# Ch8- Least Upper Bounds

1.

i) $A = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

$\sup A = 1 \in A$
$\inf A = 0 \notin A$. No least element.

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ii) $A = \{\frac{1}{n} : n \in \mathbb{Z} \text{ and } n \neq 0\}$

$= \{\dots, -\frac{1}{3}, -\frac{1}{2}, -1, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$

$\sup A = 1 \in A$
$\inf A = -1 \in A$

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iii) $A = \{x : x = 0 \text{ or } x = \frac{1}{n}, n \in \mathbb{N}\}$
$= \{0, 1, 1/2, 1/3, \dots\}$

$\sup A = 1 \in A$
$\inf A = 0 \in A$

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iv) $\{x : 0 \leq x \leq \sqrt{2}, x \text{ rational}\}$

$\forall \varepsilon > 0 \ \exists n \in \mathbb{N} : 0 < \frac{1}{n} < \varepsilon$.

Thus $\forall \varepsilon > 0, \varepsilon \neq \inf A$

$\inf A = 0 \in A$

$\sup A = \sqrt{2} \notin A$. No greatest element.

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v) $A = \{x : x^2 + x + 1 \geq 0\}$

$\Delta = 1 - 4 = -3 \rightarrow \forall x \ x^2 + x + 1 > 0$

$A = \mathbb{R}$, unbounded.

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vi) $A = \{x : x^2 + x - 1 < 0\}$

$\Delta = 1 + 4 = 5 \rightarrow x_{root} = \frac{-1 \pm \sqrt{5}}{2}$

$\forall x, x \in \left(\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right) \rightarrow x^2 + x - 1 < 0$

$A = \{x : x \in \left(\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)\}$

$\sup A = \frac{-1+\sqrt{5}}{2} \notin A$

$\inf A = \frac{-1-\sqrt{5}}{2} \notin A$

No least or greatest element.

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vii) $A = \{x : x < 0 \text{ and } x^2 + x - 1 < 0\}$

$= \{x : x \in \left(\frac{-1-\sqrt{5}}{2}, 0\right]$

$\sup A = 0 \in A$ = greatest el.

$\inf A = \frac{-1-\sqrt{5}}{2} \notin A$

no least el.

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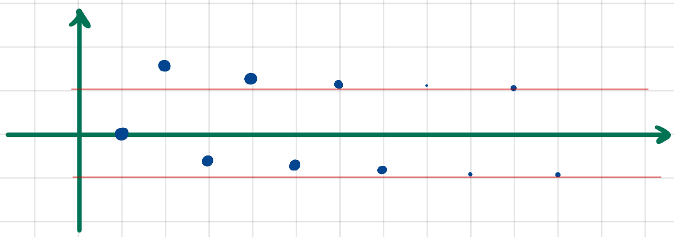
viii) $A = \{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\}$

$= \{0, \frac{1}{2} + 1, \frac{1}{3} - 1, \frac{1}{4} + 1, \dots\}$

$= \{0, 3/2, -2/3, 5/4, -4/5, 7/6, -6/7, \dots\}$



$\sup A = \frac{3}{2} \in A$ = greatest el.

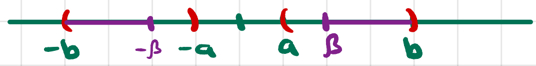
$\inf A = -1 \notin A$

no least el.

2. a)

$A \neq \emptyset$ bounded below
$-A = \{-x : x \in A\}$ → $-A \neq \emptyset$
$-A$ bounded above
$-\sup(-A) = \inf(A)$

**Proof**




Since $A \neq \emptyset$, then $\exists e \in \mathbb{R}, e \in A$. Hence, $-e \in -A$, so $-A \neq \emptyset$.

A bounded below → $\exists z \in \mathbb{R}, \forall x \in A, z \le x$

$\forall x \in A, -x \le -z$

→ $-A$ bounded above.

By P13, since $-A \neq \emptyset$ is bounded above then A has least upper bound $\alpha$.

ie $\forall x \in A, x \le \alpha$

Therefore $\forall x \in A, -x \ge -\alpha$.

$-\alpha$ is a lower bound for A.

Is it the greatest lower bound?

Assume $\beta$ is a lower bound of A and $\beta \ge -\alpha$

Assume $\beta > \alpha$.

Let e be some element in A

Then $\forall x_1 \in A, -\alpha < \beta \le x_1$

→ $\forall x_1 \in A, -x_1 \le -\beta < \alpha$

Thus $\forall x_2 \in -A, x_2 \le \beta < \alpha$
Thus $\beta$ is upper bound of $-A$.
$\alpha$ not $\sup(A)$
$\perp$.

Therefore, $\beta \le -\alpha$. Ie, $-\alpha = \inf(A)$

Therefore A has a greatest lower bound, and $\inf(A) = -\sup(-A)$ ■

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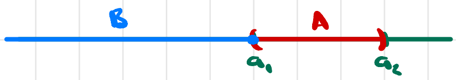
Note that this proves the "Greatest lower bound property".

A is set real numbers
$A \neq \emptyset$ → A has greatest lower bound
A bounded below

b)

$A \neq \emptyset$ bounded below
$B = \{x : x \text{ is lower bound of } A\}$ $\longrightarrow$ $B \neq \emptyset$
$B$ bounded above
$\sup B = \inf A$




A bounded below $\rightarrow \exists y \in \mathbb{R}, \forall x, x \in A \rightarrow y \leq x$.
Thus $y \in B$ and $B \neq \emptyset$.

Since A bounded below and $A \neq \emptyset$, A has a greatest lower bound (part a)), call it $\beta$.
Therefore $\forall y \in \mathbb{R}$, if $y$ is lower bound of A then $y \leq \beta$.
Thus $\beta$ is an upper bound of B.

Thus B has $\alpha = \sup(B)$.

Assume $\alpha \neq \beta$.

Case 1: $\beta < \alpha$, ie $\inf(A) < \sup(B)$

Then $\exists x_1$, $x_1 \in (\beta, \alpha) = (\inf A, \sup B) \wedge x_1 \in A$

Then $x_1$ is a lower bound for A, but $x > \inf(A)$. $\perp$

Case 2: $\alpha < \beta$, ie $\sup(B) < \inf(A)$

$\exists x_1$, $x_1 \in (\sup(B), \inf(A))$.

Then, $x_1$ not a lower bound because $x_1 > \sup(B)$, hence $x_1 \notin B$.

But since $x_1 < \inf(A)$ then $x_1$ is a lower bound of A, hence $x \in B$.

$\perp$.

In both possible cases, $\perp$.

Hence $\alpha = \beta$, ie $\sup B = \inf A$.

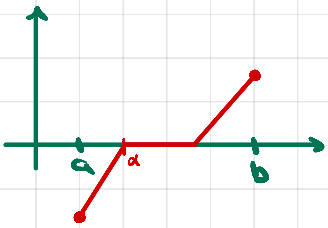
**3.**

$f$ cont. on $[a,b]$

$f(a) < 0 < f(b)$

a) The proof of Theorem 7-1 used a set, $A = \{x : a \leq x \leq b : f$ negative on $[a,x]\}$, and proved that it has a least upper bound $\alpha$, for which $f(\alpha) = 0$.

However, when we think of $x$ such that $f(x) = 0$ and $x > \alpha$, we can find counterexamples to the claim that there is necessarily a second smallest $x$ for which $f(x) = 0$.




$\exists x \in \mathbb{R}, x \in [a,b] \wedge f(x) = 0 \wedge [\forall y, y \in [a,b] \wedge f(y) = 0 \rightarrow y \leq x]$
ie, there is a largest $x$ in $[a,b]$ with $f(x) = 0$.

**Proof**

Let $A = \{x : a \leq x \leq b : f(x)$ positive in $[x,b]\}$

$b \in A$, so $A \neq \emptyset$.

Since $f$ cont. on $[a,b]$ then $\lim_{x \to b^-} f(x) = f(b) > 0$ and $\lim_{x \to a^+} f(x) = f(a) < 0$.

Hence, $\exists \delta > 0$ s.t. $\forall x, x \in (a, a+\delta) \rightarrow f(x) < 0$.
$\rightarrow x$ is a lower bound for $A$.
Therefore, $A$ has a greatest lower bound, call it $\alpha = \inf A$. Then $a < \alpha < b$.

Assume $f(\alpha) > 0$.

Then $\exists \delta > 0, \forall x \in \mathbb{R}, \alpha - \delta < x < \alpha + \delta \rightarrow f(x) > 0$
But then $\exists x_1 \in (\alpha - \delta, \alpha), f(x_1) > 0, \forall y, y \in [x_1, b] \rightarrow f(y) > 0$. Hence $x_1 \in A$, and $x_1 < \alpha$, so $\alpha$ isn't the greatest lower bound.
$\perp$.

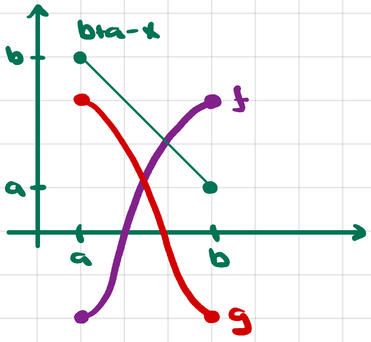
Similarly, if we assume $f(\alpha) < 0$ then $\perp$.

Hence $f(\alpha) = 0$.
Ie, there is some $\alpha \in [a,b]$ such that $f(\alpha) = 0$.
Additionally, since $f(\alpha) = 0$ and $\forall x, x \geq \alpha \rightarrow x \in A$ we have that $\forall x, x \geq \alpha \rightarrow f(x) > 0$.
Therefore $\alpha$ is the largest $x$ in $[a,b]$ s.t. $f(x) = 0$.

## Alternative Proof

Consider the function $h(x) = b + a - x$

Then $h(a) = b$ and $h(b) = a$




Let $g(x) = f(h(x)) = f(b + a - x)$

Then $g(a) = f(b)$ and $g(b) = f(a)$.

It's possible to show that there exists a smallest $x$ in $(a, b)$ s.t. $g(x) = 0$.

Let $x_1$ be that smallest $x$.

Since $\forall x, y,\ x, y \in [a,b] \wedge x < y \rightarrow h(x) > h(y)$

Then

since $\forall y \in [a,b],\ g(y) = 0 \rightarrow y \geq x_1$

Then $\forall y \in [a,b],\ g(y) = 0 \rightarrow h(y) \leq h(x_1)$

Thus

$h(x_1)$ is the largest value in $[a,b]$ such that $f(h(x_1)) = 0$.

Therefore, there is a largest $x$ in $[a,b]$ s.t. $f(x) = 0$.

