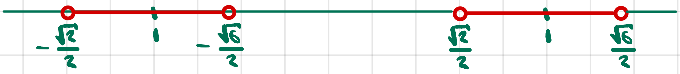
Ch.4 - Graphs

1. i) $|x-3|<1$

ii) $|x-3|\le 1$

iii) $|x-a|<\epsilon$

iv) $|x^2-1|<\frac{1}{2}$



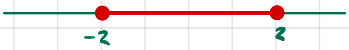
$\sqrt{(x^2-1)^2}<\frac{1}{2}$

$-\frac{1}{2}<x^2-1<\frac{1}{2}$

$\frac{1}{2}<x^2<\frac{3}{2}$

$\frac{\sqrt{2}}{2}<x<\frac{\sqrt{6}}{2} \quad \cup \quad -\frac{\sqrt{6}}{2}<x<-\frac{\sqrt{2}}{2}$

v) $\frac{1}{1+x^2}\ge\frac{1}{5}$



$5\ge 1+x^2$

$x^2\le 4$

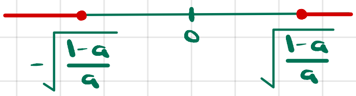
$-2\le x\le 2$

vi) $\frac{1}{1+x^2}\le a$

case 1: $a>0 \Rightarrow 1+x^2\ge\frac{1}{a} \Rightarrow x^2\ge\frac{1-a}{a}$

$\Rightarrow$ since $x^2\ge 0$, $1-a\ge 0 \Rightarrow a\le 1 \Rightarrow 0<a\le 1$

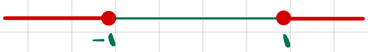
$\Rightarrow x>\sqrt{\frac{1-a}{a}},\ x<-\sqrt{\frac{1-a}{a}},\ 0<a\le 1$



case 2: $a\le 0 \Rightarrow 0<\frac{1}{1+x^2}\le a\le 0$, false

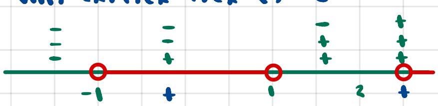
$\Rightarrow$ there are no solutions if $a<0$

vii) $x^2+1\ge 2$



$x^2\ge 1 \Rightarrow x\ge 1 \cup x\le -1$

viii) $(x+1)(x-1)(x-2)>0$



2. a) consider $[0,b]$, $b>0$

Prove $x \in [0,b] \Rightarrow x = tb$, for some $t \in [0,1]$

$0 \leq x \leq b$

$0 \leq \frac{x}{b} \leq 1$

$x = \frac{x}{b} \cdot b$

$t = \frac{x}{b} \Rightarrow x = tb$

t is a ratio representing the proportion of b that is x.

The midpoint of $[0,b]$ is $x = 0.5b$

b) $x \in [a,b] \Rightarrow x = (1-t)a + tb$, for some t w/ $0 \leq t \leq 1$

$a \leq x \leq b \Rightarrow 0 \leq x-a \leq b-a$

$\Rightarrow$ by a), $x-a = t(b-a)$, $t = \frac{x-a}{b-a}$

$\Rightarrow x = a + t(b-a)$

midpoint of $[a,b]$: $x = a + 0.5(b-a) = \frac{a+b}{2}$

1/3 point: $x = a + \frac{1}{3}(b-a) = \frac{2}{3}a + \frac{1}{3}b$

c) $0 \leq t \leq 1 \Rightarrow (1-t)a + tb \in [a,b]$

$(1-t)a + tb = a + t(b-a)$

$b>a, t \geq 0 \Rightarrow a \leq a + t(b-a)$

$t \leq 1 \Rightarrow a + t(b-a) \leq b$

$\Rightarrow a \leq a + t(b-a) \leq b$

$\Rightarrow a + t(b-a) = (1-t)a + tb \in [a,b]$

d) Prove points of open interval (a,b) are of form $(1-t)a + tb$ for $0<t<1$

$x \in (0,b) \Rightarrow 0 < x < b \Rightarrow 0 < \frac{x}{b} < 1$

$\Rightarrow x = tb, t = \frac{x}{b}$

$x \in (a,b) \Rightarrow a < x < b \Rightarrow 0 < x-a < b-a$

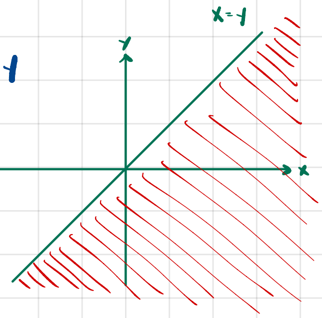
$\Rightarrow 0 < \frac{x-a}{b-a} < 1$

$\Rightarrow x - a = t(b-a)$, $t = \frac{x-a}{b-a}$

$\Rightarrow x = a + t(b-a)$, $0 < t < a$

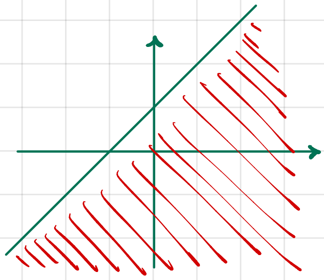
3.

i) $x > y$

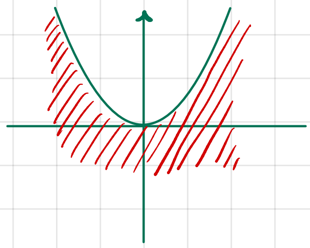


ii) $x + a > y + b$

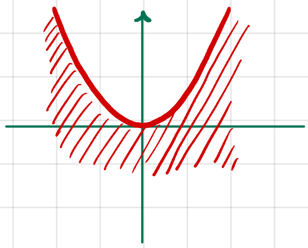
$y < x + (a-b)$



iii) $y < x^2$



iii) $y < x^2$



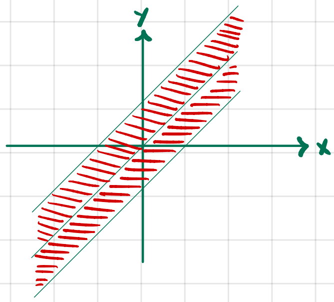
v) $|x-y| < 1$

$y > x \Rightarrow y - x < 1$

$\Rightarrow y < x + 1$

$y \leq x \Rightarrow x - y < 1$

$\Rightarrow y > x - 1$



vi) $|x+y| < 1$

$x + y > 0 \Rightarrow y > -x$

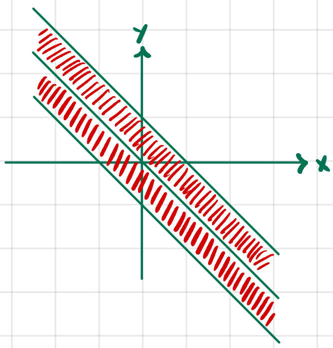
$\Rightarrow x + y < 1 \Rightarrow y < 1 - x$

$\Rightarrow -x < y < -x + 1$

$x + y < 0 \Rightarrow y < -x$

$\Rightarrow -(x+y) < 1$

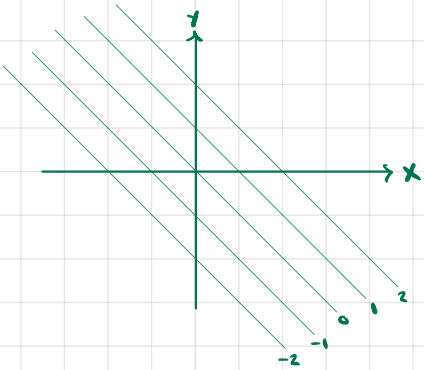
$y > -1 - x$



vii) $x+y \in \mathbb{Z}$

$x+y = i, \ \forall i \in \mathbb{Z}$

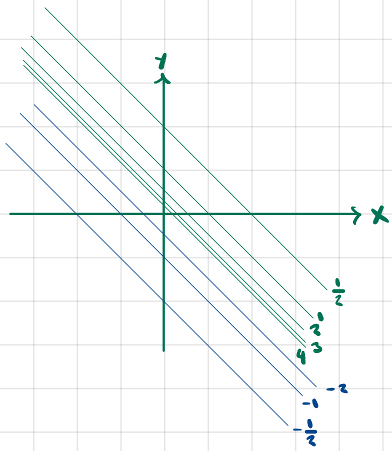
$y = i - x, \ \forall i \in \mathbb{Z}$



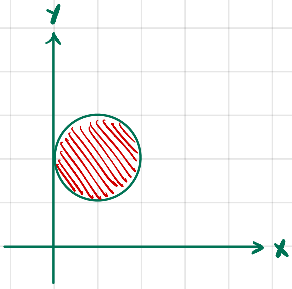
viii) $\frac{1}{x+y} \in \mathbb{Z}$

$\frac{1}{x+y} = i, \ i \in \mathbb{Z}, i \neq 0$

$x+y = \frac{1}{i} \Rightarrow y = \frac{1}{i} - x$

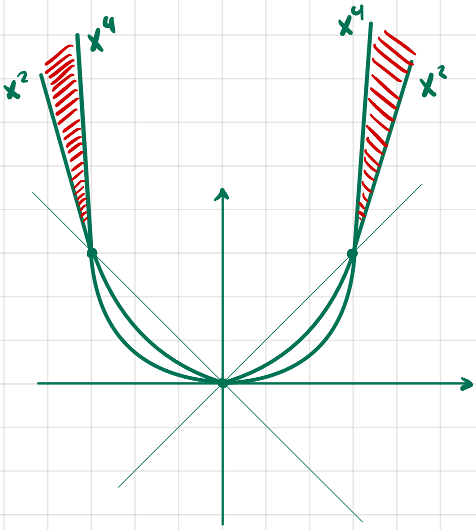


ix) $(x-1)^2 + (y-2)^2 < 1$

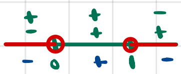


x) $x^2 < y < x^4$

$x^2 < x^4 \Rightarrow x < x^2 \cup x > -x^2$

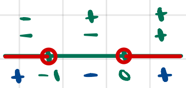


$x < x^2 \Rightarrow x(1-x) < 0$

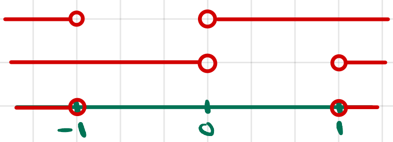


$\Rightarrow x < 0 \cup x > 1$

$x > -x^2 \Rightarrow x(1+x) > 0$



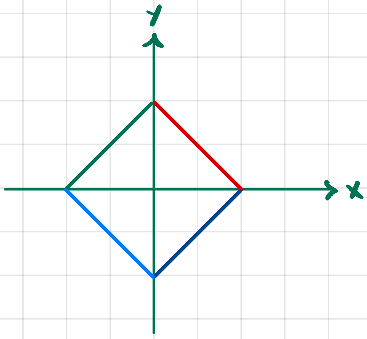
$\Rightarrow x < -1 \cup x > 0$



$\Rightarrow x > 1 \cup x < -1$

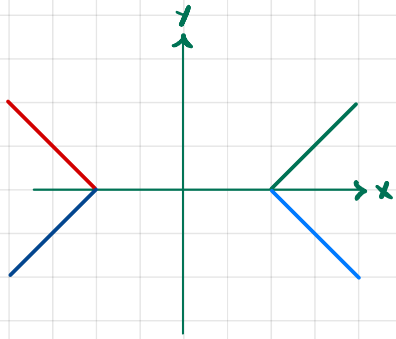
4. (i) $|x|+|y|=1$

$x>0, y>0 \Rightarrow x+y=1 \Rightarrow y=1-x$
$x>0, y<0 \Rightarrow x-y=1 \Rightarrow y=-1+x$
$x<0, y>0 \Rightarrow -x+y=1 \Rightarrow y=1+x$
$x<0, y<0 \Rightarrow -x-y=1 \Rightarrow y=-1-x$



ii) $|x|-|y|=1$

$x>0, y>0 \Rightarrow x-y=1 \Rightarrow y=x-1$
$x>0, y<0 \Rightarrow x+y=1 \Rightarrow y=1-x$
$x<0, y>0 \Rightarrow -x-y=1 \Rightarrow y=-x-1$
$x<0, y<0 \Rightarrow -x+y=1 \Rightarrow y=1+x$



iii) $|x-1|=|y-1|$

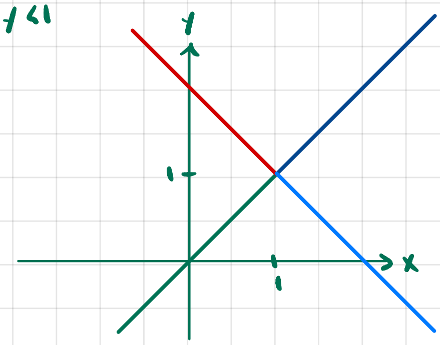
$|x-1|=\sqrt{(x-1)^2}=\sqrt{(y-1)^2}=|y-1|$

