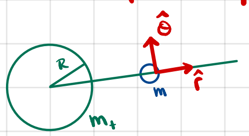
14.9 - Worked Examples

Ex 14.2 - Escape Velocity of Toro



$R = 5\,km$

$m_t = 2 \cdot 10^{15}\,kg$

$\vec{r} = r(t)\,\hat{r}$

$\frac{d\vec{r}}{dt} = r'\hat{r} + r\frac{d\hat{r}}{dt} = r'\hat{r} + r\theta'\hat{\theta}$

$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\frac{d\hat{r}}{dt} = -\sin\theta \cdot \theta'\,\hat{i} + \cos\theta\,\theta'\,\hat{j} = \theta'\hat{\theta}$

$F_{t,m} = -\frac{Gm_t m}{r_{t,m}^2}\,\hat{r}$

$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$

$$W_g = \int_{r_i}^{r_f} \vec{F}_{t,m} \cdot d\vec{r} = \int_{r_i}^{r_f} \frac{-Gm_t m}{r_{t,m}^2}\,dr = \frac{Gm_t m}{r}\Bigg|_{r_i}^{r_f} = Gm_t m\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

Define $U(r) = \frac{-Gm_t m}{r} \Rightarrow U(\infty) = 0$

$K_i = \frac{1}{2}mv_i^2$

$$W_g = -\Delta U_g = \Delta K \Rightarrow \frac{1}{2}m(v_f^2 - v_i^2) = -\left[-Gm_t m\left(\frac{1}{r_f} - \frac{1}{r_i}\right)\right]$$

$$\Rightarrow \frac{1}{2}m(v_f^2 - v_i^2) - Gm_t m\left(\frac{1}{r_f} - \frac{1}{r_i}\right) = \Delta E_{mech} = 0$$

$$\lim_{r_f \to \infty} \Delta E_{mech} = 0 = \frac{1}{2}\cancel{m}(v_f^2 - v_i^2) + \frac{Gm_t \cancel{m}}{r_i}$$

$$v_f^2 - v_i^2 = \frac{-2Gm_t}{r_i} \Rightarrow v_i = \sqrt{v_f^2 + \frac{2Gm_t}{r_i}}$$

$$v_i = \sqrt{0^2\,(m/s)^2 + \frac{2 \cdot 6.67 \cdot 10^{-11}\,\frac{Nm^2}{kg^2} \cdot 2 \cdot 10^{15}\,kg}{5 \cdot 10^3\,m}} = 7.3047\ldots\,m/s$$

$K_i = \frac{1}{2} \cdot m \cdot 7.3047 = 26.68m\ J$

$U_i = \frac{-6.67 \cdot 10^{-11} \cdot 2 \cdot 10^{15} \cdot m}{5 \cdot 10^3} = -26.68m\ J$

$\Rightarrow E_{mech_i} = 0$

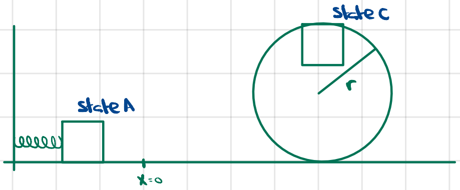
$K_f = 0$

$U_f = 0$

$\Rightarrow E_{mech_f} = 0$

The object starts w/ v_i and is displaced to a radius of ∞. Kinetic energy goes down, potential goes up as F_g does negative work.

Ex 14.3 Spring-Block-Loop-the-Loop



no friction

A: $K_A = 0$

$U_{el_A} = \frac{1}{2}kx^2$

$U_{g_A} = 0$

B: $K_B = \frac{1}{2}mv_B^2$

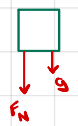
$U_{el_B} = 0$

$U_{g_B} = mg \cdot 2r$

a) $\Delta E_m = 0 \Rightarrow \underbrace{\left(0 - \frac{1}{2}kx^2\right)}_{\Delta U_{el}} + \underbrace{\left(\frac{1}{2}mv_B^2 - 0\right)}_{\Delta K} + \underbrace{\left(mg\,2r - 0\right)}_{\Delta U_g} = 0$

$\frac{1}{2}mv_B^2 = K_B = \frac{1}{2}kx^2 - 2mgr \Rightarrow v_B = \frac{1}{m}\left(kx^2 - 4mgr\right)$

b) B:



