

Ch 10 - Differentiation

Theorem 1 If constant $\exists f(x) = c \rightarrow \forall a f'(a) = 0$

Proof

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0$$

Theorem 2 If identity $\exists f(x) = x \rightarrow \forall a f'(a) = 1$

Proof

$$f'(a) = \lim_{h \rightarrow 0} \frac{a+h-a}{h} = 1$$

Theorem 3 If f and g diffata $\rightarrow f+g$ also diffata

$$(f+g)'(a) = f'(a) + g'(a)$$

Proof

$$(f+g)'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) + g(a+h) - f(a) - g(a)}{h} = f'(a) + g'(a)$$

Theorem 4 If f and g diffata $\rightarrow fg$ also diffata

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a)$$

Proof

$$\begin{aligned} (fg)'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(a+h)(g(a+h) - g(a)) + g(a)(f(a+h) - f(a))}{h} \\ &= f(a)g'(a) + g(a)f'(a) \end{aligned}$$

Note that one has to compute $\lim_{h \rightarrow 0} f(a+h) = f(a)$. This follows from f being continuous, since it is diff.

Theorem 5 $g(x) = cf(x)$ \rightarrow $\frac{g(x+h)-g(x)}{h} = c \frac{f(x+h)-f(x)}{h}$

Proof

$$g'(x) = \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} = cf'(x)$$

Alten., apply Th. 4

$$h(x) = c$$

$$g(x) = h(x)f(x)$$

$$g'(x) = h'f + hf' = hf' = cf'(x)$$

Special Cases

$$(-f)(x) = -1 \cdot f(x) \rightarrow (-f)'(x) = -f'(x)$$

$$(f-g)(x) = (f+(-g))(x) \rightarrow (f-g)(x) = f'(x) - g'(x)$$

Theorem 6 $f(x) = x^n, n \in \mathbb{N} \rightarrow \forall a \quad f'(a) = na^{n-1}$

Proof

$$A = \{n : f(x) = x^n, n \in \mathbb{N} \rightarrow \forall a \quad f'(a) = na^{n-1}\}$$

$$f(x) = x \rightarrow f'(x) = 1 = 1 \cdot x^{1-1} \rightarrow 1 \in A$$

$$\text{Assume } k \in A. \text{ Then } f(x) = x^k \rightarrow \forall a \quad f'(a) = ka^{k-1}.$$

$$\text{Let } f(x) = x^{k+1} = x \cdot x^k.$$

$$\begin{aligned} \text{By Th. 4, } f'(x) &= 1 \cdot x^k + x \cdot kx^{k-1} \\ &= x^k(1+k) \\ &= (k+1)x^{(k+1)-1} \end{aligned}$$

T.F. $A = \mathbb{N}$.

Theorem 7 g diff at a

$$g(a) \neq 0 \rightarrow \left(\frac{1}{g}\right)'(a) = \frac{-g'(a)}{g(a)^2}$$

Proof

$$g'(a) = \lim_{h \rightarrow 0} \frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} = \lim_{h \rightarrow 0} \frac{g(a) - g(a+h)}{hg(a)g(a+h)} = \lim_{h \rightarrow 0} \frac{[g(a+h) - g(a)]}{h} \cdot \lim_{h \rightarrow 0} \frac{1}{g(a)g(a+h)}$$

$$= \frac{-g'(a)}{g(a)^2}$$

Note $\lim g(a+h) = g(a)$ because g is cont. since it is diff.

Theorem 8 f, g diff at a

$$g(a) \neq 0 \rightarrow \left(\frac{f}{g}\right)'(a) = \frac{f'g - fg'}{g^2}$$

Proof

$$\frac{f}{g} = f \cdot \frac{1}{g}$$

By Th 4, $(f \cdot \frac{1}{g})'(a) = f'(a) \left(\frac{1}{g}\right)(a) + f(a) \left(\frac{1}{g}\right)'(a)$

$$= \frac{f'(a)}{g(a)} - \frac{f(a)g'(a)}{g(a)^2} = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$$

With this result we can extend Theorem 6, true for \mathbb{N} , to being true to \mathbb{Z} .

$$f(x) = x^{-n} = \frac{1}{x^n} \rightarrow f'(x) = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1}$$

$$f(x) \cdot 1 = x^0 \rightarrow f'(x) \cdot 0 = 0 \cdot x^{-1}$$

Note that we are "interpreting" $1 \approx x^0$ and $0 \approx 0 \cdot x^{-1}$.

At $x=0$ these interpretations don't work, does 0^0 need to be defined, $0 \cdot 0^{-1}$ is meaningless.

Following results presented w/o proof

$$\sin'(x) = \cos(x)$$

$$\cos'(x) = -\sin(x)$$

Chain Rule Proof Intro

$$(f \circ g)'(a) = \lim_{h \rightarrow 0} \frac{f(g(a+h)) - f(g(a))}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(g(a+h)) - f(g(a))}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \cdot \frac{g(a+h) - g(a)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(f \circ g)(a+h) - (f \circ g)(a)}{g(a+h) - g(a)}$$

$$= \lim_{h \rightarrow 0} \frac{f[g(a) + g(a+h) - g(a)] - f(g(a))}{g(a+h) - g(a)} \cdot \lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h}$$

let $h(h) = g(a+h) - g(a)$. Then

$$= \lim_{h \rightarrow 0} \frac{f(g(a) + h(h)) - f(g(a))}{h(h)} \cdot g'(a)$$

Chain Rule Proof Sketch

We want to prove $(f \circ g)'(a) = f'(g(a))g'(a)$.

