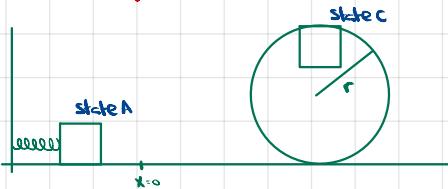


Pset 8 - Potential En. and En. Conservation

Problem 1 - Spring-Loop-the-Loop

Ex 14.3 Spring-Block-Loop-the-Loop



no friction

$$A: K_A = 0$$

$U_{elA} = \frac{1}{2}kx^2$ initially there was only elastic potential energy

$$U_{gA} = 0$$

$$B: K_B = \frac{1}{2}mv_B^2$$

$U_{elB} = 0$ at top of loop, energy has become kinetic and gravitational potential

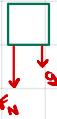
$$U_{gB} = mg \cdot 2r$$

$$\Delta E_B = 0 \Rightarrow (0 - \frac{1}{2}kx^2) + (\frac{1}{2}mv_B^2 - 0) + (mg \cdot 2r - 0) = 0$$

$$\frac{1}{2}mv_B^2 - K_B - \frac{1}{2}kx^2 - 2mgr = 0 \Rightarrow v_B = \sqrt{\frac{1}{m}(kx^2 - 4mgr)}$$

we can solve for speed in terms of initial spring compression, or vice-versa.

B:



since we know the normal force at the loop top, we know radial accel. and thus speed.

$$\vec{F}_N = -F_N \hat{r}$$

$$\vec{a}_r = -a_r \hat{r} = -r\omega^2 \hat{r} = -r\frac{v^2}{r} \hat{r} = -\frac{v^2}{r} \hat{r}$$

$$\text{At B, } \vec{F}_N = -2mgr \hat{r}$$

$$F_N + F_G = m a_r \Rightarrow -3mg = m \cdot (-a_r) = m \cdot \left(-\frac{v^2}{r}\right)$$

$$3g \cdot \frac{v^2}{r} = v_B^2 \Rightarrow v_B = \sqrt{3gr}$$

From constn. of mech. en. we know that $\frac{1}{2}mv_B^2 = \frac{1}{2}kx^2 - 2mgr$

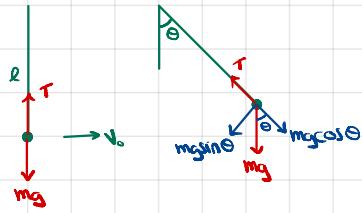
Given the knowledge of normal force at B, we know v_B . So we know total final mech. en. We can solve for x .

$$m \cdot 3gr - \frac{1}{2}kx^2 - 2mgr \Rightarrow x = \sqrt{\frac{7mgr}{k}}$$

so we can solve for initial compression.

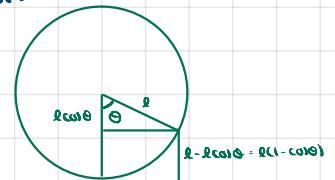
Problem 2 - Slingshot

$$W_g = \int_0^{2l} -mgdx = -mg \cdot 2l \quad \Delta E_g = 2mgl$$



The mass undergoes a circular trajectory. T is always \perp to the traj. so it does no work.

$$\begin{aligned} E_{m_i} &= \frac{mv_0^2}{2} \\ E_{m_{top}} &= \frac{mv_{top}^2}{2} + mg \cdot 2l \end{aligned} \Rightarrow \frac{v_0^2}{2} = \frac{v_{top}^2}{2} + 2gl \Rightarrow v_{top} = \pm \sqrt{v_0^2 - 4gl}$$

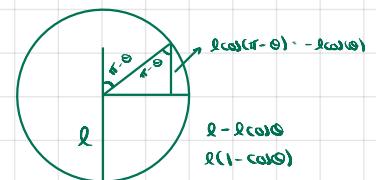


a) $\pm \min v_0$ to actually reach top position

$$v_0^2 - 4gl = 0 \Rightarrow v_0^2 = 4gl \Rightarrow v_0 = \sqrt{4gl}$$

$$E_{m_i} = \frac{mv_0^2}{2} + mg \cdot l(1 - \cos\theta)$$

$$\Rightarrow \frac{v_0^2}{2} = \frac{v_i^2}{2} + gl(1 - \cos\theta) \Rightarrow v_i = \pm \sqrt{v_0^2 - 2gl(1 - \cos\theta)}$$



