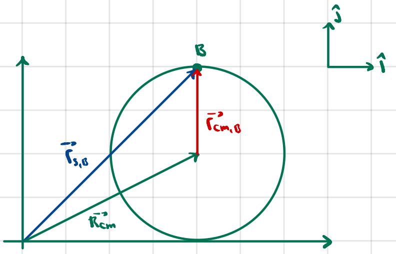
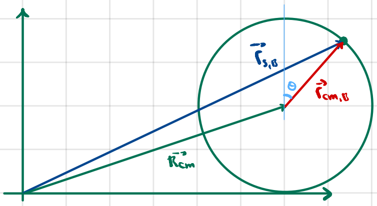
PSet 12

Problem 1 - Bicycle Wheel



$\vec{r}_0 = x_0\hat{i} + 2R\hat{j}$

$\vec{V}_{cm} = V_{cm}\hat{i} \Rightarrow \vec{R}_{cm} = (x_0 + V_{cm}t)\hat{i} + R\hat{j}$



$\vec{r}_{s,B} = \vec{R}_{cm} + \vec{r}_{cm,B}$

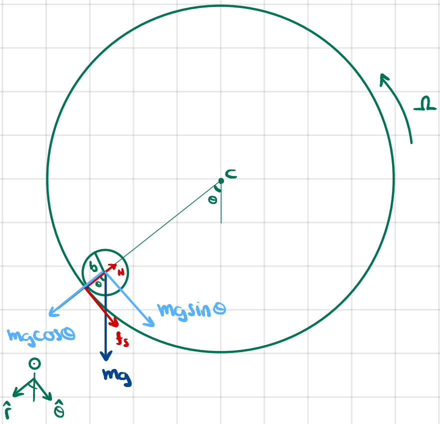
$\vec{r}_{cm,B} = R\hat{r} = R(\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$

no slipping $\Rightarrow V_{cm} = R\omega \Rightarrow \omega = \dfrac{V_{cm}}{R} \Rightarrow \theta(t) = \dfrac{V_{cm}t}{R}$

$\Rightarrow \vec{r}_{s,B} = (x_0 + V_{cm}t)\hat{i} + R\hat{j} + R\sin\theta\,\hat{i} + R\cos\theta\,\hat{j}$

$\Rightarrow \vec{r}_{s,B} = \hat{i}\left(x_0 + V_{cm}t + R\sin\left(\dfrac{V_{cm}t}{R}\right)\right) + \left(R + R\cos\left(\dfrac{V_{cm}t}{R}\right)\right)\hat{j}$

Problem 2 – Rolling Without Slipping



First, suppose no gravity, no friction, ring at rest

$-N = -mr\omega^2 \Rightarrow N = mr\omega^2$

no torques about CM $\Rightarrow \Delta L = 0$

$\vec{L}_{c,sys} = (R-b)\hat{r} \times m v_{cm}\hat{\theta} + I_{cm}\omega_{cm}\hat{k}$

$= (m(R-b)v_{cm} + I_{cm}\omega_{cm})\hat{k}$

$v_{cm} = (R-b)\cdot\omega_c$

$\vec{L}_{c,sys} = (m(R-b)^2\omega_c + I_{cm}\omega_{cm})\hat{k}$

The ball rotates with $\omega_{cm}\hat{k}$ about CM, and $\omega_c\hat{k}$ about the ring center.

$\vec{v}_{cm}$ has constant magnitude.

Angular momentum is constant.

The ball will roll around the ring forever.

Now suppose the ring has angular velocity $\Omega\hat{k}$.

Since there is no friction, nothing changes.

Add gravity.

$mg\cos\theta - N = -mr\omega_c^2$

$mg\sin\theta = m\cdot a_T = -mr\alpha_c \Rightarrow \alpha_c = \frac{-g\sin\theta}{r}$

$\vec{\tau}_{c,sys} = r\hat{r} \times mg\sin\theta\,\hat{\theta} + I_{cm}\cancel{\alpha_{cm}}^{0}\hat{k}$

$= rmg\sin\theta\,\hat{k}$

$\vec{L}_{c,sys} = r\hat{r} \times m\vec{v}_{c,cm} - I_{cm}\omega_{cm}\hat{k}$

$= r\hat{r} \times m(-r\omega_c\hat{\theta}) - I_{cm}\omega_{cm}\hat{k}$

$= (-mr^2\omega_c - I_{cm}\omega_{cm})\hat{k}$

$\frac{d\vec{L}_{c,sys}}{dt} = -mr^2\alpha_c\hat{k}$

$\cancel{r}\cancel{m}g\sin\theta = -\cancel{m}r^{\cancel{2}}\alpha_c \Rightarrow \alpha_c = -\frac{g\sin\theta}{r}$

For small θ,

$\ddot{\theta}_c + \frac{g\theta}{r} = 0$ second order, linear homog. ODE

guess $\theta(t) = e^{mt}$

$\dot{\theta} = me^{mt}$

$\ddot{\theta} = m^2e^{mt}$

$\Rightarrow e^{rt}(m^2 + \frac{g}{r}) = 0$

$m^2 = -\frac{g}{r}$

$m = \pm\sqrt{\frac{g}{r}}\,i$

one sol'n is $\theta(t) = e^{\sqrt{\frac{g}{r}}it}$

use Euler's formula

$e^{\sqrt{\frac{g}{r}}it} = \cos\left(\sqrt{\frac{g}{r}}t\right) + i\sin\left(\sqrt{\frac{g}{r}}t\right)$

$\theta_1(t) = \cos\left(\sqrt{\frac{g}{r}}t\right)$

$\theta_2(t) = \sin\left(\sqrt{\frac{g}{r}}t\right)$

are real sol'ns.

