

18.2 Setup & Recoil Problem

→ Apply momentum principle to problems of recoil

$$\vec{F}_{ext} = \frac{d\vec{P}_{sys}}{dt}$$

$$F_{ext,x} = 0 \Rightarrow P_{sys,x}(t) = P_{sys,x}(i)$$

→ Recoil: jumping off cart

consider the following scenario with the reference frame being the ground:

if we consider as reference frame a car passing by with velocity \vec{v}_c :

\vec{J} : velocity of person in moving frame (\vec{v}_p)

= vel. of person relative to the cart

As we saw in week 2

$$\vec{r} = \vec{R} + \vec{r}' \Rightarrow \vec{r}' = \vec{r} - \vec{R}$$

$$\Rightarrow \vec{v}_p = \vec{R} + \vec{J} = \vec{v}_c + \vec{J}$$

we want to solve for \vec{v}_p and \vec{v}_c in terms of \vec{J}, m_p, m_c using the momentum principle.

we treat \vec{J} as a given

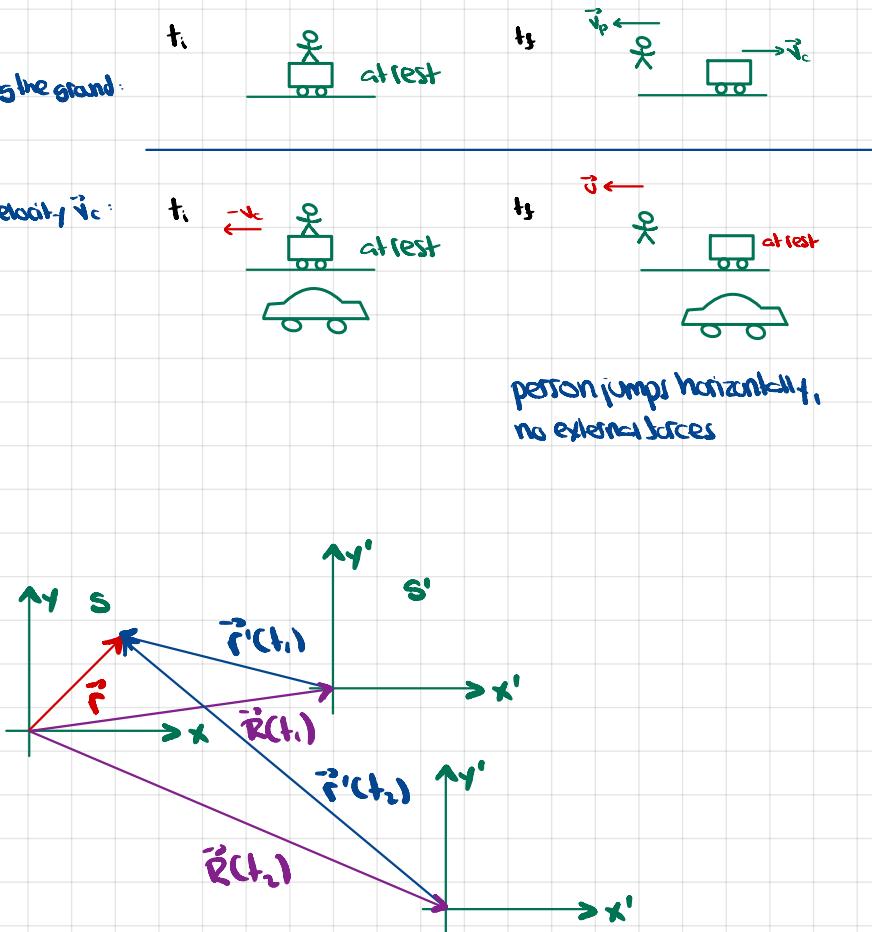
solution for the ground frame

$$\vec{P}_{sys}(i) = \vec{P}_{sys}(f)$$

$$\Rightarrow \vec{0} = m_c \vec{v}_c + m_p \vec{v}_p$$

$$\Rightarrow \vec{0} = m_c \vec{v}_c + m_p \vec{v}_c + m_p \vec{J} \Rightarrow \vec{v}_c = - \frac{m_p \vec{J}}{m_p + m_c}$$

$$\vec{v}_p = \vec{R} + \vec{J} = \vec{v}_c + \vec{J}$$



solution for the moving (car) frame

$$\vec{P}_{sys}(i) = \vec{P}_{sys}(f)$$

$$(m_p + m_c)(-\vec{v}_c) = m_p \vec{J} \Rightarrow \vec{v}_c = - \frac{m_p \vec{J}}{m_p + m_c}$$

