Ch. 18 - The Logarithm and Exponential Functions

1. (i) $f(x) = e^{e^{e^x}}$

$\log^{-1}(x) = \exp(x) = e^x$

$(\log^{-1})'(x) = \exp'(x) = \log^{-1}(x) = \exp(x) = e^x$

$h(x) = e^x \to f(x) = (h \circ h \circ h)(x)$

$f'(x) = h'((h \circ h)(x)) \cdot h'(h(x)) \cdot h'(x)$

$= e^{h(h(x))} \cdot e^{h(x)} \cdot e^x = e^{e^{e^x}} \cdot e^{e^x} \cdot e^x$

(ii) $f(x) = \log(1 + \log(1 + \log(1 + e^{1+e^{1+x}})))$

$h(x) = \log(x)$

$g(x) = 1 + \log(x)$

$m(x) = 1 + e^x$

$f(x) = h(g(g(m(m(1+x)))))$

$f'(x) = h'(g(g(m(m(1+x))))) \cdot g'(g(m(m(1+x)))) \cdot g'(m(m(1+x))) \cdot m'(m(1+x)) \cdot m'(1+x)$

$$= \frac{1}{1 + \log(1 + \log(1 + e^{1+e^{1+x}}))} \cdot \frac{1}{1 + \log(1 + e^{1+e^{1+x}})} \cdot \frac{1}{1 + e^{1+e^{1+x}}} \cdot e^{1+e^{1+x}} \cdot e^{1+x}$$

(iii) $f(x) = (\sin x)^{\sin(\sin x)} = e^{\sin(\sin x) \cdot \log(\sin x)}$

let $f_1(x) = e^x$

$f_2(x) = \sin(\sin x) \cdot \log(\sin x)$

Then $f(x) = f_1(f_2(x))$

$f'(x) = f_1'(f_2(x)) f_2'(x) = e^{\sin(\sin x) \cdot \log(\sin x)} \cdot \left[\cos(\sin x) \cdot \cos x \cdot \log(\sin x) + \sin(\sin x) \cdot \frac{\cos x}{\sin x} \right]$

Note

we defined a^x, which is any $a > 0$ to a real power, as $a^x = e^{x \log a}$.

Thus, since for every x, $\sin x$ is a number, we have

$(\sin x)^y = e^{y \log(\sin x)}$. If $y = \sin(\sin x)$ then $(\sin x)^{\sin(\sin x)} = e^{\sin x \cdot \sin(\sin x)}$, the deriv. of which we know how to compute.

(iv) $f(x) = e^{\int_0^x e^{-t^2}dt}$

$f_1(x) = e^x$

$f_2(x) = \int_0^x e^{-t^2}dt$

$f(x) = f_1(f_2(x))$

$f'(x) = e^{\int_0^x e^{-t^2}dt} \cdot e^{-x^2}$

Note

Everything flows from the definition $h(x) = \log x = \int_1^x t^{-1}dt$. From here, we get

$h'(x) = \log'(x) = \frac{1}{x} > 0$, increasing, one-one

$h(x^n) = \log(x^n) = n\log(x) \to h = \log$ unbounded on $(0, +\infty)$

$h^{-1}(x) = \log^{-1}(x)$ exists, defined on $\mathbb{R}$.

inverse fn thm $\to (h^{-1})'(x) = \frac{1}{h'(h^{-1}(x))} = \frac{1}{\log'(\log^{-1}(x))} = \log^{-1}(x) = h^{-1}(x)$

so $(h^{-1})' = h^{-1}$

Also let $x' = h^{-1}(x)$, $y' = h^{-1}(y)$. Then $h(x') = x$, $h(y') = y$, $x+y = h(x') + h(y') = h(x'+y')$

$\to h^{-1}(x+y) = x'+y' = h^{-1}(x) + h^{-1}(y)$

Hence, for rational x, $h^{-1}(x) = [h^{-1}(1)]^x$

For x irrational we define $[h^{-1}(1)]^x = h^{-1}(x)$. That is, h^{-1} is a fn with the characteristics defined above. We are taking a notation that has no defined meaning (a number $[h^{-1}(1)]$ raised to an irrational power) and defining it as h^{-1}. h^{-1} is just the inverse of $h = \int_1^x t^{-1}dt$, and we know that $(h^{-1})' = h^{-1}$.

So when we encounter something like

$[h^{-1}(1)]^{g(x)}$ this is defined as $h^{-1}(g(x))$

Also because $h^{-1}(1)$ is so central, we name it e.

$h^{-1}(x) = e^x$ by def.

(v) $f(x) = (\sin x)^{(\sin x)^{\sin x}}$

$g(x) = (\sin x)^{\sin x}$

$(\sin x)^{g(x)}$ is number $\sin x$ raised to real power $g(x)$.

But $\log^{-1}(x) = e^x \to \log^{-1}(\log(\sin x)) = \sin x = e^{\log(\sin x)}$

Hence, $g(x) = e^{\sin x \log(\sin x)}$

$$f(x) = \sin(x)^{g(x)} = \left(e^{\log(\sin x)}\right)^{g(x)} = e^{g(x)\log(\sin x)} = \log^{-1}(g(x)\log(\sin x))$$

$$f'(x) = e^{g(x)\log(\sin x)} \cdot \left(g'(x)\log(\sin x) + g(x)\cdot \frac{\cos x}{\sin x}\right)$$

$$g'(x) = \left[e^{\sin x\log(\sin x)}\right]' = e^{\sin x \log \sin x}\cdot\left(\cos x \log \sin x + \sin x \frac{\cos x}{\sin x}\right)$$

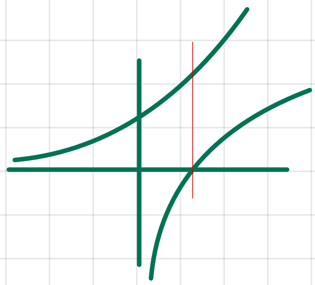
$$f'(x) = (\sin x)^{(\sin x)^{\sin x}} \cdot \left[e^{\sin x\log(\sin x)}(\cos x\log(\sin x)+\cos x)\cdot\log(\sin x) + e^{\sin x \log(\sin x)}\cdot\frac{\cos x}{\sin x}\right)$$

$$= (\sin x)^{(\sin x)^{\sin x}}\cdot\left[(\sin x)^{\sin x}(\cos x \log(\sin x)+\cos x)\cdot\log(\sin x) + (\sin x)^{\sin x}\cdot\frac{\cos x}{\sin x}\right]$$

(vi) $f(x) = \log_{(e^x)} \sin x$

$\log_a x$ is defined as the inverse of a^x.

We know a^x has an inverse defined on positive x because a^x takes on all values and



$[a^x]' = [e^{x\log a}]' = \log a \cdot e^{x \log a} = a^x \cdot \log a$ and

$0<a<1 \to \log a<0, a^x>0$ (since $e^y>0$) $\to [a^x]'<0$

$a>1 \to \log a>0, a^x>0 \to [a^x]'>0$

Let $[a^x]^{-1} = \log_a(x)$. Then

$$y = [a^x]^{-1} \to a^y = x = e^{y\log a} \to \log x = y\log a \to y = \log_a x = \frac{\log x}{\log a}$$

hence $\log_a'(x) = \frac{1}{x\log a}$

Therefore,

$$f(x) = \log_{e^x}\sin x = \frac{\log(\sin x)}{\log(e^x)}$$

$$f'(x) = \frac{\frac{\cos x}{\sin x}\log(e^x) - \frac{1}{e^x}\cdot e^x\cdot\log(\sin x)}{(\log e^x)^2} = \frac{\frac{x\cos x}{\sin x} - \log(\sin x)}{x^2}$$

(vii) $f(x) = \left[\arcsin\left(\frac{x}{\sin x}\right)\right]^{\log(\sin e^x)}$

$e^{\log[\arcsin(\frac{x}{\sin x})]} = \log^{-1}\left[\log\left[\arcsin\left(\frac{x}{\sin x}\right)\right]\right] = \arcsin\left(\frac{x}{\sin x}\right) = f_1(x)$

$f(x) = f_1(x)^{\log(\sin e^x)} = f_1(x)^{f_2(x)} = \left[\log^{-1}\left[\log\left[\arcsin\left(\frac{x}{\sin x}\right)\right]\right]\right]^{f_2(x)} = e^{\log[\arcsin(\frac{x}{\sin x})] \cdot f_2(x)}$

$f_2(x) = \log(\sin e^x)$

$f'(x) = e^{\log[\arcsin(\frac{x}{\sin x})] \cdot f_2(x)} \cdot \left[\frac{\frac{1}{\sqrt{1-(\frac{x}{\sin x})^2}}}{\arcsin(\frac{x}{\sin x})} \cdot \frac{\sin x - x\cos x}{\sin^2 x} \cdot \log(\sin e^x) + \log\left[\arcsin\left(\frac{x}{\sin x}\right)\right] \cdot \frac{1}{\sin e^x} \cdot \cos(e^x) \cdot e^x \right]$

(viii) $f(x) = (\log(3+e^4))e^{4x} + (\arcsin x)^{\log 3}$

$f'(x) = \log(3+e^4) \cdot 4e^{4x} + \log(3) \arcsin(x)^{\log 3 - 1} \frac{1}{\sqrt{1-x^2}}$

Note

Difference between a^x and a^4, $a > 0$.

a^4 is a constant. $a^4 = e^{4 \cdot \log a}$ by definition, but still just a constant.

a^x is a fn of x. $a^x = e^{x \log a}$ by def., and this is actually the inverse of $\log_a x$, which is $\frac{\log x}{\log a}$, the deriv. of which is $\frac{1}{x \log a}$.

(ix) $f(x) = (\log x)^{\log x}$

$= (\log^{-1}(\log(\log x)))^{\log x}$

$= e^{\log(\log x) \cdot \log x}$

$f'(x) = e^{\log(\log x) \cdot \log x} \cdot \left[\frac{1}{\log x} \cdot \frac{1}{x} \cdot \log x + \log(\log x) \cdot \frac{1}{x}\right]$

$= (\log x)^{\log x} \cdot \left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$

$= \frac{(\log x)^{\log x}}{x}(1 + \log(\log x))$

(x) $f(x) = x^x = (\log^{-1}(\log x))^x = e^{x \log x}$

$f'(x) = e^{x \log x}(\log x + 1) = x^x(1 + \log x)$

(xi) $f(x) = \sin(x^{\sin(x^{\sin x})}) = \sin\left[(\log^{-1}(\log x))^{\sin(x^{\sin x})}\right] = \sin\left[e^{\log x \sin(x^{\sin x})}\right]$

$$f'(x) = \cos\left[e^{\log x \sin(x^{\sin x})}\right] \cdot e^{\log x \sin(x^{\sin x})} \cdot \left[\frac{1}{x}\sin(x^{\sin x}) + \log x \cos(x^{\sin x}) \cdot x^{\sin x}\left(\frac{\sin x}{x} + \log x \cos x\right)\right]$$

$$= \cos(\sin(x^{\sin(x^{\sin x})})) \cdot x^{\sin(x^{\sin x})} \cdot \left[\frac{\sin(x^{\sin x})}{x} + \log x \cos(x^{\sin x}) \cdot x^{\sin x}\left(\frac{\sin x}{x} + \log x \cos x\right)\right]$$

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2. a) $\log \circ f$

$h(x) = \log(f(x))$

$h'(x) = \frac{1}{f(x)} \cdot f'(x)$ logarithmic derivative of f.

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b) (i) $f(x) = (1+x)(1+e^{x^2}) = 1 + e^{x^2} + x + xe^{x^2}$, $f'(x) = 1 + 2xe^{x^2} + e^{x^2} + 2x \cdot xe^{x^2} = 1 + 3e^{x^2} + 2x^2e^{x^2}$

$h(x) = \log[(1+x)(1+e^{x^2})] = \log(1+x) + \log(1+e^{x^2})$

$$h'(x) = \frac{1}{1+x} + \frac{2xe^{x^2}}{1+e^{x^2}} = \frac{1+e^{x^2}+2xe^{x^2}(1+x)}{(1+x)(1+e^{x^2})} = \frac{1+3e^{x^2}+2x^2e^{x^2}}{(1+x)(1+e^{x^2})} = \frac{f'(x)}{f(x)}$$

$\rightarrow f'(x) = h'(x)f(x) = 1 + 3e^{x^2} + 2x^2e^{x^2}$

(ii) $f(x) = \dfrac{(3-x)^{1/3}x^2}{(1-x)(3+x)^{2/3}}$

$h(x) = \log(f(x)) = \log[(3-x)^{1/3}x^2] - \log[(1-x)(3+x)^{2/3}]$

$$h'(x) = \frac{\frac{1}{3}(3-x)^{-\frac{2}{3}}(-1)x^2 + (3-x)^{1/3} \cdot 2x}{(3-x)^{1/3}x^2} - \frac{(-1)(3+x)^{2/3} + (1-x) \cdot \frac{2}{3}(3+x)^{-1/3}}{(1-x)(3+x)^{2/3}}$$

$$= -\frac{1}{3} \cdot \frac{1}{3-x} \cdot x^2 + \frac{2}{x} + \frac{1}{1-x} - \frac{2}{3}\,\frac{1}{3+x}$$

$$f'(x) = f(x) \cdot h'(x) = \frac{(3-x)^{1/3}x^2}{(1-x)(3+x)^{2/3}}\left[-\frac{1}{3} \cdot \frac{1}{3-x} \cdot x^2 + \frac{2}{x} + \frac{1}{1-x} - \frac{2}{3}\,\frac{1}{3+x}\right]$$

(iii) $f(x) = (\sin x)^{\cos x} + (\cos x)^{\sin x} = e^{\log(\sin x)\cos x} + e^{\log(\cos x)\sin x}$

$$f'(x) = e^{\log(\sin x)\cos x} \cdot \left[\frac{\cos^2 x}{\sin x} - \sin x \log(\cos x)\right] + e^{\log(\cos x)\sin x}\left[-\frac{\sin^2 x}{\cos x} + \cos x \log(\cos x)\right]$$

$$h(x) = \log f(x) = \log\left[(\sin x)^{\cos x} + (\cos x)^{\sin x}\right]$$

$$h'(x) = \frac{1}{(\sin x)^{\cos x} + (\cos x)^{\sin x}} \cdot \left[(\sin x)^{\cos x} + (\cos x)^{\sin x}\right]' = \frac{f'}{f}$$

$$= \frac{\sin x^{\cos x} \cdot \left[\frac{\cos^2 x}{\sin x} - \sin x \log(\cos x)\right] + \cos x^{\sin x}\left[-\frac{\sin^2 x}{\cos x} + \cos x \log(\cos x)\right]}{(\sin x)^{\cos x} + (\cos x)^{\sin x}}$$

(iv) $f(x) = \dfrac{e^x - e^{-x}}{e^{2x}(1+x^3)}$

$$h(x) = \log(f(x)) = \log(e^x - e^{-x}) - \log(e^{2x}(1+x^3))$$

$$h'(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} - \frac{2e^{2x} + 2e^{2x}x^3 + e^{2x}\cdot 3x^2}{e^{2x}(1+x^3)}$$

$$f'(x) = \frac{e^x - e^{-x}}{e^{2x}(1+x^3)}\left[\frac{e^x + e^{-x}}{e^x - e^{-x}} - \frac{2e^{2x} + 2e^{2x}x^3 + e^{2x}\cdot 3x^2}{e^{2x}(1+x^3)}\right]$$

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3. $\displaystyle\int_a^b \frac{f'(t)}{f(t)}\,dt \qquad f>0$ on $[a,b]$

$h(t) = \log(f(t)) \rightarrow h'(t) = f'(t)/f(t)$

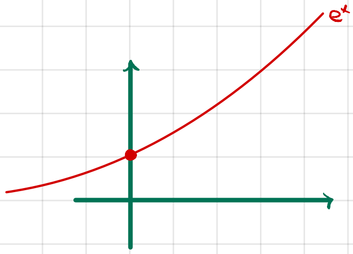
FTC2 $\rightarrow \displaystyle\int_a^b \frac{f'(t)}{f(t)}\,dt = h(b) - h(a) = \log\frac{f(b)}{f(a)}$

4. (a) $f(x) = e^{x+1}$

First let's recap what we know about the graph of e^x

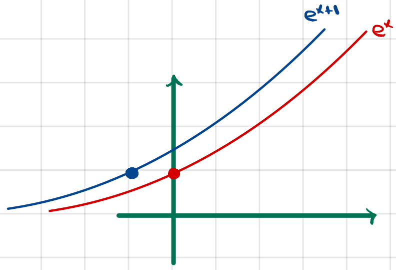
$\log^{-1}(x) = e^x$, and $\log x = \int_1^x t^{-1}dt > 0$

domain: $\mathbb{R}$
image $(0, +\infty)$
$(e^x)' = e^x > 0 \rightarrow$ increasing, one-one
$\log^{-1}(1) = e$
$\log^{-1}(0) = 1$
$(e^x)'' = e^x > 0 \rightarrow$ convex fn
$\lim_{x\to-\infty} e^x = 0$, $\lim_{x\to\infty} e^x = \infty$



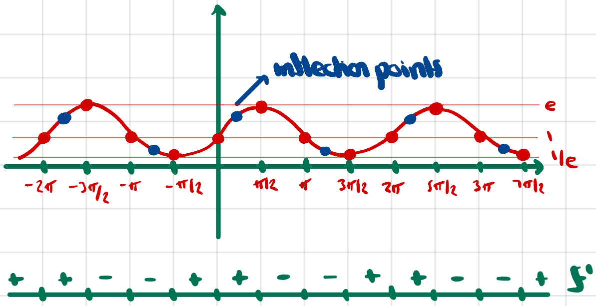
$f(1+x)$ simply shifts the graph of $f(x)$ to the left.

