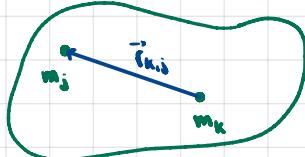


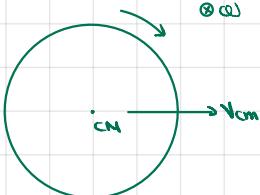
## 28.1 Rigid Bodies



$$|r_{kj}| = r_{kj} = \text{constant for all } j, k$$

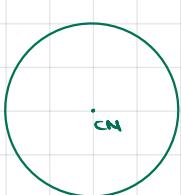
i.e., distance between any two points is fixed

## 28.2 Introduction to Rotation and Translation



ground frame

- rotation + translation
- $\vec{F}_{\text{ext}} = m_f \vec{a}_{\text{cm}}$

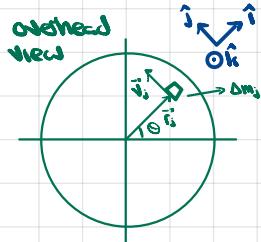
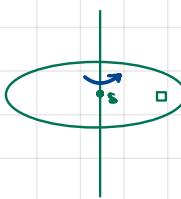


cm reference

- pure rotation around cm
- cm at rest,  $\vec{T}_{\text{cm}} = I_{\text{cm}} \cdot \vec{\omega}$  (torque)
- $K_{\text{rot}} = \frac{1}{2} I_{\text{cm}} \omega^2$  = rotational energy

rotation about cm

## 29.1 Kinetic Energy of Rotation



cut the rigid body up into many small  $dm_j$ , each with position  $\vec{r}_j$ , velocity  $\vec{v}_j$ ,  $\vec{v}_j = \vec{r} \times \vec{\omega} = r_j \omega(t) \hat{\theta} = r_j \omega_z \hat{\theta}$ , mass  $dm_j$ .

Each small piece has rotational kinetic energy,

$$K_j^{\text{rot}} = \frac{1}{2} dm_j v_j^2 = \frac{1}{2} dm_j r_j^2 \omega_z^2$$

$$K_{\text{tot.}} = \frac{1}{2} \omega_z^2 \sum_{j=1}^n dm_j r_j^2$$

Definition:  $I_z = \lim_{\substack{dm_j \rightarrow 0 \\ n \rightarrow \infty}} \sum_{j=1}^n dm_j r_j^2 = \int dm r_{z, \text{dm}}^2$   
moment of inertia about axis passing through point z,  $\perp$  to plane of rotation

$$K_{\text{tot.}} = \frac{1}{2} I_z \omega_z^2$$

$$\text{contrast this to } K_{\text{translational}} = \frac{1}{2} m V_{\text{cm}}^2$$

## 29.2 Moment of Inertia of a Rod



$$I_{\text{cm}} = \int dm r^2$$

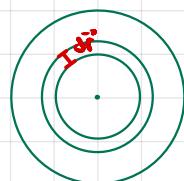
choose coord. system:



$$dm = \frac{m}{L} dx$$

$$I_{\text{cm}} = \int_{-L/2}^{L/2} \frac{m}{L} dx \cdot x^2 = \frac{m}{L} \frac{1}{3} x^3 \Big|_{-L/2}^{L/2} = \frac{1}{12} m L^2$$

## 29.3 Moment of Inertia of a Disc



uniform thin disc, mass m, radius R

$$J = \frac{m}{\pi R^2}$$

$$dm = J \cdot (\pi(r+dr)^2 - \pi r^2) = \frac{m}{\pi R^2} (\pi r^2 + 2\pi r dr + \pi dr^2 - \pi r^2)$$

$$= \frac{m}{\pi R^2} (2\pi r dr + \pi dr^2)$$

$$\lim_{dr \rightarrow 0} dr \Rightarrow dm = \frac{m}{\pi R^2} \cdot 2\pi r dr$$

$$I_{\text{cm}} = \frac{m}{\pi R^2} \int 2\pi r dr r^2 = \frac{2m}{R^2} \int_0^R r^3 dr = \frac{2m}{R^2} \cdot \frac{R^4}{4} = \frac{m R^2}{2}$$

