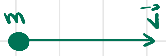
15.1 Momentum and Impulse



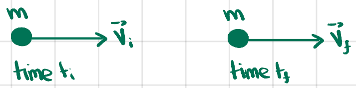
momentum $\vec{p} = m\vec{v}$ $[kg\, m \cdot s^{-1}] = [N \cdot s]$

$\rightarrow$ For single particle $\vec{F} = m\vec{a} = m\dfrac{d\vec{v}}{dt}$ most general form of 2nd law

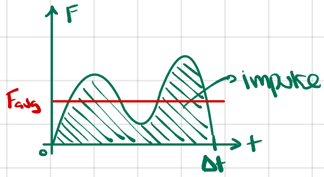
$\rightarrow$ For constant m, $\vec{F} = \dfrac{d}{dt}(m\vec{v}) = \dfrac{d}{dt}\vec{p}$ $\Rightarrow$ $\vec{F} = \dfrac{d\vec{p}}{dt}$

$$\Rightarrow \underbrace{\int_{t_1}^{t_2} \vec{F}\,dt}_{\text{impulse}} = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt}\,dt = \vec{p}(t_2) - \vec{p}(t_1) = \Delta\vec{p}$$

15.2 Impulse is a vector



$\vec{I} = \Delta\vec{p}$ $\quad kg\,m\,s^{-1}$



$$F_{avg}\Delta t = \int_0^{\Delta t} F(t)\,dt$$

integrate a vector

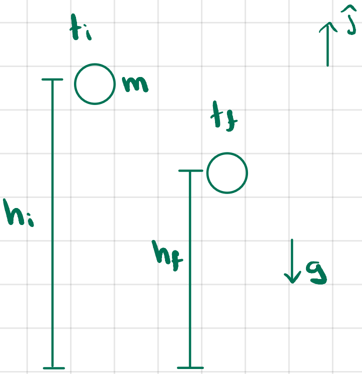
$$\vec{I} = \int_{t'=t_i}^{t'=t_f} \vec{F}(t')\,dt' = \int_{t'=t_i}^{t'=t_f} \frac{d\vec{p}}{dt'}\,dt'$$

2nd law

$$= \int_{t'=t_i}^{t'=t_f} d\vec{p} = \vec{p}(t_f) - \vec{p}(t_i) = \Delta p = \vec{I}$$

15.3 Worked Example – Bouncing Ball

Def (Average Force) $\vec{F}_{avg} = \dfrac{\int_{t_1}^{t_2} \vec{F}(t)\,dt}{t_2 - t_1}$



Let's analyze based on four different times: t_i, t_b, t_a, t_f (initial, before hitting ground, after hitting ground, final).

$t = t_i = 0$: $\vec{v} = \langle 0, 0\rangle$, $\vec{p} = \langle 0, 0\rangle$

$t = t_b$: from kinematics we know: $F = -mg = ma \Rightarrow a(t) = -g$

$\Rightarrow v(t) = -gt$

$\Rightarrow y(t) = h_i - gt^2/2 \quad \Rightarrow 0 = h_i - gt_b^2/2 \Rightarrow t_b = \sqrt{\dfrac{2h_i}{g}} \Rightarrow v_b = -gt_b = \sqrt{2gh_i}$

Also, for the upward movement:

$v(t) = v_a - gt \quad \Rightarrow 0 = v_a - gt_f \Rightarrow t_f = v_a/g$

$y(t) = v_a t - gt^2/2$

$$y(t_f) = h_f = \frac{v_a^2}{g} - \frac{v_a^2}{2g} = \frac{v_a^2}{2g} \Rightarrow v_a^2 = 2gh_f \Rightarrow v_a = \sqrt{2gh_f}$$

$$\vec{p}(t_b) - \vec{p}(t_i) = \int_{t_i}^{t_b} \vec{F}_R\,dt \Rightarrow \frac{mv_b - mv_i}{t_b - t_i} = \frac{\int_{t_i}^{t_b} \vec{F}_{R,y}\,dt}{t_b - t_i} = F_{R\,avg,y}. \text{ But } \int_{t_i}^{t_b} \vec{F}_{R,y}\,dt = \int_{t_i}^{t_b} -mg\,dt = -mg(t_b - t_i)$$