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An easier way to see the above is using slightly different fns:

$x = b + t(a - b) \quad t \in [0,1]$

$x(0) = b$
$x(1) = a$

$g(t) = f(b + t(a - b))$

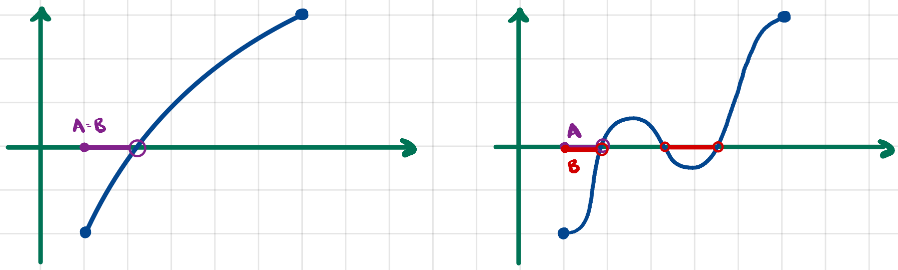
$g(0) = f(b) > 0$
$g(1) = f(a) < 0$

b)

Theorem 7-1

$f$ cont. on $[a,b]$
$f(a) < 0 < f(b)$ $\rightarrow$ $\exists x \in [a,b], f(x) = 0$

$A = \{x : a \leq x \leq b : f \text{ neg on } [a,x]\}$
$B = \{x : a \leq x \leq b : f(x) < 0\}$




Proof

$a \in B$ so $B \neq \emptyset$.
$b \notin B$, and $\forall x, x > b \rightarrow x \notin B$.
Therefore, $b$ is an upper bound for $B$.
By P13, $B$ has a least upper bound, $\alpha = \sup(B)$.

Assume $f(\alpha) > 0$.
Then $f$ is positive in an open interval around $\alpha$.
But then $\alpha$ isn't $\sup(B)$. $\perp$.

Assume $f(\alpha) < 0$.
Then $f$ is neg. in an open interval around $\alpha$.
But then $\alpha$ isn't an upper bound. $\perp$.

Therefore $f(\alpha) = 0$.

Ie $\exists x \in [a,b], f(x) = 0$.

Additionally this $x$ is s.t. $\forall y \in (\alpha, b], f(y) > 0$.

This proof locates the largest $x$ in $[a,b]$ w/ $f(x) = 0$.

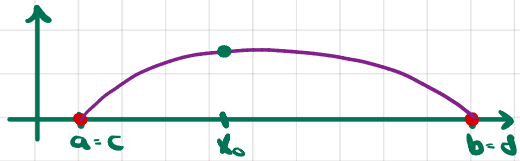
4.

a)

f cont. on $[a,b]$
$f(a)=f(b)=0$
$f(x_0)>0$, for some $x_0 \in [a,b]$

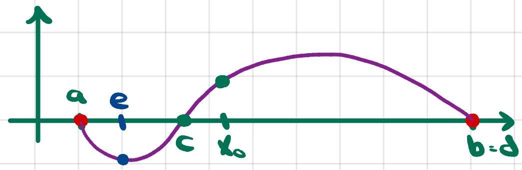
$\rightarrow \exists c,d \in \mathbb{R} \quad a \leq c < x_0 < d \leq b \ \wedge \ f(c)=f(d)=0$
$\wedge \ \forall x,\ x \in (c,d) \rightarrow f(x)>0$

Proof



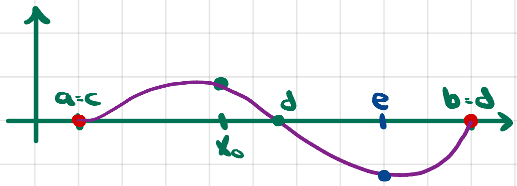

Case 1: $\forall x,\ x \in (a,b) \rightarrow f(x)>0$

let $c=a$ and $d=b$.
Then $f(c)=f(d)=0 \ \wedge \ \forall x \in (c,d) \rightarrow f(x)>0$
$\wedge \ a \leq c < x_0 < d \leq b$

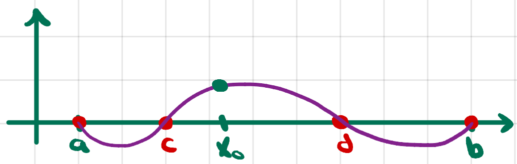



Case 2: $\exists e,\ e \in (a,x_0) \wedge f(e)<0,\ \forall x,\ x \in [x_0,b) \rightarrow f(x)>0$

By problem 3b, $\exists c,\ c \in (e,x_0) \wedge f(c)=0$




$\wedge \ \forall x,\ x \in (e,x_0) \wedge f(x)=0 \rightarrow x \leq c$, ie c is the largest x in $(e,x_0)$ s.t. $f(x)=0$. Thus $\forall x,\ x \in (c,x_0) \rightarrow f(x)>0$.




We are now in case 1: $\forall x,\ x \in (c,b) \rightarrow f(x)>0$

Thus let $d=b$ and we have the desired result.

Case 3 $\exists e,\ e \in (x_0,b) \wedge f(e)<0,\ \forall x,\ x \in (a,x_0] \rightarrow f(x)>0$

Analogous to case 2, but using the original proof of Theorem 7-1.

Case 4 $\exists e_1, e_2$

$e_1 \in (a,x_0) \wedge f(e_1)<0$

$e_2 \in (x_0,b) \wedge f(e_2)<0$

Then $\exists c_1,\ c_1 \in (e_1,x_0) \wedge f(c_1)=0 \ \wedge \ \forall x\ x \in (e_1,x_0) \wedge f(x)=0 \rightarrow x \leq c_1$

$\exists c_2,\ c_2 \in (x_0,e_2) \wedge f(c_2)=0 \ \wedge \ \forall x\ x \in (x_0,e_2) \wedge f(x)=0 \rightarrow x \geq c_2$

Let $c=c_1$ and $d=c_2$ to obtain desired result.

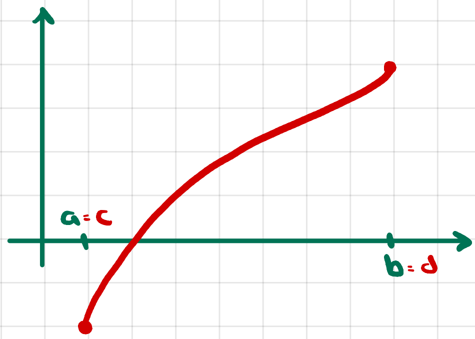
b)

$f$ cont. on $[a,b]$, $f(a) < f(b)$ $\to$ $\exists c,d: a \le c < d \le b$ s.t. $f(c) = f(a)$ $\wedge$ $f(d) = f(b)$ $\wedge$ $\forall x, x \in (c,d) \to f(a) < f(x) < f(d)$

Proof

Case 1: $\forall x, x \in (a,b) \to f(a) < x < f(b)$

Let $c = a$ and $d = b$.

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Case 2: $\exists x, x \in (a,b) \wedge f(x) < f(a)$
$\forall x, x \in (a,b) \to f(x) < f(b)$

Let $x_1$ s.t. $x_1 \in (a,b) \wedge f(x_1) < f(a)$.

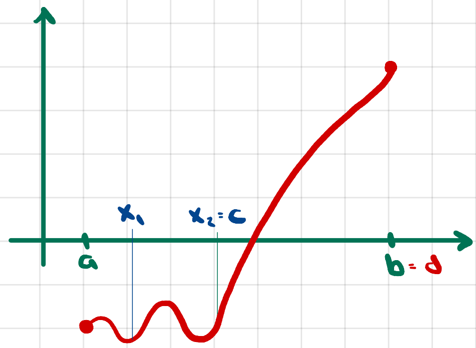
Let $g(x) = f(x) - f(a)$.

Then $g(a) = 0$, $g(b) = f(b) - f(a) > 0$, and $g(x_1) = f(x_1) - f(a) < 0$.

Apply Problem 3b to $g$ on $[x_1, b]$. Obtain $x_2 \in (x_1, b)$ s.t. $g(x_2) = 0$, and $x_2$ is the largest $x$ in $(x_1, b)$ s.t. $f(x) = 0$.

$g(x_2) = 0 \to f(x_2) = f(a)$.

Let $c = x_2$ and $d = b$.

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Case 3: $\exists x, x \in (a,b) \wedge f(x) > f(b)$
$\forall x, x \in (a,b) \to f(x) > f(a)$

Let $x_1$ s.t. $x_1 \in (a,b) \wedge f(x_1) > f(b)$

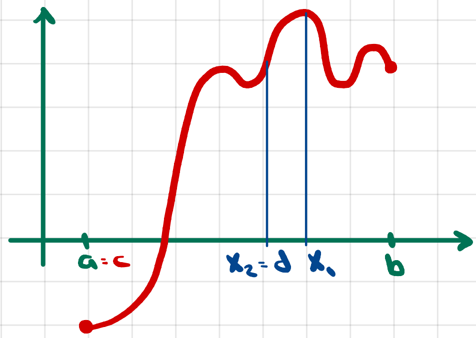
Let $g(x) = f(x) - f(b)$.

Then $g(b) = 0$, $g(a) = f(a) - f(b) < 0$, $g(x_1) = f(x_1) - f(b) > 0$.

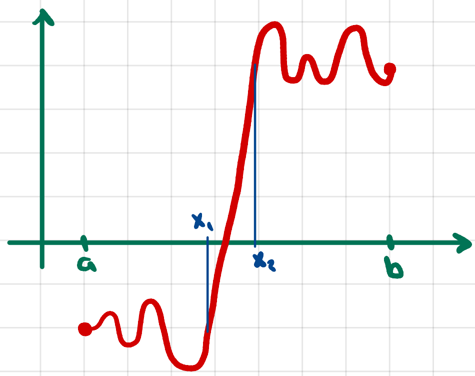
Apply Th.7.1 to $g$ on $[a, x_1]$. Obtain $x_2 \in (a, x_1)$ s.t. $g(x_2) = 0$ and $x_2$ is the smallest $x$ in $(a, x_1)$ s.t. $f(x) = 0$.

$g(x_2) = 0 \to f(x_2) = f(b)$

Let $c = a$ and $d = x_2$.



Case 4 $\exists x, x \in (a,b) \wedge f(x) < f(a)$
$\exists x, x \in (a,b) \wedge f(x) > f(b)$




Proof similar to cases 2 and 3.

Find $x_1$ using $g(x) = f(x) - f(a)$ and problem 3b.
" $x_2$ " $g(x) = f(x) - f(b)$ " theorem 7-1.

Let $c = x_1$ and $d = x_2$.

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Note: each case ends up doing the same thing: c is the largest x in $[a,b]$ s.t. $f(x) = f(a)$,

d is the smallest x in $[a,b]$ s.t. $f(x) = f(d)$.

