Chapter 2 – Problems

1) i) $1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Let A be set of all $n \in \mathbb{N}$ s.t. n satisfies i)

$1^2 = 1 = \frac{1 \cdot 2 \cdot 3}{6} \Rightarrow 1$ in A

suppose k in A

therefore $1^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$

$1^2 + \cdots + k^2 + (k+1)^2 = [k(k+1)(2k+1) + 6(k+1)^2]/6$

$= (k+1)[k(2k+1) + 6(k+1)]/6$

$= [(k+1)(k+2)(2(k+1)+1)]/6$

$\Rightarrow k+1$ in A

$\Rightarrow$ by induction, $A = \mathbb{N} \Rightarrow$ all $n \in \mathbb{N}$ satisfy i)

ii) $1^3 + \cdots + n^3 = (1 + \cdots + n)^2$

Let A be set of $n \in \mathbb{N}$ s.t. ii) is true

$1^3 = (1)^2 = 1 \Rightarrow 1$ in A

suppose k in A. Then, $1^3 + \cdots + k^3 = (1 + \cdots + k)^2$

$1^3 + \cdots + k^3 + (k+1)^3 = (1 + \cdots + k)^2 + (k+1)^3$

note that $(x+y+z)(x+y+z) = (x+y)(x+y+z) + z(x+y+z)$

$= (x+y)^2 + z(x+y) + z(x+y+z)$

$\Rightarrow (x+y)^2 = (x+y+z)^2 - z(x+y) - z(x+y+z)$

so,

$1^3 + \cdots + k^3 + (k+1)^3 = (1 + \cdots + k + k+1)^2 - (k+1)(1 + \cdots + k) - (k+1)(1 + \cdots + k+1) + (k+1)^3$

$= (1 + \cdots + k + k+1)^2 \underbrace{- [(k+1)k(k+1)]/2 - [(k+1)(k+1)(k+2)]/2 + (k+1)^3}$

$[-(k+1)^2(k+k+2)]/2 + (k+1)^3$

$= -(k+1)^2(2k+2)/2 + (k+1)^3$

$= -(k+1)^3 + (k+1)^3$

$= 0$

$\Rightarrow 1^3 + \cdots + (k+1)^3 = (1 + \cdots + k+1)^2$

$\Rightarrow A = \mathbb{N} \Rightarrow$ ii) is true for all $n \in \mathbb{N}$

2) i) $\sum_{i=1}^{n}(2i-1) = 1+3+5+\dots+(2n-1)$

$$\sum_{i=1}^{2n} i = \sum_{i=1}^{n}(2i-1) + \sum_{i=1}^{n} 2i$$

$$2n(2n+1)/2 - 2\cdot n(n+1)/2$$

$$= (4n^2+2n-2n^2-2n)/2 = 2n^2/2 = n^2$$

ii) $\sum_{i=1}^{n}(2i-1)^2 = 1^2+3^2+\dots+(2n-1)^2$

$$\sum_{i=1}^{2n} i^2 = \sum_{i=1}^{n}(2i-1)^2 + \sum_{i=1}^{n}(2i)^2$$

$$\sum_{i=1}^{n}(2i-1)^2 = \sum_{i=1}^{2n} i^2 - \sum_{i=1}^{n}(2i)^2 \rightarrow 4\sum_{i=1}^{n} i^2$$

$$= \frac{2n(2n+1)(2\cdot 2n+1)}{6} - 4\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= (2n+1)[n(4n+1) - 2n(n+1)]/3$$

$$= (2n+1)(4n^2+n-2n^2-2n)/3$$

$$= (2n+1)(2n^2-n)/3 = n(2n+1)(2n-1)/3$$

* Let A be set of $n \in \mathbb{N}$ s.t. $\sum_{i=1}^{n} 2i = 2\sum_{i=1}^{n} i$

$n=1 \Rightarrow \sum_{i=1}^{1} 2i = 2 = 2\cdot\sum_{i=1}^{1} i \Rightarrow 1$ in A

assume k in A $\Rightarrow \sum_{i=1}^{k} 2i = 2\sum_{i=1}^{k} i$

$$\sum_{i=1}^{k+1} 2i = \sum_{i=1}^{k} 2i + 2(k+1)$$

$$= 2\sum_{i=1}^{k} i + 2(k+1)$$

$$= 2\left(\sum_{i=1}^{k} i + (k+1)\right)$$

$$= 2\sum_{i=1}^{k+1} i$$

3) $0 \leq k \leq n$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{if } k \neq 0, n$$

$$\binom{n}{0} = \binom{n}{n} = 1 \quad (0! = 1)$$

for $k < 0$ or $k > n$ define $\binom{n}{k} = 0$

a) Prove $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ $\qquad \dfrac{n!}{(k-1)!(n-k+1)!} + \dfrac{n!}{k!(n-k)!}$

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{(n+1)!}{k!(n-k+1)!} \cdot \frac{k}{n+1} + \frac{(n+1)!}{k!(n-k+1)!} \cdot \frac{n-k+1}{n+1}$$

$$= \binom{n+1}{k}\left(\frac{k+n-k+1}{n+1}\right) = \binom{n+1}{k}$$

The other way around could involve reverse engineering the construction of $\binom{n+1}{k}$

$$\frac{(n+1)!}{k!(n+1-k)!} \cdot \frac{k}{n+1} = \binom{n}{k-1}$$

$$\frac{(n+1)!}{k!(n+1-k)!} \cdot \frac{n+1-k}{n+1} = \binom{n}{k}$$

$$\Rightarrow \left[\frac{(n+1)!}{k!(n+1-k)!}\right]\left(\frac{k}{n+1} + \frac{n+1-k}{n+1}\right) = \binom{n}{k-1} + \binom{n}{k}$$

With this result we have Pascal's Triangle

$$\binom{0}{0} \qquad \longrightarrow \binom{2}{1} = \binom{1}{0} + \binom{1}{1}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$(\ldots)$

=

0
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
$(\ldots)$

3) b) use c) to prove $\binom{n}{k}$ is always a natural number

Let A be set of $n \in \mathbb{N}$ s.t. all $\binom{n}{j} \in \mathbb{N}$ for $j \in \{x \in \mathbb{N} \mid x \leq n, x \geq 1\} \cup \{0\}$

1 in A because $\binom{1}{1} = 1 \in \mathbb{N}$ and $\binom{1}{0} = 1 \in \mathbb{N}$

Suppose k in A.

$$\binom{k}{j} \in \mathbb{N}, \; j \in \{0, \dots, k\}$$

Then,

$$\binom{k+1}{0} = 1 \in \mathbb{N}$$

$$\binom{k+1}{k+1} = 1 \in \mathbb{N}$$

$$\binom{k+1}{j} = \underbrace{\binom{k}{j-1} + \binom{k}{j}}_{\text{These terms both } \in \mathbb{N}}, \; j \in \{1, \dots, k\}$$

$$\Rightarrow \binom{k+1}{j} \in \mathbb{N} \quad j \in \{0, \dots, k+1\}$$

Therefore $A = \mathbb{N} \Rightarrow \binom{n}{k} \in \mathbb{N}$ for all $n, k \in \{x \in \mathbb{N} \mid x \leq n, x \geq 1\} \cup \{0\}$

c) $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

Given n distinct integers, we form k-tuples.
There are $n(n-1)(n-2)\cdots(n-k+1)$ k-tuples because:
$\rightarrow$ n possible choices for first element of the k-tuple
n-1 " " " second " " " "
(...)
n-k+1 " " " k^{th} " " " "

This is equal to $\dfrac{n!}{n-k!}$. Each k-tuple is unique in the order of distinct elements, but not in terms of the set of elements. For each k-tuple, there are k! permutations of the elements, ie k! k-tuples that contain the same elements, ie represent the same set.

So, to get the number of sets of k elements from n possible elements, we divide by k!

$$\frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

3d) Binomial Theorem

a, b any numbers, $n \in \mathbb{N} \Rightarrow (a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n-1} a b^{n-1} + b^n$

$$= \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

Using induction,

Let A be set of all $n \in \mathbb{N}$ for which the above equation holds.

$$(a+b)^1 = \sum_{i=0}^{1} \binom{1}{i} a^{1-i} b^i = 1a^1b^0 + 1a^0b^1 = a+b$$

$\Rightarrow 1$ in A

Suppose the eq holds for $n = k$.

$$(a+b)^{k+1} = (a+b)(a+b)^k$$

$$= (a+b) \cdot \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i = \sum_{i=0}^{k} \binom{k}{i} a^{k-i+1} b^i + \underbrace{\sum_{i=0}^{k} \binom{k}{i} a^{n-i} b^{i+1}}_{\substack{i=j-1 \\ = \sum_{i=1}^{k+1} \binom{k}{i-1} a^{k+1-i} b^i}}$$

$$= \binom{k}{0} a^{k+1} + \sum_{i=1}^{k} \binom{k}{i} a^{k-i+1} b^i + \sum_{i=1}^{k} \binom{k}{i-1} a^{k+1-i} b^i + \binom{k}{k} a^0 b^{k+1}$$

$$= \binom{k+1}{0} a^{k+1} + \sum_{i=1}^{k} a^{k-i+1} b^i \left[\binom{k}{i} + \binom{k}{i-1} \right] + \binom{k+1}{k+1} b^{k+1}$$

$$= \binom{k+1}{0} a^{k+1} b^0 + \sum_{i=1}^{k} \binom{k+1}{i} a^{k+1-i} b^i + \binom{k+1}{k+1} a^0 b^{k+1}$$

$$= \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i$$

e) i) $\sum_{j=0}^{n} \binom{n}{j} = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$

Alternative Proof

$2^n = (1+1)^n$

Using Binomial Theorem:

$$= \sum_{i=0}^{n} \binom{n}{i}$$

induction: A set of $n \in \mathbb{N}$ satisfying i).

