Ch 7 - Three Hard Theorems

1.

i) $f(x) = x^2$ on $(-1,1)$

Bounded above means $\exists y \in \mathbb{R}, \forall x \in (-1,1), f(x) \leq y$.

f taking on its max value on an interval means there exists a value y in the interval s.t. $f(x) \leq f(y)$ for any x in the interval.

$\forall x \in (-1,1),\ 0 < f(x) < 1$.

$\forall x \in (-1,1),\ \exists y \in (x,1)$.
Therefore $y > x$.

$0 \in (-1,1)$ and $\forall x \in (-1,1)\ f(x) \geq f(0)$

f is bounded. f takes on min value but not max value in $(-1,1)$.

Note that f is continuous on $(-1,1)$, not $[-1,1]$.

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ii) $f(x) = x^3$ on $(-1,1)$

$\forall x \in (-1,1),\ -1 < f(x) < 1$. f bounded.
f does not take on max or min in $(-1,1)$.

Note that f is continuous on $(-1,1)$, not $[-1,1]$.

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iii) $f(x) = x^2$ on $\mathbb{R}$

bounded from below, takes min value.

f cont on $\mathbb{R}$, which isn't a closed interval.

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iv) $f(x) = x^2$ on $[0,\infty)$

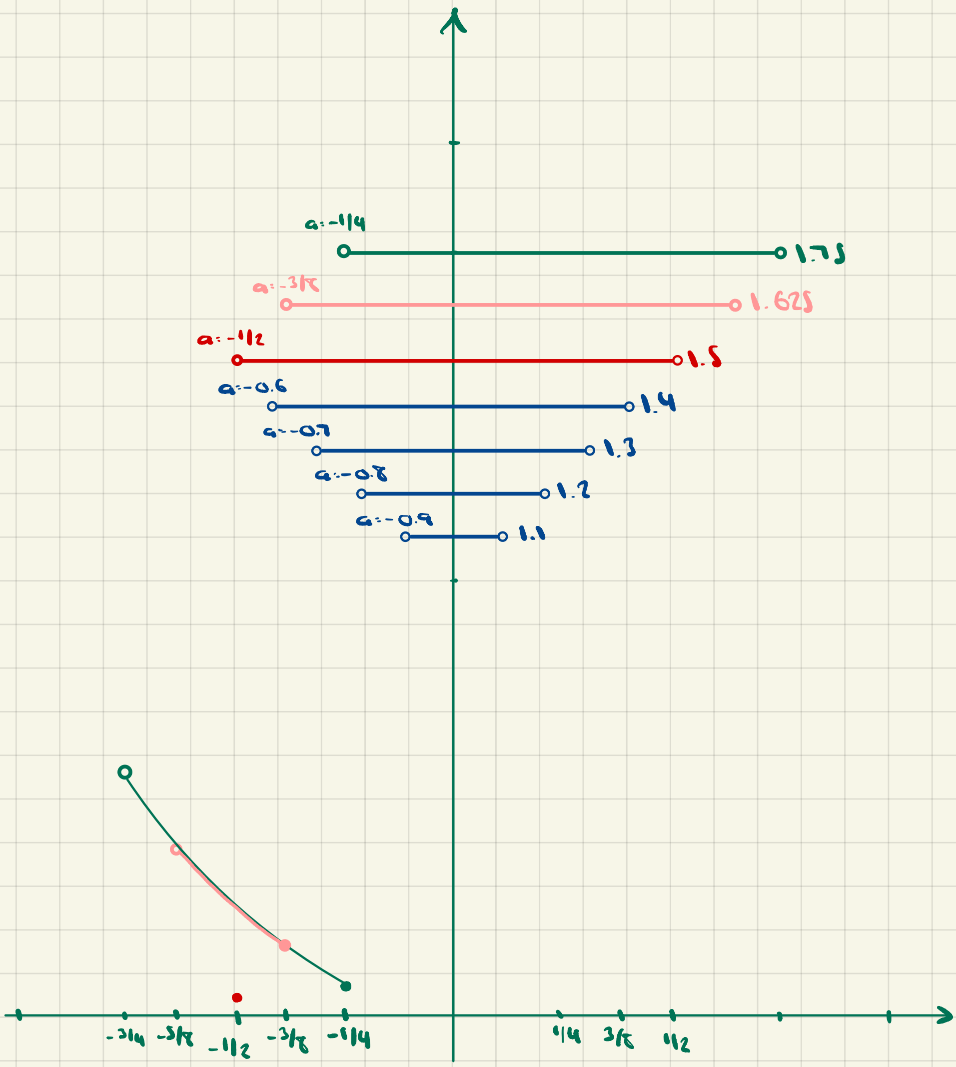
bounded from below, takes min value.

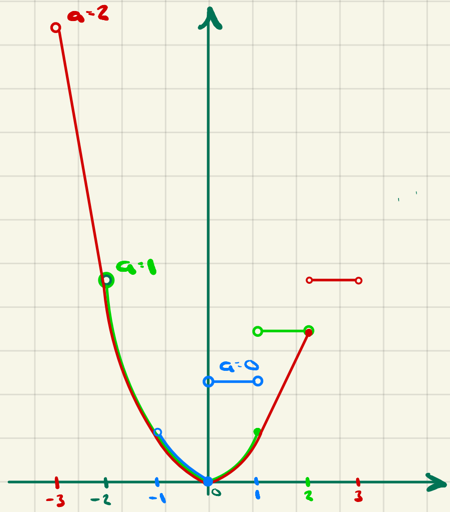
f cont. on $[0,\infty)$, which isn't a closed interval.

v) $f(x) = \begin{cases} x^2 & x \le a \\ a+2 & x > a \end{cases}$ on $(-a-1, a+1)$

Assume $a > -1$. Therefore $-a-1 < a+1$.

a	interval	$f(-a-1)$	$f(a)$	$f(a+1)$
-1	$(0,0)$			
0	$(-1,1)$	$f(-1)=1$	$f(0)=0$	$f(1)=2$
1	$(-2,2)$	$f(-2)=4$	$f(1)=1$	$f(2)=3$
2	$(-3,3)$	$f(-3)=9$	$f(2)=4$	$f(3)=4$
$-1/2$	$(-1/2,1/2)$	$f(-1/2)=1/4$	$f(-1/2)=1/4$	$f(1/2)=1.5$
$-.9$	$(-.1,.1)$	$f(-.1)=1.1$		$f(.1)=1.1$
$-.8$	$(-.2,.2)$	$f(-.2)=1.2$		$f(.2)=1.2$
$-.7$	$(-.3,.3)$	$f(-.3)=1.3$		$f(.3)=1.3$
$-1/4$	$(-.75,.75)$	$f(-.75)=.5375$	$f(1/4)=1/16$	$f(.75)=1.75$
$-3/8$	$(-5/8,5/8)$	$f(-5/8)=\frac{25}{64}=.3906$	$f(-3/8)=\frac{9}{64}=.1406$	$f(5/8)=13/8=1.625$





For all a, bounded.

$-1 < a < -\frac{1}{2}$: min = max = $a+2$

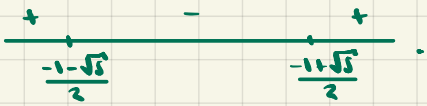
$-\frac{1}{2} \le a \le 0$: min = a^2, max = $a+2$

$0 \le a < \frac{-1+\sqrt{5}}{2}$: min = 0, max = $a+2$

$a > \frac{-1+\sqrt{5}}{2}$: min = 0

In $(a, a+1)$, $f(x) = a+2$. In $(-a-1, a]$, $f(x) = x^2$. There is no single largest value of f in $(-a-1, a]$, but if $a+2 < (-a-1)^2$ then the max of f in $(-a-1, a+1)$ isn't $a+2$ any more. This happens when $a+2 < (a+1)^2 = a^2+2a+1 \to a^2+a-1 > 0$.

$\Delta = 1+4 = 5$ $\quad a = \frac{-1 \pm \sqrt{5}}{2}$.



Ie for $a > \frac{-1+\sqrt{5}}{2}$, there is no max in the interval.

vii) $f(x) = \begin{cases} x^2 & x < a \\ a+2 & x \geq a \end{cases}$ on $[-a-1, a+1]$, $a > -1$.

Relative to vi) the difference is that even when $a+2 < (-a-1)^2$ there is a max. Therefore

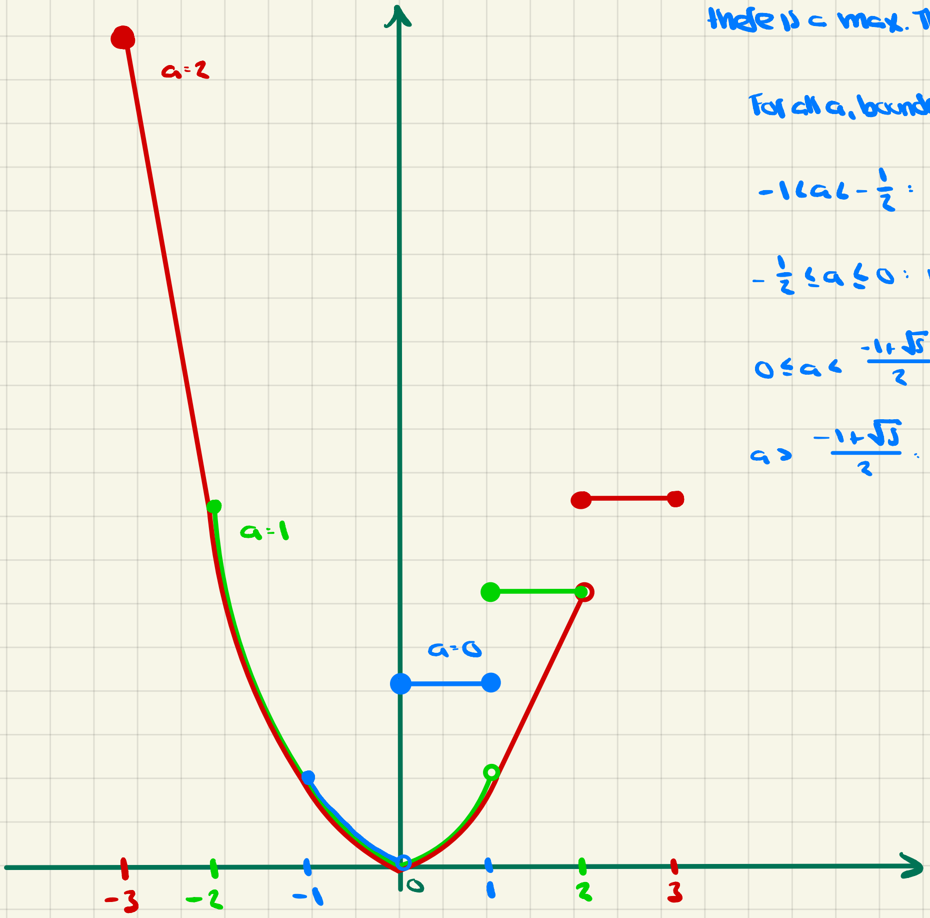
For all a, bounded.

