→ A rigid body is a body for which the distance between any two points is constant.

→ When we analyze translation of a body, we can analyze the motion of the CM using 2nd law: $\vec{F}_{ext} = m_T \cdot \vec{a}_{cm}$.

→ we can then analyze the rotational motion of the rigid body about the CM, using concepts of torque, moment of inertia, and angular acceleration, and rotational energy.

→ moment of inertia $= \int_{body} dm \cdot r^2_{s,dm}$, ie a weighted sum of squared distances to rotation axis, weights being masses.

→ rotational kinetic energy is $K^{rot} = \frac{1}{2} I_S \omega_z^2$

→ if we have the moment of inertia about an axis passing through the CM, we can find I for any other axis parallel to the CM axis:

$I_S = m_T d^2 + I_{CM}$ where d is the distance between the parallel axes

→ Torque is defined $\vec{\tau} = \vec{r} \times \vec{F}$

→ angular velocity is $\vec{\omega} = \theta'(t)\hat{n} = \omega_z \hat{n}$.

angular acceleration is $\vec{\alpha} = \theta''(t)\hat{n} = \alpha_z \hat{n}$

→ torque is the rotational equivalent of linear force "twist to an object around a specific axis"

→ torque is defined as $\vec{\tau} = \vec{r} \times \vec{F} = |\vec{r}||\vec{F}|\sin\theta$
it is a vector which arises in other computations we already know, e.g. work

$W = \int_{s_1}^{s_2} \vec{F} \cdot d\vec{s}$ = work done by variable force acting over finite linear displacement

if the displacement is circular, we can write $d\vec{s} = d\vec{\theta} \times \vec{r}$
$d\vec{\theta}$ being a vector is yet to be fully understood.

$W = \int_{s_1}^{s_2} \vec{F} \cdot d\vec{\theta} \times \vec{r}$

↘ scalar triple product $\vec{F} d\vec{\theta} \vec{r}$, also $\vec{r} \times \vec{F} \cdot d\theta$

$= \int_{s_1}^{s_2} \vec{r} \times \vec{F} \cdot d\vec{\theta} = \int_{\theta_1}^{\theta_2} \vec{\tau} \cdot d\vec{\theta}$ ↗ angular displacement

torque and angular displacement in same direction
→ $W = \int_{\theta_1}^{\theta_2} \tau \, d\theta$ where $\tau \, d\theta = \vec{\tau} \cdot d\vec{\theta} = |\vec{\tau}||d\vec{\theta}|\cos 0$

→ torque: quantitative measure of the tendency of a force to cause or change a body's rotational motion

→ Rotational Kinetic Energy

The kinetic energy of a rigid body in rotation about an axis.

Each Δm_i undergoes circular motion

$$K_{cm,i} = \frac{\Delta m_i \cdot v_{cm,i}^2}{2} = \frac{\Delta m_i \, r_i^2 \omega^2}{2}$$

$$\lim \sum_i K_{cm,i} = \frac{\omega^2}{2} \lim \sum_i \Delta m_i \cdot r_i^2 = \frac{I_{cm}\omega^2}{2}$$

→ $(\vec{\tau}_s)_z = I_s \alpha_z$

Rigid body, fixed axis rotation

Calculate torque on Δm_i

$$\vec{\tau}_{s,i} = (z_i \hat{k} + r_i \hat{r}) \times (F_{r,i}\hat{r} + F_{\theta,i}\hat{\theta} + F_{k,i}\hat{k})$$

Because rotation is about fixed axis, resultant force is in $\hat{\theta}$ dir.

$$(\vec{\tau}_{s,i})_z = r_i F_{\theta,i} \hat{k}$$

$$F_{\theta,i} = \Delta m_i \cdot r_i \alpha_z$$

$$\Rightarrow (\vec{\tau}_{s,i})_z = \Delta m_i r_i^2 \alpha_z \hat{k}$$

$$(\vec{\tau}_s)_z = \lim \sum_i (\vec{\tau}_{s,i})_z = \alpha_z \lim \sum_i \Delta m_i \cdot r_i^2 = I_s \alpha_z$$

→ Rotational Work

For rigid body undergoing rotation about a fixed axis.

Calculate ΔW_{rot} done by $F_{\theta,i}$ over a small displacement $\Delta\vec{r}$.

$\Delta\vec{r}$ is related to $\Delta\theta$

$$\Delta r_{s,i} \approx r_i \Delta\theta \hat{\theta}$$

$$\Delta W_{rot} = \vec{F}_{\theta,i} \cdot \Delta r_{s,i} \approx r_i F_{\theta,i} \Delta\theta$$

But we know that $(\vec{\tau}_{s,i})_z = r_i F_{\theta,i} \hat{k}$

$$\Rightarrow \Delta W_{rot} = (\tau_{s,i})_z \Delta\theta$$

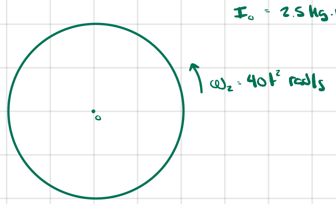
ie, the work done by $\vec{F}$ in rotating Δm_i is an expression involving the torque of F on Δm_i about the rot. axis.

