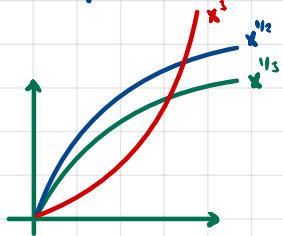


Ch 8 Appendix - Uniform Continuity

1.

a) $f(x) = x^2$, u.c. on $[0, \infty)$ for which α ?



x_2, x_3

Proof that x^2 not u.c.

$$|x^2 - y^2| = |(x-y)(x+y+y^2)|$$

$$\text{If } |x-y| < 1 \text{ then } x^2 - 2xy + y^2 < 1 \\ x^2 + 2xy + y^2 < 1 + 3xy < 4$$

$$|x^2 - y^2| < 4|x-y| < \epsilon \Leftrightarrow |x-y| < \frac{\epsilon}{4}$$

$$\text{For some } y, \text{ let } x = y + \frac{\epsilon}{\gamma}$$

$$\text{Then, } |x-y| = \frac{\epsilon}{\gamma} < \frac{\epsilon}{9}$$

$$|f(y + \frac{\epsilon}{\gamma}) - f(y)| = |y + 3\frac{\epsilon}{\gamma} + 3\frac{\epsilon^2}{64} + \frac{\epsilon^3}{\gamma^2} - y| \\ > \frac{1}{4}\frac{\epsilon^2}{64}$$

$$\frac{1}{64}\epsilon^2 > \epsilon \rightarrow \gamma > \frac{64}{\epsilon}$$

Therefore for a given $\epsilon > 0$, if $\gamma > \frac{64}{\epsilon}$ then we have

$$x - y + \frac{\epsilon}{\gamma}, \quad |x-y| < \frac{\epsilon}{9} \text{ and } |f(x) - f(y)| > \epsilon.$$

b) f cont. and bounded on $(0, 1]$, not u.c. on $(0, 1]$.

Consider $f(x) = \sin(\pi/x)$

Let $0 < \epsilon < 2$

Consider the interval $(0, \delta)$ for any $\delta > 0$.

If $x = \frac{2}{\pi n}$, $n \in \mathbb{N}$ then $\sin(\pi/x) = 1$.

Since $\lim_{n \rightarrow \infty} \frac{2}{\pi n} = 0$, we can make $x = \frac{2}{\pi n}$ arbitrarily close to 0.

In particular, $\exists N$ s.t. $n > N \Rightarrow x < \delta$.

These are therefore infinite numbers n in $(0, \delta)$ of form $\frac{2}{\pi n}$, $n \in \mathbb{N}$.
For each x , $\sin(\pi/x) = \sin(n\frac{\pi}{2}) = 1$.

Similarly we can show there are infinite $y \in (0, \delta)$ s.t. $\sin(\pi/y) = -1$.
Hence, $|\sin x - \sin y| = 2 > \epsilon$.

Thus we've shown that $\exists \epsilon > 0$ s.t. $\forall \delta > 0$, $\exists x, y \in (0, 1]$
 $|x-y| < \delta$ and $|f(x) - f(y)| > \epsilon$.

Thus $f(x) = \sin(\pi/x)$ is not u.c. on $(0, 1]$.

* note that $\sin(\pi/x)$ is not c. at 0, so it's also not u.c. on $[0, 1]$.

If we take any other interval $(a, 1]$, $a > 0$, then $\sin(\pi/x)$ is u.c. on $(a, 1]$. Just use the same $\delta > 0$ used to show continuity at a , given any $\epsilon > 0$.

c) f continuous, bounded on $[0, \infty)$, not u.c. on $[0, \infty)$.

we can come up with some sketch function like:

$$f(x) = \begin{cases} 0 & x \notin [\pi n, \pi n + 1], n \in \mathbb{N} \\ \pi n & x \in [\pi n, \pi n + 1/2], n \in \mathbb{N} \\ \pi n (\pi n + \frac{1}{2}) - \pi n x & x \in [\pi n + 1/2, \pi n + 1], n \in \mathbb{N} \end{cases}$$



