Ch.15 - Trigonometric Functions

1. (i) $f(x) = \arctan(\arctan(\arctan x))$

$$f'(x) = \frac{1}{1+\arctan^2(\arctan x)} \cdot \frac{1}{1+\arctan^2 x} \cdot \frac{1}{1+x^2}$$

(ii) $f(x) = \arcsin(\arctan(\arccos x))$

$$f'(x) = \frac{1}{\sqrt{1-\arctan^2(\arccos x)}} \cdot \frac{1}{1+\arccos^2 x} \cdot \frac{(-1)}{\sqrt{1-x^2}}$$

(iii) $f(x) = \arctan(\tan x \cdot \arctan x)$

$$f'(x) = \frac{1}{1+\tan^2 x \cdot \arctan^2 x}\left[\frac{1}{\sec^2 x} \cdot \arctan x + \tan x \cdot \frac{1}{1+x^2}\right]$$

(iv) $f(x) = \arcsin\left(\frac{1}{\sqrt{1+x^2}}\right)$

But $\frac{1}{\sqrt{1+x^2}} = \sqrt{\arctan'(x)}$

$$(1+x^2)^{-\frac{1}{2}} \quad -\frac{1}{2}(1+x^2)^{-\frac{3}{2}} \cdot 2x$$

$$f'(x) = \frac{1}{\sqrt{1-\frac{1}{1+x^2}}} \cdot \left(-\frac{1}{2}\right)(1+x^2)^{-\frac{3}{2}} \cdot 2x$$

$$= \sqrt{\frac{1+x^2}{\cancel{x^2}}} \cdot (-\cancel{x}) \frac{1}{(1+x^2)^{3/2}}$$

$$= \frac{-1}{1+x^2}$$

2. (i) $\lim_{x\to 0} \frac{\sin x - x + x^3/6}{x^3}$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$

$f'(x) = \cos x - 1 + \frac{x^2}{2}$

$g'(x) = 3x^2$

$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \frac{0}{0}$

$f''(x) = -\sin x + x$

$g''(x) = 6x$

$\lim_{x\to 0} \frac{f''(x)}{g''(x)} = \frac{0}{0}$

$f'''(x) = -\cos x + 1$

$g'''(x) = 6$

$\lim_{x\to 0} \frac{f'''(x)}{g'''(x)} = 0$

By L'Hôpital's Rule, $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$

(ii) $\lim_{x\to 0} \frac{\sin x - x + x^3/6}{x^4}$

$f^{(4)}(x) = \sin x$

$g^{(4)}(x) = 24$

$\lim_{x\to 0} \frac{f^{(4)}(x)}{g^{(4)}(x)} = 0$

By L'Hôpital's Rule, $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$

(iii) $\lim_{x\to 0} \frac{\cos x - 1 + x^2/2}{x^2} = \lim_{x\to 0} \frac{-\sin x + x}{2x} = \lim_{x\to 0} \frac{-\cos x + 1}{2} = 0$

(iv) $\lim_{x\to 0} \frac{\cos x - 1 + x^2/2}{x^4} = \lim_{x\to 0} \frac{-\sin x + x}{4x^3} = \lim_{x\to 0} \frac{-\cos x + 1}{12x^2}$

$= \lim_{x\to 0} \frac{\sin x}{24x} = \lim_{x\to 0} \frac{\cos x}{24} = \frac{1}{24}$

(v) $\lim_{x\to 0} \frac{\arctan x - x + x^3/3}{x^3} = \lim_{x\to 0} \frac{\frac{1}{1+x^2} - 1 + x^2}{3x^2}$

$= \lim_{x\to 0} \frac{\frac{-2x}{(1+x^2)^2} + 2x}{6x} = \lim_{x\to 0} \frac{\frac{-2(1+x^2)^2 + 2x \cdot 2(1+x^2) \cdot 2x}{(1+x^2)^4} + 2}{6}$

$= 0$

(vi) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right) = \lim_{x\to 0} \frac{\sin x - x}{x \sin x}$

$= \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\cos x} = \lim_{x\to 0} \frac{-\sin x}{\cos x + \cos x - x\sin x}$

$= \lim_{x\to 0} \frac{-\sin x}{2\cos x - x\sin x} = 0$

3. $f(x) = \begin{cases} \dfrac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

(a) $f'(0)$

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{\sin(h)}{h} - 1}{h} = \lim_{h \to 0} \frac{\sin(h) - h}{h^2} = \frac{0}{0} = \lim_{h \to 0} \frac{\cos(h) - 1}{2h} = \frac{0}{0}$$

$$= \lim_{h \to 0} \frac{-\sin(h)}{2} = 0$$

(b) $f''(0)$

For $x \neq 0$, $f'(x) = \dfrac{x\cos x - \sin x}{x^2}$

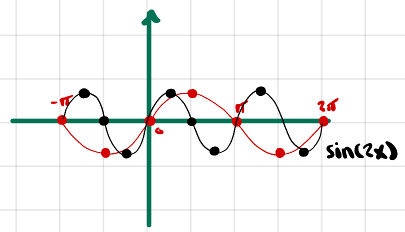
$$f''(0) = \lim_{h \to 0} \frac{f'(h) - f'(0)}{h} = \lim_{h \to 0} \frac{\frac{h\cos(h) - \sin(h)}{h^2}}{h} = \lim_{h \to 0} \frac{h\cos(h) - \sin(h)}{h^3} = \frac{0}{0}$$

$$= \lim_{h \to 0} \frac{\cancel{\cos(h)} - \cancel{h}\sin(h) - \cancel{\cos(h)}}{3h^{\cancel{2}}} = \frac{0}{0} = \lim_{h \to 0} \frac{-\cos(h)}{3} = -\frac{1}{3}$$

4. (a) $f(x) = \sin(2x)$

$\sin(2x)$ just means a new fn that modifies $\sin(x)$ by compressing the graph towards zero.

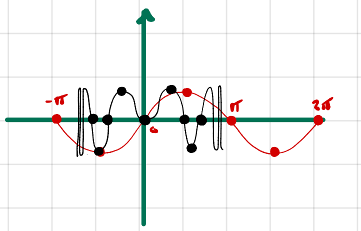
$f'(x) = 2\cos(2x)$

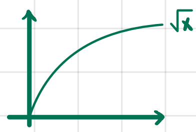


(b) $f(x) = \sin(x^2)$

$f'(x) = 2x\cos(x^2)$

The farther out from 0 the more compressed.





$f(\sqrt{h\pi}) = \sin(h\pi) = 0$

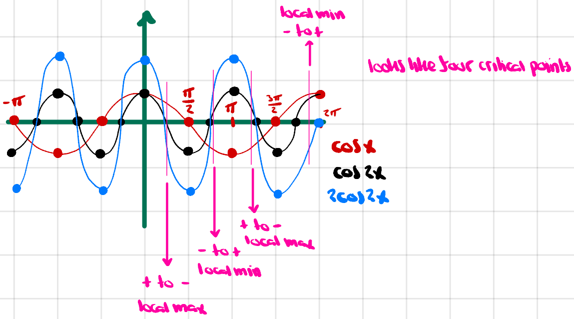
Let $g(h) = \sqrt{h\pi}$

MVT $\to \exists x,\ x \in (h, h+1) \wedge \sqrt{(h+1)\pi} - \sqrt{h\pi} = \dfrac{\pi}{2\sqrt{x\pi}} \to \sqrt{h+1} - \sqrt{h} = \dfrac{1}{2\sqrt{x}} < \dfrac{1}{2\sqrt{h}}$

ie, the distance between successive roots of f gets smaller and smaller.

(c) $f(x) = \sin x + \sin 2x$

$f'(x) = \cos x + 2\cos(2x) = 0$



$\cos(x) + 2(\cos^2(x) - \sin^2(x))$

$\cos(x) + 2(\cos^2(x) - 1 + \cos^2 x)$

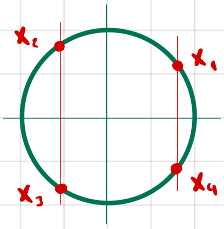
$\cos(x) + 2(2\cos^2(x) - 1)$

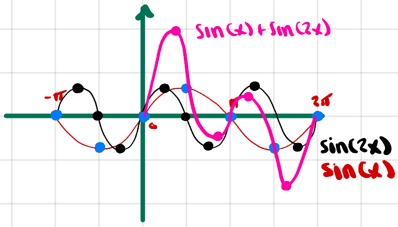
$\cos(x) + 4\cos^2(x) - 2 = 0$

$4\cos^2 x + \cos x - 2 = 0$

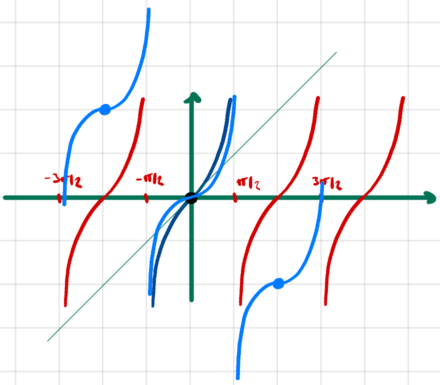
$\Delta = 1 - 4 \cdot 4 \cdot (-2) = 33$

$\cos x = \dfrac{1 \pm \sqrt{33}}{8}$ $\nearrow \approx .84$ $\searrow \approx -.59$

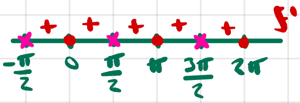




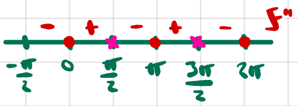
(d) $f(x) = \tan(x) - x$



$f'(x) = \dfrac{1}{\cos^2(x)} - 1 = \sec^2 x - 1 = \tan^2 x = 0 \rightarrow \sin^2(x) = 0 \rightarrow \sin(x) = 0 \rightarrow x = 0$ or $x = \pi$



$f''(x) = \dfrac{-2\cos(x)(-\sin x)}{\cos^4(x)} = \dfrac{2\sin(x)\cos(x)}{\cos^4(x)} = \dfrac{\sin 2x}{\cos^4(x)}$

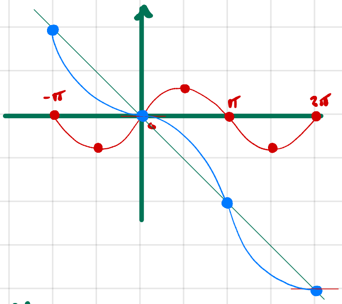


$f'(0) = 0$

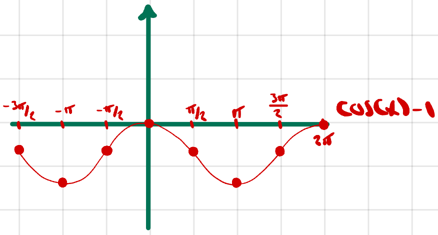
$f(k\pi) = \tan(k\pi) - k\pi = -k\pi$

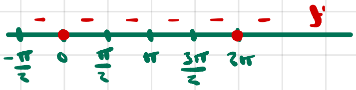
and $k\pi$ is always an inflection point.

(e) $f(x) = \sin(x) - x$

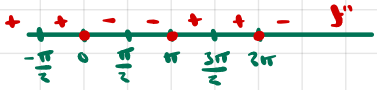


$f'(x) = \cos(x) - 1$





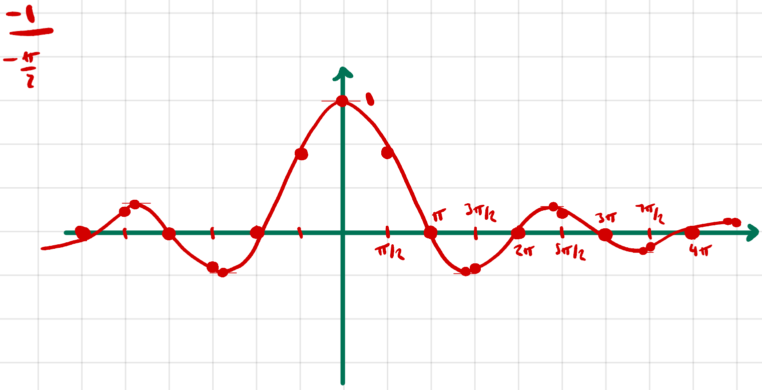
$f''(x) = -\sin(x)$



$f'(k\pi) = -2$

$f'(k \cdot 2\pi) = 0$

(f) $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$



From problem 3 we know

$f'(0) = 0$

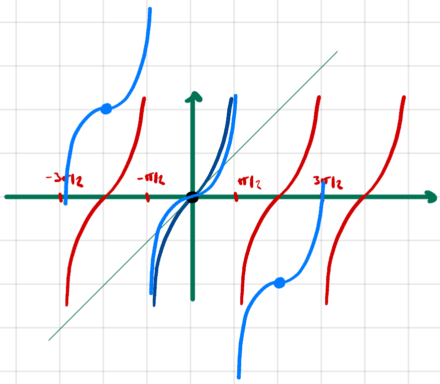
$$f'(x) = \frac{x\cos x - \sin x}{x^2} \quad x \neq 0$$

$$f''(0) = -\frac{1}{3}$$

$$f''(x) = \frac{\cancel{\cos x} - x\sin x - \cancel{\cos x}}{x^4} = -\frac{\sin x}{x^3} \quad x \neq 0$$

$$f'(x) = 0 \rightarrow \frac{x\cos x - \sin x}{x^2} = 0 \rightarrow x = \tan x$$

From part (d) where we had $g(x) = \tan x - x$ we saw that there was one root in every $\left(\frac{k\pi}{2}, \frac{(k+2)\pi}{2}\right)$

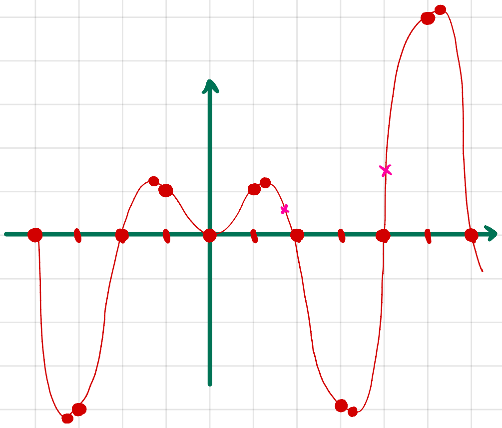


(g) $f(x) = x\sin x$

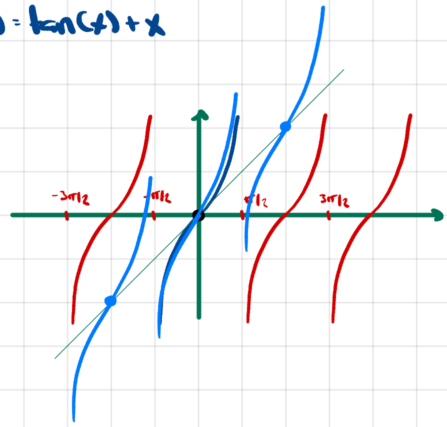
$f'(x) = \sin x + x\cos x = 0 \rightarrow x = -\frac{\sin x}{\cos x} = -\tan x$ or $\tan x = -x$

$f(-x) = -x\sin(-x) = -x(-\sin x) = x\sin x = f(x)$ (even)

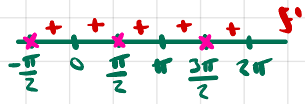
$f''(x) = \cos x + \cos x - x\sin x = 2\cos x - x\sin x$



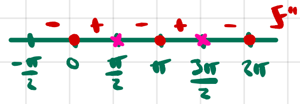
$f(x) = \tan(x) + x$



$f'(x) = \frac{1}{\cos^2(x)} + 1 > 0 \rightarrow$ no critical points



$f''(x) = \frac{-2\cos(x)(-\sin x)}{\cos^4(x)} = \frac{2\sin(x)\cos(x)}{\cos^4(x)} = \frac{\sin 2x}{\cos^4(x)}$



$f'(0) = 2$

$f(k\pi) = \tan(k\pi) + k\pi = k\pi$

and $k\pi$ is always an inflection point.