$$\sum_{j=0}^{1} \binom{1}{j} = \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2^1 \Rightarrow 1 \text{ in } A$$

suppose $\sum_{j=0}^{k} \binom{k}{j} = 2^k$

$$\sum_{j=0}^{k+1} \binom{k+1}{j} = \binom{k+1}{k+1} + \sum_{j=0}^{k} \binom{k+1}{j} = \binom{k+1}{k+1} + \sum_{j=0}^{k} \left[\binom{k}{j} + \binom{k}{j-1} \right]$$

$$= 2^k + \binom{k+1}{k+1} + \left[\underbrace{\cancel{\binom{k}{-1}}}_{0} + \binom{k}{0} + \dots + \binom{k}{k-1} \right] = 2^k + \binom{k}{0} + \dots + \binom{k}{k}, \text{ because } \binom{k}{k} = \binom{k+1}{k+1}$$

$$= 2^k + \sum_{i=0}^{k} \binom{k}{i} = 2 \cdot 2^k = 2^{k+1} \Rightarrow A = \mathbb{N}$$

3)e) ii) $\sum_{j=0}^{n}(-1)^j\binom{n}{j} = \binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\dots\pm\binom{n}{n}=0$

induction

Let A be set of $n \in \mathbb{N}$ satisfying ii).

$$\sum_{j=0}^{1}(-1)^j\binom{1}{j} = \binom{1}{0}-\binom{1}{1} = 1-1=0 \Rightarrow 1 \text{ in } A$$

Assume $\sum_{j=0}^{k}(-1)^j\binom{k}{j}=0$

Then

$$\sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j} = \sum_{j=0}^{k+1}(-1)^j\left[\binom{k}{j}+\binom{k}{j-1}\right]$$

$$= \cancelto{0}{(-1)^{k+1}\binom{k}{k+1}} + \cancelto{0}{\sum_{j=0}^{k}(-1)^j\binom{k}{j}} + \sum_{j=0}^{k+1}(-1)^j\binom{k}{j-1}$$

$$= \cancelto{0}{(-1)^0\binom{k}{-1}} + (-1)^1\binom{k}{0} + \dots + (-1)^{k+1}\binom{k}{k}$$

$$= (-1)\cdot\sum_{j=0}^{k}(-1)^j\binom{k}{j} = 0$$

Alternative Proof

$$0=(1-1)^n = \sum_{i=0}^{n}\binom{n}{i}(-1)^i$$

iii) $\sum_{i \text{ odd}}\binom{n}{i} = \binom{n}{1}+\binom{n}{3}+\dots = 2^{n-1}$

induction: A set $n \in \mathbb{N}$ satisfying iii.

$\binom{1}{1}=1=2^0$, 1 in A

Assume $\sum_{i\text{ odd}}\binom{k}{i}=2^{k-1}$

Then

$$\sum_{i \text{ odd}}\binom{k+1}{i} = \text{ if } k+1 \text{ odd } = \binom{k+1}{1}+\binom{k+1}{3}+\dots+\binom{k+1}{k+1}$$

$$=\binom{k}{1}+\binom{k}{0}+\binom{k}{3}+\binom{k}{2}+\dots+\binom{k}{k}+\binom{k}{k+1}$$

$$=\sum_{i=0}^{k}\binom{k}{i}+\cancelto{0}{\binom{k}{k+1}} = 2^k \text{, from 3e)}$$

if k+1 even

$$=\underbrace{\binom{k+1}{1}}_{\binom{k}{0}+\binom{k}{1}}+\binom{k+1}{3}+\dots+\underbrace{\binom{k+1}{k}}_{\binom{k}{k-1}+\binom{k}{k}} = \sum_{i=0}^{k}\binom{k}{i}=2^k$$

Alternative Proof

From i), $\sum_{j=0}^{n}\binom{n}{j}=2^n$

From ii), $\sum_{j=0}^{n}(-1)^j\binom{n}{j}=0$

i) − ii) $=2\sum_{i\text{ odd}}\binom{n}{i}=2^n$

$\Rightarrow \sum_{i \text{ odd}}\binom{n}{i}=2^{n-1}$

3) e) iv) $\sum_{i \text{ even}} \binom{n}{i} = 2^{n-1}$

This results as corollary to 3e iii).

$$\sum_{i=0}^{k} \binom{n}{i} = \sum_{i \text{ odd}} \binom{n}{i} + \sum_{i \text{ even}} \binom{n}{i}$$

$$2^k = 2^{n-1} + \sum_{i \text{ even}} \binom{n}{i}$$

$$\sum_{i \text{ even}} \binom{n}{i} = 2^{n-1}$$

4) a) $\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}$

$$(1+x)^n(1+x)^m = \sum_{i=0}^{n} \binom{n}{i} x^i \cdot \sum_{j=0}^{m} \binom{m}{j} x^j = \sum_{k=0}^{n+m} \binom{n+m}{k} x^k$$

The right-hand side has terms of form cx^k, $k = 0, \ldots n+m$.
The left-hand side has form: $(a_0x^0 + \cdots + a_nx^n)(b_0x^0 + \cdots + b_mx^m) = a_ib_jx^{i+j}$
For each $a_ib_jx^{i+j}$, there is a corresponding righthand side term $\binom{n+m}{k}x^k$
we need to determine which lefthand side summation equals each of these right hand side terms.
For each k, we need $i+j=k$:

$$\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} x^k = \binom{n+m}{k} x^k$$

$$\Rightarrow \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}$$

Alternative:

$$\sum_{k=0}^{n+m} \boxed{\binom{n+m}{k}} x^k = \sum_{i=0}^{n} \binom{n}{i} x^i \cdot \sum_{j=0}^{m} \binom{m}{j} x^j = \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} x^{i+j}$$

let $k = i+j$

$$\sum_{k=0}^{n+m} \boxed{\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}} x^k$$

Also,

$\binom{n+m}{k}$ is the number of ways of choosing k items from $n+m$ items.
This can also be accomplished by splitting the items into two groups, with n and m items. Then, choose i items from the group of n and $k-i$ items from the group of m. For each i, there are $\binom{n}{i}\binom{m}{k-i}$ ways of choosing. And we can pick 0 to k items from the n group, k to 0 items from the m group.

4b) $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

using 4a):

$$\binom{2n}{n} = \binom{n+n}{n} = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$$

$$\binom{n}{i} = \frac{n!}{(n-i)!\,i!} = \binom{n}{n-i} \Rightarrow \binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$$

5) a) $1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$ if $r \neq 1$

r is given and $\neq 1$.

Let A be set of $n \in \mathbb{N}$ s.t. the result above is true.

$$n=1 \Rightarrow 1+r = \frac{1-r^2}{1-r} = \frac{(1+r)(1-r)}{1-r} = 1+r \Rightarrow 1 \text{ in } A$$

assume k in A : $1+r+\cdots+r^k = \frac{1-r^{k+1}}{1-r}$

then,

$$1+r+r^2+\cdots+r^k+r^{k+1} = \frac{1-r^{k+1}}{1-r} + r^{k+1} = \frac{1-\cancel{r^{k+1}}+\cancel{r^{k+1}}-r^{k+2}}{1-r} = \frac{1-r^{k+2}}{1-r}$$

$\Rightarrow$ $k+1$ in A

By induction, $A = \mathbb{N}$.

b) $S = 1+r+\cdots+r^n$

$$rS = r(1+r+\cdots+r^n) = r+r^2+\cdots+r^{n+1} = (1+r+\cdots+r^n) + r^{n+1} - 1$$

$$rS = S + r^{n+1} - 1$$

$$S(1-r) = 1-r^{n+1} \Rightarrow S = \frac{1-r^{n+1}}{1-r}$$

6)

ex: $\sum_{i=1}^{n} i^2$

$$(k+1)^3 - k^3 = \sum_{i=0}^{3} \binom{3}{i} k^i - k^3 = \cancel{\binom{3}{0}k^3} + \binom{3}{1}k^2 + \binom{3}{2}k + \binom{3}{3} \cancel{- k^3}$$

$$= 3k^2 + 3k + 1$$

$$\begin{array}{ll}
k=1 & \cancel{2^3} - 1^3 = 3\cdot 1^2 + 3\cdot 1 + 1 \\
k=2 & \cancel{3^3} - \cancel{2^3} = 3\cdot 2^2 + 3\cdot 2 + 1 \\
\dots & \quad / \quad / \quad \dots \\
k=n & (n+1)^3 - \cancel{n^3} = 3\cdot n^2 + 3n + 1
\end{array}$$

$$(n+1)^3 - 1^3 = 3 \cdot \sum_{i=1}^{n} i^2 + 3 \boxed{\sum_{i=1}^{n} i} + n \Rightarrow \sum_{i=1}^{n} i^2 = \left[(n+1)^3 - 1 - 3\sum_{i=1}^{n} i - n\right] / 3$$

$$= \left((n+1)^3 - 1 - 3n(n+1)/2 - n\right) / 3$$

$$= (n+1)\left[2(n+1)^2 - 3n - 2\right] / 6$$

$$(k+1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k + 1$$

$$\begin{array}{ll}
k=1 & \cancel{2^2} - 1^2 = 2\cdot 1 + 1 \\
k=2 & \cancel{3^2} - \cancel{2^2} = 2\cdot 2 + 1 \\
\dots & \quad / \quad / \quad \dots \\
k=n & (n+1)^2 - \cancel{n^2} = 2n + 1
\end{array}$$

$n=3$
$1^2+2^2+3^2$
$1+4+9=14$

$$(n+1)^2 - 1^2 = 2 \cdot \sum_{i=1}^{n} i + n \Rightarrow \sum_{i=1}^{n} i = \frac{(n+1)^2 - (n+1)}{2} = \frac{n(n+1)}{2}$$

i) $\sum_{i=1}^{n} i^3$

$$(k+1)^4 - k^4 = \cancel{\binom{4}{0}k^4} + \binom{4}{1}k^3 + \binom{4}{2}k^2 + \binom{4}{3}k + \binom{4}{4} \cancel{- k^4} = 4k^3 + 6k^2 + 4k + 1$$

$$(n+1)^4 - 1 = 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + n$$

$$\sum_{i=1}^{n} i^3 = (n+1)^4 - (n+1) - 4\sum_{i=1}^{n} i^2 - 6\sum_{i=1}^{n} i^2 - 4\sum i$$

ii) $\sum_{i=1}^{n} i^4$

$$(n+1)^5 - 1 = \binom{5}{1}\sum k^4 + \binom{5}{2}\sum k^3 + \binom{5}{3}\sum k^2 + \binom{5}{4}\sum k + n$$

iii) $\sum_{i=1}^{n} \frac{1}{i(i+1)}$

$$\frac{1}{k} - \frac{1}{k+1} = \frac{k+1-k}{k(k+1)} = \frac{1}{k(k+1)}$$

$$\left.\begin{array}{ll}
k=1 & \frac{1}{1} - \frac{1}{2} = \frac{1}{1\cdot 2} \\
k=2 & \frac{1}{2} - \frac{1}{3} = \frac{1}{2\cdot 3} \\
k=n & \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}
\end{array}\right\} \quad 1 - \frac{1}{n+1} = \sum_{i=1}^{n} \frac{1}{i(i+1)}$$

