Combined Translation and Rotation : Energy Relationships

setup

→ rigid body, made up of particle masses

→ m_i, vel. $\vec{v}_i$ rel. to an inertial frame

$\vec{v}_i = \vec{v}_{cm} + \vec{v}_i'$ ↗ vel. of particle rel. to CM

$$K_i = \frac{m_i \vec{v}_i \cdot \vec{v}_i}{2} = \frac{m_i(\vec{v}_{cm} + \vec{v}_i')\cdot(\vec{v}_{cm} + \vec{v}_i')}{2}$$

$$= \frac{m_i(\vec{v}_{cm}\cdot\vec{v}_{cm} + \vec{v}_i'\cdot\vec{v}_i' + 2\vec{v}_{cm}\cdot\vec{v}_i')}{2}$$

$$= \frac{m_i v_{cm}^2}{2} + \frac{m_i v_i'^2}{2} + \vec{v}_{cm}\cdot\vec{v}_i'$$

sum over all particles

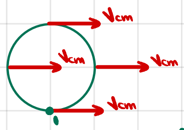
$$K = \frac{v_{cm}^2}{2}\sum m_i + \sum \frac{m_i v_i'^2}{2} + \vec{v}_{cm}\cdot \cancelto{0}{\sum \vec{v}_i'}$$

$$= \frac{Mv_{cm}^2}{2} + \sum \frac{m_i r_i^2 \omega^2}{2}$$

$$= \frac{Mv_{cm}^2}{2} + \frac{I_{cm}\omega^2}{2}$$

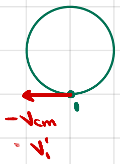
Rolling without Slipping

The entire wheel has a translation velocity $\vec{v}_{cm}$



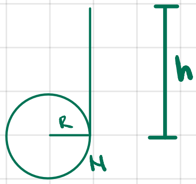
reference frame with the ground at rest
The point of the wheel that touches the ground has the same velocity as the ground;
this is what it means for that point (and the wheel) not to slip.

$\vec{v}_1 = \vec{v}_{cm} + \vec{v}_i' \Rightarrow \vec{v}_i' = -\vec{v}_{cm}$



$v_i' = R\omega \Rightarrow v_{cm} = R\omega$ (no slipping condition)

Ex 10.4 - Speed of a Primitive Yo-Yo



$E_i = Mgh$

$$E_f = \frac{Mv_{cm}^2}{2} + \frac{MR^2}{2}\cdot\frac{1}{2}\omega^2$$

no slipping ⇒ $v_{cm} = R\omega \Rightarrow \omega = \frac{v_{cm}}{R}$

$$E_f = \frac{Mv_{cm}^2}{2} + \frac{Mv_{cm}^2}{4} = \frac{3Mv_{cm}^2}{4}$$

$$\Delta E = 0 \Rightarrow v_{cm} = \sqrt{\frac{4gh}{3}}$$

Ex 10.5 Race of the Rolling Bodies

$$mgh = \frac{mv^2}{2} + \frac{I_{cm}v^2}{2R^2}$$

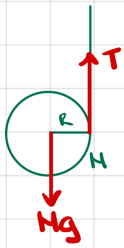
The larger the rotational kinetic energy due to moment of inertia and/or radius, the smaller v is.

we can't tell if v depends on R because I_{cm} depends on R.

For certain bodies we know I_{cm}.

solid sphere: $I_{cm} = \frac{2MR^2}{5}$

Ex 10.6 - Acceleration of a Primitive Yo-Yo



The yo-yo has both translational and rotational motion.

$$Mg - T = M\cdot a_{cm}$$

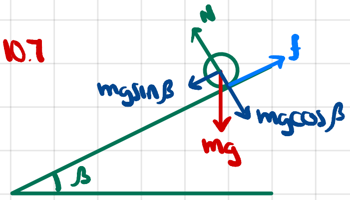
$$RT = \frac{MR^2}{2}\cdot\frac{a_{cm}}{R}$$

