examples

$f(x) = x^2$

$f(x+1) = x^2+2x+1 = f(x)+2x+1$

$g(y) = \dfrac{y^3+3y+5}{y^2+1}$

$h(c) = \dfrac{c^3+3c+5}{c^2-1} \quad c \neq \pm 1$

$g(x) = h(x) \Rightarrow (x^3+3x+5)(x^2-1) = (x^3+3x+5)(x^2+1)$

$x^2-1 \neq x^2+1 \Rightarrow -1 \neq 1$ always

$\Rightarrow x^3+3x+5 = 0$

$r(x) = x^2 \quad -17 \leq x \leq \pi/3$

$r(x+1) = r(x)+2x+1 \qquad -17 \leq x+1 \leq \pi/3$

$-17 \leq x \leq \pi/3 - 1$

$s(x) = \begin{cases} 0 & x \in Q^c \\ 1 & x \in Q \end{cases}$

$s(x+y), y \in Q$

$y \in Q \Rightarrow y = \dfrac{p}{q} \quad p,q \in Z$

$x \in Q \Rightarrow x = \dfrac{m}{n}$

$x+y = \dfrac{np+mq}{nq}$

$a,b \in Z$

Prove $a \cdot b \in Z$

strategy: Prove $1 \cdot b \in Z \ \forall b \in Z$, induction on B

Prove $a \cdot b \in Z$, induction on a

i) $A = \{n \in Z \mid 1 \cdot n \in Z\}$

$n=1 \Rightarrow 1 \cdot n = n \in Z \Rightarrow 1 \in A$

assume $k \in A \Rightarrow 1 \cdot k \in Z$

$(1+k) \cdot 1 = 1 + k \cdot 1 \in Z \Rightarrow 1+k \in A$

$n=-1 \Rightarrow 1 \cdot (-1) = -1 \in Z \Rightarrow -1 \in A$

$k \in A \Rightarrow 1 \cdot k \in Z$

$1 \cdot (k-1) = 1 \cdot k - 1 \in Z \Rightarrow k-1 \in A$

$\Rightarrow A = Z$

$A = \{n \in Z \mid n \cdot m \in Z, \forall m \in Z\}$

$n=1 \Rightarrow 1 \cdot m \in Z$ by (i) $\Rightarrow 1 \in A$

$k \in A \Rightarrow k \cdot m \in Z \ \forall m$

$(k+1)m = km + m \in Z$

backwards induction for $n<0 \Rightarrow A = Z$

ie $n \cdot m \in Z \ \forall m$ is true for $\forall n \in Z$

$\Rightarrow x+y = \dfrac{np+mq}{nq} \in Q$

$x \in Q'$

$y \in Q$

Assume $x+y \in Q$

$x + \dfrac{p}{q} = \dfrac{m}{n} \Rightarrow x = \dfrac{mq-pn}{nq} \in Q$

$\Rightarrow x+y \in Q'$

$\Rightarrow y \in Q$, $s(x+y) \in Q$ if $x \in Q$

$\in Q'$ if $x \in Q'$

$\Rightarrow s(x+y) = s(x)$

$\theta(x) = \begin{cases} 5 & x=2 \\ 36/\pi & x=17 \\ 28 & x=\pi^2/17, 36/\pi \\ 16 & x \neq 2, 17, \pi^2/17, 36/\pi \\ & x = a+b\sqrt{2} \quad a,b \in Q \end{cases}$

$\theta(\pi^2/17) = 28 = \theta(36/\pi)$

$\alpha_x(t) = t^3 + x \quad \forall t$

$\alpha_x(x) = x^3 + x = x(x^2+1) = x(f(x)+1)$

$y(x) = \begin{cases} n & n \text{ 7's in decimal exp.} \\ -\pi & \infty \text{ " " " "} \end{cases}$

$y(1/3) = y(0.33333\ldots) = 0$

$y(7/9) = -\pi$

$$\begin{array}{r} 0.77777\ldots \\ 9\,\overline{)\,7} \\ \underline{0} \\ 70 \\ \underline{63} \\ 70 \end{array} \qquad \begin{array}{r} 38.3 \\ 6\,\overline{)\,230} \\ \underline{18} \\ 50 \\ \underline{48} \\ 20 \end{array}$$

Proof: ∞ 7's in decim. exp. of $7/9$

A = set of positions with 7 in decimal exp.

1^{st} position represents how we can divide $70 \cdot 10^{-1}$
Result is 7 remainder 7
ie we can divide the $70 \cdot 10^{-1}$ into nine portions of $7 \cdot 10^{-1}$
and $7 \cdot 10^{-1}$ remain

2^{nd} position: how to divide the remainder $7 \cdot 10^{-1} = 70 \cdot 10^{-2}$

The division steps are

A = set of pos for which result is the whole part of $70/9 = 7$

$n = 1$, position 1 $\Rightarrow$ $9 \mid 7 = 7 \cdot 9 + 7 \Rightarrow$ remainder $7 = 70 \cdot 10^{-1}$

result is $\dfrac{70 \cdot 10^{-1}}{9 \cdot 10^{-1}} = \dfrac{70}{9}$. we need to divide the remainder

$70 \cdot 10^{-1}$ among 9, so each whole part is $9 \cdot 10^{-1}$

assume $h \in A \Rightarrow$ result is whole part of $70/9 = 7$.

the remainder of h^{th} pos is 7.

result of $(h+1)^{th}$ pos is the division $\dfrac{70 \cdot 10^{-(h+1)}}{9 \cdot 10^{-(h+1)}} = \dfrac{70}{9}$, whole part 7

$\Rightarrow A = \mathbb{N} \Rightarrow$ all positions are 7.

$f(s(a)) = \begin{cases} f(0) & a \in \mathbb{Q}^c \\ f(1) & a \in \mathbb{Q} \end{cases}$

$= \begin{cases} 0 & a \in \mathbb{Q}^c \\ 1 & a \in \mathbb{Q} \end{cases}$

$= s(a)$

Problems

1. $f(x) = \frac{1}{1+x}$ $\quad D_f = \{x : x \neq -1\}$

i) $f(f(x)) = \dfrac{1}{1 + \frac{1}{1+x}} = \dfrac{1}{\frac{2+x}{1+x}} = \dfrac{1+x}{2+x}$, $x \neq -2$

ii) $f\left(\frac{1}{x}\right) = \dfrac{1}{1 + \frac{1}{x}} = \dfrac{1}{\frac{x+1}{x}} = \dfrac{x}{1+x}$ $\quad x \neq -1, x \neq 0$

iii) $f(cx) = \dfrac{1}{1+cx}$ $\quad 1 + cx \neq 0 \Rightarrow x \neq \frac{-1}{c}$

iv) $f(x+y) = \dfrac{1}{1+x+y}$, $x + y \neq -1$

v) $f(x) + f(y) = \dfrac{1}{1+x} + \dfrac{1}{1+y} = \dfrac{2+x+y}{(1+x)(1+y)}$ $\quad x \neq -1, y \neq -1$

vi) $f(cx) = \dfrac{1}{1+cx} = \dfrac{1}{1+x}$

$c = 0 \quad \frac{1}{1} = \frac{1}{1+0} \quad x = 0$

$c = 1 \quad \frac{1}{1+x} = \frac{1}{1+x} \quad \forall x$

$c = 2 \quad \frac{1}{1+2x} = \frac{1}{1+x} \Rightarrow x = 2x \Rightarrow x = 0$