5.

a)

$y - x > 1 \rightarrow \exists k \in \mathbb{Z},\ x < k < y$

Proof

$[x] \leq x < [x+1]$

$[x+1] \leq x+1 < [x+1]+1 = [x+2]$

Assume $y > x+1$.

Then either $y \in (x+1, [x+2])$ or $y \in [[x+2], \infty)$.

Case 1: $y \in (x+1, [x+2])$

Then $x < [x+1] \leq x+1 < y < [x+2]$
Therefore $k = [x+1] \in \mathbb{Z}$ and $x < k < y$

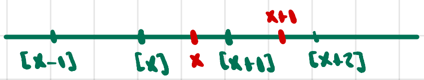
Case 2: $y \in [[x+2], \infty)$

Then $x < [x+1] < [x+2] < y$
$k = [x+1] \in \mathbb{Z}$ and $x < k < y$

In both cases $\exists k \in \mathbb{Z},\ x < k < y$.

Therefore,

$y - x > 1 \rightarrow \exists k \in \mathbb{Z},\ x < k < y$




■

b)

$x < y \rightarrow \exists r, r \in \mathbb{Q} \wedge x < r < y$

Proof

$y - x > 0$

Therefore $\exists n \in \mathbb{N},\ \frac{1}{n} < y - x$.

$ny - nx > 1$

By part a), $\exists k \in \mathbb{Z}$ s.t. $nx < k < ny$.

$x < \frac{k}{n} < y,\ \frac{k}{n} \in \mathbb{Q}$

$\exists r \in \mathbb{Q},\ x < r < y$

■

This proof used theorem 8-3, the proof of which depends on Th. 8-2, which depends on the newly introduced P13.

c)

$r, s \in \mathbb{Q}$, $r < s$ $\rightarrow$ $\exists m \in \mathbb{R} - \{\mathbb{Q}\}$, $r < m < s$

Proof

Let $f(x) = x^2$.

$f(1) = 1 < 2 = f(\sqrt{2})$

Since $f$ cont., $\exists x \in (1, 2)$, $f(x) = x^2 = 2$.

Thus $1 < \sqrt{2} < 2$ and $\frac{1}{2} < \frac{1}{\sqrt{2}} < 1$.

Since $\sqrt{2}$ irrational, so is $\frac{1}{\sqrt{2}}$.

Let $n = \frac{1}{\sqrt{2}}(s - r)$.

Then $n$ irrational and $r < n < s$.

d)

$x < y$ $\rightarrow$ $\exists n$, $x < n < y$ and $n$ irrational

Proof

$\exists x_1, x_2 \in \mathbb{Q}$ s.t. $x < x_1 < x_2 < y$ (b)

$\exists n \in \mathbb{R} - \mathbb{Q}$ s.t. $x < x_1 < n < x_2 < y$ (c)

6.

Let A be set of real numbers.

A dense $\rightarrow$ every open interval contains a point of A

$\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are both dense.

a)

A dense

$f$ cont. $\rightarrow \forall x \in \mathbb{R}\ f(x) = 0$

$\forall x \in A\ f(x) = 0$

Proof

Assume $\exists x_1 \in \mathbb{R}, f(x_1) \neq 0$.

Then either $f(x_1) > 0$ or $f(x_1) < 0$

Case 1: $f(x_1) > 0$

$f$ cont, so $\lim_{x \to x_1} f(x) = f(x_1)$

By def. this means

$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in \mathbb{R},\ |x - x_1| < \delta \rightarrow |f(x) - \ell| < \epsilon$.

Let $\epsilon = \frac{f(x_1)}{2}$. Then $\forall \delta > 0, \exists x \in (x_1 - \delta, x_1 + \delta), x \in A$.

Hence $f(x) = 0$ and $|f(x) - f(x_1)| = f(x_1) > \epsilon$.

Hence $\exists \epsilon > 0\ \forall \delta > 0\ \exists x,\ x \in (x_1 - \delta, x_1 + \delta) \wedge f(x) > \epsilon$

$\lim_{x \to x_1} f(x) \neq f(x_1)$

$\perp$.

Case 2 is analogous and leads to $\perp$.

Therefore $\perp$.

Therefore, $\forall x \in \mathbb{R}\ f(x) = 0$.

b)

$f, g$ cont.

$A$ dense $\rightarrow \forall x \; f(x) = g(x)$

$\forall x, x \in A \rightarrow f(x) = g(x)$

Proof

Let $h(x) = f(x) - g(x)$.

Then F cont and $\forall x, x \in A \rightarrow h(x) = 0$.

By part a), $\forall x \; h(x) = 0$

Therefore $\forall x \; f(x) = g(x)$.

c)

$f, g$ cont.

$A$ dense $\rightarrow \forall x \; f(x) \geq g(x)$

$\forall x, x \in A \rightarrow f(x) \geq g(x)$

Proof

Let $h(x) = f(x) - g(x)$

Assume $\exists x_1 \in \mathbb{R}, f(x_1) < g(x_1)$

Then $h(x_1) < 0$.

Since h cont., $h < 0$ in an open interval around $x_1$.

Since A dense there is an $x_2$ in the open interval that is member of A.

But then $f(x_2) \geq g(x_2)$

$f(x_2) - g(x_2) = h(x_2) \geq 0$

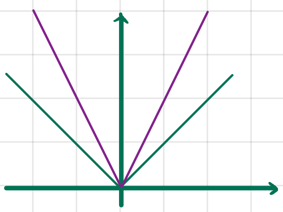
$\perp$

Hence $\forall x \in \mathbb{R} \; f(x) \geq g(x)$

■

If we replace $\geq$ with $>$ the statement isn't true any longer.

Counterexample



$f(x) = |2x|, \; g(x) = |x|$

$A = \mathbb{R} - \mathbb{Q}$

$0 \notin A, \; f(0) = g(0) = 0$

$\forall x \in \mathbb{R}, x \in A \rightarrow f(x) > g(x)$

$\forall x \in \mathbb{R}, x \notin A \rightarrow f(x) \geq g(x)$

$\forall x \in \mathbb{R}, \; f(x) \geq g(x)$

7.

$f$ cont.

$\forall x \forall y\ f(x+y) = f(x) + f(y)$ $\rightarrow$ $\exists c \in \mathbb{R}, \forall x\ f(x) = cx$

## Proof

In problem 3-16 we proved that

$\forall x \forall y\ f(x+y) = f(x) + f(y) \rightarrow \exists c, \forall x \in \mathbb{Q}, f(x) = f(1)x$

Let $g(x) = f(1)x$.

Then, $f$ and $g$ are continuous and $\forall x, x \in \mathbb{Q} \rightarrow f(x) = g(x)$, ie $f = g$ for all $x$ in the dense set $\mathbb{Q}$.

By problem 6b, $f = g$ for all $x$.

---

## Alternative Proof

Let $x_1$ be an irrational number.

Assume $f(x_1) \neq cx_1$.

Case 1: $f(x_1) > cx_1$.

Let $h(x) = f(x) - cx$

Then $h(x)$ cont. and $h(x_1) > 0$.

Hence $h$ is positive in an open interval around $x_1$.

But such an interval contains a rational number, say $x_2$.

Then $h(x_2) > 0$, so $f(x_2) > cx_2$.

But since $x_2 \in \mathbb{Q}$, $f(x_2) = cx_2$.

$\perp$.

Case 2: $f(x_1) < cx_1$

Analog to case 1.

$\perp$.

Therefore $\perp$.

Therefore $f(x_1) = cx_1$.

$\forall x\ f(x) = cx$.

8.

$\forall a,b,\ a<b \to f(a)<f(b)$

a)

$\exists \lim\limits_{x\to a^-} f(x) \ \wedge\ \exists \lim\limits_{x\to a^+} g(x)$

Proof

Let $A = \{f(x) : x<a\}$

Then $A$ is bounded above by $f(a)$.

Since $A \neq \emptyset$, then $\exists \sup A = \alpha$.

Let $\varepsilon>0$.

Then $\exists x_1,\ x_1<a \wedge f(x_1) > \alpha-\varepsilon$, because otherwise $\alpha$ isn't $\sup A$.

Let $\delta = a - x_1$

Then, $\forall x \quad a-\delta<x<a \to x_1<x<a$

$\to f(x) > f(x_1)$

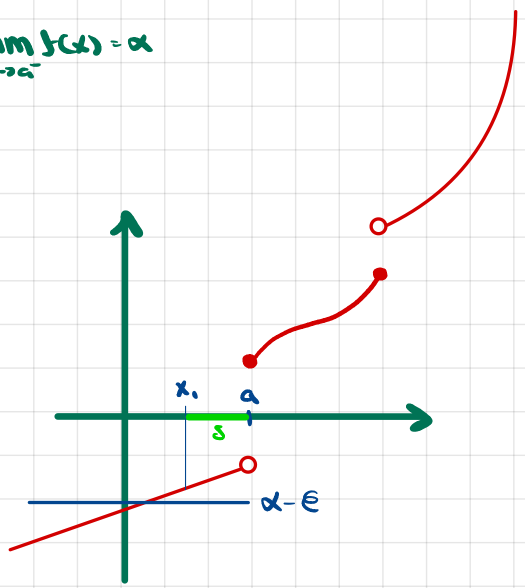
Therefore, $\alpha-\varepsilon < f(x) \leq \alpha$

$|f(x)-\alpha| < \varepsilon$.

Therefore,
$\exists\delta>0\ \forall x,\ x\in(a-\delta,a) \to |f(x)-\alpha|<\varepsilon$

Therefore
$\forall\varepsilon>0\ \exists\delta>0\ \forall x,\ x\in(a-\delta,a) \to |f(x)-\alpha|<\varepsilon$

$\exists \lim\limits_{x\to a^-} f(x) = \alpha$




Let $B = \{f(x) : x>a\}$

Then $B$ bounded below by $f(a)$.

Since $B \neq \emptyset$, it has a greatest lower bound, $\inf B = \beta$.

Let $\varepsilon>0$

Since $\beta = \inf B$, there exists an $x_1 \in B$ s.t. $f(x_1) < \beta+\varepsilon$. If there were not such an $x_1$, then $\beta+\varepsilon$ would be a lower bound greater than $\beta$, a $\perp$.

