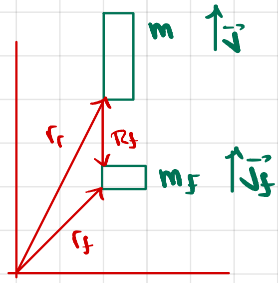
1135 Derivation of Rocket Eq.



$\vec{r_f} = \vec{r_r} + \vec{R_f}$

$\vec{v_f} = \vec{v_r} + \vec{u}$

time	momentum of system rocket + fuel
t	$m(t)v(t)$
$t+\Delta t$	$(m(t)+\Delta m_r)v(t+\Delta t) + \Delta m_f v_f$

$\Delta m_f = -\Delta m_r$

momentum principle

$$\vec{F}^{ext} = \frac{d\vec{p}^{sys}}{dt}$$

$$\vec{p}(t+\Delta t) - \vec{p}(t) = (m(t) + \cancel{\Delta m_r})\vec{v}(t+\Delta t) - \cancel{\Delta m_r \vec{v}(t+\Delta t)} - \Delta m_r \vec{u} - m(t)\vec{v}(t)$$

$$\frac{\Delta p}{\Delta t} = \frac{m(t)v(t+\Delta t) - \Delta m_r u - m(t)v(t)}{\Delta t} = m(t)\frac{\Delta v}{\Delta t} - \frac{\Delta m_r}{\Delta t}\vec{u}$$

$$\lim_{\Delta t \to 0} \frac{\Delta \vec{p}}{\Delta t} = \frac{d\vec{p}}{dt} = m(t)\frac{d\vec{v}}{dt} - u\frac{dm}{dt}$$

$$\vec{F}^{ext} = m(t)\frac{d\vec{v}}{dt} - \vec{u}\frac{dm}{dt}$$

Scenario

Free space ($\vec{F}^{ext} = 0$)

initial mass m_0

exhaust velocity $= u$

rocket mass when momentum is maximum.

$$m(t)\frac{dv}{dt} = u\frac{dm}{dt} \Rightarrow u m^{-1}(t)\,dm = dv \Rightarrow \int_{v(t_0)}^{v(t_f)} dv = \int_{m_0}^{m_f} u m^{-1}\,dm \Rightarrow v(t_f) - v(t_0) = u\ln\left(\frac{m_f}{m_0}\right)$$

$= v(t_f) = v(t_0) + u\ln(m_f/m_0)$

Interpretation: there is no external force ⇒ momentum of system is constant

Initial momentum is zero: rocket is at rest. Then part of its mass is accelerated in the positive direction (the rocket) and part in the opposite direction (the fuel). The fuel velocity is constant. The two momentums cancel out.

$\Delta m_r v_f = (m + \Delta m_r)v_r$

We break time into Δt pieces. In each piece, this equation holds: the momentum of the rocket minus a small piece of fuel is equal to the momentum of the small piece of fuel.

$\underbrace{\Delta m_r v_r}_{<0} + \underbrace{\Delta m_r u}_{>0} = \underbrace{(m + \Delta m_r)}_{>0} v_r$

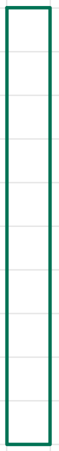
$\Delta m_r u = m v_r$

$|m| >> |\Delta m_r| \Rightarrow |u| >> |v_r|$

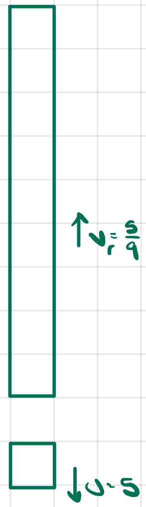
Note that $\Delta m_r < 0$

Assume Δm_r and u are always the same for each Δt. m is smaller on each Δt because Δm_r of fuel is ejected each Δt. v must be increasing to keep momentum of system constant.