6) iv) $\sum_{i=1}^{n} \frac{2i+1}{i^2(i+1)^2} = \frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \dots + \frac{2n+1}{n^2(n+1)^2}$

$$\frac{1}{i^2} - \frac{1}{(i+1)^2} = \frac{i^2+2i+1-i^2}{i^2(i+1)^2} = \frac{2i+1}{i^2(i+1)^2}$$

$i=1 \quad \frac{1}{1^2} - \cancel{\frac{1}{2^2}} =$

$i=2 \quad \cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}} =$

$i=n \quad \cancel{\frac{1}{n^2}} - \frac{1}{(n+1)^2} =$

$$1 - \frac{1}{(n+1)^2} = \sum_{i=1}^{n} \frac{2i+1}{i^2(i+1)^2}$$

$p=1 \quad \sum_{k=1}^{n} k = 1+2+\dots+n = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$

$p=2 \quad \sum_{k=1}^{n} k^2 = 1^2+2^2+\dots+n^2 =$

7) $\sum_{k=1}^{n} k^p$ can always be written in form $\frac{n^{p+1}}{p+1} + An^p + Bn^{p-1} + Cn^{p-2} + \dots$

induction

$p=1 \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$

assume true for all $i \in \mathbb{N}$ $i < p$

$$(k+1)^{p+1} = \sum_{i=0}^{p+1} \binom{p+1}{i} k^i$$

$(k+1)^{p+1} - k^{p+1} = \binom{p+1}{1} k^p$ + terms involving lower powers of k

adding for $k=1,\dots,n$

$2^{p+1} - 1^{p+1} = (p+1) \cdot 1^p + \dots$

$3^{p+1} - 2^{p+1} = (p+1) 2^p + \dots$

$(n+1)^{p+1} - n^{p+1} = (p+1) n^p + \dots$

$(n+1)^{p+1} - 1 = (p+1) \sum_{k=1}^{n} k^p$ + terms involving $\sum k^r$ $r < p$

There is a term $\binom{p+1}{p+1} k^0$ on the right that cancels

by assumption $\sum k^r$ terms can be written as expression involving n^s $s \leq p$

$\Rightarrow \sum k^p =$

7) $\sum_{k=1}^{n} k^p = \frac{n^{p+1}}{p+1} + An^p + Bn^{p-1} + \cdots$

$(n+1)^2 - 1 = 2\sum k + n \Rightarrow n^2 + 2n = 2\sum k + n \Rightarrow \sum k = \frac{1}{2}n^2 + \frac{1}{2}n$

$(n+1)^3 - 1 = 3\sum k^2 + 3\sum k + n \Rightarrow n^3 + 3n^2 + 3n = 3\sum k^2 + 3\sum k + n = 3\sum k^2 + 3\left(\frac{1}{2}n^2 + \frac{1}{2}n\right) + n$

$\Rightarrow \sum k^2 = \left[n^3 + n^2\left(3 - \frac{3}{2}\right) + n\left(3 - \frac{3}{2} - 1\right)\right] / 3 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$

$(n+1)^4 - 1 = n^4 + 4n^3 + 6n^2 + 4n = 4\sum k^3 + 6\sum k^2 + 4\sum k + n = 4\sum k^3 + 6\left[\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\right] + 4\left[\frac{1}{2}n^2 + \frac{1}{2}n\right] + n$

$\sum k^3 = \left[n^4 + n^3\left[4 - \frac{6}{3}\right] + n^2\left[6 - \frac{6}{2} - \frac{4}{2}\right] + n\left[4 - \frac{6}{6} - \frac{4}{2} - 1\right]\right] / 4$

$$\sum_{k=1}^{n} k^p = \frac{1}{\binom{p+1}{1}} \cdot \left[n^{p+1} + n^p\left[\binom{p+1}{1} - \binom{p+1}{2} \cdot \underbrace{\frac{1}{\binom{p}{1}}}_{\text{1st term of } \sum k^{p-1}}\right] + n^{p-1}\left[\binom{p+1}{2} - \binom{p+1}{2}\left[\underbrace{\frac{\binom{p}{1} - \binom{p}{2}/\binom{p-1}{1}}{\binom{p}{1}}}_{\text{2nd term of } \sum k^{p-1}}\right] - \binom{p+1}{3}\underbrace{\frac{1}{\binom{p-1}{1}}}_{\text{1st term of } \sum k^{p-2}}\right]\right.$$

$$\left. + n^{p-2}\left[\binom{p+1}{3} - \text{3rd term of } \sum k^{p-1} \times \binom{p+1}{2} - \text{2nd term of } \sum k^{p-2} \times \binom{p+1}{3} - \text{1st term of } \sum k^{p-3} \times \binom{p+1}{4}\right]\right]$$

8) Every natural number is either even or odd

Def

even number : $\{2k ; k \in \mathbb{Z}\}$
odd " : $\{2k+1 ; k \in \mathbb{Z}\}$

$A = \{n ; n \in \mathbb{N}, n = 2k \text{ or } n = 2k+1, k \in \mathbb{Z}\}$

$1 = 2 \cdot 0 + 1 \Rightarrow 1$ in A
assume $n \in A$
$\Rightarrow n = 2k$ or $n = 2k+1, k \in \mathbb{Z}$

$n = 2k \to n+1 = 2k+1 \Rightarrow n+1 \in A$

$n = 2k+1 \Rightarrow n+1 = 2k+2 = 2(k+1) \Rightarrow n+1 \in A$

9) $A \subset \mathbb{N}$
$n_0 \in A$ $\Rightarrow$ $n \in A, \forall n \geq n_0$
$k \in A \Rightarrow k+1 \in A$

Let $B = \{n \in \mathbb{N} ; n + n_0 \in A\}$

$n_0 \in A \Rightarrow n_0 + 1 \in A \Rightarrow 1 \in B$

assume $k \in B \Rightarrow n_0 + k \in A \Rightarrow n_0 + k + 1 \in A \Rightarrow k+1 \in B$

$\Rightarrow B = \mathbb{N}$

$\Rightarrow$ $n \geq n_0 \Rightarrow n \in A$

10) well-ordering principle $\Rightarrow$ mathem. induction

Let $A \subset \mathbb{N}, 1 \in A, k \in A \Rightarrow k+1 \in A$

Let $C = \{n \in \mathbb{N} ; n \notin A\}$

Assume C is non-empty. $\Rightarrow$ C has least element.

Let k be least element of $C \Rightarrow k \neq 1$ because $1 \in A$
$\Rightarrow k-1 \in A \Rightarrow k \in A \Rightarrow k \notin C$
$\Rightarrow$ contradiction
$\Rightarrow C = \emptyset \Rightarrow A = \mathbb{N}$

11) mathem. induction => complete induction

$A \subset \mathbb{N}$
$1 \in A$
$1, \dots, h \in A \Rightarrow h+1 \in A$

Let $B = \{n \in \mathbb{N};\ 1, \dots, n \in A\}$

$1 \in A \Rightarrow 1 \in B$
assume $h \in B \Rightarrow 1, \dots, h \in A \Rightarrow h+1 \in A \Rightarrow 1, \dots, h+1 \in A \Rightarrow h+1 \in B$
$\Rightarrow B = \mathbb{N} \Rightarrow A = \mathbb{N}$

12) a)

i) a rational $\Rightarrow a = \frac{m}{n},\ m, n \in \mathbb{Z}$

b irrational

assume $a+b$ is rational $\Rightarrow \frac{m}{n} + b = \frac{p}{q},\ m, n, p, q \in \mathbb{Z} \Rightarrow b = \frac{pn - mq}{nq} \in \mathbb{Z}$

$\Rightarrow$ contradiction

$\Rightarrow a+b \in \mathbb{Q}'$

ii) Now assume $a, b \in \mathbb{Q}'$

Take $b = 1 - a \in \mathbb{Q}'$

$a+b = a + 1 - a = 1 \in \mathbb{Q}$

$\Rightarrow a+b,\ a, b \in \mathbb{Q}'$ is not necessarily irrational

b) $a \in \mathbb{Q}, b \in \mathbb{Q}'$

assume $ab = \frac{m}{n},\ m, n \in \mathbb{Z}$

if $a \neq 0$,

since $a = p/q,\ p, q \in \mathbb{Z} \Rightarrow b = a^{-1} \cdot \frac{m}{n} = \frac{qm}{pn} \in \mathbb{Q}$, contradiction

$\Rightarrow ab \in \mathbb{Q}'$

if $a = 0$ then $ab = 0 \in \mathbb{Q}$

12) c) $a = 2^{1/4}$

$a^2 = 2^{1/2} \in Q'$

$a^4 = 2 \in Q$

d) $a, b \in Q'$

$a + b = r_1$
$ab = r_2$
$r_1, r_2 \in Q$

$a = r_1 - b$

$(r_1 - b)b = r_2$

$b^2 - r_1 b + r_2 = 0$

$\Delta = r_1^2 - 4r_2 \geq 0 \Rightarrow r_1^2 \geq 4r_2$

Guess a value for r_1: $r_1 = 0 \Rightarrow a = -b \Rightarrow -b^2 = r_2 \Rightarrow b^2 = -r_2 \Rightarrow b = \pm\sqrt{-r_2}$

Guess an $r_2 \leq 0$, say $r_2 = -3 \Rightarrow b = \pm\sqrt{3}$

So, $a = \sqrt{3}, b = -\sqrt{3}$ and $a = -\sqrt{3}, b = \sqrt{3}$

13)

