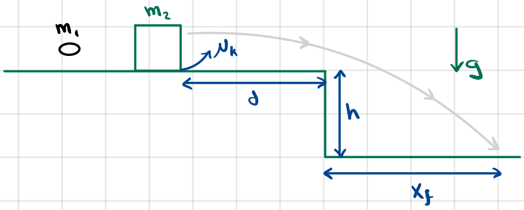
Problem Set 5

Problem 1 - Stopping a Bullet



$\to$ let's consider a system bullet + block

initially, $\vec{p}^{sys} = \langle m_1 v_{1,t_1,x}, 0 \rangle$

right after instantaneous collision, $\vec{p}^{sys} = \langle (m_1+m_2) v_{sys,t_2,x}, 0 \rangle$

between t_1 and t_2, $\vec{F}^{ext} = 0 \Rightarrow \int_{t_1}^{t_2} \vec{F}^{ext} dt = \Delta\vec{p} = \vec{p}(t_2) - \vec{p}(t_1) \Rightarrow m_1 v_{1,t_1,x} = (m_1+m_2) v_{sys,t_2,x}$

$\Rightarrow v_{sys,t_2,x} = \dfrac{m_1}{m_1+m_2} v_{1,t_1,x}$ velocity of system in x direction right after collision (before friction acts), as fn of $m_1, m_2, v_{1,i,x}$

Let's consider $t_2 = t_0 = 0$, and let's call $t_1 = i$ $\Rightarrow v_{sys,0,x} = \dfrac{m_1}{m_1+m_2} v_{1,i,x}$

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the system is subject to an external force $\vec{F}_k = \langle -\mu_k (m_1+m_2) g, 0 \rangle$ for time t_k, during which it moves a distance d.

$F_x^{ext} = F_k = m_{sys} \cdot a(t) \Rightarrow -\mu_k g(m_1+m_2) = (m_1+m_2) a(t) \Rightarrow a_x(t) = -\mu_k g$

$\Rightarrow v_x(t) = \dfrac{m_1}{m_1+m_2} v_{1,i,x} - \mu_k g t \Rightarrow v_x(t) = v_{sys,0,x} - \mu_k g t$

$x(t) = v_{sys,0,x} t - \dfrac{\mu_k g}{2} t^2$

find t_k, time to move distance d in x-direction under friction: $x(t) = d \Rightarrow \dfrac{\mu_k g}{2} t_k^2 - v_{sys,0,x} t_k + d = 0$

$\Delta = v_{sys,0,x}^2 - 4 \dfrac{\mu_k g}{2} d$ $\to$ For a real solution to exist we need $\Delta \geq 0 \Rightarrow \left[\dfrac{m_1}{m_1+m_2} v_{1,i,x}\right]^2 \geq 2\mu_k g d$

a) $v_{1,i,x} \geq \sqrt{\dfrac{m_1+m_2}{m_1} \cdot 2\mu_k g d}$ $\to$ min speed of bullet for bullet block system to move distance d after collision. note we will assume this is enough to fall off the surface. To be completely accurate, the system must move a certain distance beyond d for the system to fall off.

$t_k = \dfrac{v_{sys,0,x} - \left[v_{sys,0,x}^2 - 2\mu_k g d\right]^{1/2}}{\mu_k g}$ $\to$ time it takes for system to move distance d in x direction under effect of friction as fn of $m_1, m_2, v_{1,i,x}, \mu_k, g, d$

$\int_0^{t_k} \vec{F}^{ext} dt = \int_0^{t_k} \langle -\mu_k (m_1+m_2) g, 0 \rangle dt = \vec{p}(t_k) - \vec{p}(t_0)$

$\Rightarrow \int_0^{t_k} -\mu_k g(m_1+m_2) = (m_1+m_2) v_{sys,t_k,x} - m_1 v_{1,i,x}$

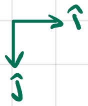
$-\mu_k g(m_1+m_2) t_k = (m_1+m_2) v_{sys,t_k,x} - m_1 v_{1,i,x}$

Because $v_{sys,0,x}$ is positive, a_x is negative, v_x decreases, and in theory becomes zero and then negative. This is why we have two solutions: the system moves distance d and beyond, then returns with negative velocity to the same position d. We are interested here in the first t_k, with the minus sign.

