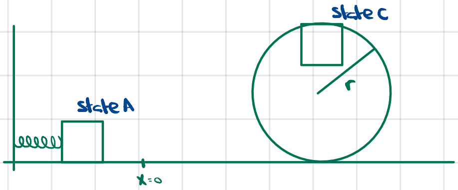
Pset 8 - Potential En. and En. Conservation

Problem 1 - Spring-Loop-the-Loop

Ex 14.3 Spring-Block-Loop-the-Loop



no friction

A: $K_A = 0$

$U_{el_A} = \frac{1}{2}kx^2$ initially there was only elastic potential energy

$U_{g_A} = 0$

B: $K_B = \frac{1}{2}mv_B^2$

$U_{el_B} = 0$ at top of loop, energy has become kinetic and gravitational potential

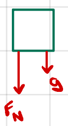
$U_{g_B} = mg\cdot 2r$

$$\Delta E_m = 0 \Rightarrow \underbrace{\left(0 - \tfrac{1}{2}kx^2\right)}_{\Delta U_{el}} + \underbrace{\left(\tfrac{1}{2}mv_B^2 - 0\right)}_{\Delta K} + \underbrace{\left(mg\cdot 2r - 0\right)}_{\Delta U_g} = 0$$

$$\frac{1}{2}mv_B^2 = K_B = \frac{1}{2}kx^2 - 2mgr \Rightarrow v_B = \frac{1}{m}\left(kx^2 - 4mgr\right)$$

we can solve for speed in terms of initial spring compression, or vice-versa.

B:



since we know the normal force at the loop top, we know total accel. and thus speed.

$\vec{F_N} = -F_{N_r}\hat{r}$

$\vec{a_r} = -a_r\hat{r} = -r\omega^2\hat{r} = -r\frac{v^2}{r^2}\hat{r} = -\frac{v^2}{r}\hat{r}$

At B, $\vec{F_N} = -2mg\hat{r}$

$F_N + F_G = m\cdot a_r \Rightarrow -3\cancel{m}g = m\cdot(-a_r) = \cancel{m}\cdot\left(-\frac{v^2}{r}\right)$

$3g = \frac{v^2}{r} \Rightarrow v_B = \pm\sqrt{3rg}$

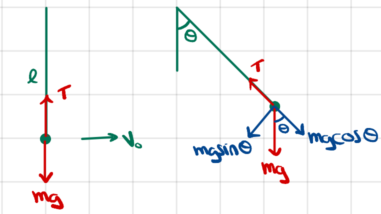
From conserv. of mech. en. we know that $\frac{1}{2}mv_B^2 = \frac{1}{2}kx^2 - 2mgr$

Given the knowledge of normal force at B, we know v_B. So we know total final mech. en. we can solve for x.

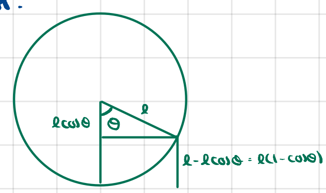
$m\cdot 3rg = kx^2 - 4mgr \Rightarrow x = \sqrt{\frac{7mgr}{k}}$ So we can solve for initial compression.

Problem 2 – Slingshot

$$W_g = \int_0^{2\ell} -mg\,dx = -mg\cdot 2\ell \qquad \Delta U_g = 2mg\ell$$



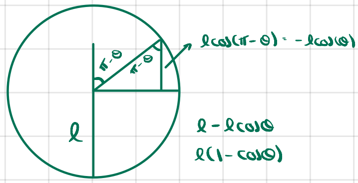
The mass undergoes a circular trajectory. T is always $\perp$ to the traj. so it does no work.



$$E_{m_i} = \frac{mv_0^2}{2}$$

$$E_{m_{Top}} = \frac{mv_{Top}^2}{2} + mg\cdot 2\ell \Rightarrow \frac{v_0^2}{2} = \frac{v_{Top}^2}{2} + 2g\ell \Rightarrow v_{Top} = \pm\sqrt{v_0^2 - 4g\ell}$$

a) * min v_0 to actually reach top position



$$v_0^2 - 4g\ell = 0 \Rightarrow v_0^2 = 4g\ell \Rightarrow v_0 = \sqrt{4g\ell}$$

$$E_{m_1} = \frac{mv_1^2}{2} + mg\ell(1-\cos\theta)$$

$$\Rightarrow \frac{v_0^2}{2} = \frac{v_1^2}{2} + g\ell(1-\cos\theta) \Rightarrow v_1 = \pm\sqrt{v_0^2 - 2g\ell(1-\cos\theta)}$$

