Ch 14 - The Fundamental Theorem of Calculus

1. (i) $F(x) = \int_a^{x^3} \sin^3 t \, dt$

$h_1(x) = x^3, \quad h_2(x) = \int_a^x \sin^3 t \, dt$

$F(x) = h_2(h_1(x))$

$F'(x) = h_2'(h_1(x)) \cdot h_1'(x) = \sin^3(t)\Big|_{t=x^3} \cdot 3x^2 = \sin^3(x^3) \cdot 3x^2$

(ii) $F(x) = \displaystyle\int_3^{\int_1^x \sin^3 t \, dt} \frac{1}{1+\sin^6 t + t^2} \, dt$

$h_1(x) = \int_1^x \sin^3 t \, dt \rightarrow h_1'(x) = \sin^3(x)$

$h_2(x) = \displaystyle\int_3^x \frac{1}{1+\sin^6 t + t^2} \, dt \rightarrow h_2'(x) = \frac{1}{1+\sin^6 x + x^2}$

$F(x) = h_2(h_1(x)) \rightarrow F'(x) = h_2'(h_1(x)) h_1'(x) = \dfrac{\sin^3(x)}{1+\sin^6\left(\int_1^x \sin^3 t \, dt\right) + \left(\int_1^x \sin^3 t \, dt\right)^2}$

(iii) $F(x) = \displaystyle\int_{15}^x \left(\underbrace{\int_8^y \frac{1}{1+t^2+\sin^2 t} \, dt}_{\text{This is a fn of } y \text{: for each } y \text{ the integral has a value}} \right) dy$

we then integrate this fn.

$h_1(x) = \int_{15}^x f(t) \, dt$

$f(y) = \displaystyle\int_8^y \frac{1}{1+t^2+\sin^2 t} \, dt$

$F(x) = \int_{15}^x f(y) \, dy \rightarrow F'(x) = f(x) = \displaystyle\int_8^x \frac{1}{1+t^2+\sin^2 t} \, dt$

(iv) $F(x) = \displaystyle\int_x^b \frac{1}{1+t^2+\sin^2 t} \, dt = \int_a^b \frac{1}{1+t^2+\sin^2 t} \, dt - \int_a^x \frac{1}{1+t^2+\sin^2 t} \, dt$

$F'(x) = -\dfrac{1}{1+x^2+\sin^2 x}$

(v) $F(x) = \int_a^b \frac{x}{1+t^2+\sin^2 t}\,dt = x\int_a^b \frac{1}{1+t^2+\sin^2 t}\,dt$

$\rightarrow F'(x) = \int_a^b \frac{1}{1+t^2+\sin^2 t}\,dt$

(vi) $F(x) = \sin\left(\int_0^x \sin\left(\int_0^y \sin^3 t\,dt\right) dy\right)$

$f(y) = \int_0^y \sin^3 t\,dt$

$h_1(x) = \sin x \rightarrow h_1'(x) = \cos x$

$h_2(x) = h_1(f(x)) = \sin\left(\int_0^x \sin^3 t\,dt\right)$

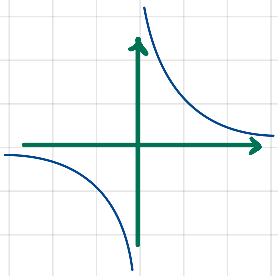
$F(x) = h_1\left(\int_0^x h_1(f(y))\,dy\right) = h_1\left(\int_0^x h_2(y)\,dy\right)$

$F'(x) = h_1'\left(\int_0^x h_2(y)\,dy\right)\cdot h_2(x) = \cos\left(\int_0^x h_2(y)\,dy\right)\cdot \sin\left(\int_0^x \sin^3 t\,dt\right)$

$= \cos\left(\int_0^x \sin\left(\int_0^y \sin^3 t\,dt\right) dy\right)\cdot \sin\left(\int_0^x \sin^3 t\,dt\right)$ ✓

(vii) $F(x) = \int_1^x \frac{1}{t}\,dt$

$F'(x) = \frac{1}{x}$.



$F' > 0$ for $x > 0$ → F one-one on $(0,\infty)$ → By Th. 12-1, f^{-1} is a function on $(0,\infty)$

$F' < 0$ for $x < 0$ → " " " " $(-\infty, 0)$ " " " " " " " $(-\infty, 0)$

Since $\frac{1}{x}$ is integrable on $[a,b]$ w/ $a,b>0$ or $a,b<0$, then by 13-7, F is continuous on $[a,b]$

Furthermore, F is diff. at any $y = F^{-1}(x) \in [a,b]$ and $F' \neq 0$.

T.F. by Th. 12-5, F^{-1} is diff. at d and

$$(F^{-1})'(x) = \frac{1}{F'(F^{-1}(x))} = \frac{1}{\frac{1}{F^{-1}(x)}} = F^{-1}(x)$$

Theorem 5 Let f be cont. one-one fn defined on interval. Suppose f diff at $f^{-1}(b)$, w/ derivative $f'(f^{-1}(b)) \neq 0$. Then, f^{-1} diff. at b and $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$

viii) $F(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\, dt = \int_0^x f(t)\, dt$

Note that $f(t)$ undefined at 1 and -1 and unbounded in any interval containing 1 or -1. Domain $(-1,1)$.

$F'(x) = \frac{1}{\sqrt{1-x^2}} > 0$ in domain $\to$ F one-one $\to$ F^{-1} is fn.

Also, F is diff. in entire domain and $F' \neq 0$.

Therefore, F^{-1} is diff. on $(-1,1)$ and

$$F^{-1\prime}(x) = \frac{1}{F'(F^{-1}(x))} = \frac{1}{\frac{1}{\sqrt{1-F^{-1}(x)^2}}} = \sqrt{1-F^{-1}(x)^2} \checkmark$$

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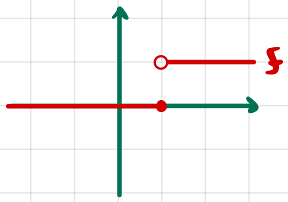
2. $F(x) = \int_0^x f$

(i)



$$f(x) = \begin{cases} 0 & x \le 1 \\ 1 & x > 1 \end{cases}$$

f is integrable and continuous on $(-\infty, 1]$.

Let $g(x) = c$. Then $g'(x) = 0 = f(x)$ on $(-\infty, 1]$.

Hence, for $x \in (-\infty, 1]$ we have $F(x) = 0$.

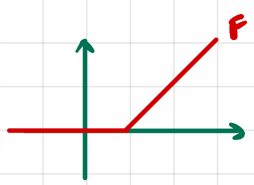
f is int. on $[1, \infty)$. f is cont. on any $(1, \infty)$.

Let $g(x) = x$. Then $g'(x) = 1 = f(x)$ on $(1, \infty)$.

Hence, for $x \in (1, \infty)$ we have $F(x) = \int_0^1 f + \int_1^x f = 0 + (x-1) = x-1$

Thus

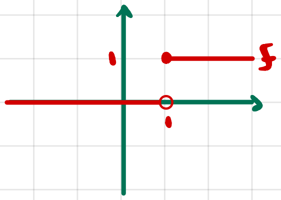
$$F(x) = \begin{cases} 0 & x \le 1 \\ x-1 & x > 1 \end{cases}$$



$$F'(x) = \begin{cases} 0 \\ 1 \end{cases}$$

$\to$ F not diff. at $x = 1$.

(ii) $f(x) = \begin{cases} 0 & x<1 \\ 1 & x \geq 1 \end{cases}$



f intl. on $(-\infty, x)$ for any $x<1$.

Thus on any $(-\infty, x]$ we have $F(x)=0$ by FTC2.

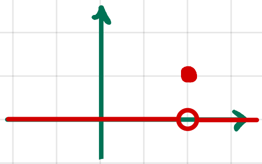
On $[1,\infty)$, FTC2 tells us $F(x) = x-1$.

$$F(x) = \begin{cases} 0 & x<1 \\ x-1 & x \geq 1 \end{cases}$$

F not diff at 1.

$$F'(x) = \begin{cases} 0 & x<1 \\ 1 & x>1 \end{cases}$$

(iii) $f(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$



On any interval $(-\infty,1)$ or $(1,\infty)$, we can use FTC2.

$$x \leq 1 \to F(x) = \int_0^x 0\,dx = 0$$

$$x \geq 1 \to F(x) = \int_0^1 0\,dx + \int_1^x 0\,dx = 0$$

we can add point $x=1$ to either interval because it doesn't affect integrability or the value of an integral.

As we saw in ch 13, f is actually integrable on $\mathbb{R}$. F is defined everywhere.

$F(x) = 0 \quad x \in \mathbb{R}$

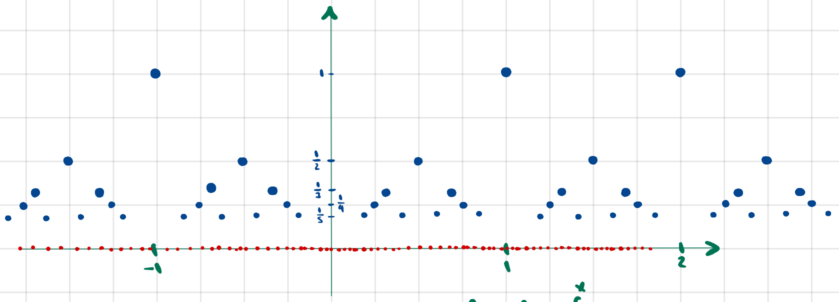
$\to F'(x) = 0$

T.F. $F'(x) = f(x)$ for $x \neq 1$.

(iv) $f(x) = \begin{cases} 0 & x \text{ irrational} \\ 1/q & x = p/q \text{ in lowest terms} \end{cases}$

In prob. 13-34 we proved that f is int. on $[0,1]$.

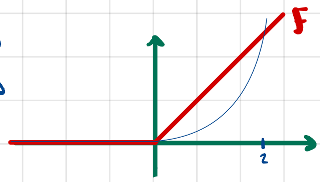
It can be shown to be integrable on any interval.



Since all lower sums equal 0, then $\sup\{L(f,P)\} = \int_0^x f = 0$ for any $[0,x]$.

Hence, $F(x) = 0$, $F'(x) = 0$, so $F'(x) = f(x)$ for all irrational x.

(v) $f(x) = \begin{cases} 0 & x \leq 0 \\ x & x \geq 0 \end{cases}$



f is cont. everywhere.

On $(-\infty, 0]$, $g(x) = c$ is s.t. $g'(x) = f(x)$

Thus for $x \in (-\infty, 0]$, $F(x) = \int_0^x f = g(x) - g(0) = 0$

For $x \in [0, \infty)$, $g(x) = \frac{x^2}{2}$ is s.t. $g' = f$.

