Ch6 - Continuous Functions

f arbitrary fn

not necess. true that $\lim_{x\to a} f(x) = f(a)$

- $\nexists \lim_{x\to a} f(x)$
- f not defined at a
- $\lim_{x\to a} f(x) \neq f(a)$

we consider such cases as abnormal.

Functions without such peculiarities are "continuous."

Definition f is continuous at a if $\lim_{x\to a} f(x) = f(a)$

Theorem 1 f, g continuous at a then

(1) $f+g$ continuous at a
(2) $f\cdot g$ continuous at a

$g(a) \neq 0$ then (3) $\frac{1}{g}$ continuous at a

Proof

f, g cont. at a $\leftrightarrow$ $\lim_{x\to a} f(x) = f(a)$
$\lim_{x\to a} g(x) = g(a)$

$\lim_{x\to a} (f+g)(x) = f(a) + g(a) = (f+g)(a)$

$\lim_{x\to a} (fg)(x) = f(a)g(a) = (fg)(a)$

$g(a) \neq 0 \rightarrow \lim_{x\to a} \frac{1}{g(x)} = \frac{1}{g(a)}$

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we can derive useful results w/ theorem 1:

$f(x) = c$ and $f(x) = x$ are continuous at a, for every a

Therefore,

$$f(x) = \frac{b_n x^n + b_{n-1}x^{n-1} + \dots + b_0}{c_m x^m + c_{m-1}x^{m-1} + \dots + c_0}$$

is continuous at every point in its domain.

Note: For a fn f continuous at a,

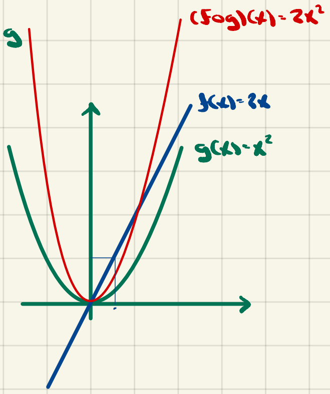
$\lim_{x\to a} f(x) = f(a)$, which means

$\forall \varepsilon > 0 \ \exists \delta \ \forall x \ 0 < |x-a| < \delta \rightarrow |f(x) - f(a)| < \varepsilon$

However, we can allow x to be a now:

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \ |x-a| < \delta \rightarrow |f(x) - f(a)| < \varepsilon$

Theorem 2: g cont. at a, f cont. at $g(a)$, then $f\circ g$ cont. at a.



Proof:

f cont. at $g(a)$ then $\forall \varepsilon > 0 \ \exists \delta' > 0$

$\forall y \ |y - g(a)| < \delta' \rightarrow |f(x) - f(g(a))| < \varepsilon$

if $y = g(x)$ then $\forall \varepsilon > 0 \ \exists \delta' > 0$

$\forall y \ |g(x) - g(a)| < \delta' \rightarrow |f(g(x)) - f(g(a))| < \varepsilon$

from cont. of g, $\forall \varepsilon > 0 \ \exists \delta > 0$

$\forall x \ |x-a| < \delta \rightarrow |g(x) - g(a)| < \delta'$

Therefore,

$\forall x \ |x-a| < \delta \rightarrow |f(g(x)) - f(g(a))| < \varepsilon$

$\rightarrow \lim_{x\to a} (f\circ g)(x) = (f\circ g)(a)$

$\rightarrow f\circ g$ continuous at a

Specific Cases

f continuous on (a,b) : f cont for $\forall x \in (a,b)$

" " " $\mathbb{R} = (-\infty, \infty)$: f cont $\forall x \in (-\infty, \infty)$

f continuous on $[a,b]$ if

(1) f cont. on (a,b)

(2) $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$

Functions continuous on an interval are usually regarded as especially well-behaved

- continuity might be specified as the first condition which a "reasonable" function ought to satisfy.

Theorem 3

f cont. at a $\longrightarrow$ $f(x) > 0$ for all x in some interval containing a

$f(a) > 0$ ie $\exists \delta > 0\ \forall x\ |x-a| < \delta \to f(x) > 0$

Also

$\lim_{x \to a} f(x) = f(a)$ $\longrightarrow$ $\exists \delta > 0\ \forall x\ |x-a| < \delta \to f(x) < 0$

$f(a) < 0$

Proof

Let $\lim_{x \to a} f(x) = f(a)$ and $f(a) > 0$.

Then $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x\ |x-a| < \delta \to |f(x) - f(a)| < \varepsilon \ \to\ -\varepsilon < f(x) - f(a) < \varepsilon$

Choose $\varepsilon = \dfrac{f(a)}{2}$. Then, $-\dfrac{f(a)}{2} < f(x) - f(a) < \dfrac{f(a)}{2}$

$\longrightarrow \quad 0 < \dfrac{f(a)}{2} < f(x) < \dfrac{3f(a)}{2}$