Tension Force

$$mg\cos\theta - T = -m\cdot\frac{v_0^2 - 2g\ell(1-\cos\theta)}{\ell} \Rightarrow T = \frac{mg\ell\cos\theta + mv_0^2 - 2mg\ell(1-\cos\theta)}{\ell} = \frac{mv_0^2 - 2mg\ell + 3mg\ell\cos\theta}{\ell}$$

$$\Rightarrow T = \frac{m}{\ell}(v_0^2 + 3g\ell\cos\theta - 2g\ell)$$

$$T(\pi) = \frac{m}{\ell}(v_0^2 - 3g\ell - 2g\ell) = \frac{m}{\ell}(v_0^2 - 5g\ell)$$

$$T(\pi) \geq 0 \Rightarrow v_0^2 - 5g\ell \geq 0 \Rightarrow v_0 \geq \sqrt{5g\ell}$$
$$\leq -\sqrt{5g\ell}$$

$$v_{top} \text{ when } T(\pi) = 0 \Rightarrow v_0 = \sqrt{5g\ell} \Rightarrow v_{Top} = \sqrt{g\ell}$$

$$v_0 = \sqrt{4g\ell} \Rightarrow T(\pi) = \frac{m}{\ell}(-g\ell) = -mg$$

But T must be >0

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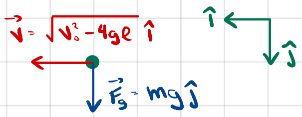
b)



$$E_{m_0} = \frac{m(v_0^2 - 4g\ell)}{2} + mg\cdot 2\ell$$

$$E_{m_f} = \frac{m|v_f|^2}{2}$$

$$v_f = \sqrt{v_0^2 - 4g\ell}\,\hat{\imath} + v_y\hat{\jmath}$$

$$\Rightarrow |v_f|^2 = v_0^2 - 4g\ell + v_y^2$$

$$\Delta E_m = 0 = \frac{\cancel{m}(v_0^2 - 4g\ell + v_y^2)}{2} - \frac{\cancel{m}(v_0^2 - 4g\ell)}{2} - \cancel{m}g\cdot 2\ell$$

here we used kinematics

$a_y = g$ $v_y = gt$ $y = \frac{gt^2}{2}$	$y = 2\ell = \frac{gt^2}{2} \Rightarrow t = \sqrt{\frac{4\ell}{g}}$ $v_x = \sqrt{v_0^2 - 4g\ell}$	$x = \int_0^x v_x\,dt = t\sqrt{v_0^2 - 4g\ell}$ $x(\sqrt{4\ell/g}) = \sqrt{\frac{4\ell(v_0^2 - 4g\ell)}{g}}$

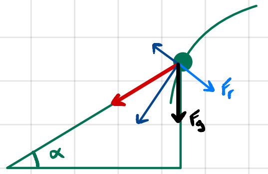
$$\cancel{v_0^2} - 4\cancel{g\ell} + v_y^2 - \cancel{v_0^2} + 4\cancel{g\ell} - 4g\ell = 0$$

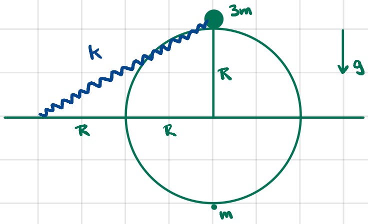
$$v_{y_f} = \sqrt{4g\ell}$$ here we used energy conservation to find final vertical speed

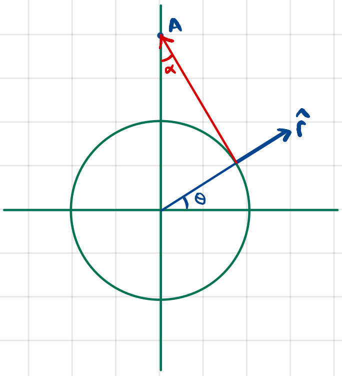
$$v_y = gt = \sqrt{4g\ell} \Rightarrow t = \sqrt{4\ell/g}$$

(...)

