Chapter 21 - Rigid Body Dynamics

System of particles

-> if system is treated as point particle of mass m_T at CM moving at $\vec{v}_{cm}$ we have

$$\vec{F}^{ext} = \frac{dp^{sys}}{dt} = \frac{d}{dt}(m_T \vec{v}_{cm})$$

(Translational Eq. of Motion)

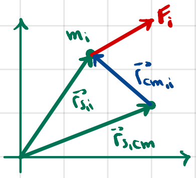
For system of particles

$$\vec{\tau}_S^{ext} = \sum_i^N \vec{r}_i \times \vec{F}_i$$

* assumption that all internal torques cancel in pairs

-> S origin of frame O

$$\vec{r}_{S,cm} = \vec{R}_{cm}$$



sub in $\vec{r}_{S,i} = \vec{R}_{cm} + \vec{r}_{cm,i}$

$$\vec{\tau}_S^{ext} = \sum (\vec{r}_{S,cm} + \vec{r}_{cm,i}) \times \vec{F}_i$$

$$= \sum \vec{r}_{S,cm} \times \vec{F}_i + \sum \vec{r}_{cm,i} \times \vec{F}_i$$

$$= \vec{r}_{S,cm} \times \vec{F}^{ext} + \sum \vec{r}_{cm,i} \times \vec{F}_i$$

$$= \vec{\tau}_{S,cm}^{ext} + \vec{\tau}_{cm}^{ext}$$

$\vec{\tau}_{cm}^{ext}$: external torque about CM in CM frame

$\vec{\tau}_{S,cm}^{ext}$: external torque about S with all forces acting on CM

External torque about S can be decomposed into two parts

$$\vec{\tau}_S^{ext} = \vec{\tau}_{S,cm}^{ext} + \vec{\tau}_{cm}^{ext}$$

This result is linked to another result derived previously.

$$\vec{L}_{S,sys} = \vec{L}_{S,cm} + \vec{L}_{cm}$$

$$= \vec{L}_S^{orbital} + \vec{L}_S^{spin}$$

$$= \vec{r}_{S,cm} \times \vec{p}_{sys} + \sum_i^N \vec{r}_{cm,i} \times m_i \vec{v}_{cm,i}$$

we also know that

$$\vec{\tau}_S^{ext} = \frac{d\vec{L}_{S,sys}}{dt}$$

$$\Rightarrow \vec{\tau}_{S,cm}^{ext} + \vec{\tau}_{cm}^{ext} = \frac{d\vec{L}_S^{orbital}}{dt} + \frac{d\vec{L}_S^{spin}}{dt}$$

$$\frac{d\vec{L}_S^{orbital}}{dt} = \frac{d}{dt}\vec{r}_{S,cm} \times \vec{p}_{sys} + \vec{r}_{S,cm} \times \frac{d\vec{p}_{sys}}{dt}$$

$$= \cancel{\vec{v}_{S,cm} \times m_{sys} \vec{v}_{S,cm}} \rightarrow 0$$

$$+ \vec{r}_{S,cm} \times m_{sys} \vec{a}_{cm}$$

$$= \vec{r}_{S,cm} \times \vec{F}^{ext} = \vec{\tau}_{S,cm}^{ext}$$

The time derivative of orbital angular momentum about S is the external torque about S if the system were a point particle at CM.

$$\frac{d\vec{L}_S^{spin}}{dt} = \cancel{\sum \vec{v}_{cm,i} \times m \vec{v}_{cm,i}} \rightarrow 0$$

$$+ \sum \vec{r}_{cm,i} \times m_i \vec{a}_{cm,i}$$

$$= \vec{\tau}_{cm}^{ext}$$

21.5 Work-Energy Theorem

We've previously derived a result for a system of particles.

$$K = \frac{\sum m_i v_{S,i}^2}{2} = K_{S,cm} + K_{cm}$$

Considering a rigid body as system of particles

$$K = K_{trans} + K_{rot}$$

Here is another derivation.

$$W_{trans}^{ext} = \int_i^f \vec{F}^{ext} \cdot d\vec{r} = \int_i^f \frac{dm\vec{v}_{cm}}{dt} \cdot d\vec{R}_{cm}$$

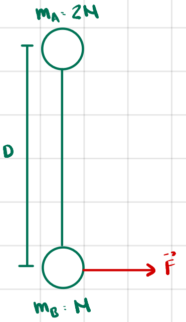
$$= m \int_i^f \frac{d\vec{v}_{cm}}{dt} \cdot \vec{v}_{cm}\, dt$$

$$= \frac{m}{2} \int_i^f d(\vec{v}_{cm} \cdot \vec{v}_{cm})$$

$$= \frac{m(v_{cm,f}^2 - v_{cm,i}^2)}{2}$$

$$= \Delta K_{trans}$$

Example 21.1 – Angular Impulse



Rigid body system

we know:

→ motion of rigid body can be decomposed into translation of CM about inertial frame and rotation about CM.

→ more specifically, for any infinitesimal dm_i on the rigid body

$$d\vec r_i = d\vec R_{cm} + d\vec r_{cm,i}$$

where $d\vec R_{cm}$ is translation of the CM and $d\vec r_{cm,i}$ is a rotational displacement.

