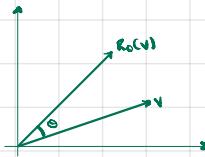
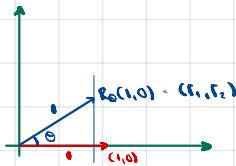


## Ch4 Appendix 1 - Vectors

1.



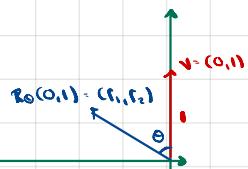
a) shows  $\text{Ro}((1,0)) = (\cos\theta, \sin\theta)$   
 $\text{Ro}((0,1)) = (-\sin\theta, \cos\theta)$



$$||v|| = [t_1^2 + t_2^2]^{1/2}$$

since we rotated only, the distance to origin should be the same.

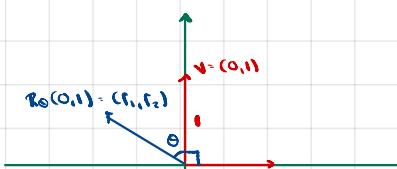
we can see that by geometry  $t_1 = \cos\theta$ ,  $t_2 = \sin\theta$



$$t_1 = -\cos(90^\circ - \theta) = -\sin\theta$$

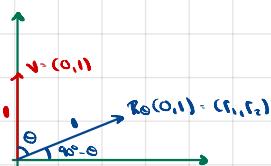
$$t_2 = \sin(90^\circ - \theta) = \cos\theta$$

Alternatively,  $\text{Ro}(0,1)$  is a rotation of  $90^\circ + \theta$  from  $(1,0)$



$$t_1 = \cos(90^\circ + \theta) = -\sin\theta$$

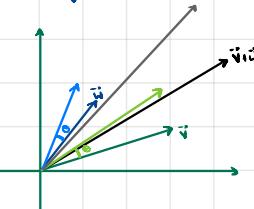
$$t_2 = \sin(90^\circ + \theta) = \cos\theta$$



$$t_1 = \cos(90^\circ - \theta) = \sin\theta$$

$$t_2 = \sin(90^\circ - \theta) = \cos\theta$$

b) explain  $\text{Ro}(\vec{v} + \vec{w}) = \text{Ro}(\vec{v}) + \text{Ro}(\vec{w})$



Rotation of a sum of two vectors is the same as the sum of the rotations of the two vectors.

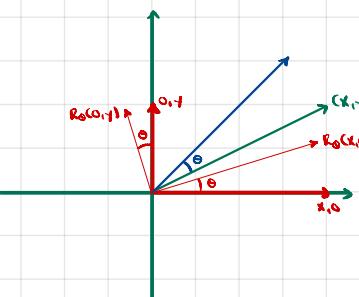
Rotation of multiplication of vector by scalar is the same as rotation of vector, multiplied by scalar

These are properties of the rotation in operation.

It is not true for addition. For example,  $\text{N}(\vec{v}) = ||\vec{v}||$ .

$$\begin{aligned} \text{N}(\vec{v} + \vec{w}) &= \|\vec{v} + \vec{w}\| = [(v_1 + w_1)^2 + (v_2 + w_2)^2]^{1/2} \\ &+ [v_1^2 + v_2^2]^{1/2} + [w_1^2 + w_2^2]^{1/2} = \text{N}(\vec{v}) + \text{N}(\vec{w}) \end{aligned}$$

c)  $\text{Ro}(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$



note that if  $\vec{w} \cdot \vec{v} = 0$  then  $\|\vec{w}\| = \|\vec{v}\| = (\alpha_1^2 v_1^2 + \alpha_2^2 v_2^2)^{1/2} = \alpha_1 \|\vec{v}\|$

$$\vec{v} = (x,y) = (x,0) + (0,y)$$

$$\text{Ro}(\vec{v}) = \text{Ro}((x,0) + (0,y)) = \text{Ro}(x,0) + \text{Ro}(0,y)$$

