

examples

$$f(x) = x^2$$

$$f(x+1) = x^2 + 2x + 1 = f(x) + 2x + 1$$

$$g(y) = \frac{y^3 + 3y + 5}{y^2 + 1}$$

$$h(c) = \frac{c^3 + 3c + 5}{c^2 - 1} \quad c \neq \pm 1$$

$$g(x) = h(x) \Rightarrow (x^3 + 3x + 5)(x^2 - 1) = (x^3 + 3x + 5)(x^2 + 1)$$

$$x^2 - 1 \neq x^2 + 1 \Rightarrow -1 \neq 1 \text{ always}$$

$$\Rightarrow x^3 + 3x + 5 = 0$$

$$r(x) = x^2 \quad -\pi \leq x \leq \pi/3$$

$$r(x+1) = r(x) + 2x + 1 \quad -\pi \leq x+1 \leq \pi/3 \\ -\pi \leq x \leq \pi/3 - 1$$

$$s(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{Q}' \end{cases}$$

$$s(x+y), y \in \mathbb{Q}$$

$$y \in \mathbb{Q} \Rightarrow y = \frac{p}{q} \quad p, q \in \mathbb{Z}$$

$$x \in \mathbb{Q} \Rightarrow x = \frac{m}{n}$$

$$x+y = \frac{np+mq}{nq}$$

$$a, b \in \mathbb{Z}$$

$$\text{Prove } a, b \in \mathbb{Z}$$

strategy: Prove $\forall b \in \mathbb{Z} \quad \forall b \in \mathbb{Z}$, induction on B

Prove $a, b \in \mathbb{Z}$, induction on a

$$\text{i) } A = \{n \in \mathbb{Z} \mid 1 \cdot n \in \mathbb{Z}\}$$

$$n=1 \Rightarrow 1 \cdot n = n \in \mathbb{Z} \Rightarrow 1 \in A$$

assume $k \in A \Rightarrow 1 \cdot k \in \mathbb{Z}$

$$(1+k) \cdot 1 = 1 + k \cdot 1 \in \mathbb{Z} \Rightarrow 1+k \in A$$

$$n=-1 \Rightarrow 1 \cdot (-1) = -1 \in \mathbb{Z} \Rightarrow -1 \in A$$

$$n \in A \Rightarrow 1 \cdot n \in \mathbb{Z}$$

$$1 \cdot (n-1) = 1 \cdot n - 1 \in \mathbb{Z} \Rightarrow n-1 \in A$$

$$\Rightarrow A = \mathbb{Z}$$

$$A = \{n \in \mathbb{Z} \mid n \cdot m \in \mathbb{Z}, \forall m \in \mathbb{Z}\}$$

$$n=1 \Rightarrow 1 \cdot m \in \mathbb{Z} \text{ by (i)} \Rightarrow 1 \in A$$

$$k \in A \Rightarrow k \cdot m \in \mathbb{Z} \quad \forall m$$

$$(k+1)m = km + m \in \mathbb{Z}$$

backwards induction for $n < 0 \Rightarrow A = \mathbb{Z}$

i.e. $n \cdot m \in \mathbb{Z} \quad \forall m$ is true for $\forall n \in \mathbb{Z}$

$$\Rightarrow x+y = \frac{np+mq}{nq} \in \mathbb{Q}$$

$$x \in \mathbb{Q}'$$

$$y \in \mathbb{Q}$$

Assume $x+y \in \mathbb{Q}$

$$x + \frac{p}{q} = \frac{m}{n} \Rightarrow x = \frac{mq-pn}{nq} \in \mathbb{Q}$$

$$\Rightarrow x+y \in \mathbb{Q}'$$

$$\Rightarrow y \in \mathbb{Q}, s(x+y) \in \mathbb{Q} \text{ if } x \in \mathbb{Q} \\ \in \mathbb{Q}' \text{ if } x \in \mathbb{Q}'$$

$$\Rightarrow s(x+y) = s(x)$$

$$\Theta(x) = \begin{array}{ll} 5 & x=2 \\ 36/\pi & x=\pi \\ 28 & x=\pi^2/\pi, 36/\pi \\ 16 & x+2, \pi, \pi^2/\pi, 36/\pi \\ & x=a+b\sqrt{2} \quad a, b \in \mathbb{Q} \end{array}$$

$$\Theta(\pi^2/\pi) = 28 = \Theta(36/\pi)$$

$$\alpha_x(t) = t^2 + x \quad \forall t$$

$$\alpha_x(x) = x^3 + x = x(x^2 + 1) = x(f(x) + 1)$$

$$f(x) = \begin{cases} n & n \text{ 7's in decimal exp.} \\ -\pi & \infty \dots \end{cases}$$

$$f(1/3) = f(0.3333\dots) = 0$$

$$f(7/9) = -\pi$$

$$\begin{array}{r} 0.7777\dots \\ 9 \sqrt{7} \\ \underline{-63} \\ 70 \\ \underline{-63} \\ 70 \end{array}$$

$$\begin{array}{r} 38.3 \\ 6 \sqrt{230} \\ \underline{-18} \\ 50 \\ \underline{-48} \\ 20 \end{array}$$

Proof: ∞ 7's in decimal exp. of $7/9$

A = set of positions with 7 in decimal exp.

