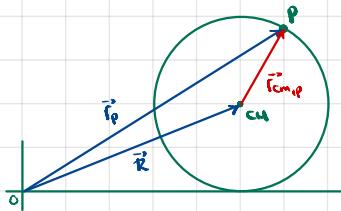


35.1 Translation and Rotation of a wheel

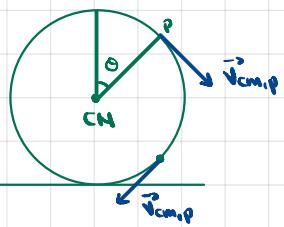


Two reference frames at origins O and CM.

$$\vec{v}_p = \vec{R} + \vec{v}_{cm,p}$$

$$\vec{v}_p = \vec{v} + \vec{v}_{cm,p}$$

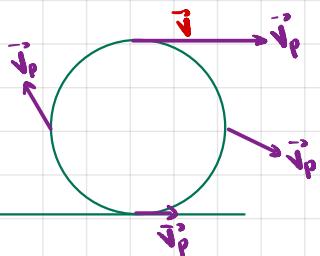
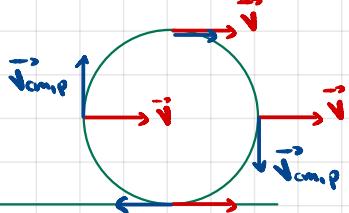
35.2 Rolling Wheel in CM Frame



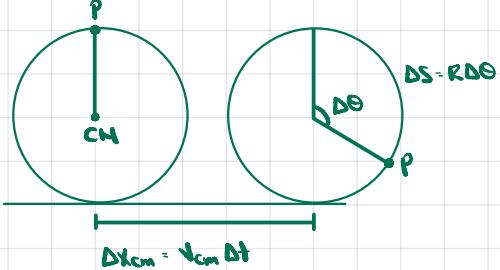
$$v_{cm,p} = R |\theta'(t)|$$

From CM frame, P has a circular motion.

35.3 Rolling Wheel in the Ground Frame



35.4 Rolling w/o Slipping, Skidding



rolling without slipping condition

$$\Delta x_{cm} = \Delta s$$

$$v_{cm} \Delta t = R \Delta \theta$$

$$v_{cm} = R \frac{\Delta \theta}{\Delta t}$$

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta \theta}{\Delta t} = \omega$$

$\Rightarrow v_{cm} = R \omega$ - tangential speed in CM frame

Slipping

$$\Delta s > \Delta x_{cm}$$

$$R \Delta \theta > v_{cm} \Delta t$$

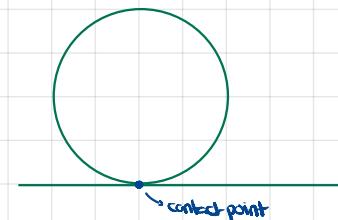
$$\Rightarrow v_{cm} < R \omega$$

Skidding

$$\Delta s < \Delta x_{cm}$$

$$v_{cm} > R \omega$$

35.5 Contact Point of a wheel Rolling without Slipping



$$|v_{cm,pl}| = R \omega$$

$$|v_{cm}| = v_{cm}$$

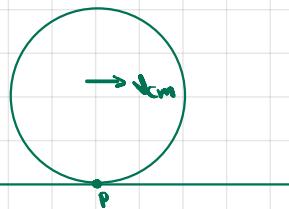
$$\text{no slipping} \Rightarrow |v_{cm,pl}| = |v_{cm}|$$

$$\Rightarrow \vec{v}_p = \vec{0}$$

contact point is instantaneously at rest w.r.t. respect to the ground.