$$\Rightarrow \frac{-mg\cancel{(t_b - t_i)}}{\cancel{(t_b - t_i)}} = F_{avg,y} = \frac{mv_b - mv_i}{t_b - t_i}$$

$$\vec{p}(t_a) - \vec{p}(t_b) = \int_{t_b}^{t_a} \vec{F}_R\,dt \Rightarrow \frac{mv_a - mv_b}{t_a - t_b} = \frac{\int_{t_b}^{t_a} (N - mg)\,dt}{t_a - t_b}. \quad \Delta t = t_a - t_b.$$

$$\frac{\int_{t_b}^{t_a} (N - mg)\,dt}{\Delta t} = \frac{\int_{t_b}^{t_a} N\,dt}{\Delta t} - \frac{\int_{t_b}^{t_a} mg\,dt}{\Delta t} = N_{avg} - mg = \frac{mv_a - mv_b}{\Delta t} \Rightarrow N_{avg} = \frac{m(v_a - v_b)}{\Delta t} + mg$$

$$\Rightarrow N_{avg} = \frac{m\left(\sqrt{2gh_f} - \left(-\sqrt{2gh_i}\right)\right)}{\Delta t} + mg = \frac{m\left(\sqrt{2gh_f} + \sqrt{2gh_i}\right)}{\Delta t} + mg$$

15.4 Momentum of a System of Particles

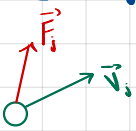
N particles, $j = 1, \dots N$

j^{th} particle  mass m_j

momentum $\vec{p}_j = m_j \vec{v}_j$

$$\vec{p}_{sys} = \sum_{j=1}^{N} m_j \vec{v}_j = \sum_{j=1}^{N} \vec{p}_j \quad \text{a vector sum}$$

Suppose force $\vec{F}_j$ acting on j^{th} particle



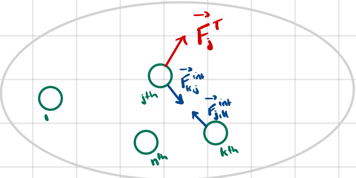
$$\vec{F} = \sum_{j=1}^{N} \vec{F}_j$$

Apply 2nd law

$$\vec{F}_j = \frac{d}{dt}\vec{p}_j$$

$$\Rightarrow \vec{F} = \sum_{j=1}^{N} \frac{d\vec{p}_j}{dt} = \frac{d}{dt}\left[\sum_{j=1}^{N} \vec{p}_j\right] = \frac{d}{dt}\vec{p}_{sys}$$

15.5 Force on a System of Particles



$$\vec{F} = \frac{d\vec{p}_{sys}}{dt}$$

total external force

$$\underbrace{\vec{F}_j^T}_{\text{total force on } j^{th} \text{ particle}} = \vec{F}_j^{T,ext} + \underbrace{\sum_{\substack{k=1 \\ k \neq j}}^{N} \vec{F}_{k,j}^{int}}_{\vec{F}_j^{T,int}} = \vec{F}_j^{T,ext} + \vec{F}_j^{T,int}$$

total internal force

$$\vec{F} = \sum_{j=1}^{N} \vec{F}_j = \sum_{j=1}^{N} \left(\vec{F}_j^{ext} + \vec{F}_j^{int}\right)$$

3rd Law

$$\vec{F}_{j,k}^{int} + \vec{F}_{k,j}^{int} = 0 \quad \text{internal forces cancel in pairs}$$

$$\sum_{j=1}^{N} \vec{F}_j^{int} = \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} \vec{F}_{k,j}^{int} = \text{sum of pairs} = 0$$

$$\Rightarrow \vec{F} = \vec{F}^{ext} + \vec{0}$$

$$\Rightarrow \vec{F}^{ext} = \frac{d}{dt}\vec{p}_{sys}$$

Interpretation / Recap

Up to this point we've used the formula $\vec{F} = m\vec{a}$. If we define the concept of momentum as $\vec{p} = m\vec{v}(t)$ then 2nd law becomes $\vec{F} = m\frac{d\vec{v}}{dt} = \frac{d}{dt}\vec{p}(t)$.
$\vec{F}$ is the rate of change of momentum relative to time.

$$\int_{t_1}^{t_2} \vec{F}\,dt = \vec{p}(t_2) - \vec{p}(t_1) = \Delta\vec{p}$$

If we have a system of N particles, the resultant force on the system has been defined as the sum of all forces acting on each particle. Since force is just rate of change of momentum, total resultant force is the sum of j resultant forces, which is sum of j derivatives of momentum.

