18.2 Setup a Recoil Problem

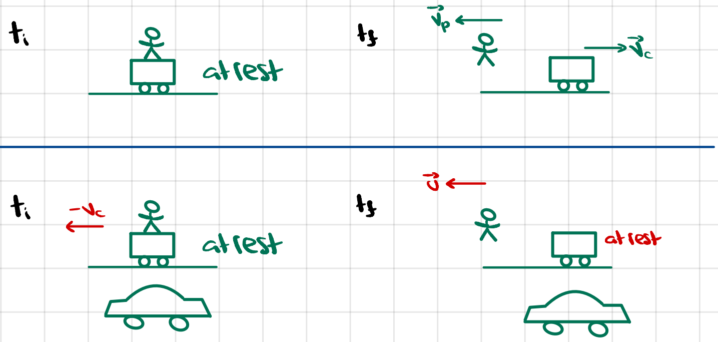
→ Apply momentum principle to problems of recoil

$\vec{F}_{ext} = \frac{d}{dt}\vec{P}_{sys}$

$F_{ext,x} = 0 \Rightarrow P_{sys,x}(f) = P_{sys,x}(i)$

→ Recoil: jumping off cart

consider the following scenario with the reference frame being the ground:



if we consider as reference frame a car passing by with velocity $\vec{V}_c$:

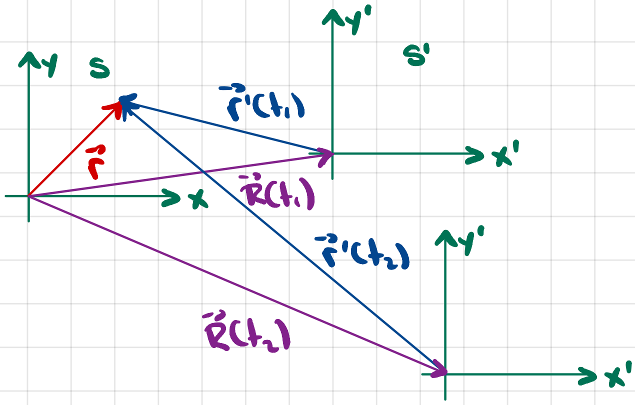
$\vec{U}$: velocity of person in moving frame ($\vec{V}_c$)
= vel. of person relative to the cart

person jumps horizontally, no external forces

As we saw in week 2

$$\vec{r} = \vec{R} + \vec{r}' \Rightarrow \vec{r}' = \vec{r} - \vec{R}$$

$$\Rightarrow \vec{V}_p = \vec{R} + \vec{U} = \vec{V}_c + \vec{U}$$



we want to solve for $\vec{V}_p$ and $\vec{V}_c$ in terms of $\vec{U}, m_p, m_c$ using the momentum principle.
we treat $\vec{U}$ as a given

Solution for the ground frame

$\vec{P}_{sys}(i) = \vec{P}_{sys}(f)$

$\Rightarrow \vec{0} = m_c\vec{V}_c + m_p\vec{V}_p$

$\vec{V}_p = \vec{R} + \vec{U} = \vec{V}_c + \vec{U}$

$\Rightarrow \vec{0} = m_c\vec{V}_c + m_p\vec{V}_c + m_p\vec{U} \Rightarrow \vec{V}_c = -\frac{m_p\vec{U}}{m_p + m_c}$

Solution for the moving (car) frame

$\vec{P}_{sys}(i) = \vec{P}_{sys}(f)$

$(m_p + m_c)(-\vec{V}_c) = m_p\vec{U} \Rightarrow \vec{V}_c = \frac{-m_p\vec{U}}{m_p + m_c}$

19.1 Rocket Problem 1

Momentum Principle

$$\vec{F}^{ext}(t) = \frac{d\vec{p}^{sys}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{p}^{sys}(t+\Delta t) - \vec{p}^{sys}(t)}{\Delta t}$$

Continuous Mass Transfer

(1) choose reference frame
(2) identify the state at time t and $t+\Delta t$
(3) draw momentum diagrams

→ System: rocket + fuel, stays the same between t and $t+\Delta t$

Analyze system momentum at times t and $t+\Delta t$.

time t $\uparrow \vec{v}_r(t)$ $m_r(t)$

$$\vec{p}^{sys}(t) = m_r(t)\vec{v}_r(t)$$

mass rocket includ. fuel

time $t+\Delta t$ $m_r(t+\Delta t)$ $\uparrow v_r(t+\Delta t)$; Δm_f $\uparrow v_f$

$$\vec{p}_{sys}(t+\Delta t) = m_r(t+\Delta t)v_r(t+\Delta t) + \Delta m_f \cdot \vec{v}_f$$

