Problems, Prologue

1) i) $ax = a,\ a \neq 0 \Rightarrow x = 1$

$a^{-1} \cdot ax = a \cdot a^{-1}$, from P7 a^{-1} exists since $a \neq 0$

$x = 1$

ii) $x^2 - y^2 = (x-y)(x+y)$

use distributive law

$x(x+y) + (-y)(x+y)$

$x \cdot x + xy + (-y)x + (-y)y$

$x^2 + xy + (-(xy)) + (-(y \cdot y))$

$x^2 - y^2$

$x^2 - y^2 = x^2 + xy - xy - y^2$

$= x(x+y) + (-y)(x+y)$

$= (x+y)(x-y)$

iii) $x^2 = y^2$ then $x = y$ or $x = -y$

$|x|^2 = |y|^2$

$|x| = \sqrt{x^2}$

$|x|^2 = x^2$

$x^2 = y^2 \Rightarrow 1 = y \cdot y \cdot x^{-1} \cdot x^{-1}$

$(y \cdot x^{-1}) \cdot (y \cdot x^{-1}) = 1$

$y \cdot x^{-1} = (y \cdot x^{-1})^{-1}$, so the number yx^{-1} is equal to its inverse

if $a = a^{-1}$ then $a \cdot a = 1 \Rightarrow a^2 = 1$

$|a|^2 = 1$

$|a| = 1$ but $|a| = \begin{cases} a & a \geq 0 \\ -a & a \leq 0 \end{cases}$ so $a = 1$ or $a = -1$

Therefore, $yx^{-1} = \pm 1$

$x^2 - y^2 = 0$

$x^2 - xy + xy - y^2 = 0$

$x(x-y) + y(x-y)$

$(x-y)(x+y) = 0$

$x - y = 0$ or $x + y = 0$

$x = y$ or $x = -y$

if x and y have same sign, then $yx^{-1} = 1$ and y here is the inverse of x^{-1}, x.

inverse has same sign as number

$a \cdot a^{-1} = 1$

if $a > 0$, then $a^{-1} > 0$

if $a < 0$, then $a^{-1} < 0$

which property justifies this?

if x and y have different signs, $yx^{-1} = -1$

so, $y = -x$ ie $x = -y$

iv) $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

use distributive law

$x(x^2 + \cancel{xy} + \cancel{y^2}) + (-y)(\cancel{x^2} + \cancel{xy} + y^2)$

$x^3 - y^3$

v) $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

$$x^4 - y^4 = (x-y)(x^3 + x^2y + xy^2 + y^3)$$

$$= (x-y) \cdot \sum_{i=0}^{3} x^i y^{3-i} = x\sum_{i=0}^{3} x^i y^{3-i} + (-y)\sum_{i=0}^{3} x^i y^{3-i} = x^4 + x\sum_{i=0}^{2} x^i y^{3-i} - y^4 - y\sum_{i=1}^{3} x^i y^{3-i}$$

$$= x^4 - y^4 + \sum_{i=0}^{2} x^{i+1} y^{3-i} - \sum_{i=1}^{3} x^i y^{4-i}$$

if $j = i-1$ then $\sum_{j=0}^{2} x^{j+1} y^{3-j}$

replace 4 with n and 3 with $n-1$

v) $x^n - y^n = (x-y)\sum_{i=0}^{n-1} x^{n-1-i} y^i$

$= x\sum_{i=0}^{n-1} x^{n-1-i} y^i - y\sum_{i=0}^{n-1} x^{n-1-i} y^i$

$= x^n + x\sum_{i=1}^{n-1} x^{n-1-i} y^i - y^n - y\sum_{i=0}^{n-2} x^{n-1-i} y^i$

$= x^n + y^n + \cancel{\sum_{i=1}^{n-1} x^{n-i} y^i} - \cancel{\sum_{i=0}^{n-2} x^{n-1-i} y^{i+1}}$

$\sum_{i=1}^{n-1} x^{n-i} y^i = x^{n-1} y^1 + x^{n-2} y^2 + \dots + x y^{n-1}$

$\sum_{i=0}^{n-2} x^{n-1-i} y^{i+1} = x^{n-1} y + x^{n-2} y^2 + \dots + x y^{n-1}$

vi) $x^3+y^3=(x+y)(x^2-xy+y^2)$

$x^3-y^3=(x-y)(x^2+xy+y^2)$

$x^3+y^3=x^3+x^2y+xy^2-yx^2-xy^2-y^3+2y^3$

$=x^3+x^2y-x^2y+xy^2-xy^2+y^3$
$=x(x^2-xy+y^2)+y(x^2-xy+y^2)$
$=(x^2-xy+y^2)(x+y)$

book solution
in x^3-y^3, if y is $-y$ then
$x^3-(-y)^3=x^3+y^3$
only works if the exponent is odd
$=(x+y)(x^2-xy+y^2)$
note: we used the additive inverse in place of the original y in x^3-y^3 and obtained the expression we wanted: x^3+y^3

general case

n terms, exponents from 0 to n-1

$x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\dots+x^2y^{n-3}+xy^{n-2}+y^{n-1})$

$x^n+y^n=x^n+x^{n-1}y+n^{n-2}y^2+\dots+x^3y^{n-3}+x^2y^{n-2}+xy^{n-1}$
$+y^n-yx^{n-1}-y^2x^{n-2}-\dots-x^2y^{n-2}-xy^{n-1}$

2) let $x=y$

$x^2=xy$
$x^2-y^2=xy-y^2$
$x^2-xy+xy-y^2=xy-y^2$
$(x-y)(x+y)=y(x-y)$

if we divide by $x-y$, we multiply by $(x-y)^{-1}$ then implicitly, $x-y\neq 0 \Rightarrow x\neq y$
so we can't do it.

3) i) $\frac{a}{b}=\frac{ac}{bc}$ if $b,c\neq 0$

$a\cdot b^{-1}=a\cdot b^{-1}\cdot 1=a\cdot b^{-1}\cdot(c\cdot c^{-1})$

ii) $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ if $b,d\neq 0$

$a\cdot b^{-1}+c\cdot d^{-1}=a\cdot b^{-1}\cdot d\cdot d^{-1}+cd^{-1}\cdot b\cdot b^{-1}$
$=b^{-1}d^{-1}(ad+cb)$

iii) $(ab)^{-1}=a^{-1}b^{-1}$ if $a,b\neq 0$

$(ab)^{-1}\cdot ab=1$
$(ab)^{-1}=a^{-1}b^{-1}$

iv) $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{db}$ if $b,d\neq 0$

$(a\cdot b^{-1})\cdot(c\cdot d^{-1})$
associative law for multiplic. $(a\cdot b^{-1}\cdot c)\cdot d^{-1}$
$=(a\cdot c)\cdot(d^{-1}\cdot b^{-1})$

v) $\frac{a}{b}\Big/\frac{c}{d}=\frac{ad}{bc}$ $b,c,d\neq 0$

$(a\cdot b^{-1})\cdot(c\cdot d^{-1})^{-1}$

$(a\cdot b^{-1})\cdot c^{-1}\cdot(d^{-1})^{-1}=a\cdot b^{-1}\cdot c^{-1}\cdot d=\frac{ad}{bc}$

vi) if $b,d\neq 0$ then $\frac{a}{b}=\frac{c}{d} \iff ad=bc$

$ab^{-1}=cd^{-1}$
$ab^{-1}b=cd^{-1}b=a$
$ad=cd^{-1}bd=cb$

$ad=bc$
$(ad)\cdot b^{-1}\cdot d^{-1}=(bc)b^{-1}d^{-1}$
$=a\cdot b^{-1}=c\cdot d^{-1}$

$\frac{a}{b}=\frac{b}{a}\Rightarrow ab^{-1}=ba^{-1}$

$ab^{-1}b=a=ba^{-1}b$
$a\cdot a=b\cdot b$
$a^2=b^2$, we know from 1) iii) that $a=\pm b$

