

## 9.3 Worked Examples in Circular Motion

### 9.1 Geostationary Orbit

$$F_{\text{es}} = -\frac{Gm_e \cancel{v_s^2}}{r^2} = -\cancel{v_s r \omega^2} \Rightarrow r = \sqrt[3]{\frac{Gm_e}{\omega^2}}$$

$$m_e = 5.98 \cdot 10^{24} \text{ kg}$$

$$G = 6.67 \cdot 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2}$$

Geostationary Satellite has period  $T = 23 \text{h} 56 \text{min} 4 \text{sec}$

$$T = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{2\pi}{T}$$

using Napple:

$$T = 86164 \text{ s}$$

$$\omega = \pi/43082 \text{ rad/s}$$

$$r = 4.22 \cdot 10^7 \text{ m}$$

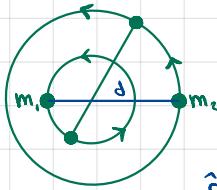
$$\Rightarrow \frac{r}{R_E} = 6.62 \Rightarrow r = 6.62 R_E$$

$$R_E = 6.67 \cdot 10^6 \text{ m} \text{ on average}$$

$$* v = r\omega = 4.22 \cdot 10^7 \cdot \frac{\pi}{43082} = 3075 \text{ m/s}$$

$$r - R_E = 3.580350164 \cdot 10^7 \text{ m} \approx 35803 \text{ km above the Earth.}$$

## 9.2 Double Star System



Two stars, always at fixed distance  $d$  apart, both in uniform circular motion

Expectation: same angular speed  $\omega \Rightarrow$  same period.  $m_2$  has lesser speed because of its larger radius.

$$\vec{F}_{2,1} = -\frac{Gm_1m_2}{d^2}\hat{r}_1 = -m_1\omega^2\hat{r}_1 = -m_1r_1\omega^2\hat{r}_1$$

$$\Rightarrow \omega^2 = \frac{Gm_2}{r_1d^2}$$

$$\Rightarrow \frac{m_2}{r_1} = \frac{m_1}{r_2} \Rightarrow m_1r_1 = m_2r_2$$

$$\vec{F}_{1,2} = -\frac{Gm_1m_2}{d^2}\hat{r}_2 = -m_2\omega^2r_2\hat{r}_2$$

$$\Rightarrow \omega^2 = \frac{Gm_1}{r_2d^2}$$

Also, geometrically we can see that  $d = r_1 + r_2$ .

Given  $m_1, m_2$ , and same  $\omega$ , we know the proportion between the radii. E.g. if  $m_1 = 2m_2$  then  $r_2 = 2r_1$   
 $\Rightarrow 2r_1 + r_2 = d \Rightarrow r_2 = d/3, r_1 = 2d/3$ . We have three equations and three unknowns ( $\omega, r_1, r_2$ ).

$$m_1r_1 = m_2(d-r_1) = m_2d - m_2r_1 \Rightarrow r_1(m_1+m_2) = m_2d \Rightarrow r_1 = \frac{m_2}{m_1+m_2}d, r_2 = \frac{m_1}{m_1+m_2}d$$

$$\Rightarrow \omega = \sqrt{\frac{Gm_2(m_1+m_2)}{d^2m_2d}} = \sqrt{\frac{G(m_1+m_2)}{d^3}}$$

$$T = \frac{2\pi}{\omega} = \sqrt{\frac{4\pi^2d^3}{G(m_1+m_2)}}$$

Note: previously, we analyzed a uniform circular motion of a mass  $m_2$  around a central point mass  $m_1$ , with an implicit assumption that  $m_2 \ll m_1$ . We obtained Kepler's law:

$$r^3 = \frac{Gm_1T^2}{4\pi^2} \Rightarrow T = \sqrt{\frac{4\pi^2r^3}{Gm_1}}$$

when we consider the acceleration of both masses, we still have Kepler's law: cube of distance proportional to square of period. But now we have both masses affecting this relationship.

Let's take an example: Earth and Moon. If we assume the moon has a circular orbit caused by earth's gravitational force, we obtain

$$T = \sqrt{\frac{4\pi^2(3.82 \cdot 10^8)^3}{6.67 \cdot 10^{-11}(5.98 \cdot 10^{24})}} = 2.3488 \cdot 10^6$$

If we take into account the the Earth also orbits the moon, with all our assumptions above (constant distance, same  $\omega$ ), then we have a "double star" system and the period is

$$T = \sqrt{\frac{4\pi^2(3.82 \cdot 10^8)^3}{6.67 \cdot 10^{-11}(5.98 \cdot 10^{24} + 7.36 \cdot 10^{22})}} = 2.3348 \cdot 10^6$$

The difference is  $14322.591$  seconds =  $0.1657$  days.