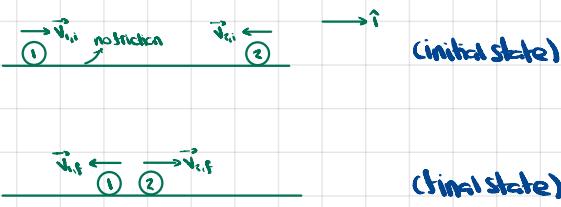


26.1 Momentum in Collisions



$$\vec{F}_{ext} \cdot \Delta t = \vec{p}_{sys,f} - \vec{p}_{sys,i}$$

physics description

assumptions: $\vec{F}_{ext} = 0 \Rightarrow \vec{p}_{sys,f} = \vec{p}_{sys,i}$, momentum of system is constant.

$$m_1 \vec{v}_{1,i} + m_2 \vec{v}_{2,i} = m_1 \vec{v}_{1,f} + m_2 \vec{v}_{2,f}$$

component eq. for x direction is

$$m_1 v_{1,x,i} + m_2 v_{2,x,i} = m_1 v_{1,x,f} + m_2 v_{2,x,f}$$

26.2 Kinetic Energy in Collisions

elastic collision: $\Delta K_{sys} = 0$

inelastic " : $\Delta K_{sys} < 0$

superelastic " : $\Delta K_{sys} > 0$

$$\frac{1}{2} m_1 v_{1,i}^2 + \frac{1}{2} m_2 v_{2,i}^2 = \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2$$

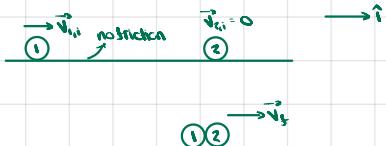
$$m_1 v_{1,x,i}^2 + m_2 v_{2,x,i}^2 = m_1 v_{1,x,f}^2 + m_2 v_{2,x,f}^2 \quad (\text{kin. en. eq.})$$

recall we have a momentum eq.:

$$m_1 v_{1,x,i} + m_2 v_{2,x,i} = m_1 v_{1,x,f} + m_2 v_{2,x,f}$$

given $m_i, v_{1,x,i}, v_{2,x,i}$ we can solve for $v_{1,x,f}$ and $v_{2,x,f}$

26.3 Totally Inelastic Collisions



$$m_1 v_{1,x,i} = (m_1 + m_2) v_{1,x,f}$$

$$v_{1,x,f} = \frac{m_1}{m_1 + m_2} v_{1,x,i}$$

$$\Delta K = \frac{1}{2} (m_1 + m_2) v_{1,f}^2 - \frac{1}{2} m_1 v_{1,x,i}^2$$

$$= \frac{1}{2} (m_1 + m_2) \left(\frac{m_1}{m_1 + m_2} v_{1,x,i} \right)^2 - \frac{1}{2} m_1 v_{1,x,i}^2$$

$$= \frac{1}{2} m_1 v_{1,x,i}^2 \left(\frac{m_1}{m_1 + m_2} - 1 \right)$$

$$= k_i \left(-\frac{m_2}{m_1 + m_2} \right)$$

27.1 Worked Example: Elastic 1D Collision



Frame: Laboratory

Ex: $m_2 = 2m_1$, elastic collision

$$m_1 v_{1,i} = m_1 v_{1,f} + m_2 v_{2,f}$$

$$\frac{m_1 v_{1,i}^2}{2} = \frac{m_1 v_{1,f}^2}{2} + \frac{m_2 v_{2,f}^2}{2} \Rightarrow v_{1,i}^2 = v_{1,f}^2 + 2v_{2,f}^2$$

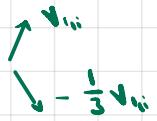
$$v_{2,f} = \frac{m_1 (v_{1,i} - v_{1,f})}{2m_1} = \frac{v_{1,i} - v_{1,f}}{2}$$

$$v_{1,f}^2 = v_{1,f}^2 + \frac{(v_{1,i} - v_{1,f})^2}{2}$$

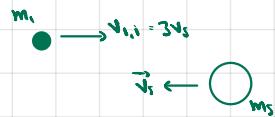
$$\Rightarrow 2v_{1,i}^2 = 2v_{1,f}^2 + v_{1,i}^2 + v_{1,f}^2 - 2v_{1,i}v_{1,f}$$

$$3v_{1,f}^2 = 2v_{1,i}v_{1,f} - v_{1,i}^2$$

$$v_{1,f} = \frac{2v_{1,i} \pm [4v_{1,i}^2 + 12v_{1,i}]^{1/2}}{6} = \frac{2v_{1,i} \pm 4v_{1,i}}{6}$$