Tension Force

$$\begin{aligned} mg\cos\theta - T &= -m \cdot \frac{v_i^2 - 2gl(1 - \cos\theta)}{l} \Rightarrow T = \frac{mg\cos\theta + mv_i^2 - 2mgl(1 - \cos\theta)}{l} = \frac{mv_0^2 - 2mgl + 3mgl\cos\theta}{l} \\ \Rightarrow T &= \frac{m}{l} (v_0^2 + 3gl\cos\theta - 2gl) \end{aligned}$$

$$T(\pi) = \frac{m}{l} (v_0^2 - 3gl - 2gl) = \frac{m}{l} (v_0^2 - 5gl)$$

$$T(\pi) \geq 0 \Rightarrow v_0^2 - 5gl \geq 0 \Rightarrow v_0 \geq \sqrt{5gl} \leq -\sqrt{5gl}$$

$$v_{top} \text{ when } T(\pi) = 0 \Rightarrow v_0 = \sqrt{5gl} \Rightarrow v_{top} = \sqrt{gl}$$

$$v_0 = \sqrt{4gl} \Rightarrow T(\pi) = \frac{m}{l} (-gl) = -mg$$

But T must be > 0

$$\begin{aligned} b) \quad &\vec{v} = \sqrt{v_0^2 - 4gl} \hat{i} \quad \vec{F}_g = mg \hat{j} \\ &\vec{F}_T = T \hat{i} \quad \vec{v}_i = \sqrt{v_0^2 - 4gl} \hat{i} \\ E_{m_i} &= \frac{m(v_0^2 - 4gl)}{2} + mg \cdot 2l \end{aligned}$$

$$E_{m_T} = \frac{m|v_T|^2}{2}$$

$$v_T = \sqrt{v_0^2 - 4gl} \hat{i} + v_T \hat{j}$$

$$\Rightarrow |v_T|^2 = v_0^2 - 4gl + v_T^2$$

$$\Delta E_{m,T} = \frac{m(v_0^2 - 4gl + v_T^2)}{2} - \frac{m(v_0^2 - 4gl)}{2} - mg \cdot 2l$$

$a_x = g$	$t = 2l = \frac{gt^2}{2} \Rightarrow t = \sqrt{\frac{4l}{g}}$	$x = \int_0^t v_x dt = t \sqrt{v_0^2 - 4gl}$
$v_x = gt$	$v_x = \sqrt{v_0^2 - 4gl}$	$x(\sqrt{4lg}) = \sqrt{\frac{4l(v_0^2 - 4gl)}{g}}$
$t = \frac{gt^2}{2}$		

here we used kinematics

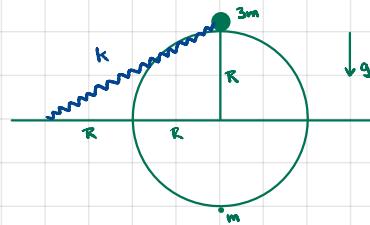
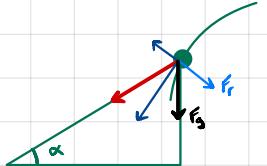
$$v_0^2 - 4gl + v_T^2 - v_0^2 + 4gl - 4gl = 0$$

$$v_{T,y} = \sqrt{4gl} \quad \text{here we used energy conservation to find final vertical speed}$$

$$v_T = gt = \sqrt{4gl} \Rightarrow t = \sqrt{4lg}$$

(...)

Problem 3 - Objects on a Ring



$$\vec{r} \cdot \hat{R} \hat{r} = \hat{r} \cdot [\cos\theta \hat{i} + \sin\theta \hat{j}]$$

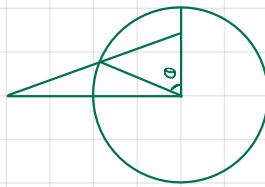
$$= R \cos\theta \hat{i} + R \sin\theta \hat{j}$$

$$\vec{r}_A = 2R \hat{j}$$

$$\vec{r}_A - \vec{r} = -R \cos\theta \hat{i} + (2R - R \sin\theta) \hat{j}$$

$$|\vec{r}_A - \vec{r}| = (R^2 \cos^2\theta + 4R^2 - 4R^2 \sin\theta + R^2 \sin^2\theta)^{1/2}$$

$$= (5R^2 - 4R^2 \sin\theta)^{1/2}$$



Strategy: the forces acting on m , are the gravitational force, a spring force, and a normal force from the ring.

$$W_g + W_s + W_n = \Delta K$$

But $W_n = 0$ because it is always \perp to the trajectory.

