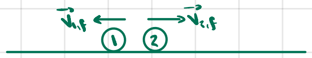
26.1 Momentum in Collisions

(initial state)



(final state)

$$\vec F_{ext}\cdot\Delta t = \vec p_{sys,f} - \vec p_{sys,i}$$

physics — description

assumptions: $\vec F_{ext} = 0 \Rightarrow \vec p_{sys,f} = \vec p_{sys,i}$, momentum of system is constant.

$$m_1\vec v_{1,i} + m_2\vec v_{2,i} = m_1\vec v_{1,f} + m_2\vec v_{2,f}$$

component eq. for x direction is

$$m_1 v_{1,x,i} + m_2 v_{2,x,i} = m_1 v_{1,x,f} + m_2 v_{2,x,f}$$

26.2 Kinetic Energy in Collisions

elastic collision: $\Delta K_{sys} = 0$

inelastic ": $\Delta K_{sys} < 0$

superelastic ": $\Delta K_{sys} > 0$

$$\frac{1}{2}m_1 v_{1,i}^2 + \frac{1}{2}m_2 v_{2,i}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2$$

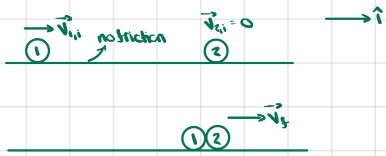
$$m_1 v_{1,x,i}^2 + m_2 v_{2,x,i}^2 = m_1 v_{1,x,f}^2 + m_2 v_{2,x,f}^2 \quad \text{(kin. en. eq)}$$

recall we have a momentum eq.:

$$m_1 v_{1,x,i} + m_2 v_{2,x,i} = m_1 v_{1,x,f} + m_2 v_{2,x,f}$$

given m_i, $v_{1,x,i}$, $v_{2,x,i}$ we can solve for $v_{1,x,f}$ and $v_{2,x,f}$

26.3 Totally Inelastic Collisions



$$m_1 v_{1,x,i} = (m_1 + m_2) v_{x,f}$$

$$v_{x,f} = \frac{m_1}{m_1+m_2} v_{1,x,i}$$

$$\Delta K = \frac{1}{2}(m_1+m_2) v_{x,f}^2 - \frac{1}{2} m_1 v_{1,x,i}^2$$

$$= \frac{1}{2}(m_1+m_2)\left(\frac{m_1}{m_1+m_2} v_{1,x,i}\right)^2 - \frac{1}{2} m_1 v_{1,x,i}^2$$

$$= \frac{1}{2} m_1 v_{1,x,i}^2 \left(\frac{m_1}{m_1+m_2} - 1\right)$$

$$= K_i\left(-\frac{m_2}{m_1+m_2}\right)$$

27.1 Worked Example: Elastic 1D Collision

Frame: laboratory

Ex: $m_2 = 2m_1$, elastic collision

$$m_1 v_{1,i} = m_1 v_{1,f} + m_2 v_{2,f}$$

$$\frac{m_1 v_{1,i}^2}{2} = \frac{m_1 v_{1,f}^2}{2} + \frac{m_2 v_{2,f}^2}{2} \Rightarrow v_{1,i}^2 = v_{1,f}^2 + 2 v_{2,f}^2$$

$$v_{2,f} = \frac{m_1(v_{1,i} - v_{1,f})}{2m_1} = \frac{v_{1,i} - v_{1,f}}{2}$$

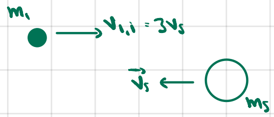
$$v_{1,i}^2 = v_{1,f}^2 + \frac{(v_{1,i} - v_{1,f})^2}{2}$$

$$\Rightarrow 2v_{1,i}^2 = 2v_{1,f}^2 + v_{1,i}^2 + v_{1,f}^2 - 2 v_{1,i} v_{1,f}$$

$$3v_{1,f}^2 - 2v_{1,i}v_{1,f} - v_{1,i}^2$$

$$v_{1,f} = \frac{2v_{1,i} \pm \left[4v_{1,i}^2 + 12 v_{1,i}^2\right]^{1/2}}{6} = \frac{2v_{1,i} \pm 4v_{1,i}}{6} \begin{cases} \nearrow v_{1,i} \\ \searrow -\frac{1}{3}v_{1,i} \end{cases}$$

