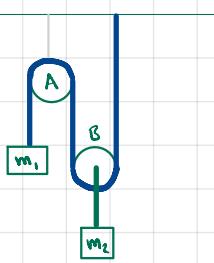
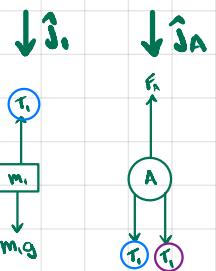


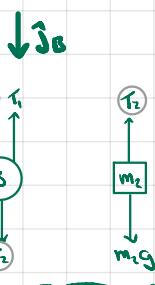
12.1 Pulley Problems



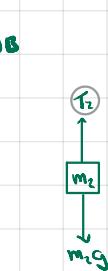
\$\downarrow \hat{j}_1\$



\$\downarrow \hat{j}_A\$



\$\downarrow \hat{j}_B\$



* Assumptions

$$m_A = m_B = 0$$

rope is slipping, massless \$\Rightarrow T\$ uniform

$$m_1g - T_1 = m_1a_1$$

$$m_2g - 2T_1 = m_2a_2$$

Three unknowns, two equations

$$(T_1, a_1, a_2)$$

we need an extra equation, a constraint eq.



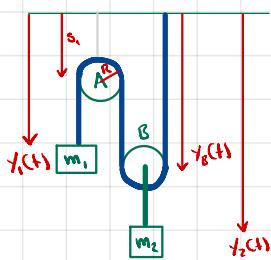
\$\uparrow 2T_1\$



\$\downarrow m_2g\$

Let's look at three L-C-E's to obtain constraint equations.

1) Analytically,



$$\frac{d^2s_1(t)}{dt^2} = a_1, \quad \frac{d^2s_2(t)}{dt^2} = a_2$$

s_1 : length of the string = constant

$$= s_1(t) - s_1 + \pi R + (s_2(t) - s_1) + \pi R + s_2(t)$$

$$s_2 = \text{constant} = s_2(t) - s_2(t) \Rightarrow \frac{d^2s_2(t)}{dt^2} - \frac{d^2s_1(t)}{dt^2} = a_2(t) - a_1(t) = 0$$

$$\Rightarrow a_2(t) = a_1(t)$$

$$\frac{d^2s_1}{dt^2} = 0 = \frac{d^2s_1(t)}{dt^2} + 2 \frac{d^2s_2(t)}{dt^2} = 0 \Rightarrow a_1(t) = -2a_2(t) = -2a_1(t)$$

or now have

$$a_1 = -2a_2$$

$$m_1g - T_1 = m_1a_1 \Rightarrow T_1 = m_1(g + 2a_1)$$

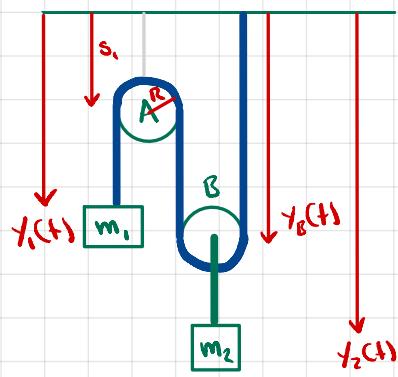
$$m_2g - 2T_1 = m_2a_2 \Rightarrow T_1 = m_2(g - a_2)$$

$$\Rightarrow m_1g + 2m_1a_2 = m_2g - m_2a_2 \Rightarrow a_2(2m_1 + m_2) = g(m_1 + m_2) \Rightarrow a_2 = \frac{g(m_1 + m_2)}{2m_1 + m_2}$$

$$\Rightarrow a_1 = -\frac{2g(m_1 + m_2)}{2m_1 + m_2}$$

$$T_1 = m_1(g - a_1) = m_1 \left[\frac{2m_1g + gm_2 + 2m_1g + 2m_2g}{2m_1 + m_2} \right] = \frac{m_1(4m_1g + 3m_2g)}{2m_1 + m_2}$$

2) Visual Argument



m_2 and B move together as a unit.