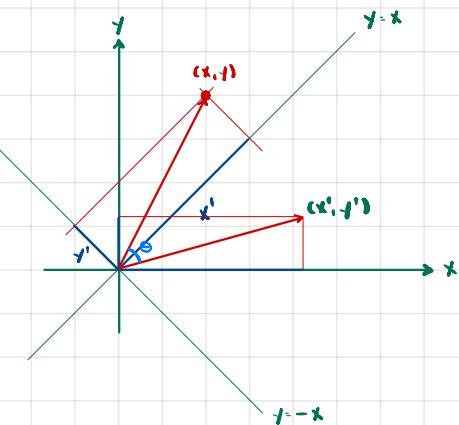
$$= \text{Ro}(x(1,0)) + \text{Ro}(y(0,1))$$

$$= x\text{Ro}(1,0) + y\text{Ro}(0,1)$$

$$= x(\cos\theta, \sin\theta) + y(-\sin\theta, \cos\theta)$$

$$= (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

d)



$$R_\theta(x_0, y_0) = (x_1, y_1) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$$

$$R_{-\theta}(x_1, y_1) = (x'_1, y'_1)$$

$$= (x \cos(-\theta) - y \sin(-\theta), x \sin(-\theta) + y \cos(-\theta))$$

$$= (x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta)$$

$$\theta = 45^\circ \Rightarrow (x'_1, y'_1) = \left( \frac{x+y}{\sqrt{2}}, \frac{y-x}{\sqrt{2}} \right)$$

$$2. \quad \vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2$$

a) given  $\vec{v}$ , find  $\vec{w} : \vec{v} \cdot \vec{w} = 0$

$$v_1 w_1 + v_2 w_2 = 0$$

$$v_1 w_1 = -v_2 w_2$$

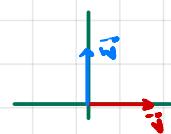
$$v_1 = 0 \Rightarrow w_2 = 0$$

$$\text{ie } \vec{v} = (0, v_2) \Rightarrow \vec{w} = (w_1, 0)$$



$$v_2 = 0 \Rightarrow w_1 = 0$$

$$\text{ie } \vec{v} = (v_1, 0) \Rightarrow \vec{w} = (0, w_2)$$



$$v_1 = 0, v_2 = 0$$

$$\Rightarrow w_1 = -\frac{v_2}{v_1} w_2$$

$$w_2 = -\frac{v_1}{v_2} w_1$$

ie  $(w_1, w_2)$  is somewhere on this line

$\Rightarrow$  the lines on which  $\vec{v}$  and  $\vec{w}$  lie are perpendicular

$$b) \quad \vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 = w_1 v_1 + w_2 v_2 = \vec{w} \cdot \vec{v}$$

$$\vec{v} \cdot (\vec{u} + \vec{z}) = \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{z}$$

$$v_1(u_1 + z_1) + v_2(u_2 + z_2)$$

$$= v_1 u_1 + v_2 u_2 + v_1 z_2 + v_2 z_2$$

$$= \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{z}$$

$$a(\vec{v} \cdot \vec{u}) = (a\vec{v}) \cdot \vec{u} = \vec{v} \cdot (a\vec{u})$$

$$a(v_1 w_1 + v_2 w_2)$$

$$= av_1 w_1 + aw_2 v_2$$

$$= (av_1) w_1 + (aw_2) v_2 = (a\vec{v}) \cdot \vec{u}$$

$$= v_1(av_1) + v_2(av_2) = \vec{v} \cdot (a\vec{u})$$

$$c) \vec{v} \cdot \vec{v} \geq 0$$

$$\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 \geq 0$$

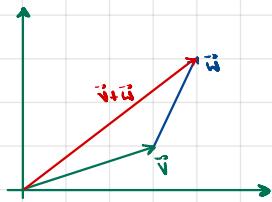
$$\vec{v} \cdot \vec{v} = 0 \Leftrightarrow \vec{v} = \vec{0}$$

