

Problem Set 5

Problem 1 - Stopping a Bullet

→ let's consider a system bullet + block

initially, $\vec{p}_{\text{sys}} = \langle m_1 v_{i,i,x}, 0 \rangle$

right after instantaneous collision, $\vec{p}_{\text{sys}} = \langle (m_1 + m_2) v_{\text{sys},t_1,x}, 0 \rangle$

$$\text{between } t_1 \text{ and } t_2, \vec{F}_{\text{ext}} = 0 \Rightarrow \int_{t_1}^{t_2} \vec{F}_{\text{ext}} dt = \Delta \vec{p} = \vec{p}(t_2) - \vec{p}(t_1) \Rightarrow m_1 v_{i,i,x} = (m_1 + m_2) v_{\text{sys},t_2,x}$$

$$\Rightarrow v_{\text{sys},t_2,x} = \frac{m_1}{m_1 + m_2} v_{i,i,x} \quad \text{velocity of system in x direction right after collision (before friction acts, as fn of } m_1, m_2, v_{i,i,x})$$

$$\text{Let's consider } t_2 = t_0 = 0, \text{ and let's call } t_1 = t \Rightarrow v_{\text{sys},0,x} = \frac{m_1}{m_1 + m_2} v_{i,i,x}$$

the system is subject to an external force $\vec{F}_h = \langle -\mu_k(m_1 + m_2)g, 0 \rangle$ for time t_h , during which it moves a distance d .

$$\vec{F}_x^{\text{ext}} \cdot F_h = m_1 v_{i,i,x} \cdot a_x(t) \Rightarrow -\mu_k g(m_1 + m_2) = (m_1 + m_2) a_x(t) \Rightarrow a_x(t) = -\mu_k g$$

$$\Rightarrow v_x(t) = \frac{m_1}{m_1 + m_2} v_{i,i,x} - \mu_k g t \Rightarrow v_x(t) = v_{\text{sys},0,x} - \mu_k g t$$

$$x(t) = v_{\text{sys},0,x} t - \frac{\mu_k g t^2}{2}$$

$$\text{find } t_h, \text{ time to move distance } d \text{ in x-direction under friction: } x(t) = d \Rightarrow \frac{\mu_k g t_h^2}{2} - v_{\text{sys},0,x} t_h + d = 0$$

$$\Delta = v_{\text{sys},0,x}^2 - 4 \frac{\mu_k g}{2} d \Rightarrow \text{For a real solution to exist we need } \Delta \geq 0 \Rightarrow \left[\frac{m_1}{m_1 + m_2} v_{i,i,x} \right]^2 \geq 2 \mu_k g d$$

$$\text{a) } v_{i,i,x} \geq \sqrt{\frac{m_1 + m_2}{m_1} \cdot 2 \mu_k g d}$$

→ min speed of bullet for bullet-block system to move distance d after collision.
note we will assume this is enough to fall off the surface. To be completely accurate,
the system must move a certain distance beyond d for the system to fall off.

$$t_h = \frac{v_{\text{sys},0,x} - [\sqrt{v_{\text{sys},0,x}^2 - 2 \mu_k g d}]}{\mu_k g} \Rightarrow \text{time it takes for system to move distance } d \text{ in x direction under effect of friction.}$$

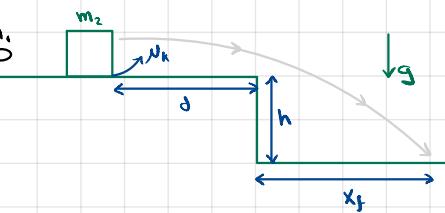
as fn of $m_1, m_2, v_{i,i,x}, \mu_k, g, d$

$$\int_0^{t_h} \vec{F}_{\text{ext}} dt = \int_0^{t_h} \langle -\mu_k(m_1 + m_2)g, 0 \rangle dt = \vec{p}(t_h) - \vec{p}(t_0)$$