→ every $m \in Z$ can be written as either $3n, 3n+1$ or $3n+2$ for some $n \in Z$
Proof:
A: set of all $m \in Z$ for which $m = 3n$ or $m = 3n+1$ or $m = 3n+2$

$1 = 3 \cdot 0 + 1 \Rightarrow 1 \in A$
assume $k \in A$
$k = 3n \Rightarrow k+1 = 3n+1$
$k = 3n+1 \Rightarrow k+1 = 3n+2$
$k = 3n+2 \Rightarrow k+1 = 3n+3 = 3(n+1) = 3n_1, n_1 \in Z$

$\Rightarrow k+1 = 3n$ or $k+1 = 3n+1$ or $k+1 = 3n+2$

→ $\forall m \in Z$, $m = 3n$ or $m = 3n+1$ or $m = 3n+2$ for some $n \in Z$

→ if k^2 is divisible by 3 then k is too
proof:
$k = 3n, n \in Z \Rightarrow k^2 = 9n^2$, divisible by 3
$k = 3n+1, n \in Z \Rightarrow k^2 = 9n^2 + 6n + 1 = 3(3n^2 + 2n) + 1$
$k = 3n+2, n \in Z \Rightarrow k^2 = 9n^2 + 12n + 4 = 3(3n^2 + 4n + 1) + 1$
} not divisible by 3

→ if $k^2 \in Z$ divisible by 3, k is of form $3n, n \in Z \Rightarrow k$ divisible by 3

Assume $\sqrt{3}$ is rational, i.e. $\sqrt{3} = p/q$, $p, q \in \mathbb{N}$

$\Rightarrow p^2 = 3q^2 \Rightarrow p^2$ divisible by 3 $\Rightarrow p$ divisible by 3 $\Rightarrow p = 3p'$, $p' \in \mathbb{N}$

$\Rightarrow (3p')^2 = 3q^2 \Rightarrow q^2 = 3p'^2 \Rightarrow q^2$ and q divisible by 3

$\Rightarrow p$ and q have common factor $\Rightarrow$ contradiction $\Rightarrow \sqrt{3}$ is not rational

$\sqrt{5}$

All $k \in \mathbb{Z}$ can be written as either $5n, 5n+1, 5n+2, 5n+3$, or $5n+4$ for some $n \in \mathbb{Z}$.
Proof of this is analogous to proof for writing all integers as $3n, 3n+1$ or $3n+2$.

$$\left.\begin{array}{lll} k = 5n,\ n \in \mathbb{Z} & \Rightarrow k^2 = 25n^2 & \\ k = 5n+1,\ n \in \mathbb{Z} & \Rightarrow k^2 = 25n^2 + 10n + 1 & = 5(5n^2 + 2n) + 1 \\ k = 5n+2,\ n \in \mathbb{Z} & \Rightarrow k^2 = 25n^2 + 20n + 4 & = 5(5n^2 + 4n) + 4 \\ k = 5n+3,\ n \in \mathbb{Z} & \Rightarrow k^2 = 25n^2 + 30n + 9 & = 5(5n^2 + 6n + 1) + 4 \\ k = 5n+4,\ n \in \mathbb{Z} & \Rightarrow k^2 = 25n^2 + 40n + 16 & = 5(5n^2 + 8n + 3) + 1 \end{array}\right\} \text{not divisible by } 5$$

$\Rightarrow k^2$ divisible by 5 $\Rightarrow k = 5n$, $n \in \mathbb{Z} \Rightarrow k$ divisible by 5

Similarly we can show k^2 divisible by 6 $\Rightarrow k = 6n$, $n \in \mathbb{Z} \Rightarrow k$ divisible by 6

Showing that $\sqrt{5}$ and $\sqrt{6}$ are irrational is analogous to the proof for $\sqrt{3}$, by contradiction.

$\sqrt{4}$

The proofs that $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{6}$ are irrational rely on the result derived above regarding divisibility of k^2 and k.
We can't derive that result for $\sqrt{4}$.

$$\begin{array}{lll} k = 4n & \Rightarrow k^2 = 16n^2 & \\ k = 4n+1 & \Rightarrow k^2 = 16n^2 + 8n + 1 & = 4(4n^2 + 2n) + 1 \\ k = 4n+2 & \Rightarrow k^2 = 16n^2 + 16n + 4 & = 4(4n^2 + 4n + 1) \\ k = 4n+3 & \Rightarrow k^2 = 16n^2 + 24n + 9 & = 4(4n^2 + 6n + 2) + 1 \end{array}$$

We see that if k^2 is divisible by 4, k could be $4n$ or $4n+2$, and the latter is not divisible by 4.

Assume $\sqrt{4}$ is rational, i.e., $\sqrt{4} = p/q$ $p, q \in \mathbb{N}$

$p^2 = 4q^2 \Rightarrow p = 4p'$ or $p = 4p' + 2$

$p = 4p' \Rightarrow p$ div. by 4 $\Rightarrow q^2 = 4p'^2 = q$ div. by 4 $\Rightarrow q = 4q'$ or $q = 4q' + 2$

$q = 4q' \Rightarrow p$ and q have common factor, 4 $\Rightarrow$ contradiction

$q = 4q' + 2 \Rightarrow (4q'+2)^2 = 4p'^2 \Rightarrow$ This is ok, i.e. $q = 4q' + 2$, $p = 4p'$

$p = 4p' + 2 \Rightarrow (4p'+2)^2 = 4q^2 \Rightarrow q^2 = 4p'^2 + 4p' + 1 = 4(p'^2 + p') + 1$, non-div. by 4 $\Rightarrow q \neq 4n, n \in \mathbb{Z} \Rightarrow$ ok, with p and q non-div. by 4

14 $\sqrt{2} + \sqrt{6} \in \mathbb{Q}'$

We've shown that $\sqrt{6} \in \mathbb{Q}'$ in 13). Let's show that $\sqrt{2} \in \mathbb{Q}'$.

$\sqrt{2} \in \mathbb{Q}'$, proof:

Any integer can be written as $2n$ or $2n+1$, $n \in \mathbb{Z}$

proof: Let A be set of $n \in \mathbb{N}$ satisfying above statement.

$1 = 2 \cdot 0 + 1 \Rightarrow 1 \in A$

assume $k \in A \Rightarrow k = 2n$ or $k = 2n+1$, $n \in \mathbb{N}$

$\Rightarrow k+1 = 2n+1$ or $k+1 = 2n+2 = 2(n+1) = 2m$, $m \in \mathbb{N}$

$\Rightarrow k+1 \in A$

$\Rightarrow$ by induction, $A = \mathbb{N}$ so for every $k \in \mathbb{N}$, $k = 2n$ or $k = 2n+1$

k^2 div by $2 \Rightarrow k$ div by 2

proof

$k = 2n, n \in \mathbb{Z} \Rightarrow k^2 = 4n^2$ div. by 2

$k = 2n+1, n \in \mathbb{Z} \Rightarrow k^2 = 4n^2 + 4n + 1 = 4(n^2+n)+1$ not div by 2

Given $k \in \mathbb{Z}$, $k^2 = 4n^2$ or $k^2 = 4n^2 + 4n + 1$ and only $4n^2$ is divisible by 2, and k is then also div by 2.

Assume $\sqrt{2} = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$, p and q have no common factor, ie $\sqrt{2} \in \mathbb{Q}$

$p^2 = q^2 \cdot 2 \Rightarrow p^2$ divisible by $2 \Rightarrow p$ div. by $2 \Rightarrow p = 2 \cdot m$, $m \in \mathbb{Z}$

$(2m)^2 = q^2 \cdot 2 \Rightarrow q^2 = 2m^2 \Rightarrow q^2$ div by $2 \Rightarrow q$ div by 2

$\Rightarrow p, q$ have common factor $\Rightarrow$ contradiction $\Rightarrow \sqrt{2} \notin \mathbb{Q}$

a)

From 12a) we know that $a, b \in \mathbb{Q}'$ does not imply $a+b$ is $\mathbb{Q}'$.

Assume $\sqrt{2} + \sqrt{6} \in \mathbb{Q} \Rightarrow \sqrt{2} + \sqrt{6} = \frac{p}{q}$, $p, q \in \mathbb{Z}$

$2 + 2 \cdot \sqrt{2} \cdot \sqrt{6} + 6 = p^2/q^2 \Rightarrow 8 + 4\sqrt{3} = p^2/q^2 \Rightarrow \sqrt{3} = \frac{p^2 - 8q^2}{4q^2}$

$p \in \mathbb{Z} \Rightarrow p \cdot p \in \mathbb{Z}$

$8q^2 \in \mathbb{Z}$, $4q^2 \in \mathbb{Z}$ $\Rightarrow \frac{p^2 - 8q^2}{4q^2} \in \mathbb{Q} \Rightarrow \sqrt{3} \in \mathbb{Q} \Rightarrow$ contradiction

b) Assume $\sqrt{2} + \sqrt{3} \in \mathbb{Q} \Rightarrow \sqrt{2} + \sqrt{3} = p/q \quad p,q \in \mathbb{Z}$

$\Rightarrow 2 + \sqrt{6} + 3 = p^2/q^2$

$\sqrt{6} = \frac{p^2 - 5q^2}{q^2} \in \mathbb{Q} \Rightarrow$ false

15

a) $x = p + \sqrt{q},\ p,q \in \mathbb{Q} \Rightarrow x^m = a + b\sqrt{q},\ a,b \in \mathbb{Q}$

$m \in \mathbb{N}$

A = set of $n \in \mathbb{N}$ s.t $x^n = a + b\sqrt{q},\ a,b \in \mathbb{Q}$

$x^1 = a + b\sqrt{q} \Rightarrow 1 \in A$

assume $n \in A \Rightarrow x^n = a + b\sqrt{q}\ \ a,b \in \mathbb{Q}$

$x^{n+1} = (p + \sqrt{q})(a + b\sqrt{q})$

$= pa + pb\sqrt{q} + a\sqrt{q} + bq$

$= \underbrace{pa + bq}_{\mathbb{Q}} + \sqrt{q}\underbrace{(pb + a)}_{\mathbb{Q}}$