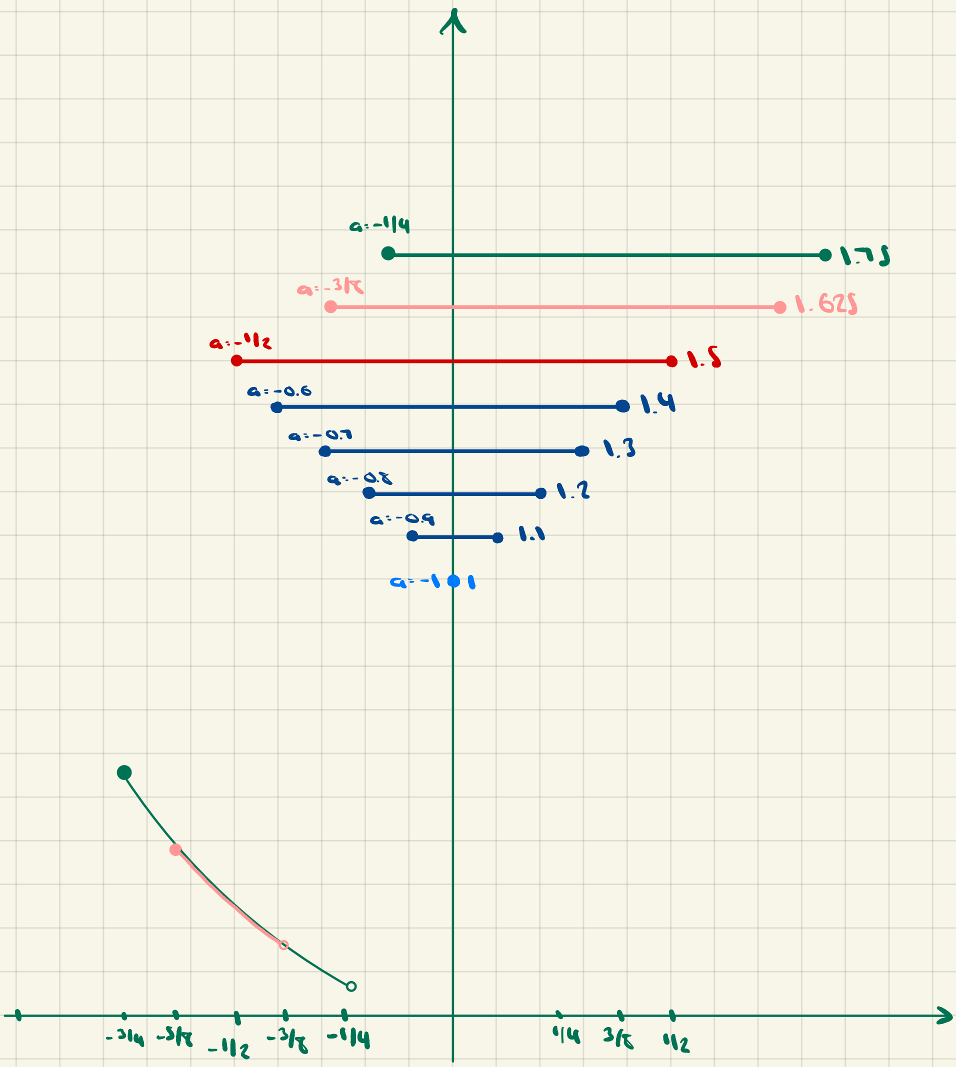
$-1 < a < -\frac{1}{2}$: min = max = $a+2$

$-\frac{1}{2} \leq a \leq 0$: min = a^2, max = $a+2$

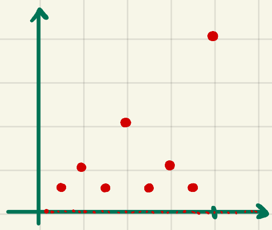
$0 \leq a < \frac{-1+\sqrt{5}}{2}$: min = 0, max = $a+2$

$a > \frac{-1+\sqrt{5}}{2}$: min = 0, max = $(-a-1)^2 = (a+1)^2$





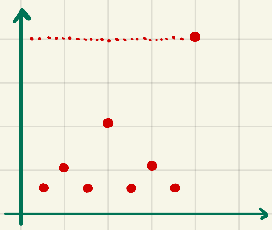
vii) $f(x) = \begin{cases} 0 & x \text{ irrational} \\ 1/q & x = p/q \text{ lowest terms} \end{cases}$ on $[0,1]$



bounded, max and min

f not cont. on $[0,1]$.

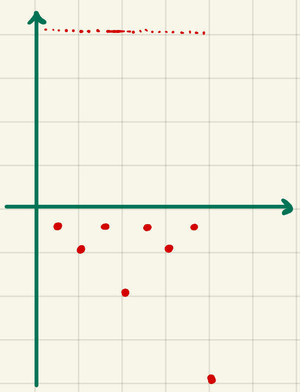
viii) $f(x) = \begin{cases} 1 & x \text{ irrational} \\ 1/q & x = p/q \text{ lowest terms} \end{cases}$ on $[0,1]$



bounded, max. 1

f not cont. on $[0,1]$.

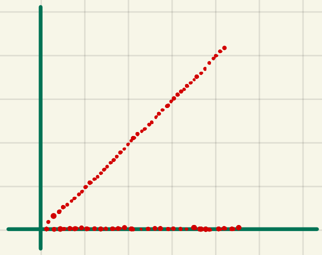
ix) $f(x) = \begin{cases} 1 & x \text{ irrational} \\ -1/q & x = p/q \text{ lowest terms} \end{cases}$ on $[0,1]$



bounded, min, max

f not cont. on $[0,1]$.

x) $f(x) = \begin{cases} x & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$ on $[0,a]$



bounded for all a. Takes on min. 0

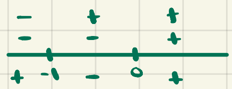
if a rational takes on max. a

f not cont on $[0,a]$.

xi) $f(x) = \sin^2(\cos x + \sqrt{a+a^2})$ on $[0,a^3]$

bounded.

$a + a^2 \geq 0 \rightarrow a(1+a) \geq 0$



$a \leq -1$ or $a \geq 0$

$g(x) = \cos(x) + \sqrt{a+a^2}$ cont on $\mathbb{R}$

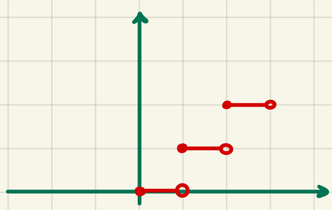
$h(x) = \sin^2(x)$ cont on $\mathbb{R}$

Th 6-2 $\rightarrow h(g(x)) = f(x)$ cont on $\mathbb{R}$.

Th 2.6: f bounded.

Th 3.7: max, min on $[0,a^3]$.

xiii) $f(x) = [x]$ on $[0,a]$



bounded, max $[a]$, min 0

2.

i) $f(x) = x^3 - x + 3$

x	f
-1	3
0	3
1	3
2	9
-2	-3

Therefore if $n = -2$ then

$f(n) = -3 < 0 < 3 = f(n+1)$

Since f is also cont. on $\mathbb{R}$ and in particular on $[n, n+1]$, we can use Th. 1: $\exists x \in [n, n+1], f(x) = 0$.

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ii) $f(x) = x^5 + 5x^4 + 2x + 1$

x	f
-1	$-1+5-2+1 = 3$
0	1
1	$1+5+2+1 = 9$
-2	$-32 + 5 \cdot 16 - 4 + 1 = -32 + 80 - 3 = 45$
-3	$-243 + 5 \cdot 81 + 2(-3) + 1 = -243 + 405 - 6 + 1 = 157$
-4	$-1024 + 5 \cdot 256 - 8 + 1 = 249$
-5	$-3125 + 5 \cdot 625 - 10 + 1 = -9$

$n = -5$

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iii) $f(x) = x^5 + x + 1$

$f(-1) = -1 - 1 + 1 = -1$

$f(0) = 1$

Choose $n = -1$.

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iv) $f(x) = 4x^2 - 4x + 1$

$f(x) = 4(x^2 - x) + 1$

$= 4(x^2 - x + \frac{1}{4}) + 1 - 1$

$= 4(x - \frac{1}{2})^2 \geq 0$

$\forall x, f(x) \geq 0$

$f(1/2) = 0$

Therefore, $n = 0$.

3.

i) $\exists x, \; x^{179} + \dfrac{163}{1 + x^2 + \sin^2 x} = 119$

Proof

Let $f(x) = x^{179} + \dfrac{163}{1 + x^2 + \sin^2 x} - 119$

$1 + x^2 + \sin^2 x$ is never zero.

$f(x)$ is defined and continuous for all $x \in \mathbb{R}$.

$f(0) = 163 - 119 = 44$

$\lim_{x \to -\infty} f(x) = -\infty$

$\to \exists N \in \mathbb{R}, \forall M \in \mathbb{R}, \forall x \in \mathbb{R}, x < N \to f(x) < M$.

Choose an $M < 0$, and an x s.t. $x < N \to f(x) < M = 0$.

Then, $f(x) < 0 < f(0) = 44$

By Th. 1, $\exists y \in [x, 0], f(y) = 0$.

Thus $y^{179} + \dfrac{163}{1 + y^2 + \sin^2 y} - 119 = 0$

Thus, $y^{179} + \dfrac{163}{1 + y^2 + \sin^2 y} = 119$.

Therefore, $\exists x, \; x^{179} + \dfrac{163}{1 + x^2 + \sin^2 x} = 119$

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ii) $\exists x \in \mathbb{R}, \sin x = x - 1$

Proof

Let $f(x) = \sin(x) - x + 1$

$f(0) = 1$

$f(3\pi/2) = -3\pi/2$

Lemma: f cont. on $[a, b]$, $f(a) > 0 > f(b)$ $\to \exists x \in [a, b], f(x) = 0$

Since f cont. on $\mathbb{R}$, by the lemma above

$\exists x \in [0, 3\pi/2], f(x) = \sin x - x + 1 = 0$

$\to \sin x = x - 1$

4.

a) in problem 3-7 we proved

a) $\forall$ polynomial f $\forall a \in \mathbb{R}$ $\Rightarrow$ $\exists$ polyn. fn g and $b \in \mathbb{R}$ s.t. $f(x) = (x-a)g(x) + b$ $\forall x$

b) $f(a) = 0$ $\Rightarrow$ $f(x) = (x-a)g(x)$ for some polyn. $g(x)$

c) f n degree polyn. $\Rightarrow$ f has at most n roots, ie numbers a with $f(a) = 0$

d) for each n $\Rightarrow$ $\exists$ polyn of degree n with n roots

$n-k \geq 0$ is even.

Thus

a polyn. of degree n has at most n roots.

since $n-k \geq 0$, then if $k \geq 0$ we have $0 \leq k \leq n$.

since $n-k \leq n$, it is possible that there is an n^{th} degree polyn. with $k \leq n$ roots (though we haven't proven such a polyn. must exist if $k < n$).

Let $f(x) = (x-a_1)(x-a_2)\dots(x-a_k)g(x)$ be polyn. of degree n.

$g(x) = 1 + x^{n-k}$.

Then

$g(x)$ has no roots because $\forall x, 1 + x^{n-k} > 0$.

But f has k roots $a_1, \dots, a_k$.

Therefore we have a polyn. of degree n with exactly k roots

$f(x) = (x-a_1) \cdot \dots \cdot (x-a_k)(1 + x^{n-k})$

b) Note:

$f(x) = (x-a)^m g(x)$

a not root of g → a root of f has multiplicity m

Let f be polyn. deg. n with k roots, including multiplicities

Then n−k is even

Proof

Let f be polyn. deg n, with k roots

We've proved that f has at most n roots, so $k \leq n$, and $n-k \geq 0$.

f can be written as $f(x) = h(x)g(x)$, where $\deg(h) = k$, h has k roots, and $\deg(g) = n-k$ and g has no roots.

results from 3-7b

Assume n−k odd. Then by Th 9, g(x) has a root. ⊥

Therefore n−k even.

Also, n−k is even ↔ n and k both even or both odd.

e.g. $\deg(f) = n$ even, then k must be even.

$\deg(f) = n$ odd, then k must be odd.

A 3rd deg. polyn. can have either one root of mult. 1 or three roots.

A 2nd deg polyn. can have 0 roots, 1 root w/ mult. 2, or 2 roots of mult. 1.

examples

$f(x) = 4x^2+4x+1 = 4(x+1/2)^2$

$\Delta = 16-16 = 0 \rightarrow x_{root} = -\frac{1}{2}$.