General sol'n:

$\theta(t) = c_1\cos\left(\sqrt{\frac{g}{r}}t\right) + c_2\sin\left(\sqrt{\frac{g}{r}}t\right)$

$\Rightarrow$ the wheel behaves like a simple harmonic oscillator.

$N = m(g\cos\theta + r\omega_c^2)$

$\omega_c = \dot{\theta}(t)$

$= -c_1\sqrt{\frac{g}{r}}\sin\left(\sqrt{\frac{g}{r}}t\right)$
$+ c_2\sqrt{\frac{g}{r}}\cos\left(\sqrt{\frac{g}{r}}t\right)$

Now add friction.

$(mg\sin\theta + f_k)\hat{\theta} = m\cdot\vec{a}_{g,cm}$

$\vec{a}_{g,cm} = \frac{mg\sin\theta + f_s}{m}\hat{\theta}$

$\vec{v}_{g,cm} = \frac{(mg\sin\theta + f_s)t}{m}\hat{\theta}$

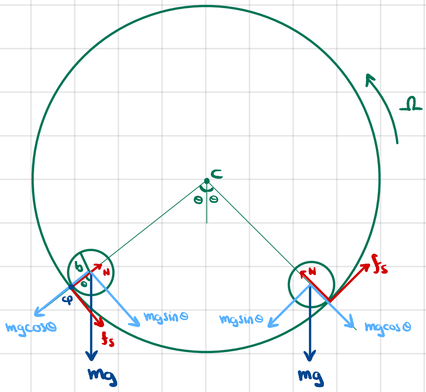
$\vec{\tau}_{c,sys} = r\hat{r} \times (mg\sin\theta + f_s)\hat{\theta} + b\hat{r} \times f_s\hat{\theta}$

$= \hat{k}(rmg\sin\theta + rf_s + bf_s)$

$\vec{\tau}_{cm,sys} = I_{cm}\alpha_{cm}\hat{k}$

$= bf_s\hat{k}$

$\Rightarrow \alpha_{cm} = \frac{bf_s}{I_{cm}}$



Though there is lots to explore here, the question simply asks for $\vec{v}_{g,cm}$ when the ball rolls without slipping and $\omega_{cm} < 0$, ie clockwise rotation about CM.

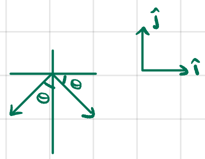
no slipping $\Rightarrow \vec{v}_{r,cm} = -\vec{v}_{cm,cp}$

$= -\omega_{cm}\hat{k} \times b\hat{r} = -b\omega_{cm}\hat{\theta}$

$\vec{v}_{g,cm} = \vec{v}_{g,r} + \vec{v}_{r,cm}$

$= \Omega\hat{k} \times R\hat{r} - b\omega_{cm}\hat{\theta}$

$= (\Omega R - b\omega_{cm})\hat{\theta}$



$\hat{r} = -\sin\theta\,\hat{i} - \cos\theta\,\hat{j}$

$\hat{\theta} = \cos\theta\,\hat{i} - \sin\theta\,\hat{j}$

$d\hat{r}/dt = -\cos\theta\,\theta'\hat{i} + \sin\theta\,\theta'\hat{j}$

$= -\theta'(\cos\theta\,\hat{i} - \sin\theta\,\hat{j})$

$= -\theta'\hat{\theta}$

$= -\omega_c\hat{\theta}$

$d\hat{\theta}/dt = -\sin\theta\,\theta'\hat{i} - \cos\theta\,\theta'\hat{j}$

$= \theta'(-\sin\theta\,\hat{i} - \cos\theta\,\hat{j})$

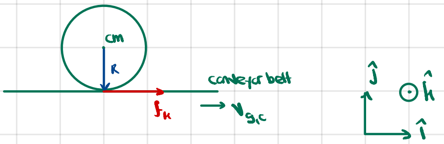
$= \theta'\hat{r}$

$= \omega_c\hat{r}$

$\vec{r}_{c,cm} = r\hat{r}$, $r = R-b$

$\vec{v}_{c,cm} = -r\omega_c\hat{\theta}$

$\vec{a}_{c,cm} = -r\alpha_c\hat{\theta} - r\omega_c^2\hat{r}$



$\vec\tau_{cm,s/s} = R\cdot f_k\,\hat k = I_{cm}\,\alpha_{cm}\,\hat k$

$\Rightarrow \alpha_{cm} = \dfrac{Rf_k}{I_{cm}}$

$f_k\hat\imath = m a_{g,cm}\hat\imath \Rightarrow \vec a_{g,cm} = \dfrac{f_k}{m}\hat\imath$

$\vec v_{g,cm} = \dfrac{f_k}{m}t\,\hat\imath$

$\vec\omega_{cm} = \dfrac{Rf_k t}{I_{cm}}\hat k$

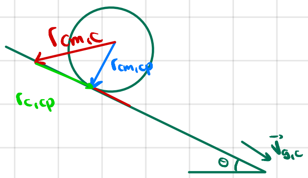
$v_{g,cm} = v_{g,c} + v_{c,cm}$

$\Rightarrow \dfrac{f_k}{m}t = v_{g,c} + v_{c,cm}$

$v_{g,cp} = v_{g,c} + v_{c,cp}$

$\vec v_{g,cp} = \vec\omega\times\vec r_{cp} = \dfrac{Rf_k t}{I_{cm}}\hat k\times R(-\hat j) = \dfrac{R^2 f_k t}{I_{cm}}\hat\imath$

The conveyor belt moves with constant speed rel. to ground $\vec v_{g,c}$, ball starts at rest. Difference in speed $\Rightarrow$ kinetic friction f_k accelerates ball and applies torque $\Rightarrow$ rotation.
we know both α_{cm} and $a_{g,cm}$.