$c = n \quad \frac{1}{1+nx} = \frac{1}{1+x} \Rightarrow nx = x \Rightarrow x = 0$

$c \in \mathbb{R} \Rightarrow f(cx) = f(x)$ for $x = 0$

vii) $f(cx) = f(x)$ for two different x

$c = 1 \Rightarrow f(cx) = f(x) \ \forall x$

2. $g(x) = x^2$

$h(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{Q}^c \end{cases}$

i) $h(y) \leq y$

$y \in \mathbb{Q} \Rightarrow h(y) = 0 \leq y$ if y is positive rational, ie $\{y : y \in \mathbb{Q}, y \geq 0\}$

$y \in \mathbb{Q}^c \Rightarrow h(y) = 1 \leq y$ if y is irrational ≥ 1, ie $\{y : y \in \mathbb{Q}^c, y > 1\}$

ii) $h(y) \leq g(y)$

$y \in \mathbb{Q} \Rightarrow h(y) = 0 \leq y^2 \Rightarrow y \in \mathbb{R}$

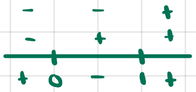
$y \in \mathbb{Q}^c \Rightarrow h(y) = 1 \leq y^2 \Rightarrow y > 1$ or $y < -1$ ie $|y| > 1$

$\Rightarrow h(y) \leq g(y)$ for $\{y : y \in \mathbb{Q}$ and $-1 \leq y \leq 1$, or $\forall y : |y| > 1\}$

iv) $g(w) \leq w$

$w^2 \leq w$

$w(w-1) \leq 0$



$\{w : 0 \leq w \leq 1\}$

v) $g(g(\epsilon)) = g(\epsilon)$

$g(\epsilon)^2 = (\epsilon^2)^2 = \epsilon^2 \Rightarrow \epsilon^4 = \epsilon^2$

$\epsilon^2(\epsilon^2 - 1) = 0$

$\epsilon = 0$ or $\epsilon^2 = 1 \Rightarrow \epsilon = \pm 1$

3. i) $f(x) = \sqrt{1-x^2}$

$1 - x^2 < 0 \Rightarrow f(x)$ isn't a real number

$x^2 > 1 \Rightarrow x > 1$ or $x < -1 \Rightarrow f(x)$ isn't defined

ii) $f(x) = \sqrt{1 - \sqrt{1-x^2}}$

$1 - x^2 \geq 0 \Rightarrow x^2 \leq 1 \Rightarrow -1 \leq x \leq 1$

$1 - \sqrt{1-x^2} \geq 0 \Rightarrow \sqrt{1-x^2} \leq 1 \Rightarrow 0 \leq 1 - x^2 \leq 1 \Rightarrow -1 \leq -x^2 \leq 0$

$1 \geq x^2 \geq 0$

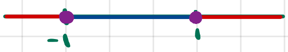
$-1 \leq x \leq 1$

Domain $= \{x : -1 \leq x \leq 1\}$

iii) $f(x) = \dfrac{1}{x-1} + \dfrac{1}{x-2}$ $\quad D = \{x : x \neq 1, x \neq 2\}$

iv) $f(x) = \sqrt{1-x^2} - \sqrt{x^2-1}$ $\quad 1 - x^2 \geq 0 \Rightarrow x^2 \leq 1 \Rightarrow -1 \leq x \leq 1$

$x^2 - 1 \geq 0 \Rightarrow x^2 \geq 1 \Rightarrow x \geq 1$ or $x \leq -1$



$D = \{-1, 1\}$

v) $f(x) = \sqrt{1-x} + \sqrt{x-2}$ $\quad 1 - x \geq 0 \Rightarrow x \leq 1$

$x - 2 \geq 0 \Rightarrow x \geq 2$

$D = \{\}$

4. $S(x) = x^2 \quad P(x) = 2^x \quad s(x) = \sin(x)$

i) $(S \circ P)(y) = S(P(y)) = S(2^y) = (2^y)^2 = 4^y$

ii) $(S \circ s)(y) = S(s(y)) = S(\sin y) = \sin^2(y)$

iii) $(S \circ P \circ s)(t) + (s \circ P)(t) = S(P(s(t))) + s(P(t)) = S(P(\sin t)) + s(2^t)$

$= S(2^{\sin t}) + \sin(2^t) = (2^{\sin t})^2 + \sin(2^t) = 4^{\sin t} + \sin(2^t)$

iv) $s(t^3) = \sin(t^3)$

5. $S(x) = x^2 \qquad P(x) = 2^x \qquad s(x) = \sin(x)$

i) $f(x) = 2^{\sin x} = P \circ s$

ii) $f(x) = \sin(2^x) = s \circ P$

iii) $f(x) = \sin(x^2) = s \circ S$

iv) $f(x) = \sin^2 x = S \circ s$

v) $f(t) = 2^{2^t} = P \circ P$

vi) $f(u) = \sin(2^u + 2^{u^2}) = S \circ (P + P \circ S)$

vii) $f(y) = \sin(\sin(\sin(2^{2^{2^{\sin y}}}))) = S \circ S \circ S \circ P \circ P \circ P \circ S$

viii) $f(a) = 2^{\sin^2 a} + \sin(a^2) + 2^{\sin(a^2 + \sin a)}$

$= P \circ S \circ s + s \circ S + P \circ s \circ (S + s)$

6. a) $x_1, \dots, x_n$ distinct numbers

Find f_i polynomial of degree $n-1$

1 at x_i

0 at x_j $\quad i \neq j$

x_1, x_2

$$f_1 = \frac{x - x_2}{x_1 - x_2} = \frac{1}{x_1 - x_2} x - \frac{x_2}{x_1 - x_2}$$

$$f_2 = \frac{x - x_1}{x_2 - x_1}$$

$$f_i = \frac{x - x_j}{x_i - x_j} \Rightarrow \begin{matrix} f(x_i) = 1 \\ f(x_j) = 0 \end{matrix}$$

x_1, x_2, x_3

$$f_1 = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} = \frac{x^2 - x(x_3 + x_2) + x_2 x_3}{(x_1 - x_2)(x_1 - x_3)}$$

$$= \frac{1}{\prod_{j \neq 1}(x_1 - x_j)} x^2 - \frac{(x_2 + x_3)}{\prod_{j \neq 1}(x_1 - x_j)} x + \frac{x_2 x_3}{\prod_{j \neq 1}(x_1 - x_j)}$$

$f_1(x_2) = f_1(x_3) = 0$

$f_1(x_1) = 1$

$x_1, \dots, x_n$

$$f_i = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}$$

b) $a_1, \dots, a_n$ given

find f degree $n-1$, $f(x_i) = a_i$

$$f_i = \frac{a_i \prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}$$

now we want a single function for which we choose the values $f(x_i)$, ie $f(x_i) = a_i$.

f_i is a_i on one input and zero on all others.

