

Ch. 5 Limits

Provisional Def: $f(x)$ approaches limit l near a , if we can make $f(x)$ as close as we like to l by requiring that x be sufficiently close to, but unequal to, a .

Example

$$f(x) = 3x$$

$$|3x - l| < \epsilon \Rightarrow -\epsilon < 3x - l < \epsilon$$

$$\Rightarrow l - \epsilon < 3x < l + \epsilon$$

$$\Rightarrow \frac{l - \epsilon}{3} < x < \frac{l + \epsilon}{3}$$

$$\text{equivalently, } -\frac{\epsilon}{3} < x - \frac{l}{3} < \frac{\epsilon}{3}$$

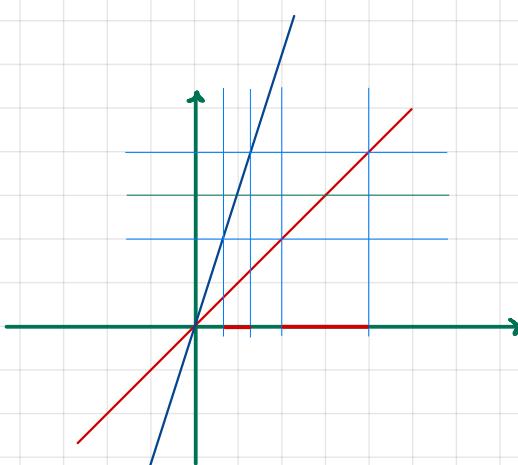
$$l=15, \epsilon=0.1 \Rightarrow 5 - \frac{1}{30} < x < 5 + \frac{1}{30}$$

$$\Rightarrow -\frac{1}{30} < x - 5 < \frac{1}{30}$$

If $f(x) = 3 \cdot 10^6 x$ then

$$l - \epsilon < 3 \cdot 10^6 x < l + \epsilon$$

$$-\frac{\epsilon}{3 \cdot 10^6} < x - \frac{l}{3 \cdot 10^6} < \frac{\epsilon}{3 \cdot 10^6}$$



$$f(x) = x^2$$

$$|x^2 - 9| < \epsilon$$

$$|x-3||x+3| < \epsilon$$

$$\text{Assume } |x-3| < 1 \Rightarrow -1 < x-3 < 1 \Rightarrow 2 < x < 4$$

$$\Rightarrow 5 < x+3 < 7 \Rightarrow |x+3| < 7$$

$$\Rightarrow |x^2 - 9| = |x-3||x+3| < 7|x-3| < \epsilon$$

$$\Rightarrow |x-3| < \frac{\epsilon}{7} \text{ provided } |x-3| < 1$$

Let's do these calc. using $|x-3| < 10$

$$\Rightarrow -10 < x-3 < 10 \Rightarrow -7 < x < 13 \Rightarrow -4 < x+3 < 16$$

$$\Rightarrow |x+3| < 16$$

$$\Rightarrow |x^2 - 9| = |x-3||x+3| < 16|x-3| < \epsilon$$

$$\Rightarrow |x-3| < \frac{\epsilon}{16}$$

more general argument to show $f(x) = x^2$ approaches a^2 near $x=a$ for any a .

Assume $|x-a| < 1$

$$\text{problem 1-12} \Rightarrow |x|-|a| \leq |x-a| < 1$$

$$\Rightarrow |x| < 1+|a|$$

$$\Rightarrow |x+a| \leq |x|+|a| < 2|a|+1$$

$$\frac{x}{a} \leq \frac{|x|}{|a|} \Rightarrow |x+a| \leq |x|+|a| = |x| + \frac{|x|}{|a|} \cdot |a| = |x| + |x| \cdot \frac{1}{|a|} + |a| = |x| \left(1 + \frac{1}{|a|}\right) + |a|$$

$$\Rightarrow |x^2 - a^2| = |x-a||x+a| < |x-a|(2|a|+1)$$

$$\Rightarrow |x^2 - a^2| < \epsilon \text{ for } |x-a|(2|a|+1) < \epsilon$$

$$\Rightarrow |x-a| < \frac{\epsilon}{2|a|+1} \text{ provided that } |x-a| < 1$$

$$\Rightarrow |x-a| < \min\left(1, \frac{\epsilon}{2|a|+1}\right)$$

Consider $f(x) = \frac{1}{x}$, $x \neq 0$

$$\left| \frac{1}{x} - \frac{1}{3} \right| = \left| \frac{3-x}{3x} \right| = \frac{1}{3} \cdot \frac{1}{|x|} \cdot |x-3|$$