1st position remainder has to be divisible by $70 \cdot 10^{-1}$

Result is 7 remainder 7

i.e. we can divide the $70 \cdot 10^{-1}$ into nine parts of $7 \cdot 10^{-1}$

and $7 \cdot 10^{-1}$ remain

2nd position: have to divide the remainder $7 \cdot 10^{-1} = 70 \cdot 10^{-2}$

The division steps are

A = set of pos for which result is the whole part of $70/9 = 7$

n=1, position 1 $\Rightarrow 9/7 = 7 \cdot 9 + 7 \Rightarrow$ remainder 7 = $70 \cdot 10^{-1}$

result is $\frac{70 \cdot 10^{-1}}{9 \cdot 10^{-1}} = \frac{70}{9}$. We need to divide the remainder

$70 \cdot 10^{-1}$ among 9, so each whole part is $9 \cdot 10^{-1}$

assume h $\in A \Rightarrow$ result is whole part of $70/9 = 7$.

the remainder of hth pos is 7.

result of (h+1)th pos is the division $\frac{70 \cdot 10^{-(h+1)}}{9 \cdot 10^{-(h+1)}} = \frac{70}{9}$, whole part 7

$\Rightarrow A = N \Rightarrow$ all positions are 7.

$$f(s(a)) = \begin{cases} f(0) & a \in Q^c \\ f(1) & a \in Q \end{cases}$$

$$= \begin{cases} 0 & a \in Q^c \\ 1 & a \in Q \end{cases}$$

$$= s(a)$$

Problems

$$1. f(x) = \frac{1}{1+x} \quad D_f = \{x : x \neq -1\}$$

$$\text{ii) } f(f(x)) = \frac{1}{1 + \frac{1}{1+x}} = \frac{1}{\frac{2+x}{1+x}} = \frac{1+x}{2+x}, \quad x \neq -2$$

$$\text{iii) } f(\frac{1}{x}) = \frac{1}{1 + \frac{1}{x}} = \frac{1}{\frac{x+1}{x}} = \frac{x}{1+x} \quad x \neq -1, x \neq 0$$

$$\text{iv) } f(cx) = \frac{1}{1+cx} \quad 1+cx \neq 0 \Rightarrow x \neq -\frac{1}{c}$$

$$\text{v) } f(x+1) = \frac{1}{1+x+1}, \quad x+1 \neq -1$$

$$\text{vi) } f(x) + f(y) = \frac{1}{1+x} + \frac{1}{1+y} = \frac{2+x+y}{(1+x)(1+y)} \quad x \neq -1, y \neq -1$$

$$\text{vii) } f(cx) = \frac{1}{1+cx} = \frac{1}{1+x}$$

$$c=0 \quad \frac{1}{1} = \frac{1}{1+0} \quad x=0$$

$$c=1 \quad \frac{1}{1+x} = \frac{1}{1+x} \quad \forall x$$

$$c=2 \quad \frac{1}{1+2x} = \frac{1}{1+x} \Rightarrow x=2x \Rightarrow x=0$$

$$c=n \quad \frac{1}{1+nx} = \frac{1}{1+x} \Rightarrow nx=x \Rightarrow x=0$$

$$c \in \mathbb{R} \Rightarrow f(cx) = f(x) \text{ for } x \neq 0$$

$$\text{viii) } f(cx) = f(x) \text{ for two different } x$$

$$c=1 \Rightarrow f(cx) = f(x) \quad \forall x$$

$$2. g(x) = x^2$$

$$h(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{Q}^c \end{cases}$$

$$\text{i) } h(y) \leq y$$

$$y \in \mathbb{Q} \Rightarrow h(y) = 0 \leq y \quad \text{if } y \text{ is positive rational, i.e. } \{y : y \in \mathbb{Q}, y \geq 0\}$$

$$y \in \mathbb{Q}^c \Rightarrow h(y) = 1 \leq y \quad \text{if } y \text{ is irrational} \geq 1, \text{ i.e. } \{y : y \in \mathbb{Q}^c, y \geq 1\}$$