The sum of f_i fns will be a_i at each x_i.

$$f = \sum_{i=1}^{n} f_i = \sum_{i=1}^{n} \left[\frac{a_i \prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)} \right]$$

7) a) Prove

$\forall$ polynomial f, $\forall a \in \mathbb{R}$ $\Rightarrow$ $\exists$ polyn. fn g and $b \in \mathbb{R}$ s.t. $f(x) = (x-a)g(x)+b \ \forall x$

example:

$f(x) = x^3-3x+1$

$x-a = x-1$

$$g(x) = \frac{f(x)}{x-a} = \frac{x^3-3x+1}{x-1}$$

$$\begin{array}{r|l} & x^2+x-2 \\ \hline x-1 & x^3 \quad -3x+1 \\ & x^3-x^2 \\ \hline & x^2-3x+1 \\ & x^2-x \\ \hline & -2x+1 \\ & -2x+2 \\ \hline & -1 \end{array}$$

$(x^2+x-2)(x-1)-1 = f(x)$

$= g(x)(x-a)+b$

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = \sum_{i=0}^{n} a_i x^i$$

example

$f(x) = a_2x^2 + a_1x + a_0$

$$\frac{f(x)}{x-a} \Rightarrow \begin{array}{r|l} & a_2x+(a_1+a) \\ \hline x-a & a_2x^2 + a_1x + a_0 \\ & a_2x^2 - ax \\ \hline & x(a_1+a)-a_0 \\ & x(a_1+a)-a(a_1+a) \\ \hline & a_0+a(a_1+a) \end{array}$$

$f(x) = (x-a)(a^2x+a_1+a) + (a_0+a(a_1+a))$

$= (x-a)g(x)+b$

general case

$A = \{n : f_n(x) = \sum_{i=0}^{n} a_i x^i$ can be written as $(x-a)g(x)+b\}$

$n=1 \Rightarrow f_1(x) = a_1x+a_0 = (x-a)g(x)+b$

$$\begin{array}{r|l} & a_1 \\ \hline x-a & a_1x+a_0 \\ & a_1x-aa_1 \\ \hline & a_0+aa_1 \end{array}$$

$g(x)x - ag(x) + b$

$g(x) = a_1$

$(x-a)a_1 + b$

$a_1x - a_1a + b$

$-a_1a + b = a_0$

$b = a_0 - a_1a$

$\Rightarrow f_1(x) = (x-a)a_1 + a_0 + aa_1$

$b = a_0 + aa_1$

$g(x) = a_1$

$\Rightarrow 1 \in A$

Assume $k \in A \Rightarrow f_k(x) = \sum_{i=0}^{k} a_i x^i = (x-a)g(x)+b$

Let $h(x) = f_k(x) + a_{k+1}x^{k+1} = \sum_{i=0}^{k+1} a_i x^i$

$= (x-a)g(x) + b + a_{k+1}x^{k+1}$

$a_{k+1}x^{k+1} = x \cdot a_{k+1}x^k = x[(x-a)\ell(x)+b_1]$

because $m(x) = a_{k+1}x^k$ is a k^{th} order polynomial.

$= (x-a)\ell(x)x + b_1x$

$\Rightarrow h(x) = (x-a)(g(x)+\ell(x)x) + b + b_1x$

But $b+b_1x$ is a first degree polynomial

$b+b_1x = (x-a)n(x) + b_2 = n(x)x - an(x) + b_2$

$\Rightarrow n(x) = b_1$

$\Rightarrow b+b_1x = (x-a)b_1 + b + ab_1$

$b+b_1x = (x-a)m(x) + b + ab_1$

$\Rightarrow h(x) = (x-a)(g(x)+\ell(x)x+b_1) + b + ab_1$

call $g_{k+1}(x) = g(x)+\ell(x)x+b_1$

$\Rightarrow h(x) = \sum_{i=1}^{n} a_i x^i = (x-a)g_{k+1}(x) + b + ab_1$

$\Rightarrow 1+k \in A$

$\Rightarrow A = \mathbb{N}$

Alternative proof of 1st induction step

we have

Assume $k \in A \Rightarrow f_k(x) = \sum_{i=0}^{k} a_i x^i = (x-a)g(x) + b$

$f_{k+1}(x) = f_k(x) + a_{k+1}(x-a)^{k+1}$ is a $(k+1)^{th}$ degree polynomial.

$f_{k+1}(x) - a_{k+1}(x-a)^{k+1} = f_k(x)$ is k^{th} degree so

$f_{k+1}(x) - a_{k+1}(x-a)^{k+1} = (x-a)g(x) + b$

$f_{k+1}(x) = (x-a)(g(x) + a_{k+1}(x-a)^{k}) + b$

$= (x-a)g_1(x) + b$

$g_1(x) = g(x) + a_{k+1}(x-a)^k$

$\Rightarrow 1+k \in A$

$\Rightarrow A = \mathbb{N}$

---

b) Prove $f(a) = 0 \Rightarrow f(x) = (x-a)g(x)$ for some polyn. $g(x)$

$n = 1 \Rightarrow f(x) = a_1 x + a_0$

$f(a) = a_1 a + a_0 = 0$

$\Rightarrow a_0 = -a_1 a$

$\Rightarrow f(x) = a_1 x - a_1 a = a_1(x-a)$

assume $k \in A$

$\Rightarrow f_k = \sum_{i=0}^{k} a_i x^i$

$f_k(a) = \sum_{i=0}^{k} a_i a^i = 0 \Rightarrow f(x) = (x-a)g(x)$

$f_{k+1}(x) = f_k(x) + a_{k+1}(x-a)^{k+1}$

$= (x-a)g(x) + a_{k+1}(x-a)^{k+1}$

$= (x-a)(g(x) + a_{k+1}(x-a)^k)$

$h(x) = g(x) + a_{k+1}(x-a)^k$ is k^{th} degree polyn.

$\Rightarrow k+1 \in A$

$\Rightarrow A = \mathbb{N}$

Alternatively, from a) we have

$f(x) = (x-a)g(x) + b \Rightarrow 0 = f(a) = b$

$\Rightarrow f(x) = (x-a)g(x)$

Note that

$a_{k+1}(x-a)^{k+1}$

$= a_{k+1} \sum_{i=0}^{k+1} \binom{k+1}{i} x^{k+1-i} a^i$

so $f(x) + a_{k+1}(x-a)^{k+1}$

$= \sum_{i=0}^{k} a_i x^i + a_{k+1} \sum_{i=0}^{k+1} \binom{k+1}{i} x^{k+1-i} a^i$

$= \sum_{i=0}^{k} x^i a_i a_{k+1} a^{k+1-i} + a_{k+1} x^{k+1} = \sum_{i=0}^{k+1} \alpha_i x^i$, a general k^{th} degree polynomial.

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c) Prove f n degree polyn. $\Rightarrow$ f has at most n roots, ie numbers a with $f(a) = 0$

Assume $f(a_1) = 0$

$\Rightarrow f(x) = (x-a_1)g_1(x)$ for some polynomial $g_1(x)$.

Note that $\deg(f(x)) = \deg(g_1(x) \cdot x)$ because

$g_1(x) = \sum_{i=0}^{d} a_i x^i$ has degree d

$x \cdot g_1(x) = \sum_{i=0}^{d} a_i x^{i+1}$ has degree $d+1$, $d+1 = n \Rightarrow d = n-1$.

$\Rightarrow \deg(g_1(x)) = n-1$

If $g_1(x)$ has a root a_2 then $g_1(a_2) = (x-a_2)g_2(x)$ where

$g_2(x)$ is a degree $(n-2)$ polyn.

$\Rightarrow f(x) = (x-a_1)(x-a_2)g_2(x)$

We can do this procedure of decomposing $g_i(x)$ n times.