$f'(k\pi) = 2$

f has a root in each $\left(\frac{k\pi}{2}, \frac{(k+2)\pi}{2}\right)$,

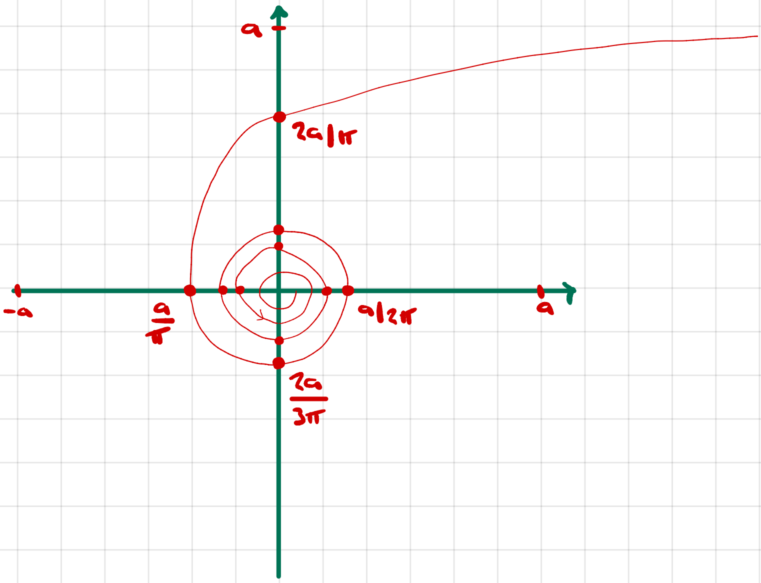
$k = 2n+1$ for $n \in \mathbb{Z}$.

5. $f(\theta) = \dfrac{a}{\theta}$

graph of f in polar coordinates is hyperbolic spiral

$(r, \theta) = (f(\theta), \theta) = (a/\theta, \theta)$

θ	r	$r_{a=1}$
0	$\to \pm\infty$	∞
$\pi/2$	$2a/\pi$	$2/\pi \approx 2/3$
π	a/π	$1/\pi \approx 1/3$
$3\pi/2$	$2a/3\pi$	$2/3\pi \approx 2/9$
2π	$a/2\pi$	$1/2\pi \approx 1/6$
$5\pi/2$	$2a/5\pi$	$2/5\pi \approx 2/15$



$x(\theta) = r\cos\theta = \dfrac{a\cos\theta}{\theta}$

$y(\theta) = r\sin\theta = \dfrac{a\sin\theta}{\theta}$

$\lim\limits_{\theta\to 0} y(\theta) = \dfrac{0}{0} = \lim\limits_{\theta\to 0} a\cos\theta = a$

$\lim\limits_{\theta\to 0} x(\theta) = \infty$

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6. let $g(x) = \cos(x+y)$

$g'(x) = -\sin(x+y)$

$g''(x) = -\cos(x+y)$

$g'' + g = 0$

$g(0) = \cos y$

$g'(0) = -\sin y$

and by Th. 4 we have

$g = (-\sin y)\sin + \cos y \cos \quad \forall x$

since this is true for all y we have $\forall x \forall y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

7. (a) $\sin 2x = \sin(x+x) = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - 1 + \cos^2 x = 2\cos^2 x - 1$

$\sin 3x = \sin(2x+x) = \sin 2x\cos x + \sin x\cos 2x$

$= 2\sin x\cos^2 x + \sin(x)(2\cos^2 x - 1)$

$= 4\sin x\cos^2 x - \sin x$

or

$2\sin x\cos^2 x + \sin x(\cos^2 x - \sin^2 x)$

$= 3\sin x\cos^2 x - \sin^3 x$

$\cos 3x = \cos(2x+x) = \cos 2x\cos x - \sin 2x\sin x$

$= 2\cos^3 x - \cos x - 2\sin^2 x\cos x$

or $(\cos^2 x - \sin^2 x)\cos x - 2\sin^2 x\cos x$

$= \cos^3 x - 3\sin^2 x\cos x = \cos^3 x - 3\cos x + 3\cos^3 x$

$= 4\cos^3 x - 3\cos x$

(b) $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\cos(\pi/2) = \cos(2\cdot\pi/4)$

$= \cos^2\frac{\pi}{4} - \sin^2\frac{\pi}{4}$

$= 2\cos^2\frac{\pi}{4} - 1$

$= 0$

$\rightarrow \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

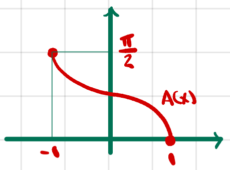
$* \; A(x) = \frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}\,dt$

Note that in form. we have implicitly used to choose the positive square root

$\pi = 2\cdot\int_{-1}^{1}\sqrt{1-x^2}\,dx \rightarrow \int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{\pi}{2}$

But $\sqrt{1-x^2}$ is even and so $\int_{-1}^{0}\sqrt{1-x^2}\,dx = \int_{0}^{1}\sqrt{1-x^2}\,dx = \frac{\pi}{4}$

Here is the graph of $A(x)$



$\cos x = 0 \rightarrow A(\cos x) = A(0) = \frac{x}{2} = \frac{\pi}{4} \rightarrow x = \frac{\pi}{2}$

ie $\cos(\pi/2) = 0$

we also know that $\cos x = -1 \rightarrow A(\cos x) = A(-1) = \frac{\pi}{2} = \frac{x}{2} \rightarrow x = \pi$

ie $\cos(\pi) = -1$. Similarly, $\cos(0) = 1$

Also, $\cos'(x) = -\sin(x) < 0$ in $[0,\pi]$.

Hence, $\cos(x) < 0$ for $x \in (\pi/2, \pi)$ and

$\cos(x) > 0$ for $x \in (0, \pi/2)$

T.F., at $\frac{\pi}{4}$, cos is positive.

$\sin(\pi/2) = \sin(2\cdot\pi/4)$

$= 2\sin\frac{\pi}{4}\cos\frac{\pi}{4} = 1$

$= \sqrt{2}\sin\pi/4$

$\rightarrow \sin\pi/4 = \frac{\sqrt{2}}{2}$

$\tan(\pi/4) = 1$

$$\frac{\sin(\pi/4)}{\cos(\pi/4)} = 1$$

$\sin(\pi/6) = \frac{1}{2}$

$$\cos(\pi/2) = \cos(3\cdot\pi/6) = 4\cos^3(\pi/6) - 3\cos(\pi/6) = 0$$

$$\cos(\pi/6)\left(4\cos^2(\pi/6) - 3\right) = 0$$

Now, $0 < \frac{\pi}{6} < \frac{\pi}{2} \to \cos(\pi/6) > 0$

$\to \cos^2(\pi/6) = \frac{3}{4} \to \cos(\pi/6) = \frac{\sqrt{3}}{2}$

$\to \sin(\pi/6) = \sqrt{1 - 3/4} = \frac{1}{2}$

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8. (a) $A\sin(x+B)$ can be written $a\sin x + b\cos x$

Let $f(x) = A\sin(x+B)$. Then,

$f'(x) = A\cos(x+B)$

$f''(x) = -A\sin(x+B)$

Hence, $f'' + f = 0$

$f(0) = A\sin(B)$

$f'(0) = A\cos(B)$

By Th. 4,

$$f(x) = f'(0)\sin x + f(0)\cos x$$

$$= A\cos(B)\sin(x) + A\sin(B)\cos(x)$$

$$= a\sin x + b\cos x$$

Now assume we are given a, b. Then, there are A and B s.t.

$$a\sin x + b\cos x = A\sin(x+B)$$

Let B s.t. $\frac{a}{\cos B} = \frac{b}{\sin B} \to \tan B = \frac{b}{a}$, if $a \neq 0$.

$\to B = \tan^{-1}(b/a)$

Define A as

$$A = \frac{a}{\cos B} = \frac{b}{\sin B}$$

Then $a = A\cos B$
$b = A\sin B$

and $a^2 + b^2 = A^2 \to A = \sqrt{a+b}$

Then

$$A\sin(x+B) = A\sin x\cos B + A\sin B\cos x$$

$$= A\left(\frac{a}{A}\sin x + \frac{b}{A}\cos x\right)$$

$$= a\sin x + b\cos x$$

If $a = 0$ then pick $B = \frac{\pi}{2}$.

(c) $f(x) = \sqrt{3}\sin x + \cos x$

let $a = \sqrt{3}$, $b = 1$.

By part b), $A = \sqrt{3+1} = 2$, $B = \tan^{-1}(\sqrt{3}/3)$

and we can write

$f(x) = A\sin(x+B)$ for all x

How do we find $\tan^{-1}(\sqrt{3}/3) = x$?

$\tan x = \frac{\sqrt{3}}{3} = \frac{\sin x}{\cos x}$

$\sqrt{3}\cos x = 3\sqrt{1-\cos^2 x}$

$3\cos^2 x = 9 - 9\cos^2 x$

$12\cos^2 x = 9$

$\cos^2 x = \frac{3}{4}$

$\cos x = \frac{\sqrt{3}}{2}$

By 7b, $x = \pi/6$.

Hence, $B = \pi/6$ and $f(x) = 2\sin(x + \pi/6)$

9. (a) $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$, if $x, y, (x+y)$ are not $k\pi + \frac{\pi}{2}$

Proof

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \sin y \cos x}{\cos x \cos y - \sin x \sin y} = \frac{\dfrac{\sin x \cos y + \sin y \cos x}{\cos x \cos y}}{\dfrac{\cos x \cos y - \sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}}{1 - \dfrac{\sin x \sin y}{\cos x \cos y}} = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

(b) $\arctan x + \arctan y = \arctan\left(\dfrac{x+y}{1-xy}\right)$

Proof

From

$$\tan(x_1 + y_1) = \frac{\tan x_1 + \tan y_1}{1 - \tan x_1 \tan y_1} \quad (1)$$

let x_1 and y_1 be the arctan's of x and y.

Recall that arctan has domain $\mathbb{R}$ and image $(-\pi/2, \pi/2)$

T.F. $\arctan x$ and $\arctan y$ are in $(-\pi/2, \pi/2)$, x and y are in $\mathbb{R}$.

The restrictions we had for (1) were that $\tan x_1$, $\tan x_2$, and $\tan(x_1 + y_1)$ be defined, ie $x_1, y_1, x_1 + y_1$ not $k\pi + \frac{\pi}{2}$.

Therefore, $\arctan x$, $\arctan y$, and $\arctan x + \arctan y \neq k\pi + \frac{\pi}{2}$

The only violation possible is $\arctan x + \arctan y = \pm \frac{\pi}{2}$, so this is the only restriction.

$$\tan(\arctan x + \arctan y) = \frac{x+y}{1-xy}$$

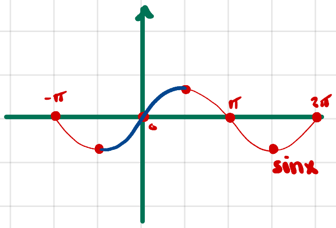
Take the arctan of each side

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$$

10. $\arcsin\alpha + \arcsin\beta = \arcsin\left(\alpha\sqrt{1-\beta^2} + \beta\sqrt{1-\alpha^2}\right)$ $\quad \alpha,\beta \in [-1,1]$

Proof

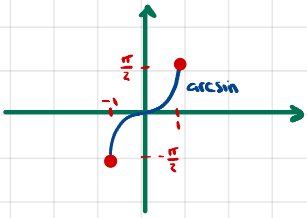
$\sin(\arcsin\alpha + \arcsin\beta) = \alpha\cos(\arcsin\beta) + \beta\cos(\arcsin\alpha)$



What is $\cos(\arcsin x)$?

$[\sin x]' = \cos x$ for $x \in \mathbb{R}$

if $x \in (-1,1)$ then we can say



$[\sin x]' = \cos x = \dfrac{1}{\arcsin'(\sin x)}$ if $\arcsin'(\sin x) \neq 0$, which is true in $(-1,1)$.

hence

$[\sin(\arcsin\beta)]' = \cos(\arcsin\beta) = \dfrac{1}{\arcsin'(\beta)} = \sqrt{1-\beta^2}$

also $\cos(\arcsin\alpha) = \sqrt{1-\alpha^2}$

$\sin(\arcsin\alpha + \arcsin\beta) = \alpha\sqrt{1-\beta^2} + \beta\sqrt{1-\alpha^2}$

Now we want to take the arcsin of each side

Note that $\alpha,\beta \in [-1,1]$ and $\arcsin\alpha, \arcsin\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

arcsin is the inverse of $\sin(x)$, $x \in [-\pi/2, \pi/2]$.

Thus, $\arcsin\alpha + \arcsin\beta \in [-\pi,\pi]$. We need to consider what happens for all such possible values of $\arcsin\alpha + \arcsin\beta$.