$|\vec r_{cm,i}|$ is constant, $\vec r_{cm,i}\cdot\vec v_{cm,i} = 0$

$\vec\omega = \vec r_{cm,i}\times\vec v_{cm,i}$

Also

$$\vec\tau^{ext}_{S,sys} = \vec\tau^{ext}_{S,cm} + \vec\tau^{ext}_{cm}$$

$$\vec L_{S,sys} = \vec L_{S,cm} + \vec L_{cm}$$

→ we also have 2nd law equation for the system

$$\vec F^{ext} = m_T\cdot\vec a_{cm}$$

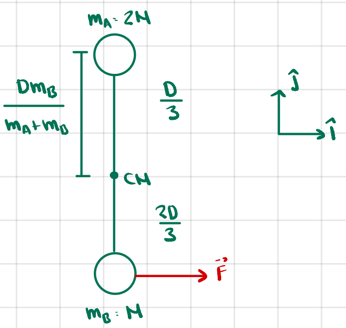
→ $\vec F$ applies an impulse on the system, changing its linear momentum. It also applies an angular impulse, changing angular momentum about the CM.

Find CM position:

$$\frac{m_A(y_A\hat j) + m_B(-(D-y_A))\hat j}{m_A+m_B} = 0$$

$$\Rightarrow m_A y_A - D m_B + m_B y_A = 0$$

$$\Rightarrow y_A = \frac{D m_B}{m_A+m_B} = \frac{D\cdot M}{3M} = \frac{D}{3}$$



$$\vec F^{ext} = F\hat i = m_T\cdot\vec a_{cm}$$

$$\int_{t_i}^{t_f} F\hat i\,dt = m_T(\vec V_{cm,f} - \vec V_{cm,i})$$

$$F\Delta t\,\hat i = m_T\vec V_{cm,f}$$

$$\vec V_{cm,f} = \frac{F\Delta t}{m_A+m_B}\hat i$$

$$\vec L_{cm,sys} = \vec L_{cm,cm} + \vec L_{cm}$$

$$= 0 + r_A\hat j\times m_A\cdot(\omega\hat k\times r_A\hat j)$$

$$+ r_B\hat j\times m_B(\omega\hat k\times r_B\hat j)$$

$$= r_A\hat j\times m_A r_A\omega(-\hat i) + r_B\hat j\times m_B r_B\omega(-\hat i)$$

$$= m_A r_A^2\omega\hat k + m_B r_B^2\omega\hat k$$

$$= I_{cm,A}\vec\omega + I_{cm_B}\vec\omega$$

$$= \left(\frac{m_A D^2 m_B^2\omega}{(m_A+m_B)^2} + \frac{m_B D^2 m_A^2\omega}{(m_A+m_B)^2}\right)\hat k$$

$$= \left(\frac{2MD^2M^2\omega}{9M^2} + \frac{MD^2 4M^2\omega}{9M^2}\right)\hat k$$

$$\Rightarrow \vec L_{cm,sys} = \frac{2D^2\omega M}{3}\hat k$$

$$\vec\tau_{cm,sys} = \cancel{\vec\tau_{cm,cm}}^{\,0} + \vec\tau_{cm,sys}$$

$$= r_B\hat j\times F\hat i = -r_B F\hat k$$

$$= \frac{2DF}{3}\hat k$$

$$\int_{t_i}^{t_f}\frac{2DF}{3}\hat k\,dt = \frac{2DF\Delta t}{3}$$

$$\frac{\cancel{2}\cancel{D}F\Delta t}{\cancel{3}} = \frac{\cancel{2}D^{\cancel{2}}\omega M}{\cancel{3}}$$

$$\omega = \frac{F\Delta t}{DM}$$

Interpretation

In this calculation we used the CM frame.

Torque about CM in CM frame is $2DF\hat k/3$, which acted for Δt applying an angular impulse $\frac{2DF\Delta t\hat k}{3}$ increasing ang. momentum from 0 to $2D^2\omega M\hat k/3$.

$\Rightarrow \omega = F\Delta t/DM$

If instead we do calculations from m_B frame.

$$\vec L_{B,sys} = \vec L_B^{orb} + \vec L_B^{spin}$$

$$= \frac{2D}{3}\hat j\times 3M\vec V_{cm}$$

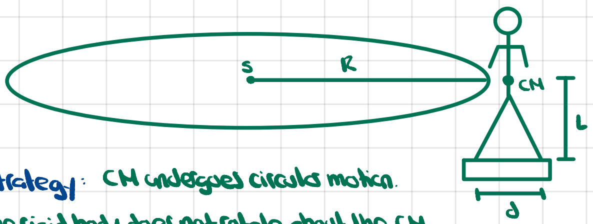
$$+ 2M\cdot\frac{D^2}{9}\omega\hat k + M\cdot\frac{4D^2}{9}\omega\hat k$$

$$= 2D\hat j\times M\frac{F\Delta t}{3M}\hat i + \frac{2MD^2}{3}\omega\hat k$$

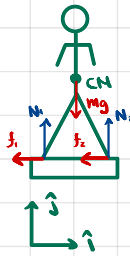
equate to 0 because there is no torque rel. to B so $\Delta\vec L_B = \vec 0$

$$\Rightarrow \frac{\cancel{2}\cancel{D}F\Delta t}{\cancel{3}} = \frac{\cancel{2}MD^{\cancel{2}}\omega}{\cancel{3}} \Rightarrow \omega = \frac{F\Delta t}{MD}$$

