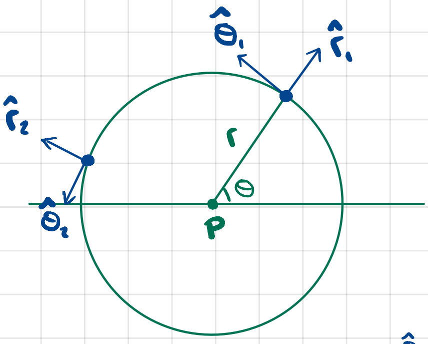
Week 3 Notes

8.1 Polar Coordinates

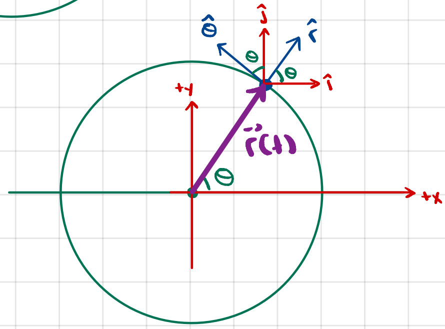


$\hat{r}, \hat{\theta}$ set of unit vectors at points (r, θ)

$\hat{r}_1 \neq \hat{r}_2,\ \hat{\theta}_1 \neq \hat{\theta}_2$

8.2 Position and Velocity Vectors

Relation with Cartesian Coord.



$$\hat{r} = \cos\theta\ \hat{i} + \sin\theta\ \hat{j}$$

$$\hat{\theta} = -\sin\theta\ \hat{i} + \cos\theta\ \hat{j}$$

when $\theta = \theta(t)$ we have

$$\hat{r}(t) = \cos\theta(t)\ \hat{i} + \sin\theta(t)\ \hat{j}$$

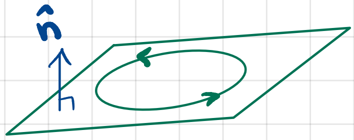
$$\vec{r}(t) = \text{position vector} = r\,\hat{r}(t) = r\cos\theta(t)\ \hat{i} + r\sin\theta(t)\ \hat{j}$$

$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t) = r\left(-\sin\theta(t)\frac{d\theta(t)}{dt}\right)\hat{i} + r\cos\theta(t)\frac{d\theta(t)}{dt}\ \hat{j}$$

$$= r\frac{d\theta(t)}{dt}\Big(\underbrace{-\sin\theta(t)\ \hat{i} + \cos\theta(t)\ \hat{j}}_{\hat{\theta}}\Big) = r\frac{d\theta}{dt}\ \hat{\theta} \equiv v_\theta\hat{\theta},\quad v_\theta = r\frac{d\theta}{dt}$$

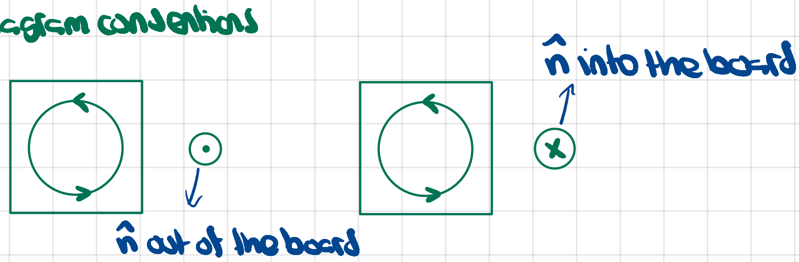
8.3 Angular Velocity

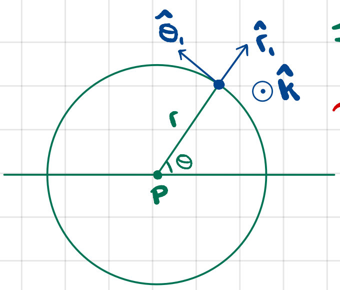
→ given circular motion on a plane as in the picture below, the motion is clockwise when seen from below and counterclockwise when seen from above.



→ to define a positive direction, we use the right hand rule: curl fingers in direction of movement, thumb points in positive direction, ie direction of vector normal to the plane in positive direction.

