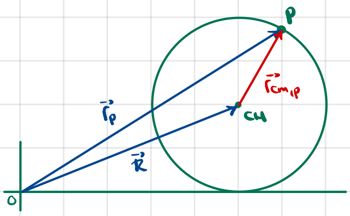
35.1 Translation and Rotation of a Wheel

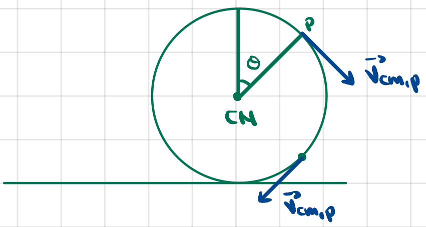


Two reference frames w/ origins at O and CM.

$\vec{r}_p = \vec{R} + \vec{r}_{cm,p}$

$\vec{v}_p = \vec{V} + \vec{v}_{cm,p}$

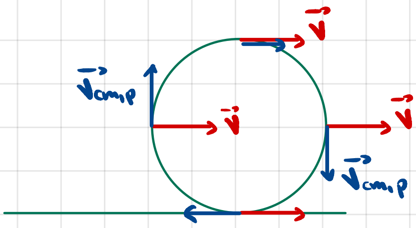
35.2 Rolling Wheel in CM Frame

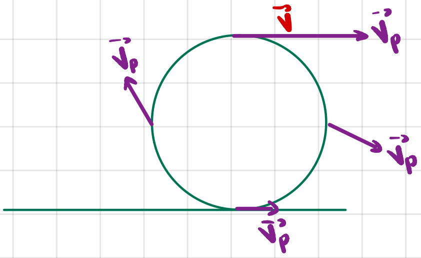


$v_{cm,p} = R|\theta'(t)|$

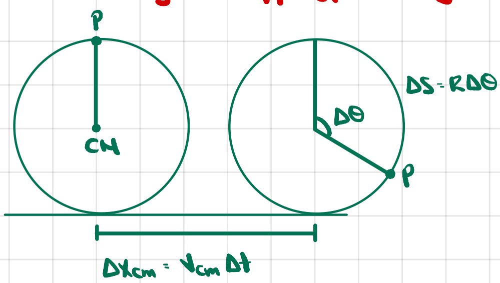
From CM frame, P has a circular motion.

35.3 Rolling Wheel in the Ground Frame





35.4 Rolling w/o Slipping, Skidding



Rolling without slipping condition

$\Delta X_{cm} = \Delta S$

$V_{cm}\Delta t = R\Delta\theta$

$V_{cm} = R\dfrac{\Delta\theta}{\Delta t}$

$\lim\limits_{\Delta t \to 0} \dfrac{\Delta\theta}{\Delta t} = \omega$

$\Rightarrow V_{cm} = R\omega$: tangential speed in CM frame

Slipping

$\Delta S > \Delta X_{cm}$

$R\Delta\theta > V_{cm}\Delta t$

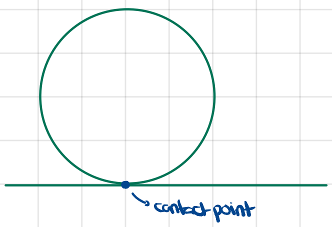
$\Rightarrow V_{cm} < R\omega$

Skidding

$\Delta S < \Delta X_{cm}$

$V_{cm} > R\omega$

35.5 Contact Point of a Wheel Rolling Without Slipping



$\vec{v}_p = \vec{V}_{cm} + \vec{v}_{cm,p}$

$|\vec{v}_{cm,p}| = R\omega$

$|\vec{V}_{cm}| = V_{cm}$

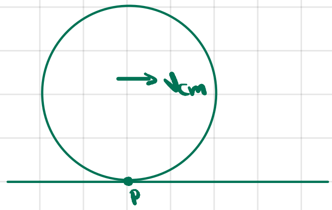
no slipping $\Rightarrow |\vec{v}_{cm,p}| = |\vec{V}_{cm}|$

$\Rightarrow \vec{v}_p = \vec{0}$

contact point is instantaneously at rest w/ respect to the ground.