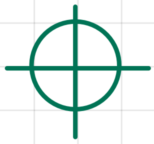
Hence, if $g(x) = e^x$ then $f(x) = g(x+1) = e^{x+1}$ has graph

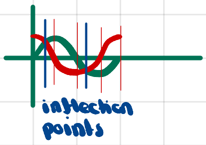


(b) $f(x) = e^{\sin x}$ $\quad f'(x) = \cos x\, e^x \quad f''(x) = -\sin x \cdot e^x + \cos x\, e^x = e^x(\cos x - \sin x)$

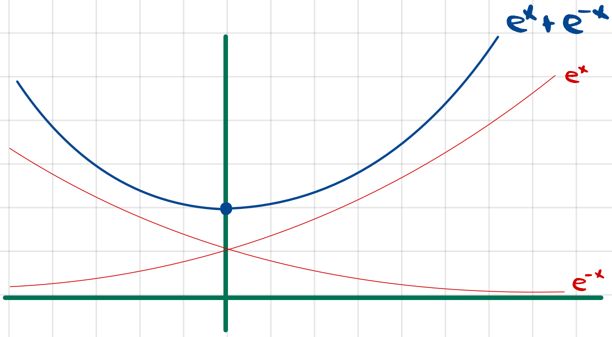
x	$\sin x$	$e^{\sin x}$
0	0	1
$\frac{\pi}{2}$	1	e
π	0	1
$3\pi/2$	-1	$1/e$







(c) $f(x) = e^x + e^{-x} > 0$



$f'(x) = e^x - e^{-x} = \dfrac{e^{2x}-1}{e^x}$

$e^{2x} = 1 \rightarrow 2x = 0 \rightarrow x = 0$

$e^{2x} > 1 \rightarrow 2x > 0 \rightarrow x > 0$
$e^{2x} < 1 \rightarrow 2x < 0 \rightarrow x < 0$

$f''(x) = e^x + e^{-x} = f(x) > 0$

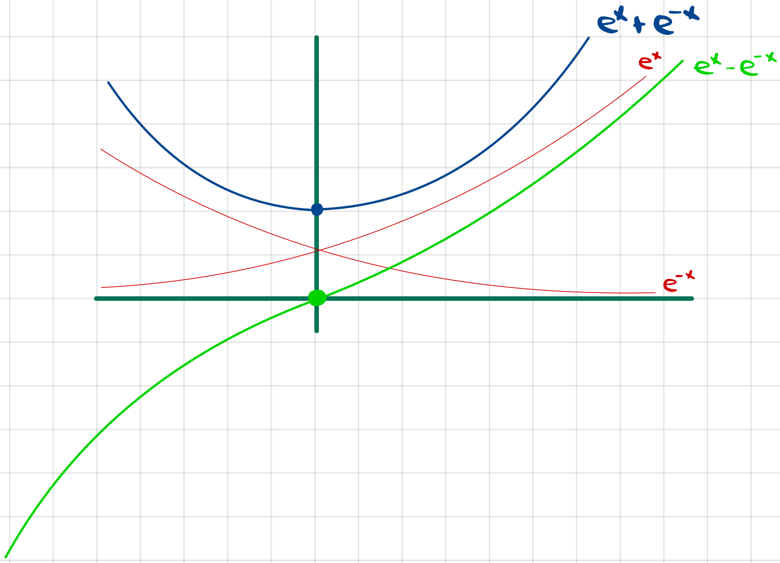
(d) $f(x) = e^x - e^{-x}$

$f'(x) = e^x + e^{-x} > 0$

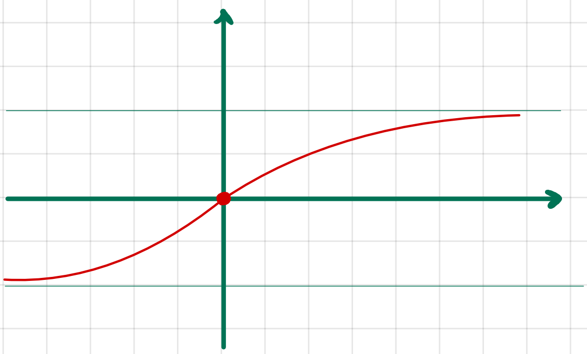
$f''(x) = e^x - e^{-x} = f(x)$

$\underset{0}{\overset{-\qquad\qquad +}{\longrightarrow}}\ f''$

$f(0) = 0$



(e) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x}-1}{e^{2x}+1} = \dfrac{e^{2x}+1-2}{e^{2x}+1} = 1 - \dfrac{2}{e^{2x}+1}$



$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 1$

$f'(x) = -\dfrac{-2e^{2x}\cdot 2}{(e^{2x}+1)^2} = \dfrac{4e^{2x}}{(e^{2x}+1)^2} > 0$

$f''(x) = \dfrac{8e^{2x}(e^{2x}+1)^{\cancel{2}} - 8e^{2x}\cancel{(e^{2x}+1)}e^{2x}\cdot 2}{(e^{2x}+1)^{\cancel{4}\,3}}$

$= \dfrac{8e^{2x}(e^{2x}+1-2e^{2x})}{(e^{2x}+1)^3}$

$= \dfrac{8e^{2x}(1-e^{2x})}{(e^{2x}+1)^3}$ $\qquad \underset{0}{\overset{+\qquad\qquad -}{\longrightarrow}}\ f''$

$1 - e^{2x} > 0 \to e^{2x} < 1 \to 2x < \log 1 = 0 \to x < 0$

$f(0) = 0$

$f'(0) = 1$

5. (i) $\lim_{x\to0} \frac{e^x-1-x-x^2/2}{x^2} = \frac{0}{0}$

$= \lim_{x\to0} \frac{e^x-1-x}{2x} = \frac{0}{0} = \lim_{x\to0} \frac{e^x-1}{2} = 0$

(ii) $\lim_{x\to0} \frac{e^x-1-x-x^2/2-x^3/6}{x^3} = \frac{0}{0}$

$= \lim \frac{e^x-1-x-x^2/2}{3x^2} = \lim \frac{e^x-1-x}{6x} = \lim_{x\to0} \frac{e^x-1}{6} = 0$

(iii) $\lim_{x\to0} \frac{e^x-1-x-x^2/2}{x^3} = \frac{0}{0}$

$= \lim_{x\to0} \frac{e^x-1-x}{3x^2} = \frac{0}{0} = \lim_{x\to0} \frac{e^x-1}{6x} = \lim_{x\to0} \frac{e^x}{6} = \frac{1}{6}$

(iv) $\lim_{x\to0} \frac{\log(1+x)-x+x^2/2}{x^2} = \frac{0}{0}$

$= \lim_{x\to0} \frac{\frac{1}{1+x}-1+x}{2x} = \lim_{x\to0} \frac{-(1+x)^{-2}+1}{2} = 0$

(v) $\lim_{x\to0} \frac{\log(1+x)-x+x^2/2}{x^3}$

$= \lim_{x\to0} \frac{\frac{1}{1+x}-1+x}{3x^2} = \lim_{x\to0} \frac{-(1+x)^{-2}+1}{6x} = \lim_{x\to0} \frac{2(1+x)^{-3}}{6} = \frac{1}{3}$

(vi) $\lim_{x\to0} \frac{\log(1+x)-x+x^2/2-x^3/3}{x^3}$

$= \lim_{x\to0} \frac{\frac{1}{1+x}-1+x-x^2}{3x^2} = \lim_{x\to0} \frac{-(1+x)^{-2}+1-2x}{6x} = \lim_{x\to0} \frac{2(1+x)^{-3}-2}{6} = 0$

6\. (i) $\lim_{x\to 0}(1-x)^{1/x} = \lim_{x\to 0} e^{\frac{1}{x}\log(1-x)} = e$

note that e^x is continuous everywhere, therefore, for any x $e^{\frac{1}{x}\log(1-x)}$ is cont. there.

Hence, $\lim e^{\frac{1}{x}\log(1-x)} = e^{\lim_{x\to 0}\frac{1}{x}\log(1-x)}$

$$\lim_{x\to 0}\frac{1}{x}\log(1-x) = \frac{0}{0} = \lim_{x\to 0}\frac{\frac{1}{1-x}}{1} = 1$$

(ii) $\lim_{x\to \pi/4}(\tan x)^{\tan 2x}$

$(\tan x)^{\tan 2x} = e^{\log(\tan x)\tan 2x}$ noting that near $\pi/4$, $\tan(x)>0$, $\tan(2x)>0$ if the limit is from below

$$\lim_{x\to\frac{\pi}{4}}[\log(\tan x)\tan(2x)] = \lim_{x\to\frac{\pi}{4}}\frac{\log(\tan x)}{\frac{1}{\tan(2x)}} = \frac{0}{0} = \lim_{x\to\frac{\pi}{4}}\frac{\frac{1}{\cos^2(x)\tan(x)}}{\frac{\frac{-2}{\cos^2(2x)}}{(\tan(2x))^2}}$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{-\cos^2(2x)\tan^2(2x)}{2\cos^2(x)\tan(x)} = \lim_{x\to\frac{\pi}{4}}\frac{-\sin^2(2x)}{2\cos(x)\sin(x)} = \frac{-1}{\cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}}\cdot\frac{\sqrt{2}}{2}} = -1$$

$\rightarrow \lim_{x\to\pi/4}(\tan x)^{\tan 2x} = e^{-1}$

(iii) $\lim_{x\to 0}(\cos x)^{1/x^2}$

$$= \lim_{x\to 0} e^{\log(\cos x)\frac{1}{x^2}} = e^{\lim_{x\to 0}\log(\cos x)\cdot\frac{1}{x^2}}$$

$$\lim_{x\to 0}\log(\cos x)\cdot\frac{1}{x^2} = \frac{0}{0}$$

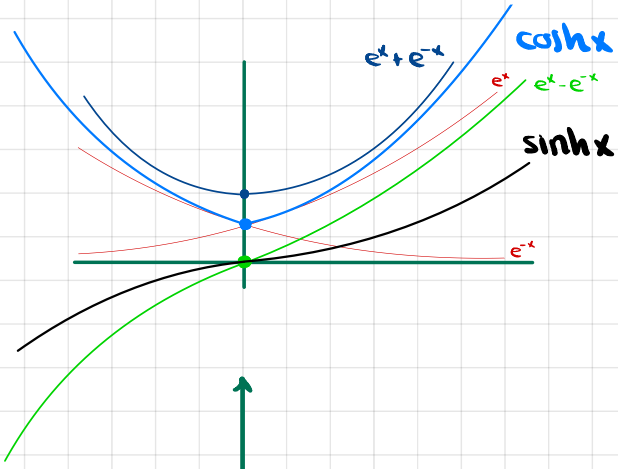
$$= \lim_{x\to 0}\frac{\frac{-\sin x}{\cos x}}{2x} = \lim_{x\to 0}\frac{-\sin x}{2x\cos x} = \frac{0}{0} = \lim_{x\to 0}\frac{-\frac{1}{\cos^2 x}}{2} = \lim_{x\to 0}\frac{-1}{2\cos^2 x} = -\frac{1}{2}$$

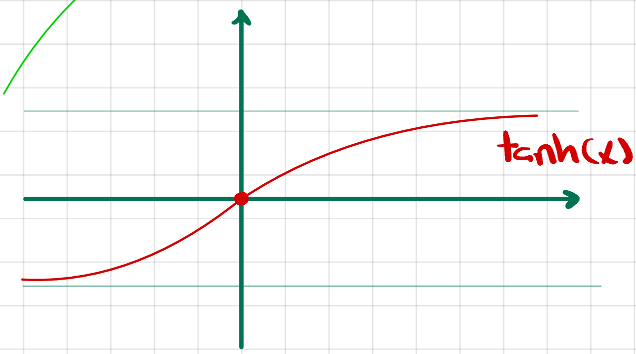
$\rightarrow \lim_{x\to 0}(\cos x)^{1/x^2} = e^{-\frac{1}{2}}$

7. $\sinh x = \dfrac{e^x - e^{-x}}{2}$ hyperbolic sine

$\cosh x = \dfrac{e^x + e^{-x}}{2}$ hyperbolic cosine

$\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = 1 - \dfrac{2}{e^{2x}+1}$ hyperbolic tangent





8. (a) $\cosh^2 - \sinh^2 = 1$

Proof

$$\frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4} = \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - \cancel{e^{2x}} + 2 - \cancel{e^{-2x}}}{4} = 1$$

(b) $\tanh^2 + \dfrac{1}{\cosh^2} = 1$

Proof

$$\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 + \frac{4}{(e^x + e^{-x})^2} = \frac{e^{2x} - 2 + e^{-2x} + 4}{(e^x + e^{-x})^2} = 1$$

