week 1 - Kinematics

Kinematics: study of the geometry of motion

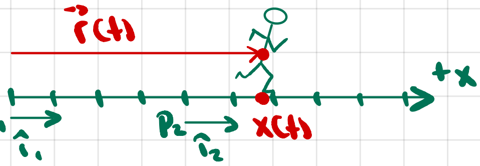
$\updownarrow$ $\vec{F} = m\vec{a}$, connects two separate questions

why? Causes of motion: dynamics

$$\underbrace{\vec{F}}_{\text{why cause?}} = \underbrace{m\vec{a}}_{\text{How? Geom. Description}}$$

1D motion

→ choose coordinate system



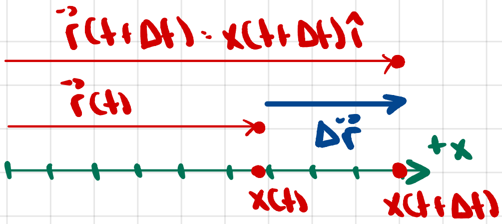
1 origin
2 axis, positive coordinate direction
3 unit vector

$\hat{i}_1 = \hat{i}_2 \quad |\hat{i}| = 1$

* there is a unit vector at every coordinate.
in this coord. system the unit vector happens to be the same everywhere, so we can drop subscripts

→ position function: $x(t)$
" vector: $\vec{r}(t) = x(t)\hat{i}$

→ displacement

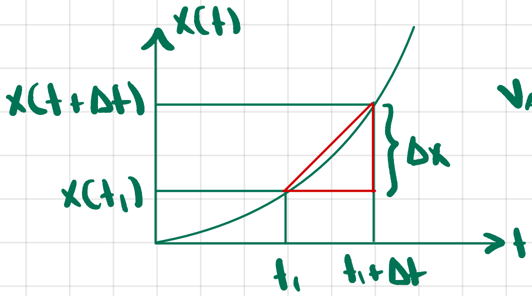


displacement vector — component of displacement vector

$$\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t) = (x(t+\Delta t) - x(t))\hat{i} = \Delta x\hat{i}$$

→ average velocity for time interval $[t, t+\Delta t]$ $\quad \vec{v}_{avg} \equiv \dfrac{\Delta\vec{r}}{\Delta t} = \dfrac{\Delta x}{\Delta t}\hat{i}$

→ velocity at some specific time t_1



V_{AVG} = (slope of line between two points)$\hat{i}$

$$\vec{v}(t_1) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}\hat{i} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}\hat{i} = \text{slope of tangent line at time } t_1$$

= instantaneous velocity at $t = t_1$

$$\Rightarrow \vec{v}(t) = \left(\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}\right)\hat{i} = \frac{dx}{dt}\hat{i}$$

↳ component of instantaneous velocity at time t, $v(t)$

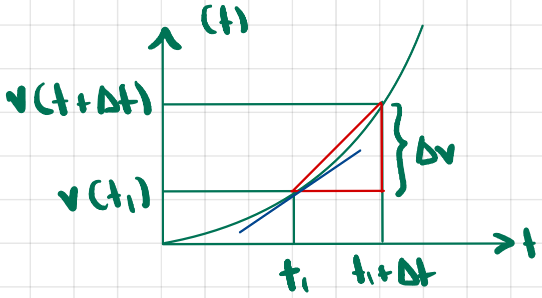
object in free fall: under the influence of gravitational force

$F_{grav} = m\vec{a}$

non-zero $\Rightarrow$ non-zero, ie when you throw an object up in the air, even at the top of the trajectory where velocity is zero, acceleration is non-zero because F_{grav} is non-zero.

"acceleration is a very abstract concept"

Acceleration in 1D



$\Delta \vec{v} = (v(t+\Delta t) - v(t))\hat{i} = \Delta v\hat{i}$

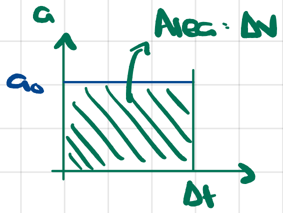
$a_{AVG} = \frac{\Delta v}{\Delta t}$

$\lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}\hat{i}$ = instantaneous acceleration = $a_x(t)\hat{i}$

Integration

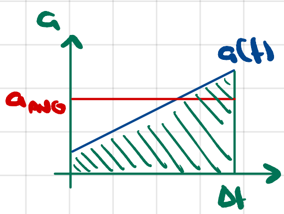
Acceleration $a(t) = a_0$ = constant

$a_0 = \frac{\Delta v}{\Delta t} \Rightarrow \Delta v = a_0 \Delta t$

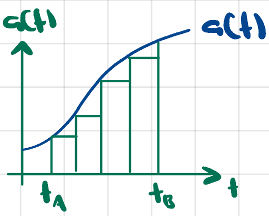


For linearly changing accel.