Hence $F(x) = \frac{x^2}{2} - 0 = \frac{x^2}{2}$

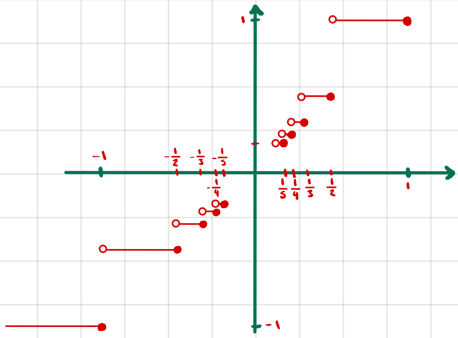
$F(x) = \begin{cases} 0 & x \leq 0 \\ x^2/2 & x \geq 0 \end{cases}$

$F'(x) = \begin{cases} 0 & x \leq 0 \\ x & x \geq 0 \end{cases}$

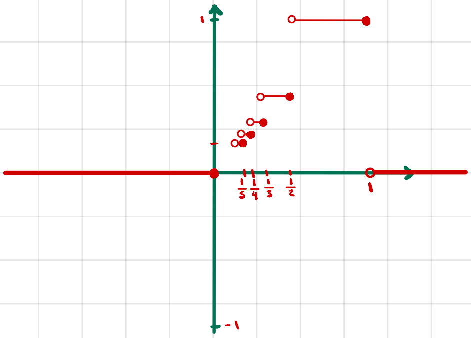
$F' = f$ for all x

(vi) $f(x) = \begin{cases} 0 & x \le 0 \text{ or } x > 1 \\ 1/[1/x] & 0 < x \le 1 \end{cases}$

Let's recall the graph of $\dfrac{1}{[\frac{1}{x}]}$.



Then f is



As shown in 13-7(vi), f is integrable.

for $x \le 0$ or $x > 1$, $F(x) = c \rightarrow F'(x) = 0 = f(x)$.

Let $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$

$$F(x) = \sum_{i=n+1}^{\infty} \int_{1/(i+1)}^{1/i} f + \int_{1/(n+1)}^{x} f = \sum_{i=n+1}^{\infty} \int_{1/(i+1)}^{1/i} \frac{1}{i+1}\,dx + \int_{1/(n+1)}^{x} \frac{1}{n+1}\,dx$$

FTC2

$$= \sum_{i=n+1}^{\infty} \frac{1}{i+1}\left(\frac{1}{i} - \frac{1}{i+1}\right) + \frac{1}{n+1}\left(x - \frac{1}{n+1}\right)$$

$$F'(x) = \frac{1}{n+1} = f(x)$$

$$F(1/n) = \sum_{i=n}^{\infty} \frac{1}{i+1}\left(\frac{1}{i} - \frac{1}{i+1}\right)$$

Let $h > 0$

$$\frac{F(1/n + h) - F(1/n)}{h} = \frac{\int_0^{1/n+h} f - \int_0^{1/n} f}{h} = \frac{\int_{1/n}^{1/n+h} f}{h} = \frac{\left(\frac{1}{n-1}\right)\left(\cancel{\frac{1}{n}} + h - \cancel{\frac{1}{n}}\right)}{h} = \frac{1}{n-1}$$

Now let $h < 0$

$$\frac{F(1/n + h) - F(1/n)}{h} = \frac{\int_0^{1/n+h} f - \int_0^{1/n} f}{h} = \frac{-\int_{1/n+h}^{1/n} \left(\frac{1}{n} + h\right)}{h} = \frac{-\left(\frac{1}{n}\right)\left(\cancel{\frac{1}{n}} - \cancel{\frac{1}{n}} - h\right)}{h} = \frac{1}{n}$$

Therefore $F'^{+}(1/n) \neq F'^{-}(1/n) \rightarrow F$ not diff. at $1/n$, $n \in \mathbb{N}$

(vii)

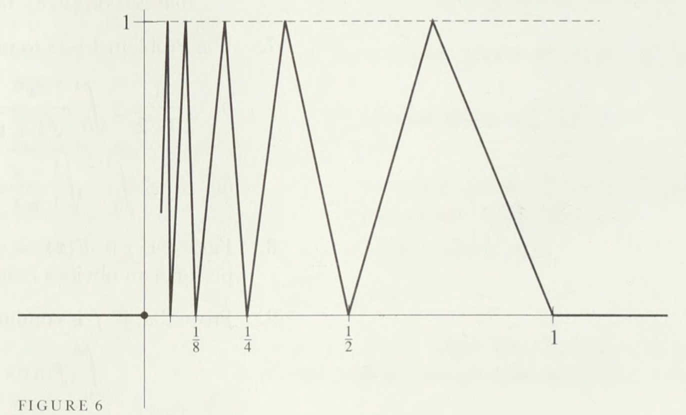


FIGURE 6

f is continuous everywhere except $x=0$.

Thus, by FTC1, $F'(x)=f(x)$ for all $x \neq 0$.

What happens at $x=0$? Is F diff. at $x=0$?

If it is then $F'_-(0)=F'_+(0)$, ie $\lim_{h\to 0^-}\frac{F(h)}{h}=0=\lim_{h\to 0^+}\frac{F(h)}{h}$.

Let $h>0$.

There exists $n\in\mathbb{N}$ s.t. $\frac{1}{2^n}<h$.

The sum of areas of triangles formed by f up to $\frac{1}{2^n}$ is $F(1/2^n)=\dfrac{1\cdot\sum_{i=n}^{\infty}\left(\frac{1}{2^i}-\frac{1}{2^{i+1}}\right)}{2}$

$=\dfrac{1/2^n}{2} \to \dfrac{F(1/2^n)}{1/2^n}=\dfrac{1}{2}$

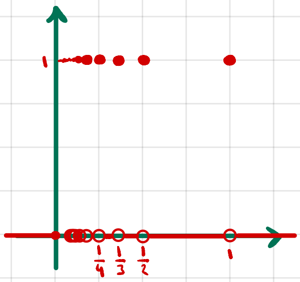
That is, for any $h>0$ there is always a number x s.t. $0<x<h$ and $\frac{F(x)}{x}=\frac{1}{2}$.

Therefore, $\lim_{h\to 0^+}\frac{F(h)}{h}\neq 0=\lim_{h\to 0^-}\frac{F(h)}{h}$.

Therefore $\lim_{h\to 0}\frac{F(h)}{h}=F'(0)$ is not defined, ie doesn't exist.

F' is not diff. at 0 hence 0 is the only point where $F'\neq f$.

(viii) $f(x) = \begin{cases} 1 & x = \frac{1}{n},\ n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$



f is cont. on intervals $(-\infty, 0]$, $(1, \infty)$, and $(1/_{n+1}, 1/n)$ for $n \in \mathbb{N}$.

By FTC1, $F'(x) = f(x)$ on these intervals.

f is integrable on $[0,1]$

Proof

Let P be any partition of $[0,1]$.

$L(f,P) = \sum m_i \Delta t_i = 0$

Let I be set of all numbers i s.t. Δt_i contains $1/n$, $n \in \mathbb{N}$.
Note that I contains infinite points.

$U(f,P) = \sum_{i \in I} 1 \cdot \Delta t_i$

For any $\epsilon > 0$, choose P s.t. $\sum_{i \in I} \Delta t_i < \epsilon$. Then $U(f,P) < \epsilon$.

Thus for any $\epsilon > 0$ there is a partition s.t. $U(f,P) - L(f,P) < \epsilon$, and f is integrable on $[0,1]$.

■

Therefore, F continuous on $[0,1]$.

Also, for all points $y = 1/n$ for some $n \in \mathbb{N}$, F' exists for all x in interval containing y and $\lim_{x \to y} F'(x) = 0$.

By Th. 11-7, since F' can't have a jump discontinuity, $F'(y) = \lim_{x \to y} F'(x) = 0$

But then $F'(y) = 0 \neq f(y) = 1$

$F'(x) = f(x)$ at all points except those points $y = 1/n$, $n \in \mathbb{N}$.

■

3. (i) $\int_0^x \frac{1}{1+t^2}\,dt + \int_0^{1/x} \frac{1}{1+t^2}\,dt = T(x)$

$\frac{1}{1+t^2}$ is continuous everywhere.

Therefore, $F(x) = \int_0^x \frac{1}{1+t^2}$ is s.t. $F'(x) = \frac{1}{1+x^2}$

Let $g(x) = \frac{1}{x}$.

Then $h(x) = F(g(x)) = \int_0^{1/x} \frac{1}{1+t^2}\,dt$.

$h'(x) = F'(g(x)) \cdot g'(x) = \frac{1}{1+\frac{1}{x^2}} \cdot \frac{(-1)}{x^2} = \frac{\frac{-1}{x^2}}{\frac{1+x^2}{x^2}} = -\frac{1}{1+x^2}$

$T'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$ $\rightarrow$ T is constant for all x.

(ii) $\int_{-\cos x}^{\sin x} \frac{1}{\sqrt{1-t^2}}\,dt$, $x \in (0, \pi/2)$

Note that

$f(t) = \frac{1}{\sqrt{1-t^2}}$ is unbounded below near ± 1. We are restricting x to $(0, \pi/2)$, so $-1 < -\cos x < 0$ and $0 < \sin x < 1$.

Given any $x \in (0, \pi/2)$, f is bounded on $[-\cos x, \sin x]$. f is also cont. on this interval.

Let $g(x) = \int_0^x f$.

Let $h(x) = \int_x^0 f = \int_a^0 f - \int_a^x f$, $a \in (-1, 1)$, in which f is integrable.

$h(-\cos x) = \int_a^0 f - \int_a^{-\cos x} f$

$T(x) = \int_{-\cos x}^{\sin x} \frac{1}{\sqrt{1-t^2}}\,dt = \int_{-\cos x}^0 f + \int_0^{\sin x} f = h(-\cos x) + g(\sin x) = \int_a^0 f - g(-\cos x) + g(\sin x)$

$T'(x) = -g'(-\cos x)(\sin x) + g'(\sin x)\cos x$

$= -\frac{\sin x}{\sqrt{1-\cos^2 x}} + \frac{\cos x}{\sqrt{1-\sin^2 x}}$

$= -1 + 1$

$= 0$

4. $(f^{-1})'(0)$

(i) $f(x) = \int_0^x (1+\sin(\sin t))\,dt$

Let $g(t) = 1+\sin(\sin t)$

$-1 \le \sin t \le 1 \rightarrow -1 < \sin(-1) \le \sin(\sin t) \le \sin 1 < 1$

Then $g(t) > 0$

$f(x) = \int_0^x g(t)$

$f'(x) = g(x) > 0 \rightarrow f$ is increasing

$\rightarrow f$ one-one

$\rightarrow f^{-1}$ is a fn

f is also continuous and $f'(x) = 1+\sin(\sin x) \neq 0$

Hence by Th. 12-5, f^{-1} is diff. and for any x s.t. $f^{-1}(x) = y$,

$$(f^{-1})'(x) = \frac{1}{f'(y)} = \frac{1}{1+\sin(\sin(f^{-1}(x)))}$$

Note that $f^{-1}(0) = 0$ because $f(0) = \int_0^0 g = 0$

Hence, $(f^{-1})'(0) = 1$

f increasing $\rightarrow f$ one-one

Proof

Let x, y and assume $x \neq y$.

Case 1: $x < y$. Then $f(x) < f(y)$

$\rightarrow f(x) \neq f(y)$

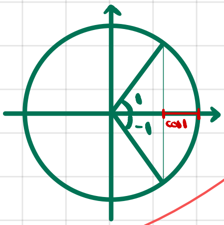
Case 2: $x > y$. Then $f(x) > f(y)$

$\rightarrow f(x) \neq f(y)$

In all cases, $f(x) \neq f(y)$

(ii) $f(x) = \int_1^x \cos(\cos t)\,dt$

1. Let $g(t) = \cos(\cos t)$
2. Then, $-1 \le \cos 1 \le 1$
3. $0 < \cos(-1) = \cos(1) \le g(t) \le 1$ Proof:



4. g is cont for all x. Proof:

$g(t)$ continuous on $\mathbb{R}$

Proof

$h_1(x) = \cos x$

$g(x) = (h_1 \circ h_1)(x) = \cos(\cos x)$

By Th. 6-2 since h_1 cont at all x and h_1 cont at all $\cos x$, then $h_1 \circ h_1$ cont at all x.

