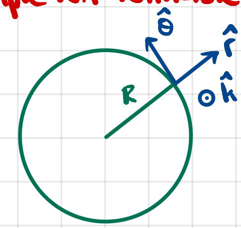
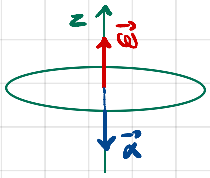
Example 16.1 Turntable



$R = 13\,cm$
$m = 1.2\,kg$

initial state

$f_0 = 33$ cycles $\cdot min^{-1}$ (rpm) $= 33/60$ cycles $\cdot s^{-1} = 0.55 Hz$



$\vec{\omega} = \omega_z \hat{k} = \theta'(t)\hat{k}$
$\vec{\alpha} = \alpha_z \hat{k} = \theta''(t)\hat{k}$

rotation in counterclockwise direction relative to positive z-axis: positive z-component of angular velocity

→ rate of rotation decreasing → $\alpha_z < 0$ → $\vec{\alpha}$ points towards $-\hat{k}$.

→ tangential velocity: $\vec{v} = \vec{\omega} \times \vec{r} = \omega_z \hat{k} \times R\hat{r} = \omega_z R\hat{\theta}$

→ note that

$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$
$d\hat{r}/dt = -\theta'\sin\theta\,\hat{i} + \theta'\cos\theta\,\hat{j}$
$= \theta'(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$
$= \theta'\hat{\theta}$

$\vec{r} = R\hat{r}$ → $\vec{v} = R\theta'\hat{\theta}$
$= R\omega_z\hat{\theta}$

→ $\vec{a} = -R\omega_z^2\hat{r} + R\alpha_z\hat{\theta}$

the motor is turned off.

$a_\theta = R\alpha_z$
$v_\theta = v_0 + R\alpha_z t$

$\omega_{z_0} = f_0 \cdot 2\pi = 0.55 \cdot 2\pi = 3.5\ rad \cdot s^{-1}$
$v_0 = R\omega_{z_0} = 0.13 \cdot 3.5 = 0.455\ m/s$

$v_\theta = 0.455 + R\alpha_z t = 0.455 + 0.13 \cdot \alpha_z \cdot t$

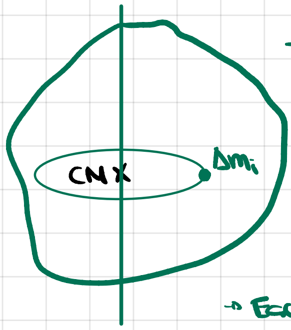
$0 = 0.455 + 0.13 \cdot \alpha_z \cdot 8$ → $\alpha_z = -0.4375\ rad \cdot s^{-2}$

note $\vec{\alpha} = -0.4375\hat{k}$

$\vec{v} = R\,\omega_z(t)\hat{\theta}$

$\omega_z(t) = 3.5 - 0.4375t$

16.3 Rotational Kinetic Energy and Moment of Inertia



→ Fixed-axis rotation about the an axis passing through CM.
→ choose z-axis along rotation axis.

→ Divide body into Δm_i

→ Each Δm_i undergoes circular motion about CM w/ z-compon. of angular vel. ω_z, radius $r_{cm,i}$.

→ $\vec{v}_{cm,i} = r_{cm,i}\,\omega_{cm}\hat{\theta}$

Def (Rotational Kinetic Energy)

$$K_{cm,i} = \frac{1}{2}\Delta m_i v_{cm,i}^2 = \frac{1}{2}\Delta m_i r_{cm,i}^2 \omega_{cm}^2$$

Add rot. kin en. for all Δm_i

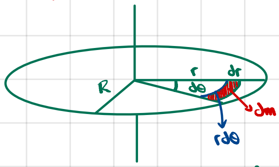
$$K_{cm} = \lim_{\substack{i\to\infty \\ \Delta m_i \to 0}} \sum_{i=1}^{n} K_{cm,i} = \lim_{\substack{i\to\infty \\ \Delta m_i \to 0}} \left(\sum_i \frac{1}{2}\Delta m_i r_{cm,i}^2\right)\omega_{cm}^2$$

$$= \frac{\omega_{cm}^2}{2}\int_{body} dm\, r_{dm}^2$$

Def (Moment of Inertia) $I = \int_{body} dm\, r_{dm}^2$

$$\Rightarrow K_{cm} = \frac{I_{cm}\,\omega_{cm}^2}{2}$$

Example 16.2 Moment of Inertia of Uniform Disc



choose cylindrical coordinates

area element: $da = r\,dr\,d\theta$

mass per unit area $= \frac{dm}{da} = \frac{m_{total}}{Area} = \frac{M}{\pi R^2} = \sigma$

$$dm = \sigma \cdot da = \frac{M}{\pi R^2} \cdot r\,dr\,d\theta$$

$$I_{cm} = \frac{M}{\pi R^2}\int_0^R\int_0^{2\pi} r^3\,d\theta\,dr = \frac{MR^2}{2}$$

Alternatively, use a different area element: a ring of radius r and width dr.