CASE 1

Therefore, for $-\pi/2 \leq \arcsin\alpha + \arcsin\beta \leq \pi/2$ we have

$\arcsin\alpha + \arcsin\beta = \arcsin\left(\alpha\sqrt{1-\beta^2} + \beta\sqrt{1-\alpha^2}\right)$

CASE 2

If $\frac{\pi}{2} < \arcsin\alpha + \arcsin\beta \leq \pi$ then it is no longer true that

$\arcsin(\sin(\arcsin\alpha + \arcsin\beta)) = \sin(\arcsin\alpha + \arcsin\beta)$

However, since $\sin(x) = \sin(\pi - x)$

$\sin(\pi - \arcsin\alpha - \arcsin\beta) = \alpha\sqrt{1-\beta^2} + \beta\sqrt{1-\alpha^2}$

$\pi - \arcsin\alpha - \arcsin\beta = \arcsin\left(\alpha\sqrt{1-\beta^2} + \beta\sqrt{1-\alpha^2}\right)$

$\arcsin\alpha + \arcsin\beta = \pi - \arcsin\left(\alpha\sqrt{1-\beta^2} + \beta\sqrt{1-\alpha^2}\right)$

CASE 3

If $-\pi \leq \arcsin\alpha + \arcsin\beta < -\pi/2$ then since $\sin(-\pi - x) = \sin(x)$

$\sin(-\pi - \arcsin\alpha - \arcsin\beta) = \alpha\sqrt{1-\beta^2} + \beta\sqrt{1-\alpha^2}$

$-\pi - \arcsin\alpha - \arcsin\beta = \arcsin\left(\alpha\sqrt{1-\beta^2} + \beta\sqrt{1-\alpha^2}\right)$

$\arcsin\alpha + \arcsin\beta = -\pi - \arcsin\left(\alpha\sqrt{1-\beta^2} + \beta\sqrt{1-\alpha^2}\right)$

11. m,n any numbers then

$$\sin(mx)\sin(nx) = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x] \quad \text{(i)}$$

$$\sin(mx)\cos(nx) = \frac{1}{2}[\sin(m+n)x - \sin(m-n)x] \quad \text{(ii)}$$

$$\cos(mx)\cos(nx) = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x] \quad \text{(iii)}$$

Proof

$\cos[(m-n)x] = \cos(mx-nx)$
$= \cos(mx)\cos(-nx) - \sin(mx)\sin(-nx)$
$= \cos(mx)\cos(nx) + \sin(mx)\sin(nx)$

$\cos[(m+n)x] = \cos(mx+nx)$
$= \cos(mx)\cos(nx) - \sin(mx)\sin(nx)$

$\frac{1}{2}[\cos(m-n)x - \cos(m+n)x] = \frac{1}{2}\cdot 2\sin(mx)\sin(nx) = \sin(mx)\sin(nx)$ (i)

$\frac{1}{2}[\cos(m-n)x + \cos(m+n)x] = \frac{1}{2}\cdot 2\cos(mx)\cos(nx) = \cos(mx)\cos(nx)$ (iii)

$\sin[(m-n)x] = \sin(mx-nx)$
$= \sin(mx)\cos(-nx) + \sin(-nx)\cos(mx)$
$= \sin(mx)\cos(nx) - \sin(nx)\cos(mx)$

$\sin[(m+n)x] = \sin(mx+nx)$
$= \sin(mx)\cos(nx) + \sin(nx)\cos(mx)$

$\frac{1}{2}[\sin(m+n)x - \sin(m-n)x] = \frac{1}{2}\, 2\sin(nx)\cos(mx) = \sin(nx)\cos(mx)$ (ii)

12. $m, n \in \mathbb{N}$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

Proof

$m = n$

$$\int_{-\pi}^{\pi} \sin^2(mx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos 0 - \cos 2mx]\,dx = \frac{1}{2}\cdot\left[2\pi - \frac{\sin(2mx)}{2m}\Big|_{-\pi}^{\pi}\right] = \pi - \frac{1}{2}(0-0) = \pi$$

$m \neq n$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[\cos(mx-nx) - \cos(mx+nx)]\,dx \quad = \frac{1}{2}\left[\frac{\sin(mx-nx)}{m-n} - \frac{\sin(mx+nx)}{m+n}\right]\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2}[0-0] = 0$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

Proof

$m = n$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \int_{-\pi}^{\pi}\cos^2(mx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[\cos(2mx) + \cos 0]\,dx$$

$$= \frac{1}{2}\left[\frac{\cos(2mx)}{2m}\Big|_{-\pi}^{\pi} + 2\pi\right]$$

$$= \pi + \left(\frac{1}{2m} - \frac{1}{2m}\right)$$

$$= \pi$$

$m \neq n$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}(\cos(m+n)x + \cos(m-n)x)\,dx \quad = \frac{1}{2}\left[\frac{\sin[(m+n)x]}{m+n} + \frac{\sin[(m-n)x]}{m-n}\right]\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2}[0+0] = 0$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0$$

Proof

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[\sin(m+n)x - \sin(m-n)x]\,dx$$

$$= \frac{1}{2}\left[-\frac{\cos[(m+n)x]}{m+n} + \frac{\cos[(m-n)x]}{m-n}\right]\Big|_{-\pi}^{\pi}$$

$= 0$, since $\cos(\alpha x)$ is even.

13. (a)

f int. on $[-\pi, \pi]$.

min of $\int_{-\pi}^{\pi}[f(x) - a\cos(nx)]^2 dx$

occurs when $a = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$

Proof

Let $h(a) = \int_{-\pi}^{\pi}[f(x) - a\cos(nx)]^2 dx$

$$= \int_{-\pi}^{\pi}\left[f(x)^2 - 2f(x)\cdot a\cos(nx) + a^2\cos^2(nx)\right]dx$$

$$= \int_{-\pi}^{\pi} f(x)^2 dx - a\cdot 2\int_{-\pi}^{\pi} f(x)\cos(nx)dx + a^2\int_{-\pi}^{\pi}\cos^2(nx)dx$$

$$h'(a) = -2\int_{-\pi}^{\pi} f(x)\cos(nx)dx + 2a\underbrace{\int_{-\pi}^{\pi}\cos^2(nx)dx}_{\pi\text{, see prob. 12}} = 0$$

$$\rightarrow a = \frac{\int_{-\pi}^{\pi} f(x)\cos(nx)dx}{\pi}$$

Note that h is a quadratic polyn. in a, thus h is cont. e.w.

Local extrema occur at critical points, of which there is just one.

$\lim_{a\to\infty} h(a) = \lim_{a\to-\infty} h(a) = \infty$

Hence,

$$a = \frac{\int_{-\pi}^{\pi} f(x)\cos(nx)dx}{\pi}$$ is global min.

Note also that

$$h''(a) = 2\int_{-\pi}^{\pi}\cos^2(nx)dx > 0$$

■

min of $\int_{-\pi}^{\pi}[f(x) - a\sin(nx)]^2 dx$ occurs at $a = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$

Proof

Let $h(a) = \int_{-\pi}^{\pi}[f(x) - a\sin(nx)]^2 dx = \int_{-\pi}^{\pi}\left[f(x)^2 - 2f(x)\cdot a\sin(nx) + a^2\sin^2(nx)\right]dx$

$= \int_{-\pi}^{\pi} f(x)^2 dx - a\cdot 2\int_{-\pi}^{\pi} f(x)\sin(nx)dx + a^2\int_{-\pi}^{\pi}\sin^2(nx)dx$, a quadratic in a.

$$h'(a) = -2\int_{-\pi}^{\pi} f(x)\sin(nx)dx + 2a\int_{-\pi}^{\pi}\sin^2(nx)dx$$

$$\rightarrow a = \frac{\int_{-\pi}^{\pi} f(x)\sin(nx)dx}{\pi}$$ global min

$h''(a) = 2\pi > 0$

(b) Define

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad n=0,1,2,\dots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \qquad n=1,2,3,\dots$$

c_i, d_i any numbers.

Then,

$$\int_{-\pi}^{\pi}\Big(f(x) - \underbrace{\Big[\frac{c_0}{2} + \sum_{n=1}^{N}(c_n\cos(nx) + d_n\sin(nx))\Big]}_{\text{linear comb. of } \cos(nx) \text{ and } \sin(nx),\ n=1,\dots,N}\Big)^2 dx$$

$$= \int_{-\pi}^{\pi}\Big[f(x)^2 - 2f(x)\Big[\frac{c_0}{2} + \sum_{n=1}^{N}(c_n\cos(nx)+d_n\sin(nx))\Big] + \Big[\frac{c_0}{2} + \sum_{n=1}^{N}(c_n\cos(nx)+d_n\sin(nx))\Big]^2\Big]dx$$

$$= \int_{-\pi}^{\pi}(f(x))^2dx - c_0\int_{-\pi}^{\pi}f(x)\,dx - \int_{-\pi}^{\pi}\sum_{n=1}^{N}\big[2c_nf(x)\cos(nx) + 2d_nf(x)\sin(nx)\big]$$
$$+ \int_{-\pi}^{\pi}\Big[\frac{c_0^2}{4} + c_0\sum_{n=1}^{N}(c_n\cos(nx) + d_n\sin(nx)) + \Big[\sum_{n=1}^{N}(c_n\cos(nx)+d_n\sin(nx))\Big]^2\Big]$$

$\to 0, \int_{-\pi}^{\pi}\cos(nx)dx = 0$ $\quad \to 0, \int_{-\pi}^{\pi}\sin(nx)dx = 0$

$$= \int_{-\pi}^{\pi}(f(x))^2dx - \pi c_0a_0 - 2\pi\sum_{n=1}^{N}(c_na_n + d_nb_n) + \frac{c_0^2\pi}{2} + \sum_{n=1}^{N}\Big[\int_{-\pi}^{\pi}c_n^2\cos(nx)^2 + \int_{-\pi}^{\pi}d_n^2\sin(nx)^2\Big]$$

note that every other term is of the form

$c_nd_m\int_{-\pi}^{\pi}\cos(nx)\sin(mx)dx = 0$ by problem 12.

$$= \int_{-\pi}^{\pi}(f(x))^2dx - \pi c_0a_0 - 2\pi\sum_{n=1}^{N}(c_na_n + d_nb_n) + \frac{c_0^2\pi}{2} + \sum_{n=1}^{N}(c_n^2\pi + d_n^2\pi)$$

add and subtract some terms to complete squares $\Big\{ + \pi\big(\sum(a_n^2+b_n^2)\big) - \sum(a_n^2+b_n^2)\big) \quad + \frac{\pi a_0^2}{2} - \frac{\pi a_0^2}{2}$

$$= \int_{-\pi}^{\pi}(f(x))^2dx + \pi\Big(\sum_{n=1}^{N}c_n^2 - 2\sum_{n=1}^{N}c_na_n + \sum_{n=1}^{N}a_n^2\Big) + \pi\Big(\sum_{n=1}^{N}d_n^2 - 2\sum_{n=1}^{N}d_nb_n + \sum_{n=1}^{N}b_n^2\Big)$$
$$+ \pi\Big(\frac{a_0^2}{2} - \frac{a_0}{\sqrt{2}}\cdot\frac{c_0}{\sqrt{2}}\cdot 2 + \frac{c_0^2}{2}\Big) - \frac{\pi a_0^2}{2} - \pi\sum(a_n^2+b_n^2)$$

$$= \underbrace{\int_{-\pi}^{\pi}(f(x))^2dx}_{\text{a number}} + \underbrace{\pi\sum_{n=1}^{N}(c_n-a_n)^2 + \pi\sum_{n=1}^{N}(d_n-b_n)^2 + \pi\Big(\frac{a_0}{\sqrt{2}} - \frac{c_0}{\sqrt{2}}\Big)^2}_{\text{this part we can choose by picking } c_n \text{ and } d_n,\ n=0,\dots,N} \underbrace{- \frac{\pi a_0^2}{2} - \pi\sum_{n=1}^{N}(a_n^2+b_n^2)}_{\text{this is a number}}$$

note that a_n, b_n are numbers. we can pick c_n and d_n.

if we pick $c_n = a_n$ and $b_n = d_n$ then the middle portion above (which is ≥ 0) is minimized at 0.

Thus the entire expr. is minimized.

14. (a) $\sin x + \sin y$

$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$

$\sin(a-b) = \sin(a)\cos(-b) + \sin(-b)\cos(a)$

$= \sin(a)\cos(b) - \sin(b)\cos(a)$

$\sin(a+b) + \sin(a-b) = 2\sin(a)\cos(b)$

Given any numbers x and y, we can always find a and b s.t. $x = a+b$ and $y = a-b$.

$x = a+b \to a = x-b$

$y = a-b \to y = x-2b \to b = \dfrac{x-y}{2}$

$\to a = x-b = \dfrac{2x-x+y}{2} = \dfrac{x+y}{2}$

Thus,

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

Also,

$\sin(a+b) - \sin(a-b) = 2\sin(b)\cos(a)$

So,

$$\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

(b) $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

$\cos(a-b) = \cos(a)\cos(-b) - \sin(a)\sin(-b)$
$= \cos(a)\cos(b) + \sin(a)\sin(b)$

$\cos(a+b) + \cos(a-b) = 2\cos(a)\cos(b)$

$\cos(a+b) - \cos(a-b) = -2\sin(a)\sin(b)$

T.F.

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

15. (a) $\cos(2x) = \cos^2 x - \sin^2 x = \cos^2 x - (1-\cos^2 x) = 2\cos^2 x - 1$

$\to \cos^2 x = \dfrac{1+\cos(2x)}{2}$

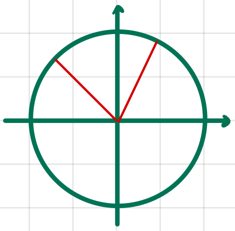
$\cos(2x) = \cos^2 x - \sin^2 x = 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x$

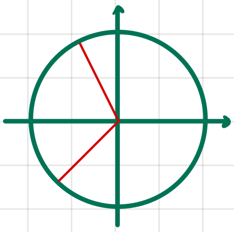
$\to \sin^2 x = \dfrac{1-\cos(2x)}{2}$

(b) $\cos\dfrac{x}{2} = \sqrt{\dfrac{1+\cos x}{2}} \qquad 0 \leq x \leq \dfrac{\pi}{2}$

Proof

$\cos(x) = \cos(2\cdot\frac{x}{2}) \to \cos^2(x/2) = \dfrac{1+\cos x}{2} \to \underbrace{\cos(x/2) = \pm\sqrt{\dfrac{1+\cos x}{2}}}$





the positive sq. root occurs when $\cos(x/2) \geq 0$ which means

$-\frac{\pi}{2} \leq \frac{x}{2} \leq \frac{\pi}{2} \quad \to -\pi \leq x \leq \pi$

if $\pi < x < 2\pi$ then $\frac{\pi}{2} \leq \frac{x}{2} \leq \pi$

and $\cos(x/2) \leq 0$ so we would need to take the neg. sq. root.
Same for $-2\pi \leq x \leq -\pi$.

