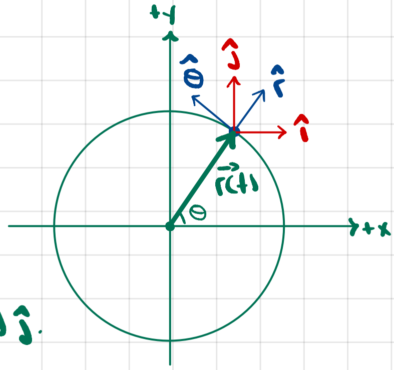
6.1 Circular Motion



→ motion of object : circular orbit of radius r

→ position vector: $\vec{r}(t) = r\hat{r}(t)$

→ two sets of unit vectors: $(\hat{i}, \hat{j}), (\hat{r}, \hat{\theta})$

→ obtaining $\hat{r}$ and $\hat{\theta}$ in $(\hat{i}, \hat{j})$ coordinates means projecting them onto $\hat{i}$ and $\hat{j}$.

$\hat{r}\cdot\hat{i} = \cos\theta \qquad \hat{r}\cdot\hat{j} = \cos(90° - \theta) = \sin\theta \quad \Rightarrow \quad \hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\hat{\theta}\cdot\hat{i} = \cos(90° + \theta) = -\sin\theta \qquad \hat{\theta}\cdot\hat{j} = \cos(\theta) \quad \Rightarrow \hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

note that though omitted in the eq. above, $\theta = \theta(t)$ for our circular orbit motion.

→ At each point in the trajectory, there is a pair $(\hat{r}(t), \hat{\theta}(t))$ of unit vectors.

How do they change in time?

$$\frac{d\hat{r}}{dt} = -\sin\theta\,\frac{d\theta}{dt}\,\hat{i} + \cos\theta\,\frac{d\theta}{dt}\,\hat{j} = \frac{d\theta(t)}{dt}\,\hat{\theta}(t)$$

$$\frac{d}{dt}\hat{\theta}(t) = -\cos\theta\,\frac{d\theta}{dt}\,\hat{i} - \sin\theta\,\frac{d\theta}{dt}\,\hat{j} = -\frac{d\theta(t)}{dt}\,\hat{r}(t)$$

→ velocity vector, tangent to the trajectory

z-comp of angular velocity

velocity direction

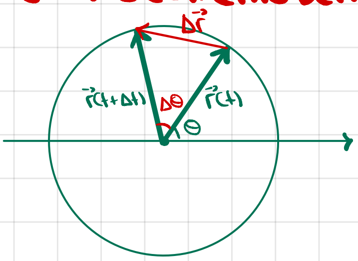
$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \frac{d(r\hat{r}(t))}{dt} = r\,\frac{d\hat{r}(t)}{dt} = r\,\frac{d\theta}{dt}\,\hat{\theta} = r\omega_z\,\hat{\theta} = v_\theta\,\hat{\theta}$$

speed along the arc

aka tangential component of velocity

→ Angular speed: magnitude of rate of change of angle w/ respect to time $\omega \equiv \left|\frac{d\theta}{dt}\right|$

6.2.1 Geometric Derivation of Velocity



we would like to derive what $\vec{v}(t)$ is.

$|\Delta\vec{r}| \approx$ arc length $r|\Delta\theta|$

$$v \equiv |\vec{v}(t)| = \lim_{\Delta t \to 0} \frac{|\Delta\vec{r}|}{\Delta t} = r\cdot\lim_{\Delta t \to 0} \frac{|\Delta\theta|}{\Delta t} = r\left|\frac{d\theta}{dt}\right| = r\omega$$

As $\Delta t \to 0$, $\Delta\vec{r}$ has direction that approaches the direction of the tangent to the circle at $\vec{r}(t)$.

In the limit $\Delta\vec{r} \perp \vec{r}(t)$, $\vec{v}(t) \perp \vec{r}(t)$.

6.3 Tangential and Radial Acceleration

$\vec{a}(t) = a_r \hat{r}(t) + a_\theta \hat{\theta}(t)$

$\vec{v}(t) = r\,\theta'(t)\,\hat{\theta}(t)$

Assume that due to the presence of some tangential force, $\vec{v}(t)$ is changing.

In particular, due to the tangential nature of the force, the magnitude of $\vec{v}(t)$ is changing $\Rightarrow \theta'(t)$ is not constant.

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = r\theta''(t)\hat{\theta}(t) + r\theta'(t)(-\cos\theta\ \theta'(t)\hat{\imath} - \sin\theta\ \theta'(t)\hat{\jmath})$$

$$= r\theta''(t)\ \hat{\theta}(t) - r\theta'(t)^2\ \hat{r}(t)$$

$$= r\theta''(t)\ \hat{\theta}(t) - r\omega^2\ \hat{r}(t)$$

$$= a_\theta\ \hat{\theta} - a_r\ \hat{r}$$

6.4 Period and Frequency for Uniform Circular Motion

If the total tangential force is zero then $F = ma$ in the $\hat{\theta}$ direction (ie the tangential direction) means

$a_\theta = 0 \Rightarrow r\theta''(t) = 0 \Rightarrow \theta''(t) = 0 \Rightarrow \theta'(t) = \text{constant}$