19.1 Rocket Problem 1

Momentum Principle

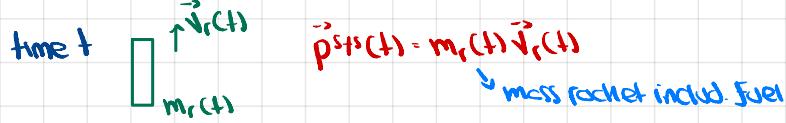
$$\vec{F}_{\text{ext}}(t) = \frac{d\vec{p}_{\text{sys}}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\vec{p}_{\text{sys}}(t + \Delta t) - \vec{p}_{\text{sys}}(t)}{\Delta t}$$

Continuous Mass Transfer

- (1) choose reference frame
- (2) identify the state at time t and $t + \Delta t$
- (3) draw momentum diagrams

→ System: rocket + fuel, stays the same between t and $t + \Delta t$

Analyze system momentum at times t and $t + \Delta t$.



time $t + \Delta t$

$\square m_r(t + \Delta t)$ $\uparrow \vec{v}_r(t + \Delta t)$

$\square \Delta m_f$ $\uparrow \vec{v}_f$

$$\vec{p}_{\text{sys}}(t + \Delta t) = m_r(t + \Delta t) \vec{v}_r(t + \Delta t) + \Delta m_f \vec{v}_f$$

\vec{j} = velocity of fuel relative to rocket

$\vec{v}_f = \vec{j} + \vec{v}_r(t + \Delta t)$

↗ velocity of fuel rel. to initial frame based on the second frame (the rocket)

before we make on lets simplify notation:

$$\begin{cases} m_r(t) = m_r \\ m_r(t + \Delta t) = m_r(t) + \Delta m_r = m_r + \Delta m_r \\ \Delta m_f = \Delta m_r \end{cases}$$

Now we write our momentum eqs again

$$\vec{p}_{\text{sys}}(t) = m_r \vec{v}_r(t)$$

here we've rewritten things slightly, using new notation and substituting $\Delta m_f = -\Delta m_r$ and $\vec{v}_f = \vec{j} + \vec{v}_r(t + \Delta t)$.

$$\vec{p}_{\text{sys}}(t + \Delta t) = (m_r + \Delta m_r) \vec{v}_r(t + \Delta t) - \Delta m_r (\vec{j} + \vec{v}_r(t + \Delta t)) = m_r \vec{v}_r(t + \Delta t) - \Delta m_r \vec{j}$$

Apply momentum principle

↗ At t, \vec{F}_{ext} is rate of change of momentum, ie the average change in an infinitesimal Δt

$$\vec{F}_{\text{ext}} = \lim_{\Delta t \rightarrow 0} \frac{(m_r + \Delta m_r) \vec{v}_r(t + \Delta t) - \Delta m_r (\vec{j} + \vec{v}_r(t + \Delta t)) - m_r \vec{v}_r(t)}{\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} \frac{m_r (\vec{v}_r(t + \Delta t) - \vec{v}_r(t))}{\Delta t} - \lim_{\Delta t \rightarrow 0} \frac{\Delta m_r \vec{j}}{\Delta t}$$

$$\vec{F}_{\text{ext}} = m_r \frac{d\vec{v}_r}{dt} - \frac{dm_r}{dt} \vec{j}$$

(Rocket Equation)

↙ rocket accel.

↙ note that mass of rocket is changing

19.5 Thrust and External Forces

$$\Delta m_r = -\Delta m_F \Rightarrow \frac{dm_r}{dt} = -\frac{dm_F}{dt}$$

$$\vec{F}_{ext} + \frac{dm_r}{dt} \vec{j} = m_r \frac{d\vec{v}_r}{dt} \Rightarrow \vec{F}_{ext} - \frac{dm_F}{dt} \vec{j} = m_r \frac{d\vec{v}_r}{dt}$$

We can rethink our system as just the rocket: $\vec{F}_{ext} + \vec{F}_{thrust} = m_r \frac{d\vec{v}_r}{dt}$, $\vec{F}_{thrust} = \frac{dm_F}{dt} \vec{j}$

with \vec{j} let's look at components:

$$F_{thrust} j = -\frac{dm_F}{dt} (-v j) = \frac{dm_F}{dt} v j$$

$$\vec{j} = -v \hat{j}$$

positive force opposite in dir. to gravity

$$\Rightarrow -mg + \frac{dm_F}{dt} v = m_r \frac{dV_r}{dt} \quad (\text{Rocket eq. in components})$$

$$\vec{F}_{ext} = -mg \hat{j}$$

19.6 Solution for No External Forces

$$\vec{F}_{ext} + \frac{dm_r}{dt} \vec{j} = m_r \frac{d\vec{v}_r}{dt}$$

special cases

i) $\vec{F}_{ext} = 0$ Rocket moves w/ terminal gravitational force
 $\rightarrow \vec{v}_r(t)$



