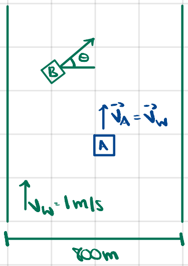
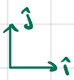
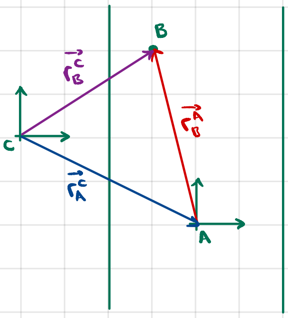
Pset 6

Problem 1




a)




$\vec{r}_B^{\,c} = \vec{r}_A^{\,c} + \vec{r}_B^{\,A}$

$\vec{v}_B^{\,c} = \vec{v}_A^{\,c} + \vec{v}_B^{\,A}$

$\vec{v}_B^{\,A} = \langle 0.5, 0 \rangle$

$\vec{v}_A^{\,c} = \langle 0, 1 \rangle$

$\hat{j}$ direction

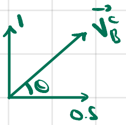
$\Rightarrow v_{B,y}^{c} = v_{A,y}^{c} + 0 \Rightarrow v_{B,y}^{c} = v_{A,y}^{c} = 1$

rel. to A, B isn't moving in the $\hat{j}$ dir $\Rightarrow$ A and B have same velocity in $\hat{j}$ dir

$\hat{i}$ direction

$v_{B,x}^{c} = v_{A,x}^{c} + v_{B,x}^{A} = 0 + 0.5 \Rightarrow v_{B,x}^{c} = 0.5$

$\Rightarrow \vec{v}_B^{\,c} = \langle 0.5, 1 \rangle$




$\tan\theta = 2 \Rightarrow \theta = \tan^{-1} 2 \approx 1.10\text{ rad} \approx 63°$

b) $\vec{r}_B^{\,c} = \int_0^t \vec{v}_B^{\,c}\, dt = \langle 0.5t, t \rangle$

Alternatively, $\tan\theta = 2 = \dfrac{d_y}{800} \Rightarrow d_y = 1600\,m$

$0.5t = 800 \Rightarrow t = 1600\,s \Rightarrow r_{B,y}^{c}(1600) = 1600\,m$

c) 1600 s

## Problem 2

system: two astronauts, ball

A


$P_A$ $\quad m_2 v_2 + (m_1 + m_3) v_1$

B


$P_B$ $\quad m_2 v_2 + m_3 v_3 + m_1 v_{1,f}$

$v_3 = v_1 - u$

$\Rightarrow m_2 v_2 + m_3 v_1 - m_3 u + m_1 v_{1,f}$

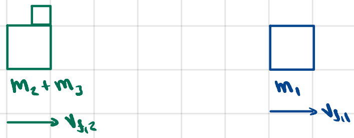
$P_A = P_B \Rightarrow \cancel{m_2 v_2} + (m_1 + \cancel{m_3}) v_1 = \cancel{m_2 v_2} + \cancel{m_3 v_1} - m_3 u + m_1 v_{1,f}$

$m_1 v_1 = -m_3 u + m_1 v_{1,f}$

a) $\quad v_{1,f} = \dfrac{m_1 v_1 + m_3 u}{m_1} \Rightarrow v_{1,f} = v_1 + \dfrac{m_3}{m_1} u$

b)

C




$p_C = (m_2 + m_3) v_{f,2} + \cancel{m_1 v_{f,1}} = m_2 v_2 + m_3 v_1 - m_3 u + \cancel{m_1 v_{1,f}}$

$v_{f,2} = v_1 + \dfrac{m_3}{m_1} u$

$\Rightarrow (m_2 + m_3) v_1 + \dfrac{(m_2 + m_3) m_3}{m_1} u = m_2 v_2 + m_3 v_1 - m_3 u$

$u \left[ \dfrac{(m_2 + m_3) m_3 + m_1 m_3}{m_1} \right] = m_2 v_2 + \cancel{m_3 v_1} - m_2 v_1 - \cancel{m_3 v_1}$

$u \left[ \dfrac{m_3 (m_1 + m_2 + m_3)}{m_1} \right] = m_2 v_2 - m_2 v_1$

$u = \dfrac{m_1 (m_2 v_2 - m_2 v_1)}{m_3 (m_1 + m_2 + m_3)} = \quad u = \dfrac{m_1 m_2 (v_2 - v_1)}{m_3 (m_1 + m_2 + m_3)}$