$$\text{Assume } |x-3| < 1 \Rightarrow 2 < x < 4 \Rightarrow \frac{1}{4} < \frac{1}{x} < \frac{1}{2}$$

$$\Rightarrow x > 0 \Rightarrow \frac{1}{|x|} < \frac{1}{2}$$

$$\Rightarrow \left| \frac{1}{x} - \frac{1}{3} \right| < \frac{1}{6} |x-3| < \epsilon$$

$$\left| \frac{1}{x} - \frac{1}{3} \right| < \epsilon \Rightarrow |x-3| < 6\epsilon \text{ provided } |x-3| < 1$$

$$|x-3| < \min(1, 6\epsilon)$$

Consider $f(x) = \frac{1}{x}$ on the limit $L = -\frac{1}{3}$.

$$\left| \frac{1}{x} - \left(-\frac{1}{3}\right) \right| < \epsilon$$

$$\Rightarrow \left| \frac{1}{x} + \frac{1}{3} \right| \cdot \left| \frac{3+x}{3x} \right| = \frac{1}{3} \cdot \frac{1}{|x|} \cdot |x+3| < \epsilon$$

$$\text{Assume } |x+3| < 1 \Rightarrow -1 < x+3 < 1 \Rightarrow -4 < x < -2$$

$$\Rightarrow -\frac{1}{2} < \frac{1}{x} < -\frac{1}{4} \Rightarrow \frac{1}{|x|} < \frac{1}{2}$$

$$\Rightarrow \left| \frac{1}{x} + \frac{1}{3} \right| < \frac{1}{3} \cdot \frac{1}{2} \cdot |x+3| < \epsilon$$

$$\Rightarrow |x+3| < 6\epsilon \text{ provided } |x+3| < 1$$

in general

$$\left| \frac{1}{x} - \frac{1}{a} \right| < \epsilon$$

So we need to assume $|x-a| < 1$ but $|a| > 1$.

$$\Rightarrow -1 < x-a <$$

$$-1 < a-1 \Rightarrow a-1 < x < a+1 \Rightarrow \text{checked}$$

But $\frac{1}{x}$ is not defined at $x=0$, so we couldn't be able to say that $\left| \frac{1}{x} - \frac{1}{a} \right| < \epsilon$ character $|x-a| < \delta$ for $\delta > a$.

Instead we assume $|x-a| < \frac{|a|}{2}$



$$\Rightarrow -\frac{|a|}{2} < x-a < \frac{|a|}{2}$$

$$a - \frac{|a|}{2} < x < a + \frac{|a|}{2}$$

$$a > 0 \Rightarrow a = |a| \Rightarrow a - \frac{|a|}{2} = |a| - \frac{|a|}{2} = |a| \cdot \frac{|a|}{2} = \frac{|a|^2}{2}$$

$$\text{Similarly, } a + \frac{|a|}{2} = \frac{3|a|}{2}$$

$$\Rightarrow \frac{|a|}{2} < x < \frac{3|a|}{2} \Rightarrow x \neq 0$$

$$a < 0 \Rightarrow |a| = -a \Rightarrow a - |a|$$

$$a - \frac{|a|}{2} = -|a| - \frac{|a|}{2} = -\frac{3|a|}{2}$$

$$a + \frac{|a|}{2} = -|a| + \frac{|a|}{2} = -\frac{|a|}{2}$$

$$\Rightarrow -\frac{3|a|}{2} < x < \frac{|a|}{2} \Rightarrow x \neq 0$$

$$a - \frac{|a|}{2} < x < a + \frac{|a|}{2} \Rightarrow \frac{2}{2|a|+|a|} < \frac{1}{|x|} < \frac{2}{2a-|a|}$$

$$a > 0 \Rightarrow a \cdot |a| = \frac{2}{3|a|} < \frac{1}{|x|} < \frac{2}{|a|} \Rightarrow \frac{1}{|x|} < \frac{2}{|a|}$$

$$a < 0 \Rightarrow a \cdot -|a| = -\frac{2}{3|a|} < \frac{1}{|x|} < -\frac{2}{|a|} < 0$$

$$\Rightarrow \frac{1}{|x|} < \frac{2}{|a|}$$

$$\left| \frac{1}{x} - \frac{1}{a} \right| = \left| \frac{a-x}{ax} \right| \cdot \frac{1}{|a|} \cdot \frac{1}{|x|} |x-a| < \frac{2}{|a|^2} |x-a| < \epsilon$$