Example 21.2 - Person on Railroad Car Moving in Circle



Strategy: CM undergoes circular motion.
The rigid body does not rotate about the CM.
There is no angular acceleration of CM, so there is no torque at CM rel. to curve center, and $\Delta \vec{L}_{S,cm} = 0$.
There are torques about the CM, but the net $\vec{\tau}_{cm}$ is zero because there is also no angular acceleration (or velocity).
$\vec{\tau}_{cm} = \vec{0}$ is one equation involving N_1, N_2.
2nd law in $\hat{j}$ direction relates N_1, N_2 to mg.



$$\vec{\tau}_{cm} = \left(\frac{d}{2}\hat{i} - L\hat{j}\right) \times \left(N_2\hat{j} - \mu N_2 \hat{i}\right) + \left(-\frac{d}{2}\hat{i} - L\hat{j}\right) \times \left(N_1\hat{j} - \mu N_1 \hat{i}\right)$$

$$= \frac{dN_2}{2}\hat{k} - L\mu N_2 \hat{k} - \frac{dN_1}{2}\hat{k} - L\mu N_1 \hat{k}$$

$$= \hat{k}\left(\frac{d}{2}(N_2 - N_1) - L\mu(N_1 + N_2)\right) = \vec{0}$$

$$\frac{d}{2}(N_2 - N_1) = L\mu(N_2 + N_1)$$

$$N_1 + N_2 - mg = 0 \Rightarrow N_1 + N_2 = mg$$

$$\vec{\tau}_{S,cm} = R\hat{i} \times \left(-mg\hat{j} + N_1\hat{j} + N_2\hat{j}\right) = \vec{0}$$

$$\Rightarrow (-Rmg + RN_1 + RN_2)\hat{k} = 0$$

$$\Rightarrow N_1 + N_2 = mg$$

Friction at the feet causes radial acceleration.

$$-(f_1 + f_2)\hat{r} = -R\omega^2\hat{r}$$

$$\Rightarrow \mu(N_1 + N_2) = R\omega^2$$

$\vec{L}_{cm,sys} = \vec{0}$ because the person is at rest in CM frame.

$$\vec{L}_{S,sys} = \vec{L}_{S,cm} = I_{S,cm}\,\omega\hat{k} = mR^2\omega\hat{k}$$

$$\frac{d}{2}(N_2 - N_1) = L\mu(N_2 + N_1)$$

$$N_1 + N_2 = mg$$

$$\mu(N_1 + N_2) = R\omega^2$$

$$\mu mg = R\omega^2$$

$$\omega = \sqrt{\frac{\mu mg}{R}}$$

$$\frac{d}{2}(N_2 - mg + N_2) = L\mu mg$$

$$dN_2 - \frac{dmg}{2} = L\mu mg$$

$$N_2 = \frac{dmg + 2L\mu mg}{2d}$$

$$\Rightarrow N_2 = \frac{mg(d + 2L\mu)}{2d} = \frac{mg}{2} + \frac{mgL\mu}{d}$$

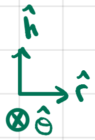
$$= \frac{mg}{2} + \frac{RL\omega^2}{d} = \frac{mg}{2} + \frac{Lv^2}{Rd}$$

$$N_1 = \frac{2dmg - mgd - mg2L\mu}{2d}$$

$$\Rightarrow N_1 = \frac{mg(d - 2L\mu)}{2d} = \frac{mg}{2} - \frac{mgL\mu}{d}$$

Textbook Solution

use cylindrical coordinates



use two dynamical equations of motion

$$\vec F_{ext} = \frac{d\vec p_{sys}}{dt} = \frac{d}{dt}(m_T \vec V_{cm}) = m\vec a_{cm}$$

$$\vec\tau_{cm,z} = I_{cm}\vec\alpha_{cm,z}$$

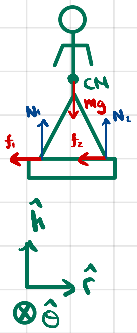
$$\vec\alpha_{cm,z} = \vec 0$$

2nd law $\hat r$ dir

$$-(f_1+f_2)\hat r = -mR\omega^2\hat r = -m\frac{v^2}{R}\hat r$$

$$\Rightarrow f_1+f_2 = \frac{mv^2}{R}$$

2nd law $\hat h$ $\quad N_1+N_2=mg$



rotational eq. of motion

$$\vec\tau_{cm,foot_1} = \vec 0$$

$$= \left(-\frac{d}{2}\hat r - L\hat k\right)\times\left(-f_1\hat r + N_1\hat h\right)$$

$$= -\frac{d}{2}N_1(-\hat\theta) + Lf_1\hat\theta$$

$$= \left(Lf_1 + \frac{dN_1}{2}\right)\hat\theta$$

$$\vec\tau_{cm,foot_2} = \left(\frac{d}{2}\hat r - L\hat h\right)\times\left(-f_2\hat r + N_2\hat h\right)$$