→ diagram conventions





In this coordinate system there's a third direction $\hat{k}$

Def: angular velocity $\vec{\omega} = \frac{d\theta}{dt}\hat{k} = \omega_z$

angular speed $\omega = |\vec{\omega}| = \left|\frac{d\theta}{dt}\right|$

$$\Rightarrow v(t) = r\frac{d\theta}{dt}\hat{\theta} = r\omega_z\hat{\theta}$$

9.1 Uniform Circular Motion

$$v(t) = r\frac{d\theta}{dt}\hat{\theta}$$

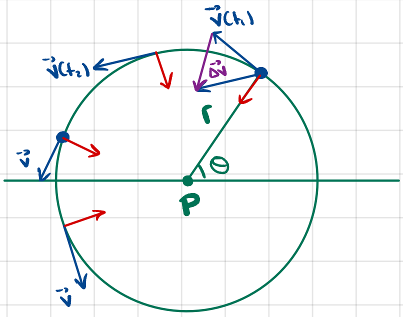
special case: $\frac{d\theta}{dt} = \text{constant}$ $\Rightarrow$ $|v(t)| = r\left|\frac{d\theta}{dt}\right| = \text{constant}$ (uniform circular motion)

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left[r\frac{d\theta}{dt}\left(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}\right)\right]$$

$$= r\overset{0,\text{ because }\theta'(t)=\text{constant}}{\cancel{\frac{d^2\theta}{dt^2}}}\hat{\theta}(t) + r\frac{d\theta}{dt}\left(-\cos\theta\frac{d\theta}{dt}\hat{\imath} - \sin\theta\frac{d\theta}{dt}\hat{\jmath}\right)$$

$$= -r\left(\frac{d\theta}{dt}\right)^2\hat{r}(t)$$

9.2 Direction of Acceleration



$$\vec{a} = -r\,\theta'(t)^2\,\hat{r}(t)$$

$$\vec{a} = a_r\hat{r} \qquad a_r = -r\,\theta'(t)^2 < 0$$

$$\Delta\vec{v} = \vec{v}(t_2) - \vec{v}(t_1)$$

$$\vec{a} = \lim_{\Delta t \to 0}\frac{\Delta\vec{v}}{\Delta t}$$

10.1 Circular Motion - Acceleration

$\vec{v} = r\frac{d\theta}{dt}\hat{\theta}$

$\vec{a} = \frac{d\vec{v}}{dt} = r\theta''(t)\hat{\theta} + \underbrace{r\theta'(t)\frac{d\hat{\theta}}{dt}}$ calculated previously $= a_r = -r\theta'(t)^2$

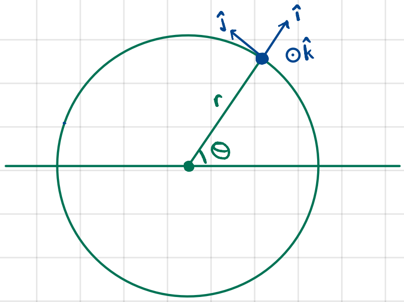
$= a_\theta\hat{\theta} + a_r\hat{r}$

tangential radial

$a_\theta = r\theta''(t)$

$\theta''(t) > 0 \Rightarrow \theta'(t)$ increasing

10.2 Angular Acceleration



recall $\vec{\omega} = \frac{d\theta}{dt}\hat{k}$, angular velocity $= \omega_z\hat{k}$

$\vec{\alpha} = \frac{d^2\theta}{dt^2}\hat{k}$, angular acceleration

$= \alpha_z\hat{k}$, $\alpha_z = \frac{d^2\theta}{dt^2} = \frac{d\omega_z}{dt}$

cases

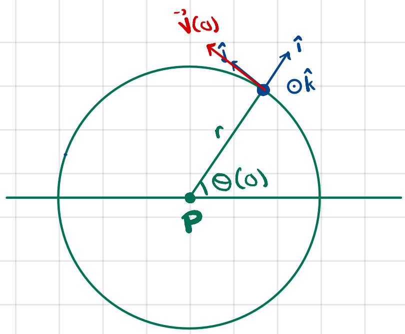
1. $\omega_z > 0$
   a. $\alpha_z > 0$, $\frac{d\omega_z}{dt} > 0 \Rightarrow$ ccw, speeding up
   b. $\alpha_z < 0$, $\frac{d\omega_z}{dt} < 0 \Rightarrow$ ccw initially, slowing down to $\omega = 0$, then $|\omega_z|$ increases with motion now cw

2. $\omega_z < 0$
   a. $\alpha_z < 0$, $\frac{d\omega_z}{dt} < 0 \Rightarrow$ cw motion, speeding up
   b. $\alpha_z > 0$, $\frac{d\omega_z}{dt} > 0 \Rightarrow$ cw motion initially, angular speed decreasing to zero, motion becomes ccw, angular speed increases.