After the n^{th} time, $f(x) = (x-a_1) \cdot (x-a_2) \cdot \dots \cdot (x-a_n) \cdot c$

$f(a) \neq 0$ for any $a \neq a_1, \dots a_n \Rightarrow f$ does not have more

Note that it is possible that one of the $g_i(x)$ has no roots.

$f(x) = (x-a_1)(x-a_2) \cdots g_i(x)$ in that case.

d) show: for each n ⇒ ∃ polyn of degree n with n roots

$f(x) = \prod_{i=1}^{n} (x - a_i)$, n^{th} degree polynomial.

⇒ $a_1, \dots, a_n$ are roots

assume n even.

$f(x) = x^n + a_0, \quad a_0 > 0$

⇒ $f(x) > 0 \; \forall x$ because $x^n > 0$
⇒ $x^n + a_0 = 0 \Rightarrow x^n = -a_0$, impossible.
⇒ $x^n + a_0 \neq 0$.

assume n odd

$f(x) = x^n + a_0$
$x^n + a_0 = 0 \Rightarrow x = \sqrt[n]{-a_0}$

γ) $f(x) = \frac{ax+b}{cx+d}$

For what a,b,c,d does f(x) satisfy

$f(f(x)) = x \quad \forall x$

$$f(f(x)) = \frac{\frac{a(ax+b)}{cx+d} + b}{\frac{c(ax+b)}{cx+d} + d} = \frac{a(ax+b) + b(cx+d)}{c(ax+b) + d(cx+d)} = x$$

$cx+d \neq 0$

$\frac{c(ax+b)}{cx+d} + d \neq 0 \Rightarrow c(ax+b) + d(cx+d) \neq 0$

⇒ $a^2x + ab + bcx + bd = cax^2 + cbx + dcx^2 + d^2x$

$x^2(ca + dc) + x(cb + d^2 - cb - a^2) - ab - bd = 0$

$x^2(ac + cd) + x(d^2 - a^2) - (ab + bd) = 0$

$ac + cd = 0$
$d^2 - a^2 = 0$
$ab + bd = 0$

$d^2 = a^2 \Rightarrow d = \pm a$

$ac + cd = 0$
$d^2 - a^2 = 0$
$ab + bd = 0$

case 1: $d = a$

⇒ $2ac = 0$
$2ab = 0$

case 1.1 $a = d = 0$

⇒ $f(x) = \frac{b}{cx}, \quad c \neq 0$

case 1.2 $a \neq 0$, $c = b = 0$

⇒ $f(x) = \frac{ax}{d} = x$

case 2 $d = -a$

⇒ $f(x) = \frac{ax+b}{cx-a}$, $cx - a \neq 0 \Rightarrow x \neq \frac{a}{c}$

Note that for f(f(x)) to make sense we need

$f(x) \neq \frac{a}{c} \Rightarrow \frac{ax+b}{cx-a} \neq \frac{a}{c} \Rightarrow acx + cb \neq acx - a^2$

⇒ $cb \neq -a^2$

⇒ $a^2 + cb \neq 0$

9. a) A any set of real numbers

$A = \{x : x \in \mathbb{R}\}$

$$C_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Find $C_{A\cap B}$, $C_{A\cup B}$, $C_{\mathbb{R}-A}$

$$C_{A\cap B} = \begin{cases} 1 & x \in A\cap B \\ 0 & x \notin A\cap B \end{cases}$$

$x \in A\cap B \Rightarrow x \in A$ and $x \in B$

$\Rightarrow C_A(x) = C_A(B) = 1$

$\Rightarrow C_A(x)\cdot C_B(x) = 1$

$x \notin A\cap B \Rightarrow x \notin A$ or $x \notin B$ or $(x \notin A$ and $x \notin B)$

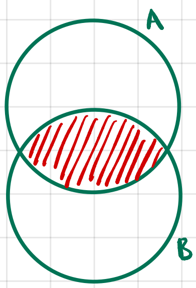
$x \notin A\cup B$

$\Rightarrow C_A(x) = 0$ or $C_B(x) = 0$ or $C_{A\cup B}(x) = 0$

$\Rightarrow C_A(x)\cdot C_B(x) = 0$

$\Rightarrow C_{A\cap B} = C_A(x)\,C_B(x)$

	$x\in B$	$x\notin B$
$x \in A$	1	0
$x \notin A$	0	0



$$C_{A\cup B} = \begin{cases} 1 & x \in A\cup B \\ 0 & x \notin A\cup B \end{cases}$$

$x \in A\cup B$: $x \in A$ or $x \in B$

$\Rightarrow C_A(x) = 1$ or $C_B(x) = 1$

$C_A(x) + C_B(x) - C_A(x)C_B(x) = 1$

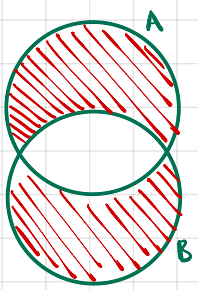
$x \notin A\cup B \Rightarrow x \notin A$ and $x \notin B$

$C_A(x) = 0$ and $C_B(x) = 0$

$\Rightarrow C_A(x) + C_B(x) - C_A(x)C_B(x) = 0$

$C_{A\cup B}(x) = C_A(x) + C_B(x) - C_A(x)C_B(x)$

	$x\in B$	$x\notin B$
$x \in A$	1	1
$x \notin A$	1	0



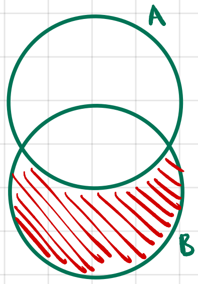
$C_{\mathbb{R}-A}(x) = 1 \Rightarrow x \notin A \Rightarrow C_A(x) = 0$

$\Rightarrow 1 - C_A(x) = 1 = C_{\mathbb{R}-A}(x)$

Extra

$B - A = \{x : x \in B, x \notin A\}$

$$C_{B-A}(x) = \begin{cases} 1 & x \in B-A \\ 0 & x \notin B-A \end{cases}$$



$x \in B-A$

$\Rightarrow x \in B$ and $x \notin A$

$\Rightarrow C_B(x) = 1$ and $C_A(x) = 0$

$x \notin B-A$

$\Rightarrow x \in A$ or $x \notin B$

$\Rightarrow C_A(x) = 1$ or $C_B(x) = 0$

$x \notin A$

$\Rightarrow C_A(x) = 0$

$\Rightarrow 1 - C_A(x) = 1$

$= C_{-A}(x)$

$C_{B-A} = C_B(x)(1 - C_A(x))$

	$x\in B$	$x\notin B$
$x \in A$	0	0
$x \notin A$	1	0

---

b) $f(x) = 0$ or 1 for each x.

Define $A = \{x \in \mathbb{R} : f(x) = 1\}$

$$\Rightarrow f(x) = C_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

c) show $f = f^2 \iff f = C_A$ for some set A

$\Rightarrow$ $f(f-1) = 0 \Rightarrow f = 1$ or $f = 0$

ie $f(x)$ can only be one of two values given any x

$\Rightarrow$ from b) we know $f = C_A$.

$$\Leftarrow \quad f = C_A = \begin{cases} 1 & x \text{ in } A \\ 0 & x \text{ in } B \end{cases}$$

$\Rightarrow f = 1 = f^2$ if $x \in A$

$f = 0 = f^2$ if $x \notin A$

$\Rightarrow f = f^2$

10. a) $f = g^2$

If f and g were real numbers, then $f \geq 0$ because $g \cdot g = g^2 \geq 0$.