For a given particle, the resultant force is the sum of multiple forces but we can categorize them as either external or internal. When we consider all j particles, internal forces in aggregate represent a set of 3rd law pairs of forces that all cancel out. Therefore, total resultant force, is actually the sum of the external forces.

So, first we derived $\vec{F} = \sum \frac{d\vec{p}_{sys}}{dt}$ and then we derived $\vec{F} = \sum_{j=1}^{N} \vec{F}_j^{ext}$

$$\Rightarrow \vec{F} = \sum_{j=1}^{N} \vec{F}_j^{ext} = \vec{F}^{ext} = \frac{d}{dt}\vec{p}_{sys}$$

16.1 Conservation of Momentum

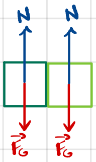
$$I = \int_{t_i}^{t_f} \vec{F}^{ext}\, dt = \Delta \vec{p}_{sys} = \vec{p}(t_f) - \vec{p}(t_i)$$

$$I = 0 \Rightarrow \vec{p}(t_f) = \vec{p}(t_i)$$

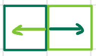
this is a vector eq. but sometimes we can set up our coord. system so we only have to consider one dimension

Example 1

collision:

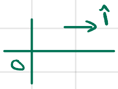


external forces are $\vec{F}_G$ and N on each block: they add to zero $\Rightarrow \vec{F}^{ext} = 0$



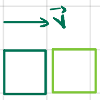
if our system is the two blocks, the forces between the blocks are internal

coord. system



identify initial and final states

initial state: just before collision

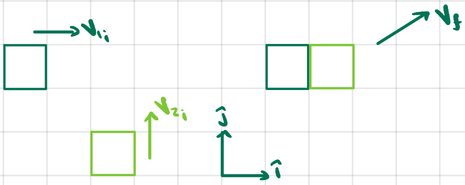


$$\vec{p}_i = m_1 v_{1i,x}\hat{\imath} + m_2 \cancel{v_{2i,x}}^{\,0}\hat{\imath}$$

final state: blocks moving together

$$\vec{p}_f = (m_1+m_2)v_{f,x}\hat{\imath}$$

Example 2: collision in 2D



$$\int_{t_i}^{t_f} F_x^{ext}\, dt = p_{f,x} - p_{i,x}$$

$$\int_{t_i}^{t_f} F_y^{ext}\, dt = p_{f,y} - p_{i,y}$$

$$p_{i,x} = m_1 v_{1i,x} + m_2 \cancel{v_{2i,x}}$$

$$\Rightarrow \vec{p}_i = m_1 v_{1i,x}\hat{\imath} + m_2 v_{2i,y}\hat{\jmath}$$

$$p_{i,y} = m_1 \cancel{v_{1i,y}}^{\,0} + m_2 v_{2i,y}$$

$$p_{f,x} = (m_1+m_2)v_{f,x}$$

$$\Rightarrow \vec{p}_f = (m_1+m_2)v_{f,x}\hat{\imath} + (m_1+m_2)v_{f,y}\hat{\jmath}$$

$$p_{f,y} = (m_1+m_2)v_{f,y}$$

$$\Rightarrow \quad m_1 v_{1i,x} = (m_1+m_2)v_{f,x}, \qquad m_2 v_{2i,y} = (m_1+m_2)v_{f,y}$$

What happens if there is friction?

there is impulse from an external force, friction $\Rightarrow \Delta p^{sys} \neq 0$. momentum not conserved.

* if we restrict our analysis to a very short Δt, from right before to right after collision, then $\int F_f\, dt$ is very small so $\Delta p^{sys} \approx 0$.