Problem 3 - objects on a Ring







$\vec r = R\hat r \qquad \hat r = \cos\theta\,\hat i + \sin\theta\,\hat j$

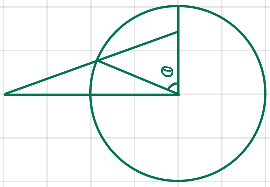
$= R\cos\theta\,\hat i + R\sin\theta\,\hat j$

$\vec r_A = 2R\hat j$

$\vec r_A - \vec r = -R\cos\theta\,\hat i + (2R - R\sin\theta)\hat j$

$|\vec r_A - \vec r| = (R^2\cos^2\theta + 4R^2 - 4R^2\sin\theta + R^2\sin^2\theta)^{1/2}$

$= (5R^2 - 4R^2\sin\theta)^{1/2}$



Strategy: the forces acting on m, are the gravitational force, a spring force, and a normal force from the ring.

$W_g + W_s + W_N = \Delta K$

But $W_N = 0$ because it is always $\perp$ to the trajectory.

Only conservative forces are in play, so $\Delta E_m = 0$.

$U_{g_0} = 3mg\cdot 2R$

$U_{g_b} = 0$ $\Rightarrow \Delta U_g = -6mgR$

To find U_{s_0} we need to know the distance between the spring's fixed point and the initial m, position. $U_{s_0} = U_{s_b}$ because the distance is the same.

$\Rightarrow \Delta U_s = 0$

$\Delta K = \frac{3mv_b^2}{2}$

Gravitational pot. becomes kinetic energy.

$\frac{3mv_b^2}{2} = 6mgR \Rightarrow v_b = \sqrt{4gR}$

Apply conservation of momentum

$3m\sqrt{4gR} = 4mv_c \Rightarrow v_c = \frac{3}{4}\sqrt{4gR}$

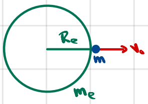
To reach A w/ zero speed

$$\Delta U_g + \Delta U_s + \Delta K = 0 \Rightarrow 4mgR + \frac{k\cdot 9R^2}{2} - \frac{k\cdot 5R^2}{2} - \frac{4m\cdot\frac{9}{16}\cdot 4gR}{2} = 0$$

$\Rightarrow m = \frac{4kR}{g}$

Problem 4 – Orbital collisions

a)



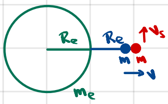
m is fired with initial speed v_0. The gravitational force does negative work on m, and the latter's potential gravit. en increases. If m keeps going until an infinite distance, and only then has zero speed and zero pot. gravit. en., then the initial speed was the magnitude of escape velocity.

$$W_g = \Delta K \Rightarrow \int_{R_e}^{R_f} -\frac{Gm_em}{r^2}\,dr = \frac{Gm_em}{r}\Big|_{R_e}^{R_f} = Gm_em\left(\frac{1}{R_f}-\frac{1}{R_e}\right)$$

$$\lim_{R_f\to\infty} W_g = -\frac{Gm_em}{R_e} = -\frac{mv_e^2}{2}$$

$$\Rightarrow v_e = \sqrt{\frac{2Gm_e}{R_e}} = \sqrt{\frac{2\cdot 6.67\cdot 10^{-11}\cdot 5.972\cdot 10^{24}}{6371\cdot 10^3}} = 11182.37\ m/s$$

b)



$$Gm_e\cancel{m}\left(\frac{1}{2R_e}-\frac{1}{R_e}\right) = \frac{\cancel{m}(v^2-v_e^2)}{2}$$

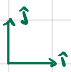
$$\Rightarrow v = \sqrt{v_e^2 + 2Gm_e\left(-\frac{1}{2R_e}\right)} = \sqrt{v_e^2 - \frac{Gm_e}{R_e}} = \sqrt{\frac{Gm_e}{R_e}}$$

$$\Rightarrow v_f = \sqrt{\frac{Gm_e}{R_e}}$$

c) satellite, circular orbit, radius $2R_e$

$$a_R = \frac{v^2}{2R_e} \Rightarrow m\cdot a_R = \cancel{m}\cdot\frac{v^2}{\cancel{2}R_e} = \frac{Gm_e\cancel{m}}{{}_2\cancel{4}R_e^{\cancel{2}}} \Rightarrow v = \sqrt{\frac{Gm_e}{2R_e}}$$

d)



$$\vec{p}_i = m\sqrt{\frac{Gm_e}{R_e}}\,\hat{\imath} + m\sqrt{\frac{Gm_e}{2R_e}}\,\hat{\jmath}$$

$$\vec{p}_f = 2m\vec{v}$$

$$\Delta\vec{p} = 0 \Rightarrow \left(2mv_x - m\sqrt{\frac{Gm_e}{R_e}}\right)\hat{\imath} + \left(2mv_y - m\sqrt{\frac{Gm_e}{2R_e}}\right) = \vec{0}$$