$x-1<0 \Rightarrow |x-1|=1-x$
$y-1<0 \Rightarrow |y-1|=1-y$ $\Rightarrow x<1, y<1$
$\Rightarrow 1-x=1-y \Rightarrow y=x$

$x-1<0, y-1>0 \Rightarrow x<1, y>1$
$\Rightarrow 1-x=y-1 \Rightarrow y=2-x$

$x-1>0, y-1<0$
$\Rightarrow x-1=1-y \Rightarrow y=2-x$

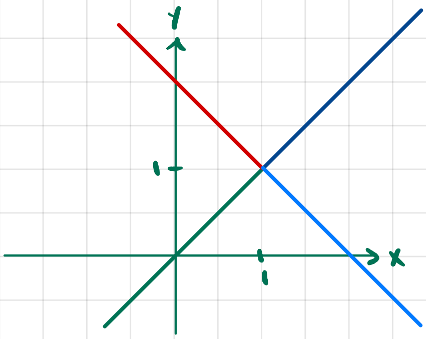
$x-1>0, y-1>0 \Rightarrow x>1, y>1$
$\Rightarrow x-1=y-1 \Rightarrow x=y$



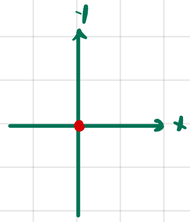
iv) $|1-x|=|y-1|$

$1-x\geq 0 \Rightarrow x\leq 1$
$y-1\geq 0 \Rightarrow y\geq 1$

$x\leq 1, y\geq 1 \Rightarrow 1-x=y-1 \Rightarrow y=2-x$
$x\leq 1, y<1 \Rightarrow 1-x=1-y \Rightarrow x=y$
$x>1, y\geq 0 \Rightarrow x-1=y-1 \Rightarrow x=y$
$x>1, y<1 \Rightarrow x-1=1-y \Rightarrow y=2-x$



v) $x^2+y^2=0 \Rightarrow x=y=0$

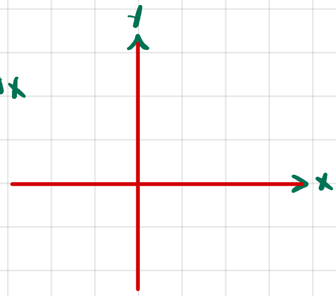


vi) $xy=0 \Rightarrow x=0$ or $y=0$

note: f, fn $\Rightarrow f.0=0.f=0$,
where $0(x)=0\ \forall x$

proof
$f(x).0(x)+f(x).0(x)$
distributive law for fns, P9
$=f(x)(0(x)+0(x))$

$=f(x).0(x)=0\ \forall x$



How many roots does ax^2+bx+c have?

$$a>0 \Rightarrow \quad ax^2 + 2\sqrt{a}\,x\cdot\frac{b}{2\sqrt{a}} + \left(\frac{b}{2\sqrt{a}}\right)^2 + c - \left(\frac{b}{2\sqrt{a}}\right)^2$$

$$\left(\sqrt{a}\,x + \frac{b}{2\sqrt{a}}\right)^2 = \left(\frac{b}{2\sqrt{a}}\right)^2 - c = \frac{b^2-4ac}{4a}$$

$b^2-4ac<0 \Rightarrow$ no solutions because left side $\geq 0\ \forall x$ and right side <0.

$$b^2-4ac=0 \Rightarrow \sqrt{a}\,x = -\frac{b}{2\sqrt{a}} \Rightarrow x = \frac{-b}{2a} \quad \text{one root}$$

$$b^2-4ac>0 \Rightarrow \sqrt{a}\,x + \frac{b}{2\sqrt{a}} = \frac{2ax+b}{\cancel{2}\cancel{\sqrt{a}}} = \pm\sqrt{\frac{b^2-4ac}{\cancel{4}\cancel{a}}}$$

$$\Rightarrow x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad \text{two roots}$$

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viii) $x^2-2x+y^2=4$

solution 1: long

$\Rightarrow x^2-2x+y^2-4=0$

$$\begin{aligned}\Delta &= 4-4\cdot1\cdot(y^2-4)\\ &= 4+16-4y^2\\ &= 20-4y^2\end{aligned}$$

$20-4y^2<0 \Rightarrow$ no solutions in x

$4y^2>20 \Rightarrow y^2>5 \Rightarrow y>\sqrt{5} \cup y<\sqrt{5}$

$20-4y^2=0 \Rightarrow$ one solution in x

$\Rightarrow y=\pm\sqrt{5} \Rightarrow x=\frac{2}{2}=1$

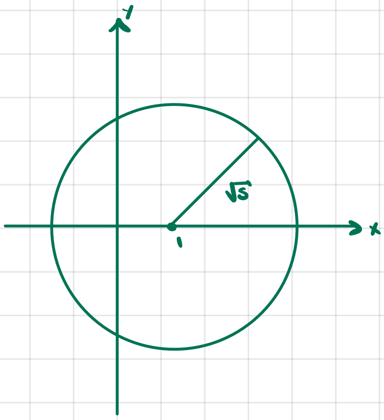
$$20-4y^2\geq0 \Rightarrow x = \frac{2\pm\sqrt{20-4y^2}}{2} = \frac{2\pm2\sqrt{5-y^2}}{2}$$

$$= 1\pm\sqrt{5-y^2}$$

$\Rightarrow x-1=\pm\sqrt{5-y^2}$

$\Rightarrow (x-1)^2=5-y^2$

$\Rightarrow (x-1)^2+y^2=(\sqrt{5})^2$ (a circle)



solution 2: complete square

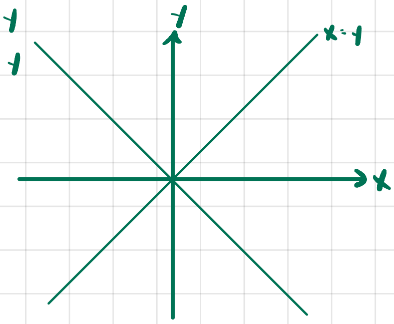
$x^2-2x+y^2=4$

$\Rightarrow x^2-2x+1+y^2=5$

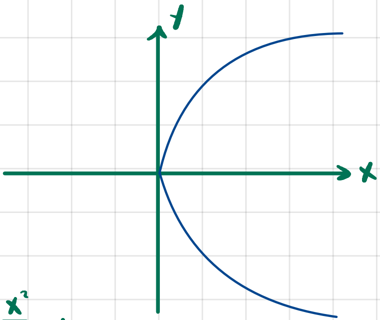
$(x-1)^2+y^2=5$, a circle

viii) $x^2 = y^2$

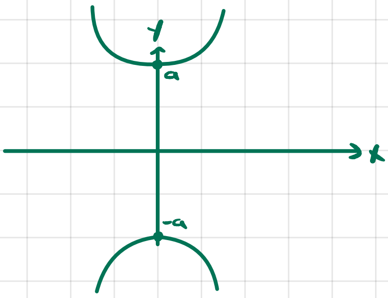
$\pm x = \pm y \Rightarrow x = y$
$-x = y$



5. i) $x = y^2 \Rightarrow y = \pm\sqrt{x}$



ii) $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$



$x = 0 \Rightarrow y^2 = a^2 \Rightarrow y = \pm a$

$y = 0 \Rightarrow x^2 = -b^2 \Rightarrow$ no sol'n $\Rightarrow y \neq 0$

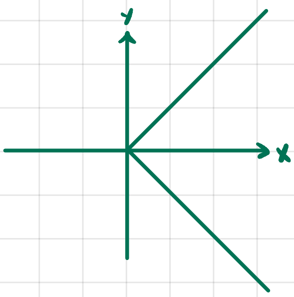
$y^2 = a^2 + \frac{a^2}{b^2}x^2$

$y = \pm\sqrt{a^2 + \frac{a^2}{b^2}x^2}$

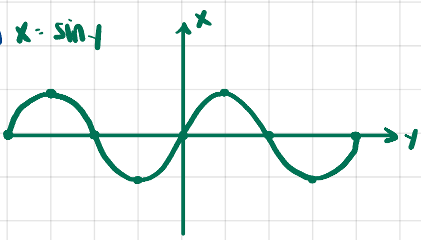
$x = 0 \Rightarrow$ the square root is minimized
$\Rightarrow y = \pm a$

iii) $x = |y|$

$y > 0 \Rightarrow x = y$
$y < 0 \Rightarrow x = -y$



iv) $x = \sin y$

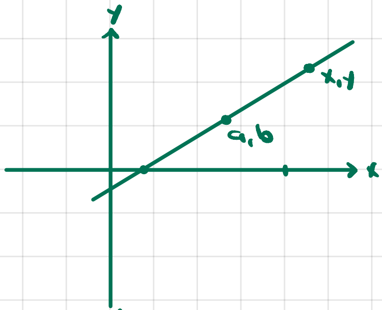


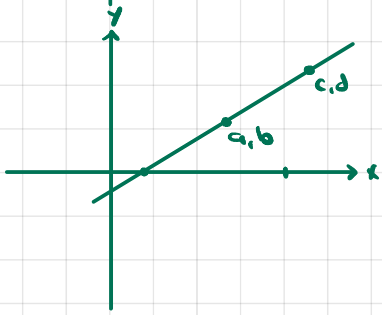
6. a) $f(a) = b$

points $(x, f(x))$ that form slope m with a point (a,b)

$$\text{slope} = m = \frac{y-b}{x-a} = \frac{f(x)-b}{x-a}$$

$\Rightarrow f(x) = b + m(x-a)$
This is a linear function of x.





b) $a \neq c$

slope of line through (a,b), (c,d)

$$\text{slope} = m = \frac{d-b}{c-a}$$

points $(x, f(x))$ forming slope m with (a,b)

$$f(x) = b + \frac{d-b}{c-a}(x-a)$$

c) $f(x) = mx + b$
$g(x) = m'x + b'$

parallel $\Leftrightarrow m = m'$

7) $A, B, C \in \mathbb{R}$

A, B not both zero

Show: $\{(x,y) : Ax + By + C = 0\}$ is a straight line.

vertical line $\Rightarrow$ $x = \text{constant} \; \forall y$

case 1: $A, B, C \neq 0$

$\Rightarrow By = -Ax - C \Rightarrow y = -\frac{A}{B}x - \frac{C}{B}$, linear fn.

case 2: $A = 0$

$By = -C \Rightarrow y = -\frac{C}{B}$, linear, horizontal fn

case 3: $B = 0$

$\Rightarrow x = \frac{-C}{A}$, linear fn of y, vertical line

b) every straight line can be described as $\{(x,y) : Ax + By + C = 0\}$

A straight line means constant slope.

