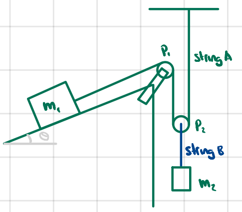
Problem set 4

Problem 1 - Block and Pulley System

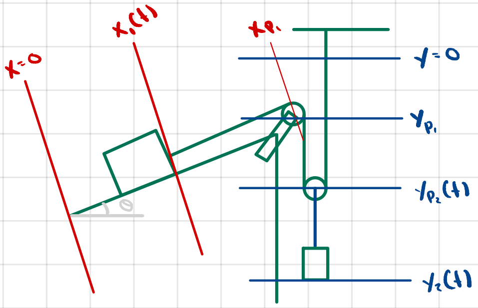


→ 2nd law equations

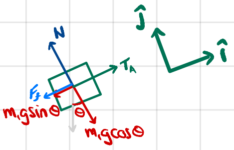
$T_A - m_1 g\sin\theta - \mu_s N = m_1 a_1$

$N - m_1 g\cos\theta = 0$

$m_2 g - 2T_A = m_2 a_2$



→ T_A, N, a_1, a_2 unknowns

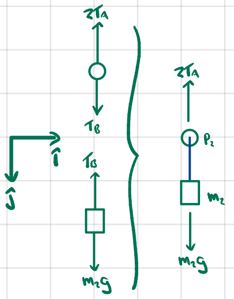


→ constraints

$\ell_A = \text{constant} = x_{P_1} - x_1(t) + \text{pulley 1 portion} + y_{P_2}(t) - y_{P_1} + \text{pulley 2 portion} + y_{P_2}(t)$

$= -x_1(t) + 2y_{P_2}(t) + C$

$\Rightarrow 0 = -a_1 + 2a_2 \Rightarrow a_1 = 2a_2$



→ All eq. together

$T_A - m_1 g\sin\theta - \mu_s N = m_1 a_1$

$N - m_1 g\cos\theta = 0$

$m_2 g - 2T_A = m_2 a_2$

$a_1 = 2a_2$

a) m_1 starts sliding up the block <=> friction acting on m_1 is at a max but resultant is zero on m_1. Any increase in tension causes accel. up.

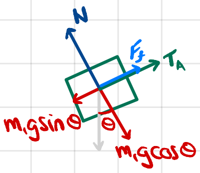
$a_1 = a_2 = 0$

$N = m_1 g\cos\theta \Rightarrow T_A = m_1 g\sin\theta + \mu_s m_1 g\cos\theta \Rightarrow T_A = m_1 g(\sin\theta + \mu_s\cos\theta)$

$2T_A = m_2(g - a_2) \Rightarrow m_2 = 2T_A/g$

$\Rightarrow m_2 = 2m_1(\sin\theta + \mu_s\cos\theta)$

b) m_1 starts sliding down the block <=> friction at max but now working with tension against acceleration down the block, resultant zero. m_2 increases T_A. We keep increasing m_2 until T_A + friction just balance $m_1 g\sin\theta$.



$T_A + \mu_s N - m_1 g\sin\theta = 0$

$N = m_1 g\cos\theta \qquad \Rightarrow T_A = m_1 g(\sin\theta - \mu_s\cos\theta)$

$m_2 g - 2T_A = 0$

$\Rightarrow m_2 = 2T_A/g = 2m_1(\sin\theta - \mu_s\cos\theta)$

* when is friction zero?

$T_A - m_1 g\sin\theta = 0 \Rightarrow T_A = m_1 g\sin\theta$

$\Rightarrow m_2 = 2m_1\sin\theta$

m_1 slides down | m_1 does not move | m_1 slides up

$m_2 = 2m_1(\sin\theta - \mu_s\cos\theta)$ | $2m_1\sin\theta$ $(F_f = 0)$ | $2m_1(\sin\theta + \mu_s\cos\theta)$

c) $T_A - m_1 g\sin\theta - \mu_k N = m_1 a_1 \Rightarrow T_A = m_1(a_1 + g\sin\theta) + \mu_k m_1 g\cos\theta = m_1(a_1 + g\sin\theta + \mu_k g\cos\theta)$

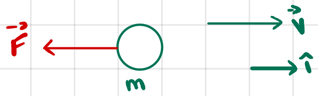
$N - m_1 g\cos\theta = 0$

$m_2 g - 2T_A = m_2 a_2 \qquad \Rightarrow a_2 = g - \frac{2T_A}{m_2} = g - 2\frac{m_1}{m_2}(a_1 + g\sin\theta + \mu_k g\cos\theta)$