$\vec{F}_N = -F_{N_r}\hat{r}$

$\vec{a}_r = -a_r\hat{r} = -r\omega^2\hat{r} = -r\frac{v^2}{r^2}\hat{r} = -\frac{v^2}{r}\hat{r}$

At B, $\vec{F}_N = -2mg\hat{r}$

$F_N + F_G = m \cdot a_r \Rightarrow -3\cancel{m}g = m \cdot (-a_r) = \cancel{m} \cdot \left(-\frac{v^2}{r}\right)$

$3g = \frac{v^2}{r} \Rightarrow v_B = \pm\sqrt{3rg}$

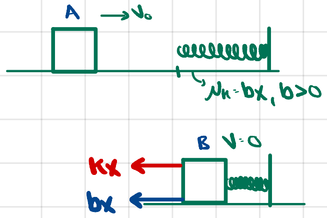
c) From conserv. of mech. en. we know that $\frac{1}{2}mv_B^2 = \frac{1}{2}kx^2 - 2mgr$

Given the knowledge of normal force at B, we know v_B. So we know total final mech. en. We can solve for x.

$m \cdot 3rg = kx^2 - 4mgr$

$x = \sqrt{\frac{7mgr}{k}}$

Ex 14.4 Mass-Spring on Rough surface



$E_{m_A} = K_A + U_{el_A} = \frac{mv_0^2}{2}$

$E_{m_B} = \frac{kx^2}{2}$

$W_f + U_{el} = \Delta K$

$W_f = \Delta E_m$

$W_f = \int_0^x -bxmg\,dx = -bmg\frac{x^2}{2} = \frac{kx^2}{2} - \frac{mv_0^2}{2} = \Delta E_m$

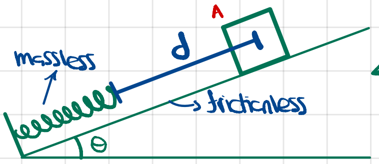
$\Rightarrow -bmgx^2 = kx^2 - mv_0^2$

$x^2(k + bmg) = mv_0^2$

$x^2 = \frac{mv_0^2}{k+bmg}$

$\Rightarrow \Delta E_m = -bmg \cdot \frac{1}{2} \cdot \frac{mv_0^2}{k+bmg}$

Ex 14.5 Cart-Spring on an Inclined Plane



A: initial pos.
B: right before compression
C: compressed, speed zero

a) initially, zero speed $\Rightarrow K_A = 0$. Consider single dimension, that of movement.
set $x = 0$ to point of initial contact w/ spring.

At A, $E_{mech} = K_A + U_{gA} + U_{elA} = 0 + mgd\sin\theta + 0$

$W_g = \Delta K = -\Delta U \quad \int_{-d}^{0} mg\sin\theta\, dx = mg\sin\theta \cdot x \Big|_{-d}^{0} = \cancel{m}gd\sin\theta = \frac{\cancel{m}v^2}{2} \Rightarrow v = \sqrt{2gd\sin\theta}$

At B, $E_{mech} = K_B + U_{gB} + U_{elB} = mgd\sin\theta + 0 + 0$

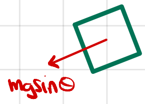
From B to C,

$W_g + W_{el} = \Delta K$

$-\Delta U_g - \Delta U_{el} = \Delta K \Rightarrow \Delta E_m = 0$

$\int_0^x mg\sin\theta\, dx + \int_0^x -kx\, dx = mgx\sin\theta - \frac{kx^2}{2} = -mgd\sin\theta$

note $\Delta U_g = -mgx\sin\theta$
$\Delta U_{el} = \frac{kx^2}{2}$



$F = m\cdot a \Rightarrow mg\sin\theta = m\cdot a \Rightarrow a = g\sin\theta$

$v(t) = gt\sin\theta$

$x(t) = -d + g\sin\theta\,\frac{t^2}{2}$

$x(t) = 0 \Rightarrow d = \frac{t^2}{2}g\sin\theta \Rightarrow t = \sqrt{\frac{2d}{g\sin\theta}}$

$v_d = g\sin\theta \cdot \sqrt{\frac{2d}{g\sin\theta}} = \sqrt{2dg\sin\theta}$

↓ gravit. pot. en.
↑ el. pot. en.
↓ kin. en.