4) i) $\overset{a}{4-x} < \overset{b}{3-2x}$

$4-x-(3-2x)<0 \quad a-b<0$

$1+x<0$

$x-(-1)<0 \Rightarrow -(x-(-1))>0 \Rightarrow -1-x \text{ in } P \Rightarrow -1>x$

$a-b<0 \qquad -(a-b)>0 \qquad b-a \text{ in } P \qquad b>a$

ii) $5-x^2<8 \qquad a<b$

$8-(5-x^2)>0 \qquad b-a \text{ in } P,\ b-a>0$

$3+x^2>0$

$x^2>-3$ but we know that $x^2>0$ for all x, so x any number

iii) $5-x^2<-2$

$x^2>7$

$\sqrt{x^2}>\sqrt{7}$

$|x|>\sqrt{7}$ if $x\geq 0$, then $x>\sqrt{7}$

$x\leq 0$, then $-x>\sqrt{7} \Rightarrow x<\sqrt{7}$

iv) $(x-1)(x-3)>0$

$a\cdot b>0$

(1) $a>0, b>0$

$x-1>0 \Rightarrow x>1$

$x-3>0 \Rightarrow x>3$

$\Rightarrow x>3$

(2) $a<0, b<0$

$x<1$

$x<3$

$\Rightarrow x<1$

v) $x^2-2x+2>0$

$(x-1)^2>-1$ valid for all x since $a^2>0\ \forall a$

vi) $x^2+x+1>2$

$x>\frac{-1+\sqrt{5}}{2}$ or $x<\frac{-1-\sqrt{5}}{2}$

$x^2+x-1>0$

$\Delta=1-4\cdot1\cdot(-1)=5$

$x_0=\frac{-1\pm\sqrt{5}}{2}$

vii) $x^2-x+10>16$

$x^2-x-6>0$

$x>2$ or $x<-3$

$\Delta=1-4\cdot1\cdot(-6)=25$

$x_0=\frac{-1\pm5}{2} \nearrow 2 \quad \searrow -3$

viii) $x^2+x+1>0$

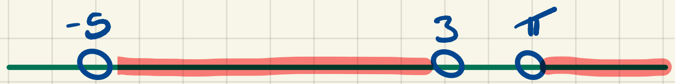
$\Delta = 1-4\cdot 1\cdot 1 = -3 \Rightarrow$ no roots

$$\left(x^2+2\cdot\tfrac{1}{2}\cdot x+\tfrac{1}{4}\right)+\tfrac{3}{4}$$

$$\left(x+\tfrac{1}{2}\right)^2+\tfrac{3}{4}>0 \quad \forall x$$

ix) $(x-\pi)(x+5)(x-3)>0$

$+ + +$	$x>\pi,\ x>-5,\ x>3$	$\Rightarrow x>\pi$
$- - +$	$x<\pi,\ x<-5,\ x>3$	$\Rightarrow$ not possible
$+ - -$	$x>\pi,\ x<-5,\ x<3$	$\Rightarrow$ " "
$- + -$	$x<\pi,\ x>-5,\ x<3$	$\Rightarrow -5<x<3$



x) $(x-\sqrt[3]{2})(x-\sqrt{2})>0$

$+ +$ $\quad x>\sqrt[3]{2},\ x>\sqrt{2} \Rightarrow x>\sqrt{2}$

$- -$ $\quad x<\sqrt[3]{2}$

xi) $2^x<8$

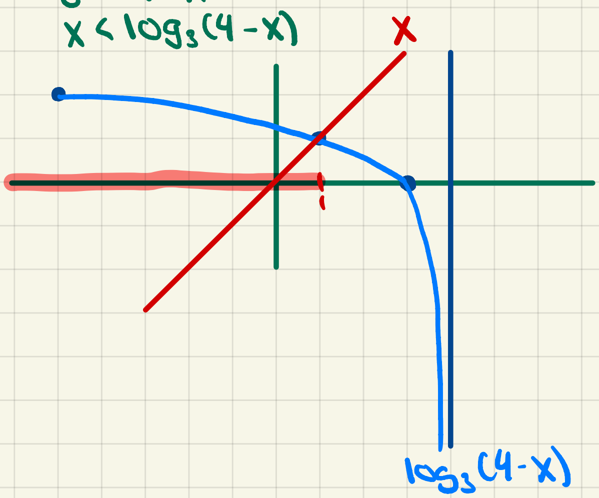
$\log_2 2^x < \log_2 8$

$x<3$

xii) $x+3^x<4$

$3^x<4-x$

$x<\log_3(4-x)$



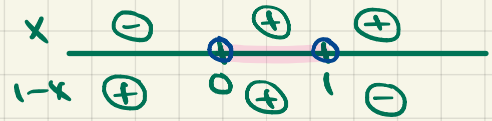
xiii) $\frac{1}{x}+\frac{1}{1-x}>0 \qquad x\neq 0,\ 1-x\neq 0 \Rightarrow x\neq 1$

Solution 1

$$\frac{1-x+x}{x(1-x)} = \frac{1}{x(1-x)}>0$$

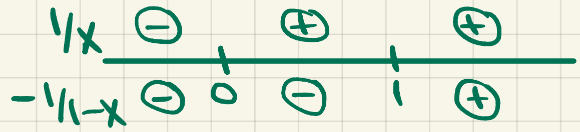
$x(1-x)>0$

$\Rightarrow x>0$ and $x<1$



Solution 2

$$\frac{1}{x} > -\frac{1}{1-x}$$



c1) $x<0 \Rightarrow$ both sides negative

$1< \underbrace{-x/(1-x)}_{<0,\text{ could stop here}}$

$(1-x)$ is >0

$(1-x)<-x \Rightarrow 1<0$

c2) $x>1$, analogous to c1 $\Rightarrow 1<0$

c3) $0<x<1$

$\frac{1}{x}>0,\ -\frac{1}{1-x}<0$

$1>-\frac{x}{1-x},\quad 1-x>0$

$1-x>-x \Rightarrow 0<1$

Note: tricky in solution 2 is to figure out when the inequality is flipped when multiplying by an inverse

xiv) $\frac{x-1}{x+1}>0 \qquad x\neq -1$

$+ +$ $\quad x-1>0 \Rightarrow x>1$; $\ x+1>0 \Rightarrow x>-1 \quad \Rightarrow x>1$

$- -$ $\quad x<1$; $\ x<-1 \quad \Rightarrow x<-1$

5)

i) if $a<b$ and $c<d$ then $a+c<b+d$

$b-a>0$
$d-c>0$
$(b-a)+(d-c)>0$
$\Rightarrow (b+d)-(a+c)>0 \Rightarrow b+d>a+c$

ii) if $a<b$ then $-b<-a$

$b-a>0$

from i) we can add

$0+a<b$

iii) if $a<b$ and $c>d$ then $a-c<b-d$

$b-a>0$
$c-d>0$

$(b-a)+(c-d)>0$
$b-d+(-(a-c))>0$
$b-d>a-c$

iv) $a<b,\ c>0 \Rightarrow ac<bc$

$b-a>0$
$c(b-a)>0$
$bc-ac>0 \Rightarrow bc>ac$

v) $a<b,\ c<0 \Rightarrow ac>bc$

$b-a>0$
$c(b-a)<0$
$bc-ac<0$
$ac>bc$

vi) $a>1 \Rightarrow a^2>a$

$a>1 \Rightarrow a>0,\ 1<a$
so by iv), $1\cdot a<a\cdot a$

vii) $0<a<1 \Rightarrow a^2<a$

$0<a<1 \Rightarrow a\cdot a<1\cdot a$

viii) $0\leq a<b,\ 0\leq c<d \Rightarrow ac<bd$

if $a=0$ or $c=0$ then $ac=0<bd$

if $a\neq 0$ and $c\neq 0$

$0<ac<bc$ by iv)

$0<bc<bd$ by iv)