$g(x)^2$ is also $\geq 0 \ \forall x$.
$\Rightarrow f(x) \geq 0 \ \forall x$

b) $f = \frac{1}{g}$

$\Rightarrow g(x) \neq 0 \ \forall x$
no matter what value $g(x)$ takes, $1/g \neq 0$
$\Rightarrow f(x) \neq 0 \ \forall x$

c) $x(t)^2 + b(t)x(t) + c(t) = 0 \ \forall t$

$b(t) - 4c(t) = 0 \ \forall t \Rightarrow$ there is one solution $x(t) = \frac{-b(t)}{2}$, for $\forall t$

d) $a(t)x(t) + b(t) = 0 \ \forall t$

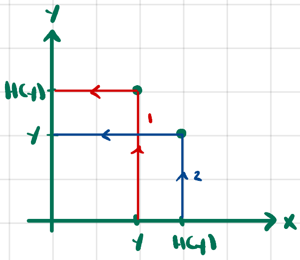
$a(t)x(t) = -b(t)$

$a(t) = 0 \Rightarrow b(t) = 0$

$a(t) \neq 0 \Rightarrow x(t) = -\frac{b(t)}{a(t)}$

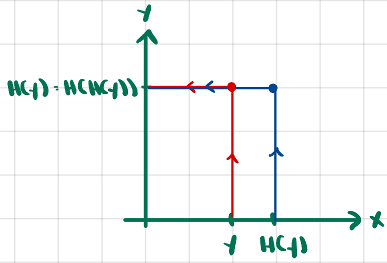
If $a(t) \neq 0 \ \forall t$ then given $a(t)$ and $b(t)$ there is a single fn $x(t) = -b(t)/a(t)$ that solves the initial eq.
If $a(t) = 0$ at some t, then for that value of t, $x(t)$ is arbitrary. Therefore, given such $a(t)$, and a $b(t)$, there are infinite $x(t)$ fns that are solutions.

11. a) $H(H(y)) = y$



$y \xrightarrow{} H(y) \xrightarrow{1} H(H(y))$
$= y$

$\xrightarrow{2} H(H(H(y))) \xrightarrow{3} H(H(H(H(y))))$
$= H(y) \qquad = y$



$\underbrace{H(H(H(\cdots H(y))\cdots))}_{80 \text{ times}} = H(y)$

b) $\underbrace{H(H(H(\cdots H(y))\cdots))}_{81 \text{ times}} = y$

c) $H(H(y)) = H(y)$

$H(H(H(y))) = H(H(y)) = H(y)$
we can compose as many times as we want, the result is $H(y)$

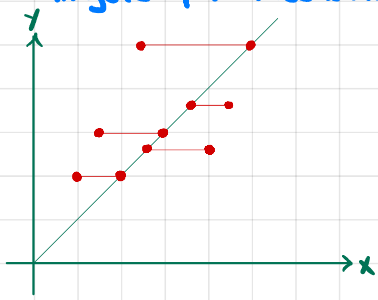
$A = \{n : H \circ \cdots \circ H = H, \ n \text{ times composed}\}$

$n = 1 \Rightarrow H \circ H = H(H(y)) = H(y)$ by initial assumption; $1 \in A$
assume $h \in A \Rightarrow H \circ \cdots \circ H = H$, h times composed.
$(\underbrace{H \circ \cdots \circ H}_{h \text{ times}}) \circ H = \underbrace{H \circ \cdots \circ H}_{h+1 \text{ times}} = H \circ H = H(y)$

$h+1 \in A$, $A = \mathbb{N}$

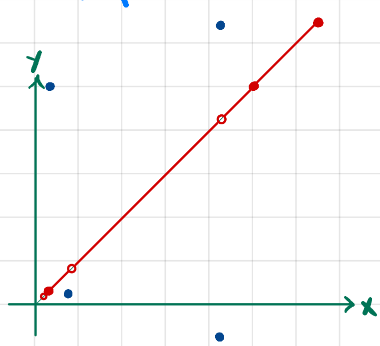
d) $H(H(x)) = H(x) \ \forall x$
$H(1) = 36$
$H(2) = \pi/3$
$H(13) = 47$
$H(36) = 36$
$H(\pi/3) = \pi/3$
$H(47) = 47$

in general, $h(x)$ looks like

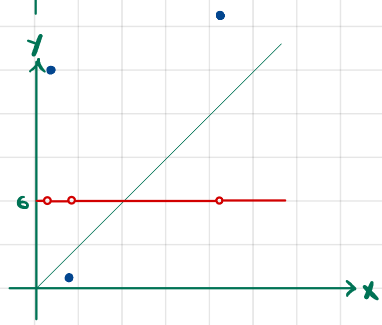


These specific cases fit item d):

$$H(x) = \begin{cases} 36 & x = 1 \\ \pi/3 & x = 2 \\ 47 & x = 13 \\ x & x \neq 1, 2, 13 \end{cases}$$



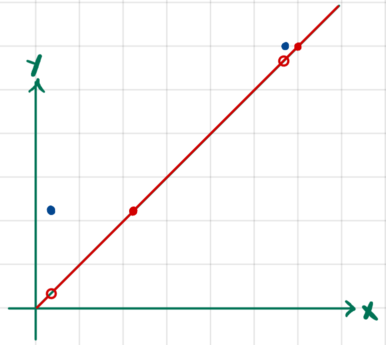
$$H(x) = \begin{cases} 36 & x = 1 \\ \pi/3 & x = 2 \\ 47 & x = 13 \\ 6 & x \neq 1, 2, 13 \end{cases}$$



e) $H(H(x)) = H(x) \ \forall x$

$H(1) = 7$

$H(17) = 18$



12. a) $f+g$

f even, g even $\Rightarrow (f+g)(x) = f(x)+g(x)$
$= f(-x)+g(-x)$
$= (f+g)(-x)$
$\Rightarrow f+g$ even

---

f even, g odd $\Rightarrow (f+g)(x) = f(-x) - g(-x)$
$= (f-g)(-x)$
$\Rightarrow$ not odd or even

---

by symmetry, f odd g even $\Rightarrow (f+g)(x) = (g-f)(-x)$
$\Rightarrow$ not odd or even

---

f odd, g odd $\Rightarrow (f+g)(x) = -f(-x) - g(-x)$
$= -(f(-x)+g(-x))$
$= -(f+g)(-x)$
$\Rightarrow f+g$ odd

$f+g$	g: even	g: odd
f: even	even	neither
f: odd	neither	odd

b) $f \cdot g$

both even $\Rightarrow (f\cdot g)(x) = f(-x)\cdot g(-x)$
$\Rightarrow$ even

both odd $\Rightarrow (f\cdot g)(x) = (-f(-x))(-g(-x))$
$= f(-x)g(-x)$
$\Rightarrow$ even

even, odd $\Rightarrow (f\cdot g)(x) = f(-x)\cdot(-g(-x))$
$= -f(-x)g(-x)$
$\Rightarrow$ odd

$f \cdot g$	g: even	g: odd
f: even	even	odd
f: odd	odd	even

c) $f \circ g$

both even $\Rightarrow (f\circ g)(x) = f(g(x))$
$= f(g(-x))$
$= (f\circ g)(-x)$
$\Rightarrow f\circ g$ even

both odd $\Rightarrow (f\circ g)(x) = f(g(x))$
$= f(-g(-x))$
$= -f(g(-x))$
$= -(f\circ g)(x)$
$\Rightarrow f\circ g$ odd

even, odd $\Rightarrow (f\circ g)(x) = f(g(x))$
$= f(-g(-x))$
$= f(g(-x))$
$= (f\circ g)(-x)$
$\Rightarrow f\circ g$ even

odd, even $\Rightarrow (f\circ g)(x)$
$= f(g(-x))$
$= (f\circ g)(x)$
$\Rightarrow f\circ g$ even

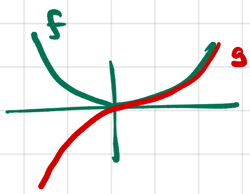
---

d) Prove every fn f can be written $f(x) = g(|x|)$ for infinitely many fns g.

