Ch. 12 - Inverse Functions

Definition: A fn is **one-one** (read one-to-one) if $f(a) \neq f(b)$ whenever $a \neq b$.

Definition: For any fn f, the **inverse** of f, denoted f^{-1}, is the set of all pairs (a,b) for which (b,a) is in f.

Theorem: f^{-1} is a fn $\leftrightarrow$ f is one-one

Proof

Assume f is one-one.

Let (a,b) and (a,c) be two pairs in f^{-1}.

Then (b,a) and (c,a) are in f.

Ie $f(b) = a$ and $f(c) = a$

But since f is one-one, by contrapositive, $f(b) = f(c) \rightarrow b = c$

Hence for any pairs (a,b) and (a,c) in f^{-1}, $b = c$. This was our definition of fn in ch.3. Thus f^{-1} is a fn.

Now assume f^{-1} is a fn.

Assume $f(b) = f(c)$

$(b, f(b))$ and $(c, f(c)) = (c, f(b))$ are in f.

Therefore, $(f(b), b)$ and $(f(b), c)$ are in f^{-1}.

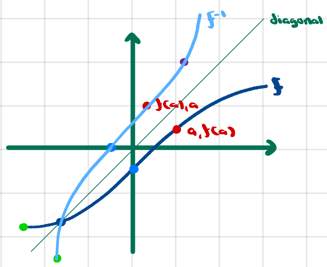
$\rightarrow b = c$.

$f(b) = f(c) \rightarrow b = c$, the contrapositive of the def. of one-one.

■

Note

$(f^{-1})^{-1} = f$. Therefore, f one-one then f^{-1} one-one since $(f^{-1})^{-1} = f$ one-one



$b = f(a)$ means (a,b) in f. But then (b,a) in f^{-1}, so $a = f^{-1}(b)$

Therefore

f, f^{-1} one-one $\rightarrow [b = f(a) \leftrightarrow a = f^{-1}(b)]$

$f(f^{-1}(x)) = x$ for all x in domain of f^{-1}, ie $f \circ f^{-1} = I$

$f^{-1}(f(x)) = x$ for all x in domain of f, ie $f^{-1} \circ f = I$

Theorem f increasing $\leftrightarrow f^{-1}$ increasing

Proof

$x_1 = f^{-1}(y_1) \longleftrightarrow y_1 = f(x_1)$

$x_2 = f^{-1}(y_2) \longleftrightarrow y_2 = f(x_2)$

Assume f increasing.

Let y_1 and y_2 be in f^{-1}'s domain.

Then for some x_1, x_2 in f's domain we have $y_1 = f(x_1)$ and $y_2 = f(x_2)$

Assume $y_1 < y_2$.
Then $f(x_1) < f(x_2)$ so $x_1 < x_2$.

But $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$.

Hence $f^{-1}(y_1) < f^{-1}(y_2)$

$y_1 < y_2 \to f^{-1}(y_1) < f^{-1}(y_2)$

$\forall y_1, y_2 \; y_1 < y_2 \to f^{-1}(y_1) < f^{-1}(y_2)$

f increasing $\to f^{-1}$ increasing

Theorem f decreasing $\to f^{-1}$ decreasing

Proof

f decreasing $\to f$ one-one $\to f^{-1}$ fn.

Let $y_1 = f(x_1)$, $y_2 = f(x_2)$ in f^{-1}'s domain.

Assume $y_1 = f(x_1) < f(x_2) = y_2$.
Then, since f decreasing, $x_1 > x_2$.

$\to f^{-1}(y_1) > f^{-1}(y_2)$

$y_1 < y_2 \to f^{-1}(y_1) > f^{-1}(y_2)$

f^{-1} decreasing

Theorem: f increasing $\leftrightarrow -f$ decreasing

Proof

f increasing $\leftrightarrow \forall x, y \; x < y \to f(x) < f(y) \leftrightarrow -f(x) > -f(y) \leftrightarrow -f$ decreasing

Theorem 2 if f cont. and one-one on an interval, then f either increasing or decreasing on that interval.