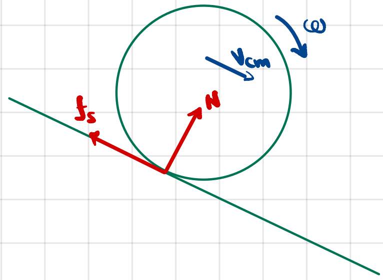
36.1 Friction on a Rolling Wheel

Horizontal Surface



$f_s = 0$ (static friction)

Inclined Surface



f_s produces torque

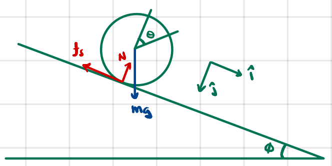
$V_{cm} = R\omega$

$a_{cm} = R\alpha$ non-zero angular acceleration

$\tau_{cm} = I_{cm}\alpha$

$\vec{\tau}_{cm} = \vec{r}_{cm,f_s} \times \vec{f}_s$

36.2 Worked Example - Wheel Rolling Without Slipping Down Inclined Plane - Torque Method



$\hat{i}$: $mg\sin\phi - f_s = m a_x$

$\hat{k}$: $f_s R = I_{cm}\alpha$

$v_{cm} = R\omega$

$a_{cm} = R\alpha$

$a_x = a_{cm}$

3 eq., 3 unknowns: a_{cm}, f_s, α

$$mg\sin\theta - \frac{I_{cm}\cdot a_{cm}}{R^2} = m a_{cm}$$

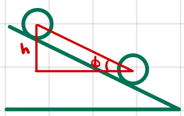
$$a_{cm} = \frac{mg\sin\phi}{m + \frac{I_{cm}}{R^2}}$$

1D Kinematics

$$x_{cm} = \frac{a_{cm}t^2}{2} = s$$

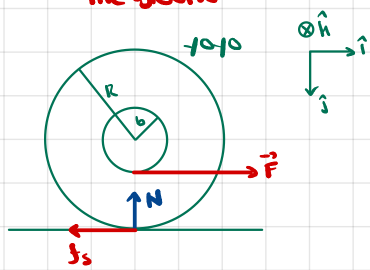
$$v_{cm} = a_{cm}t = \sqrt{2 s a_{cm}}$$

$$s = \frac{h}{\sin\phi}$$



$$\Rightarrow v_{cm} = \sqrt{\frac{2h}{\sin\phi}\cdot\frac{mg\sin\phi}{m + \frac{I_{cm}}{R^2}}}$$

36.4 Worked Example - Yo-yo pulled along the ground



$\vec{F} = F\hat{i}$

$\vec{f_s} = -f_s\hat{i}$

$\vec{F} - \vec{f_s} = m\vec{a} \Rightarrow F - f_s = m a_x$

$$\vec{\tau} = b\hat{j}\times F\hat{i} + R\hat{j}\times(-f_s\hat{i})$$
$$= -bF\hat{k} + Rf_s\hat{k} = (Rf_s - bF)\hat{k}$$

2 eq

$F - f_s = m a$ — acceleration of the system ie a_{cm}

$R f_s - bF = I_{cm}\alpha$ — expression for torque for rigid body undergoing rotation about a fixed axis

$a = R\alpha$

unknowns: a, α, f_s

$$\alpha = \frac{a}{R} = \frac{F - f_s}{mR}$$

$$Rf_s - bF = \frac{I_{cm}(F - f_s)}{mR}$$

$$f_s\left(R + \frac{I_{cm}}{mR}\right) = F\left(b + \frac{I_{cm}}{mR}\right)$$

$$F = \frac{f_s\left(R + \frac{I_{cm}}{mR}\right)}{b + \frac{I_{cm}}{mR}}$$

$$F_{max} = \mu mg\,\frac{R + \frac{I_{cm}}{mR}}{b + \frac{I_{cm}}{mR}}$$

$\downarrow$

max force such that yo-yo rolls w/o slipping

36.5

$$\vec{\tau}_P = \vec{\tau}_{P,cm} + \vec{\tau}_{cm,sys}$$

$$= \vec{r}_{P,cm}\times\vec{F} + \sum \vec{r}_{cm,j}\times\vec{F}_j$$

choose $P = CM$

$$\Rightarrow \vec{\tau}_{cm,sys} = \sum \vec{r}_{cm,j}\times\vec{F}_j$$

Also, $\vec{\tau}_{cm,sys} = \dfrac{d\vec{L}_{cm,sys}}{dt}$

$$\vec{L}_{P,sys} = \vec{r}_{P,cm}\times m\vec{v}_{P,cm} + I_{cm}\vec{\omega}$$

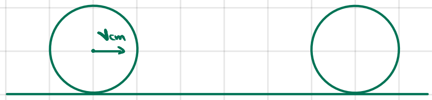
$P = CM \Rightarrow \vec{L}_{cm,sys} = I_{cm}\vec{\omega}$

