

# Pset 9 - Collision Theory

## Problem 1 - Ping Pong Ball and Bowling Ball Collision

elastic collision

frictionless surface



$$v_{p,b,i}^{\text{rel}} = -v_{p,b,f}^{\text{rel}}$$

$$v_{p,o} + v_{b,o} = v_{p,f} - v_{b,f} \quad (\text{energy-momentum principle})$$

$$m_p v_{p,o} + m_b v_{b,o} = m_p v_{p,f} + m_b v_{b,f} \quad (\text{conservation of momentum})$$

$$= m_p (v_{p,o} + v_{b,o} + v_{b,f}) + m_b v_{b,f}$$

$$v_{b,f} (m_b + m_p) = \cancel{m_p v_{p,o} + m_b v_{b,o}} - \cancel{m_p (v_{p,o} + v_{b,o})}$$

$$v_{b,f} = \frac{v_{b,o} (m_b - m_p)}{m_p + m_b} \Rightarrow v_{b,f} = \frac{m_b - m_p}{m_b + m_p} v_{b,o}$$

$$v_{p,f} = \frac{m_b - m_p}{m_b + m_p} v_{b,o} + v_{b,o} + v_{p,o}$$

$$= \frac{(m_b - m_p) v_{b,o} + (m_b + m_p) v_{b,o}}{m_b + m_p} + v_{p,o}$$

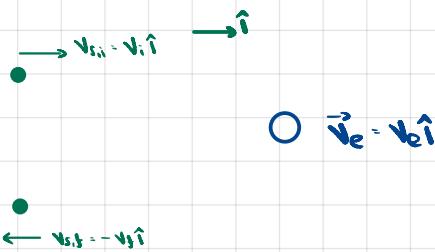
$$v_{p,f} = \frac{2m_b}{m_b + m_p} v_{b,o} + v_{p,o}$$

### Note

$$\lim_{m_b \rightarrow \infty} v_{p,f} = 2v_{b,o} + v_{p,o}$$

$$\lim_{m_b \rightarrow \infty} v_{b,f} = v_{b,o}$$

## Problem 2 - Satellite Flyby



Let's consider this an elastic collision.

$$v_{sat,i}^{rel} = -v_{sat,f}^{rel}$$

$$v_i - v_e = -(-v_f - v_e) \Rightarrow 3v_e = v_f + v_e \Rightarrow v_f = 2v_e$$

In this calculation we approximated the velocity of earth as having stayed constant in the collision. In reality, earth's speed must increase.

$$m_s v_i + m_e v_{e,i} = -m_s v_f + m_e v_{e,f} \quad (\text{conservation of momentum})$$

$$v_{e,f} = \frac{m_s(v_i + v_f) + m_e v_{e,i}}{m_e}$$

$$v_{e,f} = v_{e,i} + \frac{m_s}{m_e}(v_i + v_f) > v_{e,i} \quad \text{if } v_i, v_f > 0 \quad (1)$$

$$v_i - v_{e,i} = -(-v_f - v_{e,f}) = v_f + v_{e,f}$$

$$v_f = v_i - (v_{e,f} + v_{e,i}) \quad (\text{cm. momentum principle}) \quad (2)$$

unknowns:  $v_f, v_{e,f}$

(2)  $\rightarrow$  (1)

$$v_{e,f} = v_{e,i} + \frac{m_s}{m_e}(2v_i - v_{e,f} - v_{e,i})$$

$$v_{e,f}(1 + \frac{m_s}{m_e}) = v_{e,i}(1 - \frac{m_s}{m_e}) + 2v_i \frac{m_s}{m_e}$$

$$v_{e,f} \frac{m_e + m_s}{m_e} = v_{e,i} \frac{m_e - m_s}{m_e} + 2v_i \frac{m_s}{m_e}$$

$$v_{e,f} = \frac{m_e - m_s}{m_e + m_s} v_{e,i} + \frac{2m_s}{m_e + m_s} v_i$$

$$v_f = v_i - v_{e,i} - \left( \frac{m_e - m_s}{m_e + m_s} v_{e,i} + \frac{2m_s}{m_e + m_s} v_i \right)$$

$$= \frac{(m_e - m_s)v_i}{m_e + m_s} - \frac{v_{e,i}(m_e + m_s)}{m_e + m_s} - \frac{(m_e - m_s)v_{e,i}}{m_e + m_s}$$