If we have a displacement $\Delta y_2 = \Delta y_B$, the rope on either side of B also gets displaced by Δy_2 , so the total displacement of rope is $2\Delta y_2$.

m_1 is displaced Δy_1 due to the displacement of the rope

$$\Rightarrow \Delta y_1 = -2\Delta y_2.$$

3) Differentials

$$L_1 = y_1(t) + 2y_B(t) + 2\pi r - 2s,$$

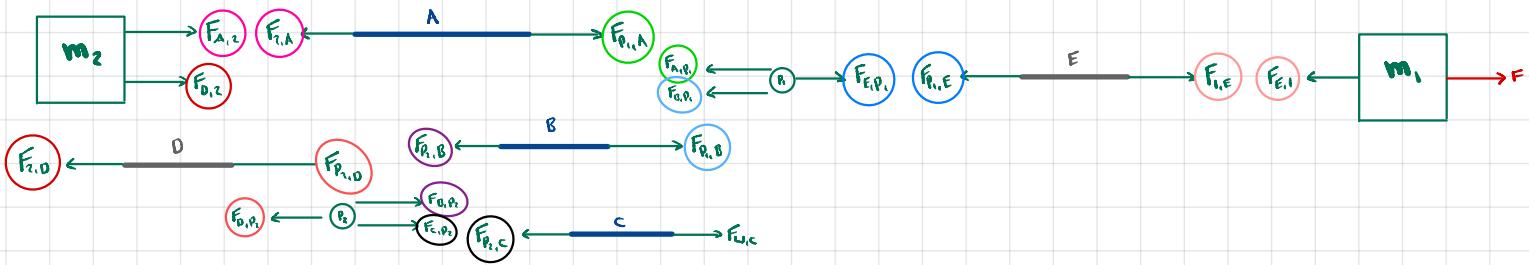
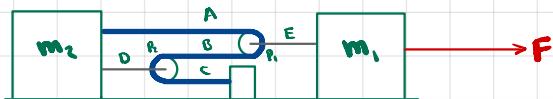
$$L_2 = y_2(t) - y_B(t)$$

$$\delta L_2 - \delta L_1 - \delta y_B = 0 \Rightarrow \delta y_2 = \delta y_B$$

$$\delta L_1 = \delta y_1(t) + 2\delta y_B(t) = 0 \Rightarrow \delta y_1(t) = -2\delta y_B(t) \Rightarrow \frac{\delta y_1}{\delta t} = -2 \frac{\delta y_B}{\delta t} \Rightarrow \frac{d^2 y_1}{dt^2} = -2 \frac{d^2 y_B}{dt^2}$$

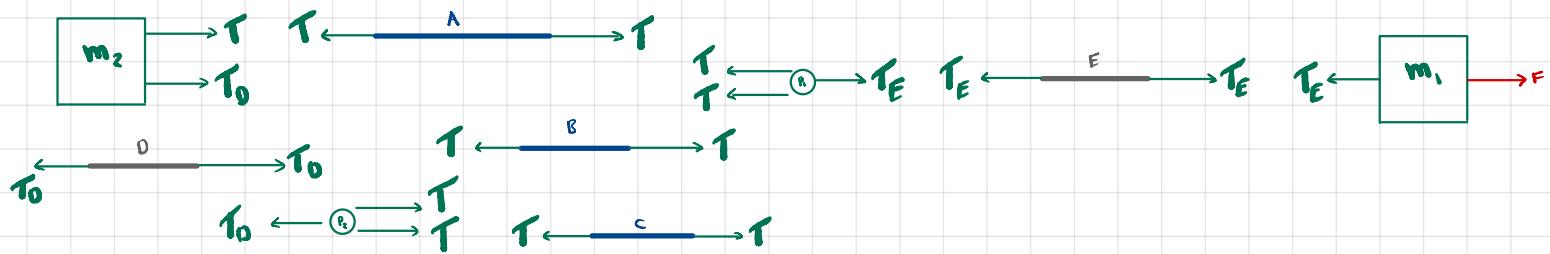
$$\Rightarrow a_1 = -2a_2$$

12.5 Worked Example - 2 Blocks, 2 Pulleys



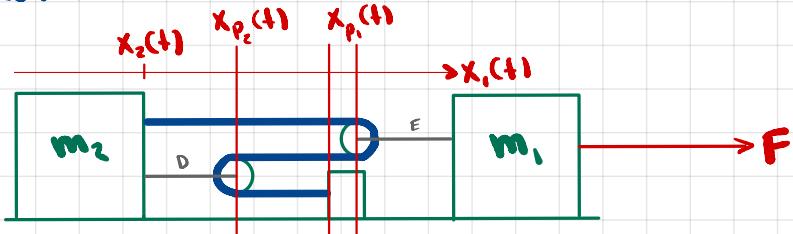
Note we assume the ropes are massless.