Consider two points on the straight line $(x_1, y_1), (x_2, y_2)$.

if $x_1 \neq x_2$ the slope is

$$\frac{y_2 - y_1}{x_2 - x_1} = m$$

Any other point (x,y) on the line forms

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} = m$$

$m = 0 \Rightarrow y = y_1$, ie $A = C = 0$, $B = 1$

$m \neq 0 \Rightarrow y - y_1 = mx - mx_1$

$mx - y - mx_1 + y_1 = 0$

$A = m$

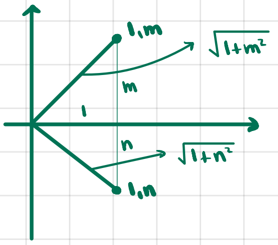
$B = -1$

$C = -mx_1 + y_1$

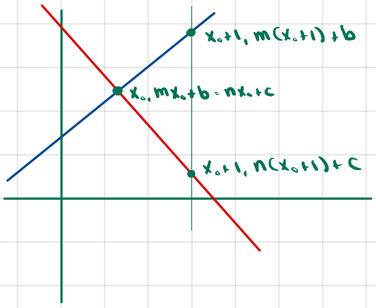
8. a)

$f(x) = mx + b$
$g(x) = nx + c$

Prove: graphs are $\perp$ iff $mn = -1$



if $\perp$ then $1+\cancel{m^2}+1+\cancel{n^2} = (m-n)^2 = \cancel{m^2}+\cancel{n^2}-2mn$
$\Rightarrow mn = -1$



$$(\cancel{x_0}+1-\cancel{x_0})^2 + (m(\cancel{x_0}+1)+\cancel{b} - m\cancel{x_0}-\cancel{b})^2$$
$$+ \ (\cancel{x_0}+1-\cancel{x_0})^2 + (n(\cancel{x_0}+1)+\cancel{c} - n\cancel{x_0}+\cancel{c})^2$$
$$= (m(x_0+1)+b-n(x_0+1)-c)^2$$

$$1+m^2+1+n^2 = [(x_0+1)(m-n)+(b-c)]^2$$

b) Two straight lines

$\{(x,y): Ax+By+C=0\}$
$\{(x,y): A'x+B'y+C'=0\}$

Prove $\perp \iff AA'+BB'=0$

$$y = -\frac{C}{B} - \frac{A}{B}x$$

$$y = -\frac{C'}{B'} - \frac{A'}{B'}x$$

$$\frac{AA'}{BB'} = -1 \Rightarrow AA'+BB' = 0$$

9. In problem 1-19 we proved

i) $x_1y_1 + x_2y_2 = \sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}$ iff $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$, $\lambda \geq 0$
or if $y_1 = y_2 = 0$

also if $\nexists \lambda$ s.t. $(x_1, x_2) = \lambda(y_1, y_2)$ then $(\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2 > 0$

ii) $2xy \leq x^2+y^2$ ie $(x-y)^2 \geq 0$

$\Rightarrow x_1y_1 + x_2y_2 \leq \sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}$

Prove $\sqrt{(x_1+y_1)^2+(x_2+y_2)^2} \leq \sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2}$

Solution 1

$$x_1(x_1+y_1) + x_2(x_2+y_2) \leq \sqrt{(x_1+y_1)^2+(x_2+y_2)^2}\cdot\sqrt{x_1^2+x_2^2}$$

$$y_1(x_1+y_1) + y_2(x_2+y_2) \leq \sqrt{(x_1+y_1)^2+(x_2+y_2)^2}\cdot\sqrt{y_1^2+y_2^2}$$

Add inequalities

$$(x_1+y_1)^2 + (x_2+y_2)^2 \leq \sqrt{(x_1+y_1)^2+(x_2+y_2)^2}\left(\sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2}\right)$$

$$\frac{(x_1+y_1)^2+(x_2+y_2)^2}{\sqrt{(x_1+y_1)^2+(x_2+y_2)^2}} = \sqrt{(x_1+y_1)^2+(x_2+y_2)^2} \leq \sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2}$$

Solution 2

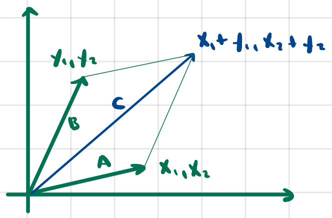
$$(x_1+y_1)^2 + (x_2+y_2)^2 = x_1^2+x_2^2+y_1^2+y_2^2 + 2x_1y_1 + 2x_2y_2$$

$$= x_1^2+x_2^2+y_1^2+y_2^2 + 2(x_1y_1+x_2y_2)$$

$$\leq x_1^2+x_2^2+y_1^2+y_2^2 + 2\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}$$

$$= \left[\sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2}\right]^2$$

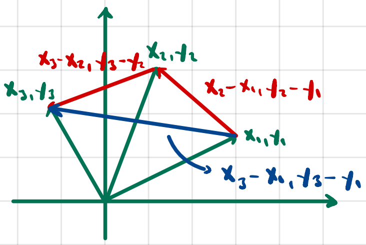
$$\Rightarrow \sqrt{(x_1+y_1)^2+(x_2+y_2)^2} \leq \sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2}$$



$C \leq A+B$

b) Prove $\sqrt{(x_3-x_1)^2+(y_3-y_1)^2} \le \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} + \sqrt{(x_3-x_2)^2+(y_3-y_2)^2}$

In a) let



$x_{1a} = x_2 - x_1$

$x_{2a} = y_2 - y_1$

$y_{1a} = x_3 - x_2$

$y_{2a} = y_3 - y_2$

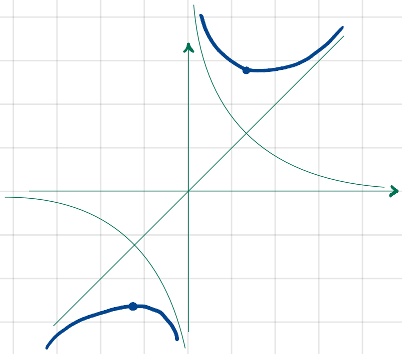
Apply result from a)

$$\left[(\cancel{x_2}-x_1+x_3-\cancel{x_2})^2 + (\cancel{y_2}-y_1+y_3-\cancel{y_2})^2\right]^{1/2} \le \left[(x_2-x_1)^2+(y_2-y_1)^2\right]^{1/2} + \left[(x_3-x_2)^2+(y_3-y_2)^2\right]^{1/2}$$

$$\Rightarrow \sqrt{(x_3-x_1)^2+(y_3-y_1)^2} \le \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} + \sqrt{(x_3-x_2)^2+(y_3-y_2)^2}$$

---

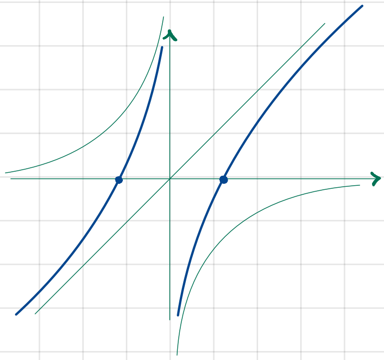
10. i) $f(x) = x + \frac{1}{x}$



$f(-x) = -x - \frac{1}{x} = -(x + \frac{1}{x}) = -f(x)$

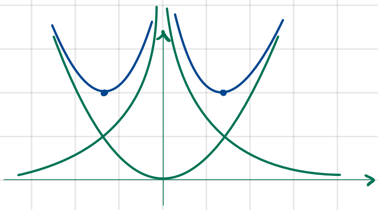
$\Rightarrow f(x) = -f(-x)$, f is odd

ii) $f(x) = x - \frac{1}{x}$

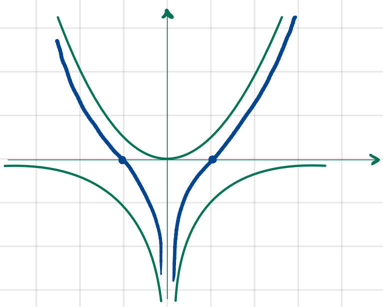


$f(-x) = -x + \frac{1}{x} = -(x - \frac{1}{x}) = -f(x)$, odd fn

iii) $f(x) = x^2 + \frac{1}{x^2}$

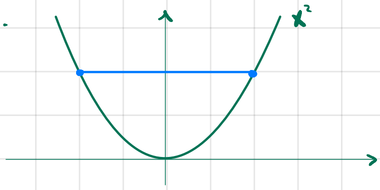


iv) $f(x) = x^2 - \frac{1}{x^2}$



11. i) f even

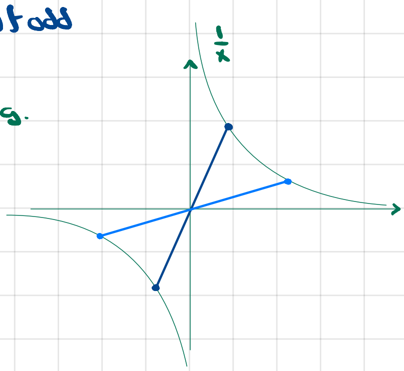
e.g.



symmetric about y axis

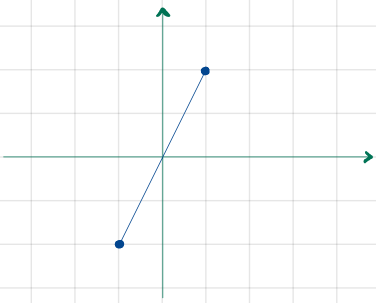
ii) f odd

e.g.



symmetric about the origin

iii) f nonnegative

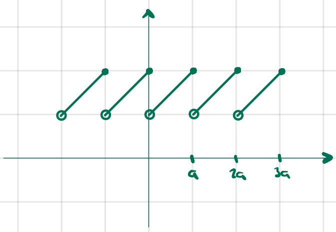


A fn can't be symmetric about the origin if fn nonneg.
no odd fns are nonnegative.

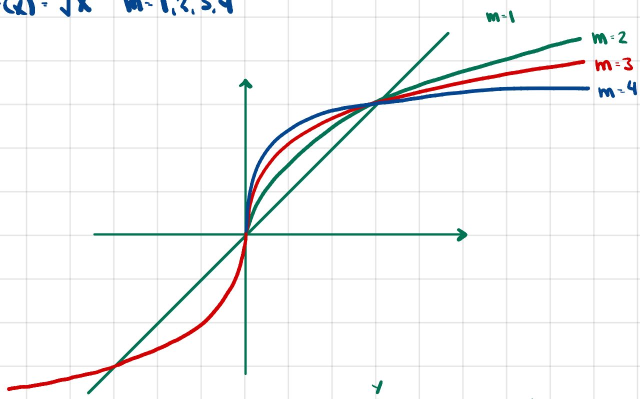
odd $\Rightarrow f(x) = -f(-x)$

$f(x) > 0 \Rightarrow -f(-x) > 0 \Rightarrow f(-x) < 0$

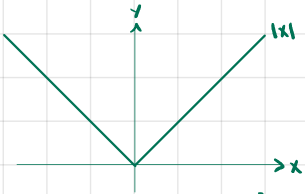
iv) $f(x) = f(x+a)\ \forall x$, ie a periodic fn, w/ period a



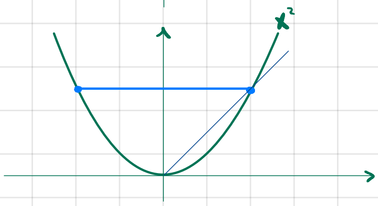
12. $f(x) = \sqrt[m]{x}$ $m = 1, 2, 3, 4$



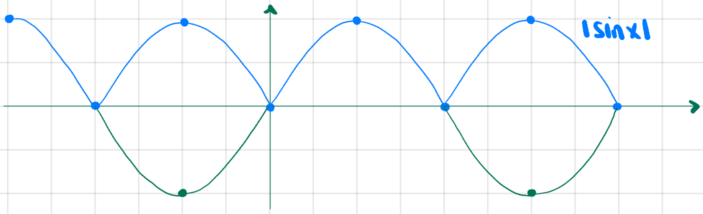
13. a) $f(x) = |x|$

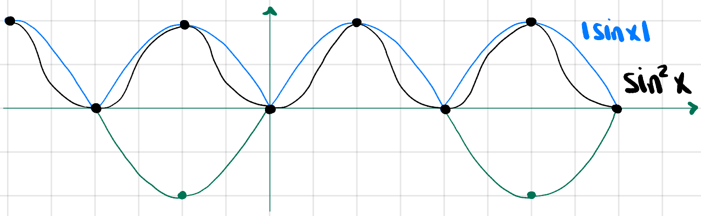


$f(x) = x^2$



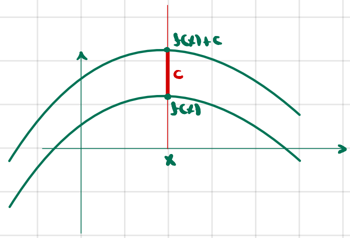
b) $f(x) = |\sin x|$



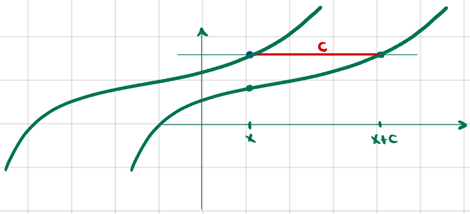


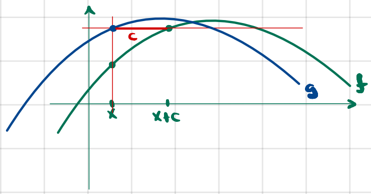
14. i) $g(x) = f(x) + c$

g shifted up or down at each x by $|c|$ units



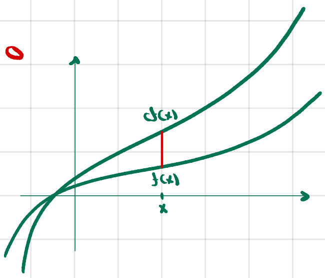
ii) $g(x) = f(x+c)$

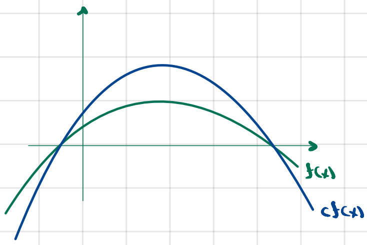




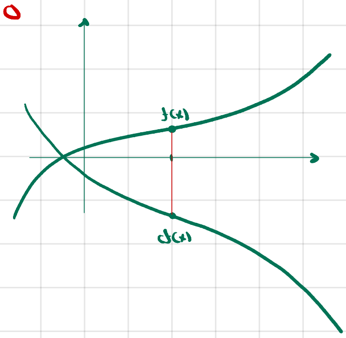
iii) $g(x) = cf(x)$

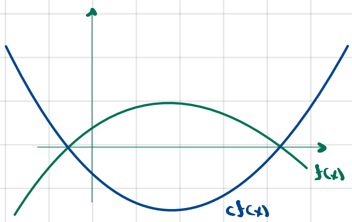
$c > 0$



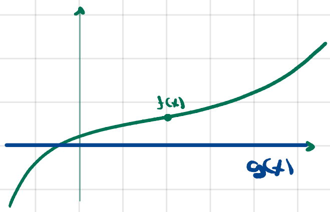


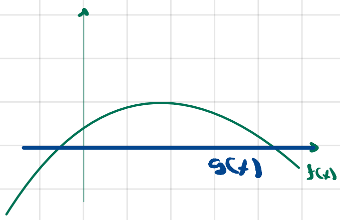
$c < 0$





$c = 0$





$g(x) = f(x) + (c-1)f(x)$

$c \neq 0 \Rightarrow$ The larger $|f(x)|$ the larger the difference $|g(x) - f(x)|$