$\Rightarrow$

$$a_{cm} = \frac{2}{3}g$$

$$T = \frac{1}{3}Mg$$

Ex 10.7



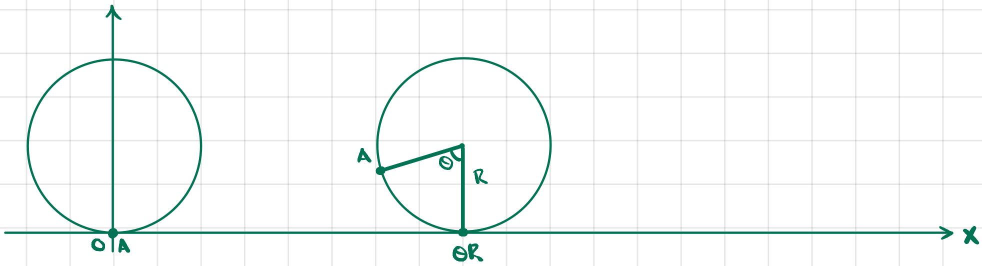
$$mg\sin\beta - f = m\cdot a$$

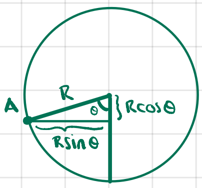
$$f\cancel{R} = \frac{2}{5}m\cancel{R^2}\cdot\frac{a}{\cancel{R}}$$

$$\Rightarrow f = \frac{2ma}{5}$$

$$\cancel{m}g\sin\beta = m\left(a + \frac{2a}{5}\right) = \cancel{m}\cdot\frac{7a}{5}$$

$$a = \frac{5g\sin\beta}{7}$$





$$\vec{r}_{OA} = \langle \theta R - R\sin\theta,\ R - R\cos\theta \rangle$$
$$= \langle R(\theta - \sin\theta),\ R(1-\cos\theta) \rangle$$

$$\vec{v}_A(\theta) = \langle R(1-\cos\theta),\ R\sin\theta \rangle$$

$$\vec{v}_A(2\pi) = \langle 0, 0 \rangle$$

10.5 Angular Momentum

$\vec{L}_o = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$

→ depends on choice of origin

→ net force acts on a particle ⇒ velocity and momentum change, so angular momentum may also change.

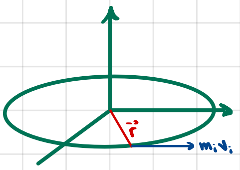
→ $\frac{d\vec{L}}{dt} = \vec{v} \times m\vec{v} + \vec{r} \times m\vec{a} = \vec{r} \times \vec{F} = \vec{\tau}$

rate of change of angular momentum equals torque of net force.

Angular Momentum of a Rigid Body

→ rigid body rotating about an axis with ω.

→ thin slice located in xy plane

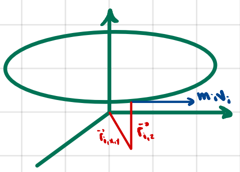


$L_i = r m_i v_i = m_i r_i^2 \omega$, positive $\hat{k}$ direction

* sin of angle between $\vec{r}_i$ and $\vec{v}_i$ is 90°

$L = \sum_i L_i = \sum_i m_i r_i^2 \omega = I_{slice,z}\,\omega$

If we try this for another slice not on the xy plane, the $\vec{r}_i$ vectors have a z-component.



The angular momentum of m_i rel. to z-axis has a component perpendicular to z.

For a rigid body that is symmetric about the axis of rotation, such momentum vectors cancel out by points on opposite sides of the axis.

⇒ For rigid bodies symmetric about rotation axis, angular momentum is a vector with same direction as axis of rotation, and magnitude $I_z\omega$. Therefore, $\vec{L} = I_z\vec{\omega}$

Also, $\vec{\tau}^{ext} = \frac{d\vec{L}}{dt} = I_z\alpha$ in this case.

10.7 Gyroscopes and Precession