5. $f'(x) = \cos(\cos x)) > 0$ Proof: by FTC1
6. f one-one Proof: f increasing $\to$ f one-one
7. f^{-1} is fn Proof: f one-one $\to$ f^{-1} is fn
8. $(f^{-1})'$ is defined at all x. Proof: $f' \neq 0$ for all x $\to$ $(f^{-1})'(f(x))$ defined for all x
9. $(f^{-1})'(x) = \dfrac{1}{(f)'(f^{-1}(x))} = \dfrac{1}{\cos(\cos(f^{-1}(x)))}$ Proof: by Th. 12-5
10. $(f^{-1})(0) = 1$ Proof: $f(1) = \int_1^1 g = 0 \to f^{-1}(f(1)) = 1 = f^{-1}(0)$
11. $(f^{-1})'(0) = \dfrac{1}{\cos(\cos 1)}$

5. (i) $\int_0^x tg(t)\,dt = x + x^2$

1. Let $f(x) = \int_0^x tg(t)\,dt = x + x^2$

2. f is cont at all x.

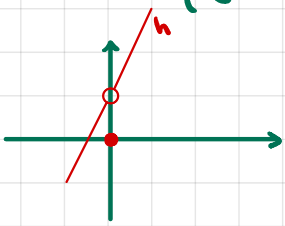
3. $f'(x) = xg(x) = 1 + 2x$ Proof: FTC1

4. $g(x) = \frac{1}{x} + 2$

note that g isn't defined at $x = 0$.

But $f(0) = \int_0^0 tg(t)\,dt = 0$ so we can define $g(t) = \begin{cases} \frac{1}{x} + 2 & x \neq 0 \\ 0 & x = 0 \end{cases}$

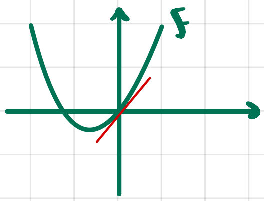
Then $h(t) = tg(t) = \begin{cases} 1 + 2t & t \neq 0 \\ 0 & t = 0 \end{cases}$



Note that $\int_0^x (1+2t)\,dt = x + x^2$, however we can't write $(1+2t) = tg(t)$ for all t.

The discontinuity of h at $x = 0$ doesn't affect integrability or the value of the integral.

Note that $f'(0) = 1 \neq h(0) = 0$



That is, because h is not continuous at 0, we can't apply the FTC1 at $x = 0$.

(ii) $f(x) = \int_0^{x^2} t g(t) dt = x + x^2$

1. let

$h(x) = x g(x)$

$f_1(x) = \int_0^x h$

$f_2(x) = x^2$

2. Then $f = f_1 \circ f_2$

$f'(x) = f_1'(f_2(x)) f_2'(x)$

note that f is diff. everywhere.

3. If h is cont. at x then

$f'(x) = x^2 g(x^2) \cdot 2x = 2x^3 g(x^2) = 1 + 2x$

$\rightarrow g(x^2) = \frac{1}{2x^3} + \frac{1}{x^2}$

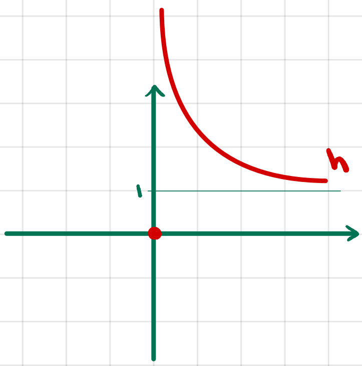
Let $y = x^2$. Then $g(y) = \frac{1}{2y\sqrt{y}} + \frac{1}{y}$

g is not defined at 0.

$f(0) = \int_0^0 h = 0$.

Let's define $g(x) = \begin{cases} \frac{1}{2x\sqrt{x}} + \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Then $h(x) = \begin{cases} \frac{1}{2\sqrt{x}} + 1 & x > 0 \\ 0 & x = 0 \end{cases}$



$\int_0^{x^2} h = (\sqrt{x} + x)\Big|_0^{x^2} = x + x^2$

Note that there is something weird w/ this solution since h is not bounded near 0, and hence shouldn't be integrable on $[0, x]$.

6. (a) All cont. fns satisfying $\int_0^x f = (f(x))^2 + C$, C constant $\neq 0$

Assume f has at most one 0.

$\int_0^0 f = f(0)^2 + C = 0 \rightarrow f(0)^2 = -C \rightarrow f(0) = \pm\sqrt{-C}$, $C<0$

f cont. then FTC1 $\rightarrow f(x) = 2f(x)f'(x)$

$\rightarrow f(x)(1-2f'(x)) = 0$

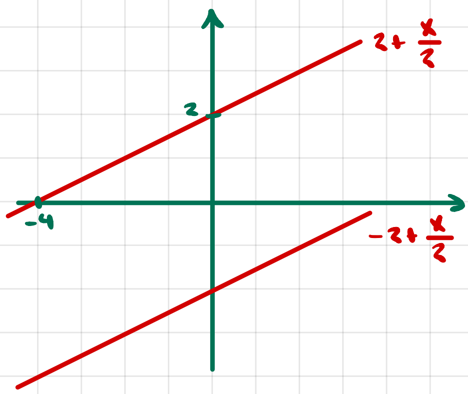
f is zero at most at one point, x_0.

$f(x_0) = 0$.

At all other x, $f(x) \neq 0$ so $f'(x) = \frac{1}{2}$

$\rightarrow f(x) = \frac{1}{2}x + b \rightarrow f(0) = b = \pm\sqrt{-C}$, $C<0$.

$f(x) = \frac{x}{2} \pm \sqrt{-C} = \frac{x}{2} + h$, $h \in \mathbb{R}$



Example

$C = -4$

$f(x) = \frac{x}{2} \pm \sqrt{4} = \frac{x}{2} \pm 2$

$\int_0^x \left(\frac{x}{2} \pm 2\right) dx = \left(\frac{x^2}{4} \pm 2x\right)\Big|_0^x$

$= \frac{x^2}{4} \pm 2x + 4 - 4$

$= \left(\frac{x}{2} \pm 2\right)^2 - 4$

$= (f(x))^2 - 4$

why can't c be 0?

$C = 0 \rightarrow f(x) = \frac{x}{2}$

$\int_0^x \frac{x}{2} dx = \frac{x^2}{4} = \left(\frac{x}{2}\right)^2 = (f(x))^2 + C$

(b) Solution that is 0 on $(-\infty, b]$, w/ $b<0$, but nonzero for $x>b$.

We now dropped the assumption that f is zero at only one point.

As before, f cont. then FTC1 $\to f(x) = 2f(x)f'(x) \to f(x)(1-2f'(x)) = 0$

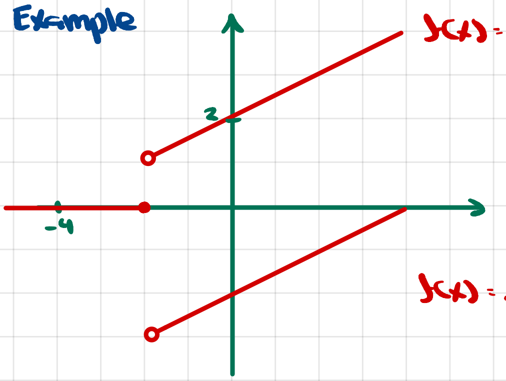
If f is zero on $(-\infty, b]$, then for $x>b$ we have $f'(x) = \frac{1}{2}$.

As before,

$\to f(x) = \frac{1}{2}x + d \to f(0) = d = \pm\sqrt{-c},\ c<0.$

$f(x) = \frac{x}{2} \pm \sqrt{-c},\ x>b.$

Example



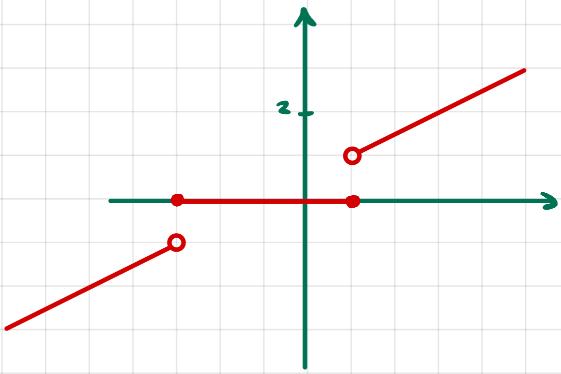
$f(x) = \begin{cases} 0 & x \le b \\ \frac{x}{2}+2 & x>b \end{cases}$

$f(x) = \begin{cases} 0 & x \le b \\ \frac{x}{2}-2 & x>b \end{cases}$

(c) $C=0$. Any $[a,b]$ w/ $a<0<b$. Solution s.t. f is zero on $[a,b]$, non-zero elsewhere.

As before, f cont. then FTC1 $\to f(x) = 2f(x)f'(x) \to f(x)(1-2f'(x)) = 0$

On $[a,b]$, $f(x)=0$. On $(-\infty, a) \cup (b, +\infty)$, $f'(x) = \frac{1}{2} \to f(x) = \frac{x}{2} + f(0) = \frac{x}{2}$



7. Recall problem 13-23

(a) f integrable on $[a,b]$
$m \le f(x) \le M$ for all x in $[a,b]$ $\to \int_a^b f(x)\,dx = (b-a)\mu$ for some μ with $m \le \mu \le M$.

(b) f cont. on $[a,b]$ $\to \int_a^b f(x)\,dx = (b-a)f(\xi)$ for some ξ in $[a,b]$.

(d) f cont. on $[a,b]$
g integr. and nonneg. on $[a,b]$ $\to \int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$ for some ξ in $[a,b]$ (MVT for Integrals)

(e) f cont. on $[a,b]$
g integr. and nonpos. on $[a,b]$ $\to \int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$ for some ξ in $[a,b]$ (MVT for Integrals)

(i) $\frac{1}{7\sqrt{2}} \le \int_0^1 \frac{x^6}{\sqrt{1+x^2}}\,dx \le \frac{1}{7}$

Proof

Let $f(x) = \frac{1}{\sqrt{1+x^2}}$ and $g(x) = x^6$.

Both are continuous and g is nonnegative.

By 13-23d), $\exists \xi \in [0,1] \wedge \int_0^1 \frac{x^6}{\sqrt{1+x^2}}\,dx = \frac{1}{\sqrt{1+\xi^2}} \int_0^1 x^6 = \frac{1}{\sqrt{1+\xi^2}} \cdot \frac{1}{7}$

Since $1 \le \frac{1}{\sqrt{1+\xi^2}} \le \frac{1}{\sqrt{2}}$ we have $\frac{1}{7\sqrt{2}} \le \int_0^1 \frac{x^6}{\sqrt{1+x^2}}\,dx \le \frac{1}{7}$

(ii) $\frac{3}{8} \leq \int_0^{1/2} \sqrt{\frac{1-x}{1+x}}\, dx \leq \frac{\sqrt{3}}{4}$

Proof

Let $f(x) = \frac{1}{\sqrt{1+x}}$ and $g(x) = \sqrt{1-x}$.

Both continuous on $[0, 1/2]$, g nonnegative.