$f(x) = -f(x)$

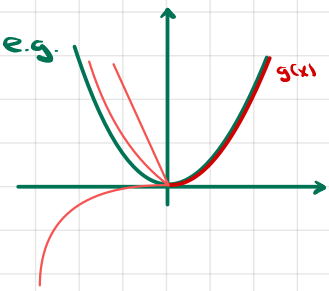
consider $g(x) = \begin{cases} f(x) & x \geq 0 \\ \text{any fn} & x < 0 \end{cases}$

$\Rightarrow g(|x|) = f(x) \ \forall x$



$\Rightarrow$ Because we can choose any fn for $g(x)$ with $x<0$, there are infinitely many fns $g(x)$ such that $f(x) = g(|x|)$.

e.g.



13. a) Prove any fn f with domain $\mathbb{R}$ can be written $f = E + O$, where E even, O odd.

Solution 1

$f - E$ odd means

$(f-E)(x) = -(f-E)(-x)$

$\Rightarrow f(x) - E(x) = -(f(-x) - E(-x))$

$= -f(-x) + E(-x)$

$\Rightarrow f(x) + f(-x) = E(x) + E(-x) = 2E(x)$

$$E(x) = \frac{f(x)+f(-x)}{2}$$

Given $E(x)$, $f(x)$ is not unique here.

$f - O$ even means

$(f-O)(x) = (f-O)(-x)$

$f(x) - O(x) = f(-x) - O(-x)$

$\Rightarrow f(x) - f(-x) = O(x) - O(-x) = 2O(x)$

$$\Rightarrow O(x) = \frac{f(x)-f(-x)}{2}$$

Given $O(x)$, $f(x)$ is not unique here either.

Given $E(x)$ and $O(x)$ as

$$O(x) = \frac{f(x)-f(-x)}{2}$$

$$E(x) = \frac{f(x)+f(-x)}{2}$$

There is a single $f(x)$ that satisfies both equations,

$f(x) = E(x) + O(x)$

Solution 2

Assume $f(x) = E(x) + O(x)$

$\Rightarrow f(-x) = E(-x) + O(-x)$

$= E(x) - O(x)$

$f(x) = E(x) + O(x)$

$f(-x) = E(x) - O(x)$

Given a fn f, this is a system of two variables and two equations.

$$\Rightarrow E(x) = \frac{f(x)+f(-x)}{2}, \quad O(x) = \frac{f(x)-f(-x)}{2}$$

14. f any fn.

Define $|f|(x) = |f(x)|$. Find $\max(f,g)$, $\min(f,g)$ in terms of $|f|(x)$.

$$\max(f,g)(x) = \max(f(x),g(x)) = \begin{cases} f(x) & \text{if } f(x) > g(x),\ f(x)-g(x) > 0 \\ g(x) & \text{if } f(x) < g(x),\ f(x)-g(x) < 0 \end{cases}$$

$$\max(f,g)(x) = \frac{f(x)+g(x)}{2} + \frac{|f(x)-g(x)|}{2}$$

$$\min(f(x),g(x)) = \frac{f(x)+g(x)}{2} - \frac{|f(x)-g(x)|}{2}$$

Proof (max)

case 1: $f(x) \geq g(x)$

$$\max(f,g)(x) = \frac{f(x)+\cancel{g(x)}}{2} + \frac{f(x) - \cancel{g(x)}}{2} = f(x)$$

case 2: $f(x) \leq g(x)$

$$\max(f,g)(x) = \frac{\cancel{f(x)}+g(x)}{2} - \frac{(\cancel{f(x)} - g(x))}{2} = g(x)$$

15. a) $f = \max(f,0) + \min(f,0)$

Using the result of problem 14

$$\frac{f}{2} + \cancel{\frac{|f|}{2}} + \frac{f}{2} - \cancel{\frac{|f|}{2}} = f$$

---

b) Prove that any fn f can be written $f = g - h$, g and h nonnegative, in infinitely many ways.

Let $g(x) = f(x) + |f(x)| + n$, $n \geq 0 \in \mathbb{R}$

$\Rightarrow f(x) = g(x) - |f(x)| - n$

Let $h(x) = |f(x)| + n \geq 0$

$\Rightarrow f(x) = g(x) - h(x)$, $g(x), h(x) \geq 0$ and there are infinitely many pairs $g(x), h(x)$ as we vary n.

---

16. $f(x+y) = f(x) + f(y) \ \forall x, y$

a) Prove $f(x_1 + \cdots + x_n) = f(x_1) + \cdots + f(x_n)$

$A = \{n : f(x_1 + \cdots + x_n) = f(x_1) + \cdots + f(x_n)\}$

$n = 1 \Rightarrow f(x_1) = f(x_1) \Rightarrow 1 \in A$

Assume $k \in A \Rightarrow f(\sum_{i=1}^{k} x_i) = \sum_{i=1}^{k} f(x_i)$

Then

$$f(\sum_{i=1}^{k+1} x_i) = f(\sum_{i=1}^{k} x_i + x_{k+1}) = f(\sum_{i=1}^{k} x_i) + f(x_{k+1})$$
$$= \sum_{i=1}^{k+1} f(x_i)$$

$\Rightarrow k+1 \in A$

b) Prove $\exists c$ s.t. $f(x) = cx \ \forall x \in \mathbb{Q}$

$f(x+y) = f(x) + f(y)$ valid for $\forall x$

$\sum_{i=1}^{x} 1 = x \ \Rightarrow x \in \mathbb{N}$

$x \in \mathbb{N} \Rightarrow f(x) = f(\sum_{i=1}^{x} 1) = x f(1)$

$\Rightarrow f(x) = f(1)x$, $x \in \mathbb{N}$ valid only for $x \in \mathbb{N}$

$\rightarrow$ what if $x \in \mathbb{Z}$?

$x \in \mathbb{N} \Rightarrow -x \in \mathbb{Z}$

$-x = -\sum_{i=1}^{x} 1$

$\Rightarrow f(-x) = f(-\sum_{i=1}^{x} 1) = x f(-1)$

$f(1) = f(-1+1+1) = f(-1) + 2f(1)$

$\Rightarrow f(-1) = -f(1)$

$\Rightarrow f(-x) = -x f(1)$, $x \in \mathbb{N}$

$\Rightarrow f(x) = f(1)x$, $x \in \mathbb{Z}$

What if $x \in \mathbb{Q}$?