$\Rightarrow ac<bd$

ix) $0\leq a<b \Rightarrow a^2<b^2$

special case of viii) where $c=a,\ d=b$

x) $a,b\geq 0,\ a^2<b^2 \Rightarrow a<b$

assume $a<b$ is false
then
1) $a=b \Rightarrow a^2=b^2$ X

2) $a>b \Rightarrow 0\leq b\leq a \Rightarrow b^2<a^2$ X

Therefore, $a<b$

6)

a) $0 \leq x < y \Rightarrow x^n < y^n$ $n = 1, 2, 3, \ldots$

From 5ix), $x^2 < y^2$
so we have $0 \leq x^2 < y^2$

From 5viii) $0 \leq x < y$, $0 \leq x^2 < y^2 \Rightarrow x \cdot x^2 < y \cdot y^2 \Rightarrow x^3 < y^3$

Keep applying 5viii) with $0 \leq x < y$, $0 \leq x^n < y^n$, $n = 1, 2, 3, \ldots$

b) $x < y$, n odd $\Rightarrow x^n < y^n$

1) $0 \leq x < y \Rightarrow x^n < y^n$ by 6a

2) $x < 0, y > 0 \Rightarrow x^n = x^{n-1} \cdot x$

$x^{n-1} > 0, x < 0 \Rightarrow x^n < 0 \quad \Rightarrow x^n < y^n > 0$

3) $x < 0, y < 0, x < y$

$\Rightarrow 0 \leq -y < -x \Rightarrow (-y)^n < (-x)^n$, for n odd

$(-y)^n = -y^n$ for n odd, same for x, so $-y^n < -x^n$

$\Rightarrow x^n < y^n$, n odd

c) $x^n = y^n$, n odd $\Rightarrow x = y$

$x < y$, n odd $\Rightarrow x^n < y^n$ by 6b, contrad.

$y < x$, n odd $\Rightarrow y^n < x^n$ by 6b, contrad.

$\Rightarrow x = y$ because one of $x = y, x > y, x < y$ must hold
* because of Trichotomy law, the fact that $x - y$ is
a number, and the definitions of $>, <$.

d) $x^n = y^n$, n even $\Rightarrow x = y$ or $x = -y$

7. $0<a<b \Rightarrow a<\sqrt{ab}<\frac{a+b}{2}<b$

$(a-b)^2>0 \Leftrightarrow a-2\sqrt{ab}+b>0 \Leftrightarrow \sqrt{ab}<\frac{a+b}{2}$

$0<a<b \Rightarrow 0<a<2a<a+b \Rightarrow 0<a<\frac{a+b}{2}$

$0<a^2<ab \Rightarrow 0<a<\sqrt{ab}$

$0<a<b \Rightarrow 0<a<a+b<2b \Rightarrow 0<\frac{a}{2}<a<\frac{a+b}{2}<b$

$\Rightarrow 0<a<\sqrt{ab}<\frac{a+b}{2}<b$

8)

P10) apply P'10 with $b=0$

$\Rightarrow \forall a$, one and only one of following holds:

$a=0$
$a>0$
$a<0$

Alternatively,

suppose $a>0$ and $-a>0 \Rightarrow a+(-a)>0$ contradicting def. of $-a$.

also if $-a=0$ then this also violates the def of $-a$, as $a+(-a)=a$

therefore $-a<0$ from P'10

P11) $\forall a,b,c$ $a<b$, $c<d$

using P'12, $a+c<b+c$

again P'12, $b+c<b+d$

$\Rightarrow a+c<b+c<b+d$

If $a=0$ and $c=0$ and $b>0, d>0 \Rightarrow 0<b+d$

P12) use P'13 with $a=0$: $\forall b,c$, if $0<b$ and $0<c$ then $0<bc$

9)

i) $\sqrt{2}+\sqrt{3}-\sqrt{5}+\sqrt{7}>0$

ii) $0 \le |a| \le |a|+|b| \quad \forall a,b$

if $b\ge 0$ then $0\le |a+b| = |a|+|b|$
if $b<0$ then $0 \le |a+b| < |a|+|b|$

$\Rightarrow \forall b \ |a+b| \le |a|+|b|$

$\Rightarrow \forall b \ |a+b|-|a|-|b| \le 0$

so, eliminate outer absolute operator:

$|(|a+b|-|a|-|b|)| = |a|+|b|-|a+b|$

iii) a+b is a number, call it d

now we are in the case of ii

$|(|d|+|c|-|d+c|)| \ge 0$

$= |d|+|c|-|d+c| = |a+b|+|c|-|a+b+c|$

iv) $|x^2-2xy+y^2| = |(x-y)^2| = (x-y)^2$

v) $|(|\sqrt{2}+\sqrt{3}|-|\sqrt{5}-\sqrt{7}|)|$

$\sqrt{2}+\sqrt{3}-(\sqrt{7}-\sqrt{5})$

10) i) $|a+b|-|b|$

$a+b\ge 0 \Rightarrow a \ge -b$
$b \ge 0 \Rightarrow a+b-b = a$
$b<0 \Rightarrow a+b+b = a+2b$
$a+b<0 \Rightarrow a<-b$
$b\ge 0 \Rightarrow -a-b-b = -a-2b$
$b<0 \Rightarrow -a-b+b = -a$

ii) $|(|x|-1)|$
$x\ge 0 \Rightarrow |x-1|$
$x \ge 1 \Rightarrow x-1$ if $x\ge 1$
$0\le x<1 \Rightarrow 1-x$
$x<0 \Rightarrow |-x-1|$
$-x-1\ge 0 \Rightarrow x\le -1 \Rightarrow -x-1$
$-x-1<0 \Rightarrow x>-1 \Rightarrow x+1$

iii) $|x|-|x^2| = |x|-x^2$
$x\ge 0 \Rightarrow x-x^2$
$x<0 \Rightarrow -x-x^2$

10) iv) $a-\underbrace{|(a-|a|)|}$

$a\ge 0 \Rightarrow 0$
$a<0 \Rightarrow 2a<0$

so, $a\ge 0 \Rightarrow a$
$a<0 \Rightarrow a+2a = 3a$

11) i) $|x-3|=8$

$x\ge 3, \ x=11$
$x<3, \ 3-x=8 \Rightarrow x=-5$

ii) $|x-3|<8$

$x\ge 3 \Rightarrow x-3<8 \Rightarrow x<11 \Rightarrow [3,11[$

$x<3 \Rightarrow 3-x<8 \Rightarrow x>-5 \Rightarrow]-5,3[$
$\Rightarrow]-5,11[$

iii) $|x+4|<2$

$x+4\ge 0 \Rightarrow x\ge -4$
$x+4<2 \Rightarrow x<-2$
$[-4,-2[$

$x+4<0 \Rightarrow x<-4$
$-x-4<2 \Rightarrow x>-6$
$]-6,-4[$

$\Rightarrow]-6,-2[$

iv) $|x-1|+|x-2|>1$

	1		2	
$-$		$+$		$+$
$-$		$-$		$+$

$x\le 1 \Rightarrow 1-x-x+2>1$
$2x<2$
$x<1$

$1<x<2$
$x-1-x+2>1 \Rightarrow 1>1$
actually $x-1-x+2=1$ always

$x\ge 2$
$x-1+x-2 = 2x-3>1$

$x>2$

so, $x<1$ or $x>2$

11) v) $|x-1|+|x+1|<2$

	-1		1	
$-$		$-$		$+$
$-$		$+$		$+$

$x<-1 \Rightarrow 1-x-x-1 = -2x<2 \Rightarrow x>-1$ contrad.