27.5 Worked Example - Gravitational Slingshot



$v_{1,i} = 3v_s\hat i$

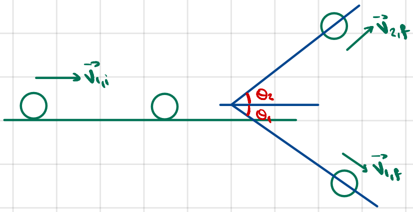
$v_{s,i} = -v_s\hat i$

$v_{rel,i} = 3v_s\hat i - (-v_s)\hat i = 4v_s\hat i$

$v_{rel,f} = v_{1,f}\hat i - (-v_s)\hat i = -4v_s\hat i$

$\Rightarrow v_{1,f}\hat i = -5v_s\hat i$

27.6 2D Collisions



$m_1\vec{v}_{1,i} = m_1\vec{v}_{1,f} + m_2\vec{v}_{2,f}$

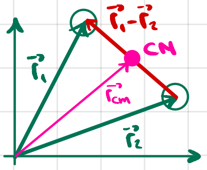
$m_1 v_{1,i} = m_1 v_{1,f}\cos\theta_1 + m_2 v_{2,f}\cos\theta_2$

$0 = m_1 v_{1,f}\sin\theta_1 + m_2 v_{2,f}\sin\theta_2$

elastic ⇒ $\Delta K = 0$ $\quad \frac{m_1 v_{1,i}^2}{2} = \frac{m_1 v_{1,f}^2}{2} + \frac{m_2 v_{2,f}^2}{2}$

3 equations, 4 unknowns: $v_{1,f}, v_{2,f}, \theta_1, \theta_2$

D.D.2.1 Position in CM Frame



$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$

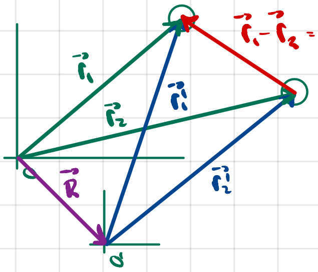
$\vec{r}_1' = \vec{r}_1 - \vec{r}_{cm} = \frac{m_2}{m_1+m_2}(\vec{r}_1 - \vec{r}_2)$

$\vec{r}_2' = \vec{r}_2 - \vec{r}_{cm} = \frac{-m_1}{m_1+m_2}(\vec{r}_1 - \vec{r}_2)$

reduced mass $= \mu = \frac{m_1 m_2}{m_1+m_2}$

⇒ $\vec{r}_1' = \frac{\mu}{m_1}\vec{r}_{12}$ $\qquad \vec{r}_2' = \frac{-\mu}{m_2}\vec{r}_{12}$

D.D.2.2 Relative Velocity is Independent of Ref. Frame



$\vec{r}_1 = \vec{R} + \vec{r}_1'$
$\vec{r}_2 = \vec{R} + \vec{r}_2'$ ⇒ $\vec{r}_1 - \vec{r}_2 = \vec{r}_1' - \vec{r}_2'$ ⇒ $\vec{v}_1 - \vec{v}_2 = \vec{v}_1' - \vec{v}_2'$

D.D.2.3 1D Elastic Collision Velocities in CM Frame

we know that

- in a 1D elastic collision, $\vec{v}_{1,2,i} = -\vec{v}_{1,2,f}$
  ⇒ $|\vec{v}_{1,2,i}| = |\vec{v}_{1,2,f}|$
- velocities in CM reference frame are fns of $\vec{v}_{1,2}$:
  $\vec{v}_1' = \frac{\mu}{m_1}\vec{v}_{1,2}$ $\qquad \vec{v}_2' = \frac{-\mu}{m_2}\vec{v}_{1,2}$

⇒ Therefore, $|\vec{v}_{1,i}'| = |\vec{v}_{1,f}'|$ and $|\vec{v}_{2,i}'| = |\vec{v}_{2,f}'|$

$\vec{v}_{1,i}' = -\vec{v}_{1,f}'$, $\vec{v}_{2,i}' = -\vec{v}_{2,f}'$

speeds of objects don't change in CM frame, only the direction of velocity changes.