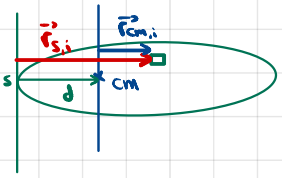
$$da_{ring} = \pi(r+dr)^2 - \pi r^2 = 2\pi r\,dr + \pi(dr)^2$$

as $dr \to 0$, ignore $\pi(dr)^2$. → $da_{ring} = 2\pi r\,dr$

$$dm = \frac{M}{\pi R^2} \cdot 2\pi r\,dr \rightarrow I_{cm} = \int_{disc}\frac{M}{\cancel{\pi} R^2} \cdot 2\cancel{\pi} r \cdot r^2\,dr$$

$$= \frac{2M}{R^2}\int_0^R r^3\,dr = \frac{MR^2}{2}$$

Example 16.3 Rotational Kinetic Energy of Disk



M, R, ω

$I_{cm} = \frac{MR^2}{2}$

$$I_S = \lim_{i \to \infty} \sum_{i=1}^{n} r_{s,i}^2 \Delta m_i = \lim \sum \left(d^2 \Delta m_i + \Delta m_i r_{cm,i}^2 + \cancel{2d \Delta m_i r_{cm,i}} \right) \to 0$$
$$= \lim \left(d^2 M_T + \sum \Delta m_i r_{cm,i}^2 \right)$$
$$= d^2 M_T + I_{cm}$$

$r_{s,i} = d + r_{cm,i}$

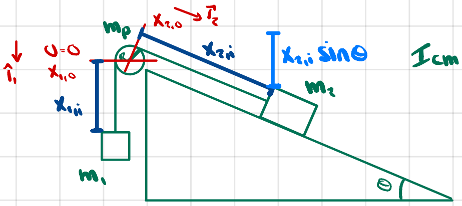
$r_{s,i}^2 = d^2 + r_{cm,i}^2 + 2d r_{cm,i}$

$$\Rightarrow I_S = R^2 M + \frac{MR^2}{2} = \frac{3}{2} MR^2$$

$$K_S^{rot} = \frac{1}{2} \cdot \frac{3}{2} MR^2 \cdot \omega^2$$

16.4 Conservation of Energy for Fixed Axis Rotation

Example 16.4 Energy and Pulley System



$I_{cm} = \frac{m_p R^2}{2}$

gravity acts on m_1 and m_2.
in the initial state, m_1 and m_2 have potential energy

$$E_{m_i} = -m_1 g x_{1,i} - m_2 g x_{2,i} \sin\theta$$

objects are released from rest, m_2 slides down. the cord moves without slipping on the disk. This means the disk is rotating at the same tangential speed as the cord's speed. The wheel has rotational kinetic energy.
after m_2 moves $d = x_2 - x_{2,i}$:

$$E = U + K = -mg x_1 - mg x_2 \sin\theta + \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} + \frac{I_p \omega^2}{2}$$

rope connects $m_1, m_2 \Rightarrow v \equiv v_1 = v_2$ ie the blocks move at same speed, and this is also the tangential speed of the rope as it moves around the wheel/pulley.

$$\Rightarrow v_{tan} = R\omega = v \quad \Rightarrow \omega = \frac{v}{R}$$

$$\Rightarrow E = -m_1 g x_1 - m_2 g x_2 \sin\theta + \frac{v^2}{2}\left(m_1 + m_2 + \frac{I_p}{R^2}\right)$$