Since we are assuming $0 \leq x/2 \leq \pi/2$ then we take the pos. sq. root.

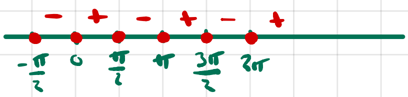
(c) $\displaystyle\int_a^b \sin^2 x\, dx = \int_a^b \frac{1-\cos(2x)}{2} dx$

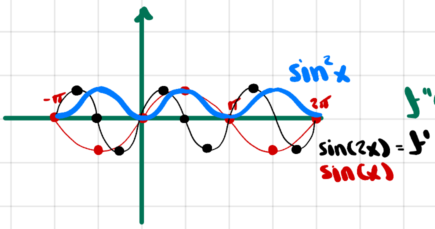
$= \dfrac{b-a}{2} - \dfrac{1}{2}\left.\dfrac{\sin(2x)}{2}\right|_a^b = \dfrac{b-a}{2} - \dfrac{\sin(2b)-\sin(2a)}{4}$

$\displaystyle\int_a^b \cos^2 x\, dx = \int_a^b \frac{1+\cos(2x)}{2} = \frac{b-a}{2} + \frac{\sin(2b)-\sin(2a)}{4}$

(d) $f(x) = \sin^2 x$

$f'(x) = 2\sin x\cos x$
$= \sin(2x)$





$f''(x) = 2\cos^2 x - 2\sin^2 x = 2\cos(2x)$

16. note: $y = \arctan x \to x = \tan y = \frac{\sin y}{\cos y} = \frac{\sin y}{\sqrt{1-\sin^2 y}} = \sin y \cdot \arcsin'(\sin y)$

$\sin(\arctan x) = x\sqrt{1-\sin^2 y}$

$\sin(\arctan x)^2 = x^2(1-\sin(\arctan x)^2)$

$\sin(\arctan x)^2(1+x^2) = x^2$

$\sin(\arctan x)^2 = \frac{x^2}{1+x^2}$

$\sin(\arctan x) = \sqrt{\frac{x^2}{1+x^2}}$

$\cos(\arctan x) = \sqrt{1-\sin^2(\arctan x)} = \sqrt{\frac{1}{1+x^2}}$

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17. $x = \tan\left(\frac{u}{2}\right)$

$\arctan x = \frac{u}{2}$

$\sin(\arctan x) = \sqrt{\frac{x^2}{1+x^2}} = \sin(u/2) = \pm\sqrt{\frac{1-\cos u}{2}} \to \frac{2x^2}{1+x^2} = 1-\cos u$

$\sin(x) = \sin(2 \cdot x/2) = 2\sin(x/2)\cos(x/2)$

$\cos(x) = \cos(2 \cdot x/2) = \cos^2(x/2) - \sin^2(x/2)$
$= 1 - 2\sin^2(x/2)$

$\to \sin^2(x/2) = \frac{1-\cos x}{2}$

$\sin(x/2) = \pm\sqrt{\frac{1-\cos x}{2}}$

$\to \cos u = \frac{1-x^2}{1+x^2}$

$\sin u = \sqrt{1-\cos^2 u} = \sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}} = \frac{\sqrt{4x^2}}{1+x^2} = \frac{2x}{1+x^2}$

$\cancel{1}+2x^2+\cancel{x^4} - (\cancel{1}-2x^2+\cancel{x^4})$

another way

$\arctan x = \frac{u}{2} \to u = 2\arctan x \to \sin u = \sin(2\arctan x) = 2\sin(\arctan x)\cos(\arctan x)$

$= 2\sqrt{\frac{x^2}{1+x^2}}\sqrt{\frac{1}{1+x^2}} = \frac{2x}{1+x^2}$

18. (a) $\sin(x+\pi/2) = \cos x$

$\sin(x+\pi/2) = \sin x \cdot 0 + \cos x \cdot 1 = \cos(x)$

(b) From part a), $\sin(x+\pi/2) = \cos x$, for all x.

we'd like to take arcsin of both sides.

If $-\pi/2 \le x+\pi/2 \le \pi/2$ then $x \in [-\pi, 0]$ and

$x + \pi/2 = \arcsin(\cos x) \qquad -\pi \le x \le 0$

If $x+\pi/2 \in [\pi/2, 3\pi/2]$ then $x \in [0,\pi]$ and

$\sin(x+\pi/2) = \sin(\pi - x - \pi/2) = \cos(x)$

and since $(\pi - x - \pi/2) \in [-\pi/2, \pi/2]$ we have

$\pi - x - \pi/2 = \pi/2 - x = \arcsin(\cos x)$

At this point, we have

$$\arcsin(\cos x) = \begin{cases} x + \pi/2 & -\pi \le x \le 0 \\ \pi - x - \pi/2 & 0 \le x \le \pi \end{cases}$$

If $x = 2\pi h + x'$ for $x' \in [-\pi, 0]$ then

$x + \pi/2 = 2\pi h + \pi/2 + x' \in [-\pi/2 + 2\pi h, \pi/2 + 2\pi h]$

and since $x' + \pi/2 \in [-\pi/2, \pi/2]$

$\sin(x+\pi/2) = \sin(2\pi h + \pi/2 + x') = \sin(x' + \pi/2) = \sin(x - 2\pi h + \pi/2)$

$\arcsin(\cos x) = x - 2\pi h + \pi/2$

If $x = 2\pi h + x'$ for $x' \in [0,\pi]$ then

$x + \pi/2 = 2\pi h + \pi/2 + x' \in [2\pi h + \pi/2, 2\pi h + 3\pi/2]$

$x' + \pi/2 \in [\pi/2, 3\pi/2]$

$\sin(x+\pi/2) = \sin(2\pi h + x' + \pi/2) = \sin(x' + \pi/2) = \sin(x - 2\pi h + \pi/2)$

$= \sin(\pi - x + 2\pi h - \pi/2) = \sin(\pi/2 - x + 2\pi h)$, since $\pi/2 - x + 2\pi h = \pi/2 - x' \in [-\pi/2, \pi/2]$

hence, $\pi/2 - x + 2\pi h = \arcsin(\cos x)$

$$\arcsin(\cos x) = \begin{cases} x - 2\pi h + \pi/2 & x \in [-\pi + 2\pi h, 2\pi h] \\ \pi/2 - x + 2\pi h & x \in [2\pi h, \pi + 2\pi h] \end{cases}$$

* note $\cos : \mathbb{R} \to [-1,1]$

$\arcsin : [-1,1] \to [-\pi/2, \pi/2]$

$\arcsin \circ \cos : \mathbb{R} \to [-\pi/2, \pi/2]$

Now let's find $\arccos(\sin x)$

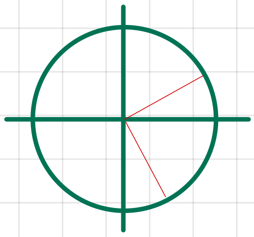
Note that $\cos(x-\pi/2) = \cos(x)\cancelto{0}{\cos(-\pi/2)} - \sin(x)\sin(-\pi/2)$
$= \sin(x)$

$x-\pi/2 \in [0,\pi] \rightarrow x \in [\pi/2, 3\pi/2]$

$\arccos(\sin x) = x - \pi/2$

$x-\pi/2 \in [-\pi,0] \rightarrow x \in [-\pi/2,\pi/2] \wedge 0 \leq \pi/2 - x \leq \pi$

Since $\cos(x) = \cos(-x)$ then



$\cos(x-\pi/2) = \cos(\pi/2 - x) = \sin x$

$\arccos(\sin x) = \pi/2 - x$

$x - \pi/2 = x' - \pi/2 + 2\pi h,\ x'-\pi/2 \in [0,\pi] \rightarrow \frac{\pi}{2} \leq x' \leq \frac{3\pi}{2}$

$\cos(x-\pi/2) = \cos(x' - \pi/2 + 2\pi h) = \cos(x' - \pi/2) = \cos(x - 2\pi h - \pi/2)$

$\arccos(\sin(x)) = x - \pi/2 - 2\pi h$

$x-\pi/2 = x' - \pi/2 + 2\pi h,\ x'-\pi/2 \in [-\pi,0] \rightarrow x' \in [-\pi/2,\pi/2]$

$\cos(x-\pi/2) = \cos(x'-\pi/2+2\pi h) = \cos(\underbrace{x'+2\pi-\pi/2}_{[\pi,2\pi]}+2\pi h) = \cos(\underbrace{x'+\frac{3\pi}{2}}_{[\pi,2\pi]}+2\pi h)$

$= \cos(x'+3\pi/2) = \cos(x - 2\pi h + 3\pi/2)$

$= \cos(\underbrace{2\pi - x + 2\pi h - 3\pi/2}_{[0,\pi]}) = \cos(\pi/2 - x + 2\pi h)$

$\rightarrow \arccos(\sin x) = \pi/2 - x + 2\pi h$

Hence,

$$\arccos(\sin x) = \begin{cases} x - \pi/2 - 2\pi h & 2\pi h \leq x \leq \pi + 2\pi h \\ \pi/2 - x + 2\pi h & 2\pi h - \pi \leq x \leq 2\pi h \end{cases}$$

Proof that $\cos(x) = \cos(-x)$

Case 1: $0 \leq -x \leq \pi$

$\to -\pi \leq x \leq 0 \to \pi \leq x + 2\pi \leq 2\pi$

$\to \cos(x) = \cos(2\pi(-1) + x + 2\pi) = \cos(x + 2\pi)$

$= \cos(2\pi - x - 2\pi) = \cos(-x)$

Case 2: $\pi \leq -x \leq 2\pi \to -2\pi \leq x \leq -\pi \to 0 \leq x + 2\pi \leq \pi$

$\to \cos(x) = \cos(2\pi(-1) + x + 2\pi) = \cos(x + 2\pi)$

Also, $\cos(-x) = \cos(2\pi + x)$

Case 3: $-x = 2\pi h + x'$, $x' \in [0, 2\pi]$

$\cos(-x) = \cos(x') = \cos(-x') = \cos(-x' - 2\pi h) = \cos(x)$

In all three cases, $\cos(x) = \cos(-x)$.

Recall the def of cos and sin

$A(\cos x) = \frac{x}{2}$	$0 \leq x \leq \pi$
$\cos x = \cos(2\pi - x)$	$\pi \leq x \leq 2\pi$
$\cos x = \cos x'$	$x = 2\pi h + x'$, $x' \in [0, 2\pi]$
$\sin x = \sqrt{1 - \cos^2 x}$	$0 \leq x \leq \pi$
$\sin x = -\sin(2\pi - x)$	$\pi \leq x \leq 2\pi$
$\sin x = \sin x'$	$x = 2\pi h + x'$, $x' \in [0, 2\pi]$

Proof that $\sin(x) = -\sin(-x)$

Case 1: $0 \leq -x \leq \pi \to -\pi \leq x \leq 0 \to \pi \leq x + 2\pi \leq 2\pi$

$x = (x + 2\pi) + 2\pi(-1)$, and hence

$\sin(x) = \sin(x + 2\pi) = -\sin(2\pi - x - 2\pi) = -\sin(-x)$

Case 2: $\pi \leq -x \leq 2\pi \to -2\pi \leq x \leq -\pi \to 0 \leq x + 2\pi \leq \pi$

By def. of sin, $\sin(-x) = -\sin(2\pi + x)$

$x = (x + 2\pi) + 2\pi(-1)$, hence

$\sin(x) = \sin(x + 2\pi) = \sin(-x)$

Case 3: $-x = 2\pi h + x'$, $x' \in [0, 2\pi]$

$\sin(-x) = \sin(x') = -\sin(-x') = -\sin(-x' - 2\pi h) = -\sin(x)$

In all three cases, $\sin(x) = -\sin(-x)$

Proof that $\sin(\pi - x) = \sin(x)$

$$\sin(\pi - x) = \cancelto{0}{\sin(\pi)}\cos(-x) + \sin(-x)\cos(\pi)$$
$$= -\sin(-x)$$
$$= \sin(x)$$

$$\cos(\pi - x) = \cos(\pi)\cos(-x) - \cancelto{0}{\sin(\pi)}\sin(-x)$$
$$= -\cos(-x)$$
$$= -\cos(x)$$

19. (a) $\int_0^1 \frac{1}{1+t^2}\,dt$

$\arctan'(x) = \frac{1}{1+x^2}$

Since $\frac{1}{1+x^2}$ is integrable, by FTC2

$$\int_0^1 \frac{1}{1+t^2}\,dt = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

(b) $\int_0^\infty \frac{1}{1+t^2}\,dt$

$\arctan'(x) = \frac{1}{1+x^2}$ is diff. everywhere.

Therefore

$$F(x) = \int_0^x \arctan'(t)\,dt$$

$$F'(x) = \arctan(x)$$

and

$$\int_1^\infty \arctan'(t)\,dt = \lim_{n\to\infty} \int_1^n \arctan'(t)\,dt = \lim_{n\to\infty} \left[\arctan(n) - \arctan(1)\right] = \frac{\pi}{2} - \frac{\pi}{4}$$

hence

$$\int_0^\infty \frac{1}{1+t^2}\,dt = \frac{\pi}{2}$$

Proof that $\lim_{x\to\frac{\pi}{2}^-} \tan x = \infty$

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\sin x}{\cos x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\sqrt{1-\cos^2 x}}{\cos x} = \lim_{x\to\frac{\pi}{2}^-} \sqrt{\frac{1-\cos^2 x}{\cos^2 x}} = \infty$$

Proof that $\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$

Let $\epsilon > 0$.

