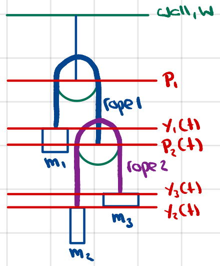
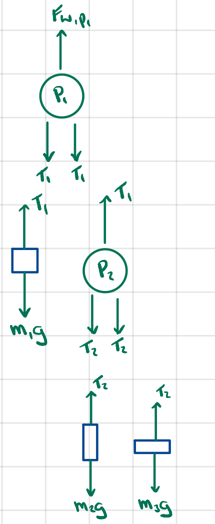
Example 8.9



$\ell_1 = y_1(t) - P_1 + \pi R + P_2(t) - P_1 = \text{constant} \Rightarrow 0 = a_1 + a_{P_2} \Rightarrow a_1 = -a_{P_2}$

$\ell_2 = y_2(t) - P_2(t) + \pi R + y_3(t) - P_2(t) = \text{constant} \Rightarrow a_2 - a_{P_2} + a_3 - a_{P_2} = 0 \Rightarrow a_2 + a_3 = 2a_{P_2} = -2a_1$

$\Rightarrow a_2 + a_3 = -2a_1$



$m_1 g - T_1 = m_1 a_1$
$2T_2 - T_1 = 0 \Rightarrow T_1 = 2T_2$
$m_2 g - T_2 = m_2 a_2$
$m_3 g - T_2 = m_3 a_3$
$a_2 + a_3 = -2a_1$
$g(m_2+m_3) - T_1 = (m_2+m_3)a_{P_2}$

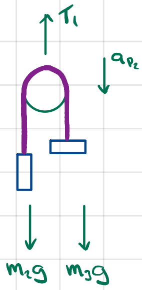
$\Rightarrow$

1. $T_1 = m_1(g - a_1)$
2. $T_1 = 2T_2$
3. $T_2 = m_2(g - a_2)$
4. $T_2 = m_3(g - a_3)$
5. $T_1 = (m_2+m_3)(g - a_{P_2})$
6. $a_2 + a_3 = -2a_1$
7. $a_1 = -a_{P_2}$

unknowns: $T_1, T_2, a_1, a_2, a_3, a_{P_2}$

Solving the system of equations
- 1., 5., and 7. are 3 eq. in 3 unknowns T_1, a_1, a_{P_2}.
- Then, 2. gives T_2
  3. gives a_2
  4. gives a_3

We did not need 6.



- If we only use 2nd law eq. for individual objects (i.e. we drop eq. 5), we have 6 eq and 6 unknowns, so we could still solve though it'd be more complicated.

Using the first path of solving the eq.

$(m_2+m_3)(g+a_1) = m_1(g-a_1) \Rightarrow g(m_2+m_3) + a_1(m_2+m_3) = m_1 g - m_1 a_1 \Rightarrow a_1(m_2+m_3+m_1) = g(m_1-m_2-m_3)$

$\Rightarrow a_1 = \dfrac{g(m_1-m_2-m_3)}{m_1+m_2+m_3}$, $a_{P_2} = -a_1$, $T_1 = m_1 g - m_1 a_1 = \dfrac{m_1 g(m_1+m_2+m_3) - m_1 g(m_1-m_2-m_3)}{m_1+m_2+m_3}$

$= \dfrac{\cancel{m_1^2 g} + m_1 m_2 g + m_1 m_3 g - \cancel{m_1^2 g} + m_1 m_2 g + m_1 m_3 g}{m_1+m_2+m_3} = \dfrac{2g(m_1 m_2 + m_1 m_3)}{m_1+m_2+m_3}$

Solution using only individual object force diagrams

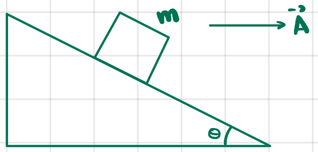
$m_1 g - T_1 = m_1 a_1$ | $a_1 = g - \dfrac{2T_2}{m_1}$
$2T_2 - T_1 = 0 \Rightarrow T_1 = 2T_2 \Rightarrow T_1 = 2T_2 \Rightarrow$
$m_2 g - T_2 = m_2 a_2$ | $a_2 = g - \dfrac{T_2}{m_2}$
$m_3 g - T_2 = m_3 a_3$
$a_2 + a_3 = -2a_1$ | $a_3 = g - \dfrac{T_2}{m_3}$