$\Rightarrow n+1 \in A$

$\Rightarrow A = \mathbb{N}$

b) $(p - \sqrt{q})^m = a - b\sqrt{q}$

$m=1 \quad p - \sqrt{q} = a - b\sqrt{q}$

$m=n \quad (p - \sqrt{q})^n = a - b\sqrt{q}$

$(p - \sqrt{q})^{n+1} = (p - \sqrt{q})(a - b\sqrt{q}) = pa - pb\sqrt{q} - a\sqrt{q} + bq$

$= \underbrace{(pa + bq)}_{\mathbb{Q}} - \sqrt{q}\underbrace{(pb + a)}_{\mathbb{Q}}$

16

a) $m, n \in \mathbb{N}$ $\quad \frac{m^2}{n^2} < 2 \Rightarrow \frac{(m+2n)^2}{(m+n)^2} > 2$

Also

$$\frac{(m+2n)^2}{(m+n)^2} - 2 < 2 - \frac{m^2}{n^2}$$

$$\frac{(m+2n)^2}{(m+n)^2} > 2 \Rightarrow \frac{m^2+4nm+4n^2}{m^2+2mn+n^2} > 2$$

$$\Rightarrow m^2 + 4\cancel{n}m + 4n^2 > 2m^2 + 4\cancel{n}m + 2n^2$$

$$m^2 < 2n^2 \Rightarrow \frac{m^2}{n^2} < 2$$

$$\frac{(m+2n)^2}{(m+n)^2} - 2 = \frac{m^2+4nm+4n^2-2m^2-4mn-2n^2}{(m+n)^2} = \frac{2n^2-m^2}{(m+n)^2}$$

$(m+n)^2 < n^2$

if $2n^2 - m^2 > 0$ then $\frac{2n^2-m^2}{(m+n)^2} < \frac{2n^2-m^2}{n^2} = 2 - \frac{m^2}{n^2}$

b) $\frac{(m+2n)^2}{(m+n)^2} < 2$ (... analogous to a) $m^2 > 2n^2 \Rightarrow \frac{m^2}{n^2} > 2$

For the second inequality:

$\frac{(m+2n)^2}{(m+n)^2} - 2 = (\dots) = \frac{2n^2-m^2}{(m+n)^2}$, but now we have $2n^2 - m^2 < 0$

so, if we decrease the denominator, the fraction gets more negative:

$$\frac{2n^2-m^2}{(m+n)^2} > \frac{2n^2-m^2}{n^2} = 2 - \frac{m^2}{n^2}$$

a) $m, n \in \mathbb{N}$

$\frac{m^2}{n^2} < 2 \Rightarrow \frac{(m+2n)^2}{(m+n)^2} > 2$

Also

$$\frac{(m+2n)^2}{(m+n)^2} - 2 < 2 - \frac{m^2}{n^2}$$

Interpretation

There is a rational number $\frac{m^2}{n^2}$, smaller than 2, and a rational number, $\frac{(m+2n)^2}{(m+n)^2}$, larger than 2, such that the second number is closer to 2 than the first, albeit from different sides of 2.

We can also say: there is a rational $\frac{m}{n} < \sqrt{2}$, which means there is another rational $\frac{m+2n}{m+n} > \sqrt{2}$, and the squares of these rational numbers have a relationship involving distance from the number 2.

b) $m, n \in \mathbb{N}$

$\frac{m^2}{n^2} > 2 \Rightarrow \frac{(m+2n)^2}{(m+n)^2} < 2$

$m^2 + \cancel{4nm} + 4n^2 < 2m^2 + \cancel{4nm} + 2n^2$

$2n^2 < m^2 \Rightarrow \frac{m^2}{n^2} > 2$

Given rational $m^2/n^2 > 2$, we can form another rational $(m+2n)^2/(m+n)^2 < 2$. Ie, given m^2/n^2 on one side of 2, we can form another rational on the other side of 2. Furthermore, these two rational numbers have the following relationship:

$$\underset{\ominus}{\frac{(m+2n)^2}{(m+n)^2} - 2} > \underset{\ominus}{2 - \frac{m^2}{n^2}}$$

Both sides are negative but the right side is more negative. Therefore m^2/n^2 is more distant from 2 than $(m+2n)^2/(m+n)^2$.

c) $\frac{m}{n} < \sqrt{2} \Rightarrow \exists \frac{m'}{n'}$ with $\frac{m}{n} < \frac{m'}{n'} < \sqrt{2}$

strategy: Given $\left(\frac{m}{n}\right)^2 < 2$, we know there is $\frac{(m+2n)^2}{(m+n)^2} > 2$, with the latter closer to 2 than the former.

Given $\frac{(m+2n)^2}{(m+n)^2} = \left(\frac{m_1}{n_1}\right)^2 > 2$ we know there is $\left(\frac{m_1+2n_1}{m_1+n_1}\right)^2 < 2$, and that the latter is closer to 2 than the former, which was closer to 2 than the original $\left(\frac{m}{n}\right)^2$.

$$\left(\frac{m}{n}\right)^2 < 2 \Rightarrow \frac{m}{n} < \sqrt{2}$$

$$\left(\frac{m_1}{n_1}\right)^2 = \frac{(m+2n)^2}{(m+n)^2} > 2 \Rightarrow \frac{m+2n}{m+n} > \sqrt{2}$$

$$\left(\frac{m_1}{n_1}\right)^2 > 2 \Rightarrow \boxed{\begin{array}{l} \dfrac{(m_1+2n_1)^2}{(m_1+n_1)^2} < 2 \\ \dfrac{m_1+2n_1}{m_1+n_1} < \sqrt{2} \end{array}}$$

$$\frac{(m+2n)^2}{(m+n)^2} - 2 < 2 - \frac{m^2}{n^2} \Rightarrow 2 - \frac{(m+2n)^2}{(m+n)^2} > \frac{m^2}{n^2} - 2$$

$$\frac{(m_1+2n_1)^2}{(m_1+n_1)^2} - 2 > 2 - \frac{m_1^2}{n_1^2}$$

$$\frac{(3m+4n)^2}{(2m+3n)^2} - 2 > 2 - \frac{(m+2n)^2}{(m+n)^2} > \frac{m^2}{n^2} - 2$$

$$2 - \frac{(m_1+2n_1)^2}{(m_1+n_1)^2} < 2 - \frac{m^2}{n^2}$$

$$\boxed{\frac{(m_1+2n_1)^2}{(m_1+n_1)^2} > \frac{m^2}{n^2}}$$

17

$p \in \mathbb{N}$ is prime if we cannot write $p = ab$, $a, b \in \mathbb{N}$ unless these are p and 1.

for convenience, 1 is not prime.

we want to prove with induction that we can always apply the following procedure to factor a non-prime number into a product of prime numbers:

Take non-prime $n > 1 \Rightarrow n = a \cdot b$ $a, b \in \mathbb{N}$
if a or b not prime they can be written as $c \cdot d$ $c, d \in \mathbb{N}$
continue with this and at some point all factors are prime.

a) Let A = set of $n \in \mathbb{N} - \{1\}$ that can be written as product of prime factors.

2 is prime $\Rightarrow$ single factor prime $\Rightarrow$ in A

assume $2, 3, \ldots k$ in A

if $k+1$ is prime $\Rightarrow$ single factor prime $\Rightarrow$ in A

if $k+1$ not prime $\Rightarrow k+1 = ab, a, b < k+1 \Rightarrow a, b$ in $A \Rightarrow k+1$ in A

b) For $n \in \mathbb{N}$, prove $\sqrt{n} \in \mathbb{Q}'$ unless $n = m^2, m \in \mathbb{N}$

Assume $\sqrt{n} \in \mathbb{Q} \Rightarrow \sqrt{n} = \frac{a}{b}$, $a, b \in \mathbb{N}$ since $n \in \mathbb{N} \Rightarrow \sqrt{n} > 0$

$a = b\sqrt{n}$

$a^2 = b^2 \cdot n$, and we know that the factorization into primes is unique for each number.
Therefore, the same factors appear on both sides. There are two of each factor in a^2, also in b^2. Therefore the same applies for $n \Rightarrow n$ is a square.

c) $\sqrt[k]{n} \in \mathbb{Q}'$ unless $n = m^k$

Assume $\sqrt[k]{n} = \frac{a}{b} \Rightarrow a = \sqrt[k]{n} \cdot b \Rightarrow a^k = n b^k$

factors is the same on both sides.

k even $\Rightarrow$ k of each factor in a^k and b^k. whatever factor in n must also then appear k times
$\Rightarrow n = m^k$

d) Assume there are only n prime numbers $p_1, \ldots, p_n$.
$p_1 \cdot p_2 \cdots p_n$ is not prime; since p_1 is 2, $p_1 \cdots p_n$ is even, ie divis. by 2.
$p_1 \cdots p_n + 1$ is not div. by any primes smaller than it. Therefore either it is prime, or there is one or more primes between p_n and $p_1 - p_n$ that factorize $p_1 - p_n + 1$.
Either way, there must be more than $p_1, -, p_n$ as primes.

18

a) $x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$

$\Rightarrow x \in \mathbb{Q}'$ unless $x \in \mathbb{Z}$

$a_{n-1}, \dots, a_0 \in \mathbb{Z}$

Assume $x \in \mathbb{Q} \Rightarrow x = p/q$, $p, q \in \mathbb{Z}$

$\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \dots + a_0 = 0$

$p^n + a_{n-1}p^{n-1}q + a_{n-2}p^{n-2}q^2 + \dots + a_0 q^n = 0$

For $n=1$ and $a_0 = -\sqrt{h}$, $h \in \mathbb{N}$

$p - \sqrt{h} \cdot q = 0 \Rightarrow \frac{p}{q} = \sqrt{h}$

note the assumptions here: $p, q \in \mathbb{Z} \Rightarrow p/q \in \mathbb{Q} \Rightarrow \sqrt{h} \in \mathbb{Q}$

according to 17b, we know that $\sqrt{h}$ is actually necessarily either a natural number or an irrational number.