Let $\delta = x_1 - a$

Then $\forall x,\ x\in(a,a+\delta) \to x\in(a,x_1)$

$\to f(a) < f(x) < f(x_1) < \beta+\varepsilon$

$\to |f(x)-\beta|<\varepsilon$

Therefore,

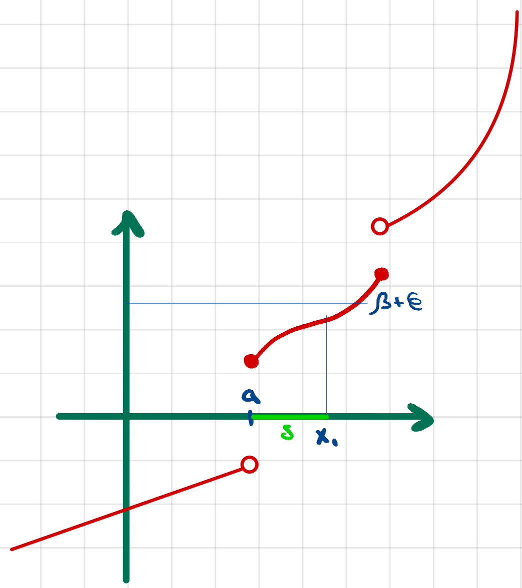
$\exists\delta>0\ \forall x,\ x\in(a,a+\delta) \to |f(x)-\beta|<\varepsilon$

Hence

$\forall\varepsilon>0\ \exists\delta>0\ \forall x,\ x\in(a,a+\delta) \to |f(x)-\beta|<\varepsilon$

Therefore

$\exists \lim\limits_{x\to a^+} f(x) = \beta$



b) $f$ never has removable discontinuity.

Proof

recall: $\exists \lim_{x\to a} f(x) \neq f(a) \iff f$ has removable discontinuity at $a$

Let $a$ be a number.

Assume $\exists \lim_{x\to a} f(x) \neq f(a)$

Then $\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \ell$

Since $\ell \neq f(a)$, we have two cases

Case 1: $f(a) > \ell$

let $\varepsilon = \dfrac{f(a)-\ell}{2} > 0$.

Then $\exists \delta > 0 \ \forall x \ x \in (a, a+\delta) \to |f(x) - \ell| < \dfrac{f(a)-\ell}{2} \to \ell - \dfrac{f(a)-\ell}{2} < f(x) < \ell + \dfrac{f(a)-\ell}{2}$

$\to \quad \dfrac{3\ell - f(a)}{2} < \ell < f(x) < \dfrac{\ell + f(a)}{2} < f(a).$

Thus we have: $\exists x, x > a \wedge f(x) > f(a)$. $\perp$.

Case 2: $f(a) < \ell$

Anal. to case 1. $\perp$.

Therefore $\perp$.

Therefore $\neg(\exists \lim_{x\to a} f(x) \neq f(a))$.

Therefore, $\neg$( $f$ has removable discontinuity at $a$)

Therefore $\forall a, \neg$( $f$ has removable discontinuity at $a$)

---

* $\neg(\exists \lim_{x\to a} f(x) \neq f(a))$

$\iff \neg(\exists \ell \in \mathbb{R}\ (\ell = \lim_{x\to a} f(x) \wedge \ell \neq f(a)))$

$\iff \forall \ell \in \mathbb{R}\ (\ell \neq \lim_{x\to a} f(x) \vee \ell = f(a))$

$\iff \nexists \lim_{x\to a} f(x) \vee f$ cont at $a$

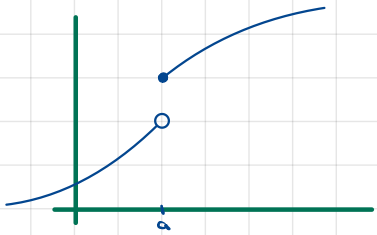
c)

$f$ satisfies conclusion of IVT. $\rightarrow$ $f$ cont.

Proof

Assume $f$ satisfies conclusion of IVT, ie $\forall a, b$, if $f(a) < f(b)$ then on $[a,b]$ $f$ takes on every value in $[f(a), f(b)]$

Assume $f$ is not cont. at a point $a$




From a), let

$$A = \{f(x) : x < a\}$$
$$B = \{f(x) : x > a\}$$

then $\lim_{x \to a^-} f(x) = \sup(A)$

$\lim_{x \to a^+} f(x) = \inf(B)$

From b) we know the discontinuity at $a$ isn't removable.

Hence $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

$\sup(A) \neq \inf(B)$

But then $\forall y \; (y \in (\sup A, \inf B) \rightarrow \nexists x, f(x) = y)$

But then $f$ doesn't satisfy IVT.

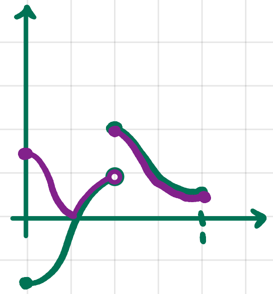
$\perp$

Therefore $\forall a$ $f$ cont. at $a$.

9.

$f$ bounded on $[0,1]$

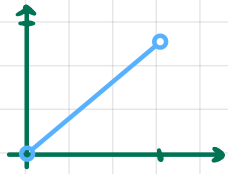
$|||f||| = \sup\{|f(x)| : x \text{ in } [0,1]\}$



a) $\forall c$, $|||cf||| = |c| \cdot |||f|||$

Proof

Note that $f$ is not necess. cont. on $[0,1]$, and so neither is $|f|$.
Therefore, these fns don't necess. take on a max or min value in $[0,1]$. E.g.



$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

Let

$m_1 = \sup\{f(x) : x \text{ in } [0,1]\}$

$m_2 = \inf\{f(x) : x \text{ in } [0,1]\}$

$|||f||| = \max(|m_1|, |m_2|)$

$$|||cf||| = \max(|cm_1|, |cm_2|) = \frac{|cm_1| + |cm_2| + \big||cm_1| - |cm_2|\big|}{2}$$

$$= \frac{|c||m_1| + |c||m_2| + \big||c||m_1| - |c||m_2|\big|}{2}$$

$$= \frac{|c||m_1| + |c||m_2| + \big||c|(|m_1| - |m_2|)\big|}{2}$$

$$= \frac{|c||m_1| + |c||m_2| + |c|\big||m_1| - |m_2|\big|}{2}$$

$$= |c| \frac{|m_1| + |m_2| + \big||m_1| - |m_2|\big|}{2}$$

$$= |c| \max(|m_1|, |m_2|)$$

$$= |c|\, |||f|||$$

b) $|||f+g||| \leq |||f||| + |||g|||$

Proof

Note that $|||f||| = \sup\{|f(x)| : x \text{ in } [0,1]\}$

$= \sup\{\,|||f(x)||| : x \text{ in } [0,1]\}$

$= |||\,|f|\,|||$

That is

$|||f||| = |||\,|f|\,|||$

Therefore,

$\forall x \in [0,1],\ |f(x)| \leq |||f|||$

Also,

$|f+g|(x) = |f(x)+g(x)| \leq |f(x)| + |g(x)| \leq |||f(x)||| + |||g(x)|||$ (1)

Since $f$ and $g$ are bounded on $[0,1]$, so is $f+g$. Since $\{f(x)+g(x) : x \text{ in } [0,1]\} \neq \emptyset$,

this set has a least upper bound: $|||f+g||| = \sup\{|f(x)+g(x)| : x \in [0,1]\}$

$\forall x, x \in [0,1] \to |f(x)+g(x)| \leq |||f+g|||$

Assume $|||f+g||| > |||f||| + |||g|||$

Then $\exists x \in [0,1],\ |||f||| + |||g||| < f(x) + g(x) \leq |||f+g|||$.
(if such an $x$ did not exist then $|||f||| + |||g||| = |||f+g|||$.)

Then $\exists x \in [0,1],\ f(x) + g(x) > |||f||| + |||g|||$.
This contradicts (1).
$\perp$.

Therefore $|||f+g||| \leq |||f||| + |||g|||$

Assume $\nexists x,\ x \in [0,1] \wedge |||f+g||| < |f(x)| + |g(x)| < |||f||| + |||g|||$

Ie $\neg(\exists x, x \in [0,1] \wedge |||f+g||| < |f(x)| + |g(x)| \wedge |f(x)| + |g(x)| < |||f||| + |||g|||)$

$\forall x,\ x \notin [0,1] \vee |||f+g||| \geq (|f|+|g|)(x) \vee (|f|+|g|)(x) \geq |||f||| + |||g|||)$

Assume $x \in [0,1]$. Therefore, two cases:

Case 1: $\forall x\ |||f+g||| \geq (|f|+|g|)(x)$

$|||f+g|||$ is upper bound for $|f(x)| + |g(x)|$.

But $|||f+g||| \leq |||f||| + |||g||| = \sup\{|f(x)| + |g(x)| : x \text{ in } [0,1]\}$.

Thus $|||f+g||| = |||f||| + |||g|||$

Case 2: $\forall x\ (|f|+|g|)(x) \geq |||f||| + |||g|||$

$\perp$.

Thus $|||f+g||| = |||f||| + |||g|||$

Hence $\forall x\, x \in [0,1] \to (|||f+g||| = |||f||| + |||g|||)$

## Alternative Proof

$|f(x)+g(x)| \leq |f(x)|+|g(x)|$, $\forall x \in [0,1]$

$|||f+g||| = \sup\{|f(x)+g(x)| : x \text{ in } [0,1]\}$

$\forall \epsilon > 0 \ \exists x_0,\ x_0 \in [0,1] \wedge |||f+g||| - |f(x_0)+g(x_0)| < \epsilon$

$\rightarrow |||f+g||| - |f(x_0)| - |g(x_0)| < \epsilon$

$\rightarrow |||f+g||| - |||f||| - |||g||| < \epsilon$

This is true for all $\epsilon > 0$.

If $|||f+g||| > |||f||| + |||g|||$ then if we set $\epsilon = |||f+g||| - |||f||| - |||g|||$, (1) can't be true for $\epsilon$'s lesser than this chosen $\epsilon$.

Therefore $|||f+g||| \leq |||f||| + |||g|||$.

10.

$\alpha > 0 \rightarrow \forall x, x = h\alpha + x', h \in \mathbb{Z}, 0 \le x' < \alpha$

Proof

Let $d = x - \alpha$

Case 1: $d < 0$
Then $x < \alpha$
choose $h = 0, x' = x$
Then $x = h\alpha + x' = x$

Case 2: $d > 0$
Then $x > \alpha$
Let $A = \{y : y\alpha < x\}$
$1 \in A$ since $\alpha < y$.

$\frac{x}{\alpha}$ is an upper bound of A.

Then A has least upper bound, $\gamma = \sup A$.