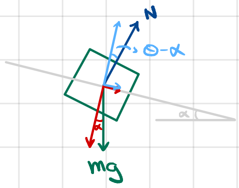
sub a_1, a_2, a_3 into constraint eq. $\quad g - \dfrac{T_2}{m_2} + g - \dfrac{T_2}{m_3} + 2g - \dfrac{4T_2}{m_1} = 0$

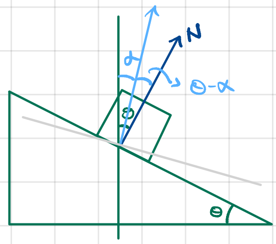
$\Rightarrow \dfrac{-T_2 m_1 m_3 - T_2 m_1 m_2 - 4T_2 m_2 m_3}{m_1 m_2 m_3} = -4g \Rightarrow T_2 = \dfrac{4g m_1 m_2 m_3}{m_1 m_3 + m_1 m_2 + 4m_2 m_3}$

Example 8.10 Accelerating Wedge

Note: this was my attempt at a solution. It is correct I believe, but the textbook solution on the next page is much simpler, more straightforward, and more general.







→ The textbook solution obtains equations for x and y components of acceleration by applying the 2nd law in those directions.

→ Here we assume that because accelerations are constant along both x and y directions, the resultant acceleration will be along a line passing through the initial point of the block and forming an angle $0 \le \alpha \le \theta$ with the horizontal direction.

For this to occur, namely acceleration along such a line, we need:

$$N\cos(\theta-\alpha) = mg\cos\alpha$$

$$N\sin(\theta-\alpha) + mg\sin\alpha = ma_\alpha$$

$$\Rightarrow N = \frac{mg\cos\alpha}{\cos(\theta-\alpha)}$$

Here we are determining: given slope of angle θ and that acceleration of the block occurs at an angle α, what is the required normal force on

Given such an N, we have

$$a_\alpha(\alpha,\theta,g,m) = \frac{N}{m}\sin(\theta-\alpha) + g\sin\alpha = g\cos\alpha\frac{\sin(\theta-\alpha)}{\cos(\theta-\alpha)} + g\sin\alpha = \frac{g\cos\alpha\sin(\theta-\alpha) + g\sin\alpha\cos(\theta-\alpha)}{\cos(\theta-\alpha)}$$

Given m and g, a slope θ, a normal force that together with gravitational force is a contributor to acceleration at an angle α, the magnitude of the acceleration is a_α.

As we derive on the next page we have a constraint eq.:

$a_{b,y} = \tan\theta(A_{w,x} - a_{b,x})$ (constraint eq.) $\Rightarrow a_{\alpha_y} = \tan\theta(A_{w,x} - a_{\alpha_x}) \Rightarrow a_\alpha\sin\alpha = \tan\theta(A_{w,x} - a_\alpha\cos\alpha) \Rightarrow A_{w,x} = \dfrac{a_\alpha[\cos\alpha\tan\theta + \sin\alpha]}{\tan\theta}$

recap:

$$a_\alpha(\alpha,\theta,g) = \frac{g\cos\alpha\sin(\theta-\alpha) + g\sin\alpha\cos(\theta-\alpha)}{\cos(\theta-\alpha)} \qquad A_{w,x}(\alpha,\theta) = \frac{a_\alpha[\cos\alpha\tan\theta + \sin\alpha]}{\tan\theta}$$

*** Alternative solution (next page)**

$$a_{b,x}(g,A_{w,x},\theta) = \frac{g + A_{w,x}\tan\theta}{\cot\theta + \tan\theta}$$

$$a_{b,y}(g,A_{w,x},\theta) = \frac{A_{w,x} - g\tan\theta}{\cot\theta + \tan\theta}$$

examples:

$\alpha = 0$, $\theta = \frac{\pi}{4}$, ie no vertical acceleration.