$\Rightarrow v_{sys,t_k,x} = \dfrac{m_1 v_{1,i,x}}{m_1+m_2} - \mu_k g t_k$ $\to$ velocity of system in x direction when fall starts as fn of $m_1, m_2, v_{1,i,x}, \mu_k, g, d$

Next we analyze the motion of the system in free fall.
Let's reset the time subscripts. $t = t_0 = 0$ is now the start of the fall.

$$\vec{v}_0 = \vec{v}(0) = \langle m_1 v_{1,i,x} - \mu_k g(m_1+m_2)t_k, 0\rangle$$



$$\vec{a}(t) = \langle 0, -g\rangle$$

$$\vec{v}(t) = \langle m_1 v_{1,i,x} - \mu_k g(m_1+m_2)t_k, -gt\rangle = \langle v_{sys,t_h,x}, gt\rangle$$

$$\vec{r}(t) = \langle v_{sys,t_h,x}\, t, gt^2/2\rangle = \langle x(t), y(t)\rangle$$

$$y(t) = h \Rightarrow gt^2 = 2h \Rightarrow t_h = \sqrt{\frac{2h}{g}}$$

$$x(t_h) = v_{sys,t_h,x}\, t_h \Rightarrow x(t_h) = \sqrt{\frac{2h}{g}}\,(m_1 v_{1,i,x} - \mu_k g(m_1+m_2)t_k)$$

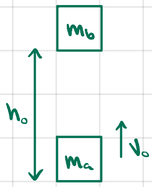
b) The bullet/block system hits the ground at a distance of $\sqrt{2gh}\,(m_1 v_{1,i,x} - \mu_k g(m_1+m_2)t_k)$ from the bottom edge of the surface, where

$$t_k = \frac{v_{sys,0,x} \pm [v_{sys,0,x}^2 - 2\mu_k g d]^{1/2}}{\mu_k g}$$

and

$$v_{sys,0,x} = \frac{m_1}{m_1+m_2} v_{1,i,x}$$

Problem 2 - Acrobat and Clown



System: $m_a + m_b$

external force: gravity

$p_{sys_0} = m_a v_0$

$p_{sys_1} = (m_a + m_b) v_1$

$$\int_0^{t_1} -m_a g\,dt = p_{sys}(t_1) - p_{sys}(t_0) = (m_a+m_b)v_1 - m_a v_0$$

$$-m_a g t_1 = (m_a+m_b)v_1 - m_a v_0$$

$$v_1 = \frac{m_a(v_0 - g t_1)}{m_a + m_b}$$

we don't know what t_1 is but it depends on h_0.

$a(t) = -g$

$v(t) = v_0 - gt$

$y(t) = v_0 t - gt^2/2$

$y(t) = h_0 \Rightarrow gt^2/2 - v_0 t + h_0 = 0$

$\Delta = v_0^2 - 4\frac{g}{2}h_0 \geq 0 \Rightarrow v_0 \geq \sqrt{2gh_0}$

$$t_1 = \frac{v_0 \pm [v_0^2 - 2gh_0]^{1/2}}{g}$$

$$\int_{t_1}^{t_f} -(m_a+m_b)g\,dt = p_{sys}(t_f) - p_{sys}(t_1) = (m_a+m_b).0 - (m_a+m_b)v_1$$

$$-\cancel{(m_a+m_b)}g(t_f - t_1) = -\cancel{(m_a+m_b)}v_1$$

$$gt_f = v_1 + gt_1 \Rightarrow t_f = t_1 + \frac{v_1}{g}$$

$\frac{v_1}{g}$ seconds pass after the grab. We want to know how high the system moves in that time.