$v_r = \frac{1}{m}\Delta m_r u$

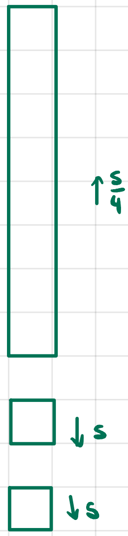


$t=0$
$m_r = 10$
$v_r = 0$
$m_r v_r = 0$



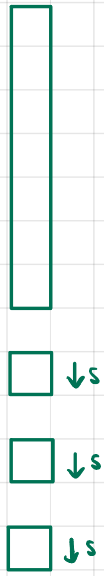
$t=1$
$m_r = 9$
$m_1 = 1$
$m_1 \cdot v_1 + m_r v_r = 0$
$1 \cdot (-5) + 9 \cdot v_r = 0$
$\Rightarrow v_r = \frac{5}{9}$

$V_{1g} = V_r + V_1 = \frac{5}{9} - 5$
$= -40/9$



$t=2$
$m_r = 8$
$m_1 = m_2 = 1$
$1(-5) + 1(-5) + 8 v_r = 0$
$v_r = 10/8 = 5/4$

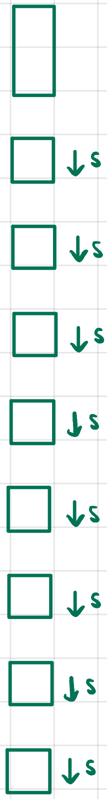
$V_{2g} = \frac{5}{4} - 5 = -\frac{15}{4}$
$= -3.75$



$t=3$

$m_r = 7$
$3 \cdot 1(-5) + 7 v_r = 0$
$v_r = \frac{15}{7}$

$V_{3g} = \frac{15}{7} - 5 = -\frac{20}{7}$



$t = 8$
$m_r = 2$
$8 \cdot 1(-5) + 2 v_r = 0$
$v_r = \frac{40}{2} = 20$

$t = 9$
$m_r = 1$
$v_r = \frac{9 \cdot 5}{1} = 45$

$v_r = \frac{vt}{(10-t)}$ $\Rightarrow$ v_r is exponential (decay) in t. v_r is proportional to v.

t	v_r
1	10/9
2	20/8
3	30/7
(...)	
8	80/2
9	90

Back to problem 1135:

$v(t) - v(t_0) = v \ln\left(\frac{m_t}{m_0}\right)$

$v(t_0) = 0 \Rightarrow v(t) = v \ln(m_t/m_0)$

$p(t) = m_t v_t = m_t v \ln(m_t/m_0)$

If we think of momentum as a function of m:

$p(m) = m v \ln(m/m_0)$

$\frac{dp}{dm} = v \ln(m/m_0) + \cancel{m} v \cdot \frac{\cancel{m_0}}{\cancel{m}} \cdot \frac{1}{\cancel{m_0}} = 0 \Rightarrow \ln(m/m_0) = -1 \Rightarrow m = \frac{m_0}{e}$

Recap

Rocket Eq. $\vec{F}^{ext} = m(t)\frac{d\vec{v}}{dt} - \vec{u}\frac{dm}{dt}$, $\quad u\frac{dm}{dt} = m\frac{dv}{dt}$, $\quad F^{Thrust} = u\frac{dm}{dt}$

No external force $\Rightarrow$ $v(m_f) = v(t_0) + u\ln(m_f/m_0)$

momentum $p(t) = m_f \cdot u\ln(m_f/m_0)$

$v'(m_f) = \frac{u}{m_f} \quad \Rightarrow v'(m_0) = \frac{u}{m_0}$

The relationship between $v(t_f)$ and m_f is logarithmic: $v(t_f) \sim -\ln(m_f)$. Because m_f is falling, v_r is increasing. Initially $(m_f = 0)$, the slope is $-1/2$: $\Delta v/\Delta m \sim -1/2$, but as far as $p = m_f\, u\ln(m_f/m_0)$ is concerned what matters to know if p increases or decreases when we change m_f is the relationship between $\frac{\Delta v}{v}$ and $\frac{\Delta m_f}{m_f}$.

As we decrease m_f by Δm_f, as long as $\frac{\Delta v}{v}$ is larger than $\frac{\Delta m_f}{m}$ $p(t)$ will increase. At a certain m_f, $\frac{\Delta m_f}{m}$ starts to be larger than $\frac{\Delta v}{v}$ so each time we decrease m_f the proportional change in m_f is larger than the proportional change in v and so p decreases.