$\Rightarrow \vec\alpha_{cm} = \dfrac{Rf_k}{I_{cm}}\hat k$

$\vec a_{g,cm} = \dfrac{f_k}{m}\hat\imath$

$\vec v_{g,cm} = \dfrac{f_k}{m}t\,\hat\imath$

$r_{cm,cp} = r_{cm,c} + r_{c,cp}$

$\vec\omega_{cm} = \dfrac{Rf_k t}{I_{cm}}\hat k$

No slipping means that

$\vec v_{c,cm} = R\hat j\times\omega_{cm}\hat k$
$= -R\omega_{cm}\hat\imath$

we also have an expression for $\vec v_{c,cm}$

$\vec v_{c,cm} = \left(\dfrac{f_k}{m}t - v_{g,c}\right)\hat\imath$

Speed rel. to conveyor belt decreases in time until it reaches $-R\omega_{cm}$. The contact point is at rest w/ the belt, there is no more friction.

$\dfrac{f_k t}{m} - v_{g,c} = -\dfrac{R^2 f_k t}{I_{cm}}$

$\Rightarrow t\left[\dfrac{f_k}{m} + \dfrac{f_k R^2}{I_{cm}}\right] = v_{g,c}$

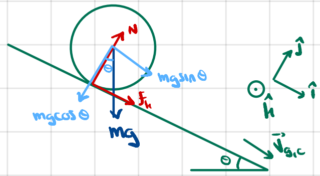
$\Rightarrow t = \dfrac{v_{g,c}}{\dfrac{f_k}{m} + \dfrac{f_k R^2}{I_{cm}}}$

$\vec v_{cm,cp} = \vec v_{cm,c} + \vec v_{c,cp}$

$v_{cm,cp} = R\hat j\times\omega_{cm}\hat k$
$= -R\omega_{cm}\hat\imath$

$\vec v_{cm,c} = -\vec v_{cm,c} = R\omega_{cm}\hat\imath$

$\Rightarrow \vec v_{c,cp} = \vec 0$ when not slipping.



Torque as before $\vec\tau_{cm,s/s} = R\cdot f_k\hat k = I_{cm}\alpha_{cm}\hat k \Rightarrow \alpha_{cm} = \dfrac{Rf_k}{I_{cm}}$

2nd law: $(f_k + mg\sin\theta)\hat\imath = m\vec a_{g,cm} \Rightarrow \vec a_{g,cm} = \dfrac{f_k + mg\sin\theta}{m}$

$\vec\omega_{cm} = \dfrac{Rf_k t}{I_{cm}}\hat k$

$\vec v_{g,cm} = \dfrac{(f_k + mg\sin\theta)t}{m}\hat\imath$

$\vec v_{c,cm} = \vec v_{c,cp} + \vec v_{cp,cm}$
$= -\vec v_{cm,cp} = -(-R\hat j)\times$

no slipping: $\vec v_{c,cm} = \cancel{\vec v_{c,cp}}^{\,0} + \vec v_{cp,cm} = -\vec v_{cm,cp} = -\left[\omega_{cm}\hat k\times(-R\hat j)\right]$

$= -\left[-R\omega_{cm}(-\hat\imath)\right] = -R\omega_{cm}\hat\imath$

$\vec v_{c,cm} = \vec v_{g,cm} - \vec v_{g,c} = \dfrac{(f_k + mg\sin\theta)t}{m}\hat\imath - v_{g,c}\hat\imath$

$\Rightarrow -\dfrac{R^2 f_k t}{I_{cm}} = -v_{g,c} + \dfrac{(f_k + mg\sin\theta)t}{m}$

$\Rightarrow t = \dfrac{v_{g,c}}{\dfrac{R^2 f_k}{I_{cm}} + \dfrac{f_k + mg\sin\theta}{m}} < \dfrac{v_{g,c}}{\dfrac{f_k}{m} + \dfrac{f_k R^2}{I_{cm}}}$

The ball takes less time to reach no slip velocity. However, $mg\sin\theta$ still present means the ball will continue to accelerate.

$\vec v_{c,cp} = \vec v_{cm,cp} - \vec v_{cm,c} = \vec v_{cm,cp} + \vec v_{c,cm}$

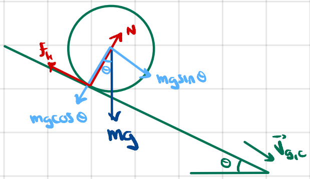
$= -R\hat j\times\omega_{cm}\hat k + \dfrac{(f_k + mg\sin\theta)t}{m}\hat\imath - v_{g,c}\hat\imath$

$= R\omega_{cm}\hat\imath + \dfrac{(f_k + mg\sin\theta)t}{m}\hat\imath - v_{g,c}\hat\imath$

$= \hat\imath\left[-v_{g,c} + \dfrac{R^2 f_k t}{I_{cm}} + \dfrac{(f_k + mg\sin\theta)t}{m}\right]$

This equation is only valid up until no slipping happens. After this point we must start the analysis again because we once again have kinetic friction, but in the opposite direction.