16.2 Momentum Diagrams

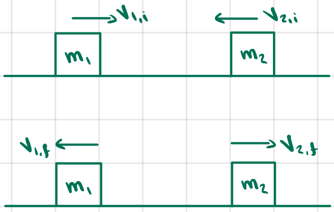
setup: $\int \vec{F}^{ext} dt = \vec{p}_f - \vec{p}_i$

initial state

final state

(1) choose system

(2) choose reference frame



$F^{ext} = 0$, no friction

ref frame: ground

$$0 = \vec{p}_f - \vec{p}_i \Rightarrow m_1\vec{v}_{1,f} + m_2\vec{v}_{2,f} = m_1\vec{v}_{1,i} + m_2\vec{v}_{2,i}$$

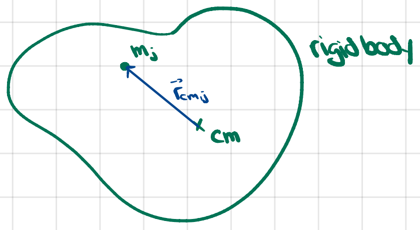
(3) choose coord. system

e.g. $\longrightarrow \hat{i}$

$\vec{v}_{1,i} = v_{1,i,x}\hat{i}$, $\vec{v}_{2,i} = v_{2,i,x}\hat{i}$, etc

$$\Rightarrow m_1 v_{1,x,f} + m_2 v_{2,f,x} = m_1 v_{1,i,x} + m_2 v_{2,i,x}$$

17.1 Definition of Center of Mass



m_j: rigid body composed of small pieces with mass m_j

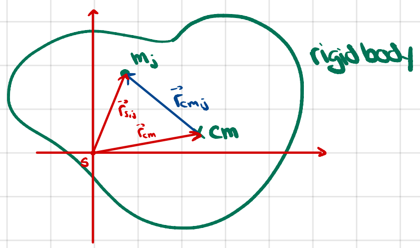
$\vec{r}_{cm,j}$: position of m_j rel. to cm

$$\sum_{j=1}^{N} m_j \vec{r}_{cm,j} = \vec{0} \qquad \text{(definition of center of mass)}$$

ie center of mass is the point for which the weighted average of position vector to all other points is zero when the weights are the masses of the points.

→ we can't calculate where CM is with this def.

→ calculating center of mass



Given an arbitrary point s in the rigid body, use that point as origin.

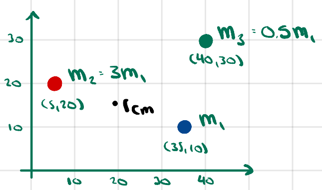
$$\vec{r}_{s,j} = \vec{r}_{cm} + \vec{r}_{cm,j} \qquad j = 1, \dots, N$$

$$m_j \vec{r}_{s,j} = m_j \vec{r}_{cm} + m_j \vec{r}_{cm,j} \qquad j = 1, \dots, N$$

$$\sum_{j=1}^{N} m_j \vec{r}_{s,j} = \vec{r}_{cm} \sum_{j=1}^{N} m_j + \underbrace{\sum_{j=1}^{N} m_j \vec{r}_{cm,j}}_{= 0 \text{ by our def of center of mass}}$$

$$\Rightarrow \vec{r}_{cm} = \frac{\sum_{j=1}^{N} m_j \vec{r}_{s,j}}{\sum_{j=1}^{N} m_j} = \frac{\sum_{j=1}^{N} m_j \vec{r}_{s,j}}{m_{total}}$$

17.2 Worked Example: Center of Mass of Three Objects



$$\vec{r}_{cm} = \frac{1}{m_{sys}} \sum_{j=1}^{N} m_j \vec{r}_j$$

$$= \frac{1}{4.5 m_1} \left[m_1 \langle 35, 10 \rangle + 3m_1 \langle 5, 20 \rangle + 0.5 m_1 \langle 40, 30 \rangle \right]$$

$$\Rightarrow \frac{m_1 (35 + 15 + 20)\,\hat{i}}{4.5 m_1} = \frac{70}{4.5}\,\hat{i}$$

$$\frac{m_1 (10 + 60 + 15)\,\hat{j}}{4.5 m_1} = \frac{85}{4.5}\,\hat{j}$$

$$\Rightarrow \vec{r}_{cm} = 15.5\,\hat{i} + 18.9\,\hat{j}$$

17.3 Center of Mass of a Continuous System

discrete particles forming a rigid body $\Rightarrow \vec{r}_{cm} = \dfrac{\sum m_j \vec{r}_j}{\sum m_j}$