$$= \frac{(m_e - m_s)v_i}{m_e + m_s} - \frac{2m_s v_{e,i}}{m_e + m_s}$$

$$\lim_{m_s \rightarrow 0} v_f = v_i - 2v_{e,i}$$

$$v_{e,f} = \frac{m_e - m_s}{m_e + m_s} v_{e,i} + \frac{2m_s v_i}{m_e + m_s}$$

$$\lim_{m_s \rightarrow 0} v_{e,f} = v_{e,i}$$

Note that

$$v_{e,f} = \frac{m_e v_{e,i} + m_s (2v_i - v_{e,i})}{m_e + m_s}$$

$$v_i = v_{e,i} + v_{e,f} + v_f$$

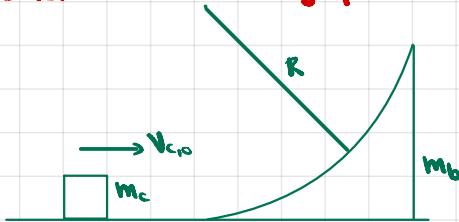
$$= v_{e,i} + v_{e,f}^{rel} \geq v_{e,i}$$

$$\Rightarrow 2v_i \geq 2v_{e,i}$$

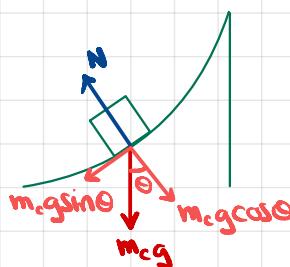
$$\Rightarrow 2v_i - v_{e,i} \geq v_{e,i}$$

$$\therefore v_{e,f} \geq v_{e,i}$$

### Problem 3 - Block Sliding Up Curved Surface



$$m_c < m_b$$



a) The cube starts a circular trajectory relative to the block at initial velocity  $v_{c,0}\hat{\theta}$

$$\vec{F}_g = m_c g \cos \theta \hat{r} - m_c g \sin \theta \hat{\theta}$$

$$\vec{r} = R\hat{r}$$

$$d\vec{r} = R d\hat{r} - R d\theta \hat{\theta}$$

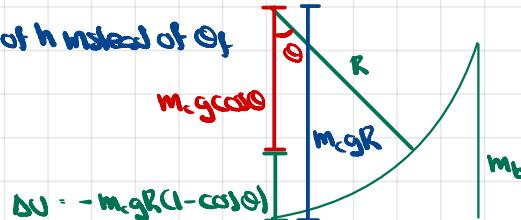
$$\vec{F}_g \cdot d\vec{r} = -m_c g \sin \theta R d\theta$$

$$\int \vec{F}_g \cdot d\vec{r} = -m_c g R \int \sin \theta d\theta = -m_c g R (-\cos \theta) \Big|_0^{\theta_f} \\ = m_c g R (\cos \theta_f - 1) < 0$$

System is block + cube.  $F_g$  is the only external force.

$$\Delta E_m = 0 \Rightarrow \Delta K + \Delta U = 0 \Rightarrow \frac{(m_b + m_c)v_i^2}{2} - \frac{m_c v_{c,0}^2}{2} + m_c g R (1 - \cos \theta_f) = 0$$

Note that we can write this in terms of  $h$  instead of  $\theta_f$ .



$$\Delta E_m = \frac{(m_b + m_c)v_i^2}{2} - \frac{m_c v_{c,0}^2}{2} + m_c g h = 0$$

Unknowns:  $v_f$ ,  $\theta_f$  or  $h$

Initial momentum:  $m_c v_{c,0} \hat{i}$

Final momentum:  $(m_c + m_b)v_f \hat{i}$

$$m_c v_{c,0} = (m_c + m_b)v_f \Rightarrow v_f = \frac{m_c}{m_c + m_b} v_{c,0}$$

Sub into  $\Delta E_m = 0$ , solve for  $\theta_f$  or  $h$

$$\Rightarrow h = \frac{v_{c,0}^2 \cdot m_b}{2g(m_b + m_c)}$$

$$\theta_f = \arccos \left[ \frac{2gR(m_b + m_c) - v_{c,0}^2 m_b}{2gR(m_b + m_c)} \right]$$

$$\hat{r} = \cos \theta \hat{i} + \sin \theta \hat{j}$$

$$\hat{\theta} = -\sin \theta \hat{i} + \cos \theta \hat{j}$$

$$\frac{d\hat{\theta}}{dt} = -\cos \theta \theta' \hat{i} - \sin \theta \theta' \hat{j} \\ = \theta'(-\hat{r}) \\ = -\theta' \hat{r}$$

$$\frac{d\hat{r}}{dt} = -\sin \theta \theta' \hat{i} + \cos \theta \theta' \hat{j} \\ = \theta' \hat{\theta}$$