$a_{AVG} = \frac{\Delta v}{\Delta t} \Rightarrow \Delta v = a_{avg}\Delta t$ = area under $a(t)$



More generally



For each strip, $\Delta v = a_{avg}(t)\Delta t$ = area of strip

$v_B - v_A \approx \sum_{j=1}^{n} a(t_j)\Delta t$ = area of n strips

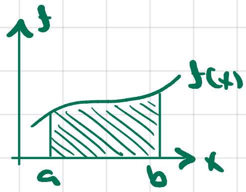
$v_B - v_A = \lim_{\substack{\Delta t \to 0 \\ n \to \infty}} \sum_{j=1}^{n} a(t_j)\Delta t = \int_{t_A}^{t_B} a(t)\,dt$ = area under $a(t)$

consider $G(x)$ with derivative $\frac{dG}{dx} = f(x)$

Note $\frac{d}{dx}(G(x)+C) = \frac{dG}{dx} = f(x)$

Antiderivative of $f(x)$, $\underbrace{\int f(x)\,dx}_{\text{indefinite integral}} = G(x) + C$

Definite integral: $\int_a^b f(x)dx = G(b) - G(a)$ = area under $f(x)$ in $[a,b]$



$\frac{dv}{dt} = a(t) \qquad dv = a(t)\,dt$

$\int_{v_0}^{v_1} dv = \int_{t_0}^{t_1} a(t)dt \qquad v_0 = v(t_0),\ v_1 = v(t_1)$

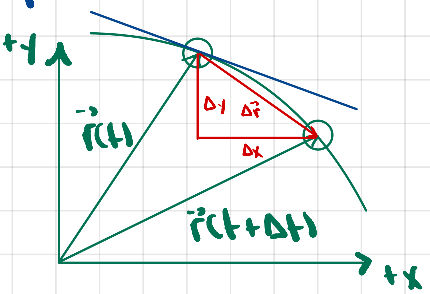
$v_1 - v_0 = \int_{t_0}^{t_1} a(t)dt \Rightarrow v(t_1) = v(t_0) + \int_{t_0}^{t_1} a(t)dt \Rightarrow v(t) = v_0 + \int_{t'=t_0}^{t'=t} a(t')dt'$

Given $v(t)$, $\quad x(t) = x_0 + \int_{t_0}^{t} v(t)dt$

2D Projectile Motion

First, some initial considerations:

position vector $\vec{r}(t)$ and change in $\vec{r}(t)$, $\Delta\vec{r}$



$r(t) = x(t)\hat{i} + y(t)\hat{j}$

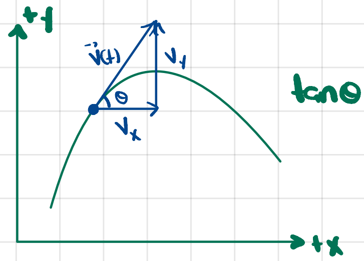
$r(t+\Delta t) = \vec{r}(t) + \Delta\vec{r}$

$\lim_{\Delta t \to 0} \Delta\vec{r}$ has direction tangent to the orbit

$\vec{v}$ is instantaneous rate of change of $\vec{r}(t)$

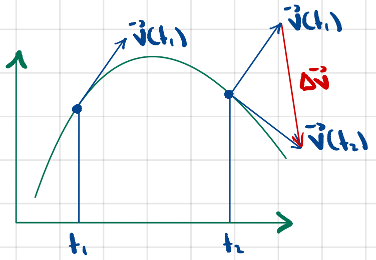
$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \left[\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}\right]\hat{i} + \left[\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}\right]\hat{j} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = v_x\hat{i} + v_y\hat{j}$$

speed is the magnitude of $\vec{v}$ $\qquad |\vec{v}| = (v_x^2 + v_y^2)^{1/2}$



$\tan\theta = \frac{v_y}{v_x} \Rightarrow \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$

acceleration is instant. rate of change of $\vec{v}(t)$



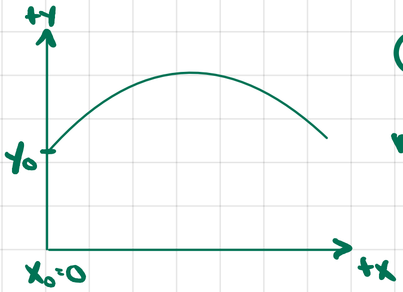
$\Delta\vec{v} = \vec{v}(t_2) - \vec{v}(t_1), \quad \vec{v} = v_x\hat{i} + v_y\hat{j}$

