

## 6.1 Circular Motion

→ motion of object: circular orbit of radius  $r$

→ position vector:  $\vec{r}(t) = r\hat{r}(t)$

→ two sets of unit vectors:  $(\hat{i}, \hat{j}), (\hat{r}, \hat{\theta})$

→ obtaining  $\hat{r}$  and  $\hat{\theta}$  in  $(\hat{i}, \hat{j})$  coordinates means projecting them onto  $\hat{i}$  and  $\hat{j}$ .

$$\hat{r} \cdot \hat{i} = \cos \theta \quad \hat{r} \cdot \hat{j} = \cos(90^\circ - \theta) = \sin \theta \Rightarrow \hat{r} = \cos \theta \hat{i} + \sin \theta \hat{j}$$

$$\hat{\theta} \cdot \hat{i} = \cos(90^\circ + \theta) = -\sin \theta \quad \hat{\theta} \cdot \hat{j} = \cos(\theta) \Rightarrow \hat{\theta} = -\sin \theta \hat{i} + \cos \theta \hat{j}$$

Note that though omitted in the eq. above,  $\theta = \theta(t)$  for our circular orbit motion.

→ At each point in the trajectory, there is a pair  $(\hat{r}(t), \hat{\theta}(t))$  of unit vectors.

How do they change in time?

$$\frac{d\hat{r}}{dt} = -\sin \theta \frac{d\theta}{dt} \hat{i} + \cos \theta \frac{d\theta}{dt} \hat{j} = \frac{d\theta(t)}{dt} \hat{\theta}(t)$$

$$\frac{d\hat{\theta}(t)}{dt} = -\cos \theta \frac{d\theta}{dt} \hat{i} - \sin \theta \frac{d\theta}{dt} \hat{j} = -\frac{d\theta(t)}{dt} \hat{r}(t)$$

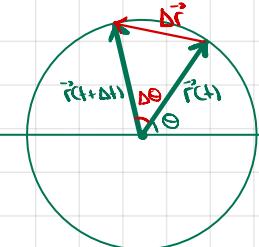
→ velocity vector, tangent to the trajectory

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \frac{d(r\hat{r}(t))}{dt} = r \frac{d\hat{r}(t)}{dt} = r \frac{d\theta}{dt} \hat{\theta} = \underbrace{r \omega_z \hat{\theta}}_{\substack{\text{z-comp of angular velocity} \\ \text{velocity direction}}} = v_0 \hat{\theta}$$

$v_0 \hat{\theta}$   
speed along the arc  
aka tangential component of velocity

→ Angular speed: magnitude of rate of change of angle w.r.t. time  $\omega = \left| \frac{d\theta}{dt} \right|$

### 6.2.1 Geometric Derivation of Velocity



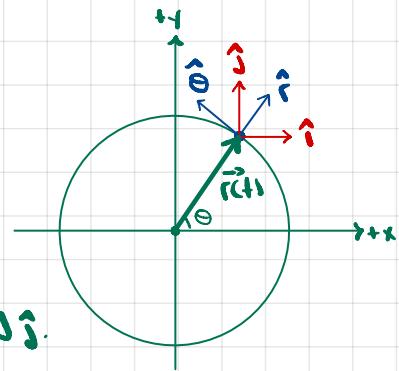
We would like to derive what  $\vec{v}(t)$  is.

$$|\Delta \vec{r}| \approx \text{arc length } r |\Delta \theta|$$

$$v = |\vec{v}(t)| = \lim_{\Delta t \rightarrow 0} \frac{|\Delta \vec{r}|}{\Delta t} = r \lim_{\Delta t \rightarrow 0} \frac{|\Delta \theta|}{\Delta t} = r \left| \frac{d\theta}{dt} \right| = r \omega$$

As  $\Delta t \rightarrow 0$ ,  $\Delta \vec{r}$  has direction that approaches the direction of the tangent to the circle at  $(t)$ .

In the limit  $\Delta \vec{r} \perp \vec{r}(t)$ ,  $\vec{v}(t) \perp \vec{r}(t)$ .



## 6.3 Tangential and Radial Acceleration

$$\vec{a}(t) = a_r \hat{r}(t) + a_\theta \hat{\theta}(t)$$

$$\vec{v}(t) = r \dot{\theta}(t) \hat{\theta}(t)$$

Assume that due to the presence of some tangential force,  $\vec{v}(t)$  is changing.

In particular, due to the tangential nature of the force, the magnitude of  $\vec{v}(t)$  is changing  $\Rightarrow \dot{\theta}(t)$  is not constant.

$$\begin{aligned}\vec{a}(t) &= \frac{d\vec{v}(t)}{dt} = r \dot{\theta}''(t) \hat{\theta}(t) + r \dot{\theta}'(t) (-\cos \theta \hat{i} - \sin \theta \hat{j}) \\ &= r \dot{\theta}''(t) \hat{\theta}(t) - r \dot{\theta}'(t)^2 \hat{r}(t) \\ &= r \dot{\theta}''(t) \hat{\theta}(t) - r \omega^2 \hat{r}(t) \\ &= a_\theta \hat{\theta} - a_r \hat{r}\end{aligned}$$

## 6.4 Period and Frequency for Uniform Circular Motion

If the total tangential force is zero then  $F = ma$  in the  $\hat{\theta}$  direction (ie the tangential direction) means

$$a_\theta = 0 \Rightarrow r \dot{\theta}''(t) = 0 \Rightarrow \dot{\theta}''(t) = 0 \Rightarrow \dot{\theta}'(t) = \text{constant}$$

$$v(t) = |\vec{v}(t)| = r |\dot{\theta}'(t)| = \text{constant}, \text{ ie speed is constant} \Rightarrow \omega = \dot{\theta}'(t) \text{ constant}$$