$\deg(f) = 2 = n$

$k = 2$

$n-k = 0$

$f(x) = x(x^2+1)$

$n = 3$

$x_{root} = 0$

$k = 1$

$n-k = 2$, even

$f(x) = x^3+1 = (x+1)(x^2-x+1)$

$n = 3$

$x_{1r} = -1$

$n-k = 2$ even

$$\begin{array}{r} x^2-x \\ x+1 \,\big)\, x^3+1 \\ x^3+x^2 \\ \hline 1-x^2 \\ -x^2-x \\ \hline 1+x \end{array}$$

$x^3+1 = (x+1)(x^2-x) + (1+x)$

$= (1+x)(x^2-x+1)$

$\Delta = -3 < 0$

no roots

$f(x) = x(x^3+1) = x(x+1)(x^2-x+1)$

$n = 4$

$x_{1r} = 0$

$x_{2r} = -1$

$k = 2$

$n-k = 2$, even

$f(x) = x^3(x+1)$

$n = 4$

$x_{1r} = 0$

$x_{2r} = -1$

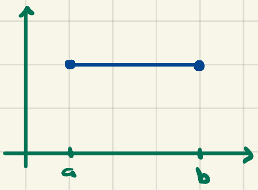
$k = 4$

$n-k = 0$

5.

f cont on $[a,b]$

$f(x)$ always rational



Assume $f(a) \neq f(b)$.

Without loss of generality, assume $f(a) < f(b)$ and let $f(a) < c < f(b)$, c irrational.
By Th.4 $\exists x \in [a,b], f(x) = c$. $\perp$

Therefore $f(a) = f(b)$.

Furthermore,

Assume $\exists x \in [a,b]$, $f(x) \neq f(a) = f(b)$.
Then f cont. on $[a,x]$, and $f(a) \neq f(x)$.
But then, by the same argument above, $f(a) = f(x)$.

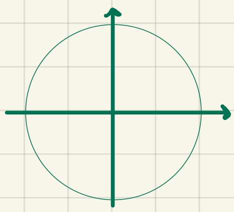
Therefore $f(x) =$ constant on $[a,b]$.

6.

f cont. on $[-1,1]$ $\rightarrow$ $\forall x \; f(x) = \sqrt{1-x^2}$

$x^2 + (f(x))^2 = 1 \quad \forall x \in \mathbb{R}$ $\quad$ or else $\forall x \; f(x) = -\sqrt{1-x^2}$

Proof



Given an x, since $\alpha = (f(x))^2 = 1 - x^2 \geq 0$ in $[-1,1]$, by Th.8 we know that α has a square root.
Therefore for each $x \in [-1,1]$, $f(x)$ is either $\sqrt{1-x^2}$ or $-\sqrt{1-x^2}$.

Let $x_1, x_2 \in (-1,1)$. Without loss of generality assume $x_1 < x_2$.

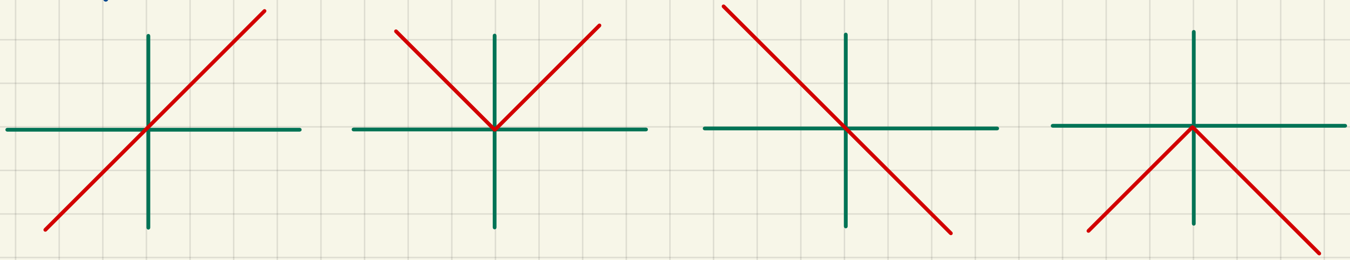
If $f(x_1) = \sqrt{1-x_1^2}$ and $f(x_2) = -\sqrt{1-x_2^2}$ then by Th.1 $\exists x \in [x_1, x_2]$, $f(x) = 0$.
But then $x^2 + (f(x))^2 = x^2 = 1$ w/ $-1 < x < 1$. This is impossible.
The same impossibility is obtained if $f(x_1) = -\sqrt{1-x_1^2}$ and $f(x_2) = \sqrt{1-x_2^2}$.

Thus $f(x_1)$ and $f(x_2)$ have same sign, ie $\forall x \; f(x) = \sqrt{1-x^2}$ or $\forall x \; f(x) = -\sqrt{1-x^2}$

7) $\forall x \ (f(x))^2 = x^2$, f continuous.

$y^2 = x^2$

$y = \pm x$



8.

f, g cont. $\qquad \forall x \ f(x) = g(x)$

$f^2 = g^2 \quad \rightarrow \quad$ or

$\forall x \ f(x) \neq 0 \qquad \forall x \ f(x) = -g(x)$

Proof

suppose $f(x_1) > 0$ and $f(x_2) < 0$, $x_1 < x_2$.

Then by Th.1 $\exists x \in [x_1, x_2], f(x) = 0$. $\perp$

Therefore $\forall x \ f(x) > 0$ or $\forall x \ f(x) < 0$.

statement true for individual x.

$\forall x \ f^2(x) = g^2(x)$

$\forall x \ (f(x) = g(x)$ or $f(x) = -g(x))$

Case 1: $\forall x \ f(x) > 0$.

Let $x_1 \in \mathbb{R}$

Case 1.1: $f(x_1) = g(x_1)$.

Then $g(x_1) > 0$.

Let $x_2 \in \mathbb{R}$.

Assume $f(x_2) = -g(x_2)$

Then $-g(x_2) > 0$.

$\rightarrow g(x_2) < 0$.

Then $\exists x \in [x_1, x_2], g(x) = 0$

$\rightarrow f(x) = 0$.

$\perp$

$f(x_2) = g(x_2)$

$\forall x \in \mathbb{R} \ f(x) = g(x)$

Case 1.2 $f(x_1) = -g(x_1)$

Then $-g(x_1) > 0$

$g(x_1) < 0$

Let $x_2 \in \mathbb{R}$.

Assume $f(x_2) = g(x_2)$.

Then $g(x_2) > 0$.

$\rightarrow \exists x \in [x_1, x_2], g(x) = 0$.

$\rightarrow f(x) = 0$

$\perp$

$f(x_2) = -g(x_2)$

$\forall x \in \mathbb{R} \ f(x) = -g(x)$.

Therefore,

$\forall x \in \mathbb{R} \ f(x) = -g(x)$ or $\forall x \in \mathbb{R} \ f(x) = g(x)$

Case 2: $\forall x \ f(x) < 0$

Let $x_1 \in \mathbb{R}$.

Case 2.1 $f(x_1) = g(x_1)$

Then $g(x_1) < 0$.

Let $x_2 \in \mathbb{R}$

Assume $f(x_2) = -g(x_2)$

Then $g(x_2) > 0$.

$\exists x \in [x_1, x_2] \ g(x) = 0$

$\rightarrow f(x) = 0$

$\perp$

$f(x_2) = g(x_2)$

$\forall x \in \mathbb{R} \ f(x) = g(x)$

Case 2.1 $f(x_1) = -g(x_1)$

Then $g(x_1) > 0$.

Let $x_2 \in \mathbb{R}$.

Assume $f(x_2) = g(x_2)$

$g(x_2) > 0$

$\exists x \in [x_1, x_2] \ g(x) = 0$

$\rightarrow f(x) = 0$

$\perp$

$f(x_2) = -g(x_2)$

$\forall x \in \mathbb{R} \ f(x) = -g(x)$

Therefore $\forall x \in \mathbb{R} \ f(x) = -g(x)$ or $\forall x \in \mathbb{R} \ f(x) = g(x)$

We thus conclude our proof by cases with:

$\forall x \in \mathbb{R} \ f(x) = -g(x)$ or $\forall x \in \mathbb{R} \ f(x) = g(x)$

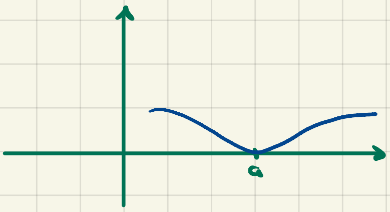
9.

a)

f cont

$f(x)=0$ only for $x=a$

$f(x)>0$ for some $x>a$ as well as for some $x<a$.



If f becomes negative then f must be 0 at least twice.
This contradicts an assumption, so f must be >0 for $x \neq a$.

Proof

Let $x_1 \in \mathbb{R}$, $x_1 \neq a$

Assume $f(x_1)<0$

case 1: $x_1<a$

By assumption $\exists x_2<a$ s.t. $f(x_2)>0$.

w/o loss of gen. assume $x_1<x_2$.

By Th.1 $\exists x \in [x_1,x_2]$, $f(x)=0$.

$\perp$

case 2: $x_1>a$

Analogous.

$\perp$

Therefore $f(x_1)>0$.

Therefore, $\forall x \in \mathbb{R}$ $x \neq a \rightarrow f(x)>0$.

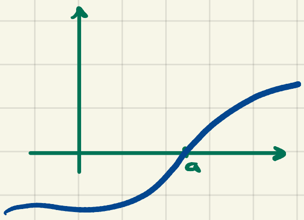
b)

f cont

$f(x)=0$ only for $x=a$

$f(x)>0$ for some $x>a$

$f(x)<0$ for some $x<a$



$x<a \rightarrow f(x)<0$

$x>a \rightarrow f(x)>0$

Proof

Let $x_1 \in \mathbb{R}$

Assume $x_1<a$.

Assume $f(x_1)>0$.

$\exists x_2,\ x_2<a \wedge f(x_2)<0$

$\exists x \in [x_1,x_2]$ $f(x)=0$

$\perp$

$f(x_1) \leq 0$

Assume $f(x_1)=0$. $\perp$

$f(x_1)<0$

$x_1<a \rightarrow f(x_1)<0$

$\forall x$ $x<a \rightarrow f(x)<0$

Similarly if we assume $x_1>a$ we reach $\forall x$ $x>a \rightarrow f(x)>0$.

c)

$x^3+x^2y+xy^2+y^3$

x and y not both zero

$$x^3+x^2y+xy^2+y^3 = x^3+3x^2y+3xy^2+y^3-2x^2y-2xy^2$$
$$= (x+y)^3-2xy(x+y)$$
$$= (x+y)((x+y)^2-2xy)$$
$$= (x+y)(x^2+y^2)$$

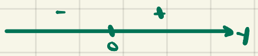
$(x\neq 0 \wedge y\neq 0) \vee (x=0 \wedge y\neq 0) \vee (x\neq 0 \wedge y=0)$

$x+y>0 \rightarrow y>-x$

$x^2+y^2 \geq 0$ for all x,y

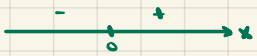
Case 1: $x=0 \wedge y\neq 0$

The expression becomes y^3.

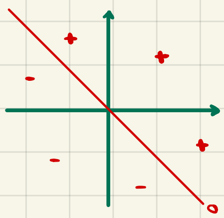


Case 2: $x\neq 0 \wedge y=0$

Analogous to case 1.