$$\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 = 0 \Leftrightarrow v_1^2 = -v_2^2 \Leftrightarrow v_1 = v_2 = 0 \Rightarrow \vec{v} = \vec{0}$$

$$\text{norm } \|\vec{v}\| \equiv \sqrt{\vec{v} \cdot \vec{v}}$$

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2} = \sqrt{(v_1 - 0)^2 + (v_2 - 0)^2}, \text{ distance from } (0,0)$$

$$d) \|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\| \quad \text{ie } \sqrt{(v_1 + w_1)^2 + (v_2 + w_2)^2} \leq \sqrt{v_1^2 + v_2^2} + \sqrt{w_1^2 + w_2^2}$$



This is the triangle inequality which we proved  
in problem 4-9

recall

$$\begin{aligned} (x_1 + t_1)^2 + (x_2 + t_2)^2 &= x_1^2 + x_2^2 + t_1^2 + t_2^2 + 2x_1t_1 + 2x_2t_2 \\ &= x_1^2 + x_2^2 + t_1^2 + t_2^2 + 2(x_1t_1 + x_2t_2) \\ &\leq x_1^2 + x_2^2 + t_1^2 + t_2^2 + 2\sqrt{x_1^2 + x_2^2} \sqrt{t_1^2 + t_2^2} \\ &= [\sqrt{x_1^2 + x_2^2} + \sqrt{t_1^2 + t_2^2}]^2 \\ &\Rightarrow \sqrt{(x_1 + t_1)^2 + (x_2 + t_2)^2} \leq \sqrt{x_1^2 + x_2^2} + \sqrt{t_1^2 + t_2^2} \end{aligned}$$

Here we used Schurz Inequality.  
For equality here, either

$\vec{v} = \lambda \vec{w}$  or at least one vector is zero

In the context of this problem,  $\|\vec{v} + \vec{w}\| = \|\vec{v}\| + \|\vec{w}\|$

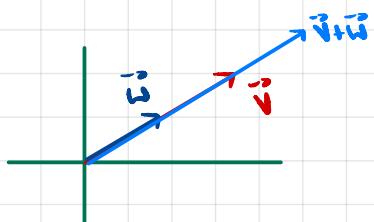
if either

$$i) \vec{v} \text{ and/or } \vec{w} = \vec{0}$$



$$c) \vec{v} + \vec{w}$$

$$ii) \vec{v} = \lambda \vec{w}$$



$$e) \vec{v} \cdot \vec{w} = \frac{\|\vec{v} + \vec{w}\|^2 - \|\vec{v} - \vec{w}\|^2}{4}$$

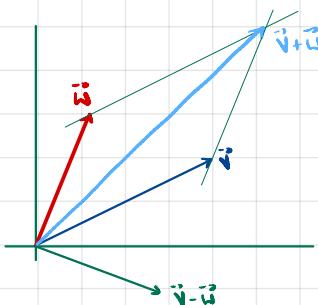
$$\|\vec{v} + \vec{w}\|^2 = (v_1 + w_1)^2 + (v_2 + w_2)^2$$

$$\|\vec{v} - \vec{w}\|^2 = (v_1 - w_1)^2 + (v_2 - w_2)^2$$

$$\begin{aligned} \|\vec{v} + \vec{w}\|^2 - \|\vec{v} - \vec{w}\|^2 &= v_1^2 + 2v_1w_1 + w_1^2 + v_2^2 + 2v_2w_2 + w_2^2 \\ &\quad - (v_1^2 - 2v_1w_1 + w_1^2 + v_2^2 - 2v_2w_2 + w_2^2) \end{aligned}$$