Assume $\gamma < \frac{x}{\alpha}$. Then $\gamma\alpha < x$

Then, by def. of $\gamma = \sup A$, $\nexists y, (\gamma < y < \frac{x}{\alpha}) \wedge y \in A$. (1)

But consider $\frac{\gamma + \frac{x}{\alpha}}{2} = \beta$

$$\gamma = \frac{\gamma + \gamma}{2} < \frac{\gamma + \frac{x}{\alpha}}{2} = \frac{\alpha\gamma + x}{2\alpha} < \frac{2x}{2\alpha} = \frac{x}{\alpha}$$

ie $\gamma < \beta < \frac{x}{\alpha}$

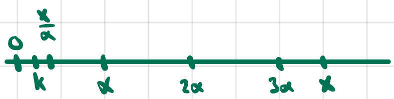
Also $\alpha\beta = \alpha \frac{\gamma + x/\alpha}{2} = \frac{\alpha\gamma + x}{2} < x \rightarrow \beta \in A$

Thus $\gamma < \beta < \frac{x}{2} \wedge \beta \in A$, contradicting (1)

$\perp$.

Therefore, $\gamma \ge \frac{x}{\alpha}$. However, if $\gamma > \frac{x}{\alpha}$ then $\gamma$ isn't $\sup A$. $\perp$.

Therefore $\sup A = \gamma = \frac{x}{\alpha}$.




Let $h = [\frac{x}{\alpha}]$, which we know exists for all $\mathbb{R}$.

$h \le \frac{x}{\alpha} < h+1$
$h\alpha \le x < (h+1)\alpha$
$0 \le x - h\alpha < \alpha$

Let $x' = x - h\alpha$. Then $x = h\alpha + x'$, with $h \in \mathbb{Z}$ and $0 \le x' < \alpha$.

11.

a)

$a_1, a_2, a_3, \ldots$ sequence of $a_n > 0$

$a_{n+1} \leq \frac{a_n}{2}$ $\rightarrow \forall \varepsilon > 0 \; \exists n, a_n < \varepsilon$

Proof

$\forall n \in \mathbb{N}, a_n \leq \frac{a_1}{2^n}$

Proof

$A = \{n : a_{n+1} \leq \frac{a_1}{2^n}\}$

$1 \in A$ since $a_2 \leq \frac{a_1}{2}$

Assume $k \in A$.
Then $a_{k+1} \leq \frac{a_1}{2^k}$.

$a_{k+2} \leq \frac{a_{k+1}}{2} = \frac{a_1}{2^{k+1}}$

So $k+1 \in A$.

By induction we conclude that $A = \mathbb{N}$.

Therefore $\forall n \in \mathbb{N}$, $a_{n+1} \leq \frac{a_1}{2^n}$.

$\lim_{n \to \infty} \frac{a_1}{2^n} = 0$

$\forall \varepsilon > 0 \; \exists N > 0 \; \forall n \; n > N \rightarrow 0 < a_{n+1} \leq \frac{a_1}{2^n} < \varepsilon$

Alternative Proof

$\forall n \in \mathbb{N}, \; 2^n > n$

Proof

$A = \{n \in \mathbb{N} : 2^n > n\}$
$1 \in A$ since $2^1 = 2 > 1$.
assume $k \in A$. then $2^k > k$
$2 \cdot 2^k = 2^{k+1} > 2k \geq k+1$
$A = \mathbb{N}$
By induction, $\forall n \in \mathbb{N}, 2^n > n$.

Therefore $\forall n \in \mathbb{N}$, $\frac{1}{2^n} < \frac{1}{n}$ ■

Hence, since $a_{n+1} \leq \frac{a_1}{2^n} < \frac{a_1}{n}$

Then $\forall \varepsilon > 0$, $\exists n_1 \in \mathbb{N}$ s.t. $\frac{1}{n_1} < \frac{\varepsilon}{a_1}$

$\rightarrow \frac{a_1}{n_1} < \varepsilon$.

Therefore

$a_{n_1+1} < \varepsilon$

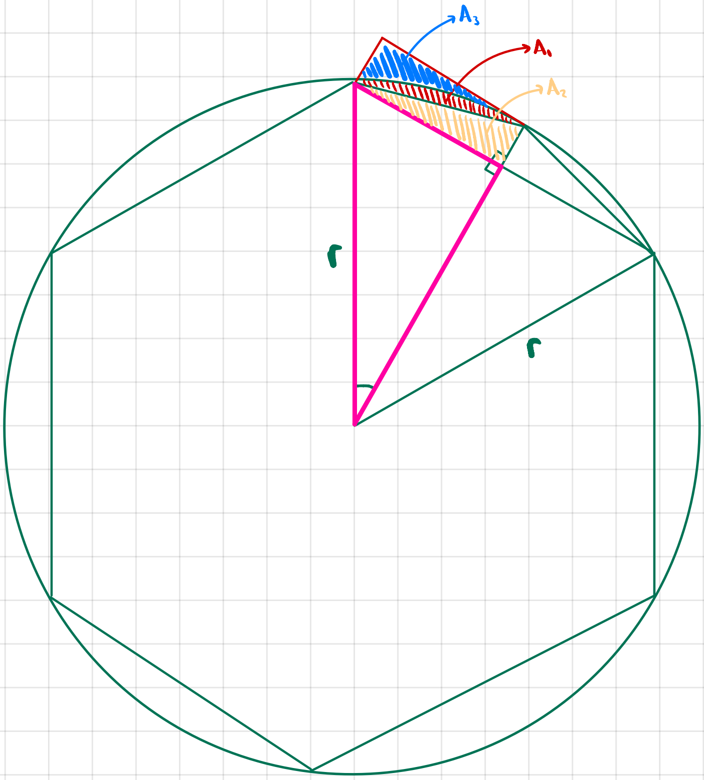
b)

P is regular polygon inscribed in circle

P' " " " " " , twice as many sides as P.

$A(\text{circle}) - A(P') < \dfrac{A(\text{circle}) - A(P)}{2}$

Proof




$A_c - A(P) = A_1 + A_2$

$A_c - A(P') = A_1$

From the picture we can see that $A_2$ is more than half of $(A_1 + A_2)$.

Note that $A_2 = \dfrac{A_3}{2}$ and $A_1 < \dfrac{A_3}{2} = A_2$

Thus

$2A_1 < A_1 + A_2 \rightarrow A_1 < \dfrac{A_1 + A_2}{2}$

T.F.

$A_c - A(P') = A_1 < \dfrac{A_1 + A_2}{2} = \dfrac{A_c - A(P)}{2}$

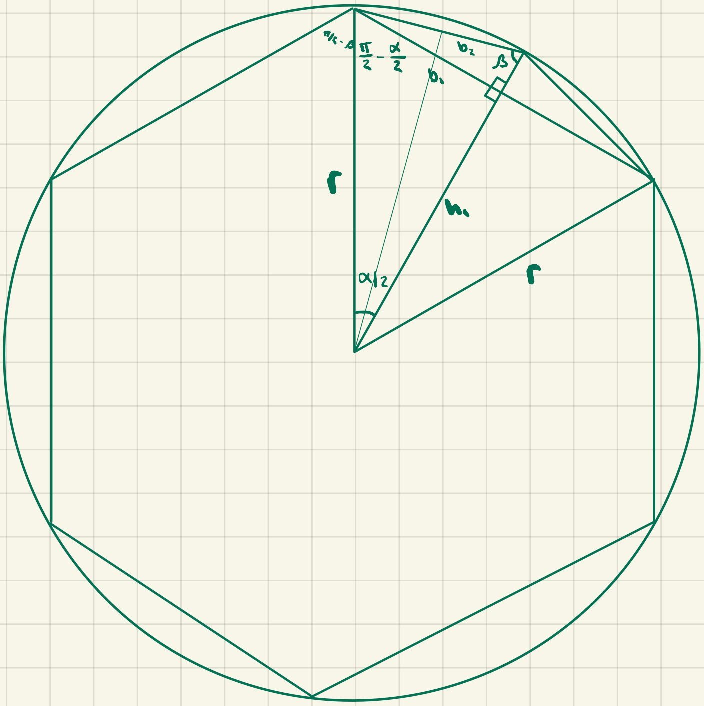
# First Attempt (probably wrong)

b)

P is regular polygon inscribed in circle

P' " " " " " ", twice as many sides as P.

$$A(\text{circle}) - A(P') < \frac{A(\text{circle}) - A(P)}{2}$$




Proof

$$A_1 = \frac{b_1 \cdot h_1}{2}$$

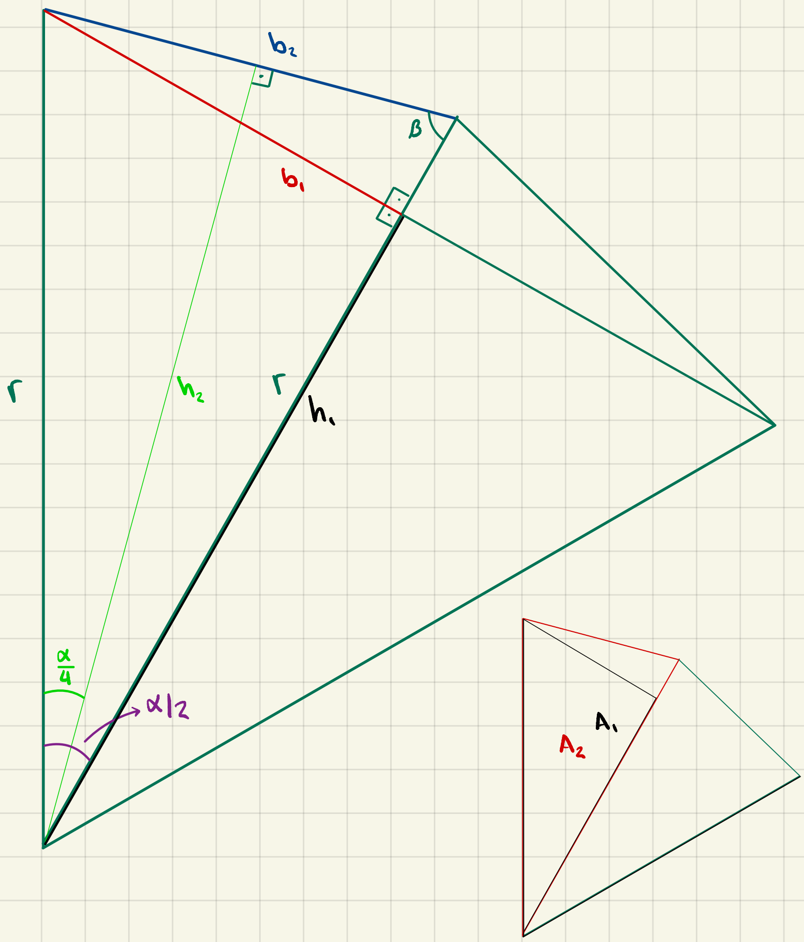
$$A_2 = \frac{b_2 \cdot h_2}{2}$$

$$\sin\left(\frac{\alpha}{2}\right) = \frac{b_1}{r} \rightarrow b_1 = r\sin\left(\frac{\alpha}{2}\right) \qquad (1)$$

$$\cos\left(\frac{\alpha}{2}\right) = \frac{h_1}{r} \rightarrow h_1 = r\cos\left(\frac{\alpha}{2}\right)$$

$$A_1 = \frac{r^2 \sin(\alpha/2)\cos(\alpha/2)}{2}$$

$$\sin\beta = \frac{b_1}{b_2} \rightarrow b_1 = b_2 \sin\beta \qquad (2)$$




From (1) and (2)

$$r\sin(\alpha/2) = b_2 \sin\beta$$

$$b_2 = \frac{r\sin(\alpha/2)}{\sin\beta} \qquad (3)$$

$$\sin\beta = \frac{h_2}{r} \rightarrow h_2 = r\sin\beta \qquad (4)$$

From (3) and (4)

