Ch.19 - Integration in Elementary Terms

Every time compute a derivative (e.g.

$F(x) = x \cdot \log x - x$
$F'(x) = \log x + 1 - 1 = \log x$

the FTC2 gives us a statement about integrals

$\int_a^b \log(x)\,dx = F(x)\Big|_a^b, \quad a,b>0$

In general, if $F' = f$, then F is called a **primitive** of f.

Every cont. f has a primitive $\int_a^x f$, but we want to express the primitive in terms of

addition, multiplication, division, and composition from

the rational functions

the trigonometric fns and their inverses

and log and exp

If a fn can be expressed in this way then it is called an **elementary fn**

Usually such fns cannot be found.

e.g. there is no elementary F s.t. $F'(x) = e^{-x^2}$ for all x

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Integration is the process of finding elementary primitives of given elementary fns

The symbol $\int f$ or $\int f(x)\,dx$ means "**primitive of f**"; more precisely "**the collection of primitives of f**"

Example

$\int x^3\,dx = \frac{x^4}{4} + C$

means $F(x) = \frac{x^4}{4} + C$ and $F'(x) = x^3$

A fn in $\int f$ is called an **indefinite integral** of f, while $\int_a^b f$ is called a **definite integral** of f

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A table of indefinite integrals has entries such as

$\int a\,dx = ax$
$\int \cos x\,dx = \sin x$
(...)

Also

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

which means

"a primitive of $f+g$ can be found by adding a primitive of f to a primitive of g"

Thus we start w/ primitives of certain basic fns, and build primitives of other fns based on the basic prim.

Note

$$\int_a^b [f(x)+g(x)] = F(x)\Big|_a^b \quad \text{where } F'(x) = f(x)+g(x)$$

But (1) tells us that one such F is $F(x) = \int f(x)\,dx + \int g(x)\,dx$

Thus $\int_a^b [f(x)+g(x)] = \left[\int f(x)\,dx + \int g(x)\,dx\right]\Big|_a^b$

What is $[\int f]\Big|_a^b$? It is the primitive represented by $\int f$ evaluated at b minus $\int f$ evaluated at a. But this is $\int_a^b f$.

Thus $\int_a^b [f(x)+g(x)] = \left[\int f(x)\,dx + \int g(x)\,dx\right]\Big|_a^b = \int_a^b f + \int_a^b g$

also

$$\int c f(x)\,dx = c \cdot \int f(x)\,dx$$

Theorem 1 (Integration by Parts)

f', g' cont. $\rightarrow$ $\int f g' = fg - \int f' g$

$$\int_a^b f(x) g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x) g(x)\,dx$$

Proof

$$(fg)' = f'g + fg'$$

$$fg' = (fg)' - f'g$$

$\int f g' = fg - \int f'g$ $\rightarrow$ this means that $[fg - \int f'g]' = fg'$

also

$$\int_a^b f g' = \left[fg - \int f'g\right]\Big|_a^b = fg\Big|_a^b - \int_a^b f'g$$

Theorem 2 (The Substitution Formula)

f, g' cont., then

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) \cdot g'$$

$$\int_{g(a)}^{g(b)} f(u)\,du = \int_a^b f(g(x))\,g'(x)\,dx$$

Proof

Let F be primitive of f.

Then $\int_{g(a)}^{g(b)} f = F(g(b)) - F(g(a))$

$(F \circ g)' = (F' \circ g) \cdot g' = (f \circ g) \cdot g'$

$\rightarrow$ $F \circ g$ is primitive of $(f \circ g) \cdot g'$

Hence,

$$\int_a^b (f \circ g) \cdot g' = (F \circ g)\Big|_a^b = (F \circ g)(b) - (F \circ g)(a) = F(g(b)) - F(g(a))$$

T.F.

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) \cdot g'$$

$\blacksquare$

Examples

$$\int_a^b \sin^5(x) \cdot \cos(x)\,dx$$

Let $f(x) = x^5$, $g(x) = \sin(x)$. Then $\sin^5(x)\cos(x) = f(\sin x) \cdot (\sin x)' = f(g(x)) \cdot g'(x)$

