8.6 Drag Forces in Fluids

liquid or gas

→ solid object moving through fluid experiences a resistive force called the **drag force**, opposing its motion

complicated, depends on properties of object (speed, size, shape) and fluid (density, viscosity, compressibility)

→ for objects moving rapidly through air the drag force can be modelled as

cross sectional area

$$F_{drag} = \frac{1}{2} C_D A \rho v^2$$

drag coeff., dimensionless number, property of the object

* this model does not apply to all fluids, e.g. oil, honey, water, which all have different viscosities

* at low speeds, we model air drag differently: linear relationship w/ speed, proportional to viscosity η
when the object is a sphere of radius R, we have: $\vec{F}_{drag} = -6\pi\eta R\vec{v}$ (Stokes')

Example 8.5 Drag Force at Low Speeds

i) $mg - F_{drag} = m\dfrac{dv(t)}{dt}$

$\Rightarrow mg - 6\pi\eta R v(t) = m\dfrac{dv(t)}{dt} \Rightarrow g - \dfrac{6\pi\eta R}{m} v(t) = \dfrac{dv(t)}{dt} \Rightarrow g - \gamma v(t) = \dfrac{dv(t)}{dt}$

$\Rightarrow -\gamma\left(v(t) - \dfrac{g}{\gamma}\right) = \dfrac{dv(t)}{dt} \Rightarrow -\gamma dt = \dfrac{dv(t)}{v(t) - g/\gamma}$

$$\int_{v(0)=0}^{v(t)} \frac{dv}{v - g/\gamma} = \int_0^t -\gamma dt$$

$$\ln(v - g/\gamma)\Big|_0^{v(t)} = -\gamma t$$

$$\ln(v(t) - g/\gamma) - \ln(-g/\gamma) = -\gamma t$$

$$\ln\left[\frac{v(t) - g/\gamma}{-g/\gamma}\right] = -\gamma t$$

$$\frac{v(t) - g/\gamma}{-g/\gamma} = e^{-\gamma t} \Rightarrow v(t) = -\frac{g}{\gamma}e^{-\gamma t} + g/\gamma$$

$$\Rightarrow v(t) = \frac{g}{\gamma}(1 - e^{-\gamma t})$$

ii) $v(t \to \infty) = \dfrac{g}{\gamma} = \dfrac{gm}{6\pi\eta}$

Note $\dfrac{dv}{dt} = \dfrac{g}{\gamma}(-e^{-\gamma t})-\gamma = ge^{-\gamma t}$

$t \to \infty \Rightarrow v = g/\gamma,\ dv/dt \to 0$

iii) $t \to \infty \Rightarrow dv/dt = 0 \Rightarrow F_{drag} = m\cancel{g} = 6\pi\eta R v(t=\infty) = 6\pi\eta R \dfrac{\cancel{g}}{\gamma} \Rightarrow \eta = \dfrac{m\gamma}{6\pi R}$

or $\eta = \dfrac{m}{6\pi R v_{term}}$

iv) $dv(t) = [g - \gamma v(t)]dt$

$$\int_{v(0)}^{v(t)} dv(t) = \int_0^t g dt - \gamma\int_0^t v(t) dt$$

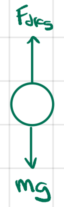
$$v(t) - v(0) = gt - \gamma y(t)$$

$v(0) = 0 \Rightarrow v(t) = gt - \gamma y(t) \Rightarrow y(t) = \dfrac{gt - v(t)}{\gamma} = \dfrac{gt}{\gamma} - \dfrac{g}{\gamma^2}(1 - e^{-\gamma t})$

Example 8.6 Drag Forces at High Speeds

setup:

object: mass m, $\vec{v}(0) = \vec{v}_0$, cross-sectional area A perpendicular to motion



fluid: density ρ

$F_{drag} = \frac{1}{2} C_D A \rho v^2$

$mg - F_{drag} = m\frac{dv}{dt}$

$mg - \frac{1}{2} C_D A \rho v^2 = m\frac{dv}{dt} \Rightarrow g - \frac{C_D A \rho}{2m} v^2 = \frac{dv}{dt}$ $\quad \gamma = \frac{C_D A \rho}{2m}$