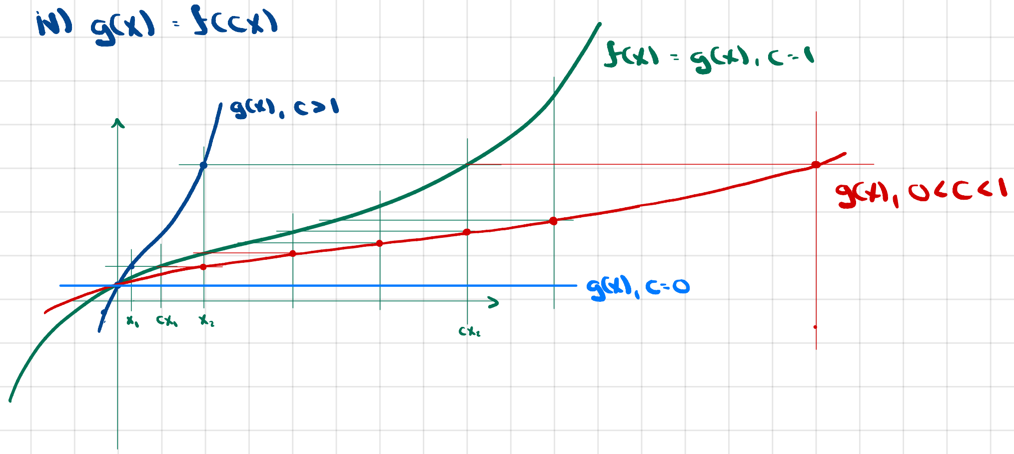
$g(x) - f(x) = (c-1)f(x)$

$c > 1 \Rightarrow |g(x) - f(x)| = (c-1)|f(x)|$

$c < 1 \Rightarrow |g(x) - f(x)| = (1-c)|f(x)|$

f and g coincide only when $f(x) = g(x) = 0$, the roots.

iv) $g(x) = f(cx)$



$c = 0 \Rightarrow g(x) = f(0)$

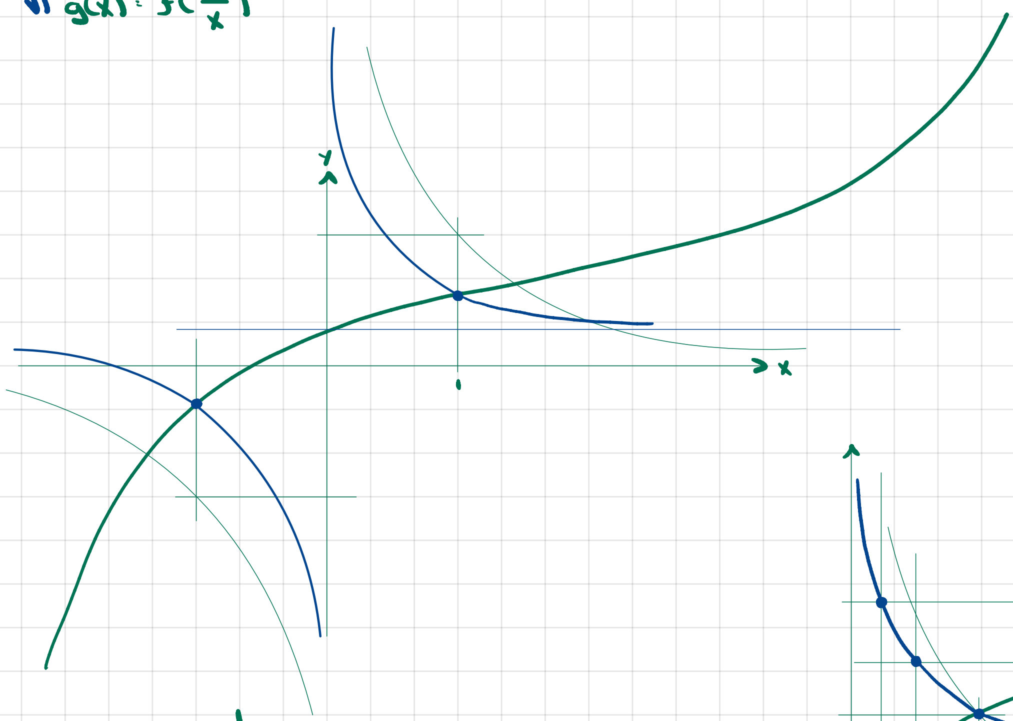
$c = 1 \Rightarrow g(x) = f(x)$

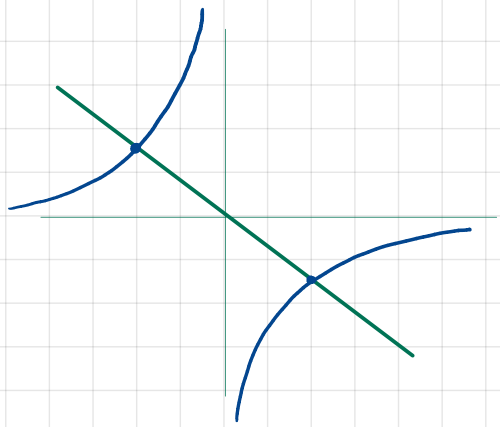
$0 < c < 1 \Rightarrow$ g "stretches" f along the x-axis direction

$c > 1 \Rightarrow$ g "compresses" f along the x axis

For $c < 0$ we have three analogous cases, but w/ respect to the fn $-f(x)$.

v) $g(x) = f(\frac{1}{x})$





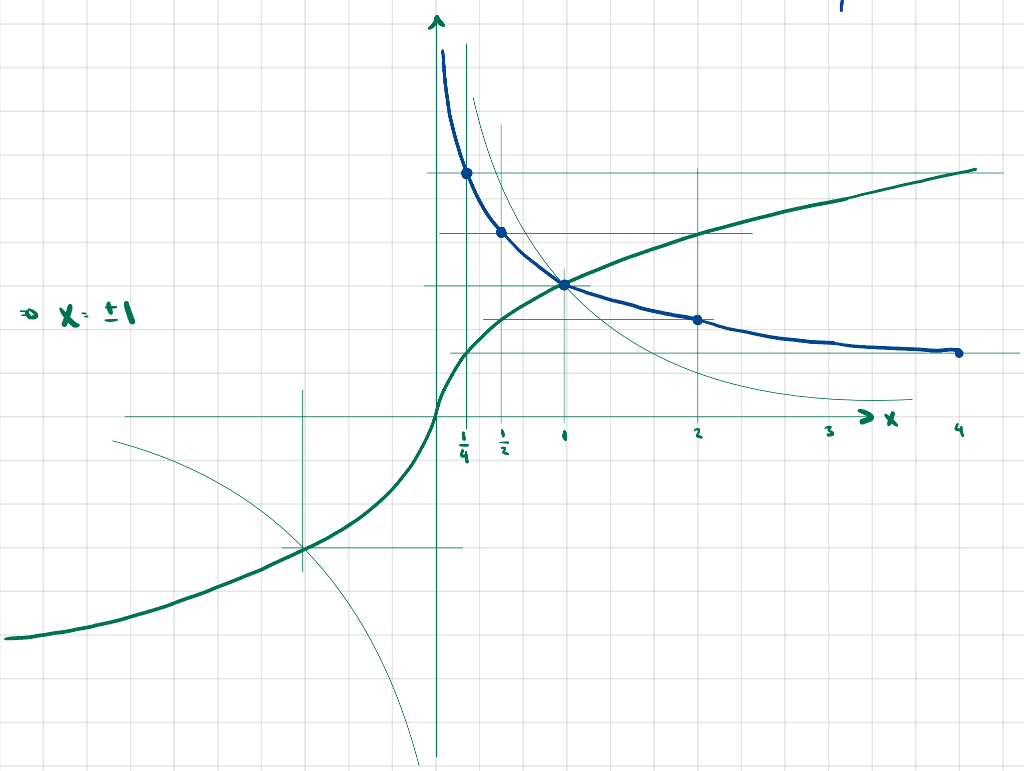
$g(x) = f(x) = \frac{1}{x} = x \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

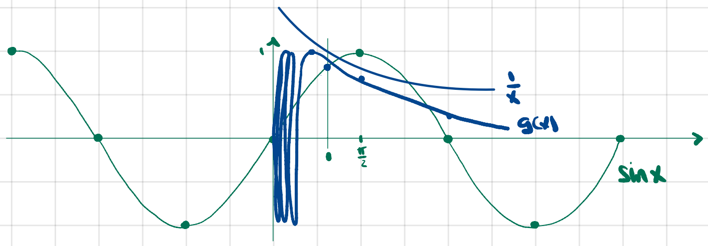
$g(0^+) = f(\frac{1}{0^+}) = f(\infty)$

$g(0^-) = f(\frac{1}{0^-}) = f(-\infty)$

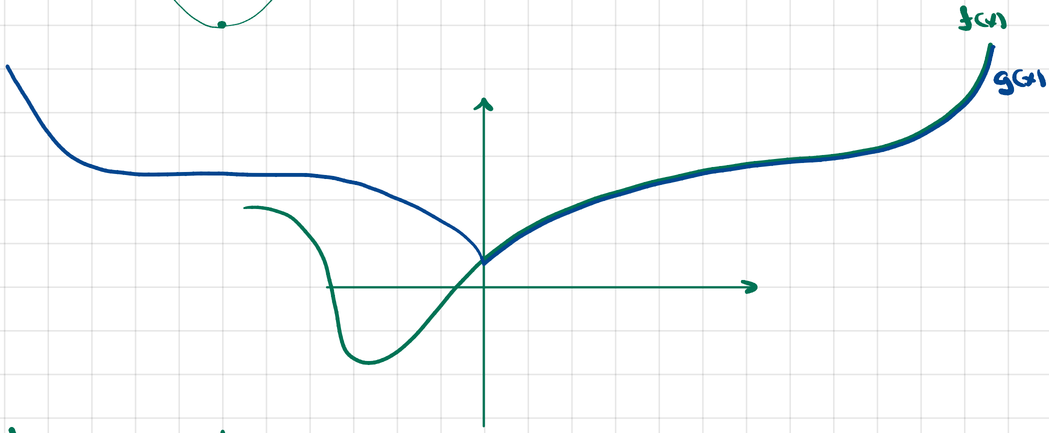
$g(\infty) = f(\frac{1}{\infty}) = f(0)$