$$\text{ii) } h(y) \leq g(y)$$

$$y \in \mathbb{Q} \Rightarrow h(y) = 0 \leq y^2 \Rightarrow y \in \mathbb{R}$$

$$y \in \mathbb{Q}^c \Rightarrow h(y) = 1 \leq y^2 \Rightarrow y \geq 1 \text{ or } y \leq -1 \text{ i.e. } |y| \geq 1$$

$$\Rightarrow h(y) \leq g(y) \text{ for } \{y : y \in \mathbb{Q} \text{ and } -1 \leq y \leq 1, \text{ or } \forall y : |y| \geq 1\}$$

$$\text{iv)} g(w) \leq w$$

$$w^2 \leq w$$

$$w(w-1) \leq 0$$

$$\begin{array}{c|ccc} & - & + & + \\ \hline - & - & + & + \\ + & 0 & - & + \\ \hline & + & - & + \end{array}$$

$$\{w : 0 \leq w \leq 1\}$$

$$\text{v)} g(g(\epsilon)) = g(\epsilon)$$

$$g(\epsilon)^2 = (\epsilon^2)^2 = \epsilon^2 \Rightarrow \epsilon^4 = \epsilon^2$$

$$\epsilon^2(\epsilon^2 - 1) = 0$$

$$\epsilon = 0 \text{ or } \epsilon^2 = 1 \Rightarrow \epsilon = \pm 1$$

$$\text{3. ii) } f(x) = \sqrt{1-x^2}$$

$$1-x^2 \leq 0 \Rightarrow f(x) \text{ isn't a real number}$$

$$x^2 \geq 1 \Rightarrow x \geq 1 \text{ or } x \leq -1 \Rightarrow f(x) \text{ isn't defined}$$

$$\text{iii) } f(x) = \sqrt{1-\sqrt{1-x^2}}$$

$$1-x^2 \geq 0 \Rightarrow x^2 \leq 1 \Rightarrow -1 \leq x \leq 1$$

$$1-\sqrt{1-x^2} \geq 0 \Rightarrow \sqrt{1-x^2} \leq 1 \Rightarrow 0 \leq 1-x^2 \leq 1 \Rightarrow -1 \leq -x^2 \leq 0$$

$$1 \geq x^2 \geq 0$$

$$-1 \leq x \leq 1$$

$$\text{Domain} = \{x : -1 \leq x \leq 1\}$$

$$\text{iv) } f(x) = \frac{1}{x-1} + \frac{1}{x-2} \quad D = \{x : x \neq 1, x \neq 2\}$$

$$\text{v) } f(x) = \sqrt{1-x^2} - \sqrt{x^2-1} \quad 1-x^2 \geq 0 \Rightarrow x^2 \leq 1 \Rightarrow -1 \leq x \leq 1$$

$$x^2-1 \geq 0 \Rightarrow x^2 \geq 1 \Rightarrow x \geq 1 \text{ or } x \leq -1$$

$$D = \{-1, 1\}$$

$$\text{vi) } f(x) = \sqrt{1-x} + \sqrt{x-2} \quad 1-x \geq 0 \Rightarrow x \leq 1$$

$$x-2 \geq 0 \Rightarrow x \geq 2$$

$$D = \emptyset$$

$$\text{4. } s(x) = x^2 \quad p(x) = 2^x \quad sc(x) = \sin(x)$$

$$\text{i) } (s \circ p)(y) = s(p(y)) = s(2^y) = (2^y)^2 = 4^y$$

$$\text{ii) } (s \circ s)(y) = s(s(y)) = s(\sin y) = \sin^2(y)$$

$$\text{iii) } (s \circ p \circ s)(t) = (s \circ p)(s(t)) = s(p(s(t))) = s(p(\sin t)) = s(2^{\sin t}) = \sin(2^t) = (2^{\sin t})^2 + \sin(2^t) = 4^{\sin t} + \sin(2^t)$$

$$\text{iv) } s(t^3) = \sin(t^3)$$

$$5. S(x) = x^2 \quad P(x) = 2^x \quad S(x) = \sin(x)$$

i) $f(x) = 2^{\sin x}$ - Pos

$$\text{iii) } f(x) = \sin(2^x) = \text{soP}$$

$$\text{iii) } f(x) = \sin(x^2) = \text{ so s}$$

$$\text{iv) } f(x) = \sin^2 x = -\sin x$$

$$y) f(t) = z^2 = \text{Pop}$$

$$y_1) f(y) = \sin(z^0 + z^{0^2}) = \sin(p + p_0 s)$$

$$\text{viii) } f(y) = \sin(\sin(\sin(\sin(2^{2^{\frac{y}{\ln 2}}})})) = \text{SoSoSoPoPoPoS}$$

$$\text{viii) } f(a) = 2^{\sin^2 a} + \sin(a^2) + 2^{\sin(a^2 + \sin a)} \\ = P_0 S_0 S_0 + S_0 S_0 + P_0 S_0 (S_0 + S_0)$$