$$|x-a| < \frac{\epsilon}{2} |a|^2$$

$$\text{e.g. } a=3 \Rightarrow \left| \frac{1}{x} - \frac{1}{3} \right| < \epsilon \Leftrightarrow |x-3| < \frac{9\epsilon}{2} = 4.5\epsilon$$

$$\text{provided } |x-3| < \frac{3}{2} = 1.5$$



$$\epsilon < \frac{1}{2} \Rightarrow 4.5\epsilon < 1.5 \Rightarrow \text{smaller interval than initially assumed for } |x-3|$$

example $f(x) = \sqrt{|x|} \sin\left(\frac{1}{x}\right)$

$$|\sqrt{|x|} \sin\left(\frac{1}{x}\right) - 0| < \epsilon \Rightarrow |\sqrt{|x|} \sin\left(\frac{1}{x}\right)| < \epsilon$$

$|\sin\left(\frac{1}{x}\right)| \leq 1 \forall x \neq 0$ (and we are considering only $x \neq 0$, not equal to 0)

$$\Rightarrow |\sqrt{|x|} \sin\left(\frac{1}{x}\right)| \leq \sqrt{|x|} \forall x \neq 0$$

$$\sqrt{|x|} < \epsilon \forall x \neq 0 \Rightarrow |\sqrt{|x|} \sin\left(\frac{1}{x}\right)| < \epsilon$$

$$|x| < \epsilon^2 \forall x \neq 0 \Rightarrow |\sqrt{|x|} \sin\left(\frac{1}{x}\right)| < \epsilon$$

example $f(x) = \sin\left(\frac{1}{x}\right)$

$$|f(x) - 0| < \epsilon \Rightarrow |\sin\left(\frac{1}{x}\right)| < \epsilon$$

$$\text{let } \epsilon = \frac{1}{2}$$

Consider any interval containing $x=0$.

$$x^* = \frac{1}{\frac{\pi}{2} + 2\pi n} \text{ is in the interval for some } n.$$

$$f(x^*) - f\left(\frac{\pi}{2} + 2\pi n\right) = 1 > \frac{1}{2}.$$

I.e., it is not true that for any ϵ we can have

$|\sin\left(\frac{1}{x}\right)| < \epsilon$ if x is within some interval containing zero. As we showed above, $\epsilon = \frac{1}{2}$ is an example where this isn't possible

• 0 is not the limit of f as x approaches 0.

We can show that there is no limit at $x=0$, i.e. f does not approach any value as x approaches zero.

$$|f(x) - l| < \epsilon \Rightarrow$$

$$\text{let } \epsilon = \frac{1}{2} \Rightarrow |\sin(1/x) - l| < \frac{1}{2}$$

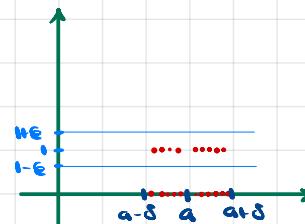
$$\Rightarrow l - \frac{1}{2} < \sin(1/x) < l + \frac{1}{2}$$

In any interval containing $x=0$ we can find an x such that $f(x) = 1$, and another with $f(x) = -1$.

No matter which l we choose, $|\sin(1/x) - l|$ will be at least 1 for one of the points. I.e. $\epsilon = \frac{1}{2}$. I.e., it is impossible to get $\sin(1/x)$ arbitrarily close to l by having x arbitrarily close to zero.

⇒ There is no limit of $\sin(1/x)$ at $x=0$.

example: $f(x) = \begin{cases} 0, & x \in \mathbb{Q}' \\ 1, & x \in \mathbb{Q} \end{cases}$



no matter what a is we cannot get $f(x)$ as close to l as we want simply by taking values from a small enough interval around a .

⇒ There is no limit at a .

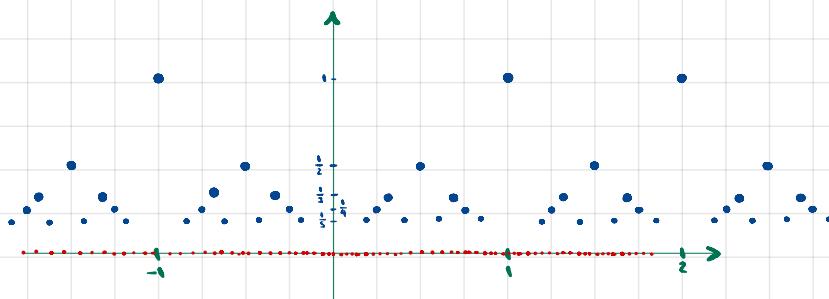
Def: The fn f approaches the limit l near a means: For every $\epsilon > 0$ there is some $\delta > 0$ such that $\forall x$ if $|x-a| < \delta$ then $|f(x) - l| < \epsilon$.