$$\Rightarrow \int_0^{t_h} -\mu_k g(m_1 + m_2) = (m_1 + m_2) v_{\text{sys},t_h,x} - m_1 v_{i,i,x}$$

$$-\mu_k g(m_1 + m_2) t_h = (m_1 + m_2) v_{\text{sys},t_h,x} - m_1 v_{i,i,x}$$

$$\Rightarrow v_{\text{sys},t_h,x} = \frac{m_1 v_{i,i,x} - \mu_k g t_h}{m_1 + m_2} \rightarrow \text{velocity of system in x direction when ball starts to fall off as fn of } m_1, m_2, v_{i,i,x}, \mu_k, g, d$$



Next we analyze the motion of the system in free fall.

Let's reset the time subscripts. $t = t_0 = 0$ is now the start of the fall.

$$\vec{v}_0 = \vec{v}(0) = \langle m_1 v_{1,i,x} - \mu g(m_1 + m_2) t_h, 0 \rangle$$



$$\vec{a}_{\text{all}} = \langle 0, -g \rangle$$

$$\vec{v}(t) = \langle m_1 v_{1,i,x} - \mu g(m_1 + m_2) t_h, -gt \rangle = \langle v_{\text{sys},t,x}, gt \rangle$$

$$\vec{r}(t) = \langle v_{\text{sys},t,h,x} t, gt^2/2 \rangle = \langle x(t), y(t) \rangle$$

$$y(t) = h \Rightarrow gt^2 = 2h \Rightarrow t_h = \sqrt{\frac{2h}{g}}$$

$$x(t_h) = v_{\text{sys},t_h,x} t_h \Rightarrow x(t_h) = \sqrt{\frac{2h}{g}} (m_1 v_{1,i,x} - \mu g(m_1 + m_2) t_h)$$

b) The bullet/block system hits the ground at a distance of $\sqrt{2gh} (m_1 v_{1,i,x} - \mu g(m_1 + m_2) t_h)$ from the bottom edge of the surface, where

$$t_h = \frac{\sqrt{v_{\text{sys},0,x}^2 + [v_{1,i,x} - 2\mu gd]^{1/2}}}{\mu g}$$

and

$$v_{\text{sys},0,x} = \frac{m_1}{m_1 + m_2} v_{1,i,x}$$

Problem 2 - Adiabatic and Closer

System: $m_a + m_b$

External Force: gravity

$$P_{SIS_0} = m_a V_0$$

$$P_{SIS_1} = (m_a + m_b) V_1$$

$$\int_0^{t_1} -m_a g dt = P_{SIS}(t_1) - P_{SIS}(t_0) = (m_a + m_b) V_1 - m_a V_0$$

$$-m_a g t_1 = (m_a + m_b) V_1 - m_a V_0$$

$$V_1 = \frac{m_a(V_0 - g t_1)}{m_a + m_b}$$

We don't know what t_1 is but it depends on h_0 .

$$a(t) = -g$$

$$v(t) = V_0 - gt$$

$$y(t) = V_0 t - \frac{gt^2}{2}$$

$$y(t) = h_0 \Rightarrow \frac{gt^2}{2} - V_0 t + h_0 = 0$$

$$\Delta = V_0^2 - 4gh_0 \geq 0 \Rightarrow V_0 \geq \sqrt{2gh_0}$$

$$t_1 = \frac{V_0 \pm \sqrt{V_0^2 - 2gh_0}}{g}$$

$$\int_{t_1}^{t_2} -(m_a + m_b) g dt = P_{SIS}(t_2) - P_{SIS}(t_1) = (m_a + m_b) \cdot 0 - (m_a + m_b) V_1$$

$$-(m_a + m_b) g (t_2 - t_1) = -(m_a + m_b) V_1$$

$$g t_2 = V_1 + g t_1 \Rightarrow t_2 = t_1 + \frac{V_1}{g}$$

$\frac{V_1}{g}$ seconds pass after the grab. We want to know how high the system moves in that time.

