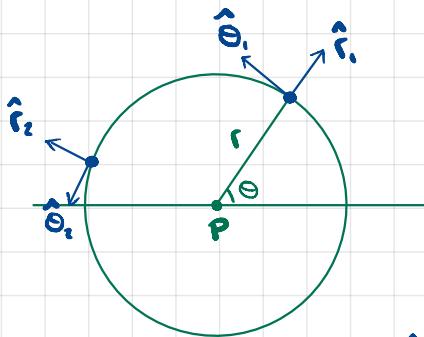


Week 3 Notes

8.1 Polar Coordinates



$\hat{r}, \hat{\theta}$ set of unit vectors at point $s_1(r, \theta)$

$$\hat{r}_1 + \hat{r}_2, \hat{\theta}_1 + \hat{\theta}_2$$

8.2 Position and Velocity Vectors

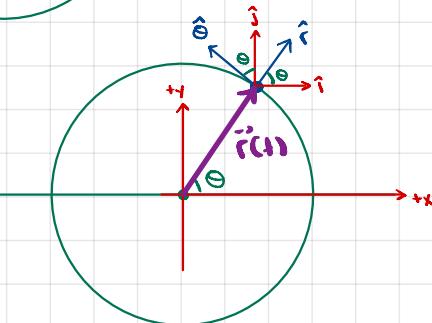
Relation with Cartesian Coord.

when $\Theta = \Theta(t)$ we have

$$\hat{r}(t) = \cos \Theta(t) \hat{i} + \sin \Theta(t) \hat{j}$$

$$\vec{r}(t) - \text{position vector} = r \hat{r}(t) = r \cos \Theta(t) \hat{i} + r \sin \Theta(t) \hat{j}$$

$$\begin{aligned} \vec{v}(t) - \frac{d\vec{r}(t)}{dt} &= r \left(-\sin \Theta(t) \frac{d\Theta(t)}{dt} \right) \hat{i} + r \cos \Theta(t) \frac{d\Theta(t)}{dt} \hat{j} \\ &= r \underbrace{\frac{d\Theta(t)}{dt}}_{\hat{\Theta}} \left(-\sin \Theta(t) \hat{i} + \cos \Theta(t) \hat{j} \right) = r \frac{d\Theta}{dt} \hat{\Theta} = v_\theta \hat{\Theta}, \quad v_\theta = r \frac{d\Theta}{dt} \end{aligned}$$

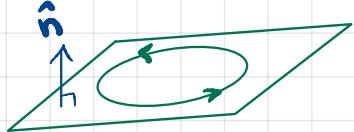


$$\hat{r} = \cos \Theta \hat{i} + \sin \Theta \hat{j}$$

$$\hat{\Theta} = -\sin \Theta \hat{i} + \cos \Theta \hat{j}$$

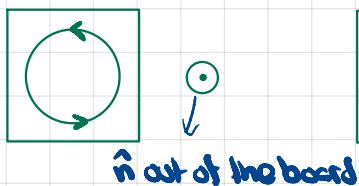
8.3 Angular Velocity

→ given circular motion on a plane as in the picture below, the motion is clockwise when seen from below and counterclockwise when seen from above.

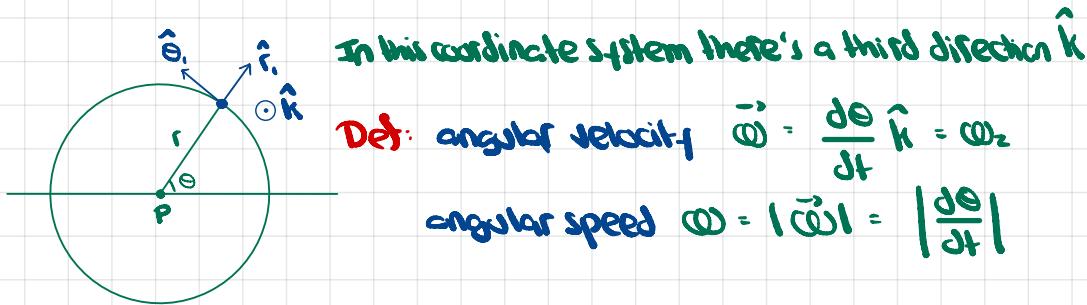


→ to define a positive direction, we use the right hand rule: curl fingers in direction of movement, thumb points in positive direction, i.e. direction of vector normal to the plane in positive direction.

