

8.6 Drag Forces in Fluids

liquid or gas

- solid object moving through fluid experiences a resistive force called the **drag force**, opposing its motion
- for objects moving rapidly through air the drag force can be modeled as

$$F_{\text{drag}} = \frac{1}{2} C_d A \rho v^2$$

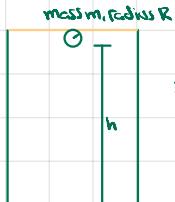
C_d : drag coeff., dimensionless number, property of the object

complicated, depends on properties of object (speed, size, shape) and fluid (density, viscosity, compressibility)

* this model does not apply to all fluids, e.g. oil, honey, water, which all have different viscosities

* at low speeds, we model air drag differently: linear relationship of speed, proportional to viscosity η
when the object is a sphere of radius R , we have: $F_{\text{drag}} = -6\pi\eta R \dot{v}$ (Stokes')

Example 8.5 Drag Force at low speeds



$$\text{ii) } mg - F_{\text{drag}} = m \frac{dv(t)}{dt}$$

$$\Rightarrow mg - 6\pi\eta R v(t) - m \frac{dv(t)}{dt} \Rightarrow g - \frac{6\pi\eta R}{m} v(t) - \frac{dv(t)}{dt} \Rightarrow g - \gamma v(t) - \frac{dv(t)}{dt}$$

$$\Rightarrow -\gamma \left(v(t) - \frac{g}{\gamma} \right) - \frac{dv(t)}{dt} \Rightarrow -\gamma dt = \frac{dv(t)}{v(t) - g/\gamma}$$

$$\int_{v(0)=0}^{v(t)} \frac{dv}{\sqrt{g - \gamma v}} = \int_0^t -\gamma dt$$

$$\ln(v - g/\gamma) \Big|_0^{v(t)} = -\gamma t$$

$$\ln(v(t) - g/\gamma) - \ln(-g/\gamma) = -\gamma t$$

$$\ln \left[\frac{v(t) - g/\gamma}{-g/\gamma} \right] = -\gamma t$$

$$\frac{v(t) - g/\gamma}{-g/\gamma} e^{-\gamma t} \Rightarrow v(t) = -\frac{g}{\gamma} e^{-\gamma t} + g/\gamma$$

$$\Rightarrow v(t) = \frac{g}{\gamma} (1 - e^{-\gamma t})$$

$$\text{ii) } v(t \rightarrow \infty) = \frac{g}{\gamma} = \frac{gm}{6\pi\eta}$$

$$\text{Note } \frac{dv}{dt} = \frac{g}{\gamma} (-e^{-\gamma t}) - \gamma = g e^{-\gamma t}$$

$$t \rightarrow \infty \Rightarrow v = g/\gamma, dv/dt = 0$$

$$\text{iii) } t \rightarrow \infty \Rightarrow dv/dt = 0 \Rightarrow F_{\text{drag}} = mg = 6\pi\eta R v(t \rightarrow \infty) = 6\pi\eta R g \frac{g}{\gamma} \Rightarrow \eta = \frac{mg}{6\pi R g}$$

$$\text{or } \eta = \frac{m}{6\pi R v_{\text{term}}}$$

$$\text{i) } dv(t) = [g - \gamma v(t)] dt$$

$$\int_{v(0)}^{v(t)} dv(t) = \int_0^t g dt - \gamma \int_0^t v(t) dt$$

$$v(t) - v(0) = gt - \gamma \int_0^t v(t) dt$$

$$v(0) = 0 \Rightarrow v(t) = gt - \gamma \int_0^t v(t) dt \Rightarrow v(t) = \frac{gt - v(t)}{\gamma} \cdot \frac{gt}{\gamma} - \frac{g}{\gamma^2} (1 - e^{-\gamma t})$$