(c) $\sinh(x+y) = \sinh x \cdot \cosh y + \cosh x \sinh y$

Proof

$$\sinh(x+y) = \frac{e^{x+y} - e^{-(x+y)}}{2} = \frac{e^x e^y - e^{-x} e^{-y}}{2} = \frac{2(e^x e^y - e^{-x} e^{-y})}{4} + \frac{e^x e^{-y} - e^x e^{-y} + e^{-x} e^y - e^{-x} e^y}{4}$$

$$= \frac{(e^x e^y + e^x e^{-y} - e^{-x} e^y - e^{-x} e^{-y}) + (e^x e^y - e^x e^{-y} + e^{-x} e^y - e^{-x} e^{-y})}{4}$$

$$= \frac{(e^x - e^{-x})}{2}\frac{(e^y + e^{-y})}{2} + \frac{(e^x + e^{-x})}{2}\frac{(e^y - e^{-y})}{2}$$

$$= \sinh x \cdot \cosh y + \cosh x \cdot \sinh y$$

(d) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Proof

$\cosh x \cosh y + \sinh x \sinh y$

$$= \frac{(e^x + e^{-x})(e^y + e^{-y}) + (e^x - e^{-x})(e^y - e^{-y})}{4}$$

$$= \frac{e^{x+y} + \cancel{e^{x-y}} + \cancel{e^{y-x}} + e^{-x-y} + e^{x+y} - \cancel{e^{x-y}} - \cancel{e^{y-x}} + e^{-x-y}}{4}$$

$$= \frac{e^{x+y} + e^{-x-y}}{2}$$

$= \cosh(x+y)$

(e) $\sinh' = \cosh$

Proof

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$(\sinh)'(x) = \frac{2(e^x + e^x)}{4} = \cosh'$$

(f) $\cosh' = \sinh$

Proof

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\cosh'(x) = \frac{2(e^x - e^{-x})}{4} = \sinh x$$

(g) $\tanh' = \dfrac{1}{\cosh^2}$

Proof

$$\tanh(x) = 1 - \frac{2}{e^{2x}+1}$$

$$\tanh'(x) = \frac{2e^{2x} \cdot 2}{(e^{2x}+1)^2}$$

$$\frac{1}{\cosh^2(x)} = \frac{4}{(e^x + e^{-x})^2} = \frac{4}{\left[\dfrac{e^{2x}+1}{e^x}\right]^2} = \frac{4e^{2x}}{(e^{2x}+1)^2}$$

Alternatively

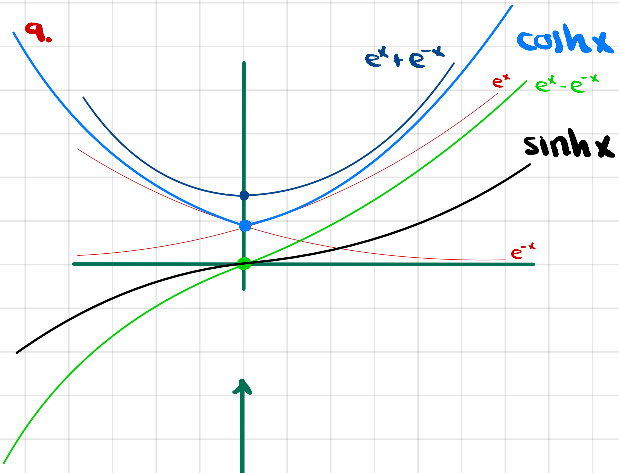
$$\tanh^2(x) = 1 - \frac{1}{\cosh^2 x}$$

$$2\tanh(x)\tanh'(x) = \frac{\cancel{2\cosh(x)}\cosh'(x)}{(\cosh x)^{\cancel{4}\,2}} = \cancel{2}\,\frac{\sinh x}{\cancel{\cosh x}}\tanh'(x)$$

$$\tanh'(x) = \frac{\cancel{\cosh'(x)}}{\cancel{\sinh x}\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x}$$

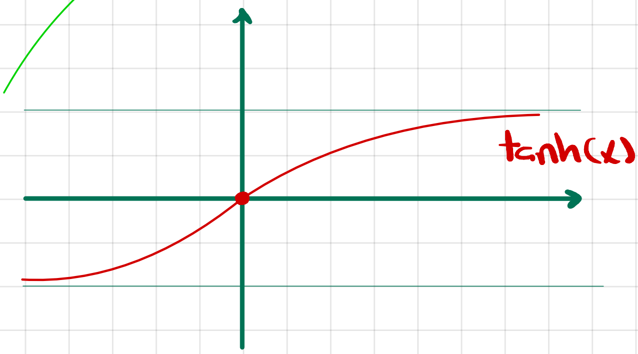
9.



$\operatorname{argsinh} = (\sinh)^{-1} : \mathbb{R} \to \mathbb{R}$

$\operatorname{argcosh} = (\cosh)^{-1} : [1, +\infty) \to [0, +\infty)$

$\operatorname{argtanh} = (\tanh)^{-1} : (-1, 1) \to \mathbb{R}$



(a) $\sinh(\cosh^{-1} x) = \sqrt{x^2 - 1}$

Proof

$\cosh^2(\cosh^{-1} x) - \sinh^2(\cosh^{-1} x) = 1$

$x^2 - \sinh^2(\cosh^{-1} x) = 1$

$\sinh^2(\cosh^{-1}(x)) = x^2 - 1$

$\sinh(\cosh^{-1}(x)) = \sqrt{x^2 - 1}$, the positive root because $\cosh^{-1} \geq 0 \to \sinh \geq 0$

(b) $\cosh(\sinh^{-1}(x)) = \sqrt{1 + x^2}$

Proof

$\cosh^2(\sinh^{-1}(x)) - \sinh^2(\sinh^{-1}(x)) = 1$

$\cosh^2(\sinh^{-1}(x)) - x^2 = 1$

$\cosh(\sinh^{-1}(x)) = \sqrt{1 + x^2}$, as

(c) $(\sinh^{-1})'(x) = \dfrac{1}{\sqrt{1+x^2}}$ $x \in \mathbb{R}$

Proof

$\cosh(\sinh^{-1}(x)) = \sqrt{1+x^2}$

$\underbrace{\sinh(\sinh^{-1}(x))}_{x} \cdot (\sinh^{-1})'(x) = \dfrac{x}{\sqrt{1+x^2}}$

$(\sin^{-1})'(x) = \dfrac{1}{\sqrt{1+x^2}}$

(d) $(\cosh^{-1})'(x) = \dfrac{1}{\sqrt{x^2-1}}$ for $x > 1$

Proof

$\cosh^{-1}$ is defined on $[1, +\infty)$.

$\sinh(\cosh^{-1}x) = \sqrt{x^2-1}$, $x \in [1, +\infty)$

$\underbrace{\cosh(\cosh^{-1}x)}_{x} \cdot (\cosh^{-1})'(x) = \dfrac{x}{\sqrt{x^2-1}}$

$(\cosh^{-1})'(x) = \dfrac{1}{\sqrt{x^2-1}}$ $x > 1$

(e) $(\tanh^{-1})'(x) = \dfrac{1}{1-x^2}$ * tanh is fn on $\mathbb{R} \to (-1,1)$

Proof

inverse fn thm

$(\tanh^{-1})'(x) = \dfrac{1}{\tanh'(\tanh^{-1}(x))} = \cosh^2(\tanh^{-1}(x))$ $x \in (-1,1)$

$\tanh^2(\tanh^{-1}(x)) + \dfrac{1}{\cosh^2(\tanh^{-1}x)} = 1$

$x^2 + \dfrac{1}{(\tanh^{-1})'(x)} = 1$

$(\tanh^{-1})'(x) = \dfrac{1}{1-x^2}$ $x \in (-1,1)$

10. a) explicit Formulas for $\sinh^{-1}$, $\cosh^{-1}$, $\tanh^{-1}$

Let $y = \sinh^{-1}(x)$

Then $x = \sinh(y) = \dfrac{e^y - e^{-y}}{2}$

$$2x = \frac{e^{2y}-1}{e^y} \to 2xe^y = e^{2y}-1 \to (e^y)^2 - 2xe^y - 1 = 0$$

$$\Delta = 4x^2 - 4\cdot(-1) = 4x^2+4 = 4(x^2+1)$$

$$e^y = \frac{2x \pm 2\sqrt{x^2+1}}{2} = x \pm \sqrt{x^2+1}$$

But $e^y > 0$. $x - \sqrt{x^2+1} < 0$. T.F.

$e^y = x + \sqrt{1+x^2} > 0$ $\quad * \; 0 \le x + |x| = x + \sqrt{x^2} < x + \sqrt{x^2+1}$

$$y = \log(x + \sqrt{1+x^2})$$

$$\sinh^{-1}(x) = \log(x + \sqrt{1+x^2})$$

$$\sinh(\cosh^{-1}(x)) = \sqrt{x^2-1} = \frac{e^{\cosh^{-1}x} - e^{-\cosh^{-1}x}}{2} \qquad x \in [1,\infty)$$

$$e^{2\cosh^{-1}x} - 1 = 2e^{\cosh^{-1}x}\sqrt{x^2-1}$$

$$e^{2\cosh^{-1}x} - 2e^{\cosh^{-1}x}\sqrt{x^2-1} - 1 = 0$$

$$\Delta = 4(x^2-1) - 4(-1) = 4(x^2-1+1) = 4x^2$$

$$e^{\cosh^{-1}x} = \frac{2\sqrt{x^2-1} \pm 2x}{2} = \sqrt{x^2-1} \pm x$$

But $e^{\cosh^{-1}x} > 0$ and $x > 0 \to 0 = |x| - x = \sqrt{x^2} - x > \sqrt{x^2-1} - x \to \sqrt{x^2-1} - x < 0$

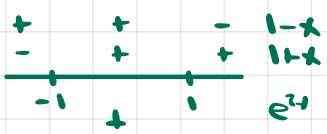
T.F. $e^{\cosh^{-1}x} = \sqrt{x^2-1} + x$

$$\cosh^{-1}(x) = \log(x + \sqrt{x^2-1})$$

$y = \tanh^{-1}(x) \qquad x \in (-1,1)$

$$x = \tanh(y) = 1 - \frac{2}{e^{2y}+1} \to x(e^{2y}+1) = e^{2y}+1-2 \to e^{2y}x + x = e^{2y}-1$$

$$\to e^{2y}(1-x) = 1+x \to e^{2y} = \frac{1+x}{1-x} \to 2y = \log\left(\frac{1+x}{1-x}\right)$$



$$\to y = \tanh^{-1}(x) = \log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\left(\log(1+x) - \log(1-x)\right)$$

(b)

$$\int_a^b \frac{1}{\sqrt{1+x^2}}\,dx$$

if $f(x) = \sinh^{-1}(x)$ then $f'(x) = (\sinh^{-1})'(x) = \frac{1}{\sqrt{1+x^2}}$

$\frac{1}{\sqrt{1+x^2}}$ is continuous hence integrable.

FTC2 $\to \int_a^b \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1}(b) - \sinh^{-1}(a)$

$\int_a^b \frac{1}{\sqrt{x^2-1}}\,dx$ for $a,b > 1$ or $a,b < -1$

For $x > 1$, $(\cosh^{-1})'(x) = \frac{1}{\sqrt{x^2-1}}$

Note

$\cosh : \mathbb{R} \to [1,\infty)$

$\cosh^{-1} : [1,\infty) \to [0,\infty)$

$(\cosh^{-1})' : [1,\infty) \to (0,\infty)$

Thus, if $a,b > 1$ then FTC2 $\to \int_a^b \frac{1}{\sqrt{x^2-1}}\,dx = (\cosh^{-1})'(b) - (\cosh^{-1})'(a)$

Let $f(x) = \frac{1}{\sqrt{x^2-1}}$. f is even.

Hence $\int_a^b f = \int_{-b}^{-a} f$ if $a,b < 0$.

Thus, for $a,b < -1$

$$\int_a^b \frac{1}{\sqrt{x^2-1}}\,dx = \int_{-b}^{-a} \frac{1}{\sqrt{x^2-1}}\,dx = (\cosh^{-1})'(-a) - (\cosh^{-1})'(-b)$$

In summary,

$$\int_a^b \frac{1}{\sqrt{x^2-1}}\,dx = \begin{cases} (\cosh^{-1})'(b) - (\cosh^{-1})'(a) & a,b > 1 \\ (\cosh^{-1})'(-a) - (\cosh^{-1})'(-b) & a,b < -1 \end{cases}$$

$\int_a^b \frac{1}{1-x^2}\,dx$ for $|a|,|b| < 1$

Note that tanh is defined on $\mathbb{R}$, maps to $(-1,1)$. $\tanh^{-1}$ maps $(-1,1)$ to $\mathbb{R}$, and for $x \in (-1,1)$ $(\tanh^{-1})'(x) = \frac{1}{1-x^2}$.

Since $\frac{1}{1-x^2}$ cont. on $(-1,1)$ then for $a,b \in (-1,1)$ we have by FTC2

$$\int_a^b \frac{1}{1-x^2}\,dx = (\tanh^{-1})'(b) - (\tanh^{-1})'(a)$$

If we had written $\frac{1}{1-x^2} = \frac{1}{2}\left[\frac{1}{1-x} + \frac{1}{1+x}\right]$ then since $[-\log(1-x)]' = \frac{1}{1-x}$ and $[\log(1+x)]' = \frac{1}{1+x}$ we have

$$\int_a^b \frac{1}{1-x^2}\,dx = \frac{1}{2}\int_a^b \frac{1}{1-x}\,dx + \frac{1}{2}\int_a^b \frac{1}{1+x}\,dx = \frac{1}{2}\left[-\log(1-b) + \log(1-a) + \log(1+b) - \log(1+a)\right]$$

$$= \frac{1}{2}\left(\log(1+b) - \log(1-b)\right) + \frac{1}{2}\left(\log(1+a) - \log(1-a)\right)$$

$$= \tanh^{-1}(b) - \tanh^{-1}(a)$$

---

11. $F(x) = \int_2^x \frac{1}{\log t}\,dt$ not bounded on $[2,\infty)$

Proof

Let $f(x) = \frac{1}{\log x}$.

$$f'(x) = \frac{-\frac{1}{x}}{(\log x)^2} = \frac{-1}{x(\log x)^2}$$

	$-$	$-$	-1
	$+$	$+$	$(\log x)^2$
	$+$	$+$	x
0	$-$ 1	$-$	$f' < 0$

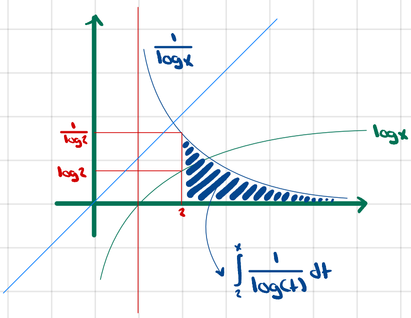
$$f(2) = \frac{1}{\log 2}$$

Let's show that for $x > 2$, $f(x)$ lies above $\frac{1}{x}$. Then, $\int_2^x f(t)\,dt > \int_2^x t^{-1}\,dt$ and $\lim_{x\to\infty}\int_2^x f(t)\,dt > \lim_{x\to\infty}\int_2^x t^{-1}\,dt = \infty$.

Assume $x > 0$.

Both x and e^x are increasing but $0 < e^0 = 1$ and for all $x > 0$ we have $[e^x]' = e^x > 1 = [x]'$

Hence $e^x > x \rightarrow x > \log x \rightarrow \frac{1}{\log x} > \frac{1}{x}$



12. f nondecreasing on $[1,\infty)$.

$F(x) = \int_1^x \frac{f(t)}{t}\,dt \quad x \geq 1$

$\rightarrow \left[f \text{ bounded on } [1,\infty) \leftrightarrow \frac{F}{\log} \text{ bounded on } [1,\infty) \right]$

Proof

Assume f bounded on $[1,\infty)$ $\rightarrow$ $\exists N$ s.t. $-N \leq f(x) \leq N$ for all $x \in [1,\infty)$.

$$-N = \frac{-N \int_1^x t^{-1}dt}{\int_1^x t^{-1}dt} \leq \frac{\int_1^x \frac{f(t)}{t}dt}{\int_1^x t^{-1}dt} \leq \frac{N\int_1^x t^{-1}dt}{\int_1^x t^{-1}dt} = N$$

$$\rightarrow -N \leq \frac{F(x)}{\log(x)} \leq N$$

$$\rightarrow \frac{F(x)}{\log(x)} \text{ bounded on } [1,\infty).$$

T.F. f bounded on $[1,\infty)$ $\rightarrow$ $\frac{F(x)}{\log(x)}$ bounded on $[1,\infty)$.