$-1<x<1 \Rightarrow 1-x+x+1 = 2$ always >2 X

$x \geq 1 \Rightarrow x-1+x+1 \Rightarrow 2x<2 \Rightarrow x<1$ X

no solutions

vi) $|x-1|+|x+1|<1$

again, three cases

i) $x<-1$, $-2x<1$ $x>-\frac{1}{2}$ X

ii) $-1<x<1$, $=2$ always X

iii) $x\geq 1$, $2x<1$, $x<\frac{1}{2}$ X

no solutions

vii) $|x-1|\cdot|x+1|=0$

$|x-1|=0 \Rightarrow x=1$

$|x+1|=0 \Rightarrow x=-1$

viii) $|x-1||x+2|=3$

	-2		1	
$-$		$-$		$+$
$-$		$+$		$+$

$x<-2$

$(1-x)(-x-2)$

$= -x-2+x^2+2x = x^2+x-2 = 3$

$x^2+x-5=0$

$\Delta = 1-4\cdot1\cdot(-5) = 21$ $x = \frac{-1\pm\sqrt{21}}{2}$

$x = \frac{-1-\sqrt{21}}{2}$

$-2<x<1$

$(1-x)(x+2)$

$= x+2-x^2+2x = -x^2+3x+2 = 3$

$x^2-3x+1=0$

$\Delta = 3-4\cdot1\cdot1 = -1$

no solutions

$x>1$

$(x-1)(x+2) = x^2+2x-x-2$

$= x^2+x-2 = 3$

$x^2+x-5=0$

$x = \frac{-1+\sqrt{21}}{2}$

12)

i) $|xy| = |x| \cdot |y|$

$xy \geq 0$

$\Rightarrow$ i) $x \geq 0, y \geq 0$

$xy = |x| \cdot |y|$

ii) $x < 0, y < 0$

$xy = (-x)(-y) = |x| \cdot |y|$

$xy < 0$

$\Rightarrow$ i) $x < 0, y > 0$

$-xy = (-x)y = |x||y|$

ii) $x > 0, y < 0$

$-xy = x(-y) = |x||y|$

ii) $\left|\frac{1}{x}\right| = \frac{1}{|x|}$ if $x \neq 0$

$x^{-1} > 0 \Rightarrow x \cdot x^{-1} = 1$ and $x > 0$

$|1/x| = 1/x = 1/|x|$

$x^{-1} < 0 \Rightarrow x \cdot x^{-1} = 1 \Rightarrow x < 0$

$|1/x| = 1/(-x) = 1/|x|$

iii) $|x|/|y| = |x/y|$ if $y \neq 0$

$x \geq 0, y > 0 \Rightarrow x/y = |x/y|$

$x < 0, y > 0 \Rightarrow -x/y = -(x/y) = |x/y|$

$x \geq 0, y < 0 \Rightarrow x/(-y) = -(x/y) = |x/y|$

$x < 0, y < 0 \Rightarrow -x/-y = x/y = |x/y|$

iv) $|x - y| \leq |x| + |y|$

$$|x-y| = \begin{cases} x - y & \text{if } x \geq y \\ y - x & \text{if } x < y \end{cases}$$

$x - y \leq |x| - y = |x| + (-y) \leq |x| + |y|$

because $x \leq |x|$ — because $-y \leq |y|$

$y - x \leq |y| - x = |y| + (-x) \leq |y| + |x|$

bec. $y \leq |y|$ — bec. $-x \leq |x|$

Alternatively

$|x - y| = |x + (-y)| \leq ||x| + |y|| = |x| + |y|$

$x \leq |x|, -y \leq |y|$

v) $|x| - |y| \leq |x - y|$

$|x| = |y - (y - x)| \leq |y| + |y - x|$

$\Rightarrow |x| - |y| \leq |y - x|$

vi) $|(|x| - |y|)| \leq |x - y|$

$$||x| - |y|| = \begin{cases} |x| - |y| & |x| \geq |y| \\ |y| - |x| & |x| < |y| \end{cases}$$

v) $\rightarrow \forall x, y$ $|x| - |y| \leq |x - y|$ and $|y| - |x| \leq |y - x| = |x - y|$

$\rightarrow ||x| - |y|| \leq |x - y|$

vii) $|x + y + z| \leq |x| + |y| + |z|$

$|x + y + z| \leq |x + y| + |z| \leq |x| + |y| + |z|$

$$|x + y + z| = \begin{cases} x + y + z & \text{if } \geq 0 \\ -(x + y + z) & \text{if } < 0 \end{cases}$$

if $x + y + z \geq 0$

$x \leq |x|, y \leq |y|, z \leq |z|$

also $x = |x|$ if $x > 0$

(...)

so, $x + y + z = |x| + |y| + |z|$ if $x, y, z \geq 0$

if $x + y + z < 0$

$|x + y + z| = -(x + y + z) < |x| + |y| + |z| > 0$

13) $\max(x,y) = \dfrac{x+y+|y-x|}{2}$

$\min(x,y) = \dfrac{x+y-|y-x|}{2}$

MAX

1) $y>x$

$\max(x,y) = \dfrac{x+y+y-x}{2} = y$

$y<x$

$\max(x,y) = \dfrac{x+y+x-y}{2} = x$

analogous for min

$\max(x,y,z) = \max(x, \max(y,z))$

$\max(x, \max(y,z)) = (x + m_{yz} + |m_{yz} - x|)/2$

$= \left(x + \dfrac{z+y+|y-z|}{2} + \left|\dfrac{z+y+|y-z|}{2} - x\right|\right)/2$

$= \left(\dfrac{2x+z+y+|y-z|}{2} + \left|\dfrac{z+y-2x+|y-z|}{2}\right|\right)/2$

14)

a) $|a| = |-a|$

$a \geq 0$

$|a| = a$

$|-a| = -(-a) = a$

$a<0$

$|a| = -a$

$|-a| = -a$

b) $-b \leq a \leq b \iff |a| \leq b$

$\Rightarrow \quad -b \leq a \leq b$

if $a>0$ then $|a| = a \leq b$

$a<0$ then $|a| = -a \leq b$ $\Rightarrow |a| \leq b$

$\Leftarrow$ $|a| \leq b$, first, $b \geq 0$ since $|a| \geq 0$

$a>0 \Rightarrow a \leq b$, also $a>-b$ since $a \geq 0$ and $b \geq 0 \Rightarrow -b \leq 0$

$a<0 \Rightarrow a<b$ since $a<0, b \geq 0$; $-a<b$ by assumption, $\Rightarrow a>-b$

Therefore if $b = |a|$: $-|a| \leq a \leq |a| \iff |a| \leq |a|$

14) c) $|a+b| \leq |a| + |b|$

$|a| \leq b \Rightarrow -b \leq a \leq b$

$\Rightarrow |a| \leq |a| \Rightarrow -|a| \leq a \leq |a|$

so,

$-|a| \leq a \leq |a|$

$-|b| \leq b \leq |b|$

$a+b \leq |a|+|b|$

$a+b \geq -|a|-|b|$

$\Rightarrow -(|a|+|b|) \leq a+b \leq |a|+|b|$

$\Rightarrow$ by 14b) $|a+b| \leq |a|+|b|$

15) x and y not both 0 then

$$x^2+xy+y^2>0$$

$$x^4+x^3y+x^2y^2+xy^3+y^4>0$$

→ if either is zero, trivial solution

→ if neither zero

1) $x^3-y^3=(x-y)(x^2+xy+y^2)$

$x^3-y^3>0 \Rightarrow x^3>y^3$

$\Rightarrow x>y \Rightarrow x-y>0$

$\Rightarrow x^2+xy+y^2>0$

$x^3-y^3<0 \Rightarrow x^3<y^3 \Rightarrow x<y$

$\Rightarrow x-y<0$

$\Rightarrow x^2+xy+y^2>0$

$x^3-y^3=0 \Rightarrow x=y$

$\Rightarrow x^2+x\cdot x+x^2=3x^2>0$

summary: using the factorization, we know that x^3-y^3 has to be <0, >0, or $=0$ and this has implications for the factors. Usually one factor is what we are investigating.

