Ch 10 - Differentiation

Theorem 1 f constant fn $f(x)=c$ $\rightarrow$ $\forall a\ f'(a)=0$

Proof

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{0}{h}=0$$

Theorem 2 f the identity fn $f(x)=x$ $\rightarrow$ $\forall a\ f'(a)=1$

Proof

$$f'(a)=\lim_{h\to 0}\frac{a+h-a}{h}=1$$

Theorem 3 f and g diff at a $\rightarrow$ $f+g$ also diff at a

$$(f+g)'(a)=f'(a)+g'(a)$$

Proof

$$(f+g)'(a)=\lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(a)-g(a)}{h}=f'(a)+g'(a)$$

Theorem 4 f and g diff at a $\rightarrow$ fg also diff at a

$$(fg)'(a)=f'(a)g(a)+f(a)g'(a)$$

Proof

$$(fg)'(a)=\lim_{h\to 0}\frac{f(a+h)g(a+h)-f(a)g(a)}{h}$$

$$=\lim_{h\to 0}\frac{f(a+h)g(a+h)-f(a+h)g(a)+f(a+h)g(a)-f(a)g(a)}{h}$$

$$=\lim_{h\to 0}\frac{f(a+h)(g(a+h)-g(a))+g(a)(f(a+h)-f(a))}{h}$$

$$=f(a)g'(a)+g(a)f'(h)$$

note that we had to compute $\lim_{h\to 0}f(a+h)=f(a)$. This follows from f being continuous, since it is diff.

Theorem 5 $g(x) = cf(x)$, f diff at a $\rightarrow$ g diff at a, $g'(x) = cf'(x)$

Proof

$$g'(x) = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} = cf'(x)$$

Altern., apply Th.4

$h(x) = c$

$g(x) = h(x)f(x)$

$g'(x) = h'f + hf' = hf' = cf'(x)$

Special cases

$(-f)(x) = -1 \cdot f(x) \rightarrow (-f)'(x) = -f'(x)$

$(f-g)(x) = (f+(-g))(x) \rightarrow (f-g)'(x) = f'(x) - g'(x)$

Theorem 6 $f(x) = x^n$, $n \in \mathbb{N} \rightarrow \forall a \; f'(a) = na^{n-1}$

Proof

$A = \{n : f(x) = x^n, n \in \mathbb{N} \rightarrow \forall a \; f'(a) = na^{n-1}\}$

$f(x) = x \rightarrow f'(x) = 1 = 1 \cdot x^{1-1} \rightarrow 1 \in A$

Assume $k \in A$. Then $f(x) = x^k \rightarrow \forall a \; f'(a) = ka^{k-1}$.

Let $f(x) = x^{k+1} = x \cdot x^k$.

By Th. 4, $f'(x) = 1 \cdot x^k + x \cdot kx^{k-1}$

$= x^k(1+k)$

$= (k+1)x^{(k+1)-1}$

T.F. $A = \mathbb{N}$.

Theorem 7 g diff at a, $g(a) \neq 0$ $\rightarrow$ $\frac{1}{g}$ diff at a, $\left(\frac{1}{g}\right)'(a) = \frac{-g'(a)}{g(a)^2}$

Proof

$$g'(a) = \lim_{h\to 0} \frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} = \lim_{h\to 0} \frac{g(a) - g(a+h)}{hg(a)g(a+h)} = \lim_{h\to 0} \frac{-[g(a+h)-g(a)]}{h} \cdot \lim_{h\to 0} \frac{1}{g(a)g(a+h)}$$

$$= \frac{-g'(a)}{g(a)^2}$$

note $\lim g(a+h) = g(a)$ because g is cont. since it is diff.

Theorem 8 f, g diff at a, $g(a) \neq 0$ $\rightarrow$ f/g diff at a, $\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - g'(a)f(a)}{g(a)^2}$

Proof

$$\frac{f}{g} = f \cdot \frac{1}{g}$$

By Thm 4, $\left(f \cdot \frac{1}{g}\right)'(a) = f'(a)\left(\frac{1}{g}\right)(a) + f(a)\left(\frac{1}{g}\right)'(a)$

$$= \frac{f'(a)}{g(a)} - \frac{f(a)g'(a)}{g(a)^2} = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$$

With this result we can extend Theorem 6, true for $\mathbb{N}$, to being true for $\mathbb{Z}$.

$$f(x) = x^{-n} = \frac{1}{x^n} \rightarrow f'(x) = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1}$$

$$f(x) = 1 = x^0 \rightarrow f'(x) = 0 = 0 \cdot x^{-1}$$

Note that we are "interpreting" 1 as x^0 and 0 as $0 \cdot x^{-1}$.
At $x = 0$ these interpretations aren't valid: 0^0 needs to be defined, $0 \cdot 0^{-1}$ is meaningless.