$a_\alpha(0, \pi/4, g, m) = g$

$A_{w,x}(0, \pi/4) = a_\alpha = g$

Alternative solution

$a_{b,x}(g, g, \pi/4) = g$

$a_{b,y}(g, g, \pi/4) = 0$

now let's go the other way around

1.1+3/(cos θ

$A_{w,x} = 5$, $\theta = \pi/3 \Rightarrow a_{b,x}(9.8, 5, \pi/3) = 7.9938$

$a_{b,y}(9.8, 5, \pi/3) = -5.1855$

Check the resultant acceleration's angle with horizontal.

$\vec{a} = \langle 7.9938, -5.1855\rangle \qquad \cos\alpha = \dfrac{\vec{a}\cdot\hat{i}}{|\vec{a}|} \Rightarrow \alpha = 0.5754$ rad

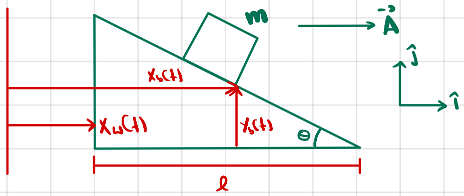
$|\vec{a}| = 9.5284$

$$a_\alpha = \frac{a_{\alpha x}}{\cos\alpha} = \frac{a_{b,x}}{\cos\alpha} = \frac{a_{b,y}}{\sin\alpha} \Rightarrow \tan\alpha = \frac{a_{b,y}}{a_{b,x}} \Rightarrow \alpha = \tan^{-1}\frac{a_{b,y}}{a_{b,x}}$$

$\Rightarrow \alpha = -0.5754$ rad

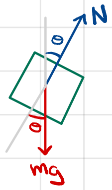
$a_\alpha = -9.5285$

Example 8.10 - Textbook Solution



$$\tan\theta = \frac{y_b(t)}{\ell - (x_b(t) - x_w(t))} \Rightarrow y(t) = [\ell - (x_b(t) - x_w(t))]\tan\theta$$

Differentiate twice: $a_{b,y} = \tan\theta\,(A_{w,x} - a_{b,x})$ (constraint eq.)



$$N\sin\theta = ma_{b,x} \qquad \Rightarrow N = \frac{ma_{b,x}}{\sin\theta}$$

$$N\cos\theta - mg = ma_{b,y}$$

$$\frac{\cancel{m}a_{b,x}}{\sin\theta}\cos\theta - \cancel{m}g = \cancel{m}a_{b,y} \qquad \text{sub in the constraint}$$

$$\frac{a_{b,x}}{\tan\theta} - g = \tan\theta\,(A_{w,x} - a_{b,x})$$

$$a_{b,x}\left[\frac{1}{\tan\theta} + \tan\theta\right] = g + A_{w,x}\tan\theta$$

$$\Rightarrow a_{b,x} = \frac{g + A_{w,x}\tan\theta}{\cot\theta + \tan\theta}$$

$$\Rightarrow a_{b,y} = \tan\theta\,\frac{A_{w,x}\cot\theta + \cancel{A_{w,x}\tan\theta} - g - \cancel{A_{w,x}\tan\theta}}{\cot\theta + \tan\theta}$$

$$a_{b,y} = \frac{A_{w,x} - g\tan\theta}{\cot\theta + \tan\theta}$$

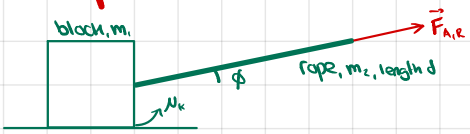
With $\theta = 45°$ we have

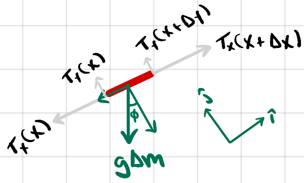
$$a_{b,x} = \frac{g + A_{w,x}}{2} \qquad a_{b,y} = \frac{A_{w,x} - g}{2}$$

Edge Cases

$$a_{b,y} = 0 \Rightarrow A_{w,x} = g\tan\theta \Rightarrow a_{b,x} = \frac{g(1+\tan^2\theta)}{\cot\theta + \tan\theta} = g\tan\theta$$