compression start, B

$K_B = \frac{m\cdot 4dg\sin\theta}{2} = 2mg\sin\theta$

$\Delta K + \Delta U_g + \Delta U_{el} = 0$

$-mgd\sin\theta - mgx\sin\theta + \frac{kx^2}{2} = 0$

$\Rightarrow kx^2 - 2mg\sin\theta\, x - 2mgd\sin\theta = 0$

$x = \frac{2mg\sin\theta \pm \sqrt{4m^2g^2\sin^2\theta + 4k\cdot 2mgd\sin\theta}}{2k} \quad \frac{\sqrt{m^2g^2\sin^2\theta + 2kmgd\sin\theta}}{k}$

real sol'ns $\cancel{4}m^2g^{\cancel{2}}\cancel{\sin\theta} > \cancel{4}k\cdot 2\cancel{mg}\cancel{\sin\theta}\, d \Rightarrow mg > 2kd$

* Note: we started with $\quad \Delta K + \Delta U_g + \Delta U_{el} = 0$

$-mgd\sin\theta - mgx\sin\theta + \frac{kx^2}{2} = 0$

$\Delta K = -mgd\sin\theta$ is the chg in kinetic energy whether the initial velocity is positive or negative, ie if the mass is compressing the spring or if the spring is pushing the mass as it stretches. In the latter case, $W_g = \int_0^x mg\sin\theta\, dx = mgx\sin\theta$, x is negative, $W_g < 0$.
we'd still have $-mgd\sin\theta - mgx\sin\theta + \frac{kx^2}{2} = 0$. There is a negative x compatible w/ this scenario and it is the sol'n above w/ negative root.

b) now we introduce friction

$E_{m_A} = mgd\sin\theta$

$W_f + W_g + W_{el} = \Delta K$

$$\int_{-d}^{x} -\mu_k mg\cos\theta\,dx + \int_{-d}^{x} mg\sin\theta\,dx + \int_{0}^{x} -kx\,dx = 0$$

$$-\mu_k mg\cos\theta(x+d) + mg\sin\theta(x+d) - \frac{kx^2}{2} = 0$$

$$kx^2 + 2\mu_k mg\cos\theta(x+d) - 2mg\sin\theta(x+d) = 0$$

$$kx^2 + x\cdot 2mg(\mu_k\cos\theta - \sin\theta) + 2mgd(\mu_k\cos\theta - \sin\theta)$$

$$x = \frac{-2mg(\mu_k\cos\theta - \sin\theta) \pm \sqrt{4m^2g^2(\mu_k\cos\theta - \sin\theta)^2 - 4k\cdot 2mgd(\mu_k\cos\theta - \sin\theta)}}{2k}$$

$$= \frac{mg(\sin\theta - \mu_k\cos\theta) \pm \sqrt{m^2g^2(\sin\theta - \mu_k\cos\theta)^2 + 2kmgd(\sin\theta - \mu_k\cos\theta)}}{k}$$

$$W_f = \Delta E_m = \int_{-d}^{x_f} -\mu_k mg\cos\theta\,dx$$

$$= -\mu_k mg\cos\theta\left[x_f + d\right],\ x_f =$$

Note, from A to B:

$K_A = 0$ $\quad K_B = \frac{mv_B^2}{2}$

$U_A = mgd\sin\theta$ $\quad U_B = 0$

$$W_f = \Delta E_m \Rightarrow \int_{-d}^{0} -\mu_k mg\cos\theta\,dx = mgd\sin\theta - \frac{mv_B^2}{2}$$

$$\Rightarrow \mu_k d\cancel{m}g\cos\theta = \cancel{m}gd\sin\theta - \frac{\cancel{m}v_B^2}{2}$$

$$\Rightarrow v_B = \sqrt{2dg(\sin\theta - \mu_k\cos\theta)}$$

$\sin\theta - \mu_k\cos\theta < 1$

$\sin\theta < 1 + \mu_k\cos\theta$

for $\theta \in [0, \pi/2]$ we have $1 + \mu_k\cos\theta > 1$ and $\sin\theta \leq 1\ \forall\theta$, so v_B in this scenario is smaller than before