Now assume $\frac{F(x)}{\log(x)}$ bounded on $[1,\infty)$

$$\exists N \text{ s.t. } \quad -N \leq \frac{\int_1^x \frac{f(t)}{t}dt}{\int_1^x t^{-1}dt} \leq N$$

$$\forall x, x > 1 \rightarrow -N\int_1^x t^{-1}dt \leq \int_1^x \frac{f(t)}{t}dt \leq N\int_1^x t^{-1}dt \quad (1)$$

Assume f not bounded on $[1,\infty)$.

$\forall N\ \exists x, x \in [1,\infty) \wedge f(x) > N$

Let $N = N$ and x_1 s.t. $x_1 \in [1,\infty) \wedge f(x_1) > N$

Then, since f nondecreasing, $\forall x, x > x_1 \rightarrow f(x) \geq f(x_1) > N$

Hence, $\int_{x_1}^x \frac{f(t)}{t}dt > N\int_{x_1}^x t^{-1}dt$

Let $x > x_1 > 1$. Then

$$\int_1^x \frac{f(t)}{t}dt = \int_1^{x_1} \frac{f(t)}{t}dt + \int_{x_1}^x \frac{f(t)}{t}dt > N\int_{x_1}^x t^{-1}dt$$

which contradicts (1).

By proof by contradiction, f bounded on $[1,\infty)$.

T.F. $\frac{F(x)}{\log(x)}$ bounded on $[1,\infty)$ $\rightarrow$ f bounded on $[1,\infty)$

Solution Manual Proof (This is an attempt to understand it)

Assume f bounded on $[1,\infty)$. Then, $|f| \le M$ on $[1,\infty)$ and $F(x) = \int_1^x \frac{f(t)}{t}dt \le |F(x)| = \int_1^x \frac{|f(t)|}{t}dt \le M\int_1^x t^{-1}dt = M\log x$

$\rightarrow \frac{|F(x)|}{\log x} \le M$ for $x \ge 1$

T.F. f bounded on $[1,\infty)$ $\rightarrow$ $\frac{F(x)}{\log(x)}$ bounded on $[1,\infty)$.

Assume $f \ge 0$ on $[1,\infty)$. Since f nondecreasing, then

$$F(x) = \int_1^x \frac{f(t)}{t}dt \le f(x)\int_1^x t^{-1}dt = f(x)\log x$$

$$\rightarrow \frac{F(x)}{\log x} \le f(x)$$

?????? not sure what conclusion they reach here.

Let $b>1$, $f(b)=0$ and $x \ge b$.

$$F(x) = \int_1^b \frac{f(t)}{t}dt + \int_b^x \frac{f(t)}{t}dt$$

$$\le \int_1^b \frac{f(t)}{t}dt + f(x)\int_b^x t^{-1}dt \quad = \int_1^b \frac{f(t)}{t}dt + f(x)\left(\int_1^x t^{-1}dt - \int_1^b t^{-1}dt\right)$$

$$= \int_1^b \frac{f(t)}{t}dt + f(x)(\log x - \log b)$$

$$\rightarrow \frac{F(x)}{\log x} = \frac{1}{\log x}\int_1^b \frac{f(t)}{t}dt + \frac{\log x - \log b}{\log x} f(x)$$

$$= A(x) + B(x)f(x)$$

Note that

$A(x) = \frac{1}{\log x}\int_1^b \frac{f(t)}{t}dt$ is bounded: $\int_1^b \frac{f(t)}{t}dt \le 0$ because by assump. $f(b)=0$ and f nondecreasing, $A(x)$ gets smaller in magnitude (i.e. increases as x increases). Thus $A(x)$ bounded above by zero and below by $\frac{1}{\log b}\int_1^b t^{-1}dt$. We can also just say: $|A(x)|$ is bounded.

$B(x)$ is also bounded. $0 < \frac{\log x - \log b}{\log x} < 1$ and $\lim_{x\to\infty} \frac{\log x - \log b}{\log x} = \lim_{x\to\infty} \frac{1/x}{1/x} = 1$

Hence,

$f(x) = \frac{1}{B(x)}\frac{F(x)}{\log x} - \frac{A(x)}{B(x)}$. Since $\frac{1}{B(x)}$, $A(x)$, and $\frac{F(x)}{\log x}$ are bounded so is $f(x)$.

13. (a) $\lim_{x\to\infty} a^x \quad 0<a<1$

$a^x = e^{x\log a}$

$0<a<1 \to \log a = \int_1^a t^{-1}dt < 0$

$\to a^x = e^{-x|\log a|} = \dfrac{1}{e^{x|\log a|}} = 0$

(b) $\lim_{x\to\infty} \dfrac{x}{(\log x)^n} = \dfrac{\infty}{\infty} = \lim_{x\to\infty} \dfrac{1}{n(\log x)^{n-1}\cdot \frac{1}{x}} = \lim_{x\to\infty} \dfrac{x}{n(\log x)^{n-1}} = (\dots) = \lim_{x\to\infty} \dfrac{1}{n!\cdot\frac{1}{x}} = \lim_{x\to\infty} \dfrac{x}{n!} = \infty$

Alternatively,

$y = \log x \to x = e^y$

$\lim_{x\to\infty} \dfrac{x}{(\log x)^n} = \lim_{y\to\infty} \dfrac{e^y}{y^n} = \infty$ by Th 6.

(c) $\lim_{x\to\infty} \dfrac{(\log x)^n}{x} = \lim_{y\to\infty} \dfrac{y^n}{e^y} = 0$

(d) $\lim_{x\to 0^+} x(\log x)^n$

$x(\log x)^n = x(-(\log 1 - \log x))^n = \dfrac{(-1)^n(\log\frac{1}{x})^n}{\frac{1}{x}}$

$\lim_{x\to 0^+} \dfrac{(-1)^n(\log\frac{1}{x})^n}{\frac{1}{x}} = \dfrac{\infty}{\infty}$

$= (-1)^n \lim_{x\to 0^+} \dfrac{(\log 1/x)^n}{1/x} = (-1)^n \cdot \lim_{x\to\infty} \dfrac{(\log x)^n}{x} = 0$

(e) $\lim_{x\to 0^+} x^x$

$x^x = e^{x\log x}$

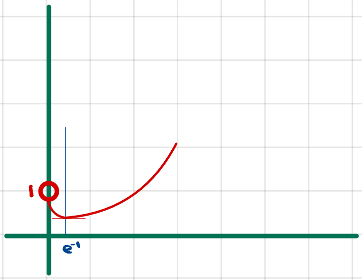
$\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} \dfrac{\log x}{\frac{1}{x}} = \dfrac{-\infty}{\infty} = \lim_{x\to 0^+} \dfrac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} (-x) = 0$

$\to \lim_{x\to 0^+} x^x = e^{\lim_{x\to 0^+} x\log x} = e^0 = 1$

14. $f(x) = x^x \quad x > 0$

$f(x) = x^x = e^{x\log x}$

$f'(x) = e^{x\log x}(\log x + 1) = x^x(1 + \log x)$



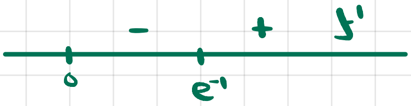
$\lim_{x\to 0^+} x^x = 1$ (Problem 13e)

$\lim_{x\to\infty} x^x = e^{\lim_{x\to\infty} x\log x} = \infty$

$1 + \log(x) = 0 \to \log(x) = -1$

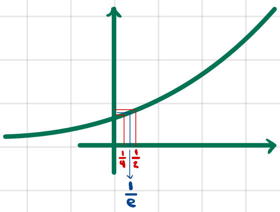
$\to x = \log^{-1}(-1) = \exp(-1) = e^{-1}$ critical point at e^{-1}

$\to f'(e^{-1}) = 0$



$f(e^{-1}) = \dfrac{1}{e^{1/e}} < 1$

$e^{1/e} = \log^{-1}(1/e) = \exp(1/e) > 1$



$f''(x) = e^{x\log x}(\log x + 1)^2 + e^{x\log x}\cdot\dfrac{1}{x}$

$= e^{x\log x}\left(\dfrac{1}{x} + (\log x + 1)^2\right) > 0$ convex

$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} e^{x\log x}(\log x + 1) = -\infty$

$=$

$\lim_{x\to 0^+} \dfrac{x^x}{\dfrac{1}{\log x}} = \dfrac{1}{0^-} = -\infty$

15. (a) min of $f(x)=\frac{e^x}{x^n}$, $x>0$

$f'(x)=\frac{e^x x^n - n x^{n-1} e^x}{x^{2n}}=0 = e^x x^{-n} - n e^x x^{-n-1} = \frac{e^x}{x^{n+1}}(x-n)$

n is critical point.

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^x}{x^n} = \infty$, $\lim_{x\to\infty} \frac{e^x}{x^n}=\infty$ (Th.6) $\to$ the critical point is a min

$\to \forall x, x>n \to f(x)>f(n) \to f(x)>\frac{e^n}{x^n}$

(b) $f'(x)=\frac{e^x(x-n)}{x^{n+1}} \to f'(x)>\frac{e^{n+1}}{(n+1)^{n+1}}$ for $x>n+1$

Proof

$f'(n)=0$

$f'(n+1)=\frac{e^{n+1}}{(n+1)^{n+1}}$

$x>n+1 \to (x-n)>1 \to f'(x)>\frac{e^x}{x^{n+1}}$

By part a), $\frac{e^x}{x^{n+1}}$ has min at $n+1$.

Hence,

$f'(x)>\frac{e^x}{x^{n+1}}>\frac{e^{n+1}}{(n+1)^{n+1}}$ for $x>n+1$

$f(x)=\int_1^x f'(x)dx > \int_1^x \frac{e^{n+1}}{(n+1)^{n+1}}dt = (x-1)\frac{e^{n+1}}{(n+1)^{n+1}}$

$\lim_{x\to\infty}(x-1)\frac{e^{n+1}}{(n+1)^{n+1}}=\infty \to \lim_{x\to\infty} f(x)=\infty$

16. $f(x) = \dfrac{e^x}{x^n}$

Problem is → min at $x = n$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{e^x}{x^n} = \infty$$

$$\lim_{x \to \infty} \frac{e^x}{x^n} = \infty \text{ (Th.6)}$$

$$e^x x^n (xn + x - n^2 - n)$$

$$f'(x) = \frac{e^x}{x^{n+1}}(x-n)$$

$$f''(x) = \frac{[e^x(x-n) + e^x]x^{n+1} - e^x(x-n)(n+1)x^n}{x^{2n+2}}$$

$$= \frac{e^x x^{n+2} - ne^x x^{n+1} + \cancel{e^x x^{n+1}} - ne^x x^{n+1} - \cancel{e^x x^{n+1}} + e^x n^2 x^n + e^x n x^n}{x^{2n+2}}$$

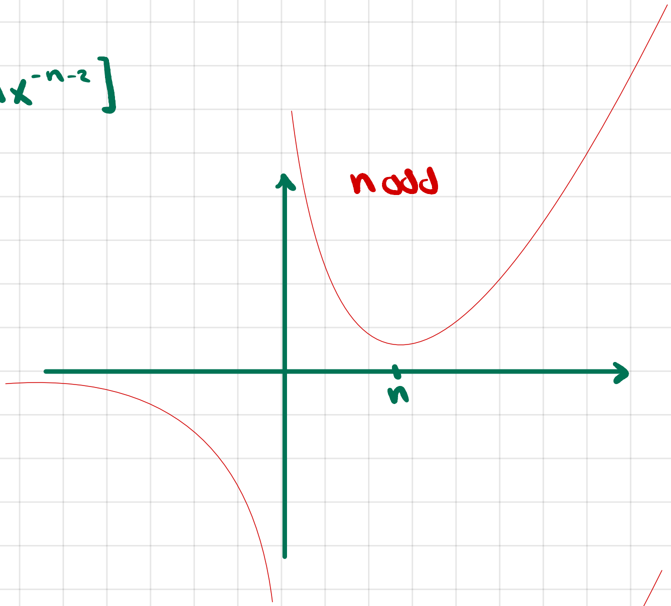
$$= \frac{e^x x^{n+2} - 2ne^x x^{n+1} + e^x n^2 x^n + e^x n x^n}{x^{2n+2}}$$

$$= e^x \left[x^{-n} - 2n x^{-n-1} + n^2 x^{-n-2} + n x^{-n-2} \right]$$

$$= \frac{e^x}{x^{n+2}} (x^2 - 2nx + n^2 + n)$$

$$\Delta = 4n^2 - 4n^2 - 4n = -4n$$

→ $f''(x) > 0$ → f convex



$$\lim_{x \to 0^-} \frac{e^x}{x^n}$$

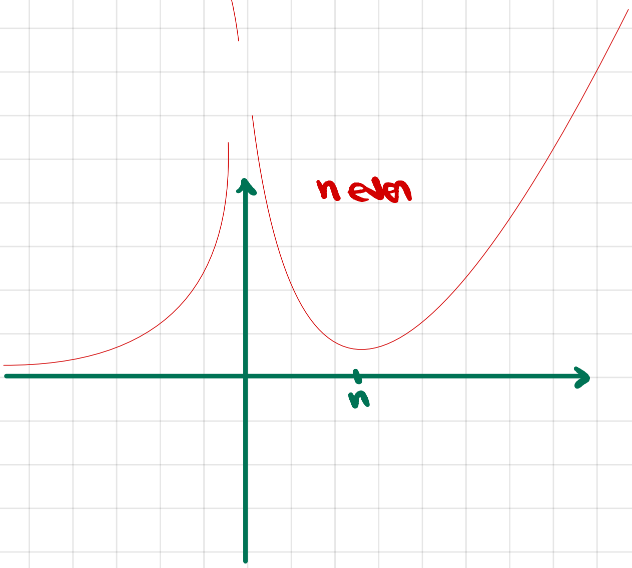
if n even then $= \infty$

if n odd then $= -\infty$

$$\lim_{x \to -\infty} \frac{e^x}{x^n}$$

if n even then $= 0^+$

if n odd then $= 0^-$



17. (a) $\lim_{y\to 0} \frac{\log(1+y)}{y}$

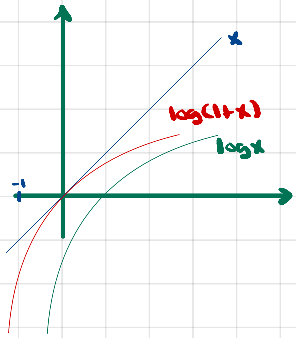
$$\log'(1) = \lim_{h\to 0} \frac{\log(1+h) - \log(1)}{h}$$

$$= \lim_{h\to 0} \frac{\log(1+h)}{h}$$

$$= \lim_{h\to 0} \frac{\int_1^{1+h} t^{-1}\,dt}{h}$$

Proof inside the proof of FTC1

$$= \frac{1}{1} = 1$$



Alternatively,

since $\lim_{y\to 0} \frac{\log(1+y)}{y} = \log'(1)$

and $\log'(x) = \frac{1}{x}$ then $\lim_{y\to 0} \frac{\log(1+y)}{y} = 1$

Or just use L'Hôpital's Rule

$$\lim_{y\to 0} \frac{\log(1+y)}{y} = \lim_{y\to 0} \frac{1/(1+y)}{1} = 1$$

(b) $\lim_{x\to\infty} x\log(1+\frac{1}{x}) = \lim_{x\to\infty} \frac{\log(1+\frac{1}{x})}{\frac{1}{x}}$

