

$$\vec{\tau}_{S,i} = z_i \hat{h} + r_i \hat{r}$$

$$\vec{\tau}_{S,i} = \vec{\tau}_{S,i} \times \vec{F}_i$$

$$= (z_i \hat{h} + r_i \hat{r}) \times (F_{r,i} \hat{r} + F_{\theta,i} \hat{\theta} + F_{z,i} \hat{z})$$

$$= z_i F_{r,i} (-\hat{\theta}) + z_i F_{\theta,i} (-\hat{r}) + r_i F_{\theta,i} \hat{h} + r_i F_{z,i} (-\hat{\theta})$$

Fixed axis rotation  $\Rightarrow$  resultant torque on  $\Delta m_i$  is in  $\hat{\theta}$  direction  
there must be other forces canceling the  $F_{r,i}$  and  $F_{z,i}$  components.

$$(\vec{\tau}_{S,i})_z = r_i F_{\theta,i} \hat{h}$$

$$F_{\theta,i} = \Delta m_i \cdot a = \Delta m_i \cdot r_i \alpha_z$$

$$(\vec{\tau}_{S,i})_z = \Delta m_i r_i^2 \alpha_z$$

$$(\vec{\tau}_S)_z = \lim_{n \rightarrow \infty} \sum_{i=1}^n (\vec{\tau}_{S,i})_z = I_S \alpha_z$$

$$\vec{\tau}_S = \vec{\tau}_S^{\text{ext}} + \vec{\tau}_S^{\text{int}}$$

$$\vec{\tau}_S^{\text{ext}} = \sum_{i=1}^n \vec{\tau}_{S,i}^{\text{ext}} = \sum \vec{\tau}_{S,i} \times \vec{F}_i^{\text{ext}}$$

$$\vec{\tau}_S^{\text{int}} = \sum_{i=1}^n \sum_{j \neq i} \vec{\tau}_{S,i}^{\text{int}} \times \vec{F}_{S,j}^{\text{int}}$$

$$3^{\text{rd}} \text{ law} \Rightarrow \vec{F}_{S,i} = -\vec{F}_{i,S}$$

$$\Rightarrow \vec{\tau}_{S,i}^{\text{int}} + \vec{\tau}_{S,j}^{\text{int}} = \vec{\tau}_{S,i} \times \vec{F}_{S,j}^{\text{int}} + \vec{\tau}_{S,j} \times \vec{F}_{S,i}^{\text{int}} \\ = (\vec{\tau}_{S,i} - \vec{\tau}_{S,j}) \times \vec{F}_{S,j}^{\text{int}}$$

$= 0$  assuming that the pair of 3<sup>rd</sup> law forces act along the line joining the two particles. This is the 3<sup>rd</sup> law without extra assumption.

$$\Rightarrow \vec{\tau}_S = \vec{\tau}_S^{\text{ext}}$$

$$\Rightarrow (\vec{\tau}_S^{\text{ext}})_z = I_S \alpha_z \quad \text{similar to } \vec{F} = m \vec{a}$$

## 17.4.2 Torque Acts at Center of Gravity

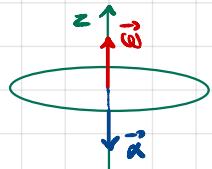
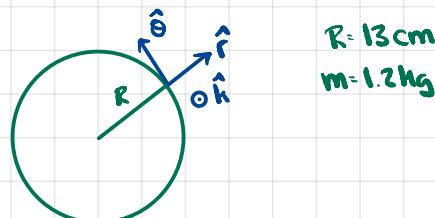
Setup: rigid body, N particles, static equilibrium  
coord. system origin O

$$\vec{\tau}_O = \sum_{i=1}^N \vec{\tau}_{O,i} = \sum \vec{r}_i^* \times \vec{F}_{O,i} = \sum \vec{r}_i^* \times m_i \vec{g} = \vec{\tau}_{CM}$$

$$\vec{\tau}_O = (\sum m_i \vec{r}_i^*) \times \vec{g} = M_T \cdot \vec{r}_{CM} \times \vec{g} = \vec{r}_{CM} \times M_T \vec{g}$$

Torque due to gravity on each point-like particle is equivalent to torque due to gravity on center of mass. Note the implicit assumption that  $\vec{g}$  is the same at every point in the rigid body.

### Example 17.9



$$I_S = 1.01 \cdot 10^{-2} \text{ kg} \cdot \text{m}^2$$

$$\omega_0 = 33 \text{ rpm} = 0.55 \text{ Hz}$$

The turntable is spinning with  $\omega_0 = \omega_0 \cdot 2\pi = 1.10\pi$  rad/s  
The motor is turned off and a torque is applied by friction.