→ diagram conventions



\hat{n} into the board



$$\Rightarrow v(t) = r \frac{d\theta}{dt} \hat{\Theta} = r \omega_z \hat{\Theta}$$

9.1 Uniform Circular Motion

$$\vec{v}(t) = r \frac{d\theta}{dt} \hat{\theta}$$

special case: $\frac{d\theta}{dt} = \text{constant} \Rightarrow |\vec{v}(t)| = r \left| \frac{d\theta}{dt} \right| = \text{constant}$ (Uniform Circular Motion)

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt} \left[r \frac{d\theta}{dt} (-\sin\theta \hat{i} + \cos\theta \hat{j}) \right]$$

$$= r \frac{d^2\theta}{dt^2} \hat{\theta}(t) + r \frac{d\theta}{dt} \left(-\cos\theta \frac{d\theta}{dt} \hat{i} - \sin\theta \frac{d\theta}{dt} \hat{j} \right)$$

$$= -r \left(\frac{d\theta}{dt} \right)^2 \hat{r}(t)$$

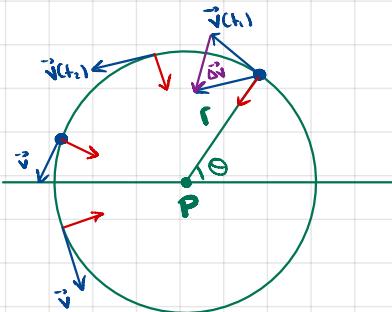
9.2 Direction of Acceleration

$$\vec{a} = -r \dot{\theta}^2 \hat{r}(t)$$

$$\vec{a} = a_r \hat{r} \quad a_r = -r \dot{\theta}^2 < 0$$

$$\Delta \vec{v} = \vec{v}(t_2) - \vec{v}(t_1)$$

$$\vec{a}_c = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}}{\Delta t}$$



10.1 Circular Motion - Acceleration

$$\vec{v} = r \frac{d\theta}{dt} \hat{\theta}$$

calculated previously: $a_r = -r\theta'(t)^2$

$$\vec{a} = \frac{d\vec{v}}{dt} = r\theta''(t)\hat{\theta} + r\theta'(t) \frac{d\hat{\theta}}{dt}$$

\downarrow
tangential \downarrow
radial

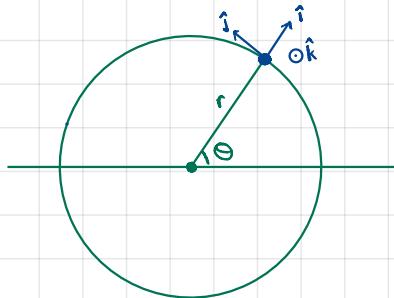
$$a_\theta = r\theta''(t) \Rightarrow \theta''(t) > 0 \Rightarrow \theta'(t) \text{ increasing}$$

10.2 Angular Acceleration

recall $\vec{\omega} = \frac{d\theta}{dt} \hat{k}$, angular velocity = $\omega_z \hat{k}$

$\vec{\alpha} = \frac{d^2\theta}{dt^2} \hat{k}$, angular acceleration

$$= \alpha_z \hat{k}, \quad \alpha_z = \frac{d^2\theta}{dt^2} = \frac{d\omega_z}{dt}$$



Cases

1. $\omega_z > 0$

a. $\alpha_z > 0, \frac{d\omega_z}{dt} > 0 \Rightarrow \text{ccw, speeding up}$

b. $\alpha_z < 0, \frac{d\omega_z}{dt} < 0 \Rightarrow \text{ccw initially, slowing down to } \omega_z = 0, \text{ then } (\omega_z < 0 \text{ induces) with motion now cw}$

2. $\omega_z < 0$

a. $\alpha_z < 0, \frac{d\omega_z}{dt} < 0 \Rightarrow \text{cw motion, speeding up}$

b. $\alpha_z > 0, \frac{d\omega_z}{dt} > 0 \Rightarrow \text{cw motion initially, angular speed decreasing to zero, motion becomes ccw, angular speed increases.}$

