L-x)$$

Alternatively, just differentiate $T(x) = \frac{mg}{L} (L-x)$ bond previously.

$$\frac{dT(x)}{dx} = -\frac{mg}{L}$$

Example 8.11 - Capstan



Apply 2nd law

$$N - T(\theta) \sin(\Delta\theta/2) - T(\theta + \Delta\theta) \sin(\Delta\theta/2) = 0$$

$$\Rightarrow N = \sin(\Delta\theta/2)(T(\theta) + T(\theta + \Delta\theta))$$

$$T(\theta + \Delta\theta) \cos(\Delta\theta/2) - T(\theta) \cos(\Delta\theta/2) - \mu_s N = 0$$

$$\cos(\Delta\theta/2)(T(\theta + \Delta\theta) - T(\theta)) = \mu_s \sin(\Delta\theta/2)(T(\theta) + T(\theta + \Delta\theta))$$

For small $\Delta\theta$, $\cos(\Delta\theta/2) \approx 1$, $\sin(\Delta\theta/2) \approx \Delta\theta/2$.

Using these approximations,

$$1 (T(\theta + \Delta\theta) - T(\theta)) = \mu_s \frac{\Delta\theta}{2} (T(\theta) + T(\theta + \Delta\theta))$$

$$\frac{T(\theta + \Delta\theta) - T(\theta)}{\Delta\theta} = \frac{\mu_s}{2} (T(\theta + \Delta\theta) + T(\theta))$$

Take limit as $\Delta\theta \rightarrow 0$

$$\frac{dT}{d\theta} = T(\theta) \mu_s \quad \text{first order linear diff eq.}$$

Solve diff eq.

$$\frac{1}{T(\theta)} dT = \mu_s d\theta \Rightarrow \int_{T(0)}^{T(\Theta_{AB})} \frac{1}{T} dT \cdot \int_0^{\Theta_{AB}} \mu_s d\theta = \ln[T(\Theta_{AB})] - \ln[T(0)] = \mu_s \Theta_{AB} \Rightarrow \ln \left[\frac{T(\Theta_{AB})}{T(0)} \right] = \mu_s \Theta_{AB}$$

At $\theta = 0$ we have $T(0) = T_A$, equal to the force applied by the sailor.

At $\theta = \Theta_{AB}$, $T(\Theta_{AB}) = T_B$, the force applied by the load.

$$\Rightarrow \frac{T_B}{T_A} = e^{\mu_s \Theta_{AB}} \Rightarrow T_A = T_B e^{-\mu_s \Theta_{AB}}$$

⇒ Force applied by sailor goes down the higher Θ_{AB} is, i.e. the more times you wrap the rope around the capstan.