By 13-23d), $\exists \xi \in [0, 1/2]$ ∧ $\int_0^{1/2} \sqrt{\frac{1-x}{1+x}}\, dx = \frac{1}{\sqrt{1+\xi}} \cdot \int_0^{1/2} \sqrt{1-x}\, dx$

Now if $h(x) = -\frac{(1-x)^{3/2}}{\frac{3}{2}}$ then $h'(x) = g(x)$.

Hence, by FTC2,

$$\int_0^{1/2} \sqrt{1-x}\, dx = -\frac{2}{3}\left((1/2)^{3/2} - 1\right) = -\frac{2}{3}\left(\frac{1}{2\sqrt{2}} - 1\right) = -\frac{\cancel{2}}{3}\left(\frac{1-2\sqrt{2}}{\cancel{2}\sqrt{2}}\right)$$

$= \frac{2\sqrt{2}-1}{3\sqrt{2}}$. Also $\sqrt{\frac{2}{3}} \leq \frac{1}{\sqrt{1+\xi}} \leq 1$

Hence,

$$0.3518 \approx \frac{2\sqrt{2}-1}{3\sqrt{2}} \sqrt{\frac{2}{3}} \leq \int_0^{1/2} \sqrt{\frac{1-x}{1+x}}\, dx \leq \frac{2\sqrt{2}-1}{3\sqrt{2}} \approx 0.4309$$

However, the above, though true is not what was asked.

Rewrite $\int_0^{1/2} \sqrt{\frac{1-x}{1+x}}\, dx = \int_0^{1/2} \frac{\sqrt{1-x}}{\sqrt{1-x}} \sqrt{\frac{1-x}{1+x}}\, dx = \int_0^{1/2} \frac{1-x}{\sqrt{1-x^2}}\, dx$

Let $f(x) = \frac{1}{\sqrt{1-x^2}}$ and $g(x) = 1-x$. Then since both cont. on $[0, 1/2]$ and g nonneg., by 13-23d) we have

$\exists \xi \in [0, 1/2]$ ∧ $\int_0^{1/2} \sqrt{\frac{1-x}{1+x}}\, dx = \frac{1}{\sqrt{1-\xi^2}} \cdot \int_0^{1/2} (1-x)\, dx = \frac{1}{\sqrt{1-\xi^2}} \cdot \frac{3}{8}$

Since $1 \leq \frac{1}{\sqrt{1-\xi^2}} \leq \sqrt{\frac{4}{3}}$

$$\frac{3}{8} \leq \int_0^{1/2} \sqrt{\frac{1-x}{1+x}}\, dx \leq \frac{2}{\sqrt{3}} \cdot \frac{3}{8} = \frac{\sqrt{3}}{4}$$

8. $F(x) = \int_0^x x f(t)\,dt$

$= x\int_0^x f(t)\,dt$

$F'(x) = \int_0^x f + x f(x)$

---

9. f cont. $\rightarrow \int_0^x f(u)(x-u)\,du = \int_0^x \left(\int_0^u f(t)\,dt\right) du$

Proof

Let $h(x) = \int_0^x f(u)(x-u)\,du = x\int_0^x f(u)\,du - \int_0^x u f(u)\,du$

$g(x) = \int_0^x \left(\int_0^u f(t)\,dt\right) du$

$h'(x) = \int_0^x f + x f(x) - x f(x) = \int_0^x f$

$g'(x) = \int_0^x f$

$\rightarrow h(x) = g(x) + c$ for all x

$h(0) = 0 = 0 + c \rightarrow c = 0$

$\rightarrow h(x) = g(x)$

---

10. $\int_0^x f(u)(x-u)^2\,du = 2\int_0^x \left(\int_0^{u_2} \left(\int_0^{u_1} f(t)\,dt\right) du_1\right) du_2$

Proof

$h(x) = \int_0^x f(u)(x-u)^2\,du = \int_0^x [f(u)(x-u)](x-u)\,du$

By problem 9, $h(x) = \int_0^x \left(\int_0^u [f(t)(x-t)]\,dt\right) du = \int_0^x \left(\int_0^u [f(t)(u-t+x-u)]\,dt\right) du$

$= \int_0^x \left[\int_0^u f(t)(u-t)\,du + \int_0^u f(t)(x-u)\,dt\right] du$

$\int_0^u f(t)(u-t)\,du = \int_0^u \left(\int_0^{u_1} f(t)\,dt\right) du_1$, by problem 9

$\int_0^u f(t)(x-u)\,dt = (x-u)\int_0^u f(t)\,dt$

$\rightarrow h(x) = \int_0^x \left[\int_0^u \left(\int_0^{u_1} f(t)\,dt\right) du_1 + (x-u)\int_0^u f(t)\,dt\right] du$

$= \int_0^x \left(\int_0^u \left(\int_0^{u_1} f(t)\,dt\right) du_1\right) du + \int_0^x (x-u)\left(\int_0^u f(t)\,dt\right) du$

By problem 9 once more, $\int_0^x (x-u)\left(\int_0^u f(t)\,dt\right) du = \int_0^x \left(\int_0^u \left(\int_0^{u_1} f(t)\,dt\right) du_1\right) du$

Hence, $h(x) = 2\int_0^x \left(\int_0^u \left(\int_0^{u_1} f(t)\,dt\right) du_1\right) du$

11. Function f s.t. $f'''(x) = \frac{1}{\sqrt{1+\sin^2 x}}$

f''' cont., so FTC1 $\to$ $f''(x) = F_1(x) = \int_{a_1}^{x} f'''$ is s.t. $F_1'(x) = f'''(x)$

If f'' cont then FTC1 $\to$ $f'(x) = F_2(x) = \int_{a_2}^{x} f''$ is s.t. $F_2'(x) = f''(x)$.

If f' " " " $f(x) = F_3(x) = \int_{a_3}^{x} f'$ is s.t. $F_3'(x) = f'(x)$

Thus,

$$f(x) = \int_{a_1}^{x}\int_{a_2}^{y}\int_{a_3}^{z} f'''(t)\,dt$$

---

12. f is periodic w/ period a if $f(x+a) = f(x)$ for all x

(a) f periodic w/ period a, int. on $[0,a]$. $\to$ $\int_0^a f = \int_b^{b+a} f$ for all b

Proof

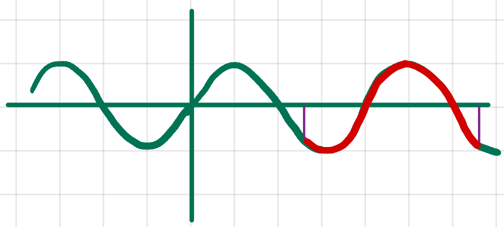
$\forall x,\ x \in [0,a] \to f(x+a) = f(x)$

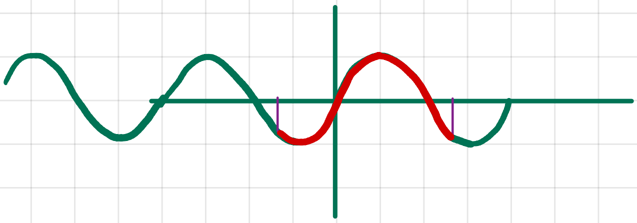
consider $\int_b^{b+a} f(x)\,dx$.

By problem 13-14, we can shift f left by a and change the int. limits by a and the integral stays the same

$$\int_0^a f(x+a)\,dx = \int_b^{b+a} f(x)\,dx$$

$$\to \int_0^a f(x)\,dx = \int_b^{b+a} f(x)\,dx$$





(b) f not periodic, f' is periodic.

Assume g periodic and $g(x) > 0$ for all x.

Let $f(x) = \int_0^x g(x)\,dx$.

Then $f' = g$, therefore f' periodic, $f' > 0$, f increasing.

$\forall x \forall y,\ x < y \to f(x) < f(y)$

in particular, for any $a > 0$, $f(x+a) > f(x)$ so f not periodic.

(c) f' periodic w/ period a, $f(a) = f(0)$ $\to$ f periodic w/ period a

Proof

$f(x) = \int_0^x f' + f(0)$

$f(a) = \int_0^a f' + f(0) = f(0) \to \int_0^a f' = 0$

$f(x+a) = \int_0^{x+a} f' + f(0) = \int_0^x f' + \int_x^{x+a} f' + f(0)$

$= f(x) + \int_x^{x+a} f'$

$= f(x) + \int_0^a f'$ (by part (a))

$= f(x)$

$\to f$ periodic

Alternative Proof

let $g(x) = f(x+a)$.

Then $g'(x) = f'(x+a) = f'(x)$

$f(a) = f(0) \to g(0) = f(a) = f(0)$

But then $g = f$, ie $f(x+a) = f(x)$ for all x.

(d) f' periodic w/ period a, f periodic w/ period b $\to$ $f(a) = f(0)$

Proof

$f'(x+a) = f'(x)$ (f' periodic)

$f(x) = \int_0^x f' + f(0)$ (anti-deriv.)

$f(x+b) = \int_0^{x+b} f' + f(0) = f(x)$ (f periodic)

$\to \int_0^x f' + f(0) = \int_0^{x+b} f' + f(0) \to \int_x^{x+b} f' = 0$

Alternative Proof

let $g(x) = f(x+a) - f(x)$

$\to g'(x) = f'(x+a) - f'(x) = 0$, because f' periodic.

T.F. g is constant and $g(x) = g(0) \to f(x+a) - f(x) = f(a) - f(0)$

$\to f(x+a) = f(x) + f(a) - f(0)$

$\to f(na) = f((n-1)a) + f(a) - f(0)$

$= f((n-2)a + a) + f(a) - f(0)$

$= f((n-2)a) + 2f(a) - 2f(0)$

$= f(0) + nf(a) - nf(0)$

$= n(f(a) - f(0)) + f(0)$

f bounded $\to f(a) = f(0)$

13. $\int_0^b \sqrt[n]{x}\,dx = \int_0^b x^{1/n}\,dx$

Let $g(x) = \frac{n}{n+1} x^{\frac{n+1}{n}}$

Then $g'(x) = x^{\frac{1}{n}}$

FTC2 $\rightarrow \int_0^b \sqrt[n]{x}\,dx = \frac{n}{n+1}\left(b^{\frac{n+1}{n}} - 0\right) = \frac{n b^{\frac{n+1}{n}}}{n+1}$

14. Recall 12-21

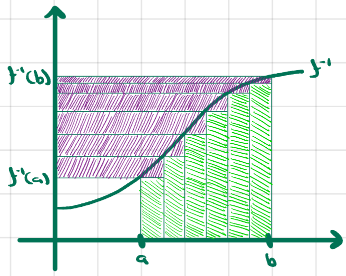
21. f diff, one-one

$f' \neq 0$

$f = F'$

$G(x) = x f^{-1}(x) - F(f^{-1}(x))$ $\rightarrow G'(x) = f^{-1}(x)$

This say that if we know an antiderivative of f then we can build an antiderivative of f^{-1}.

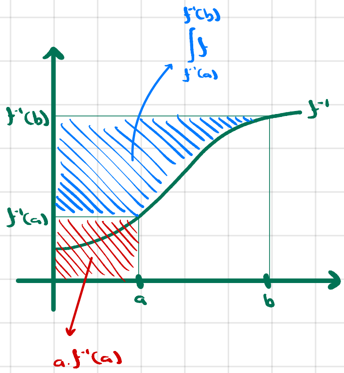


Recall the proofs in 13-21. For f increasing,

(a) $P = \{t_0, \ldots, t_n\}$ partition of $[a,b]$

$\rightarrow L(f^{-1}, P) + U(f, P') = b f^{-1}(b) - a f^{-1}(a)$

$P' = \{f^{-1}(t_0), \ldots, f^{-1}(t_n)\}$



(b) $\int_a^b f^{-1} = f^{-1}(b)b - f^{-1}(a)a - \int_{f^{-1}(a)}^{f^{-1}(b)} f$

Now, given a diff. one-one f for which $f' \neq 0$ and an antiderivative F, let's find an antiderivative of f^{-1}.