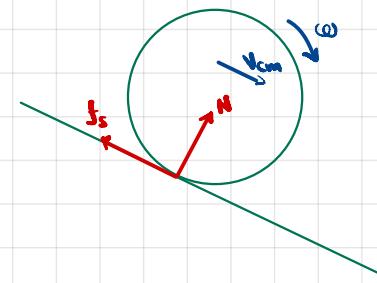
36.1 Friction on a Rolling Wheel

Horizontal Surface



$$f_s = 0 \text{ (static friction)}$$

Inclined Surface



f_s produces torque

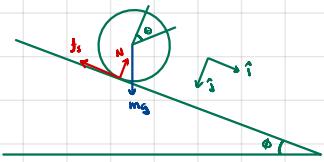
$$v_{cm} = R \omega$$

$a_{cm} = R \alpha$ non-zero angular acceleration

$$T_{cm} = I_{cm} \alpha$$

$$\vec{T}_{cm} = \vec{r}_{cm,pl} \times \vec{f}_s$$

36.2 Worked Example - Wheel Rolling Without Slipping Down Inclined Plane - Torque Method



$$i: mg \sin \phi = f_s = m a_x$$

$$ii: f_s R = I_{cm} \alpha$$

$$v_{cm} = R\omega$$

$$a_{cm} = R\alpha$$

$$a_x = a_{cm}$$

3 eq., 3 unknowns: a_{cm}, f_s, α

$$mg \sin \theta = \frac{I_{cm} \cdot a_{cm}}{R^2} = m a_{cm}$$

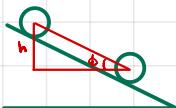
$$a_{cm} = \frac{mg \sin \phi}{m + \frac{I_{cm}}{R^2}}$$

1D Kinematics

$$x_{cm} = \frac{a_{cm} t^2}{2} = s$$

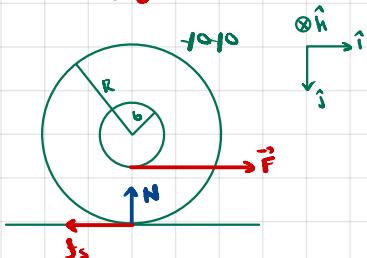
$$v_{cm} = a_{cm} t = \sqrt{2s a_{cm}}$$

$$s = \frac{h}{\sin \phi}$$



$$\Rightarrow v_{cm} = \sqrt{\frac{2h}{\sin \phi} \cdot \frac{mg \sin \phi}{m + \frac{I_{cm}}{R^2}}}$$

36.4 Worked Example - Yo-yo pulled along the ground



$$\vec{F} = F\hat{i}$$

$$\vec{f}_s = -f_s \hat{i}$$

$$\vec{F} - \vec{f}_s = m \vec{a} \Rightarrow F - f_s = m a_x$$

$$\vec{F} = b\hat{j} \times F\hat{i} + R\hat{j} \times (-f_s\hat{i}) \\ = -bF\hat{k} + RF\hat{i} = (RF_s - bF)\hat{k}$$

2 eq

$$F - f_s = m a_x \quad \text{acceleration of the system ie } a_{cm}$$

$$R\vec{f}_s - b\vec{F} = I_{cm} \alpha$$

expression for torque for rigid body undergoing rotation about a fixed axis
 $\alpha = R\alpha$

unknowns: a, α, f_s

$$\alpha = \frac{a}{R} = \frac{F - f_s}{mR}$$

$$R\vec{f}_s - b\vec{F} = I_{cm} \frac{(F - f_s)}{mR}$$

$$f_s(R + \frac{I_{cm}}{mR}) = F(b + \frac{I_{cm}}{mR})$$

$$F = \frac{f_s(R + \frac{I_{cm}}{mR})}{b + \frac{I_{cm}}{mR}}$$

$$F_{max} = \mu mg = \frac{R + \frac{I_{cm}}{mR}}{b + \frac{I_{cm}}{mR}}$$

max force such that yo-yo rolls onto slipping

36.5

$$\vec{T}_p = \vec{T}_{p,cm} + \vec{T}_{cm,sys}$$

$$= \vec{T}_{p,cm} \times \vec{F} + \sum \vec{T}_{cm,j} \times \vec{F}_j$$

choose p = CM

$$\Rightarrow \vec{T}_{cm,sys} = \sum \vec{T}_{cm,j} \times \vec{F}_j$$

$$\text{Also, } \vec{T}_{cm,sys} = \frac{d \vec{L}_{cm,sys}}{dt}$$

$$\vec{T}_{p,sys} = \vec{T}_{p,cm} \times m \vec{v}_{p,cm} + I_{cm} \vec{\omega}$$

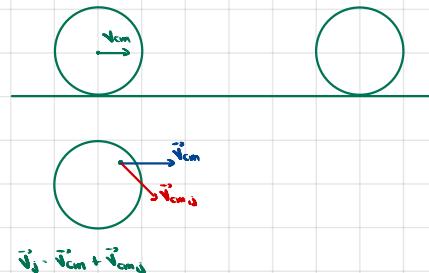
$$p = CM \Rightarrow \vec{T}_{cm,sys} = I_{cm} \vec{\omega}$$

$$\frac{d \vec{L}_{cm,sys}}{dt} = I_{cm} \alpha$$

$$\text{Rotational Dynamics: } \vec{T}_{cm,sys} = \frac{d \vec{L}_{cm,sys}}{dt}$$