2) $x^5-y^5=(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$

$x^5-y^5>0 \Rightarrow x>y \Rightarrow x-y>0$

$\Rightarrow x^4+\dots+y^4>0$

(analogous to part 1)

16) a) $(x+y)^2=x^2+y^2$ only when $x=0$ or $y=0$

$(x+y)^3=x^3+y^3$ only when $x=0$ or $y=0$ or $x=-y$

i) $(x+y)^2=(x+y)(x+y)=x^2+2xy+y^2$

$=x^2+y^2 \Rightarrow 2xy=0 \Rightarrow x=0$ or $y=0$

ii) $(x+y)(x^2+2xy+y^2)=x^3+2x^2y+xy^2+yx^2+2xy^2+y^3$

$=x^3+y^3 \Rightarrow 2x^2y+xy^2+yx^2+2xy^2=0$

$xy(2x+y+x+2y)=0$

$xy(3x+3y)=0$

$\Rightarrow x=0$ or $y=0$ or $x=-y$

b) $x^2+2xy+y^2=(x+y)^2\geq 0$

show $4x^2+6xy+4y^2>0$ unless $x=0, y=0$

$4(x^2+2xy+y^2)-2xy$

$4(x+y)^2-2xy$

$2xy=(x+y)^2-x^2-y^2$

$4(x+y)^2-(x+y)^2+x^2+y^2$

$=3(x+y)^2+x^2+y^2$, which is >0 $\forall x,y$ $x\neq 0, y\neq 0$

c) $(x+y)^4=x^4+y^4$

$(x+y)^2(x+y)^2$

$(x^2+2xy+y^2)(x^2+2xy+y^2)$

$x^4+2x^3y+x^2y^2+$

$2x^3y+4x^2y^2+2xy^3$

$+y^2x^2+2xy^3+y^4$

$=x^4+y^4+4x^3y+6x^2y^2+4xy^3$

$4x^3y+6x^2y^2+4xy^3=0$

$xy(4x^2+6xy+4y^2)=0$

$x=0$ or $y=0$ or $x=0, y=0$

16) d) $(x+y)^5 = x^5 + y^5$ when?

$(x+y)^5 = (x+y)^4(x+y)$

$(x^4 + y^4 + 4x^3y + 6x^2y^2 + 4xy^3)(x+y)$

$x^5 + y^4x + 4x^4y + 6x^3y^2 + 4x^2y^3$
$+ 4y^4x + x^4y + 4x^3y^2 + 6x^2y^3 + y^5$

$= x^5 + 5xy^4 + 10x^2y^3 + 10x^3y^2 + 5x^4y + y^5$

$= x^5 + y^5 + 5xy(y^3 + 2xy^2 + 2x^2y + x^3)$

$5xy(x^3 + 2x^2y + 2xy^2 + y^3) = 0$

$x = 0$ or $y = 0$ or $x^3 + 2x^2y + 2xy^2 + y^3 = 0$

$x^3 + 3x^2y + 3xy^2 + y^3 - x^2y - xy^2 = 0$

$(x+y)^3 = x^2y + xy^2 = xy(x+y)$

which is true if $x + y = 0 \Rightarrow x = -y$

also if $x+y \neq 0$, $(x+y)^2 = xy$

$\Rightarrow x^2 + xy + y^2 = 0$ but from (5) this expression is >0
unless $x = y = 0$

so, $x = y = 0$ or $x = -y$ or $x = 0$ or $y = 0$

$\Rightarrow$ $x = 0$ or $y = 0$ or $x = -y$

we can now guess as to when $(x+y)^n = x^n + y^n$
so far we have

$(x+y)^2 = x^2 + y^2$ if $x = 0$ or $y = 0$

$(x+y)^3 = x^3 + y^3$ if $x = 0$ or $y = 0$ or $x = -y$

$(x+y)^4 = x^4 + y^4$ if $x = 0$ or $y = 0$

$(x+y)^5 = x^5 + y^5$ if $x = 0$ or $y = 0$ or $x = -y$

so we guess, $(x+y)^n = x^n + y^n$ when
if n even, x or y is zero

if n odd, x or y is zero or $x = -y$

17) a) smallest possible $2x^2-3x+4$

$2(x^2-1.5x+2)$

$2(x^2-2\cdot1\cdot0.75x+9/16+23/16)$

$2(x^2-2\cdot1\cdot0.75x+0.75^2)+46/16$

$2(x-3/4)^2+23/8$

$x=3/4$ gives smallest value of $23/8$

b) $x^2-3x+2y^2+4y+2$

$(x^2-2\cdot1\cdot1.5x+1.5^2)-1.5^2+2(y^2+2\cdot1\cdot1y+1)$

$(x-1.5)^2+2(y+1)^2-1.5^2$

$x=1.5$, $y=-1$, smallest value is -1.5^2

c) $x^2+4xy+5y^2-4x-6y+7$

$x^2+4x(y-1)+5y^2-6y+7$

$[x^2+4x(y-1)+4(y-1)^2]$

$+5y^2-6y+7-4(y-1)^2$

$(x+2(y-1))^2+\underbrace{5y^2-6y+7-4y^2+8y-4}$

$y^2+2y+3 \Rightarrow y^2+2y+1+2 \Rightarrow (y+1)^2+2$

$(x+2(y-1))^2+(y+1)^2+2$

smallest value is 2 when $y=-1$
and $(x+(-4))^2=0 \Rightarrow x=4$

18) a) $b^2-4ac\geq 0$

$x^2+bx+c=0$

i) $\left(\frac{-b+\sqrt{b^2-4c}}{2}\right)^2+b\left(\frac{-b+\sqrt{b^2-4c}}{2}\right)+c$

$=(b^2-4c-2b\sqrt{b^2-4c}+b^2)/4+(-b^2+b\sqrt{b^2-4c}+2c)/2$

$=[\cancel{b^2}\cancel{-4c}\cancel{-2b\sqrt{b^2-4c}}\cancel{+b^2}\cancel{-2b^2}\cancel{+2b\sqrt{b^2-4c}}\cancel{+4c}]/4$

$=0$

ii) For $(-b-\sqrt{b^2-4c})/2$ analogous.

18) b) $b^2-4c<0$

$x^2+bx+c>0$

$x^2+bx+0.25b^2+c-0.25b^2$

$\underbrace{(x+0.5b)^2}_{>0}\underbrace{-(b^2-4c)/4}_{>0}$

c) $x^2+xy+y^2>0$ if $x\neq0, y\neq0$

$\forall y\neq0$, $y^2-4y^2=-3y^2<0$
so, from c), $x^2+xy+y^2>0\ \forall x$
analogous to show $\forall x\neq0$, $x^2+xy+y^2>0\ \forall y$

d) $x^2+\alpha xy+y^2>0$ whenever $x\neq0$ or $y\neq0$

if $y=0$, then $\forall x\neq0$ satisfies the inequality.
$\forall y\neq0$, $\alpha^2y^2-4y^2=y^2(\alpha^2-4)=b^2-4c$

if $y^2(\alpha^2-4)<0$ then $x^2+\alpha xy+y^2>0$
$\Rightarrow \alpha^2-4<0 \Rightarrow \alpha^2<4 \Rightarrow \alpha<2, \alpha>-2$

$\Rightarrow \forall y\neq0$, if $\alpha\in]-2,2[$ then ineq. is $>0\ \forall x$

analogously we reach,

$\forall x\neq0$ if α in $]-2,2[$, ineq is >0 for $\forall y$

e) x^2+bx+c, ax^2+bx+c, $a>0$

i) $x^2+bx+(\frac{b}{2})^2+c-(\frac{b}{2})^2$

$\underbrace{(x+\frac{b}{2})^2}_{\geq0}+(c-(\frac{b}{2})^2)$

if $x=-b/2$, the value of the expr. is $c-\frac{b^2}{4}$

ii) $(ax^2+2\cdot\sqrt{a}\cdot x\cdot\frac{b}{2\sqrt{a}}+(\frac{b}{2\sqrt{a}})^2)+c-(\frac{b}{2\sqrt{a}})^2$