Proof

if $a<b<c$ in an interval, then either

(i) $f(a)<f(b)<f(c)$
(ii) $f(a)>f(b)>f(c)$

Proof

Case 1: $f(a)<f(c)$

Assume $f(b)<f(a)$.
IVT $\to \exists d,\ d\in(b,d) \wedge f(d)=f(a)$. $\perp$.
$f(b)>f(a)$

Assume $f(b)>f(c)$
IVT $\to \exists d,\ d\in(a,b) \wedge f(d)=f(c)$. $\perp$.
$f(b)<f(c)$

$f(a)<f(b)<f(c)$

Case 2: $f(a)>f(c)$

Analogous steps to show $f(a)>f(b)>f(c)$ ■

if $a<b<c<d$ are four points in the interval, then either

(i) $f(a)<f(b)<f(c)<f(d)$
(ii) $f(a)>f(b)>f(c)>f(d)$

Just apply (i) to $a<b<c$ and $b<c<d$. ■

given any interval $[a,d]$

Case 1: $f(a)<f(d)$
Pick any two points s.t. $a<b<c<d$
(2) $\to f(a)<f(b)<f(c)<f(d)$
Since this is true for any two points in (a,b), f increasing on $[a,d]$.

Case 2: $f(a)>f(d)$
Analogous steps as case 1 $\to f$ decreas. on $[a,b]$.

Case 3: $f(a)=f(d)$. $\perp$.

■

If f is continuous and increas./decreas. and defined on some interval we can precisely say what the domain of f^{-1} is.

f cont. increas. defined on $[a,b]$

IVT $\to$ f takes on every value in $[f(a), f(b)]$ = domain f^{-1}.

f cont. increas. defined on (a,b)

Case 1: f becomes arbitrarily large in (a,b)

Then, for some $c \in (a,b)$, f takes on all values $> f(c)$ by IVT.

Case 2: f bounded above.

$A = \{f(x) : c \leq x < b\}$ bounded above

$\to \exists \sup A = \alpha$

Let $f(c) < y < \alpha$.

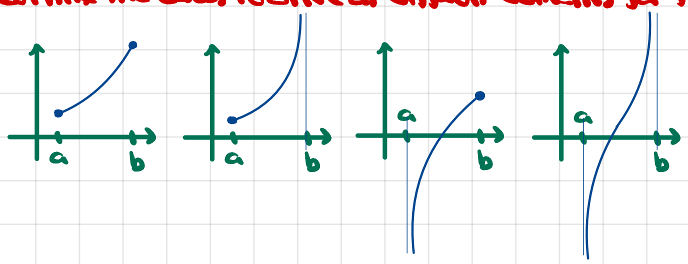
f takes on some value $f(x) > y$ because otherwise y would be a lower bound ... $y < \alpha$.

IVT $\to$ f takes on value y, but it cannot take on α because if it did this would happen in a open interval at, say, x_1, and there would be $x_1 < x_2 < b$ s.t. $f(x_2) > \alpha$. $\perp$.

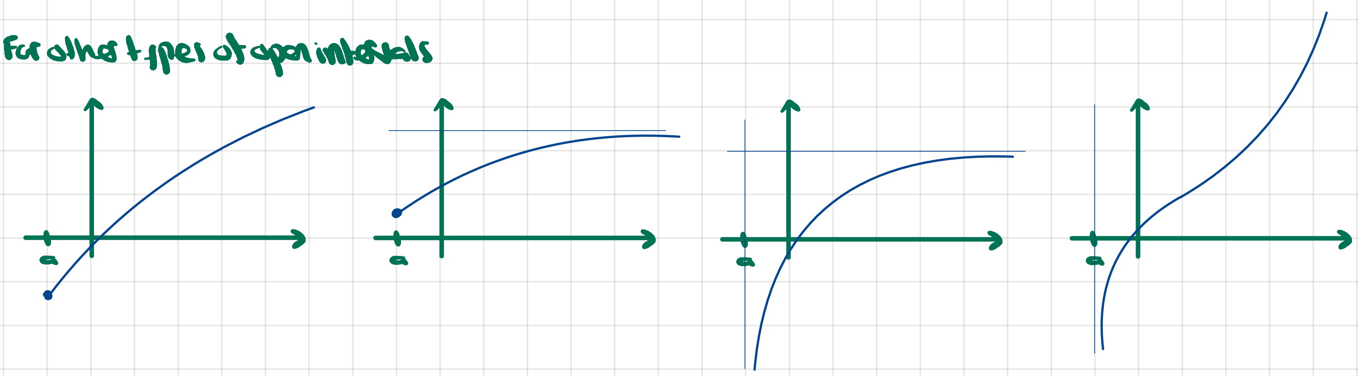
Hence the domain of f^{-1} either includes all x s.t. $f(x) \geq f(c)$ or it includes $[f(c), \alpha)$.