$\Rightarrow$ $a(t) = -g$

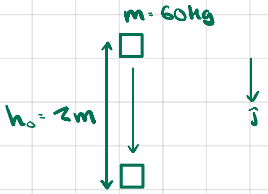
$v(t) = v_1 - gt$ * we could also have found t_f here: $v = 0 \Rightarrow t = \frac{v_1}{g}$

$y(t) = h_0 + v_1 t - gt^2/2$

$$y(v_1/g) = h_0 + \frac{v_1^2}{g} - \frac{v_1^2}{g^{\cancel{2}}}\cdot\frac{\cancel{g}}{2} = h_0 + \frac{v_1^2}{2g}$$

$\Rightarrow$ max height $= h_0 + \frac{v_1^2}{2g}$

Problem 3 - Compressive Strength of Bones



Force/Area to break bone = $\frac{F}{A} = 1.6\times10^8\ N/m^2$

$$\int_0^t mg\,dt = mv - 0 \Rightarrow mgt = mv \Rightarrow v = gt$$

t depends of course on the height h_0: $a(t) = g,\ v(t) = gt,\ y(t) = -h_0 + gt^2/2$ $\Rightarrow y(t) = 0 \Rightarrow gt^2 = 2h_0 \Rightarrow t = \sqrt{\frac{2h_0}{g}} \Rightarrow v = \sqrt{2gh_0}$

Just before collision, $v = \sqrt{2gh_0}$, $a = g$

During the collision, the center of mass (cm), which was accelerating at a rate of g, now decelerates bringing v from $\sqrt{2gh_0}$ to 0 in a distance of $d = 1cm$.

For the period of collision and deceleration:

$$a(t) = g - a$$
$$v(t) = \sqrt{2gh_0} + (g-a)t$$
$$y(t) = t\sqrt{2gh_0} + (g-a)t^2/2$$

Let's call t_e end of collision time:

$$v(t_e) = 0 \Rightarrow t_e = \frac{\sqrt{2gh_0}}{a-g}$$

$$y(t_e) = 0.01 = \frac{2gh_0}{a-g} + \frac{g-a}{\cancel{2}}\cdot\frac{\cancel{2}gh_0}{(a-g)^2} = \frac{2gh_0}{a-g} - \frac{gh_0}{a-g} = \frac{gh_0}{a-g}$$

$$\Rightarrow a - g = 100gh_0 \Rightarrow a = g(1+100h_0) \Rightarrow t_e = \frac{\sqrt{2gh_0}}{100gh_0}$$

a) $a = 9.8(1+100\cdot2) = 9.8\cdot201 = 1969.8$

collision time $= \frac{\sqrt{2\cdot9.8\cdot2}}{100\cdot9.8\cdot2} = 0.003194 \approx 0.0032$ secs

b) $\vec{F}_{ext} = m^T\vec{A}_{cm} = 60\,kg\cdot(-1960\ m/s^2) = -117600\ N$

alternatively we could calculate average force as

$$\frac{\int_{t'=0}^{t'=t_e} F^{ext}dt'}{t_e - 0} = \frac{\cancel{p(t_e)}^{0} - p(0)}{t_e - 0} = F^{ext}_{avg} = -\frac{p(0)}{t_e} = -\frac{60\sqrt{2\cdot9.8\cdot2}}{0.0032} = -117600\ N$$

$p(t_e) = 0$ $\quad p(0) = m\cdot\sqrt{2gh_0} = 60\sqrt{2\cdot9.8\cdot2}$

c) During the 0.0032 secs of collision, the person was under the effect of a gravitational force of $60\cdot9.8 = 588N$ that accelerated the person at a rate of $9.8\ m/s^2$. The person was also under the effect of a ground force of $60(-1969.8) = -118188\ N$ that accelerated the person $-1969.8\ m/s^2$. The resultant force was thus $-118188 + 588 = -117600\ N$ which accelerated the person $-1900\ m/s^2$ in ≈ 0.0032 s to bring the person from a speed of 6.26 m/s to 0 in a distance of 0.01m.
The ratio of the two external forces is $F_G/F_{ground} = 1/201$. If we had disregarded g in the calculations above we'd have that the acceleration required to stop the person in 0.01m would be $-1960\ m/s^2$ instead of 1969.8 m/s. Total average force would be -117600 N instead of 118188 N.

