Ch.13 Appendix – Riemann Sums

1. f, g cont. on $[a,b]$

$P = \{t_0, \dots, t_n\}$ partition of $[a,b]$

$A = \{x_i : x_i \in [t_{i-1}, t_i]\}$

$B = \{v_i : v_i \in [t_{i-1}, t_i]\}$

$S_n = \sum_{i=1}^{n} f(x_i) g(v_i) \Delta t_i$ (not a Riemann sum for fg)

$\rightarrow$ S_n is within ϵ of $\int_a^b fg$ if Δt_i small enough

Proof

A Riemann sum for fg is $R_n = \sum_{i=1}^{n} f(x_i) g(x_i) \Delta t_i$.

Since f and g are cont. on $[a,b]$, they are u.c. on $[a,b]$.

In particular, $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $\forall x \forall y,\ x \in [a,b] \wedge y \in [a,b] \wedge |x-y| < \delta \rightarrow |g(x) - g(y)| < \epsilon$.

Given any $\epsilon > 0$, let $\epsilon_1 = \dfrac{\epsilon}{\int_a^b f + \epsilon}$ and let $\delta_1 > 0$ s.t.

$\forall x \forall y,\ x \in [a,b] \wedge y \in [a,b] \wedge |x-y| < \delta_1 \rightarrow |g(x) - g(y)| < \epsilon_1$

If we choose the partition s.t. $\Delta t_i < \delta$, then $|x_i - v_i| < \delta$, so $|g(x_i) - g(v_i)| < \epsilon_1$

$$S_n - R_n = \sum_{i=1}^{n} (g(v_i) - g(x_i)) f(x_i) \Delta t_i$$
$$< \epsilon \sum_{i=1}^{n} f(x_i) \Delta t_i$$

But $\sum_{i=1}^{n} f(x_i) \Delta t_i$ is a Riemann sum for f.

By Theorem 1, we can make this sum arbitrarily close to $\int_a^b f$.

Ie. $\exists \delta_2 > 0$ s.t. if $\Delta t_i < \delta_2$ then $\left| \sum_{i=1}^{n} f(x_i) \Delta t_i - \int_a^b f \right| < \epsilon$.

$\rightarrow \sum_{i=1}^{n} f(x_i) \Delta t_i < \int_a^b f + \epsilon$

Therefore, if $\delta = \min(\delta_1, \delta_2)$ then $S_n - R_n < \dfrac{\epsilon}{\int_a^b f + \epsilon} \cdot \left(\int_a^b f + \epsilon\right) = \epsilon$

Alternative Proof

f, g cont. on $[a,b] \to f, g$ bounded on $[a,b]$ $\wedge$ f, g u.c. on $[a,b]$

$\exists M$ s.t. $\forall x, x \in [a,b] \to |f(x)| \le M$

For any $\epsilon > 0$ let $\epsilon_1 = \dfrac{\epsilon}{M(b-a)}$ and δ s.t. $|x-y| < \delta \to |g(x) - g(y)| < \epsilon_1$

Let $P = \{t_0, \dots, t_n\}$ be partition of $[a,b]$.

If $\Delta t_i < \delta$ then

$$|R_n - S_n| = \left| \sum_{i=1}^{n} [f(x_i) g(x_i)] \Delta t_i - \sum_{i=1}^{n} [f(x_i) g(v_i)] \Delta t_i \right|$$

$$= \left| \sum_{i=1}^{n} f(x_i) [g(x_i) - g(v_i)] \Delta t_i \right|$$

$$\le M \cdot \frac{\epsilon}{M(b-a)} \sum_{i=1}^{n} \Delta t_i$$

$$= \epsilon$$

2. f, g cont., non-neg. on $[a,b]$

For partition P consider $\rightarrow$ S_n is within ϵ of $\int_a^b \sqrt{f+g}$ if all Δt_i small enough.