* ω_z, α_z are like v, a in 1D motion

10.3 Worked Example (m/solution)

$$\vec{\alpha}(t) = \frac{1}{r}(A - Bt)\hat{k}. \quad \omega_z = \theta'(t), \alpha_z = \theta''(t) = \theta'(t) \cdot \frac{1}{r}(A - Bt)$$

$$a_\theta = r\theta''(t) = A - Bt$$

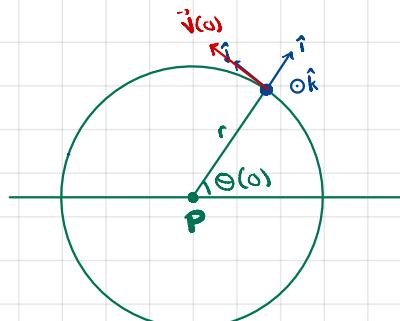
$$t_0 = 0, \theta(t_0), v_0(t_0) = v_0 = r\theta'(0) \Rightarrow \theta'(0) = \frac{v_0}{r}$$

arc length defined $s(t) = r(\theta(t) - \theta(0))$

$$\theta'(t) = \frac{1}{r}(At - B\frac{t^2}{2}) + \frac{v_0}{r}$$

$$\theta(t) = \frac{1}{r}(\frac{At^2}{2} - \frac{Bt^3}{6}) + \frac{v_0}{r}t + \theta(0)$$

$$s(t) = \frac{At^2}{2} - \frac{Bt^3}{6} + v_0 t$$



→ Lecture Solution

$$a_\theta(t) = A - Bt = r \theta''(t)$$

$$v_\theta(t) - v_\theta(t_0) = \int_{t_0=0}^t a_\theta(t) dt = At - \frac{Bt^2}{2}$$

$$v_\theta(t) = v_\theta(0) + At - \frac{Bt^2}{2}$$

$$\underbrace{r\theta(t) - r\theta(0)}_{\text{change in arc length}} = \int_0^t v_\theta(t) dt = r \int_0^t \theta'(t) dt$$

$$= \int_0^t [v_\theta(0) + At - \frac{Bt^2}{2}] dt = v_\theta(t) + \frac{1}{2}At^2 - \frac{1}{6}Bt^3$$

11.1 Newton's Second Law and Circular Motion

recall

$$\vec{F} = \vec{ma}$$

why?
dynamics
how?
geometric

vector equation

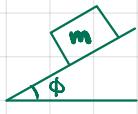
$$F_r = m a_r$$

$$F_r = -m r \omega^2$$

$$F_\theta = m a_\theta$$

$$F_\theta = m r \theta'(t) = m r \alpha_z$$

11.2 Worked Example: Car on a Banked Turn



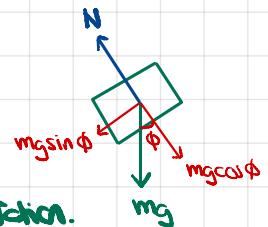
$$\begin{aligned}\vec{r}(t) &= r \hat{r}(t) \\ \dot{\vec{r}}(t) &= r \dot{\theta}(t) \hat{\theta}(t) \\ \ddot{\vec{r}}(t) &= r \theta''(t) \hat{\theta}(t) - r \theta'(t)^2 \hat{r}(t)\end{aligned}$$

A car's engine generates friction of the wheels which create a friction force with the ground. Friction propels the car forward. If the friction coeff. is zero, the car cannot accelerate tangentially due to the car's engine.
 $\Rightarrow a_\theta = 0$.

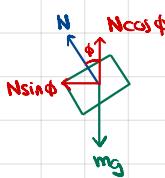
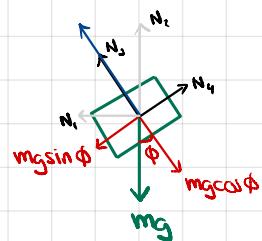
\vec{a} is a vector so it can have nonzero components other than in direction $\hat{\theta}$.

If the car did not have tangential velocity, $\dot{\theta}_0 = 0 \Rightarrow \theta'(t) = 0$

Acceleration would be positive in the direction parallel to the slope the car is on. In the absence of friction, the car would slide down the slope with acceleration.