$\omega_{cm}(t_{ns}) =$



$$\vec{\tau}_{cm,sys} = -R\hat{j}\times(-f_k\hat{i}) = -Rf_k\hat{k} = I_{cm}\alpha_{cm}\hat{k}$$

$$\Rightarrow \vec{\alpha}_{cm} = \frac{-Rf_k}{I_{cm}}\hat{k}$$

$$\vec{\omega}_{cm} = \left(\omega_{ns} - \frac{Rf_k t}{I_{cm}}\right)\hat{k}$$

$$(mg\sin\theta - f_k)\hat{i} = m a_{g,cm}\hat{i} \Rightarrow \vec{a}_{g,cm} = \frac{mg\sin\theta - f_k}{m}\hat{i}$$

$$\vec{v}_{g,cm} = \left(v_{g,cm,ns} + \frac{(mg\sin\theta - f_k)t}{m}\right)\hat{i}$$

$$\vec{v}_{c,cm} = \vec{v}_{g,cm} - \vec{v}_{g,c}$$

smaller than before

$$= \left(v_{g,cm,ns} + \frac{(mg\sin\theta - f_k)t}{m}\right)\hat{i} - v_{g,c}\hat{i}$$

$$\vec{v}_{c,cp} = \vec{v}_{cm,cp} - \vec{v}_{cm,c} = \vec{v}_{cm,cp} + \vec{v}_{c,cm}$$

$$= \left[R\omega_{ns} - \frac{R^2 f_k}{I_{cm}}t + v_{g,cm,ns} + \frac{(mg\sin\theta - f_k)t}{m} - v_{g,c}\right]\hat{i}$$

$$\vec{v}_{cm,cp} = \left(\omega_{ns} - \frac{Rf_k t}{I_{cm}}\right)\hat{k}\times(-R\hat{j}) = \left(-R\omega_{ns} + \frac{R^2 f_k t}{I_{cm}}\right)(-\hat{i})$$

$$= \left(R\omega_{ns} - \frac{R^2 f_k t}{I_{cm}}\right)\hat{i}$$

$$a_{c,cp} = \frac{R^2 f_k}{I_{cm}} + \frac{mg\sin\theta - f_k}{m}$$

$$= f_k\left(\frac{R^2}{I_{cm}} - \frac{1}{m}\right) + g\sin\theta$$

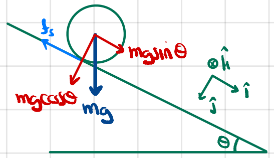
$$= f_k\left(\frac{5}{2m} - \frac{1}{m}\right) + g\sin\theta$$

$$= f_k\cdot\frac{3}{2m} + g\sin\theta > 0$$

$$\vec{v}_{g,cp} = \vec{v}_{g,c} + \vec{v}_{c,cp}$$

$$= \vec{v}_{g,c} + 0$$

$$= \vec{v}_{g,c}$$



$(mg\sin\theta - f_s)\,\hat{i} = m\vec{a}_{c,cm}$

$\vec{a}_{c,cm} = \dfrac{mg\sin\theta - f_s}{m}\,\hat{i}$

$\vec{\tau}_{cm,s,fs} = R\hat{j} \times (-f_s)\hat{i} = Rf_s\hat{k} = I_{cm}\alpha_{cm}\hat{k}$

$\Rightarrow \vec{\alpha}_{cm} = \dfrac{Rf_s}{I_{cm}}\,\hat{k}$

no slipping $\Rightarrow \vec{v}_{c,cm} = -\vec{v}_{cm,cp} = -\omega_{cm}\hat{k} \times R\hat{j}$

$= \omega_{cm} R\,\hat{i}$

$\Rightarrow \vec{a}_{c,cm} = \alpha_{cm} R\,\hat{i}$

$\Rightarrow \alpha_{cm} R = \dfrac{mg\sin\theta - f_s}{m}$

$\dfrac{R^2 f_s}{I_{cm}} = \dfrac{mg\sin\theta - f_s}{m}$

$f_s\left(\dfrac{R^2}{I_{cm}} + \dfrac{1}{m}\right) = g\sin\theta$

$f_s = \dfrac{g\sin\theta}{\dfrac{R^2}{I_{cm}} + \dfrac{1}{m}}$

$I_{cm} = \dfrac{2mR^2}{5}$ (solid sphere)

$\Rightarrow f_s = \dfrac{g\sin\theta}{\dfrac{5}{2m} + \dfrac{1}{m}} = \dfrac{g\sin\theta}{\dfrac{7}{2m}}$

$= \dfrac{2mg\sin\theta}{7}$

$f_{s_{max}} = \mu mg\cos\theta$

$f_s \leq f_{s_{max}} \Rightarrow 2\cancel{mg}\sin\theta \leq 7\mu\cancel{mg}\cos\theta$

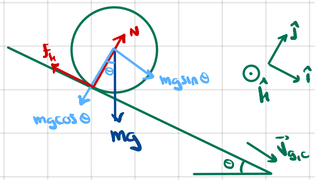
$\Rightarrow \tan\theta \leq \frac{7}{2}\mu$

$\vec{v}_{c,cp} = \vec{v}_{c,cm} + \vec{v}_{cm,cp}$

$= \dfrac{mg\sin\theta - f_s}{m}t\,\hat{i} - \dfrac{R^2 f_s t}{I_{cm}}\,\hat{i}$

$=$ (... sub in f_s)

$= 0$



$(mg\sin\theta - f_s)\,\hat{i} = m\vec{a}_{g,cm}$

$\vec{a}_{g,cm} = \dfrac{mg\sin\theta - f_s}{m}\,\hat{i}$ $\qquad \vec{v}_{g,cm} = \left(v_{g,cm,0} + \dfrac{(mg\sin\theta - f_s)t}{m}\right)\hat{i}$

$\vec{\tau}_{cm,s,fs} = (-R\hat{j}) \times (-f_s)\hat{i} = -Rf_s\hat{k} = I_{cm}\alpha_{cm}\hat{k}$

$\Rightarrow \vec{\alpha}_{cm} = \dfrac{-Rf_s}{I_{cm}}\,\hat{k}$

$\vec{\omega}_{cm} = \left(\omega_{cm,0} - \dfrac{R f_s t}{I_{cm}}\right)\hat{k}$

no slipping $\Rightarrow \vec{v}_{c,cm} = -\vec{v}_{cm,cp} = -\omega_{cm}\hat{k} \times (-R\hat{j})$

$= R\omega_{cm}(-\hat{i}) = -R\omega_{cm}\hat{i}$

$\Rightarrow \vec{v}_{c,cm} = -R\left(\omega_{cm,0} - \dfrac{Rf_s t}{I_{cm}}\right)\hat{i}$

Note difference vs the static conveyor case: here we have $\vec{v}_{c,cm}$, not $\vec{v}_{g,cm}$. We can't yet directly relate this to $\vec{a}_{g,cm}$.