$$= \frac{d}{2}N_2(-\hat\theta) + Lf_2\hat\theta$$

$$= \hat\theta\left(Lf_2 - \frac{N_2 d}{2}\right)$$

$$\vec\tau_{cm,f_1} + \vec\tau_{cm,f_2} = \vec 0$$

$$\Rightarrow Lf_1 + \frac{N_1 d}{2} + Lf_2 - \frac{N_2 d}{2} = 0$$

Equations

Torque: $Lf_1 + \frac{N_1 d}{2} + Lf_2 - \frac{N_2 d}{2} = 0$

2nd law: $f_1+f_2 = \frac{mv^2}{R}$

$$N_1+N_2 = mg$$

unknowns: N_1, N_2, f_1, f_2

$$L(f_1+f_2) = \frac{d}{2}(N_2-N_1)$$

$$\frac{Lmv^2}{R} = \frac{d}{2}(N_2 - mg + N_2)$$

$$dN_2 = \frac{Lmv^2}{R} + \frac{mgd}{2}$$

$$N_2 = \frac{mg}{2} + \frac{Lmv^2}{Rd}$$

$$N_1 = \frac{mg}{2} - \frac{Lmv^2}{Rd}$$

$$N_1 = 0 \Rightarrow \frac{Lmv_{max}^2}{Rd} = \frac{mg}{2}$$

$$\Rightarrow v_{max} = \sqrt{\frac{Rgd}{2L}}$$

$$\Rightarrow N_2 = \frac{mg}{2} + \frac{\cancel{L}m}{\cancel{R}\cancel{d}}\cdot\frac{\cancel{R}g\cancel{d}}{2\cancel{L}}$$

$$= mg$$

What happens if $v > v_{max}$?

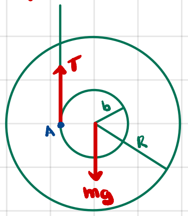
N_1 cannot be < 0 so $N_{2_{max}} = mg$.

The torque eq. doesn't balance out anymore.

$$L(f_1+f_2) - \frac{mgd}{2} > 0$$

$$\Rightarrow \vec\alpha_{cm} > 0$$

Example 31.3 Torque, Rotation, and Translation



$\otimes\hat{k}$ $\rightarrow \hat{i}$ $\downarrow \hat{j}$

Tension applies a torque about CM. This changes angular momentum.

Alternatively mg applies torque about A changing ang. momentum.

mg − T accelerates the CM.

$\vec{\tau}_{cm,sys} = \cancel{\vec{\tau}_{cm,cm}}^{\,0} + \vec{\tau}_{cm,sys}$

$= (-b\hat{i}) \times (-T\hat{j}) = bT\hat{k} = I_{cm}\alpha\,\hat{k}$

$\Rightarrow bT = \frac{mR^2}{2} \cdot \alpha$

$mg - T = m \cdot a \Rightarrow T = m(g-a)$

$a = b\alpha$

Three equations, T, a, α unknowns

$T = m(g - b\alpha)$

$\Rightarrow b\cancel{m}(g - b\alpha) = \frac{\cancel{m}R^2\alpha}{2}$

$bg - b^2\alpha = \frac{R^2\alpha}{2}$

$\Rightarrow \alpha\left(\frac{R^2}{2} + b^2\right) = bg$

$\alpha = \frac{2bg}{R^2 + 2b^2}$

$a = \frac{2b^2 g}{R^2 + 2b^2}$

$T = m\left(\frac{g(R^2 + 2b^2 - 2b^2)}{R^2 + 2b^2}\right)$

$= \frac{mgR^2}{R^2 + 2b^2}$

After a length ℓ has been unwound

$v = \frac{2b^2 gt}{R^2 + 2b^2}$

$y = \frac{b^2 g t^2}{R^2 + 2b^2}$

$t_\ell = \sqrt{\frac{\ell(R^2 + 2b^2)}{b^2 g}}$

$v_\ell = 2\sqrt{\frac{\ell b^2 g}{R^2 + 2b^2}}$

$\omega = \frac{2}{\cancel{b}}\sqrt{\frac{\ell \cancel{b^2} g}{R^2 + 2b^2}} = \sqrt{\frac{4g\ell}{R^2 + 2b^2}}$

we could also have integrated α

$\omega = \frac{2bgt}{R(R + 2b)}$

and subbed in t_ℓ

Alternatively we solve using energy

$E_i = mg\ell$

$E_f = \frac{mv_f^2}{2} + \frac{mR^2}{2} \cdot \frac{v_f^2}{b^2} \cdot \frac{1}{2}$