$(\sqrt{a}x+\frac{b}{2\sqrt{a}})^2+c-(\frac{b}{2\sqrt{a}})^2$

so, $x=-\frac{b}{2a} \Rightarrow$ min value of $\frac{-b^2+4ac}{4a}$

Recap of Derivations related to quadratic expression

given ax^2+bx+c

we'd like to find analytical expressions for
value of x when the expression is zero (roots), and when the expression has min value

First, we factorize

$$ax^2 + 2\sqrt{a}x\cdot\frac{b}{2\sqrt{a}} + \left(\frac{b}{2\sqrt{a}}\right)^2 + c - \left(\frac{b}{2\sqrt{a}}\right)^2$$

$$\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 + c - \left(\frac{b}{2\sqrt{a}}\right)^2$$

set the expression to zero to find roots

$$\left(\sqrt{a}\cdot x + \frac{b}{2\sqrt{a}}\right)^2 = \frac{b^2}{4a} - c = \frac{b^2-4ac}{4a}$$

$$\frac{2ax+b}{2\sqrt{a}} = \pm\frac{\sqrt{b^2-4ac}}{2\sqrt{a}}, \quad b^2-4ac \geq 0$$

$$2ax+b = \pm\sqrt{b^2-4ac}$$

$$\boxed{x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

* Note alternative factorization

$ax^2+bx+c =$

$= a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$

18e)il has min x as $c - \frac{b^2}{4}$

$= \frac{c}{a} - \frac{b^2}{a^2\cdot 4} \Rightarrow$

min for whole expre. is

$a\left(\frac{c}{a} - \frac{b^2}{4a^2}\right) = c - \frac{b^2}{4a}$, same

as alternative deriv. of this at bottom of page.

note the assumptions required to derive this: $a>0$, $b^2-4ac \geq 0$

we know more about b^2-4ac by looking at:

$$\underbrace{(\sqrt{a}x + b/2\sqrt{a})^2}_{\geq 0} + \underbrace{c-(b/2\sqrt{a})^2}_{=\frac{4ac-b^2}{4a}}$$

$4a \to > 0$ $\Rightarrow 4ac-b^2 > 0$ $\Rightarrow$ the whole initial expr. is >0

$\Rightarrow b^2-4ac<0$

so, if $b^2-4ac<0$ (with $a>0$),
$ax^2+bx+c>0$

if $b^2-4ac \geq 0$, we have at least one root.

Finally, notice that the minimum value for $\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2$ is zero. This happens when

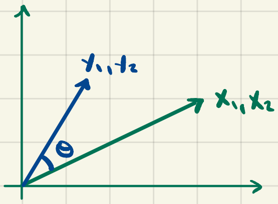
$\sqrt{a}x = -b/2\sqrt{a} \Rightarrow \boxed{x = -\frac{b}{2a}}$. The expression has it's minimum value at that value of x,

that minimum is $c - \left(\frac{b}{2\sqrt{a}}\right)^2 = \frac{-b^2+4ac}{4a}$

19. Schwarz Inequality $x_1y_1 + x_2y_2 \le \sqrt{x_1^2+x_2^2}\,\sqrt{y_1^2+y_2^2}$

a) $x_1 = \lambda y_1$
$x_2 = \lambda y_2$ $\quad \Rightarrow x_1y_1 + x_2y_2 = \sqrt{x_1^2+x_2^2}\,\sqrt{y_1^2+y_2^2}$
$\lambda \ge 0$

vector interpretation



$\langle x_1,x_2\rangle\langle y_1,y_2\rangle = |\vec{x}||\vec{y}|\cos\theta$

$\cos\theta < 1 \Rightarrow x_1y_1+x_2y_2 \le |\vec{x}||\vec{y}|$

i) $x_1y_1+x_2y_2 = \lambda y_1^2 + \lambda y_2^2 = \lambda(y_1^2+y_2^2) = \lambda\sqrt{y_1^2+y_2^2}\,\sqrt{y_1^2+y_2^2} = \sqrt{x_1^2+x_2^2}\,\sqrt{y_1^2+y_2^2}$

$x_1 = \lambda y_1 \Rightarrow y_1 = \dfrac{x_1}{\lambda}$

$\sqrt{x_1^2+x_2^2} = \lambda\sqrt{y_1^2+y_2^2}$

ii) $y_1=y_2=0 \Rightarrow x_1y_1+x_2y_2 = 0 = 0\sqrt{x_1^2+x_2^2}$

iii) $\nexists \lambda$ s.t. $x_1=\lambda y_1,\ x_2=\lambda y_2$

vector interpretation

ie for $\vec{x}=\langle x_1,x_2\rangle, \vec{y}=\langle y_1,y_2\rangle$, $c_1\vec{x}+c_2\vec{y} = 0 = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. For there to be non-trivial sol'n, determinant must be 0. $\Rightarrow x_1y_2 - x_2y_1 = 0 \Rightarrow \dfrac{x_1}{y_1} = \dfrac{x_2}{y_2} = \lambda$.

$\Rightarrow$ Since we assume there is no such λ, the only soln is $(0,0)$. $\Rightarrow \vec{x},\vec{y}$ linearly indep.

For some λ, $(\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2 = \lambda^2(y_1^2+y_2^2) - 2\lambda(x_1y_1+x_2y_2) + (x_1^2+x_2^2) > 0$, a second degree polyn. in λ, given x_1,y_1,x_2,y_2. The ineq. represents all the x_1,x_2,y_1,y_2 (ie vectors $\vec{x}$ and $\vec{y}$) we can choose such that they are lin. indep. Note if we instead choose $x_1=\lambda y_1$ and $x_2=\lambda y_2$, then $\vec{x}$ and $\vec{y}$ are lin. dep. and the eq. has roots, ie the ineq. is an equality.

$\Delta = b^2-4ac = 4(x_1y_1+x_2y_2)^2 - 4(y_1^2+y_2^2)(x_1^2+x_2^2) < 0$

$\Rightarrow (x_1y_1+x_2y_2)^2 < (y_1^2+y_2^2)(x_1^2+x_2^2)$

$\Rightarrow x_1y_1+x_2y_2 < \sqrt{x_1^2+x_2^2}\,\sqrt{y_1^2+y_2^2}$

Note $|\lambda\langle y_1,y_2\rangle - \langle x_1,x_2\rangle|^2 = \langle \lambda y_1 - x_1, \lambda y_2 - x_2\rangle \cdot \langle \lambda y_1 - x_1, \lambda y_2 - x_2\rangle$

$= (\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2 > 0 \Rightarrow$ The norm of a linear combination of $\vec{x}$ and $\vec{y}$ is > 0, no matter what $\lambda \ne 0$ is. This is a way of saying they are l.i.

b) Use $2xy \le x^2 + y^2$ with $x = \frac{x_i}{\sqrt{x_1^2+x_2^2}}$, $y = \frac{y_i}{\sqrt{y_1^2+y_2^2}}$ with $i=1$ then $i=2$.

$$\frac{2x_1y_1}{\sqrt{(x_1^2+x_2^2)(y_1^2+y_2^2)}} \le \frac{x_1^2}{x_1^2+x_2^2} + \frac{y_1^2}{y_1^2+y_2^2}$$

$$\frac{2x_2y_2}{\sqrt{(x_1^2+x_2^2)(y_1^2+y_2^2)}} \le \frac{x_2^2}{x_1^2+x_2^2} + \frac{y_2^2}{y_1^2+y_2^2}$$