Example 8.3 - Tension in a Massive Rope



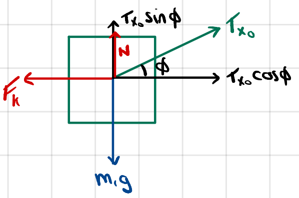


$T_x(x+\Delta x) - T_x(x) - g\Delta m \sin\phi = 0 \Rightarrow T_x(x+\Delta x) - T_x(x) = \frac{g\Delta x\, m_2 \sin\phi}{d}$

$\Delta m = \frac{\Delta x}{d} m_2$

$$\Rightarrow \frac{T_x(x+\Delta x)-T_x(x)}{\Delta x} = \frac{g m_2 \sin\phi}{d} \Rightarrow \frac{dT_x(x)}{dx} = \frac{g m_2 \sin\phi}{d} \Rightarrow dT_x = \frac{g m_2 \sin\phi}{d}\,dx \Rightarrow \int_{T(0)}^{T(x)} dT_x = \int_0^x \frac{g m_2 \sin\phi}{d}\,dx \Rightarrow T_x(x) - T_x(0) = \frac{g m_2 \sin\phi}{d}x$$

What is $T_x(0)$, ie tension in x direction at contact point with m_1?



$$\begin{cases} T_{x_0}\sin\phi + N - m_1 g = 0 \Rightarrow N = m_1 g - T_{x_0}\sin\phi \\ T_{x_0}\cos\phi - \mu_k N = m_1 a \end{cases}$$

unknowns: T_{x_0}, N parameters: ϕ, μ_k, m_1, a

$\Rightarrow T_{x_0}\cos\phi - \mu_k m_1 g + \mu_k T_{x_0}\sin\phi = m_1 a$

$T_{x_0}(\cos\phi + \mu_k \sin\phi) = m_1(a + \mu_k g) \Rightarrow T_{x_0} = \frac{m_1(a+\mu_k g)}{\cos\phi + \mu_k\sin\phi}$

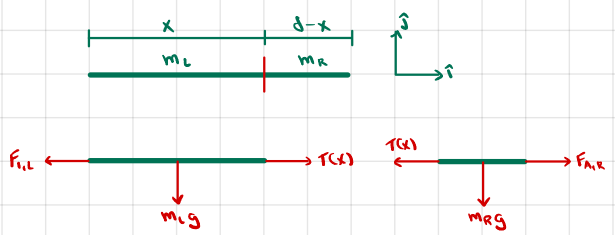
$$\Rightarrow T_x(x) = \frac{m_1(a+\mu_k g)}{\cos\phi + \mu_k\sin\phi} + \frac{g m_2 \sin\phi}{d}x$$

There is, however, a component of $g\Delta m$ in the y direction. Because there is no acceleration in the y direction, there must be a component of tension force in that direction

$$T_y(x+\Delta x) + T_y(x) - g\Delta m\cos\phi = 0 \Rightarrow T_y(x) - T_y(0) = \frac{g m_2 \cos\phi}{d}x, \quad T_{y_0} = 0 \Rightarrow T_y(x) = \frac{g m_2\cos\phi}{d}x$$

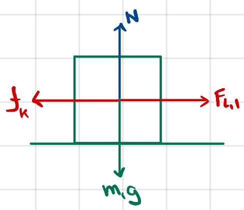
Textbook Solution

→ Assume ϕ very small: pulling and tension forces essentially act in horizontal direction.



2nd law

1. $F_{A,R} - T(x) = m_R a_R = \frac{d-x}{d} m_2 a_R$
2. $T(x) - F_{1,L} = m_L a_L = \frac{x}{d} m_2 a_L$



2nd law 3. $F_{L,1} - \mu_k m_1 g = m_1 a_1$

3rd law 4. $F_{L,1} = F_{1,L}$

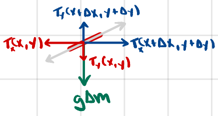
4., 3. into 2. $\Rightarrow T(x) - \mu_k m_1 g - m_1 a_1 = \frac{x}{d} m_2 a_L$

But $a = a_1 \equiv a_L$ because rope and block move together.

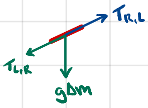
$$\Rightarrow T(x) = \mu_k m_1 g + a\left(m_1 + \frac{x}{d} m_2\right)$$

Is the second solution a special case of the first?