Total work is an integral over angular displacement.

$$W_{rot} = \lim \sum_i \Delta W_{rot} = \int_{\theta_i}^{\theta_f} \tau_{s,z} \, d\theta$$

Ex 10.9

$I_o = 2.5\ kg\cdot m^2$



The turbine is symmetric about the axis of rotation.

$\vec{L}_i = r_i m_i v_i \hat{k} = r_i m_i \cdot r_i \omega_{z,i} \hat{k} = m_i r_i^2 \cdot \omega_z \hat{k}$

$\sum \vec{L}_i = \hat{k}\omega_z \sum m_i r_i^2 = I_z \omega_z \hat{k}$

The turbine isn't a slice on the xy plane.

$\vec{L}_i$ for parts of the turbine not on xy-plane is

$\vec{L}_i = (r_{i,x}\hat{i} + r_{i,y}\hat{j} + r_{i,z}\hat{k}) \times m_i \vec{v}_i$

$= \underbrace{r_{i,z}\hat{k}\times m_i\vec{v}_i}_{\text{cancels out over all } i} + (r_{i,x}\hat{i}+r_{i,y}\hat{j})\times m_i\vec{v}_i$

$\Rightarrow \vec{L} = \sum r m_i v_i \hat{k} = \sum m_i r_i^2 \omega \hat{k} = I_z\omega\hat{k}$

$\vec{L} = 2.5\cdot 40t^2\hat{k} = 100t^2\hat{k}$

$\vec{L}(3) = 900\ kg\cdot m^2/s$

b) $\frac{d\vec{L}}{dt} = 200t\hat{k} = \vec{\tau}$

note Net torque is the vector $200t\hat{k}$.

Net torque is increasing, and so is the rate of change of angular momentum.

For this symmetric rigid body we also have

$\frac{d\vec{L}}{dt} = \vec{\tau} = I_z\alpha_z = 2.5\cdot 80t = 200t$

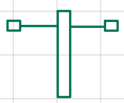
Ex 10.10

one revolution in 2 s $\Rightarrow$ 0.5 Hz = f

$I_{z,out} = 3$

$I_{z,in} = 2.2$

A: arms outstretched, 5kg dumbbells each hand



$f = 0.5$ Hz $\Rightarrow \omega_{z,A} = \pi$ rad/s

strategy: no external torques $\Rightarrow$ angular momentum conserved

initial angular momentum calculated based on I and ω_z.

with dumbbells pulled in, we know angular mom. and I in this new config so we solve for $\omega_{z,B}$.

$I_{A,z} = I_{z,out} + 2I_{dumb}$

$= 3 + 2\cdot 5\cdot 1^2 = 13$

$\vec{L}_A = I_{A,z}\cdot\omega_z\hat{k} = 13\cdot\pi\hat{k}$

$I_{B,z} = I_{z,in} + I_{B,dumb}$

$= 2.2 + 2\cdot 5\cdot 0.2^2 = 2.2 + 0.4 = 2.6$

$\vec{L}_{B,z} = I_{B,z}\cdot\omega_{B,z}\hat{k}$

$= 2.6\,\omega_{B,z}\hat{k}$

$\Delta\vec{L} = \vec{0} \Rightarrow 2.6\,\omega_{B,z} = 13\pi$

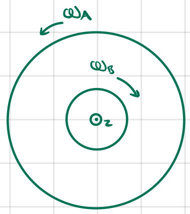
$\Rightarrow \omega_{B,z} = 5\pi$ rad/s

Ex 10.11

system: flywheel + plate

when brought together, they exchange torques.

The entity w/ higher angular speed receives negative torque, decreasing angular momentum.



Torques are internal. Angular momentum stays constant

$\vec{L}_{A,z} = I_A\omega_A\hat{k}$

$\vec{L}_{B,z} = I_B\omega_B\hat{k}$

$\vec{L}_{sys,z} = (I_A\omega_A + I_B\omega_B)\hat{k}$

Final state: $\vec{L}_{sys,z,f} = \omega(I_A + I_B)\hat{k}$

note that I imposed that this is the final state. For this to happen, work was done to speed one up and slow the other down.

$\Delta\vec{L} = 0 \Rightarrow \omega = \frac{I_A\omega_A + I_B\omega_B}{I_A + I_B}$

$K_i = \frac{I_A\omega_A^2}{2} + \frac{I_B\omega_B^2}{2}$

$K_f = \frac{(I_A + I_B)\,\omega^2}{2}$

$= \frac{\cancel{(I_A + I_B)}}{2}\,\frac{(I_A\omega_A + I_B\omega_B)^2}{(I_A + I_B)^{\cancel{2}}}$

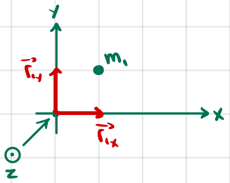
$= \frac{I_A^2\omega_A^2 + 2I_AI_B\omega_A\omega_B + I_B^2\omega_B^2}{2(I_A + I_B)}$

$K_f - K_i = \frac{\cancel{I_A^2\omega_A^2} + 2I_AI_B\omega_A\omega_B + \cancel{I_B^2\omega_B^2}}{2(I_A + I_B)} - \frac{\cancel{I_A^2\omega_A^2} + I_AI_B\omega_A^2 + \cancel{I_B^2\omega_B^2} + I_AI_B\omega_B^2}{2(I_A + I_B)}$