* ω_z, α_z are like v, a in 1D motion

10.3 Worked Example (m+solution)

$\vec{\alpha}(t) = \frac{1}{r}(A - Bt)\hat{k}$. $\omega_z = \theta'(t)$, $\alpha_z = \theta''(t) \Rightarrow \theta''(t) = \frac{1}{r}(A - Bt)$



$a_\theta = r\theta''(t) = A - Bt$

$t_0 = 0$, $\theta(t_0)$, $v_\theta(t_0) = v_0 = r\theta'(0) \Rightarrow \theta'(0) = \frac{v_0}{r}$

arclength defined $s(t) = r(\theta(t) - \theta(0))$

$\theta'(t) = \frac{1}{r}(At - B\frac{t^2}{2}) + \frac{v_0}{r}$

$\theta(t) = \frac{1}{r}(\frac{At^2}{2} - \frac{Bt^3}{6}) + \frac{v_0}{r}t + \theta(0)$

$s(t) = \frac{At^2}{2} - \frac{Bt^3}{6} + v_0 t$

→ Lecture Solution

$$a_\theta(t) = A - Bt = r\theta''(t)$$

$$V_\theta(t) - V_\theta(t_0) = \int_{t_0=0}^{t} a_\theta(t)\,dt = At - \frac{Bt^2}{2}$$

$$V_\theta(t) = V_\theta(0) + At - \frac{Bt^2}{2}$$

$$\underbrace{r\theta(t) - r\theta(0)}_{\text{change in arc length}} = \int_0^t V_\theta(t)\,dt = r\int_0^t \theta'(t)\,dt$$

$$= \int_0^t \left[V_\theta(0) + At - \frac{Bt^2}{2}\right]dt = V_\theta(t) + \frac{1}{2}At^2 - \frac{1}{6}Bt^3$$

11.1 Newton's Second Law and Circular Motion

recall $\qquad \vec{F} = m\vec{a} \qquad$ vector equation

$\qquad$ why? $\qquad$ how?

$\qquad$ dynamics $\qquad$ geometric

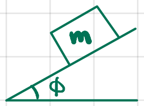
$$F_r = ma_r$$

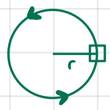
$$F_r = -mr\omega^2$$

$$F_\theta = ma_\theta$$

$$F_\theta = mr\theta''(t) = mr\alpha_z$$

11.2 Worked Example: Car on a Banked Turn

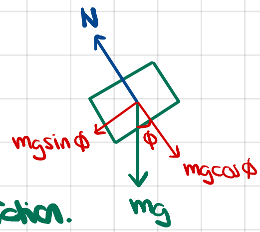




$$\vec{r}(t) = r\hat{r}(t)$$
$$\vec{v}(t) = r\theta'(t)\hat{\theta}(t)$$
$$\vec{a}(t) = r\theta''(t)\hat{\theta}(t) - r\theta'(t)^2\hat{r}(t)$$

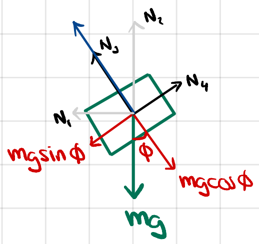
A car's engine generates rotation of the wheels which create a friction force with the ground. Friction propels the car forward. If the friction coeff. is zero, the car cannot accelerate tangentially due to the car's engine. $\Rightarrow a_\theta = 0$.

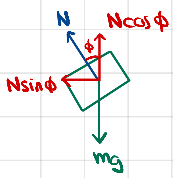
$\vec{a}$ is a vector so it can have nonzero components other than in direction $\hat{\theta}$.



If the car did not have tangential velocity, $v_\theta = 0 \Rightarrow \theta'(t) = 0$
Acceleration could be positive in the direction parallel to the slope the car is on. In the absence of friction, the car would slide down the slope with acceleration.

If the car is turning on the banked curve without slipping down the slope, acceleration is radial





The normal vector has to be such that added to mg the resultant force points to the center of the circular motion.