6. a) x_1, \dots, x_n distinct numbers

Find }_i polynomial of degree n-1

1 at x_i
0 at x_j

x₁ x₂

$$S_1 = \frac{x - x_2}{x_1 - x_2} = \frac{1}{x_1 - x_2} x - \frac{x_2}{x_1 - x_2}$$

$$x_2 = \frac{x - x_1}{x_2 - x_1}$$

$$f_{x_i} = \frac{x - x_3}{x_1 - x_3} \Rightarrow \begin{cases} f(x_i) = 1 \\ f(x_i) = 0 \end{cases}$$

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$$f_1 = \frac{(x-x_1)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{x^2 - x(x_3+x_2) + x_2x_3}{(x_1-x_2)(x_1-x_3)}$$

$$= \frac{1}{\pi(x_1-x_3)} x^2 - \frac{(x_2+x_3)}{\pi(x_1-x_3)} x + \frac{x_2 x_3}{\pi(x_1-x_3)}$$

$$f_1(t_3) = f_2(t_3) = 0$$

$$t(t_0) = 1$$

x_{t+1}, x_0

$$f_{i,j} = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}$$

b) a_1, \dots, a_n given

Find } degree n-1, } $f(x_i) = a_i$

$$f_i = \frac{a_i \pi(x - x_i)}{\prod_{j \neq i} (x_i - x_j)}$$

Now we want a single function f which can choose the values $s_i(x_i)$, i.e. $f(x_i) = s_i$.

j_i is a_i on one input and zero on all others.

The sum of S_i 's will be a_j at each x_i .

$$f = \sum_{i=1}^n f_i = \sum_{i=1}^n \left[\frac{a_i \prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)} \right]$$

7) a) Prove

\forall polynomial f \exists polyn. g and $b \in \mathbb{R}$
 $\forall a \in \mathbb{R}$ s.t. $f(x) = (x-a)g(x) + b \forall x$

example:

$$f(x) = x^3 - 3x + 1$$

$$x-a = x-1$$

$$g(x) = \frac{f(x)}{x-a} = \frac{x^3 - 3x + 1}{x-1}$$

$$\begin{array}{r} x^2 + x - 2 \\ x-1 \left[\begin{array}{r} x^3 & -3x + 1 \\ x^3 - x^2 \\ \hline x^2 - 3x + 1 \\ x^2 - x \\ \hline -2x + 1 \\ -2x + 2 \\ \hline -1 \end{array} \right] \end{array}$$

$$(x^2 + x - 2)(x-1) - 1 = f(x)$$

$$= g(x)(x-a) + b$$

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = \sum_{i=0}^n a_i x^i$$

example

$$f(x) = a_2 x^2 + a_1 x + a_0$$

$$\frac{f(x)}{x-a} \Rightarrow x-a \left[\begin{array}{r} a_2 x + (a_1 + a_0) \\ a_2 x^2 + a_1 x + a_0 \\ a_2 x^2 - a x \\ \hline x(a_1 + a_0) - a_0 \\ x(a_1 + a_0) - a(a_1 + a_0) \\ \hline a_0 + a(a_1 + a_0) \end{array} \right]$$

$$f(x) = (x-a)(a_2 x + a_1 + a_0) + (a_0 + a(a_1 + a_0))$$

$$= (x-a)g(x) + b$$

general case

$$A = \{n : f_n(x) = \sum_{i=0}^n a_i x^i \text{ can be written as } (x-a)g(x) + b\}$$

$$n=1 \Rightarrow f_1(x) = a_1 x + a_0 = (x-a)g_1(x) + b$$

$$x-a \left[\begin{array}{r} a_1 x + a_0 \\ a_1 x - a a_1 \\ \hline a_0 + a a_1 \end{array} \right]$$

$$g_1(x) = a_1$$

$$(x-a) \in_1 + b$$

$$ax - a, a + b$$

$$-a, a + b - a$$

$$b = a_0 - a a_1$$

$$\Rightarrow f_1(x) = (x-a)a_1 + a_0 + a a_1$$

$$b = a_0 + a a_1$$

$$g(x) = a_1$$

$$\Rightarrow 1 \in A$$

$$\text{Assume } k \in A \Rightarrow f_k(x) = \sum_{i=0}^k a_i x^i = (x-a)g(x) + b$$

$$\text{let } h(x) = f_k(x) + a_{k+1} x^{k+1} = \sum_{i=0}^{k+1} a_i x^i$$

$$= (x-a)g(x) + b + a_{k+1} x^{k+1}$$

$$a_{k+1} x^{k+1} = x \cdot a_{k+1} x^k = x[(x-a)l(x) + b_1]$$

because $m(x) = a_{k+1} x^k$ is a k^{th} order polynomial.