$$\vec{r} = r\hat{r}$$

$$\vec{v} = \vec{r} \frac{d\hat{r}}{dt} = r(-\sin \theta \cdot \theta' \hat{i} + \cos \theta \cdot \theta' \hat{j}) \\ = r\theta' \hat{\theta}$$

$$\vec{a} = \vec{r}\theta''\hat{\theta} + \vec{r}\theta'(-\theta')\hat{r} \\ = -r\theta'^2 \hat{r} + r\theta'' \hat{\theta}$$

$$b) P_{tot} = (m_b + m_c) v_{f,i}$$

$$P_{tot} = m_b v_{f,b} + m_c v_{f,c}$$

$$\Delta E_m = -m_c g h + \frac{m_b v_{f,b}^2}{2} + \frac{m_c v_{f,c}^2}{2} - \frac{(m_b + m_c) v_{f,i}^2}{2}$$

$$\Delta p = m_b v_{f,b} + m_c v_{f,c} - (m_b + m_c) v_{f,i}$$

unknowns:  $v_{f,b}, v_{f,c}$

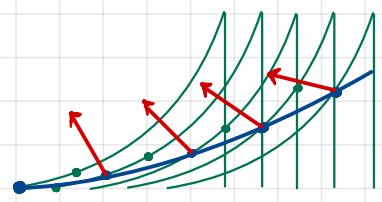
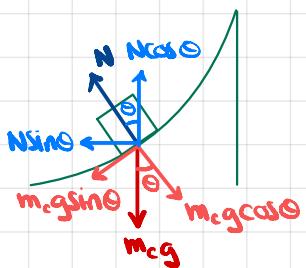
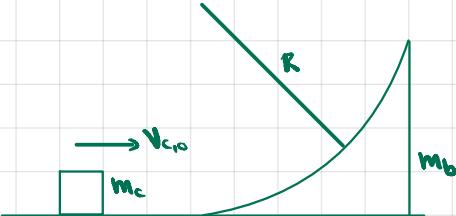
We are equating momentum and kinetic energy at the top of the cube's trajectory and at the bottom. One solution is

$$v_{f,b} = \frac{2m_c v_{c,0}}{m_b + m_c} \quad v_{f,c} = -\frac{(m_b - m_c) v_{c,0}}{m_b + m_c}$$

The other sol'n is  $v_{f,b} = 0, v_{f,c} = v_{c,0}$

$$v_{f,i} = \frac{m_c}{m_c + m_b} v_{c,0}$$

$$h = \frac{v_{c,0}^2 \cdot m_b}{2g(m_b + m_c)}$$



$$N_{x,t} = m_c \dot{v}_{c,x,t} = m_c \frac{d v_{c,x,t}}{dt} = m_c \ddot{v}_{c,x,t}$$

$$N_{y,t} - m_c g = m_c \dot{v}_{c,y,t} = m_c \frac{d v_{c,y,t}}{dt} = m_c \ddot{v}_{c,y,t}$$

$$\int_0^t (N_y - m_c g) dt = m_c v_{c,y,t}$$

$\left. \right\} \text{momentum}$

$$m_c v_{c,0} = m_c v_{c,x,t} + m_b v_{b,x,t}$$

$$\frac{m_c v_{c,0}^2}{2} = \frac{m_b v_{b,x,t}^2}{2} + \frac{m_c v_{c,t}^2}{2}$$

$$\Rightarrow m_b \cdot v_{b,x,t} \dot{v}_{b,x,t} = -m_c v_{c,x,t} \dot{v}_{c,x,t}$$

$$m_b (v_{b,x,t}^2 + v_{b,y,t}^2)^{\frac{1}{2}} \cdot \frac{1}{2} (v_{b,x,t}^2 + v_{b,y,t}^2)^{-\frac{1}{2}} (2 v_{b,x,t} \dot{v}_{b,x,t} + 2 v_{b,y,t} \dot{v}_{b,y,t})$$

$$= m_b \cdot (v_{b,x,t} \dot{v}_{b,x,t} + v_{b,y,t} \dot{v}_{b,y,t}) = -m_c (v_{c,x,t} \dot{v}_{c,x,t} + v_{c,y,t} \dot{v}_{c,y,t})$$