$g(-\infty) = f(\frac{1}{-\infty}) = f(0^-)$



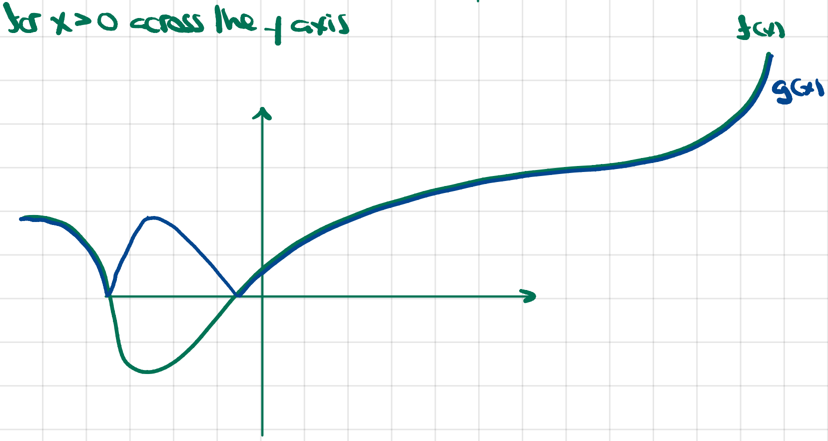


vi) $g(x) = f(|x|)$



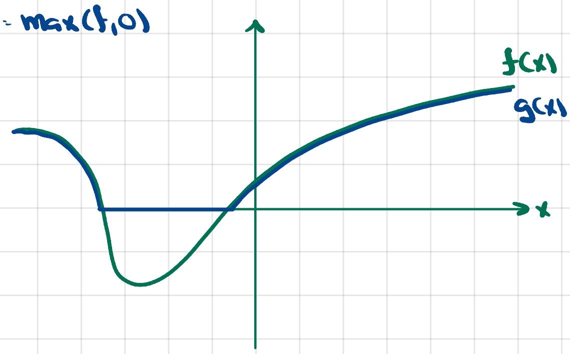
g reflects the portion of f for $x > 0$ across the y axis

vii) $g(x) = |f(x)|$



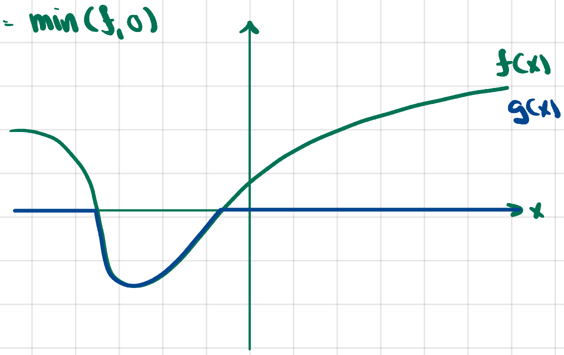
g matches f if $f(x) \geq 0$. It reflects the negative portions of f across the x-axis.

viii) $g(x) = \max(f, 0)$



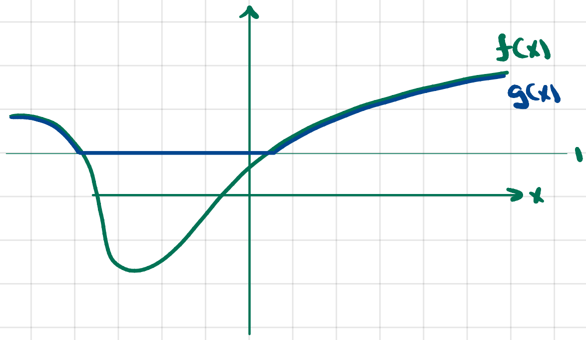
g keeps positive parts of f. where f is <0, g is 0.

ix) $g(x) = \min(f, 0)$



g keeps negative parts of f and is zero everywhere else.

x) $g(x) = \max(f,1)$



g keeps portions of f s.t. $f(x) > 0$ and is 1 everywhere else.

15. $f(x) = ax^2 + bx + c = a\left(x + \frac{b}{a}x + \frac{c}{a}\right) = a\,g(x)$

$$g(x) = x^2 + 2x \cdot \frac{b}{2a} + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2$$

$$= \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}$$

$$= \underbrace{\left(x + \frac{b}{2a}\right)^2}_{>0} + \underbrace{\frac{4ac - b^2}{4a^2}}_{\text{constant}}$$

$\Rightarrow g(x)$ min when $x = -\frac{b}{2a} \Rightarrow g(x_{min}) = \frac{4ac - b^2}{4a^2}$

roots

$$g(x) = 0 \Rightarrow \frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

case 1: $b^2 - 4ac = 0 \Rightarrow$ one root at $x_r = -\frac{b}{2a} = x_{min}$

$g(x) \geq 0 \ \forall x$

case 2: $b^2 - 4ac > 0 \Rightarrow$ two roots

$$x_r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$x > x_{r_+},\ x < x_{r_-} \Rightarrow g(x) > 0$

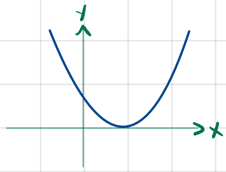
$x_{r_-} < x < x_{r_+} \Rightarrow g(x) < 0$

case 3: $b^2 - 4ac < 0 \Rightarrow$ no roots

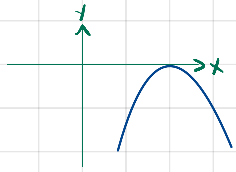
$g(x) > 0 \ \forall x$

Now consider $f(x)$

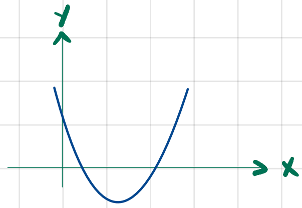
case 1: $a > 0,\ b^2 - 4ac = 0$



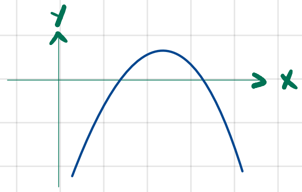
case 2: $a < 0,\ b^2 - 4ac = 0$



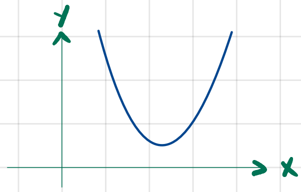
case 3: $a > 0,\ b^2 - 4ac > 0$



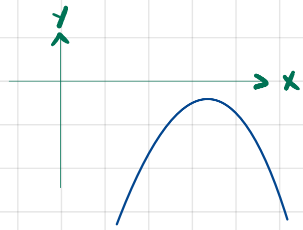
case 4: $a < 0,\ b^2 - 4ac > 0$



case 5: $a > 0,\ b^2 - 4ac < 0$



case 6: $a < 0,\ b^2 - 4ac < 0$



16. Assume A and C not both zero.

Show $\{(x,y) : Ax^2 + Bx + Cy^2 + Dy + E = 0\}$

is either

- parabola
- ellipse
- hyperbola
- or degenerate cases
  - two lines
  - one line
  - a point
  - ϕ

case 1: $C = 0$ $\quad Ax^2 + Bx + Dy + E = 0$

$D \neq 0 \Rightarrow y = -\frac{A}{D}x^2 - \frac{B}{D}x - \frac{E}{D}$

$= ax^2 + bx + c$ parabola

$A = 0 \Rightarrow y = -\frac{B}{D}x - \frac{E}{D} = bx + c$ line

$A = 0, B = 0 \Rightarrow y = -\frac{E}{D} = c$ line

$D = 0 \Rightarrow Ax^2 + Bx + E = 0$

we are back in the cases of problem 15.

0 roots $\Rightarrow$ ϕ
1 " $\Rightarrow$ line $x = x_r$
2 " $\Rightarrow$ two lines $x = x_{r_1}, x = x_{r_2}$

case 2: $A = 0$ $\quad Bx + Cy^2 + Dy + E = 0$

symmetric to case 1.

Case 3: $A, C \neq 0$

$$A\left(x^2 + \frac{B}{A}x\right) + C\left(y^2 + \frac{D}{C}y\right) + E = 0$$

$$= A\left(x^2 + 2x\frac{B}{2A} + \left(\frac{B}{2A}\right)^2\right) + c\left(y^2 + 2y\frac{D}{2C} + \left(\frac{D}{2C}\right)^2\right) + E - A\left(\frac{B}{2A}\right)^2 - c\left(\frac{D}{2C}\right)^2$$

$$= A\left(x + \frac{B}{2A}\right)^2 + C\left(y + \frac{D}{2C}\right)^2 = A\left(\frac{B}{2A}\right)^2 + c\left(\frac{D}{2C}\right)^2 - E$$

$$\Rightarrow A\left(x + \frac{B}{2A}\right)^2 + C\left(y + \frac{D}{2C}\right)^2 = F$$

$F = 0 \Rightarrow$ single point (circle of radius 0)

$F < 0 \Rightarrow \phi$

$F > 0,\ A, C > 0 \Rightarrow \dfrac{\left(x + \frac{B}{2A}\right)^2}{C} + \dfrac{\left(y + \frac{D}{2C}\right)^2}{A} = \dfrac{F}{AC} \Rightarrow$ ellipse

$A = C > 0 \Rightarrow$ circle

$F > 0,\ A, C < 0 \Rightarrow |A|\left(x + \frac{B}{2A}\right)^2 + |C|\left(y + \frac{D}{2C}\right)^2 = -F \Rightarrow \phi$

$A > 0, C < 0$

$$\Rightarrow \frac{\left(x + \frac{B}{2A}\right)^2}{C} + \frac{\left(y + \frac{D}{2C}\right)^2}{A} = \frac{F}{AC}$$

$$\frac{\left(x + \frac{B}{2A}\right)^2}{F/A} + \frac{\left(y + \frac{D}{2C}\right)^2}{F/C} = 1$$

$F \neq 0 \Rightarrow$ hyperbola

$F = 0 \Rightarrow \left(x + \frac{B}{A}\right) = \pm\left(y + \frac{D}{2C}\right)\sqrt{\frac{C}{A}}$

$x = -\frac{B}{A} \pm \frac{D}{2\sqrt{AC}} \pm \sqrt{\frac{C}{A}}\,y$

$\Rightarrow$ two intersecting lines

16. Assume A and C not both zero.

Show $\{(x,y) : Ax^2+Bx+Cy^2+Dy+E=0\}$

is either
- parabola
- ellipse
- hyperbola
- or degenerate cases
  - two lines
  - one line
  - a point
  - ϕ

* There are 3^5 possible cases

---

case 1: $C=0$ $\quad Ax^2+Bx+Dy+E=0$ $\quad 3^4$ cases

$\Rightarrow y = -\frac{A}{D}x^2 - \frac{B}{D}x - \frac{E}{D}$

$= ax^2+bx+c$ parabolas

$A=0 \quad y = -\frac{B}{D}x - \frac{E}{D} = bx+c$ line

$A=0, B=0: \quad y = -\frac{E}{D} = c$ line

$D=0 \Rightarrow Ax^2+Bx+E=0$

we are back in the cases of problem 15.

0 roots $\Rightarrow \phi$
1 " $\Rightarrow$ line
2 " $\Rightarrow$ two lines

---

case 2: $A=0$ $\quad Bx+Cy^2+Dy+E=0$ $\quad 3^4-3^3$ cases

symmetric w/ case 1.

$3^5-(3^4+3^4-3^3) = 3^4+3^3 = 81+27 = 108$
cases left

Case 3: $A>0, C>0, B=D=0$ $\quad$ 3 cases

$\frac{x^2}{C} + \frac{B}{AC}x + \frac{y^2}{A} + \frac{D}{AC}y + \frac{E}{AC} = 0$

$\frac{x^2}{C} + \frac{y^2}{A} = -\frac{E}{AC}$

$\rightarrow E>0 \Rightarrow \phi$

$\rightarrow E=0 \Rightarrow (x,y)=(0,0)$

$\rightarrow E<0 \Rightarrow \frac{x^2}{-E/A} + \frac{y^2}{-E/C} = 1$

$\frac{x^2}{a} + \frac{y^2}{b} = 1$ ellipse

* circle if $A=C$

Case 4: $A>0, C<0, B=D=0$ $\quad$ 3 cases

$\frac{x^2}{C} + \frac{y^2}{A} = -\frac{E}{AC}$

$E=0 \Rightarrow (x,y)=(0,0)$

$E>0 \Rightarrow \frac{y^2}{-E/C} + \frac{x^2}{-E/A} = 1$

$\frac{y^2}{a} - \frac{x^2}{b} = 1$ hyperbola

$E<0 \Rightarrow \phi$

Case 5: $A<0, C>0, B=D=0$ $\quad$ 3 cases

By symmetry w/ case 4 we have either single point (0,0), empty set, or hyperbola.

Case 6 $\quad A<0, C<0, B=D=0$ $\quad$ 3 cases

$\frac{x^2}{C} + \frac{y^2}{A} = -\frac{E}{AC}$

$E=0 \Rightarrow (0,0)$
$E<0 \Rightarrow \phi$
$E>0 \Rightarrow \frac{y^2}{-E/C} + \frac{x^2}{-E/A} = 1$

$\Rightarrow \frac{x^2}{a} + \frac{y^2}{b} = 1$ ellipse

What's left are cases of B and D not both zero.