d) $\frac{|F|}{A} = \frac{118188\ N}{0.00032\ m^2} = 3.693\times10^8\ N/m^2 > 1.6\times10^8\ N/m^2 \Rightarrow$ person breaks ankle

The result would be similar disregarding g: $\frac{117600}{0.00032} = 3.675\times10^8\ N/m$

Problem 4 - Center of Mass of a Rod

a) uniform rod ⇒ constant density $\lambda(x) = \delta$

$$\int_0^L \lambda(x)\,dx = m \Rightarrow \delta L = m \Rightarrow$$

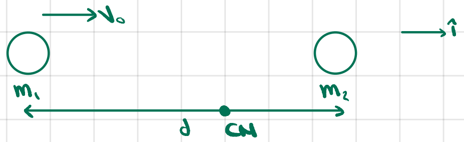
$$\vec{r}_{cm} = \frac{\int_{rod} \vec{r}_{dm}(x)\,dm}{\int_{rod} dm} = \frac{\int_0^L x\hat{\imath}\,\delta\,dx}{\int_0^L \delta\,dx} = \frac{\delta x^2/2\,\big|_0^L\,\hat{\imath}}{\delta x\,\big|_0^L} = \frac{L^2}{2L}\hat{\imath} = \frac{L}{2}\hat{\imath}$$

b) $\lambda(x) = \frac{\lambda_0}{L^3}x^3$

$$m = \int_0^L \frac{\lambda_0}{L^3}x^3\,dx = \frac{\lambda_0}{L^3}\frac{L^4}{4} = \frac{\lambda_0 L}{4} \Rightarrow \lambda_0 = \frac{4m}{L}$$

$$\vec{r}_{cm} = \frac{\int_0^L x\hat{\imath}\,\frac{\lambda_0}{L^3}x^3\,dx}{m} = \frac{\frac{4m}{L^4}\frac{L^5}{5}\hat{\imath}}{m} = \frac{4L}{5}\hat{\imath}$$

Problem 5 - Two Particles Colliding



$$\vec{F}_{21} = \frac{\beta}{x(t)^3} = m_1 a(t) = m_1 \frac{d^2 x(t)}{dt^2}$$

system: m_1, m_2

There are no net external forces.

$$F^{ext} = m_T \cdot a_{cm} = 0 \Rightarrow a_{cm} = 0$$

$$v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$x_{cm} = \frac{m_1(x_{1,0} + v_1 t) + m_2(x_{2,0} + v_2 t)}{m_1 + m_2}$$

Take the origin to be m_1's initial position.

$$x_{cm} = \frac{m_1 v_1 t + m_2(d + v_2 t)}{m_1 + m_2}$$

$$= \frac{m_2 d}{m_1 + m_2} + \frac{m_1 v_1 t + m_2 v_2 t}{m_1 + m_2}$$

$$x_{cm,0} = \frac{m_1 \cdot 0 + m_2 d}{m_1 + m_2}$$

$$= \frac{m_2 d}{m_1 + m_2}$$

$$x_{cm,t_1} = \frac{m_2 d}{m_1 + m_2} + \frac{m_1 v_1 t_1 + m_2 v_2 t_1}{m_1 + m_2}$$

$$\Delta x_{cm} = \frac{m_1 v_1 t_1 + m_2 v_2 t_1}{m_1 + m_2}$$

$$L = x_{cm,0} + \Delta x_{cm} = x_{cm,t_1}$$