If the car is turning on the banked curve without slipping down the slope, acceleration is radial



The normal vector has to be such that added to mg the resultant force points to the center of the circular motion.

With such a configuration, we apply 2nd law

$$\vec{F} = m \vec{a}$$

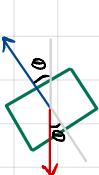
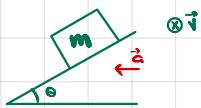
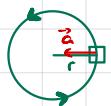
$$\hat{N} \quad N \cos \phi - mg = 0 \Rightarrow N = \frac{mg}{\cos \phi}$$

$$\hat{r} \quad -N \sin \phi = m a_r = -m r \theta'(t)^2$$

$$\Rightarrow \cancel{mg} \frac{\sin \phi}{\cos \phi} = \cancel{m r \theta'(t)^2} \Rightarrow \theta'(t) = \sqrt{\frac{g}{r} \tan \phi}$$

$$\Rightarrow v(t) = |\dot{\vec{r}}(t)| = |r \dot{\theta}(t) \hat{\theta}| = r |\theta'(t)| = r \sqrt{\frac{g}{r} \tan \phi} = \sqrt{rg \tan \phi}$$

Lecture Solution



$$\vec{F} = m\vec{a}$$

$$\hat{i} \quad -N\sin\theta = -m\frac{v^2}{r}$$

$$* a_r = -r\theta''(t)^2, \vec{v}(t) = r\theta'(t)\hat{\theta}$$

$$\Rightarrow a_r = -\frac{r(r\theta''(t))^2}{r} = -\frac{v_\theta(t)^2}{r} = -\frac{v(t)^2}{r}, v(t) = \dot{v}(t)$$

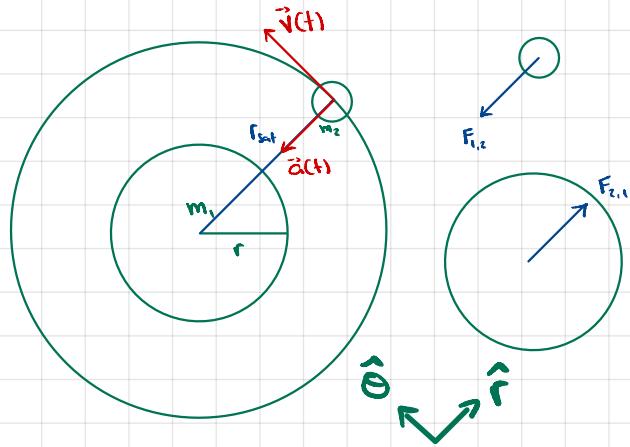
$$\hat{k} \quad N\cos\theta - mg = 0$$

$$\begin{cases} N\sin\theta = \frac{mv^2}{r} \\ N\cos\theta - mg = 0 \end{cases} \Rightarrow \tan\theta = \frac{v^2}{rg} \Rightarrow v = \sqrt{rg\tan\theta}$$

Questions

→ Why is the normal force larger than $v=0$ than than $v=0$? It seems like the car, in its motion is constantly pushing tangentially against the curvature of the road

PS. 3.1 - Orbital Circular Motion



$$\vec{F}_{c,2} = m_2 \cdot \vec{\alpha}_2$$

Since we have uniform circular motion, $\vec{\alpha}$ has only a radial component.

$\vec{\alpha} = r_{sat} \Theta'(t)$. When we plug $\vec{\alpha}_2$ into the second law, we can solve for the unknown r_{sat} .

$$\frac{Gm_1 m_2}{r_{sat}^2} = m_2 r_{sat} \Theta'(t)^2$$

. Decoupling $\Theta = \Theta'(t)$ because we know

the period of the orbit, T .

$$T = \frac{2\pi r_{sat}}{v}, \quad v = r_{sat} \Theta'(t) \Rightarrow \Theta'(t) = \frac{2\pi r_{sat}}{T} = \frac{2\pi}{T}$$

$$\Rightarrow r_{sat} = \sqrt[3]{\frac{Gm_1}{\frac{4\pi^2}{T^2}}} = \sqrt[3]{\frac{Gm_1 T^2}{4\pi^2}}$$