Such a function G would be of form $G(x) = \int_a^x f^{-1} + C$

From 13-21b),

$$G(x) = f^{-1}(x)x - f^{-1}(a)a - \int_{f^{-1}(a)}^{f^{-1}(x)} f$$

f one-one, cont. $\leftrightarrow$ f increasing or f decreas.

$\forall x\, \forall y,\ x \neq y \rightarrow f(x) \neq f(y)$

Let a, b, c be numbers.

Case 1: $f(a) < f(b)$. Assume $f(c) < f(b)$.

IVT $\rightarrow \forall x, x \in (f(c), f(b)) \rightarrow$

$\exists y, y \in (a,b) \wedge f(y) = f(x)$

IVT $\rightarrow \exists z, z \in (b,c) \wedge f(z) = f(x)$

Thus, $y \neq z \wedge f(y) = f(z)$. $\perp$.

Thus $f(a) < f(b) < f(c)$

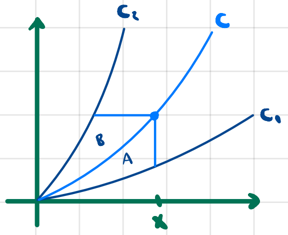
Case 2: $f(a) > f(b)$. Analogously. $\perp$.

Thus, $f(a) > f(b) > f(c)$

T.F. Either $f(a) < f(b) < f(c)$ or

$f(a) > f(b) > f(c)$.

15.



(a) $C_1 : f_1(x) = x^2,\ x \geq 0$

$C : f(x) = 2x^2,\ x \geq 0$

Area A: $\int_0^x 2x^2\,dx - \int_0^x x^2\,dx = \int_0^x x^2\,dx = \frac{x^3}{3}$

Let C_2 be the graph of $g(x)$. Then

Area B: $\int_0^{2x^2} \sqrt{\frac{x}{2}}\,dx - \int_0^{2x^2} g^{-1}(x)\,dx = \frac{x^3}{3}$

$\int_0^{2x^2} g^{-1}(x)\,dx = \int_0^{2x^2} \sqrt{\frac{x}{2}}\,dx - \frac{x^3}{3}$

Differentiate both sides

$g^{-1}(2x^2)\cdot 4x = \sqrt{\frac{2x^2}{2}}\cdot 4x - \frac{3x^2}{3}$

$g^{-1}(2x^2) = x - \frac{x^2}{4x} = x - \frac{x}{4}$

$g^{-1}(f(x)) = \frac{3x}{4}$

$y = 2x^2 \to x = \sqrt{\frac{y}{2}} \to g^{-1}(y) = \frac{3}{4}\sqrt{\frac{y}{2}}$

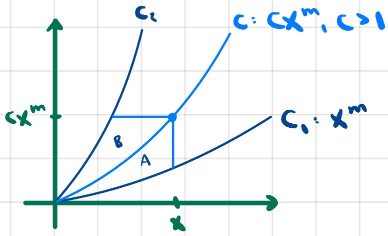
$y = g(x)$

$\to x = \frac{3}{4}\sqrt{\frac{g(x)}{2}} \to \frac{4\sqrt{2}\,x}{3} = \sqrt{g(x)}$

$\to g(x) = \frac{32}{9}x^2$

(b) $C_1 : f(x) = x^m$

$C : f(x) = cx^m,\ c > 1$



Area A: $\int_0^x x^m(c-1)\,dx = (c-1)\frac{x^{m+1}}{m+1}$

Area B $\int_0^{cx^m} (y/c)^{1/m}\,dy - \int_0^{cx^m} g^{-1}(y)\,dy = (c-1)\frac{x^{m+1}}{m+1}$

$\int_0^{cx^m} g^{-1}(y)\,dy = \int_0^{cx^m} (y/c)^{1/m}\,dy - (c-1)\frac{x^{m+1}}{m+1}$

Differentiate

$g^{-1}(cx^m)\cdot cmx^{m-1} = \left[\frac{\cancel{c}x^m}{\cancel{c}}\right]^{1/m}\cdot cmx^{m-1} - (c-1)x^m$

$g^{-1}(cx^m) = x + \frac{(1-c)}{cm}x = x\left(1 + \frac{(1-c)}{cm}\right)$

$= x\left(\frac{cm-c+1}{cm}\right)$

$y = cx^m \to x = (y/c)^{1/m}$

$\to g^{-1}(y) = \left(\frac{y}{c}\right)^{1/m}\left(\frac{cm-c+1}{cm}\right)$

$= \frac{y^{1/m}}{c^{\frac{1+m}{m}}}\ \frac{(cm-c+1)}{m}$

$y = g(x) \to x = \frac{g(x)^{1/m}}{c^{\frac{1+m}{m}}}\ \frac{(cm-c+1)}{m}$

$g(x) = \left(\frac{mx}{cm-c+1}\right)^m \cdot c^{1+m}$

$= \left(\frac{cm-c+1}{cm}\right)^{-m} x^m c^{1+m}\cdot c^{-m}$

$= \left(\frac{cm-c+1}{cm}\right) x^m c$

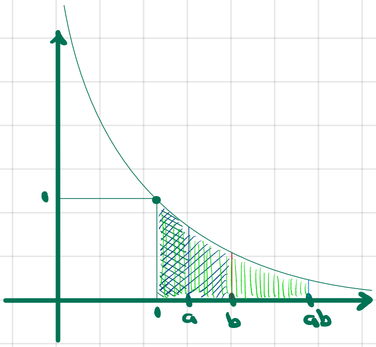
16. (a) $F(x) = \int_1^x \frac{1}{t}\,dt$

$G(x) = \int_b^{bx} \frac{1}{t}\,dt$

$\frac{1}{t}$ is cont on (a, ∞) and $(-\infty, -a)$ for any $a > 0$.

By FTC1, $F'(x) = \frac{1}{x}$, $x \geq 1$

and $G'(x) = \frac{1}{bx} b = \frac{1}{x}$, $x \geq 1$



(b) Recall 13-15

$a, b > 1 \rightarrow \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{t}\,dt = \int_1^{ab} \frac{1}{t}\,dt$

Proof

$\int_1^{ab} \frac{1}{t}\,dt - \int_1^b \frac{1}{t}\,dt = \int_b^{ab} \frac{1}{t}\,dt$

Thus, we are trying to prove $\int_1^a \frac{1}{t}\,dt = \int_b^{ab} \frac{1}{t}\,dt$ (1)

If (1) is true then the derivatives of both sides are the same everywhere.

Differentiate (1) wrt to a: $\frac{1}{a} = \frac{1}{ab} \cdot b = \frac{1}{a}$

However, at this point all we know is that $\int_1^a \frac{1}{t}\,dt$ and $\int_b^{ab} \frac{1}{t}\,dt$ differ by a constant.

But for $a = 1$ we have

$\int_1^1 1/t\,dt = 0 = \int_b^b 1/t\,dt + C \rightarrow C = 0$.

17. Recall 11-60c)

c) f differentiable on $[a,b]$

$f'(a) < c < f'(b)$ $\rightarrow \exists x, x \in (a,b) \wedge f'(x) = c$ (Darboux's Theorem)

And the Intermediate Value Theorem (IVT)

Let f be cont. on $[a,b]$. Then f takes on every value between $f(a)$ and $f(b)$ between a and b.

In other words

f cont. on $[a,b]$

$f(a) < c < f(b)$ $\rightarrow \exists x, x \in (a,b) \wedge f(x) = c$

Proof

Let f be cont. on $[a,b]$.

Define $F(x) = \int_a^x f$. Then,

$F'(a) = f(a)$
$F'(b) = f(b)$

Let c s.t. $F'(a) = f(a) < c < f(b) < F'(b)$

By Darboux's Theorem, $\exists x, x \in (a,b) \wedge F'(x) = f(x) = c$

---

18. h cont.

f, g diff.

$F(x) = \int_{f(x)}^{g(x)} h(t)\,dt$ $\rightarrow F'(x) = h(g(x)) \cdot g'(x) - h(f(x)) \cdot f'(x)$

Proof

$$F(x) = \int_{f(x)}^{g(x)} h(t)\,dt = \int_{f(x)}^{c} h + \int_{c}^{g(x)} h, \quad c \in (f(x), g(x))$$

$$= -\int_{c}^{f(x)} h + \int_{c}^{g(x)} h$$

$$F'(x) = -h(f(x))f'(x) + h(g(x))g'(x)$$

19. f int. on $[a,b]$

$c \in (a,b)$

$F(x) = \int_a^x f \quad a \le x \le b$

(a) f diff at $c \to F$ diff. at c

Proof

1. f cont at c. Proof: f diff at c

2. F diff at c (and $F'(c) = f(c)$) Proof: FTC I

(b) f diff at $c \to F'$ cont. at c

counterexample

Let $f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ — start w/ f

$\lim_{x \to 0} f(x) = 0 = f(0) \to$ cont. at 0 — f cont at 0

$f'(x) = 2x \sin(1/x) + \cancel{x^2} \cos(1/x) \dfrac{(-1)}{\cancel{x^2}} \quad x \neq 0$

$= 2x \sin(1/x) - \cos(1/x)$

$\lim_{x \to 0} f'(x)$ does not exist.

on the other hand $\lim_{h \to 0} \dfrac{h^2 \sin(1/h)}{h} = 0 \to f'(0) = 0$ — $f'(0)$ exists,

$\to f'$ not cont at 0. — but f' discont. at 0

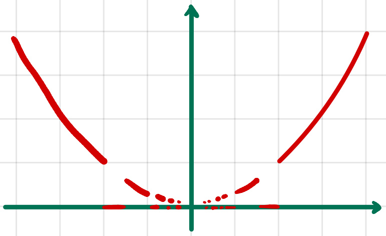
$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

f' integrable $\to \exists F$ s.t. $F(x) = \int_{-1}^x f'$ and $F' = f'$. — Apply FTC I to f. obtain $F = \int_{-1}^x f'$.

T.F. F is diff. at 0 but F' is not cont. at 0. — $F' = f'$ so it is not cont at 0.

counterexample 2

$$f(x) = \begin{cases} 0 & x \in \left(\frac{1}{n+1}, \frac{1}{n}\right] \\ x^2 & x \in \left(\frac{1}{n}, \frac{1}{n-1}\right] \end{cases}$$



$\lim_{x \to 0} f(x) = 0 = f(0) \rightarrow f$ cont. at 0

f is cont on each interval $\left(\frac{1}{n}, \frac{1}{n+1}\right)$

$F(x) = \int_0^x f$ is cont. everywhere, but not diff. at any point $1/n$, $n \in \mathbb{N}$ since

$$\lim_{x \to \frac{1}{n}^+} F'(x) = x^2 \neq 0 = \lim_{x \to \frac{1}{n}^-} F'(x)$$

Therefore, since F' is not defined at infinitely many points in any interval around 0, it is not continuous at 0.

(c) f' cont. at c $\rightarrow$ F' cont. at c

f' cont. at c then f' exists in some interval around c.

f diff. on $[c-\delta, c+\delta]$.