There are no non-conservative external forces $\Rightarrow \Delta E_m = 0$

$$E_{m_i} = E \Rightarrow -m_1 g x_{1,i} - m_2 g x_{2,i} \sin\theta$$
$$= -m_1 g x_1 - m_2 g x_2 \sin\theta + \frac{v^2}{2}\left(m_1 + m_2 + \frac{I_p}{R^2}\right)$$

$$-m_1 g (x_{1,i} - x_1) + m_2 g \sin\theta (x_2 - x_{2,i})$$
$$= \frac{v^2}{2}\left(m_1 + m_2 + \frac{I_p}{R^2}\right)$$

there is a constraint equation

$$d = x_2 - x_{2,i} = x_{1,i} - x_1$$

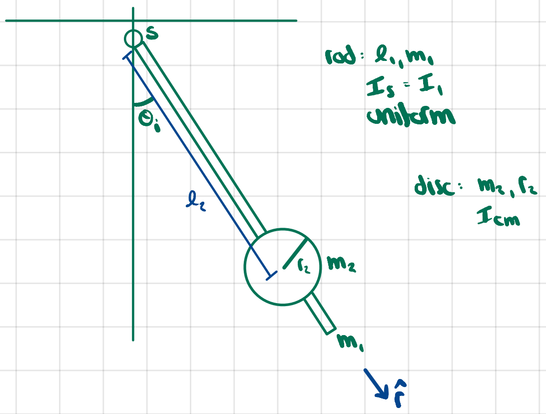
$$\Rightarrow gd(-m_1 + m_2 \sin\theta) = \frac{v^2}{2}\left(m_1 + m_2 + \frac{I_p}{R^2}\right)$$

$$\Rightarrow v = \sqrt{\frac{2gd(-m_1 + m_2 \sin\theta)}{m_1 + m_2 + I_p/R^2}}$$

if $I_p = \frac{m_p R^2}{2}$ then

$$v = \sqrt{\frac{2gd(-m_1 + m_2 \sin\theta)}{m_1 + m_2 + m_p/2}}$$

Example 16.5 Physical Pendulum

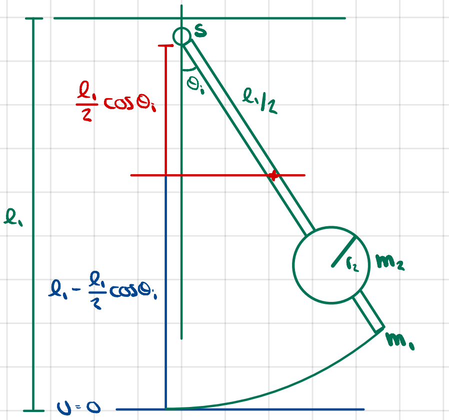


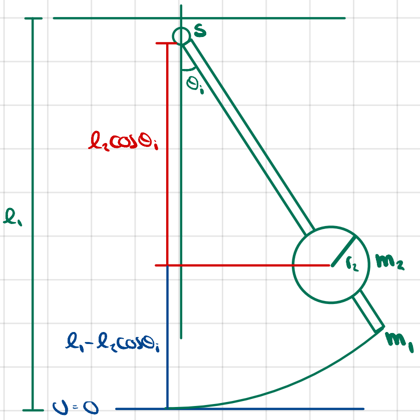
a) $I_{s\,disc} = I_{cm} + m_2 \cdot l_2^2$

$$I_{s\,pend} = \int_{body} dm\, r^2 = \int_{rod} dm\, r^2 + \int_{disc} dm\, r^2 = I_1 + I_{cm} + m_2 l_2^2$$

b) $$l_{cm} = \frac{\frac{l_1}{2} \cdot m_1 + l_2 \cdot m_2}{m_1 + m_2}$$

c) gravity acts on the entire rigid body, which will gain rotational kinetic energy. Apply conservation of mechanical energy, consider energy of rod and disc separately.





$$E_i = m_1 g\left(l_1 - \frac{l_1}{2}\cos\theta_i\right) + m_2 g\left(l_1 - l_2\cos\theta_i\right)$$

At the bottom, $\theta_f = 0$, angular velocity is ω_f

$$E_f = m_2 g(l_1 - l_2) + m_1 g \frac{l_1}{2} + \frac{I_s \omega_f^2}{2}, \quad I_s = I_1 + I_{cm} + m_2 l_2^2$$

$E_i = E_f \Rightarrow$

$$m_1 g\left(l_1 - \frac{l_1}{2}\cos\theta_i\right) + m_2 g\left(\cancel{l_1} - l_2\cos\theta_i\right) = m_2 g\left(\cancel{l_1} - l_2\right) + m_1 g\frac{l_1}{2} + \frac{I_s\omega_f^2}{2}$$

$$\Rightarrow m_1 g \frac{l_1}{2}\left(1 - \cos\theta_i\right) + m_2 g l_2\left(1 - \cos\theta_i\right) = \frac{I_s\omega_f^2}{2}$$

$$(1-\cos\theta_i) g \left(\frac{m_1 l_1}{2} + m_2 l_2\right) = \frac{I_s\omega_f^2}{2}$$

$$\Rightarrow \omega_f = \sqrt{\frac{2\left(\frac{m_1 l_1}{2} + m_2 l_2\right) g (1-\cos\theta_i)}{I_1 + I_{cm} + m_2 l_2^2}}$$

note that here

$$(1-\cos\theta_i) g \left(\frac{m_1 l_1}{2} + m_2 l_2\right) = \frac{I_s\omega_f^2}{2}$$