$$\vec{v}_f = \vec{j} + \vec{v}_r(t+\Delta t)$$

$$\vec{v}_r(t+\Delta t)$$

$$\vec{v}_r(t)$$

$$\Delta m_g = -\Delta m_r$$

$$\text{exhaust speed } \vec{j} = -v \hat{i}$$

$$\vec{v}_r = v \hat{i}$$

$$\text{Rocket eq.: } \frac{dm_r}{dt} (-v) \hat{i} = m_r \frac{d\vec{v}_r}{dt} \hat{i}$$

$$-\frac{dm_r}{dt} v = m_r \frac{d\vec{v}_r}{dt}$$

$$-\frac{1}{m_r} v dm_r = d\vec{v}_r \Rightarrow \int_{m_r(t_i)}^{m_r(t_f)} -\frac{v}{m_r} dm_r = \int_{v_r(t_i)}^{v_r(t_f)} d\vec{v}_r$$

$$-v \ln(m_r(t_f)/m_r(t_i)) = v_r(t_f) - v_r(t_i)$$

$$v_r(t_f) - v_r(t_i) = v \ln \left[\frac{m_r(t_i)}{m_r(t_f)} \right]$$

19.7 Solution with External Forces



$$\vec{F}_{ext} = -m_r g \hat{j} \neq 0$$

$$-m_r g + \frac{dm_r}{dt} (-v) = m_r \frac{d\vec{v}_r}{dt} \Rightarrow -m_r g dt - v dm_r = m_r d\vec{v}_r$$

$$-gdt - \frac{dm_r}{m_r} v = d\vec{v}_r$$

$$\Rightarrow \int_{t_i}^{t_f} -gdt - v \int_{m_r(t_i)}^{\frac{m_r(t_f)}{m_r(t_i)}} \frac{1}{m_r} dm_r = \int_{v_r(t_i)}^{v_r(t_f)} d\vec{v}_r$$

$$= -g(t_f - t_i) - v \ln \left[\frac{m_r(t_f)}{m_r(t_i)} \right] = v_r(t_f) - v_r(t_i)$$

$$\text{At } t_i = 0, v_r(0) = 0, m_r(0) = m_{r,i} + m_g$$

$$\text{At } t_f, v_r(t_f), m_r(t_f) = m_{r,i}$$

$$-gt_f + v \ln \left[\frac{m_{r,i} + m_g}{m_{r,i}} \right] = v_r(t_f)$$

P.S. 6.1 Rocket sled

Problem Data

$$m_r = m_{r,0} + m_{r,f} = 2m_0$$

$$\vec{v}(0) = \vec{v}_0 \hat{i}$$

$$\vec{v}_{fr} = \vec{v}_0 + v\hat{i}$$

$$\vec{v}_f = \vec{v}_{fr} + \vec{v}_r = \vec{v}_0 + \vec{v}_r$$



Let's do the entire derivations here.

reference frame: ground

$$\text{initial momentum: } \vec{p}_{sys}(t) = m_r(t) \vec{v}_r(t)$$

$$\text{at } t + \Delta t \quad \vec{p}_{sys}(t + \Delta t) = m_r(t + \Delta t) \vec{v}_r(t + \Delta t) + \Delta m_f \vec{v}_f(t + \Delta t)$$

rewrite some terms differently

$$\begin{aligned} m_r(t + \Delta t) &= m_r(t) + \Delta m_r \Rightarrow \vec{p}_{sys}(t + \Delta t) = (m_r(t) + \Delta m_r) \vec{v}_r(t + \Delta t) - \Delta m_r (\vec{v}_r(t + \Delta t)) \\ \Delta m_f &= -\Delta m_r \end{aligned}$$

$$= m_r(t) \vec{v}_r(t + \Delta t) - v \Delta m_r$$

$$\begin{aligned} \text{momentum principle: } \vec{F}_{ext} = \frac{d\vec{p}_{sys}(t)}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{\vec{p}_{sys}(t + \Delta t) - \vec{p}_{sys}(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{m_r(t) \vec{v}_r(t + \Delta t) - v \Delta m_r - m_r(t) \vec{v}_r(t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} m_r(t) \frac{\vec{v}_r(t + \Delta t) - \vec{v}_r(t)}{\Delta t} - \lim_{\Delta t \rightarrow 0} \frac{\Delta m_r}{\Delta t} = m_r(t) \frac{d\vec{v}_r(t)}{dt} - v \frac{dm_r(t)}{dt} \end{aligned}$$

a) Since there is no friction force or gravitational force in the x-direction, $\vec{F}_{ext} = 0$.