* when f does not approach l at a then the negation of the above definition occurs:

There is some $\epsilon > 0$ such that for every $\delta > 0$ there is some x such that $0 < |x-a| < \delta$ but not $|f(x) - l| < \epsilon$.

example

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q}' \\ \frac{p}{q} & x = \frac{p}{q} \text{ in lowest terms } (p, q \in \mathbb{Z}, \text{ no common factors, } q > 0) \end{cases}$$



Theorem 1: A function cannot approach two different limits near a

In other words, if f approaches l near a , and f approaches m near a then $l=m$.

Proof:

f approaches l near a

$$\forall \epsilon > 0, \exists \delta_1 > 0, |x-a| < \delta_1 \Rightarrow |f(x) - l| < \epsilon \quad (1)$$

f approaches m near a

$$\forall \epsilon > 0, \exists \delta_2 > 0, |x-a| < \delta_2 \Rightarrow |f(x) - m| < \epsilon \quad (2)$$

$$\Rightarrow \forall \epsilon > 0, |x-a| < \min(\delta_1, \delta_2) \Rightarrow |f(x) - m| < \epsilon \text{ and } |f(x) - l| < \epsilon$$

$$\text{Call } \min(\delta_1, \delta_2) = \delta$$

Assume $m \neq l$

$$|x-a| < \delta \Rightarrow m - \epsilon < f(x) < m + \epsilon$$

$$|x-a| < \delta \Rightarrow l - \epsilon < f(x) < l + \epsilon$$

$$\text{choose } \epsilon = \frac{|m-l|}{2}$$

$$|x-a| < \delta \Rightarrow |f(x) - m| < \frac{|m-l|}{2}$$

$$|f(x) - l| < \frac{|m-l|}{2}$$

$$|m-l| = |m-f(x)+f(x)-l|$$

$$= |(m-f(x)) + (f(x)-l)|$$

$$\leq |m-f(x)| + |f(x)-l|$$

$$< \frac{|m-l|}{2} + \frac{|m-l|}{2} = |m-l|$$

$$\Rightarrow |m-l| < |m-l|, \text{ a contradiction}$$

$$\Rightarrow \text{The statement: } \forall \epsilon > 0, \exists \delta = \min(\delta_1, \delta_2), |x-a| < \delta \Rightarrow |f(x) - m| < \epsilon, |f(x) - l| < \epsilon, \forall \epsilon$$

is false. We found a ϵ for which there is no δ such that $|x-a| < \delta \Rightarrow |f(x) - m| < \epsilon, |f(x) - l| < \epsilon$

→ Because limit of f at a is unique, we can denote it as $\lim_{x \rightarrow a} f(x)$

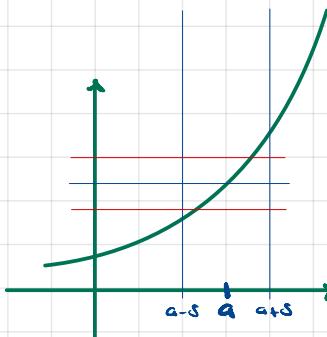
$$\lim_{x \rightarrow a} f(x) = l \quad \text{"the limit of } f(x) \text{ as } x \text{ approaches } a \text{ is } l"$$

" f approaches l near a "

→ It is possible that near a , f doesn't approach a limit.

i.e. $\lim_{x \rightarrow a} f(x) = l$ is false for every l

This means $\lim_{x \rightarrow a} f(x)$ does not exist



$$\lim_{x \rightarrow a} x + t^3 = l \text{ s.t. } |x - a| < \delta \Rightarrow |x + t^3 - l| < \epsilon$$

If we have a guess for what the limit is, we can verify if it is:

$$|x + t^3 - l| = |x - a| < \delta \text{ by assumption}$$

$$\text{Given } \epsilon > 0, \text{ choose } \delta = \frac{\epsilon}{2} \Rightarrow |x + t^3 - (l + t^3)| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} (x + t^3) = l + t^3$$

