

19.7 Angular Momentum and Torque for Fixed Axis Rotation

we know that $\vec{L}_{S,z}^{\text{ext}} = I_S \vec{\alpha}$

this is a special case of $\vec{L}_{S,z}^{\text{ext}} = \frac{d}{dt} \vec{L}_{S,z}^{\text{int}}$

Setup

→ rigid body, rotating around fixed axis passing through point S

→ fixed axis is the z-axis

→ angular velocity: $\vec{\omega} = \omega(t) \hat{h} = \omega_z \hat{h}$

→ acceleration: $\vec{\alpha} = \alpha(t) \hat{h} = \alpha_z \hat{h}$

Divide the rigid body

Δm_j moving in circle of radius $r_{S,j}^\perp$

$\vec{r}_{S,j}$ position of Δm_j

\vec{p}_j is tangent to the circle

$$\vec{L}_{S,j} = \vec{r}_{S,j} \times \vec{p}_j$$

$$\vec{r}_{S,j} = \vec{r}_{S,j}^{\parallel} + \vec{r}_{S,j}^\perp$$

$$\vec{v}_j = \vec{r}_{S,j}^\perp \omega_z \hat{\theta}$$

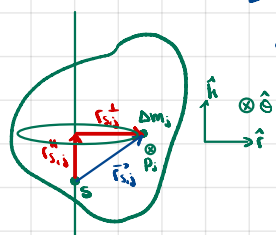
$$\Rightarrow \vec{p}_j = \Delta m_j \vec{v}_j = \Delta m_j \vec{r}_{S,j}^\perp \omega_z \hat{\theta}$$

$$\Rightarrow \vec{L}_{S,j} = \vec{r}_{S,j} \times \vec{p}_j$$

$$= (\vec{r}_{S,j}^\perp \hat{r} + r_{S,j}^{\parallel} \hat{h}) \times \Delta m_j \vec{r}_{S,j}^\perp \omega_z \hat{\theta}$$

$$= \Delta m_j r_{S,j}^\perp \omega_z \hat{h} + \Delta m_j r_{S,j}^{\parallel} r_{S,j}^\perp \omega_z (-\hat{r})$$

→ z-component of L about S



$$(L_{S,j})_z = \Delta m_j (r_{S,j}^\perp)^2 \omega_z$$

$$L_{S,z}^{\text{int}} = \sum (L_{S,j})_z = \sum \Delta m_j (r_{S,j}^\perp)^2 \omega_z$$

For a continuous mass distribution

$$L_{S,z}^{\text{int}} = \int_{\text{body}} dm (r_{dm})^2 \omega_z$$

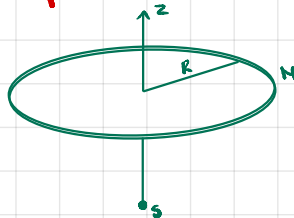
$$= I_S \omega_z$$

$$\Rightarrow \vec{L}_{S,z}^{\text{ext}} = \frac{d}{dt} \vec{L}_{S,z}^{\text{int}}$$

For z-direction

$$\tau_{S,z}^{\text{ext}} = \frac{dL_{S,z}^{\text{int}}}{dt} = \frac{d}{dt} (I_S \omega_z) = I_S \alpha_z$$

Example 19.6



Apply formula: $L_{S,z} = I_S \omega_z$
 $= MR^2 \omega_z$

Alternatively

The ring is made up of pairs of particle masses.

Their angular momentum about S is

$$2 \Delta m_j R^2 \omega \hat{h}$$

$$L_S = \lim_{n \rightarrow \infty} \sum_{i=1}^n 2 \Delta m_i R^2 \omega$$

$$= \int_{\text{ring}} dm \cdot 2R^2 \omega$$

$$dm = \frac{R d\theta}{2\pi R} M$$

$$= \int_0^{2\pi} \frac{M}{2\pi} \cdot 2R^2 \omega d\theta = \frac{MR^2 \omega}{\pi} \cdot \pi = MR^2 \omega$$