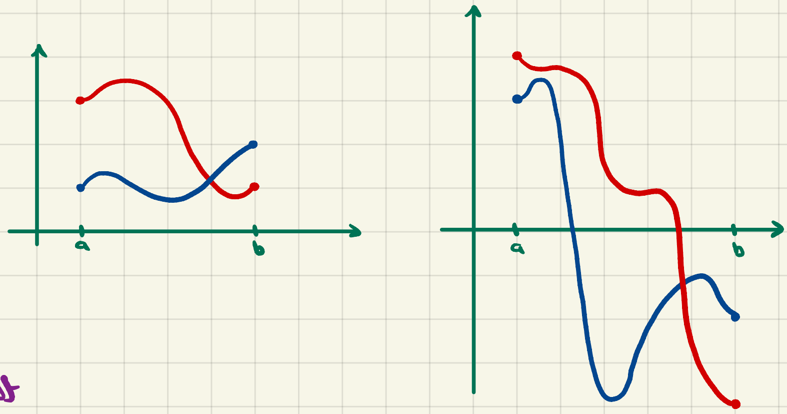


Case 3: $x\neq 0 \wedge y\neq 0$



10.

f,g cont. on $[a,b]$
$f(a)<g(a)$ $\rightarrow$ $f(x)=g(x)$ for some $x\in[a,b]$
$f(b)>g(b)$



Proof

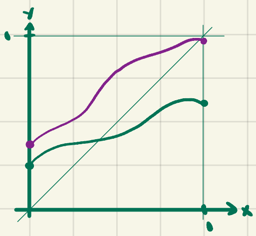
Let $h(x)=f(x)-g(x)$

$h(a)=f(a)-g(a)<0$
$h(b)=f(b)-g(b)>0$
h cont. on $[a,b]$.

By Th. 1, $\exists x\in[a,b], h(x)=f(x)-g(x)=0$
$\rightarrow f(x)=g(x)$.

11.

f cont. on $[0,1]$
$f(x)$ in $[0,1]$ for each x $\rightarrow$ $f(x)=x$ for some x



Proof

Let $h(x)=x$.
Then $h(0)\leq f(0)$, $h(1)\geq f(1)$.

If $h(0)=f(0)=0$ or $h(1)=f(1)=1$ then $\exists x\in[0,1], f(x)=x$.

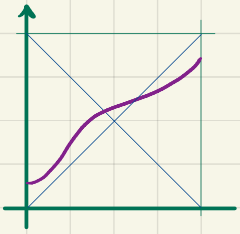
If $h(0)<f(0)$ and $h(1)>f(1)$ then by problem 10 $f(x)=h(x)=x$ for some $x\in[0,1]$.

12.

a)

f cont. on $[0,1]$ $\rightarrow$ f intersects the line $y = 1-x$ in $[0,1]$.

$f(x)$ in $[0,1]$ for each x



Proof

Let $h(x) = 1-x$.

Then

$h(0) = 1$

$f(0) \leq h(0)$

$h(1) = 0$

$f(1) \geq h(1)$

Therefore, by problem 10, $\exists x \in [0,1]$ $f(x) = h(x) = 1-x$.

b)

g cont. on $[0,1]$ $\rightarrow$ $\exists x \in [0,1]$ $f(x) = g(x)$

$g(0) = 0,\ g(1) = 1$ or $g(0) = 1, g(1) = 0$

Proof

Case 1: $g(0) = 0, g(1) = 1$

$f(0) \geq g(0)$

$f(1) \leq g(1)$

Case 1.1: $f(0) = g(0)$ or $f(1) = g(1)$

$\exists x \in [0,1]$ $f(x) = g(x)$

Case 1.2: $f(0) > g(0)$ and $f(1) < g(1)$

by prob. 9 $\exists x \in [0,1], f(x) = g(x)$.

Case 2: $g(0) = 1, g(1) = 0$

Analogous to case 1.

Therefore $\exists x \in [0,1], f(x) = g(x)$.

13.

a)

$$f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

f cont on $[-1,1]$?

$\nexists \lim_{x \to 0} \sin(1/x)$

Therefore f not cont at $x=0$.

f satisfies conclusion of IVT on $[-1,1]$

Proof

In $[-1,1]$, the max value of f is 1, min is -1.

$\frac{1}{x}$ is cont. on $(0,\infty)$.

In particular, on $[2/\pi, 2/3\pi]$

$f(2/\pi) = 1$

$f(2/3\pi) = -1$

$\to \forall c \in [-1,1], \exists x \in [2/\pi, 2/3\pi], f(x) = c.$

Therefore, for any two values of f in $[-1,1]$, f takes on all values in between.

Alternatively,

Let $x_1, x_2 \in [-1,1]$

Case 1: $0 < x_1 < x_2 < 1$

f cont. on $[x_1, x_2]$.

Therefore by Th. 4 $\forall c, f(x_1) < c < f(x_2)$

$\to \exists x \in [x_1, x_2], f(x) = c.$

Case 2: $-1 < x_1 < x_2 < 0$

Analogous to case 1.

Case 3: $x_1 < 0 < x_2$

f takes on all values between -1 and 1 in $[x_1, x_2]$.

Therefore it takes all values in $[f(x_1), f(x_2)]$ since $\forall x, f(x) \in [-1,1]$.

b)

f satisfies cond. of IVT
f takes on each value only once $\longrightarrow$ f continuous

Proof

Assume f is defined at a, but not continuous at a.

Then, as proved in 6-9, for some $\epsilon > 0$, there are numbers x arbitrarily close to a w/ $f(x) < f(a) - \epsilon$ or $f(x) > f(a) + \epsilon$.

Let $\epsilon_1 > 0$ be such an ϵ, and x_1 such that $f(x_1) > f(a) + \epsilon_1$ or $f(x_1) < f(a) - \epsilon_1$.

We need to consider two cases, but they are analogous.

Let's consider the case where $f(x_1) > f(a) + \epsilon_1$.

By IVT, $\forall c, f(a) < c < f(x_1), \exists x \in [a, x_1], f(x) = c.$

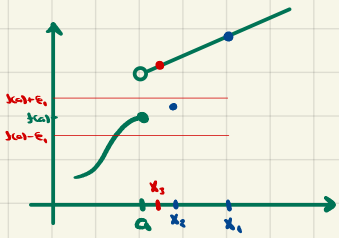
Let $c \in (f(a), f(a) + \epsilon)$. Then, $f(a) < c < f(a) + \epsilon < f(x_1)$.

Therefore, $\exists x_2$ s.t. $x_2 \in [a, x_1]$ and $f(x_2) = c < f(a) + \epsilon_1$

However, again by 6-9b, there is x_3 arbitrarily close to a (e.g. $a < x_3 < x_2$) s.t. $f(x_3) > f(a) + \epsilon_1$

But then, $x_3 < x_2 < x_1$ and $f(x_2) < f(a) + \epsilon_1 < f(x_3)$ and $f(x_2) < f(a) + \epsilon_1 < f(x_1)$

By IVT, f takes on value $f(a) + \epsilon_1$ in $[x_3, x_2]$ and in $[x_2, x_1]$, contradicting the premise that f takes on each value only once.



Therefore, f continuous.

c)

f satisfies cond. of IVT
f takes on each value only finitely many times $\rightarrow$ f continuous

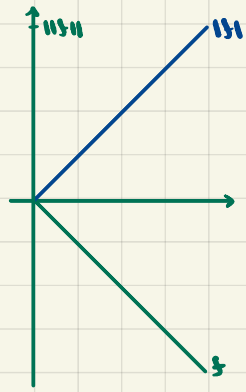
In 6-9 we showed that there are x arbitrarily close to a, such that $f(x) > f(a) + \epsilon$ or $f(x) < f(a) - \epsilon$.

Let n be the maximum number of times f takes on a value.

If we assume f is discontinuous at a, we can repeat the steps used in part b) $n+1$ times to reach the result that for some $\epsilon > 0$, f takes on the value $f(a) + \epsilon$ more than n times, a contradiction.

---

14. f cont. on $[0,1]$
$\|f\| =$ max of $|f|$ on $[0,1]$



a) $\forall c \quad \|cf\| = |c| \cdot \|f\|$

Let $f(x_1)$ be the max of f on $[0,1]$. We know $\exists x_1$ by Th. 3.
$\forall x \in [0,1],\ f(x) \leq f(x_1)$.
Let $f(x_2)$ be the min of f on $[0,1]$. We know $\exists x_2$ by Th. 7.
$\forall x \in [0,1],\ f(x) \geq f(x_2)$.

Then $\|f\| = \max(|f(x_1)|, |f(x_2)|) = \dfrac{|f(x_1)| + |f(x_2)| + \big||f(x_1)| - |f(x_2)|\big|}{2}$

Then, $\forall x \in [0,1],\ |f(x)| \leq \|f\|$

Similarly, $\|cf\| = \max(|cf(x_1)|, |cf(x_2)|) = \dfrac{|cf(x_1)| + |cf(x_2)| + \big||cf(x_1)| - |cf(x_2)|\big|}{2}$

$$= \frac{|c||f(x_1)| + |c||f(x_2)| + \big||c||f(x_1)| - |c||f(x_2)|\big|}{2}$$

$$= \frac{|c||f(x_1)| + |c||f(x_2)| + \big||c|(|f(x_1)| - |f(x_2)|)\big|}{2}$$

$$= \frac{|c||f(x_1)| + |c||f(x_2)| + |c|\big||f(x_1)| - |f(x_2)|\big|}{2}$$

$$= |c| \frac{|f(x_1)| + |f(x_2)| + \big||f(x_1)| - |f(x_2)|\big|}{2}$$

$$= |c| \, \|f\|$$

b) $\|f+g\| \le \|f\| + \|g\|$

Let $f(x_{f_{min}})$ be the minimum of f on $[0,1]$.

Let $f(x_{f_{max}})$ " " maximum " " " " .

Then, $\forall x \in [0,1]$, $f(x) \le |f(x)| \le \max(|f(x_{f_{min}})|, |f(x_{f_{max}})|) = \|f\|$

Let $f(x_{g_{min}})$ and $f(x_{g_{max}})$ be defined analogously for g.

Let $f(x_{f+g_{min}})$ and $f(x_{f+g_{max}})$ be defined analogously for $f+g$.

This proof turned out way more complicated than necessary.

Then

$$\|f\| = \max(|f(x_{f_{min}})|, |f(x_{f_{max}})|) \ge \max(|f(x_{f+g_{min}})|, |f(x_{f+g_{max}})|)$$

$$\|g\| = \max(|g(x_{g_{min}})|, |g(x_{g_{max}})|) \ge \max(|f(g_{f+g_{min}})|, |f(x_{f+g_{max}})|)$$

$$\|f\| + \|g\| \ge \max(|f(x_{f+g_{min}})|, |f(x_{f+g_{max}})|) + \max(|f(g_{f+g_{min}})|, |f(x_{f+g_{max}})|)$$

$$= \Big[|f(x_{f+g_{min}})| + |g(x_{f+g_{min}})| + |f(x_{f+g_{max}})| + |g(x_{f+g_{max}})| + \big||f(x_{f+g_{min}})| - |f(x_{f+g_{max}})|\big| + \big||g(x_{f+g_{min}})| - |g(x_{f+g_{max}})|\big|\Big]/2$$

$|x| + |y| \ge |x+y|$

$$\ge \Big[|f(x_{f+g_{min}}) + g(x_{f+g_{min}})| + |f(x_{f+g_{max}}) + g(x_{f+g_{max}})| + \big|\underline{|f(x_{f+g_{min}})| + |g(x_{f+g_{min}})|} - (|f(x_{f+g_{max}})| + |g(x_{f+g_{max}})|)\big|\Big]/2$$

$|x| + |y| \ge |x+y|$

$$\ge \Big[|f(x_{f+g_{min}}) + g(x_{f+g_{min}})| + |f(x_{f+g_{max}}) + g(x_{f+g_{max}})| + \big|\underbrace{|f(x_{f+g_{min}}) + g(x_{f+g_{min}})|}_{a} - \underbrace{(|f(x_{f+g_{max}})| + |g(x_{f+g_{max}})|)}_{b}\big|\Big]/2$$

$|a-b| \ge \big||a| - |b|\big|$

$$\ge \Big[|f(x_{f+g_{min}}) + g(x_{f+g_{min}})| + |f(x_{f+g_{max}}) + g(x_{f+g_{max}})| + \big|\underline{|f(x_{f+g_{min}}) + g(x_{f+g_{min}})|} - \underline{\big||f(x_{f+g_{max}})| + |g(x_{f+g_{max}})|\big|}\big|\Big]/2$$