For ex, 3rd law: $F_{A,2} = F_{2,A}$, 2nd law: $F_{P,A} - F_{2,A} = 0 \Rightarrow F_{P,A} = F_{2,A}$



$$\begin{aligned} 2T &\leftarrow R \xrightarrow{E} m_1 \rightarrow F \\ m_2 &\xrightarrow{D} R \xrightarrow{T_0} 3T \end{aligned} \quad \left. \begin{aligned} F - 2T &= m_1 a_1 \\ 3T &= m_2 a_2 \end{aligned} \right\} 2 \text{ eq. 3 unknowns } T, a_1, a_2$$

Constraints



$$l_0 = \text{constant} = x_{p_2}(t) - x_2(t) \Rightarrow a_{p_2} = a_2$$

$$l_E = \text{constant} = x_1(t) - x_{p_1}(t) \Rightarrow a_1 = a_{p_1}$$

$$l = \text{constant} = x_{p_1}(t) - x_2(t) + \pi R + x_{p_1}(t) - x_{p_2}(t) + \pi R + w - x_{p_2}(t)$$

$$= 2x_{p_1}(t) - 2x_{p_2}(t) - x_2(t) + 2\pi R + w = 2x_1(t) + C_1 - 2x_2(t) + C_2 - x_2(t) + 2\pi R + w$$

$$= 2x_1(t) - 3x_2(t) + 2\pi R + w + C$$

$$\Rightarrow 2a_1 - 3a_2 = 0 \Rightarrow a_1 = \frac{3}{2}a_2 \Rightarrow \text{extra equation} \rightarrow 3 \text{ eq. 3 unknowns}$$

$$F - 2T = m_1 a_1 \Rightarrow F - 2T = m_1 \cdot \frac{3a_2}{2}$$

$$3T - m_2 a_2 \Rightarrow T = \frac{m_2 a_2}{3}$$

$$a_1 = \frac{3}{2} a_2$$

$$\Rightarrow F = \frac{2}{3} m_2 a_2 + \frac{3}{2} m_1 a_2 \Rightarrow 6F = 4m_2 a_2 + 9m_1 a_2$$

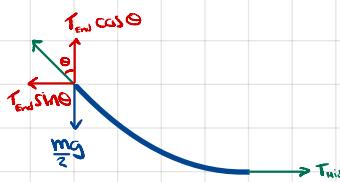
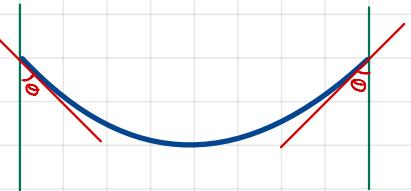
$$\Rightarrow a_2 (9m_1 + 4m_2) = 6F$$

$$\Rightarrow a_2 = \frac{6F}{9m_1 + 4m_2}$$

$$a_1 = \frac{3}{2} a_2$$

$$T = \frac{m_2 a_2}{3}$$

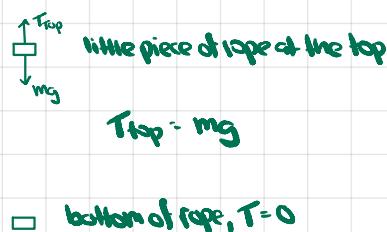
13.1 Rope Hanging Between Trees



$$T_E \cos \theta - \frac{mg}{2} = 0 \Rightarrow T_E = \frac{mg}{2 \cos \theta}$$

$$T_H - T_E \sin \theta = 0 \Rightarrow T_H = \frac{mg \tan \theta}{2}$$

13.2 Differential Analysis of a Massive Rope

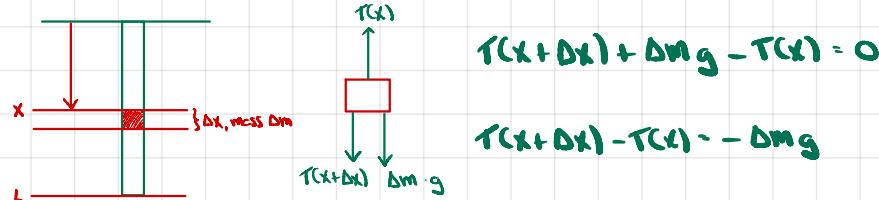


At some position x what is the tension?