$\vec{v}_{g,cm} = \vec{v}_{g,c} + \vec{v}_{c,cm} = v_{g,c}\hat{i} - R\left(\omega_{cm,0} - \dfrac{Rf_s t}{I_{cm}}\right)\hat{i}$

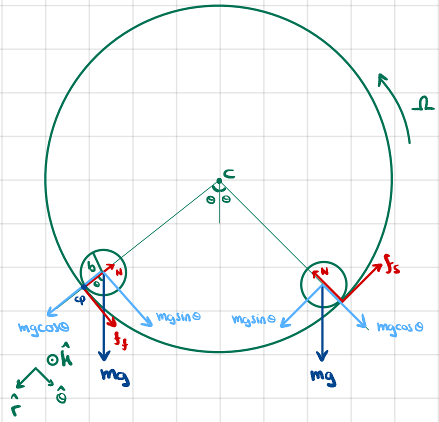
$= \hat{i}\left[v_{g,c} - R\left(\omega_{cm,0} - \dfrac{Rf_s t}{I_{cm}}\right)\right]$

what $\vec{v}_{g,cm}$ must be for no slipping to occur

$\Rightarrow$

$v_{g,c} - R\left(\omega_{cm,0} - \dfrac{Rf_s t}{I_{cm}}\right) = v_{g,cm,0} + \dfrac{(mg\sin\theta - f_s)t}{m}$

$v_{g,cm}$ according to 2nd law forces



$$(mg\sin\theta + f_f)\hat{\theta} = m \cdot \vec{a}_{g,cm} \Rightarrow \vec{a}_{g,cm} = \frac{mg\sin\theta + f_f}{m}\hat{\theta}$$

$$\vec{\tau}_{cm,s/s} = b\hat{r} \times f_f \hat{\theta} = bf_f\hat{h}$$

$$\Rightarrow bf_f\hat{h} = \frac{dL_{cm,s/s}}{dt} = I_{cm}\alpha_{cm}\hat{h}$$

$$\vec{L}_{cm,s/s} = I_{cm}\omega_{cm}\hat{h}$$

$$\Rightarrow \vec{\alpha}_{cm} = \frac{bf_f}{I_{cm}}\hat{h}$$

$$(mg\cos\theta - N)\hat{r} = -mr\omega_c^2\hat{r} \Rightarrow \vec{N} = -N\hat{r} = -m(r\omega_c^2 + g\cos\theta)$$

no slipping: $\vec{v}_{r,cm} = -v_{cm,cp} = -\omega_{cm}\hat{h} \times b\hat{r}$

$$= -b\omega_{cm}\hat{\theta} \Rightarrow \vec{v}_{r,cm} = -b\omega_{cm}\hat{\theta}$$

$$\vec{v}_{g,cm} = \vec{v}_{g,r} + \vec{v}_{r,cm} = \vec{\omega}_r \times \vec{r}_{c,r} + \vec{v}_{r,cm}$$

$$= \Omega\hat{h} \times R\hat{r} + \vec{v}_{r,cm} = \Omega R\hat{\theta} - b\omega_{cm}\hat{\theta}$$

$$\vec{v}_{g,cm} = \hat{\theta}(\Omega R - b\omega_{cm})$$

From torque eq., $\vec{v}_{g,cm} = \dfrac{(mg\sin\theta + f_f)t}{m}\hat{\theta}$

$$\vec{\alpha}_{cm} = \frac{bf_f t}{}$$

$$\vec{\tau}_{c,s/s} = r\hat{r} \times (mg\sin\theta + f_f)\hat{\theta} + bf_f\hat{h}$$
$$= r(mg\sin\theta + f_f)\hat{h} + bf_f\hat{h}$$

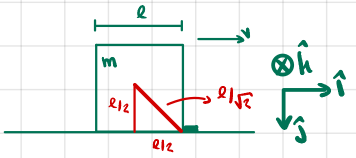
$$\vec{L}_{c,s/s} = r\hat{r} \times m\vec{v}_{c,cm} + I_{cm}\omega_{cm}\hat{h}$$
$$= rmv_{c,cm}\hat{h} + I_{cm}\omega_{cm}\hat{h} = I_c\omega_c\hat{h}$$

$$\frac{d\vec{L}_{c,s/s}}{dt} = rma_{c,cm}\hat{h} + I_{cm}\alpha_{cm}\hat{h} = I_c\alpha_c\hat{h}$$

$$\Rightarrow \cancel{r}ma_{c,cm}\hat{h} = \cancel{r}(mg\sin\theta + f_f)\hat{h}$$

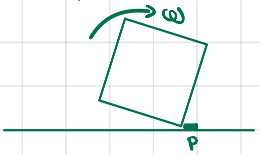
$$\Rightarrow a_{c,cm} = \frac{mg\sin\theta + f_f}{m}$$