\*  $\vec{\omega} = \dot{\theta}(t) \hat{z} = \omega_z \hat{z}$  = angular velocity vector, rad s<sup>-1</sup>

angular speed =  $\omega = \dot{\theta}(t) = |\vec{\omega}|$

$$\vec{v} = \vec{\omega} \times \vec{r} = \dot{\theta}(t) \hat{z} \times r \hat{r} = r \dot{\theta}(t) \hat{e}$$

$$\text{angular acceleration} = \frac{d\vec{\omega}}{dt} = \ddot{\theta}(t) \hat{z} = \alpha_z \hat{z}$$

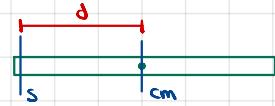
$$\alpha = |\ddot{\theta}| = |\ddot{\theta}(t)|$$

## Alten. Calculation

$$\iint_0^{2\pi} \iint_0^R \frac{m}{4\pi R^2} \cdot r dr d\theta \cdot r^2 dm$$

$$= \frac{m}{\pi R^2} \int_0^{2\pi} \frac{R^4}{4} d\theta = \frac{m R^2}{4\pi} \cdot 2\pi = \frac{m R^2}{2}$$

## 29.4 Parallel Axis theorem

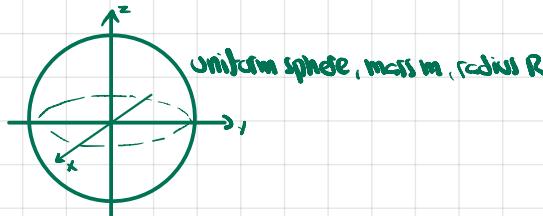


$$I_S = I_{cm} + md^2$$

$$Ex: I_{cm} = \frac{mL^2}{12}$$

$$I_S = \frac{mL^2}{12} + m \frac{L^2}{4} = \frac{mL^2}{3}$$

## 29.5 Moment of Inertia of a Sphere



$$I_{cm,z} = \int_{sphere} dm (x^2 + z^2)$$

$$I_{cm,x} = \int_{sphere} dm (y^2 + z^2)$$

$$I_{cm,y} = \int_{sphere} dm (x^2 + y^2)$$

$$By symmetry, I_{cm,x} = I_{cm,y} = I_{cm,z} = I_{cm}$$

$$3I_{cm} = I_{cm,x} + I_{cm,y} + I_{cm,z}$$

$$= \int_{sphere} dm \cdot 3(x^2 + y^2 + z^2)$$

$r^2$  of a sphere of thickness  $dr$

$$dm = S dV$$

$$\text{volume density} = \frac{m}{\frac{4}{3}\pi R^3}$$

$$dV = \text{volume of spherical shell, of thick } 4\pi r^2 dr$$

$$dm = \frac{3m}{4\pi R^3} \cdot 4\pi r^2 dr = \frac{3mr^2}{R^3}$$

$$\Rightarrow 3I_{cm} = 2 \int dm r^2 = 2 \int_0^R \frac{3mr^2 dr}{R^3} \cdot r^2$$

$$\Rightarrow I_{cm} = \frac{2m}{R^3} \int_0^R r^4 dr = \frac{2m R^5}{5R^3} = \frac{3m R^2}{5}$$

## Alten. Calculation

$$\text{mass density} = \frac{m}{\frac{4}{3}\pi R^3} = \frac{3m}{4\pi R^3}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^R \underbrace{r^2 \sin^2 \theta \left( \frac{3m}{4\pi R^3} \cdot r^2 \sin^2 \theta dr d\theta d\phi \right)}_{\text{distance to } z \text{ axis squared}} dm$$