$\Rightarrow$  Uniform Circular Motion

$$\vec{a}(t) = a_r(t) \hat{r}(t) = -r \omega^2 \hat{r}(t)$$

Period: Amount of time to complete one circular orbit

$$2\pi r = Tr = T(\omega) \Rightarrow T = \frac{2\pi}{\omega}, \text{ units } \frac{[L]}{[T]} = [T]$$

Frequency: Reciprocal of period  $1/T$ , units  $1/[T]$ , in SI this is  $\text{s}^{-1} \equiv \text{Hz}$

$$f = \frac{\omega}{2\pi}$$

Recall the centripetal acceleration  $a_r(t) = -r \omega^2(t)^2$ . Because of the relationships between  $\omega, T, f$ ,  $v$ , and  $r$  we can express  $a_r(t)$  in different ways.

$$|a_r(t)| = \frac{v^2}{r} = r \cdot 4\pi^2 f^2 = r \frac{4\pi^2}{T^2}. \text{ There are other forms too.}$$

## 6.S.1 Angular Velocity

→ we will always choose a right-handed cylindrical coordinate system

$z$ -axis points up  $\Rightarrow \theta$  increases in  $ccc$  direction

→  $\vec{\omega}$  is directed along  $z$ -axis

$$\vec{\omega} = \frac{d\theta}{dt} \hat{k} = \omega_z \hat{k} \quad [\text{rad s}^{-1}]$$

$$\omega = \text{angular speed} = |\vec{\omega}| = |\omega_z| = |d\theta/dt|$$

$$\rightarrow \vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ 0 & 0 & \omega_z \\ r & 0 & 0 \end{vmatrix} = -(-r\omega_z)\hat{\theta} = r\omega_z \hat{\theta} = r \frac{d\theta}{dt} \hat{\theta}$$

### Ex 6.2

$$\theta(t) = At - Bt^3, A, B > 0$$

$$a) \theta'(t) = A - 3Bt^2$$

$$\vec{\omega}(t) = (A - 3Bt^2) \hat{k}$$

$$b) \vec{v} = r\theta'(t) \hat{\theta} = r(A - 3Bt^2) \hat{\theta}$$

$$c) \vec{\omega}(t) = \vec{0} \Rightarrow A - 3Bt^2 = 0 \Rightarrow t = \sqrt{\frac{A}{3B}}$$

$$d) t < t_1 \Rightarrow \omega_z > 0 \Rightarrow \text{dir}(\vec{\omega}) \cdot \hat{k}$$

$$t > t_1 \Rightarrow \omega_z < 0 \Rightarrow \text{dir}(\vec{\omega}) \cdot -\hat{k}$$

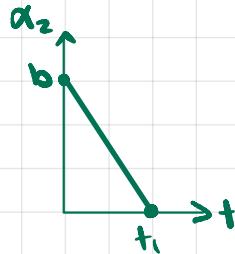
## 6.5.2 Angular Acceleration

$$\text{angular acceleration} = \vec{\alpha} = \Theta''(t) \hat{k} = \alpha_z \hat{k} \quad [\text{rad} \cdot \text{s}^{-2}]$$

$$\alpha = |\vec{\alpha}| = |\Theta''(t)|$$

Ex 6.3 point-like object, circular motion

$$\alpha_z(t) = \begin{cases} b\left(1 - \frac{t}{t_1}\right) & 0 \leq t \leq t_1, \\ 0 & t > t_1, \end{cases} \quad b > 0$$



$$\begin{aligned} \text{a) } \omega_z(t) &= \begin{cases} \omega_{z_0} + \int_0^t [b - \frac{b}{t_1}t] dt & 0 \leq t \leq t_1, \\ \omega_z(t_1) + \int_{t_1}^t 0 dt & t > t_1, \end{cases} \\ &= \begin{cases} \omega_{z_0} + bt - \frac{bt^2}{2t_1} & 0 \leq t \leq t_1, \\ 0 & t > t_1, \end{cases} \end{aligned}$$

interpretation:

$\omega_z = \Theta'(t)$  starts at  $\omega_{z_0}$

$\alpha_z$  starts at  $b > 0$  and decreases to 0.

$\Rightarrow \Theta'(t)$  is increasing, but increasing at a progressively lower rate. At  $t_1$ ,  $\Theta'(t_1)$  stops increasing, so angular velocity stays constant.

Note that no assumptions are made on the sign of  $\omega_{z_0}$ . We don't know if motion is cw or ccw.

$$\text{At } t=t_1, \omega_z(t_1) = \omega_{z_0} + bt_1 - \frac{bt_1^2}{2t_1} = \omega_{z_0} + \frac{2bt_1^2 - bt_1^2}{2t_1} = \omega_{z_0} + \frac{bt_1}{2}$$

$$\text{b) } \omega_z(t) = r\Theta'(t)$$

$$\begin{aligned} \Theta'(t) &= \frac{\omega_z(t)}{r} \Rightarrow \Theta(t) = \frac{1}{r} \int \omega_z(t) dt + \Theta_0 \quad 0 \leq t \leq t_1, \\ &\quad = \frac{1}{r} \left[ \omega_{z_0}t + \frac{bt^2}{2} - \frac{bt^3}{6t_1} \right] + \Theta_0 \end{aligned}$$

$$\Theta(t_1) = \Theta_0 + \frac{\omega_{z_0}t_1}{r} + \frac{2bt_1^2}{6}$$