$1 = \sum_{i=1}^{x} \frac{1}{x} = x \cdot \frac{1}{x} = 1$, $x \in \mathbb{Z}$

$f(1) = f(\sum_{i=1}^{x} \frac{1}{x}) = x \cdot f(\frac{1}{x})$

$\Rightarrow f(\frac{1}{x}) = f(1) \cdot \frac{1}{x}$, $x \in \mathbb{Z}$

$\frac{y}{x} = y \cdot \frac{1}{x}$, $y, x \in \mathbb{Z}$

$f(\frac{y}{x}) = f(\sum_{i=1}^{y} \frac{1}{x}) = y f(\frac{1}{x}) = f(1)\frac{y}{x}$

$\Rightarrow f(\frac{y}{x}) = f(1)\frac{y}{x}$

$\Rightarrow f(x) = f(1)x$, $x \in \mathbb{Q}$

17. $f(x)=0\ \forall x \Rightarrow f(x+y)=f(x)+f(y)\ \forall x,y$

$f(x\cdot y)=f(x)\cdot f(y)\ \forall x,y$

Suppose $f(x)$ not always zero, but still has the two properties above.

---

a) $f(0+1)=f(1)=f(1)+f(0)$

$\Rightarrow f(0)=0$

$f(1\cdot 1)=f(1)^2=f(1)$

$\Rightarrow f(1)(f(1)-1)=0$

$\Rightarrow f(1)=1$ or $f(1)=0$

$f(1\cdot x)=f(1)\cdot f(x)=f(x)$

if $f(1)=0$ then $f(x)=0\ \forall x$, but this isn't true by assumption $\Rightarrow f(1)=1$

---

b) From 16b) we know that

$f(x+y)=f(x)+f(y)\ \forall x,y \Rightarrow f(x)=f(1)x\ \forall x\in\mathbb{Q}$

$f(1)=1 \Rightarrow f(x)=x\ \forall x\in\mathbb{Q}$

---

c) Given any $x>0$, we have $x=d^2$.

$f(x)=f(d^2)=f(d)^2>0$

---

d) $x>y \Rightarrow x-y>0$

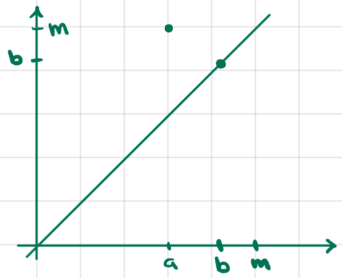
$\Rightarrow f(x-y)>0$ by c)

$\Rightarrow f(x)+f(-y)>0$

$\Rightarrow f(x)-f(y)>0$

$\Rightarrow f(x)>f(y)$

e) we know that $f(x)=x\ \forall x\in\mathbb{Q}$



Assume there is an $a\in\mathbb{R}$ such that $f(a)=m>a$

Consider the x coordinate $a+(m-a)=m$

Between a and m there is a rational number b and $f(b)=b$

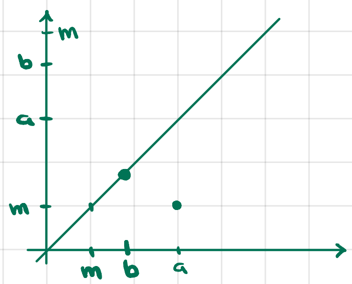
Thus we have

$a<b$ and $f(a)=m>b=f(b)$

violating our conclusion from d).

$\Rightarrow f(a)$ cannot be $>a$.

Analogously, if $f(a)=m<a$, we consider the x coordinate



$a-(a-m)=m$. There is a $b\in\mathbb{Q}$ and $m<b<a$, $f(b)=b$.

Thus,

$b<a$

$f(b)=b>m=f(a)$

$\Rightarrow$ contradiction

$\Rightarrow f(a)$ cannot be $<a$

$\Rightarrow f(a)=a$

18. $f(x)g(y) = h(x)k(y) \quad \forall x,y$

$h(x) \neq 0,\ g(y) \neq 0,\ k(y) \neq 0,\ f(x) \neq 0$

$\Rightarrow \dfrac{f(x)}{h(x)} = \dfrac{k(y)}{g(y)}$

consider x_1 and x_2

$\dfrac{f(x_1)}{h(x_1)} = c_1 = \dfrac{k(y)}{g(y)} \quad \forall y$

$\dfrac{f(x_2)}{h(x_2)} = c_2 = \dfrac{k(y)}{g(y)} \quad \forall y$

$\Rightarrow k(y) = c_1 g(y) \quad \forall y$

$k(y) = c_2 g(y) \quad \forall y$

$\Rightarrow c_1 = c_2$

$\Rightarrow \dfrac{f(x_1)}{h(x_1)} = \dfrac{f(x_2)}{h(x_2)} = c_x \neq 0 \quad \forall x_1, x_2$

$\Rightarrow \dfrac{f(x)}{h(x)} = c_x \neq 0 \quad \forall x$

Similarly, $\dfrac{k(y)}{g(y)} = c_y \neq 0 \quad \forall y$

$c_x h(x) g(y) = h(x) c_y g(y)$

$\Rightarrow c_x = c_y$

$\Rightarrow \dfrac{f(x)}{h(x)} = \dfrac{k(y)}{g(y)} = c \neq 0$

Assume $f(x_0) = 0$

$\Rightarrow h(x) = 0$ or $k(y) = 0 \quad \forall x,y$

Assume $k(y_1) \neq 0$

$\Rightarrow h(x) = 0 \quad \forall x$

Assume $h(x_1) \neq 0$

$\Rightarrow k(y) = 0 \quad \forall y$

Therefore if $f(x_0) = 0$ for some x_0 then either h or k must be identically 0

The precise conditions for $f(x)g(y) = h(x)k(y)$ are therefore

1) f has root $\Rightarrow h(x) = 0 \ \forall x$, g and k free

or $k(y) = 0 \ \forall y$, g and h free

2) g has root $\Rightarrow h(x) = 0 \ \forall x$, f and k free

or $k(y) = 0 \ \forall y$, f and h free

3) h has root $\Rightarrow f(x) = 0 \ \forall x$, g and k free

or $g(y) = 0 \ \forall y$, f and k free

4) k has root $\Rightarrow f(x) = 0 \ \forall x$, g and h free

or $g(y) = 0 \ \forall y$, f and h free

5) no roots

$\Rightarrow \dfrac{f(x)}{h(x)} = \dfrac{k(y)}{g(y)} = c \quad \forall x,y,\ c \neq 0$

19. a) $\nexists f, g$ with either of properties

i) $f(x) + g(y) = xy \;\; \forall x, y$
ii) $f(x) \cdot g(y) = x + y \;\; \forall x, y$

i) $f(0) + g(y) = 0 \Rightarrow g(y) = -f(0) \;\; \forall y$

$f(x) + g(0) = 0 \Rightarrow f(x) = -g(0) \;\; \forall x$

$\Rightarrow g(y) = g(0) \;\; \forall y$
$f(x) = f(0) \;\; \forall x$

$f(x) + g(1) = x \Rightarrow f(0) + g(1) = 0$
$\Rightarrow f(0) = -g(1) = f(x)$
$\Rightarrow f(x) + g(1) = -g(1) + g(1) = 0 \neq x$ contradiction

ii) $f(0)g(0) = 0$
$\Rightarrow$ either $f(0) = 0$ or $g(0) = 0$ or both.
Assume $f(0) = 0$

$f(0)g(y) = y + 0 = 0 \Rightarrow \underbrace{y = 0 \;\; \forall y}$ FALSE

contradiction because y can be any number.