Add

$$\frac{2(x_1y_1+x_2y_2)}{\sqrt{(x_1^2+x_2^2)(y_1^2+y_2^2)}} \le 2 \Rightarrow x_1y_1 + x_2y_2 \le \sqrt{(x_1^2+x_2^2)(y_1^2+y_2^2)}$$

where does the initial ineq. come from? $2xy \le x^2 + y^2$

$$(x-y)^2 = x^2 - 2xy + y^2 \ge 0 \Rightarrow 2xy \le x^2 + y^2$$

$$2\frac{\langle x_1, x_2\rangle}{\sqrt{x_1^2+x_2^2}} \cdot \frac{\langle y_1, y_2\rangle}{\sqrt{y_1^2+y_2^2}} \le \frac{x_1^2+x_2^2}{x_1^2+x_2^2} + \frac{y_1^2+y_2^2}{y_1^2+y_2^2}$$

$$\Rightarrow \frac{2(x_1y_1+x_2y_2)}{\sqrt{(x_1^2+x_2^2)(y_1^2+y_2^2)}} \le 2$$

c) Prove $(x_1^2+x_2^2)(y_1^2+y_2^2) = (x_1y_1+x_2y_2)^2 + (x_1y_2 - x_2y_1)^2$

$$(x_1^2+x_2^2)(y_1^2+y_2^2) = x_1^2y_1^2 + x_1^2y_2^2 + x_2^2y_1^2 + x_2^2y_2^2$$

$$= (x_1^2y_1^2 + 2x_1x_2y_1y_2 + x_2^2y_2^2) + (x_1^2y_2^2 - 2x_1x_2y_1y_2 + x_2^2\cancel{y}_1^2)$$

$$= (x_1y_1+x_2y_2)^2 + (x_1y_2 - x_2y_1)^2 \ge (x_1y_1+x_2y_2)^2$$

Take square root $\Rightarrow x_1y_1 + x_2y_2 \le \sqrt{(x_1^2+x_2^2)(y_1^2+y_2^2)}$

Note that we've proved thus far regarding Schwarz Inequality:

- $\vec{x} = \lambda\vec{y} \Rightarrow$ equality holds in Schwarz Ineq.
- if one vector is $\vec{0}$, holds trivially.

1) if $\vec{x}$ and $\vec{y}$ l.i. then strict ineq.
2) proof starting at $(x-y)^2 \ge 0$, x and y unit vectors
3) " " " $(x_1^2+x_2^2)(y_1^2+y_2^2)$
$= (x_1y_1+x_2y_2)^2 + (x_1y_2 - x_2y_1)^2$

Lets deduce the equality case from each proof

1) $(\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2 = 0$

$$\Rightarrow x_1y_1 + x_2y_2 = \sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}$$

ie there is a λ such that $\vec{x} = \langle x_1, x_2\rangle$ and $\vec{y} = \langle y_1, y_2\rangle$ have a lin. comb. $= \vec{0}$, with $\lambda \ne 0$.
What sort of vectors $\vec{x}$ and $\vec{y}$ are these? The ones for which Schwarz equality holds.

2) $(\vec{x} - \vec{y})^2 = 0 \Rightarrow \vec{x} = \vec{y}$, and Schwarz Equality

$\vec{x} = \vec{y}$ means we have the same unit vector

But $\frac{x_i}{\sqrt{x_1^2+x_2^2}} = \frac{y_i}{\sqrt{y_1^2+y_2^2}} \Rightarrow x_i = \lambda y_i$

$$\lambda = \sqrt{\frac{x_1^2+x_2^2}{y_1^2+y_2^2}}$$

3) $(x_1^2+x_2^2)(y_1^2+y_2^2) =$
$(x_1y_1+x_2y_2)^2 + (x_1y_2 - x_2y_1)^2$
$= (x_1y_1+x_2y_2)^2$
$\Rightarrow (x_1y_2 - x_2y_1)^2 = 0$
$\Rightarrow \frac{x_1}{y_1} = \frac{x_2}{y_2} = \lambda$

b) $2xy \leq x^2 + y^2$

$x = x_i / \sqrt{x_1^2 + x_2^2}$ $\quad y = y_i / \sqrt{y_1^2 + y_2^2}$

i) we know that $x^2 + 2xy + y^2 \geq 0$ because $(x-y)^2 \geq 0$

$$i=1 \Rightarrow 2 \frac{x_1 y_1}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} \leq \frac{x_1^2}{x_1^2 + x_2^2} + \frac{y_1^2}{y_1^2 + y_2^2}$$

$$i=2 \Rightarrow \frac{2 x_2 y_2}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} \leq \frac{x_2^2}{x_1^2 + x_2^2} + \frac{y_2^2}{y_1^2 + y_2^2}$$

Adding the two inequalities:

$$\frac{2x_1y_1 + 2x_2y_2}{\sqrt{}\cdot\sqrt{}} \leq \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2} + \frac{y_1^2 + y_2^2}{y_1^2 + y_2^2} = 2$$

$$\Rightarrow \frac{2(x_1y_1 + x_2y_2)}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} \leq 2$$

$$\Rightarrow x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

c) $(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2$

$$x_1^2y_1^2 + x_1^2y_2^2 + x_2^2y_1^2 + x_2^2y_2^2$$

$$(x_1y_1)^2 + 2x_1y_1x_2y_2 + (x_2y_2)^2 + x_1^2y_2^2 + x_2^2y_1^2 - 2x_1y_1x_2y_2$$

$$(x_1y_1 + x_2y_2)^2 + (x_1y_2)^2 - 2x_1y_1x_2y_2 + (x_2y_1)^2$$

$$= (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2 \geq (x_1y_1 + x_2y_2)^2$$

therefore, taking $\sqrt{}$:

$$\sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)} \geq x_1y_1 + x_2y_2$$

d) equality of Schwarz Ineq.

i) in c), we assumed a $\lambda \; \exists$ s.t. $x_1 = \lambda y_1, x_2 = \lambda y_2$ and got equality i
also if $y_1 = y_2 = 0$, we get $0 = 0 \; \forall x_1, x_2$

ii) we started from a known relationship between numbers: $x^2 - 2xy + y^2 \geq 0$
we assumed arbitrary values for x_1, x_2, y_1, y_2 and got the inequality.

we could start with $x^2 - 2xy + y^2 = 0$ and get the eq. in the Schwarz Ineq if we have $(x-y)^2 = 0$
$\Rightarrow x = y$. In this case $\frac{x_i}{\sqrt{x_1^2 + x_2^2}} = \frac{y_i}{\sqrt{y_1^2 + y_2^2}}$

But this means our assumed values imply that $x_i = \lambda y_i$ where

$$\lambda = \left[\frac{(x_1^2 + x_2^2)}{(y_1^2 + y_2^2)}\right]^{1/2}$$

Finally, $y_1 = y_2 = 0$ means that

$x_i = y_i = 0$ so we get the equality derived from the initial expr. as well.

iii) In c) we started with
$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (\ldots)$

$= (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2$

If $x_1y_2 - x_2y_1 = 0$ we obtain Schwarz Equality.

This happens if $y_1 = y_2 = 0 \; \forall x_1, x_2$ or if:

1) $y_1 \neq 0 \Rightarrow x_2 = \left(\frac{x_1}{y_1}\right) y_2$

$x_1 = \frac{x_2 y_1}{y_2} = \left(\frac{x_1}{y_1}\right) y_1$, $\lambda = x_1 / y_1$

2) $y_2 \neq 0 \Rightarrow x_1 = \left(\frac{x_2}{y_2}\right) y_1$

$x_2 = \frac{x_1}{y_1} y_2 = \frac{x_2}{y_2} y_2$ $\quad \lambda = x_2 / y_2$

20. Prove

$$|x-x_0| < \frac{\epsilon}{2}, \quad |y-y_0| < \frac{\epsilon}{2} \Rightarrow \begin{array}{l} |(x+y)-(x_0+y_0)| < \epsilon \\ |(x-y)-(x_0-y_0)| < \epsilon \end{array}$$

$$x \leq |x|,\ y \leq |y| \Rightarrow |x+y| \leq ||x|+|y|| = |x|+|y|$$

$$\begin{aligned} |(x+y)-(x_0+y_0)| &= |(x-x_0)+(y-y_0)| \\ &\leq |x-x_0| + |y-y_0| \\ &= \epsilon \end{aligned}$$

$$|x-y| = |x+(-y)| \leq ||x|+|y|| = |x|+|y|$$

$$\begin{aligned} |(x-y)-(x_0-y_0)| &= |(x-x_0)-(y-y_0)| \\ &\leq |x-x_0| + |y-y_0| \\ &= \epsilon \end{aligned}$$