Problem 3 - Cuboid Block Collision at Low Ridge



no friction

inelastic, instantaneous collision



a) collision forces at point P apply no torque about P

$$\Rightarrow \vec{\tau}_{P,sys,collision} = \frac{d\vec{L}_{P,sys}}{dt} = \vec{0}$$

$$\vec{L}_{P,sys,A} = \left(-\frac{\ell}{2}\hat{i} - \frac{\ell}{2}\hat{j}\right) \times m v_{x,i}\hat{i}$$

$$= \frac{\ell m v_{x,i}}{2}\hat{k}$$

$$\vec{L}_{P,sys,B} = I_P \omega \hat{k}$$

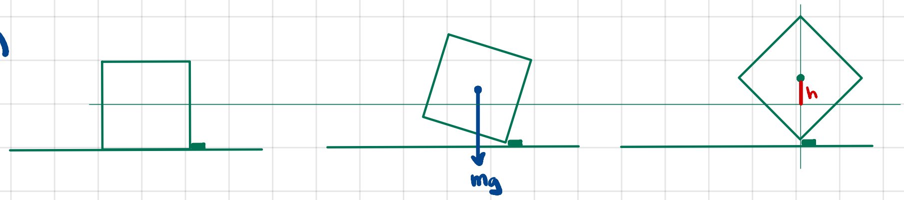
$$\Delta\vec{L} = \vec{0} \Rightarrow \omega = \frac{\ell m v_{x,i}}{2 I_P}$$

$$I_P = I_{cm} + m\frac{\ell^2}{2} = \frac{m\ell^2}{6} + \frac{m\ell^2}{2}$$

$$= \frac{2m\ell^2}{3}$$

$$\Rightarrow \omega_i = \frac{3\cancel{\ell}\cancel{m}v_{x,i}}{4\cancel{m}\ell^{\cancel{2}}} = \frac{3v_{x,i}}{4\ell}$$

b)



There is torque about P due to gravity.

$$E_{m_i} = \frac{I_P\omega_i^2}{2}$$

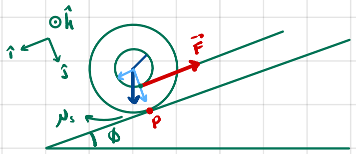
$$E_{m_f} = mg\left(\frac{\ell}{\sqrt{2}} - \frac{\ell}{2}\right)$$

$$\frac{I_P\omega_i^2}{2} = mg\left(\frac{\ell}{\sqrt{2}} - \frac{\ell}{2}\right)$$

$$\Rightarrow \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}\cancel{m}\cancel{\ell^2}}{\cancel{3}} \cdot \frac{\overset{3}{\cancel{9}}v_{x,i}^2}{16\cancel{\ell^2}} = \cancel{m}g\left(\frac{\ell}{\sqrt{2}} - \frac{\ell}{2}\right)$$

$$v_{x,i} = \sqrt{\frac{16}{3}g\left(\frac{\ell}{\sqrt{2}} - \frac{\ell}{2}\right)}$$

Problem 4 - Yo-yo Rolling on Inclined Plane



First, consider case of no friction

Torque: $\vec{\tau}_{cm,sys} = \frac{R}{3}\hat{\jmath} \times (-F\hat{\imath}) = \frac{RF}{3}\hat{k} = I_{cm}\alpha_{cm}\hat{k}$

$\Rightarrow \vec{\alpha}_{cm} = \frac{RF}{3I_{cm}}\hat{k}$ angular accel. due to torque

2nd law: $mg\sin\theta - F = m \cdot a_{cm} \Rightarrow a_{cm} = \frac{mg\sin\theta - F}{m}\hat{\imath}$

accel. of CM due to net force

no slipping: $\vec{V}_{cm} = -\vec{V}_{cm,P}$, ie $\vec{V}_{cm} + \vec{V}_{cm,P} = \vec{V}_P = 0$

↗ velocity at each instant compatible w/ no slipping

$\Rightarrow \vec{V}_{cm} = -\omega_{cm}\hat{k} \times R\hat{\jmath} = \omega_{cm}R\hat{\imath}$

$$\vec{V}_{cm} = \frac{(mg\sin\theta - F)t}{m}\hat{\imath} = \omega_{cm}R\hat{\imath} = \frac{R^2Ft}{3I_{cm}}\hat{\imath}$$

$\Rightarrow (mg\sin\theta - F) \cdot 3I_{cm} = mR^2F$

$F(mR^2 + 3I_{cm}) = mg \cdot 3I_{cm}\sin\theta$

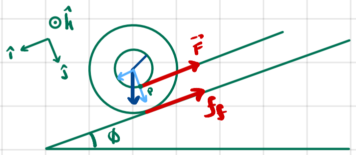
$$F_{ns} = \frac{3mg\sin\theta \cdot I_{cm}}{mR^2 + 3I_{cm}}$$ ↗ value of F compatible w/ no slipping.

$F > F_{ns} \Rightarrow \uparrow \alpha_{cm}, \downarrow a_{cm} \Rightarrow$ too much torque, $|\vec{V}_{cm}| < |\omega_{cm}R|$
$\Rightarrow$ slipping

$F < F_{ns} \Rightarrow \downarrow \alpha_{cm}, \uparrow a_{cm} \Rightarrow$ too little torque, too little rotation
$\Rightarrow$ skidding

extreme case: $F = 0 \Rightarrow$ no rotation only CM translation, extreme form of skidding.

