# Combined Translation and Rotation : Energy Relationships

## setup

→ rigid body, made up of particle masses

→ $m_i$, vel. $\vec{v}_i$ rel. to an inertial frame

$\vec{v}_i = \vec{v}_{cm} + \vec{v}_i'$ ↗ vel. of particle rel. to CM

$$K_i = \frac{m_i \vec{v}_i \cdot \vec{v}_i}{2} = \frac{m_i(\vec{v}_{cm} + \vec{v}_i')\cdot(\vec{v}_{cm} + \vec{v}_i')}{2}$$

$$= \frac{m_i(\vec{v}_{cm}\cdot\vec{v}_{cm} + \vec{v}_i'\cdot\vec{v}_i' + 2\vec{v}_{cm}\cdot\vec{v}_i')}{2}$$

$$= \frac{m_i v_{cm}^2}{2} + \frac{m_i v_i'^2}{2} + \vec{v}_{cm}\cdot\vec{v}_i'$$

sum over all particles

$$K = \frac{v_{cm}^2}{2}\sum m_i + \sum \frac{m_i v_i'^2}{2} + \vec{v}_{cm}\cdot \cancel{\sum \vec{v}_i'} \nearrow 0$$

$$= \frac{M v_{cm}^2}{2} + \sum \frac{m_i r_i^2 \omega^2}{2}$$

$$= \frac{M v_{cm}^2}{2} + \frac{I_{cm}\omega^2}{2}$$

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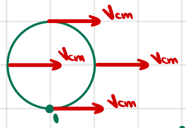
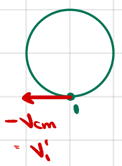
## Rolling without slipping

The entire wheel has a translation velocity $\vec{v}_{cm}$



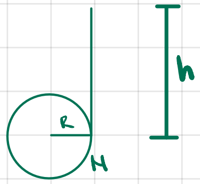

reference frame with the ground at rest
The point of the wheel that touches the ground has the same velocity as the ground; this is what it means for that point (and the wheel) not to slip.

$\vec{v}_1 = \vec{v}_{cm} + \vec{v}_i' \Rightarrow \vec{v}_i' = -\vec{v}_{cm}$




$v_i' = R\omega \Rightarrow v_{cm} = R\omega$ (no slipping condition)

## Ex 10.4 - Speed of a Primitive Yo-Yo




$E_i = Mgh$

$$E_f = \frac{M v_{cm}^2}{2} + \frac{MR^2}{2}\cdot\frac{1}{2}\omega^2$$

no slipping $\Rightarrow v_{cm} = R\omega \Rightarrow \omega = \frac{v_{cm}}{R}$

$$E_f = \frac{M v_{cm}^2}{2} + \frac{M v_{cm}^2}{4} = \frac{3 M v_{cm}^2}{4}$$

$$\Delta E = 0 \Rightarrow v_{cm} = \sqrt{\frac{4gh}{3}}$$

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## Ex 10.5 Race of the Rolling Bodies

$$mgh = \frac{mv^2}{2} + \frac{I_{cm} v^2}{2R^2}$$

The larger the rotational kinetic energy due to moment of inertia and/or radius, the smaller $v$ is.

we can't tell if $v$ depends on $R$ because $I_{cm}$ depends on $R$.

For certain bodies we know $I_{cm}$.

solid sphere : $I_{cm} = \frac{2MR^2}{5}$

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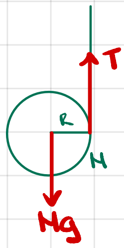
## Ex 10.6 - Acceleration of a Primitive Yo-Yo



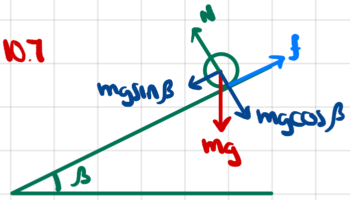

The yo-yo has both translational and rotational motion.