Thus,

$$\int_a^b \sin^5(x) \cdot \cos(x)\,dx = \int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du = \int_{\sin a}^{\sin b} u^5\,du = \frac{u^6}{6}\Big|_{\sin a}^{\sin b} = \frac{\sin^6 b - \sin^6 a}{6}$$

$$\int_a^b \tan x\,dx = -\int_a^b \frac{-\sin x}{\cos x}\,dx$$

Let $f(x) = \frac{1}{x}$, $g(x) = \cos x$. Then $\frac{1}{\cos x}(-\sin x) = f(g(x)) \cdot g'(x)$

$$\int_a^b \tan x\,dx = -\int_a^b \frac{-\sin x}{\cos x}\,dx = -\int_a^b f(g(x))g'(x)\,dx = -\int_{\cos a}^{\cos b} u^{-1}\,du = -\log(u)\Big|_{\cos a}^{\cos b} = \log(\cos a) - \log(\cos b)$$

$$\int_a^b \frac{1}{x\log x}\,dx = \int_a^b \frac{1}{\log x}\cdot\frac{1}{x}\,dx = \int_a^b f(g(x))g'(x)\,dx = \int_{\log a}^{\log b} u^{-1}\,du = \log(\log b) - \log(\log a)$$

$f(x) = \frac{1}{x}$

$g(x) = \log(x)$

Shortcuts

$$\int_a^b f(\underbrace{g(x)}_{u})\underbrace{g'(x)\,dx}_{du} = \int_{g(a)}^{g(b)} f(u)\,du$$

$$\int_a^b \sin^5 x \cos x\,dx = \int_{\sin a}^{\sin b} u^5\,du$$

once we identify $g(x)$ we set

$u = g(x)$

$du = g'(x)\,dx$

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let's recap this procedure

$$\int_a^b \sin^5 x \cos x\,dx$$

$u = g(x) = \sin x$

$du = g'(x)\,dx = \cos x\,dx$

and we obtain a definite integral $\int_{\sin a}^{\sin b} u^5\,du = \frac{u^6}{6}\Big|_{\sin a}^{\sin b}$

what if we want the indef. int. $\int \sin^5 x \cos x\,dx$?

we can do the process up to $\frac{u^6}{6}$, and then just sub $g(x)$ back in: $\frac{\sin^6 x}{6}$

$\int \sin^5 x \cos x\,dx = \int u^5\,du = \frac{u^6}{6} = \frac{\sin^6 x}{x}$

$$\int \frac{1}{x\log x}\,dx = \int u^{-1}\,du = \log u = \log(\log x)$$

$u = g(x) = \log x$

$du = g'(x)\,dx = x^{-1}\,dx$

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{2x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\log(1+x^2)$$

$f(x) = \frac{1}{x}$

$g(x) = 1+x^2$

$g'(x) = 2x$

$\int \arctan x \, dx = \int \underset{g'}{1} \cdot \underset{f}{\arctan x} \, dx$

$= x\arctan x - \int \frac{x}{1+x^2}$

$= x\arctan x - \frac{1}{2}\log(1+x^2)$

$* \int fg' = fg - \int f'g$

$\int \sec^2 x \tan^5 x \, dx = \int u^5 du = \frac{\tan^6 x}{6}$

$f(x) = x^5$

$g(x) = \tan x$

$g'(x) = \sec^2 x$

$\int \cos x \, e^{\sin x} dx = \int e^u du = e^{\sin x}$

$f(x) = e^x$

$g(x) = \sin x$

$g'(x) = \cos x$

$\int \frac{e^x}{\sqrt{1-e^{2x}}} dx$

$\arcsin x$ is $\sin^{-1} x$

$(\sin^{-1})'(x) = \frac{1}{\sin'(\sin^{-1}(x))} = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\sqrt{1-x^2}}$

$\sin^2(\sin^{-1} x) + \cos^2(\sin^{-1}(x)) = 1$

$\cos^2(\sin^{-1} x) = 1 - x^2$

$\cos(\sin^{-1} x) = \sqrt{1-x^2}$ because $x \in (-\pi/2, \pi/2)$

$\int \frac{e^x}{\sqrt{1-e^{2x}}} dx$

T.F.

$\int \frac{e^x}{\sqrt{1-e^{2x}}} dx = \int \frac{1}{\sqrt{1-u^2}} du = \int \arcsin' u \, du = \arcsin(e^x)$

$f(x) = \arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$

$g(x) = e^x$

$g'(x) = e^x$

$$\int \frac{1+e^x}{1-e^x}dx = \int \frac{1+e^x}{1-e^x}\cdot\frac{1}{e^x}\cdot e^x dx = \int \frac{1+u}{1-u}\cdot\frac{1}{u}du = \int \frac{2u+(1-u)}{(1-u)\cdot u}du = \int\left[\frac{2}{1-u}+\frac{1}{u}\right]du = -2\log(1-u)+\log(u)$$

$u = e^x$

$du = e^x dx$

$$= -2\log(1-e^x)+x$$

Another shortcut

$u = e^x$

$x = \log u$

$dx = \frac{1}{u}du$

$$\int \frac{1+e^x}{1-e^x}dx = \int \frac{1+u}{1-u}\,\frac{1}{u}du$$

Why does this work?