Kepler's Law: The cube of the distance between two objects proportional to square of period of orbit

b) $v(t) = |v(t)|$? The orbit motion here is uniform circular motion. $v(t) = \text{speed} = \text{constant}$.

$$v(t) = r_{sat} \Theta'(t) \hat{\theta}$$

$$|v(t)| = r_{sat} |\Theta'(t)| = r_{sat} |\omega| = r_{sat} \cdot \frac{2\pi}{T} = \sqrt[3]{\frac{8\pi^3}{T^3} \cdot \frac{T^2}{4\pi^2} \cdot Gm_1} = \sqrt[3]{\frac{2\pi}{T} \cdot Gm_1}$$

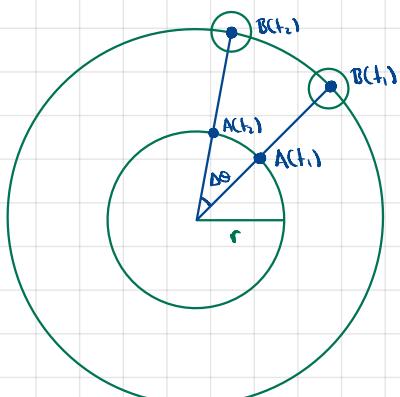
Alternatively, solve 2nd equation for v by using $\vec{\alpha} = v^2/r$

$$\frac{Gm_1 m_2}{r_{sat}^2} = m_2 \frac{v^2}{r_{sat}} \Rightarrow v = \sqrt{\frac{Gm_1}{r_{sat}}}$$

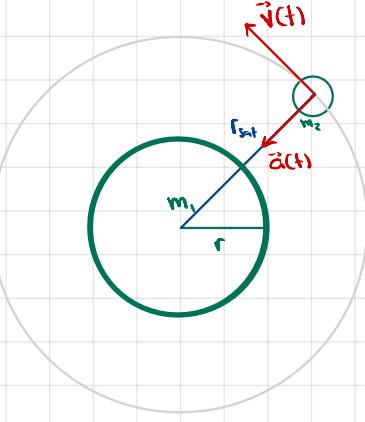
c) geostationary orbit

A person on the asteroid is on a uniform circular motion with period T_a . T_a must equal T_g for the satellite to be geostationary.

$$\Rightarrow r_{sat} = \sqrt[3]{\frac{Gm_1 T_g^2}{4\pi^2}}$$



A Walkthrough of Circular Orbital Motion



m_1 and m_2 attract each other by way of a pair of gravitational forces. Let's consider in particular the force of m_1 on m_2 . It accelerates m_2 in the direction $\vec{i}_1 - \vec{i}_2$, ie from m_2 's position to m_1 's position.

$\vec{z}(t)$ is the derivative of $\vec{v}(t)$, ie the rate of change of $\vec{v}(t)$, meaning that each component of $\vec{z}(t)$ is the rate of change of the corresponding component of $\vec{v}(t)$. Taken together, such rates of change added to \vec{v} represent where \vec{v} could be if it changed from t onwards with constant acceleration \vec{z} .

So far we're talking about these concepts in general. However, when we define position to be $\vec{r}(t) = r \vec{i}(t)$, we've set the trajectory \vec{r} always somewhere on a circle of radius r . This imposes structure on $\vec{v}(t)$ and $\vec{z}(t)$.

\vec{r} is a function of t , but it is easily described as a function of $\Theta(t)$ in rectangular coordinates.

Note that Θ and \vec{r} change in time, and when we write $\vec{r} = r \cos(\Theta) \hat{i} + r \sin(\Theta) \hat{j}$ we are expressing \vec{r} as a function of fixed coord. system.

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = r \frac{d\Theta(t)}{dt} \hat{\Theta}. \text{ Already we see the structure. } \vec{v} \text{ is always in } \hat{\Theta} \text{ direction, always } \perp \text{ to } \vec{r}.$$

$$\vec{z}(t) = r \Theta''(t) \hat{\Theta}(t) - r \Theta'(t)^2 \hat{r}(t)$$

\vec{r} depends on $\Theta(t)$ so $\vec{v}(t)$ depends on the rate of change of Θ . For any particular r , you can move as fast or slow as you want around the circle: $\Theta(t)$ determines how fast. Some says for $\vec{z}(t)$: Θ determines its components. In particular, if the rate of change of Θ is constant then \vec{z} has only a radial component.