Only conservative forces are in play, so $\Delta E_m = 0$.

$$U_{g_0} = 3mg \cdot 2R \Rightarrow \Delta U_g = -6mgR$$

$$U_{g_b} = 0$$

To find U_{g_b} , we need to know the distance between the spring's fixed point and the initial m , position. $U_{g_0} = U_{g_b}$ because the distance is the same.

$$\Rightarrow \Delta U_g = 0$$

$$\Delta K = \frac{3mV_b^2}{2}$$

Gravitational pot. becomes kinetic energy.

$$\frac{3mV_b^2}{2} = 6mgR \Rightarrow V_b = \sqrt{4gR}$$

Apply conservation of momentum

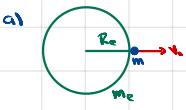
$$3m\sqrt{4gR} \cdot 4mV_c \Rightarrow V_c = \frac{3}{4}\sqrt{4gR}$$

To reach A w/ zero speed

$$\Delta U_g + \Delta U_s + \Delta K = 0 \Rightarrow 4mgR + \frac{k \cdot 9R^2}{2} - \frac{k \cdot 5R^2}{2} - \frac{4m \cdot \frac{9}{16} \cdot 4gR}{2} = 0$$

$$\Rightarrow m = \frac{4kR}{g}$$

Problem 4 - Orbital Collisions



m is fired with initial speed v_0 . The gravitational force does negative work on m , and the latter's potential grav. en increases. If m keeps going with an infinite distance, and v_0 has zero speed and zero pot. grav. en., then the initial speed can be measured at escape velocity.

$$W_g = \Delta K \Rightarrow \int_{R_p}^{R_f} -\frac{Gm_m}{r^2} dr = -\frac{Gm_m}{r} \Big|_{R_p}^{R_f} = Gm_m \left(\frac{1}{R_f} - \frac{1}{R_p} \right)$$

$$\lim_{R_f \rightarrow \infty} W_g = -\frac{Gm_m}{R_p} = -\frac{mv_e^2}{2}$$

$$\Rightarrow v_e = \sqrt{\frac{2Gm}{R_p}} = \sqrt{\frac{2 \cdot 6.67 \cdot 10^{-11} \cdot 5.972 \cdot 10^{24}}{6371 \cdot 10^3}} = 11182.37 \text{ m/s}$$

b)

$$Gm_m \left(\frac{1}{2R_p} - \frac{1}{R_p} \right) = \frac{v_e^2}{2}$$

$$\Rightarrow v_e = \sqrt{v_e^2 + 2Gm_e \left(-\frac{1}{2R_p} \right)} = \sqrt{v_e^2 - \frac{Gm_e}{R_p}} + \sqrt{\frac{Gm_e}{R_p}}$$

$$\Rightarrow v_e = \sqrt{\frac{Gm_e}{R_p}}$$

c) Satellite, circular orbit, radius $2R_p$

$$c_R = \frac{v^2}{2R_p} \Rightarrow m c_R \cdot \cancel{v} \cdot \cancel{v} = \frac{Gm_e}{2R_p} \Rightarrow v = \sqrt{\frac{Gm_e}{2R_p}}$$



$$\vec{p}_i = m \sqrt{\frac{Gm_e}{R_p}} \hat{i} + m \sqrt{\frac{Gm_e}{2R_p}} \hat{j}$$

$$\vec{p}_k = 2m \hat{k}$$

$$\vec{p} \cdot \vec{0} \Rightarrow (2mv_k - m \sqrt{\frac{Gm_e}{R_p}}) \hat{i} + (2mv_j - m \sqrt{\frac{Gm_e}{2R_p}}) \hat{j} = \vec{0}$$

$$\Rightarrow v_k = \frac{1}{2} \sqrt{\frac{Gm_e}{R_p}}$$

$$v_j = \frac{1}{2\sqrt{2}} \sqrt{\frac{Gm_e}{R_p}}$$

$$\Rightarrow \vec{v} = \left\langle \frac{1}{2} \sqrt{\frac{Gm_e}{R_p}}, \frac{1}{2\sqrt{2}} \sqrt{\frac{Gm_e}{R_p}} \right\rangle$$