no nonconservative forces doing work

$\Rightarrow \Delta E = 0$

$v_f^2 + \frac{R^2 v_f^2}{2b^2} = 2g\ell$

$v_f^2\left(\frac{2b^2 + R^2}{2b^2}\right) = 2g\ell$

$v_f = \sqrt{\frac{4b^2 g\ell}{R^2 + 2b^2}}$

$\Rightarrow \omega_f = \frac{v_f}{b} = \sqrt{\frac{4g\ell}{R^2 + 2b^2}}$

$\int_0^{t_\ell} \left(mg - \frac{mgR^2}{R^2 + 2b^2}\right) dt$

$= \int_0^{t_\ell} \frac{mg \cdot 2b^2}{R^2 + 2b^2}\, dt$

$= \frac{mg \cdot 2b^2}{R^2 + 2b^2}\sqrt{\frac{\ell(R^2 + 2b^2)}{b^2 g}}$

$= \sqrt{\frac{4m^2 g b^2 \ell}{R^2 + 2b^2}} = \Delta p_y$

$p_{y_f} = m \cdot 2 \cdot \sqrt{\frac{\ell b^2 g}{R^2 + 2b^2}}$

$= \sqrt{\frac{4m^2 g b^2 \ell}{R^2 + 2b^2}}$

$\int_0^{t_\ell} b\, \frac{mgR^2}{R^2 + 2b^2}\, dt$

$= \frac{\cancel{b}mgR^2}{R^2 + 2b^2}\sqrt{\frac{\ell(R^2 + 2b^2)}{\cancel{b^2}g}}$

$= \sqrt{\frac{m^2 g R^4 \ell}{R^2 + 2b^2}} = \Delta L_{cm}$

$\vec{L}_{cm,sys} = \frac{mR^2}{\cancel{2}}\sqrt{\frac{\cancel{4}g\ell}{R^2 + 2b^2}}$

$= \sqrt{\frac{m^2 g R^4 \ell}{R^2 + 2b^2}}$

Recap of Steps

draw forces and possibly torque diagram

write equation for torque of system about a chosen point

$\vec{\tau}_{s,sys} = \vec{\tau}_{s,cm} + \vec{\tau}_{cm}$

write 2nd law equations for the CM.

$\vec{F}^{ext} = m_T \cdot \vec{a}_{cm}$

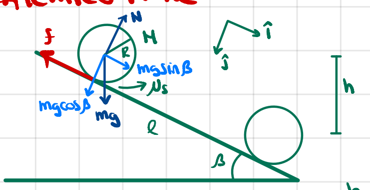
use constraint equations such as $\vec{a}_{cm} = b\vec{\alpha}$

solve for unknown forces, acceleration and angular accel.

Kinematics can now be used to insert time into the analysis. We can now find velocity, position, angular velocity, angular displacement, and find the time it takes to reach specified values of these variables.

In the case of speed and displacement, we can use the work energy theorem.

Ex 21.4 – Cylinder Rolling Down Inclined Plane



$\sin\beta = \frac{h}{l}$

Solution 1

$\vec{\tau}_{cm,s/s} = R\hat{j}\times(-f\hat{i}) = Rf\hat{k}$

$\Rightarrow Rf = \frac{mR^2}{2}\alpha$ unknowns: f, N, α, a_{cm}

$mg\cos\beta = N$

$mg\sin\beta - f = m\cdot a_{cm}$

$a_{cm} = R\alpha$

$f = \frac{mR}{2}\cdot\frac{a}{R} = \frac{ma}{2}$

$\cancel{m}g\sin\beta = \frac{3\cancel{m}a_{cm}}{2}$

$a_{cm} = \frac{2}{3}g\sin\beta$

$f = \frac{1}{3}mg\sin\beta$

$\alpha = \frac{2g\sin\beta}{3R}$

$V_{cm} = \frac{2g\sin\beta\cdot t}{3}$

$x_{cm} = \frac{g\sin\beta\, t^2}{3} = l$

$\Rightarrow t_l = \sqrt{\frac{3l}{g\sin\beta}}$

$V_{cm_l} = \frac{2g\sin\beta}{3}\sqrt{\frac{3l}{g\sin\beta}}$

$= \sqrt{\frac{4lg\sin\beta}{3}} = \sqrt{\frac{4gh}{3}}$

V_{cm} changes due to $F_{gx} - f$.
Rotation occurs due to $\tau_{cm,f}$.
The no-slip condition is $V_{cm} = R\omega$, which are assumed true when we wrote $a_{cm} = R\alpha$.

we reached $f = \frac{1}{3}mg\sin\beta$ for the equations to balance w/ no slip.
Experimentally we have another eq

$F_{max} = \mu N = \mu mg\cos\beta$

$\Rightarrow \frac{\cancel{m}\cancel{g}\sin\beta}{3} \leq \mu\cancel{m}\cancel{g}\cos\beta$

$\Rightarrow \tan\beta \leq 3\mu_s$ ie $\mu_s \geq \frac{\tan\beta}{3}$

Solution 2: Energy Methods

Work by non-conservative friction equals change in mechanic energy.
The motion of the rigid body is a CM translation and a rotation about the CM.

$W_T = \int_{x_i}^{x_f}\left(mg\sin\beta - \frac{mg\sin\beta}{3}\right)dx$

$= \frac{2mg\sin\beta}{3}$

$E_{m_i} = mgh$ gravitational potential energy

$E_{m_f} = \frac{mv^2}{2} + \frac{m\cancel{R^2}}{2}\cdot\frac{v^2}{\cancel{R^2}}\cdot\frac{1}{2}$

translational kinetic energy

rotational kinetic energy

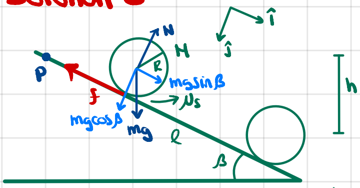
$W_f = 0$, because the contact point is at rest relative to the surface.