8 possibilities × 4 cases involving A, B, B = D = 0, E × 3 cases

for E = 96 cases

---

case 7: $A > 0, C > 0, B > 0, D = 0, E$

$$\frac{x^2}{C} + \frac{B}{AC}x + \frac{y^2}{A} + \frac{E}{AC} = 0$$

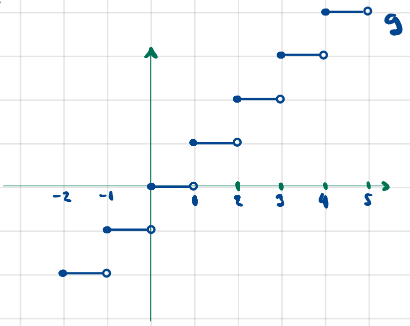
$$\frac{x^2}{C} + 2 \cdot \frac{x}{\sqrt{C}} \cdot \frac{B}{2A\sqrt{C}} + \left(\frac{B}{2A\sqrt{C}}\right)^2 - \left(\frac{B}{2A\sqrt{C}}\right)^2 + \frac{y^2}{A} + \frac{E}{AC}$$

$$\left(\frac{x}{\sqrt{C}} + \frac{B}{2A\sqrt{C}}\right)^2$$

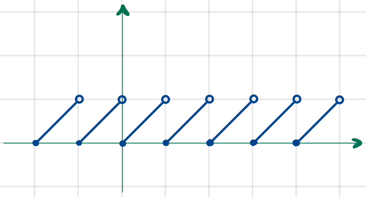
$Ax^2 + Bx + Cy^2 + Dy + E = 0$

17. $[x]$ = largest $\mathbb{Z} \leq x$

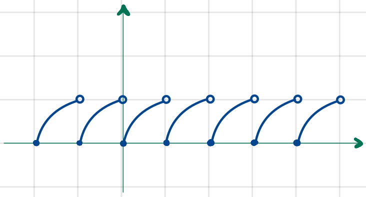
i) $f(x) = [x]$



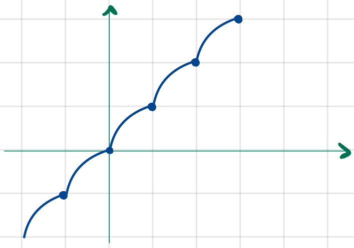
ii) $f(x) = x - [x]$



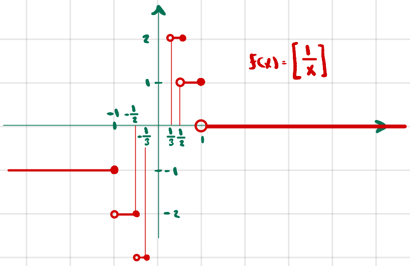
iii) $f(x) = \sqrt{x - [x]}$



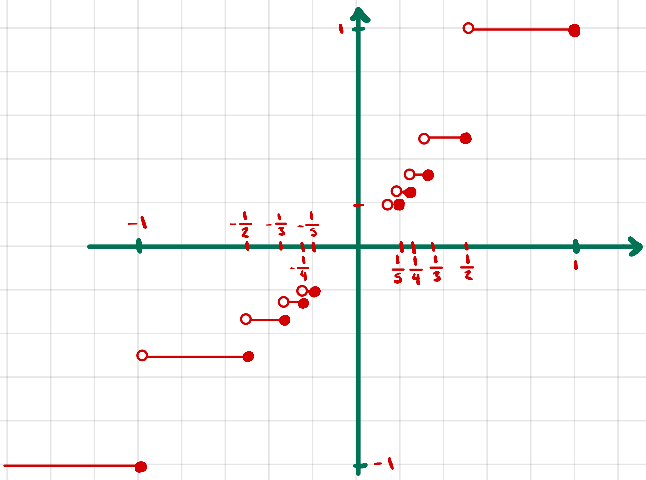
iv) $f(x) = [x] + \sqrt{x - [x]}$



v) $f(x) = \left[\frac{1}{x}\right]$

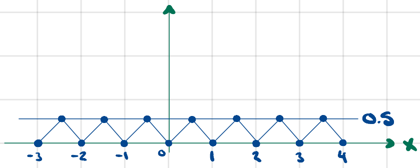


vi) $f(x) = \dfrac{1}{\left[\frac{1}{x}\right]}$

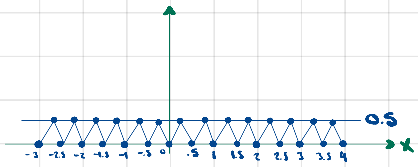


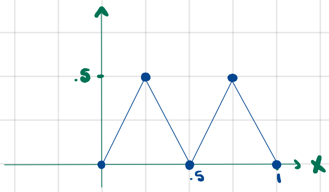
18. $\{x\}$ distance from x to nearest integer

i) $f(x) = \{x\}$

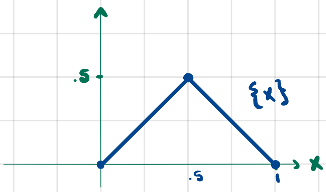


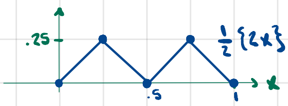
ii) $f(x) = \{2x\}$

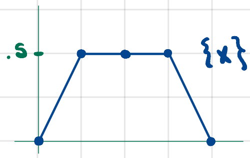


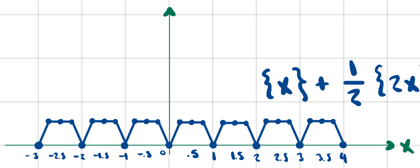


iii) $f(x) = \{x\} + \frac{1}{2}\{2x\}$

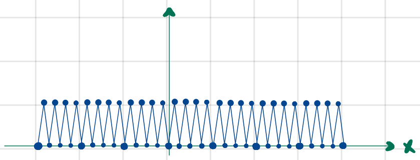


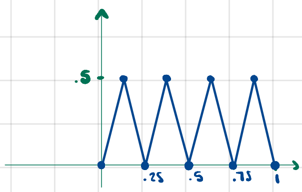




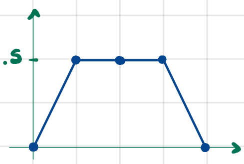


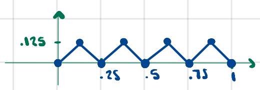
iv) $f(x) = \{4x\}$

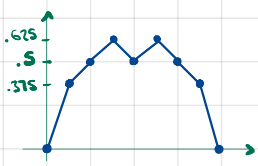




v) $f(x) = \{x\} + \frac{1}{2}\{2x\} + \frac{1}{4}\{4x\}$

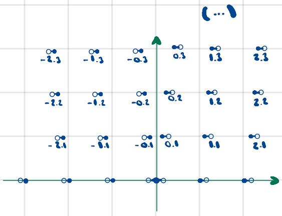


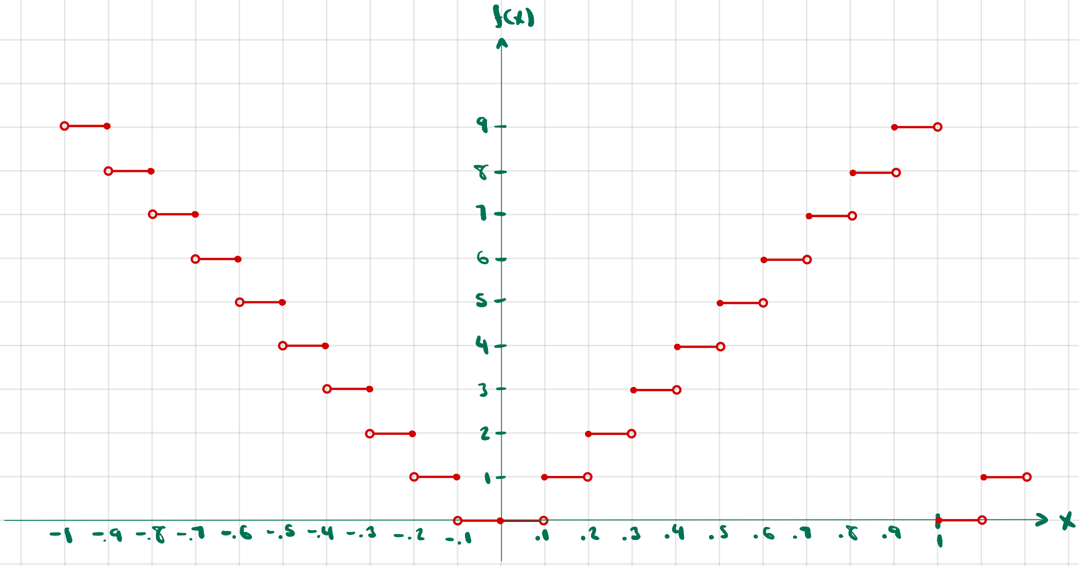




19.

i) $f(x)$ = 1st number of decimal expansion of x

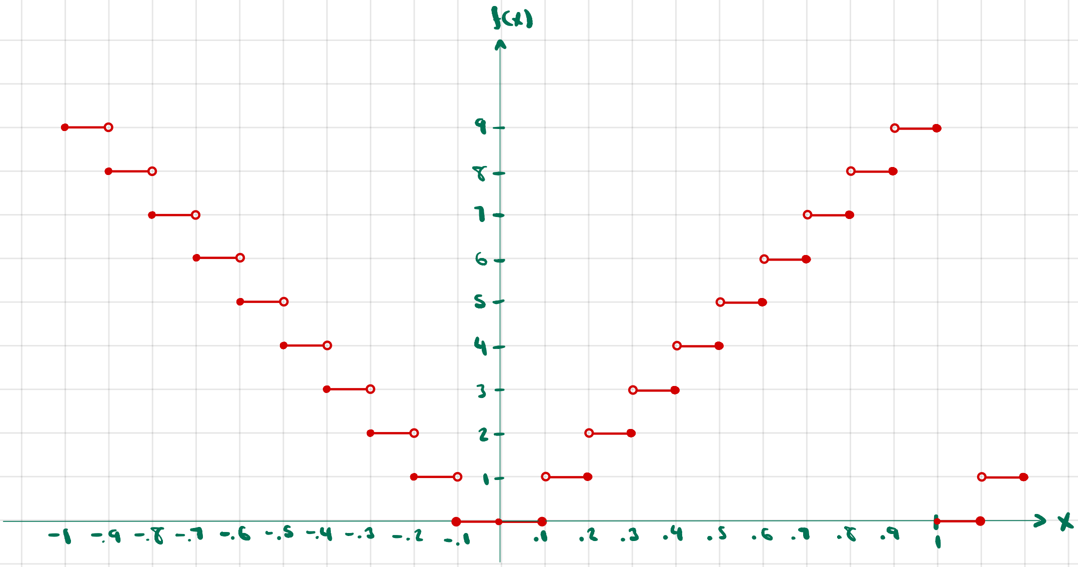




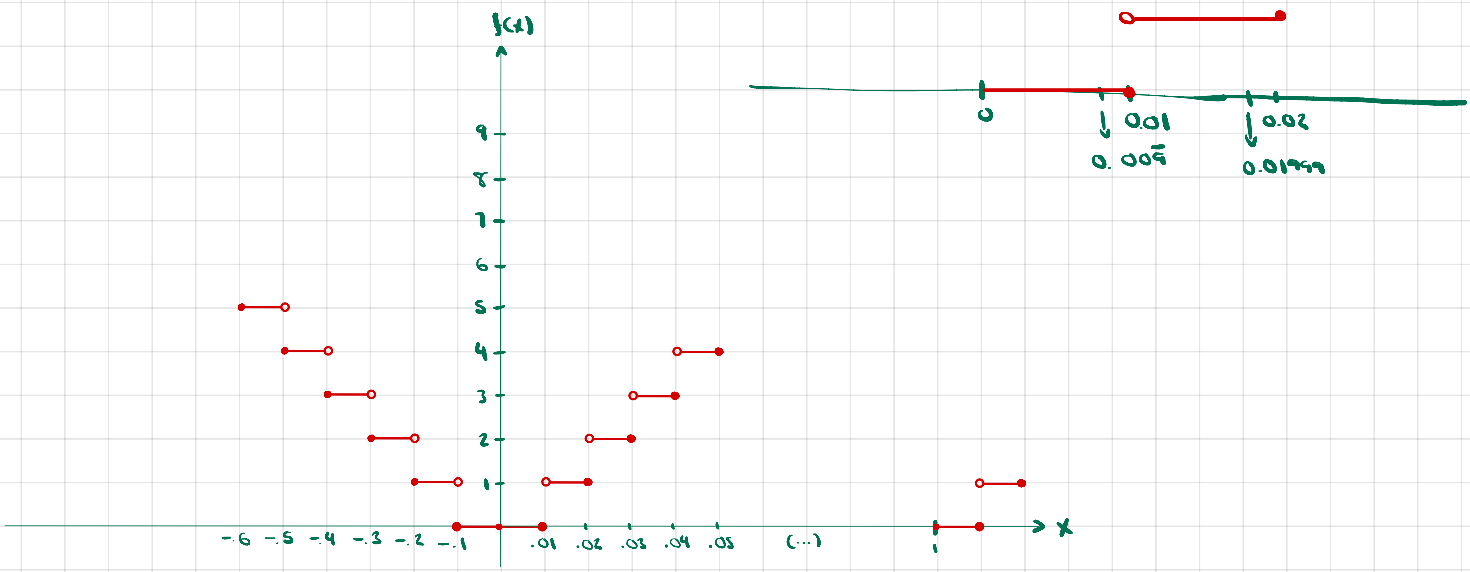
However, we are considering that, e.g., $0.0\bar{9} = 1$.