Let $y = \frac{1}{x}$

Then, $= \lim_{y\to 0} \frac{\log(1+y)}{y} = 1$ by (a)

Altern., use L'Hôpital

$$\lim_{x\to\infty} x\log(1+\frac{1}{x}) = \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot (-1)\cdot\frac{1}{x^2}}{\frac{-1}{x^2}} = 1$$

(c) $e = \lim_{x\to\infty} (1+\frac{1}{x})^x = e^{\lim_{x\to\infty} \log(1+1/x)\cdot x} = e^1 = e$

$$(1+\frac{1}{x})^x = e^{\log(1+\frac{1}{x})\cdot x}$$

$$\lim_{x\to\infty} \frac{\log(1+\frac{1}{x})}{\frac{1}{x}} = \frac{0}{0} = \lim_{x\to\infty} \frac{\frac{-1/x^2}{1+1/x}}{-1/x^2} = \lim_{x\to\infty} \frac{1}{1+1/x}$$

$$= 1$$

Altern. $e = \log^{-1}(1) = \log^{-1}[\lim_{x\to\infty} x\log(1+1/x)]$

$$= \lim_{x\to\infty} \log^{-1}(x\log(1+1/x))$$

$$= \lim_{x\to\infty} e^{x\log(1+1/x)}$$

$$= \lim_{x\to\infty} (1+\frac{1}{x})^x$$

(d) $e^a = \lim_{x\to\infty} (1+\frac{a}{x})^x$

$$e^a = \left(\lim_{x\to\infty} (1+\frac{1}{x})^x\right)^a = \lim_{x\to\infty} (1+\frac{1}{x})^{ax} = \lim_{ax\to\infty} (1+\frac{1}{x})^{ax}$$

Let $y = ax$. Then $x = \frac{y}{a}$.

$$= \lim_{y\to\infty} (1+\frac{a}{y})^y$$

(e) $\log b = \lim_{x\to\infty} x(b^{1/x}-1)$

$$\lim_{x\to\infty} x(b^{1/x}-1) = \lim_{x\to\infty} \frac{b^{1/x}-1}{1/x} = \lim_{x\to\infty} \frac{e^{\frac{1}{x}\log b}-1}{1/x} = \frac{0}{0} = \lim_{x\to\infty} \frac{e^{\frac{1}{x}\log b}\cdot \log b \cancel{\frac{(-1)}{x^2}}}{\cancel{\frac{-1}{x^2}}} = \lim_{x\to\infty} \log b \cdot e^{\frac{1}{x}\log b} = \log b$$

Also n.,

$$f(x) = e^{bx} \to f'(x) = be^{bx} \to f'(0) = b$$

$$\lim_{y\to 0} \frac{e^{by}-1}{y} = \frac{0}{0} = \lim_{y\to 0} \frac{b}{1} = b$$

Thus

$$\lim_{x\to\infty} x(e^{b/x}-1) = \lim_{x\to\infty} \frac{e^{b\cdot\frac{1}{x}}-1}{1/x} = b$$

$$\lim_{x\to\infty} x(e^{\frac{\log b}{x}}-1) = \lim (b^{1/x}-1)x = \log b$$

18. $f(x) = (1+\frac{1}{x})^x, \ x > 0$

From Problem 17c, $\lim_{x\to\infty} (1+\frac{1}{x})^x = e.$

$f(x) = e^{\log(1+\frac{1}{x})\cdot x}$

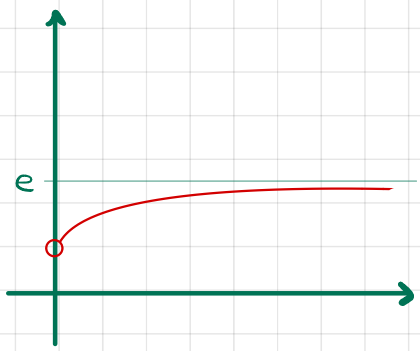
$\lim_{x\to 0^+} f(x) = e^{\lim_{x\to 0^+} x\log(1+\frac{1}{x})} = 1$

$$\lim_{x\to 0^+} x\log(1+1/x) = \lim_{x\to 0^+} \frac{\log(1+\frac{1}{x})}{\frac{1}{x}} = \frac{\infty}{\infty} = \lim_{x\to 0^+} \frac{\frac{-1/x^2}{1+1/x}}{-1/x^2} = \lim_{x\to 0^+} \frac{1}{1+1/x} = 0$$

$$f'(x) = e^{\log(1+\frac{1}{x})\cdot x} \cdot \left[\frac{\frac{-1}{x^2}}{1+\frac{1}{x}} \cdot x + \log(1+\frac{1}{x}) \right] = \underbrace{e^{\log(1+\frac{1}{x})\cdot x}}_{>0} \cdot \underbrace{\left[\frac{-1}{1+x} + \log(1+\frac{1}{x}) \right]}_{>0} > 0$$

$g(x) = \frac{-1}{1+x} + \log(1+\frac{1}{x})$

$$g'(x) = \frac{1}{(1+x)^2} - \frac{1}{x(x+1)} = \frac{x-(x+1)}{x(x+1)^2} = \frac{-1}{x(x+1)^2}$$



$+ \quad | \quad - \quad g'$

0

$\lim_{x\to 0^+} g(x) = \infty$

$\lim_{x\to\infty} g(x) = 0$

Thus, g is decreasing and as $x\to\infty$ g goes to 0.

$\to g(x) > 0$

$\to f'(x) > 0$

19. a% interest per annum

initial investment = I

one-year yield = $I(1+\frac{a}{100})$

n-year yield = $I(1+\frac{a}{100})^n$

n-year yield, interest given twice a year = $I(1+\frac{a}{100}\cdot\frac{1}{2})^{2n}$

n-year yield, interest given k times a year = $I(1+\frac{a}{100}\cdot\frac{1}{k})^{kn}$

$f(x) = (1+\frac{c}{x})^x = e^{\log(1+\frac{c}{x})\cdot x}$

$\lim_{x\to\infty} f(x) = e^c$ (Problem 17d)

$f'(x) > 0$

$f(1) = 1+c$

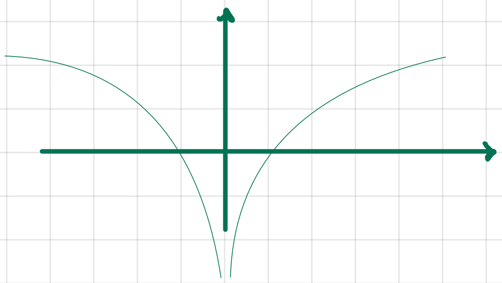
x is the number of times interest is given.

$c = \frac{a}{100}$

-> An initial investment of 1 dollar yields $e^{a/100}$ dollars after a year.

20. (a) $f(x) = \log|x|,\ x \neq 0 \ \rightarrow\ f'(x) = \frac{1}{x},\ x \neq 0$

Proof



$$f(x) = \begin{cases} \log x & x > 0 \\ \log(-x) & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 1/x & x > 0 \\ 1/x & x < 0 \end{cases}$$

(b) $f(x) \neq 0$ for all $x \ \rightarrow\ (\log|f|)' = \frac{f'}{f}$

$$h(x) = \log|f(x)| = \begin{cases} \log(f(x)) \\ \log(-f(x)) \end{cases}$$

$$h'(x) = \begin{cases} \frac{f'(x)}{f(x)} \\ \frac{-f'(x)}{-f(x)} = \frac{f'(x)}{f(x)} \end{cases}$$

Alternatively,

let $g(x) = \log|x|$

Then $h(x) = \log|f(x)| = g(f(x))$

$$h'(x) = g'(f(x))f'(x) = \frac{1}{f(x)} \cdot f'(x)$$

21. $f' = cf$ on some interval

(a) $f \neq 0 \to |f(x)| = \ell e^{cx}$, $\ell > 0 \to f(x) = ke^{cx}$, for some k

Proof

$\frac{f'}{f} = c \to (\log|f(x)|)' = c$

FTC2 $\to \int_a^x (\log|f(x)|)' dx = \log|f(x)| - \log|f(a)| = \int_a^x c\,dx = c(x-a)$

$\log|f(x)| = cx - ca + \log|f(a)| = cx + d$

$\log^{-1}$ is defined everywhere so

$|f(x)| = e^d e^{cx} = ke^{cx}$

(b) only assumption: $f' = cf$ on some interval (f can be zero on the interval)

Let $[a,b]$ be the interval.

f is diff hence cont. on the interval.

If $f = 0$ on the entire interval then $f' = 0$.

But if $f(x) \neq 0$ for some point, let's say $f(x) > 0$, then by continuity it is > 0 on some (m,n)

By part (a), on (m,n), $f(x) = ke^{cx}$, $k > 0$. Hence $f(x) > 0$ on (m,n)

Assume $f(n) = 0$. Then f is discontinuous at n. $\perp$.

Proof: $\lim_{x \to n} ke^{cx} = ke^{cn} > 0 \neq 0$.

Assume $f(m) = 0$. Then f is discont. at m. $\perp$.

$\to f > 0$ on $[m,n]$

This means that f must be ke^{cx} on the entire $[a,b]$.

A similar argument shows that if $f(x) < 0$ for some point then $f(x) = ke^{cx} < 0$ on some (m,n) containing the point, and because of continuity, $f(m)$ and $f(n) < 0$.

Thus $f < 0$ on all of $[a,b]$.

(c) $g(x) = \dfrac{f(x)}{e^{cx}}$

If we show that g is constant then we will have proved that $f(x) = ke^{cx}$.

$$g'(x) = \frac{f'(x)\cancel{e^{cx}} - f(x)c\cancel{e^{cx}}}{(e^{cx})^{\cancel{2}}} = \frac{f'(x) - cf(x)}{e^{cx}} = 0$$

$$\rightarrow g(x) = k = \frac{f(x)}{e^{cx}} \rightarrow f(x) = ke^{cx}$$

(d) $f' = fg'$ for some g $\rightarrow$ $f(x) = ke^{g(x)}$ for some k

Proof

$$h(x) = \frac{f(x)}{e^{g(x)}}$$

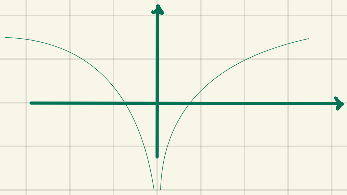
$$h'(x) = \frac{f'(x)\cancel{e^{g(x)}} - f(x)\cancel{e^{g(x)}}g'(x)}{(e^{g(x)})^{\cancel{2}}}$$

$$= \frac{f'(x) - f(x)g'(x)}{e^{g(x)}} = 0$$

$$\rightarrow h(x) = k \rightarrow f(x) = ke^{g(x)}$$

Problem Recap

$f(x) = \log|x|$



$f'(x) = \frac{1}{x}$

$h(x) = \log|f(x)|$

$h'(x) = \frac{f'(x)}{f(x)}$

If $h'(x) = c$ then $f'(x) = cf(x)$, ie rate of change proportional to fn value.

Note that for a fn such as $f(x) = x^n$ we have $f'(x) = nx^{n-1}$ and $\frac{f'(x)}{f(x)} = \frac{nx^{n-1}}{x^n} = \frac{n}{x}$, so the ratio is decreasing, even

though, since $f''(x) = n(n-1)x^{n-2}$ is positive for $n \geq 2$.

A fn like $f(x) = x^{100}$ increases faster than its derivative $100x^{99}$?

In problem 20b we show that there is a specific composite of a fn f s.t its derivative is the ratio $\frac{f'}{f}$.

That fn is $\log|f(x)|$. (If $f \neq 0$ for all x, that is).

Note that this ratio can be variable.

But what if $f'(x) = cf(x)$, for c some number?

If $f(x) \neq 0$ everywhere, then $\frac{f'(x)}{f(x)} = c$. But we know an antiderivative of $\frac{f'}{f}$ in this case: $\log|f(x)|$.

Hence $(\log|f(x)|)' = f'(x)/f(x) = c$, and FTC2 $\to$ $\log|f(x)| - \log|f(a)| = c(x-a)$ $\to$ $\log|f(x)| = cx + d$

$\to |f(x)| = e^d e^{cx} = ke^{cx}$, $k > 0$.

What if $f'(x) = cf(x)$ but $f(x)$ is not necessarily $\neq 0$ everywhere?

A few notes on why

If $g(x) = cx + d$ then $g'(x) = c$.

Also, $\log|f(x)| = \int_1^{|f(x)|} t^{-1}\,dt$ is a composition $f_1 \circ f_2$, where $f_1(x) = \log x$ and $f_2(x) = |f(x)|$.

Note that $f_2(x) = |f(x)|$ must be $\neq 0$ otherwise $\log|f(x)|$ is not defined.

f_1 is continuous on $(0,\infty)$, since t^{-1} is integrable on $(0,\infty)$

Since t^{-1} is continuous on $(0,\infty)$, then f_1 is diff. on $(0,\infty)$ and $f_1'(x) = \frac{1}{x}$.

What about $f_2(x) = |f(x)|$?

Since $f' = cf$ on some interval then f is diff. on this interval. Since $f \neq 0$ on the interval then either $f > 0$ or $f < 0$. Either way,

$|f|$ is diff on the interval.

Hence, on this interval, we can differentiate $f_1 \circ f_2$ using the chain rule.

$\log|f(x)| = f_1(f_2(x))$

From Problem 20b, $\log'|f(x)| = \dfrac{f'(x)}{f(x)}$

$g'(x)$ is a constant fn $g'(x) = c$.

The prime notation indicates that g' is the derivative of a fn g.

If g' is integrable on $[a,b]$ then

$$\int_a^b g'(t)\,dt = g(b) - g(a)$$

but this just says

$$\int_a^b c\,dt = g(b) - g(a)$$

Now if $g(x) = cx + d$ then $g'(x) = c$

so $g(b) - g(a) = cb + d - (ca + d) = c(b-a)$

22. $A(t)$ = amount of atoms at time t

$A'(t) = cA(t)$ for some c

(a) $A_0 = A(0) > 0$

By 21(b), $A(t) = ke^{ct}$

$A(0) = k$

$\rightarrow A(t) = A_0 e^{ct}$

(b) $\exists \tau$ s.t. $A(t+\tau) = \dfrac{A(t)}{2}$

Proof

$A(t) = A_0 e^{ct}$

$A(t_2) = A_0 e^{ct_2} = \dfrac{A_0 e^{ct}}{2}$

$\rightarrow e^{c(t_2 - t)} = \dfrac{1}{2}$

$c(t_2 - t) = -\log 2$

$t_2 = t - \dfrac{\log 2}{c}$

---

23. Surrounding temp = M

$T'(t) = c(T - M) = cg(t)$

But $(T-M)' = T'$ so

$(T-M)' = c(T-M)$

$\rightarrow T(t) - M = ke^{ct}$

$T(t) = M + ke^{ct}$

$T(0) = M + k \rightarrow k = T_0 - M$

$T(t) = M + (T_0 - M)e^{ct}$

24. $f(x) = \int_0^x f(t)\,dt \to f = 0$

f is integrable because $\int_0^x f(t)\,dt$ defined.

But then f is continuous by 13-8.

