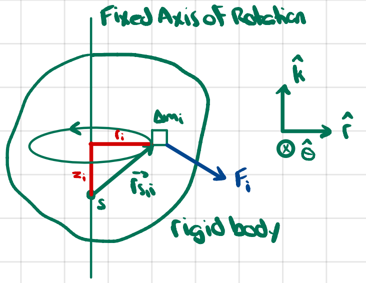



$\vec r_{s,i} = z_i\hat k + r_i\hat r$

$\vec\tau_{s,i} = \vec r_{s,i}\times\vec F_i$

$= (z_i\hat k + r_i\hat r)\times(F_{r,i}\hat r + F_{\theta,i}\hat\theta + F_{z,i}\hat k)$

$= z_iF_{r,i}(-\hat\theta) + z_iF_{\theta,i}(-\hat r) + r_iF_{\theta,i}\hat k + r_iF_{z,i}(-\hat\theta)$

Fixed axis rotation ⇒ resultant force on $\Delta m_i$ is in $\hat\theta$ direction
There must be other forces canceling the $F_{z,i}$ and $F_{r,i}$ components.

$(\vec\tau_{s,i})_z = r_iF_{\theta,i}\hat k$

$F_{\theta,i} = \Delta m_i\cdot a = \Delta m_i\cdot r_i\alpha_z$

$(\tau_{s,i})_z = \Delta m_i r_i^2\alpha_z$

$(\vec\tau_s)_z = \lim_{n\to\infty}\sum_{i=1}^{n}(\tau_{s,i})_z = I_s\alpha_z$

$\vec\tau_s = \vec\tau_s^{ext} + \vec\tau_s^{int}$

$\vec\tau_s^{ext} = \sum_{i=1}^{n}\vec\tau_{s,i}^{ext} = \sum\vec r_{s,i}\times\vec F_i^{ext}$

$\vec\tau_s^{int} = \sum_{i=1}^{n}\sum_{j\neq i}\vec r_{s,i}\times\vec F_{j,i}^{int}$

3rd law ⇒ $\vec F_{j,i} = -\vec F_{i,j}$

⇒ $\vec\tau^{int}_{s,j,i} + \vec\tau^{int}_{s,i,j} = \vec r_{s,i}\times\vec F^{int}_{j,i} + \vec r_{s,j}\times\vec F^{int}_{i,j}$

$= (\vec r_{s,i} - \vec r_{s,j})\times\vec F^{int}_{j,i}$

$= 0$ assuming that the pair of 3rd law forces act along the line joining the two particles. This is the 3rd law with an extra assumption.

⇒ $\vec\tau_s = \vec\tau_s^{ext}$

⇒ $(\vec\tau_s^{ext})_z = I_s\alpha_z$   Similar to $\vec F = m\vec a$

## 17.4.2 Torque Acts at Center of Gravity

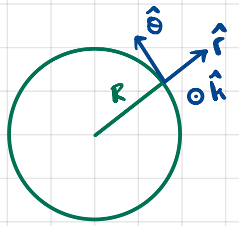
setup: rigid body, N particles, static equilibrium
coord. system origin O

$\vec\tau_O = \sum_{i=1}^{N}\vec\tau_{O,i} = \sum\vec r_i\times\vec F_{g,i} = \sum\vec r_i\times m_i\vec g = \vec 0$

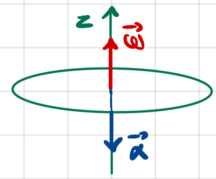
$\vec\tau_O = (\sum m_i\vec r_i)\times\vec g = M_T\cdot\vec r_{cm}\times\vec g = \vec r_{cm}\times M_T g$

Torque due to gravity on each point-like particle is equivalent to torque due to gravity on center of mass. Note the implicit assumption that $\vec g$ is the same at every point in the rigid body.

## Example 17.9




R: 13 cm
m = 1.2 kg




$I_s = 1.01\cdot10^{-2}\ \text{kg}\cdot\text{m}^2$

$f_i = 33\ \text{rpm} = 0.55\ \text{Hz}$

The turntable is spinning with $\omega_0 = f_i\cdot 2\pi = 1.10\pi$ rad/s
The motor is turned off and a torque is applied by India.

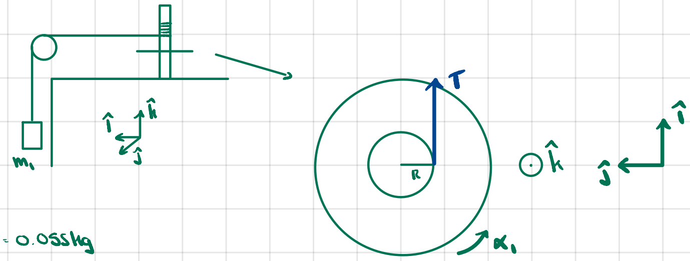
$\vec\tau_{s,f} = I_s\alpha_z\hat k$

$\vec\alpha = \alpha_z\hat k$
$\vec\omega = (\omega_0 + \alpha_z t)\hat k$
$\vec\theta = (\omega_0 t + \alpha_z t^2/2)\hat k$