Then, $\frac{\pi}{2} - \epsilon < \frac{\pi}{2}$

Let $x > \tan(\pi/2 - \epsilon)$. Then $\frac{\pi}{2} > \arctan(x) > \frac{\pi}{2} - \epsilon$

That is, $\forall \epsilon > 0\ \exists N > 0\ \forall x\ x > N \rightarrow \left|\arctan(x) - \frac{\pi}{2}\right| < \epsilon$

$\rightarrow \lim_{x\to\infty} \arctan(x) = \pi/2$

20. $\lim_{x\to\infty} x\sin(1/x) = \lim_{x\to\infty} \frac{\sin(1/x)}{\frac{1}{x}} = \frac{0}{0}$

Note that

$$\lim_{x\to\infty} \frac{\cos(x)\,\frac{-1}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \cos(x) \text{ which doesn't exist} \to \text{can't use L'Hôpital}$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\cos(x)}{1} = 1$$

$\lim_{x\to\infty} \sin(1/x)$

$\forall M>0 \quad x>M \to \frac{1}{x} < \frac{1}{M}$

$\forall \epsilon>0$ let $\frac{1}{M} = \epsilon \to M = \frac{1}{\epsilon}$. Then $x>M \to \frac{1}{x} < \epsilon$

$\to \lim_{x\to\infty} 1/x = 0$

sin is continuous everywhere. Hence

$$\lim_{x\to\infty} \sin(1/x) = \sin(\lim_{x\to\infty} 1/x) = \sin 0 = 0$$

21. (a) Define

$$\sin^\circ(x) = \sin\left(\frac{\pi x}{180}\right)$$

$$\cos^\circ(x) = \cos\left(\frac{\pi x}{180}\right)$$

Recall that previously, $\sin^\circ x$ had x in degrees and $\sin^r x$ had x in radians.

Because 2π rad $= 360°$, we had $1° = \frac{2\pi}{360}$ rad and

$\sin^\circ x = \sin^r\left(\frac{2\pi}{360}x\right)$ with x in degrees.

$\sin^{\circ\prime}$ tells us the rate of change of $\sin^\circ$ w/ respect to degrees.

r.o.c. of rad rel. to degrees

$$\sin^{\circ\prime}(x) = \cos\left(\frac{\pi x}{180}\right)\cdot\frac{\pi}{180} = \frac{\pi\cos^\circ(x)}{180}$$

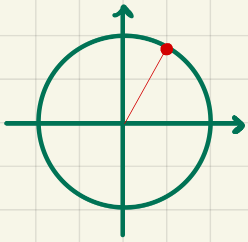
$$\cos^{\circ\prime}(x) = -\sin\left(\frac{\pi x}{180}\right)\cdot\frac{\pi}{180} = \frac{-\pi\sin^\circ(x)}{180}$$

(b) $$\lim_{x\to 0}\frac{\sin^\circ(x)}{x} = \frac{0}{0} = \lim_{x\to 0}\frac{\cos\left(\frac{\pi x}{180}\right)\cdot\frac{\pi}{180}}{1} = \frac{\pi}{180}$$

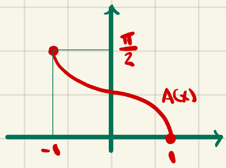
$$\lim_{x\to\infty} x\sin^\circ(1/x) = \lim_{x\to 0^+}\frac{1}{x}\sin^\circ(x) = \lim_{x\to 0^+}\frac{\sin\left(\frac{\pi x}{180}\right)}{x} = \frac{0}{0} = \lim\frac{\cos\left(\frac{\pi x}{180}\right)\cdot\frac{\pi}{180}}{1} = \frac{\pi}{180}$$

Altern.

$$= \lim_{x\to 0^+}\frac{\pi}{180}\cdot\frac{\sin\left(\frac{\pi x}{180}\right)}{\frac{\pi x}{180}} = \frac{\pi}{180}$$

22.  For $x \in [-1,1]$, we defined $A(x) = \frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}\,dt$

the graph of which is



We defined $\cos(x)$ for $x \in [0,\pi]$ by $A(\cos x) = \frac{x}{2}$ and $\sin x = \sqrt{1-\cos^2 x}$.

$x \in [0,\pi] \to \frac{x}{2} \in [0, \pi/2]$

The image of A is $[0,\pi/2]$. Hence, for every $\frac{x}{2} \in [0,\pi/2]$ there is a $y \in [-1,1]$

s.t. $A(y) = \frac{x}{2}$. Furthermore, y is unique since A is decreasing (hence one-one).

Therefore $y = \cos x$ is defined and unique for every $x \in [0,\pi]$.

Now, $\sin^2 x + \cos^2 x = 1$, so each point $(\cos x, \sin x)$ is on the unit circle.

Now, for $\pi < x \leq 2\pi$ we have $\cos(x) = \cos(2\pi - x)$ and $\sin(x) = -\sin(2\pi - x) = -\sqrt{1-\cos^2(2\pi - x)}$

Note that $2\pi - x \in [0,\pi]$. Therefore, for $x \in [\pi, 2\pi]$ we have $\cos(x) \in [-1,1]$,

and $\sin(x) \in [-1,0]$

Suppose (x,y) is a point on the unit circle, ie $x^2 + y^2 = 1$.

Then $y = \pm\sqrt{1-x^2}$, $1 - x^2 \geq 0 \to x^2 \leq 1 \to x \in [-1,1]$

$\to \sqrt{1-x^2} \in [0,1] \to y \in [-1,1]$

For any given x there is a $z \in [0,\pi]$ s.t. $\cos(z) = x$ and a $w \in [\pi, 2\pi]$ s.t. $x = \cos(w)$.

Thus $x = \cos(z) = \cos(w)$

If $y \geq 0$ then $y = \sqrt{1-\cos^2(z)} = \sqrt{1-\cos^2(w)} = \sin(z) = -\sin(w)$

If $y \leq 0$ then $y = -\sqrt{1-\cos^2(z)} = -\sin(z) = -\sqrt{1-\cos^2(w)} = \sin(w)$

Thus every point (x,y) is of form $(\cos(\theta), \sin(\theta))$

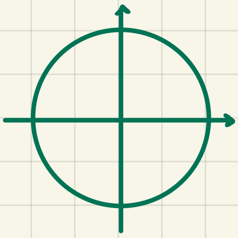
23. (a) π is max. possible length of an interval on which sin is one-one.

Such an interval is of form

$$[2k\pi - \tfrac{\pi}{2}, 2k\pi + \tfrac{\pi}{2}] \quad \text{or} \quad [2k\pi + \tfrac{\pi}{2}, 2(k+1)\pi - \tfrac{\pi}{2}]$$

Proof

Any interval containing $\frac{\pi}{2} + 2\pi k$ or $\frac{3\pi}{2} + 2\pi k$ as an interior point is not one-one.



Proof Assume $A = (\frac{\pi}{2} + 2\pi k - \delta, \frac{\pi}{2} + 2\pi k + \delta)$

Then for any $\delta_1 < \delta$ we have $\frac{\pi}{2} + 2\pi k \pm \delta_1 \in A$ and $\sin(\pi/2 + 2\pi k + \delta_1) = \sin(\pi/2 + 2\pi k - \delta_1)$.

$*$ $\sin(\pi/2 + x) = \cos x$
$\sin(\pi/2 - x) = \cos(-x)$ $\rightarrow \sin(\pi/2 + x) = \sin(\pi/2 - x)$

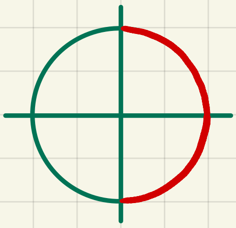
The result for $\frac{3\pi}{2} + 2\pi k$ is proved analogously.

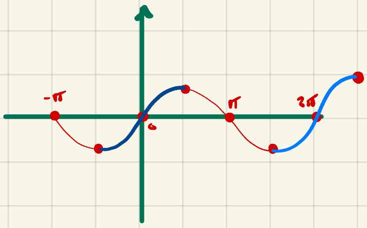
The largest possible intervals not containing $\frac{\pi}{2} + 2\pi k$ or $\frac{3\pi}{2} + 2\pi k$ as interior points are of form

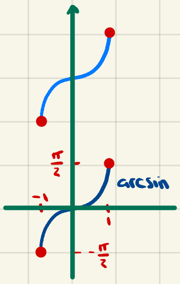
$$[2k\pi - \tfrac{\pi}{2}, 2k\pi + \tfrac{\pi}{2}] \quad \text{or} \quad [2k\pi + \tfrac{\pi}{2}, 2(k+1)\pi - \tfrac{\pi}{2}]$$

and sin is one-one on them.

(b) $g(x) = \sin x$, $x \in (-\pi/2 + 2k\pi, \pi/2 + 2k\pi)$





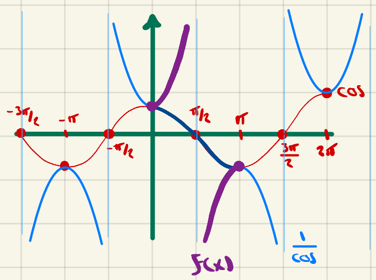


There is one fn g for each k.
Relative to arcsin, g^{-1} differs by $2\pi k$.

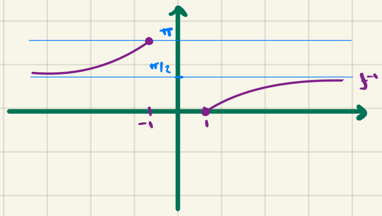
Therefore, $(g^{-1})' = \arcsin'$

24. $f(x) = \sec(x) = \frac{1}{\cos x}$ $\quad 0 \le x \le \pi$

$\frac{1}{\cos x}$ not defined at $x = \frac{\pi}{2} + \pi k,\ k \in \mathbb{Z}$.



The image of f is $[1, \infty) \cup (-\infty, -1]$, and this is the domain of f^{-1}.



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25. $|\sin x - \sin y| < |x - y|$ for all $x \neq y$

let $x \neq y$.

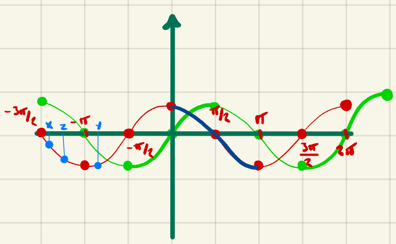
MVT $\to$ $\exists c,\ c \in (x, y) \wedge \frac{\sin x - \sin y}{x - y} = \sin'(c) = \cos(c)$

$\to -1 \le \frac{\sin x - \sin y}{x - y} \le 1$ equality only if $c = \pi k,\ k \in \mathbb{Z}$

$\left|\frac{\sin x - \sin y}{x - y}\right| \le 1 \to |\sin x - \sin y| \le |x - y|$

let $x < y$ be any numbers, and let $c \in (x, y)$ s.t. $\frac{\sin y - \sin x}{y - x} = \sin'(c) = \cos(c)$

choose z s.t. $\pi k \notin (x, z)$. Then



$|\sin y - \sin x| = |\sin y - \sin z + \sin z - \sin x|$

$= |(y - z)\cos(c_1) + (z - x)\cos(c_2)|$

where $c_1 \in (z, y)$ and $c_2 \in (x, z)$, and therefore $c_2 \neq k\pi \to |\cos(c_2)| < 1$ and $|\cos(c_1)| \le 1$

$\to |\sin y - \sin x| \le |(y - z)\cos(c_1)| + |(z - x)\cos(c_2)|$

$= |y - z||\cos(c_1)| + |z - x||\cos(c_2)|$

$\le |y - z| + |z - x|$

$= y - x$

$= |y - x|$

Alternative Proofs

From problem 14, $\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$.

$$\left|2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)\right| = 2\left|\sin\frac{x-y}{2}\right|\left|\cos\frac{x+y}{2}\right| \leq 2\left|\sin\frac{x-y}{2}\right|$$

Assume $\sin(y) \leq \sin(x)$.

$$|\sin y - \sin x| = \sin y - \sin x = \int_x^y \cos t\,dt = \int_x^y dt - \int_x^y dt + \int_x^y \cos t\,dt = (y-x) - \int_x^y (1-\cos t)\,dt$$

$$= (y-x) - 2\int_x^y \frac{1-\cos t}{2} = (y-x) - 2\int_x^y \sin^2(t/2)\,dt$$

Let $h(x) = \sin^2(x/2) \geq 0$ and not identically zero in any interval.

Then $\int_a^b h(x)\,dx > 0$. Hence $\int_x^y \sin^2(t/2)\,dt > 0$ and thus, $|\sin y - \sin x| < y - x = |y-x|$

recall $\sin^2(x)$. $\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x \to \sin^2 x = \frac{1-\cos 2x}{2}$

thus $\sin^2(x/2) = \frac{1-\cos(x)}{2}$

Def: $r > 0$, $S \subset \mathbb{R}$

S is r-discrete $\leftrightarrow [x, y \in S \wedge x \neq y \to |x-y| \geq r]$

Let $r > 0$ and $S \subset \mathbb{R}$ r-discrete.

$f : \mathbb{R} \to \mathbb{R}$
$f' > 0$ for all $x \in \mathbb{R} \setminus S$ $\quad \to f$ strictly increasing

26. $\lim_{\lambda\to\infty} \int_a^b f(x)\sin(\lambda x)\,dx$

(a) $\lim_{\lambda\to\infty} \int_c^d \sin\lambda x\,dx = 0$

* $f(x) = \sin(\lambda x)$

Proof

$\lambda \to \infty$ means the graph is compressed extreme

$f(x) = \dfrac{-\cos(\lambda x)}{\lambda} \to f'(x) = \sin(\lambda x)$

f is cont. everywhere.