But then f is diff. by FTC1 applied to f, and

$f'(x) = f(x)$

By Problem 21(b), $f(x) = ke^x = f'(x)$.

$f(0) = 0 = k \to f(x) = 0$

---

25. f continuous.

(i) $\int_0^x f = e^x$ (1)

If (1) is true for all x then it is true for $x = 0$.

$\int_0^0 f = 0 = e^0 = 1$. $\perp$.

Therefore, there is no f satisfying (1).

Alternatively,

f cont $\to \left(\int_0^x f\right)' = f(x)$ (FTC1)

$\to f(x) = e^x$

But, $\int_0^x e^x\,dx = e^x - e^0 = e^x - 1 = e^x \to 0 = -1$. $\perp$.

(ii) $\int_0^{x^2} f = 1 - e^{2x^2}$

$\left(\int_0^{x^2} f\right)' = f(x^2) \cdot 2x = -4xe^{2x^2}$

$\to f(x^2) = -2e^{2x^2}$ $x \neq 0$

$\to f(x) = -2e^{2x}$ $x > 0$

$\int_0^{x^2} -2e^{2x}\,dx = -2\left.\frac{e^{2x}}{2}\right|_0^{x^2} = -(e^{2x^2} - 1) = 1 - e^{2x^2}$

26. $f'(t) = f(t) + \int_0^1 f(t)\,dt$ (1)

$\hookrightarrow$ f diff.

$\hookrightarrow$ f integrable on [0,1] constant term

$f'(t) = f(t) + c,$

$f''(t) = f'(t) \rightarrow f'(t) = ke^t \rightarrow f(t) = ke^t + c$

$$(1) \rightarrow f'(t) = \cancel{ke^t} = ke^t + c + \int_0^1 (ke^t + c)\,dt$$

$$= ke^t + c + ke^t\Big|_0^1 + ct\Big|_0^1$$

$$= \cancel{ke^t} + c + ke - k + c$$

$$2c = k - ke = k(1-e)$$

$$c = \frac{k(1-e)}{2}$$

$$\rightarrow f(t) = ke^t + \frac{k(1-e)}{2}$$

---

27. $(f(x))^2 = \int_0^x f(t)\,\frac{t}{1+t^2}\,dt$ (1)

f continuous

$\frac{t}{1+t^2}$ cont., t.f. $f(t)\frac{t}{1+t^2}$ cont.

$$2f(x)f'(x) = f(x)\cdot\frac{x}{1+x^2}$$

If $f(x) \neq 0$ then

$f'(x) = \frac{x}{2(1+x^2)}$ cont. everywhere

But $[\log(1+x^2)]' = \frac{2x}{1+x^2}$ $x \in \mathbb{R}$,

so $[\frac{1}{4}\log(1+x^2)]' = \frac{x}{2(1+x^2)} = f'(x)$

Hence,

$$f(x) - f(a) = \frac{1}{2}\int_a^x \frac{x}{1+x^2}\,dx = \frac{1}{4}\cdot\log(1+x^2)\Big|_a^x$$

$$= \frac{1}{4}\log(1+x^2) - \frac{1}{4}\log(1+a^2)$$

$$\rightarrow f(x) = \frac{1}{4}\log(1+x^2) + c$$

Consider the point $x=0$. (1) $\rightarrow (f(0))^2 = 0 \rightarrow f(0) = 0$.

$\rightarrow f(0) = 0 = c$

Now, wherever $f(x) \neq 0$, $f'(x) > 0$.

There are other possibilities for $f(x)$.

$f(x)$ could be 0 at all $x < a$, w/ $a \geq 0$.

Then

$$f(x) - \cancel{f(a)}^{\,0} = \frac{1}{2}\int_a^x \frac{x}{1+x^2}\,dx = \frac{1}{4}\cdot\log(1+x^2)\Big|_a^x$$

$$f(x) = \frac{1}{4}\left(\log(1+x^2) - \log(1+a^2)\right)$$

for $x \geq a$

ie

$$f(x) = \begin{cases} 0 & x \leq a \\ \frac{1}{4}\left(\log(1+x^2) - \log(1+a^2)\right) & x \geq a \end{cases}$$

28. (a) f, g cont. on $[a,b]$

g nonnegative

$$f(x) \le C + \int_a^x fg \quad \text{for some } C \text{ and } a \le x \le b$$

Then

$$f(x) \le C e^{\int_a^x g}$$ (Gronwall's Inequality)

Proof

Let $h(x) = \left(C + \int_a^x fg\right) e^{-\int_a^x g}$

f, g cont $\to$ $f \cdot g$ cont. $\to$ $\int_a^x fg$ diff.

g cont. $\to$ $\int_a^x g$ diff.

$$h'(x) = f(x)g(x)e^{-\int_a^x g} + \left(C + \int_a^x fg\right) e^{-\int_a^x g} \cdot (-g(x))$$

$$= \underbrace{e^{-\int_a^x g}}_{>0} \underbrace{g(x)}_{\ge 0} \underbrace{\left[f(x) - C - \int_a^x fg\right]}_{\le 0 \text{ by assumption}} \le 0$$

$\to$ h is non-increasing.

$h(a) = C \cdot e^0 = C$

Therefore for any $x \in [a,b]$, $h(x) = \left(C + \int_a^x fg\right) e^{-\int_a^x g} \le C \quad \to C + \int_a^x fg \le C e^{\int_a^x g}$

But then, using the third assumption, $f(x) \le C + \int_a^x fg$

(b) f, g nonnegative

g continuous

f differentiable $\rightarrow f = 0$

$f'(x) = g(x) f(x)$

$f(0) = 0$

Proof

$$f(x) = \int_0^x f'(x)\,dx = \int_0^x g(x)f(x)\,dx \leq 0 + \int_0^x g(x)f(x)\,dx$$

By Gronwall's Ineq.,

$$f(x) \leq 0 \cdot e^{\int_a^x g} = 0$$

But f is nonnegative so $f(x) = 0$.

Alternatively

$f'(0) = 0$

g cont. $\rightarrow \int_a^x g$ exists and $\left(\int_a^x g\right)' = g(x)$

T.F.

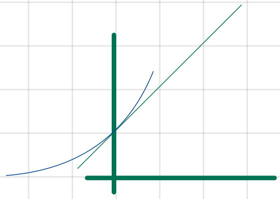
$$f'(x) = \left(\int_a^x g\right)' f(x)$$

Problem 21d $\rightarrow f(x) = k e^{\int_a^x g}$ for some k.

$f(0) = 0 \rightarrow k = 0$

29. (a) $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^n}{n!} \le e^x \quad x \ge 0$

$n=1 \to 1+\frac{x^1}{1!} = 1+x$



Let $f(x) = e^x$

$g(x) = 1+x$

$f'(x) = e^x$

$g'(x) = 1$

For $x>0$, $e^x > 1$ so $f'(x) > g'(x) = 1$

Hence for any $x>0$, $f(x) > 1+x$

$n=k$. Assume $1+\sum_{i=1}^{k} \frac{x^i}{i!} \le e^x$

Let $g(x) = 1+\sum_{i=1}^{k+1} \frac{x^i}{i!}$

Then $g'(x) = 1+\sum_{i=1}^{k} \frac{x^i}{i!} \le e^x = f'(x)$

By Problem 11-30, $f(x) \ge g(x)$ for $x \ge 0$.

$\to \; e^x \ge 1+\sum_{i=1}^{k+1} \frac{x^i}{i!} \quad x \ge 0$

By induction, $e^x \ge 1+\sum_{i=1}^{n} \frac{x^i}{i!}$ for all $n \in \mathbb{N}$.

(b) $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$

Proof

Since $\forall n, n \in \mathbb{N} \to 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^n}{n!}+\frac{x^{n+1}}{(n+1)!} \le e^x$

Then,

$$\frac{1}{x^n}+\frac{1}{x^{n-1}}+\frac{1}{2!}\cdot\frac{1}{x^{n-2}}+\frac{1}{3!}\cdot\frac{1}{x^{n-3}}+\dots+\frac{1}{n!}+\frac{x}{(n+1)!} \le \frac{e^x}{x^n}$$

$$\lim_{x\to\infty}\left[\frac{1}{x^n}+\frac{1}{x^{n-1}}+\frac{1}{2!}\cdot\frac{1}{x^{n-2}}+\frac{1}{3!}\cdot\frac{1}{x^{n-3}}+\dots+\frac{1}{n!}+\frac{x}{(n+1)!}\right] = \infty \le \lim_{x\to\infty}\frac{e^x}{x^n}$$

30. $\lim\limits_{x\to\infty} \frac{e^x}{x^n} = \frac{\infty}{\infty} = \lim\limits_{x\to\infty} \frac{e^x}{nx^{n-1}} = \frac{\infty}{\infty} = (\dots) = \lim\limits_{x\to\infty} \frac{e^x}{n!} = \infty$

31. (a) $\lim\limits_{x\to\infty} e^{-x^2}\int_0^x e^{t^2}dt$

$= \lim\limits_{x\to\infty} \frac{\int_0^x e^{t^2}dt}{e^{x^2}}$ $\quad \frac{\text{area under } e^{t^2}}{\text{rate of change of area under } e^{t^2}} \quad = \frac{\infty}{\infty}$

If $f(x) = \int_0^x e^{t^2}dt$ then $f'(x) = e^{x^2}$.

Thus, we want to compute $\lim\limits_{x\to\infty} \frac{f(x)}{f'(x)}$

Using L'Hôpital,

$= \lim\limits_{x\to\infty} \frac{e^{x^2}}{2xe^{x^2}} = \lim\limits_{x\to\infty} \frac{1}{2x} = 0$

(b) $\lim\limits_{x\to\infty} e^{-x^2}\int_x^{x+\frac{1}{x}} e^{t^2}dt = \lim\limits_{x\to\infty} \frac{\int_x^{x+\frac{1}{x}} e^{t^2}dt}{e^{x^2}}$

$e^{x^2} > 0$ for all x.

$\int_0^{x+\frac{1}{x}} e^{t^2}dt > \int_x^{x+\frac{1}{x}} e^{t^2}dt$

$\frac{\int_0^{x+\frac{1}{x}} e^{t^2}dt}{e^{x^2}} > \frac{\int_x^{x+\frac{1}{x}} e^{t^2}dt}{e^{x^2}}$

$0 = \lim\limits_{x\to\infty} \frac{\int_0^{x+\frac{1}{x}} e^{t^2}dt}{e^{x^2}} > \lim\limits_{x\to\infty} \frac{\int_x^{x+\frac{1}{x}} e^{t^2}dt}{e^{x^2}} > 0$

$\to \lim\limits_{x\to\infty} \frac{\int_x^{x+\frac{1}{x}} e^{t^2}dt}{e^{x^2}} = 0$

$\frac{\int_x^{x+\frac{1}{x}} e^{t^2}dt}{e^{x^2}} = \frac{\int_0^{x+\frac{1}{x}} e^{t^2}dt}{e^{x^2}} - \frac{\int_0^{x} e^{t^2}dt}{e^{x^2}}$

$\lim\limits_{x\to\infty} \frac{\int_x^{x+\frac{1}{x}} e^{t^2}dt}{e^{x^2}} = \lim\limits_{x\to\infty} \frac{\int_0^{x+\frac{1}{x}} e^{t^2}dt}{e^{x^2}} - \cancel{\lim\limits_{x\to\infty} \frac{\int_0^{x} e^{t^2}dt}{e^{x^2}}}^{\,0} = \frac{\infty}{\infty}$

Using L'Hôpital's Rule

$= \lim\limits_{x\to\infty} \frac{e^{(x+\frac{1}{x})^2}}{2xe^{x^2}} = \lim\limits_{x\to\infty} \frac{e^{2+\frac{1}{x^2}}}{2x} = 0$

$$\lim_{x\to\infty} e^{-x^2}\int_x^{x+\frac{\log x}{x}} e^{t^2}\,dt$$

$$= \lim_{x\to\infty} \frac{\int_0^{x+\frac{\log x}{x}} e^{t^2}\,dt}{e^{x^2}} - \cancel{\lim_{x\to\infty} \frac{\int_0^{x} e^{t^2}\,dt}{e^{x^2}}} \to 0$$

$$= \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{e^{\left(x+\frac{\log x}{x}\right)^2}\left(1+\frac{1-\log x}{x^2}\right)}{2xe^{x^2}} = \lim_{x\to\infty} \frac{e^{\left(2\log x+\frac{\log^2 x}{x^2}\right)}\left(1+\frac{1-\log x}{x^2}\right)}{2x}$$

$$= \lim_{x\to\infty} \frac{e^{2\log x}\cdot e^{\frac{\log^2 x}{x^2}}\left(1+\frac{1-\log x}{x^2}\right)}{2x} = \lim_{x\to\infty} \frac{\left(x^2+1-\log x\right)e^{\frac{\log^2 x}{x^2}}}{2x} = \lim_{x\to\infty}\Big(\underbrace{\frac{x}{2}}_{\infty}+\underbrace{\frac{1}{2x}}_{0}-\underbrace{\frac{\log x}{2x}}_{0}\Big)\underbrace{e^{\frac{\log^2 x}{x^2}}}_{1}$$

$$= \infty$$

$$\lim_{x\to\infty} e^{-x^2}\int_x^{x+\frac{\log x}{2x}} e^{t^2}\,dt = \lim_{x\to\infty} \frac{\int_0^{x+\frac{\log x}{2x}} e^{t^2}\,dt}{e^{x^2}} - \cancel{\lim_{x\to\infty} \frac{\int_0^{x} e^{t^2}\,dt}{e^{x^2}}} \to 0 \quad = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{e^{\left(x+\frac{\log x}{2x}\right)^2}}{2xe^{x^2}}\left(1+\frac{\frac{1}{\cancel{x}}\cdot 2\cancel{x}-2\log x}{4x^2}\right) = \lim_{x\to\infty} \frac{e^{\cancel{\log x}+\left(\frac{\log x}{2x}\right)^2}}{2\cancel{x}}\left(1+\frac{1-\log x}{2x^2}\right)$$

$$= \lim_{x\to\infty} \frac{e^{\left(\frac{\log x}{2x}\right)^2}}{2}\left(1+\frac{1-\log x}{2x^2}\right)$$

$$= \frac{1}{2}$$

38.