$\omega_0 + 8\alpha_z = 0 \Rightarrow \alpha_z = -\frac{\omega_0}{8} = -\frac{1.10\pi}{8}$

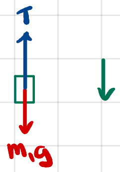
⇒ $\vec\tau_{s,f} = 1.01\cdot10^{-2}\,\frac{(-1.10\pi)}{8}\,\hat k$

Ex 17.11 Experimental Method for Determining Moment of Inertia




$m_1 = 0.055\,kg$

$R = 0.01277\,m$




$$(-R\hat{j}) \times (T\hat{i}) + \vec{\tau}_f = I_s \alpha_1 \hat{k} \quad , \quad \vec{\tau}_f = \tau_f \hat{k}$$

$$(m_1 g - T)\hat{i} = m_1 a \hat{i} = m_1 R \alpha_1 \hat{i}$$

$$\Rightarrow RT + \tau_f = I_s \alpha_1$$
$$m_1 g - T = m_1 R \alpha_1$$

unknowns: $T, I_s, \tau_f$

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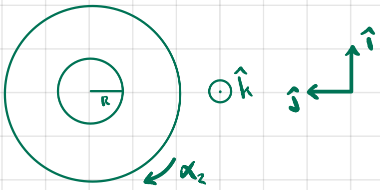



$$\vec{\tau}_f = I_s \vec{\alpha}_2$$

$$\vec{\alpha}_2 = -\alpha_2 \hat{k}$$
$$\vec{\omega} = (\omega_0 - \alpha_2 t)\hat{k}$$

$$\omega = 95 - \alpha_2 t$$
$$0 = 95 - \alpha_2 \cdot 2.82 \Rightarrow \alpha_2 = 33.68\ rad/s^2$$

$$\Rightarrow \vec{\tau}_f = -I_s \cdot 33.68\,\hat{k}$$

---

$$RT + \tau_f = I_s \alpha_1$$
$$m_1 g - T = m_1 R \alpha_1 \Rightarrow T = m_1(g - R\alpha_1)$$
$$\tau_f = -I_s \cdot 33.68$$

$$R m_1 (g - R\alpha_1) - 33.68 I_s = I_s \alpha_1$$

$$I_s = \frac{R m_1 (g - R\alpha_1)}{\alpha_1 + 33.68} = \frac{0.01277 \cdot 0.055(9.8 - 0.01277 \cdot 80.50)}{80.50 + 33.68} = 5.3 \cdot 10^{-5}\ kg \cdot m^2$$

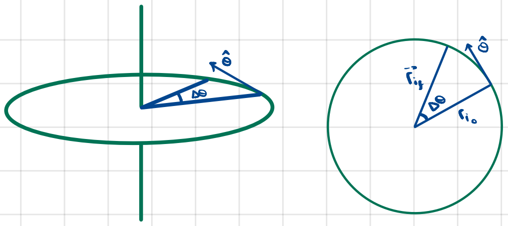
$$\alpha_1 \approx \frac{95}{1.18} \approx 80.50\ rad/s^2$$

## 17.5.1 Rotational Work

-> rigid body rotating about axis

-> each $\Delta m_i$ moves in circle of radius $(r_{S,i})_\perp$ undergoing $\Delta\theta$ angular displacement under action of $\vec{F}_{\theta,i} = F_{\theta,i}\hat{\theta}$

-> $\Delta \vec{r}_{S,i} \approx r_i \Delta\theta \hat{\theta}$




-> work done by tangential force

$$\Delta W_{rot,i} = \vec{F}_{\theta,i} \cdot \Delta\vec{r}_{S,i} \approx (F_{\theta,i}\hat{\theta}) \cdot (r_i \Delta\theta \hat{\theta}) = r_i F_{\theta,i} \Delta\theta$$

For a force $\vec{F}$ acting on $\Delta m_i$ undergoing fixed axis rotation, the latter axis being in dir. $\hat{n}$, we prev. derived

$$(\vec{\tau}_{S,i})_z = r_i F_{\theta,i} \hat{n}$$

$$\Rightarrow \Delta W_{rot,i} = (\tau_{S,i})_z \Delta\theta$$

$$W_{rot} = \sum_i \Delta W_{rot,i} = (\tau_S)_z \Delta\theta$$

$$\Delta\theta \to d\theta,\ \Delta W_{rot} \to dW_{rot} \Rightarrow dW_{rot} = \vec{\tau}_{S,z}\, d\theta$$

$$W_{rot} = \int dW_{rot} = \int_{\theta_i}^{\theta_f} \tau_{S,z}\, d\theta$$

Work is just force times displacement. Displacement is now related to an angular displacement. We find the resulting expression contains torque on the $i^{th}$ small mass.
Summing over all small masses, work done on the rigid body is the integral of torque over the angular displacement.

## 17.5.2 Rotational Work-Kinetic Energy Theorem

$$dW_{rot} = \tau_{S,z}\, d\theta = I_S \alpha_z\, d\theta$$

$$\alpha_z = \frac{d\omega_z}{dt} \qquad \omega_z = \frac{d\theta}{dt}$$