$= \frac{I_AI_B(2\omega_A\omega_B - \omega_A^2 - \omega_B^2)}{2(I_A + I_B)} = \frac{-I_AI_B(\omega_A - \omega_B)^2}{2(I_A + I_B)} < 0$

some calculations

→



$\vec{v}_1 = \vec{0} \Rightarrow \vec{p}_1 = \vec{0} \Rightarrow \vec{L}_{1,z} = \vec{0}$

$\vec{v}_1 = v_1\hat{i} \Rightarrow \vec{L}_{1,z} = \vec{r}_1 \times v_1\hat{i} = (r_{1x}\hat{i} + r_{1y}\hat{j}) \times v_1\hat{i}$

$= r_{1y}v_1(-\hat{k})$

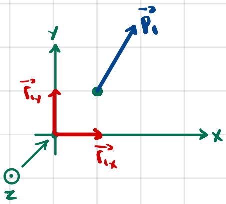
$= -r_{1y}v_1\hat{k}$

→ m_1 can have any $\vec{r}_{1x}$ and the angular momentum is the same in this example.

$V_1 = V_{1x}\hat{i} + V_{1y}\hat{j} \Rightarrow \vec{L}_{1,z} = (r_{1x}\hat{i} + r_{1y}\hat{j}) \times (V_{1x}\hat{i} + V_{1y}\hat{j})$

$= r_{1x}V_{1y}\hat{k} + r_{1y}V_{1x}(-\hat{k})$

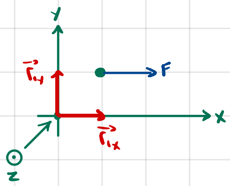
$= (r_{1x}V_{1y} - r_{1y}V_{1x})\hat{k}$



if $r_{1x} = r_{1y}$, then $\vec{L}_{1,z}$ is in $+z$ direction if $V_{1,y} > V_{1,x}$.

$\vec{V}_{1,0} = \vec{0} \Rightarrow \vec{L}_{1,z} = \vec{0}$

external force acts on system.



$\vec{F} = m_1\vec{a}$

$F\hat{i} = m_1 a\hat{i}$

$a = \frac{F}{m_1}$

F does work ⇒ $\Delta K > 0$

F provides an impulse ⇒ $\Delta p > 0$

what about torque?

$\vec{\tau}_{F,z} = (r_{1x}\hat{i} + r_{1y}\hat{j}) \times (F\hat{i})$

$= r_{1y}F(-\hat{k}) = -r_{1y}F\hat{k}$

$\vec{a} = \frac{F}{m_1}\hat{i}$

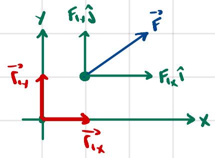
$\vec{v} = \frac{Ft}{m_1}\hat{i}$

$\vec{L}_{1,z} = (r_{1x}\hat{i} + r_{1y}\hat{j}) \times m_1\frac{Ft}{m_1}\hat{i}$

$= -r_{1y}Ft\hat{k}$

angular momentum is decreasing in time, because velocity is changing.

momentum in the direction perpendicular

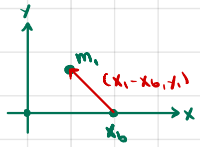


$\vec{\tau}_{F_1 z} = (r_{1x}\hat{i} + r_{1y}\hat{j}) \times (F_{1x}\hat{i} + F_{1y}\hat{j})$

$= r_{1x}F_{1y}\hat{k} - r_{1y}F_{1x}\hat{k}$

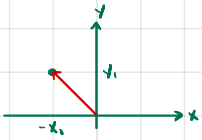
$= (r_{1x}F_{1y} - r_{1y}F_{1x})\hat{k}$

if $r_{1x} = r_{1y}$, whether torque is in positive z-dir depends on $F_{1y} > F_{1x}$.



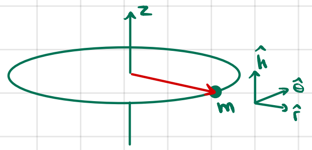
$\vec{L}_{1,x_0} = ((x_1 - x_0)\hat{i} + y_1\hat{j}) \times m_1v_1\hat{i}$

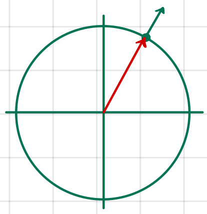
$= -m_1v_1y_1\hat{k}$



$\vec{L}_{x_1,z} = (-x_1\hat{i} + y_1\hat{j}) \times m_1v_1\hat{i}$

$= -m_1v_1y_1\hat{k}$





m has circular motion about z-axis.

→ constant angular velocity $\vec{\omega} = \omega_z\hat{k}$

⇒ $\vec{V} = \vec{\omega} \times \vec{r} = \omega_z\hat{k} \times r\hat{r} = \omega_z r\hat{\theta}$

* $\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\frac{d\hat{r}}{dt} = \theta'(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$

$= \theta'\hat{\theta}$

→ because there is change in direction of velocity, there is acceleration in radial direction. we don't know what the origin of this force is, but we know that

* $\hat{r} \times \hat{k} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= \hat{r}\cdot 0 - \hat{\theta}\cdot 1 + \hat{k}\cdot 0$

$= -\hat{\theta}$

$\vec{a} = \omega_z r\frac{d\hat{\theta}}{dt} = -r\omega_z^2\hat{r}$

$\vec{F} = -m\cdot r\omega_z^2\hat{r}$ ie we know what the force is, not its origin.