Problem 5 - Exponential Potential Energy Diagram

$U(x) = -U_0 e^{-\frac{x^2}{a^2}}$, potential fn of particle mass m moving in one dimension

a)

$$f(x) = \frac{1}{a^2} e^{-\frac{x^2}{a^2}}$$

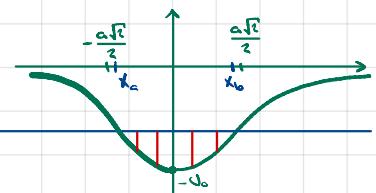
$$f' = -2x e^{-\frac{x^2}{a^2}}$$

$$f'' = -2e^{-\frac{x^2}{a^2}} + 4x^2 e^{-\frac{x^2}{a^2}} = e^{-\frac{x^2}{a^2}}(4x^2 - 2)$$

$$4x^2 - 2 = x^2 - \frac{1}{2} \Rightarrow x = \pm \frac{\sqrt{2}}{2}$$

$U(x)$ is a potential fn for $F(x)$.

$$F = \frac{d}{dx} U = \frac{d}{dx} (-\Delta U)$$



$$f(x) = -U_0 e^{-\frac{x^2}{a^2}}$$

$$y = U_0 \cdot \frac{2x}{a^2} e^{-\frac{x^2}{a^2}}$$

$$y' = \frac{2U_0}{a^2} e^{-\frac{x^2}{a^2}} - \frac{4x^2}{a^4} U_0 e^{-\frac{x^2}{a^2}} = \frac{2U_0}{a^2} e^{-\frac{x^2}{a^2}} \left(1 - \frac{2x^2}{a^2}\right)$$

$$2x^2 = a^2 \Rightarrow x = \pm \frac{\sqrt{2}}{2}$$

$$\text{b) } W_F = \int_{x_i}^x F(x) dx = -\Delta U = -\left(-U_0 e^{-\frac{x^2}{a^2}} + U_0 e^{-\frac{x_i^2}{a^2}}\right) = U_0 \left(e^{-\frac{x^2}{a^2}} - e^{-\frac{x_i^2}{a^2}}\right)$$

$$= U_0 \left(\frac{1}{e^{\frac{x^2}{a^2}}} - \frac{1}{e^{\frac{x_i^2}{a^2}}}\right)$$

$$F(x) = \frac{d}{dx} \left[U_0 \left(\frac{1}{e^{\frac{x^2}{a^2}}} - \frac{1}{e^{\frac{x_i^2}{a^2}}}\right)\right] = U_0 \left(\frac{-2x}{a^2}\right) e^{-\frac{x^2}{a^2}}$$

$$= \frac{-2U_0 x}{a^2} e^{-\frac{x^2}{a^2}}$$

c) At $x = \pm a$ the particle stops and reverses direction of motion.

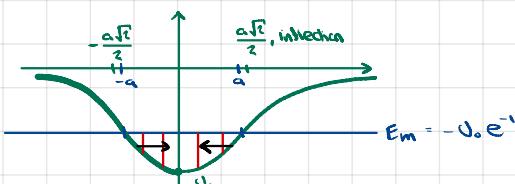
→ At $x = \pm a$, only potential energy, no kinetic.

choosing a speed at $x=0$ determines total mechanical energy, and we need enough mech en such that

$$E_{m0} = U_0 = -U_0 + K_0$$

$$U_0 - U(0) = -U_0 e^{-1} = -U_0 + \frac{mv_0^2}{2} \Rightarrow \frac{mv_0^2}{2} = U_0(1 - e^{-1})$$

$$\Rightarrow v_0 = \sqrt{\frac{2U_0}{m}(1 - e^{-1})}$$



$x > 0 \Rightarrow$ potential increasing, negative with being done,
negative force.

At $x_b, K_b = 0$ so $v_b = 0$.