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21. Prove

$$|x-x_0| < \min\left(\frac{\epsilon}{2(|y_0|+1)}, 1\right), \quad |y-y_0| < \frac{\epsilon}{2(|x_0|+1)} \Rightarrow |xy - x_0y_0| < \epsilon$$

$$|x| - |x_0| \leq |x-x_0| < 1$$

$$\Rightarrow |x| < 1 + |x_0| \Rightarrow |y-y_0| < \frac{\epsilon}{2|x|}$$

$$|x-x_0| < \frac{\epsilon}{2(|y_0|+1)}$$

$$(x-x_0)(y-y_0)$$

$$= xy - x_0y - xy_0 + x_0y_0$$

$$= (xy - x_0y_0) + 2x_0y_0 - x_0y - xy_0$$

$$\begin{aligned} \Rightarrow xy - x_0y_0 &= (x-x_0)(y-y_0) + x_0y + xy_0 - 2x_0y_0 \\ &= x(y-y_0) - x_0(\cancel{y}-y_0) + \cancel{x_0y} + xy_0 - 2x_0y_0 \\ &= x(y-y_0) + y_0(x-x_0) \end{aligned}$$

$$|xy - x_0y_0| = |x(y-y_0) + y_0(x-x_0)|$$

$$\leq |x||y-y_0| + |y_0||x-x_0|$$

$$\leq |x|\frac{\epsilon}{2|x|} + \frac{|y_0|}{1+|y_0|}\frac{\epsilon}{2}$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

20) $|x-x_0| < \frac{\varepsilon}{2}, \; |y-y_0| < \frac{\varepsilon}{2} \Rightarrow |(x+y)-(x_0+y_0)| < \varepsilon$

$|(x-y)-(x_0-y_0)| < \varepsilon$

$|(x+y)-(x_0+y_0)|$
$= |(x-x_0)+(y-y_0)| \leq |x-x_0|+|y-y_0| < \varepsilon$

$|a| = |-a|$

$|(x-y)-(x_0-y_0)|$
$= |(x-x_0)-(y-y_0)| = |(x-x_0)+(y_0-y)| \leq |x-x_0|+|y_0-y| = |x-x_0|+|y-y_0| < \varepsilon$

21) $|x-x_0| < \min\left(\frac{\varepsilon}{2(|y_0|+1)}, 1\right)$

$|y-y_0| < \frac{\varepsilon}{2(|x_0|+1)}$ $\Rightarrow |xy-x_0y_0| < \varepsilon$

$|x|-|x_0| \leq |x-x_0| < 1 \Rightarrow |x| \leq 1+|x_0|$

$|y-y_0| < \frac{\varepsilon}{2|x|} \Rightarrow |y-y_0||x| < \frac{\varepsilon}{2}$ (1)

$|x-x_0| < \varepsilon/2(|y_0|+1) \Rightarrow |x-x_0|(|y_0|+1) < \frac{\varepsilon}{2}$ (2)

Sum (1) and (2)

$|y-y_0||x| + |x-x_0|(|y_0|+1) < \varepsilon$

$|xy - \cancel{xy_0} + \cancel{xy_0} - x_0y_0| + |x-x_0| < \varepsilon$

$|xy-x_0y_0| < \varepsilon$

Alternative solution

$|x|-|x_0| \leq |x-x_0| < 1 \Rightarrow |x| < 1+|x_0|$

$|xy-x_0y_0| = |x(y-y_0)+y_0(x-x_0)| \leq |x||y-y_0| + |y_0||x-x_0|$

$< |x| \cdot \frac{\varepsilon}{2|x|} + \underbrace{|y_0| \cdot \frac{\varepsilon}{2(|y_0|+1)}}_{\frac{|y_0|}{|y_0|+1} \cdot \frac{\varepsilon}{2} < \frac{\varepsilon}{2}} \Rightarrow |xy-x_0y_0| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$

22) $y_0 \neq 0 \quad |y-y_0| < \min\left(\frac{|y_0|}{2}, \varepsilon\frac{|y_0|^2}{2}\right) \Rightarrow y \neq 0$ and $\left|\frac{1}{y} - \frac{1}{y_0}\right| < \varepsilon$

$$|y_0| = |y + (y_0 - y)| \leq |y| + |y_0 - y|$$

$$|y-y_0| < \frac{|y_0|}{2} \Rightarrow |y_0| < |y| + \frac{|y_0|}{2} \Rightarrow |y_0|/2 < |y|$$

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| = \left|\frac{y_0 - y}{y y_0}\right| = \left|\frac{y-y_0}{y y_0}\right| = \frac{|y-y_0|}{|y||y_0|} < \frac{|y-y_0|}{|y_0|^2/2}$$

$$< \frac{\varepsilon|y_0|^2}{2} \Big/ |y_0|^2/2 = \varepsilon$$

23) $y_0 \neq 0$

$y \neq 0$

$|y-y_0| < ?$ $\quad \Rightarrow \left|\frac{x}{y} - \frac{x_0}{y_0}\right| < \varepsilon$

$|x-x_0| < ?$

From 21 we have:

$$\left|x \cdot \frac{1}{y} - x_0 \cdot \frac{1}{y_0}\right| < \varepsilon \quad \text{if } |x-x_0| < \min\left(\frac{\varepsilon}{2(|\frac{1}{y_0}|+1)}, 1\right)$$

$$|1/y + 1/y_0| < \frac{\varepsilon}{2(|x_0|+1)}$$

From 22 we have:

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| < \frac{\varepsilon}{2(|x_0|+1)} \quad \text{if } |y-y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon}{2(|x_0|+1)} \cdot \frac{|y_0|^2}{2}\right)$$

$$= \min\left(\frac{|y_0|}{2}, \frac{\varepsilon|y_0|^2}{4(|x_0|+1)}\right)$$

Recap Problems 21-23

$\rightarrow \quad xy - x_0 y_0 = x(y - y_0) + y_0(x - x_0)$

$\rightarrow \quad y_0 = y_0 + (y - y_0)$

25)

number: 0 or 1

operations

$+$	$b=0$	$b=1$
$a=0$	0	1
$a=1$	1	0

$\cdot$	$b=0$	$b=1$
$a=0$	0	0
$a=1$	0	1

P1) $(a+b)+c = a+(b+c)$

$a+b$		$c=0$	$c=1$
t_{00}	0	0	1
t_{01}	1	1	0
t_{10}	1	1	0
t_{11}	0	0	1

$b+c$		$a=0$	$a=1$
t_{00}	0	0	1
t_{01}	1	1	0
t_{10}	1	1	0
t_{11}	0	0	1

P2) $a+0 = 0+a = a$

	$a=0$	$a=1$
0	0	1

P3) $a+(-a) = (-a)+a = 0$

$a=0, -a=0$

$a=1, -a=1$

P4) $0+0=0$

$1+0 = 0+1 = 1$

$1+1=0$

P5)

ab		c = 0	c = 1
00	0	0	0
01	0	0	0
10	0	0	0
11	1	0	1

$(ab)\cdot c$, $a(bc)$ is the same: replace ab w/ bc, c with a

P6) $a\cdot 1 = 1\cdot a = a$

a	1
0	0
1	1

P7) $a\cdot a^{-1} = a^{-1}\cdot a = 1, \forall a \neq 0$

$a = 1 \quad a^{-1} = 1$

P8) $a\cdot b = b\cdot a \quad \forall a,b$

$0\cdot 0 = 0$

$0\cdot 1 = 1\cdot 0 = 0$

$1\cdot 1 = 1$

P9) $a(b+c) = ab + ac$

a	b	c	a(b+c)	ab+ac
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
0	1	1	0	0
1	1	1	0	0