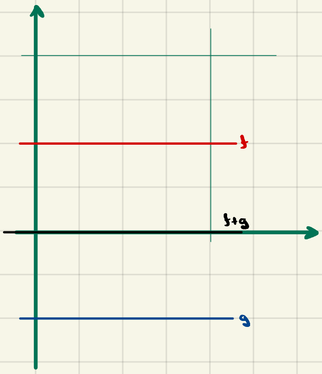
$$\ge \Big[|f(x_{f+g_{min}}) + g(x_{f+g_{min}})| + |f(x_{f+g_{max}}) + g(x_{f+g_{max}})| + \big||f(x_{f+g_{min}}) + g(x_{f+g_{min}})| - \big||f(x_{f+g_{max}}) + g(x_{f+g_{max}})|\big|\big|\Big]/2$$

$$= \Big[|f(x_{f+g_{min}}) + g(x_{f+g_{min}})| + |f(x_{f+g_{max}}) + g(x_{f+g_{max}})| + \big||f(x_{f+g_{min}}) + g(x_{f+g_{min}})| - |f(x_{f+g_{max}}) + g(x_{f+g_{max}})|\big|\Big]/2$$

$$= \max(|(f+g)(x_{f+g_{min}})|, |(f+g)(x_{f+g_{max}})|)$$

$$= \|f+g\|$$

examples of $\|f+g\| < \|f\| + \|g\|$



$c > 0$

$f(x) = c$

$g(x) = -c$

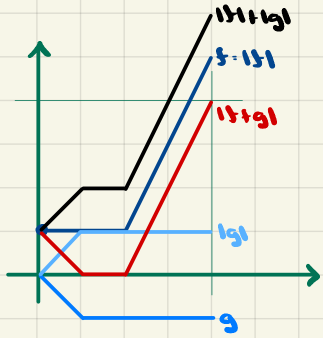
$(f+g)(x) = 0$

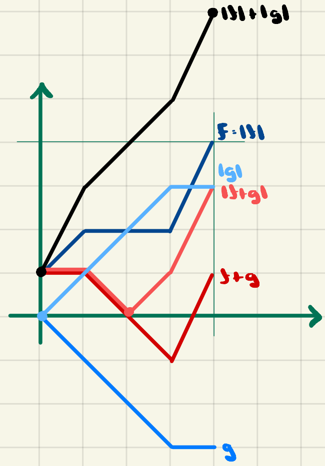
$\|f+g\| = 0$

$\|f\| = c$

$\|g\| = c$

$\|f\| + \|g\| = 2c > \|f+g\|$





Alternative proof

Let $x \in [0,1]$. Then

$$|f+g|(x) = |f(x)+g(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|$$

Note f and g are cont. so f+g and $|f+g|$ take on max and min in $[0,1]$.

Let x_0 s.t. $\|f+g\| = |f+g|(x_0) \leq |f|(x_0) + |g|(x_0) \leq \|f\| + \|g\|$

---

c) $\|h-f\| \leq \|h-g\| + \|g-f\|$

Proof

Let $f_1 = h-g$

$g_1 = g-f$

Then

$f_1 + g_1 = h-g+g-f = h-f$

From part b),

$\|f_1 + g_1\| \leq \|f_1\| + \|g_1\|$

$\|h-f\| \leq \|h-g\| + \|g-f\|$

15.

ϕ cont.

$$\lim_{x\to\infty}\frac{\phi(x)}{x^n}=\lim_{x\to-\infty}\frac{\phi(x)}{x^n}=0$$

a) n odd $\to \exists x,\ x^n+\phi(x)=0$

Theorem 9

If n is odd then any equation

$$x^n+a_{n-1}x^{n-1}+\cdots+a_0=0$$

has a root.

Theorem 8.2 If $n\in\mathbb{N}$ is odd then every number has an n^{th} root.

Proof

Assume n odd. (Conditional Proof Ass.)

premise: $\lim_{x\to\infty}\frac{\phi(x)}{x^n}=0$

$\leftrightarrow \forall\epsilon>0\ \exists N>0\ \forall x\left(x>N\to\left|\frac{\phi(x)}{x^n}\right|<\epsilon\right)$ (Def limit)

$\leftrightarrow \forall\epsilon>0\ \exists N>0\ \forall x\left(x>N\to -\epsilon<\frac{\phi(x)}{x^n}<\epsilon\right)$

Choose $\epsilon=1$. Then $\exists N>0\ \forall x\left(x>N\to -1<\frac{\phi(x)}{x^n}<1\right)$ ($\exists$ Elim Ass. because $\exists\epsilon,\ \epsilon=1$)

Let N s.t. $N>0$ and $\forall x\left(x>N\to -1<\frac{\phi(x)}{x^n}<1\right)$ ($\exists$ Elim Ass.)

Let x_1 s.t. $x_1>N>0$ ($\exists$ Elim Ass., because $\exists x, x>N$).

Then $x_1^n>0$ and $-x_1^n<\phi(x)<x_1^n$

Since n odd then $-x_1^n=(-x_1)^n<\phi(x)<x_1^n$

$\to (-x_1)^n<-\phi(x)<x_1^n$

Let $f(x)=x^n$. Then $f(-x_1)<-\phi(x)<f(x_1)$

Since f cont in $\mathbb{R}$, $\exists x\in[-x_1,x_1], f(x)=x^n=-\phi(x)$

$\to x^n+\phi(x)=0$

Therefore, $\exists x, x^n+\phi(x)=0$. ($\exists$ Elim)

Therefore, $\exists x, x^n+\phi(x)=0$. ($\exists$ Elim)

Therefore, $\exists x, x^n+\phi(x)=0$. ($\exists$ Elim)

Therefore n odd $\to \exists x, x^n+\phi(x)=0$.

Alternative Proof

$$\lim_{x\to\infty} \frac{\phi(x)}{x^n} = 0$$

$\Leftrightarrow \forall \epsilon > 0 \ \exists N > 0 \ \forall x \left(x > N \rightarrow \left| \frac{\phi(x)}{x^n} \right| < \epsilon \right)$

Let $\epsilon = \frac{1}{2}$

Let N s.t. $\forall x \left(x > N \rightarrow \left| \frac{\phi(x)}{x^n} \right| < \frac{1}{2} \right)$

Let b s.t. $b > N$

Then $\left| \frac{\phi(b)}{b^n} \right| < \frac{1}{2}$

$-\frac{1}{2} < \frac{\phi(b)}{b^n} < \frac{1}{2}$

$\frac{1}{2} < 1 + \frac{\phi(b)}{b^n}$

$0 < \frac{b^n}{2} < b^n\left(1 + \frac{\phi(b)}{b^n}\right) = b^n + \phi(b)$

Similarly

$\lim_{x\to-\infty} \frac{\phi(x)}{x^n} = 0 \Leftrightarrow \forall \epsilon > 0 \ \exists N_2 < 0 \ \forall x \left(x < N_2 \rightarrow \left| \frac{\phi(x)}{x^n} \right| < \epsilon \right)$

Let $\epsilon = \frac{1}{2}$

Let $N_2 < 0$ s.t. $\forall x \left(x < N_2 \rightarrow \left| \frac{\phi(x)}{x^n} \right| < \frac{1}{2} \right)$

Let a s.t. $a < N_2$.

Then $\left| \frac{\phi(a)}{a^n} \right| < \frac{1}{2}$

$-\frac{1}{2} < \frac{\phi(a)}{a^n} < \frac{1}{2}$

since n odd, $a^n < 0$.

$\frac{1}{2} < 1 + \frac{\phi(a)}{a^n} < \frac{3}{2}$

$0 > \frac{a^n}{2} > a^n\left(1 + \frac{\phi(a)}{a^n}\right) > \frac{3a^n}{2}$

Ie, $a^n\left(1 + \frac{\phi(a)}{a^n}\right) = a^n + \phi(a) < 0$

Thus we have

$a^n + \phi(a) = f(a) < 0 < f(b) = b^n + \phi(b)$

$\exists x \in [a,b],\ f(x) = x^n + \phi(x) = 0.$

b)

ϕ cont.

$$\lim_{x\to\infty} \frac{\phi(x)}{x^n} = \lim_{x\to-\infty} \frac{\phi(x)}{x^n} = 0$$

n even

$\forall x\ \exists y,\ y^n + \phi(y) \leq x^n + \phi(x)$

Proof

$$\lim_{x\to\infty} \frac{\phi(x)}{x^n} = 0 \quad \leftrightarrow \quad \forall \epsilon > 0\ \exists N_1 > 0\ \forall x \left(x > N_1 \to \left| \frac{\phi(x)}{x^n} \right| < \epsilon \right)$$

$$\lim_{x\to-\infty} \frac{\phi(x)}{x^n} = 0 \quad \leftrightarrow \quad \forall \epsilon > 0\ \exists N_2 < 0\ \forall x \left(x < N_2 \to \left| \frac{\phi(x)}{x^n} \right| < \epsilon \right)$$

Let $N_{max} = \max(|N_1|, |N_2|)$. Then $\forall \epsilon > 0\ \exists N > 0\ \forall x \left(|x| > N \to \left| \frac{\phi(x)}{x^n} \right| < \epsilon \right)$

Let $\epsilon = \frac{1}{2}$.
Let $N > 0$ s.t. $\forall x \left(|x| > N \to \left| \frac{\phi(x)}{x^n} \right| < \frac{1}{2} \right)$
Let $N_1 > 2\phi(0)$.
Let $M = \max(N, N_1)$.

Then $M > 2\phi(0)$ and $\forall x \left(|x| > M \to \left| \frac{\phi(x)}{x^n} \right| < \frac{1}{2} \right)$

Let x_1 s.t. $|x_1| > M$. Then $-\frac{1}{2} < \frac{\phi(x_1)}{x_1^n} < \frac{1}{2}$.

$$\frac{x_1^n}{2} < x_1^n + \phi(x_1) < \frac{3x_1^n}{2}$$

But $x_1^n > M > 2\phi(0)$ so $\frac{x_1^n}{2} > \phi(0)$.
Therefore

$$\phi(0) < x_1^n + \phi(x_1)$$

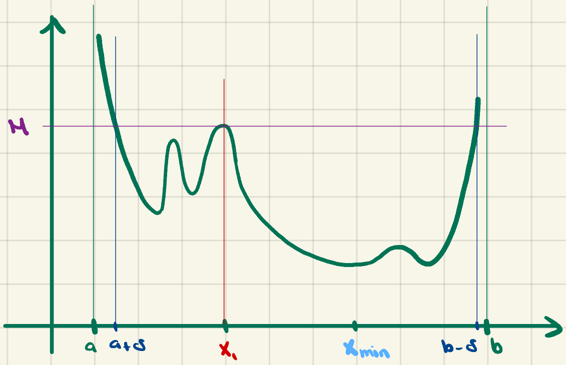
Therefore $\forall x\ \ x > M$ or $x < -M \ \to\ x^n + \phi(x) > \phi(0)$.