Up to now we've been using

$u = g(x)$

If g is one-one on the integration interval in x, then we have

$x = g^{-1}(u)$

and

$$\frac{dx}{du} = (g^{-1})'(u) = \frac{1}{g'(g^{-1}(u))}$$

If we have an integral $\int f(g(x))\,dx$ (note the missing $g'(x)$), then if we write

$dx = (g^{-1})'(u)\,du$

we get

$$\int f(u)\,(g^{-1})'(u)\,du$$

Alternatively, we could use another substit., with du in terms of dx now

$u = g(x)$

$du = g'(x)dx$

$$\int f(g(x))dx = \int f(g(x))\frac{1}{g'(x)}\,g'(x)dx$$

$$= \int f(u)\cdot\frac{1}{g'(g^{-1}(u))}du = \int f(u)\,(g^{-1})'(u)\,du$$

Just a note on substit. rule

$$\int_a^b f(g(x))g'(x)\,dx$$

$h(x) = f(g(x))g'(x)$

we integrate h from a to b.

The sub. rule says this is the same as integrating f from g(a) to g(b).

The shortcuts are simple ways to implement this

$$\int_a^b f(\underbrace{g(x)}_{u})\underbrace{g'(x)\,dx}_{du} = \int_{g(a)}^{g(b)} f(u)\,du$$

Example

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx = \int \frac{(u^2-1)^{\cancel{2}}}{\cancel{u}} \cdot \frac{2\cancel{u}}{\cancel{u^2-1}}\,du = 2\int (u^2-1)\,du = \frac{2u^3}{3} - 2u = \frac{2}{3}(e^x+1)^{3/2} - 2(e^x+1)^{1/2}$$

choose the substit.

$u = \sqrt{e^x+1}$ u in terms of x, $u = g(x)$

$u^2 = e^x + 1$

$u^2 - 1 = e^x$

$x = \log(u^2-1)$ x in terms of u, $x = g^{-1}(u)$

$dx = \frac{2u}{u^2-1}\,du$ $(g^{-1})'(u)\,du$

$$\int\sqrt{1-x^2}\,dx = \int\sqrt{1-\sin^2 u}\,\cos(u)\,du = \int \cos^2 u\,du = \int \frac{1+\cos(2u)}{2}\,du = \frac{u}{2} + \frac{\sin(2u)}{4}$$

$x = \sin u$ $(u = \arcsin x)$

$dx = \cos(u)\,du$

$$= \frac{\arcsin(x)}{2} + \frac{\sin(2\arcsin x)}{4}$$

$$= \frac{\arcsin(x)}{2} + \frac{2\sin(\arcsin x)\cdot\cos(\arcsin x)}{4}$$

$$= \frac{\arcsin(x)}{2} + \frac{x\cos(\arcsin x)}{2}$$

$$= \frac{\arcsin(x)}{2} + \frac{x\sqrt{1-x^2}}{2}$$

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$$

arcsin x is $\sin^{-1} x$

$$f(x) = \arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$$

$g(x) = e^x = u$
$g'(x) = e^x$, $du = g'(x)dx = e^x dx$

$$\int \frac{\overbrace{e^x\,dx}^{du}}{\sqrt{1-\underbrace{e^{2x}}_{u^2}}} = \int f(u)\,du = \int \arcsin'(u)\,du = \arcsin(u) = \arcsin(e^x)$$

Can we use a different substit.?

$$u = e^x \to x = \log u \to dx = \frac{1}{u}\,du$$

sub in u

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{u}{\sqrt{1-u^2}} \cdot \frac{1}{u}\,du = \int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin(u) = \arcsin(e^x)$$

sub in $\frac{1}{x}du$, based on $x = g^{-1}(u)$

Trigonometric Integrals

$$\sin^2(x) = \frac{1-\cos 2x}{2}$$

$$\cos^2(x) = \frac{1+\cos 2x}{2}$$

used to solve

$$\int \sin^n x\,dx$$

$$\int \cos^n x\,dx$$

when n even.

when n odd then

$$\int \sin^{n-1} x \cdot \sin x\,dx$$

same strategy for

$$\int \sin^n x \sin^m x\,dx$$

Finally

$$\int \frac{1}{\cos x}\,dx = \log(\sec x + \tan x)$$

Reduction Formulas

Consider Rational Fns

$p(x) = a_n x^n + \cdots + a_0$
$q(x) = b_m x^m + \cdots + b_0$

Assume $a_n = b_m = 1$.