The same analysis for f bounded below or becoming arbitrarily large negative tells us that the domain of f^{-1} either includes all x s.t. $f(x) \leq f(c)$ or it includes just $[\beta, f(c)]$ where β is the infimum of $\{f(x) : a < x \leq c\}$

We can mix the cases to arrive at different domains for f^{-1}



For other types of open intervals



Theorem 3 f cont. and one-one on an interval $\rightarrow f^{-1}$ also cont.

Proof

Since f is cont. and one-one, then by Th.2 f is either increasing or decreasing in the interval.

Assume f increasing. Now, since f is one-one we know from Th.1 that f^{-1} is an increasing fn.

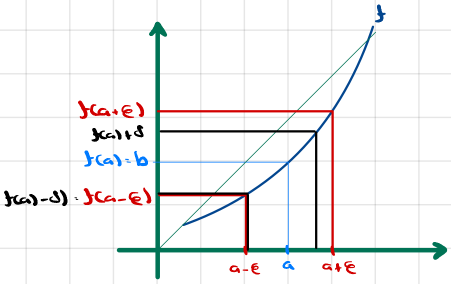
Assume the interval is open.

We want to show that f^{-1} is cont., ie $\lim_{x \to b} f^{-1}(x) = f^{-1}(b)$ for each b in the domain of f^{-1}.

For each b, there is some a in f's domain s.t. $f(a) = b$.

$\forall \epsilon > 0$ we need a $\delta > 0$ s.t. $\forall x \; |x - f(a)| < \delta \rightarrow |f^{-1}(x) - f^{-1}(f(a))| < \epsilon \rightarrow |f^{-1}(x) - a| < \epsilon$

in words, if we keep x close to $b = f(a)$ then f^{-1} will be close to $f^{-1}(b) = a$



if $a - \epsilon < a < a + \epsilon$ then $f(a-\epsilon) < f(a) < f(a+\epsilon)$

Let $\delta = \min(f(a+\epsilon) - f(a), f(a) - f(a-\epsilon))$.

Then $f(a-\epsilon) \leq f(a) - \delta \wedge f(a+\epsilon) \geq f(a) + \delta$

Therefore, if $f(a) - \delta < x < f(a) + \delta$ then $f(a-\epsilon) < x < f(a+\epsilon)$

But f^{-1} increasing implies $f^{-1}(f(a-\epsilon)) = a - \epsilon < f^{-1}(x) < a + \epsilon = f^{-1}(f(a+\epsilon))$

Thus we have shown that $\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; |x - b| < \delta \rightarrow |f^{-1}(x) - f^{-1}(b)| < \epsilon$

ie $\lim_{x \to b} f^{-1}(x) = f^{-1}(b)$

At this point we know

f one-one $\leftrightarrow$ f^{-1} is function

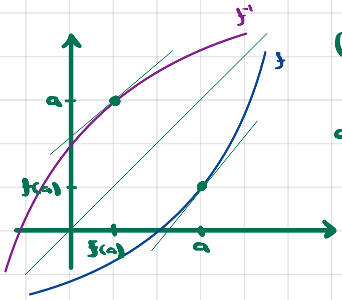
f increasing $\leftrightarrow$ f^{-1} increasing

f cont. and one-one on interval $\rightarrow$ f either increasing or decreasing on that interval

$\rightarrow$ f^{-1} cont. on the interval

Now we want to know about differentiability of f^{-1}.

The following picture indicates what is likely true in general for f^{-1}



$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}$$

or $$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

Theorem 4 f cont. and one-one defined on an interval
$f'(f^{-1}(a)) = 0$ $\rightarrow$ f^{-1} not diff at a

Proof

$f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}

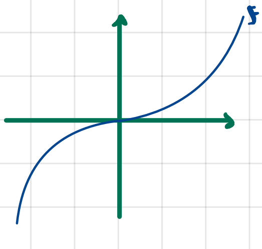
If f^{-1} were diff. at a, the Chain Rule would imply

$$f'(f^{-1}(a)) \cdot (f^{-1})'(a) = 1$$

$$0 \cdot (f^{-1})'(a) = 1$$

$\perp$.

Example $f(x) = x^3$



$f'(0) = 0$
$f^{-1}(0) = 0$ $\rightarrow$ f^{-1} not diff at 0

Theorem 5 Let f be cont. one-one fn defined on interval.

Suppose f diff at $f^{-1}(b)$, w/ derivative $f'(f^{-1}(b)) \neq 0$.

Then,

f^{-1} diff. at b and $(f^{-1})'(b) = \dfrac{1}{f'(f^{-1}(b))}$

Proof