* $\vec{r} = r\hat{r}$

$\vec{v} = r\frac{d\hat{r}}{dt} = r\omega\hat{\theta}$

$\vec{L}_{m,z} = r\hat{r} \times m\omega_z r\hat{\theta} = m\omega_z r^2\hat{k}$

note that $\vec{p} = m\omega_z r\hat{\theta}$. Linear momentum has a constant magnitude but changing dir., because $\vec{v}$ changes dir.

* $\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$\frac{d\hat{\theta}}{dt} = \theta'(-\cos\theta\,\hat{i} - \sin\theta\,\hat{j})$

$= -\theta'\hat{r} = -\omega\hat{r}$

also, $\vec{L}_{m,z} = (mr^2)\omega_z\hat{k} = I_{m,z}\omega_z\hat{k}$

→ suppose that ω isn't constant $\vec{\omega} = \omega(t)\hat{k}$

$\vec{V} = \vec{\omega} \times \vec{r} = \omega(t)\hat{k} \times r\hat{r} = \omega(t)r\hat{\theta}$

$\vec{a} = \alpha(t)r\hat{\theta} - r\omega^2(t)\hat{r}$

$\vec{L}_{m,z} = r\hat{r} \times m\omega(t)r\hat{\theta} = m\omega(t)r^2\hat{k}$, angular momentum is fn of time, ie it is changing.

$\frac{d\vec{L}_{m,z}}{dt} = mr^2\omega'(t)\hat{k} = I_{m,z}\alpha_z\hat{k} = \vec{\tau}_z$

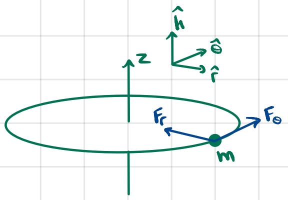
$F_r\hat{r} + F_\theta\hat{\theta} + F_z\hat{k} = m\cdot(\alpha(t)r\hat{\theta} - r\omega^2(t)\hat{r})$

(cont.) $F_r\hat{r} + F_\theta\hat{\theta} + F_z\hat{h} = m\cdot(\alpha(t) r\hat{\theta} - r\omega^2(t)\hat{r})$

$\Rightarrow F_r = -mr\omega^2(t)$

$F_\theta = m\alpha(t) r$

$F_z = 0$



$\vec{\tau} = r\hat{r} \times (F_r\hat{r} + F_\theta\hat{\theta}) = rF_\theta\hat{h} = mr^2\alpha(t)\hat{h}$

we derived the expr. for torque directly from the cross product as above, and previously by diff angular momentum.

For the cross product we needed to know the resultant force, which depended on knowing the acceleration vector, which we knew from the assumptions about angular velocity and a circular motion.

For angular momentum we similarly needed the velocity vector, obtained from angular velocity vector and position vector (which contains the assumption of circular motion.

Let's look at kinetic energy of the two circular motion cases

1) constant ω

$$K = \frac{m\vec{v}\cdot\vec{v}}{2} = \frac{m\omega_z^2 r^2}{2} = \frac{1}{2} I_z \omega_z^2$$

2) $\vec{\omega} = \omega_z(t)\hat{h}$

$$K = \frac{m\cdot r^2\omega(t)^2}{2}$$

K changes because work is being done by $\vec{F}$.

$$\Delta W_{tot} = (r\cdot\Delta\theta\cdot\hat{\theta})\cdot(F_r\hat{r} + F_\theta\hat{\theta})$$

$$= r\Delta\theta F_\theta = (rF_\theta)\Delta\theta = \tau_z\Delta\theta$$

$$\sum \Delta W_{tot} = \sum \tau_z \Delta\theta$$

$$\lim_{n\to\infty} \sum \Delta W_{tot} = \int_{\theta_i}^{\theta_f} \tau_z \, d\theta$$

$$\int_0^{2\pi} mr^2\alpha(t)\, d\theta$$

$$\int_0^{2\pi} mr^2\alpha(t)\, d\theta$$

can't solve if we don't know $\alpha(t)$

Assume $\alpha(t) = t$

$\Rightarrow \omega(t) = \frac{t^2}{2}$

$\Rightarrow \theta(t) = \frac{t^3}{6}$

$\Rightarrow t = \sqrt[3]{6\theta}$

$\Rightarrow \alpha(t) = \sqrt[3]{6\theta}$

$$W = mr^2 \int_0^{2\pi} \sqrt[3]{6\theta}\, d\theta$$

$$= mr^2 \sqrt[3]{6} \int_0^{2\pi} \theta^{1/3}\, d\theta$$

$$= \left.\frac{\sqrt[3]{6}\, mr^2 \theta^{4/3}}{4/3}\right|_0^{2\pi}$$

$$= \frac{3\sqrt[3]{6}\, mr^2 \cdot \sqrt[3]{16\pi^4}}{4}$$

$$= \frac{3mr^2\cdot \cancel{2}\sqrt[3]{12\pi^4}}{\cancel{4}_2}$$

$$= \frac{3mr^2\sqrt[3]{12\pi^4}}{2}$$

now check kinetic energy

$\omega(0) = 0 \Rightarrow v(0) = 0 \Rightarrow K_i = 0$

$t(\theta) = \sqrt[3]{6\theta}$

$t(2\pi) = \sqrt[3]{12\pi}$

$\vec{v}(t) = \omega(t) r\hat{\theta} = \frac{t^2 r}{2}\hat{\theta}$

$v(t) = \frac{t^2 r}{2}$

$$v(\sqrt[3]{12\pi}) = \frac{\sqrt[3]{144\pi^2}\, r}{2}$$

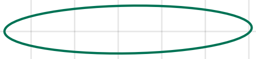
$$K_f = \frac{m\left(\frac{\sqrt[3]{144\pi^2}\, r}{2}\right)^2}{2}$$