For the general problem,

$p^n = -q(a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q + \dots + a_0 q^{n-1})$

$\Rightarrow p^n$ is divisible by q

Assume $q \neq \pm 1 \Rightarrow p/q \notin \mathbb{Z}$ and q is product of prime factors

$\Rightarrow p^n$ divisible by the prime factors of q

$\Rightarrow p$ divisible by prime factors of q

$\Rightarrow$ we can simplify p/q by dividing num. and den. by q; thus p/q is an integer, contradicting our assumption

* my solution posted on SE.

18

b) Prove $\sqrt{6}-\sqrt{2}-\sqrt{3} \in \mathbb{Q}'$

$x = \sqrt{6}-(\sqrt{2}+\sqrt{3})$

$x^2 = 6+(\sqrt{2}+\sqrt{3})^2-2\sqrt{6}(\sqrt{2}+\sqrt{3})$

$= 6+2+3+2\sqrt{6}-2\sqrt{6}(\sqrt{2}+\sqrt{3})$

$= 11+2\sqrt{6}(1-(\sqrt{2}+\sqrt{3}))$

$(x^2-11)^2 = 24(1-(\sqrt{2}+\sqrt{3}))^2 = 24(1+(\sqrt{2}+\sqrt{3})^2-2(\sqrt{2}+\sqrt{3}))$

$= 24(1+2+3+2\sqrt{6}-2\sqrt{2}-2\sqrt{3})$

$= 24(6+2(\sqrt{6}-\sqrt{2}-\sqrt{3}))$

$= 24(6+2x)$

$(x^2-11)^2-48x-144=0$

From part a we know that $x \in \mathbb{Z}$ or $x \in \mathbb{Q}'$.

Proof that $0<x<1 \Rightarrow x \in \mathbb{Q}'$

$0 > \sqrt{6}-\sqrt{2}-\sqrt{3} > -1$

$\sqrt{6} < \sqrt{2}+\sqrt{3}$ | $\sqrt{6}+1 > \sqrt{2}+\sqrt{3}$
$6 < 2+3+2\sqrt{6}$ | $6+1+2\sqrt{6} > 2+3+2\sqrt{6}$
$1 < 2\sqrt{6}$ | $7>5$

$\Rightarrow \sqrt{6}-\sqrt{2}-\sqrt{3} \in]-1,0[\Rightarrow x \notin \mathbb{Z} \Rightarrow x \in \mathbb{Q}'$

* Proofs thus far

even: $\{2k ; k \in \mathbb{Z}\}$
odd: $\{2k+1 ; k \in \mathbb{Z}\}$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$$

$$\sum_{i=1}^{n} (2i-1) = n^2$$

$$\sum_{i=1}^{n} (2i-1)^2 = \frac{n(2n+1)(2n-1)}{3}$$

$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ $\Rightarrow$ Pascal's Triangle

$\binom{n}{k}$ always $\in \mathbb{N}$

Binomial Thm: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$

$$\sum_{i=1}^{n} \binom{n}{i} = 2^n$$

$$\sum_{i=1}^{n} (-1)^i \binom{n}{i} = 0$$

$$\sum_{i=\text{odd}}^{n} \binom{n}{i} = \sum_{i=\text{even}}^{n} \binom{n}{i} = 2^{n-1}$$

$$\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r} \quad r \neq 1$$

$\sum_{k=1}^{n} k^p$ can always be written $\frac{n^{p+1}}{p+1} + An^p + Bn^{p-1} + Cn^{p-3} + \ldots$

every $n \in \mathbb{N}$ is either even or odd

19 Prove $h > -1 \Rightarrow (1+h)^n \geq 1+nh \quad \forall n$

Binomial Thm: $(1+h)^n = \sum_{i=0}^{n} \binom{n}{i} h^i = 1 + \sum_{i=1}^{n} \binom{n}{i} h^i$

Induction

$n=1 \Rightarrow (1+h)^1 = 1+h$

assume $n=k \Rightarrow (1+h)^k \geq 1+kh$

multiply each side by $(1+h)$: $(1+h)(1+h)^k \geq (1+kh)(1+h)$ if $1+h>0$

$n=k+1 \Rightarrow (1+h)^{k+1} = (1+h)(1+h)^k$

$\geq (1+h)(1+kh) = 1+h+kh+kh^2 = 1+h(k+1)+kh^2$

$kh^2 \geq 0$ so $(1+h)^{k+1} \geq 1+h(k+1)+kh^2 \geq 1+h(k+1)$

note that this holds for $h>-1$

If $h=0$ then $(1+h)^n = 1$ and $1+kh = 1$. The result holds trivially, $1=1$.

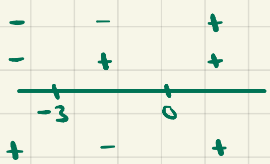
$h<-1 \Rightarrow (1+h)<0$

$(1+h)^1 = 1+h < 0$

$(1+h)^2 > 0$
$1+2h<0$ $\Rightarrow (1+h)^2 > (1+2h)$

$(1+h)^3<0$ $\quad (1+h)^3 = 1+3h+3h^2+h^3$
$1+3h<0$

$3h^2+h^3 = h^2(3+h)$



$\Rightarrow (1+h)^3 = 1+3h+3h^2+h^3 \geq 1+3h$ if $h \leq -3$ or $h \geq 0$

what if we assume $(1+h)^k \geq 1+hk$, $h<-1$?

$(1+h)^{k+1} = (1+h)(1+h)^k$

$(1+h)^k(1+h) \underset{}{\overset{\nearrow \text{switch sign}}{\leq}} (1+hk)(1+h) = 1+h(k+1)+kh^2$

we don't obtain the relationship we're looking for.

20. Fibonacci seq.

$a_1 = 1$

$a_2 = 1$

$a_n = a_{n-1} + a_{n-2} \quad n \geq 3$

Prove $a_n = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$

Induction: A set of n for which $a_n = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$ is n^{th} Fibonacci number

$n=1 \Rightarrow \dfrac{\frac{\cancel{1}+\sqrt{5}}{2} - \frac{\cancel{1}-\sqrt{5}}{2}}{\sqrt{5}} = \dfrac{2\sqrt{5}}{2\sqrt{5}} = 1 = a_1$

$n=2 \Rightarrow \dfrac{\cancel{1}+2\sqrt{5}+\cancel{5} - (\cancel{1}-2\sqrt{5}+\cancel{5})}{4\sqrt{5}} = \dfrac{4\sqrt{5}}{4\sqrt{5}} = 1 = a_2$

Assume $a_k = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}}$ for $k < n$

we want to show that $a_n = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$

we know that $a_n = a_{n-1} + a_{n-2}$, and by assumption

$a_{n-1} = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}}$, $a_{n-2} = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}}{\sqrt{5}}$

$a = \dfrac{1+\sqrt{5}}{2} \qquad b = \dfrac{1-\sqrt{5}}{2}$

$a_{n-1} + a_{n-2} = \dfrac{a^{n-1} - b^{n-1} + a^{n-2} - b^{n-2}}{\sqrt{5}} = \dfrac{a^{n-2}(1+a) - b^{n-2}(1+b)}{\sqrt{5}}$

$1+a = \dfrac{3+\sqrt{5}}{2} = \left(\dfrac{1+\sqrt{5}}{2}\right)^2 = \dfrac{1}{4} + \dfrac{\sqrt{5}}{2} + \dfrac{5}{4} = \dfrac{3}{2} + \dfrac{\sqrt{5}}{2} = a^2$

$1+b = \dfrac{3-\sqrt{5}}{2} = \left(\dfrac{1-\sqrt{5}}{2}\right)^2 = \dfrac{1}{4} - \dfrac{\sqrt{5}}{2} + \dfrac{5}{4} = \dfrac{3}{2} - \dfrac{\sqrt{5}}{2} = b^2$

$\Rightarrow a_{n-1} + a_{n-2} = \dfrac{a^{n-2} \cdot a^2 - b^{n-2} \cdot b^2}{\sqrt{5}} = \dfrac{a^n - b^n}{\sqrt{5}} = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$

Recap: we know the fib. seq. we asserted that there is a formula for the n^{th} element. we proved using induction as follows:

- verify formula valid for $n=1$, $n=2$
- assume formula valid for $k <$ some n
- obtain formula for a_n based on formulas assumed valid.
- since we can obtain the formula for an arbitrary n, all elements of the seq. have the formula.

21. Schwarz Inequality $\sum x_i y_i \le \sqrt{\sum x_i^2}\sqrt{\sum y_i^2}$

Proof 1

$\vec{x} = \langle x_1, \dots, x_n \rangle$, $\vec{y} = \langle y_1, \dots, y_n \rangle$ l.i. $\Rightarrow$ $c_1\vec{x} + c_2\vec{y} = \vec{0} \iff c_1 = c_2 = 0$ Definition of linear indep.

$\Rightarrow \lambda\vec{x} - \vec{y} \neq \vec{0}$ a specific linear combination w/ $c_1, c_2 \neq 0$

$\Rightarrow |\langle \lambda x_1 - y_1, \dots, \lambda x_n - y_n \rangle|^2 > 0$ resulting vector has length $\neq 0$

$\Rightarrow (\lambda x_1 - y_1)^2 + \dots + (\lambda x_n - y_n)^2 > 0$

$= \lambda^2 \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - 2\lambda \sum_{i=1}^{n} x_i y_i > 0$ what sorts of vectors does this?