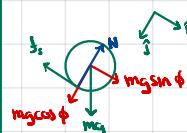
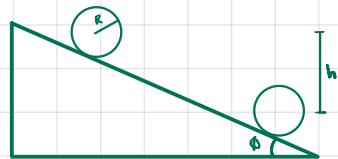
$$\text{Linear Dynamics: } \vec{F} = m, \vec{a}_{cm}$$

37.1 Kinetic Energy of Translation and Rotation



$$\begin{aligned} K &= \sum \frac{1}{2} m_j (\vec{v}_{cm} + \vec{v}_{cm,j}) (\vec{v}_{cm} + \vec{v}_{cm,j}) \\ &= \sum \frac{m_j}{2} \vec{v}_{cm}^2 + \sum m_j \vec{v}_{cm,j} \vec{v}_{cm} + \sum \frac{m_j}{2} \vec{v}_{cm,j}^2 \\ &= \frac{m_c \vec{v}_{cm}^2}{2} + \frac{\sum m_j \vec{v}_j^2}{2} \\ &= \frac{m_c \vec{v}_{cm}^2}{2} + \frac{I_{cm} \omega^2}{2} \\ &= K_{trans} + K_{rot} \end{aligned}$$

37.2 Worked Example: Wheel Rolling Without Slipping Down Inclined Plane



$$\text{Torque: } \vec{T}_{cm,sys} = R\hat{j} \times (-f_s\hat{i}) = R\hat{s} \vec{k}$$

$$E_{mi} = mgh$$

$$E_{fr} = \frac{m v_{cm,t}^2}{2} + \frac{I_{cm} \omega^2}{2}$$

$$v_{cm,t} = R\omega$$

$$\Rightarrow \frac{m v_{cm,t}^2}{2} + \frac{I_{cm} v_{cm,t}^2}{2R^2} = mgh$$

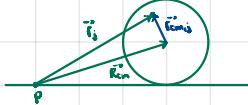
$$v_{cm,t}^2 (m + \frac{I_{cm}}{R^2}) = 2mgh$$

$$v_{cm,t} = \sqrt{\frac{2mgh}{m + \frac{I_{cm}}{R^2}}}$$

$$E_{mi} = 0 \quad E_{fr} = -mgh + \frac{I_{cm} \omega^2}{2}$$

$$\omega = \sqrt{\frac{2mgh}{I_{cm}}}$$

37.3 Angular Momentum of Translation and Rotation



$$\vec{L}_p = \sum \vec{L}_{ij} + \vec{L}_{cm}$$

$$\vec{L}_p = \vec{R}_{cm} + \vec{L}_{cm}$$

$$\vec{L}_p = \sum (\vec{R}_{cm} + \vec{r}_{cm,j}) \times m_j (\vec{v}_{cm} + \vec{v}_{cm,j})$$

$$= \sum \vec{R}_{cm} \times m_j \vec{v}_{cm} + (\sum m_j \vec{r}_{cm,j}) \times \vec{v}_{cm}$$

$$+ \sum \vec{R}_{cm} \times m_j \vec{v}_{cm,j} + \sum \vec{r}_{cm,j} \times m_j \vec{v}_{cm,j}$$

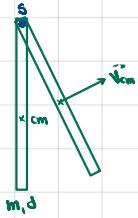
$$= \vec{R}_{cm} \times (\sum m_j) \vec{v}_{cm} + \vec{R}_{cm} \times \sum m_j \vec{v}_{cm,j}$$