FTC2 $\to \int_c^d \sin(\lambda x)\,dx = -\frac{1}{\lambda}\left(\cos(\lambda d) - \cos(\lambda c)\right)$

$-\frac{1}{\lambda}\cdot 2 \leq -\frac{1}{\lambda}(\cos\lambda d - \cos\lambda c) \leq -\frac{1}{\lambda}\cdot 2$

$\to \lim_{\lambda\to\infty} \int_c^d \sin\lambda x\,dx = 0$ (squeeze theorem)

(b) recall

fn s defined on $[a,b]$ is called a step function if there is partition $P = \{t_0, \dots, t_n\}$ of $[a,b]$ such that s is constant on each (t_{i-1}, t_i). Note that the values of f at t_i are arbitrary.

let s be step fn on $[a,b]$. Then $\lim_{\lambda\to\infty} \int_a^b s(x)\sin\lambda x\,dx = 0$

Proof

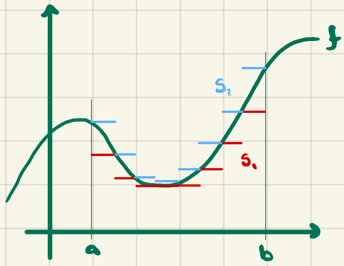
s is defined based on a partition $P = \{t_0, \dots, t_n\}$ of $[a,b]$

$\lim_{\lambda\to\infty} \int_a^b s(x)\sin(\lambda x)\,dx = \lim_{\lambda\to\infty} \sum_{i=1}^{n} s_i \int_{t_{i-1}}^{t_i} \sin(\lambda x)\,dx = 0$

(c) recall results proved in 13-26

(a) f integrable on $[a,b]$ $\rightarrow$ $\forall \epsilon > 0$ $\exists$ step fn $s_1 \leq f$ w/ $\int_a^b f - \int_a^b s_1 < \epsilon$

$\exists$ step fn $s_2 \geq f$ w/ $\int_a^b s_2 - \int_a^b f < \epsilon$



(b) $\forall \epsilon > 0$, $\exists s_1 \leq f$ and $s_2 \geq f$ s.t $\int_a^b s_2 - \int_a^b s_1 < \epsilon$ $\rightarrow$ f integrable

Assume f integrable on $[a,b]$. Given λ, let $P_\lambda = \{t_0, \dots, t_{n_\lambda}\}$ be a partition of $[a,b]$ s.t. for any $i = 1, \dots, n_\lambda$ we have that $\forall x, x \in [t_{i-1}, t_i] \rightarrow \sin(\lambda x) \geq 0$ or $\forall x, x \in [t_{i-1}, t_i] \rightarrow \sin(\lambda x) < 0$.
Let s_1 and s_2 be step fns for P_λ s.t. $s_1 < f$ and $s_2 > f$, ie $s_1 < f < s_2$. Then for any $[t_{i-1}, t_i]$ either

1) $\sin(\lambda x) \geq 0$ $\rightarrow$ $s_1(x)\sin(\lambda x) < f(x)\sin(\lambda x) < s_2(x)\sin(\lambda x)$

2) $\sin(\lambda x) \leq 0$ $\rightarrow$ $s_2(x)\sin(\lambda x) < f(x)\sin(\lambda x) < s_1(x)\sin(\lambda x)$

In either case if we integrate and take the limit

$$0 = \lim_{\lambda \to \infty} s_1 \int_{t_{i-1}}^{t_i} \sin \lambda x \, dx < \lim_{\lambda \to \infty} \int_{t_{i-1}}^{t_i} f(x) \sin \lambda x \, dx < \lim_{\lambda \to \infty} s_2 \int_{t_{i-1}}^{t_i} \sin \lambda x \, dx = 0$$

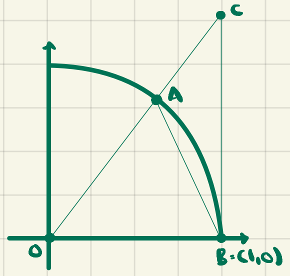
$$0 = \lim_{\lambda \to \infty} s_1 \int_{t_{i-1}}^{t_i} \sin \lambda x \, dx < \lim_{\lambda \to \infty} \int_{t_{i-1}}^{t_i} f(x) \sin \lambda x \, dx < \lim_{\lambda \to \infty} s_2 \int_{t_{i-1}}^{t_i} \sin \lambda x \, dx = 0$$

$$\rightarrow \lim_{\lambda \to \infty} \int_{t_{i-1}}^{t_i} f(x) \sin \lambda x \, dx = 0$$

$$\int_a^b f(x) \sin(\lambda x) \, dx = \sum_{i=1}^{n_\lambda} \int_{t_{i-1}}^{t_i} f(x) \sin(\lambda x) \, dx = 0$$

$$\lim_{\lambda \to \infty} \int_a^b f(x) \sin(\lambda x) \, dx = \lim_{\lambda \to \infty} \sum_{i=1}^{n_\lambda} \int_{t_{i-1}}^{t_i} f(x) \sin(\lambda x) \, dx = 0$$

27.



(a) $0 < x < \pi/2 \to \frac{\sin x}{2} < \frac{x}{2} < \frac{\sin x}{2\cos x}$

Proof

$$\text{Area}(OCB) = \frac{CB}{2} = \frac{\tan x}{2}$$

$$\text{Area}(OAB) = \frac{\sin x}{2}$$

$$\frac{\sin x}{2} < \frac{x}{2} < \frac{\tan x}{2}$$

(b) $\cos(x) < \frac{\sin(x)}{x} < 1$

Proof

From a), $\sin x < x < \frac{\sin x}{\cos x}$

$\cos x > 0,\ x > 0.$

$$\cos x < \frac{\sin x}{x} < 1$$

$$\lim_{x \to 0} \cos x = 1$$

$$\to \lim_{x \to 0} \frac{\sin x}{x} = 1$$

(c) $\lim_{x \to 0} \frac{1-\cos x}{x} = \frac{0}{0} = \lim_{x \to 0} \frac{\sin x}{1} = 0$

Altern.

$$\lim_{x \to 0} \frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)}$$

$$= \lim_{x \to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$$

$$= 1 \cdot 0 = 0$$

(d) $\sin'$

$$\sin'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \left[-\sin(x) \frac{(1-\cos(h))}{h} + \cos(x) \frac{\sin(h)}{h} \right]$$

$$= -\sin(x) \cancel{\lim_{h \to 0} \frac{1-\cos h}{h}}^{\,0} + \cos(x) \lim_{h \to 0} \frac{\sin h}{h}$$

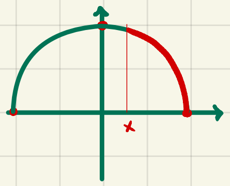
$$= \cos(x)$$

28. $f(x) = \sqrt{1-x^2} \quad x \in [-1,1]$

$L(x)$ = length of f on $[x,1]$

(a) $L(x) = \int_x^1 \frac{1}{\sqrt{1-t^2}}\,dt$

Proof



$f'(x) = \frac{-x}{\sqrt{1-x^2}}$ $\quad$ $+ \quad 0 \quad - \quad f'$

$$f''(x) = \frac{-\sqrt{1-x^2} - x\frac{(-x)}{\sqrt{1-x^2}}}{1-x^2} = \frac{-1-x^2+x^2}{(1-x^2)^{3/2}} = \frac{-1}{(1-x^2)^{3/2}} < 0$$

top half of unit circle

In Problem 13-25 we started with the length $\ell(f,P)$ of a polygonal curve inscribed in the graph of f, and defined the length of f on $[a,b]$ as the sup of $\ell(f,P)$.

We showed that this number is $\int_a^b \sqrt{1+(f')^2}$ if the integrand is in fact integrable.

$f'(x)^2 = \frac{x^2}{1-x^2}$

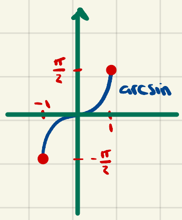
$1+f'(x)^2 = \frac{1}{1-x^2}$

$\sqrt{1-(f'(x))^2} = \frac{1}{\sqrt{1-x^2}}$

Therefore, $\int_x^1 \frac{1}{\sqrt{1-t^2}}\,dt$ is in fact the length of f on $[x,1]$, if $\frac{1}{\sqrt{1-x^2}}$ is integrable on $[x,1]$.

recall that $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$, $x \in (-1,1)$

recall the graph of arcsin



arcsin' is unbounded near 1. Hence $\int_x^1 \arcsin'(t)\,dt$ is improper and is defined

$$\lim_{\varepsilon \to 1^-} \int_x^{\varepsilon} \arcsin'(t)\,dt = \lim_{\varepsilon \to 1^-} \left[\arcsin(\varepsilon) - \arcsin(x)\right] = \frac{\pi}{2} - \arcsin(x)$$

(b) $L'(x) = \frac{-1}{\sqrt{1-x^2}}$ $x \in (-1,1)$

$L(x) = \int_x^1 \frac{1}{\sqrt{1-t^2}}\,dt = \int_0^1 \frac{1}{\sqrt{1-t^2}}\,dt - \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt$

$\frac{1}{\sqrt{1-t^2}}$ is cont. on $(-1,1)$, hence by FTC I

$L'(x) = \frac{-1}{\sqrt{1-x^2}}$ $x \in (-1,1)$

(c) $\pi = L(-1)$

$L(\cos(x)) = x$ $x \in [0,\pi]$ $\rightarrow$ $\cos'(x) = -\sin(x)$, $\sin'(x) = \cos(x)$ $x \in (0,\pi)$

$\sin(x) = \sqrt{1-\cos^2 x}$

Proof

$L(x) = \int_x^1 \frac{1}{\sqrt{1-t^2}}\,dt$ $x \in [-1,1]$

$L'(x) = \frac{-1}{\sqrt{1-x^2}}$ $x \in (-1,1)$

$L' < 0 \rightarrow L$ decreasing, one-one $\rightarrow L^{-1}$ is a fn

L cont $\rightarrow L^{-1}$ cont.

Note that $L(-1) = \pi$, $L(1) = 0$, and L decreasing.

The domain of L^{-1} is $[0,\pi]$, the image is $[-1,1]$.

$L^{-1}(L(\cos x)) = L^{-1}(x)$ $x \in [0,\pi]$

$\cos(x) = L^{-1}(x)$

$(L^{-1})'(x) = \frac{1}{L'(L^{-1}(x))} = \frac{1}{\frac{-1}{\sqrt{1-\cos^2 x}}} = -\sin(x)$ $x \in [0,\pi]$

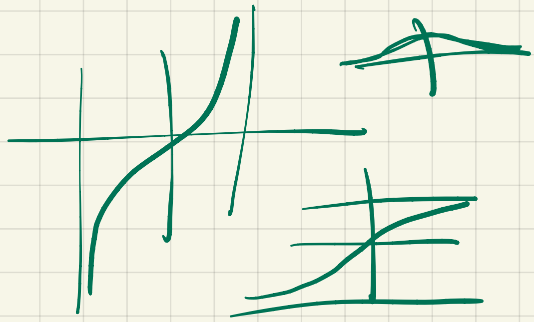
$\sin'(x) = \frac{-2\cos(x)\cos'(x)}{2\sqrt{1-\cos^2 x}} = \frac{-\cos(x)(-\sin x)}{\sin(x)} = \cos(x)$ $x \in [0,\pi]$

29. (a) $\alpha(x) = \int_0^x (1+t^2)^{-1}\,dt$ → α odd, increasing

$\lim_{x\to\pm\infty} \alpha(x)$ exist

$\lim_{x\to\infty} \alpha(x) = -\lim_{x\to-\infty} \alpha(x)$

$\pi = 2\lim_{x\to\infty} \alpha(x)$ → α^{-1} defined on $(-\pi/2, \pi/2)$



Proof

$\alpha(x) = \int_0^x (1+t^2)^{-1}\,dt$

$\frac{1}{1+t^2}$ is differentiable and even.

FTC1 → $\alpha'(x) = (1+x^2)^{-1} > 0$ for all x. → α increasing

T.F. $\alpha(x)$ is odd.

$\lim_{x\to\infty} \alpha(x) = \lim_{x\to\infty} \int_0^x (1+t^2)^{-1}\,dt$ exists because

$\int_0^x (1+t^2)^{-1}\,dt = \underbrace{\int_0^1 (1+t^2)^{-1}\,dt}_{} + \int_1^x (1+t^2)^{-1}\,dt$ exists

cont., hence integrable on $[0,1]$.

$\int_1^x \frac{1}{t^2}\,dt = -t^{-1}\Big|_1^x = -\frac{1}{x} + 1$ → $\int_1^\infty x^{-2}\,dx = 1$

Since $0 < \frac{1}{1+t^2} < \frac{1}{t^2}$ then $\int_1^\infty (1+t^2)^{-1}\,dt$ exists.

$\lim_{x\to-\infty} \alpha(x) = \lim_{x\to-\infty} \int_0^x (1+t^2)^{-1}\,dt$ exists because

$= \lim_{x\to-\infty} -\int_x^0 (1+t^2)^{-1}\,dt = \lim_{x\to-\infty} -\Big[\underbrace{\int_x^{-1} (1+t^2)^{-1}\,dt}_{} + \underbrace{\int_{-1}^0 (1+t^2)^{-1}\,dt}_{\text{cont on } [-1,0] \text{ hence int.}}\Big]$

$\lim_{x\to-\infty} \int_x^{-1} t^{-2}\,dt = \lim_{x\to-\infty} \left(-t^{-1}\Big|_x^{-1}\right) = \lim_{x\to-\infty} \left(1 + \frac{1}{x}\right) = 1$

$0 < \frac{1}{1+t^2} < \frac{1}{t^2}$ on $(-\infty, 0)$ → $\int_{-\infty}^{-1} (1+t^2)^{-1}\,dt$ exists

Since α is odd then $\alpha(x) = \int_0^x f(t)\,dt = -\int_0^{-x} f(t)\,dt = -\alpha(-x)$ and

$\lim_{x\to\infty} \int_0^x f(t)\,dt = -\lim_{x\to\infty} \int_0^{-x} f(t)\,dt = -\lim_{x\to-\infty} \int_0^x f(t)\,dt = \frac{\pi}{2}$

Finally, α is increasing (hence, one-one) and $\alpha' \neq 0$ so α^{-1} is defined on the image of α which is $(-\pi/2, \pi/2)$.

---

f diff
f' odd → f even

Proof

Let $f(x)$ be a differ. fn and

$g(x) = f(x) - f(-x)$

Then $g'(x) = f'(x) + f'(-x)$

If f' is odd, ie if $f'(x) = -f'(-x)$, then

$g'(x) = 0$ → g is constant

Since $g(0) = 0$ then $g(x) = 0$.

Hence, $f(x) = f(-x)$, ie f is even.