$\Delta = 4\left(\sum_{i=1}^{n} x_i y_i\right)^2 - 4\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 < 0 \Rightarrow \sum x_i y_i < \sqrt{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2}$

Proof 2

Let $\vec{x}$ and $\vec{y}$ be unit vectors in $\mathbb{R}^n$.

$$\vec{x} = \frac{\langle x_1, \dots, x_n \rangle}{\sqrt{\sum x_i^2}} \qquad \vec{y} = \frac{\langle y_1, \dots, y_n \rangle}{\sqrt{\sum y_i^2}}$$

$(\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y}) = |\vec{x} - \vec{y}|^2 \ge 0$

$\Rightarrow \vec{x}\cdot\vec{x} - 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y} \ge 0 \Rightarrow 2\vec{x}\vec{y} \le \vec{x}\cdot\vec{x} + \vec{y}\cdot\vec{y}$

$$\frac{2\sum x_i y_i}{\sqrt{\sum x_i^2}\sqrt{\sum y_i^2}} \le \frac{\sum x_i^2}{\sum x_i^2} + \frac{\sum y_i^2}{\sum y_i^2} = 2$$

$\Rightarrow \sum x_i y_i \le \sqrt{\sum x_i^2}\sqrt{\sum y_i^2}$

$(x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2)$

$= x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2 + 2x_1y_1x_2y_2 + 2x_1y_1x_3y_3 + 2x_2y_2x_3y_3$

$+ x_1^2 y_2^2 + x_1^2 y_3^2 - 2x_1y_1x_2y_2 - 2x_1y_1x_3y_3 - 2x_2y_2x_3y_3$

$+ x_2^2 y_1^2 + x_2^2 y_3^2$

$+ x_3^2 y_1^2 + x_3^2 y_2^2$

Proof 3

$$\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} x_i^2 y_i^2 + \sum_{i=1}^{n}\sum_{j\neq i} x_i^2 y_j^2$$

Add and subtract $2\sum_{i=1}^{n}\sum_{j>i} x_i y_i x_j y_j$

$$\left[\sum_{i=1}^{n} x_i^2 y_i^2 + 2\sum_{i=1}^{n}\sum_{j>i} x_i y_i x_j y_j\right] + \sum_{i=1}^{n}\sum_{j>i} x_i^2 y_j^2 - 2\sum_{i=1}^{n}\sum_{j>i} x_i y_i x_j y_j$$

There are $C_{n,2}$ combinations (i,j), $j>i$

$$C_{n,2} = \binom{n}{2} = \frac{n!}{(n-2)!\,2!}$$

Take one such comb. (i,j). We have

$$x_i^2 y_j^2 + x_j^2 y_i^2 - 2x_i y_j x_j y_i = (x_i y_j - x_j y_i)^2 \ge 0$$

$= \left(\sum_{i=1}^{n} x_i y_i\right)^2 + \text{positive terms} \ge \left(\sum_{i=1}^{n} x_i y_i\right)^2$

$\Rightarrow \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 \ge \left(\sum_{i=1}^{n} x_i y_i\right)^2$

$\Rightarrow \sqrt{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2} \ge \sum_{i=1}^{n} x_i y_i$

22. $a_1, \dots, a_n \geq 0$

arithmetic mean $A_n = \dfrac{a_1 + \cdots + a_n}{n}$

geometric mean $G_n = \sqrt{a_1 \cdots a_n}$

satisfy $G_n \leq A_n$

a) We have a set of n elements $S_0 = \{a_1, \dots, a_n\}$

If $a_i = A_n$ for every i then $G_n = \sqrt{(A_n)^n} = A_n$.

If any element is $< A_n$ then there must be at least one other element $> A_n$. Assume $a_1 < A_n$ and $a_2 > A_n$.

Form a new set S_1 by replacing a_1 w/ $\bar{a}_1 = A_n$ and a_2 w/ $\bar{a}_2 = a_1 + a_2 - \bar{a}_1$

$S_1 = \{\bar{a}_1, \bar{a}_2, a_3, \dots, a_n\}$

$$A_n' = \frac{\cancel{A_n} + a_1 + a_2 - \cancel{A_n} + a_3 + \cdots + a_n}{n} = A_n$$

$$G_n' = \sqrt{\bar{a}_1 \cdot \bar{a}_2 \cdot a_3 \cdots a_n}$$

we can show $\bar{a}_1 \cdot \bar{a}_2 \geq a_1 a_2$. Therefore, $G_n' \geq G_n$.

The current set S_1 either has all elem. equal to A_n or at least one $< A_n$ and one $> A_n$. Assume $a_3 < A_n$ and $a_4 > A_n$.
we apply the same steps to reach

$S_2 = \{\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, a_5, \dots, a_n\}$

$\bar{a}_3 = A_n$
$\bar{a}_4 = a_3 + a_4 - \bar{a}_3$

$$A_n'' = \frac{\cancel{A_n} + a_1 + a_2 - \cancel{A_n} + \cancel{A_n} + a_3 + a_4 - \cancel{A_n} + \cdots + a_n}{n} = A_n$$

$$G_n'' = \sqrt{\bar{a}_1 \cdot \bar{a}_2 \cdot \bar{a}_3 \cdot \bar{a}_4 \cdot a_5 \cdots a_n} \geq \sqrt{\bar{a}_1 \cdot \bar{a}_2 \cdot a_3 \cdots a_n} = G_n' \geq G_n$$

Note that at this point at least two elements, $\bar{a}_1$ and $\bar{a}_3$ are equal to A_n. Each iteration replaces an elem. w/ A_n, the arithm. mean stays the same, and the geom. mean increases. However, when all elem equal A_n, then the geom. mean equals A_n.
we end up with

$$G_n < G_n' < G_n'' < G_n''' < \cdots < G_n^{(n)} = A_n \Rightarrow G_n < A_n$$

* Assume $a_1 < A_n$, $a_2, \dots, a_{n-1} \leq A_n$

$$A_n = \frac{a_n + \sum_{i=1}^{n-1} a_i}{n} \Rightarrow nA_n - \sum_{i=1}^{n-1} a_i = a_n$$

But $\sum_{i=1}^{n-1} a_i < (n-1)A_n$

$\Rightarrow a_n = nA_n - \sum_{i=1}^{n-1} a_i > nA_n - (n-1)A_n = A_n$
$\Rightarrow a_n > A_n$

* $\dfrac{\bar{a}_1 + a}{2} = \dfrac{a_1 + a_2}{2} \Rightarrow a = a_1 + a_2 - \bar{a}_1$

---

assume $a_1 < A_n$. Then $a_i > A_n$ for some i, say $a_2 > A_n$
show $\bar{a}_1 \bar{a}_2 \geq a_1 a_2$
sub in expr. for $\bar{a}_1$ and $\bar{a}_2$ into $\bar{a}_1 \bar{a}_2 \geq a_1 a_2$

$\bar{a}_1 \bar{a}_2 \geq a_1 a_2 \Rightarrow A_n^2 - A_n(a_1 + a_2) + a_1 a_2 \leq 0$
the expr. can be factored
$\Rightarrow a_1(a_2 - A_n) + A_n(A_n - a_2) \leq 0$

$$(A_n - a_2)\left(A_n + \frac{a_1(a_2 - A_n)}{A_n - a_2}\right) = \underbrace{(A_n - a_2)}_{<0}\underbrace{(A_n - a_1)}_{>0} \leq 0$$

$a_1 = A_n$ or $a_2 = A_n \Rightarrow (A_n - a_2)(A_n - a_1) = 0$

$S_0 = \{a_1, \dots, a_n\}$

$a_1, a_2 < A_n$

$$A_n = \frac{a_1 + a_2 + \sum_{i=3}^{n} a_i}{n}$$

$$nA_n - \sum_{i=3}^{n} a_i = a_1 + a_2$$

i) assume $a_i = A_n \quad i = 3, \dots, n$

$\Rightarrow a_1 + a_2 = nA_n - (n-2)A_n = 2A_n$

which can't be true if $a_1, a_2 < A_n$.

$\Rightarrow a_1 = 2A_n - a_2$

iii) assume $a_i = A_n \quad i = 4, \dots, n$

$$A_n = \frac{a_1 + a_2 + a_3 + \sum_{i=4}^{n} a_i}{n}$$

$\Rightarrow a_1 + a_2 + a_3 = nA_n - \sum_{i=4}^{n} a_i = nA_n - (n-3)A_n = 3A_n$

$a_1 = 3A_n - a_2 - a_3$

cases: a_2	a_3	a_1
$< A_n$	$< A_n$	$> A_n$
$< A_n$	$> A_n$	depends
$> A_n$	$< A_n$	depends
$> A_n$	$> A_n$	$< A_n$

b) Prove $G_n \le A_n$, $n = 2^k$

induction on k

$k = 1 \Rightarrow n = 2. \quad S_1 = \{a_1, a_2\}$

if $a_1 = a_2$: $\frac{a_1 + a_2}{2} = A_2$ then $G_2 = \sqrt{A_n^2} = A_n$

if $a_1 \ne A_n$ then $a_1 = 2A_2 - a_2$

Form new set $S_2 = \{\bar{a}_1, \bar{a}_2\}$, $\bar{a}_1 = A_2$, $\bar{a}_2$

$= a_1 + a_2 - \bar{a}_1 = \frac{a_1 + a_2}{2} = A_2$

$G_2' = \sqrt{\bar{a}_1 \bar{a}_2} > \sqrt{a_1 a_2} = G_2$

$A_2' = A_2$

since $S_2 = \{A_2, A_2\}$, $G_2' = A_2 \quad \Rightarrow G_2 < G_2' = A_2$

assume $G_n \le A_n$, $n = 2^k$

$S_0 = \{a_1, \dots, a_{n = 2^{k+1}}\} = \{a_1, \dots, a_{2^k}, \dots a_{2^{k+1}}\}$

$$G_{2^{k+1}} = \sqrt[2^{k+1}]{a_1 \cdot \dots \cdot a_{2^k} \cdot a_{2^k+1} \cdot \dots \cdot a_{2^{k+1}}}$$

$$= \left[(a_1 \cdots a_{2^k})(a_{2^k+1} \cdots a_{2^{k+1}})\right]^{\frac{1}{2^k \cdot 2}}$$

$$= \sqrt{\sqrt[2^k]{a_1 \cdot \dots \cdot a_{2^k}} \sqrt[2^k]{a_{2^k+1} \cdots a_{2^{k+1}}}}$$ This is the geometric mean of two elements

$$\le \frac{\sqrt[2^k]{a_1 \cdot \dots \cdot a_{2^k}} + \sqrt[2^k]{a_{2^k+1} \cdots a_{2^{k+1}}}}{2} \qquad (G_2 \le A_2)$$

$$\le \frac{\frac{\sum_{i=1}^{2^k} a_i}{2^k} + \frac{\sum_{i=1}^{2^k} a_{2^k+i}}{2^k}}{2}$$ (using our induction assumption)

$$= \frac{\sum_{i=1}^{2^{k+1}} a_i}{2^{k+1}} = A_{2^{k+1}}$$

$\Rightarrow G_{2^{k+1}} \le A_{2^{k+1}}$

c) general n, let $2^m > n$. Apply b) to $a_1, \dots, a_n, \underbrace{A_n, \dots, A_n}_{2^m - n \text{ times}}$ to prove $G_n \le A_n$

$$G_{2^m} \le A_{2^m} \Rightarrow \left[a_1 \cdot \dots \cdot a_n \cdot A_n^{2^m - n}\right]^{1/2^m} \le \frac{a_1 + \dots + a_n + (2^m - n)A_n}{2^m}$$

$$\Rightarrow a_1 \cdot \dots \cdot a_n A_n^{2^m - n} \le \left[\frac{a_1 + \dots + a_n + (2^m - n)A_n}{2^m}\right]^{2^m}$$

$$\frac{a_1 + \dots + a_n}{n} = A_n \Rightarrow a_1 + \dots + a_n = nA_n$$

$$\Rightarrow a_1 \cdots a_n \cdot A_n^{2^m - n} \le \left[\frac{\cancel{nA_n} + (2^m - \cancel{n})A_n}{2^m}\right]^{2^m} = A_n^{2^m}$$

$$\Rightarrow a_1 \cdots a_n \le A_n^n \Rightarrow (a_1 \cdots a_n)^{1/n} \le A_n$$