But $0 \in [-M, M]$, and $f(x) = x^n + \phi(x)$ is cont. on $[-M, M]$.
Therefore f takes on its minimum value on that interval, say at x_2.
Therefore $\forall x \in [-M, M]\ \ x^n + \phi(x) \geq x_2^n + \phi(x_2)$
In particular, $0^n + \phi(0) = \phi(0) \geq x_2^n + \phi(x_2)$
Therefore $\forall x \in \mathbb{R}\ \ x^n + \phi(x) > \phi(0) \geq x_2^n + \phi(x_2)$

16.

a)

f cont on (a,b)
$\lim_{x\to a^+} f(x) = \lim_{x\to b^-} f(x) = \infty$ $\rightarrow$ f has a minimum on all of (a,b)



Proof

Let $x_1 \in (a,b)$

Let $M = f(x_1)$

Since $\lim_{x\to a^+} f(x) = \infty$, $\exists \delta_1 > 0 \ \forall x \ 0 < x-a < \delta_1 \rightarrow f(x) > M$

Since $\lim_{x\to b^-} f(x) = \infty$, $\exists \delta_2 > 0 \ \forall x \ 0 < b-x < \delta_2 \rightarrow f(x) > M$

Let $\delta = \min(\delta_1, \delta_2)$. Then

$\forall x \ \ a < x < a+\delta$ or $b-\delta < x < b \ \rightarrow \ f(x) > M$

Consider the following intervals

$(a, a+\delta)$
$[a+\delta, b-\delta]$
$(b-\delta, b)$

Their union is (a,b). Since f is cont. on $[a+\delta, b-\delta]$,

$\exists x_{min} \in [a+\delta, b-\delta] \ \forall x \in [a+\delta, b-\delta] \ f(x) \geq f(x_{min})$

But $x_1 \in [a+\delta, b-\delta]$, so $\forall x$, $x \in (a, a+\delta)$ or $x \in (b-\delta, b) \ \rightarrow f(x) > f(x_1) \geq f(x_{min})$

Therefore $\forall x \in (a,b) \ f(x) \geq f(x_{min})$

b)

f cont on $(-\infty, \infty)$

$\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to -\infty} f(x) = \infty$ $\rightarrow$ f has a minimum on all of (a, b)

Proof

$\forall M > 0 \; \exists N > 0 \; \forall x \;\; x > N \rightarrow f(x) > M$

$\forall M > 0 \; \exists N < 0 \; \forall x \;\; x < N \rightarrow f(x) > M$

Let $x_1 \in \mathbb{R}$. Let $M = f(x_1)$.

$\exists N > 0 \;\; \forall x \;\; |x| > N \rightarrow f(x) > M$

Consider the intervals

$$(-\infty, -N), \; [-N, N], \; (N, \infty)$$

There is an $x_{min} \in [-N, N]$ s.t. $\forall x \in [-N, N]$, $f(x) \geq f(x_{min})$.

But $x_1 \in [-N, N]$, so $\forall x$, $x \in (-\infty, -N)$ or $x \in (N, \infty)$ $\rightarrow$ $x > f(x_1) \geq f(x_{min})$.
Therefore,

$$\forall x \in \mathbb{R}, \; f(x) \geq f(x_{min})$$

17.

f any polynomial. $\rightarrow$ $\exists y, \forall x\ |f(y)| \le |f(x)|$

Proof

Polynomial Fn f is polyn. if $\forall a_0, \dots, a_n \in \mathbb{R}$ s.t. $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$

highest power: degree of f

since f polynomial then f cont in $\mathbb{R}$.

by 6-10a, f cont at $a \rightarrow |f|$ cont at a

Thus $|f|$ contin. in $\mathbb{R}$.

Note that $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} a_n x^n$

In 5-16a we proved that: $\lim\limits_{x\to a} f(x) = \ell \rightarrow \lim\limits_{x\to a} |f(x)| = |\ell|$

Therefore

$$\lim_{x\to\infty} |f(x)| = |\lim_{x\to\infty} f(x)|$$

$$\lim_{x\to-\infty} |f(x)| = |\lim_{x\to-\infty} f(x)|$$

since $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = \pm\infty$ then

$$\lim_{x\to\infty} |f(x)| = \lim_{x\to-\infty} |f(x)| = \infty$$

we thus have the assumptions made in 16b:

$|f|$ cont on $\mathbb{R}$

$$\lim_{x\to\infty} |f(x)| = \lim_{x\to-\infty} |f(x)| = \infty$$

Therefore $|f|$ has a minimum on $(-\infty, \infty)$.

Ie $\exists y \in \mathbb{R}, \forall x\ |f(y)| \le |f(x)|$

Proof

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$$

$$= a_n x_n \left(1 + \frac{a_{n-1}}{a_n}\cdot\frac{1}{x} + \frac{a_{n-2}}{a_n}\frac{1}{x^2} + \dots + \frac{a_0}{x^n}\right)$$

$$\lim_{x\to\infty} f(x) = \left(\lim_{x\to\infty} a_n x_n\right)\left(\lim_{x\to\infty}\left(1 + \frac{a_{n-1}}{a_n}\cdot\frac{1}{x} + \frac{a_{n-2}}{a_n}\frac{1}{x^2} + \dots + \frac{a_0}{x^n}\right)\right)$$

$$= \lim_{x\to\infty} a_n x_n$$

* $\lim\limits_{x\to\infty} f(x) = \infty$, $\lim\limits_{x\to\infty} g(x) = c > 0$ $\rightarrow$ $\lim\limits_{x\to\infty} f(x)g(x) = \infty$

$\forall M_1 > 0\ \exists N_1 > 0\ \forall x\ x > N_1 \rightarrow f(x) > M_1$

$\forall \epsilon > 0\ \exists N_2 > 0\ \forall x\ x > N_2 \rightarrow |g(x) - c| < \epsilon$

$\forall M > 0$ let $M_1 > 0$ and $0 < \epsilon < c$ such that $M_1(c - \epsilon) = M$.

let $N = \max(N_1, N_2)$

Then

$\forall x\ x > N \rightarrow f(x)g(x) > M_1(c - \epsilon) = M > 0$

That is, we've shown

$\forall M > 0\ \exists N > 0\ \forall x\ x > N \rightarrow f(x)g(x) > M$

$$\lim_{x\to\infty} f(x)g(x) = \infty$$

Alternative Proof

$f(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0$

$$= b_n x^n \left(1 + \frac{b_{n-1}}{b_n}\cdot\frac{1}{x} + \frac{b_{n-2}}{b_n}\frac{1}{x^2} + \cdots + \frac{b_0}{b_n}\frac{1}{x^n}\right)$$

$$= b_n x^n \left(1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right)$$

note: $\left|\frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right| \le \frac{|a_{n-1}|}{|x|} + \cdots + \frac{|a_0|}{|x^n|}$

here we start the same steps used to prove Theorem 9.

Let $M = \max(1, 2n|a_{n-1}|, \ldots, 2n|a_0|)$

Then $\forall x$ s.t. $|x| > M$

$$|x|^k > |x|$$

$$\frac{|a_{n-k}|}{|x^k|} < \frac{|a_{n-k}|}{|x|} < \frac{|a_{n-k}|}{2n|a_{n-k}|} = \frac{1}{2n}$$

Thus,

$$\left|\frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right| < n \cdot \frac{1}{2n} = \frac{1}{2}$$

$$-\frac{1}{2} < \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n} < \frac{1}{2}$$

$$\frac{1}{2} < 1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}$$

$$\frac{bx^n}{2} < bx^n\left(1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right) = f(x)$$

Therefore

$$|f(x)| = \left|bx^n\left(1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right)\right| > \left|\frac{bx^n}{2}\right|$$

$|x| > M$ then this expr. for $|f(x)|$, ie $|f(x)|$ lower than $\frac{1}{2}$ fn of the highest order term $b_n x^n$.

choose some x_1 such that $|x_1| > M$ and also $|x_1^n| \ge \frac{2f(0)}{|b|}$

Then, $\forall x,\ |x| > x_1 \rightarrow |f(x)| > f(0)$

Thus, the minimum on $[-x_1, x_1]$ is the minimum on $\mathbb{R}$.

15.

f cont.

$\forall x\ f(x) > 0$ $\quad\longrightarrow\quad$ $\exists y\ \forall x\ f(y) \geq f(x)$

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = 0$

Proof

Def. of limit at ∞

$\forall \varepsilon > 0\ \exists N > 0\ \forall x\ |x| > N \to |f(x)| < \varepsilon$.

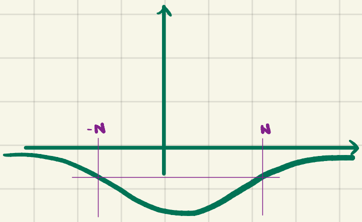
Let $x_1 \in \mathbb{R}$ and $f(x_1) > 0$.

Let $\varepsilon = \frac{f(x_1)}{2} < f(x_1)$. Then

since $\forall x\ f(x) > 0$

$\exists N > 0\ \forall x\ |x| > N \to 0 < f(x) < \frac{f(x_1)}{2} < f(x_1)$

Note that if $f(x)$ could be neg. one could have the situation



in which there is no max.

since f cont.

Also $\exists y \in [-N, N]$, $\forall x \in [-N, N]$, $f(x) \leq f(y)$

In particular $x_1 \in [-N, N]$, thus $f(y) \geq f(x_1)$.

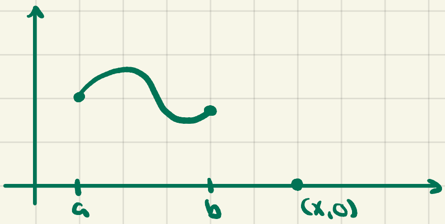
Hence $\forall x \in \mathbb{R}\ f(y) \geq f(x)$.

19.

a)

f cont. on $[a,b]$
$x \in \mathbb{R}$
$\rightarrow$ $\exists y \in [a,b], \forall z \in [a,b], D((z,f(z)),(x,0)) \geq D((y,f(y)),(x,0))$
ie there is a point on graph of f in $[a,b]$ that is closest to $(x,0)$.

Proof



Distance between two points $(x,0)$ and $(z,f(z))$

$D(z) = \sqrt{(x-z)^2 + f^2(z)}$, $z \in [a,b]$

$D(z)$ cont. in $[a,b]$

Proof

$(x-z)^2$ and $f^2(z)$ cont. in $[a,b]$ $\rightarrow$ $(x-z)^2 + f^2(z)$ cont. in $[a,b]$

Therefore $\lim_{z \to y} [(x-z)^2 + f^2(z)] = (x-y)^2 + f^2(y) \geq 0$, $y \in [a,b]$

$g(x) = \sqrt{x}$ cont. in $[0,\infty)$.

Therefore, $\forall y \in [a,b]$, $\lim_{z \to y} [(x-z)^2 + f^2(z)] = l \geq 0$, and g cont. at l.

Therefore by 6-12a, $\lim_{z \to y} g\left(\sqrt{(x-z)^2 + f^2(z)}\right) = g\left(\lim_{z \to y} \sqrt{(x-z)^2 + f^2(z)}\right) = \sqrt{(x-y)^2 + f^2(y)}$

ie, $\lim_{z \to y} D(z) = D(y)$.

Therefore $D(z)$ cont. in $[a,b]$.

Therefore, D takes on minimum value in $[a,b]$.

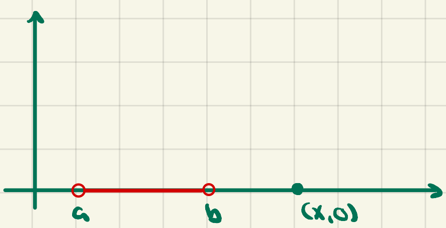
Ie

$\forall x \in \mathbb{R}\ \exists z \in [a,b]\ \forall y \in [a,b]\ D(z) \leq D(y)$

---

b)

f cont. on (a,b)
$x \in \mathbb{R}$



$D(z) = \sqrt{(x-z)^2} = |z-x|$ cont. in (a,b).

There is no minimum value of $D(z)$ in (a,b).

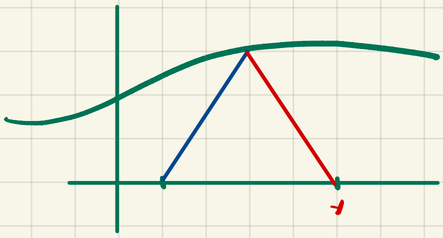
$\forall z \in [a,b], \exists z_1 \in (z,b)$ and $D(z_1) < D(z)$.

c)

f cont. on $\mathbb{R}$
$x \in \mathbb{R}$ $\rightarrow$ $\exists y \in \mathbb{R}, \forall z \in \mathbb{R}, D((z, f(z)), (x,0)) \geq D((y, f(y)), (x,0))$

ie there is a point $(y, f(y))$ on graph of f in $\mathbb{R}$ that is closest to $(x,0)$.