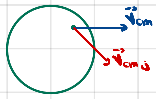
$$\frac{d\vec{L}_{cm,sys}}{dt} = I_{cm}\alpha$$

Rotational Dynamics: $\vec{\tau}_{cm,sys} = \dfrac{d\vec{L}_{cm,sys}}{dt}$

Linear Dynamics: $\vec{F} = m_T\vec{a}_{cm}$

37.1 Kinetic Energy of Translation and Rotation

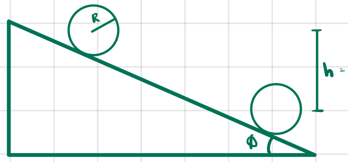


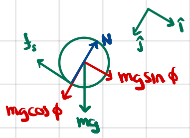


$$\vec{v}_j = \vec{v}_{cm} + \vec{v}_{cm,j}$$

$$K = \sum \frac{1}{2} m_j(\vec{v}_{cm} + \vec{v}_{cm,j})(\vec{v}_{cm} + \vec{v}_{cm,j})$$
$$= \sum \frac{m_j}{2}\vec{v}_{cm}^{\,2} + \cancelto{0}{\sum m_j\vec{v}_{cm,j}\vec{v}_{cm}} + \sum \frac{m_j}{2} v_{cm,j}^2$$
$$= \frac{m_T\vec{v}_{cm}^{\,2}}{2} + \sum \frac{m_j r_j^2\omega^2}{2}$$
$$= \frac{m_T v_{cm}^2}{2} + \frac{I_{cm}\omega^2}{2}$$
$$= K_{trans} + K_{rot}$$

37.2 Worked Example: Wheel Rolling Without Slipping Down Inclined Plane





Torque: $\vec{\tau}_{cm,sys} = R\hat{j}\times(-f_s\hat{i}) = Rf_s\hat{k}$

$E_{m_i} = mgh$

$$E_{m_f} = \frac{m v_{cm,f}^2}{2} + \frac{I_{cm}\omega^2}{2}$$

$v_{cm,f} = R\omega$

$$\Rightarrow \frac{m v_{cm,f}^2}{2} + \frac{I_{cm}v_{cm,f}^2}{2R^2} = mgh$$

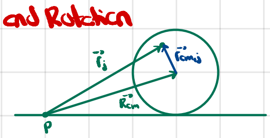
$$v_{cm,f}^2\left(m + \frac{I_{cm}}{R^2}\right) = 2mgh$$

$$v_{cm,f} = \sqrt{\frac{2mgh}{m + \frac{I_{cm}}{R^2}}}$$

$E_{m_i} = 0$ $\quad E_{m_f} = -mgh + \dfrac{I_{cm}\omega^2}{2}$

$$\omega = \sqrt{\frac{2mgh}{I_{cm}}}$$

37.3 Angular Momentum of Translation and Rotation



$\vec{L}_P = \sum_j \vec{r}_j \times m_j \vec{v}_j$

$\vec{r}_j = \vec{R}_{cm} + \vec{r}_{cm,j}$

$\vec{L}_P = \sum_j (\vec{R}_{cm} + \vec{r}_{cm,j}) \times m_j (\vec{V}_{cm} + \vec{v}_{cm,j})$

$= \sum_j \vec{R}_{cm} \times m_j \vec{V}_{cm} + (\cancelto{0}{\sum_j m_j \vec{r}_{cm,j}}) \times \vec{V}_{cm}$

$+ \sum_j \vec{R}_{cm} \times m_j \vec{v}_{cm,j} + \sum_j \vec{r}_{cm,j} \times m_j \vec{v}_{cm,j}$

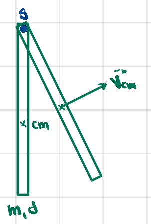
$= \vec{R}_{cm} \times (\sum_j m_j) \vec{V}_{cm} + \vec{R}_{cm} \times \cancelto{0}{\sum_j m_j \vec{v}_{cm,j}}$