$\vec{u}$ = velocity of fuel relative to rocket

velocity of fuel rel. to initial frame based on the second frame (the rocket)

$$\vec{v}_f = \vec{u} + \vec{v}_r(t+\Delta t)$$

before we move on lets simplify notation:

$$\begin{cases} m_r(t) = m_r \\ m_r(t+\Delta t) = m_r(t) + \Delta m_r = m_r + \Delta m_r \\ \Delta m_f = -\Delta m_r \end{cases}$$

now we write our momentum eqs again

$$\vec{p}^{sys}(t) = m_r\vec{v}_r(t)$$

here we've rewritten things slightly using new notation and subbing in $\Delta m_f = -\Delta m_r$ and $\vec{v}_f = \vec{u} + \vec{v}_r(t+\Delta t)$.

$$\vec{p}^{sys}(t+\Delta t) = (m_r + \cancel{\Delta m_r})v_r(t+\Delta t) - \Delta m_r(\vec{u} + \cancel{\vec{v}_r(t+\Delta t)}) = m_r v_r(t+\Delta t) - \Delta m_r \vec{u}$$

Apply momentum principle

At t, F^{ext} is rate of change of momentum, ie the average change in an infinitesimal dt

$$\vec{F}^{ext} = \lim_{\Delta t \to 0} \frac{(m_r + \cancel{\Delta m_r})v_r(t+\Delta t) - \Delta m_r(\vec{u} + \cancel{\vec{v}_r(t+\Delta t)}) - m_r\vec{v}_r(t)}{\Delta t}$$

$$= \lim \frac{m_r(v_r(t+\Delta t) - v_r(t))}{\Delta t} - \lim_{\Delta t \to 0} \frac{\Delta m_r}{\Delta t}\vec{u}$$

$$\vec{F}^{ext} = m_r \frac{dv_r}{dt} - \frac{dm_r}{dt}\vec{u}$$

(Rocket Equation)

rocket accel.

rate that mass of rocket is changing

19.5 Thrust and External Forces

$$\Delta m_r = -\Delta m_f \Rightarrow \frac{dm_r}{dt} = -\frac{dm_f}{dt}$$

$$\vec{F}^{ext} + \frac{dm_r}{dt}\vec{u} = m_r\frac{d\vec{v}_r}{dt} \Rightarrow \vec{F}^{ext} - \frac{dm_f}{dt}\vec{u} = \underbrace{m_r\frac{d\vec{v}_r}{dt}}_{m_r \vec{a}_r}$$

we can rethink our system as just the rocket: $\vec{F}^{ext} + \vec{F}^{thrust} = m_r\frac{d\vec{v}_r}{dt}$, $\vec{F}^{thrust} = -\frac{dm_f}{dt}\vec{u}$

with $\uparrow \hat{j}$ lets look at components:

$$F_y^{thrust}\hat{j} = -\frac{dm_f}{dt}(-u\hat{j}) = \frac{dm_f}{dt}u\hat{j}$$

positive force opposite in dir. to gravity

$$\vec{u} = -u\hat{j}$$

$$\vec{F}^{ext} = -mg\hat{j}$$

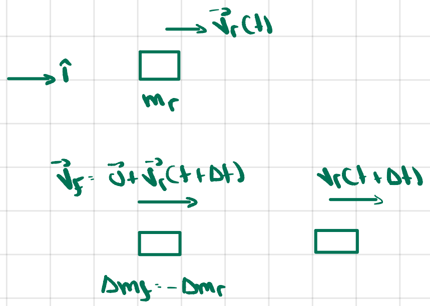
$$\Rightarrow -mg + \frac{dm_f}{dt}u = m_r\frac{dv_{r,y}}{dt}$$ (rocket eq. in components)