Proof



$D(z) = \sqrt{(x-z)^2 + f^2(z)}$, $z \in \mathbb{R}$

$D(z)$ cont. in $\mathbb{R}$.

$\lim_{z \to \infty} D(z) = \lim_{z \to -\infty} D(z) = \infty$

By T-16b, D has a minimum on $(-\infty, \infty) = \mathbb{R}$.

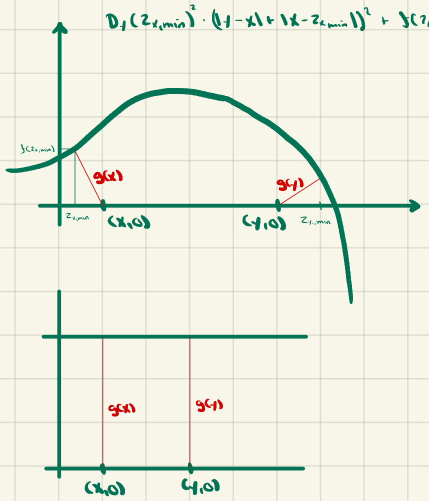
---

d)

f cont. on $\mathbb{R}$
$x \in \mathbb{R}$ $\rightarrow$ $g(y) \leq g(x) + |x-y|$
$g(x) = \min_z \sqrt{(x-z)^2 + f(z)^2}$ $\quad$ g cont.

$g(y) = D_y(z_{y,min}) \leq D_y(z_{x,min})$

$D_y(z_{x,min})^2 = (|y-x| + |x - z_{x,min}|)^2 + f(z_{x,min})^2$



Proof

Let $x_1 \in \mathbb{R}$. We know that $\exists z_{x_1,min} \in \mathbb{R}, \forall y \in \mathbb{R}, D_{x_1}(z_{x_1,min}) \leq D_{x_1}(y)$, where $D_{x_1}(z) = \sqrt{(x_1 - z)^2 + f^2(z)}$.
Then $g(x_1) = D_{x_1}(z_{x_1,min})$

$\forall y_1 \in \mathbb{R}, \exists z_{y_1,min} \in \mathbb{R}, \forall y \in \mathbb{R}$ $D_{y_1}(z_{y_1,min}) \leq D_{y_1}(y)$, where $D_{y_1}(z) = \sqrt{(y_1 - z)^2 + f^2(z)}$.
Then $g(y_1) = D_{y_1}(z_{y_1,min})$

Therefore since $z_{x_1,min} \in \mathbb{R}$, we have $g(y_1) = D_{y_1}(z_{y_1,min}) \leq D_{y_1}(z_{x_1,min})$ $\quad$ (1)

Consider the points $(x_1, 0)$, $(y_1, 0)$, and $(z_{x_1,min}, f(z_{x_1,min}))$.
By the triangle inequality, we have:

$$\sqrt{(z_{x_1,min} - y_1)^2 + f^2(z_{x_1,min})} \leq \sqrt{(z_{x_1,min} - x_1)^2 + f^2(z_{x_1,min})} + \sqrt{(x_1 - y_1)^2}$$

$$D_{y_1}(z_{x_1,min}) \leq D_{x_1}(z_{x_1,min}) + |x - y| = g(x_1) + |x - y| \quad (2)$$

by (1) and (2), $g(y_1) \leq g(x_1) + |x - y|$.

Hence $\forall y \forall x$ $g(y) \leq g(x) + |x - y|$.

As for continuity of g

$\forall y \forall x$ $g(y) - g(x) \leq |x - y|$

Therefore we have
$g(y) - g(x) \leq |x-y| \rightarrow -|x-y| \leq g(x) - g(y)$
$g(x) - g(y) \leq |x - y|$

Hence $|g(y) - g(x)| \leq |x - y|$

$\forall \varepsilon > 0 \ \exists \delta = \varepsilon$
$|x - a| < \delta \rightarrow |g(x) - g(a)| < \varepsilon$

Therefore

$\lim_{y \to a} g(y) = g(a)$

$\rightarrow g$ cont. on $\mathbb{R}$.

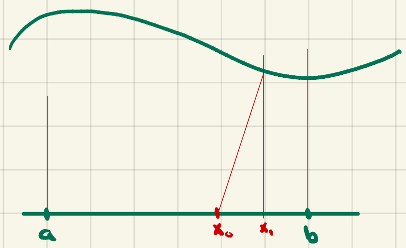
e)

$\exists x_0, x_1$ s.t

$x_0, x_1 \in [a,b]$

$\forall x_0', x_1'$ in $[a,b]$ $\longrightarrow$ $D((x_0,0),(x_1,f(x_1))) \leq D((x_0',0),(x_1',f(x_1')))$

in other words, there is a minimum distance from $[a,b]$ on the x axis to the graph of f.



Proof

In d) we showed that $g(x) = \min_z D_x(z)$ is continuous in $\mathbb{R}$.

Therefore it is cont. in $[a,b]$. Hence it takes on a min in $[a,b]$.

20.

a)

f cont. on $[0,1]$

$f(0)=f(1)$ $\quad \rightarrow \exists x,\ f(x)=f(x+\frac{1}{n})$

$n \in \mathbb{N}$

Proof

Let $n \in \mathbb{N}$.

let $g(x)=f(x)-f(x+\frac{1}{n})$

Assume $\forall x \in \mathbb{R},\ g(x) \neq 0$

Then either $\forall x\ f(x) > f(x+1/n)$ or $\forall x\ f(x) < f(x+1/n)$

Case 1: $\forall x\ f(x) > f(x+1/n)$

$f(0) > f(1/n) > f(2/n) > \cdots > f(1)$

$\perp$

Case 2: $\forall x\ f(x) < f(x+1/n)$

$f(0) < f(1/n) < f(2/n) < \cdots < f(1)$

$\perp$

Therefore $\perp$.

Therefore $\exists x\ g(x)=0$.

Ie $\exists x\ f(x)=f(x+1/n)$

Therefore $\forall n \in \mathbb{N}\ \exists x \in \mathbb{R}\ f(x)=f(x+1/n)$

b)

$0<a<1$

$\forall n \in \mathbb{N},\ a \neq 1/n$

Find f cont. on $[0,1]$, $f(0)=f(1)$, but does not satisfy $f(x)=f(x+a)$ for any x.

$$\frac{1}{n+1} < a < \frac{1}{n} \qquad (1)$$

Define f on $[0,a]$, subject only to following conditions

$f(0)=0$

$f(a)>0$

$f(1-na)=-nf(a)$

Because of (1),

$$\frac{n}{n+1} < na < 1$$

$$1 < -na < -\frac{n}{n+1}$$

$$0 < 2 < 1-na < \frac{1}{n+1} < a$$

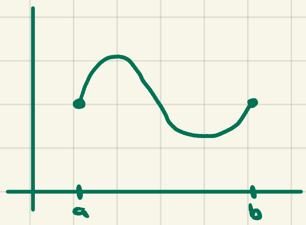
we see that the numbers 0, $1-na$, and a are distinct. Therefore we can define f as above.

Then define f on $[ka,(k+1)a]$ by $f(ka+x)=f(x)+ka$.

21.

a) Prove:

$\nexists f$ s.t. f defined on $\mathbb{R}$, f cont., and f takes on every value twice.



Hint:

Given f continuous on $[a,b]$, if f takes each value exactly twice then

if $a<b$ and $f(a)=f(b)$ then either

1) $\forall x \in (a,b)$ $f(x)>f(a)$ and $\forall x$, $x<a$ or $x>b \to f(x)<f(a)$

or

2) $\forall x \in (a,b)$ $f(x)<f(a)$ and $\forall x$, $x<a$ or $x>b \to f(x)>f(a)$

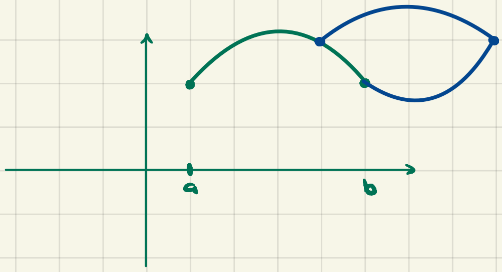
Proof of Hint

$f(a)=f(b)$, so f takes on the value $f(a)$ twice.

If for some $x,y \in (a,b)$ we have $f(x)<f(a)<f(y)$ or $f(y)<f(a)<f(x)$, then $\exists z \in (a,b)$ s.t. $f(z)=f(a)$.
But then f takes on value $f(a)$ more than twice.

Therefore $\forall x \in [a,b]$ either $f(x)>f(a)$ or $f(x)<f(a)$.

Note also that
$f(x)>f(a)$ in (a,b) then $f(x)<f(a)$ for $x<a$ or $x>b$,
and $f(x)<f(a)$ in (a,b) then $f(x)>f(a)$ for $x<a$ or $x>b$



Main Proof

Assume f takes on every value exactly twice.

Let $f(a)=f(b)$.

Case 1: $\forall x \in (a,b)$ $f(x)>f(a)$

Then f has a max in $\mathbb{R}$. Therefore f doesn't take on values bigger than the max. $\perp$

Case 2: $\forall x \in (a,b)$ $f(x)<f(a)$

Then f has a min in $\mathbb{R}$. Therefore f doesn't take on values smaller than the min. $\perp$

Therefore f does not take on every value twice.

b) $\nexists f$ s.t. f cont. and takes on each value either 0 or 2 times.

Proof:

Assume for proof by contrad. that $\exists f$ s.t. f cont. and takes on each value either 0 or 2 times.

Let $x_1 \in \mathbb{R}$. Then there is a value $f(x_1)$, and so $\exists x_2 \in \mathbb{R}$ s.t. $f(x_2) = f(x_1)$.

Since f cont. in $\mathbb{R}$ it is cont. in $[x_1, x_2]$.

As shown in a), either $\forall x, x \in [x_1, x_2] \to f(x) > f(x_1)$ or $\forall x, x \in [x_1, x_2] \to f(x) < f(x_1)$.

Case 1: $\forall x, x \in (x_1, x_2) \to f(x) > f(x_1)$

Then $\forall x, x < x_1$ or $x > x_2 \to f(x) < f(x_1)$

Also, $\exists x_{max} \in [x_1, x_2]$ s.t. $\forall x \in [x_1, x_2]$ $f(x) \leq f(x_{max})$

Assume $\exists x_{max_2} \in [x_1, x_2]$ s.t. $f(x_{max_2}) = f(x_{max})$.

Then either: $\forall x, x \in [x_{max}, x_{max_2}] \to f(x) > f(x_{max})$

or $\forall x, x \in [x_{max}, x_{max_2}] \to f(x) < f(x_{max})$

Either case leads to $\perp$.

Thus $\nexists x_{max_2} \in [x_1, x_2]$ s.t. $f(x_{max_2}) = f(x_{max})$.

That is, there is only one value x_{max} s.t. $\forall x \in [x_1, x_2], f(x) \leq f(x_{max})$.

In fact, $\forall x \in \mathbb{R}, f(x) \leq f(x_{max})$.

Thus f takes on value $f(x_{max})$ only once.

$\perp$.

Case 2: $\forall x, x \in (x_1, x_2) \to f(x) < f(x_1)$

Analogous to case 1 using min. in $[x_1, x_2]$

$\perp$

Therefore $\perp$ in both possible cases

Therefore, $\nexists f$ s.t. f cont. and takes on each value either 0 or 2 times.

Then $\forall x \in \mathbb{R}$ s.t. $x < a$ or $x > b$, $f(x) < f(a)$.

Note that if two values $x_1, x_2 \in \mathbb{R}$ are both smaller than a, then $f(x_1) \neq f(x_2)$. Same result is true if x_1, x_2 are both larger than b.

Thus, for $x_1, x_2 \in \mathbb{R}$ s.t. $f(x_1) = f(x_2) < f(a) = f(b)$, it must be that $x_1 < a$ and $x_2 > b$ or $x_2 < a$ and $x_1 > b$.