$+ \sum_j \vec{r}_{cm,j} \times m_j \vec{v}_{cm,j}$

$= \vec{L}_{P,cm} + \vec{L}_{cm,sys}$

$= \vec{L}_P^{\,orbital} + \vec{L}_P^{\,spin}$

37.4 Summary of Angular Momentum and Kinetic Energy



$\vec{L}_{S,m_i} = \vec{r}_{S,i} \times m_i \vec{v}_i$

$= r_{S,i}\hat{r} \times m_i v_i \hat{\theta}$

$= r_{S,i} m_i v_i \hat{k}$

$\vec{L}_{S,sys} = \sum r_{S,i} m_i v_i \hat{k}$

$= \sum r_{S,i} m_i r_{S,i} \omega \hat{k}$

$= \omega \sum m_i r_{S,i}^2 \hat{k}$

$= I_S \omega \hat{k}$

Alternatively,

$\vec{L}_{S,sys} = \vec{L}_{S,cm} + \vec{L}_{cm,sys}$

$= \vec{r}_{cm} \times m\vec{V}_{cm} + I_{cm}\omega\hat{k}$

$= r_{cm}\hat{r} \times m V_{cm}\hat{\theta} + I_{cm}\omega\hat{k}$

$= (m r_{cm}^2 \omega + I_{cm}\omega)\hat{k}$

$= \omega (I_{cm} + m r_{cm}^2)$

$= I_S \omega$

Energy

$K_i = \frac{m_i v_i^2}{2} = \frac{m_i r_i^2 \omega^2}{2}$

$K = \lim \sum_i K_i = \frac{I_S \omega^2}{2}$

Alternatively,

$K = K_{trans} + K_{rot}$

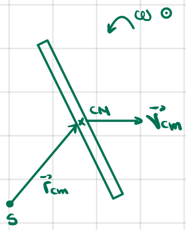
$= \frac{m V_{cm}^2}{2} + \frac{I_{cm}\omega^2}{2}$

$V_{cm} = r_{S,cm}\omega$

$\Rightarrow K = \frac{m r_{S,cm}^2 \omega^2}{2} + \frac{I_{cm}\omega^2}{2}$

$= \frac{\omega^2}{2}(I_{cm} + m r_{S,cm}^2)$

$= \frac{I_S \omega^2}{2}$



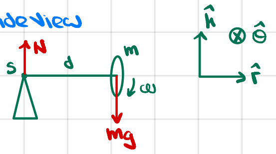
$\vec{L}_{S,sys} = \vec{L}_{S,cm} + \vec{L}_{cm,sys}$

$= \vec{r}_{cm} \times m\vec{V}_{cm} + I_{cm}\omega$

$K_{sys} = \frac{m_T V_{cm}^2}{2} + \frac{I_{cm}\omega^2}{2}$

D.D.3.1 Gyroscopes I

Side View



system undergoing precession: gyroscope

Top View

Torque

$\vec{\tau}_s = d\hat{r} \times (-mg\hat{k})$

$= dmg\hat{\theta}$

Case 1: not spinning

$\vec{L}_{s,\omega,i} = 0$

$\Delta\vec{L} = \vec{\tau}_s \Delta t$, ie torque due to gravity provides an angular impulse, which points in same direction as torque : $\hat{\theta}$.

$\Rightarrow$ the wheel rotates towards earth.

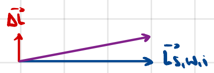
Case 2: spinning

$\vec{L}_{s,\omega,i} \neq 0$

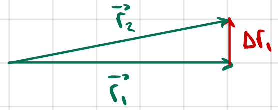
The wheel has angular momentum about CM. In the depiction above $\vec{L}_{cm,\omega,i}$ points in $\hat{r}$ dir.

There is still torque due to gravity.

$\vec{L}_{s,\omega,i} + \Delta\vec{L}$ looks like



Torque due to gravity changes the direction of angular momentum vector.



$|\Delta\vec{r}_1| \ll \vec{r}_1$

$\Delta\vec{r}_1 \cdot \vec{r}_1 = 0$

$\vec{r}_2 = \vec{r}_1 + \Delta\vec{r}_1$

For small $\Delta\theta$

$$r_2 \approx \frac{r_1}{\cos(\Delta\theta)} \approx r_1 \equiv r$$

$\Rightarrow$ When there is a large vector r_1 and a small perpendicular vector is added to it, the effect is that r_1 rotates w/o change in length.

Also $|\Delta r| = |r| \sin(\Delta\theta) \approx |r| \Delta\theta$

$$\Rightarrow \left|\frac{\Delta\vec{r}}{\Delta t}\right| = |r| \frac{\Delta\theta}{\Delta t}$$

$$\longrightarrow \left|\frac{d\vec{r}}{dt}\right| = r\frac{d\theta}{dt} = r\Omega$$