$\Delta E_m = W_f = 0$

$\frac{\cancel{m}v^2}{2} + \frac{\cancel{m}v^2}{4} - \cancel{m}gh = 0$

$\frac{3v^2}{4} = gh \Rightarrow v = \sqrt{\frac{4gh}{3}}$

Solution 3



$\sin\beta = \frac{h}{l}$

Torque about P

$\vec\tau_{P,mg} = \vec\tau_{P,cm,mg} + \cancel{\vec\tau_{cm,mg}}^{\,0}$

$= \vec r_{P,cm}\times\vec F_g$

$= (x_{P,cm}\hat i - y_{P,cm}\hat j)\times mg(\sin\beta\,\hat i + \cos\beta\,\hat j)$

$= (x_{P,cm}\cos\beta + R\sin\beta)mg\,\hat k$

$\vec\tau_{P,f} = \vec\tau_{P,cm,f} + \vec\tau_{cm,f}$

$= (x_{P,cm}\hat i - R\hat j)\times(-f\hat i) + R\hat j\times(-f\hat i)$

$= -Rf\hat k + Rf\hat k = \vec 0$

$\vec\tau_{P,N} = (x_{P,cm}\hat i - R\hat j)\times(-N\hat j) + R\hat j\times N\hat j$

$= -Nx_{P,cm}\hat k$

$\vec\tau_{P,s/s} = \vec\tau_{P,N} + \vec\tau_{P,f} + \vec\tau_{P,F_g}$

$= [(x_{P,cm}\cos\beta + R\sin\beta)mg - mg\cos\beta\cdot x_{P,cm}]\hat k$

$= Rmg\sin\beta\,\hat k$

$\vec L_{P,s/s} = \vec L_{P,cm} + \vec L_{cm,s/s}$

$= (x_{P,cm}\hat i - R\hat j)\times mV_{cm}\hat i + \frac{mR^2}{2}\cdot\omega\hat k$

$= \left(RmV_{cm} + \frac{mR^2\omega}{2}\right)\hat k$

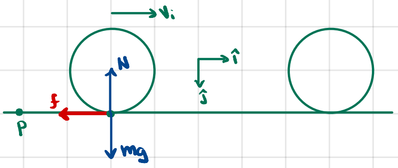
$\frac{d\vec L_{P,s/s}}{dt} = \left(Rma_{cm} + \frac{mR^2\alpha}{2}\right)\hat k$

$\cancel{Rm}a_{cm} + \frac{\cancel{mR}a_{cm}}{2} = \cancel{Rm}g\sin\beta$

$\frac{3}{2}a_{cm} = g\sin\beta$

$\Rightarrow a_{cm} = \frac{2g\sin\beta}{3}$

Example 21.5 - Bowling Ball



If there is friction there is torque about CM and hence rotation and change in angular momentum.

Slipping means $v_{cm} > R\omega$

Friction also generates negative acceleration. At some point v_{cm} equals $R\omega$ and there is no slipping.

$\vec{\tau}_{P,f} = \vec{r}_{P,f} \times \vec{f} = \vec{0}$

$\vec{\tau}_{P,mg} = \vec{\tau}_{P,cm,mg} + \vec{\tau}_{cm,mg}$

$= (x_{P,cm}\hat{i} - R\hat{j}) \times mg\hat{j} + \vec{0}$

$= x_{P,cm} mg \hat{k}$

$\vec{\tau}_{P,N} = \vec{\tau}_{P,cm,N} + \vec{\tau}_{cm,N}$

$= (x_{P,cm}\hat{i} - R\hat{j}) \times (-N\hat{j}) + R\hat{j} \times (-N\hat{j})$

$= -N x_{P,cm} \hat{k}$

$\Rightarrow \vec{\tau}_{P,sys} = (x_{P,cm} mg - x_{P,cm} N)\hat{k}$

2nd law:

$N = mg$

$\Rightarrow \vec{\tau}_{P,sys} = x_{P,cm}(mg - N)\hat{k} = \vec{0}$

No torque about any fixed point along contact surface.

$\Rightarrow \vec{L}_{P,i} = \vec{L}_{P,f}$

$\vec{L}_{P,i} = \vec{L}_{P,cm} + \vec{L}_{cm,sys}$

$= (x_{P,cm}\hat{i} - R\hat{j}) \times m v_{cm,i} + I_{cm}\cancel{\omega_i}^{0}$

$= v_{cm,i} Rm \hat{k}$

$\vec{L}_{P,f} = (x_{P,cm}\hat{i} - R\hat{j}) \times m v_{cm,f}\hat{i} + I_{cm}\omega_f \hat{k}$

$v_{cm,f} = \omega_f \cdot R$

$I_{cm} = \frac{2mR^2}{5}$

$\Rightarrow \vec{L}_{P,f} = v_{cm,f} Rm \hat{k} + I_{cm}\omega_f \hat{k}$

$= (\omega_f R^2 m + \frac{2mR^2\omega_f}{5})\hat{k}$

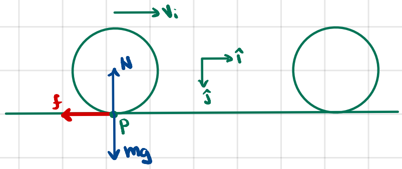
$= \frac{7\omega_f mR^2}{5}\hat{k}$

$\vec{L}_{P,i} = \vec{L}_{P,f} \Rightarrow \cancel{m}\cancel{R} v_{cm,i} = \frac{7\omega_f \cancel{m}R^{\cancel{2}}}{5}$

$\Rightarrow \omega_f = \frac{5v_{cm,i}}{7R}$

$v_{cm,f} = \frac{5v_{cm,i}}{7}$

Can p be the moving contact point?