$$= \frac{mr^2\sqrt[3]{144^2\pi^4}}{8}$$

$$= \frac{mr^2\cdot 12\sqrt[3]{12\pi^4}}{8}$$

$$= \frac{3mr^2\sqrt[3]{12\pi^4}}{2}$$

rigid bodies



Dish, radius R, uniform, mass m

Density = $\frac{m}{\pi R^2}$

CM: $\sum \vec{r}_{cm,i} m_i = \vec{0}$

$\Rightarrow \sum r_{cm,i,x} m_i = 0$

$\sum r_{cm,i,y} m_i = 0$

But we don't know the center of mass position so we don't know $\vec{r}_{cm,i}$.

Assume another reference frame, origin at point S.

$\vec{r}_{cm,i} + \vec{r}_{S,cm} = \vec{r}_{S,i}$

$\Rightarrow \vec{r}_{cm,i} = \vec{r}_{S,i} - \vec{r}_{S,cm}$

$\sum \vec{r}_{S,i} m_i - \sum \vec{r}_{S,cm} m_i = 0$

$\Rightarrow \vec{r}_{S,cm} \cdot M = \sum \vec{r}_{S,i} m_i$

$\Rightarrow \vec{r}_{S,cm} = \frac{\sum \vec{r}_{S,i} m_i}{\sum m_i}$

For the dish,

$$\vec{r}_{S,cm} = \lim_{\substack{N\to\infty \\ m_i\to 0}} \frac{\sum \vec{r}_{S,i} m_i}{\sum m_i} = \frac{\int_{body} \vec{r}\, dm}{m}$$

$dm = \rho r\, dr\, d\theta$

$$\vec{r}_{S,cm} = \frac{\int_0^{2\pi}\int_0^R r\hat{r} \cdot \rho r\, dr\, d\theta}{m} = \frac{\rho \int_0^{2\pi}\int_0^R r^2 \hat{r}\, dr\, d\theta}{m}$$

$$= \frac{\rho}{m}\frac{R^3}{3}\int_0^{2\pi}\hat{r}\, d\theta = \frac{\rho R^3}{3m}(-\hat{j} - (-\hat{j})) = \vec{0}$$

$\hat{r} = \cos\theta\, \hat{i} + \sin\theta\, \hat{j}$

$\frac{d\hat{r}}{d\theta} = -\sin\theta\, \hat{i} + \cos\theta\, \hat{j} = \hat{\theta}$

$\int \hat{r}\, d\theta = \sin\theta\, \hat{i} - \cos\theta\, \hat{j} = -\hat{\theta}$

Assume the dish is rotating about the axis passing through the CM, perpend. to the dish.

1) constant ω_z

every point on the dish has the same angular speed, but linear speed is $r\omega_z$.

Angular Momentum

→ of a mass particle Δm_i on the dish.

→ from previous calculations, this is

$\vec{L}_{cm,i} = I_{cm,i}\,\omega_z$

$= \Delta m_i r_i^2 \omega_z$

$$\vec{L}_{cm} = \lim_{\substack{N\to\infty \\ \Delta m_i\to 0}} \sum \Delta m_i r_i^2 \omega_z = \int_{disk} r^2\, dm \cdot \omega_z = I_{cm}\cdot\omega_z$$

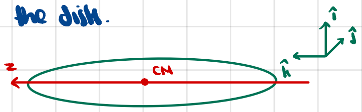
$$I_{cm} = \int_0^{2\pi}\int_0^R r^2 \cdot \rho r\, dr\, d\theta$$

$$= \frac{\rho R^4}{4}\cdot 2\pi = \frac{m}{\cancel{\pi R^2}}\cdot\frac{R^{\cancel{4}\,2}}{\cancel{4}\,2}\cdot\cancel{2\pi}$$

$$= \frac{mR^2}{2}$$

$\Rightarrow \vec{L}_{cm} = \frac{mR^2}{2}\omega_z\hat{k}$

This is the angular momentum of a uniform disk rotating about an axis through the CM perpend. to the disk.



Consider a new rotation axis as above

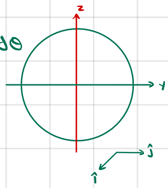
The derivation of the angular momentum is the same as previously: each Δm_i undergoes circular motion with ω_z.