19.6 Solution for No External Forces

$$\vec{F}^{ext} + \frac{dm_r}{dt}\vec{u} = m_r\frac{d\vec{v}_r}{dt}$$

Special Cases

1) $\vec{F}^{ext} = 0$ rocket far away from any gravitational force



exhaust speed $\vec{u} = -u\hat{i}$

$$\vec{v}_r = v_r\hat{i}$$

Rocket eq: $\frac{dm_r}{dt}(-u)\hat{i} = m_r\frac{dv_r}{dt}\hat{i}$

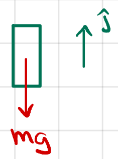
$$-dm_r u = m_r dv_r$$

$$-\frac{1}{m_r}u\, dm_r = dv_r \Rightarrow \int_{m_r(t_i)}^{m_r(t_f)} -\frac{u}{m_r}dm_r = \int_{v_r(t_i)}^{v_r(t_f)} dv_r$$

$$-u\ln(m_r(t_f)/m_r(t_i)) = v_r(t_f) - v_r(t_i)$$

$$v_r(t_f) - v_r(t_i) - u\ln\left[\frac{m_r(t_f)}{m_r(t_i)}\right]$$

19.7 Solution with External Forces



$$\vec{F}^{ext} = -m_r g\hat{j} \neq 0$$

$$-m_r g + \frac{dm_r}{dt}(-u) = m_r\frac{dv_r}{dt} \Rightarrow -m_r g\,dt - u\,dm_r = m_r\,dv_r$$

$$-g\,dt - \frac{dm_r}{m_r}u = dv_r$$

$$\Rightarrow \int_{t_i}^{t_f} -g\,dt - u\int_{m_r(t_i)}^{m_r(t_f)} \frac{1}{m_r}dm_r = \int_{v_r(t_i)}^{v_r(t_f)} dv_r$$

$$= -g(t_f - t_i) - u\ln\left[\frac{m_r(t_f)}{m_r(t_i)}\right] = v_r(t_f) - v_r(t_i)$$

At $t_i = 0$, $v_r(0) = 0$, $m_r(0) = m_{r,d} + m_f$ (dry mass; fuel mass)

At t_f, $v_r(t_f)$, $m_r(t_f) = m_{r,d}$

$$-gt_f + u\ln\left[\frac{m_{r,d} + m_f}{m_{r,d}}\right] = v_r(t_f)$$

P.S.6.1 Rocket Sled

Problem Data

$m_r = m_{r,d} + m_{r,f} = 2m_0$

$\vec{v}(0) = v_0\hat{i}$

$\vec{v}_{f_r} = \vec{u} = \pm u\hat{i}$

$\vec{v}_f = \vec{v}_{f_r} + \vec{v}_r = \vec{u} + \vec{v}_r$

Let's do the entire derivations here.

reference frame: ground

time t momentum: $\vec{p}^{sys}(t) = m_r(t)\vec{v}_r(t)$

" $t+\Delta t$ " : $\vec{p}_{sys}(t+\Delta t) = m_r(t+\Delta t)\vec{v}_r(t+\Delta t) + \Delta m_f \cdot \vec{v}_f(t+\Delta t)$

rewrite some terms differently

$m_r(t+\Delta t) = m_r(t) + \Delta m_r$

$\Delta m_f = -\Delta m_r$

$\Rightarrow \vec{p}^{sys}(t+\Delta t) = (m_r(t) + \cancel{\Delta m_r})\vec{v}_r(t+\Delta t) - \Delta m_r(u + \vec{v}_r(\cancel{t+\Delta t}))$

$= m_r(t)v_r(t+\Delta t) - u\Delta m_r$

momentum principle: $\vec{F}^{ext} = \dfrac{d}{dt}\vec{p}^{sys}(t) = \lim\limits_{\Delta t\to 0}\dfrac{\vec{p}^{sys}(t+\Delta t) - \vec{p}^{sys}(t)}{\Delta t} = \lim\limits_{\Delta t\to 0}\dfrac{m_r(t)v_r(t+\Delta t) - u\Delta m_r - m_r(t)\vec{v}_r(t)}{\Delta t}$

$= \lim\limits_{\Delta t\to 0} m_r(t)\dfrac{\vec{v}_r(t+\Delta t) - \vec{v}_r(t)}{\Delta t} - \lim\limits_{\Delta t\to 0}\vec{u}\dfrac{\Delta m_r}{\Delta t} = m_r(t)\dfrac{d\vec{v}_r(t)}{dt} - \vec{u}\dfrac{dm_r(t)}{dt}$

a) Since there is no friction force or gravitation force in the x-direction, $\vec{F}^{ext} = 0$.