## 29.6 Definition of Parallel Axis Theorem

Take a rigid body. Consider two points, and parallel axes passing through them:

$$I_{cm} = \int_{body} dm r_{cm,x}^2$$

$$I_s = \int_{body} dm r_s^2$$

$$r_s = d + r_{cm,x} \Rightarrow r_s^2 = d^2 + r_{cm,x}^2 + 2dr_{cm,x}$$

$$\Rightarrow I_s = \int_{body} dm d^2 + \int_{body} dm r_{cm,x}^2 + 2d \int_{body} dm r_{cm,x}$$

$$\vec{r}_{cm} \text{ is defined } \int dm \vec{r}_{cm} = 0 \Rightarrow \int dm r_{cm,x} = 0$$

$$\Rightarrow I_s = d^2 m + I_{cm}$$

## 30.1 Introduction to Torque and Rotational Dynamics

$$\vec{F} \cdot \vec{N} = \vec{r}_{cm} \cdot \vec{F}$$

### Charles' Theorem

Rotational equiv. to Newton's 2nd law

$$\text{Rotational Eq. of Motion: } \vec{\tau} \cdot \vec{I} \ddot{\vec{\alpha}}$$

$\vec{\tau}$  = torque

I = moment of inertia

$\ddot{\vec{\alpha}}$  = angular acceleration

### Inertia

$$m$$

10m

$$F = ma$$

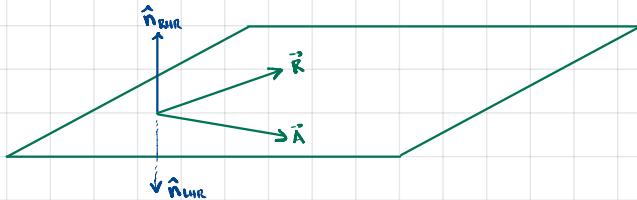
m represents inertia in translational motion

moment of inertia plays this role in rotational motion

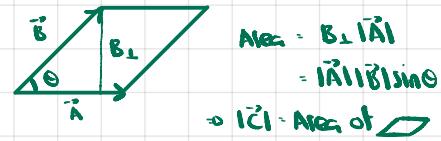
$I = \sum_{body} m_i r_i^2$  But in rot. motion, it's not just how much mass, but also how that mass is distributed.

## 30.2 Cross Product Review

Vector Product (cross product)  $\vec{A} \times \vec{B} = \vec{C}$



$$\vec{C} = |\vec{A}| |\vec{B}| \sin\theta \hat{n}_{\text{RHR}} \quad 0 \leq \theta < \pi$$



## 30.3 Cross Product in Cartesian Coord.



define  $\hat{i} \cdot \hat{i} \times \hat{j}$  (right-handed coord. sys.)

$(i, j, k)$  || a cyclic order, e.g.  $(j, i, k)$  anticlockwise

$$\begin{aligned} \hat{j} \times \hat{k} &= \hat{i} & (\hat{j}, \hat{k}, \hat{i}) \\ \hat{k} \times \hat{i} &= -\hat{j} & (\hat{k}, \hat{i}, \hat{j}) \end{aligned}$$

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$$\begin{aligned} \hat{i} \times \hat{j} &= -\hat{i} \\ \hat{i} \times \hat{k} &= -\hat{j} \\ \hat{j} \times \hat{i} &= -\hat{k} \end{aligned}$$

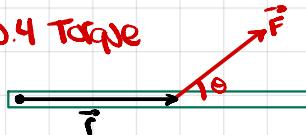
$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$$\begin{aligned} A &= A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \\ B &= B_x \hat{i} + B_y \hat{j} + B_z \hat{k} \end{aligned}$$

$$\begin{aligned} \vec{A} \times \vec{B} &= A_x B_y \hat{i} + A_x B_z (-\hat{j}) + A_y B_z (-\hat{k}) + A_y B_x \hat{i} \\ &\quad + A_z B_x \hat{j} + A_z B_y (-\hat{i}) \end{aligned}$$

$$\begin{aligned} &= \hat{i}(A_x B_y - A_y B_x) + \hat{i}(A_y B_z - A_z B_y) \\ &\quad + \hat{j}(A_z B_x - A_x B_z) \end{aligned}$$

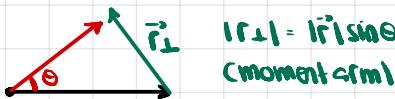
## 30.4 Torque



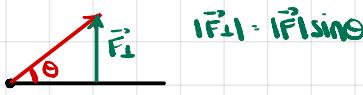
$$\vec{T} = \vec{r} \times \vec{F} = |\vec{r}| |\vec{F}| \sin\theta$$