If we don't know what l is

$$\text{Consider 1. } \lim_{x \rightarrow a} f(x)$$

$$2. \lim_{h \rightarrow 0} f(a+h)$$

(1) is the number l s.t. $\forall \epsilon, \exists \delta$ s.t. $|x - a| < \delta \Rightarrow |f(x) - l| < \epsilon$

(2) is the number l s.t. $\forall \epsilon, \exists \delta$ s.t. $|h| < \delta \Rightarrow |f(a+h) - l| < \epsilon$

Let $h = x - a$. Then $x = h+a$

(1) is now l s.t. $|h| < \delta \Rightarrow |f(a+h) - l| < \epsilon$

Theorem 2

$$\lim_{x \rightarrow a} f(x) = l$$

$$\lim_{x \rightarrow a} (f+g)(x) = l+m \quad (1)$$

$$\lim_{x \rightarrow a} g(x) = m$$

$$\lim_{x \rightarrow a} (f \cdot g)(x) = l \cdot m \quad (2)$$

$$m \neq 0 \Rightarrow \lim_{x \rightarrow a} \frac{1}{g(x)} = \frac{1}{m} \quad (3)$$

To prove these, we need following results from Ch. 1 problems:

Lemma

$$1. |x - x_0| < \frac{\epsilon}{2}, |t - t_0| < \frac{\epsilon}{2} \Rightarrow |(x+t) - (x_0+t_0)| < \epsilon$$

$$2. |x - x_0| < \min\left(1, \frac{\epsilon}{2(|t_0|+1)}\right) \Rightarrow |x-t-x_0-t_0| < \epsilon$$

$$|t-t_0| < \frac{\epsilon}{2(|x_0|+1)}$$

$$3. y_0 \neq 0, |t-t_0| < \min\left(\frac{|t_0|}{2}, \frac{\epsilon|t_0|^2}{2}\right)$$

$$\Rightarrow y_0 \neq 0, \left|\frac{1}{t} - \frac{1}{t_0}\right| < \epsilon$$

Proof (1)

$$\forall \epsilon, \exists \delta_1 \text{ s.t. } |x - a| < \delta_1 \Rightarrow |f(x) - l| < \epsilon$$

$$\forall \epsilon, \exists \delta_2 \text{ s.t. } |x - a| < \delta_2 \Rightarrow |g(x) - m| < \epsilon$$

$$\text{Let } \delta = \min(\delta_1, \delta_2)$$

$$\forall \epsilon, |x - a| < \delta \Rightarrow |f(x) - l| < \frac{\epsilon}{2}, |g(x) - m| < \frac{\epsilon}{2}$$

By part (1) of the lemma,

$$|f(x) + g(x) - (l+m)| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} (f+g)(x) = l+m$$

Proof of (2)

By assumption

$$\forall \epsilon > 0, \exists \delta_1, \delta_2 \text{ s.t.}$$

$$|x - a| < \delta_1 \Rightarrow |f(x) - l| < \min\left(1, \frac{\epsilon}{2(|m|+1)}\right)$$

$$|x - a| < \delta_2 \Rightarrow |g(x) - m| < \frac{\epsilon}{2(|l|+1)}$$

$$\text{Let } \delta = \min(\delta_1, \delta_2)$$

$$\Rightarrow |x - a| < \delta \Rightarrow |f(x) - l| < \min\left(1, \frac{\epsilon}{2(|m|+1)}\right)$$

$$|g(x) - m| < \frac{\epsilon}{2(|l|+1)}$$

by lemma part (2)

$$|f(x)g(x) - lm| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} (f \cdot g)(x) = lm$$

Proof of (3)

$$\forall \epsilon > 0$$

$$\text{Let } \epsilon_0 = \min\left(\frac{|m|}{2}, \frac{\epsilon|m|^2}{2}\right), m \neq 0$$

$$\Rightarrow \exists \delta > 0 \text{ s.t. } |x - a| < \delta \Rightarrow |g(x) - m| < \epsilon_0$$

$$\Rightarrow g(x) \neq 0, \left|\frac{1}{g(x)} - \frac{1}{m}\right| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{1}{g(x)} = \frac{1}{m}, g(x) \neq 0$$

$$\rightarrow \text{Prove } \lim_{x \rightarrow a} \frac{x^3 + 7x^5}{x^2 + 1} = \frac{a^3 + 7a^5}{a^2 + 1}$$

by using theorem 2.

