Pset 9 - Collision Theory

Problem 1 - Ping Pong Ball and Bowling Ball Collision

elastic collision
frictionless surface

$v^{rel}_{p,b,i} = -v^{rel}_{p,b,f}$

$v_{p,0} + v_{b,0} = v_{p,f} - v_{b,f}$ (energy-momentum principle)

$m_p v_{p,0} + m_b v_{b,0} = m_p v_{p,f} + m_b v_{b,f}$ (conservation of momentum)

$= m_p(v_{p,0} + v_{b,0} + v_{b,f}) + m_b v_{b,f}$

$v_{b,f}(m_b + m_p) = \cancel{m_p v_{p,0}} + m_b v_{b,0} - m_p(\cancel{v_{p,0}} + v_{b,0})$

$v_{b,f} = \dfrac{v_{b,0}(m_b - m_p)}{m_p + m_b} \Rightarrow v_{b,f} = \dfrac{m_b - m_p}{m_b + m_p} v_{b,0}$

$v_{p,f} = \dfrac{m_b - m_p}{m_b + m_p} v_{b,0} + v_{b,0} + v_{p,0}$

$= \dfrac{(m_b - m_p)v_{b,0} + (m_b + m_p)v_{b,0}}{m_b + m_p} + v_{p,0}$

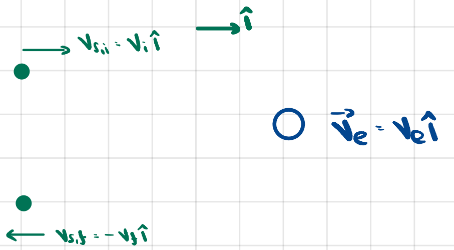
$v_{p,f} = \dfrac{2m_b}{m_b + m_p} v_{b,0} + v_{p,0}$

Note

$\lim_{m_b \to \infty} v_{p,f} = 2v_{b,0} + v_{p,0}$

$\lim_{m_b \to \infty} v_{b,f} = v_{b,0}$

Problem 2 - Satellite Flyby



Let's consider this an elastic collision.

$$v^{rel}_{s,e,i} = -v^{rel}_{s,e,f}$$

$$v_i - v_e = -(-v_f - v_e) \Rightarrow 3v_e = v_f + v_e \Rightarrow v_f = 2v_e$$

In this calculation we approximated the velocity of earth as having stayed constant in the collision. In reality, earth's speed must increase.

$$m_s v_i + m_e v_{e,i} = -m_s v_f + m_e v_{e,f} \qquad \text{(conservation of momentum)}$$

$$v_{e,f} = \frac{m_s(v_i + v_f) + m_e v_{e,i}}{m_e}$$

$$v_{e,f} = v_{e,i} + \frac{m_s}{m_e}(v_i + v_f) > v_{e,i} \quad \text{if } v_i, v_f > 0 \qquad (1)$$

$$v_i - v_{e,i} = -(-v_f - v_{e,f}) = v_f + v_{e,f}$$

$$v_f = v_i - (v_{e,f} + v_{e,i}) \qquad \text{(cm. momentum principle)} \qquad (2)$$

unknowns: v_f, $v_{e,f}$

(2) → (1)

$$v_{e,f} = v_{e,i} + \frac{m_s}{m_e}(2v_i - v_{e,f} - v_{e,i})$$

$$v_{e,f}\left(1 + \frac{m_s}{m_e}\right) = v_{e,i}\left(1 - \frac{m_s}{m_e}\right) + 2v_i\frac{m_s}{m_e}$$

$$v_{e,f}\frac{m_e + m_s}{m_e} = v_{e,i}\frac{m_e - m_s}{m_e} + 2v_i\frac{m_s}{m_e}$$

$$v_{e,f} = \frac{m_e - m_s}{m_e + m_s}v_{e,i} + \frac{2m_s}{m_e + m_s}v_i$$

$$v_f = v_i - v_{e,i} - \left(\frac{m_e - m_s}{m_e + m_s}v_{e,i} + \frac{2m_s}{m_e + m_s}v_i\right)$$

$$= \frac{(m_e - m_s)v_i}{m_e + m_s} - \frac{v_{e,i}(m_e + \cancel{m_s}) - (m_e - \cancel{m_s})v_{e,i}}{m_e + m_s}$$