Let's analyze this expression in two ways

$$1) \vec{T} = |\vec{F}| (|\vec{r}| \sin\theta) \quad \text{moment arm}$$



$$2) \vec{T} = (|\vec{F}| \sin\theta) |\vec{r}|$$



## 30.5 Torque From Gravity

$$\begin{aligned} \vec{T} &= \sum_{j=1}^n (\vec{r}_{S,j} \times m_j \vec{g}) \\ &= \left( \sum_{j=1}^n \vec{r}_{S,j} m_j \right) \times \vec{g} \end{aligned}$$

$$\Rightarrow \vec{T}_T = m_T \cdot \vec{r}_{S,T} \times \vec{g}$$

## 31.1 Relationship Between Torque and Angular Acceleration

$$\begin{aligned} \text{Free body diagram: } &\vec{F}_{ext}^{\text{ext}} \quad \vec{r} \quad \hat{r} \quad \theta \quad \vec{r}_{S,i} \\ \vec{T}_S &= \sum_{j=1}^n \vec{r}_{S,j} \times \vec{F}_{ext}^{\text{ext}} \\ \text{Coord: } &\vec{r}_{S,i} \end{aligned}$$

$$\vec{F}_{ext}^{\text{ext}} = F_{ext}^{\text{ext}} \hat{r} + F_{ext}^{\text{ext}} \hat{\theta} + F_{ext}^{\text{ext}} \hat{k}$$

$\vec{r}_{S,i} \times \vec{F}_{ext}^{\text{ext}}$  is not in the  $\hat{z}$  ( $\hat{c} \times \hat{h}$ ) direction. We ignore those terms.

$$\Rightarrow \vec{T}_{S,i} = \sum_{j=1}^n \vec{r}_{S,j} \times (\vec{F}_{ext}^{\text{ext}} \hat{r} + \vec{F}_{ext}^{\text{ext}} \hat{\theta})$$

Note, however, that  $\vec{r}_{S,j} \times \vec{F}_{ext}^{\text{ext}} \hat{r} = 0$

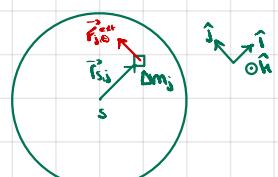
$$\Rightarrow \vec{T}_{S,i} = \sum_{j=1}^n \vec{r}_{S,j} \times \vec{F}_{ext}^{\text{ext}} \hat{\theta} = \sum_{j=1}^n r_{S,j} F_{ext} \hat{\theta}$$

$$\text{2nd Law: } F_{ext} = \Delta m_s \cdot a_{s,\theta} = \Delta m_s \cdot (r_{S,i} \alpha_z)$$

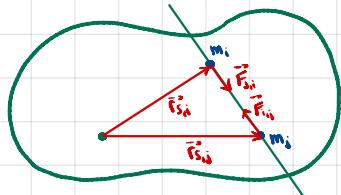
$$\Rightarrow \vec{T}_{S,i} = \left( \sum_{j=1}^n \Delta m_s r_{S,j}^2 \right) \alpha_z \hat{k}$$

$$\lim_{\Delta m_s \rightarrow 0} \sum \Delta m_s r_{S,j}^2 = \int_{\text{body}} dm r_{S,j}^2$$

$$\Rightarrow \vec{T}_{S,z} = I_S \ddot{\alpha}, \quad \ddot{\alpha} = \alpha_z \hat{k}$$