	First solution		Second solution
$\phi = 0$	$T_x(x) = m_1(a+\mu_k g)$ $T_y(x) = \frac{g m_2}{d}x$	$\Rightarrow T_{HORIZ}(x) = T_x(x)\cos\phi + T_y(x)\sin\phi = T_x(x)$ $T_{VERT}(x) = T_x(x)\sin\phi + T_y(x)\cos\phi = T_y(x)$	$T_{HORIZ}(x) = \mu_k m_1 g + a\left(m_1 + \frac{x}{d}m_2\right)$ $T_{VERT} = 0$
$\phi = \pi/4$	$T_x(x) = \frac{m_1(a+\mu_k g)}{\frac{\sqrt{2}}{2}(1+\mu_k)}$ $T_y(x) = \frac{g m_2 \sqrt{2}\,x}{2d}$	$T_{HORIZ}(x) = T_{VERT}(x)$ $= \frac{m_1(a+\mu_k g)}{1+\mu_k} + \frac{g m_2 x}{2d}$	not applicable



$$\tan\phi = \frac{\Delta y}{\Delta x}$$



$$T_x(x+\Delta x, y+\Delta y) - T_x(x,y) = 0$$

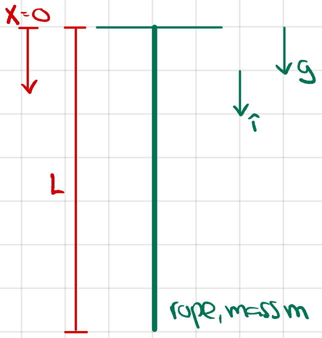
$$T_y(x+\Delta x, y+\Delta y) - T_y(x,y) - g\Delta m = 0$$

$$\Delta m = \frac{\sqrt{\Delta x^2 + \Delta y^2}}{d} m_2$$

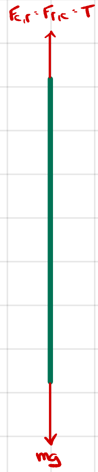
$$\Rightarrow T_y(x+\Delta x, y+\Delta y) - T_y(x,y) = \frac{\sqrt{\Delta x^2 + \Delta y^2}}{d} m_2 g$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{T_y(x+\Delta x, y+\Delta y) - T_y(x,y)}{\sqrt{\Delta x^2 + \Delta y^2}} = D_u T_y(x,y) = \frac{m_2 g}{d} = \nabla T_y(x,y) \cdot \vec{u} = \frac{dT_y}{dx} dx + \frac{dT_y}{dy} dy \,, \quad \frac{dy}{dx} = \tan\phi$$

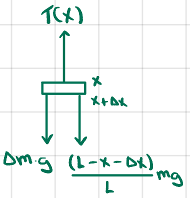
Example 8.4 Tension in a Suspended Rope



a) $mg - T = 0 \Rightarrow T = mg$



b) Let's consider a piece of rope located at position x, with mass Δm.

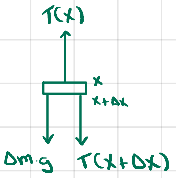


$$\Delta m \cdot g + \frac{L-x-\Delta x}{L} mg - T(x) = 0$$

$$\Delta m = \frac{\Delta x}{L} m \Rightarrow T(x) = \frac{\Delta x\, mg}{L} + \frac{L-x-\Delta x}{L} mg = \frac{mg}{L}(L-x)$$

$$T(x) = \frac{L-x}{L} mg$$

c)



$$\frac{\Delta x}{L} mg + T(x+\Delta x) - T(x) = 0 \Rightarrow \lim_{\Delta x \to 0} \frac{T(x+\Delta x) - T(x)}{\Delta x} = -\frac{mg}{L} \Rightarrow \frac{dT(x)}{dx} = -\frac{mg}{L}$$