Following results presented w/o proof

$\sin'(a) = \cos(a)$
$\cos'(a) = -\sin(a)$

Chain Rule Proof Intro

$$(f \circ g)'(a) = \lim_{h \to 0} \frac{(f \circ g)(a+h) - (f \circ g)(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{h}$$

$g(a+h) - g(a)$ could be 0 for $h \neq 0$ rendering this step meaningless
e.g. if $g(x) = c$. Also $g(x) = \begin{cases} x^2 \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$

$$= \lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \cdot \frac{g(a+h) - g(a)}{h}$$

$$= \lim_{h \to 0} \; (f \circ g)(a+h) - (f \circ g)(a)$$

$$= \lim_{h \to 0} \frac{f[g(a) + g(a+h) - g(a)] - f(g(a))}{g(a+h) - g(a)} \cdot \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

let $k(h) = g(a+h) - g(a)$. Then

$$= \lim_{h \to 0} \frac{f(g(a) + k(h)) - f(g(a))}{k(h)} \cdot g'(a)$$

Chain Rule Proof Strategy

we want to prove $(f \circ g)'(a) = f'(g(a))\,g'(a)$.

$$(f \circ g)'(a) = \lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{h} = \lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \cdot \frac{g(a+h) - g(a)}{h}$$

$$= \underbrace{\lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)}}_{\text{we need to prove that this equals } f'(g(a))} \cdot \underbrace{\lim_{h \to 0} \frac{g(a+h) - g(a)}{h}}_{g'(a)}$$

we do this by defining a fn $\phi(h) = \begin{cases} \dfrac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} & \text{if } g(a+h) - g(a) \neq 0 \\ f'(g(a)) & \text{if } g(a+h) - g(a) = 0 \end{cases}$

and proving that it is cont. at 0, ie that $\displaystyle\lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} = f'(g(a))$.

Theorem 9 Chain Rule

g diff at a, f diff at $g(a)$ $\to$ $f \circ g$ diff at a, $(f \circ g)'(a) = f'(g(a))\, g'(a)$

Proof

Define ϕ

$$\phi(h) = \begin{cases} \dfrac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} & \text{if } g(a+h) - g(a) \neq 0 \\ f'(g(a)) & \text{if } g(a+h) - g(a) = 0 \end{cases}$$

Intuitively, near $h=0$, $\phi(h)$ is either $f'(g(a))$ or it is close to $f'(g(a))$. We just prove $\lim_{h\to 0} \phi(h) = f'(g(a))$, thus ϕ is cont. at 0.

f diff at $g(a)$ $\to$ $\displaystyle\lim_{h\to 0} \frac{f(g(a)+h) - f(g(a))}{h} = f'(g(a))$

Thus $\forall \varepsilon > 0\ \exists \delta' > 0$ s.t. $\forall h$

$$0 < |h| < \delta' \to \left| \frac{f(g(a)+h) - f(g(a))}{h} - f'(g(a)) \right| < \varepsilon \qquad (1)$$

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g is diff and hence cont. at a. Thus

$$\exists \delta > 0\ \forall h\quad |h| < \delta \to |g(a+h) - g(a)| < \delta' \qquad (2)$$

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nothing special so far, simple definitions of differentiability.

Let h s.t. $|h| < \delta$, let $k = g(a+h) - g(a) \neq 0$.

$(2) \to |k| < \delta'$

rewrite $\phi(h)$ with k

$$\phi(h) = \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} = \frac{f(g(a)+k) - f(g(a))}{k}$$

$(1) \to |\phi(h) - f'(g(a))| < \varepsilon$

$\to \displaystyle\lim_{h\to 0} \phi(h) = f'(g(a))$

since g is continuous, we can say something about $g(a+h) - g(a)$. This term appears in the denominator and numerator of $\phi(h)$. In fact when we sub it in, this term performs the role of h in a classic derivative expr, $[f(x+h) - f(x)]/h$.

Then

assume $|h| < \delta$. then $|g(a+h) - g(a)| < \delta'$

then $|\phi(h) - f'(g(a))| < \varepsilon$.

Hence $\displaystyle\lim_{h\to 0} \phi(h) = f'(g(a))$

$$(f\circ g)'(a) = \lim_{h\to 0} \frac{f(g(a+h)) - f(g(a))}{h} = \lim_{h\to 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \cdot \frac{g(a+h) - g(a)}{h}$$

$$= \phi(h) \cdot g'(a)$$

$$= f'(g(a))\, g'(a)$$

Some Examples

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f'(0) = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0$$

Now with the chain rule, we can compute $f'(x)$ for $x \neq 0$.

$$f'(x) = 2x \sin {}^1/x - x^2 \cos({}^1/x) \cdot ({}^1/x^2), \quad x \neq 0.$$

Thus $f'(x) = \begin{cases} 2x\sin({}^1/x) - \cos({}^1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} (2x\sin({}^1/x) - \cos({}^1/x))$$

As we know, $\lim \cos({}^1/x)$ does not exist.

Hence $f'(x)$ is not cont. at 0.

$$f(x) = \begin{cases} x^3 \sin({}^1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f'(0) = \lim \frac{h^3 \sin({}^1/h)}{h} = \lim_{h \to 0} h^2 \sin({}^1/x) = 0$$

$$f'(x) = 3x^2 \sin({}^1/x) + x^3 \cos({}^1/x)(-{}^1/x^2)$$
$$= 3x^2 \sin({}^1/x) - x\cos({}^1/x)$$

$$\lim_{h \to 0} \frac{h\cos({}^1/h)}{h} = \lim_{h \to 0} \cos({}^1/h), \text{ which doesn't exist.}$$

Hence $x\cos({}^1/x)$ is not differentiable at 0, therefore neither is $f'(x)$.

$\lim_{x \to 0} f'(x) = 0$, hence $f'(x)$ is cont. at 0.