$$= (x-a)l(x)x + b_1 x$$

$$\Rightarrow h(x) = (x-a)(g(x) + l(x)x) + b + b_1 x$$

But $b + b_1 x$ is a first degree polynomial

$$b + b_1 x = (x-a)n(x) + b_2 = n(x)x - a n(x) + b_2$$

$$\Rightarrow n(x) = b_2$$

$$\Rightarrow b + b_1 x = (x-a)b_2 + b + ab_2$$

$$b + b_1 x = (x-a)m(x) + b + ab_2$$

$$\Rightarrow h(x) = (x-a)(g(x) + l(x)x + b_2) + b + ab_2$$

$$\text{call } g_{k+1}(x) = g(x) + l(x)x + b_2$$

$$\Rightarrow h(x) = \sum_{i=0}^n a_i x^i = (x-a)g_{k+1}(x) + b + ab_2$$

$$\Rightarrow 1+k \in A$$

$$\Rightarrow A = \mathbb{N}$$

Alternative proof of 1st induction step

we have

$$\text{Assume } h \in A \Rightarrow f_h(x) = \sum_{i=0}^h a_i x^i = (x-a)g(x) + b$$

$f_{hn}(x) = f_h(x) + a_{hn}(x-a)^{hn}$ is hn^{th} degree polynomial.

$$f_{hn}(x) = a_{hn}(x-a)^{hn} - f_h(x) \text{ is } hn^{\text{th}} \text{ degree so}$$

$$f_{hn}(x) = a_{hn}(x-a)^{hn} = (x-a)g(x) + b$$

$$f_{hn}(x) = (x-a)(g(x) + a_{hn}(x-a)^h) + b$$

$$= (x-a)g_1(x) + b$$

$$g_1(x) = g(x) + a_{hn}(x-a)^h$$

$$\Rightarrow hn \in A$$

$$\Rightarrow A = \mathbb{N}$$

b) Prove $f(a) = 0 \Rightarrow f(x) = (x-a)g(x)$ for some polyn. $g(x)$

$$n=1 \Rightarrow f(x) = a_1 x + a_0$$

$$f(a) = a_1 a + a_0 = 0$$

$$\Rightarrow a_0 = -a_1 a$$

$$\Rightarrow f(x) = a_1 x - a_1 a = a_1(x-a)$$

assume $h \in A$

$$\Rightarrow f_h = \sum_{i=0}^h a_i x^i$$

$$f_h(a) = \sum_{i=0}^h a_i a^i = 0 \Rightarrow f(x) = (x-a)g(x)$$

$$f_{hn}(x) = f_h(x) + a_{hn}(x-a)^{hn}$$

$$= (x-a)g(x) + a_{hn}(x-a)^{hn}$$

$$= (x-a)(g(x) + a_{hn}(x-a)^h)$$

$h(x) = g(x) + a_{hn}(x-a)^h$ is h^{th} degree polyn.

$$\Rightarrow hn \in A$$

$$\Rightarrow A = \mathbb{N}$$

Alternatively, from a) we have

$$f(x) = (x-a)g(x) + b \Rightarrow 0 = f(a) = b$$

$$\Rightarrow f(x) = (x-a)g(x)$$

Note that

$$a_{hn}(x-a)^{hn}$$

$$= a_{hn} \sum_{i=0}^{hn} \binom{hn}{i} x^{hn-i} a^i$$

$$\text{so } f(x) + a_{hn}(x-a)^{hn}$$

$$= \sum_{i=0}^h a_i x^i + a_{hn} \sum_{i=0}^{hn} \binom{hn}{i} x^{hn-i} a^i$$

$$= \sum_{i=0}^h x^i a_i a^{h-i} + a_{hn} x^{hn} = \sum_{i=0}^{hn} a_i x^i, \text{ a general } hn^{\text{th}}$$

degree polynomial.

c) Prove f n degree polyn. $\Rightarrow f$ has at most n roots, i.e. numbers a with $f(a) = 0$

Assume $f(a_1) = 0$

$$\Rightarrow f(x) = (x-a_1)g_1(x) \text{ for some polynom. } g_1(x)$$

Note that $\deg(f(x)) = \deg(g_1(x)) \cdot 1$ because

$$g_1(x) = \sum_{i=0}^d a_i x^i \text{ has degree } d$$

$$x \cdot g_1(x) = \sum_{i=0}^d a_i x^{i+1} \text{ has degree } d+1, d+1 = n \Rightarrow d = n-1.$$

$$\Rightarrow \deg(g_1(x)) = n-1$$

If $g_1(x)$ has a root a_2 then $g_1(a_2) = (x-a_2)g_2(x)$ where

$g_2(x)$ is a degree $(n-2)$ polyn.

$$\Rightarrow f(x) = (x-a_1)(x-a_2)g_2(x)$$

We can do this procedure of decomposing $g_1(x)$ n times.