$\vec{L}_z = I_z\omega_z\hat{k}$

$$I_z = \int_0^{2\pi}\int_0^R (r\cos\theta)^2 \cdot \rho r\, dr\, d\theta$$

$$= \rho\int_0^{2\pi}\int_0^R r^3\cos^2\theta\, dr\, d\theta$$

$$= \frac{\rho R^4}{4}\int_0^{2\pi}\cos^2\theta\, d\theta$$



$\int\cos^2\theta\, d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\Rightarrow I_z = \frac{\rho R^4}{4}\left[\frac{2\pi}{2}\right] = \frac{\pi\rho R^4}{4}$

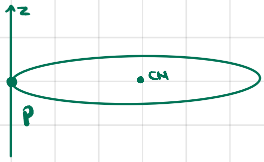
For uniform disc, $\rho = \frac{m}{\pi R^2}$

$\Rightarrow I_z = \frac{\cancel{\pi} R^{\cancel{4}\,2}}{4}\cdot\frac{m}{\cancel{\pi R^2}} = \frac{mR^2}{4}$

$\Rightarrow \vec{L}_z = \frac{mR^2}{4}\omega_z\hat{k}$

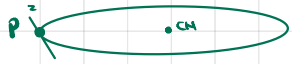
Angular momentum depends on moment of inertia, which depends on mass distribution. For the same angular speed, rotation about diameter has less angular momentum than rotation about perpendicular through CM because of the higher moment of inertia about the latter.

Consider point P. There are multiple (infinite) axes passing through P.



$I_z = I_{cm} + mR^2 = \frac{3mR^2}{2}$

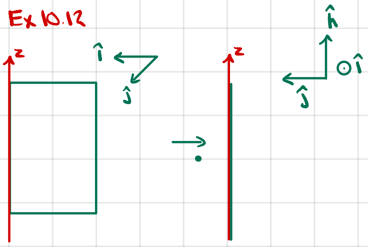
$\vec{L}_z = \frac{3mR^2}{2}\omega_z\hat{k}$



$I_z = I_{cm,x} + mR^2 = \frac{mR^2}{4} + mR^2$

$= \frac{5mR^2}{4}$

Ex 10.12



system: bullet + door

$$I_{door,z} = \int_0^w \int_0^h x^2 \rho \, dz \, dx$$

$$= \int_0^w x^2 \rho h \, dx$$

$$= \frac{\rho h w^3}{3}, \quad \rho = \frac{m}{wh}$$

$$= \frac{m \cancel{h} w^{\cancel{3}\,2}}{3 \cancel{w} \cancel{h}} = \frac{mw^2}{3}$$

right before collision, A

$$\vec{L}_{B,z,A} = -\frac{W}{2}\hat{i} \times m_B \cdot (-v_{B,i}\hat{j})$$

$$= \frac{m_B v_{B,i} W}{2}\hat{k} = \vec{L}_{sys,z,A}$$

right after collision, B

$$I_{sys,z,B} = \frac{mW^2}{3} + m_B \cdot \frac{W^2}{4}$$

$$= \frac{W^2(4m + 3m_B)}{12}$$

$$\vec{L}_{sys,z,B} = I_{sys,z,B} \cdot \omega_z \hat{k}$$

$$= \frac{W^2(4m + 3m_B)}{12} \cdot \omega_z \hat{k}$$

There are no external torques. $\Delta\vec{L} = \vec{0}$

$$\frac{W^{\cancel{2}}(4m + 3m_B)}{\cancel{12}\, 6} \cdot \omega_z = \frac{m_B v_{B,i} \cancel{W}}{\cancel{2}}$$

$$\omega_z = \frac{6 m_B v_{B,i}}{W(4m + 3m_B)}$$

$W = 1m$

$m = 15kg$

$m_B = 0.01\ kg$

$v_{B,i} = 400 m/s$

$\Rightarrow \omega_z = 0.39\ rad/s$

$$K_{sys,i} = \frac{0.01 \cdot 400^2}{2}$$

$$= \frac{10^{-2} \cdot 16 \cdot 10^4}{2} = 8 \cdot 10^2$$

$= 800\ J$

$$K_{sys,f} = \left(\frac{15 \cdot 1^2}{3} + 0.01 \cdot \frac{1^2}{4}\right) \cdot \frac{0.39^2}{2}$$

$= 0.38\ J$

$$\Rightarrow \frac{K_i}{K_f} = 2093$$

Linear momentum is defined

$\vec{p} = m\vec{v}$

$$\Rightarrow \frac{d\vec{p}}{dt} = m\vec{a}$$

Force is $\frac{d\vec{p}}{dt}$ by definition $\vec{F} = m\vec{a}$.

In a system of particles, $\vec{p}_{sys} = \sum \vec{p}_i$.

$\Rightarrow \frac{d\vec{p}_{sys}}{dt} = \sum \vec{F}_i$, ie by definition, the resultant force on the system equals rate of change in momentum.

$$\sum \vec{F}_i = \vec{F}_{sys} = m_{sys} \cdot \vec{a}_{sys} = \frac{d\vec{p}_{sys}}{dt}$$

Note also that

$$\vec{p}_{sys} = \sum m_i \vec{v}_i = m_{sys} \cdot \vec{v}_{sys}$$

$$\Rightarrow v_{sys} = \frac{\sum m_i \vec{v}_i}{\sum m_i} = \vec{v}_{cm}$$

$\Rightarrow \vec{p}_{sys} = m_{sys} \vec{v}_{cm}$

so $\frac{d\vec{p}_{sys}}{dt} = m_{sys} \cdot \vec{a}_{cm} = \vec{F}_{sys}$

ie the acceleration generated by the sum of all forces on the system is equivalent to a fictitious single resultant force acting on the center of mass.