Friction did negative work, so mechanical energy was lost. Work done by gravity stayed the same, ie the loss of U_g was the same. Therefore the loss in mech. en. was fully in lost kinetic energy.

$$K_B = \frac{m\cdot\cancel{2}dg(\sin\theta - \mu_k\cos\theta)}{\cancel{2}}$$

$$= mdg(\sin\theta - \mu_k\cos\theta)$$

$U_{g_B} = 0$

$U_{el_B} = 0$

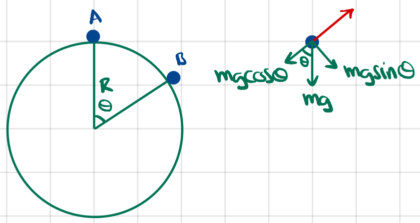
$$\Delta E_m = mdg(\sin\theta - \mu_k\cos\theta) - mgd\sin\theta$$

$$= -mgd\mu_k\cos\theta$$

14.6 object sliding on a sphere

$\hat r = \cos\theta\,\hat\imath + \sin\theta\,\hat\jmath$

$\frac{d\hat r}{dt} = -\sin\theta\,\theta'\,\hat\imath + \cos\theta\,\theta'\,\hat\jmath = \theta'\hat\theta$



$\int_c \vec F_g \cdot d\vec r$

$\vec F_g = -mg\cos\theta\,\hat r + mg\sin\theta\,\hat\theta$

$\vec r = R\hat r$

$d\vec r = R\,d\hat r = R\cdot d\theta\,\hat\theta$

$\vec F_g\, d\vec r = mgR\sin\theta\, d\theta$

work done by gravity

$$W_g = \int_0^\theta mgR\sin\theta\, d\theta = -mgR\cos\theta\,\Big|_0^\theta = mgR(1-\cos\theta)$$

$$N - mg\cos\theta = -m\frac{v^2}{R} = -\frac{m\cdot 2g\cancel{R}(1-\cos\theta)}{\cancel{R}}$$

$N = mg(\cos\theta - 2 + 2\cos\theta$

$= mg(3\cos\theta - 2)$

$K_A = 0 \quad U_{g_A} = 0$

$K_B = \frac{mv_B^2}{2} \quad U_{g_B} = -mgR(1-\cos\theta_B)$

↓g potential, ↑ kinetic

$$\Delta E_{mech} = 0 \Rightarrow \frac{\cancel{m}v_B^2}{2} - \cancel{m}gR(1-\cos\theta_B) = 0 \Rightarrow v_B = \sqrt{2gR(1-\cos\theta_B)}$$

The object is undergoing circular motion. It has both tangential and radial acceleration. Radial force is decreasing and tangential force is increasing as θ increases.

$\uparrow v \rightarrow \uparrow\omega \rightarrow \uparrow a_r = R\omega^2 \rightarrow \uparrow F_r$

At some point radial force will go below the force required to keep the mass on the circular motion.

Given v_B, we have radial acceleration

$$a_{r_B} = R\left(\frac{v_B}{R}\right)^2 = \frac{v_B^2}{R} = 2g(1-\cos\theta)$$

$F_R = \cancel{mg}\cos\theta = \cancel{m}\cdot 2\cancel{g}(1-\cos\theta)$

$\cos\theta = 2(1-\cos\theta)$

$3\cos\theta = 2$

$\cos\theta = \frac{2}{3}$

$\theta = \cos^{-1} 2/3$

b) $v = \sqrt{2gR(1-2/3)}$

$= \sqrt{\frac{2}{3}gR}$

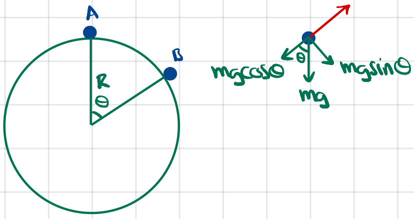
$mg\cos\theta = m\frac{v_B^2}{R} \Rightarrow v_B^2 = Rg\cos\theta$ (2nd law at departure from sphere)

$\frac{mv_B^2}{2} + mgR(\cos\theta - 1) = 0$ (energy conservation)