$\Rightarrow f(0.1) = f(0.0\bar{9}) = 0$

$f(0.2) = f(0.1\bar{9}) = 1$



ii) $f(x)$ = 2^{nd} number in decimal expansion of x



iii) $f(x)$ = numbers of 7's in decimal expansion of x if this number is finite, 0 otherwise

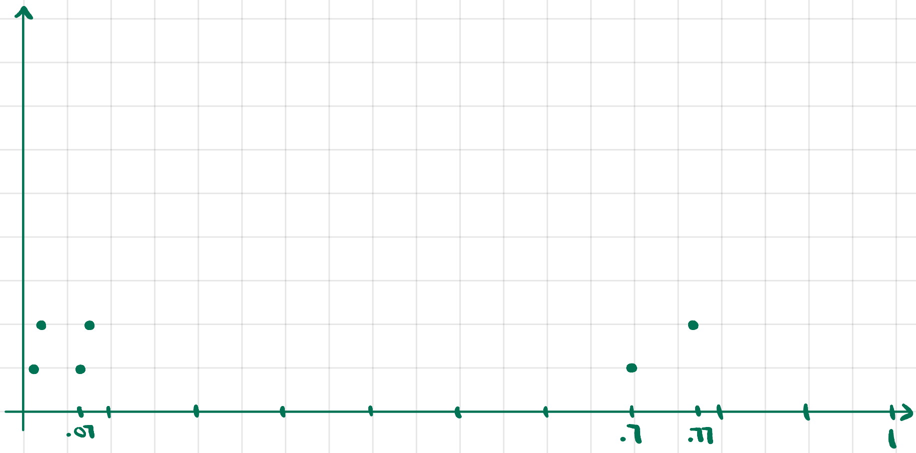
Infinite 7's

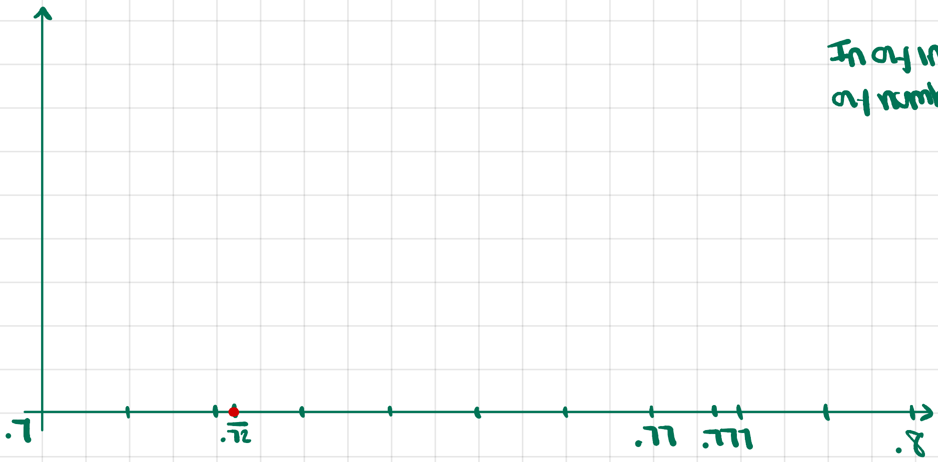
irrational numbers w/ infinite 7's

rational w/ repeating decimal repres. e.g. $0.\bar{7}$, $0.\overline{74}$, $0.\overline{347}$

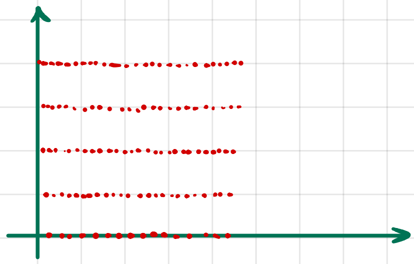
Finite 7's

0.7	0.77
0.07	0.077
0.007	0.0077





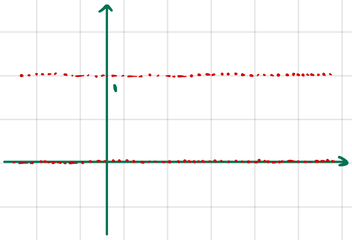
In any interval there are infinite numbers w/ any number of 7's, from zero to ∞.



$.\overline{72} \Rightarrow 10^2 \cdot .\overline{72} = 72 + .\overline{72} \Rightarrow 72 = .\overline{72}(10^2 - 1)$

$\Rightarrow .\overline{72} = \frac{72}{99}$

iv) $f(x)$: 0 if number of 7's in dec. exp. of x is finite, 1 otherwise.

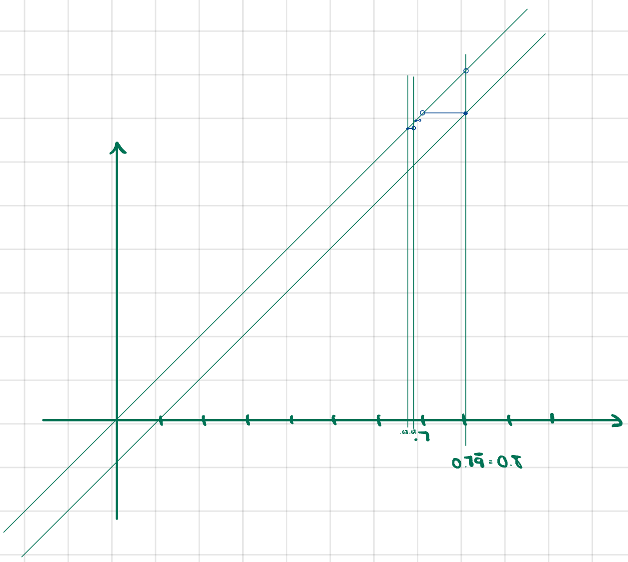


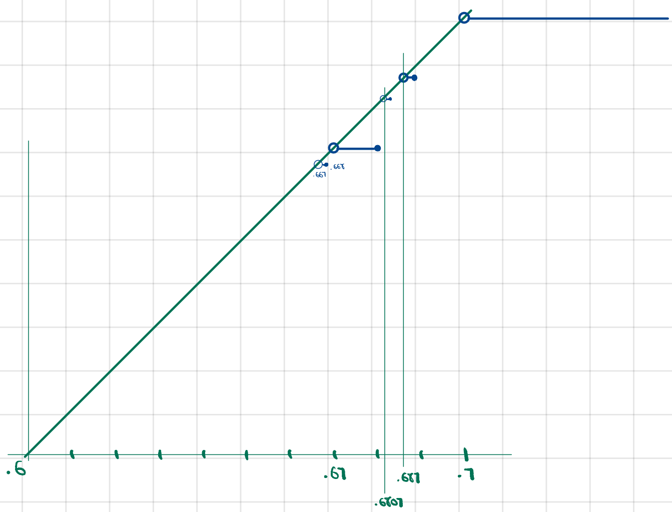
note: initially I read the problem incorrectly. I read that only one digit could be set to zero: the one after the first 7.
we remove at most 0.07 from numbers w/ a 7 in decimal expansion

v) $f(x)$ = number obtained by replacing all digits in decimal expansion of x which come after first 7, if any, by 0.

$f(0.77) = 0.7$
$f(0.07) = 0.07$
$f(0.\overline{72}) = 0.7$
$f(0.7\bar{9}) = 0.7$

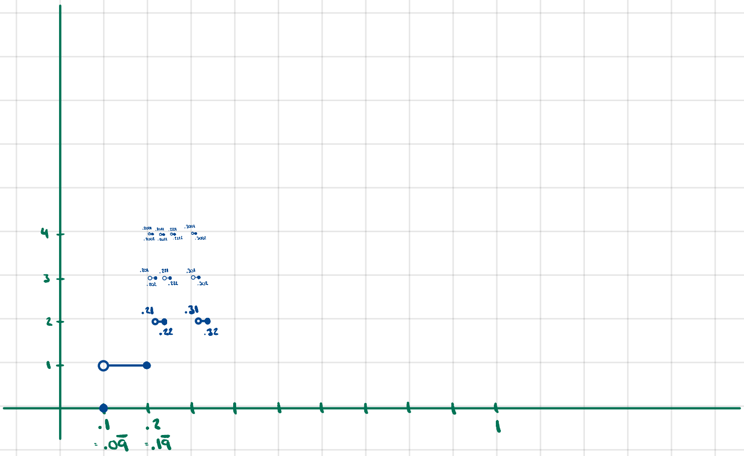
$0.0\bar{9}$ is max subtraction.

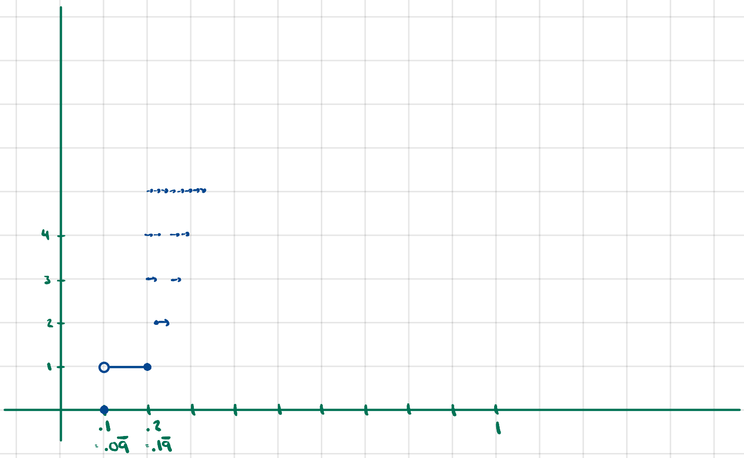




F is composed of infinite intervals like the ones above, with points on the line $y = x$ for numbers x with no 7s in them.

vi) $f(x) = \begin{cases} 0 & \text{if 1 never appears in decimal expansion of } x \\ n & \text{if 1 first appears in } n^{th} \text{ place} \end{cases}$





20.

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q}' \\ \dfrac{1}{q} & x = \dfrac{p}{q} \in \mathbb{Q} \text{ in lowest terms} \end{cases}$$

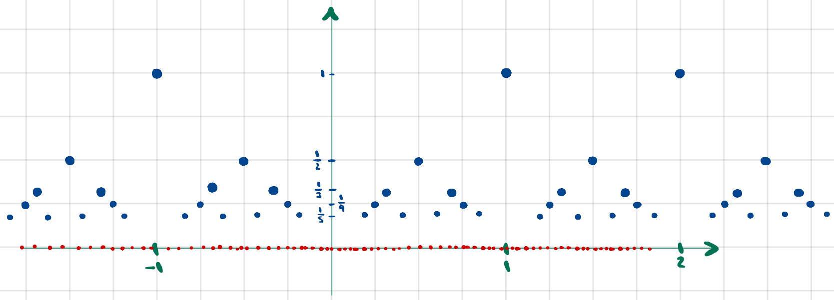
no common factor, $q > 0$

$q = 1 \Rightarrow x = p \in \mathbb{Z}$, $f(x) = 1$, $x = -1, 1, 2$

$q = 2 \Rightarrow x = \dfrac{p}{2}$, $p \in \mathbb{Z}$, $f(x) = \dfrac{1}{2}$, $x = -\dfrac{3}{2}, -\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{3}{2}$

$q = 3 \Rightarrow x = \dfrac{p}{3}$, $p \in \mathbb{Z}$, $f(x) = \dfrac{1}{3}$, $x = -\dfrac{5}{3}, -\dfrac{4}{3}, -\dfrac{2}{3}, -\dfrac{1}{3}, \dfrac{1}{3}, \dfrac{2}{3}, \dfrac{4}{3}$

$q = 4 \Rightarrow x = \dfrac{p}{4}$, $p \in \mathbb{Z}$, $f(x) = \dfrac{1}{4}$, $x = -\dfrac{5}{4}, -\dfrac{3}{4}, -\dfrac{1}{4}, \dfrac{1}{4}, \dfrac{3}{4}$