Now add friction



1) no $\vec{F}$, no friction $\Rightarrow$ skidding, zero rotation
2) no $\vec{F} \Rightarrow$ if μ_s high enough, friction increases to level compatible with no slipping: $a_{cm} = \frac{R}{3}\alpha_{cm}$. Contact point is always at rest rel. to surface but there is a force from contact point onto surface.
3) $F > 0$ ie pulling on string $\Rightarrow \uparrow$ torque, $\uparrow \alpha_{cm}$

Torque: $\vec{\tau}_{cm,sys} = \frac{R}{3}\hat{\jmath} \times (-F\hat{\imath}) + R\hat{\jmath} \times (-f_s)\hat{\imath}$

$= \frac{RF}{3}\hat{k} + Rf_s\hat{k} = I_{cm}\alpha_{cm}\hat{k} \Rightarrow \vec{\alpha}_{cm} = \frac{R(\frac{F}{3} + f_s)}{I_{cm}}\hat{k}$

2nd law: $(mg\sin\theta - F - f_s)\hat{\imath} = m \cdot a_{cm}\hat{\imath}$

$$\vec{a}_{cm} = \frac{mg\sin\theta - F - f_s}{m}\hat{\imath}$$

no slipping: $\vec{V}_{cm} = -\omega_{cm}\hat{k} \times R\hat{\jmath} = \omega_{cm}R\hat{\imath}$

$$\Rightarrow \frac{R(\frac{F}{3} + f_s)\cancel{t}}{I_{cm}} \cdot R = \frac{(mg\sin\theta - F - f_s)\cancel{t}}{m}$$

$R^2m(\frac{F}{3} + f_s) = I_{cm}(mg\sin\theta - F - f_s)$

$F\left[\frac{R^2m}{3} + I_{cm}\right] = I_{cm}(mg\sin\theta - f_s) - R^2mf_s$

$$F = \frac{I_{cm}mg\sin\theta - f_s(I_{cm} + R^2m)}{\frac{R^2m}{3} + I_{cm}}$$

If $F - mg\sin\theta > 0$ then f_s will point in same direction as $mg\sin\theta$, ie f_s will be negative.

If $f_{s_{max}} = \mu mg\cos\theta$ is the largest magnitude of static friction then

$$F_{max} = \frac{I_{cm}mg\sin\theta + f_{s_{max}}(I_{cm} + R^2m)}{\frac{R^2m}{3} + I_{cm}}$$

$$= \frac{3I_{cm}mg\sin\theta + 3\mu mg\cos\theta(I_{cm} + mR^2)}{mR^2 + 3I_{cm}}$$

$$> \frac{3mg\sin\theta \cdot I_{cm}}{mR^2 + 3I_{cm}}$$

in the case of no friction

$F = F_{ns} \Rightarrow$

$$\vec{a}_{cm} = \frac{mg\sin\theta \cdot R^2}{mR^2 + 3I_{cm}}\,\hat{\imath}$$

$$\vec{\alpha}_{cm} = \frac{mg\sin\theta \cdot R}{mR^2 + 3I_{cm}}\,\hat{h}$$

For a solid sphere, $I_{cm} = \frac{2mR^2}{5}$

$$\vec{a}_{cm} = \frac{5g\sin\theta}{11}\,\hat{\imath}$$

$$\vec{\alpha}_{cm} = \frac{5g\sin\theta}{11R}\,\hat{h}$$

$$\vec{F}_{ns} = -\frac{6mg\sin\theta}{11}\,\hat{\imath}$$

in the case of friction and a solid sphere

$$\vec{a}_{cm} = \left(\frac{5g\sin\theta}{11} - \frac{10g\mu\cos\theta}{11}\right)\hat{\imath}$$

$$\vec{\alpha}_{cm} = \left(\frac{5g\sin\theta}{11R} - \frac{10g\mu\cos\theta}{11R}\right)\hat{h}$$

$$\vec{F}_{max} = -\left(\frac{6mg\sin\theta}{11} + \frac{21mg\mu\cos\theta}{11}\right)\hat{\imath}$$

if $\theta = 45°$ and $\mu = 1$ (rubber)

$$\vec{a}_{cm} = -\frac{5g}{11}\frac{\sqrt{2}}{2} < 0 \Rightarrow \text{accel. up the slope}$$

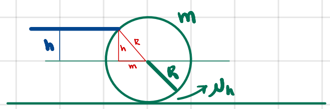
For the yo-yo, $I_{cm} = \frac{mR^2}{2}$

$$\Rightarrow \vec{F}_{max} = -\frac{3mg(\sin\theta + 3\mu\cos\theta)}{5}\,\hat{\imath}$$

$$\vec{a}_{cm} = \left(\frac{2g\sin\theta}{5} - \frac{4g\mu\cos\theta}{5}\right)\hat{\imath}$$

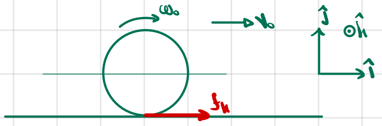
$$\vec{\alpha}_{cm} = \left(\frac{2g\sin\theta}{5R} - \frac{4g\mu\cos\theta}{5R}\right)\hat{h}$$

Problem 5 - Billiards



The cue stick applied an angular impulse $\int_{t_i}^{t_f} \vec{\tau}_{cm}\, dt$, changing angular momentum. There is also impulse $\int_{t_i}^{t_f} \vec{F}\, dt$, changing momentum.

After the collision the ball has v_0, ω_0. The ball slides because $v_0 > R\omega_0$



The ball accelerates at some point after the collision.
The only force acting in x-dir after collision is friction, so it must be in +x direction.
This is due to the contact point having negative velocity relative to the ground.

