

Ch 4 Appendix 2 – Polar Coordinates

Assume $\epsilon < 1$.

graph of r in polar coord. is collection of all points P w/ polar coord. (r, θ) satisfying $r = f(\theta)$.

i.e. collection of all points w/ polar coord. $(f(\theta), \theta)$

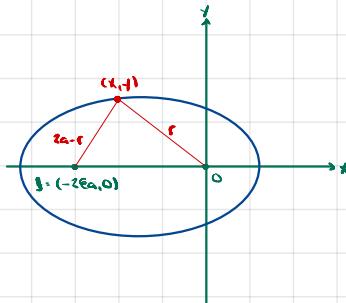
i.e. the graph of the eq. $r = f(\theta)$

equation of ellipse in polar coord.

distance between foci = $0 < 2\epsilon a < 2a$ in ellipse

$$\Rightarrow 0 < \epsilon < 1$$

$$(x, y) \text{ to } O = r = (x^2 + y^2)^{1/2} \Rightarrow r^2 = x^2 + y^2 \quad (1)$$



$$(x, y) \text{ to } \{ \text{foci}\} = 2a - r \text{ by assumption of } f_0(x, y) + f_1(x, y) = 2a = f_0(x, y) = 2a - r$$

$$\Rightarrow (2a - r)^2 = (x + 2\epsilon a)^2 + y^2$$

$$4a^2 - 4ar + r^2 = x^2 + 4\epsilon^2 a^2 + y^2 \quad (2)$$

subtract (1) from (2)

$$4a^2 - 4ar + r^2 - x^2 - y^2 = 4\epsilon^2 a^2$$

$$a - r - xe - \epsilon^2 a$$

$$= a(1 - \epsilon^2) - ex$$

$$= \perp - ex \quad \perp = a(1 - \epsilon^2)$$

$$r = \perp - e \cos \theta$$

$$r(1 + e \cos \theta) = \perp$$

$$r = \frac{\perp}{1 + e \cos \theta}$$

Consider the ellipse defined in Ch. 4, with foci at $(c, 0)$ and $(-c, 0)$ and equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If we move it left by c , we get the ellipse defined above, with foci at $(-2c, 0)$ and $(0, 0)$, assuming we choose $c = \epsilon a$.

Conversely, given the ellipse $r = \frac{\perp}{1 + e \cos \theta}$, $\perp = a(1 - \epsilon^2)$

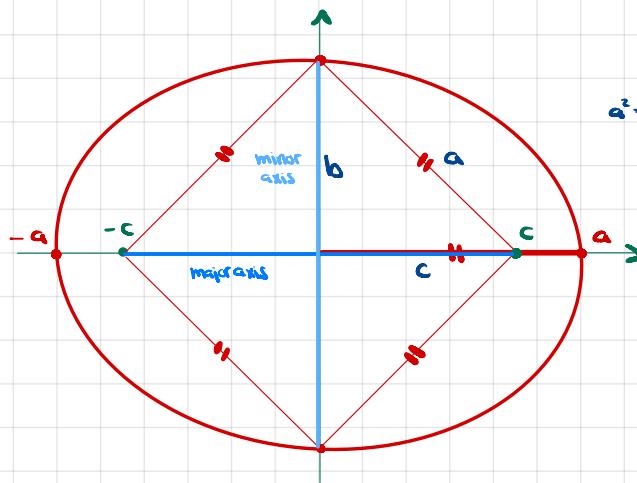
If we want the eq. in Cartesian coord centered on the ellipse's midpoint, we have

$$a = \frac{\perp}{1 - \epsilon^2}$$

$$c = \epsilon a$$

$$b = \sqrt{a^2 - c^2} = \sqrt{a^2 - \epsilon^2 a^2} = a \sqrt{1 - \epsilon^2} = \frac{\perp}{\sqrt{1 - \epsilon^2}}$$

We thus obtain lengths of major and minor axes, a and b , immediately from \perp and ϵ .



$$a^2 - b^2 + c^2 = b^2 = a^2 - c^2$$

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

is the eccentricity of the ellipse, determines "shape" (ratio of major to minor axes)

$$= \frac{\text{distance center to focus}}{\text{distance center to vertex}}$$

Fix a, b, c .