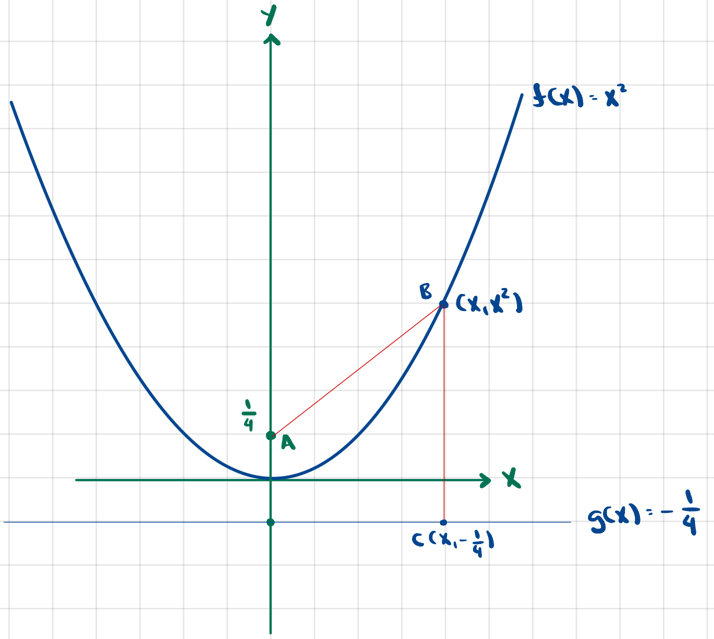
21. a) $f(x) = x^2$

$$AB = \left(x^2 + (x^2 - \tfrac{1}{4})^2\right)^{1/2} = \left(x^2 + x^4 - \frac{x^2}{2} + \frac{1}{16}\right)^{1/2}$$

$$= \left(x^4 + \frac{x^2}{2} + \frac{1}{16}\right)^{1/2}$$

$$BC = \left((x^2 + \tfrac{1}{4})^2\right)^{1/2} = \left(x^4 + \frac{x^2}{2} + \frac{1}{16}\right)^{1/2}$$

$\Rightarrow AB = BC$, ie any point (x, x^2) is equidistant from $(0, 1/4)$ and the line $g(x) = -\frac{1}{4}$.



b) consider a point $A = (x, y)$, equidistant from P and g.

$$AP = \sqrt{(x-\alpha)^2 + (y-\beta)^2}$$

$$Ag = \sqrt{(y-\gamma)^2}$$

$$AP = Ag \Rightarrow (x-\alpha)^2 + (y-\beta)^2 = (y-\gamma)^2$$

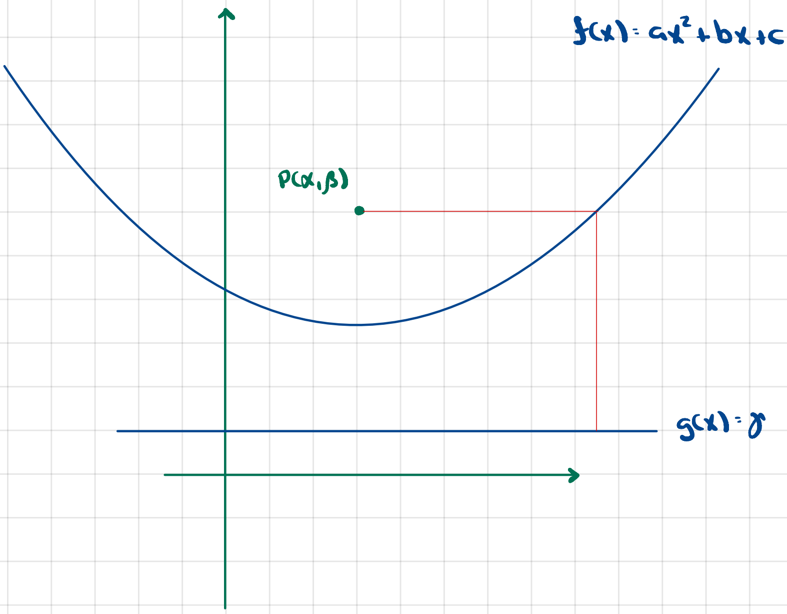
$$x^2 - 2x\alpha + \alpha^2 + \cancel{y^2} - 2y\beta + \beta^2 = \cancel{y^2} - 2y\gamma + \gamma^2$$

$$x^2 - 2\alpha x + \alpha^2 + \beta^2 - \gamma^2 = 2y\beta - 2y\gamma = 2y(\beta - \gamma)$$

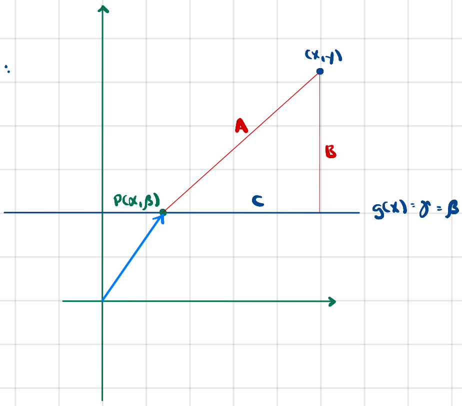
by assumption, $\beta \neq \gamma$

$$\Rightarrow y = \frac{1}{2(\beta-\alpha)} x^2 - \frac{2\alpha}{2(\beta-\alpha)} x + \frac{\alpha^2 + \beta^2 - \gamma^2}{2(\beta-\alpha)}$$

$$\Rightarrow y = f(x) = ax^2 + bx + c$$



* if $\beta = \gamma$ we would be looking for points (x, y) such that $A = B$ in the figure:



ie $\sqrt{(x-\alpha)^2 + (y-\beta)^2} = \sqrt{(y-\beta)^2}$

$\Rightarrow x = \alpha$, ie points on the y axis.

* note that by the Triangle inequality

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} \leq \sqrt{(y-\beta)^2} + \sqrt{(x-\alpha)^2}$$

$$A \leq B + C$$

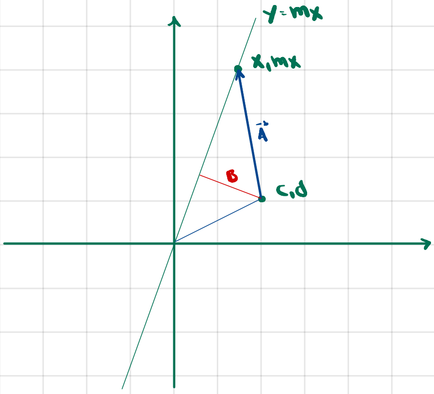
$\vec{A} = (x-\alpha, y-\beta)$
$\vec{B} = (0, y-\beta)$
$\vec{C} = (x-\alpha, 0)$

equality only if $\vec{B} = \lambda \vec{C}$ or one of $\vec{B}, \vec{C}$ is $\vec{0}$.

22. a) show square of distance from (c,d) to (x, mx) is

$x^2(m^2+1) + x(-2md-2c) + d^2 + c^2$

$\vec{A} = (x, mx) - (c, d)$
$= (x-c, mx-d)$



$\|\vec{A}\| = \sqrt{(x-c)^2 + (mx-d)^2}$

$f(x) = \|\vec{A}\|^2 = x^2 - 2xc + c^2 + m^2x^2 - 2mdx + d^2$
$= x^2(m^2+1) + x(-2md-2c) + d^2 + c^2$

2^{nd} degree polynomial in x, representing square of distance from point (c,d) to abitrary point on $f(x) = mx$.

$m^2+1 > 0 \Rightarrow$ parabola goes to $\pm\infty$ as $x \to \pm\infty$.

$$x_{min} = -\frac{(-2md-2c)}{2(m^2+1)} = \frac{md+c}{m^2+1}$$

$$f(x_{min}) = \left(\frac{md+c}{m^2+1}\right)^2 (m^2+1) - \frac{md+c}{m^2+1} \cdot 2(md+c) + d^2 + c^2 = \frac{(md+c)^2}{m^2+1} - 2\frac{(md+c)^2}{m^2+1} + d^2 + c^2$$

$$= -\frac{(md+c)^2}{m^2+1} + d^2 + c^2 = \frac{-(\cancel{m^2d^2} + 2mdc + \cancel{c^2}) + \cancel{m^2d^2} + m^2c^2 + d^2 + \cancel{c^2}}{m^2+1}$$

$$= \frac{m^2c^2 - 2mdc + d^2}{m^2+1} = \frac{(mc-d)^2}{m^2+1}$$

$$\Rightarrow \text{min distance} = \frac{|mc-d|}{\sqrt{m^2+1}}$$

Note

$\Delta = (-2md-2c)^2 - 4(m^2+1)(d^2+c^2)$
$= \cancel{4m^2d^2} + 8mdc + \cancel{4c^2} - \cancel{4m^2d^2} - 4m^2c^2 - 4d^2 - \cancel{4c^2}$

$= 8mdc - 4m^2c^2 - 4d^2$
$= -4(m^2c^2 - 2mcd + d^2)$
$= -4(mc-d)^2 \leq 0$

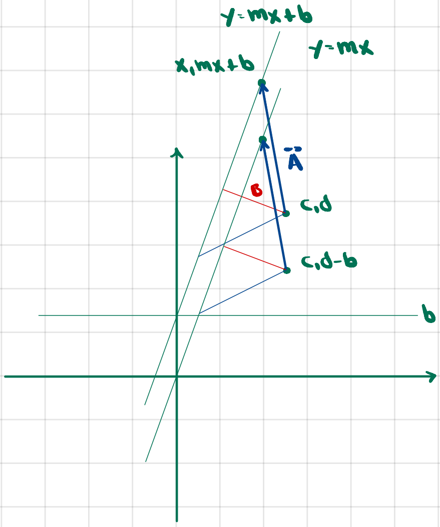
$\Rightarrow$ either zero roots or one root

Assume $m > 0$

$mc - d = 0 \Rightarrow mc = d \Rightarrow$ either $c = d = 0$ or
c and d have same sign when there is one root. This happens when $m = \frac{d}{c}$
$\Rightarrow f(x) = \frac{d}{c}x$, $f(c) = d$, (c,d) on f, so there is one point where the distance is zero.

$\Rightarrow (c,d)$ in quadrants II or IV then no roots no matter what.

b) distance from (c,d) to $f(x) = mx+b$



$$\vec{A} = (x, mx+b) - (c,d) = (x-c, mx+b-d) = (x-c, mx-(d-b))$$
$$= (x, mx) - (c, d-b)$$

$$\|\vec{A}\|^2 = x^2(m^2+1) + x(-2m(d-b) - 2c) + (d-b)^2 + c^2$$

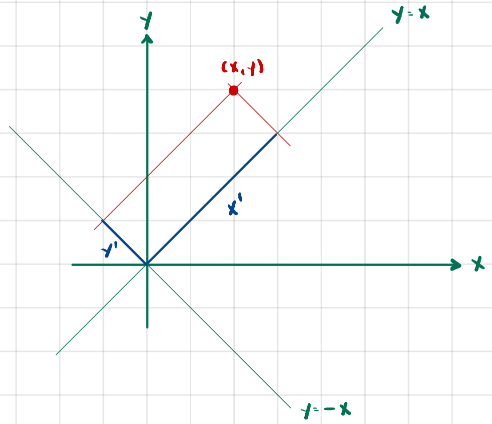
This expression is the same as in a), with d now being d-b.

Therefore by part a)

$$\min \|A\| = \frac{|mc-(d-b)|}{\sqrt{m^2+1}}$$

$$= \frac{|mc+b-d|}{\sqrt{m^2+1}}$$

23.



a)

distance from (x, y) to $f(x) = x$

$$y' = \frac{|xm - y|}{\sqrt{m^2+1}} = \frac{|x-y|}{\sqrt{2}} = \frac{y-x}{\sqrt{2}}$$ because $y > x$ in graph above

distance from (x, y) to $f(x) = -x$

$$x' = \frac{|-x-y|}{\sqrt{2}} = \frac{x+y}{\sqrt{2}}$$

b) show $\left\{(x,y) : \left(\frac{x'}{\sqrt{2}}\right)^2 - \left(\frac{y'}{\sqrt{2}}\right)^2 = 1\right\} = \left\{(x,y) : xy = 1\right\}$

$$\left(\frac{x'}{\sqrt{2}}\right)^2 = \frac{(x+y)^2}{4}$$

$$\left(\frac{y'}{\sqrt{2}}\right)^2 = \frac{(y-x)^2}{4}$$

$$\left(\frac{x'}{\sqrt{2}}\right)^2 - \left(\frac{y'}{\sqrt{2}}\right)^2 = \frac{(x+y)^2}{4} - \frac{(y-x)^2}{4} = \frac{\cancel{x^2} + 2xy + \cancel{y^2} - (\cancel{y^2} - 2xy + \cancel{x^2})}{4} = xy$$

$$\Rightarrow \left(\frac{x'}{\sqrt{2}}\right)^2 - \left(\frac{y'}{\sqrt{2}}\right)^2 = xy$$