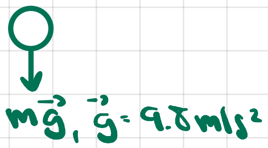
$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt} + \frac{dv_y}{dt} = \frac{d^2x(t)}{dt^2}\hat{i} + \frac{d^2y(t)}{dt^2}\hat{j}$$

$$= a_x\hat{i} + a_y\hat{j}$$

magnitude acceleration $= a = (a_x^2 + a_y^2)^{1/2}$

Calculations for 2D Projectile Motion





Given Force (gravity), obtain acceleration

$$\vec{F} = \langle 0, -mg \rangle = m\langle a_x, a_y \rangle$$

$$\Rightarrow a_x = 0 \qquad a_y = -g$$

Given $\vec{a}$, obtain $\vec{v}$

$$\vec{v} = \int \vec{a}(t)\,dt = \left\langle \int a_x(t)\,dt, \int a_y(t)\,dt \right\rangle = \langle v_{x,0}, v_{y,0} - gt \rangle$$

Given $\vec{v}$, obtain $\vec{r}$

$$\vec{r} = \int \vec{v}(t)\,dt = \left\langle \int v_x\,dt, \int v_y\,dt \right\rangle = \left\langle \int v_{x,0}\,dt, \int (v_{y,0} - gt)\,dt \right\rangle$$

$$= \left\langle x_0 + v_{x,0}t, y_0 + v_{y,0}t - \frac{gt^2}{2} \right\rangle$$

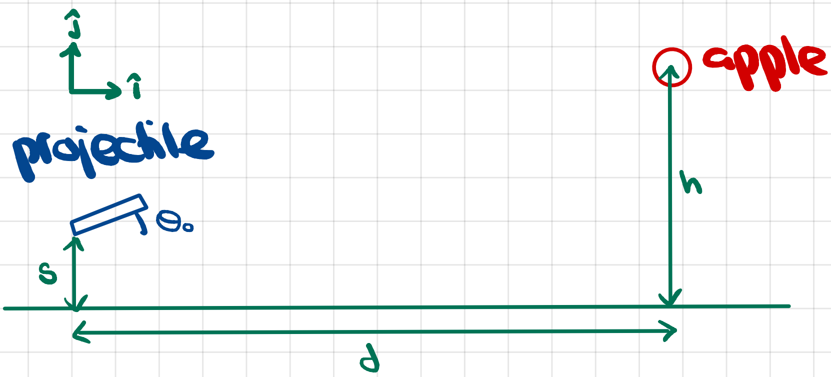
With $x_0 = 0$,

$$\vec{r}(t) = \left\langle v_{x,0}t, y_0 + v_{y,0}t - \frac{gt^2}{2} \right\rangle$$

To obtain $y = f(x)$, substitute for t

$$x(t) = v_{x,0}t \Rightarrow t = \frac{x}{v_{x,0}} \Rightarrow y(x) = y_0 + \frac{v_{y,0}}{v_{x,0}}x - \frac{gx^2}{2v_{x,0}^2}$$

PS.1.2 - Shooting the Apple



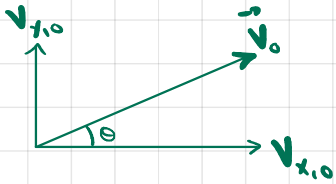
Motion of Apple

$\vec{F} = \vec{g} = \langle 0, -mg \rangle \Rightarrow \vec{a} = \langle 0, -g \rangle$

$\vec{v} = \langle 0, -gt \rangle$

$\vec{r}_a(t) = \langle d, h - \frac{gt^2}{2} \rangle$

Motion of Projectile



$v_{x,0} = |\vec{v}_0|\cos\theta$

$v_{y,0} = |\vec{v}_0|\sin\theta$

$\vec{v}_0 = \langle v_{x,0}, v_{y,0} \rangle$

$F = \vec{g} = \langle 0, -mg \rangle \Rightarrow \vec{a} = \langle 0, -g \rangle$

$\vec{v} = \langle v_{x,0}, v_{y,0} - gt \rangle$

$\vec{r}_p(t) = \langle v_{x,0}t, s + v_{y,0}t - \frac{gt^2}{2} \rangle$

For projectile to intercept apple, position vectors must coincide at some time t.