D.D.2.4 Worked Example: 1D Elastic Collision in CM

$\vec{v}_{1,f}' = -\vec{v}_{1,i}'$

$\vec{v}_{1,i}' = \frac{\mu}{m_1}\vec{v}_{1,2,i} = \frac{\mu}{m_1}\vec{v}_{1,i} = \frac{m_2}{m_1+m_2}\vec{v}_{1,i} = \frac{2}{3}v_{1,i}$

⇒ $\vec{v}_{1,f}' = -\frac{2}{3}v_{1,i}$

$\vec{v}_{2,f}' = -\vec{v}_{2,i}' = -\left(-\frac{m_1}{m_1+m_2}\vec{v}_{1,i}\right) = \frac{1}{3}\vec{v}_{1,i}$

$\vec{v}_{cm,i} = \frac{m_1 v_{1,i}}{m_1+m_2} = \frac{1}{3}v_{1,i}$

$\vec{v}_{1,f} = \vec{v}_{1,f}' + \vec{v}_{cm} = -\frac{2}{3}v_{1,i} + \frac{1}{3}v_{1,i} = -\frac{1}{3}v_{1,i}$

$\vec{v}_{2,f} = \frac{1}{3}v_{1,i} + \frac{1}{3}v_{1,i} = \frac{2}{3}v_{1,i}$

D.D.2.5 Kinetic Energy in Different Reference Frames

$K_{cm} = \frac{m_1 v_{1,i}'^2}{2} + \frac{m_2 v_{2,i}'^2}{2}$

$K_{ground} = \frac{m_1 v_{1,i}^2}{2} + \frac{m_2 v_{2,i}^2}{2}$

$= \frac{m_1}{2}(v_{1,i}' + v_{cm})^2 + \frac{m_2}{2}(v_{2,i}' + v_{cm})^2$

$= \frac{m_1}{2}(v_{1,i}'^2 + v_{cm}^2 + 2v_{1,i}'v_{cm}) + \frac{m_2}{2}(v_{2,i}'^2 + v_{cm}^2 + 2v_{2,i}'v_{cm})$

$= \frac{m_1 v_{1,i}'^2}{2} + \frac{m_2 v_{2,i}'^2}{2} + (m_1+m_2)v_{cm}^2 + v_{cm}\cancelto{0}{(m_1 v_{1,i}' + m_2 v_{2,i}')}$

$= K_{cm} + \frac{m_1+m_2}{2}v_{cm}^2$

D.D.2.6 Kinetic Energy in CM Frame

$K_{cm} = \frac{m_1}{2}\left(\frac{\mu}{m_1}v_{1,2}\right)^2 + \frac{m_2}{2}\left(-\frac{\mu}{m_2}v_{1,2}\right)^2$

$= \frac{1}{2}\frac{\mu^2 v_{1,2}^2}{m_1} + \frac{1}{2}\frac{\mu^2 v_{1,2}}{m_2} = \frac{1}{2}\mu^2 v_{1,2}^2\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$

$= \frac{\mu v_{1,2}^2}{2}$

$\Delta K_{cm} = \frac{\mu}{2}(v_{1,2,f}^2 - v_{1,2,i}^2)$

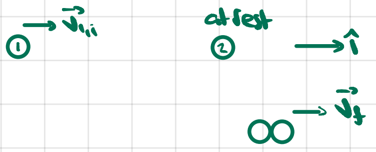
elastic collision ⇒ $v_{1,2,i}^2 = v_{1,2,f}$

totally inelastic ⇒ $v_{1,2,f} = 0$

D.D. 2.7 Change in Kinetic Energy

$$K_{grand} = K_{cm} + \frac{m_1+m_2}{2} v_{cm}^2 \Rightarrow \Delta K_{grand} = \Delta K_{cm}$$

lab frame



cm frame

$$K_{cm} = \frac{\mu v_{1,2}^2}{2}$$

$$\Delta K_{cm} = -\frac{\mu v_{1,2}^2}{2} = \Delta K_{grand}$$

Recap of Results

we can always write the eq. for conservation of momentum

$m_1\vec{v}_{1,i} + m_2\vec{v}_{2,i} = m_1\vec{v}_{1,f} + m_2\vec{v}_{2,f}$

we can work with the component in x-direction

$m_1 v_{1,x,i} + m_2 v_{2,x,i} = m_1 v_{1,x,f} + m_2 v_{2,x,f}$

for elastic collisions, we write $\Delta K = 0 \Rightarrow K_i = K_f$

$\frac{1}{2}m_1v_{1,i}^2 + \frac{1}{2}m_2v_{2,i}^2 = \frac{1}{2}m_1v_{1,f}^2 + \frac{1}{2}m_2v_{2,f}^2$

we can solve for $v_{1,f}$ and $v_{2,f}$ to obtain the 1D energy momentum principle