Assume $n < m$.

* if $n \geq m$ then we can use division to obtain a polyn. and a rational fn with the equivalent of $n < m$ met.

We want to integrate p/q.

To do this we need two results.

Theorem Every polynomial fn

$q(x) = x^m + b_{m-1} x^{m-1} + \cdots + b_0$

can be written as the product

$q(x) = (x-\alpha_1)^{r_1} \cdots (x-\alpha_k)^{r_k} (x^2+\beta_1 x+\gamma_1)^{s_1} \cdots (x^2+\beta_\ell x+\gamma_\ell)^{s_\ell}$

where $r_1 + \cdots + r_k + 2(s_1 + \cdots + s_\ell) = m$

assumptions

$x - \alpha_i$ and $x^2 + \beta_i + \gamma_i$ can all be assumed distinct

each quadratic factor cannot be factored further
* this means

$\beta_i^2 - 4\gamma_i < 0$

Theorem If $n < m$ and

$$p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$$
$$q(x) = x^m + b_{m-1}x^{m-1} + \dots + b_0$$
$$= (x-\alpha_1)^{r_1} \cdots (x-\alpha_k)^{r_k}(x^2+\beta_1 x+\gamma_1)^{s_1} \cdots (x^2+\beta_l x+\gamma_l)^{s_l}$$

then $p(x)/q(x)$ can be written in the form

$$\frac{p(x)}{q(x)} = \left[\frac{\alpha_{1,1}}{x-\alpha_1} + \dots + \frac{\alpha_{1,r_1}}{(x-\alpha_1)^{r_1}}\right]$$
$$+ \left[\frac{\alpha_{k,1}}{x-\alpha_k} + \dots + \frac{\alpha_{k,r_k}}{(x-\alpha_k)^{r_k}}\right]$$
$$+ \left[\frac{b_{1,1}x + c_{1,1}}{x^2+\beta_1 x+\gamma_1} + \dots + \frac{b_{1,s_1}x + c_{1,s_1}}{(x^2+\beta_1 x+\gamma_1)^{s_1}}\right]$$
$$+ \left[\frac{b_{l,1}x + c_{l,1}}{x^2+\beta_l x+\gamma_l} + \dots + \frac{b_{l,s_l}x + c_{l,s_l}}{(x^2+\beta_l x+\gamma_l)^{s_l}}\right]$$

The important thing to remember is that if a problem has been reduced to the integration of a rational function, it is then certain that an elementary primitive exists, even if it is difficult or impossible to find the factors of $q(x)$ required to even begin the process of integration by partial fractions.

Note

$$\int \frac{1+e^x}{1-e^x}\,dx$$

Guess $g(x) = e^x$. Then $g'(x) = e^x$.

Put this factor in

$$\int \frac{1+e^x}{1-e^x} \cdot \frac{1}{e^x} \cdot e^x \,dx$$

$\frac{1+e^x}{1-e^x} \cdot \frac{1}{e^x}$ is our $f(g(x))$, so the original integral, by the substit. formula, equals

$\int \frac{1+u}{1-u}\frac{1}{u}\,du$ which we find is $-2\log(1-e^x)+x = F(g(x))$

Thus $[F(g(x))]' = f(g(x))g(x)$

The goal of the manipulations was to make the integrand $F'(g(x))g'(x) = f(g(x))g'(x)$.

Now, let's see the shortcut.

$u = e^x$

$x = \log(u)$

$dx = \frac{1}{u}\,du$

Ie, if our $g(x)$ is $u = e^x$, then finding the inverse and diff. tells us what the extra factor should be.