---

time	momentum rocket	momentum fuel
t	$m\,v(t)$	$(m_f(t) + \Delta m_f)\,v(t)$
$t+\Delta t$	$m\,v(t+\Delta t)$	$\Delta m_f v_f = \Delta m_f(v(t)+u)$

$\Delta p = m\Delta v \Rightarrow \frac{dp}{dt} = m\frac{dv}{dt}$

$F_{f,r} = m\frac{dv}{dt}$

$F_{f,r}$ constant $\Rightarrow$ $dv = \frac{F}{m}dt \Rightarrow$

$v(t) = \frac{F}{m}t + v_0$

$\Delta p = \cancel{\Delta m_f v(t)} + \Delta m_f u - m_f(t)v(t) - \cancel{\Delta m_f v(t)}$

$\frac{\Delta p}{\Delta t} = \frac{\Delta m_f}{\Delta t}u - \frac{m_f(t)v(t)}{\Delta t}$

$\frac{\Delta p}{\Delta t} = \frac{\Delta m_f}{\Delta t}u - m_f(t)\cdot\left(-\frac{\Delta p}{\Delta t}\right)\cdot\frac{1}{m}\cdot\frac{1}{\Delta t}t$

$\frac{\Delta p}{\Delta t}\left(1 - \frac{m_f(t)}{m\Delta t}\cdot t\right) = \frac{\Delta m_f}{\Delta t}u$

1136

Derive rocket equation with gravity

$\vec{F}_{ext} = m(t)\frac{d\vec{v}}{dt} - \vec{u}\frac{dm}{dt}$

$-mg = m\frac{dv}{dt} - u\frac{dm}{dt}$ $\frac{dp}{dt} < 0$, the rocket + fuel system loses momentum because of gravity.

$-mg + u\frac{dm}{dt} = m\frac{dv}{dt}$ however, if we take the system as just the rocket, we can think of $u\frac{dm}{dt}$ as a force: the thrust force.
Thrust is larger than gravity if $\frac{dv}{dt} > 0$.

$-mg\,dt + u\,dm = m\,dv$

$-g\,dt + \frac{u}{m}dm = dv$

$\int_{t_0}^{t_f} -g\,dt + u\int_{m_0}^{m_f} m^{-1}dm = \int_{v(t_0)}^{v(t_f)} dv \Rightarrow v(t_f) = v(t_0) + u\ln\left(\frac{m_f}{m_0}\right) - g(t_f - t_0)$

a)

Assumptions

mass is ejected at a constant rate: $\frac{dm}{dt} = c$

(thrust force)
momentum of ejected mass $(dm \cdot u)$ equals force applied on entire rocket by gravity

at $t=0$, $a(0) = \frac{dv}{dt}(0) = 0$, $m = m_0 \Rightarrow$ (RE) $-m_0 g = m_0 \cdot 0 - u\frac{dm}{dt} \Rightarrow u\frac{dm}{dt} = m_0 g \Rightarrow dm = \frac{m_0 g}{u}dt$

$\Rightarrow m_f - m_0 = \frac{m_0 g}{u}t_f \Rightarrow m_f = m_0\left(1 + \frac{g}{u}t_f\right)$

sub eq. describing m_f into the diff. eq. of motion

$-mg = m\frac{dv}{dt} - u\frac{dm}{dt} \Rightarrow -m_0\left(1 - \frac{g}{u}t_f\right)g = m_0\left(1 - \frac{g}{u}t_f\right)a(t) - \cancel{u}\left(\frac{m_0 g}{\cancel{u}}\right)$

$\cancel{m_0}\left(1 - \frac{gt_f}{u}\right)a(t) = \frac{g^2 t_f \cancel{m_0}}{u} \Rightarrow a(t) = \frac{g^2 t_f}{u - gt_f}$

b) $v(t) = \int \frac{g^2 t}{u - gt}dt = -gt\ln(u-gt) + g\int \ln(u-gt)\,dt$

$w = g^2 t \quad dw = g^2 dt$

$dv = \frac{1}{u-gt}dt \quad v = -\frac{1}{g}\ln(u-gt)$

$\int \ln(u-gt)\,dt = t\ln(u-gt) + \int \frac{gt}{u-gt}dt$

$w = \ln(u-gt) \quad dw = \frac{-g}{u-gt}dt$

$dv = dt \quad v = t$

$\int \frac{gt}{u-gt}dt = \int \frac{(-dm)(u-m)/g}{m} = \int \frac{m-u}{gm}dm = \int\left(\frac{1}{g} - \frac{u}{gm}\right)dm = \frac{m}{g} - \frac{u}{g}\ln(m) + C$

$m = u - gt \quad dm = -g\,dt$

$t = \frac{u-m}{g}$

$= \frac{u-gt}{g} - \frac{u}{g}\ln(u-gt) + C = \frac{u}{g} - t - \frac{u}{g}\ln(u-gt) = \frac{u}{g}\left(1 - \ln(u-gt)\right) - t =$