$\frac{mgR\cos\theta}{2} + mgR(\cos\theta - 1) = 0$

$3mgR\cos\theta = 2mgR$

$\cos\theta = \frac{2}{3} \Rightarrow \theta = \cos^{-1}(2/3)$



particle moves under influence of $F_T = mg\sin\theta$. Tangential accel. is $g\sin\theta$

$\theta = \dfrac{s(t)}{R}$

$a_T = g\sin\theta \Rightarrow \dfrac{d^2s}{dt} = g\sin\left(\dfrac{s(t)}{R}\right)$ (DE of motion) (1)

Initial conditions: $\left[\dfrac{ds}{dt}\right]_{t=0} = 0, \; s(0) = 0$

2nd Law: $N - mg\cos\theta = ma_R = -m\dfrac{v^2}{R}$

The particle leaves the sphere when $N = 0 \Rightarrow \cancel{m}g\cos\theta = \cancel{m}\dfrac{v^2}{R}$

$\Rightarrow \left(\dfrac{ds}{dt}\right)^2 - Rg\cos\left(\dfrac{s}{R}\right) = 0$ (2)

multiply (1) by $\dfrac{ds}{dt} \Rightarrow \dfrac{ds}{dt}\cdot\dfrac{d^2s}{dt^2} - \dfrac{ds}{dt}g\sin\left(\dfrac{s(t)}{R}\right) = 0$

$= \dfrac{ds}{dt}\cdot\dfrac{d}{dt}\left(\dfrac{ds}{dt}\right) + \dfrac{d}{dt}\left[gR\cos\left(\dfrac{s}{R}\right)\right] = 0$

$= \dfrac{d}{dt}\left[\dfrac{1}{2}\left(\dfrac{ds}{dt}\right)^2 + gR\cos\left(\dfrac{s}{R}\right)\right] = 0$

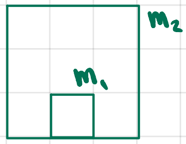
$\Rightarrow \dfrac{1}{2}\left(\dfrac{ds}{dt}\right)^2 + gR\cos\left(\dfrac{s}{R}\right) = \text{constant}$

For $t = 0 \Rightarrow 0^2 + gR\cos(0) = gR = \text{constant}$

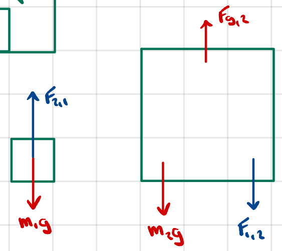
$\Rightarrow \dfrac{1}{2}\left(\dfrac{ds}{dt}\right)^2 + gR\cos\left(\dfrac{s}{R}\right) = gR$ (3)

$2\cdot(3) - (2)$

$= \; gR\cdot 3\cos\left(\dfrac{s}{R}\right) = 2gR \Rightarrow \cos(s/R) = \dfrac{2}{3}$



a) At rest

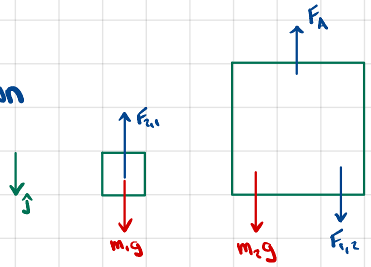


$F_{2,1}-m_1g=0 \Rightarrow F_{2,1}=m_1g$

$F_{2,1}=F_{1,2}=m_1g$

$F_{g,2}-m_2g-m_1g=0 \Rightarrow F_{g,2}=g(m_1+m_2)$

b) Accelerating down



$m_1g-F_{2,1}=m_1\cdot a \Rightarrow F_{2,1}=m_1(g-a)$

$m_2g+F_{1,2}-F_A=m_2\cdot a$

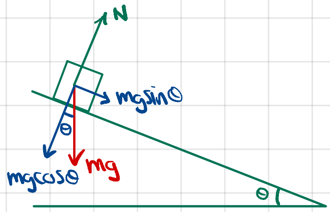
$F_{2,1}=F_{1,2}$

$\Rightarrow m_2g+m_1(g-a)-F_A=m_2\cdot a$

$F_A=m_2(g-a)+m_1(g-a)$

$=(g-a)(m_1+m_2)$

c)



