9.3 Worked Examples in Circular Motion

9.1 Geosynchronous Orbit

$$F_{e,s} = -\frac{G m_e \cancel{m_s}}{r^2} = -\cancel{m_s} r\omega^2 \Rightarrow r = \sqrt[3]{\frac{G m_e}{\omega^2}}$$

$m_e = 5.98\cdot10^{24}$ kg

$G = 6.67\cdot10^{-11}$ $N\cdot m^2\cdot kg^{-2}$

Geostationary Satellite has period $T = 23h56min4sec$

$T = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{2\pi}{T}$

using Maple:

$T = 86164$ s

$\omega = \pi/43082$ rad/s

$r = 4.22\cdot10^7$ m

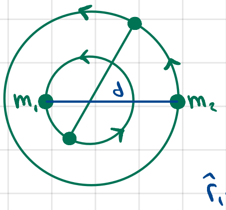
$\Rightarrow \frac{r}{R_E} = 6.62 \Rightarrow r = 6.62 R_E$

$R_E = 6.67\cdot10^6$ m on average

* $v = r\omega = 4.22\cdot10^7\cdot\frac{\pi}{43082} = 3075$ m/s

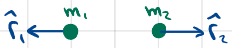
$r - R_E = 3.580350164\times10^7$ m $\approx$ 35803 km above the Earth.

9.2 Double Star System



Two stars, always at fixed distance d apart, both in uniform circular motion

Expectations: same angular speed ω $\Rightarrow$ same period. m_2 has larger speed because of its larger radius.



$$\vec{F}_{2,1} = -\frac{G\cancel{m_1}m_2}{d^2}\hat{r}_1 = -m_1 a_{r_1}\hat{r}_1 = -\cancel{m_1}r_1\omega^2\hat{r}_1$$

$$\Rightarrow \omega^2 = \frac{Gm_2}{r_1 d^2}$$

$$\Rightarrow \frac{m_2}{r_1} = \frac{m_1}{r_2} \Rightarrow m_1 r_1 = m_2 r_2$$

$$\vec{F}_{1,2} = -\frac{Gm_1\cancel{m_2}}{d^2}\hat{r}_2 = -\cancel{m_2}r_2\omega^2\hat{r}_2$$

$$\Rightarrow \omega^2 = \frac{Gm_1}{r_2 d^2}$$

Also, geometrically we can see that $d = r_1 + r_2$.

Given m_1, m_2, and same ω, we know the proportion between the radiuses. E.g. if $m_1 = 2m_2$ then $r_2 = 2r_1$
$\Rightarrow 2r_2 + r_2 = d \Rightarrow r_2 = d/3,\ r_1 = 2d/3$. We have three equations and three unknowns ω, r_1, r_2.

$$m_1 r_1 = m_2(d - r_1) = m_2 d - m_2 r_1 \rightsquigarrow r_1(m_1 + m_2) = m_2 d \Rightarrow r_1 = \frac{m_2}{m_1+m_2}d,\ r_2 = \frac{m_1}{m_1+m_2}d$$

$$\Rightarrow \omega = \sqrt{\frac{G\cancel{m_2}(m_1+m_2)}{d^2\cancel{m_2}d}} = \sqrt{\frac{G(m_1+m_2)}{d^3}}$$

$$T = \frac{2\pi}{\omega} = \sqrt{\frac{4\pi^2 d^3}{G(m_1+m_2)}}$$

Note: previously, we analyzed a uniform circular motion of a mass m_2 around a central point mass m_1, with an implicit assumption that $m_2 \ll m_1$. We obtained Kepler's law:

$$r^3 = \frac{Gm_1T^2}{4\pi^2} \Rightarrow T = \sqrt{\frac{4\pi^2 r^3}{Gm_1}}.$$

When we consider the acceleration of both masses, we still have Kepler's law: cube of distance proportional to square of period. But now we have both masses affecting this relationship.

Let's take an example: Earth and Moon. If we assume the moon has a circular orbit caused by earth's gravitational force, we obtain

$$T = \sqrt{\frac{4\pi^2\cdot(3.82\cdot10^8)^3}{6.67\cdot10^{-11}(5.98\cdot10^{24})}} = 2.3488\times10^6$$

If we take into account the the Earth also orbits the moon, with all our assumptions above (constant distance, same ω), then we have a "double star" system and the period is

$$T = \sqrt{\frac{4\pi^2\cdot(3.82\cdot10^8)^3}{6.67\cdot10^{-11}(5.98\cdot10^{24}+7.36\cdot10^{22})}} = 2.3345\times10^6$$

The difference is 14322.591 seconds = 0.1657 days.