Problem 3

zero gravitational field

$m_{r,i} = m_{r,d} + m_{r,f}$

$m_{r,i} = 2.81 \cdot 10^7$ kg

$m_{f,i} = 2.46 \cdot 10^7$ kg

$m_{r,d} = 0.35 \cdot 10^7$ kg

$u = 3000$ m/s

Fuel burned at constant rate $\Rightarrow \frac{dm_{r,i}}{dt} =$ constant

Burn time = 510 s

a) model of change in momentum between $t$ and $t+\Delta t$

$t$: $m(t)v(t)$

$t+\Delta t$ $(m(t)+\Delta m)v(t+\Delta t) - \Delta m(v(t+\Delta t)+u)$

$\Delta p = 0 = (m(t)+\cancel{\Delta m})v(t+\Delta t) - \Delta m(v(\cancel{t+\Delta t})+u) - m(t)v(t)$

$\frac{\Delta p}{\Delta t} = m(t)\frac{\Delta v}{\Delta t} - \frac{\Delta m}{\Delta t}u = 0 \Rightarrow m(t)\frac{\Delta v}{\Delta t} = \frac{\Delta m}{\Delta t}u$

$\Rightarrow v(t) = v(0) - u\ln\left(\frac{m}{m_0}\right)$

After 510 s, mass is $m_{r,d}$.

$v(510) = -3000\ln\left(\frac{m_{r,d}}{m_{r,i}}\right) = -3000\ln\left(\frac{0.35}{2.81}\right) = 6249.019$ m/s

b) Two burn stages

stage 1

$u = 3000$ m/s

$m_{f,1,i} = 2.03 \cdot 10^7$ kg

burn time = 150 s

After stage 1, total rocket mass is $0.78 \cdot 10^7$ kg

$v(t) - v(0) = -u\ln(m_f/m_0) = -3000\ln\left(\frac{0.78}{2.81}\right) = 3844.94$ m/s

c) Discarded parts have mass $1.4 \cdot 10^6$ kg

initial stage two dry mass = $2.1 \cdot 10^6$ kg

" " " fuel mass = $4.3 \cdot 10^6$ kg

burn time = 360 s

$v(t) - v(0) = -3000\ln\left(\frac{2.1 \cdot 10^6}{6.4 \cdot 10^6}\right) = 3343.08$ m/s

d) Final speed = 7188.02 m/s

e) In a) all the fuel $(2.46 \cdot 10^7$ kg$)$ was used to accelerate a dry mass of $0.35 \cdot 10^7$ kg.

In the two stage setup, $2.03 \cdot 10^7$ kg of fuel was used to accelerate $0.35 \cdot 10^7$ of dry mass, then $0.43 \cdot 10^7$ kg of fuel was used to accelerate a substantially lighter $0.21 \cdot 10^7$ kg of dry mass.

$\Rightarrow$ more acceleration $\Rightarrow \uparrow$ final velocity

# Problem 4 - Falling Drop

raindrop mass $m_0$


$$\frac{dm_r}{dt} = k m_r v_r$$

$m_r$ : instantaneous mass

$k \ [m^{-1}]$

a)

| time | momentum |
|---|---|
| $t$ | $m_r(t)v_r(t)$ |
| $t+\Delta t$ | $(m_r(t)+\Delta m_r)v_r(t+\Delta t)$ |

$$\Delta p = (m_r(t)+\Delta m_r)v_r(t+\Delta t) - m_r(t)v_r(t)$$
$$= m_r(t)\Delta v_r + \Delta m_r v_r(t+\Delta t)$$

$$\frac{\Delta p}{\Delta t} = m_r(t)\frac{\Delta v_r}{\Delta t} + \frac{\Delta m_r}{\Delta t}v_r(t+\Delta t)$$

$$F^{ext} = \frac{dp}{dt} = m_r(t)\frac{dv_r}{dt} + \frac{dm_r}{dt}v_r(t)$$

$$m_r(t)g = m_r(t)\frac{dv_r}{dt} + \frac{dm_r}{dt}v_r(t)$$

$$\Rightarrow \cancel{m_r(t)}g = \cancel{m_r(t)}\frac{dv_r}{dt} + k\cancel{m_r(t)}v_r^2(t) \Rightarrow \frac{dv_r}{dt} = g - kv_r^2(t)$$