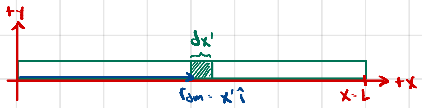
to consider and analyze the body as a continuous mass, we start with the discrete case where the body is composed of small pieces of mass Δm_j and position $\vec{r}_{dm}$.

$$\lim_{N\to\infty} \sum_{j=1}^{N} m_j = \int_{body} dm$$

$$\lim_{N\to\infty} \sum_{j=1}^{N} m_j \vec{r}_j = \int_{body} \vec{r}_{dm}\, dm$$

$$\Rightarrow \vec{r}_{cm} = \frac{\int_{body} \vec{r}_{dm}\, dm}{\int_{body} dm}$$

17.5 Center of Mass of a Uniform Rod



rod has mass m

(1) coordinate system

(2) identification of dm

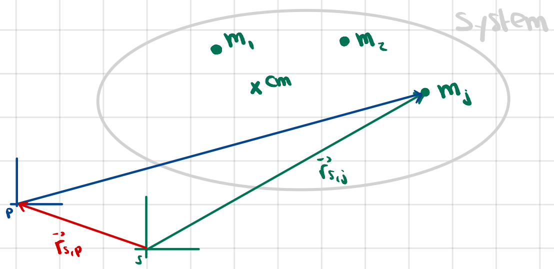
(3) introduce integration variable x'

linear mass density

since the rod is uniform, $dm = \dfrac{dx'}{L} m = \dfrac{m}{L} dx' = \lambda dx'$

$$r_{cm} = \frac{\int_{x'=0}^{x'=L} \left(\frac{m}{L} dx'\right) x'\hat{i}}{\int_{x'=0}^{x'=L} \frac{m}{L} dx'} = \frac{\frac{1}{2}\cancel{\frac{m}{L}} L^{\cancel{2}}\hat{i}}{\cancel{\frac{m}{L}}\cancel{L}} = \frac{1}{2} L\,\hat{i}$$

17.6 Velocity and Acceleration of the Center of Mass



we can calculate the coordinates of CM relative to different coord. systems.

$$\vec{r}_{s,cm} = \frac{\sum m_j \vec{r}_{s,j}}{\sum m_j} \qquad \vec{r}_{p,cm} = \frac{\sum m_j \vec{r}_{p,j}}{\sum m_j}$$

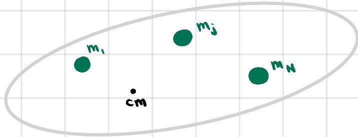
$$\vec{r}_{s,j} = \vec{r}_{s,p} + \vec{r}_{p,j} \Rightarrow \frac{d\vec{r}_{s,j}}{dt} = \frac{d\vec{r}_{p,j}}{dt}$$

$\Rightarrow \vec{v}$ independent of choice of coordinate system

$$\vec{V}_{cm} = \frac{\sum m_j \vec{v}_j}{\sum m_j}$$

$$\vec{A}_{cm} = \frac{\sum m_j \vec{a}_j}{\sum m_j}$$

17.7 Reduction of a System to a Point Particle



$$\vec{F}^{ext} = \frac{d}{dt}\vec{p}^{sys}$$

$$\vec{p}^{sys} = \sum_{j=1}^{N} m_j \vec{v}_j$$

$$\frac{d}{dt}\vec{p}^{sys} = \sum m_j \vec{a}_j$$

But since $\vec{A}_{cm} = \dfrac{\sum m_j \vec{a}_j}{\sum m_j}$ we have $m^T \vec{A}_{cm} = \sum m_j \vec{a}_j$

$$\Rightarrow \frac{d}{dt}\vec{p}^{sys} = m^T \vec{A}_{cm}$$

$$\Rightarrow \vec{F}^{ext} = m^T \vec{A}_{cm}$$

we can think of the system as a point particle of mass m^T located at cm.