2.

a) f, g u.c. on $A \rightarrow f+g$ u.c. on A

Proof

 f u.c. on A : $\forall \epsilon > 0 \exists \delta_1 > 0 \forall x, y, x \in A \wedge y \in A \wedge |x-y| < \delta_1 \rightarrow |f(x)-f(y)| < \frac{\epsilon}{2}$ g u.c. on A : $\forall \epsilon > 0 \exists \delta_2 > 0 \forall x, y, x \in A \wedge y \in A \wedge |x-y| < \delta_2 \rightarrow |g(x)-g(y)| < \frac{\epsilon}{2}$ Therefore, if $\delta = \min(\delta_1, \delta_2)$

$$|x-y| < \delta \rightarrow |(f(x)+g(x)) - (f(y)+g(y))| < |f(x)-f(y)| + |g(x)-g(y)| < \epsilon$$

Therefore,

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, y, x \in A \wedge y \in A \wedge |x-y| < \delta \rightarrow |(f(x)+g(x)) - (f(y)+g(y))| < \epsilon.$$

 $f+g$ u.c. on A .b) f, g u.c. bounded on $A \rightarrow fg$ u.c. on A

Proof

since f, g bounded in $A, \exists M$ s.t. $\forall x, y \in A \rightarrow |f(x)| < M \wedge |g(x)| < M$ Since f, g u.c., choose $\epsilon, \delta > 0$ s.t. $\forall x, y$

$$x \in A, y \in A \rightarrow |f(x)-f(y)| < \frac{\epsilon}{2M}$$

$$|g(x)-g(y)| < \frac{\epsilon}{2M}$$

Then,

$$|(f(x)g(x)) - (f(y)g(y))|$$

$$= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)|$$

$$= |f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))|$$

$$\leq |N \frac{\epsilon}{2M} + N \frac{\epsilon}{2M}|$$

$$= \epsilon$$

c) $f(x) = x$ and $g(x) = \sin x$ are both u.c. on $[0, +\infty)$.

$f \cdot g$ is not u.c. on $[0, \infty)$.

As x increases, the amplitude of one cycle of $f \cdot g$ grows without bound though the period stays the same. The \ln grows at a rate more rapid than at the same section of each subsequent cycle. No single $\delta > 0$ can suffice to constrain f to be ϵ distance from $g(x)$, $f(x)$.

d) f u.c. on A

$$\begin{aligned} g \text{ u.c. on } B \\ \forall x, y \in A \rightarrow f(x) \in B \end{aligned} \quad \rightarrow g \circ f \text{ u.c. on } A$$

Proof

Let $\epsilon > 0$.

Then $\exists \delta_1 > 0$, $\forall x, y \in B \wedge |x - y| < \delta_1 \rightarrow |g(x) - g(y)| < \epsilon$

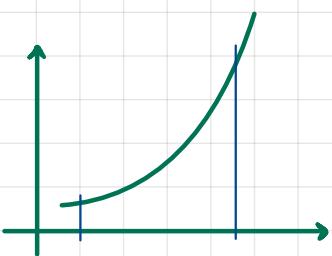
Given δ_1 , $\exists \delta > 0$ s.t. $\forall x, y \in A \wedge |x - y| < \delta \rightarrow |f(x) - f(y)| < \delta_1$.

Thus

$\forall x, y \in A \wedge |x - y| < \delta \rightarrow |f(x) - f(y)| < \delta_1 \wedge f(x) \in B \wedge f(y) \in B$

$$\rightarrow |g(f(x)) - g(f(y))| < \epsilon$$

3. Theorem 1 If cont. on $[a, b] \rightarrow f$ u.c. on $[a, b]$



$\Rightarrow f$ u.c. on $[a, b] \Leftrightarrow f$ ϵ -good on $[a, b]$ for every $\epsilon > 0$

let $\epsilon > 0$.

Assume f not ϵ -good on $[a, b]$.