$$0 < \sqrt{a^2 - b^2} < a \Rightarrow 0 < \frac{\sqrt{a^2 - b^2}}{a} < 1$$

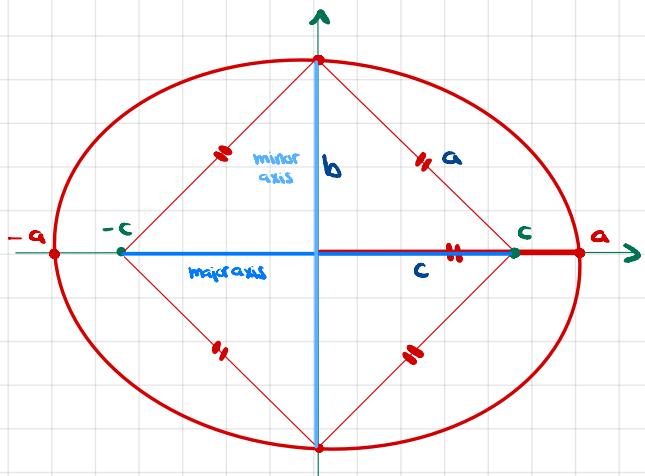
$$\Rightarrow 0 < e < 1$$

$$c \text{ closer to } a \Rightarrow b^2 + a^2 - c^2 \geq 0$$

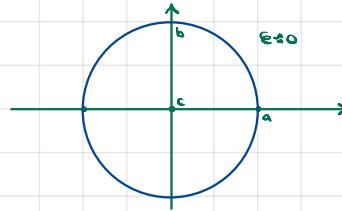
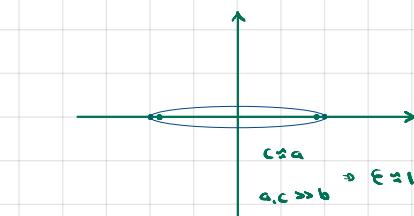
$$\Rightarrow b \approx 0$$

$$\Rightarrow e \approx 1$$

very flat ellipse



$$b \approx a \Rightarrow c \approx 0 \Rightarrow e \approx 0$$

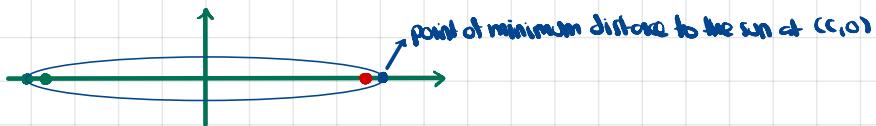


λ determines "size": $\uparrow \lambda$ means $\uparrow r$ for each θ

example Halley's comet has elliptical path around sun with sun at one focus point.

$$e = 0.967 : \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \frac{b^2}{a^2}}$$

$$c = 0.967a$$



$$\text{min distance} = a - c = 0.587 \text{ AU}$$

$$a - 0.967a = 0.587 \Rightarrow a = \frac{0.587}{0.033} = 17.78 \text{ AU}$$

$$\text{max distance} = a + c = a(1.967) = 34.98 \text{ AU} = 3.25 \times 10^9 \text{ miles}$$

$$\min (x-c)^2 + y^2 \quad \text{s.t.} \quad \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

$$g(x) = \frac{1}{a^2}x^2 + \frac{1}{a^2 - c^2}y^2 - 1 = 0$$

$$\nabla f = (2(x-c), 2y)$$

$$\nabla g = \left(\frac{2x}{a^2}, \frac{2y}{a^2 - c^2} \right)$$

$$\nabla f - \lambda \nabla g = 2(x-c) - \frac{\lambda^2 x}{a^2}$$

$$2x - \frac{\lambda^2 x}{a^2 - c^2}$$

$$\Rightarrow 2x(a^2 - c^2) - \lambda^2 x = 0 \Rightarrow x = 0$$

$$y = 0 \Rightarrow \frac{x^2}{a^2} = 1 \Rightarrow x = \pm a$$

$$\Rightarrow f(a-c) = \frac{\lambda \cdot 7a}{a^2}$$

$$\Rightarrow \lambda \cdot a(a-c) > 0$$