27.5 Worked Example - Gravitational Slingshot



$$v_{1,i} = 3v_s i$$

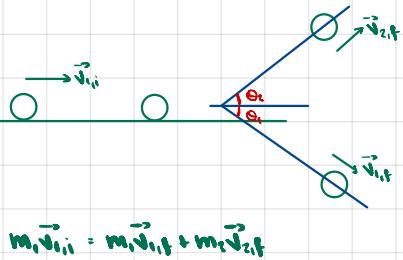
$$v_{1,f} = -v_s i$$

$$v_{2,i} = 3v_s i - (-v_s) i = 4v_s i$$

$$v_{2,f} = v_{1,f} - (-v_s) i = -4v_s i$$

$$\Rightarrow v_{2,f} = -5v_s i$$

27.6 2D Collisions



$$m_1 \vec{v}_{1,ii} = m_1 \vec{v}_{1,if} + m_2 \vec{v}_{2,if}$$

$$m_1 v_{1,ii} = m_1 v_{1,if} \cos \theta_1 + m_2 v_{2,if} \cos \theta_2$$

$$0 = m_1 v_{1,if} \sin \theta_1 + m_2 v_{2,if} \sin \theta_2$$

$$\text{elastic} \Rightarrow \Delta E = 0 \quad \frac{m_1 v_{1,ii}^2}{2} = \frac{m_1 v_{1,if}^2}{2} + \frac{m_2 v_{2,if}^2}{2}$$

3 equations, 4 unknowns: $v_{1,if}, v_{2,if}, \theta_1, \theta_2$

D.D.2.1 Position in CM Frame

$$\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\vec{r}_1' = \vec{r}_1 - \vec{r}_{cm} = \frac{m_2}{m_1 + m_2} (\vec{r}_1 - \vec{r}_2)$$

$$\vec{r}_2' = \vec{r}_2 - \vec{r}_{cm} = \frac{-m_1}{m_1 + m_2} (\vec{r}_1 - \vec{r}_2)$$

reduced mass = $\mu = \frac{m_1 m_2}{m_1 + m_2}$

$$\Rightarrow \vec{r}_1' = \frac{\mu}{m_1} \vec{r}_{12} \quad \vec{r}_2' = \frac{-\mu}{m_2} \vec{r}_{12}$$

D.D.2.2 Relative Velocity is Independent of Ref. Frame

$$\vec{v}_1 = \vec{v}_2 + \vec{v}_{12} \quad \vec{v}_1 - \vec{v}_2 = \vec{v}_1 - \vec{v}_2 \Rightarrow \vec{v}_1 - \vec{v}_2 = \vec{v}_1 - \vec{v}_2$$

$$\vec{v}_2 = \vec{v}_1 + \vec{v}_{12}$$

D.D.2.3 1D Elastic Collision Velocities in CM Frame

or know that

$$\rightarrow \text{in a 1D elastic collision, } \vec{v}_{1,2,ii} = -\vec{v}_{1,2,if}$$

$$\rightarrow |\vec{v}_{1,2,if}| = |\vec{v}_{1,2,if}|$$

\rightarrow velocities in CM reference frame are bns of $\vec{v}_{1,2,if}$:

$$\vec{v}_1' = \frac{\mu}{m_1} \vec{v}_{1,2,if} \quad \vec{v}_2' = -\frac{\mu}{m_2} \vec{v}_{1,2,if}$$

\rightarrow therefore, $|\vec{v}_{1,if}| = |\vec{v}_{1,if}|$ and $|\vec{v}_{2,if}| = |\vec{v}_{2,if}|$

$$\vec{v}_{1,ii}' = -\vec{v}_{1,if}', \quad \vec{v}_{2,ii}' = \vec{v}_{2,if}'$$

speeds of objects don't change in CM frame, only the direction of velocity changes.

D.D.2.4 Worked Example: 1D Elastic Collision in CM

$$\vec{v}_{1,ii} \rightarrow \vec{v}_{1,ii} \quad \vec{v}_{2,ii} = 0$$

$$\textcircled{O} m_1 \quad \textcircled{O} m_2 = 2m_1$$

$$\vec{v}_{1,if}' = -\vec{v}_{1,ii}$$

$$\vec{v}_{1,if}' = \frac{\mu}{m_1} \vec{v}_{1,ii} = \frac{\mu}{m_1} \vec{v}_{1,ii} = -\frac{m_2}{m_1 + m_2} \vec{v}_{1,ii} = \frac{2}{3} \vec{v}_{1,ii}$$