"cut" the rope at x , what tension force would be necessary to support the rope?

$$T(x) = \left(\frac{L-x}{L}\right)mg = (1 - \frac{x}{L})mg$$

Alternative approach: differential analysis



* if massless rope ($\Delta m = 0$) then $T(x+Delta x) = T(x)$

$$\text{Note that } \Delta m = \frac{\Delta x}{L} m$$

$$\Rightarrow T(x+Delta x) - T(x) = -\frac{\Delta x m}{L} g \Rightarrow \frac{T(x+Delta x) - T(x)}{\Delta x} = -\frac{mg}{L}$$

$$\lim_{\Delta x \rightarrow 0} \frac{T(x+Delta x) - T(x)}{\Delta x} = \frac{dT(x)}{dx} = -\frac{mg}{L}$$

$$\Rightarrow dT = -\frac{mg}{L} dx \Rightarrow \int_{T(0)}^{T(x)} dT' = -\frac{mg}{L} \int_0^x dx' \Rightarrow T(x) - T(0) = -\frac{mgx}{L}$$

$$T(0) = mg$$

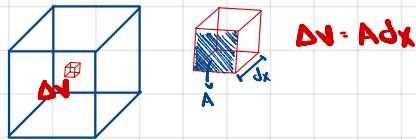
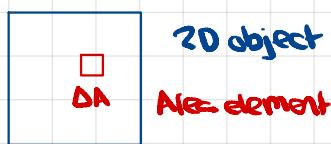
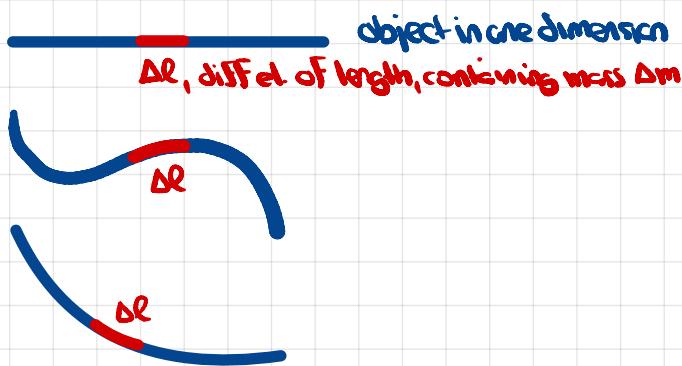
$$\Rightarrow T(x) = mg\left(1 - \frac{x}{L}\right)$$

* we reached same result but this approach is much more powerful. If the rope's mass were not uniformly distributed through the rope, it could still be modelled like this using a different expression for Δm .

13.3 Differential Elements

→ Frequently, we need to consider how the mass of an object is distributed throughout the object.

→ we define a small piece of the object and then consider the mass of that small piece



13.4 Density

We want to relate $\Delta L, \Delta A, \Delta V$ to Δm

$$1D, \text{ linear density} : \lambda = \Delta m / \Delta L$$

$$2D : \sigma = \Delta m / \Delta A$$

$$3D : \rho = \Delta m / \Delta V$$

13.6 Summary of Differential Analysis

Steps when analyzing continuous mass distribution

1. Choose Δm , small mass element

2. Analyze forces acting on Δm

3. Take limits $\Delta x \rightarrow 0 \Rightarrow$ differential equation

4. separate and integrate

5. Apply boundary condition

14.1 Intro to Resistive Forces

→ Drag forces in fluid (e.g. liquid, air)

depends on

(1) properties of the object (speed, size, shape)

(2) density, viscosity, compressibility of the fluid

Air drag

$$\text{drag force} = \frac{1}{2} C_D \rho A V^2$$

drag coeff. cross sectional area

object speed

resistive force density of air

* drag coeff.: different for different shapes

viscosity: η

$$|\vec{F}_{\text{drag}}| \sim V \sim \eta$$

→ special case: sphere of radius R

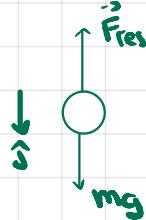
$$\vec{F}_{\text{drag}} = -6\pi\eta R \vec{V} \quad (\text{Stokes' Law})$$

14.2 Resistive Forces - Low Speed Case

$$\vec{F}_{\text{res}} = -\alpha \vec{V}$$

velocity dependent resistive force

$$\text{units of } \alpha: \frac{[F]}{[V]} = \frac{\text{kg m s}^{-2}}{\text{m s}^{-1}} = \text{kg s}^{-1}$$



e.g. marble falling in a vat of oil

$\vec{F} = m\vec{a}$, \vec{j} direction

$$mg - \alpha V_1 = m \frac{dV_1}{dt} \Rightarrow \frac{dV_1}{dt} = g - \frac{\alpha}{m} V_1 \quad (\text{linear 1st order diff eq.})$$

$$\frac{1}{g - \alpha/m} dV_1 = dt \Rightarrow \int_{V_1(0)}^{V_1(t)} \frac{dV_1}{\frac{\alpha}{m}(V_1 - \frac{mg}{\alpha})} = \int_{t=0}^{t=t} dt \Rightarrow -\frac{m}{\alpha} \ln \left[\frac{V_1 - mg/\alpha}{-mg/\alpha} \right] = t$$