If we try this w/ the factor already present,

$$\int \sin^3 x \cos x \,dx$$

$u = \sin x$
$x = \arcsin x$

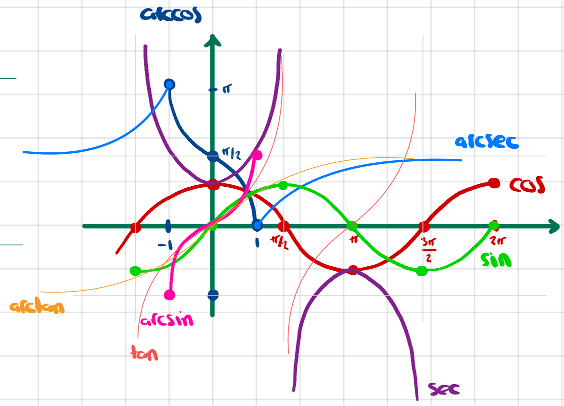
$$dx = \frac{1}{\sqrt{1-u^2}}\,du$$

$$\int u^3 \cdot \cos x \cdot \frac{1}{\sqrt{1-u^2}}\,du = \int u^3 \cdot du$$

now, $x = \arcsin u$, so $\cos x = \cos(\arcsin u) = \sqrt{1-u^2}$

$\sin^2(\arcsin x)+\cos^2(\arcsin x)=1 \qquad x\in[-1,1] \to \arcsin(x)\in[-\pi/2,\pi/2]$

$\cos^2(\arcsin x)=1-x^2 \to \cos(\arcsin x)=\sqrt{1-x^2} \qquad x\in[-1,1]$



$\sin^2(\arccos x)+x^2=1 \qquad x\in[-1,1]\to\arccos(x)\in[0,\pi]$

$\sin(\arccos x)=\sqrt{1-x^2} \qquad x\in[-1,1]$

$\tan(x)=\frac{\sin(x)}{\cos(x)} \to \tan(\arcsin x)=\frac{x}{\sqrt{1-x^2}} \qquad x\in(-1,1)$

$\tan(\arccos x)=\frac{\sqrt{1-x^2}}{x} \qquad x\in(-1,1)$

$\tan^2 x+1=\sec^2 x \qquad x\neq \frac{\pi}{2}+k\pi$

$\tan^2(\arctan x)+1=\frac{1}{\cos^2(\arctan x)}=x^2+1 \qquad x\in\mathbb{R}$

$\cos(\arctan x)=\frac{1}{\sqrt{x^2+1}} \qquad x\in\mathbb{R}$

$\sin^2(\arctan x)=1-\frac{1}{x^2+1}=\frac{x^2}{x^2+1}$

$\sin(\arctan x)=\frac{x}{\sqrt{x^2+1}} \qquad x\in\mathbb{R}$

$\tan^2(\operatorname{arcsec} x)=x^2-1 \qquad x\in(-\infty,-1]\cup[1,\infty)\wedge x\neq\frac{\pi}{2}+k\pi$

$\tan(\operatorname{arcsec} x)=\pm\sqrt{x^2-1}$

$x\in(-\infty,-1] \to \operatorname{arcsec} x\in(\pi/2,\pi] \to \tan(\operatorname{arcsec} x)=-\sqrt{x^2-1}$

$x\in[1,\infty) \to \operatorname{arcsec} x\in[0,\pi/2) \to \tan(\operatorname{arcsec} x)=\sqrt{x^2-1}$

$$\tan(\operatorname{arcsec} x)=\begin{cases}\sqrt{x^2-1} & x\in(-\infty,-1]\\ -\sqrt{x^2-1} & x\in[1,\infty)\end{cases}$$

$\tan^2(\arctan x)+1=x^2+1=\sec^2(\arctan x) \qquad x\neq\frac{\pi}{2}+k\pi$

$\arctan(x)\in(-\pi/2,\pi/2) \to \sec(\arctan x)=\sqrt{x^2+1}$

$$\sec'(x) = \left[\frac{1}{\cos x}\right]' = \tan x \cdot \sec x \qquad x \neq \frac{\pi}{2} + k\pi$$

$$\tan'(x) = \sec^2 x$$

$$\arcsin'(x) = \frac{1}{\sin'(\arcsin x)} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}} \qquad x \in (-1, 1)$$

$$\arccos'(x) = \frac{1}{\cos'(\arccos x)} = \frac{1}{-\sin(\arccos x)} = \frac{-1}{\sqrt{1-x^2}} \qquad x \in (-1, 1)$$

$$\arctan'(x) = \frac{1}{\tan'(\arctan x)} = \frac{1}{\sec^2(\arctan x)} = \cos^2(\arctan x) = \frac{1}{1+x^2}$$

$$\operatorname{arcsec}'(x) = \frac{1}{\sec'(\operatorname{arcsec} x)} = \frac{1}{\tan(\operatorname{arcsec} x) \cdot x}$$