---

f diff
f' even → f odd

Proof

$g(x) = f(x) + f(-x)$

$g'(x) = f'(x) - f'(-x)$

f' even → $f'(x) = f'(-x)$

→ $g'(x) = 0$

→ g constant

$g(0) = 0$ → $g = 0$

→ $f(x) = -f(-x)$

→ f odd

(b) $(\alpha^{-1})'(x) = 1 + [\alpha^{-1}(x)]^2$

Proof

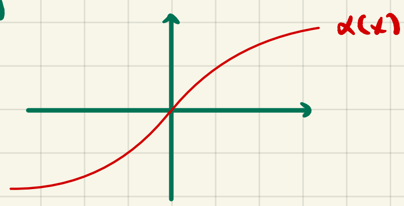
$$(\alpha^{-1})'(x) = \frac{1}{\alpha'(\alpha^{-1}(x))} = \frac{1}{\frac{1}{1+[\alpha^{-1}(x)]^2}} = 1 + [\alpha^{-1}(x)]^2$$

recap

$$\alpha(x) = \int_0^x (1+t^2)^{-1}\,dt$$

$$\alpha'(x) = (1+x^2)^{-1} > 0$$

we know α^{-1} is defined on $(-\pi/2, \pi/2)$ and



$$(\alpha^{-1})'(x) = 1 + [\alpha^{-1}(x)]^2$$

(c) $x \in (-\pi/2, \pi/2)$ define

$\tan(x) = \alpha^{-1}(x)$ here we are simply naming a fn we already know exists on $(-\pi/2, \pi/2)$

$\sin(x) = \dfrac{\tan x}{\sqrt{1+\tan^2 x}}$ and we create another fn from the previous one

(i) $\lim\limits_{x \to \frac{\pi}{2}^-} \sin(x) = 1$

Proof

$$\sin(x) = \frac{\alpha^{-1}(x)}{\sqrt{1+[\alpha^{-1}(x)]^2}} = \frac{\tan(x)}{\sqrt{1+\tan^2(x)}} = \frac{\tan(x)}{\sqrt{\tan^2(x)}}$$

$\lim\limits_{x \to \frac{\pi}{2}^-} \tan(x) = \infty$ (not sure how to show this formally here)

$\lim\limits_{x \to \pi/2} \sqrt{1+\tan^2(x)} = \infty$

Using L'Hôpital,
$$\lim_{x \to \frac{\pi}{2}^-} \sin(x) = \lim_{x \to \pi/2^-} \frac{\tan x}{\sqrt{1+\tan^2 x}} = \frac{\infty}{\infty} = \lim_{x \to \frac{\pi}{2}^-} \frac{1+[\alpha^{-1}(x)]^2}{\frac{\alpha^{-1}(x)(\alpha^{-1})'(x)}{\sqrt{1+[\alpha^{-1}(x)]^2}}} = \lim_{x \to \frac{\pi}{2}^-} \frac{(1+[\alpha^{-1}(x)]^2)^{3/2}}{\alpha^{-1}(x)(\alpha^{-1})'(x)}$$

$$= \lim \frac{(1+[\alpha^{-1}(x)]^2)^{3/2}}{\alpha^{-1}(x)(1+[\alpha^{-1}(x)]^2)} = \lim \frac{(1+[\alpha^{-1}(x)]^2)^{1/2}}{\alpha^{-1}(x)}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{1}{\sin(x)}$$

Ie, $\lim\limits_{x \to \frac{\pi}{2}^-} \sin(x) = \lim\limits_{x \to \frac{\pi}{2}^-} \dfrac{1}{\sin(x)}$. If $\lim\limits_{x \to \frac{\pi}{2}^-} \sin(x)$ exists then we have $\lim\limits_{x \to \frac{\pi}{2}^-} \sin(x) = L$, $\lim\limits_{x \to \frac{\pi}{2}^-} \dfrac{1}{\sin(x)} = \dfrac{1}{L}$,

and $L = \dfrac{1}{L}$. Therefore, $L^2 = 1 \to L = \pm 1$. However, since we are considering $x > 0$ in our limit then $\alpha(x) > 0 \to \alpha'(x) > 0$

$\to \tan(x) > 0 \to \sin(x) > 0$.

ii) $\lim_{x\to -\frac{\pi}{2}^+} \sin(x) = -1$

Proof

Analogous to (i), $\lim_{x\to -\frac{\pi}{2}^+} \sin(x) = \lim_{x\to -\pi/2^+} \frac{\tan x}{\sqrt{1+\tan^2 x}} = \frac{-\infty}{\infty} = \lim_{x\to -\frac{\pi}{2}^+} \frac{1}{\sin x}$

$\rightarrow \lim_{x\to -\pi/2^+} \sin(x) = -1$

(iii) $\sin'(x) = \begin{cases} \frac{\sin(x)}{\tan(x)} & x \neq 0 \text{ and } x \in (-\pi/2, \pi/2) \\ 1 & x = 0 \end{cases}$

Proof

$$\sin(x) = \frac{\tan x}{\sqrt{1+\tan^2 x}}$$

$$\sin'(x) = \frac{(1+\tan^2 x)\sqrt{1+\tan^2 x} - \tan x \frac{\tan x \cdot (1+\tan^2 x)}{\sqrt{1+\tan^2 x}}}{1+\tan^2 x}$$

$$= \frac{(1+\tan^2 x)^2 - \tan^2 x (1+\tan^2 x)}{(1+\tan^2 x)^{3/2}} = \frac{1}{\sqrt{1+\tan^2 x}}$$

$x = 0 \rightarrow \sin'(x) = 1$

$x \neq 0 \rightarrow \tan(x) \neq 0 \rightarrow \sin'(x) = \frac{\tan x}{\tan x \sqrt{1+\tan^2 x}} = \frac{\sin x}{\tan x}$

(iv) $\sin''(x) = -\sin(x) \quad x \in (-\pi/2, \pi/2)$

$$\sin''(x) = \frac{-\frac{\tan(x) \cdot (1+\tan^2 x)}{\sqrt{1+\tan^2 x}}}{1+\tan^2 x} = \frac{-\tan x}{\sqrt{1+\tan^2 x}} = -\sin x$$

30. Suppose there is fn f_0 not always zero w/ $f_0'' + f_0 = 0$.

(a) $f_0^2 + (f_0')^2$ is constant, and either $f_0(0) \neq 0$ or $f_0'(0) \neq 0$.

Proof

$$f_0' f_0'' + f_0' f_0 = 0$$

$$2 f_0' f_0'' + 2 f_0' f_0 = 0$$

$$[(f_0')^2 + f_0^2]' = 0$$

$\rightarrow (f_0')^2 + f_0^2$ is constant

Since $f_0(x)$ is not always zero then there is some x w/ $f_0(x) = c \neq 0$.

Then $(f_0'(x))^2 + c^2 = \text{constant} \neq 0$ because $c^2 > 0$ and $(f_0'(x))^2 \geq 0$.

Assume $f_0(0) = 0$. Then $(f_0'(0))^2 = k > 0 \rightarrow f_0'(0) \neq 0$.

Assume $f_0'(0) = 0$. Then $f_0(0)^2 = k > 0 \rightarrow f_0(0) \neq 0$.

(b) there is fn s s.t.

$$s'' + s = 0$$
$$s(0) = 0$$
$$s'(0) = 1$$

Proof

We are still under the initial assumption that there is fn f_0 not always zero w/ $f_0'' + f_0 = 0$.

Let $s = a f_0 + b f_0'$

Then $s' = a f_0' + b f_0'' = a f_0' - b f_0$ since $f_0'' = -f_0$

$s(0) = a f_0(0) + b f_0'(0) = 0$

$s'(0) = a f_0'(0) + b f_0''(0) = 1$

$s'' = a f_0'' + b f_0'''$

$s'' + s = a f_0'' + b f_0''' + a f_0 + b f_0'$

$= a\cancelto{0}{(f_0'' + f_0)} + b\cancelto{0}{(f_0''' + f_0')}$ 0, because $(f_0'' + f_0)' = 0$ $= f_0''' + f_0'$

$$\underbrace{\begin{bmatrix} f_0(0) & f_0'(0) \\ f_0'(0) & -f_0(0) \end{bmatrix}}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

determinant $= -f_0(0)^2 - f_0'(0)^2 = -(f_0(0)^2 + f_0'(0)^2) = \text{constant} \neq 0$ by part a).

Thus there is a solution.

(c) Define

$\sin = s$
$\cos = s'$

Then, cos cannot be positive for all $x>0$.

Proof

In IIAI-6 we proved

f twice differentiable
$x \geq 0 \to f(x) > 0$
f decreasing
$f'(0) = 0$

$\to \exists x, x > 0 \wedge f''(x) = 0$

Recall what s is.

$s(x) = a f_0(x) + b f_0'(x)$
$s(0) = \sin 0 = 0$
$s'(0) = \cos 0 = 1$

where f_0 is s.t. $f_0'' + f_0 = 0$, and is not always zero.

$\cos x = s'(x) = a f_0'(x) + b f_0''(x)$

Assume $\forall x, x > 0 \to \cos x > 0$.

Then $s'(x) > 0$.
Since $s(0) = 0$ then $\forall x, x > 0 \to s(x) > 0$

But it is also true that $s'' + s = 0 \to s''(x) = -s(x)$

$\to$ s' is decreasing

At this point we have

$\cos = s'$ is decreasing
$x \geq 0 \to \cos(x) > 0$
$\cos'(0) = s''(0) = -s(0) = -\sin(0) = 0$
cos is twice differentiable

By IIAI-6, $\exists x, x > 0 \wedge \cos''(x) = 0$

But $\cos''(x) = s'''(x) = -s'(x) = -\cos(x)$

So $\exists x, x > 0 \wedge \cos(x) = 0$. $\perp$.

Thus, $\exists x_0, x_0 > 0 \wedge \cos x \leq 0$.

Recap

We assume $f'' + f = 0$ has a solution other than the fn 0.

We name this assumed-to-exist sol'n f_0.

Then, this sol'n satisfies another D.E. as well

$f_0^2 + (f_0')^2 = \text{constant}$

and at 0, f_0 and f_0' cannot both be 0.

Now define a fn $s = a f_0 + b f_0'$.

What property does this fn have?

1. $s'' + s = 0$, just like f_0.

Now assume $s(0) = 0$, $s'(0) = 1$.

These should single out a fn, ie specify a and b.

We showed it is always possible to solve the linear equation that result.

At this point we've constructed this fn s s.t.

$s'' + s = 0$
$s(0) = 0$
$s'(0) = 1$

Note that

$s = -s''$ by assumption, so at least twice diff.
$s' = -s'''$ but then it must be thrice diff.

What does all this mean?

s' is continuous, $s'(0) = 1$, and $\exists x, x > 0$ s.t. $s'(x) \leq 0$.

Thus the set $\{x : x > 0 \wedge s'(x) \leq 0\}$ has an infimum.
Let's call such a number x_0.

We define π as $2x_0$.

(d) $\sin(\pi/2) = 1$

Proof

$s'' = -s \to s(x) = -s''(x)$

$s(x) = \sin(x)$

$s'(x) = \cos(x)$

$s''(x) = -s(x) = -\sin(x)$

$s'''(x) = -s'(x) = -\cos(x)$

$s^{(4)}(x) = -s''(x) = s(x) = \sin(x)$

by definition, $s'(\pi/2) = \cos(\pi/2) = 0$

As shown in part (c), $s^2 + (s')^2 = \text{constant}$.

$s^2(0) + (s'(0))^2 = 0 + 1 = 1$, so $s^2 + (s')^2 = 1$

hence, $\sin^2(\pi/2) + \cancel{\cos^2(\pi/2)} = 1$

$\to \sin^2(\pi/2) = 1$

which is it: $\sin(\pi/2) = 1$ or $\sin(\pi/2) = -1$?

we know

$\sin(0) = 0$ and on $(0, \pi/2)$, $s'(x) > 0 \to \sin(\pi/2) = 1$

$s(x) = \sin(x) = -s''(x)$

let $h(x) = s(ax)$

$h'(x) = s'(ax) \cdot a = \cos(ax) \cdot a$

we have

$s'' + s = 0$

$s(0) = 0$

$s'(0) = 1$

Therefore,

$s =$

(e) $\cos\pi, \sin\pi, \cos 2\pi, \sin 2\pi$

Recall

Lemma: suppose f'' defined everywhere and that

$f'' + f = 0$ (1)

$f(0) = 0$

$f'(0) = 0$

Then $f = 0$

Theorem 4 f'' defined everywhere

$f'' + f = 0$

$f(0) = a$

$f'(0) = b$

$\to f = b\sin + a\cos$

Proof

Let $g(x) = f(x) - bs - as'$

Then

$g'(x) = f'(x) - bs' + as$

$g''(x) = f''(x) + bs + as'$

T.F. the conditions in the lemma are satisfied by g

$g'' + g = f'' + f = 0$

$g(0) = f(0) - b \cdot 0 - a = 0$

$g'(0) = f'(0) - b + a \cdot 0 = 0$

Hence, $g = 0$, ie $f = bs + as'$

$= b\sin + a\cos$

This allows us to prove

$\sin(x+y) = \sin x\cos y + \sin y\cos x$

$\cos(x+y) = \cos x\cos y - \sin x\sin y$

Thus

$\cos\pi = \cos(\pi/2 + \pi/2) = \cos(\pi/2)^2 - \sin(\pi/2)^2 = -1$

$\sin\pi = 2\sin\pi/2\cos\pi/2 = 0$

$\cos 2\pi = \cos^2\pi - \sin^2\pi = 1$

$\sin 2\pi = 2\sin\pi\cos\pi = 0$

(5) cos and sin are periodic w/ period 2π

Proof

Recall: a fn f is periodic w/ period T if $f(x+T) = f(x)$ for all x.

$$\sin(x+2\pi) = \sin(x)\cos(2\pi) + \cancel{\sin(2\pi)}^{0}\cos(x)$$
$$= \sin x$$

$$\cos(x+2\pi) = \cos(x)\cos(2\pi) - \sin(x)\cancel{\sin(2\pi)}^{0}$$
$$= \cos x$$

31. Recall 11-24

Graph of Rational Function

Rational Fns $\frac{p}{q}$, p,q polynomial fns, q not always zero

Let $p(x) = a_n x^n + \dots + a_1 x + a_0$ and $q(x) = b_m x^m + \dots + b_1 x + b_0$.

p has at most n roots, q has at most m roots.

- roots of p, if not roots of q, are roots of $\frac{p}{q}$.

- p/q is not defined at roots of q.

we can factor out common roots in denom. and numer.

- The resulting expression is not defined at these common roots, but for other points that aren't a common root nor a root of q

we can cancel the common factors.