## Equations

$$m_b \cdot (v_{b,x,t} \dot{v}_{b,x,t} + v_{b,y,t} \dot{v}_{b,y,t}) = -m_c (v_{c,x,t} \dot{v}_{c,x,t} + v_{c,y,t} \dot{v}_{c,y,t}) = 0$$

$$N_{x,t} = m_c \dot{v}_{c,x,t}$$

$$N_{y,t} - m_c g = m_c \dot{v}_{c,y,t}$$

$$m_c v_{c,0} = m_c v_{c,x,t} + m_b v_{b,x,t}$$

$$\int_0^t (N_y - m_c g) dt = m_c v_{c,y,t}$$

$$-N_{x,t} = m_b \dot{v}_{b,x,t}$$

unknowns:  $v_{b,x}(t)$ ,  $\dot{v}_{b,x}(t)$ ,  $v_{b,y}(t)$ ,  $\dot{v}_{b,y}(t)$

$v_{c,x}(t)$ ,  $\dot{v}_{c,x}(t)$ ,  $v_{c,y}(t)$ ,  $\dot{v}_{c,y}(t)$

$N_x(t)$ ,  $N_y(t)$

### Problem 4 - Spring Block Motion



$$\text{energy conservation: } \frac{(m_1 + m_2)v_i^2}{2} - \frac{m_1 v_i^2}{2} + \frac{kx^2}{2} = 0$$

$$\text{momentum: } m_1 v_i = (m_1 + m_2) v_f \Rightarrow v_f = \frac{m_1}{m_1 + m_2} v_i$$

$$\Rightarrow \cancel{\frac{m_1 + m_2}{(m_1 + m_2)} \frac{m_1 v_i^2}{(m_1 + m_2)}} - \cancel{\frac{m_1 v_i^2}{2}} + \cancel{\frac{kx^2}{2}} = 0$$

$$m_1^2 v_i^2 - m_1 (m_1 + m_2) v_i^2 + (m_1 + m_2) kx^2$$

$$x = \sqrt{\frac{m_1 m_2 v_i^2}{(m_1 + m_2) k}} = v_i \sqrt{\frac{m_1 m_2}{(m_1 + m_2) k}}$$

$m_1$  compresses the spring. The spring force is an internal force: it decelerates  $m_1$  and accelerates  $m_2$ . At some point  $m_1$  and  $m_2$  have same speed. At this point whatever compression is occurring on  $m_1$ 's side is offset by expansion on  $m_2$ 's side. But deceleration of  $m_1$  and accel. of  $m_2$  continue.

$$m_1 v_{1f} + m_2 v_{2f} = m_1 v_i$$

$$\frac{m_1 v_{1f}^2}{2} + \frac{m_2 v_{2f}^2}{2} - \frac{m_1 v_i^2}{2}$$

There are two solutions. One is the sol'n representing the initial state:  $v_{1f} = v_i, v_{2f} = 0$ .

$$\text{The other is } v_{1f} = \frac{v_i(m_1 - m_2)}{m_1 + m_2} \quad v_{2f} = \frac{2m_1 v_i}{m_1 + m_2}$$

As the spring pushes on both masses, it changes the momentum of each mass equally. The lesser of the two masses experiences a smaller change in speed.

### Problem 5 - Bullet-Spring Collision

a)  $m_b v_b = (m_b + M) v_f \Rightarrow v_f = \frac{m_b v_b}{m_b + M}$  (conservation of momentum during totally inelastic collision)

b)  $\frac{kx^2}{2} - \frac{1}{2} \cancel{(m_b + M)} \frac{m_b^2 v_b^2}{(m_b + M)^2} = 0$  (conservation of mechanical energy during compression of spring)

$$x = \sqrt{\frac{m_b^2 v_b^2}{k(m_b + M)}} = \frac{m_b v_b}{\sqrt{k(m_b + M)}}$$

c)  $K_{\text{collide}} = \frac{\cancel{(m_b + M)} \cdot m_b^2 v_b^2}{2 \cdot \cancel{(m_b + M)^2}} = \frac{m_b^2 v_b^2}{2(m_b + M)}$

$$K_{\text{initial}} = \frac{m_b v_b^2}{2}$$

$$\frac{K_{\text{collide}}}{K_{\text{initial}}} = \frac{m_b}{m_b + M} < 1, \text{ energy is lost in a totally inelastic collision.}$$

d)  $v_z = \frac{m_b v_b}{2(m_b + M)}$

inelastic collision occurs, speed after collision is  $\frac{m_b v_b}{m_b + M}$ .

work done by friction is  $\int_0^x -\mu_k (m_b + M) g dx = -(m_b + M) g \mu_k x$

on the way back to  $x=0$ , friction does the same work.

$$-2(m_b + M) \mu_k g x = \frac{1}{2} (m_b + M) \left( \frac{m_b v_b}{2(m_b + M)} \right)^2 - \frac{1}{2} (m_b + M) \left( \frac{m_b v_b}{m_b + M} \right)^2$$

$$= -\frac{3}{8} \cancel{(m_b + M)} \cdot \frac{m_b^2 v_b^2}{(m_b + M)^2}$$

$$\Rightarrow 2(m_b + M)^2 \mu_k g x = \frac{3}{8} m_b^2 v_b^2$$

$$\Rightarrow x = \frac{3}{16} \cdot \frac{m_b^2 v_b^2}{(m_b + M)^2 \mu_k g} = \frac{3}{16 \mu_k g} \cdot v_{\text{collision}}^2$$

The block travels this distance  $x$  before returning to  $x=0$ .