If $\sum \vec{F}_i = 0$ then $\frac{d\vec{p}_{sys}}{dt} = 0$.

This is pretty much directly from the definition of momentum and its derivative.

Kinetic Energy is defined as $\frac{mv^2}{2}$.

If a force is applied on a particle mass for a displacement Δx, then the change in kinetic energy is $\int_{x_i}^{x_f} F\,dx$. Ie kinetic energy appears as the antiderivative above.

Take a collision between m_1 and m_2 in which the masses stick together. Momentum is conserved: there are no external forces. Kinetic energy, however, is not. Why?

momentum is a vector.

Ball A has initial momentum $m_A v_{A,i} \hat{i}$ and kinetic energy $\frac{m_A v_{A,i}^2}{2}$, a scalar.

Ball B is at rest.

The balls collide.

System momentum is

$(m_A v_{A,f} + m_B v_{B,f})\hat{i}$

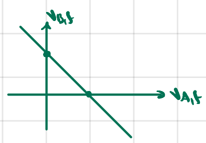
There are infinite combinations of $v_{A,f}$ and $v_{B,f}$ that make $\vec{p}_i = \vec{p}_f$.

For example, if $m_A = m_B = 1$, $v_{A,i} = 1$

$p_i = 1 \cdot 1 = 1$

$p_f = 1 \cdot v_{A,f} + 1 \cdot v_{B,f} = 1$

$v_{A,f} + v_{B,f} = 1$



Each combination has an associated system kinetic energy.

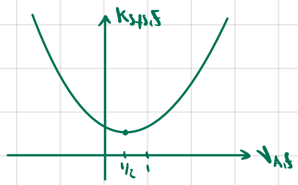
$$K_f = \frac{m_A v_{A,f}^2 + m_B(1 - v_{A,f})^2}{2}$$

$$= \frac{m_A v_{A,f}^2 + m_B - m_B \cdot 2v_{A,f} + v_{A,f}^2 m_B}{2}$$

$$= \frac{v_{A,f}^2(m_A + m_B) - 2m_B v_{A,f} + m_B}{2}$$

$$= \frac{2v_{A,f}^2 - 2 \cdot 1 v_{A,f} + 1}{2}$$

$$= v_{A,f}^2 - v_{A,f} + \frac{1}{2}$$



note that in totally inelastic collision,

$v_{A,f} = v_{B,f} = \frac{1}{2}$

and this speed minimizes kinetic energy.

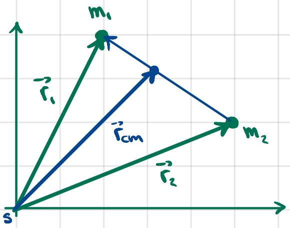
But $\vec{v}_{sys} = \sum \vec{v}_i$

→ Polar ice caps melt, the water would be redestributed over the oceans around the planet. The moment of inertia would increase as mass moves away from rotation axis. For angular momentum to stay constant, angular speed must decrease → longer days.

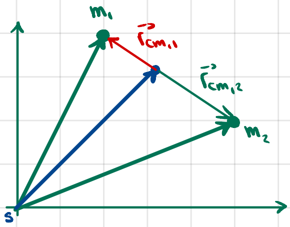
Review of Chasles' Theorem

that the motion of a rigid body can be analyzed as a translation of CM relative to an inertial frame and a rotation of the rigid body about its CM.

Define coordinate system.



Vectors in CM frame.



We are interested in describing the motion of each mass, especially their position vectors

m_1 and m_2 represent a rigid body

$|\vec r_{1,2}| = $ constant

$\Rightarrow |\vec r_{1,2}|^2 = (\vec r_1 - \vec r_2)\cdot(\vec r_1 - \vec r_2)$

$0 = 2(\vec r_1 - \vec r_2)(d\vec r_1 - d\vec r_2)$

The two particles must satisfy this condition if they are a rigid body

$\vec r_1 = \vec R_{cm} + \vec r_{cm,1}$

$r_{cm,1} = r_1 - \dfrac{m_1\vec r_1 + m_2\vec r_2}{m_1+m_2}$

$= \dfrac{m_2}{m_1+m_2}(\vec r_1 - \vec r_2) \Rightarrow \vec r_{cm,1} = \dfrac{\mu}{m_1}\vec r_{1,2}$

$\vec r_2 = \vec R_{cm} + \vec r_{cm,2}$

$\vec r_{cm,2} = \vec r_2 - \dfrac{m_1\vec r_1 + m_2\vec r_2}{m_1+m_2}$

$= \dfrac{m_1}{m_1+m_2}(\vec r_2 - \vec r_1) \Rightarrow \vec r_{cm,2} = -\dfrac{\mu}{m_2}\vec r_{1,2}$

$d\vec r_{cm,1} = \dfrac{\mu}{m_1} d\vec r_{1,2}$

$d\vec r_{cm,2} = -\dfrac{\mu}{m_2} d\vec r_{1,2}$

$\Rightarrow d\vec r_{cm,1} m_1 = -d\vec r_{cm,2} m_2$

$\vec r_i = \vec R_{cm} + \vec r_{cm,i}$

$d\vec r_i = d\vec R_{cm} + d\vec r_{cm,i}$

The infinitesimal change in the position of each particle is the change in CM frame position plus change in particle position rel. to CM.