$$Mg - T = M\cdot a_{cm} \qquad\qquad a_{cm} = \frac{2}{3}g$$

$$\Rightarrow$$

$$RT = \frac{MR^2}{2}\cdot\frac{a_{cm}}{R} \qquad\qquad T = \frac{1}{3}Mg$$

Ex 10.7



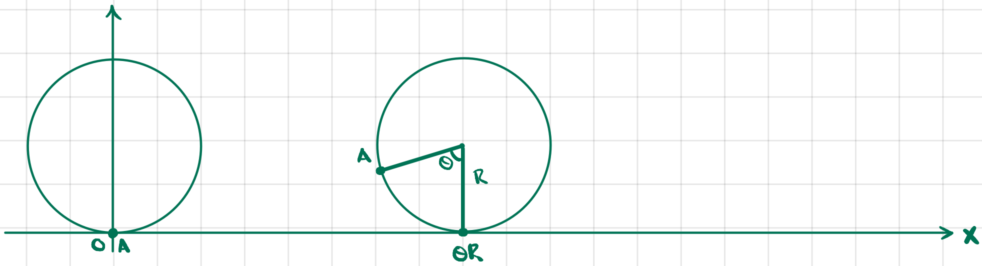

$mg\sin\beta - f = m\cdot a$

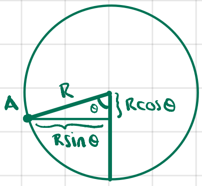
$f\cancel{R} = \frac{2}{5}m\cancel{R^2}\cdot\frac{a}{\cancel{R}}$

$\Rightarrow f = \frac{2ma}{5}$

$\cancel{m}g\sin\beta = m\left(a + \frac{2a}{5}\right) = \cancel{m}\cdot\frac{7a}{5}$

$a = \frac{5g\sin\beta}{7}$




$\vec{r}_{OA} = \langle \theta R - R\sin\theta, R - R\cos\theta \rangle$

$= \langle R(\theta - \sin\theta), R(1-\cos\theta) \rangle$

$\vec{v}_A(\theta) = \langle R(1-\cos\theta), R\sin\theta \rangle$

$\vec{v}_A(2\pi) = \langle 0, 0 \rangle$

## 10.5 Angular Momentum

$\vec{L}_o = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$

→ depends on choice of origin

→ net force acts on a particle ⇒ velocity and momentum change,
so angular momentum may
also change.

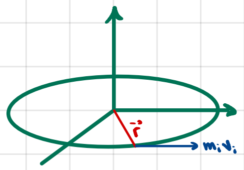
→ $\dfrac{d\vec{L}}{dt} = \vec{v} \times m\vec{v} + \vec{r} \times m\vec{a} = \vec{r} \times \vec{F} = \vec{\tau}$

rate of change of angular momentum equals torque of net force.

## Angular Momentum of a Rigid Body

→ rigid body rotating about an axis with $\omega$.

→ thin slice located in $xy$ plane

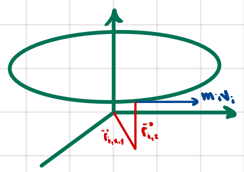



$L_i = r m_i v_i = m_i r_i^2 \omega$, positive $\hat{k}$ direction

* sin of angle between $\vec{r}_i$ and $\vec{v}_i$ is $90°$

$L = \sum_i L_i = \sum_i m_i r_i^2 \omega = I_{slice,z}\,\omega$

If we try this for another slice not on the $xy$ plane, the $\vec{r}_i$ vectors
have a $z$-component.




The angular momentum of $m_i$ rel. to $z$-axis has a component perpendicular
to $z$.

For a rigid body that is symmetric about the axis of rotation, such
momentum vectors cancel out by points on opposite sides of the axis.

⇒ For rigid bodies symmetric about rotation axis, angular momentum
is a vector with same direction as axis of rotation, and magnitude
$I_z \omega$. Therefore, $\vec{L} = I_z \vec{\omega}$

Also, $\vec{\tau}^{ext} = \dfrac{d\vec{L}}{dt} = I_z \alpha$ in this case.

# 10.7 Gyroscopes and Precession