After the n^{th} time, $f(x) = (x-a_1) \cdot (x-a_2) \cdots (x-a_n) \cdot c$

$f(a) \neq 0$ for all $a \neq a_1, \dots, a_n \Rightarrow f$ does not have more

Note that it is possible that one of the $g_i(x)$ has no roots.

$f(x) = (x-a_1)(x-a_2) \cdots g_i(x)$ in that case.

d) show: for each $n \rightarrow \exists$ poly of degree n with n roots

$$f(x) = \prod_{i=1}^n (x-a_i), n^{\text{th}} \text{ degree polynomial.}$$

$\Rightarrow a_1, \dots, a_n$ are roots

assume n even.

$$f(x) = x^n + a_0, a_0 > 0$$

$\Rightarrow f(x) > 0 \forall x$ because $x^n > 0$

$$\Rightarrow x^n + a_0 = 0 \Rightarrow x^n = -a_0, \text{ impossible.}$$

$$\Rightarrow x^n + a_0 \neq 0.$$

assume n odd

$$f(x) = x^n + a_0$$

$$x^n + a_0 = 0 \Rightarrow x = \sqrt[n]{-a_0}$$

$$i) f(x) = \frac{ax+b}{cx+d}$$

For what a, b, c, d does $f(x)$ satisfy

$$f(f(x)) = x \quad \forall x$$

$$f(f(x)) = \frac{a(\frac{ax+b}{cx+d}) + b}{c(\frac{ax+b}{cx+d}) + d} = \frac{a(ax+b) + b(cx+d)}{c(ax+b) + d(cx+d)} = x$$

$$cx+d \neq 0$$

$$\frac{c(cx+b)}{cx+d} + d \neq 0 \Rightarrow c(cx+b) + d(cx+d) \neq 0$$

$$\Rightarrow a^2x + ab + bcx + bd = cx^2 + dx + dcx^2 + d^2x$$

$$x^2(c a + d c) + x(d a + d^2 - c b - a^2) - a b - b d = 0$$

$$x^2(a c + c d) + x(d^2 - a^2) - (a b + b d) = 0$$

$$a c + c d = 0$$

$$d^2 - a^2 = 0$$

$$a b + b d = 0$$

$$d^2 - a^2 \Rightarrow d = \pm a$$

$$\text{case 1: } d = a$$

$$\Rightarrow 2ac = 0$$

$$2ab = 0$$

$$\text{case 1.1 } a = d = 0$$

$$\Rightarrow f(x) = \frac{b}{cx}, c \neq 0$$

$$\text{case 1.2 } a \neq 0, c = b = 0$$

$$\Rightarrow f(x) = \frac{ax}{d} = x$$

$$\text{case 2 } d = -a$$

$$\Rightarrow f(x) = \frac{ax+b}{cx-a}, cx-a \neq 0 \Rightarrow x \neq \frac{a}{c}$$

Note that for $f(f(x))$ to make sense we need

$$f(x) \neq \frac{a}{c} \Rightarrow \frac{ax+b}{cx-a} \neq \frac{a}{c} \Rightarrow acx + cb \neq acx - a^2$$

$$\Rightarrow cb \neq -a^2$$

$$\Rightarrow a^2 + cb \neq 0$$

$$\Rightarrow a^2 + cb = 0$$

9. a) A can set of real numbers

$$A = \{x : x \in \mathbb{R}\}$$

$$C_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Find $C_{A \cap B}$, $C_{A \cup B}$, C_{B-A}

$$C_{A \cap B} = \begin{cases} 1 & x \in A \cap B \\ 0 & x \notin A \cap B \end{cases}$$

$$x \in A \cap B \Rightarrow x \in A \text{ and } x \in B$$

$$\Rightarrow C_A(x) = C_B(x) = 1$$

$$\Rightarrow C_A(x) \cdot C_B(x) = 1$$

$$x \notin A \cap B \Rightarrow x \notin A \text{ or } x \notin B \text{ or } (x \notin A \text{ and } x \notin B)$$