(a) $\log_a' x = \lim_{h\to 0} \dfrac{\log_a(x+h) - \log_a(x)}{h} = \lim_{h\to 0} \log_a\left[1+\dfrac{h}{x}\right]^{1/h}$

Since log is continuous on $(0,\infty)$, and $\left(1+\frac{h}{x}\right)^{1/h} > 0$, then $\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{1/h} > 0$ if it exists and thus log is cont. at this limit. Hence

$$\lim_{h\to 0} \log_a\left[1+\frac{h}{x}\right]^{1/h} = \log_a\left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{1/h}\right] = \log_a\left[\lim_{1/h\to\infty}\left(1+\frac{1}{x}\cdot\frac{1}{1/h}\right)^{1/h}\right] = \log_a\left[e^{1/x}\right]$$

Note an alternative algebraic manipulation

$$\log_a\left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{1/h}\right] = \log_a\left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}\cdot\frac{1}{x}}\right] = \log_a\left[\lim_{\frac{h}{x}\to 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}\cdot\frac{1}{x}}\right]$$

Let $k = \frac{h}{x}$. Then

$$= \log_a\left[\lim_{k\to 0}(1+k)^{\frac{1}{k}\cdot\frac{1}{x}}\right] = \frac{1}{x}\log_a\left[\lim_{k\to 0}(1+k)^{\frac{1}{k}}\right]$$

Let $y = \frac{1}{k}$

$$= \frac{1}{x}\log_a\left[\underbrace{\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y}_{\text{Problem 178 } = e}\right] = \frac{1}{x}\log_a e$$

(b) $a_n = (1+\frac{1}{n})^n,\ n \in \mathbb{N} \quad \rightarrow \quad a_n = 2 + \sum_{k=2}^{n} \frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n}) \cdots (1-\frac{k-1}{n})$

Proof

$$a_n = (1+\frac{1}{n})^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{1}{n}\right)^i$$

$$= \binom{n}{0} + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots + \binom{n}{n-1}\frac{1}{n^{n-1}} + \binom{n}{n}\frac{1}{n^n}$$

$$= 1 + 1 + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \cdots + \frac{n!}{(n-1)!} \cdot \frac{1}{n^{n-1}} + \frac{n!}{n!\,0!} \cdot \frac{1}{n^n}$$

$$= 2 + \frac{1}{2!}\frac{n-1}{n} + \frac{1}{3!}\frac{(n-1)(n-2)}{n^2} + \cdots + \frac{1}{(n-1)!}\frac{(n-1)(n-2)\cdots 2 \cdot 1}{n^{n-2}} + \frac{1}{n!} \cdot \frac{(n-1)(n-2)\cdots 2 \cdot 1}{n^{n-1}}$$

$$= 2 + \frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + \cdots + \frac{1}{(n-1)!} \cdot (1-\frac{1}{n})(1-\frac{2}{n}) \cdots (1-\frac{n-2}{n})$$

$$+ \frac{1}{n!}(1-\frac{1}{n}) \cdots (1-\frac{n-1}{n})$$

Hence

$$a_{n+1} = (1+\frac{1}{n+1})^{n+1}$$

$$= 2 + \frac{1}{2!}(1-\frac{1}{n+1}) + \frac{1}{3!}(1-\frac{1}{n+1})(1-\frac{2}{n+1}) + \cdots + \frac{1}{(n-1)!}(1-\frac{1}{n+1}) \cdots (1-\frac{n-2}{n+1})$$

$$+ \frac{1}{n!}(1-\frac{1}{n+1}) \cdots (1-\frac{n-1}{n+1})$$

$$+ \frac{1}{(n+1)!}(1-\frac{1}{n+1}) \cdots (1-\frac{n}{n+1})$$

Each term in the expr. for a_{n+1} is smaller than the corresp. term in a_n.

a_{n+1} then has an extra positive term.

T.F.

$$a_{n+1} > a_n$$

(c) $\frac{1}{k!} \leq \frac{1}{2^{k-1}}$ for $k \geq 2$ $\to a_n < 3$

Proof

$$a_n = 2 + \sum_{k=2}^{n} \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

$$\leq 2 + \sum_{k=2}^{n} \frac{1}{2^{k-1}}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

$$\leq 2 + \sum_{k=2}^{n} \frac{1}{2^{k-1}} = 2 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}}$$

$$= 2 + \frac{\frac{1}{2}\left(\frac{1}{2^{n-1}} - 1\right)}{-\frac{1}{2}}$$

$$= 2 + 1 - \frac{1}{2^{n-1}}$$

$$= 3 - \frac{1}{2^{n-1}}$$

$\to a_n < 3$

$\to \{a_1, a_2, a_3, \ldots\}$ bounded above, has supremum e.

Let $\epsilon > 0$.

$\exists n, n \in \mathbb{N} \wedge e - \epsilon < a_n < e \to -\epsilon < a_n - e < 0 \to 0 < e - a_n < \epsilon$

This is true because otherwise, $e - \epsilon$ would be an upper bound for $\{a_n\}$.

Let n_0 be one such n. Then, $\forall n, n \in \mathbb{N} \wedge n > n_0 \to e - \epsilon < a_{n_0} < a_n < e \to 0 < e - a_n < \epsilon$.

Recap

$$\log_a' x = \frac{1}{x} \log_a \left[\lim (1+h)^{1/h}\right]$$

we want to compute this (note that we've actually already done this in 17c)

$$a_n = \left(1+\frac{1}{n}\right)^n = 2 + \sum_{k=2}^{n} \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

$a_n < 3$

$\{a_i\}$ has supremum, some number that we don't know, but we're calling it e.

But if $n \leq x \leq n+1$ then $\frac{1}{n+1} \leq \frac{1}{x} \leq \frac{1}{n}$ and

$$\left(1+\frac{1}{n+1}\right)^n \leq \left(1+\frac{1}{x}\right)^n \leq \left(1+\frac{1}{x}\right)^x \leq \left(1+\frac{1}{n}\right)^x \leq \left(1+\frac{1}{n}\right)^n$$

In part d), we use this to show that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.

Going back to part a), this means

$$\log_a' x = \frac{1}{e}\log_a e$$

(d) $n \le x \le n+1 \rightarrow \left(1+\frac{1}{n+1}\right)^n \le \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{n}\right)^{n+1}$

If $n \le x \le n+1$ then $\frac{1}{n+1} \le \frac{1}{x} \le \frac{1}{n}$ so

$$\left(1+\frac{1}{n+1}\right)^n \le \left(1+\frac{1}{x}\right)^n \le \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{n}\right)^x \le \left(1+\frac{1}{n}\right)^n$$

$$\left(1+\frac{1}{n+1}\right)^n = \left[\left(1+\frac{1}{n+1}\right)^{n+1}\right]^{\frac{n}{n+1}}$$

in part c) we showed $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. Also, $\lim_{n\to\infty}\frac{n}{n+1} = 1$.

Hence, for large n, $\left[\left(1+\frac{1}{n+1}\right)^{n+1}\right]^{\frac{n}{n+1}}$ is close to e.

For any $\epsilon > 0$ we have

$$e - \epsilon < \left(1+\frac{1}{n+1}\right)^n \le \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{n}\right)^n < e$$

Since $n \to \infty \rightarrow x \to \infty$ we have

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

let $y = \frac{1}{x}$. $x \to \infty \rightarrow y \to 0$

$$\lim_{y\to 0^+}(1+y)^{1/y} = e$$

Now let's show $\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$.

First we show

$$\lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)^x}{\left(1-\frac{1}{x}\right)^x} = e^2$$

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = e^{-2} \cdot \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^{-1}$$

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\left(1+\frac{1}{-x}\right)^{-x} = e$$

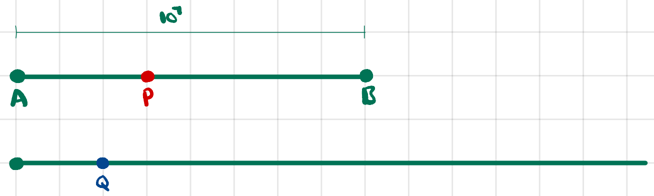
T.F.

Let $y = \frac{1}{x}$. Then $x \to -\infty \rightarrow y \to 0^-$.

$$\lim_{y\to 0^-}(1+y)^{1/y} = e$$

T.F. $\lim_{x\to 0}(1+x)^{1/x} = e$

33.



$P'(t) = 10^7 - P(t)$

$Q'(t) = 10^7$

Distance traveled by Q after time t defined to be **Napierian logarithm** of distance from P to B at time t

$$10^7 t = \text{Nap}\log[10^7 - P(t)]$$

Then,

$$\text{Nap}\log x = 10^7 \log \frac{10^7}{x}$$

Proof

$$[P(t) - 10^7]' = P'(t) = 10^7 - P(t) = -(P(t) - 10^7)$$

$$[P(t) - 10^7]' = -[P(t) - 10^7]$$

$$A'(t) = -A(t)$$

By Problem 31, $A(t) = he^{-t}$, ie $P(t) = 10^7 + he^{-t}$

$$P(0) = 10^7 + h \to h = P(0) - 10^7$$

$$P(t) = 10^7 + (P(0) - 10^7)e^{-t}$$

So,

$$10^7 t = \text{Nap}\log[\cancel{10^7} - \cancel{10^7} - (P(0) - 10^7)e^{-t}]$$

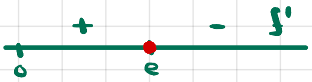
$$= \text{Nap}\log[(10^7 - P_0)e^{-t}]$$

$$\frac{(P_0 - 10^7)}{e^t} = x \to e^t = \frac{10^7 - P_0}{x} \to t = \log\left[\frac{10^7 - P_0}{x}\right]$$

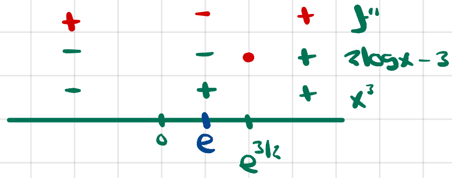
$$\text{Nap}\log x = 10^7 \cdot \log\left[\frac{10^7 - P_0}{x}\right]$$

34. (a) $f(x) = \dfrac{\log x}{x}$

$f'(x) = \dfrac{\frac{1}{x}\cdot x - \log x}{x^2} = \dfrac{1-\log x}{x^2}$



$f''(x) = \dfrac{-\frac{1}{x}x^2 - 2x(1-\log x)}{x^4} = \dfrac{-x-2x+2x\log x}{x^4} = \dfrac{2\log x - 3}{x^3}$

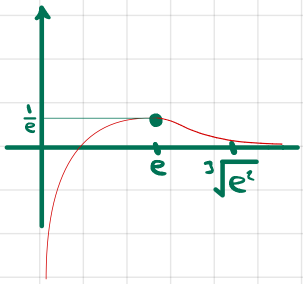


$\lim\limits_{x\to\infty} \dfrac{\log x}{x} = \lim\limits_{x\to\infty} \dfrac{1/x}{1} = 0$

e is local max.

$\lim\limits_{x\to 0^+} \dfrac{\log x}{x} = \dfrac{-\infty}{0^+} = -\infty$

$f(e) = e^{-1}$



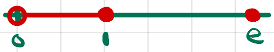
(b) e^π vs π^e

$\pi^e = e^{e\log\pi}$

let $f(x) = \dfrac{e\log\pi}{\pi}$

we know that $\dfrac{\log\pi}{\pi} < e^{-1}$. Hence $\dfrac{e\log\pi}{\pi} < 1 \to e\log\pi < \pi \to e^{e\log\pi} = \pi^e < e^\pi$

(c) $0 < x \le 1$ or $x = e \to x^y = y^x \iff y = x$



$x > 1,\ x \ne e \to$ there is exactly one $y \ne x$ s.t. $x^y = y^x$

also $x < e \to y > e$

$x > e \to y < e$

Proof

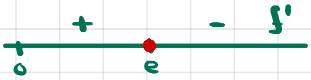
$x^y = e^{y \log x}$

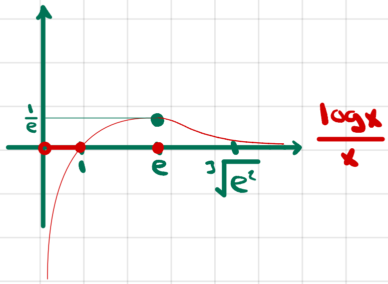
$x^y = y^x \to y \log x = x \log y \to \dfrac{\log x}{\log y} = \dfrac{x}{y} \to \dfrac{\log(x)}{x} = \dfrac{\log(y)}{y}$

$y^x = e^{x \log y}$

Let $f(x) = \dfrac{\log x}{x}$.

$f'(x) = \dfrac{\frac{1}{x} \cdot x - \log x}{x^2} = \dfrac{1 - \log x}{x^2}$





$0 < x \le 1 \to f(x) \le 0$ and $f'(x) > 0 \to f$ one-one

$\to x \ne y \to f(x) \ne f(y)$

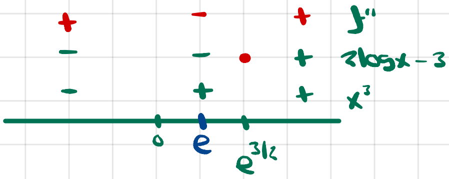
by contrapositive, $f(x) = f(y) \to x = y$

$x = e \to x$ is local and global max

Now, if $x > 1 \wedge x \ne e$ then

$0 < f(x) < e^{-1} = f(e)$

$f''(x) = \dfrac{2\log x - 3}{x^3}$



$1 < x < e \to f'(x) > 0$ and $f''(x) < 0$

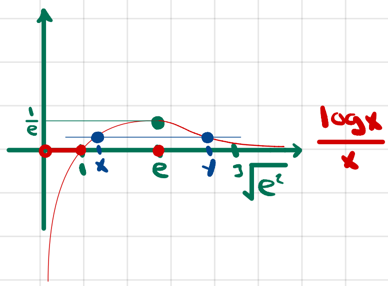
$e < x < \infty \to f'(x) < 0$ and $f''(x) > 0$

since $\lim_{x \to \infty} f(x) = 0$, we can get f as close to 0 as we want.

Hence

$1 < x < e \to f(x) \in (0, e^{-1})$

$e < x < \infty \to f(x) \in (0, e^{-1})$



Thus for any $1 < x < e$, there is a $e < y < \infty$ s.t. $f(x) = f(y)$

for any $e < y < \infty$ there is $1 < x < e$ s.t. $f(x) = f(y)$

(d) $x, y \in \mathbb{N}$

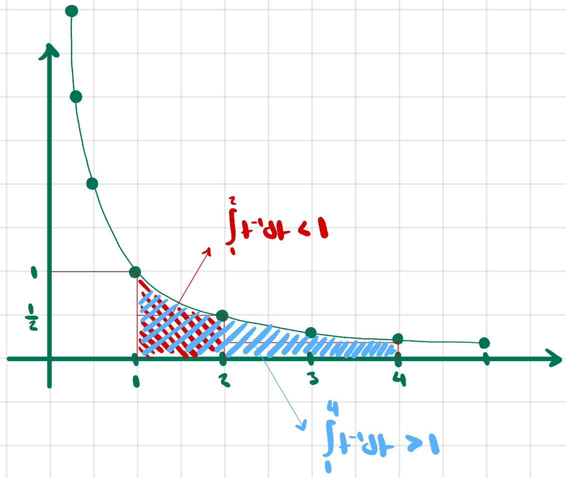
$x^y = y^x \rightarrow x = y$ or $x=2, y=4$ or $x=4, y=2$

Proof

$x^y = y^x \Leftrightarrow \frac{\log x}{x} = \frac{\log y}{y}$

Recall that $e = \log^{-1}(1) \rightarrow \log e = 1 \rightarrow \int_1^e t^{-1} dt = 1$

and graphically we see that

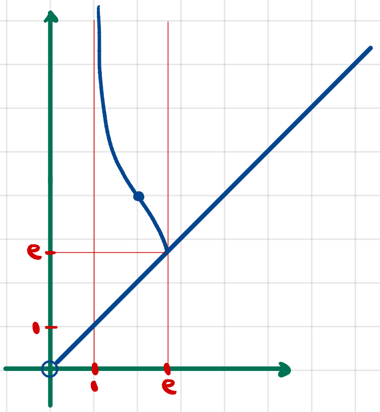


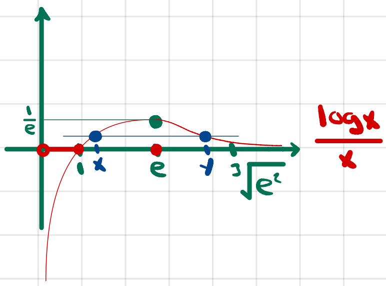
so $\int_1^2 t^{-1} dt < 1 < \int_1^e t^{-1} dt < 1 < \int_1^4 t^{-1} dt \rightarrow 2 < e < 4$

Thus, the only natural number in $(1, e)$ is 2.