$\Rightarrow g - \gamma v^2 = dv/dt \Rightarrow -\gamma\left(v^2 - \frac{g}{\gamma}\right) = \frac{dv}{dt} \Rightarrow -\gamma\, dt = \frac{1}{v^2 - g/\gamma} dv \Rightarrow \int_0^t -\gamma\, dt = \int_{v_0}^{v(t)} \frac{1}{v^2 - g/\gamma} dv$

$\Rightarrow -\int_0^t \gamma\, dt = -\frac{\gamma}{g} \underbrace{\int_{v_0}^{v(t)} \frac{1}{1 - \frac{\gamma v^2}{g}} dv}_{\text{change of variables}} \Rightarrow \int_0^t g\, dt = \int_{v_0}^{v(t)} \frac{1}{1 - \frac{\gamma v^2}{g}} dv$

$u = (\gamma/g)^{1/2} v \Rightarrow du = (\gamma/g)^{1/2} dv, \; u(0) = 0, \; u(t) = (\gamma/g)^{1/2} v(t)$

$$\int_{u(0)}^{u(t)} \frac{1}{1 - \frac{\gamma v^2}{g}} dv = (g/\gamma)^{1/2} \int_0^{u(t)} \frac{du}{1 - u^2} = (g/\gamma)^{1/2} \frac{1}{2} \ln\left[\frac{1 + u(t)}{1 - u(t)}\right]$$

$$\Rightarrow gt = \left[\frac{g}{\gamma}\right]^{1/2} \cdot \frac{1}{2} \ln\left[\frac{1 + (\gamma/g)^{1/2} v(t)}{1 + (\gamma/g)^{1/2} v(t)}\right]$$

$$2t\sqrt{g\gamma} = \ln\left[\frac{1 + (\gamma/g)^{1/2} v(t)}{1 + (\gamma/g)^{1/2} v(t)}\right] \Rightarrow e^{2t\sqrt{g\gamma}} = \frac{1 + (\gamma/g)^{1/2} v(t)}{1 + (\gamma/g)^{1/2} v(t)}$$

$$\Rightarrow e^{2t\sqrt{g\gamma}} + e^{2t\sqrt{g\gamma}} (\gamma/g)^{1/2} v(t) = 1 + (\gamma/g)^{1/2} v(t)$$

$$v(t)(\gamma/g)^{1/2}\left[1 - e^{2t\sqrt{g\gamma}}\right] = e^{2t\sqrt{g\gamma}} - 1 \Rightarrow v(t) = \left[\frac{g}{\gamma}\right]^{1/2} \frac{e^{2t\sqrt{g\gamma}} - 1}{1 - e^{2t\sqrt{g\gamma}}}$$

Here is a case with an initial horizontal v_0 through a fluid, no gravity.

$F_{drag} = -\frac{1}{2} C_D A \rho v^2 = m\frac{dv}{dt} \Rightarrow -\beta v^2 = m\, dv/dt \Rightarrow -\frac{\beta}{m} dt = \frac{1}{v^2} dv \Rightarrow \int_0^t -\frac{\beta}{m} dt = \int_{v_0}^{v(t)} v^{-2} dv \Rightarrow -\frac{\beta}{m} t = -v^{-1}\Big|_{v_0}^{v(t)} \Rightarrow -\frac{\beta t}{m} = -\left[v(t)^{-1} - v_0^{-1}\right]$

$\frac{1}{v(t)} = \frac{\beta t}{m} + \frac{1}{v_0} = \frac{v_0 \beta t + m}{v_0 m} \Rightarrow v(t) = \frac{v_0 m}{v_0 \beta t + m}$