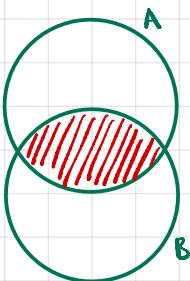
$$x \notin A \cup B$$

$$\Rightarrow C_A(x) = 0 \text{ or } C_B(x) = 0 \text{ or } C_{A \cup B}(x) = 0$$

$$\Rightarrow C_A(x) \cdot C_B(x) = 0$$

$$\Rightarrow C_{A \cap B} = C_A(x) \cdot C_B(x)$$

$$\begin{array}{cc} x \in B & x \notin B \\ \hline x \in A & 1 & 0 \\ x \notin A & 0 & 0 \end{array}$$



$$C_{A \cup B} = \begin{cases} 1 & x \in A \cup B \\ 0 & x \notin A \cup B \end{cases}$$

$$x \in A \cup B \Rightarrow x \in A \text{ or } x \in B$$

$$\Rightarrow C_A(x) = 1 \text{ or } C_B(x) = 1$$

$$C_A(x) + C_B(x) - C_A(x)C_B(x) = 1$$

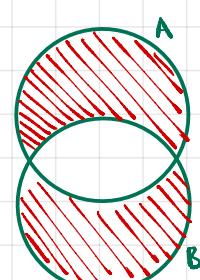
$$x \notin A \cup B \Rightarrow x \notin A \text{ and } x \notin B$$

$$C_A(x) = 0 \text{ and } C_B(x) = 0$$

$$\Rightarrow C_A(x) + C_B(x) - C_A(x)C_B(x) = 0$$

$$C_{A \cup B}(x) = C_A(x) + C_B(x) - C_A(x)C_B(x)$$

$$\begin{array}{cc} x \in B & x \notin B \\ \hline x \in A & 1 & 1 \\ x \notin A & 1 & 0 \end{array}$$



$$C_{B-A}(x) = 1 \Rightarrow x \notin A \Rightarrow C_A(x) = 0$$

$$\Rightarrow 1 - C_A(x) = 1 = C_{B-A}(x)$$

Extra

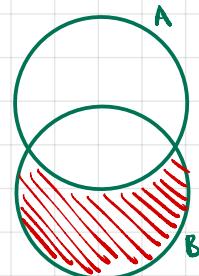
$$B-A = \{x : x \in B, x \notin A\}$$

$$C_{B-A}(x) = \begin{cases} 1 & x \in B-A \\ 0 & x \notin B-A \end{cases}$$

$$x \in B-A$$

$$\Rightarrow x \in B \text{ and } x \notin A$$

$$\Rightarrow C_B(x) = 1 \text{ and } C_A(x) = 0$$



$$x \notin B-A$$

$$\Rightarrow x \in A \text{ or } x \notin B$$

$$\Rightarrow C_A(x) = 1 \text{ or } C_B(x) = 0$$

$$x \notin A$$

$$\Rightarrow C_A(x) = 0$$

$$\Rightarrow 1 - C_A(x) = 1$$

$$= C_{-A}(x)$$

$$C_{B-A} = C_B(x)(1 - C_A(x))$$

$$\begin{array}{cc} x \in B & x \notin B \\ \hline x \in A & 0 & 0 \\ x \notin A & 1 & 0 \end{array}$$

b) $f(x) = 0 \text{ or } 1 \text{ for each } x$.

Define $A = \{x \in \mathbb{R} : f(x) = 1\}$

$$\Rightarrow f(x) = C_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

c) show $f = f^2 \Leftrightarrow f = C_A$ for some set A

$$\Rightarrow f(f-1) = 0 \Rightarrow f = 1 \text{ or } f = 0$$

i.e. $f(x)$ can only be one of two values given any x
⇒ from b) we know $f = C_A$.

$$\Leftarrow f = C_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

$$\Rightarrow f = 1 = f^2 \text{ if } x \in A$$

$$f = 0 = f^2 \text{ if } x \notin A$$

$$\Rightarrow f = f^2$$

10. a) $f = g^2$

If f and g were real numbers, then $f \geq 0$ because $g \cdot g = g^2 \geq 0$.