$a_1 = 2a_2$

$\Rightarrow a_1 = 2a_2$

Problem 2 - Velocity Dependent Force



$\vec{F} = -b e^{cv} \hat{\imath}$ b N >0

c $m^{-1}s$ >0

$v(0) = v_0$

$$F = ma \Rightarrow -b e^{cv} = m \frac{dv}{dt} \Rightarrow \frac{-b}{m} dt = e^{-cv} dv$$

$$-\frac{b}{m} t = \left. \frac{-e^{-cv}}{c} \right|_{v_0}^{v(t)} = -\frac{1}{c}\left[e^{-cv(t)} - e^{-cv_0}\right]$$

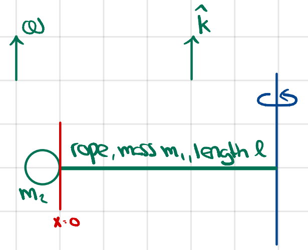
$$\frac{bc}{m} t + e^{-cv_0} = e^{-cv(t)}$$

$$\ln\left[\frac{bc}{m} t + e^{-cv_0}\right] = -cv(t) \Rightarrow v(t) = -\frac{1}{c} \ln\left[\frac{bc}{m} t + e^{-cv_0}\right]$$

$$v(t) = 0 \Leftrightarrow \frac{bct}{m} + e^{-cv_0} = 1 \Rightarrow t = \frac{m}{bc}\left(1 - e^{-cv_0}\right)$$

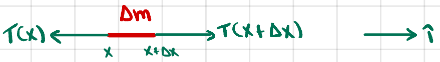
Example: $b = c = 1$, $m = 10$, $v_0 = 5$ $\Rightarrow$ $t_{v=0} = 9.93$ s

Problem 3 - Tension in Massive Rotating Rope with Object



Assumptions

- radius of rotating shaft is negligible compared to ℓ
- disregard gravity; we assume the rotation occurs at such a high ω that the rope is practically horizontal



$a_r = x\omega^2$

$\Delta m = \frac{\Delta x}{\ell} m_1$

$T(x+\Delta x) - T(x) = \Delta m \cdot a_r = \frac{\Delta x}{\ell} m_1 x \omega^2$

$$\frac{T(x+\Delta x) - T(x)}{\Delta x} = \frac{m_1 x \omega^2}{\ell}$$

$$\frac{dT(x)}{dt} = \lim_{\Delta x \to 0} \frac{T(x+\Delta x) - T(x)}{\Delta x} = \frac{m_1 x \omega^2}{\ell}$$

$$\int_{x'=0}^{x'=x} \frac{dT(x')}{dt} dx' = \frac{m_1 \omega^2}{\ell} \int_{x'=0}^{x'=x} x' dx'$$

$$\Rightarrow T(x) - T(0) = \frac{m_1 \omega^2 x}{2\ell}$$

To find $T(0) = T_0$, we assume m_2 is a point mass

$\bullet \longrightarrow T_0$ $\quad T_0 = m_2 \ell \omega^2$

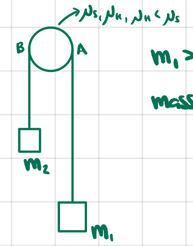
$$\Rightarrow T(x) = m_2 \ell \omega^2 + \frac{m_1 \omega^2 x}{2\ell} = T(x) = \omega^2 \left(m_2 \ell + \frac{m_1 x}{2\ell} \right)$$

If we want T as a function of $r = \ell - x \Rightarrow x = \ell - r$

$$T(r) = \omega^2 \left[m_2 \ell + \frac{m_1(\ell - r)}{2\ell} \right] = \omega^2 \left[m_2 \ell + \frac{m_1}{2} - \frac{m_1 r}{2\ell} \right] = \frac{\omega^2}{2} \left[2m_2 \ell + m_1 - \frac{m_1 r}{\ell} \right]$$

$$\Rightarrow T(r) = \frac{\omega^2}{2} \left[2m_2 \ell + m_1 \left(1 - \frac{r}{\ell}\right) \right]$$

Problem 4 - Tension in Rope Wrapped Around a Rod



$m_1 > m_2$

massless rope

From the capstan example (8.11) we know that if the rope is about to slide to the right, then the tension at any point of the rope in contact with the capstan and Θ rad around from point B (where we set $\Theta = 0$) is:

$$T(\Theta) = T_B e^{\mu_s \Theta}$$

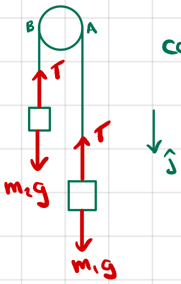
In particular $T(0) = T_B = m_2 g$, $T(\Theta_A) = T(\pi) = T_A = m_2 g e^{\pi \mu_s}$

a) Because the rope is massless and at any point between A and m_1 the only forces acting are tension forces, the tension is constant and equal to $m_1 g$. $T_A = T_1$.

b) Rope starts sliding $\Leftrightarrow$ $T(\Theta_A) = m_1 g > m_2 g e^{\Theta_A \mu_s}$ $\Rightarrow$ $m_1 > m_2 e^{\Theta_A \mu_s}$

c) Rope is sliding $\Rightarrow$ $T(\Theta_A) = m_2 g e^{\Theta_A \mu_k}$, ie smaller than before at each point on the rope in contact with the capstan.