$$S_n = \sum_{i=1}^{n} \sqrt{f(x_i) + g(v_i)}\, \Delta t_i$$

Proof

$$|S_n - R_n| = \left| \sum_{i=1}^{n} \sqrt{f(x_i)+g(v_i)}\, \Delta t_i - \sum_{i=1}^{n} \sqrt{f(x_i)+g(x_i)}\, \Delta t_i \right|$$

$$= \left| \sum_{i=1}^{n} \left(\sqrt{f(x_i)+g(x_i)+(g(v_i)-g(x_i))} - \sqrt{f(x_i)+g(x_i)} \right) \Delta t_i \right|$$

Since g is u.c. on $[a,b]$, $\forall \epsilon>0\ \exists \delta_1>0$ s.t. $\forall x, \forall y\ |x-y|<\delta_1 \rightarrow |g(x)-g(y)|<\epsilon$

Let $h(x) = \sqrt{x}$.

Since h is u.c. on $[a,b]$, $\forall \epsilon>0\ \exists \delta>0$ s.t. $\forall x, \forall y\ |x-y|<\delta \rightarrow |h(x)-h(y)|<\epsilon$

For any $\epsilon>0$, let $\epsilon_1 = \dfrac{\epsilon}{b-a}$.

Let $\delta>0$ s.t. $\forall x, \forall y\ |x-y|<\delta \rightarrow |h(x)-h(y)|<\epsilon_1$

Let $\delta_1>0$ s.t. $\forall x, \forall y\ |x-y|<\delta_1 \rightarrow |g(x)-g(y)|<\delta$

Choose P s.t. $\Delta t_i < \delta_1$. Then $|v_i - x_i| < \delta_1 \rightarrow |g(v_i)-g(x_i)| < \delta$

Notice that $f(x_i)+g(v_i) - f(x_i)-g(x_i) = \cancel{f(x_i)}+\cancel{g(x_i)}+(g(v_i)-g(x_i)) - \cancel{f(x_i)}-\cancel{g(x_i)}$

$= g(v_i)-g(x_i) < \delta$

$$\rightarrow \sqrt{f(x_i)+g(x_i)+(g(v_i)-g(x_i))} - \sqrt{f(x_i)+g(x_i)} < \frac{\epsilon}{b-a}$$

$$|S_n - R_n| < \left| \frac{\epsilon}{b-a} \sum \Delta t_i \right| = \epsilon$$

Alternative Proof

f, g cont. $\to$ $f+g$ cont. $\to$ $f+g$ bounded on $[a,b]$

$\exists M$ s.t. $f(x)+g(x) \le M$ on $[a,b]$

Now, $f+g$ is also u.c. on $[a,b]$, and so is g.

$\forall \epsilon>0\ \exists \delta>0$ s.t. $\forall x \forall y$ in $[0,M]$, $|x-y|<\delta \to |\sqrt{x}-\sqrt{y}|<\epsilon$ (1)

$\forall \delta>0\ \exists \delta_1>0$ s.t. $\forall x_i \forall v_i$ $|x_i - v_i| < \delta_1 \to |g(x_i)-g(v_i)|<\delta$ (2)

For any $\epsilon>0$, let $\epsilon_1 = \dfrac{\epsilon}{b-a}$.

let $\delta>0$ s.t. (1) for ϵ_1

let $\delta_1>0$ satisfying (2) for δ.

choose the partition s.t. $\Delta t_i < \delta_1$. Then $\forall i, i=1,\dots,n \to |x_i - v_i| < \delta_1$

$$\to \left|[f(x_i)+g(v_i)]-[f(x_i)+g(x_i)]\right| = \left|g(v_i)-g(x_i)\right| < \delta$$

$$\to \sqrt{f(x_i)+g(v_i)} - \sqrt{f(x_i)+g(x_i)} < \epsilon_1 = \frac{\epsilon}{b-a}$$

Hence,

$$|S_n - R_n| = \left|\sum_{i=1}^{n}\sqrt{f(x_i)+g(v_i)}\,\Delta t_i - \sum_{i=1}^{n}\sqrt{f(x_i)+g(x_i)}\,\Delta t_i\right|$$

$$= \left|\sum_{i=1}^{n}\left(\sqrt{f(x_i)+g(x_i)+(g(v_i)-g(x_i))} - \sqrt{f(x_i)+g(x_i)}\right)\Delta t_i\right|$$

$$< \frac{\epsilon}{b-a}\sum \Delta t_i = \epsilon$$

Let's prove a few results beforehand

Let $h(x) = \sqrt{x}$.

h(x) is continuous.