$$\Rightarrow a(t) = -g$$

$$v(t) = V_1 - gt$$

$$y(t) = h_0 + V_1 t - \frac{gt^2}{2}$$

* we could also have found t_2 here: $V_1 = 0 \Rightarrow t = \frac{V_1}{g}$

$$y(\frac{V_1}{g}) = h_0 + \frac{V_1^2}{g} - \frac{V_1^2}{g} \cdot \cancel{\frac{1}{2}} = h_0 + \frac{V_1^2}{2g}$$

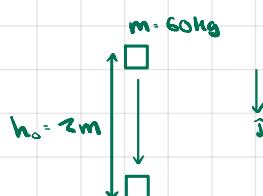
$$\Rightarrow \text{max height} = h_0 + \frac{V_1^2}{2g}$$



Problem 3 - Compressive Strength of Bones

$$\text{Force/Area to break bone} = \frac{F}{A} = 1.6 \times 10^8 \text{ N/m}^2$$

$$+ \int mg dt = mv - 0 \Rightarrow mgt = mv \Rightarrow v = gt$$



t depends of course on the height h_0 : $a(t) = g$, $v(t) = gt$, $y(t) = -h_0 + \frac{1}{2}g t^2$

$$\Rightarrow v(t) = 0 \Rightarrow gt^2 = 2h_0 \Rightarrow t = \sqrt{\frac{2h_0}{g}} \Rightarrow v = \sqrt{2gh_0}$$

Just before collision, $v = \sqrt{2gh_0}$, $a = g$

During the collision, the center of mass (cm), which was accelerating at a rate of g , now decelerates bringing v from $\sqrt{2gh_0}$ to 0 in a distance of $d = 1\text{m}$. For the period of collision and deceleration:

$$a(t) = g - a$$

$$v(t) = \sqrt{2gh_0} + (g-a)t$$

$$y(t) = t\sqrt{2gh_0} + (g-a)t^2/2$$

Let's call t_e the end of collision time:

$$v(t_e) = 0 \Rightarrow t_e = \frac{\sqrt{2gh_0}}{a-g}$$

$$y(t_e) = 0.01 = \frac{2gh_0}{a-g} + \frac{g-a}{2} \cdot \frac{\sqrt{2gh_0}}{(a-g)} = \frac{2gh_0}{a-g} - \frac{gh_0}{a-g} = \frac{gh_0}{a-g}$$

$$\Rightarrow a-g = 100gh_0 \Rightarrow a = g(1+100h_0) \Rightarrow t_e = \frac{\sqrt{2gh_0}}{100gh_0}$$

$$a) a = 9.8(1+100 \cdot 2) = 9.8 \cdot 201 = 1969.8$$

$$\text{collision time} = \frac{\sqrt{2 \cdot 9.8 \cdot 2}}{100 \cdot 9.8 \cdot 2} = 0.003194 = 0.0032 \text{ secs}$$

$$b) F_{\text{ext}} = m \ddot{A}_{\text{cm}} = 60 \text{ kg} \cdot (-1969.8 \text{ m/s}^2) = -117600 \text{ N}$$

alternatively we could calculate average force as

$$\frac{\int F_{\text{ext}} dt'}{t_e - 0} = \frac{p(t_e) - p(0)}{t_e - 0} = F_{\text{avg}} = -\frac{p(0)}{t_e} = -\frac{60 \sqrt{2 \cdot 9.8 \cdot 2}}{0.0032} = -117600 \text{ N}$$

$$p(t_e) = 0 \quad p(0) = m \cdot \sqrt{2gh_0} = 60 \sqrt{2 \cdot 9.8 \cdot 2}$$

c) During the 0.0032 secs of collision, the person was under the effect of a gravitational force of $60 \cdot 9.8 = 588 \text{ N}$ that accelerated the person at a rate of 9.8 m/s^2 . The person was also under the effect of a ground force of $60(-1969.8) = -118188 \text{ N}$ that accelerated the person -1969.8 m/s^2 . The resultant force does thus $-118188 + 588 = -117600 \text{ N}$ which accelerated the person -1900 m/s^2 in $\approx 0.0032 \text{ s}$ to bring the person from a speed of 6.26 m/s to 0 in a distance of 0.01 m .