$$\Rightarrow m_r(t) \frac{d\vec{v}_r(t)}{dt} = -v \frac{dm_r(t)}{dt} \Rightarrow m_r(t) \frac{d\vec{v}_r(t)}{dt} \uparrow = \frac{dm_r(t)}{dt} + v \uparrow$$

$$\Rightarrow m_r(t) \frac{d\vec{v}_r(t)}{dt} = \frac{dm_r(t)}{dt} \uparrow$$

$$\text{b) } d\vec{v}_r = \frac{v}{m_r(t)} dm_r \Rightarrow \int_{v_r(t_i)}^{v_r(t_f)} d\vec{v}_r = \int_{m_r(t_i)}^{m_r(t_f)} v \frac{dm_r}{m_r} \Rightarrow \vec{v}_r(t_f) - \vec{v}_r(t_i) = v \ln \left[\frac{m_r(t_f)}{m_r(t_i)} \right]$$

$$\text{c) } \vec{v}_r(t_f) = \vec{v}_r(t_i) + v \ln \left[\frac{m_r(t_f)}{m_r(t_i)} \right]$$

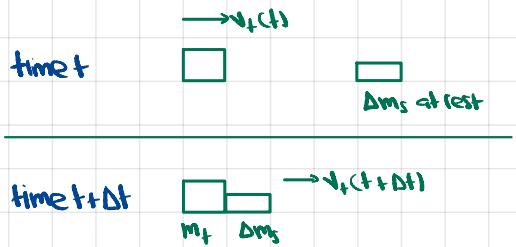
$$\vec{v}_r(t_f) = 0, \text{ i.e. the rocket stops.}$$

$$m_r(t_i) = 2m_0$$

$$m_r(t_f) = m_0$$

$$\Rightarrow \vec{v}_r(t_i) = -v \ln(1/2) \approx 0.69v$$

P.S. 6.2 Snowplow



$$v(t + \Delta t) - v(t) + \Delta v$$

momentum principle $F = \lim_{\Delta t \rightarrow 0} \frac{(m_t + \Delta m_s)v_t(t + \Delta t) - m_t v_t(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{(m_t + \Delta m_s)(v_t(t) + \Delta v_t) - m_t v_t(t)}{\Delta t}$

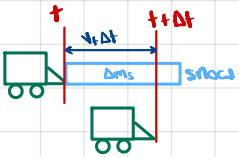
$$= \lim_{\Delta t \rightarrow 0} \frac{m_t \Delta v_t + \Delta m_s v_t(t) + \Delta m_s \Delta v_t}{\Delta t} = \frac{\Delta m_s}{\Delta t} v_t + m_t \frac{dv_t}{dt}$$

disregard this second order term

$$\Rightarrow F = \frac{\Delta m_s}{\Delta t} v_t + m_t \frac{dv_t}{dt}$$

$$F dt = \Delta m_s v_t + m_t dv_t$$

what is Δm_s ?



$$\Delta m_s = \rho A v_t \Delta t$$

cross sectional area of truck

$$\Rightarrow F dt = (\rho A v_t \Delta t) v_t + m_t dv_t \Rightarrow (F - \rho A v_t^2) dt = m_t dv_t \Rightarrow \int_{t_i}^{t_f} dt = \int_{v_i(t_i)}^{v_f(t_f)} \frac{m_t}{F - \rho A v_t^2} dv_t$$

$$\int \frac{m_t}{F - \rho A v_t^2} dv_t = \int \frac{m_t}{F - \rho A \frac{F \sin^2 \theta}{PA} [\frac{PA}{F}]^{1/2}} \cdot \frac{\cos \theta}{\cos^2 \theta} d\theta = \sqrt{\frac{F}{PA}} \frac{m_t}{F} \int \frac{\cos \theta}{\cos^2 \theta} d\theta = \frac{m_t}{\sqrt{PA F}} \int \cos^{-1} \theta d\theta \cdot \frac{m_t}{\sqrt{PA F}} \ln(\sec \theta + \tan \theta)$$

$$v_t = \frac{\sin \theta}{[\frac{PA}{F}]^{1/2}} \Rightarrow v_t^2 = \frac{F \sin^2 \theta}{PA}$$

$$dv_t \cdot \frac{\cos \theta}{[\frac{PA}{F}]^{1/2}} d\theta$$

$$\theta = \sin^{-1} \left[\sqrt{t} \sqrt{\frac{PA}{F}} \right]$$

$$\Rightarrow t_f - t_i = \frac{m_t}{\sqrt{PA F}} \ln \left[\frac{\sec(\sin^{-1} \left[\sqrt{t_i} \sqrt{\frac{PA}{F}} \right]) + \tan(\sin^{-1} \left[\sqrt{t_i} \sqrt{\frac{PA}{F}} \right])}{\sec(\sin^{-1} \left[\sqrt{t_f} \sqrt{\frac{PA}{F}} \right]) + \tan(\sin^{-1} \left[\sqrt{t_f} \sqrt{\frac{PA}{F}} \right])} \right]$$