$$= \frac{(m_e - m_s)v_i}{m_e + m_s} - \frac{2m_e v_{e,i}}{m_e + m_s}$$

$$\lim_{m_s \to 0} v_f = v_i - 2v_{e,i}$$

$$v_{e,f} = \frac{m_e - m_s}{m_e + m_s}v_{e,i} + \frac{2m_s v_i}{m_e + m_s}$$

$$\lim_{m_s \to 0} v_{e,f} = v_{e,i}$$

Note that

$$v_{e,f} = \frac{m_e v_{e,i} + m_s(2v_i - v_{e,i})}{m_e + m_s}$$

$$v_i = v_{e,i} + v_{e,f} + v_f$$

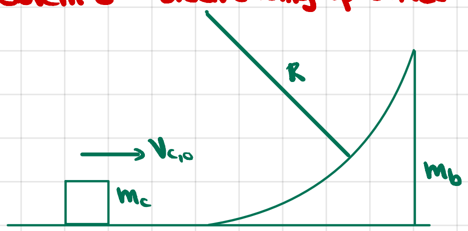
$$= v_{e,i} + v^{rel}_{e,s,f} > v_{e,i}$$

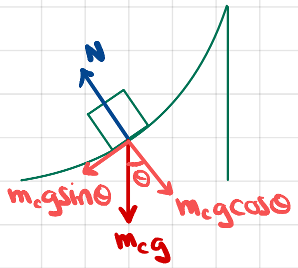
$$\Rightarrow 2v_i > 2v_{e,i}$$

$$\Rightarrow 2v_i - v_{e,i} > v_{e,i}$$

$$\therefore v_{e,f} > v_{e,i}$$

Problem 3 – Block Sliding up Curved Surface





$$\hat{r} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$
$$\hat{\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$$

$$\frac{d\hat{\theta}}{dt} = -\cos\theta\,\theta'\,\hat{\imath} - \sin\theta\,\theta'\,\hat{\jmath} = \theta'(-\hat{r}) = -\theta'\hat{r}$$

$$\frac{d\hat{r}}{dt} = -\sin\theta\,\theta'\,\hat{\imath} + \cos\theta\,\theta'\,\hat{\jmath} = \theta'\hat{\theta}$$

$m_c < m_b$

a) The cube starts a circular trajectory relative to the block w/ initial velocity $V_{c,0}\hat{\theta}$

$$\vec{F}_g = m_c g\cos\theta\,\hat{r} - m_c g\sin\theta\,\hat{\theta}$$

$$\vec{r} = r\hat{r}$$
$$\vec{v} = r\frac{d\hat{r}}{dt} = r(-\sin\theta\cdot\theta'\hat{\imath} + \cos\theta\cdot\theta'\hat{\jmath}) = r\theta'\hat{\theta}$$
$$\vec{a} = r\theta''\hat{\theta} + r\theta'(-\theta')\hat{r} = -r\theta'^2\hat{r} + r\theta''\hat{\theta}$$

$$\vec{r} = R\hat{r}$$
$$d\vec{r} = R\,d\hat{r} = R\,d\theta\,\hat{\theta}$$

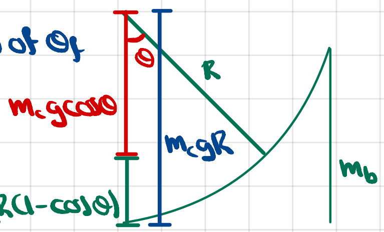
$$\vec{F}_g\cdot d\vec{r} = -m_c g\sin\theta\,R\,d\theta$$

$$\int_C \vec{F}_g\cdot d\vec{r} = -m_c gR\int_0^{\theta_f}\sin\theta\,d\theta = -m_c gR(-\cos\theta)\Big|_0^{\theta_f}$$
$$= m_c gR(\cos\theta_f - 1) < 0$$

System is block + cube. F_g is the only external force.

$$\Delta E_m = 0 \Rightarrow \Delta K + \Delta U = 0 \Rightarrow \frac{(m_b+m_c)v_f^2}{2} - \frac{m_c V_{c,0}^2}{2} + m_c gR(1-\cos\theta_f) = 0$$

note that we can write this in terms of h instead of θ_f



$\Delta U = -m_c gR(1-\cos\theta)$

$$\Delta E_m = \frac{(m_b+m_c)v_f^2}{2} - \frac{m_c V_{c,0}^2}{2} + m_c gh = 0$$

unknowns: v_f, θ_f or h

initial momentum: $m_c V_{c,0}\hat{\imath}$

Final momentum: $(m_c+m_b)V_f\hat{\imath}$

$$m_c V_{c,0} = (m_c+m_b)V_f \Rightarrow V_f = \frac{m_c}{m_c+m_b}V_{c,0}$$

sub into $\Delta E_m = 0$, solve for θ_f or h

$$\Rightarrow h = \frac{V_{c,0}^2\cdot m_b}{2g(m_b+m_c)} \qquad \theta_f = \arccos\left[\frac{2gR(m_b+m_c) - V_{c,0}^2 m_b}{2gR(m_b+m_c)}\right]$$

b) $p_{o,x} = (m_b + m_c) v_{f,1}$

$p_{f,x} = m_b v_{f,b} + m_c v_{f,c}$

$\Delta E_m = -m_c g h + \frac{m_b v_{f,b}^2}{2} + \frac{m_c v_{f,c}^2}{2} - \frac{(m_b + m_c) v_{f,1}^2}{2}$