$$+ \sum \vec{r}_{cm,j} \times m_j \vec{v}_{cm,j}$$

$$= \vec{L}_{p,cm} + \vec{L}_{cm,rel}$$

$$= \vec{L}_{p,rel} + \vec{L}_p^{spin}$$

37.4 Summary of Angular Momentum and Kinetic Energy



$$\vec{L}_{sys} = \vec{L}_{S,cm} + \vec{L}_p$$

$$= \vec{r}_{cm} \hat{r} \times m \vec{v}_{cm} \hat{\theta}$$

$$= \vec{r}_{cm} m \vec{v}_{cm} \hat{\theta}$$

$$\vec{L}_{sys} = \sum \vec{L}_{S,i} + \vec{L}_p$$

$$= \sum \vec{r}_{cm,i} m_i \vec{v}_{cm,i} \hat{\theta}$$

$$= \omega \sum m_i \vec{r}_{ci} \hat{\theta}$$

$$= I_S \omega \hat{\theta}$$

Alternatively,

$$\vec{L}_{sys} = \vec{L}_{S,cm} + \vec{L}_{cm,rel}$$

$$= \vec{r}_{cm} \times m \vec{v}_{cm} + I_{cm} \omega \hat{\theta}$$

$$= \vec{r}_{cm} \hat{r} \times m \vec{v}_{cm} \hat{\theta} + I_{cm} \omega \hat{\theta}$$

$$= (m r_{cm}^2 \omega + I_{cm} \omega) \hat{\theta}$$

$$= \omega (I_{cm} + m r_{cm}^2)$$

$$= I_S \omega \hat{\theta}$$

Energy

$$K_i = \frac{m_i v_i^2}{2} = \frac{m_i r_i^2 \omega^2}{2}$$

$$K = I_S \sum K_i = \frac{I_S \omega^2}{2}$$

Alternatively,

$$K = K_{trans} + K_{rot}$$

$$= \frac{m V_{cm}^2}{2} + \frac{I_{cm} \omega^2}{2}$$

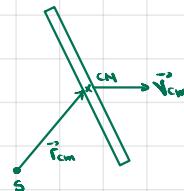
$$V_{cm} = \vec{r}_{S,cm} \omega$$

$$\Rightarrow K = \frac{m \vec{r}_{S,cm}^2 \omega^2}{2} + \frac{I_{cm} \omega^2}{2}$$

$$= \frac{\omega^2}{2} (I_{cm} + m \vec{r}_{S,cm}^2)$$

$$= \frac{I_S \omega^2}{2}$$

$$\curvearrowright \omega \circ$$



$$\vec{L}_{sys} = \vec{L}_{S,cm} + \vec{L}_{cm,rel}$$

$$= \vec{r}_{cm} \times m \vec{v}_{cm} + I_{cm} \omega \hat{\theta}$$

$$K_{sys} = \frac{m \vec{r}_{cm}^2 \omega}{2} + \frac{I_{cm} \omega^2}{2}$$

D.D.3.1 Gyroscopes 1



System undergoing precession: gyroscope



Torque

$$\vec{\tau}_s = d\vec{r} \times (-mg\hat{k}) = -dmg\hat{i}$$

Case 1: not spinning

$$\vec{L}_{s,ui} = 0$$

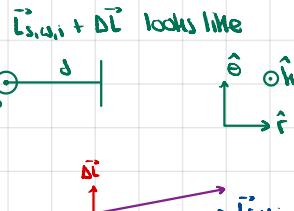
$\Delta\vec{L} = \vec{\tau}_s \Delta t$, ie torque due to gravity provides an angular impulse, which points in same direction as torque \hat{i} .
→ the wheel rotates towards earth.

Case 2: spinning

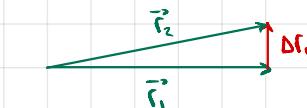
$$\vec{L}_{s,ui} \neq 0$$

The wheel has angular momentum about CM. In the depiction above $\vec{L}_{cm,ui}$ points in \hat{r} dir.

There is still torque due to gravity.



Torque due to gravity changes the direction of angular momentum vector.



$$|\Delta L_i| \ll L_i$$

$$\Delta L_i \cdot L_i = 0$$

$$\vec{L}_i = \vec{L}_i + \Delta \vec{L}_i$$

For small $\Delta\theta$

$$L_i \approx \frac{r_i}{\cos(\Delta\theta)} \approx r_i \equiv r$$

→ When there is a large vector r_i and a small perpendicular vector is added to it, the effect is that r_i rotates with no change in length.