But then f cont on $[c-\delta, c+\delta]$ and therefore by FTC1 $F'(x) = f(x)$ for all $x \in [c-\delta, c+\delta]$, hence F' cont in this interval, in particular at c.

20. Let $f(x) = \begin{cases} \cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

$F(x) = \int_0^x f$ diff at 0?

Let $g(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

Then $g'(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

Let $h(x) = \begin{cases} 2x\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

Then

$$f(x) = h(x) - g'(x)$$

$$F(x) = \int_0^x f = \int_0^x (h - g') = \int_0^x h - \int_0^x g' = \int_0^x h - g(x) + \cancel{g(0)}$$

$$= \int_0^x h - g(x)$$

FTC2: g is integrable on $[0, x]$

$g = g'$ on this interval $\rightarrow \int_0^x g' = g(x) - g(0)$

h is cont. (hence int.) on $[0, x]$.

By FTC1, the deriv. of $\int_0^x h$ at x is $h(x)$.

Hence,

$$F'(0) = h(0) - g(0) = 0$$

21. f' int. on $[0,1]$, $f(0)=0$ $\rightarrow$ $\forall x, x\in[0,1] \rightarrow |f(x)| \le \sqrt{\int_0^1 |f'|^2}$

Proof

Recall 13-39

39. f, g integrable on $[a,b]$

Cauchy-Schwarz Inequality: $\left(\int_a^b fg\right)^2 \le \left(\int_a^b f^2\right)\left(\int_a^b g^2\right)$

(a) Schwarz Ineq. is special case of Cauchy-Schwarz Ineq.

(b) Let's prove $\left(\int_a^b fg\right)^2 \le \left(\int_a^b f^2\right)\left(\int_a^b g^2\right)$ in three different ways.

(d) $\left(\int_0^1 f\right)^2 \le \left(\int_0^1 f^2\right)$

Let $g(x)=1$.

Then, the Cauchy-Schwarz Ineq. applied to f' and g on $[0,x]$ becomes

$$\left(\int_0^x f'(x)\cdot 1\,dx\right)^2 \le \left(\int_0^x f'^2(x)\,dx\right)\left(\int_0^x 1^2\,dx\right)$$

$$\left(\int_0^x f'\right)^2 \le \int_0^x f'^2$$

$$(f(x)-f(0))^2 \le \int_0^x f'^2$$

$$(f(x))^2 \le \int_0^x f'^2$$

But note that $\int_0^1 f'^2 = \int_0^x f'^2 + \int_x^1 f'^2$ and $\int_x^1 f'^2 \ge 0$ so $\int_0^x f'^2 \le \int_0^1 f'^2$

Hence,

$$[f(x)]^2 \le \int_0^1 f'^2$$

$$f(x) \le \sqrt{\int_0^1 f'^2}$$

22. f diff.

$f(0) = 0$ $\quad \rightarrow \forall x, x \geq 0 \rightarrow \int_0^x f^3 \leq \left(\int_0^x f\right)^2$

$0 < f' \leq 1$

Proof

Consider $f_1(x) = \int_0^x f^3(x)$ and $f_2(x) = \left(\int_0^x f\right)^2$

For $x = 0$, both equal 0.

If $f_1' \leq f_2'$ then we have the desired outcome (by 11-28).

$f_1'(x) = f^3(x)$

$f_2'(x) = 2\int_0^x f \cdot f(x)$

$f_1(x) \leq f_2(x) \rightarrow f^2(x) \leq 2\int_0^x f$

Again we have equality at 0.
Differentiating again.

$2f(x)f'(x)$

$\rightarrow 2f(x)f'(x) \leq 2f(x)$ since $0 < f' \leq 1$ and $f > 0$.

$2f(x)$

Hence $f_1' \leq f_2'$ and thus for $x \geq 0$ we have $f_1(x) \leq f_2(x)$

23. (a) $G' = g$
$F' = f$

Then if fn y satisfies

$g(y(x))\,y'(x) = f(x)$ for all x in some interval *

then there is c s.t.

$G(y(x)) = F(x) + c$ for all x in this interval **

Proof

$g(y(x))\,y'(x) = f(x)$

$[G(y(x))]' = f(x) = F'(x)$

Then $G(y(x)) = F(x) + c$

(b) y satisfies ** then y is solution of *.

Proof

$G(y(x)) = F(x) + c$

Differentiate: $G'(y(x))\,y'(x) = F'(x) = f(x)$

(c) $y'(x) = \dfrac{1+x^2}{1+y(x)}$

$(1+y(x))\,y'(x) = 1+x^2$

Let $g(x) = 1+x$ and $f(x) = 1+x^2$. Then,

$g(y(x))\,y'(x) = f(x)$ (1)

Let $F(x) = x + \dfrac{x^3}{3}$. Then $F' = f$ specific F and G obeying (1)

$G(x) = x + \dfrac{x^2}{2}$. Then $G' = g$.

By part a),

$G(y(x)) = F(x) + c$

$\left(y(x) + \dfrac{y(x)^2}{2}\right) = x + \dfrac{x^3}{3} + c$

$y(x)^2 + 2y(x) - 2x - \dfrac{2x^3}{3} - c = 0$

$\Delta = 4 - 4\left(-2x - \dfrac{2x^3}{3} - c\right)$

$= 4\left(1 + 2x + \dfrac{2x^3}{3} + c\right)$

$y(x) = \dfrac{-2 \pm \sqrt{4\left(1+2x+\frac{2x^3}{3}+c\right)}}{2}$

$= -1 \pm \sqrt{1 + 2x + \dfrac{2x^3}{3} + c}$

$1 + 2x + \dfrac{2x^3}{3} + c \geq 0$

$\rightarrow 2x^3 + 2x + 3 + c \geq 0$

Since this expression can be negative, these solutions are not defined on $\mathbb{R}$.

(d) $y'(x) = \dfrac{-1}{1+5(y(x))^4}$

Recall 12-14

(a) $\exists f$, f differentiable $\wedge$ $f(x)^5 + f(x) + x = 0$ for all x

We showed this by assuming there were an inverse f^{-1} and found that it would satisfy

$$f^{-1}(y) = -y^5 - y$$

w/ derivative

$$(f^{-1})'(y) = -5y^4 - 1 < 0$$

From these equations we were able to obtain various results about f

f^{-1} one-one and decreas. $\rightarrow$ f is fn and decreas.

$(f^{-1})' \neq 0 \rightarrow f'$ exists and equals

$$f'(x) = \frac{1}{(f^{-1})'(f(x))} = \frac{1}{-5f(x)^4 - 1}$$

In the current problem, we are starting w/ one such expression and asking what condition f satisfies.

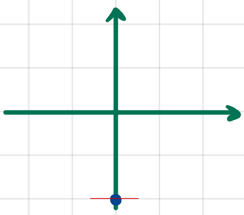
That condition is the one we started w/ in 12-14

$$y(x)^5 + y(x) + x = 0$$

However, we can be more general

$$y(x)^5 + y(x) + x = c$$

(e) $y(x)\,y'(x) = -x \qquad y(0) = -1$



$-1\,y'(0) = 0 \rightarrow y'(0) = 0$

Let $G(x) = \dfrac{y(x)^2}{2}$. Then $G'(x) = y(x)\,y'(x)$

Let $F(x) = -\dfrac{x^2}{2}$. Then $F'(x) = -x$

Here, our eq. is $G'(x) = F'(x)$

$$G(x) = F(x) + C$$

$$\frac{y(x)^2}{2} = -\frac{x^2}{2} + C$$

$\dfrac{y(0)^2}{2} = \dfrac{1}{2} = C \qquad \rightarrow y(x) = \pm\sqrt{-x^2+1}\,,\ x^2 \leq 1 \rightarrow -1 \leq x \leq 1$

24. f, f', f'', f''' exist, ie f three times differentiable.

$f'(x) \neq 0$

Schwarzian derivative of f: $\mathcal{D}f(x) = \dfrac{f'''(x)}{f'(x)} - \dfrac{3}{2}\left(\dfrac{f''(x)}{f'(x)}\right)^2$

In 10-19 we proved

$$\mathcal{D}(f \circ g) = [\mathcal{D}f \circ g] \cdot g'^2 + \mathcal{D}g \quad (1)$$

and that

$$\left.\begin{aligned} f(x) &= \frac{ax+b}{cx+d} \\ ad-bc &\neq 0 \end{aligned}\right\} \to \mathcal{D}f = 0 \quad (2)$$

therefore, by (1) and (2), given the antecedent in (2), we have from (1) that

$$D(f \circ g) = Dg$$

Now, we assume f is any fn w/ Schwarzian derivative zero.

(a) $\dfrac{f''^2}{f'^3}$ is constant

Proof

Then, $D(f \circ g) = Dg$

$$\frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 = 0$$

$$\left(\frac{f''(x)}{f'(x)}\right)^2 = \frac{2}{3}\frac{f'''(x)}{f'(x)} \to \frac{f''(x)^2}{f'(x)^3} = \frac{2}{3}\frac{f'''(x)}{f'(x)^2} \to 2f'(x)^{\cancel{3}}f'''(x) = 3\cancel{f'(x)^2}f''(x)^2$$

$$\to 2f'f''' = 3f''^2$$

$$\to 2f'f''' - 3f''^2 = 0$$

$$\left(\frac{f''^2}{f'^3}\right)' = \frac{2f''f'''f'^3 - f''^2 \cdot 3f'^2 f''}{(f')^6} = \frac{2f'^3f''f''' - 3f'^2f''^3}{(f')^6} = \frac{f'^2f''(2f'f''' - 3f''^2)}{(f')^6} = 0$$

$$\to \frac{f''^2}{f'^3} \text{ is constant}$$

(b) f is the form $f(x) = \frac{ax+b}{cx+d}$

Proof

Let $u(x) = f'(x)$. Then $u' = f''$ and part a) says $\frac{u'(x)^2}{u(x)^3} = C$

$$\to \frac{u'(x)}{u(x)^{3/2}} = u^{-\frac{3}{2}} \cdot u' = C$$

Let $G(x) = -2x^{-\frac{1}{2}}$. Then $G'(x) = x^{-\frac{3}{2}} = g(x)$.

$$g(u(x))u'(x) = u(x)^{-\frac{3}{2}} \cdot u'(x) = [G(u(x))]' = h(x) = c$$

This is expression * in problem 23a).