$$\Rightarrow \vec{v}_{1,if}' = -\frac{2}{3} \vec{v}_{1,ii}$$

$$\vec{v}_{2,if}' = -\vec{v}_{1,if}' = -\left(-\frac{m_1}{m_1 + m_2} \vec{v}_{1,ii}\right) = \frac{1}{3} \vec{v}_{1,ii}$$

$$\vec{v}_{cm,if}' = \frac{m_1 \vec{v}_{1,ii}}{m_1 + m_2} = \frac{1}{3} \vec{v}_{1,ii}$$

$$\vec{v}_{1,if} = \vec{v}_{1,if}' + \vec{v}_{cm} = -\frac{2}{3} \vec{v}_{1,ii} + \frac{1}{3} \vec{v}_{1,ii} = -\frac{1}{3} \vec{v}_{1,ii}$$

$$\vec{v}_{2,if} = \frac{1}{3} \vec{v}_{1,ii} + \frac{1}{3} \vec{v}_{1,ii} = \frac{2}{3} \vec{v}_{1,ii}$$

D.D.2.5 Kinetic Energy in Different Reference Frames

$$K_{cm} = \frac{m_1 v_{1,ii}^2}{2} + \frac{m_2 v_{2,ii}^2}{2}$$

$$K_{background} = \frac{m_1 v_{1,ii}^2}{2} + \frac{m_2 v_{2,ii}^2}{2}$$

$$= \frac{m_1}{2} (v_{1,ii}^2 + v_{cm}^2) + \frac{m_2}{2} (v_{2,ii}^2 + v_{cm}^2)$$

$$= \frac{m_1}{2} (v_{1,ii}^2 + v_{cm}^2 + 2v_{1,ii}v_{cm}) + \frac{m_2}{2} (v_{2,ii}^2 + v_{cm}^2 + 2v_{2,ii}v_{cm})$$

$$= \frac{m_1 v_{1,ii}^2}{2} + \frac{m_2 v_{2,ii}^2}{2} + (m_1 + m_2)v_{cm}^2 + v_{cm}(m_1 v_{1,ii} + m_2 v_{2,ii})$$

$$= K_{cm} + \frac{m_1 + m_2}{2} v_{cm}^2$$

D.D.2.6 Kinetic Energy in CM Frame

$$K_{cm} = \frac{m_1}{2} \left(\frac{\mu}{m_1} v_{1,2,if} \right)^2 + \frac{m_2}{2} \left(-\frac{\mu}{m_2} v_{1,2,if} \right)^2$$

$$= \frac{1}{2} \frac{\mu^2 v_{1,2,if}^2}{m_1} + \frac{1}{2} \frac{\mu^2 v_{1,2,if}^2}{m_2} = \frac{1}{2} \mu^2 v_{1,2,if}^2 \left(\frac{1}{m_1} + \frac{1}{m_2} \right)$$

$$= \frac{\mu v_{1,2,if}^2}{2}$$

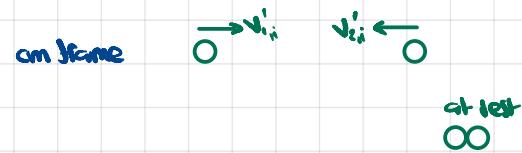
$$\Delta K_{cm} = \frac{\mu}{2} (v_{1,2,if}^2 - v_{1,2,ii}^2)$$

$$\text{elastic collision} \Rightarrow v_{1,2,ii}^2 = v_{1,2,if}^2$$

$$\text{totally inelastic} \Rightarrow v_{1,2,if} = 0$$

D.D.2.7 Change in Kinetic Energy

$$K_{\text{grund}} = K_{\text{cm}} + \frac{m_1 + m_2}{2} v_{\text{cm}}^2 \Rightarrow \Delta K_{\text{grund}} = \Delta K_{\text{cm}}$$



$$K_{\text{cm}} = \frac{M v_{\text{cm}}^2}{2}$$

$$\Delta K_{\text{cm}} = - \frac{M v_{\text{cm}}^2}{2} = \Delta K_{\text{grund}}$$

Recap of Results

We can always write the eq. for conservation of momentum

$$m_1 \vec{v}_{1,i} + m_2 \vec{v}_{2,i} = m_1 \vec{v}_{1,f} + m_2 \vec{v}_{2,f}$$

or cons with the component in x-direction

$$m_1 v_{1,x,i} + m_2 v_{2,x,i} = m_1 v_{1,x,f} + m_2 v_{2,x,f}$$

In elastic collisions, we write $\Delta K = 0 \Rightarrow K_i = K_f$

$$\frac{1}{2} m_1 v_{1,i}^2 + \frac{1}{2} m_2 v_{2,i}^2 = \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2$$

we can solve for $v_{1,f}$ and $v_{2,f}$ to obtain the 1D energy/momentum principle

$$v_{1,x,i} - v_{2,x,i} = v_{1,x,f} - v_{2,x,f} \Rightarrow v_{1,x,i} = v_{1,x,f}$$