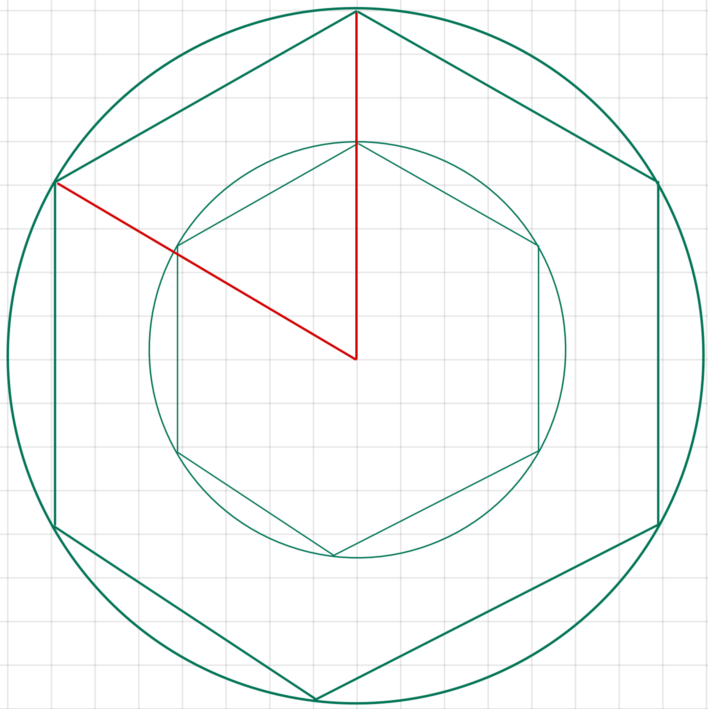
$$A_2 = \frac{r\sin(\alpha/2)}{\cancel{\sin\beta}} \cdot r\cancel{\sin\beta} \cdot \frac{1}{2}$$

$$A_2 = \frac{r^2 \sin(\alpha/2)}{2}$$

c)

Let $a_n = A(\text{circle}) - A(P_{2^{n-1}})$

where $P_{2^{n-1}}$ is a regular polygon w/ $2^{n-1}$ sides inscribed in the circle.



In b) we proved that $a_{n+1} < \frac{1}{2} a_n$.

Therefore by a), $\forall \epsilon > 0 \ \exists n \in \mathbb{N}, \ a_n < \epsilon$.

$\to A(\text{circle}) - A(\text{Polygon } 2^{n-1} \text{ sides}) < \epsilon$.

d) Given $\frac{A(P_{1,n})}{A(P_{2,n})} = \frac{\ell_1^2}{\ell_2^2}$

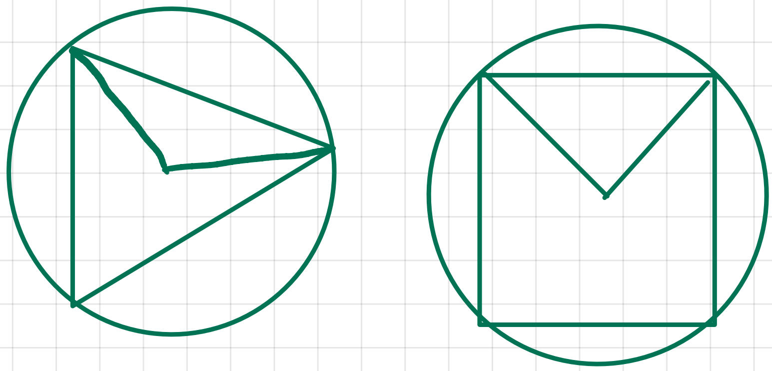
Prove $\frac{A(C_1)}{A(C_2)} = \frac{r_1^2}{r_2^2}$

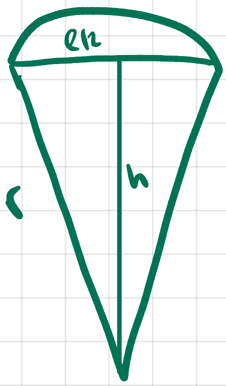
Assume $\frac{A(C_1)}{A(C_2)} \neq \frac{r_1^2}{r_2^2}$

Case 1: $\frac{A(C_1)}{A(C_2)} > \frac{r_1^2}{r_2^2}$

$A(C_1) - A(P_{1,n}) < \epsilon$

$A(C_2) - A(P_{2,n}) < \epsilon$







$$\frac{\ell_1^2}{4} + h_1^2 = r_1^2 \to h_1^2 = r_1^2 - \frac{\ell_1^2}{4}$$

$$A_1 = \ell_1 \sqrt{r_1^2 - \frac{\ell_1^2}{4}}$$

$$A_2 = \ell_2 \sqrt{r_2^2 - \frac{\ell_2^2}{4}}$$

$$\frac{A_1}{A_2} = \frac{\ell_1}{\ell_2} \frac{\sqrt{r_1^2 - \ell_1^2/4}}{\sqrt{r_2^2 - \ell_2/4}} \to \frac{r_1^2 - \ell_1^2/4}{r_2^2 - \ell_2^2/4} = \frac{A_1}{A_2} = \frac{4r_1^2 - \ell_1^2}{4r_2^2 - \ell_2^2} = \frac{\ell_1^2}{\ell_2^2}$$

$$\frac{\ell_1}{\ell_2} \downarrow \frac{\sqrt{A_1}}{\sqrt{A_2}}$$

$$= \cancel{4}r_1^2\ell_2^2 - \cancel{\ell_1^2\ell_2^2} = \cancel{-\ell_1^2\ell_2^2} + \cancel{4}r_2^2\ell_1^2$$

$$\frac{r_1^2}{r_2^2} = \frac{\ell_1^2}{\ell_2^2}$$

12.

$A, B$ nonempty sets of numbers

$\forall x \forall y,\ x \in A \wedge y \in B \rightarrow x \le y$

a) $\forall y,\ y \in B \rightarrow \sup A \le y$

Proof

Every $y \in B$ is an upper bound for $A$.

Therefore, $A$ has a $\sup A$.

Therefore $\forall y,\ y \in B \rightarrow \sup A \le y$

---

More rigorous proof

Assume $\exists y,\ y \in B \wedge y < \sup A$

Then $\forall x,\ x \in A \rightarrow x \le y < \sup A$

$y$ is an upper bound for $A$ and $y < \sup A$. $\perp$.

Therefore, $\forall y,\ y \notin B \vee y \ge \sup A$

Let $y$, $y \in B$.

Then $y \notin B \vee y \ge \sup A$

Case 1: $y \notin B$. $\perp$. Then $y \ge \sup A$.
Case 2: $y \ge \sup A$

Therefore $y \ge \sup A$.

Therefore, $\forall y,\ y \in B \rightarrow y \ge \sup A$

b) $\sup A \le \inf B$

Proof

From a), $\sup A$ is a lower bound for $B$.

Therefore $\exists \inf B$.

Therefore $\sup A \le \inf B$.

13.

A, B nonempty sets of numbers, bounded above $\rightarrow$ $\sup(A+B) = \sup A + \sup B$

$A+B = \{x+y : x \in A, y \in B\}$

**Proof**

Let $x \in A, y \in B$.
Then $x \le \sup A$, $y \le \sup B$.
$x+y \le \sup A + \sup B$
$\forall x,y \quad x \in A \wedge x \in B \rightarrow x+y \le \sup A + \sup B$

Therefore $\sup A + \sup B$ is an upper bound for $A+B$.

Therefore $\sup(A+B)$ exists and $\sup(A+B) \le \sup A + \sup B$

Let $\epsilon > 0$
Let $x, y$ s.t. $x \in A \wedge \sup A - \frac{\epsilon}{2} \le x \le \sup A$
$y \in B \wedge \sup B - \frac{\epsilon}{2} \le y \le \sup B$

Then $\sup A + \sup B - \epsilon \le x + y \le \sup(A+B)$

$\sup A + \sup B \le \sup(A+B) + \epsilon$

$\forall \epsilon > 0 \ \sup A + \sup B \le \sup(A+B) + \epsilon$

Therefore

$\forall \epsilon > 0 \ \sup(A+B) \le \sup A + \sup B \le \sup(A+B) + \epsilon$

Let $\epsilon > 0$.

Assume $\neg \exists x, x \in A \wedge \sup A - \epsilon \le x \le \sup A$

$\forall x, \ x \notin A \vee (x < \sup A - \epsilon \vee x > \sup A)$

Let $x \in A$.
Case 1: $x \notin A$. $\perp$.
Case 2: $x < \sup A - \epsilon \vee x > \sup A$
Case 2.1: $x < \sup A - \epsilon$

Case 2.2: $x > \sup A$. $\perp$

$x < \sup A - \epsilon$
$x < \sup A - \epsilon$
$\forall x, x \in A \rightarrow x < \sup A - \epsilon$
$\perp$
$\exists x, x \in A \wedge \sup A - \epsilon \le x \le \sup A$
$\forall \epsilon > 0 \ \exists x, x \in A \wedge \sup A - \epsilon \le x \le \sup A$

Similarly,
$\forall \epsilon > 0 \ \exists x, x \in B \wedge \sup B - \epsilon \le x \le \sup B$.

14.

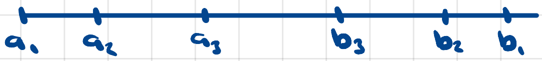
a) Sequence of closed intervals

$I_1 = [a_1, b_1],\ I_2 = [a_2, b_2], \dots$ $\longrightarrow$ $\exists x, \forall i, x \in I_i$

$a_n \le a_{n+1}$

$b_{n+1} \le b_n$

$a_n \le b_n$




Proof

Apply result from prob 12.