$$(f \circ g)'(a) = \lim_{h \rightarrow 0} \frac{f(g(a+h)) - f(g(a))}{h} = \lim_{h \rightarrow 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \cdot \frac{g(a+h) - g(a)}{h}$$

$$= \underbrace{\lim_{h \rightarrow 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)}}_{\text{we need to prove that this equals } f'(g(a))} \cdot \underbrace{\lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h}}_{g'(a)}$$

$$\text{we do this by defining } \phi(h) = \begin{cases} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} & \text{if } g(a+h) - g(a) \neq 0 \\ f'(g(a)) & \text{if } g(a+h) - g(a) = 0 \end{cases}$$

if $g(a+h) - g(a) \neq 0$

if $g(a+h) - g(a) = 0$

and proving that it is cont. at 0, ie that $\lim_{h \rightarrow 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} = f'(g(a))$.

Theorem 9 Chain Rule

$$\begin{array}{l} g \text{ diff at } a \\ f \text{ diff at } g(a) \end{array} \Rightarrow \begin{array}{l} f \circ g \text{ diff at } a \\ (f \circ g)'(a) = f'(g(a))g'(a) \end{array}$$

Proof

Define ϕ

$$\phi(h) = \begin{cases} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} & \text{if } g(a+h) - g(a) \neq 0 \\ f'(g(a)) & \text{if } g(a+h) - g(a) = 0 \end{cases}$$

Intuitively, near $h=0$, $\phi(h)$ is either $f'(g(a))$ or it is close to $f'(g(a))$. We just prove $\lim_{h \rightarrow 0} \phi(h) = f'(g(a))$, thus ϕ is cont. at 0.

$$f \text{ diff at } g(a) \Rightarrow \lim_{h \rightarrow 0} \frac{f(g(a)+h) - f(g(a))}{h} = f'(g(a))$$

thus $\forall \epsilon > 0 \exists \delta > 0$ s.t. $\forall h$

$$0 < |h| < \delta \rightarrow \left| \frac{f(g(a)+h) - f(g(a))}{h} - f'(g(a)) \right| < \epsilon \quad (1)$$

g is diff and hence cont. at a . Thus

$$\exists \delta' > 0 \forall h \quad |h| < \delta' \rightarrow |g(a+h) - g(a)| < \delta' \quad (2)$$

nothing special so far, simple definitions of differentiability.

let h s.t. $|h| < \delta$, let $k = g(a+h) - g(a) \neq 0$.

$$(2) \rightarrow |h| < \delta'$$

rewrite $\phi(h)$ with h

$$\phi(h) = \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} = \frac{f(g(a)+h) - f(g(a))}{k}$$

$$(1) \rightarrow |\phi(h) - f'(g(a))| < \epsilon$$

$$\rightarrow \lim_{h \rightarrow 0} \phi(h) = f'(g(a))$$

since g is continuous, we can say something about $g(a+h) - g(a)$. This term appears in the denominator and numerator of $\phi(h)$. In fact when we substitute, this term performs the role of h in a classic derivative expr., $[f(x+h) - f(x)]/h$.

Then

assume $|h| < \delta$, then $|g(a+h) - g(a)| < \delta'$
then $|\phi(h) - f'(g(a))| < \epsilon$.

$$\text{Hence } \lim_{h \rightarrow 0} \phi(h) = f'(g(a))$$

$$(f \circ g)'(a) = \lim_{h \rightarrow 0} \frac{f(g(a+h)) - f(g(a))}{h} = \lim_{h \rightarrow 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \cdot \frac{g(a+h) - g(a)}{h}$$

$$= \phi(h) \cdot g'(a)$$

$$= f'(g(a))g'(a)$$

Some Examples

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x=0 \end{cases}$$

$$f'(0) = \lim_{h \rightarrow 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \rightarrow 0} h \sin \frac{1}{h} = 0$$

Now with the chain rule, we can compute $f'(x)$ for $x \neq 0$.

$$f'(x) = 2x \sin(1/x) - x^2 \cos(1/x) \cdot (1/x^2), \quad x \neq 0.$$

$$\text{Thus } f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & x \neq 0 \\ 0 & x=0 \end{cases}$$

$$\lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} (2x \sin(1/x) - \cos(1/x))$$

As we know, $\lim \cos(1/x)$ does not exist.

Hence $f'(x)$ is not cont. at 0.

$$g(x) = \begin{cases} x^3 \sin(1/x) & x \neq 0 \\ 0 & x=0 \end{cases}$$

$$g'(0) = \lim_{h \rightarrow 0} \frac{h^3 \sin(1/h)}{h} = \lim_{h \rightarrow 0} h^2 \sin(1/h) = 0$$

$$g'(x) = 3x^2 \sin(1/x) + x^3 \cos(1/x) \cdot (-1/x^2) \\ = 3x^2 \sin(1/x) - x \cos(1/x)$$

$$\lim_{h \rightarrow 0} \frac{h \cos(1/h)}{h} = \lim_{h \rightarrow 0} \cos(1/h), \text{ which doesn't exist.}$$

Hence $x \cos(1/x)$ is not differentiable at 0, therefore neither is $f'(x)$.

$$\lim_{x \rightarrow 0} g'(x) = 0, \text{ hence } g'(x) \text{ is cont. at 0.}$$