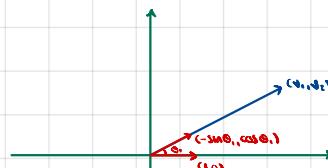
$$= 4v_1w_1 + 4v_2w_2$$

$$\Rightarrow \frac{\|\vec{v} + \vec{w}\|^2 - \|\vec{v} - \vec{w}\|^2}{4} = v_1w_1 + v_2w_2 = \vec{v} \cdot \vec{w}$$



$$3. a) R_\theta(\vec{v}) \cdot R_\theta(\vec{w}) = \vec{v} \cdot \vec{w}$$

$$R_\theta(v_1, v_2) = R_\theta(v_1, 0) + R_\theta(0, v_2)$$



$$R_\theta(w_1, w_2) = R_\theta(w_1, 0) + R_\theta(0, w_2)$$

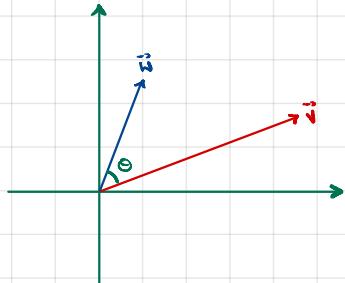
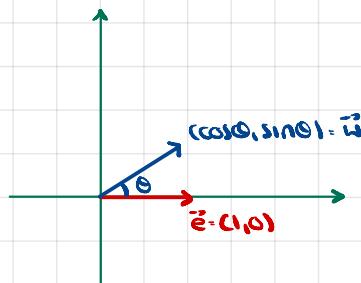
$$R_\theta(\vec{v}) \cdot R_\theta(\vec{w}) = R_\theta(v_1, 0) \cdot R_\theta(w_1, 0) + R_\theta(0, v_2) \cdot R_\theta(0, w_2)$$

$$= v_1w_1 R_\theta(1,0) \cdot R_\theta(1,0) + v_2w_2 R_\theta(0,1) \cdot R_\theta(0,1)$$

$$= v_1w_1 \|R_\theta(1,0)\|^2 + v_2w_2 \|R_\theta(0,1)\|^2$$

$$= v_1w_1 + v_2w_2$$

$$b) \vec{e} \cdot \vec{w} = \cos \theta = \|\vec{e}\| \|\vec{w}\| \cos \theta$$



$$\text{Let } \hat{v} = \frac{\vec{v}}{\|\vec{v}\|} = \left( \frac{v_1}{\|\vec{v}\|}, \frac{v_2}{\|\vec{v}\|} \right) = (v_{1n}, v_{2n})$$

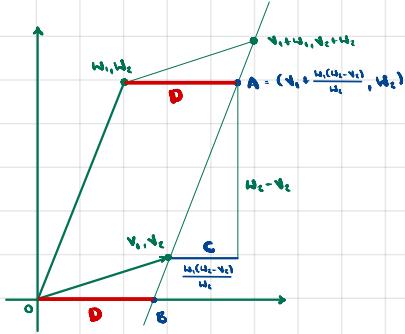
$$\hat{w} = R_\theta(\hat{v}) = \frac{1}{\|\vec{v}\|} R_\theta(\vec{v})$$

$$= \frac{1}{\|\vec{v}\|} \left( \frac{v_1}{\|\vec{v}\|} \cos \theta - \frac{v_2}{\|\vec{v}\|} \sin \theta, \frac{v_1}{\|\vec{v}\|} \sin \theta + \frac{v_2}{\|\vec{v}\|} \cos \theta \right)$$

$$\hat{v} \cdot \hat{w} = \frac{v_1^2}{\|\vec{v}\|^2} \cos \theta - \frac{v_1 v_2}{\|\vec{v}\|^2} \sin \theta + \frac{v_1 v_2}{\|\vec{v}\|^2} \sin \theta + \frac{v_2^2}{\|\vec{v}\|^2} \cos \theta = \cos \theta$$

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \hat{v} \cdot \hat{w} = \|\vec{v}\| \|\vec{w}\| \cos \theta$$

4.