Proof

If $a=0$, then for any $\epsilon>0$ choose $\delta = \epsilon^2$. Then, $|x|<\delta = \epsilon^2 \to |\sqrt{x}|<\epsilon \to \lim_{x\to 0}\sqrt{x} = 0$

If $a \in (0,\infty)$ then

$$|x-a|<\delta \to |\sqrt{x}-\sqrt{a}||\sqrt{x}+\sqrt{a}|<\delta \to |\sqrt{x}-\sqrt{a}| < \frac{\delta}{|\sqrt{x}+\sqrt{a}|}$$

T.F. if $\delta < \sqrt{a}\,\epsilon$ then we have $|x-a|<\delta \to |\sqrt{x}-\sqrt{a}|<\epsilon$.

$$\to \lim_{x\to a}\sqrt{x} = \sqrt{a}$$

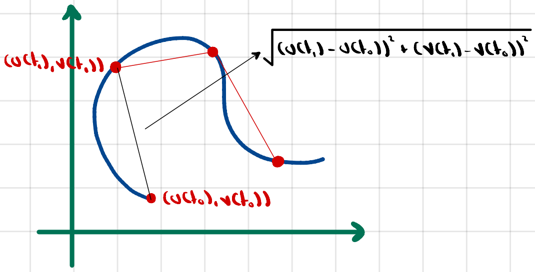
h(x) is unif. cont.

$$|\sqrt{x}-\sqrt{y}|^2 \le |\sqrt{x}-\sqrt{y}||\sqrt{x}+\sqrt{y}| = |x-y| < \epsilon^2 \quad \to |\sqrt{x}-\sqrt{y}|<\epsilon$$

3. curve c given parametrically by fns u, v on $[a,b]$.
For partition $P = \{t_0, \dots, t_n\}$ of $[a,b]$ define

$$\ell(c,P) = \sum_{i=1}^{n} \sqrt{[u(t_i)-u(t_{i-1})]^2 + [v(t_i)-v(t_{i-1})]^2}$$

length of inscribed polygonal curve



length of c = $\sup\{\ell(c,P)\}$ if the latter exists

Prove: u', v' cont. on $[a,b]$ $\rightarrow$ length of c is $\int_a^b \sqrt{u'^2 + v'^2}$

Proof

1. $\ell(c,P) = \sum_{i=1}^{n} \sqrt{[u(t_i)-u(t_{i-1})]^2 + [v(t_i)-v(t_{i-1})]^2} = \sum_{i=1}^{n} \Delta t_i \sqrt{\left(\frac{u(t_i)-u(t_{i-1})}{\Delta t_i}\right)^2 + \left(\frac{v(t_i)-v(t_{i-1})}{\Delta t_i}\right)^2}$

2. For each $i \in \{1, \dots, n\}$, $\exists x_i, x_i \in (t_{i-1}, t_i) \wedge u'(x_i) = \frac{u(t_i)-u(t_{i-1})}{\Delta t_i}$

   Proof: By MVT applied to $u(t)$, and the assumption that u' cont. on $[a,b]$, which implies u' exists on (a,b), and thus u diff on (a,b).

3. For each $i \in \{1, \dots, n\}$, $\exists y_i, y_i \in (t_{i-1}, t_i) \wedge v'(y_i) = \frac{v(t_i)-v(t_{i-1})}{\Delta t_i}$

   Proof: Analogous to proof of 2.

4. $\ell(c,P) = \sum_{i=1}^{n} \sqrt{u'(x_i)^2 + v'(y_i)^2}\, \Delta t_i$, for x_i, y_i in (t_{i-1}, t_i)

   Proof: By 2. and 3.

5. Let $I = \int_a^b \sqrt{u' + v'}$. Then we can make $\ell(c,P)$ arbitrarily close to I by choosing Δt_i small enough.

   Proof: By problem 2.