$f_k = m \cdot a_{cm} \Rightarrow a_{cm} = \frac{f_k}{m}$ friction accelerates the ball's CM

$v_{cm} = v_0 + a_{cm} t$

we know final and initial speeds → we know impulse required

$\frac{9}{7} v_0 = v_0 + a_{cm} t \Rightarrow a_{cm} t = \frac{2v_0}{7}$

$f_k t = \int_0^t f_k\, dt = \frac{2 v_0 m}{7}$

$\Rightarrow t_{ns} = \frac{2 v_0 m}{7 f_k}$ ↗ time of friction impulse

f_k accelerates the ball but also reduces angular speed.
Because the ball is initially sliding, $v_{cm} < R\omega_0$

$\vec{\tau}_{cm,sys} = -R\hat{j} \times f_k \hat{i} = R f_k \hat{k} = I_{cm} \alpha_{cm} \hat{k}$

$\vec{\alpha}_{cm} = \frac{R f_k}{I_{cm}} \hat{k}$ angular accel. caused by torque due to f_k

$\vec{\omega}_{cm} = \left(\omega_0 + \frac{R f_k t}{I_{cm}}\right)\hat{k} = \omega_{cm} \hat{k}$

$\omega_{cm,ns} = \omega_0 + \frac{R \cancel{f_k}}{I_{cm}} \cdot \frac{2 v_0 m}{7 \cancel{f_k}} = \omega_0 + \frac{2 R m v_0}{7 I_{cm}}$

no slipping $\vec{v}_{cm} = -\vec{v}_{cm,cp}$

$= -\omega_{cm} \hat{k} \times (-R\hat{j}) = -R\omega_{cm} \hat{i}$

$\Rightarrow v_{cm,ns} = -R\omega_{cm,ns}$

$\frac{9}{7} v_0 = -R(\omega_0 + \omega_{cm,ns}) = -\left(R\omega_0 + \frac{2R^2 m v_0}{7 I_{cm}}\right)$

$\Rightarrow \omega_0 = \frac{-2R^2 m v_0 - 9 v_0 I_{cm}}{7 I_{cm} \cdot R}$

Initial angular impulse

$\int_0^{t_i} \vec{\tau}_{cm,sys}\, dt = \vec{L}_{cm,sys,0} = I_{cm} \cdot \omega_0 \hat{k}$

$\vec{\tau}_{cm,sys} = (h\hat{j} - m\hat{i}) \times F\hat{i} = -Fh\hat{k}$

$\int_0^{t_i} -Fh\hat{k}\, dt = -Fht_i \hat{k} = I_{cm}\omega_0 \hat{k} \Rightarrow \omega_0 = \frac{-Fht_i}{I_{cm}}$

Initial Impulse

$\int_0^{t_i} F\, dt = \vec{p}_{sys,0} = m v_0$

$F t_i = m v_0$

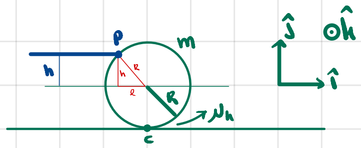
$\Rightarrow \omega_0 = \frac{-hFt_i}{I_{cm}} = \frac{-hmv_0}{I_{cm}} = -\frac{5h\cancel{m}v_0}{2\cancel{m}R^2} = -\frac{5hv_0}{2R^2}$

recall $\omega_0 = \frac{-2R^2 m v_0 - 9 v_0 I_{cm}}{7 I_{cm} \cdot R}$

$I_{cm} = \frac{2mR^2}{5} \Rightarrow \omega_0 = -\frac{2v_0}{R}$

$-\frac{2\cancel{v_0}}{\cancel{R}} = -\frac{5h\cancel{v_0}}{2R^{\cancel{2}}} \Rightarrow \frac{h}{R} = \frac{4}{5}$

solution 2



$\vec{L}_{p,sys} = \vec{0}$

$= \vec{L}_p^{orbital} + \vec{L}_p^{spin}$

$= (l\hat{i} - h\hat{j}) \times mv_0\hat{i} + I_{cm}\omega_0\hat{k}$

$= -hmv_0(-\hat{k}) + I_{cm}\omega_0\hat{k}$

$= \hat{k}(hmv_0 + I_{cm}\omega_0)$

$\Rightarrow mv_0h = -I_{cm}\omega_0$

$\vec{L}_{c,sys,i} = R\hat{j} \times mv_0\hat{i} + I_{cm}\omega_0\hat{k}$

$= -Rmv_0\hat{k} + I_{cm}\omega_0\hat{k}$

$= \hat{k}(-mv_0R - mv_0h)$

$= \hat{k}(-mv_0(R+h))$

$\vec{L}_{c,sys,f} = R\hat{j} \times mv_f\hat{i} + I_{cm}\omega_f\hat{k}$

$v_f = -R\omega_f$

$\Rightarrow \vec{L}_{c,sys,f} = -Rmv_f\hat{k} - \frac{I_{cm}v_f}{R}\hat{k}$

$= \hat{k}\left(-Rmv_f - \frac{2mR^{\cancel{2}}v_f}{5\cancel{R}}\right)$

$= \hat{k}\left(\frac{-5Rmv_f - 2Rmv_f}{5}\right)$

$= \hat{k} \cdot \left(\frac{-7mRv_f}{5}\right)$

$= \hat{k}\left(\frac{-\cancel{7}mR}{5} \cdot \frac{9v_0}{\cancel{7}}\right)$

$= \hat{k}\left(-\frac{9}{5}mRv_0\right)$

$\vec{L}_{c,sys,i} = \vec{L}_{c,sys,f}$

$\Rightarrow -mv_0(R+h) = -\frac{9}{5}mRv_0$

$\cancel{m}\cancel{v_0}h = \frac{9}{5}mRv_0 - mRv_0$

$= \frac{4}{5}\cancel{m}R\cancel{v_0} \Rightarrow \frac{h}{R} = \frac{4}{5}$