$$\Rightarrow \ln \left[\frac{V_1 - mg/\alpha}{-mg/\alpha} \right] = -\frac{\alpha}{m} t \Rightarrow \frac{V_1 - mg/\alpha}{-mg/\alpha} = e^{-\frac{\alpha}{m} t}$$

$$\Rightarrow V_1(t) = \frac{mg}{\alpha} \left(1 - e^{-\frac{\alpha}{m} t} \right)$$

$$\text{Define } e^{-\frac{\alpha}{m} t} = e^{-t/\tau} \quad \tau = \frac{m}{\alpha} \quad \Rightarrow [\tau] = \frac{\text{kg}}{\text{kg s}^{-1}} = \text{s}^{-1}$$

$$V_{1,\text{term}} \Leftrightarrow \frac{dV_1}{dt} = g - \frac{\alpha}{m} V_1 = 0 \Leftrightarrow \text{resistive force} = \text{gravitational force}$$



14.3 Resistive Forces - High Speed Case

- more complicated example of drag forces

- model: $\vec{F}_{\text{res}} = -\beta V^2 \hat{j}$

$$[\beta] = \frac{\text{kg m s}^{-2}}{\text{m}^2 \text{s}^{-2}} = \text{kg} \cdot \text{m}^{-1}$$

$\vec{F} = m\vec{a}$ in dir. \hat{j}

nonlinear 1st order diff eq.

$$mg - \beta V^2 = m \frac{dv}{dt} \Rightarrow \frac{dv}{dt} = g - \frac{\beta}{m} V^2 \Rightarrow \frac{dv}{g - \frac{\beta}{m} V^2} = dt \Rightarrow \frac{-dv}{g(1 - \frac{\beta V^2}{mg})} = -dt \Rightarrow \frac{-dv}{1 - \frac{\beta V^2}{mg}} = -gdt$$

change of variables

$$u = \sqrt{\frac{\beta}{mg}} v \Rightarrow du = \sqrt{\frac{\beta}{mg}} dv$$

$$v_0 = 0, u_0 = 0$$

$$v(t) = [\beta/mg]^{1/2} u(t)$$

$$\Rightarrow -[\frac{mg}{\beta}]^{1/2} \int_0^{u(t)} \frac{du}{1-u^2} = - \int_0^t g dt$$

$$\Rightarrow \int_0^{u(t)} \frac{du}{1-u^2} = [\beta/mg]^{1/2} \int_0^t g dt$$

$$= \frac{1}{2} \ln \left(\frac{1+u}{1-u} \right) \Big|_0^{u(t)} = -gt[\beta/mg] \Rightarrow -\ln \left(\frac{1+u}{1-u} \right) \Big|_0^{u(t)} = -2gt[\beta/mg]^{1/2}$$

$$-\ln \left[\frac{1 - [\beta/mg]^{1/2} u(t)}{1 + [\beta/mg]^{1/2} u(t)} \right] = -2gt[\beta/mg]^{1/2}$$

$$\frac{1 - [\beta/mg]^{1/2} u(t)}{1 + [\beta/mg]^{1/2} u(t)} = e^{-2gt[\beta/mg]^{1/2}}$$

$$\text{Define } \tau = [\frac{mg}{\beta}]^{1/2} \quad [\tau] = \text{s}$$

$$\Rightarrow \frac{1 - \frac{u(t)}{\tau}}{1 + \frac{u(t)}{\tau}} = e^{-\frac{t}{\tau}} \Rightarrow 1 - \frac{u(t)}{\tau} = e^{-\frac{t}{\tau}} + \frac{e^{-\frac{t}{\tau}} u(t)}{\tau}$$

$$\Rightarrow \frac{u(t)}{\tau} \left[1 + e^{-\frac{t}{\tau}} \right] = 1 - e^{-\frac{t}{\tau}} \Rightarrow u(t) = \sqrt{\frac{mg}{\beta}} \frac{(1 - e^{-\frac{t}{\tau}})}{(1 + e^{-\frac{t}{\tau}})}$$

$$* 2\tau = \sqrt{\frac{mg}{\beta}}$$

$$\Rightarrow t \rightarrow \infty \quad e^{-\frac{t}{\tau}} \rightarrow 0 \quad u(t \rightarrow \infty) = \sqrt{mg} \cdot \sqrt{\frac{mg}{\beta}} \Rightarrow \frac{du}{dt} = 0$$