Let $A = \{a_i, i \in N\}$

$B = \{b_i, i \in N\}$

Then by 12b, $\sup A \le \inf B$

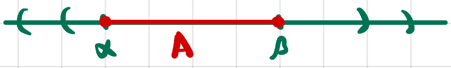
Any $x \in [\sup A, \inf B]$ is a member of all $I_i$.

b) $I_1 = (a_1, b_1),\ I_2 = (a_2, b_2), \dots$

Let $\sup A = \inf B = \alpha$

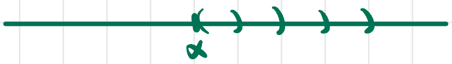

$a_n = \alpha - \frac{1}{n},\ b_n = \alpha + \frac{1}{n}$. This example works: $x = \alpha$ is in all of $I_n$.

Let $\sup A = \alpha,\ \inf B = \beta$




Any $x \in A$ is also in all of $I_n$.

If, however, $\alpha$ is an open endpoint.




$a_n = \alpha,\ b_n = \alpha + \frac{1}{n}$. This is a counterexample. For any $x \in I_n$, there is always a $y \in (\alpha, x)$ such that $y = \alpha + \frac{1}{n_1}$ because for any number $\frac{1}{n}$ there exist $n_1 \in N$ s.t. $\frac{1}{n_1} < \frac{1}{n}$.

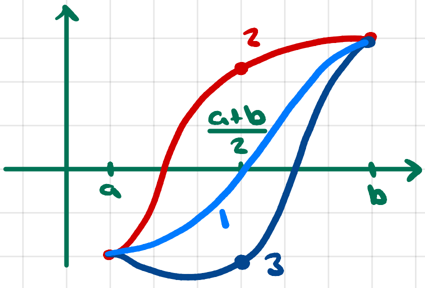
15.

$f$ cont. on $[a,b]$
$f(a) < 0 < f(b)$ $\rightarrow$ 1) $f\left(\frac{a+b}{2}\right) = 0$ OR

2) $f$ has different signs at endpoints of interval $\left[a, \frac{a+b}{2}\right]$ OR

3) $f$ has different signs at endpoints of interval $\left[\frac{a+b}{2}, b\right]$

**Proof**



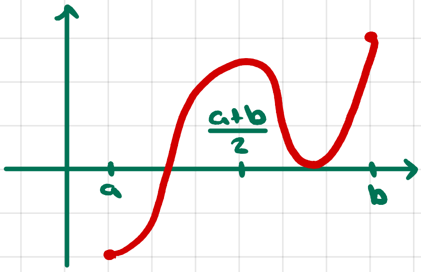

Since $f$ is defined at $\frac{a+b}{2}$, there are three possible cases

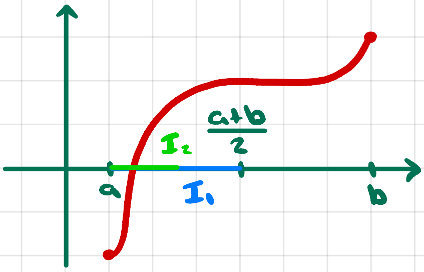
i) $f\left(\frac{a+b}{2}\right) = 0$. Therefore 1).

ii) $f\left(\frac{a+b}{2}\right) > 0$. Therefore 2).

iii) $f\left(\frac{a+b}{2}\right) < 0$. Therefore 3).

Therefore 1) $\vee$ 2) $\vee$ 3).
■



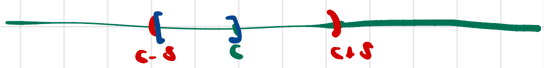

The intervals $I_n$ are nested. Therefore there is a point $c$ that is in all the intervals. Note that the point where $f$ changes sign is also in every $I_n$, by definition.

Note that each interval $I_n$ has length $\frac{b-a}{2^n}$.

Assume $f(c) < 0$.

Since $f$ is continuous at $c$, $\exists \delta > 0$ s.t. $\forall x$, $|x-c| < \delta \rightarrow f(x) < 0$.

Choose $n$ such that $I_n$ has length $\frac{b-a}{2^n} < \delta$.




Then, $\forall x \forall y$, $x \in I_n \wedge y \in I_n \rightarrow |x-y| < \frac{b-a}{2^n} < \delta$.
In particular, $|x-c| < \delta$ so
$\forall x$, $x \in I_n \rightarrow f(x) < 0$.

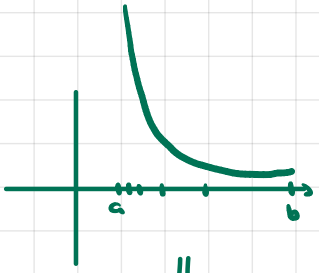
But then $f$ doesn't change sign in $I_n$. $\perp$.

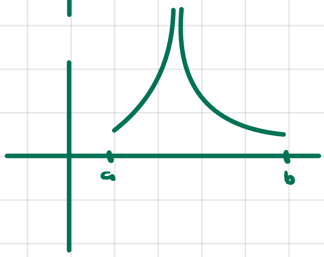
Therefore $f(c) \geq 0$.
Similarly we can show that assuming $f(c) > 0$ leads to $\perp$.

Therefore $f(c) = 0$. In conclusion, any point that is in all $I_n$ must be a root of $f$.

**16.**

$f$ cont. on $[a,b]$, unbounded on $[a,b]$.




The intervals $I_n$ are nested as described in 14a.

$\exists c$, $c \in [a,b]$ and $c$ in every $I_n$.

Since $f$ is cont at $c$,

$$\lim_{x \to c} f(x) = f(c)$$

$$\forall \varepsilon > 0 \ \exists \delta > 0, \forall x \ |x-c| < \delta \to |f(x) - f(c)| < \varepsilon$$

The length of each bisection interval is $\frac{b-a}{2^n}$.

choose $n$ such that $\frac{b-a}{2^n} < \delta$.

Then,

$$\forall x, \ |x-c| < \frac{b-a}{2^n} < \delta \to |f(x)-f(c)| < \varepsilon.$$
$$\to f(x) < f(c) + \varepsilon$$

So $f$ is bounded in $I_n$. $\perp$.

**17.**

a) $A = \{x : x < \alpha\}$

i) $x \in A \wedge y < x \rightarrow y \in A$

Proof

Since $x \in A$ then $x < \alpha$.
Since $y$ transitive, $y < \alpha$.
$\rightarrow y \in A$.

ii) $A \neq \emptyset$

Proof

$\alpha - 1 < \alpha$
$\rightarrow \alpha - 1 \in A$
$\rightarrow A \neq \emptyset$

iii) $A \neq \mathbb{R}$

Proof

$\alpha + 1 > \alpha$
$\rightarrow \alpha + 1 \notin A$
$A \neq \mathbb{R}$

iv) $x \in A \rightarrow \exists x', x' \in A \wedge x < x'$

Proof

For any $x \in A$, let $x' = \frac{x + \alpha}{2}$

Then $x < x' < \alpha$, $x' \in A$.

b) $x \in A \wedge y < x \rightarrow y \in A$
$A \neq \emptyset$
$A \neq \mathbb{R}$
$x \in A \rightarrow (\exists x' \in A \wedge x < x')$

$\rightarrow A = \{x : x < \sup A\}$

Proof

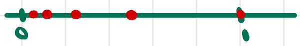
$A \neq \emptyset$ and $\alpha$ is an u.b.
$\rightarrow A$ has l.u.b. $\beta$
Assume $\beta < \alpha$
Let $b = \frac{\alpha + \beta}{2}$
Then $\beta < b < \alpha$
$\rightarrow b \in A$
$\beta$ not u.b of A. $\perp$
Therefore $\alpha$ is the l.u.b. of A.

**18.**

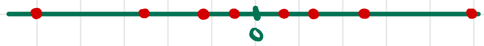
a)

i) $A = \{\frac{1}{n} : n \in \mathbb{N}\}$

Every number $\alpha \geq 0$ is an almost upper bound.
" " $\alpha \leq 0$ " " " lower bound

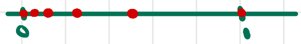



ii) $A = \{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\}$



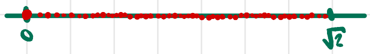

Any number $\alpha > 0$ is a.u.b.
" " $\alpha < 0$ is a.l.b.

iii) $A = \{x : x = 0 \text{ or } x = 1/n \text{ for } n \in \mathbb{N}\}$




Same as i).

iv) $A = \{x : 0 \leq x \leq \sqrt{2} \wedge x \in \mathbb{Q}\}$




Any number $\alpha \leq 0$ is a.l.b.
" " $\alpha \geq \sqrt{2}$ is a.u.b.

v) $A = \{x : x^2 + x + 1 \geq 0\}$

$\Delta = 1 - 4 = -3 \quad \rightarrow \forall x \; x^2 + x + 1 > 0$

$A = \mathbb{R}$, unbounded.

no a.l.b. nor a.u.b.

vi) $A = \{x : x^2 + x - 1 < 0\}$

$\forall x, x \in (\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}) \rightarrow x^2 + x - 1 < 0$


Any $\alpha \leq \frac{-1-\sqrt{5}}{2}$ is a.l.b.

Any $\alpha \geq \frac{-1+\sqrt{5}}{2}$ is a.u.b.

vii) $A = \{x : x<0 \text{ and } x^2+x-1<0\}$

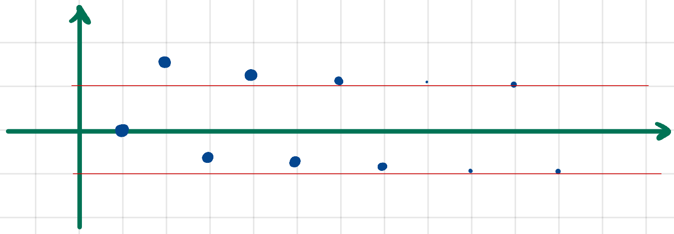
$= \{x : x \in (\frac{-1-\sqrt{5}}{2}, 0]$


All $\alpha \geq 0$ is a.u.b.
All $\alpha \leq \frac{-1-\sqrt{5}}{2}$ is a.l.b.

viii) $A = \{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\}$

$= \{0, \frac{1}{2}+1, \frac{1}{3}-1, \frac{1}{4}+1, \dots\}$

$= \{0, 3/2, -2/3, 5/4, -4/5, 7/6, -6/7, \dots\}$



Consider any element $a_n = \frac{1}{n} + (-1)^n, n \in \mathbb{N}$.

Any $\alpha > 1$ is a.u.b.
Any $\alpha \leq -1$ is a.l.b.

b) A bounded infinite set.

$\to B \neq \emptyset \wedge B$ bounded below

$B = \{x : x \text{ is a.u.b. of } A\}$

Proof

$A \neq \emptyset$, A bounded, therefore A has l.u.b. $\alpha$

$\exists \alpha, \forall x, x \in A \to x \leq \alpha$

$\alpha$ is also an a.u.b, and so is any number $y > \alpha$.

There might be values $y < \alpha$ that are a.u.b.