### 31.2 Internal Torques Cancel in Pairs



$$\vec{T}_s = \vec{T}_{s,i} + \vec{T}_{s,j} + \vec{T}_{s,k}$$

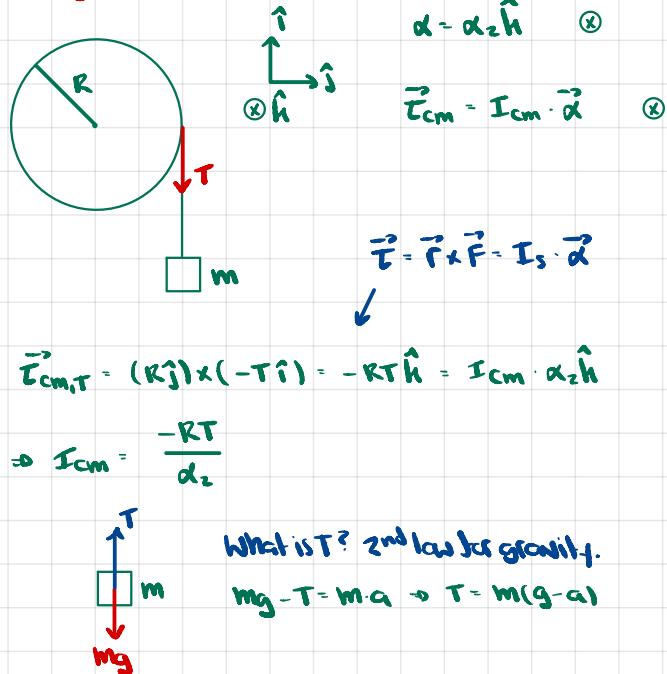
$$3^{\text{rd}} \text{ law: } \vec{F}_{ji,i} = -\vec{F}_{ij,i}$$

$$\Rightarrow \vec{T}_s = (\vec{T}_{s,i} - \vec{T}_{s,j}) + \vec{T}_{s,k}$$

$$\vec{F}_{ji,i} \parallel (\vec{r}_{ii} - \vec{r}_{ij,i}) \Rightarrow \vec{T}_s = 0$$

$\Rightarrow$  we only need to address torque due to external forces

### 31.3 Worked Example: Moment of Inertia of a Disc From a Falling Mass



The mass accelerates due to gravity. There is an opposing tension force. The acceleration is the same as that experienced on the edge of the disc. The disc edge has circular motion. Therefore we can obtain angular acceleration from tangential acceleration. This relation is a constraint eq.

$$\text{constraint condition: } a_T = R\alpha_z \Rightarrow \alpha_z = \frac{a_T}{R}$$

At this point we've used the torque equations to get formulas for moment of inertia of the disc. It applies to any setup where a force T is pulling tangentially on the disc. With 2nd law we specified where T is coming from. Given the setup and physical reality of the rope, we have a constraint that tells us how the disc is accelerating under the effect of T.

Therefore, we can further refine our formula for I\_cm

$$I_{cm} = -\frac{Rm(g-a)}{a} = -\frac{R^2 m(g-a)}{R} \\ = mR^2 \left( \frac{g}{a} - 1 \right)$$

The I\_cm depends on various factors. We could choose a, or we could choose the mass falling  $\propto$  distance h in time t to know a.

This is 1D kinematics.

$$h(t) = \frac{at^2}{2} \Rightarrow a = \frac{2h}{t^2}$$

$$\Rightarrow I_{cm} = mR^2 \left( \frac{gt^2}{2h} - 1 \right)$$

### Recap 31.3

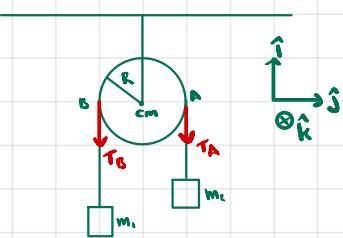
$$\text{Torque eq: } -RT = I_{cm}\alpha_z$$

$$\text{2nd Law eq: } T = m(g-a)$$

$$\text{constraint eq: } \alpha_z = \frac{a}{R}$$

unknowns: T, a,  $\alpha_z$ , I\_cm

### 31.4 Worked Example - Atwood Machine



$$m_2 > m_1$$

point A

$$\vec{\tau}_{cm,A} = \vec{r}_{cm,A} \times \vec{T}_A = (R\hat{j}) \times (-T\hat{i}) = -TR(-\hat{k}) = RT_A \hat{k}$$

$$\vec{\tau}_{cm,B} = \vec{r}_{cm,B} \vec{\alpha} = (-R\hat{j})(-T\hat{i}) = -RT_B \hat{k}$$

$$\vec{\tau}_{cm} = \vec{\tau}_{cm,A} + \vec{\tau}_{cm,B} = \hat{k} R(T_A - T_B)$$

$$\vec{\tau}_{cm} = I_{cm} \vec{\alpha} \Rightarrow R(T_A - T_B) = I_{cm} \alpha_z$$

$$I_{cm} = \frac{R(T_A - T_B)}{\alpha_z} \quad \text{4 equations}$$

$$\alpha = R\alpha_z$$

unknowns:  $T_A, T_B, \alpha, \alpha_z$

$$T_A = m_2(g - a)$$

It is assumed we know  $I_{cm}$ .