## Problem 6 - Elastic 2D Collision

$$m_1 \vec{v}_{1,i} = m_1 \vec{v}_{1,f} + m_2 \vec{v}_{2,f} \quad \Delta \vec{p} = 0$$

$$m_1 (\vec{v}_{1,i}) = m_1 (v_{1,f} \cos \theta_{1,f} \hat{i} + v_{1,f} \sin \theta_{1,f} \hat{j}) + m_2 (v_{2,f} \cos \theta_{2,f} \hat{i} - v_{2,f} \sin \theta_{2,f} \hat{j})$$

$$\cancel{m_1 v_{1,i}} = m_1 v_{1,f} \cos \theta_{1,f} + 2m_2 v_{2,f} \cos \theta_{2,f} \quad \Delta p_x = 0$$

$$0 = \cancel{m_1 v_{1,f} \sin \theta_{1,f}} - 2m_2 v_{2,f} \sin \theta_{2,f} \quad \Delta p_y = 0$$

$$\frac{m_1 v_{1,i}^2}{2} = \frac{m_1 v_{1,f}^2}{2} + \frac{2m_2 v_{2,f}^2}{2} \Rightarrow v_{1,i}^2 = v_{1,f}^2 + 2v_{2,f}^2 \quad \Delta K = 0$$

$$v_{1,i} - v_{1,f} \cos \theta_{1,f} = 2v_{2,f} \cos \theta_{2,f} \Rightarrow v_{1,i} - 2v_{2,f} \cos \theta_{2,f} = v_{1,f} \cos \theta_{1,f} \quad \text{relative momentum}$$

$$v_{1,f} \sin \theta_{1,f} = 2v_{2,f} \sin \theta_{2,f} \Rightarrow 2v_{2,f} \sin \theta_{2,f} = v_{1,f} \sin \theta_{1,f}$$

$$v_{1,i}^2 + 4v_{2,f}^2 \cos^2 \theta_{2,f} - 4v_{1,i} v_{2,f} \cos \theta_{2,f} = v_{1,f}^2 \cos^2 \theta_{1,f}$$

$$4v_{2,f}^2 \sin^2 \theta_{2,f} = v_{1,f}^2 \sin^2 \theta_{1,f}$$

$$\cancel{v_{1,i}^2 + 4v_{2,f}^2 - 4v_{1,i} v_{2,f} \cos \theta_{2,f}} = v_{1,f}^2 = \cancel{v_{1,i}^2 - 2v_{2,f}^2}$$

$$6v_{2,f}^2 - 4v_{1,i} v_{2,f} \cdot \frac{\sqrt{2}}{2} = 0$$

$$v_{2,f} (6v_{2,f} - 2\sqrt{2}v_{1,i}) = 0$$

$$\begin{cases} v_{2,f} = \frac{\sqrt{2}v_{1,i}}{3} \\ v_{2,f} = 0 \end{cases}$$

Solution 1

$$v_{2,f} = 0$$

$$v_{1,f} = \pm v_{1,i}$$

$$v_{1,f} \cdot \sin \theta_{1,f} = 0 \Rightarrow \theta_{1,f} = 0$$

i.e. there is no collision.

Solution 2

$$v_{2,f} = \frac{\sqrt{2}v_{1,i}}{3}$$

$$v_{1,f}^2 = v_{1,i}^2 - \frac{4}{9}v_{1,i}^2 = \frac{5}{9}v_{1,i}^2 \Rightarrow v_{1,f} = \pm \frac{\sqrt{5}v_{1,i}}{3}$$

$$\frac{2\sqrt{2}v_{1,i}}{3} \frac{\sqrt{2}}{1} = \frac{\sqrt{5}v_{1,i}}{3} \sin \theta_{1,f}$$

$$\sin \theta_{1,f} = \frac{2}{\sqrt{5}} \Rightarrow \theta_{1,f} \approx 110^\circ \approx 63.43^\circ$$