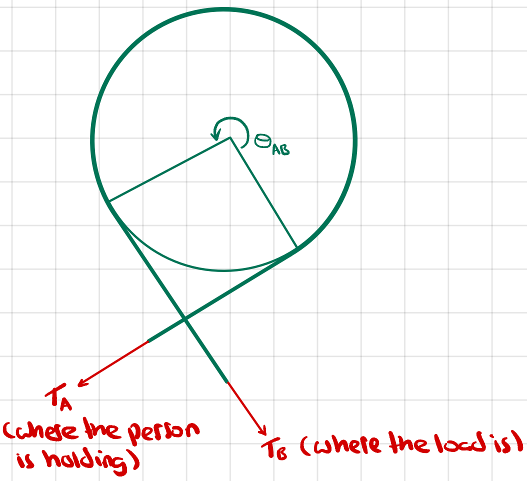
$$\Rightarrow \int_{T(0)}^{T(x)} dT = \int_0^x -\frac{mg}{L} dx \Rightarrow T(x) - T(0) = -\frac{mgx}{L}$$

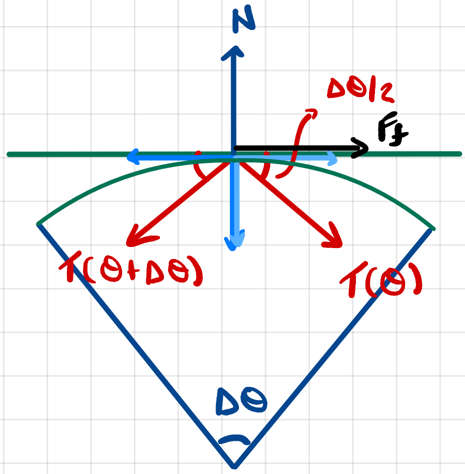
Since $T(0) = mg$, $T(x) = mg(1 - x/L) = \frac{mg}{L}(L-x)$

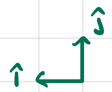
Alternatively, just differentiate $T(x) = \frac{mg}{L}(L-x)$ found previously.

$$\frac{dT(x)}{dx} = -\frac{mg}{L}$$

Example 8.11 - Capstan







Apply 2nd law

$$N - T(\theta)\sin(\Delta\theta/2) - T(\theta+\Delta\theta)\sin(\Delta\theta/2) = 0$$

$$\Rightarrow N = \sin(\Delta\theta/2)\left(T(\theta) + T(\theta+\Delta\theta)\right)$$

$$T(\theta+\Delta\theta)\cos(\Delta\theta/2) - T(\theta)\cos(\Delta\theta/2) - \mu_s N = 0$$

$$\cos(\Delta\theta/2)\left(T(\theta+\Delta\theta) - T(\theta)\right) = \mu_s \sin(\Delta\theta/2)\left(T(\theta) + T(\theta+\Delta\theta)\right)$$

For small $\Delta\theta$, $\cos(\Delta\theta/2) \doteq 1$, $\sin(\Delta\theta/2) \doteq \Delta\theta/2$.

Using these approximations,

$$1\left(T(\theta+\Delta\theta) - T(\theta)\right) = \mu_s \frac{\Delta\theta}{2}\left(T(\theta) + T(\theta+\Delta\theta)\right)$$

$$\frac{T(\theta+\Delta\theta) - T(\theta)}{\Delta\theta} = \frac{\mu_s}{2}\left(T(\theta+\Delta\theta) + T(\theta)\right)$$

Take limit as $\Delta\theta \to 0$

$$\frac{dT}{d\theta} = T(\theta)\mu_s$$ ↙ first order linear diff eq.

Solve diff eq.

$$\frac{1}{T(\theta)}dT = \mu_s d\theta \Rightarrow \int_{T(0)}^{T(\theta_{AB})} T^{-1}(\theta)\, dT = \int_0^{\theta_{AB}} \mu_s d\theta \Rightarrow \ln[T(\theta_{AB})] - \ln[T(0)] = \mu_s \theta_{AB} \Rightarrow \ln\left[\frac{T(\theta_{AB})}{T(0)}\right] = \mu_s\theta_{AB}$$

At $\theta = 0$ we have $T(0) = T_A$, equal to the force applied by the sailor.
At $\theta = \theta_{AB}$, $T(\theta_{AB}) = T_B$, the force applied by the load.

$$\Rightarrow \frac{T_B}{T_A} = e^{\mu_s\theta_{AB}} \Rightarrow T_A = T_B e^{-\mu_s\theta_{AB}}$$

$\Rightarrow$ Force applied by sailor goes down the higher θ_{AB} is, ie the more times you wrap the rope around the capstan.