23. recursive def. of a^n

$a^1 = a$

$a^{n+1} = a^n \cdot a$

prove

$a^{n+m} = a^n \cdot a^m$

$(a^n)^m = a^{nm}$

i) $A = \{m \in N \mid a^{n+m} = a^n \cdot a^m\}$

$m=1 \Rightarrow a^{n+1} = a^n \cdot a$ by definition. $\Rightarrow 1 \in A$

assume $k \in A \Rightarrow a^{n+k} = a^n \cdot a^k$

by def $a^{n+k+1} = a^{n+k} \cdot a = a^n \cdot a^k \cdot a = a^n a^{k+1}$

$\Rightarrow k+1 \in A$

ii) $A = \{m \in N \mid (a^n)^m = a^{nm}\}$

$m=1 \Rightarrow (a^n)^1 = a^n = a^{n \cdot 1} \Rightarrow 1 \in A$

assume $k \in A \Rightarrow (a^n)^k = a^{nk}$

multiply both sides by a^n

$(a^n)^k \cdot a^n = a^{nk} \cdot a^n$

by definition on each side

$(a^n)^{k+1} = a^{nk+n} = a^{n(k+1)} \Rightarrow k+1 \in A$

24.

P1 a,b,c any numbers, then $a+(b+c) = (a+b)+c$ (associative law for addition)

P4 a,b any numbers, then $a+b = b+a$ (commutative law for addition)

recursive def. of multiplication

A $1 \cdot b = b$

B $(a+1) \cdot b = a \cdot b + b$

i) Prove $a(b+c) = a \cdot b + a \cdot c$

Let $A = \{a \in N \mid a(b+c) = ab + ac\}$

$a=1 \Rightarrow 1 \cdot (b+c) \overset{A}{=} b + c \overset{A}{=} 1 \cdot b + 1 \cdot c \Rightarrow 1 \in A$

assume $k \in A \Rightarrow k(b+c) = kb + kc$

$(k+1)(b+c)$ by definition of multiplic. is

$\overset{B}{=} k(b+c) + (b+c) \overset{\text{induction assump.}}{=} kb + kc + b + c$

$= (kb+b) + (kc+c)$ P1

$= (k+1)b + (k+1)c$ B

ii) Prove $a \cdot 1 = a$

Let $A = \{a \in N \mid a \cdot 1 = a\}$

$a=1 \Rightarrow a \cdot 1 = 1 \cdot 1 = 1$ A $\Rightarrow 1 \in A$

assume $k \in A \Rightarrow k \cdot 1 = k$

$(k+1) \cdot 1 = k \cdot 1 + 1$ B

$= k+1$ ind. ass.

iii) $a \cdot b = b \cdot a$

$A = \{a \in N \mid a \cdot b = b \cdot a\}$

$a=1 \Rightarrow 1 \cdot a = a \cdot 1$, proved in ii) $\Rightarrow 1 \in A$

assume $k \in A \Rightarrow kb = bk$

$(k+1)b \overset{B}{=} kb + b \overset{\text{ind. ass.}}{=} bk + b = bk + b \cdot 1 \overset{(i)}{=} b(k+1)$

$\Rightarrow k+1 \in A$

25.

a) set A of real numbers is inductive if

(1) $1 \in A$

(2) $k \in A \Rightarrow k+1 \in A$

i) Prove $\mathbb{R}$ is inductive

$1 \in \mathbb{R}$

assume $k \in \mathbb{R}$. Then $k+1 \in \mathbb{R}$

ii) Prove $A = \{k \in \mathbb{R} \mid k > 0\}$ inductive

$1 \in A$

$k \in A \Rightarrow k > 0 \Rightarrow k+1 > 0 \Rightarrow k+1 \in A$

$\Rightarrow$ A inductive

iii) Prove $A = \{k \in \mathbb{R} \mid k > 0, k \neq \frac{1}{2}\}$ inductive

$1 > 0, 1 \neq 1/2 \Rightarrow 1 \in A$

$k \in A \Rightarrow k > 0, k \neq 1/2 \Rightarrow k+1 > 1 \Rightarrow k+1 \in A$

$\Rightarrow$ A inductive

iv) Prove $A = \{k \in \mathbb{R} \mid k > 0, k \neq 5\}$ inductive

$4 \in A \Rightarrow 4+1 = 5 \notin A \Rightarrow$ A not inductive

v) Prove A, B inductive $\Rightarrow C = \{k \in \mathbb{R} \mid k \in A, k \in B\}$ inductive

$1 \in A, 1 \in B \Rightarrow 1 \in C$

$k \in A, k \in B \Rightarrow k+1 \in A, k+1 \in B \Rightarrow k+1 \in C$

$\Rightarrow$ C inductive

b) 1 is in every inductive set. Because $k \in A \Rightarrow k+1 \in A$, 2,3,4,... are all in every induct. set. I.e. $\mathbb{N}$ is in every inductive set.

i) Prove $1 \in \mathbb{N}$

By def. of inductive set, 1 is in every ind. set. $\Rightarrow 1 \in \mathbb{N}$

ii) $k \in \mathbb{N} \Rightarrow k \in$ every inductive set

$\Rightarrow k+1 \in$ every inductive set, by def. of ind. set.

26. 3 spindles

n concentric rings of decreasing diameter

oper. 2 $s_1 \longrightarrow A$

$s_2 \longrightarrow B$

$s_1 \longrightarrow B \Rightarrow s_2 s_1$ on B

in three moves you change location of a two-item stack. $= 2^2 - 1$

$s_3 \to A$

oper 2 $(s_1, s_2) \Rightarrow s_3 s_2 s_1$ on A

one move to lay next largest, 3 moves to change two-item stack.

$s_4 \to B$

oper 3 $\begin{cases} \text{oper } 2(s_1, s_2) \Rightarrow s_2 s_1 \text{ on } C \\ s_3 \to B \\ \text{oper } 2(s_1, s_2) \Rightarrow s_2 s_1 \text{ on } B \end{cases}$

oper 3 moves a 3-item stack, takes 7 moves. $= 2^3 - 1$

oper 4 moves 4 item stack.

→ oper 3 to move 3-item stack

→ move largest piece

→ oper 3 to place 3-item stack back on top

15 moves $= 2^4 - 1$

Proposition: a stack of n rings $r_1, r_2, \dots, r_n$ with $size(r_i) < size(r_j) \iff i < j$, on a spindle, can be moved to another spindle in $2^n - 1$ moves.

$A = \{ n \in \mathbb{N} \mid$ proposition is true $\}$

$n = 1 \Rightarrow$ move the sole ring r_1 to the other spindle.

$\Rightarrow$ 1 move necessary $= 2^1 - 1$

$\Rightarrow 1 \in A$

assume $k \in A \Rightarrow 2^k - 1$ moves to move a k-ring stack

Consider a $(k+1)$ ring stack.

It takes $2^k - 1$ moves to move the topmost k rings.

An additional move to move the bottom ring.

Again $2^k - 1$ to move the k rings onto the bottom ring.

Total moves: $2^k - 1 + 1 + 2^k - 1 = 2^{k+1} - 1$

$\Rightarrow k+1 \in A$

Proposition: $2^n - 1$ is the minimum number of moves to move n ring stack to another spindle.

$A = \{ n \in \mathbb{N} \mid$ proposition is true $\}$

$n = 1 \Rightarrow 2^1 - 1 = 1$ min moves to move one ring.

$\Rightarrow 1 \in A$

assume $n = k \in A \Rightarrow 2^k - 1$ min moves to move k-ring stack.

if we have a $(k+1)$-ring stack, it takes $2^k - 1$ to move the top k, by assumption the min number of moves. Now there is a one ring stack. Min moves to move it is 1. Finally, min to move k rings is another 2^{k-1}. Total $2^{k+1} - 1$.

27. Proposition: For n professors addressed by prof X at meeting 0, all n professors resign at the n^{th} meeting.

$A = \{ n \in \mathbb{N} \mid \text{propos. true for } n \}$

denote meeting x by m_x

$n = 2$

$\Rightarrow$ at m_1 each professor thinks

i) the other professor did not find a mistake in my work

ii) therefore the other prof has concluded that I found a mistake in his work

iii) therefore the other prof. will resign

$\Rightarrow$ no one resigns at m_1

$\Rightarrow$ at m_2 each prof. thinks

i) if the other prof. did not resign, then he must have thought I would resign because he found a mistake in my work.

ii) Because a mistake in my work was found, I will resign.

$\Rightarrow$ both resign at m_2

$n = 3$

m_1: each prof. thinks

i) the other two prof. represent the $n = 2$ case and will resign at m_2

$\Rightarrow$ no one resigns at m_1 or m_2

m_3: each prof. thinks

i) the other two prof.