With such a configuration, we apply 2nd law

$$\vec{F} = m\vec{a}$$

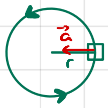
$\hat{k} \qquad N\cos\phi - mg = 0 \Rightarrow N = \dfrac{mg}{\cos\phi}$

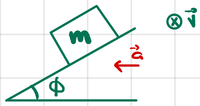
$\hat{r} \qquad -N\sin\phi = ma_r = -mr\theta'(t)^2$

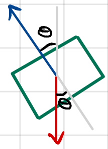
$$\Rightarrow \frac{\cancel{m}g\sin\phi}{\cos\phi} = \cancel{m}r\theta'(t)^2 \Rightarrow \theta'(t) = \sqrt{\frac{g}{r}\tan\phi}$$

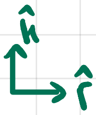
$$\Rightarrow v(t) = |\vec{v}(t)| = |r\theta'(t)\hat{\theta}| = r|\theta'(t)| = r\sqrt{\frac{g}{r}\tan\phi} = \sqrt{rg\tan\theta}$$

Lecture Solution









$$\vec{F} = m\vec{a}$$

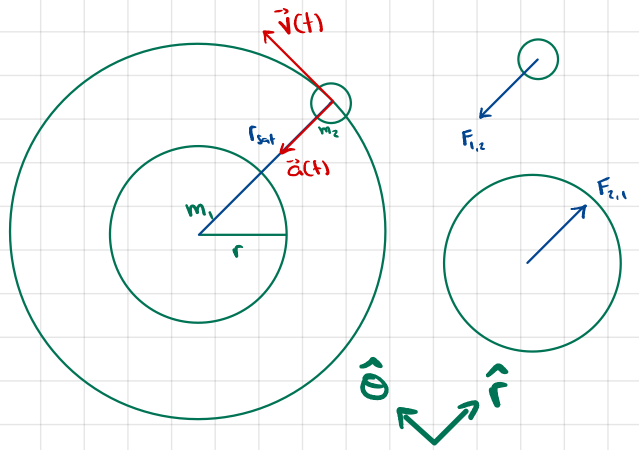
$\hat{r}$ $\quad -N\sin\theta = -\frac{mv^2}{r}$ $\qquad * \; a_r = -r\theta'(t)^2, \; \vec{v}(t) = r\theta'(t)\hat{\theta}$

$$\Rightarrow a_r = -\frac{r\,r\theta'(t)^2}{r} = \frac{-v_\theta(t)^2}{r} = \frac{-v(t)^2}{r}, \; v(t) = |\vec{v}(t)|$$

$\hat{h}$ $\quad N\cos\theta - mg = 0$

$$\begin{cases} N\sin\theta = \frac{mv^2}{r} \\ N\cos\theta = mg \end{cases} \Rightarrow \tan\theta = \frac{v^2}{rg} \Rightarrow v = \sqrt{rg\tan\theta}$$

PS.3.1 - Orbital Circular Motion



$$\vec{F}_{1,2} = m_2 \cdot \vec{a}_2$$

Since we have uniform circular motion, $\vec{a}$ has only a radial component.

$\vec{a} = r_{sat}\theta'(t)$. When we plug $\vec{a}_2$ into the second law, we can solve for the unknown r_{sat}.

$$\frac{Gm_1\cancel{m_2}}{r_{sat}^2} = \cancel{m_2}\, r_{sat}\, \theta'(t)^2$$

$$r_{sat}^3 = \frac{Gm_1}{\theta'(t)^2} \Rightarrow r_{sat} = \sqrt[3]{\frac{Gm_1}{\omega^2}}$$

. We can find $\omega = \theta'(t)$ because we know the period of the orbit, T.

$$T = \frac{2\pi r_{sat}}{v},\quad v = r_{sat}\,\theta'(t) \Rightarrow \theta'(t) = \frac{2\pi \cancel{r_{sat}}}{T\cancel{r_{sat}}} = \frac{2\pi}{T}$$

$$\Rightarrow r_{sat} = \sqrt[3]{\frac{Gm_1}{\frac{4\pi^2}{T^2}}} = \sqrt[3]{\frac{Gm_1 T^2}{4\pi^2}}$$

→ **Kepler's Law**: The cube of the distance between two objects proportional to square of period of orbit.