Since $H(x) = cx+d$ is s.t. $H'(x) = h(x) = c$, by problem 23 we know that

$$G(u(x)) = H(x) + b$$

$$-2(f'(x))^{-\frac{1}{2}} = cx+d+b$$

$$\sqrt{f'(x)} = -\frac{2}{cx+d+b} \quad \to f'(x) = \frac{4}{(cx+D)^2}$$

$$\to f(x) - f(a) = \int_a^x \frac{4}{(cx+D)^2}\,dx \quad \text{(FTC2)}$$

$$= 4 \cdot \frac{(cx+D)^{-1}}{(-1)} = -\frac{4}{cx+D}$$

$$\to f(x) = -\frac{4}{cx+d} + E = \frac{-4 + cEx + dE}{cx+d} = \frac{ax+b}{cx+d}$$

25. $\lim_{n\to\infty} \int_a^n f$ if it exists is denoted $\int_a^\infty f$ and is called an improper integral

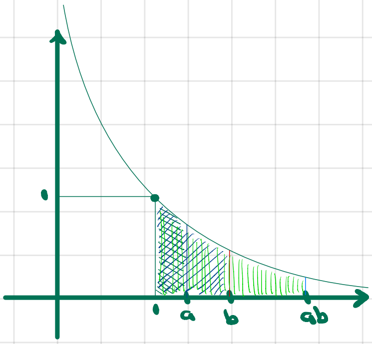
(a) $\int_1^\infty x^r dx \quad r<-1$

$$\int_1^n x^r dx = \frac{x^{r+1}}{r+1}\Big|_1^n = \frac{n^{r+1}-1^{r+1}}{r+1}$$

$$\lim_{n\to\infty} \frac{n^{r+1}-1}{r+1} = \lim_{n\to\infty} \frac{\frac{1}{n^{|r+1|}}-1}{r+1} = \frac{-1}{r+1}$$

(b) In problem 13-15 we proved

$$a,b>1 \to \int_1^a \frac{1}{t}dt + \int_1^b \frac{1}{t}dt = \int_1^{ab}\frac{1}{t}dt$$



now we want to show that

$\int_1^\infty \frac{1}{x} dx$ does not exist

Proof

$$\int_1^{2^n} \frac{1}{x}dx = \int_1^{2\cdot 2^{n-1}} \frac{1}{x}dx = \int_1^2 \frac{1}{x}dx + \int_1^{2^{n-1}} \frac{1}{x}dx = n\int_1^2 \frac{1}{x}dx$$

$$\to \lim_{n\to\infty} \int_1^{2^n} \frac{1}{x}dx = \infty \quad (1)$$

$$\to \lim_{n\to\infty} \int_1^{n} \frac{1}{x}dx = \infty$$

Because (1) means

$$\forall M>0\ \exists N\ \forall n\ n>N \to \int_1^{2^n}\frac{1}{x}dx > M$$

$$n>N \to 2^n > 2^N$$

T.F., if $n_1 = 2^n$ then

$$\forall M>0\ \exists N_1\ \forall n_1,\ n_1>N_1 \to \int_1^{n_1}\frac{1}{x}dx > M$$

$\int_0^\infty f$ exists

Then,

$0 \le g(x) \le f(x)$ for all $x \ge 0$
g int. on each $[0,N]$ $\rightarrow$ $\int_0^\infty g$ exists

Proof

Since $0 \le g(x) \le f(x)$ then $0 \le \int_0^N g \le \int_0^N f$.

Then $0 \le \lim_{N\to\infty} \int_0^N g \le \lim_{N\to\infty} \int_0^N f$

$\rightarrow$ $\int_0^\infty g$ exists

(d) $\int_0^\infty \frac{1}{1+x^2} dx$ exists.

$= \int_0^1 \frac{1}{1+x^2} dx + \int_1^\infty \frac{1}{1+x^2} dx$

$0 < \frac{1}{1+x^2} < \frac{1}{x^2}$ for all $x > 0$

In part a) we showed that $\int_1^\infty \frac{1}{x^2}$ exists.

T.F. by part c), $\int_1^\infty \frac{1}{1+x^2} dx$ exists

As for $\int_0^1 \frac{1}{1+x^2} dx$, since $\frac{1}{1+x^2}$ is continuous it is integrable

Hence $\int_0^1 \frac{1}{1+x^2} dx$ exists.

26.

(i) $\int_0^\infty \frac{1}{\sqrt{1+x^3}}\,dx$

By 25a), we know that $\int_1^\infty x^{-\frac{3}{2}}\,dx$ exists since $-\frac{3}{2}<-1$

$\int_0^1 x^{-\frac{3}{2}}\,dx$ also exists since $x^{-\frac{3}{2}}$ is cont on $[0,\infty)$.

Since

$$0<\frac{1}{\sqrt{1+x^3}}<\frac{1}{\sqrt{x^3}}$$

$$\frac{1}{\sqrt{1+x^3}}>0 \text{ for } x\geq 0$$

$$\frac{1}{\sqrt{1+x^3}} \text{ cont.} \to \text{integrable}$$

$$\int_0^\infty x^{-\frac{3}{2}}\,dx \text{ exists}$$

Then, by 25c) we know that $\int_0^\infty \frac{1}{\sqrt{1+x^3}}\,dx$ exists.

(ii) $\int_0^\infty \frac{x}{1+x^{3/2}}\,dx$

$$\frac{x}{1+x^{3/2}}=\frac{1}{\frac{1}{x}+x^{1/2}}>\frac{1}{1+x^{1/2}}>\frac{1}{2x^{1/2}}>0 \text{ for } x>1$$

T.F.

$$0<\frac{1}{2x^{1/2}}<\frac{x}{1+x^{3/2}}$$

But $\int_1^\infty x^{-\frac{1}{2}}\,dx$ doesn't exist. Therefore $\int_0^\infty \frac{x}{1+x^{3/2}}\,dx$ doesn't either.

(iii)

For $x\in(0,1)$, $x\sqrt{1+x}<x\sqrt{2}$.

T.F. $\frac{1}{\sqrt{2}x}<\frac{1}{x\sqrt{1+x}}$ and hence $\int_0^1 \frac{1}{x\sqrt{1+x}}\,dx$ does not exist.

$$\frac{1}{x\sqrt{1+x}}=\frac{1}{\sqrt{x^2+x^3}}<\frac{1}{x^{3/2}}$$

T.F. $\int_1^\infty \frac{1}{x\sqrt{1+x}}\,dx$ exists

27. New types of improper integrals

$$\int_{-\infty}^{a} f = \lim_{N\to-\infty} \int_{N}^{a} f$$

$$\int_{-\infty}^{\infty} f = \int_{0}^{\infty} f + \int_{-\infty}^{0} f \quad \text{provided the latter two exist}$$

(a) Let $f(x) = \dfrac{1}{1+x^2}$.

Then $f(-x) = f(x)$, ie f is even.

T.F. $\int_{-N}^{0} f = \int_{0}^{N} f$

Since by 23(a) we know $\int_{0}^{\infty} f$ exists then so does $\int_{-\infty}^{0} f$.

(b) $\int_{-N}^{N} x\,dx = \dfrac{N^2 - (-N)^2}{2} = 0$

Thus, $\lim_{N\to\infty} \int_{-N}^{N} x\,dx = 0$.

$$\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx = \lim_{N\to-\infty} \int_{N}^{0} x\,dx + \lim_{N\to\infty} \int_{0}^{N} x\,dx = -\infty + \infty$$

$\to \int_{-\infty}^{\infty} x\,dx$ doesn't exist.

(c) $\int_{-\infty}^{\infty} f$ exists $\to$ $\lim_{N\to\infty} \int_{-N}^{N} f$ exists and $= \int_{-\infty}^{\infty} f$.

Proof

$\int_{-\infty}^{\infty} f = \int_{-\infty}^{0} f + \int_{0}^{\infty} f$ so both the limits exist.

$$\int_{0}^{\infty} f = \lim_{N\to\infty} \int_{0}^{N} f$$

$$\int_{-N}^{N} f = \int_{-N}^{0} f + \int_{0}^{N} f$$

$$\lim_{N\to\infty} \int_{-N}^{N} f = \lim_{N\to\infty} \left(\int_{-N}^{0} f + \int_{0}^{N} f \right) = \lim_{N\to\infty} \int_{-N}^{0} f + \lim_{N\to\infty} \int_{0}^{N} f = \int_{-\infty}^{0} f + \int_{0}^{\infty} f = \int_{-\infty}^{\infty} f$$

$\lim_{N\to\infty} \int_{-N}^{N+1} f$ and $\lim_{N\to\infty} \int_{-N^2}^{N} f$ both exist and equal $\int_{-\infty}^{\infty} f$

Proof

Let's prove that $\lim_{N\to\infty} \int_{-N}^{N+1} f$ exists. split $\int_{-\infty}^{\infty} f$.

$\int_0^\infty f$ and $\int_{-\infty}^0 f$ exist since $\int_{-\infty}^{\infty} f$ does. These are numbers. We can get $\int_0^N f$ and $\int_{-N}^0 f$ as close to $\int_0^\infty f$ and $\int_{-\infty}^0 f$ as we want, by definition of limit.

That is, $\forall \epsilon\ \exists N_0\ \forall N\ N > N_0 \to \left|\int_0^\infty f - \int_0^N f\right| < \epsilon \wedge \left|\int_{-\infty}^0 f - \int_{-N}^0 f\right| < \epsilon$. By def. of $\lim_{N\to\infty} \int_0^N f$ and $\lim_{N\to\infty} \int_{-N}^0 f$

Let N_0 s.t. $N > N_0 \to \left|\int_0^\infty f - \int_0^N f\right| < \frac{\epsilon}{2} \wedge \left|\int_{-\infty}^0 f - \int_{-N}^0 f\right| < \frac{\epsilon}{2}$ choose specific ϵ, $\frac{\epsilon}{2}$

Let $N+1 > N_0$. Then $\left|\int_0^\infty f - \int_0^{N+1} f\right| < \frac{\epsilon}{2} \wedge \left|\int_{-\infty}^0 f - \int_{-N}^0 f\right| < \frac{\epsilon}{2}$

Then, for $N > N_2$ we have

$$\left|\int_{-\infty}^{\infty} f - \int_{-N}^{N+1} f\right| = \left|\int_{-\infty}^{0} f + \int_0^{\infty} f - \int_{-N}^{0} f - \int_0^{N+1} f\right| \leq \left|\int_{-\infty}^{0} f - \int_{-N}^{0} f\right| + \left|\int_0^{\infty} f - \int_0^{N+1} f\right| = \epsilon$$

The same argument can be made for showing that $\int_{-N^2}^{N} f$ exists and equals $\int_{-\infty}^{\infty} f$.

Just to summarize the argument,

$\int_0^N f$ and $\int_{-N}^0 f$ can be made arbitrarily close to $\int_0^\infty f$ and $\int_{-\infty}^0 f$, respect., by choosing N large enough (choose some N_0).

Choose $\epsilon = \epsilon/2$ and let N_0 be the threshold value.

Then if $N > N_0$ we have $N^2 > N_0$ and

$$\left|\int_0^\infty f - \int_0^{N+1} f\right| < \frac{\epsilon}{2} \wedge \left|\int_{-\infty}^0 f - \int_{-N^2}^0 f\right| < \frac{\epsilon}{2}$$

$$\left|\int_{-\infty}^{\infty} f - \int_{-N^2}^{N} f\right| = \left|\int_{-\infty}^{0} f + \int_0^{\infty} f - \int_{-N^2}^{0} f - \int_0^{N} f\right| \leq \left|\int_{-\infty}^{0} f - \int_{-N^2}^{0} f\right| + \left|\int_0^{\infty} f - \int_0^{N} f\right| = \epsilon$$

28.

(a) $a>0$. Find $\lim_{\epsilon\to 0^+}\int_\epsilon^a \frac{1}{\sqrt{x}}\,dx$, denoted $\int_0^a \frac{1}{\sqrt{x}}\,dx$

note that $\frac{1}{\sqrt{x}}$ is not bounded on $[0,a]$, no matter how $f(0)$ is defined.