$$\Rightarrow v_x = \frac{1}{2}\sqrt{\frac{Gm_e}{R_e}}$$

$$v_y = \frac{1}{2\sqrt{2}}\sqrt{\frac{Gm_e}{R_e}}$$

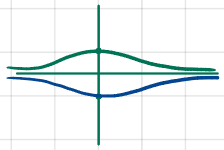
$$\Rightarrow \vec{v} = \left\langle \frac{1}{2}\sqrt{\frac{Gm_e}{R_e}},\ \frac{1}{2\sqrt{2}}\sqrt{\frac{Gm_e}{R_e}} \right\rangle$$

Problem 5 - Exponential Potential Energy Diagram

$U(x) = -U_0 e^{\frac{-x^2}{a^2}}$, potential fn of particle mass m moving in one dimension

U(x) is a potential fn for F(x).
$F = \frac{d}{dx}W = \frac{d}{dx}(-\Delta U)$

a)



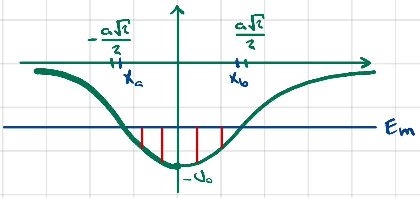
$f(x) = \frac{1}{e^{x^2}} = e^{-x^2}$

$f' = -2xe^{-x^2}$

$f'' = -2e^{-x^2} + 4x^2e^{-x^2} = e^{-x^2}(4x^2-2)$

$4x^2 = 2 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{\sqrt{2}}{2}$

$+ \quad - \quad +$ ($-\frac{\sqrt{2}}{2}$, $\frac{\sqrt{2}}{2}$)



$f(x) = -U_0 e^{\frac{-x^2}{a^2}}$

$f' = U_0 \cdot \frac{2x}{a^2} e^{\frac{-x^2}{a^2}}$

$f'' = \frac{2U_0}{a^2} e^{\frac{-x^2}{a^2}} - \frac{4x^2}{a^4} \cdot U_0 e^{\frac{-x^2}{a^2}} = \frac{2U_0}{a^2} e^{\frac{-x^2}{a^2}}\left(1 - \frac{2x^2}{a^2}\right)$

$2x^2 = a^2 \Rightarrow x = \pm\frac{a\sqrt{2}}{2}$

b) $W_F = \int_{x_1}^{x} F(x)\,dx = -\Delta U = -\left(-U_0 e^{\frac{-x^2}{a^2}} + U_0 e^{\frac{-x_1^2}{a^2}}\right) = U_0\left(e^{\frac{-x^2}{a^2}} - e^{\frac{-x_1^2}{a^2}}\right)$

$= U_0\left(\frac{1}{e^{x^2/a^2}} - \frac{1}{e^{x_1^2/a^2}}\right)$

$F(x) = \frac{d}{dx}\left[U_0\left(\frac{1}{e^{x^2/a^2}} - \frac{1}{e^{x_1^2/a^2}}\right)\right] = U_0\left(\frac{-2x}{a^2}\right)e^{\frac{-x^2}{a^2}}$

$= \frac{-2U_0 x}{a^2} e^{\frac{-x^2}{a^2}}$

c) At $x = \pm a$ the particle stops and reverses direction of motion.

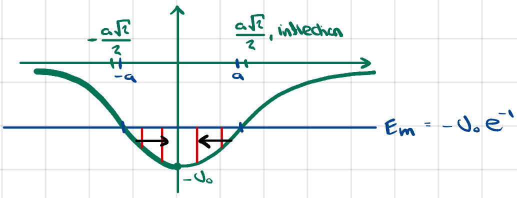
$\Rightarrow$ At $x = \pm a$, only potential energy, no kinetic.

choosing a speed at $x=0$ determines total mechanical energy, and we need enough mech en. such that

$E_{m_a} = U_a = -U_0 + K_0$

$U_a = U(a) = -U_0 e^{-1} = -U_0 + \frac{mv_0^2}{2} \Rightarrow \frac{mv_0^2}{2} = U_0(1 - e^{-1})$

$\Rightarrow v_0 = \sqrt{\frac{2U_0}{m}(1 - e^{-1})}$



$x > 0 \Rightarrow$ potential increasing, negative work being done, negative force.

At x_b, $K_b = 0$ so $v_b = 0$.