Note: With no friction, the tension on the massless rope



constraint: $\ell = y_2 - y_p + \pi r^2 + y_1 - y_p$

$\Rightarrow 0 = a_2 + a_1 \Rightarrow a_1 = -a_2$

note that in defining this constraint we assumed $\downarrow \hat{j}$ coord. system.

$$\begin{cases} m_1 g - T = m_1 a_1 \\ T - m_2 g = m_2(-a_2) \end{cases} \Rightarrow \begin{array}{l} T = m_1(g - a_1) \\ T = m_2(g - a_2) = m_2(g + a_1) \end{array} \Rightarrow m_1 g - m_1 a_1 = m_2 g + m_2 a_1 \Rightarrow a_1(m_2 + m_1) = g(m_1 - m_2)$$

$$\Rightarrow a_1 = \frac{g(m_1 - m_2)}{m_1 + m_2} \qquad a_2 = \frac{g(m_2 - m_1)}{m_1 + m_2} \qquad T = \frac{2g m_1 m_2}{m_1 + m_2}$$

$\Rightarrow$ zero acceleration if $m_1 = m_2$.

If we have friction, it "helps" one of the sides. For example say we start with $m_1 = m_2$. $T = mg$, there is no acceleration and no friction. If we now increase m_1 by Δm, we have a new force acting: static friction. At each point that the rope touches the surface, the tension on the right must equilibrate the sum of friction and tension on the left. Therefore tension increases along the capstan, in fact exponentially.

Each small change in tension (ie increase) increases on the next small piece of rope the tension to offset, the latter increases the normal force which increases the friction.

When we derived $T(\Theta)$ for a rope on a capstan, we assumed that on each infinitesimal piece of rope friction was maximum at $\mu_s N$. $T(\Theta_A)$ is the maximum tension at A that sustains no acceleration. With tension at $T(\Theta)$ the tension increases to counterbalance the combined tension and friction in the other direction. Therefore $T(\Theta_A)$ is the max weight of m_1. $m_{1_{max}} g = m_2 g e^{\Theta_A \mu_s} \Rightarrow m_{1_{max}} = m_2 e^{\Theta_A \mu_s}$.

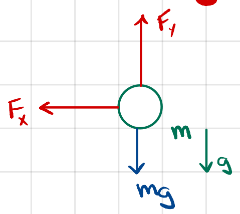
If $m_1 > m_{1_{max}}$ the rope slides and accelerates because now $\mu = \mu_k < \mu_s$ so the max friction is lower on each small piece of rope. Tension at Θ_A is now $m_1 g$, larger than $T(\Theta_A)$ calculated previously. However we need to calculate a new $T_k(\Theta_A)$, because actually $m_1 g$ could be $T_k(\Theta_A) < m_1 g < T(\Theta_A)$ and we still get sliding and acceleration.

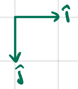
Basically the same derivation as before but with $\mu = \mu_k$ instead of μ_s gives us $T_k(\Theta_A) = T_B e^{\mu_k \Theta_A}$

If we assume $T(\Theta_A) = m_2 g e^{\pi \mu_s}$ was just breached but then restored to that value, we have:

$$T(\Theta_A) - T_k(\Theta_A) = m_1 a \Rightarrow m_2 g e^{\pi \mu_s} - m_2 g e^{\pi \mu_k} = m_1 a \Rightarrow a = \frac{m_2}{m_1} g \left(e^{\pi \mu_s} - e^{\pi \mu_k}\right)$$

Problem 5 - Drag Forces at Low Speeds





$\vec{F} = -c\vec{v}$

$\vec{v}(0) = v\hat{\imath}$

a) $\vec{F} = m\vec{a} \Rightarrow \hat{\imath}: \quad -cV_x(t) = ma_x(t) \Rightarrow a_x(t) = \frac{-cV_x(t)}{m}$

b) $\hat{\jmath}: \quad mg - cV_y(t) = ma_y(t) \Rightarrow a_y(t) = mg - cV_y(t)$

c) $\frac{dV_x}{dt} = \frac{-c}{m}V_x \Rightarrow V_x^{-1}\,dV_x = -\frac{c}{m}dt \Rightarrow \int_{V_{x_0}}^{V_x(t)} V_x^{-1}\,dV_x = \int_0^t -\frac{c}{m}dt \Rightarrow \ln(V_x(t)) - \ln(V_{x_0}) = -\frac{ct}{m}$