Example 8.6 Drag Forces at High Speeds

Setup:

object: mass m , $\vec{v}(0) = \vec{v}_0$, cross-sectional area A perpendicular to motion

fluid: density ρ

$$\text{Forces} = \frac{1}{2} C_D A \rho v^2$$

$$mg - F_{\text{drag}} = m \frac{dv}{dt}$$



$$mg - \frac{1}{2} C_D A \rho v^2 = m \frac{dv}{dt} \Rightarrow g - \frac{C_D A \rho}{2m} v^2 = \frac{dv}{dt} \quad \gamma = \frac{C_D A \rho}{2m}$$

$$\Rightarrow g - \gamma v^2 = \frac{dv}{dt} \Rightarrow -\gamma(v^2 - \frac{g}{\gamma}) = \frac{dv}{dt} \Rightarrow -\gamma dt = \frac{1}{v^2 - g/\gamma} dv \Rightarrow \int_0^t -\gamma dt = \int_{v(0)}^{v(t)} \frac{1}{v^2 - g/\gamma} dv$$

$$\Rightarrow -\int_0^t \gamma dt = -\frac{\gamma}{g} \int_{v(0)}^{v(t)} \frac{1}{1 - \frac{g}{\gamma} v^2} dv \Rightarrow \int_0^t g dt = \int_{v(0)}^{v(t)} \frac{1}{1 - \frac{g}{\gamma} v^2} dv$$

change of variables

$$u = (\gamma/g)^{1/2} v \Rightarrow du = (\gamma/g)^{1/2} dv, v(0) = 0, v(t) = (\gamma/g)^{1/2} u(t)$$

$$\int_{v(0)}^{v(t)} \frac{1}{1 - \frac{g}{\gamma} v^2} dv = (\gamma/g)^{1/2} \int_0^{u(t)} \frac{du}{1 - u^2} = (\gamma/g)^{1/2} \frac{1}{2} \ln \left[\frac{1+u(t)}{1-u(t)} \right]$$

$$\Rightarrow gt = \left[\frac{g}{\gamma} \right]^{1/2} \cdot \frac{1}{2} \ln \left[\frac{1 + (\gamma/g)^{1/2} v(t)}{1 - (\gamma/g)^{1/2} v(t)} \right]$$

$$2t\sqrt{\gamma g} = \ln \left[\frac{1 + (\gamma/g)^{1/2} v(t)}{1 - (\gamma/g)^{1/2} v(t)} \right] \Rightarrow e^{2t\sqrt{\gamma g}} = \frac{1 + (\gamma/g)^{1/2} v(t)}{1 - (\gamma/g)^{1/2} v(t)}$$

$$\Rightarrow e^{2t\sqrt{\gamma g}} + e^{2t\sqrt{\gamma g}} (\gamma/g)^{1/2} v(t) = 1 + (\gamma/g)^{1/2} v(t)$$

$$v(t)(\gamma/g)^{1/2} [1 - e^{2t\sqrt{\gamma g}}] = e^{2t\sqrt{\gamma g}} - 1 \Rightarrow v(t) = \left[\frac{g}{\gamma} \right]^{1/2} \frac{e^{2t\sqrt{\gamma g}} - 1}{1 - e^{2t\sqrt{\gamma g}}}$$

Here is a case with an initial horizontal v_0 through a fluid, no gravity.



$$F_{\text{drag}} = -\frac{1}{2} C_D A \rho v^2 = m \frac{dv}{dt} \Rightarrow -\beta v^2 = m \frac{dv}{dt} \Rightarrow -\frac{\beta}{m} dt = \frac{1}{v^2} dv \Rightarrow \int_0^t -\frac{\beta}{m} dt = \int_{v(0)}^{v(t)} \frac{1}{v^2} dv \Rightarrow -\frac{\beta}{m} t = -v^{-1} \Big|_{v(0)}^{v(t)} \Rightarrow -\frac{\beta t}{m} = -[v(t)^{-1} - v_0^{-1}]$$

$$\frac{1}{v(t)} = \frac{\beta t}{m} + \frac{1}{v_0} = \frac{v_0 \beta t + m}{v_0 m} \Rightarrow v(t) = \frac{v_0 m}{v_0 \beta t + m}$$