$$\begin{cases} \vec{r}_p(t) = \langle |\vec{v}_0|\cos\theta\, t, s + |\vec{v}_0|\sin\theta\, t - gt^2/2 \rangle = \langle vt\cos\theta, s + vt\sin\theta - \frac{gt^2}{2} \rangle \\ \vec{r}_a(t) = \langle d, h - gt^2/2 \rangle \end{cases}$$

t in which $x_p(t) = x_a(t)$

$v\cos\theta\, t = d \Rightarrow t = \dfrac{d}{v\cos\theta}$

t in which $y_p(t) = y_a(t)$

$s + v\sin\theta\, t - \cancel{gt^2/2} = h - \cancel{gt^2/2} \Rightarrow h = t v\sin\theta + s \Rightarrow t = \dfrac{h-s}{v\sin\theta}$

we need the two times to coincide and our parameters are θ and v

$$\frac{d}{\cancel{v}\cos\theta} = \frac{h-s}{\cancel{v}\sin\theta} \Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{h-s}{d} \Rightarrow \tan\theta = \frac{h-s}{d}$$

$\Rightarrow \theta = \tan^{-1}\left(\frac{h-s}{d}\right)$ Note θ doesn't depend on $\vec{v}$. As long as you aim at the apple, in theory you hit it. However this may occur at a y coordinate smaller than 0, which isn't realistic.

The apple reaches the ground at: $t = \left[\frac{2h}{g}\right]^{1/2}$

To be realistic $t \in [0, (2h/g)^{1/2}]$

Given $\theta = \tan^{-1}((h-s)/d)$, $t(v) = \dfrac{d}{v\cos\theta} \leq \left[\dfrac{2h}{g}\right]^{1/2} \Rightarrow v \geq \dfrac{d}{\cos\theta}\sqrt{\dfrac{g}{2h}}$

PS.1.3 - Braking Car

Given Info: $x(0) = x_0 = 0$

$v(0) = v_0$

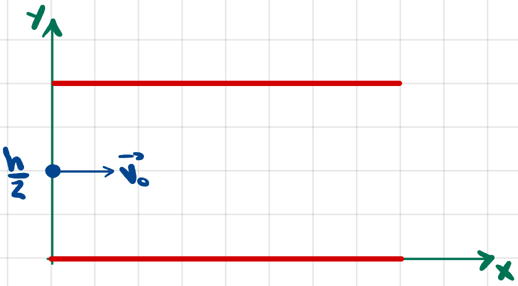
$$a(t) = \begin{cases} 0 & 0 \le t \le t_1 \\ -c(t-t_1) & t_1 < t \le t_2 \end{cases}$$

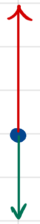
$$v(t) = \begin{cases} v_0 & 0 \le t \le t_1 \\ -\frac{ct^2}{2} + ct_1 t + v_0 & t_1 < t \le t_2 \end{cases}$$

$$x(t) = \begin{cases} v_0 t & 0 \le t \le t_1 \\ -\frac{ct^3}{6} + \frac{ct_1 t^2}{2} + v_0 t + x(t_1) & t_1 < t \le t_2 \end{cases}$$

$$\begin{cases} v_0 t & 0 \le t \le t_1 \\ -\frac{ct^3}{6} + \frac{ct_1 t^2}{2} + v_0 (t + t_1) & t_1 < t \le t_2 \end{cases}$$

PS.1.4 Sketch the Motion





$|\vec F_b| = bt$

$|\vec F_b| = m|\vec a_b| = bt \Rightarrow |\vec a_b| = \frac{bt}{m} \Rightarrow \vec a_b = \langle 0, \frac{bt}{m}\rangle$

$\vec F_b = m\cdot\vec a_b \Rightarrow \vec F_b = \langle 0, bt\rangle$

$\vec F_g = m\vec g = m\langle 0,-g\rangle = \langle 0,-mg\rangle$

$\vec v(t) = \langle x(t), y(t)\rangle$

$(\vec F_b+\vec F_g) = m(\vec a_b+\vec a_g) = m\langle 0, \frac{bt}{m} - g\rangle = \langle 0, bt-mg\rangle$

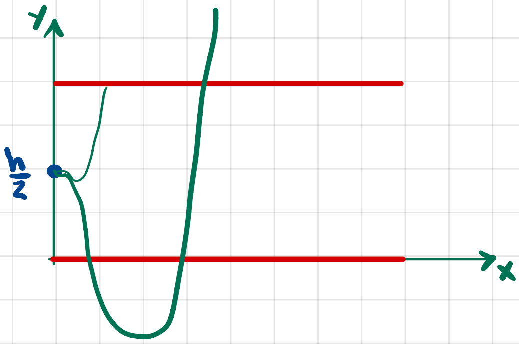
$\vec v(0) = \langle v_0, 0\rangle = \vec v_0$

a) units of b

$\vec F_b = \langle 0, bt\rangle = F_{b_y} = bt \quad kg\cdot m\cdot s^{-2} = kg\cdot m\cdot s^{-3}\cdot s$