$$\frac{dm_r}{dt} = k m_r v_r$$

$$\Rightarrow \frac{1}{g-kv_r^2}dv_r = dt$$

$$= \int \frac{1}{k(g/k - v_r^2)}dv_r = \frac{1}{k}\int\frac{(g/k)^{1/2}\cos\theta d\theta}{(g/k)(1-\sin^2\theta)} = \frac{1}{k}\left(\frac{g}{k}\right)^{1/2}\left(\frac{k}{g}\right)\int\frac{\cos\theta}{1-\sin^2\theta}d\theta = \left(\frac{k}{g}\right)^{1/2}\cdot\frac{1}{k}\ln(1-\sin^2\theta)\left(-\frac{1}{2}\right) = -\frac{\ln(1-\sin^2\theta)}{2\sqrt{gk}} = -\frac{\ln(1-(k/g)^{1/2}v_r)}{2\sqrt{gk}}$$

$$v_r = \sqrt{\frac{g}{k}}\sin\theta$$
$$dv_r = (g/k)^{1/2}\cos\theta d\theta$$

$$\Rightarrow v_r(t) = -\frac{e^{-2c\sqrt{gk}-2t\sqrt{gk}}-1}{\sqrt{k/g}} \Rightarrow v_r(t) = -\frac{e^{-2t\sqrt{gk}}-1}{\sqrt{\frac{k}{g}}}$$

$$v_r(0) = 0 \Rightarrow C = 0$$

Interpretation: mass of raindrop increases exponentially. At time $t$ mass is $m(t)$, gravitational force is $m(t)g$. At time $t+\Delta t$ mass is $m+\Delta m$, where $\Delta m$ is proportional to $km(t)v(t)$. $\Delta m$ is accelerated to $v_r(t+\Delta t)$.

F increased by $\Delta m g$

b) $$\lim_{t\to\infty} v_r(t) = \lim_{t\to\infty} -\frac{\frac{1}{e^{2t\sqrt{gk}}}-1}{\sqrt{k/g}} = \frac{1}{\sqrt{k/g}} = \sqrt{\frac{g}{k}}$$

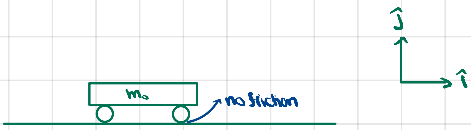
## Example

$g = 9.8 m/s^2$

$k = 0.1 kg^{-1}$

$$v(t) = -9.89 e^{-1.979t} + 9.89$$

# Problem 5 - Moving Vehicle and Falling Rain




* in the $\hat{j}$ direction

| | |
|---|---|
| $t$: | $\Delta m_r v_b$ |
| $t+\Delta t$ | $0$ |

$$\frac{\Delta p}{\Delta t} = -\frac{\Delta m_r}{\Delta t} v_b = -b v_b \Rightarrow \frac{dp}{dt} = -b v_b = F_{ext}$$

Force exerted by the cart on the rain to stop the rain in $\hat{j}$ dir.

| time | momentum |
|---|---|
| $t$ | $m_0 v(t)$ |
| $t+\Delta t$ | $(m_0 + \Delta m_r) v(t+\Delta t) = (m_0 + b\Delta t) v(t+\Delta t)$ |

a)

$$\Delta p = m_0 \Delta v + b \Delta t \, v(t+\Delta t)$$

$$\frac{\Delta p}{\Delta t} = m_0 \frac{\Delta v}{\Delta t} + b v(t+\Delta t)$$

$$\frac{dp}{dt} = m_0 \frac{dv}{dt} + b v(t) = 0$$

very similar eq. to the one from problem 4. There, dm/dt modeled exponential growth. Here dm/dt is constant.
also, there are no forces external to cart + rain in the x direction here.

exponential decay

$$\frac{dv}{dt} = -\frac{b}{m_0} v(t) \Rightarrow \frac{1}{v} dv = -\frac{b}{m_0} dt$$

$$\ln(v_f / v_0) = -\frac{b}{m_0}(t_f - t_0)$$

b) $v(t) = v_0 + e^{-\frac{b}{m_0} t}$