$$\lim_{x \rightarrow a} 7 = 7$$

$$\text{proof: } \forall \epsilon > 0, \exists \delta = |x - a| < \delta \Rightarrow |f(x) - 7| < \epsilon$$

$$\lim_{x \rightarrow a} 1 = 1$$

$$\lim_{x \rightarrow a} x = a$$

$$\text{proof: } \forall \epsilon > 0, \forall \delta < \epsilon \Rightarrow |x - a| < \delta \Rightarrow |f(x) - a| < \epsilon$$

$$\lim_{x \rightarrow a} x^2 = \lim_{x \rightarrow a} x \cdot x = \lim_{x \rightarrow a} x \cdot \lim_{x \rightarrow a} x = a \cdot a = a^2$$

$$\text{Assume } \lim_{x \rightarrow a} x^n = a^n$$

$$\lim_{x \rightarrow a} x^{n+1} = \lim_{x \rightarrow a} x \cdot x^n = \lim_{x \rightarrow a} x \cdot \lim_{x \rightarrow a} x^n = a \cdot a^n = a^{n+1}$$

$$\text{By induction, } \lim_{x \rightarrow a} x^n = a^n$$

$$\lim_{x \rightarrow a} \frac{x^3 + 7x^5}{x^2 + 1} = \lim_{x \rightarrow a} \left(\frac{1}{x^2 + 1} \right) \cdot (x^3 + 7x^5)$$

$$= \lim_{x \rightarrow a} \frac{1}{x^2 + 1} \cdot \lim_{x \rightarrow a} (x^3 + 7x^5) = \frac{1}{a^2 + 1} a^3 + 7a^5$$

$$\lim_{x \rightarrow a} \frac{1}{x^2 + 1} = \frac{1}{a^2 + 1} \text{ by Thm 2(3)}$$

$$\lim_{x \rightarrow a} (x^3 + 7x^5) = a^3 + 7a^5 \text{ Thm 2(1)}$$

$$\lim_{x \rightarrow a} (x^3 + 7x^5) = a^3 + 7a^5 \text{ Thm 2(1)}$$

Now let's actually find a δ such that $\forall \epsilon > 0$

$$|x - a| < \delta \Rightarrow \left| \frac{x^3 + 7x^5}{x^2 + 1} - \frac{a^3 + 7a^5}{a^2 + 1} \right| < \epsilon$$

$$\rightarrow \text{Find } \lim_{x \rightarrow a} (x^2 + 1)$$

$$|x^2 + 1 - (a^2 + 1)| = |x^2 - a^2| = |x - a||x + a|$$

$$\text{Assume } |x - a| < 1$$

$$|x| - |a| \leq |x - a| < 1 \Rightarrow |x| \leq |a| + 1$$

$$|x + a| \leq |x| + |a| \leq 1 + 2|a|$$

$$|x^2 - a^2| \cdot |x + a||x - a| \leq |x - a|(1 + 2|a|)$$

$$\text{if } |x - a|(1 + 2|a|) < \epsilon \text{ then } |x^2 - a^2| < \epsilon, \forall \epsilon > 0$$

$$\Rightarrow |x - a| < \frac{\epsilon}{1 + 2|a|}$$

$$\forall \epsilon > 0 \text{ we choose } 0 < \delta < \frac{\epsilon}{1 + 2|a|} \text{ then } |x^2 - a^2| < \epsilon$$

$$\rightarrow |(x^3 + 7x^5) - (a^3 + 7a^5)| < \epsilon$$

$$\rightarrow \lim_{x \rightarrow a} (x^3 + 7x^5) = a^3 + 7a^5$$

$$\text{Find } \lim_{x \rightarrow a} 7x^5$$

$$\lim_{x \rightarrow a} x^5 = a^5, \lim_{x \rightarrow a} 7 = 7, \text{ as stated previously.}$$

$$\forall \epsilon > 0$$

$$\forall \delta_1 > 0, |x - a| < \delta_1 \Rightarrow |7 - 7| < \min\left(1, \frac{\epsilon}{2(a^2 + 1)}\right)$$

$$|x - a| < \delta_2 \Rightarrow |x^5 - a^5| < \frac{\epsilon}{2(\epsilon + 1)}$$

$$|x - a| < \delta_1 \Rightarrow |7 - 7| < \min\left(1, \frac{\epsilon}{2(a^2 + 1)}\right), |x^5 - a^5| < \frac{\epsilon}{2(\epsilon + 1)}$$

$$\text{Lemma (c) } \Rightarrow |7x^5 - 7a^5| < \epsilon$$

$$\rightarrow \lim_{x \rightarrow a} 7x^5 = 7a^5$$

$$\forall \epsilon > 0, \exists \delta_1 : |x - a| < \delta_1 \Rightarrow |x^3 - a^3| < \epsilon$$