b) $v(t) = |\vec{v}(t)|$? The orbit motion here is uniform circular motion. $v(t)$ = speed = constant.

$$v(t) = r_{sat}\,\theta'(t)\,\hat{\theta}$$

$$|v(t)| = r_{sat}\,|\theta'(t)| = r_{sat}\,|\omega| = r_{sat}\cdot\frac{2\pi}{T} = \sqrt[3]{\frac{8\pi^3}{T^3}\cdot\frac{T^2}{4\pi^2}\cdot Gm_1} = \sqrt[3]{\frac{2\pi}{T}\cdot Gm_1}$$

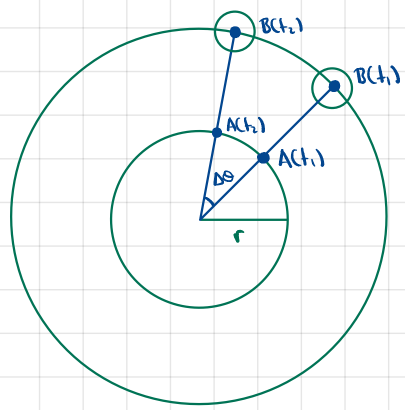
Alternatively, solve 2nd equation for v by using $\vec{a} = v^2/r$

$$\frac{Gm_1\cancel{m_2}}{r_{sat}^2} = \cancel{m_2}\frac{v^2}{r_{sat}} \Rightarrow v = \sqrt{\frac{Gm_1}{r_{sat}}}$$

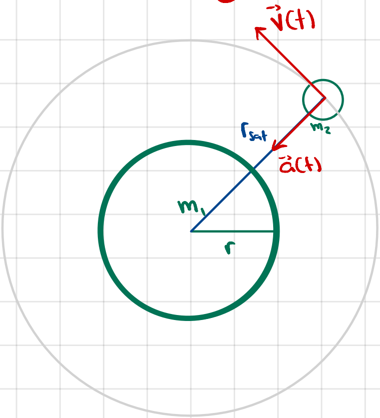
c) geostationary orbit

A person on the asteroid is on a uniform circular motion with period T_a. T must equal T_a for the satellite to be geostationary.

$$\Rightarrow r_{sat} = \sqrt[3]{\frac{Gm_1 T_a^2}{4\pi^2}}$$



A Walkthrough of Orbital Circular Motion



m_1 and m_2 attract each other by way of a pair of gravitational forces. Lets consider in particular the force of m_1 on m_2. It accelerates m_2 in the direction $\vec{r}_1 - \vec{r}_2$, ie from m_2's position to m_1's position.

$\vec{a}(t)$ is the derivative of $\vec{v}(t)$, ie the rate of change of $\vec{v}(t)$, meaning that each component of $\vec{a}(t)$ is the rate of change of the corresponding component of $\vec{v}(t)$. Taken together, such rates of change added to $\vec{v}$ represent where $\vec{v}$ would be if it changed from t onwards with constant acceleration $\vec{a}$.

So far we're talking about these concepts in general. However, when we define position to be $\vec{r}(t) = r\,\hat{r}(t)$, we've said the trajectory is always somewhere on a circle of radius r. This imposes structure on $\vec{v}(t)$ and $\vec{a}(t)$.

$\hat{r}$ is a function of t, but it's easily described as a function of $\theta(t)$ in rectangular coordinates.

Note that $\hat{\theta}$ and $\hat{r}$ change in time, and when we write $\hat{r} = \cos\theta(t)\,\hat{i} + \sin\theta(t)\,\hat{j}$ we are expressing $\hat{r}$ as a function of fixed coord. system.

$\vec{v}(t) = \dfrac{d\vec{r}}{dt} = r\dfrac{d\theta(t)}{dt}\hat{\theta}$. Already we see the structure. $\vec{v}$ is always in $\hat{\theta}$ direction, always $\perp$ to $\vec{r}$.

$\vec{a}(t) = r\theta''(t)\hat{\theta}(t) - r\theta'(t)^2\,\hat{r}(t)$

$\hat{r}$ depends on $\theta(t)$ so $v(t)$ depends on the rate of change of θ. For any particular r, you can make it fast or slow as you want around the circle: $\theta(t)$ determines how fast. Same goes for $\vec{a}(t)$: θ determines its components. In particular, if the rate of chan