$g(x)^2$ is also $\geq 0 \forall x$.

$$\Rightarrow f(x) \geq 0 \forall x$$

b) $f = \frac{1}{g}$

$$\Rightarrow g(x) \neq 0 \forall x$$

no matter what value $g(x)$ takes, $\frac{1}{g} \neq 0$
 $\Rightarrow f(x) \neq 0 \forall x$

c) $x(t)^2 + b(t)x(t) + c(t) = 0 \quad \forall t$

$$b(t) - 4c(t) = 0 \quad \forall t \Rightarrow \text{there is one solution } x(t) = -\frac{b(t)}{2}, \quad \text{for } \forall t$$

d) $a(t)x(t) + b(t) = 0 \quad \forall t$

$$a(t)x(t) = -b(t)$$

$$a(t) = 0 \Rightarrow b(t) = 0$$

$$a(t) \neq 0 \Rightarrow x(t) = -\frac{b(t)}{a(t)}$$

If $a(t) \neq 0 \forall t$ then given $a(t)$ and $b(t)$ there is a single $\ln x(t) = -b(t)/a(t)$ that solves the initial eq.

If $a(t) = 0$ at some t , then for that value of t , $x(t)$ is arbitrary. Therefore, given such $a(t)$, and $b(t)$, there are infinite $x(t)$'s that are solutions.

11. a) $H(H(y)) = y$

$$y \rightarrow H(y) \rightarrow H(H(y))$$

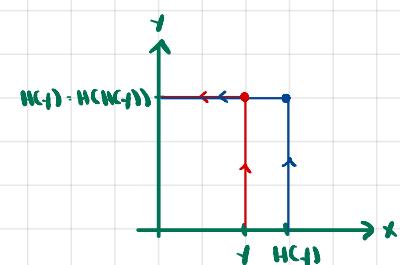
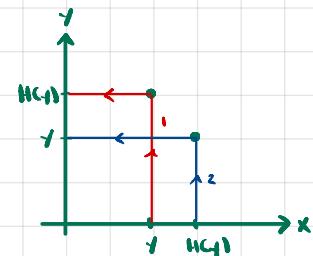
$$\rightarrow H(H(H(y))) \rightarrow H(H(H(H(y))))$$

$$= H(y) \quad = y$$

$$H(H(H(\dots H(y)\dots))) = H(y)$$

$$b) H(H(H(\dots H(y)\dots))) = y$$

$$c) H(H(y)) = H(y)$$



$$H(H(H(y))) = H(H(y)) = H(y)$$

We can compose as many times as we want,
 the result is $H(y)$

$$A = \{n: H_0 \dots \circ H = H, n \text{ times composed}\}$$

$$n=1 \Rightarrow H_0H = H(H(y)) = H(y) \text{ by initial assumption, } 1 \in A$$

assume $k \in A \Rightarrow H_0 \dots \circ H = H, k \text{ times composed.}$

$$(H_0 \dots \circ H) \circ H = H_0 \dots \circ H = H_0H = H(y)$$

$$k+1 \in A, A = \mathbb{N}$$

d) $H(H(x)) = H(x) \quad \forall x$

$$H(1) = 36$$

$$H(2) = \pi/3$$

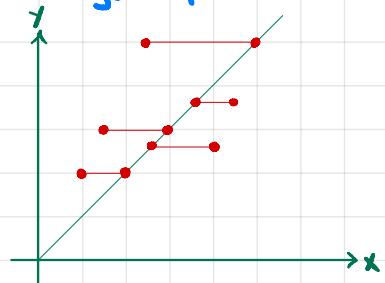
$$H(3) = 47$$

$$H(36) = 36$$

$$H(\pi/3) = \pi/3$$

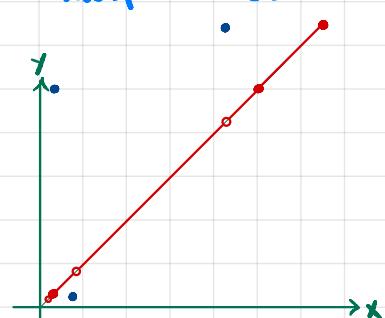
$$H(47) = 47$$

In general, $h(x)$ looks like

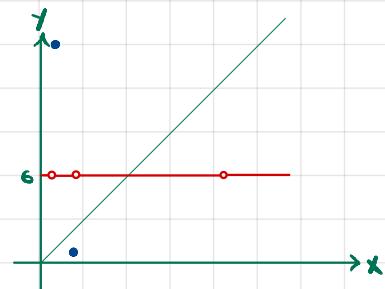


These specific cases fit item d):

$$H(x) = \begin{cases} 36 & x=1 \\ \pi/3 & x=2 \\ 47 & x=3 \\ x & x \neq 1, 2, 3 \end{cases}$$



$$H(x) = \begin{cases} 36 & x=1 \\ \pi/3 & x=2 \\ 47 & x=3 \\ 6 & x \neq 1, 2, 3 \end{cases}$$



$$e) H(H(x)) = H(x) \quad \forall x$$

$$H(1) = 7$$

$$H(7) = 18$$