$$\text{a) Prove } B = \left( \frac{v_1 w_2 - w_1 v_2}{w_2}, 0 \right)$$

$$\text{c: } \frac{w_2}{w_1} = \frac{w_2 - v_2}{c} \Rightarrow c = \frac{w_1(w_2 - v_2)}{w_2}$$

$$\text{d: } D = v_1 + \frac{w_1(w_2 - v_2)}{w_2} - w_1 \\ = \frac{w_2(v_1 - w_1) + w_1(w_2 - v_2)}{w_2}$$

$$= \frac{v_1 w_2 - w_1 v_2}{w_2}$$

$$\Rightarrow B = \left( \frac{v_1 w_2 - w_1 v_2}{w_2}, 0 \right)$$

height of parallelogram =  $w_2$

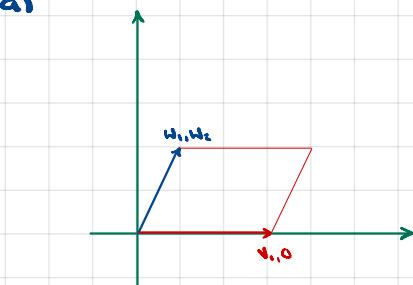
$$\text{Area} = \text{base} \times \text{height} = w_2 \cdot \frac{v_1 w_2 - w_1 v_2}{w_2}$$

$$= v_1 w_2 - w_1 v_2$$

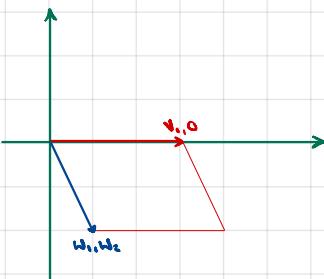
$$= \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix}$$

$$= \det(\vec{v}, \vec{w})$$

5. a)



$$\det(\vec{v}, \vec{w}) = v_1 w_2 - w_1 v_2 \\ = v_1 w_2 = \text{base} \times \text{height} = \text{Area}$$



$$w_2 < 0 \Rightarrow \det(\vec{v}, \vec{w}) = v_1 w_2 < 0$$

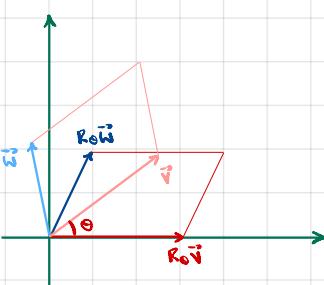
b) show  $\det(R_\theta(\vec{v}), R_\theta(\vec{w})) = \det(\vec{v}, \vec{w})$

$$R_\theta(\vec{v}) = (v_1 \cos \theta - v_2 \sin \theta, v_1 \sin \theta + v_2 \cos \theta) = (v_1 \cos \theta, v_1 \sin \theta)$$

$$R_\theta(\vec{w}) = (w_1 \cos \theta - w_2 \sin \theta, w_1 \sin \theta + w_2 \cos \theta)$$

$$\det(R_\theta \vec{v}, R_\theta \vec{w}) = v_1 w_1 \sin \theta \cos \theta + v_1 w_2 \cos^2 \theta - (v_1 w_1 \sin \theta \cos \theta - v_1 w_2 \sin^2 \theta)$$

$$= v_1 w_2 (\sin^2 \theta + \cos^2 \theta) = v_1 w_2 = \det(\vec{v}, \vec{w})$$



decompose any parallelogram by  $\theta$  so that one of the vectors is on the x-axis.  
The area of the rotated parallelogram equals the area of the original, i.e.

$$\text{Area}_\theta = \det(R_\theta \vec{v}, R_\theta \vec{w}) = \det(\vec{v}, \vec{w}) = \text{Area}$$

If the rotation from  $\vec{v}$  to  $\vec{w}$  is clockwise, then one or both vectors must be rotated clockwise such that  $\vec{v}$  is on the x-axis,  $\vec{v}$  is either in quadrant III or IV, both with  $\cos \theta < 0 \Rightarrow \det(\vec{v}, \vec{w}) < 0$  as per a).