$$\forall \epsilon > 0, \exists \delta_2 : |x - a| < \delta_2 \Rightarrow |7x^5 - 7a^5| < \epsilon$$

$$\delta = \min(\delta_1, \delta_2) \Rightarrow \forall \epsilon > 0, |x - a| < \delta \Rightarrow |x^3 - a^3| < \frac{\epsilon}{12}$$

$$\text{Lemma (1) } \Rightarrow |x^3 + 7x^5 - a^3 - 7a^5| < \epsilon$$

$$\rightarrow \lim_{x \rightarrow a} (x^3 + 7x^5) = a^3 + 7a^5$$

$$\lim_{x \rightarrow a} \frac{1}{x^2+1}$$

$$g(x) - x^2 + 1 \rightarrow \lim_{x \rightarrow a} g(x) = a^2 + 1$$

$$\forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } |x-a| < \delta \Rightarrow |g(x) - (a^2 + 1)| < \epsilon$$

* thus δ is $0 < \delta < \frac{\epsilon}{1+2|a|}$

$$\text{let } \epsilon_0 = \min\left(\frac{|a^2+1|}{2}, \frac{\epsilon_0|a^2+1|^2}{2}\right), \forall \epsilon_0 > 0$$

$$\text{Lemma (3)} \Rightarrow g(a) = 0, \lim_{x \rightarrow a} \frac{1}{x^2+1} = \frac{1}{a^2+1}$$

$$\text{i.e., } \forall \epsilon > 0, 0 < \delta < \frac{1}{1+2|a|} \min\left(\frac{|a^2+1|}{2}, \frac{\epsilon_0|a^2+1|^2}{2}\right) \Rightarrow \left|\frac{1}{x^2+1} - \frac{1}{a^2+1}\right| < \epsilon_0.$$

At this point we have

$$\lim_{x \rightarrow a} \frac{x^3 + 7x^5}{x^2 + 1} = \lim_{x \rightarrow a} \left(\frac{1}{x^2 + 1} (x^3 + 7x^5) \right) = \lim_{x \rightarrow a} f(x) \cdot g(x)$$

$$\text{with } \lim_{x \rightarrow a} f(x) = \frac{1}{a^2+1} \quad \forall \epsilon > 0, 0 < \delta < \frac{1}{1+2|a|} \min\left(\frac{|a^2+1|}{2}, \frac{\epsilon_0|a^2+1|^2}{2}\right) \Rightarrow \left|\frac{1}{x^2+1} - \frac{1}{a^2+1}\right| < \epsilon_0.$$

$$\lim_{x \rightarrow a} g(x) = a^3 + 7a^5 \quad \forall \epsilon > 0, \delta = \min(\delta_1, \delta_2) = |x-a| < \delta \Rightarrow |x^3 + 7x^5 - a^3 - 7a^5| < \epsilon$$

$$\forall \epsilon, \delta = \min\left[\frac{1}{1+2|a|}, \delta_1, \delta_2\right] \Rightarrow$$

$$|x-a| < \delta \Rightarrow \left|f(x) - \frac{1}{a^2+1}\right| < \min\left(1, \frac{\epsilon_0}{2(|a^3 + 7a^5| + 1)}\right)$$

$$\left|g(x) - a^3 - 7a^5\right| < \frac{\epsilon_0}{2\left(1 + \frac{1}{a^2+1}\right) + 1}$$

$$\text{Lemma (2)} \Rightarrow \left| \frac{x^3 + 7x^5}{x^2 + 1} - \frac{a^3 + 7a^5}{a^2 + 1} \right| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{x^3 + 7x^5}{x^2 + 1} = \frac{a^3 + 7a^5}{a^2 + 1}$$

In summary, to prove the above statement, we proved three limits separately. The final δ is the minimum of the δ 's used in the limit expressions of $x^2 + 1$, $x^3 + 7x^5$. Note that for the latter expr. f. ex., the δ is also the min of δ 's used in the limit expressions of x^3 and $7x^5$. We thus need to keep x close enough to a such that each limit required in the calculation of the limit of the entire expr. is true.

Note that the limit of, say, x^3 involves some δ which we didn't specify explicitly. We know it exists because we proved that $\lim_{x \rightarrow a} x^n = a^n$ using induction and theorem 1.