$\Rightarrow$ b has units $kg\cdot m\cdot s^{-3}$

b) trajectory



$\vec a = \langle 0, \frac{bt}{m} - g\rangle$

$\vec v = \langle v_0, \frac{bt^2}{2m} - gt\rangle$

$\vec r = \langle v_0 t, \frac{h}{2} + \frac{bt^3}{6m} - \frac{gt^2}{2}\rangle$

c) how large must b be to avoid collision with bottom plate?

$\Rightarrow v_y = 0 \Rightarrow t\left(\frac{bt}{2m} - g\right) = 0 \Rightarrow bt = 2gm \Rightarrow t = \frac{2gm}{b}$

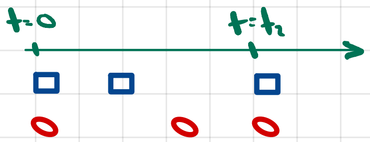
$r_y\left(\frac{2gm}{b}\right) = \frac{h}{2} + \frac{\cancel{b}}{\cancelto{3}{6m}}\cdot\frac{\cancelto{4}{8}g^3m^{\cancelto{2}{3}}}{b^{\cancelto{2}{3}}} - \frac{g\cdot\cancelto{2}{4}g^2m^2}{\cancel{2}\,b^2} = \frac{h}{2} + \frac{4g^3m^2}{3b^2} - \frac{2g^3m^2}{b^2} = 0$

$\Rightarrow \frac{3b^2h + 8g^3m^2 - 12g^3m^2}{6b^2} = \frac{3b^2h - 4g^3m^2}{6b^2} = 0 \Rightarrow 3b^2h = 4g^3m^2$

$\Rightarrow b^2 = \frac{4g^3m^2}{3h} \Rightarrow b = \left[\frac{4g^3m^2}{3h}\right]^{1/2}$ = this is what b needs to be, given g, m, and h, for the mass to change direction the instant it touches the bottom plate, which happens at $t = \frac{2gm}{b}$.

If $b \geq \left[\frac{4g^3m^2}{3h}\right]^{1/2}$ then the mass never touches the bottom plate.

PS.1.5 Pedestrian and Bike at Intersection



Initial Data

Car

$$a_c(t) = \begin{cases} b_1 & 0 \le t \le t_1 \\ 0 & t_1 < t \le t_2 \end{cases} \qquad b_1 > 0$$

$$v_c(0) = 0$$

Bike

$$v_{b_0} = v_0 \qquad v_b(t_2) = 0$$

$$a_b = -b_2 \qquad 0 \le t \le t_2 \qquad b_2 > 0$$

$$r_b(t_2) = r_c(t_2)$$

Motion of Car

$$v_c(t) = \begin{cases} b_1 t & 0 \le t \le t_1 \\ b_1 t_1 & t_1 < t \le t_2 \end{cases}$$

$$x(t) = \begin{cases} b_1 t^2/2 & 0 \le t \le t_1 \\ b_1 t_1^2/2 + b_1 t_1 t & t_1 < t \le t_2 \end{cases}$$

Motion of Bike

$$a_b(t) = -b_2 \qquad 0 \le t \le t_2$$

$$v_b(t) = v_0 - b_2 t$$

$$x_b(t) = v_0 t - b_2 t^2/2$$

b_2, given that bike stops at $t = t_2$

$$v_b(t_2) = 0 = v_0 - b_2 t_2 \Rightarrow b_2 = \frac{v_0}{t_2}$$

↗ b_2 depends on initial v and how long the braking takes

also,

$$x_c(t_2) = x_b(t_2) \Rightarrow b_1 t_1^2/2 + b_1 t_1 t_2 = v_0 t_2 - b_2 t_2^2/2$$

$$b_1 t_1^2 + 2 b_1 t_1 \frac{v_0}{b_2} = \frac{2 v_0^2}{b_2} - \frac{\cancel{b_2} v_0^2}{b_2^{\cancel{2}}} = \frac{v_0^2}{b_2}$$

$$b_1 b_2 t_1^2 + 2 b_1 t_1 v_0 = v_0^2 \Rightarrow b_2 = \frac{v_0^2 - 2 b_1 t_1 v_0}{b_1 t_1^2}$$