Recall the lemma used in the chapter's main text:

$$a < \frac{a+b}{2} < b, f \text{ cont. on } [a, b]$$

let $\epsilon > 0$ and following derive

i) x, y both in $[a, \frac{a+b}{2}]$, $|x-y| < \delta \rightarrow |f(x) - f(y)| < \epsilon$, ie f is ϵ -good on $[a, \frac{a+b}{2}]$

ii) x, y both in $[\frac{a+b}{2}, b]$, $|x-y| < \delta \rightarrow |f(x) - f(y)| < \epsilon$, ie f is ϵ -good on $[\frac{a+b}{2}, b]$

then,

$\exists \delta > 0$ s.t. x, y both in $[a, b]$, $|x-y| < \delta \rightarrow |f(x) - f(y)| < \epsilon$

ie, f is ϵ -good on $[a, b]$.

Hence our assumption is the negation of the consequent. Therefore either i) or ii) must be false.

Let I_1 be one of the halves in which f is not ϵ -good. Repeat the procedure on I_1 : bisect and let I_2 be a half where f is not ϵ -good.

The intervals I_n are nested, thus there is some x_0 that is in all of them.

Recall that as a result of how we defined the intervals, f is not u.c. on any of the nested intervals.

But f is cont. at x_0 , so we can choose $\delta > 0$ s.t.

$$\forall x, |x-x_0| < \delta \rightarrow |f(x) - f(x_0)| < \frac{\epsilon}{2}$$

$$\forall y, |y-x_0| < \delta \rightarrow |f(y) - f(x_0)| < \frac{\epsilon}{2}$$

Therefore

$$\forall x, \forall y, |x-x_0| < \delta \text{ and } |y-x_0| < \delta \rightarrow |f(y) - f(x)| < \epsilon$$

f is ϵ -good on $(x_0 - \delta, x_0 + \delta)$.

1.

f is ϵ -good on $[a, b]$

$\forall \epsilon > 0, f$ is ϵ -good on $[a, b]$.

Theorem 1 f cont. on $[a,b] \rightarrow f$ u.c. on $[a,b]$

Theorem 7-2 f cont. on $[a,b] \rightarrow f$ bounded above on $[a,b]$, i.e.
 $\exists N, \forall x \in [a,b], |f(x)| \leq N$

Theorem 1 \rightarrow Theorem 7-2

Intuitively, if f is u.c. on a interval, then for any x in the interval, within a certain distance from x , f stays within a certain distance from $f(x)$.

Assume Theorem 1.

Let f be cont. on $[a,b]$.

Then by Th. 1, f is u.c. on $[a,b]$.

For given $\epsilon > 0$, choose $\delta > 0$ s.t.

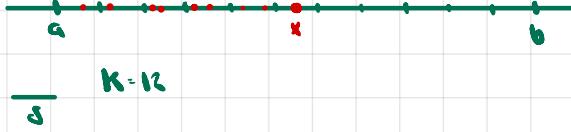
$$\forall x_1, x_2 \in [a,b], |x_1 - x_2| < \delta \rightarrow |f(x_1) - f(x_2)| < \epsilon$$

$$\text{let } M = \frac{b-a}{\delta} + 1$$

Then for any $x \in [a,b]$ there is a sequence

$$a = a_0, a_1, a_2, \dots, a_K = x$$

with $k \in \mathbb{N}$ and $|a_{i+1} - a_i| < \delta$



Thus,

$$|f(a_0) - f(a)| < \epsilon$$

$$|f(a_1) - f(a_0)| < \epsilon$$

...

$$|f(a) - f(a_K)| < \epsilon$$

$$\rightarrow |f(x) - f(a)| < K\epsilon$$

$$|f(x)| \leq |f(a)| + M\epsilon, \forall x \in [a,b]$$

Therefore f is bounded on $[a,b]$.

f cont. on $[a,b] \rightarrow f$ is bounded on $[a,b]$.

Th. 1 \rightarrow Th. 7-2