If rot. from  $\vec{v}$  to  $\vec{w}$  counterclockwise then  $\vec{w}$  is in quadrant I or II so  $\det(\vec{v}, \vec{w})$  is  $> 0$ .

6) Show  $\det(\vec{v}, \vec{w} + \vec{z}) = \det(\vec{v}, \vec{w}) + \det(\vec{v}, \vec{z})$

$$\det(\vec{v}, \vec{w} + \vec{z}) = \det(\vec{v}, \vec{z}) + \det(\vec{v}, \vec{w})$$

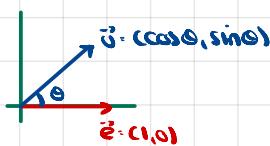
$$\det(\vec{v}, \vec{w}) = \det(a\vec{v}, \vec{w}) - \det(\vec{v}, a\vec{w})$$

$$\begin{aligned}\det(\vec{v}, \vec{w} + \vec{z}) &= v_1(w_2 + z_2) - v_2(w_1 + z_1) \\ &= v_1w_2 - v_2w_1 + v_1z_2 - v_2z_1 \\ &= \det(\vec{v}, \vec{w}) + \det(\vec{v}, \vec{z})\end{aligned}$$

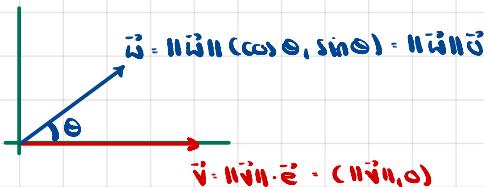
$$\begin{aligned}\det(\vec{v} + \vec{w}, \vec{z}) &= (v_1 + w_1)z_2 - (v_2 + w_2)z_1 \\ &= v_1z_2 - v_2z_1 + w_1z_2 - w_2z_1 \\ &= \det(\vec{v}, \vec{z}) + \det(\vec{w}, \vec{z})\end{aligned}$$

$$\begin{aligned}\det(\vec{v}, \vec{w}) &= a(v_1w_2 - v_2w_1) \\ &= (av_1)w_2 - (av_2)w_1 = \det(a\vec{v}, \vec{w}) \\ &= v_1(av_2) - v_2(av_1) = \det(\vec{v}, a\vec{w})\end{aligned}$$

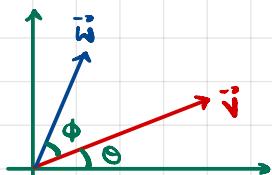
7. Show  $\det(\vec{v}, \vec{w}) = \|\vec{v}\| \|\vec{w}\| \sin \theta$



$$\det(\vec{v}, \vec{w}) = v_1w_2 - v_2w_1 = \sin \theta$$



$$\begin{aligned}\det(\vec{v}, \vec{w}) &= \det(\|\vec{v}\| \vec{e}, \|\vec{w}\| \vec{u}) = \|\vec{v}\| \|\vec{w}\| \|\vec{e}\| \cdot \vec{u} \\ &= \|\vec{v}\| \|\vec{w}\| \sin \theta\end{aligned}$$



$\vec{v} = R_\theta(\vec{v}')$  for some  $\vec{v}'$  along x-axis.

$\vec{w} = R_\phi(\vec{w}')$

$\|\vec{v}\| = \|\vec{v}'\|$ ,  $\|\vec{w}\| = \|\vec{w}'\|$  because rotation doesn't change lengths, also the angle between  $\vec{v}'$  and  $\vec{w}'$  is maintained in rotation

$$\det(\vec{v}, \vec{w}) = \det(R_\theta \vec{v}', R_\phi \vec{w}') = \det(\vec{v}', \vec{w}')$$

$$\Rightarrow \det(\vec{v}, \vec{w}) = \|\vec{v}'\| \|\vec{w}'\| \sin \phi = \|\vec{v}\| \|\vec{w}\| \sin \phi$$