$\Delta p = m_b v_{f,b} + m_c v_{f,c} - (m_b + m_c) v_{f,1}$

unknowns: $v_{f,b}, v_{f,c}$

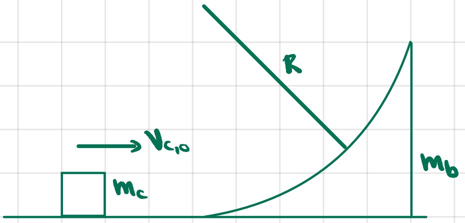
$$v_{f,1} = \frac{m_c}{m_c + m_b} v_{c,0}$$

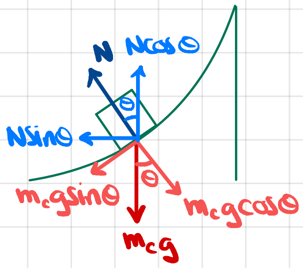
$$h = \frac{v_{c,0}^2 \cdot m_b}{2g(m_b + m_c)}$$

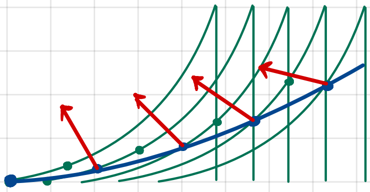
we are equating momentum and kinetic energy at the top of the cube's trajectory and at the bottom. one solution is

$$\Rightarrow v_{f,b} = \frac{2 m_c v_{c,0}}{m_b + m_c} \qquad v_{f,c} = -\frac{(m_b - m_c) v_{c,0}}{m_b + m_c}$$

the other sol'n is $v_{f,b} = 0$, $v_{f,c} = v_{c,0}$







$$N_{x,t} = m_c a_{c,x,t} = m_c \frac{dv_{c,x,t}}{dt} = m_c \dot{v}_{c,x,t}$$

$$N_{y,t} - m_c g = m_c a_{c,y,t} = m_c \frac{dv_{c,y,t}}{dt} = m_c \dot{v}_{c,y,t}$$

$$\left.\begin{aligned} &\int_0^t (N_y - m_c g)\,dt = m_c v_{c,y,t} \\ &m_c v_{c,0} = m_c v_{c,x,t} + m_b v_{b,x,t} \end{aligned}\right\} \text{momentum}$$

$$\frac{m_c v_{c,0}^2}{2} = \frac{m_b v_{b,t}^2}{2} + \frac{m_c v_{c,t}^2}{2}$$

$$\Rightarrow m_b \cdot v_{b,t} \dot{v}_{b,t} = -m_c v_{c,t} \dot{v}_{c,t}$$

$$m_b \left(v_{b,x,t}^2 + v_{b,y,t}^2\right)^{1/2} \cdot \frac{1}{2}\left(v_{b,x,t}^2 + v_{b,y,t}^2\right)^{-\frac{1}{2}} \left(2v_{b,x,t}\dot{v}_{b,x,t} + 2v_{b,y,t}\dot{v}_{b,y,t}\right)$$

$$= m_b \cdot \left(v_{b,x,t}\dot{v}_{b,x,t} + v_{b,y,t}\dot{v}_{b,y,t}\right) = -m_c\left(v_{c,x,t}\dot{v}_{c,x,t} + v_{c,y,t}\dot{v}_{c,y,t}\right)$$