$\Rightarrow \ln\left[\frac{V_x(t)}{v}\right] = -\frac{ct}{m} \Rightarrow \frac{V_x(t)}{v} = e^{-\frac{ct}{m}} \Rightarrow V_x(t) = ve^{-\frac{ct}{m}}$

$mg/c\left(1 - V_f\frac{g}{mg}\right)$

d) $\frac{dV_y(t)}{dt} = mg - cV_y(t) = c\left[\frac{mg}{c} - V_y(t)\right]$

$\Rightarrow \frac{1}{\frac{mg}{c} - V_y(t)}dV_y = c\,dt \Rightarrow \int_{V_{y_0}=0}^{V_y(t)} \frac{1}{\frac{mg}{c} - V_y(t)}dV_y = \int_0^t c\,dt$

$\Rightarrow -\ln\left[\frac{mg}{c} - V_y(t)\right] + \ln\left[\frac{mg}{c}\right] = ct \Rightarrow \ln\left[\frac{mg/c}{mg/c - V_y(t)}\right] = \ln\left[\frac{1}{1 - \frac{gV_y(t)}{m}}\right] = ct$

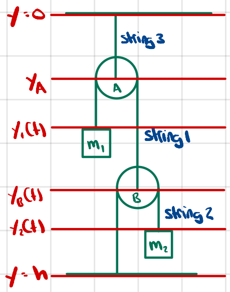
$\frac{1}{1 - \frac{gV_y(t)}{m}} = e^{ct} \Rightarrow 1 = e^{ct} - \frac{g}{m}e^{ct}V_y(t) \Rightarrow \frac{g}{m}e^{ct}V_y(t) = e^{ct} - 1$

$\Rightarrow V_y(t) = \frac{m(e^{ct} - 1)}{ge^{ct}} = \frac{m}{g} - \frac{m}{ge^{ct}}$

e) $V_x(t \to \infty) = V_{x_T} = 0$

f) $V_y(t \to \infty) = V_{y_T} = \frac{m}{g}$

Problem 6 – Two Pulleys, Two Strings, Two Blocks



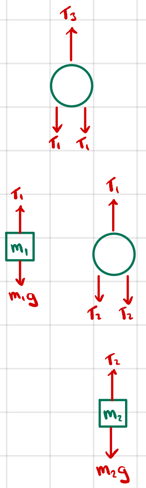
constraint equations

$\ell_1 = \text{constant} = y_1(t) - y_A + \pi r^2 + y_B(t)$

$0 = a_1 + a_B \Rightarrow a_1 = -a_B \Rightarrow a_1 = -\frac{a_2}{2}$

$\ell_2 = \text{constant} = h - y_B(t) + \pi r^2 + y_2(t) - y_B(t)$

$0 = -a_B + a_2 - a_B \Rightarrow a_2 = 2a_B$



1. $T_1 - m_1 g = m_1 a_1$
2. $T_3 - 2T_1 = 0$
3. $2T_2 - T_1 = 0$
4. $m_2 g - T_2 = m_2 a_2$
5. $a_1 = -\frac{a_2}{2}$

3. $\Rightarrow T_2 = T_1/2$

4. $\Rightarrow a_2 = g - \frac{T_2}{m_2} \Rightarrow a_2 = g - \frac{T_1}{2m_2}$

1. $\Rightarrow T_1 = m_1(a_1 + g) = m_1\left(g - \frac{a_2}{2}\right)$

$$\Rightarrow a_2 = g - \frac{m_1}{2m_2}\left(g - \frac{a_2}{2}\right) = \frac{2m_2 g - m_1 g + m_1 a_2/2}{2m_2}$$

$$\Rightarrow 2m_2 a_2 = 2m_2 g - m_1 g + \frac{m_1 a_2}{2}$$

$$\Rightarrow 4m_2 a_2 - m_1 a_2 = 4m_2 g - 2m_1 g$$

$$\Rightarrow a_2(4m_2 - m_1) = 2g(2m_2 - m_1) \Rightarrow a_2 = \frac{2g(2m_2 - m_1)}{4m_2 - m_1}$$

$$a_1 = \frac{-g(2m_2 - m_1)}{4m_2 - m_1}$$