$N=mg\cos\theta$

$mg\sin\theta=m\cdot a$

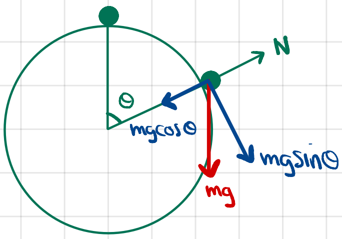
$v=\theta'(t)\cdot R$

$a=\theta''(t)\cdot R \Rightarrow \theta''(t)=\frac{a}{R}$

$N-mg\cos\theta=-\frac{mv^2}{R}$

$mg\sin\theta=m\cdot a_T=m\cdot R\cdot\theta''(t)=mR\cdot\frac{a}{R}$

d)

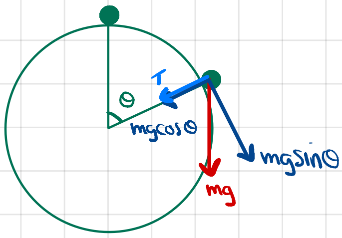


$v=\sqrt{2gR(1-\cos\theta)}$

$N=mg\cos\theta-\frac{m\cdot 2g\cancel{R}(1-\cos\theta)}{\cancel{R}}$

$=mg(3\cos\theta-2)$ $\quad\theta\in[0,\cos^{-1}(2/3)]$

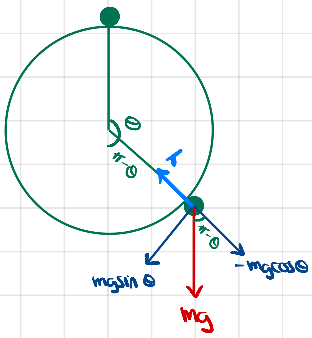
e)



$-T-mg\cos\theta=-mg\cdot 2(1-\cos\theta)$

$T=2mg-2mg\cos\theta-mg\cos\theta$

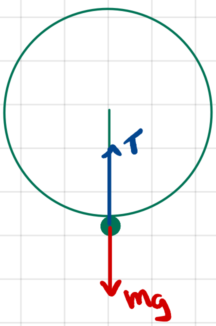
$T=mg(2-3\cos\theta)=-mg(3\cos\theta-2)$ $\quad\theta\in[\cos^{-1}(2/3),\pi/2]$



$-mg\cos\theta-T=-mg\cdot 2(1-\cos\theta)$

$T=-mg\cos\theta+2mg-2mg\cos\theta$

$=mg(2-3\cos\theta)$



$m=1,\ g=9.8,\ \theta=\pi$

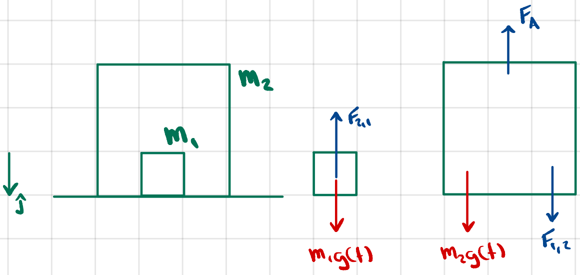
$T=49$

$F_g=9.8$

$v=6.26\ m/s$

$a_R=39.2\ m/s^2$

f) Accelerating down



$g(t) = \sin t$

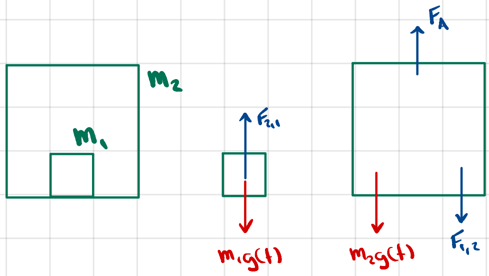
$m_1 \sin t - F_{2,1} = 0$

$F_{21} = m_1 \sin t$

$m_2 \sin t + F_{1,2} - F_A = 0$

$F_A = \sin t (m_1 + m_2)$

g) remove the ground



Free Fall $F_A = 0$

$F_{1,2} + m_2 \sin t = m_2 a$

$m_1 \sin t - F_{2,1} = m_1 a$

$F_{2,1} = m_1(\sin t - a)$

$m_1(\sin t - a) + m_2 \sin t = m_2 a$

$(\sin t - a)(m_1 + m_2) = 0$

$\Rightarrow a = \sin t$

$F_{2,1} = F_{1,2} = 0$

now imagine the ground accelerates