Proof

Assume $x_1, x_2 \in \mathbb{R}$ s.t. $x_1 < x_2 < a$ and $f(x_1) = f(x_2)$.

Case 1.1 $\forall x \in (x_1, x_2)$ $f(x) > f(x_1)$

$\forall x \in [x_2, a]$, f takes on all values in $[f(x_2), f(a)]$.

Let $x_3 \in (x_1, x_2)$, $f(x_3) > f(x_1)$

f takes on value $f(x_3)$ twice in (x_1, x_2) and an additional time in (x_2, a). $\perp$

Case 1.2 $\forall x \in (x_1, x_2)$ $f(x) < f(x_1)$

Then $\forall x, x > x_2$ or $x < x_1 \to f(x) > f(x_1)$

Let m be the min of f on $[x_1, x_2]$.

Then $\forall x$ $f(x) \geq m$.

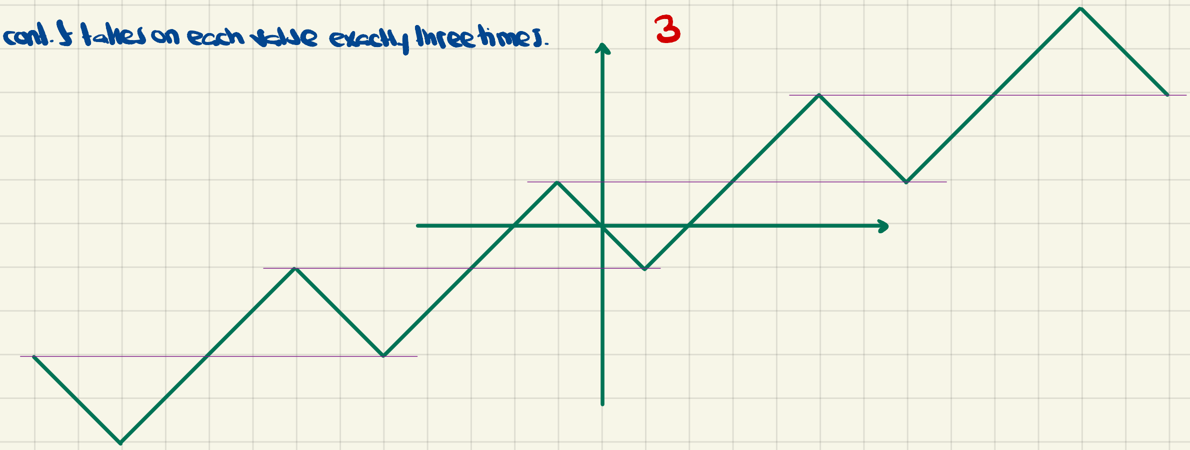
Ie f doesn't take on values smaller than m.

$\perp$

Hence, $x_1 < x_2 < a \to f(x_1) \neq f(x_2)$.

c) cont. f takes on each value exactly three times.

3



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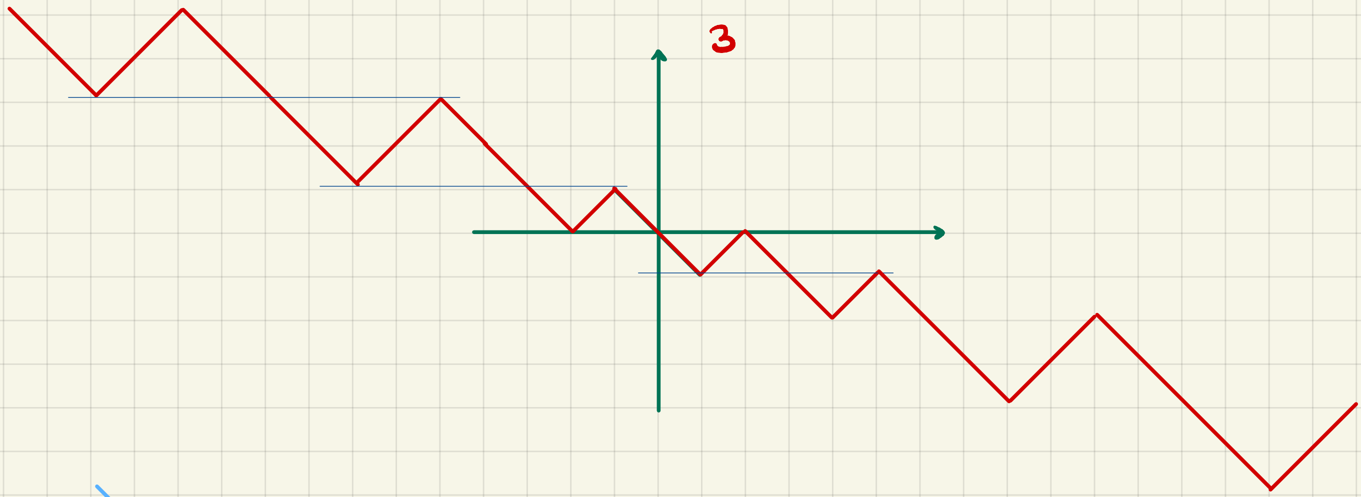

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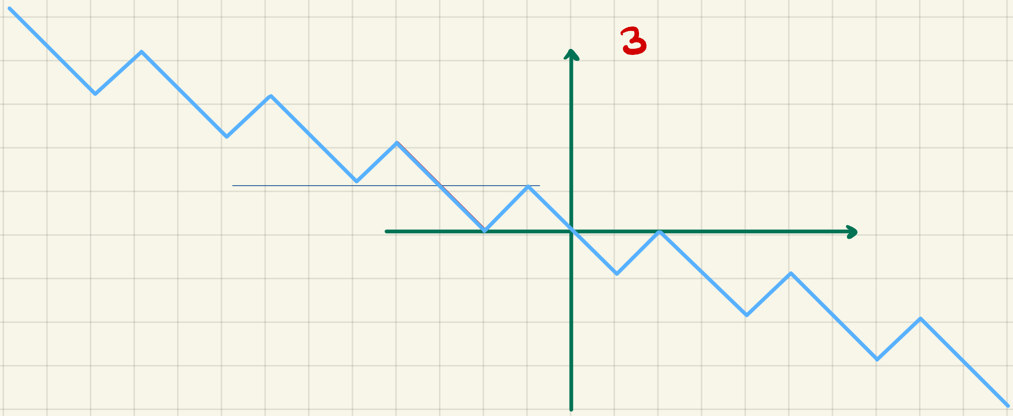
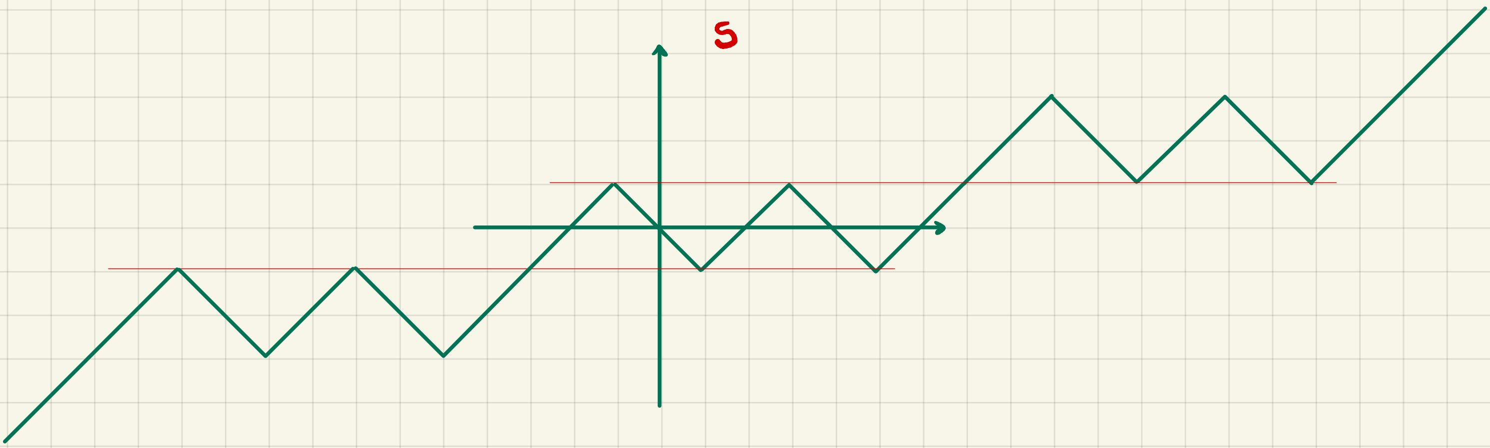


5



d) n even $\rightarrow$ $\nexists f$ s.t. f takes on every value n times.

Assume n is even.

In a) we proved the case $n=2$.

Let $n=4$. Let x_1, x_2, x_3, x_4 s.t. $f(x_1)=f(x_2)=f(x_3)=f(x_4)$

Note that $\forall x \in \mathbb{R},\ x \neq x_1 \wedge x \neq x_2 \wedge x \neq x_3 \wedge x \neq x_4 \rightarrow f(x) \neq f(x_1)$.

Let $x_1 < y_1 < y_2 < x_2$

Assume $f(y_1) < f(x_1) < f(y_2)$

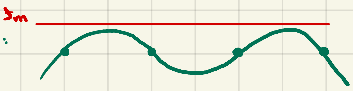
Then $\exists y \in (y_1, y_2),\ f(y) = f(x_1)$. $\perp$.

* we also obtain $\perp$ if we assume $f(y_1) > f(x_1) > f(y_2)$.

Therefore $\forall y,\ y \in [x_1, x_2] \rightarrow f(y) > f(x_1)$ or $\forall y\ y \in [x_1, x_2] \rightarrow f(y) < f(x_1)$

For each of $(x_1,x_2), (x_2,x_3), (x_3,x_4)$, if f takes on a value in an interval it takes on the value twice in that interval except at maximum or minimum. Therefore, relative to $f(x_1)$, the sign of f can be the same in at most two of such intervals. Hence we have two possible cases

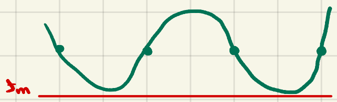
Case 1:



Note that $x < x_1$ or $x > x_4 \rightarrow f(x) < f(x_1)$.

Therefore f doesn't take on values above f_m. $\perp$.

Case 2:



Note that $x < x_1$ or $x > x_4 \rightarrow f(x) > f(x_1)$.

Therefore, f doesn't take on values below f_m. $\perp$.

n even $\rightarrow$ $\nexists f$ s.t. f takes on every value 4 times.

general case

x_1 x_2 x_3 (...) x_n

$f(x_1) = f(x_2) = \dots = f(x_n)$

For any interval (x_i, x_j), $i,j \in \{n_1 \in \mathbb{N} : n_1 \leq n\}$, $\forall x,\ x \in (x_i, x_j) \rightarrow f(x) > f(x_i)$ or $\forall x,\ x \in (x_i, x_j) \rightarrow f(x) < f(x_i)$

For any value f takes on in (x_i, x_j) it takes it on twice, except at maximum or minimum (and these must be max or min because $\forall x,\ x \in (x_i, x_j) \rightarrow f(x) \neq f(x_i) = f(x_j)$, so either the max on $[x_i, x_j]$ is the max on (x_i, x_j) or the min on $[x_i, x_j]$ is the min on (x_i, x_j).

Therefore $\frac{n}{2}$ intervals have $f > f(x_i)$, $n - 1 - \frac{n}{2}$ have $f < f(x_i)$, or vice-versa.

$\forall x\ x < x_i$ or $x > x_j \rightarrow f(x) < f(x_i)$ in the first case, $f(x) > f(x_i)$ in the second.

In the first case, f is bounded above, in the second case it's bounded below. $\perp$ both cases.

n even $\rightarrow$ $\nexists f$ s.t. f takes on every value n times.