For totally inelastic collisions, $v_{1,x,f} = 0$ ie $v_{1,x,i} = v_{1,x,f}$

again, conservation of momentum

$$m_1 v_{1,x,i} + m_2 v_{2,x,i} = (m_1 + m_2) v_{x,f}$$

$$\Rightarrow v_{x,f} = \frac{m_1 v_{1,x,i} + m_2 v_{2,x,i}}{m_1 + m_2}$$

We haven't imposed any explicit condition on ΔK but implicitly we have

$$\Delta K = \frac{(m_1 + m_2) v_{x,f}^2}{2} - \frac{m_1 v_{1,x,i}^2}{2} - \frac{m_2 v_{2,x,i}^2}{2}$$

$$\text{simplifying } = \frac{-m_1 m_2 (v_{1,x,i} - v_{2,x,i})^2}{2m_1 + 2m_2} < 0$$

\Rightarrow Kinetic Energy is lost in totally inelastic collision.

Next we defined results concerning the center of mass frame we know \vec{r}_{cm} , so we can obtain my position from CM frame given position in lab frame

$$\vec{r}_1' = \frac{\mu}{m_1} \vec{r}_{12} \quad \vec{v}_1' = \frac{\mu}{m_1} \vec{v}_{12}$$

$$\vec{r}_2' = \frac{\mu}{m_2} \vec{r}_{12} \quad \vec{v}_2' = -\frac{\mu}{m_2} \vec{v}_{12}$$

$$\text{where } \mu = \text{reduced mass} = \frac{m_1 m_2}{m_1 + m_2}$$

But $\vec{v}_{1,2}' = \vec{v}_{1,2}$, ie the relative velocity vector is the same in every reference frame (that is relatively inertial)

This gives us a new result in the 1D elastic collision case.

1D energy/momentum principle $\Rightarrow v_{1,2,i} = -v_{1,2,f}$

$$\vec{v}_1' = \frac{\mu}{m_1} \vec{v}_{1,2} \quad \vec{v}_2' = -\frac{\mu}{m_2} \vec{v}_{1,2}$$

$$\Rightarrow \vec{v}_{1,i}' = \frac{\mu}{m_1} \vec{v}_{1,2,i}$$

$$\vec{v}_{1,f}' = \frac{\mu}{m_1} \vec{v}_{1,2,f} = \frac{\mu}{m_1} (-\vec{v}_{1,2,i})$$

$$\Rightarrow \vec{v}_{1,f}' = -\vec{v}_{1,i}'$$

$$\text{likewise } \vec{v}_{2,f}' = -\vec{v}_{2,i}'$$

In CM frame, in 1D elastic collision, velocities of objects change direction but not magnitude (speed).

Next we look at kinetic energy. Start with the definitions

$$K_{cm} = \frac{m_1 v_{1,i}^2}{2} + \frac{m_2 v_{2,i}^2}{2}$$

$$K_{ground} = \frac{m_1 v_{1,i}^2}{2} + \frac{m_2 v_{2,i}^2}{2}$$

we know the relationship between $v_{1,i}$ and $v_{1,i}'$, and between $v_{2,i}$ and $v_{2,i}'$.

$$\vec{v}_1' = \frac{\mu}{m_1} \vec{v}_{1,2} \quad \vec{v}_2' = -\frac{\mu}{m_2} \vec{v}_{1,2}$$

so we can find K_{ground} in terms of K_{cm} and vice versa
sub v_1' and v_2' into K_{ground} , obtain

$$K_{ground} = K_{cm} + \frac{m_1 + m_2}{2} v_{cm}^2$$

sub v_1' and v_2' into K_{cm} , obtain

$$K_{cm} = \frac{\mu v_{1,2}^2}{2}$$

In an elastic collision, $v_{1,2,i} = -v_{1,2,f}$ so

$$\Delta K_{cm} = 0$$

In a totally inel. collision $v_{1,2,f} = 0$

$$\Delta K_{cm} = -\frac{\mu v_{1,2,i}^2}{2} < 0$$

Finally, note that $K_{ground} = K_{cm} + \text{constant}$

$$\Rightarrow \Delta K_{ground} = \Delta K_{cm}$$