$\neg \forall x\, P(x)$

$\exists x\, \neg P(x)$

$\neg \exists x\, \neg P(x)$

$\boxed{a}$

$\neg P(a)$

$\exists x\, \neg P(a)$ — $\exists$ Intro

$\perp$ — $\perp$ Intro

$P(a)$ — $\neg$ Intro

$\forall x\, P(x)$ — $\forall$ Intro

$\perp$ — $\perp$ Intro

$\exists x\, \neg P(x)$ — $\neg$ Intro

$\neg \exists x\, P(x)$

$\forall(x)\, \neg P(x)$

$\boxed{a}$

$P(a)$

$\exists x\, P(x)$ — $\exists$ Intro

$\perp$ — $\perp$ Intro

$\neg P(a)$ — $\neg$ Intro

$\forall x\, \neg P(x)$ — $\forall$ Intro

$A \leftrightarrow B$

$\neg B \rightarrow \neg A$

$(A \wedge B) \mid (\neg A \wedge \neg B)$

$\neg B$

$A \wedge B$

B

$\perp$

$\neg A$

$\neg A \wedge \neg B$

$\neg A$

$\neg A$

$\neg B \rightarrow \neg A$

At least two $\quad \exists x \exists y \, (P(x) \land P(y) \land x \neq y)$

At least three $\quad \exists x \exists y \exists z \, (P(x) \land P(y) \land P(z) \land x \neq y \land x \neq z \land y \neq z)$

At most two $\equiv \neg$ At least three

$$\neg \exists x \exists y \exists z \, (P(x) \land P(y) \land P(z) \land x \neq y \land x \neq z \land y \neq z)$$
$$\iff \forall x \forall y \forall z \; \neg (P(x) \land P(y) \land P(z) \land x \neq y \land x \neq z \land y \neq z)$$
$$\iff \forall x \forall y \forall z \; (\neg P(x) \lor \neg P(y) \lor \neg P(z) \lor x = y \lor x = z \lor y = z)$$
$$\iff \forall x \forall y \forall z \, (\neg (P(x) \land P(y) \land P(z)) \lor x = y \lor x = z \lor y = z)$$
$$\iff \forall x \forall y \forall z \, ((P(x) \land P(y) \land P(z)) \rightarrow (x = y \lor x = z \lor y = z))$$

Exactly two

$$\exists x \exists y \, (P(x) \land P(y) \land x \neq y) \land \forall x \forall y \forall z \, ((P(x) \land P(y) \land P(z)) \rightarrow (x = y \lor x = z \lor y = z))$$

At least two $\land$ At most two

also.

$$\exists x \exists y \, (P(x) \land P(y) \land x \neq y \land \forall z (P(z) \rightarrow (z = x \lor z = y)))$$

two $\exists$, one $\forall$

$$\iff \exists x \exists y \, (x \neq y \land \forall z (P(z) \leftrightarrow (z = x \lor z = y)))$$

* in general, to say $\exists^{!n} x \, P(x)$ we need n+1 quantifiers, n $\exists$ and one $\forall$.

Exactly one $\quad \exists x \, (P(x) \land \forall y \, (P(y) \rightarrow y = x))$

abbrev. $\exists^{!1} x \, P(x)$

Proof of $\exists^{!n} x \, P(x)$

prove two things

$\exists^{\geq n} x \, P(x)$ At least n

$\exists^{\leq n} x \, P(x)$ At most n

The A is B (Russellian Analysis)

$$\exists x \, (A(x) \land \forall y \, (A(y) \rightarrow x = y) \land B(x))$$

Exactly one with A, and it is also B.

Both As are Bs. $\quad \exists^{!2} x \, A(x) \land \forall x \, (A(x) \rightarrow B(x))$

Neither Ps are Bs. $\exists^{!2} x\, P(x) \wedge \forall x\, (P(x) \to \neg B(x))$

$\neg$ (The A is B)

$$\neg(\exists x\, (A(x) \wedge \forall y\, (A(y) \to y = x)) \wedge B(x))$$

$$\neg \exists x\, (A(x) \wedge \forall y\, (A(y) \to y = x) \vee \neg B(x)$$

Neither cube is not large

$$\exists x \exists y\, (Cube(x) \wedge Cube(y) \wedge \forall z (Cube(z) \to z = x \vee z = y) \wedge \neg\neg Large(x) \wedge \neg\neg Large(y))$$

Neither cube is large.

$$\exists x \exists y\, (Cube(x) \wedge Cube(y) \wedge \forall z (Cube(z) \to z = x \vee z = y) \wedge \neg Large(x) \wedge \neg Large(y))$$

14.12

$\exists x \exists y (Cube(x) \land Cube(y) \land x \neq y)$

$\forall x \forall y \forall z ((Cube(x) \land Cube(y) \land Cube(z)) \to (x = y \lor x = z \lor y = z))$

$\boxed{a}$ $\boxed{b}$ $Cube(a) \land Cube(b) \land a \neq b$

$\boxed{z}$ $Cube(z)$

$(Cube(a) \land Cube(b) \land Cube(z)) \to (a = b \lor a = z \lor b = z)$

$a = b \lor a = z \lor b = z$

$a = b$

$\perp$

$a = z \lor b = z$

$a = z \lor b = z$

$a = z \lor b = z$

$\forall z (Cube(z) \to (a = z \lor b = z))$

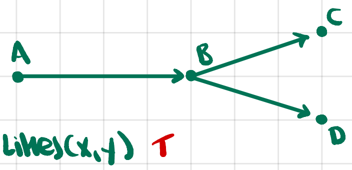
$\forall z (Cube(z) \to (a = z \lor b = z))$

14.21

Consider predicate $Likes(x, y)$

$\exists! x \exists! y\, Likes(x, y)$ means there is exactly one person with the property that there is exactly one person that the first person likes

$\exists! y \exists! x\, Likes(x, y)$ means there is exactly one person with the property that there is exactly one person that likes the first person.



$\exists! x \exists! y\, Likes(x, y)$ T

$\exists! y \exists! x\, Likes(x, y)$ F

14.23

$$\exists^{!2} y\,(y+y = y \times y)$$

$$\exists x\, \exists y\, \big((y+y = y \times y) \wedge (x+x = x \times x) \wedge \forall z\,(z+z = z \times z \rightarrow z = x \vee z = y)\big)$$

$$y+y = y \cdot y$$

$$\leftrightarrow\ 2y = y^2$$

$$\leftrightarrow\ y(y-2) = 0$$

$$\leftrightarrow\ y = 0 \vee y = 2$$