$v = r\Omega$

For rotating angular momentum vector

$$\left|\frac{dL}{dt}\right| = L\Omega$$

$$\frac{dL}{dt} = \tau$$

$\Rightarrow |\vec{\tau}| = L\Omega$

$\vec{L}_s = I\omega\hat{r}$

large vector, small perpendicular changes $\Rightarrow$ rotating vector

$\vec{\tau}_s = mgd\hat{\theta}$

$$\left|\frac{dL_s}{dt}\right| = \Omega L_s = I\omega\Omega$$

angular speed of the rotat. vector. precessional angular velocity

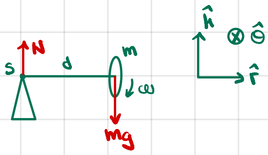
$= |\vec{\tau}_s| = mgd$

$$\Rightarrow \Omega = \frac{mgd}{I\omega}$$

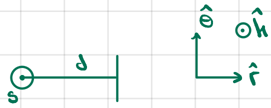
$$\frac{dL_s}{dt} = I\alpha\hat{\theta}$$

Rotation

Side View



Top View



Looking at the wheel from S, if it spins clockwise then $L_{s,\omega,i}$ points in $+\hat{r}$ dir, $\vec{\tau}_s$ points in $\hat{\theta}$, so from the top view the rope and wheel rotate counterclockwise about the pivot S, ie $\vec{\Omega} = \Omega\hat{k}$.

In the other case the wheel spins counterclockwise.

$\vec{L}_{s,\omega,i}$ points in $-\hat{r}$ dir.

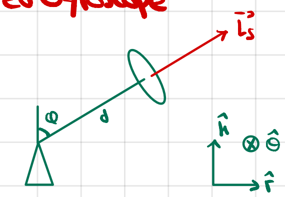
$\vec{\tau}_s$ is same.

$\vec{\Omega} = -\Omega\hat{k}$

Note that we used the approximations about vectors that lead to the result that a vector rotates. The angular momentum vector of the wheel must be very large relative to the angular momentum increments, ie the torque vector.

In practice this means $\omega \gg \Omega$ "gyroscopic approximation"

Tilted Gyroscope



$\vec{L}_s = \vec{L}_{s,z} + \vec{L}_{s,r} = L_{s,z}\hat{k} + L_{s,r}\hat{r}$

$\vec{L}_{s,z}$ = constant

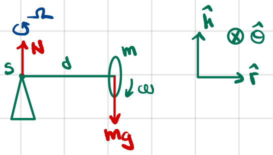
$\vec{L}_{s,r}$ is rotating

$$\left|\frac{dL_s}{dt}\right| = \left|\cancel{\frac{dL_z}{dt}}\right| + \left|\frac{dL_r}{dt}\right| = \Omega L_r = \Omega L_s \sin\phi$$

$\Rightarrow \Omega I\omega \sin\phi = mgd$

$$\Omega = \frac{mgd}{I\omega\sin\phi}$$

D.D.3.3 Gyroscopes 3 - Nutation and Total Angular Momentum



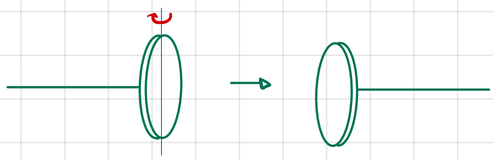
When the system is precessing, the CM is orbiting the z-axis through S. Thus, there is a component of angular momentum in the $\hat{h}$ direction.

The gyroscope dips by a very small angle. $\vec{L}_{S,\omega}$ thus has a z-component that balances the z-comp. of the angular momentum of the (circular) translational motion of the CM.

$$\vec{L}_S = \underbrace{\vec{L}_S^{orb}}_{\text{translation CM}} + \underbrace{\vec{L}_S^{spin}}_{\text{rot. about CM}}$$

$$\vec{L}_S^{orb} = d\hat{r} \times m v_{cm}\hat{\theta} = d m v_{cm}\hat{h}$$
$$= d^2 m \Omega \hat{h}$$

The spin ang. mom. has a subtlety.
The wheel is spinning about an axis through CM in $\hat{r}$ dir with ω.
But it also spins about an axis through CM in $\hat{h}$ dir with Ω.



$$\vec{L}_S^{spin} = I_1\omega\hat{r} + I_2\Omega\hat{h}$$

$$I_1 = \frac{mR^2}{2}$$

$$I_2 = \frac{mR^2}{4}$$

$$\vec{L}_S = \underbrace{I_1\omega\hat{r}}_{\text{rotating}} + \underbrace{I_2\Omega\hat{h} + d^2 m\Omega\hat{h}}_{\text{constant}}$$

This is the exact expression for $\vec{L}_S$.

The gyroscopic approx says essentially that the first term is very large relative to the other two, ie it dominates.