Note that even though at a specific r for a given angular velocity ω matches you want, in terms of the 2nd law you also need to have a force generating the acceleration. $\omega = \text{constant} \Rightarrow \vec{a} = \text{radially}. \uparrow \omega \Rightarrow \uparrow \text{magnitude of acceleration} \Rightarrow \text{some force has to become stronger.}$

Let's connect all this to circular orbital motion

At a particular r between an object and another orbiting object, gravitational force is constant. At that radius, we have $\frac{Gm_1 m_2}{r^2} = m_2 a_{2,r}$. G, m_1, m_2, r fixed $\Rightarrow a_{2,r}$ fixed.

In this setup, if ω is increased but F and thus $a_{2,r}$ stay the same, motion won't be circular anymore (at least temporarily).

position radius
 $\vec{r}(t) = r(t) \hat{r}(t)$, unit radial vector

Indicates motion, $r(t)$ is constant. Now it is variable with time.

We want to know what this general trajectory is due to impose the presence of a gravitational force created by the celestial.

$$\frac{d\vec{r}(t)}{dt} = \vec{v}(t) = \frac{dr(t)}{dt} \hat{r}(t) + r(t) \frac{d\hat{r}(t)}{dt}$$

$$\frac{d^2\vec{r}(t)}{dt^2} = \vec{a}(t) = \frac{d^2r(t)}{dt^2} \hat{r}(t) + \frac{dr(t)}{dt} \frac{d\hat{r}(t)}{dt} + \frac{dr(t)}{dt} \frac{d\hat{r}(t)}{dt} + r(t) \frac{d^2\hat{r}(t)}{dt^2}$$

Immediately we want to describe the motion using polar coordinates $\langle r(t), \theta(t) \rangle$

$$\frac{d\hat{r}}{dt} = -\sin\theta \frac{d\theta}{dt} \hat{i} + \cos\theta \frac{d\theta}{dt} \hat{j} = \frac{d\theta(t)}{dt} \hat{\theta}(t)$$

$$\frac{d\hat{\theta}(t)}{dt} = -\cos\theta \frac{d\theta}{dt} \hat{i} - \sin\theta \frac{d\theta}{dt} \hat{j} = -\frac{d\theta(t)}{dt} \hat{r}(t)$$

$$\frac{d^2\hat{r}(t)}{dt^2} = \frac{d^2\theta(t)}{dt^2} \hat{\theta}(t) + \frac{d\theta(t)}{dt} \frac{d\hat{\theta}(t)}{dt} = \frac{d^2\theta(t)}{dt^2} \hat{\theta}(t) - \left[\frac{d\theta(t)}{dt} \right]^2 \hat{r}(t)$$

$$\Rightarrow \frac{d^2\vec{r}(t)}{dt^2} = \frac{d^2r(t)}{dt^2} \hat{r}(t) + 2 \frac{dr(t)}{dt} \frac{d\theta(t)}{dt} \hat{\theta}(t) + r(t) \left[\frac{d^2\theta(t)}{dt^2} \hat{\theta}(t) - \left[\frac{d\theta(t)}{dt} \right]^2 \hat{r}(t) \right]$$

$$= \hat{r}(t) \left[\frac{d^2r(t)}{dt^2} - r(t) \left[\frac{d\theta(t)}{dt} \right]^2 \right] + \hat{\theta}(t) \left[2 \frac{dr(t)}{dt} \frac{d\theta(t)}{dt} + r(t) \frac{d^2\theta(t)}{dt^2} \right]$$

So far there are general equations valid for any motion. However we have a constraint: Newton's second law.

$$-\frac{G m_1 m_2}{r^2} \hat{r} = m \cdot \vec{a}(t)$$

$$\Rightarrow \begin{cases} -\frac{G m_1 m_2}{r(t)^2} = m_2 \left[\frac{d^2r(t)}{dt^2} - r(t) \left[\frac{d\theta(t)}{dt} \right]^2 \right] \\ 0 = 2 \frac{dr(t)}{dt} \frac{d\theta(t)}{dt} + r(t) \frac{d^2\theta(t)}{dt^2} \end{cases}$$

These are two differential equations.