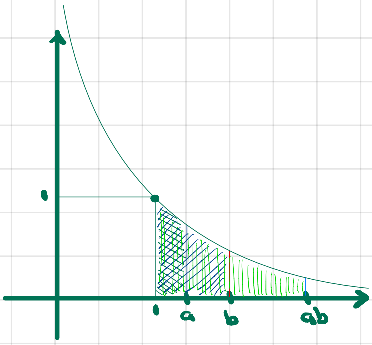
$$\lim_{\epsilon\to 0^+}\int_\epsilon^a x^{-\frac{1}{2}}\,dx = \lim_{\epsilon\to 0^+} 2x^{1/2}\Big|_\epsilon^a = \lim_{\epsilon\to 0^+} 2\left(a^{1/2}-\epsilon^{1/2}\right) = 2a^{1/2}$$

(b) $\int_0^a x^r \quad -1<r<0$

$$= \lim_{\epsilon\to 0^+}\int_\epsilon^a x^r\,dx = \lim_{\epsilon\to 0^+}\frac{x^{r+1}}{r+1}\Big|_\epsilon^a = \lim_{\epsilon\to 0^+}\frac{a^{r+1}-\epsilon^{r+1}}{r+1} = \frac{a^{r+1}}{r+1}$$ if $r+1>0 \to r>-1$

(c) $\int_0^a x^{-1}\,dx$ does not make sense

Proof



Intuitively, since $f^{-1}(x)=f(x)=\frac{1}{x}$ then $\int_1^\infty f^{-1}$ does not exist.

Since $\int_0^1 f$ includes this area, it too does not exist.

In problem 13-15 we proved

$a,b>1 \to \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{t}\,dt = \int_1^{ab}\frac{1}{t}\,dt$

Consider $\lim_{n\to\infty}\int_{\frac{1}{2^n}}^1 \frac{1}{t}\,dt = \lim_{n\to\infty}\left(\int_{\frac{1}{2}}^1 \frac{1}{t}\,dt + \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}}\frac{1}{t}\,dt\right) = \lim_{n\to\infty}\left(n\int_{1/2}^1 \frac{1}{t}\,dt\right) = \infty$

(d) $\int_a^0 |x|^r\,dx$ for $a<0$ and $-1<r<0$

note that for $-1<r<0$, $|x|^r$ is not bounded on $[a,0]$.

thus, $\int_a^0 |x|^r\,dx$ is an improper integral.

let's define it as $\lim_{\epsilon\to 0^-}\int_a^\epsilon |x|^r\,dx$.

$$= \lim_{\epsilon\to 0^-}\int_a^\epsilon (-x)^r\,dx = \lim_{\epsilon\to 0^-}\frac{(-x)^{r+1}}{r+1}(-1)\Big|_a^\epsilon = \lim_{\epsilon\to 0^-}\frac{(-1)(-\epsilon)^{r+1}-(-1)(-a)^{r+1}}{r+1} = \frac{(-a)^{r+1}}{r+1} = -\frac{a^{r+1}}{r+1}$$

(e) $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$

note that $\frac{1}{\sqrt{1-x^2}}$ is unbounded on $[-1,0]$ and $[0,1]$.

let's define $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$ as $\int_{-1}^{0} \frac{1}{\sqrt{1-x^2}}\,dx + \int_{0}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\epsilon\to -1^+} \int_{\epsilon}^{0} \frac{1}{\sqrt{1-x^2}}\,dx + \lim_{\epsilon \to 1^-} \int_{0}^{\epsilon} \frac{1}{\sqrt{1-x^2}}\,dx$

note that $\int_{-1}^{0} \frac{1}{\sqrt{1+x}}\,dx$ is improper as $\frac{1}{\sqrt{1+x}}$ grows unbounds near -1.

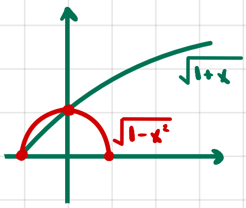
$$\int_{-1}^{0} \frac{1}{\sqrt{1+x}} = \lim_{\epsilon\to -1^+} \int_{\epsilon}^{0} \frac{1}{\sqrt{1+x}}\,dx = \lim_{\epsilon\to -1^+} 2\sqrt{1+x}\Big|_{\epsilon}^{0} = \lim_{\epsilon\to -1^+}\left(2-2\sqrt{1+\epsilon}\right) = 2$$

$\frac{1}{\sqrt{1-x^2}}$ is also unbounded near -1.

$$\int_{-1}^{0} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\epsilon\to -1^+} \int_{\epsilon}^{0} \frac{1}{\sqrt{1-x^2}}$$

but note that $\sqrt{1-x^2} \geq \sqrt{1+x}$ for $x \in [-1,0]$.

$\rightarrow 0 < \frac{1}{\sqrt{1-x^2}} \leq \frac{1}{\sqrt{1+x}}$ in this interval



T.F. $\int_{-1}^{0} \frac{1}{\sqrt{1-x^2}}\,dx$ exists.

By symmetry of $\frac{1}{\sqrt{1-x^2}}$, $\int_{0}^{1} \frac{1}{\sqrt{1-x^2}}$ also exists.

29. (a) f cont. on $[0,1]$. Compute $\lim\limits_{x\to 0^+} x\int_x^1 \frac{f(t)}{t}dt$

$$\lim_{x\to 0^+} x\int_x^1 \frac{dt}{t} = \lim_{x\to 0^+} \frac{\int_x^1 \frac{dt}{t}}{\frac{1}{x}}$$

Let $f(x) = \int_x^1 t^{-1}dt$ and $g(x) = x^{-1}$.

We know from 28b that $\lim\limits_{x\to 0^+} \int_x^1 t^{-1}dt = \infty$

Also $\lim\limits_{x\to 0^+} \frac{1}{x} = \infty$

Hence, $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} g(x) = \infty$

$f'(x) = -\frac{1}{x}$

$g'(x) = -\frac{1}{x^2}$

$\to \lim\limits_{x\to 0^+} \frac{f'(x)}{g'(x)} = \lim\limits_{x\to 0^+} x = 0$

By a version of L'Hôpital's rule

$$\lim_{x\to 0^+} x\int_x^1 \frac{dt}{t} = \lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = 0$$

But we want to compute $\lim\limits_{x\to 0^+} x\int_x^1 \frac{f(t)}{t}dt$

Since f is cont. on $[0,1]$ it is bounded there.

Hence, $\exists M$ s.t. $|f(x)| \leq M$ for $x \in [0,1]$.

$$\to -\frac{M}{x} \leq \frac{f(x)}{x} \leq \frac{M}{x}$$

$$\to -\int_x^1 \frac{M}{x}dx \leq \int_x^1 \frac{f(x)}{x}dx \leq \int_x^1 \frac{M}{x}dx$$

$$-x\int_x^1 \frac{M}{x}dx \leq x\int_x^1 \frac{f(x)}{x}dx \leq x\int_x^1 \frac{M}{x}dx$$

$$0 = \lim_{x\to 0^+}\left[-x\int_x^1 \frac{M}{x}dx\right] \leq \lim_{x\to 0^+} x\int_x^1 \frac{f(x)}{x}dx \leq \lim_{x\to 0^+} x\int_x^1 \frac{M}{x}dx = 0$$

$$\to \lim_{x\to 0^+} x\int_x^1 \frac{f(x)}{x}dx = 0$$

(b) f int. on $[0,1]$, cont. at 0. Compute $\lim_{x\to 0^+} x\int_x^1 \frac{f(t)}{t^2}\,dt$.

Since f int. on $[0,1]$ it is bounded there.

Consider $\lim_{x\to 0^+} x\int_x^1 \frac{1}{t^2}\,dt$

$$= \lim_{x\to 0^+} \frac{\int_x^1 t^{-2}\,dt}{\frac{1}{x}}$$

Then, by FTC2, $\int_x^1 t^{-2}\,dt = -t^{-1}\Big|_x^1 = -1+\frac{1}{x}$

$$\lim_{x\to 0^+} \frac{\int_x^1 t^{-2}\,dt}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{-1+\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to 0^+}(-x+1) = 1$$

Let $\lim_{x\to 0^+} f(x) = \ell$.

Given $\epsilon>0$, let $\delta>0$ s.t. $\forall t,\ 0<t<\delta \to |f(t)-\ell|<\epsilon$

Then,
$$\left|x\int_x^1 \frac{f(t)-\ell}{t^2}\,dt\right| \le x\int_x^\delta \frac{|f(t)-\ell|}{t^2}\,dt + x\left|\int_\delta^1 \frac{f(t)-\ell}{t^2}\,dt\right|$$
$$\le x\epsilon\int_x^\delta \frac{dt}{t^2} + x\left|\int_\delta^1 \frac{f(t)-\ell}{t^2}\,dt\right|$$

But $\left|x\int_x^1 \frac{f(t)-\ell}{t^2}\,dt\right| = \left|x\int_x^1 \frac{f(t)}{t^2}\,dt - x\ell\int_x^1 t^{-2}\,dt\right| = \left|x\int_x^1 \frac{f(t)}{t^2}\,dt - x\ell(-1+x)\right| = \left|x\int_x^1 \frac{f(t)}{t^2}\,dt + x\ell - \ell\right|$

So

$$\left|x\int_x^1 \frac{f(t)}{t^2}\,dt - \ell\right| - |x\ell| \le \left|x\int_x^1 \frac{f(t)}{t^2}\,dt + x\ell - \ell\right| \le \epsilon - \frac{\epsilon x}{\delta} + x\left|\int_\delta^1 \frac{f(t)-\ell}{t^2}\,dt\right|$$

$$\left|x\int_x^1 \frac{f(t)}{t^2}\,dt - \ell\right| \le \epsilon - \frac{\epsilon \cancel{x}}{\delta} + \cancel{x}\left|\int_\delta^1 \frac{f(t)-\ell}{t^2}\,dt\right| + |\cancel{x}\ell|$$

By making x small enough we can make $\left|x\int_x^1 \frac{f(t)}{t^2}\,dt - \ell\right|$ arbitrarily close to ϵ.

Therefore

$$\lim_{x\to 0^+} x\int_x^1 \frac{f(t)}{t^2}\,dt = \ell$$

30.

(a) $f(x) = \begin{cases} \dfrac{1}{\sqrt{x}} & \text{for } 0 \leq x \leq 1 \\ \dfrac{1}{x^2} & x \geq 1 \end{cases}$

$\int_0^\infty f$?

$$\int_0^\infty f = \int_0^1 f + \int_1^\infty f = \lim_{\varepsilon_1 \to 0^+} \int_{\varepsilon_1}^1 \frac{1}{\sqrt{x}}\,dx + \lim_{a \to \infty} \int_1^a x^{-2}\,dx$$

$$= 2 + 1 = 3$$

(b) $\int_0^\infty x^r\,dx$

In the problems so far, we've shown:

$$\int_1^\infty x^r\,dx = \frac{-1}{r+1} \quad \text{if } r < -1$$

if $r = -1$ the int. doesn't exist, ie $= \infty$

$$\int_0^a x^{-1/2}\,dx = 2\sqrt{a}$$

$$\int_0^a x^r\,dx = \frac{a^{r+1}}{r+1} \quad \text{if } -1 < r < 0$$

$\int_0^a x^{-1}\,dx$ doesn't exist.

Let's now consider all possible cases.

Case 1: $r > -1$

$$\int_1^\infty x^r\,dx = \lim_{a \to \infty} \int_1^a x^r\,dx = \lim_{a \to \infty} \frac{a^{r+1} - 1}{r+1} = \infty$$

Case 2: $r = -1$

By 25b, $\int_0^1 x^{-1}\,dx = \infty$, and by 28(c), $\int_0^a x^{-1}\,dx$ is ∞.

Case 3: $r < -1$

For $x \in (0,1)$ we have that for any $n > 1$

$x^n < x$

$\rightarrow \dfrac{1}{x^n} > \dfrac{1}{x}$

$\rightarrow \int_0^a x^{-n}\,dx = \infty$