Basic Properties of Numbers

P1 a, b, c any numbers, then $a+(b+c)=(a+b)+c$ (associative law for addition)

P2 a any number, then $a+0=0+a=a$ (existence of additive identity)

P3 for every number a, there is number $-a$, s.t. $a+(-a)=(-a)+a=0$ (existence of additive inverses)
$a-b$ is an abbrev. for $a+(-b)$: subtraction is defined in terms of addition

P4 a, b any numbers, then $a+b=b+a$ (commutative law for addition)

P5 a, b, c any numbers, then $a\cdot(b\cdot c)=(a\cdot b)\cdot c$ (associative law for multiplication)

P6 a any number, then $a\cdot 1=1\cdot a=a$ (existence of multiplicative identity)
moreover, $1\neq 0$

P7 for every $a\neq 0$, there is a^{-1} s.t. $a\cdot a^{-1}=a^{-1}\cdot a=1$ (existence of multiplicative inverses)

P8 a, b any numbers, then $a\cdot b=b\cdot a$ (commutative law for multiplication)

P9 if a, b and c are any numbers, then $a\cdot(b+c)=ab+ac$ (distributive law)

* we can now prove $a\cdot 0=0$ for any number a

$a\cdot 0+a\cdot 0=a(0+0)=a\cdot 0$
$a\cdot 0+a\cdot 0+(-(a\cdot 0))=a\cdot 0+(-(a\cdot 0))$
$a\cdot 0=0$

* division is defined in terms of multiplication

a/b means $a\cdot b^{-1}$
$a\cdot 0=0$, for any number a. Therefore, there is no 0^{-1} and no $a/0$.
0^{-1} and $a/0$ are undefined.

* $(-a)\cdot b=-(a\cdot b)$

proof: $(-a)\cdot b+a\cdot b=((-a)+a)b=0\cdot b$
$(-a)\cdot b=-(a\cdot b)$

* $(-a)(-b)=a\cdot b$

proof: $(-a)(-b)-(a\cdot b)=(-a)(-b)+(-b)\cdot a=(-b)(a+(-a))=0$
$(-a)(-b)=a\cdot b$

$\Rightarrow$ product of two negative numbers is positive, consequence of P1-P9

* factorization $(x-1)(x-2)=x^2-3x+2$ involves P9

$x(x-2)+(-1)(x-2)$
$x\cdot x+x(-2)+(-1)x+(-1)(-2)$
$x^2+x((-2)+(-1))+2$
$x^2+x(-3)+2$
x^2-3x+2

P10 (Trichotomy law) for every number a, one and only one of following holds:

(i) $a = 0$
(ii) a is in collection P
(iii) $-a$ is in collection P

P is collection of all positive numbers

P11 (closure under addition) if a and b are in P, then $a+b$ is in P

P12 (closure under multiplication) if a and b are in P, then $a \cdot b$ is in P

Definitions

$a > b$ if $(a-b)$ in P
$a < b$ if $b > a$
$a \geq b$ if $a > b$ or $a = b$
$a \leq b$ if $a < b$ or $a = b$

* corollaries

→ $a > 0$ if a in P

→ $a-b$, being a number, means that one of following must hold

(i) $a-b=0 \Rightarrow a=b$
(ii) $a-b$ in $P \Rightarrow a>b$
(iii) $-(a-b)$ in $P \Rightarrow b-a$ in $P \Rightarrow b>a \Rightarrow a<b$

→ if $a<b$, we have:

$b-a$ in P
$(b+c)-(a+c) = b-a$ in $P \Rightarrow b+c > a+c$

→ $a<0, b<0 \Rightarrow ab>0$
also, $a^2>0$ and therefore $1>0$ because if $a=1$ then $1^2>0$

Absolute Value

$$|a| = \begin{cases} a, & a \geq 0 \\ -a & a \leq 0 \end{cases}$$

* we know that if $a<0$ then $0-a$ in $P \Rightarrow -a$ in $P \Rightarrow -a>0$
so, $|a|$ is always in P

Theorem 1 for all numbers a,b we have $|a+b| \leq |a|+|b|$

Note $|a| = \sqrt{a^2}$

Notes on Assumptions thus far

→ we've assumed numbers are familiar objects; no justification was given
→ however they are defined, they should have P1-P12
→ P1-P12 do not account for all properties of numbers

Derivations

- inverse of x has same sign as x

  $x \cdot x^{-1} = 1$

  i) $x > 0$

- how are formulas for solving quadratic equations derived?