$$T_B = m_1(a + g)$$

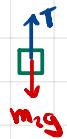
$$I_{cm} \cdot \frac{a}{R} = R(m_2(g - a) - m_1(g + a))$$

$$I_{cm} \cdot a = R^2 [m_2g - m_2a - m_1g - m_1a]$$

$$a(I_{cm} + R^2(m_1 + m_2)) = R^2 g(m_2 - m_1)$$

$$a = \frac{R^2 g(m_2 - m_1)}{I_{cm} + R^2(m_1 + m_2)}$$

+ massless, frictionless pulley



$$m_2g - T = m_2 \cdot a$$

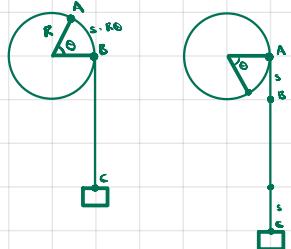


$$T - m_1g = m_1 \cdot a$$

$$g(m_2 - m_1) = a(m_2 + m_1)$$

$$a = \frac{m_2 - m_1}{m_2 + m_1} g$$

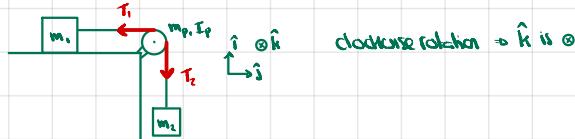
### 31.6 Non-slip condition



$$v_A = \frac{ds}{dt} = R \frac{d\theta}{dt} = v$$

$$\Rightarrow a = R \frac{d^2\theta}{dt^2} = R\alpha$$

### 31.7 Worked Example: Two Blocks and a Pulley Using Energy!



Find  $v_{3,2}$

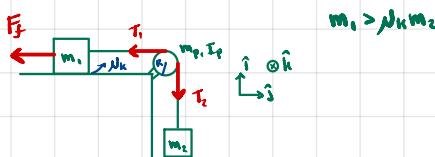
$$E_i = 0$$

$$E_f = -m_2gh + \frac{m_1v^2}{2} + \frac{m_2v^2}{2} + \frac{I_p\omega^2}{2}$$

$$= -m_2gh + \frac{v^2(m_1+m_2)}{2} + \frac{I_p}{2} \cdot \frac{v^2}{R^2}$$

$$\Delta E = 0 \Rightarrow v^2(m_1+m_2 + \frac{I_p}{R^2}) = 2m_2gh$$

$$v = \sqrt{\frac{2m_2gh}{m_1+m_2 + \frac{I_p}{R^2}}}$$



$$W_{F_f} = \Delta E_m$$

$$W_g = \int -\mu_k m_1 g dx = -\mu_k m_1 g x$$

$$\text{Torque: } (Rj) \times (-T_2\hat{i}) + (Ri) \times (-T_1\hat{j}) = I_{cm}\alpha_z\hat{k}$$

$$\Rightarrow -RT_2(-\hat{k}) - RT_1\hat{i} = I_{cm}\alpha_z\hat{k}$$

$$\Rightarrow R(T_2 - T_1) = I_{cm}\alpha_z$$

$$\text{2nd law: } T_1 - F_f = m_1 a \Rightarrow T_1 - \mu_k m_1 g - m_1 a = T_1 - m_1(a + \mu_k g)$$

$$m_2g - T_2 = m_2 a \Rightarrow T_2 = m_2(g - a)$$

$$\text{constraint: } a = \alpha R$$

$$\text{unknowns: } T_2, T_1, a, \alpha$$

$$\text{solution for } a: R(m_2g - m_2a - m_1a - m_1\mu_k g) = I_{cm} \cdot \frac{a}{R}$$

$$a(I_{cm} + R^2 m_2 + R^2 m_1) = R^2(m_2g - m_1\mu_k g)$$

$$a = \frac{R^2 g (m_2 - m_1 \mu_k)}{I_{cm} + R^2(m_2 + m_1)}$$