$\vec{\tau}_{P,sys} = \vec{\tau}_{P,cm} + \vec{\tau}_{cm,sys}$

$\vec{\tau}_{P,sys,N} = -R\hat{j} \times (-N\hat{j}) + R\hat{j} \times (-N\hat{j})$

$= \vec{0}$

$\vec{\tau}_{P,sys,f_k} = -R\hat{j} \times (-f_k\hat{i}) + R\hat{j} \times (-f_k\hat{i})$

$= -Rf_k\hat{k} + Rf_k\hat{k} = \vec{0}$

$\vec{\tau}_{P,sys,mg} = -R\hat{j} \times mg\hat{j} + \vec{0}$

$= \vec{0}$

$\Rightarrow \vec{\tau}_{P,sys} = \vec{0}$

$\vec{L}_{P,sys,i} = \vec{L}_{P,cm,i} + \vec{L}_{cm,sys,i}$

$= (-R\hat{j}) \times m v_{cm,i}\hat{i} + I_{cm}\cancel{\omega_{cm,i}}^{0}\hat{k}$

$= Rmv_{cm,i}\hat{k}$

$\vec{L}_{P,sys,f} = -R\hat{j} \times m v_{cm,f}\hat{i} + I_{cm}\omega_{cm,f}\hat{k}$

$= (Rmv_{cm,f} + I_{cm}\omega_{cm,f})\hat{k}$

$\Delta \vec{L}_{P,sys} = 0$

$\Rightarrow Rmv_{cm,i} = Rmv_{cm,f} + I_{cm}\omega_{cm,f}$

In the final state, no slipping

$\Rightarrow v_{cm,f} = R\omega_{cm,f}$

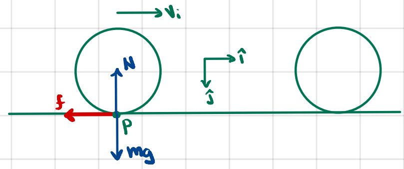
Also, $I_{cm} = \frac{2mR^2}{5}$

$\Rightarrow \cancel{R}\cancel{m} v_{cm,i} = Rmv_{cm,f} + \frac{2mR^{\cancel{2}} v_{cm,f}}{5\cancel{R}}$

$= \frac{7\cancel{R}\cancel{m} v_{cm,f}}{5}$

$\Rightarrow v_{cm,f} = \frac{5}{7}v_{cm,i}$

$\omega_{cm,f} = \frac{5v_{cm,i}}{7R}$



$\vec{\tau}_{cm,sys} = R\hat{j} \times (-f\hat{i}) = Rf\hat{k} = I_{cm}\alpha\hat{k}$

about CM system torque is now not zero anymore

$-f\hat{i} = m a_{cm}\hat{i}$

$a_{cm}\hat{i} = -\frac{f}{m}\hat{i}$ the ball is decelerating due to kinetic friction while having accelerating angular speed, ie positive angular acceleration, due to torque applied by the same kinetic friction. v_{cm} is thus decreasing and ω is increasing. At some point v_{cm} coincides with $R\omega$. At that point there is no f anymore.

$\alpha = \frac{Rf}{I_{cm}} \Rightarrow \omega = \frac{Rft}{I_{cm}}$

$v_{cm} = v_{cm,i} - \frac{ft}{m}$

When the ball stops slipping, the contact point on the ball is at rest rel. to surface. There is no longer kinetic friction and static friction is zero.

$v_{cm}(t_f) = v_{cm,i} - \frac{ft_f}{m} = R\omega(t_f) = \frac{R^2 f t_f}{I_{cm}}$

$t_f f\left(\frac{R^2}{I_{cm}} + \frac{1}{m}\right) = v_{cm,i}$

$t_f = \frac{v_{cm,i}}{f} \cdot \frac{I_{cm} m}{R^2 m + I_{cm}}$

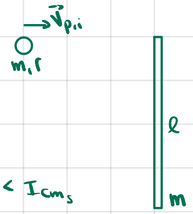
$v_{cm,f} = v_{cm,i} - \frac{v_{cm,i} I_{cm}}{R^2 m + I_{cm}}$

$= \frac{v_{cm,i} R^2 m}{R^2 m + I_{cm}} = \frac{v_{cm,i} R^2 m}{\frac{7}{5} mR^2}$