$\Rightarrow m_r(t)\dfrac{d\vec{v}_r(t)}{dt} = \vec{u}\dfrac{dm_r(t)}{dt} \Rightarrow m_r(t)\dfrac{dv_r(t)}{dt}\hat{i} = \dfrac{dm_r(t)}{dt} + u\hat{i}$

$\Rightarrow m_r(t)\dfrac{dv_r(t)}{dt} = \dfrac{dm_r(t)}{dt}u$

b) $dv_r = \dfrac{u}{m_r(t)}dm_r \Rightarrow \displaystyle\int_{v_r(t_i)}^{v_r(t_f)} dv_r = \int_{m_r(t_i)}^{m_r(t_f)} u\, m_r^{-1}(t)\, dm_r \Rightarrow v_r(t_f) - v_r(t_i) = u\ln\left[\dfrac{m_r(t_f)}{m_r(t_i)}\right]$

c) $v_r(t_f) = v_r(t_i) + u\ln\left[\dfrac{m_r(t_f)}{m_r(t_i)}\right]$

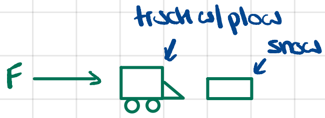
$v_r(t_f) = 0$, ie the rocket stops.

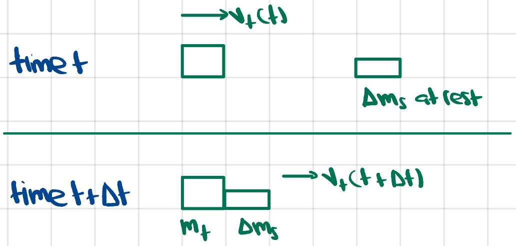
$m_r(t_i) = 2m_0$

$m_r(t_f) = m_0$

$\Rightarrow v_r(t_i) = -u\ln(1/2) \approx 0.69u$

P.S. 6.2 Snowplow





$v(t+\Delta t) = v(t) + \Delta v$

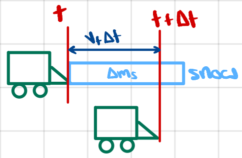
momentum principle $$F = \lim_{\Delta t \to 0} \frac{(m_t+\Delta m_s)v_t(t+\Delta t) - m_t v_t(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{(m_t+\Delta m_s)(v_t(t)+\Delta v_t) - m_t v_t(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{m_t \Delta v_t}{\Delta t} + \frac{\Delta m_s v_t(t)}{\Delta t} + \underbrace{\frac{\Delta m_s \Delta v_t}{\Delta t}}_{\text{disregard this second order term}} = \frac{dm_s}{dt} v_t + m_t \frac{dv_t}{dt}$$

$$\Rightarrow F = \frac{dm_s}{dt} v_t + m_t \frac{dv_t}{dt}$$

$$F\,dt = dm_s v_t + m_t dv_t$$

what is dm_s?



snow density

$$\Delta m_s = \rho A v_t \Delta t$$

cross sectional area of truck

$$\Rightarrow F\,dt = (\rho A v_t dt) v_t + m_t dv_t \Rightarrow (F - \rho A v_t^2)\,dt = m_t dv_t \Rightarrow \int_{t_i}^{t_f} dt = \int_{v_t(t_i)}^{v_t(t_f)} \frac{m_t}{F - \rho A v_t^2} dv_t$$

$$\int \frac{m_t}{F-\rho A v_t^2} dv_t = \int \frac{m_t}{F - \cancel{\rho A} \cdot \frac{F \sin^2\theta}{\cancel{\rho A}}} \cdot \frac{\cos\theta \, d\theta}{[\rho A/F]^{1/2}} = \sqrt{\frac{F}{\rho A}} \frac{m_t}{F} \int \frac{\cos\theta}{\cos^2\theta} d\theta = \frac{m_t}{\sqrt{\rho A F}} \int \cos^{-1}\theta \, d\theta = \frac{m_t}{\sqrt{\rho A F}} \ln(\sec\theta + \tan\theta)$$

$$v_t = \frac{\sin\theta}{[\rho A/F]^{1/2}} \Rightarrow v_t^2 = \frac{F \sin^2\theta}{\rho A} \qquad dv_t = \frac{\cos\theta \, d\theta}{[\rho A/F]^{1/2}} \qquad \theta = \sin^{-1}\left[v_t \sqrt{\frac{\rho A}{F}}\right]$$

$$\Rightarrow t_f - t_i = \frac{m_t}{\sqrt{\rho A F}} \ln \left[\frac{\sec\left(\sin^{-1}\left[v_t(t_i)\sqrt{\frac{\rho A}{F}}\right]\right) + \tan\left(\sin^{-1}\left[v_t(t_i)\sqrt{\frac{\rho A}{F}}\right]\right)}{\sec\left(\sin^{-1}\left[v_t(t_f)\sqrt{\frac{\rho A}{F}}\right]\right) + \tan\left(\sin^{-1}\left[v_t(t_f)\sqrt{\frac{\rho A}{F}}\right]\right)} \right]$$