Equations

$$m_b \cdot \left(v_{b,x,t}\dot{v}_{b,x,t} + v_{b,y,t}\dot{v}_{b,y,t}\right) = -m_c\left(v_{c,x,t}\dot{v}_{c,x,t} + v_{c,y,t}\dot{v}_{c,y,t}\right) = 0$$

$$N_{x,t} = m_c \dot{v}_{c,x,t}$$

$$N_{y,t} - m_c g = m_c \dot{v}_{c,y,t}$$

$$m_c v_{c,0} = m_c v_{c,x,t} + m_b v_{b,x,t}$$

$$\int_0^t (N_y - m_c g)\,dt = m_c v_{c,y,t}$$

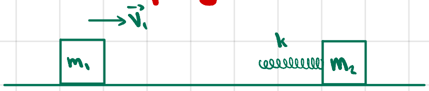
$$-N_{x,t} = m_b \cdot \dot{v}_{b,x,t}$$

unknowns: $v_{b,x}(t)$, $\dot{v}_{b,x}(t)$, $v_{b,y}(t)$, $\dot{v}_{b,y}(t)$

$v_{c,x}(t)$, $\dot{v}_{c,x}(t)$, $v_{c,y}(t)$, $\dot{v}_{c,y}(t)$

$N_x(t)$, $N_y(t)$

Problem 4 - Spring Blocks Motion



energy conservation: $\dfrac{(m_1+m_2)v_f^2}{2} - \dfrac{m_1 v_1^2}{2} + \dfrac{kx^2}{2} = 0$

momentum: $m_1 v_1 = (m_1+m_2)v_f \Rightarrow v_f = \dfrac{m_1}{m_1+m_2} v_1$

$$\Rightarrow \frac{\cancel{m_1+m_2}}{\cancel{2}} \frac{m_1^2 v_1^2}{(m_1+m_2)^{\cancel{2}}} - \frac{m_1 v_1^2}{\cancel{2}} + \frac{kx^2}{\cancel{2}} = 0$$

$$\cancel{m_1^2 v_1^2} - m_1(\cancel{m_1}+m_2)v_1^2 + (m_1+m_2)kx^2$$

$$x = \sqrt{\frac{m_1 m_2 v_1^2}{(m_1+m_2)k}} = v_1 \sqrt{\frac{m_1 m_2}{(m_1+m_2)k}}$$

m_1 compresses the spring. The spring force is an internal force: it decelerates m_1 and accelerates m_2. At some point m_1 and m_2 have same speed. At this point whatever compression is occurring on m_1's side is offset by expansion on m_2's side. But deceleration of m_1 and accel. of m_2 continue.

$$m_1 v_{1f} + m_2 v_{2f} = m_1 v_1$$

$$\frac{m_1 v_{1f}^2}{2} + \frac{m_2 v_{2f}^2}{2} = \frac{m_1 v_1^2}{2}$$

There are two solutions. One is the sol'n representing the initial state: $v_{1,f} = v_1$, $v_{2,f} = 0$.

The other is $v_{1,f} = \dfrac{v_1(m_1 - m_2)}{m_1+m_2}$ $\quad v_{2,f} = \dfrac{2m_1 v_1}{m_1+m_2}$

As the spring pushes on both masses, it changes the momentum of each mass equally. The larger of the two masses experiences a smaller change in speed.

Problem 5 - Bullet-Spring Collision

a) $m_b v_b = (m_b + M) v_f \Rightarrow v_f = \dfrac{m_b v_b}{m_b + M}$ (conservation of momentum during totally inelastic collision)

b) $\dfrac{kx^2}{\cancel{2}} - \dfrac{1}{\cancel{2}} \cancel{(m_b+M)} \dfrac{m_b^2 v_b^2}{(m_b+M)^{\cancel{2}}} = 0$ (conservation of mechanical energy during compression of spring)

$$x = \sqrt{\frac{m_b^2 v_b^2}{k(m_b+M)}} = \frac{m_b v_b}{(k(m_b+M))^{1/2}}$$

c) $$K_{collided} = \frac{\cancel{(m_b+M)} \cdot m_b^2 v_b^2}{2 \cdot (m_b+M)^{\cancel{2}}} = \frac{m_b^2 v_b^2}{2(m_b+M)}$$

$$K_{initial} = \frac{m_b v_b^2}{2}$$

$\dfrac{K_{collided}}{K_{initial}} = \dfrac{m_b}{m_b + M} < 1$, energy is lost in a totally inelastic collision.

d) $$v_2 = \frac{m_b v_b}{2(m_b+M)}$$

inelastic collision occurs, speed after collision is $\dfrac{m_b v_b}{m_b + M}$.

work done by friction is $\int_0^x -\mu_k (m_b+M) g \, dx = -(m_b+M) g \mu_k x$

on the way back to $x=0$, friction does the same work.

$$-2(m_b+M)\mu_k g x = \frac{1}{2}(m_b+M)\left(\frac{m_b v_b}{2(m_b+M)}\right)^2 - \frac{1}{2}(m_b+M)\left(\frac{m_b v_b}{m_b+M}\right)^2$$

$$= -\frac{3}{8}\cancel{(m_b+M)} \cdot \frac{m_b^2 v_b^2}{(m_b+M)^{\cancel{2}}}$$

$$\Rightarrow 2(m_b+M)^2 \mu_k g x = \frac{3}{8} m_b^2 v_b^2$$

$$\Rightarrow x = \frac{3}{16} \cdot \frac{m_b^2 v_b^2}{(m_b+M)^2 \mu_k g} = \frac{3}{16 \mu_k g} \cdot v_{collision}^2$$

The block travels twice this distance x before returning to $x=0$.