$= \frac{5}{7} v_{cm,i}$

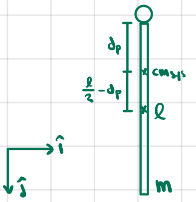
$\omega_{cm,f} = \frac{5v_{cm,i}}{7R}$

Example 21.6 Rotation and Translation object and Stick Collision



$r << \ell$

$\Rightarrow I_{cm_p} << I_{cm_s}$



a) center of mass

$$\frac{-md_p\hat{j} + (\frac{\ell}{2} - d_p)m}{2m} = 0$$

$$\Rightarrow 2md_p = \frac{m\ell}{2}$$

$$d_p = \frac{\ell}{4}$$

b) linear velocity of system

no external forces $\Rightarrow \Delta p = 0$

$\vec{p}_i = mv_{p,i}\hat{i}$

$\vec{p}_f = 2m\vec{v}_{cm,f}$

$$\Rightarrow \vec{v}_{cm,f} = \frac{v_{p,i}}{2}\hat{i}$$

c) mechanical energy

$$K_i = \frac{mv_{p,i}^2}{2}$$

$$K_f = \frac{mv_{p,i}^2}{8} + \frac{I_{cm,sys}\,\omega^2}{2}$$

$$= \frac{mv_{p,i}^2}{8}$$

$$+ \left(I_{cm} + \frac{m\ell^2}{8}\right)\left[\frac{2\ell m}{8I_{cm} + m\ell^2}\right]^2 v_{p,i}^2 \cdot \frac{1}{2}$$

$$= \frac{mv_{p,i}^2}{2} \cdot \frac{6m\ell^2 + 8I_{cm}}{8m\ell^2 + 64I_{cm}} < \frac{mv_{p,i}^2}{2}$$

The collision is totally inelastic.

Em not conserved.

d) angular momentum

$$\vec{L}_{cm,sys,i} = -\frac{\ell}{4}\hat{j} \times mv_{p,i}\hat{i} = \frac{\ell m v_{p,i}}{4}\hat{k}$$

$$\vec{L}_{cm,sys,f} = \frac{\ell m v_{p,f}}{4}\hat{k} + I_{cm_s}\omega\hat{k}$$

$$I_{cm_s} = I_{cm} + m\frac{\ell^2}{16}$$

$$v_{p,f} = \frac{\ell}{4}\omega$$

$$\Rightarrow \vec{L}_{cm,sys,f} = \left[\frac{\ell^2 m\omega}{16} + \left(I_{cm} + \frac{m\ell^2}{16}\right)\omega\right]\hat{k}$$

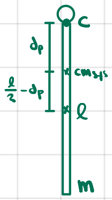
$$\Delta\vec{L} = 0 \Rightarrow \frac{\ell m v_{p,i}}{4} = \omega I_{cm} + \frac{m\ell^2}{8}\omega$$

$$\Rightarrow 2\ell m v_{p,i} = 8I_{cm}\omega + m\ell^2\omega$$

$$\Rightarrow \omega = \frac{2\ell m}{8I_{cm} + m\ell^2} v_{p,i}$$

$$v_{p,f} = \frac{\ell^2 m}{16I_{cm} + 2m\ell^2}$$

Repeat angular momentum calc about point C



$$\vec{L}_{C,sys,i} = \vec{0}$$

$$\vec{L}_{C,sys,f} = -\frac{\ell}{4}\hat{j} \times 2m\vec{v}_{c,cm,f} + \vec{L}_{cm,sys,f}$$

$$= -\frac{\ell}{4}\hat{j} \times 2m\vec{v}_{c,cm,f} + \omega\left(I_{cm} + \frac{m\ell^2}{8}\right)\hat{k}$$

$$\vec{v}_{g,cm,f} = \vec{v}_{g,c,f} + \vec{v}_{c,cm,f}$$

$$\vec{v}_{c,cm,f} = \frac{v_{p,i}}{2}\hat{i} - \frac{\ell}{4}\omega_{cm}\hat{i}$$

$$\vec{L}_{c,sys,f} = \frac{\ell}{4}\hat{j} \times 2m\left(\frac{v_{p,i}}{2}\hat{i} - \frac{\ell}{4}\omega_{cm}\hat{i}\right)$$
$$+ \omega\left(I_{cm} + \frac{m\ell^2}{8}\right)\hat{k}$$

$$= -\frac{\cancel{2}m\ell v_{p,i}}{\cancel{8}\,4}\hat{k} + \frac{\cancel{2}m\ell^2\omega_{cm}}{\cancel{16}\,8}\hat{k}$$
$$+ \omega\left(I_{cm} + \frac{m\ell^2}{8}\right)\hat{k}$$

$$-\frac{m\ell v_{p,i}}{4} + \frac{m\ell^2\omega_{cm}}{8} + \omega_{cm}I_{cm} = 0$$

$$\omega(m\ell^2 + 8I_{cm}) = 2m\ell v_{p,i}$$

$$\omega = \frac{2m\ell v_{p,i}}{m\ell^2 + 8I_{cm}}$$

e) The CM has $\vec{v}_{g,cm,f} = \frac{v_{p,i}}{2}\hat{i}$

The stick rotates at $\omega = \frac{2\ell m}{8I_{cm} + m\ell^2} v_{p,i}$

$$\vec{x}_{g,cm}(t) = \frac{v_{p,i}t}{2}$$

$$\theta(t) = \frac{2\ell m v_{p,i}t}{8I_{cm} + m\ell^2}$$

$$\theta = 2\pi \Rightarrow t_1 = \frac{\pi(8I_{cm} + m\ell^2)}{\ell m v_{p,i}}$$

$$x_{g,cm}(t_1) = \frac{\pi(8I_{cm} + m\ell^2)}{2\ell m}$$