6. I is an upper bound for $\ell(c,P)$

   Proof: Making Δt_i smaller means adding points to the partition, which (by Problem 13-25 c) increases $\ell(c,P)$.

   If $\ell(c,P) > I$ for some P then for any partition containing P, the difference between $\ell(c,P)$ and I would increase.

   But this contradicts the result from Problem 2 that we can make $\ell(c,P)$ arbitrarily close to I.

7. $I = \int_a^b \sqrt{u' + v'} = \sup\{\ell(c,P)\}$

   Proof: By Problem 2, $\forall \varepsilon > 0$, $\exists$ partition P s.t. $0 < I - \ell(c,P) < \varepsilon$ $\rightarrow$ $I - \varepsilon < \ell(c,P) < I$

   $\rightarrow \int_a^b \sqrt{u' + v'} = \sup\{\ell(c,P)\}$

   $\rightarrow$ length of c is $\int_a^b \sqrt{u' + v'}$

4. f' continuous on $[\theta_0, \theta_1]$ $\rightarrow$ Graph of f in polar coord. on this interval has length $\int_{\theta_0}^{\theta_1} \sqrt{f^2 + f'^2}$

Proof

$h(\theta) = \langle r(\theta)\cos\theta, r(\theta)\sin\theta \rangle = \langle u(\theta), v(\theta) \rangle$, $\theta \in [\theta_0, \theta_1]$

$f'(\theta) = \langle r'(\theta)\cos\theta - r(\theta)\sin\theta, r'(\theta)\sin\theta + r(\theta)\cos\theta \rangle = \langle u'(\theta), v'(\theta) \rangle$ is continuous

By problem 3, the length of f on $[\theta_0, \theta_1]$ is

$$\int_{\theta_0}^{\theta_1} \sqrt{u'^2 + v'^2} = \int_{\theta_0}^{\theta_1} \left[(r'(\theta)\cos\theta - r(\theta)\sin\theta)^2 + (r'(\theta)\sin\theta + r(\theta)\cos\theta)^2 \right]^{1/2}$$

$$= \int_{\theta_0}^{\theta_1} \left[= \begin{array}{l} r'^2\cos^2\theta - \cancel{2r'r\sin\theta\cos\theta} + r^2\sin^2\theta \\ + r'^2\sin^2\theta + \cancel{2r'r\sin\theta\cos\theta} + r^2\cos^2\theta \end{array} \right]^{1/2}$$

$$= \int_{\theta_0}^{\theta_1} \sqrt{r'^2(\theta) + r^2(\theta)}$$

* f in the problem statement corresponds to the function r in the proof above.

5. Cauchy-Schwarz inequality is consequence of Schwarz inequality. Prove using Th. 1.

Proof

Recall the Cauchy-Schwarz inequality $\left(\int_a^b fg\right)^2 \le \left(\int_a^b f^2\right)\left(\int_a^b g^2\right)$

and the Schwarz inequality $\sum x_i y_i \le \sum x_i \sum y_i$

and

Theorem 1: Suppose f is integrable on $[a,b]$. Then for every $\epsilon > 0$, there is $\delta > 0$ s.t. if $P = \{t_0, \dots, t_n\}$ is partition of $[a,b]$ with $t_i - t_{i-1} < \delta$ for all i then

$$\left| \sum f(x_i)\Delta t_i - \int_a^b f(x)dx \right| < \epsilon$$

for any Riemann sum formed by choosing $x_i \in [t_{i-1}, t_i]$

From the Schwarz Inequality we have

$$\sum f(x_i)g(x_i)\Delta t_i = \sum f(x_i)\sqrt{\Delta t_i}\, g(x_i)\sqrt{\Delta t_i} \le \sum f^2(x_i)\Delta t_i \sum g(x_i)^2 \Delta t_i$$

If we make Δt_i small enough,

$\sum f(x_i)g(x_i)\Delta t_i$ is close to $\int_a^b fg$

$\sum f^2(x_i)\Delta t_i$ is close to $\int_a^b f^2$

$\sum g(x_i)^2\Delta t_i$ is close to $\int_a^b g^2$

Hence

$$\int_a^b fg \le \int_a^b f^2 \int_a^b g^2$$