The ratio of the two external forces is $F_0 / F_{\text{ground}} = 1/201$. If we had disregarded g in the calculations above we'd have that the acceleration required to stop the person in 0.01 m would be -1960 m/s^2 instead of 1969.8 m/s^2 . Total average force could be -117600 N instead of 118188 N .

$$d) \frac{|F|}{A} = \frac{118188 \text{ N}}{0.00032 \text{ m}^2} = 3.693 \times 10^8 \text{ N/m}^2 > 1.6 \times 10^8 \text{ N/m}^2 \Rightarrow \text{person breaks ankle}$$

The result would be similar disregarding g : $\frac{117600}{0.00032} = 3.675 \times 10^8 \text{ N/m}^2$

Problem 4 - Center of Mass of Rod



a) Uniform rod \Rightarrow constant density $\lambda(x) = \lambda$

$$\int_0^L \lambda(x) dx = m \Rightarrow \lambda L = m \Rightarrow$$

$$\bar{r}_{cm} = \frac{\int_{rod} \vec{r}_{dm}(x) dm}{\int_{rod} dm} = \frac{\int_0^L x \hat{i} \lambda dx}{\int_0^L \lambda dx} = \frac{\frac{\lambda x^2}{2} \Big|_0^L \hat{i}}{\lambda L} = \frac{\frac{L^2}{2} \hat{i}}{L} = \frac{L}{2} \hat{i}$$

b) $\lambda(x) = \frac{\lambda_0}{L^3} x^3$

$$m = \int_0^L \frac{\lambda_0}{L^3} x^3 dx = \frac{\lambda_0}{L^3} \frac{L^4}{4} = \frac{\lambda_0 L}{4} = \lambda_0 = \frac{4m}{L}$$

$$\bar{r}_{cm} = \frac{\int_0^L x \hat{i} \frac{\lambda_0}{L^3} x^3 dx}{m} = \frac{\frac{4m}{L^4} \frac{L^5}{5} \hat{i}}{m} = \frac{4L}{5} \hat{i}$$

Problem 5 - Two Particles Colliding



$$\vec{F}_{21} = \frac{\beta}{x(t)^3} = m_1 a(t) = m_1 \frac{d^2 x(t)}{dt^2}$$

System: m_1, m_2

There are no net external forces.

$$F_{\text{ext}} = m_T \cdot a_{\text{cm}} = 0 \Rightarrow a_{\text{cm}} = 0$$

$$v_{\text{cm}} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$x_{\text{cm}} = \frac{m_1(x_{1,0} + v_1 t) + m_2(x_{2,0} + v_2 t)}{m_1 + m_2}$$

Take the origin to be m_1 's initial position.

$$x_{\text{cm}} = \frac{m_1 v_1 t + m_2(d + v_2 t)}{m_1 + m_2}$$

$$= \frac{m_2 d}{m_1 + m_2} + \frac{m_1 v_1 t + m_2 v_2 t}{m_1 + m_2}$$

$$x_{\text{cm},0} = \frac{m_1 \cdot 0 + m_2 d}{m_1 + m_2}$$

$$= \frac{m_2 d}{m_1 + m_2}$$

$$x_{\text{cm},t_i} = \frac{m_2 d}{m_1 + m_2} + \frac{m_1 v_1 t_i + m_2 v_2 t_i}{m_1 + m_2}$$

$$\Delta x_{\text{cm}} = \frac{m_1 v_1 t_i + m_2 v_2 t_i}{m_1 + m_2}$$

$$L - x_{\text{cm},0} + \Delta x_{\text{cm}} = x_{\text{cm},t_i}$$