$v_{1,x,i} - v_{2,x,i} = v_{1,x,f} - v_{2,x,f} \Rightarrow v_{1,2,i} = v_{1,2,f}$

---

for totally inelastic collisions, $v_{1,2,f} = 0$ ie $v_{1,x,f} = v_{2,x,f}$

again, conservation of momentum

$m_1 v_{1,x,i} + m_2 v_{2,x,i} = (m_1+m_2) v_{x,f}$

$\Rightarrow v_{x,f} = \dfrac{m_1 v_{1,x,i} + m_2 v_{2,x,i}}{m_1+m_2}$

we haven't imposed any explicit condition on ΔK but implicitly we have

$\Delta K = \dfrac{(m_1+m_2)v_{x,f}^2}{2} - \dfrac{m_1 v_{1,x,i}^2}{2} - \dfrac{m_2 v_{2,x,i}^2}{2}$

simplifying $= \dfrac{-m_1m_2(v_{1,x,i}-v_{2,x,i})^2}{2m_1+2m_2} < 0$

$\Rightarrow$ Kinetic Energy is lost in totally inelastic collision.

---

next we derived results concerning the center of mass frame
we know $\vec{r}_{cm}$, so we can obtain any position from CM frame given position in lab frame

$\vec{r}_1' = \frac{\mu}{m_1}\vec{r}_{12}$ $\qquad\Rightarrow\qquad$ $\vec{v}_1' = \frac{\mu}{m_1}\vec{v}_{1,2}$

$\vec{r}_2' = \frac{-\mu}{m_2}\vec{r}_{12}$ $\qquad\qquad$ $\vec{v}_2' = -\frac{\mu}{m_2}\vec{v}_{1,2}$

where μ = reduced mass $= \dfrac{m_1m_2}{m_1+m_2}$

But $\vec{v}_{1,2} = \vec{v}_{1,2}$, ie the relative velocity vector is the same in every reference frame (that is relatively inertial)

This gives us a new result in the 1D elastic collision case.

1D energy momentum principle $\Rightarrow v_{1,2,i} = -v_{1,2,f}$

$\vec{v}_1' = \frac{\mu}{m_1}\vec{v}_{1,2}$ $\qquad$ $\vec{v}_2' = -\frac{\mu}{m_2}\vec{v}_{1,2}$

$\Rightarrow \vec{v}_{1,i}' = \frac{\mu}{m_1}\vec{v}_{1,2,i}$

$\vec{v}_{1,f}' = \frac{\mu}{m_1}\vec{v}_{1,2,f} = \frac{\mu}{m_1}(-\vec{v}_{1,2,i})$

$\Rightarrow \vec{v}_{1,f}' = -\vec{v}_{1,i}'$

likewise $\vec{v}_{2,f}' = -\vec{v}_{2,i}'$

In CM frame, in 1D elastic collision, velocities of objects change direction but not magnitude (speed).

Next we look at kinetic energy. Start with the definitions

$K_{cm} = \dfrac{m_1v_{1,i}'^2}{2} + \dfrac{m_2v_{2,i}'^2}{2}$

$K_{ground} = \dfrac{m_1v_{1,i}^2}{2} + \dfrac{m_2v_{2,i}^2}{2}$

we know the relationship between $v_{1,i}$ and $v_{1,i}'$, and between $v_{2,i}$ and $v_{2,i}'$.

$\vec{v}_1' = \frac{\mu}{m_1}\vec{v}_{1,2}$ $\qquad$ $\vec{v}_2' = -\frac{\mu}{m_2}\vec{v}_{1,2}$

so we can find K_{ground} in terms of K_{cm} and vice versa
sub v_1' and v_2' into K_{ground}, obtain

$K_{ground} = K_{cm} + \dfrac{m_1+m_2}{2}v_{cm}^2$

sub v_1' and v_2' into K_{cm}, obtain

$K_{cm} = \dfrac{\mu v_{1,2}^2}{2}$

in an elastic collision, $v_{1,2,i} = -v_{1,2,f}$ so

$\Delta K_{cm} = 0$

in a totally inel. collision $v_{1,2,f} = 0$

$\Delta K_{cm} = -\dfrac{\mu v_{1,2,i}^2}{2} < 0$

Finally, note that $K_{ground} = K_{cm} + \text{constant}$

$\Rightarrow \Delta K_{ground} = \Delta K_{cm}$