$v(t) = |\vec{v}(t)| = r|\theta'(t)| = \text{constant}$, ie speed is constant $\Rightarrow \omega = \theta'(t)$ constant

$\Rightarrow$ Uniform Circular Motion

$\vec{a}(t) = a_r(t)\hat{r}(t) = -r\omega^2\hat{r}(t)$

Period: Amount of time to complete one circular orbit

$2\pi r = Tv = Tr\omega \Rightarrow T = \dfrac{2\pi}{\omega}$, units are $\dfrac{[L]}{[L/T]} = [T]$

Frequency: reciprocal of period $1/T$, units $1/[T]$, in SI this is $s^{-1} \equiv Hz$

$f = \dfrac{\omega}{2\pi}$

recall the centripetal acceleration $a_r(t) = -r\omega(t)^2$. Because of the relationships between ω, T, f, v, and r we can express $a_r(t)$ in different ways.

$|a_r(t)| = \dfrac{v^2}{r} = r \cdot 4\pi^2 f^2 = r\dfrac{4\pi^2}{T^2}$. There are yet other forms too.

6.5.1 Angular Velocity

→ we will always choose a right-handed cylindrical coordinate system

z-axis points up ⇒ θ increases in ccw direction

→ $\vec{\omega}$ is directed along z-axis

$$\vec{\omega} = \frac{d\theta}{dt}\hat{k} = \omega_z \hat{k} \quad [\text{rad s}^{-1}]$$

ω = angular speed = $|\vec{\omega}| = |\omega_z| = |d\theta/dt|$

→ $$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{k} \\ 0 & 0 & \omega_z \\ r & 0 & 0 \end{vmatrix} = -(-r\omega_z)\hat{\theta} = r\omega_z\hat{\theta} = r\frac{d\theta}{dt}\hat{\theta}$$

Ex 6.2

$\theta(t) = At - Bt^3$, $A, B > 0$

a) $\theta'(t) = A - 3Bt^2$

$\vec{\omega}(t) = (A - 3Bt^2)\hat{k}$

b) $\vec{v} = r\theta'(t)\hat{\theta} = r(A - 3Bt^2)\hat{\theta}$

c) $\vec{\omega}(t) = \vec{0} \Rightarrow A - 3Bt^2 = 0 \Rightarrow t = \sqrt{\frac{A}{3B}}$

d) $t < t_1 \Rightarrow \omega_z > 0 \Rightarrow \text{dir}(\vec{\omega}) = \hat{k}$

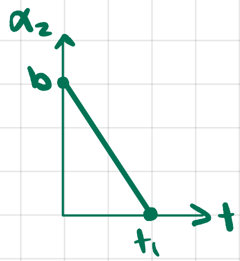
$t > t_1 \Rightarrow \omega_z < 0 \Rightarrow \text{dir}(\vec{\omega}) = -\hat{k}$

6.5.3 Angular Acceleration

angular acceleration $= \vec{\alpha} = \theta''(t)\hat{k} = \alpha_z \hat{k}$ $[rad \cdot s^{-2}]$

$\alpha = |\vec{\alpha}| = |\theta''(t)|$

Ex 6.3 point-like object, circular motion



$$\alpha_z(t) = \begin{cases} b\left(1 - \frac{t}{t_1}\right) & 0 \le t \le t_1 \\ 0 & t > t_1 \end{cases} \qquad b > 0$$

a)
$$\omega_z(t) = \begin{cases} \omega_{z_0} + \int_0^t \left[b - \frac{b}{t_1}t\right]dt & 0 \le t \le t_1 \\ \omega_z(t_1) + \int_{t_1}^t 0\, dt & t > t_1 \end{cases}$$

$$= \begin{cases} \omega_{z_0} + bt - \frac{bt^2}{2t_1} & 0 \le t \le t_1 \\ 0 & t > t_1 \end{cases}$$

interpretation:

$\omega_z = \theta'(t)$ starts at ω_{z_0}

α_z starts at $b > 0$ and decreases to 0.

$\Rightarrow$ $\theta'(t)$ is increasing, but increasing at a progressively lower rate. At t_1, $\theta'(t_1)$ stops increasing, so angular velocity stays constant.

note that no assumption was made on the sign of ω_{z_0}. We don't know if motion is cw or ccw.

At $t = t_1$, $\omega_z(t_1) = \omega_{z_0} + bt_1 - \frac{bt_1^2}{2t_1} = \omega_{z_0} + \frac{2bt_1^2 - bt_1^2}{2t_1} = \omega_{z_0} + \frac{bt_1}{2}$

b) $\omega_z(t) = r\theta'(t)$

$$\theta'(t) = \frac{\omega_z(t)}{r} \Rightarrow \theta(t) = \frac{1}{r}\int \omega_z(t)dt + \theta_0 \qquad 0 \le t \le t_1$$

$$= \frac{1}{r}\left[\omega_{z_0}t + \frac{bt^2}{2} - \frac{bt^3}{6t_1}\right] + \theta_0$$

$$\theta(t_1) = \theta_0 + \frac{\omega_{z_0}t_1}{r} + \frac{2bt_1^2}{6}$$