General Conditional Proof

$\boxed{c}$ $P(c)$ Conditional proof

$(\dots)$

$Q(c)$

Universal Introduction

$\forall x\,(P(x) \to Q(x))$

Universal Intro

$\boxed{c}$

$(\dots)$

$P(c)$

$\forall x\, P(x)$

We can think of Univ. Intro as special case of GCP:

$\boxed{c}$ $c = c$

$(\dots)$

$P(c)$

$\forall x\,(x = x \to P(x))$

$\forall x\, P(x)$

Euclid's Theorem $\forall x\, \exists y\, [y \geq x \wedge \text{Prime}(y)]$

1 **Domain:** $\forall x\; x \in \mathbb{N}$

2 **Lemma 1:** $\forall x$ HasPrimeFactorization(x)

3 **Lemma 2:** $\forall y\, (\exists x\, \exists f\, (x = \prod_{i=1}^{f} p_i \wedge \forall z\, ((z < y \wedge \text{Prime}(z)) \rightarrow \exists i\, (z = p_i))))$

ie for any natural number, there is a natural number equal to the product of the primes smaller than x.

4 **Lemma 3:** $\forall x\, (\exists f\, (x = \prod_{i=1}^{f} p_i) \rightarrow$ each p_i divides x w/ remainder $0)$

ie every natural number that is a product of factors $p_1, \dots, p_f$ is divisible by each factor.

5 **$\forall$-Generalization Assumption:** Let n be any object of domain.

6 **$\forall$ Elim 3:** $\exists x\, \exists f\; x = \prod_{i=1}^{f} p_i \wedge \forall z\, ((z < n \wedge \text{Prime}(z)) \rightarrow \exists i\, (z = p_i)))$

ie there is a number equal to the product of primes smaller than n

7 **$\exists$-Elim Assumption:** Let k be such that $\exists f\; k = \prod_{i=1}^{f} p_i \wedge \forall z\, ((z < n \wedge \text{Prime}(z)) \rightarrow \exists i\, (z = p_i)))$

ie let k be that number.

8 **Lemma 4:** $\forall x\, (\exists y\;\; y = x + 1)$

ie for every number there exists a number equal to the first number plus one

9 **$\forall$ Elim 8:** $\exists y\;\; y = k+1$

ie there is a number equal to k plus one

10 **$\exists$-Elim Assumption:** Let $m = k+1$.

ie let m be that number

11 **Lemma 5:** $\forall x$ (each prime in x's factorization divides $x+1$ w/ remainder 1)

12 **$\forall$ Elim 2:** HasPrimeFactorization(m)

13 **$\exists$-Elim Assumption:** Let p be a prime from m's factorization.

14 **$\neg$Intro Assumption:** $p < n$

15 **$\wedge$ Elim, 7:** $\forall z\, ((z < n \wedge \text{Prime}(z)) \rightarrow \exists i\, (z = p_i)))$

ie every prime smaller than n is a factor of k

16 **$\forall$ Elim 15:** $(p < n \wedge \text{Prime}(p))) \rightarrow \exists i\;\; p = p_i$

17 **$\rightarrow$ Elim 16,13,14:** p is a prime factor of k

18 **$\forall$ Elim 11:** p divides m with remainder 1

19 p is not in m's factorization.

20 **$\perp$ Intro 19,14** $\perp$

11 Lemma 5: $\forall x$ (each prime in x's factorization divides x+1 w/ remainder 1)

12 $\forall$ Elim 2: HasPrimeFactorization(m)

13 $\exists$-Elim Assumption: Let p be a prime from m's factorization.

14 $\neg$ Negation Assumption: $p < n$

15 $\wedge$ Elim, 7: $\forall z\,((z < n \wedge \text{Prime}(z)) \rightarrow \exists i\,(z = P_i))$
ie every prime smaller than n is a factor of k

16 $\forall$ Elim 15: $(p < n \wedge \text{Prime}(p)) \rightarrow \exists i\; p = p_i$

17 $\rightarrow$ Elim 16,13,14: p is a prime factor of k

18 $\forall$ Elim 11: p divides m with remainder 1

19 p is not in m's factorization.