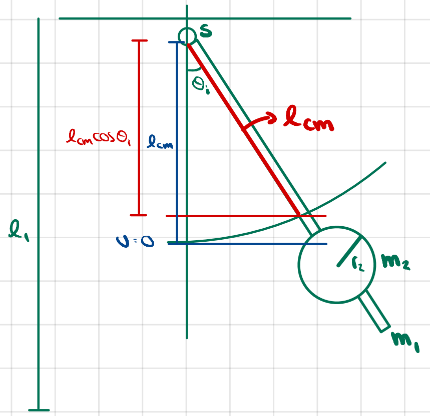
we could use

$$l_{cm} = \frac{\frac{l_1}{2} \cdot m_1 + l_2 \cdot m_2}{m_1 + m_2}$$

to obtain

$$(1-\cos\theta_i) g (m_1 + m_2) l_{cm} = \frac{I_s\omega_f^2}{2}$$

we could have done the entire analysis using only the center of mass. the result is the same eq.



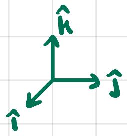
$$E_i = (m_1 + m_2) g (l_{cm} - l_{cm}\cos\theta_i)$$

$$E_f = \frac{I_{s\,pend}\,\omega_f^2}{2}$$

$$E_i = E_f \Rightarrow (m_1 + m_2) g l_{cm} (1 - \cos\theta_i) = \frac{I_s \omega_f^2}{2}$$

$$\omega_f = \sqrt{\frac{2(m_1+m_2) g l_{cm}(1-\cos\theta_i)}{I_1 + I_{cm} + m_2 l_2^2}}$$

Example 17.6

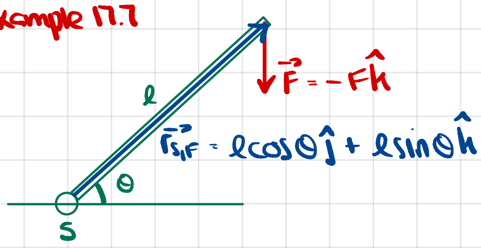


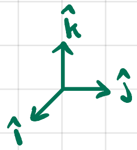
$\vec{r}_{P,F} = x\hat{\imath} \quad x>0$

$\vec{F} = F_x\hat{\imath} + F_z\hat{k} \quad F_x, F_z > 0$

$\vec{r}_{P,F} \times \vec{F} = x\hat{\imath} \times (F_x\hat{\imath} + F_z\hat{k}) = xF_z(-\hat{j})$

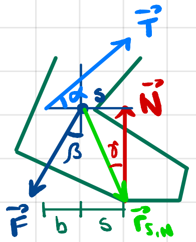
Example 17.7

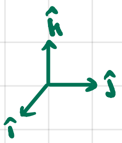




$\vec{\tau}_S = \vec{r}_{S,F} \times \vec{F} = -Fl\cos\theta\,\hat{\imath}$

Example 17.8 Torque and the Ankle





a) Torque about point S due to $\vec{T}$

$\vec{\tau}_{S,T} = \vec{r}_{S,T} \times \vec{T}$

$\vec{r}_{S,T} = -b\hat{j}$

$\vec{T} = T\cos\alpha\,\hat{j} + T\sin\alpha\,\hat{k}$

$\vec{\tau}_{S,T} = -bT\sin\alpha\,\hat{\imath}$

b) due to $\vec{F}$

$\vec{\tau}_{S,F} = \vec{0}$

c) due to $\vec{N}$

$\vec{r}_{S,N} = s\hat{j} - h\hat{k}$

$\vec{N} = \frac{mg}{2}\hat{k}$

$\vec{\tau}_{S,N} = \frac{smg}{2}\hat{\imath}$

$|\vec{\tau}_{S,N}| = \underbrace{|\vec{r}_{S,N}|\sin\gamma}_{\text{moment arm } = s}\,|\vec{N}| = \frac{smg}{2}\hat{k}$