$$\vec{\tau}_{S,f} = I_S \alpha_z \hat{h}$$

$$\vec{a} = \alpha_z \hat{h}$$

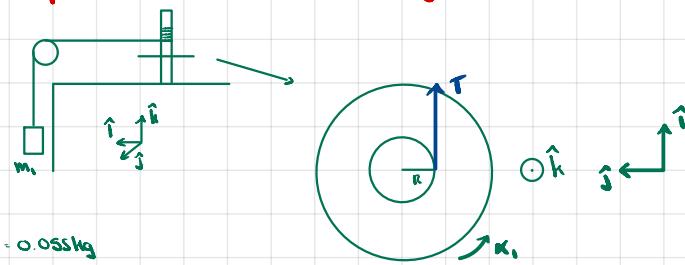
$$\vec{\omega} = (\omega_0 + \alpha_z t) \hat{h}$$

$$\vec{\theta} = (\omega_0 t + \alpha_z t^2 / 2) \hat{h}$$

$$\omega_0 + 8\alpha_z = 0 \Rightarrow \alpha_z = -\frac{\omega_0}{8} = -\frac{1.10\pi}{8}$$

$$\Rightarrow \vec{\tau}_{S,f} = 1.01 \cdot 10^{-2} \frac{(-1.10\pi)}{8} \hat{h}$$

## Ex II. Experimental Method for Determining Moment of Inertia



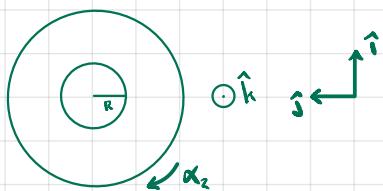
$$(-R\hat{j}) \times (T\hat{i}) + \vec{\tau}_f = I_s \alpha_1 \hat{k}, \quad \tau_f = \tau_f \hat{h}$$

$$(m_1 g - T)\hat{i} = m_1 a_1 \hat{i} = m_1 R \alpha_1 \hat{i}$$

$$\Rightarrow RT + \tau_f = I_s \alpha_1$$

$$m_1 g - T = m_1 R \alpha_1$$

unknowns:  $T, I_s, \tau_f$



$$\vec{\tau}_f = I_s \vec{\alpha}_2$$

$$\vec{\alpha}_2 = -\alpha_2 \hat{k}$$

$$\vec{\omega} = (\omega_0 - \alpha_2 t) \hat{h}$$

$$\omega = 95 - \alpha_2 t$$

$$0 = 95 - \alpha_2 \cdot 2.82 \Rightarrow \alpha_2 = 33.68 \text{ rad/s}^2$$

$$\Rightarrow \vec{\tau}_f = -I_s \cdot 33.68 \hat{k}$$

$$RT + \tau_f = I_s \alpha_1$$

$$m_1 g - T = m_1 R \alpha_1 \Rightarrow T = m_1 (g - R \alpha_1)$$

$$\tau_f = -I_s \cdot 33.68$$

$$R m_1 (g - R \alpha_1) - 33.68 I_s = I_s \alpha_1$$

$$I_s = \frac{R m_1 (g - R \alpha_1)}{\alpha_1 + 33.68} = \frac{0.01277 \cdot 0.055 (9.8 - 0.01277 \cdot 80.50)}{80.50 + 33.68} = 5.3 \cdot 10^{-5} \text{ kg} \cdot \text{m}^2$$

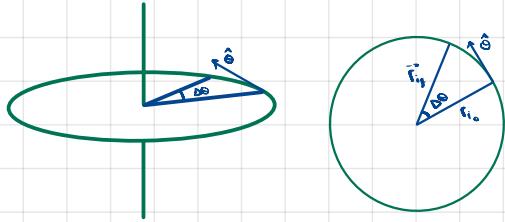
$$\alpha_1 \approx \frac{95}{1.18} = 80.50 \text{ rad/s}^2$$

## 17.5.1 Rotational Work

→ rigid body rotating about axis

→ each  $\Delta m_i$  moves in circle of radius ( $r_{s,i}$ ) undergoing  $\Delta\theta$  angular displacement under action of  $\vec{F}_{o,i} = F_{o,i}\hat{\theta}$

$$\rightarrow \Delta \vec{r}_{s,i} = r_{s,i} \Delta\theta \hat{\theta}$$



→ work done by tangential force

$$\Delta W_{rot,i} = \vec{F}_{o,i} \cdot \vec{D}\vec{r}_{s,i} \approx (F_{o,i}\hat{\theta}) \cdot (r_{s,i} \Delta\theta \hat{\theta}) = r_{s,i} F_{o,i} \Delta\theta$$

For a fixed  $\vec{F}$  acting on  $\Delta m_i$  undergoing fixed axial rotation, the forces  $\vec{F}_{s,i}$  being in dir.  $\hat{n}$ , are perp. defined

$$(\vec{F}_{s,i})_z = r_{s,i} F_{o,i} \hat{n}$$

$$\Rightarrow \Delta W_{rot,i} = (\vec{F}_{s,i})_z \Delta\theta$$

$$W_{rot} = \sum \Delta W_{rot,i} = (I_s)_z \Delta\theta$$

$$\Delta\theta \rightarrow d\theta, \Delta W_{rot} \rightarrow dW_{rot} \Rightarrow dW_{rot} = \vec{T}_{s,z} d\theta$$

$$W_{rot} = \int dW_{rot} = \int_{\theta_i}^{\theta_f} T_{s,z} d\theta$$

Work is just force times displacement. Displacement is now related to an angular displacement. We find the resulting expression contains torque on the  $i^{th}$  small mass.

Summing over all small masses, work done on the rigid body is the integral of torque over the angular displacement.

## 17.5.2 Rotational Work-Kinetic Energy theorem

$$dW_{rot} = T_{s,z} d\theta = I_s \alpha_z d\theta$$

$$\alpha_z = \frac{d\omega_z}{dt} \quad \omega_z = \frac{d\theta}{dt}$$