Problem 6 - Elastic 2D Collision

$m_1\vec{v}_{1,i} = m_1\vec{v}_{1,f} + m_2\vec{v}_{2,f}$ $\qquad \Delta\vec{p} = 0$

$m_1(v_1\hat{\imath}) = m_1(v_{1,f}\cos\theta_{1,f}\hat{\imath} + v_{1,f}\sin\theta_{1,f}\hat{\jmath}) + m_2(v_{2,f}\cos\theta_{2,f}\hat{\imath} - v_{2,f}\sin\theta_{2,f}\hat{\jmath})$

$\cancel{m_1}v_{1,i} = \cancel{m_1}v_{1,f}\cos\theta_{1,f} + 2\cancel{m_1}v_{2,f}\cos\theta_{2,f}$ $\qquad \Delta p_x = 0$

$0 = \cancel{m_1}v_{1,f}\sin\theta_{1,f} - 2\cancel{m_1}v_{2,f}\sin\theta_{2,f}$ $\qquad \Delta p_y = 0$

$\frac{m_1 v_{1,i}^2}{2} = \frac{m_1 v_{1,f}^2}{2} + \frac{2m_1 v_{2,f}^2}{2} \Rightarrow v_{1,i}^2 = v_{1,f}^2 + 2v_{2,f}^2$ $\qquad \Delta K = 0$

$v_{1,i} - v_{1,f}\cos\theta_{1,f} = 2v_{2,f}\cos\theta_{2,f} \Rightarrow v_{1,i} - 2v_{2,f}\cos\theta_{2,f} = v_{1,f}\cos\theta_{1,f}$ $\qquad$ rewrite momentum

$v_{1,f}\sin\theta_{1,f} = 2v_{2,f}\sin\theta_{2,f} \Rightarrow 2v_{2,f}\sin\theta_{2,f} = v_{1,f}\sin\theta_{1,f}$

$v_{1,i}^2 + 4v_{2,f}^2\cos^2\theta_{2,f} - 4v_{1,i}v_{2,f}\cos\theta_{2,f} = v_{1,f}^2\cos^2\theta_{1,f}$

square and add

$4v_{2,f}^2\sin^2\theta_{2,f} = v_{1,f}^2\sin^2\theta_{1,f}$

$\cancel{v_{1,i}^2} + 4v_{2,f}^2 - 4v_{1,i}v_{2,f}\cos\theta_{2,f} = v_{1,f}^2 = \cancel{v_{1,i}^2} - 2v_{2,f}^2$

$6v_{2,f}^2 - 4v_{1,i}v_{2,f}\cdot\frac{\sqrt{2}}{2} = 0$

$v_{2,f}(6v_{2,f} - 2\sqrt{2}v_{1,i}) = 0$

$$\begin{cases} v_{2,f} = \frac{\sqrt{2}v_{1,i}}{3} \\ v_{2,f} = 0 \end{cases}$$

solution 1

$v_{2,f} = 0$

$v_{1,f} = \pm v_{1,i}$

$v_{1,f}\cdot\sin\theta_{1,f} = 0 \Rightarrow \theta_{1,f} = 0$

ie, there is no collision.

solution 2

$v_{2,f} = \frac{\sqrt{2}v_{1,i}}{3}$

$v_{1,f}^2 = v_{1,i}^2 - \frac{4}{9}v_{1,i}^2 = \frac{5}{9}v_{1,i}^2 \Rightarrow v_{1,f} = \pm\frac{\sqrt{5}v_{1,i}}{3}$

$\frac{\cancel{2}\sqrt{2}\cancel{v_{1,i}}}{\cancel{3}}\cdot\frac{\sqrt{2}}{\cancel{2}} = \frac{\sqrt{5}\cancel{v_{1,i}}}{\cancel{3}}\sin\theta_{1,f}$

$\sin\theta_{1,f} = \frac{2}{\sqrt{5}} \Rightarrow \theta_{1,f} \approx 1.10\text{ rad} \approx 63.43°$