If $g(0) = 0$, same result

$f(x)g(0) = x + 0 = 0 \Rightarrow x = 0$

$\Rightarrow$ we can't have $f(x)g(y) = x + y \;\; \forall x, y$

b) $f(x+y) = g(xy) \;\; \forall x, y$

$f(x+y) = c$
$g(xy) = c$

Are these the only possibilities?

$f(0+y) = f(y) = g(0) \;\; \forall y$

$f(0+x) = f(x) = g(0) \;\; \forall x$

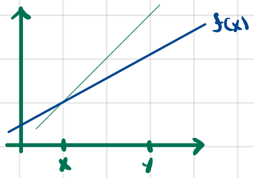
$\Rightarrow f$ is a constant function.

$f(x+y) = g(0) = g(xy)$

$\Rightarrow g$ is also a constant function

20. a) $|f(y) - f(x)| \leq |y - x|$

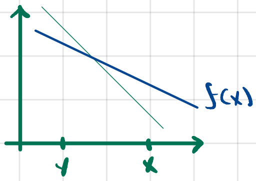
$f(y) - f(x) > 0$, $y - x > 0$, ie an increasing fn



$f(y) - f(x) \leq y - x \Rightarrow \dfrac{f(y) - f(x)}{y - x} \leq 1$

e.g. $f(x) = \dfrac{x}{2}$

$f(y) - f(x) > 0$, $y - x < 0$, ie a decreasing fn

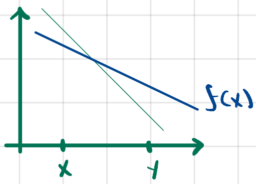


$f(y) - f(x) \leq -(y - x)$

$\Rightarrow \dfrac{f(y) - f(x)}{y - x} = \dfrac{f(x) - f(y)}{x - y} \geq -1$

e.g. $f(x) = -\dfrac{x}{2}$

$f(y) - f(x) < 0$, $y - x > 0$, ie a decreasing fn

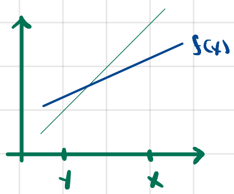


$-(f(y) - f(x)) \leq y - x$

$\Rightarrow -\dfrac{f(y) - f(x)}{y - x} \leq 1$

$\Rightarrow \dfrac{f(y) - f(x)}{y - x} \geq -1$

$f(y) - f(x) < 0$, $y - x < 0$, ie an increasing fn



$-(f(y) - f(x)) \leq -(y - x)$

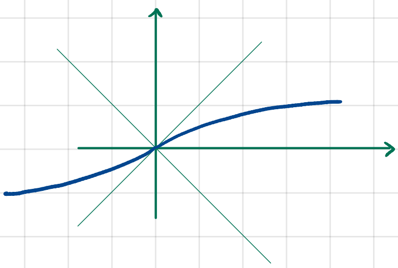
$\Rightarrow f(y) - f(x) \geq y - x$

$\Rightarrow \dfrac{f(y) - f(x)}{y - x} \leq 1$

$= \dfrac{f(x) - f(y)}{x - y} \leq 1$

$\dfrac{|f(y) - f(x)|}{|y - x|} = \left| \dfrac{f(y) - f(x)}{y - x} \right| \leq 1$

$\Rightarrow -1 \leq \dfrac{f(y) - f(x)}{y - x} \leq 1$



The slope between any two points must not exceed 1 in absolute value.

e.g. $f(x) = \dfrac{x}{2}$

b) Assume $f(y)-f(x) \le (y-x)^2 \ \forall x,y$
Prove f is constant.

Proof that $|f(y)-f(x)| \le (y-x)^2 \ \forall x,y$

$f(y)-f(x) \le (y-x)^2 \ \forall x,y$
$f(y)-f(x) \ge 0 \Rightarrow |f(y)-f(x)| \le (y-x)^2$
if $f(y)-f(x) < 0$ then $0 < f(x)-f(y) < (x-y)^2 = (y-x)^2$
$\Rightarrow |f(y)-f(x)| \le (y-x)^2$

$\Rightarrow |f(y)-f(x)| \le (y-x)^2 \ \forall x,y$

Partition the interval $[x,y]$ in n equal parts of length $\frac{y-x}{n}$.

$|f(y)-f(x)| = \left| \sum_{i=1}^{n} \left[f\left(x+k\frac{(y-x)}{n}\right) - f\left(x+(k-1)\frac{(y-x)}{n}\right) \right] \right|$

$\Rightarrow |\sum x_i| \le \sum |x_i|$
proof
case 1: $0 \le \sum x_i = |\sum x_i| \le \sum |x_i|$

case 2: $\sum x_i < 0 \Rightarrow 0 \le -\sum x_i = |\sum x_i| = \sum(-x_i) \le \sum |x_i|$

$\Rightarrow |f(y)-f(x)|$
$\le \sum_{i=1}^{n} \left| f\left(x+k\frac{(y-x)}{n}\right) - f\left(x+(k-1)\frac{(y-x)}{n}\right) \right|$

use the result we proved earlier: $|f(y)-f(x)| \le (y-x)^2 \ \forall x,y$

$\le \sum_{i=1}^{n} \left(\cancel{x} + \cancel{k\frac{(y-x)}{n}} - \cancel{x} - (\cancel{k}-1)\frac{(y-x)}{n} \right)^2$

$= \sum_{i=1}^{n} \frac{(y-x)^2}{n^2}$

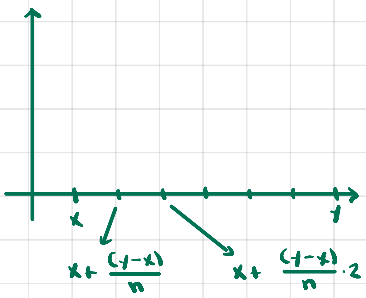
$= \frac{(y-x)^2}{n}$

$\Rightarrow |f(y)-f(x)| \le \frac{(y-x)^2}{n}$

$0 \le \frac{|f(y)-f(x)|}{(y-x)^2} \le \frac{1}{n} \quad \forall n \in \mathbb{N}$

$(y-x)^2 > 0$
$|f(y)-f(x)| \ge 0$

$\Rightarrow \frac{|f(y)-f(x)|}{(y-x)^2} = 0$



$\Rightarrow f(y) = f(x)$

$f(x)$ = constant

21. a) $f\circ(g+h) = f\circ g + f\circ h$

$(f\circ(g+h))(x) = f((g+h)(x)) = f(g(x)+h(x))$

counterexample

consider $f(x) = x^2$, $g(x) = h(x) = x$

$f(g(1)+h(1)) = f(2) = 4$

$f(g(1)) + f(h(1)) = f(1) + f(1) = 2$

$\Rightarrow 4 = (f\circ(g+h))(1) \ne (f\circ g + f\circ h)(1) = 2$

b) $(g+h)\circ f = g\circ f + h\circ f$

$((g+h)\circ f)(x) = (g+h)(f(x))$