$$\text{Also } |\Delta r| = |r| \sin(\Delta\theta) \approx |r| \Delta\theta$$

$$\Rightarrow \left| \frac{\Delta \vec{r}}{\Delta t} \right| = |r| \frac{\Delta\theta}{\Delta t}$$

$$\rightarrow \left| \frac{d\vec{r}}{dt} \right| = r \frac{d\theta}{dt} = r \omega$$

$$v = r\omega$$

For rotating angular momentum vector

$$\left| \frac{dL}{dt} \right| = L \omega$$

$$\frac{dL}{dt} = \vec{\tau}$$

$$\Rightarrow |\vec{L}| = L \omega$$

$$\vec{L}_s = I \omega \hat{r}$$

large vector, small perpendicular
 $\vec{L}_s = mgd \hat{i}$ changes → rotating vector

$$\left| \frac{dL_s}{dt} \right| = \omega L_s = I \omega \omega \hat{r}$$

angular speed of the rotating vector.
precessional angular velocity

$$= |\vec{L}_s| = mgd$$

$$\Rightarrow \omega = \frac{mgd}{I \omega}$$

$$\frac{dL_s}{dt} = I \alpha \hat{\theta}$$

Rotation

sideview



Top View



Looking at the view from S, if it spins clockwise then $L_{s,ui}$ points in $+\hat{r}$ dir, \vec{L} points in $\hat{\theta}$, so from the top view the rope and wheel rotate counterclockwise about the pivot S, ie $\vec{L} = -r\hat{i}$.

In the other case the wheel spins counterclockwise.

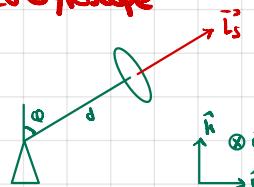
$L_{s,ui}$ points in $-\hat{r}$ dir.

\vec{L} is same.

$$\vec{L} = -r\hat{i}$$

Note that we used the approximations about vectors that lead to the result that a vector rotates. The angular momentum vector of the wheel must be very large relative to the angular momentum increments, ie the torque vector. In practice this means $\omega \gg \omega_L$ "gyroscopic approximation"

Tilted Gyroscope



$$\vec{L}_s = \vec{L}_{s,z} + \vec{L}_{s,r} = L_{s,z} \hat{i} + L_{s,r} \hat{r}$$

$$L_{s,z} = \text{constant}$$

$L_{s,r}$ is rotating

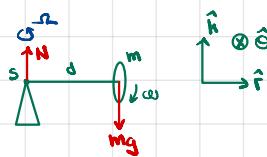
$$\left| \frac{dL_s}{dt} \right| = \left| \frac{dL_z}{dt} \right| + \left| \frac{dL_r}{dt} \right| = \omega L_r = \omega L_s \sin\phi$$

$$\Rightarrow \omega L_s \sin\phi = mgd$$

$$\omega = \frac{mgd}{I \omega \sin\phi}$$

D.D.3.3 Gyroscopes 3 - Nutation and

Total Angular Momentum



When the system is precessing, the CM is orbiting the z-axis through S. Thus, there is a component of angular momentum in the k direction.

The gyroscope dips by a very small angle. L_S thus has a z-component that balances the z-comp. of the angular momentum of the (circular) translational motion of the CM.

$$\vec{L}_S = \vec{L}_S^{\text{rot}} + \vec{L}_S^{\text{spin}}$$

transl CM ↗ rot about CM

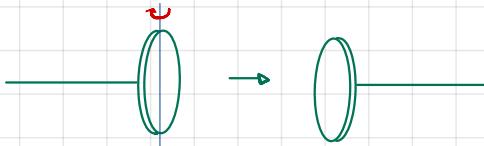
$$\vec{L}_S^{\text{rot}} = \vec{d}\hat{i} \times m v_{\text{CM}} \hat{k} = d m v_{\text{CM}} \hat{k}$$

$$= d m \omega R \hat{k}$$

The spin ang. mom. has a subtlety.

The wheel is spinning about an axis through CM in i dir with ω.

But it also spins about an axis through CM in k dir with ω.



$$\vec{L}_S^{\text{spin}} = I_1 \omega \hat{i} + I_2 \omega \hat{k}$$

$$I_1 = \frac{mR^2}{2}$$

$$I_2 = \frac{mR^2}{4}$$

$$\vec{L}_S = I_1 \omega \hat{i} + \underbrace{I_2 \omega \hat{k}}_{\text{rotating}} + \underbrace{d'm \omega R \hat{k}}_{\text{constant}}$$

This is the exact expression for \vec{L}_S .

The gyroscopic effect says essentially that the first term is very large relative to the other two, ie it dominates.