$$= \frac{1}{|x|\sqrt{x^2-1}} \qquad x \in (-\infty, -1] \cup [1, \infty)$$

$$* \ \tan(\operatorname{arcsec} x) = \begin{cases} \sqrt{x^2-1} & x \in (-\infty, -1] \\ -\sqrt{x^2-1} & x \in [1, \infty) \end{cases}$$

$$\int \cos^2 x\,dx = \int \frac{1+\cos(2x)}{2}dx = \frac{x}{2} + \frac{\sin(2x)}{4} = \frac{x}{2} + \frac{\sin(x)\cos(x)}{2}$$

$$\int \sin^2 x\,dx = \int \frac{1-\cos(2x)}{2}dx = \frac{x}{2} - \frac{\sin(2x)}{4} = \frac{x}{2} - \frac{\sin(x)\cos(x)}{2}$$

$$\int \cos^4 x\,dx = \int \cos^2 x \cdot \cos^2 x\,dx = \int \left(\frac{1+\cos(2x)}{2}\right)^2 dx = \int \left(\frac{1}{4} + 2\cdot\frac{1}{2}\cdot\frac{\cos 2x}{2} + \frac{\cos^2 2x}{4}\right)dx$$

$$= \int \left(\frac{1}{4} + \frac{\cos 2x}{2} + \frac{\cos^2 2x}{4}\right)dx$$

$$= \frac{x}{4} + \frac{1}{2}\cdot\frac{\sin 2x}{2} + \frac{1}{4}\left(\frac{x}{2} + \frac{\sin(2x)\cos(2x)}{4}\right)$$

$$= \frac{x}{4} + \frac{x}{8} + \frac{2\sin x\cos x}{4} + \frac{2\sin x\cos x\,(2\cos^2 x - 1)}{16}$$

$$= \frac{3x}{8} + \frac{\sin x\cos x}{2} + \frac{\sin x\cos^3 x}{4} - \frac{\sin x\cos x}{8}$$

$$= \frac{3x}{8} + \frac{3\sin x\cos x}{8} + \frac{\sin x\cos^3 x}{4}$$

$$\int x\sin(x)\cos(x)\,dx = \frac{x\sin^2 x}{2} - \frac{1}{2}\int \sin^2 x\,dx = \frac{x\sin^2 x}{2} - \frac{x}{4} - \frac{\sin(x)\cos(x)}{4}$$

$f = x \qquad f' = 1$

$g' = \sin x\cos x \qquad g = \frac{\sin^2 x}{2}$

$$\int \sin^2 x\cos^2 x\,dx = \int \left(\frac{1+\cos(2x)}{2}\right)\left(\frac{1-\cos(2x)}{2}\right)dx = \int \left(\frac{1}{4} - \frac{\cos^2(2x)}{4}\right)dx$$

$$= \frac{x}{4} - \frac{1}{4}\int \cos^2(2x)\,dx = \frac{x}{4} - \frac{x}{8} - \frac{\sin(2x)\cos(2x)}{16} = \frac{x}{8} - \frac{\sin(x)\cos^3(x)}{8} + \frac{\sin^3(x)\cos(x)}{8}$$

$$= \frac{x}{8} - \frac{2\sin x\cos x\,(2\cos^2 x - 1)}{16} = \frac{x}{8} + \frac{\sin x\cos x}{8} - \frac{\sin x\cos^3 x}{4}$$

$$\int \cos^2(2x)\,dx = \int \cos^2(2x)\frac{1}{2}2\,dx = \frac{1}{2}\int \cos^2(u)\,du = \frac{1}{2}\left[\frac{u}{2} + \frac{\sin(u)\cos(u)}{2}\right] = \frac{1}{2}\left[\frac{2x}{2} + \frac{\sin(2x)\cos(2x)}{2}\right]$$

$$= \frac{x}{2} + \frac{\sin(2x)\cos(2x)}{4}$$

$g(x) = 2x$

$g'(x) = 2$

$$\frac{\sin(2x)\cos(2x)}{16} = \frac{2\sin(x)\cos(x)\,(\cos^2 x - \sin^2 x)}{16} = \frac{\sin(x)\cos^3(x)}{8} - \frac{\sin^3(x)\cos(x)}{8}$$

$$\sin(2x)\cos(2x) = 2\sin x\cos x\,(\cos^2 x - \sin^2 x) = 2\sin x\cos^3 x - 2\sin^3 x\cos x$$