20 $\perp$ Intro 19,14 $\perp$

21 $\neg$ Intro 14-20 $p \geq n$

22 $\wedge$ Intro 13,21: $\text{Prime}(p) \wedge p \geq n$

23 $\exists$ Intro 22: $\exists x\,(\text{Prime}(x) \wedge x \geq n)$

24 $\exists$ Elim 13-23: $\exists x\,(\text{Prime}(x) \wedge x \geq n)$

25 $\exists$-Elim 10-24: $\exists x\,(\text{Prime}(x) \wedge x \geq n)$

26 $\exists$-Elim 7-25: $\exists x\,(\text{Prime}(x) \wedge x \geq n)$

27 $\forall$ Intro 5-26 $\forall y\,(\exists x\,(\text{Prime}(x) \wedge x \geq y))$

21 ¬ Intro 14-20 $p \geq n$

22 ∧ Intro 13, 21: $Prime(p) \wedge p \geq n$

23 ∃ Intro 22: $\exists x\,(Prime(x) \wedge x \geq n)$

24 ∃ Elim 13-23: $\exists x\,(Prime(x) \wedge x \geq n)$

25 ∃-Elim 10-24: $\exists x\,(Prime(x) \wedge x \geq n)$

26 ∃-Elim 7-25: $\exists x\,(Prime(x) \wedge x \geq n)$

27 ∀ Intro 5-26 $\forall y\,(\exists x\,(Prime(x) \wedge x \geq y))$

$\forall x\, (Prof(x) \rightarrow Teacher(J))$

$Prof(J)$

$Prof(J) \rightarrow Teacher(J)$ $\forall$ Elim

$Teacher(J)$ $\rightarrow$ Elim

$\exists x\, Teacher(x)$ $\exists$ Intro

12.16

$\forall x \forall y\, (LeftOf(x,y) \rightarrow Larger(x,y))$
$\forall x\, (Cube(x) \rightarrow Small(x))$
$\forall x\, (Tet(x) \rightarrow Large(x))$
$\forall x \forall y\, ((Small(x) \wedge Small(y)) \rightarrow \neg Larger(x,y))$

$\neg$ Intro Ass.: $\exists x \exists y\, (Cube(x) \wedge Cube(y) \wedge RightOf(x,y))$

$\exists$ Elim Ass: $Cube(C1) \wedge Cube(C2) \wedge RightOf(C1,C2)$

$\forall$ Elim: $Cube(C1) \rightarrow Small(C1)$
$Cube(C2) \rightarrow Small(C2)$

$\wedge$ Elim $Cube(C1)$
$Cube(C2)$

$\rightarrow$ Elim $Small(C1)$
$Small(C2)$
$\neg Larger(C_1, C_2)$
$\neg Larger(C_2, C_1)$

$\wedge$ Elim $RightOf(C_1, C_2)$

Axiom: $LeftOf(C_2, C_1)$

$\rightarrow$ Elim: $Larger(C_2, C_1)$

$\perp$ Intro: $\perp$

$\exists$ Elim: $\perp$

$\neg$ Intro: $\neg \exists x \exists y\, (Cube(x) \wedge Cube(y) \wedge RightOf(x,y))$

12.18

$\forall x \forall y\, (\mathrm{LeftOf}(x,y) \to \mathrm{Larger}(x,y))$
$\forall x\, (\mathrm{Cube}(x) \to \mathrm{Small}(x))$
$\forall x\, (\mathrm{Tet}(x) \to \mathrm{Large}(x))$
$\forall x \forall y\, ((\mathrm{Small}(x) \land \mathrm{Small}(y)) \to \neg \mathrm{Larger}(x,y))$

$\forall z \forall w\, ((\mathrm{Tet}(z) \land \mathrm{Cube}(w)) \to \mathrm{LeftOf}(z,w))$

INVALID

12.33

$\forall x \forall y\ \mathrm{SameShape}(x,y)$

$\forall x\, \mathrm{Cube}(x) \lor \forall x\, \mathrm{Tet}(x) \lor \forall x\, \mathrm{Dodec}(x)$

$\boxed{c}$

$\boxed{d}$

$\forall$ Elim SameShape(c,d)

A4 : Tet(c) $\lor$ Cube(c) $\lor$ Dodec(c)

T(c)

A10 : T(d)

C(c)

A8 : C(d)

D(c)

A9 : D(d)

$\lor$-Elim $(T(d) \land T(c)) \lor (C(d) \land C(c)) \lor (D(d) \land D(c))$

$\forall x\ (T(x) \land T(c)) \lor (C(x) \land C(c)) \lor (D(x) \land D(c))$