For example,

i) $A = \{1/n, n \in \mathbb{N}\}$
$\alpha = 1$
$B = \{x : x > 0\}$

$B \neq \emptyset$ because $\alpha \in B$.

Since A is bounded below, $\exists \beta, \forall x, x \in A \to x \geq \beta$.

Let $x \in B$.

Assume $x < \beta$.

Then, because x is a.u.b of A, there are finite number of elements of A larger than x. But all elements of A are larger than $\beta$, thus larger than x, and A has infinite elements $\perp$.

Therefore $x \geq \beta$

$\forall x, x \in B \to x \geq \beta$

Thus, B is bounded below.

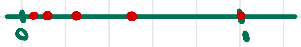
Therefore $B \neq \emptyset$ and B is bounded below

■

c) $B \neq \emptyset \wedge B$ bounded below $\to \exists \inf B$

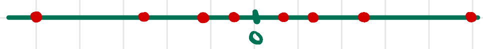
inf B is the g.l.b. of the a.u.b. of A.
This is called the limit superior of A, denoted $\overline{\lim}\, A$ or lim sup A.

i) $A = \{\frac{1}{n} : n \in \mathbb{N}\}$



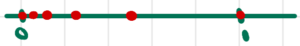

lim sup A = 0

ii) $A = \{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\}$



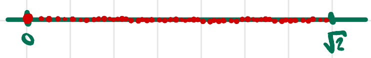

lim sup A = 0

iii) $A = \{x : x = 0 \text{ or } x = 1/n \text{ for } n \in \mathbb{N}\}$




lim sup A = 0

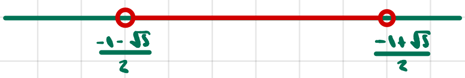
iv) $A = \{x : 0 \leq x \leq \sqrt{2} \wedge x \in \mathbb{Q}\}$




lim sup A = $\sqrt{2}$

v) $A = \{x : x^2+x+1 \geq 0\} = \mathbb{R}$, has no limit superior.

vi) $A = \{x : x^2+x-1<0\}$

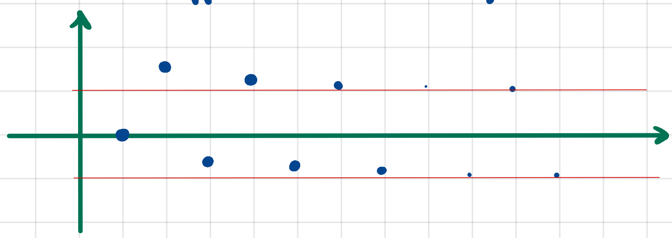



lim sup A = $\frac{-1+\sqrt{5}}{2}$

vii) $A = \{x : x < 0 \text{ and } x^2 + x - 1 < 0\}$


$\lim\sup A = 0$

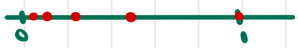
viii) $A = \{ \frac{1}{n} + (-1)^n : n \in \mathbb{N} \}$



$\lim\sup A = 1$

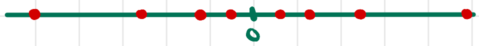
d) $\underline{\lim}$ A is the limit inferior of A and is sup C, where $C = \{x : x \text{ is a l.b of } A\}$.

i) $A = \{\frac{1}{n} : n \in \mathbb{N}\}$



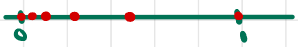

$\lim\inf A = 0$

ii) $A = \{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\}$



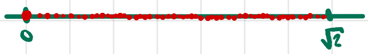

$\lim\inf A = 0$

iii) $A = \{x : x = 0 \text{ or } x = \frac{1}{n} \text{ for } n \in \mathbb{N}\}$




$\lim\inf A = 0$

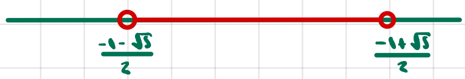
iv) $A = \{x : 0 \leq x \leq \sqrt{2} \wedge x \in \mathbb{Q}\}$




$\lim\inf A = 0$

v) $A = \{x : x^2 + x + 1 \geq 0\} = \mathbb{R}$, has no limit inferior

vi) $A = \{x : x^2 + x - 1 < 0\}$

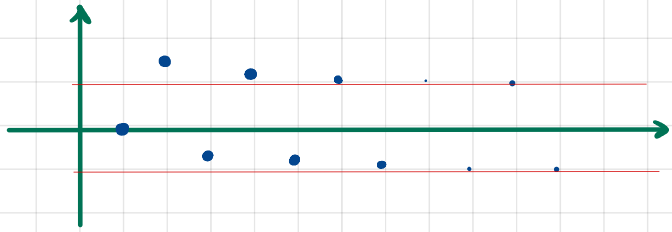



$\lim\inf A = \frac{-1-\sqrt{5}}{2}$

vii) $A = \{x : x < 0 \text{ and } x^2 + x - 1 < 0\}$


$\lim\inf A = \frac{-1-\sqrt{5}}{2}$

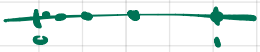
viii) $A = \{ \frac{1}{n} + (-1)^n : n \in \mathbb{N} \}$



$\lim\inf A = -1$

19. A bounded infinite set.

a) $\underline{\lim} A \le \overline{\lim} A$



Proof

Assume $\underline{\lim} A > \overline{\lim} A$

From the definitions of limit inferior and limit superior,

There is a finite number of elements of A that are smaller than $\underline{\lim} A$, and a finite number larger than $\overline{\lim} A$.

Let $\alpha$ s.t. $\overline{\lim} A < \alpha < \underline{\lim} A$

Then there is finite number of elements of A smaller than $\alpha$, but also larger than $\alpha$. Hence A is finite. $\perp$.

Hence, $\underline{\lim} A \le \overline{\lim} A$

b) $\overline{\lim} A \le \sup A$



Proof

Intuitively, all elements of A are $\le \sup A$, but a finite number are larger than $\overline{\lim} A$.

Assume $\overline{\lim} A > \sup A$

There must be infinite points $a \in A$ s.t. $\sup A < a < \overline{\lim} A$, otherwise $\sup A$ would be a.u.b. and thus $\overline{\lim} A$ couldn't actually be the limit superior of A.

But then $\sup A$ isn't the supremum of A. $\perp$.

Hence $\overline{\lim} A \le \sup A$

c) $\overline{\lim} A < \sup A \to$ A contains a largest element

Intuitively, there is finite number of elements of A between $\overline{\lim} A$ and $\sup A$. One of them is largest among this finite set.

But then it is also the largest in A since all other members of A are smaller than $\overline{\lim} A$.

Proof

Let $B = \{x : x \in A \wedge x > \overline{\lim} A\}$. Then $\forall x, x \in B \to x < \sup A$, i.e. $\sup A = \sup B$.

$B \ne \emptyset$ because otherwise $\overline{\lim} A$ would be $\sup A$, a $\perp$.

Thus because $B \ne \emptyset$ and B is finite, one of its elements is largest. Call it $\alpha$. $\forall x, x \in B \to x \le \alpha$.

But

$\forall x, x \in B \to x \in A \wedge x \ge \overline{\lim} A$

$\forall x, x \in A \to x \in B \vee x \le \overline{\lim} A$

Therefore

$\forall x, x \in A \to \lim \bar{A} < x \le \alpha$

Hence $\forall x, x \in A \to x \le \alpha$.

d)

i) $\underline{\lim} A \ge \inf A$

Intuitively, all elements of A are larger than or equal to $\inf A$, but a finite number of them are smaller than $\underline{\lim} A$.

Proof

Assume $\underline{\lim} A < \inf A$.

There must be infinite points a s.t. $a \in A$ and $\underline{\lim} A < a < \inf A$, otherwise $\inf A$ would be a.l.b. and limit inferior.

But then $\inf A$ isn't a lower bound. $\perp$.

Hence $\underline{\lim} A \ge \inf A$.

ii) $\underline{\lim} A > \inf A \to$ A contains smallest element

Intuitively, there is a finite number of elements of A between inf A and $\underline{\lim}$ A. There is a smallest among them. Since all such elements are $< \underline{\lim} A$, and all the rest of the (infinite) elements are $> \underline{\lim} A$, the smallest in the finite set is the smallest of entire A.

Proof

Let $B = \{x : x \in A \wedge x < \underline{\lim} A\}$

Assume $B = \emptyset$.
Then $\forall x, x \in A \to x \geq \underline{\lim} A$
Then $\underline{\lim} A$ is l.b. and $\underline{\lim} A > \inf A$. $\perp$

Thus $B \neq \emptyset$.

Since B is finite, it contains a smallest element $\alpha$
$\forall x, x \in B \to x \geq \alpha$.

But it is also true that
$\forall x, x \in B \to x \in A \wedge x \geq \underline{\lim} A$.

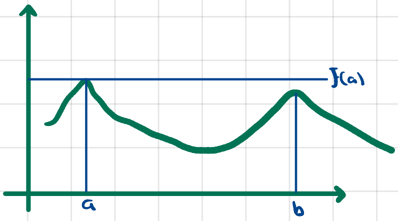
$\forall x, x \in A \to \inf A \leq x \leq \underline{\lim} A \vee x \geq \underline{\lim} A$
$\to x \in B \vee x \geq \underline{\lim} A$

Hence
$\forall x, x \in A \to \alpha \leq x \leq \underline{\lim} A$

$\forall x, x \in A \to x \geq \alpha$.

20. $f$ cont. on $\mathbb{R}$

point $x$ is shadow point of $f$ if there is $y > x$ w/ $f(y) > f(x)$




a)

Assume $f(a) > f(b)$

Consider the closed interval $I = [a,b]$.
$I$ is bounded and nonempty.
Since $f$ cont. on $I$, it takes on a max value in $I$, say at $\alpha$.

Assume $\alpha \neq a$.

Then $\alpha > a$ and $f(\alpha) \geq f(a)$.

Assume $f(\alpha) > f(a)$
Then $a$ is shadow point.
$\perp$.

Therefore $f(\alpha) = f(a) > f(b)$

Since $b$ is not a shadow point, $\forall y,\ y > b \to f(y) \leq f(b)$.
Therefore $\forall y,\ y > \alpha \to f(y) \leq f(\alpha)$
But then $\alpha$ is not a shadow point.
$\perp$.

Therefore, $\alpha = a$.

b) Since $f$ cont. on $[a,b]$ it takes on all values in $[f(a), f(b)]$.

Therefore, $\exists x,\ x \in [a,b] \wedge f(x) \in (f(a), f(b))$

$\to f(x) > f(b)$

Some such point is not a shadow point.
But by assumption all points in $(a,b)$ are shadow points.
$\perp$.
Therefore $f(a) \leq f(b)$, by contradiction.

But $f(a) = f(b)$ because o.w. $a$ is a shadow point.