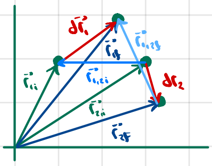
What does the rigid body condition lead to?

Case 1: $d\vec r_1 = d\vec r_2$, ie the masses have the exact same change in position, both magnitude and direction.

$\Rightarrow d\vec r_1 - d\vec r_2 = 0$

$\Rightarrow d\vec r_{1,2} = 0$

e.g.



$r_{1,2,f} = (r_{1,i} + dr_1) - (r_{2,i} - dr_2)$

$= (r_{1,i} - r_{2,i}) + (dr_1 - dr_2)$

$= r_{1,2,i} + d\vec r_{1,2}$

$d\vec r_{1,2} = 0 \Rightarrow d\vec r_{cm,1} = d\vec r_{cm,2} = \vec 0$

$\Rightarrow$ the masses are at rest relative to the CM

$\Rightarrow d\vec r_1 = d\vec r_2 = d\vec R_{cm}$

$\Rightarrow$ Case 1 implies that both masses move change position exactly the same way as the CM. They all have the same velocity vector and acceleration, and this is a purely translational motion: relative to one another the masses and CM are at rest.

We can therefore analyze the motion of the CM, and obtain the positions of the individual particles as a fixed offset to the CM position.

Case 2: $d\vec r_1 \neq d\vec r_2$

The rigid body condition tells us that

$(\vec r_1 - \vec r_2)(d\vec r_1 - d\vec r_2) = 0$

$\Rightarrow \vec r_{1,2}\cdot d\vec r_{1,2} = 0$

but $\vec r_{1,2} = \dfrac{m_1}{\mu} d\vec r_{cm,1} = -\dfrac{m_2}{\mu} d\vec r_{cm,2}$

$\Rightarrow \vec r_{1,2}\cdot d\vec r_{cm,1} = r_{1,2}\cdot d\vec r_{cm,2} = 0$

$\vec r_{1,2} = \vec r_1 - \vec r_2 = (\cancel{\vec R}_{cm} + \vec r_{cm,1}) - (\cancel{R}_{cm} + r_{cm,2})$

$\Rightarrow \vec r_{1,2} = \vec r_{cm,1,2}$ relative velocity vector is indep. of reference frame

$\Rightarrow \vec r_{cm,1,2}\cdot d\vec r_{cm,1} = \vec r_{cm,1,2}\cdot d\vec r_{cm,2} = 0$

In the CM frame, change in position of each mass is perpendicular to position.

$\vec r_{cm,1,2}\cdot\vec V_{cm,1} = \vec r_{cm,1,2}\cdot\vec V_{cm,2} = 0$

note that $\vec r_{cm,1,2} = \dfrac{m_1}{\mu}\vec r_{cm,1} = -\dfrac{m_2}{\mu}\vec r_{cm,2}$

$\Rightarrow m_1\vec r_{cm,1}\cdot\vec V_{cm,1} = -m_2\vec r_{cm,2}\vec V_{cm,2} = 0$

ie the velocity vectors are perp to position vectors.
But how do we know that this is circular motion?

$|\vec r_{1,2}| = $ constant $\Rightarrow |r_{cm,1}|, |r_{cm,2}|$ constant

$\vec r_{cm,1} = |\vec r_{cm,1}|\hat r \qquad \hat r = \cos\theta\,\hat i_{cm} + \sin\theta\,\hat j_{cm}$

How do we know θ changes?

Each mass has velocity relative to CM, and that velocity is in the $\hat\theta_{cm}$ direction, $\hat\theta = -\sin\theta\,\hat i_{cm} + \cos\theta\,\hat j_{cm}$.

$\vec V_{cm,1} = |\vec V_{cm,1}|\hat\theta$

$\vec V = \vec r\times\vec\omega \Rightarrow \vec\omega = \vec r\times\vec V \Rightarrow \omega_{cm,i} = \vec r_{cm,i}\times\vec V_{cm,i}$

$\Rightarrow \vec\omega_{cm,i} = |\vec r_{cm,i}||\vec V_{cm,i}|\hat k$

Results so far: constant distance from CM, velocity $\perp$ position,
$\Rightarrow \vec\omega_{cm,i} \neq \vec 0$

Are the $\vec\omega_{cm,i}$ the same for both masses?

we know $|r_{cm,1}| = \dfrac{\mu}{m_1}|\vec r_{cm,1,2}|$ and $|r_{cm,2}| = \dfrac{\mu}{m_2}|\vec r_{cm,1,2}|$

$\Rightarrow m_1|\vec r_{cm,1}| = m_2|\vec r_{cm,2}|$

$|d\vec r_{cm,1}| = d\theta_1\cdot|\vec r_{cm,1}|$

$|d\vec r_{cm,2}| = d\theta_2\cdot|\vec r_{cm,2}|$

$d\theta_1 = \dfrac{\mu}{m_1}\dfrac{|\vec r_{cm,1,2}|}{|\vec r_{cm,1}|} = \dfrac{\mu|\vec r_{cm,1,2}|\cancel{m_1}}{\cancel{m_1}m_2|\vec r_{cm,2}|} = d\theta_2$

$d\theta_2 = \dfrac{\mu}{m_2}\dfrac{|\vec r_{cm,1,2}|}{|\vec r_{cm,2}|}$