Important Detail

For $\lim_{x \rightarrow a} f(x)$ to be defined, $f(a)$ does not need to be defined, nor does it need to be defined at all $x \neq a$. However, there must be some $\delta > 0$ such that $f(x)$ is defined $\forall x : 0 < |x - a| < \delta$

Limits From Above/Below

$$\lim_{x \rightarrow a^+} f(x) = l \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) = l$$

$$\forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } 0 < x - a < \delta \Rightarrow |f(x) - l| < \epsilon$$

$$\lim_{x \rightarrow \infty} f(x) = l$$

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall x > N \Rightarrow |f(x) - l| < \epsilon$$

$$\lim_{x \rightarrow a} \sqrt{x}$$

$$f(x) = \sqrt{x}$$

$\forall \epsilon > 0$, we are looking for $\delta > 0$: $|x-a| < \delta \Rightarrow |\sqrt{x} - \sqrt{a}| < \epsilon$

the domain of f is $[0, +\infty)$, so $x \geq 0$.

consider $0 < \delta \leq 1$

$$|x-a| < \delta \leq 1 \Rightarrow -a-\delta < x-a < a+\delta$$

$$\Rightarrow 0 < a-\delta < x < a+\delta < 2a$$

$$|\sqrt{x} - \sqrt{a}| = \frac{|\sqrt{x} - \sqrt{a}| |\sqrt{x} + \sqrt{a}|}{|\sqrt{x} + \sqrt{a}|} = \frac{|x-a|}{|\sqrt{x} + \sqrt{a}|}$$

$$a \geq 1 \Rightarrow |\sqrt{x} - \sqrt{a}| < |x-a|$$

$$\Rightarrow a \geq 1, \forall \epsilon > 0 \Rightarrow |x-a| < \epsilon \Rightarrow |\sqrt{x} - \sqrt{a}| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} f(x) = \sqrt{a}, a \geq 1$$



let's consider $0 < \delta \leq 1$.

$$|x-a| < \delta \leq 1 \Rightarrow -a-\delta < x-a < a+\delta$$

$$\Rightarrow a-1 < a-\delta < x < a+\delta < a+1$$

Let $\delta < \min(a, 1-a) \Rightarrow$

$$|x-a| < \delta \Rightarrow a-\delta < x < a+\delta$$

$$a - \frac{1}{2} < \min(a, 1-a) - a \Rightarrow 0 < x < a < 1$$

$$a + \frac{1}{2} < \min(a, 1-a) + a \Rightarrow 2a - 1 < x < 1$$

$$\Rightarrow 0 < x < 1$$

$$|x-a| \cdot |\sqrt{x} + \sqrt{a}| \cdot |\sqrt{x} - \sqrt{a}| < \delta$$

$$\Rightarrow |\sqrt{x} - \sqrt{a}| < \frac{\delta}{|\sqrt{x} + \sqrt{a}|}$$

$$0 < a-\delta < x < a+\delta < 1$$

$$0 < x < 1 \Rightarrow x < \sqrt{a}$$

$$0 < a < 1 \Rightarrow a < \sqrt{a} \Rightarrow |\sqrt{x} - \sqrt{a}| < \frac{\delta}{x+a} < \frac{\delta}{2a-\delta}$$

$$a-\delta < x < a+\delta$$

$$\Rightarrow 2a-\delta < x+a < 2a+\delta$$

$$\forall \epsilon > 0, \text{ choose } \delta = \frac{\epsilon}{2a-\delta} \Rightarrow \delta = \frac{\epsilon}{2a-\delta} = \frac{\epsilon(1+\epsilon)}{2a} = \frac{\epsilon(1+\epsilon)}{2a} = \frac{\epsilon(1+\epsilon)}{2a}$$

$$\delta < \min\left(\frac{2a\epsilon}{1+\epsilon}, \min(a, 1-a)\right), |x-a| < \delta$$

$$\Rightarrow |\sqrt{x} - \sqrt{a}| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} \sqrt{x} = \sqrt{a}, a \geq 1$$

$$\left| \frac{|x-a|}{\sqrt{x} + \sqrt{a}} \right| < \frac{\delta}{\sqrt{a}} < \frac{\epsilon}{\sqrt{a}} \Rightarrow \delta < \sqrt{a}\epsilon$$

Therefore

$$\forall \epsilon > 0, \text{ choose } \delta = \sqrt{a}\epsilon, |x-a| < \delta \Rightarrow |\sqrt{x} - \sqrt{a}| < \epsilon$$

much simpler sol'n!