From 8-32 we know that $\lim_{x\to\infty} \frac{a_n x^n + \dots + a_1 x + a_0}{b_m x^m + \dots + b_1 x + b_0} = \lim_{x\to\infty} \frac{x^n}{x^m} \frac{(a_n + a_{n-1}x^{-1} + \dots a_0 x^{-n})}{(b_m + b_{m-1}x^{-1} + \dots + b_0 x^{-m})}$ exists iff $m \geq n$.

(a) A rational fn has a finite number of roots (unless it is zero everywhere).

sin has infinite roots: $\sin(2\pi k) = 0, k \in \mathbb{Z}$

and sin is not 0 everywhere.

(b) sin isn't defined implicitly by an algebraic eq.

ie, there do not exist rational Fns $f_0, \dots, f_{n-1}$ s.t.

$$(\sin x)^n + f_{n-1}(x)(\sin x)^{n-1} + \dots + f_0(x) = 0 \text{ for all } x$$

Proof

$x = \pi k \to \sin(\pi k) = 0 \to f_0(\pi k) = 0 \to f_0 = 0$ (since f_0 is rational, either it is 0 at a finite number of points or it is zero).

$(\sin x)^n + f_{n-1}(x)(\sin x)^{n-1} + \dots + f_1(x)\sin(x) = 0$

$= \sin x\left[\sin^{n-1}x + f_{n-1}(x)(\sin x)^{n-2} + \dots + f_1(x)\right] = 0$

$x \neq \pi k \to \sin^{n-1}x + f_{n-1}(x)(\sin x)^{n-2} + \dots + f_1(x) = 0$

But then, by continuity, $\forall x$, $\sin^{n-1}x + f_{n-1}(x)(\sin x)^{n-2} + \dots + f_1(x)$

Therefore, $f_1(x) = 0$ since $f(\pi k) = 0$.

Let's prove by induction that the set

$$A = \{ n : n \in \mathbb{N} \cup \{0\} \wedge f_n = 0 \text{ in } (\sin x)^n + f_{n-1}(x)(\sin x)^{n-1} + \dots + f_0(x) = 0 \text{ for all } x \}$$

$n = 0 \rightarrow f_0 = 0$ as we proved previously.

Assume $f_i = 0$ for $i = 0, 1, \dots, k$.

Then $(\sin x)^n + f_{n-1}(x)(\sin x)^{n-1} + \dots + f_{k+1}(x)(\sin x)^{k+1}$

$$= \sin(x)^{k+1} \left[(\sin x)^{n-(k+1)} + f_{n-1}(x)(\sin x)^{n-1-(k+1)} + f_{k+1}(x) \right]$$

$x \neq \pi k \rightarrow (\sin x)^{n-(k+1)} + f_{n-1}(x)(\sin x)^{n-1-(k+1)} + f_{k+1}(x) = 0$

But this expression is continuous except at possibly a finite set of points

But if

32. ϕ_1, ϕ_2 satisfy

$$\phi_1'' + g_1\phi_1 = 0$$
$$\phi_2'' + g_2\phi_2 = 0$$

$$g_2 > g_1$$

(a) $\phi_1''\phi_2 - \phi_2''\phi_1 - (g_2 - g_1)\phi_1\phi_2 = 0$

Proof

$\phi_1''\phi_2 + g_1\phi_1\phi_2 = 0$ (1)

$\phi_2''\phi_1 + g_2\phi_2\phi_1 = 0$ (2)

(1)-(2) $= \phi_1''\phi_2 - \phi_2''\phi_1 + g_1\phi_1\phi_2 - g_2\phi_1\phi_2 = 0$

$$\phi_1''\phi_2 - \phi_2''\phi_1 - \phi_1\phi_2(g_2 - g_1) = 0$$

(b) $\phi_1(x) > 0$, $\phi_2(x) > 0$ for all x in (a,b) $\rightarrow \int_a^b [\phi_1''\phi_2 - \phi_2''\phi_1] > 0$

Proof

$x \in (a,b) \rightarrow \phi_1\phi_2(g_2 - g_1) > 0 \rightarrow \phi_1''\phi_2 - \phi_2''\phi_1 = \phi_1\phi_2(g_2 - g_1) > 0$

T.F. $\int_a^b [\phi_1''\phi_2 - \phi_2''\phi_1] > 0$

Since $[\phi_1'\phi_2 - \phi_1\phi_2']' = \phi_1''\phi_2 + \cancel{\phi_1'\phi_2'} - \cancel{\phi_1'\phi_2'} - \phi_1\phi_2'' = \phi_1''\phi_2 - \phi_2''\phi_1$

Then, FTC2 $\rightarrow \int_a^b [\phi_1''\phi_2 - \phi_2''\phi_1] = [\phi_1'(b)\phi_2(b) - \phi_1(b)\phi_2'(b)] - [\phi_1'(a)\phi_2(a) - \phi_1(a)\phi_2'(a)]$

$$= [\phi_1'(b)\phi_2(b) - \phi_1'(a)\phi_2(a)] - [\phi_1(b)\phi_2'(b) - \phi_1(a)\phi_2'(a)] > 0$$

(c) Cannot have $\phi_1(a) = \phi_1(b) = 0$

Assume $\phi_1(a) = \phi_1(b) = 0$. Then

$\phi_1'(b)\phi_2(b) - \phi_1'(a)\phi_2(a) > 0$, from part b).

Now, $\phi_1'(a) \geq 0$ and $\phi_1'(b) \leq 0$, otherwise we'd have negative values of ϕ_1 on (a,b).

But then $\phi_1'(b)\phi_2(b) - \phi_1'(a)\phi_2(a) < 0$ (if ϕ_2 is continuous then $\phi_2(a)$ and $\phi_2(b) \geq 0$)

$\perp$

Hence $\phi_1(a) \neq 0$ or $\phi_1(b) \neq 0$.

(d) $\phi_1(a) = \phi_1(b) = 0$ is also impossible if

i) $\phi_1 > 0,\ \phi_2 < 0$ on (a,b)

ii) $\phi_1 < 0,\ \phi_2 > 0$ on (a,b)

iii) $\phi_1 < 0,\ \phi_2 < 0$ on (a,b)

In i) we have $\phi_1 > 0,\ \phi_2 < 0$ on (a,b)

$x \in (a,b) \rightarrow \phi_1\phi_2(g_2 - g_1) < 0 \rightarrow \phi_1''\phi_2 - \phi_2''\phi_1 = \phi_1\phi_2(g_2 - g_1) < 0$

$\int_a^b [\phi_1''\phi_2 - \phi_2''\phi_1] < 0 = [\phi_1'(b)\phi_2(b) - \phi_1'(a)\phi_2(a)] - [\phi_1(b)\phi_2'(b) - \phi_1(a)\phi_2'(a)] < 0$

Assume $\phi_1(a) = \phi_1(b) = 0$. Then, $\underbrace{\phi_1'(b)\phi_2(b)}_{\geq 0} - \underbrace{\phi_1'(a)\phi_2(a)}_{\leq 0} < 0$

But $\phi_1'(b) < 0,\ \phi_1'(a) > 0$ so $\phi_1'(b)\phi_2(b) - \phi_1'(a)\phi_2(a) \geq 0$. $\perp$.

In ii), $\phi_1 < 0,\ \phi_2 > 0$ on (a,b)

$[\phi_1'(b)\phi_2(b) - \phi_1'(a)\phi_2(a)] - [\phi_1(b)\phi_2'(b) - \phi_1(a)\phi_2'(a)] < 0$ by part b) calculations

Now assume $\phi_1(a) = \phi(b) = 0$.

Then $\phi_1'(b)\phi_2(b) - \phi_1'(a)\phi_2(a) < 0$.

But consider our initial assumptions (which also imply $\phi_1'(a) < 0,\ \phi_1'(b) > 0$): $\underset{+}{\phi_1'(b)}\underset{\geq 0}{\phi_2(b)} - \underset{-}{\phi_1'(a)}\underset{\geq 0}{\phi_2(a)} \geq 0$. $\perp$.

Similarly in iii,

part b) plus $\phi_1(a) = \phi_1(b) = 0$ means $\phi_1'(b)\phi_2(b) - \phi_1'(a)\phi_2(a) > 0$.

but since $\phi_1 < 0,\ \phi_2 < 0$ and $\phi_1'(a) < 0$ and $\phi_1'(b) > 0$, then $\underset{+}{\phi_1'(b)}\underset{\leq 0}{\phi_2(b)} - \underset{-}{\phi_1'(a)}\underset{\leq 0}{\phi_2(a)} \leq 0$. $\perp$.

Interpretation

We proved

ϕ_1, ϕ_2 satisfy

$$\phi_1'' + g_1 \phi_1 = 0$$
$$\phi_2'' + g_2 \phi_2 = 0$$

$$g_2 > g_1$$

Then

$\phi_1(x) > 0$, $\phi_2(x) > 0$ for all x in (a,b) $\rightarrow$ Cannot have $\phi_1(a) = \phi_1(b) = 0$

Therefore the contrapositive says

$\phi_1(a) = \phi_1(b) = 0 \rightarrow Q_1(x) \leq 0$ for some x in (a,b) or $Q_2(x) \leq 0$ for some x in (a,b)

If we assume $Q_1 > 0$ on (a,b) then a and b are consecutive roots, and the result says that ϕ_2 must have a zero in (a,b).

33. (a) $\sin[(n+\frac{1}{2})x] - \sin[(n-\frac{1}{2})x] = 2\sin(x/2)\cos(nx)$ This identity

Proof

Recall from Problem 14

$$\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

Hence

$$\sin[(n+\tfrac{1}{2})x] - \sin[(n-\tfrac{1}{2})x]$$

$$= 2\sin\left(\frac{x}{2}\right)\cos(nx)$$

(b) $\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \dfrac{\sin[(n+\frac{1}{2})x]}{2\sin(\frac{x}{2})}$ leads to this formula for $\sum_{i=1}^{n}\cos(ix) + \frac{1}{2}$ which we can later (part d) use to compute $\int_0^b \cos$ by computing lower sums.

Proof

$$\sin[(n+\tfrac{1}{2})x] - \sin[(n-\tfrac{1}{2})x] = 2\sin(x/2)\cos(nx)$$

$$\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx$$

$$= \frac{1}{2} + \frac{1}{2\sin(x/2)}\left[\cancel{\sin[(1+\tfrac{1}{2})x]} - \sin[(1-\tfrac{1}{2})x]\right]$$

$$+ \frac{1}{2\sin(x/2)}\left[\cancel{\sin[(2+\tfrac{1}{2})x]} - \cancel{\sin[(2-\tfrac{1}{2})x]}\right]$$

$$+ (\ldots)$$

$$+ \frac{1}{2\sin(x/2)}\left[\sin[(n+\tfrac{1}{2})x] - \cancel{\sin[(n-\tfrac{1}{2})x]}\right]$$

$$= \cancel{\frac{1}{2}} + \frac{1}{2\sin(x/2)}\left[\sin[(n+\tfrac{1}{2})x] - \cancel{\sin[(1-\tfrac{1}{2})x]}\right]$$

$$= \frac{\sin[(n+\frac{1}{2})x]}{2\sin(\frac{x}{2})}$$

(c) $\sin x + \sin 2x + \cdots + \sin nx = \dfrac{\sin\left(\frac{n+1}{2}x\right)\sin\left(\frac{n}{2}x\right)}{\sin\left(\frac{x}{2}\right)}$

Proof

Consider the expression $\cos\left[\left(k+\frac{1}{2}\right)x\right] - \cos\left[\left(k-\frac{1}{2}\right)x\right]$

From problem 14

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos(a-b) = \cos(a)\cos(-b) - \sin(a)\sin(-b)$$
$$= \cos(a)\cos(b) + \sin(a)\sin(b)$$

$$\cos(a+b) + \cos(a-b) = 2\cos(a)\cos(b)$$

$$\cos(a+b) - \cos(a-b) = -2\sin(a)\sin(b)$$

Using the latter result we have

$$\cos\left[\left(k+\tfrac{1}{2}\right)x\right] - \cos\left[\left(k-\tfrac{1}{2}\right)x\right] = -2\sin(kx)\cdot\sin\left(\frac{x}{2}\right)$$

$$\to \sin(kx) = \frac{\cos\left[\left(k-\frac{1}{2}\right)x\right]}{2\sin\left(\frac{x}{2}\right)} - \frac{\cos\left[\left(k+\frac{1}{2}\right)x\right]}{2\sin\left(\frac{x}{2}\right)}$$

$$\sum_{k=1}^{n}\sin(kx) = \frac{-1}{2\sin(x/2)}\cdot\Big[\cos(x/2) - \cancel{\cos(3x/2)} + \cancel{\cos(3x/2)} - \cancel{\cos(5x/2)} + \cdots + \cancel{\cos\left(\left(n-\tfrac{1}{2}\right)x\right)} - \cos\left(\left(n+\tfrac{1}{2}\right)x\right)$$

$$= \frac{\cos[(n+1/2)x] - \cos(x/2)}{2\sin(x/2)} = -2\sin\left[x\,\frac{(1+n)}{2}\right]\sin\left(\frac{nx}{2}\right)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

(d) Let P be partition of $[0,b]$ into equal subintervals.

Let $f_1(x) = \cos x$, $f_2(x) = \sin x$. Suppose $b \in [0, \pi/2]$. Then

$$L(f_1, P) = \frac{b}{n} \cdot \sum_{i=1}^{n} \cos\left(i \cdot \frac{b}{n}\right) = \frac{b}{n}\left[\frac{1}{2} + \sum_{i=1}^{n} \cos\left(i \cdot \frac{b}{n}\right)\right] - \frac{b}{2n}$$

Since $\frac{1}{2} + \cos x + \cos 2x + \dots + \cos nx = \dfrac{\sin[(n+\frac{1}{2})x]}{2\sin\left(\frac{x}{2}\right)}$ then

$$L(f_1, P) = \frac{b}{n} \frac{\sin[(n+\frac{1}{2})\frac{b}{n}]}{2\sin\left(\frac{b}{2n}\right)} - \frac{b}{2n}$$

We know that lower sums increase as we increase points of any given partition.

$$\lim_{n\to\infty} \sin\left[(n+\tfrac{1}{2})\tfrac{b}{n}\right] = \sin b$$

$$\lim_{n\to\infty} \frac{\sin(b/2n)}{b/2n} = \lim_{x\to 0} \frac{\sin x}{x} = 1$$

Hence $\lim_{n\to\infty} L(f_1, P_n) = \sin b$

Proving $\int_0^b \sin = 1 - \cos(b)$ is analogous.