$2^y = y^2 \rightarrow y = 2$ or $y = 4$

(e) $\{(x, y) : x^y = y^x\}$





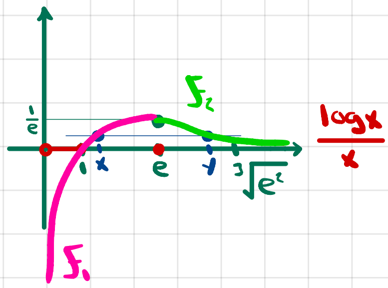
(F) $1 < x < e$ → g differentiable

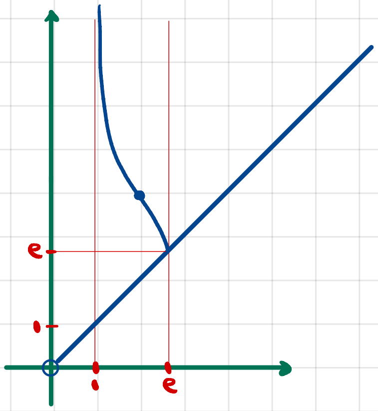
$g(x) > e$ s.t. $x^{g(x)} = g(x)^x$

Proof

let $f_1(x) = \dfrac{\log x}{x} \quad 0 < x < e$

$f_2(x) = \dfrac{\log x}{x} \quad x > e$





$g(x) = f_2^{-1} \circ f_1 = f_2^{-1}(f_1(x))$

since f_1 and f_2 are diff. so is g.

$g'(x) = (f_2^{-1})'(f_1(x)) f_1'(x)$

$(f_2^{-1})'(x) = \dfrac{1}{f_2'(f_2^{-1}(x))}$

$(f_2^{-1})'(f_1(x)) = \dfrac{1}{f_2'(f_2^{-1}(f_1(x)))} = \dfrac{1}{f_2'(g(x))}$

$f_1'(x) = \dfrac{\frac{1}{x} \cdot x - \log x}{x^2} = \dfrac{1 - \log x}{x^2} = f_2'(x)$

$g'(x) = \dfrac{1 - \log(g(x))}{g(x)^2} \cdot \dfrac{1 - \log x}{x^2}$

35. (a) $\exp = \log^{-1}$ is convex

Proof

$\exp(x) = \log^{-1}(x) \cdot e^x = [\log^{-1}(1)]^x$

$\exp'(x) = \dfrac{1}{\log'(\log^{-1}(x))} = \log^{-1}(x) = \exp(x) > 0$

$\exp''(x) = \exp(x) > 0 \rightarrow \exp$ is convex

■

log is concave

Proof

$\log x = \int_1^x t^{-1}dt$

$\log' x = x^{-1}$

$\log'' x = \dfrac{-1}{x^2} < 0 \rightarrow \log$ concave

■

(b) $\sum_{i=1}^n p_i = 1$, $p_i > 0$ $\rightarrow [\forall z_i,\ z_i > 0 \rightarrow z_1^{p_1} \cdots z_n^{p_n} < \underbrace{p_1 z_1 + \cdots + p_n z_n}_{\text{weighted average}}]$

Proof

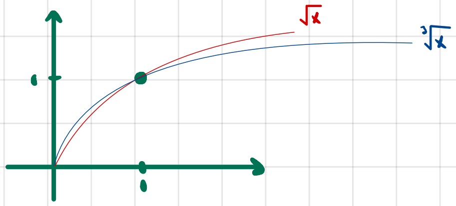
recall that we proved in chapter 11 Appendix, Problem 8

Let $n > 1$, $p_1, \dots, p_n$ positive numbers, $\sum_{i=1}^n p_i = 1$. Then

a) $\forall x_1, x_2, \dots, x_n \rightarrow \sum_{i=1}^n p_i x_i$ lies between smallest and largest x_i

b) if $t = \sum_{i=1}^{n-1} p_i$, then $\frac{1}{t}\sum_{i=1}^{n-1} p_i x_i$ lies between smallest and largest x_i

c) f convex $\rightarrow f\left(\sum_{i=1}^n p_i x_i\right) < \sum_{i=1}^n p_i f(x_i)$ (Jensen's Inequality)



Let $f(x) = e^x$ and $x_i = \log z_i$. Then Jensen's Inequality tells us that

$f\left(\sum_{i=1}^n p_i \log z_i\right) < \sum_{i=1}^n p_i f(\log z_i)$

$e^{\sum_{i=1}^n p_i \log z_i} < \sum_{i=1}^n p_i z_i$

$\prod_{i=1}^n e^{p_i \log z_i} = \prod_{i=1}^n z_i^{p_i} < \sum_{i=1}^n p_i z_i$

(c) $A_n = \dfrac{\sum_{i=1}^n a_i}{n}$, $G_n = \sqrt[n]{\prod_{i=1}^n a_i}$ $\rightarrow G_n \leq A_n$

Proof

Let $p_i = \frac{1}{n}$ in part b). Then,

$\prod_{i=1}^n z_i^{1/n} = \sqrt[n]{\prod_{i=1}^n z_i} = G_n < \sum_{i=1}^n \frac{1}{n} z_i = \dfrac{\sum_{i=1}^n z_i}{n} = A_n$

36. (a) f positive on $[a,b]$

P_n partition of $[a,b]$ into n equal intervals

$$\to \frac{1}{b-a} L(\log f, P_n) \le \log\left(\frac{1}{b-a} L(f, P_n)\right)$$

Proof

Avg value of lower sum of $\log f$ on $[a,b]$ $\le$ log of average value of lower sum of f on $[a,b]$

Consider some partition subinterval $[t_{i-1}, t_i]$.

Let $m_i = \inf\{f(x_i) : x_i \in [t_{i-1}, t_i]\}$

Let $m_{\ell,i} = \inf\{\log f(x_i) : x_i \in [t_{i-1}, t_i]\}$

Then $m_{\ell,i} = \log(m_i)$

Proof: $\forall x, x \in [t_{i-1}, t_i] \to f(x) \ge f(x_i) \to \log f(x) \ge \log f(x_i) \to \log(f(x_i))$ is lower bound for $\log f$ on $[t_{i-1}, t_i]$

Assume $\alpha > \log(f(x_i))$ and α is lower bound for $\log f$ on $[t_{i-1}, t_i]$

Then $\forall x, x \in [t_{i-1}, t_i] \to \log f(x) > \alpha \to f(x) > e^{\alpha} > e^{\log f(x_i)} = f(x_i)$

Thus e^{α} is lower bound for f, $\perp$.

$\log a + \log b = \log$

Hence, $\log(f(x_i)) = \log(m_i) = m_{\ell,i}$

$$L(f, P_n) = \frac{1}{n}\sum_{i=1}^{n} m_i$$

$$L(\log f, P_n) = \sum_{i=1}^{n} m_{\ell,i}\Delta t_i = \sum_{i=1}^{n} \log(m_i)\Delta t_i = \frac{b-a}{n}\sum_{i=1}^{n}\log(m_i) = \frac{b-a}{n}\log\left(\prod_{i=1}^{n} m_i\right)$$

$$= \frac{b-a}{n}\log\left(\frac{n}{b-a}\cdot \prod\left(\frac{b-a}{n}\right)m_i\right) < \frac{b-a}{n}\log\left(\frac{n}{b-a}\cdot \sum\left(\frac{b-a}{n}\right)m_i\right)$$

$$= \frac{b-a}{n}\log\left(\frac{n}{b-a}\cdot L(f, P_n)\right)$$

Thus

$$\frac{1}{b-a}L(f, P_n) < \frac{n}{b-a}L(f, P_n) < \log\left(\frac{n}{b-a}\cdot L(f, P_n)\right) < \log\left(\frac{1}{b-a}L(f, P_n)\right)$$

since $n \ge 1$.

(b) $f > 0$ integrable $\to \frac{1}{b-a}\int_a^b \log f \le \log\left(\frac{1}{b-a}\int_a^b f\right)$

Proof

Recall Theorem 13A-1

Theorem 1: Suppose f is integrable on $[a,b]$. Then for every $\epsilon > 0$, there is $\delta > 0$ s.t. if $P = \{t_0, \dots, t_n\}$ is partition of $[a,b]$ with $t_i - t_{i-1} < \delta$ for all i then

$$\left| \sum f(x_i)\Delta t_i - \int_a^b f(x)dx \right| < \epsilon$$

for any Riemann sum formed by choosing $x_i \in [t_{i-1}, t_i]$

That is, we can make a Riemann sum as close as we want to $\int_a^b f$ by choosing a partition w/ small enough subintervals.

Since $L(f,P) \le \int_a^b f \le U(f,P)$ for all P, and by def. of integrability of f, $\forall \epsilon > 0$, $\exists P$ s.t. $0 < U(f,P) - L(f,P) < \epsilon$, then

$$0 \le \int_a^b f - L(f,P) \le U(f,P) - L(f,P) < \epsilon$$

$$\to \left| L(f,P) - \int_a^b f \right| < \epsilon$$

Thus, specifically for the fns $\log f$ and f

$$\left| L(\log f, P) - \int_a^b f \right| < \epsilon$$

$$\left| L(f,P) - \int_a^b f \right| < \epsilon \to L(f,P) < \int_a^b f + \epsilon \to \frac{L(f,P)}{b-a} < \frac{\int_a^b f + \epsilon}{b-a} \to \log\left(\frac{L(f,P)}{b-a}\right) < \log\left(\frac{\int_a^b f}{b-a} + \frac{\epsilon}{b-a}\right)$$

$$\frac{1}{b-a}\int_a^b \log f - \log\left(\frac{1}{b-a}\int_a^b f\right)$$

$$= \left(\frac{1}{b-a}\int_a^b \log f - \frac{1}{b-a}L(\log f, P_n)\right) \quad < \frac{\epsilon}{b-a}$$

$$+ \frac{1}{b-a}L(\log f, P_n) - \log\left(\frac{1}{b-a}L(f,P_n)\right) \le 0, \text{ by part a}$$

$$+ \log\left(\frac{1}{b-a}L(f,P_n)\right) - \log\left(\frac{1}{b-a}\int_a^b f\right) \le \log\left(\frac{\int_a^b f}{b-a} + \frac{\epsilon}{b-a}\right) - \log\left(\frac{1}{b-a}\int_a^b f\right)$$

$$\le \frac{\epsilon}{b-a} + \log\left(\frac{\int_a^b f}{b-a} + \frac{\epsilon}{b-a}\right) - \log\left(\frac{1}{b-a}\int_a^b f\right)$$

$$\lim_{\epsilon \to 0}\left[\frac{1}{b-a}\int_a^b \log f - \log\left(\frac{1}{b-a}\int_a^b f\right)\right] \le 0$$

where we used continuity of log.

37. $f' = f$ $\quad$ for all x, y $\quad \to f = \exp$ or $f = 0$

$f(x+y) = f(x)f(y)$

Proof

Recall

Theorem 5 f diff. and $f'(x) = f(x)$ for all x, then there is c s.t. $f(x) = ce^x$ for all x.

$f(x+y) = ce^{x+y} = ce^x e^y$

True if $c = 0$ or, if $c \neq 0$ then $ce^x e^y = e^x e^y \to c = 1$

$\blacksquare$

Alternatively

$f(x) = ce^x$

$f(x) = f(x+0) = f(x) \cdot f(0)$

Either $f(x) = 0$ or $f(0) = 1 \to c = 1$

38. f cont.
$f(x+y)=f(x)f(y)$ for all x,y $\rightarrow$ $f=0$ or $f(x)=[f(1)]^x$

Proof

By assumption, $\forall x,y\in\mathbb{Q}\rightarrow f(x+y)=f(x)\cdot f(y)$. Let's call $f(1)=a$.

$f(1)=f(1+0)=f(1)\cdot f(0)$, so either $f(1)=0$ or $f(0)=1$.

Case 1: $f(1)=0$. Then $f(x)=f(x-1+1)=f(x-1)\cdot f(1)=0$, ie $f=0$

Case 2: $f(0)=1$

Now, $x\in\mathbb{Q}\rightarrow x$ is either

1) a natural number 1, 2, 3, ...
2) the number 0
3) $-n$ for $n\in\mathbb{N}$
4) $1/n$, $n\in\mathbb{Z}-\{0\}$
5) m/n, $m\in\mathbb{Z}, n\in\mathbb{Z}-\{0\}$

Recall problem 8-6.

Let A be set of real numbers. Then, A dense $\rightarrow$ every open interval contains a point of A.

we proved the following results

A dense f cont. $\forall x\in A\ f(x)=0$	$\rightarrow \forall x\in\mathbb{R}\ f(x)=0$
f,g cont. A dense $\forall x, x\in A\rightarrow f(x)=g(x)$	$\rightarrow \forall x\ f(x)=g(x)$
f,g cont. A dense $\forall x, x\in A\rightarrow f(x)\geq g(x)$	$\rightarrow \forall x\ f(x)\geq g(x)$

Let's consider each case.

1) $n\in\mathbb{N}$. $f(n)=f(\sum_{i=1}^{n}1)=\prod_{i=1}^{n}f(1)=\prod_{i=1}^{n}a=a^n$

$\rightarrow f(n)=a^n=[f(1)]^n$

2) $x=0$. $f(0)=1=a^0=[f(1)]^0$

3) $x=-n$, $n\in\mathbb{N}$. $1=f(0)=f(-n+n)=f(n)\cdot f(-n)\rightarrow f(-n)=\dfrac{1}{f(n)}=\dfrac{1}{a^n}=a^{-n}=[f(1)]^{-n}$

4) $x=1/n$, $n\in\mathbb{Z}-\{0\}$.

$f(1)=f(\sum_{i=1}^{n}1/n)=\prod_{i=1}^{n}f(1/n)=(f(1/n))^n\rightarrow f(1/n)=a^{1/n}=[f(1)]^{1/n}$

5) $x=m/n$, $m\in\mathbb{Z}, n\in\mathbb{Z}-\{0\}$

$f(m/n)=f(\sum_{i=1}^{m}1/n)=\prod_{i=1}^{m}f(1/n)=[f(1/n)]^m=[a^{1/n}]^m=a^{m/n}$

$\rightarrow f(m/n)=a^{m/n}=[f(1)]^{m/n}$

In every possible case, $f(x)=a^x$, $x\in\mathbb{Q}$. But the set $\mathbb{Q}$ is dense.

So we have

$\mathbb{Q}$ dense
f cont., $g(x)=a^x=e^{x\log a}$ cont.
$\forall x, x\in\mathbb{Q}\rightarrow f(x)=g(x)$

By problem 8-6b, $\forall x, f(x)=g(x)=a^x$.

39. f cont., defined on $\mathbb{R}^+$
$f(xy) = f(x) + f(y)$ for all $x, y > 0$ $\rightarrow$ $f = 0$ or $f(x) = f(e)\log x$ for $x > 0$

Proof

let $g(x) = f(e^x)$.

Then $g(x) + g(y) = f(e^x) + f(e^y) = f(e^{x+y}) = g(x+y)$

ie, $g(x+y) = g(x) + g(y)$

let's recall some previous results.

In 8-7 we proved

f cont.
$\forall x \forall y\ f(x+y) = f(x) + f(y)$ $\rightarrow$ $\exists c \in \mathbb{R}, \forall x\ f(x) = cx$

Since e^x is cont. on $(-\infty, \infty)$, $e^x > 0$, and f is cont. on $(0, +\infty)$, then $g = f \circ (e^x)$ is cont. on $\mathbb{R}$.

Thus, by the result above, $\exists c,\ c \in \mathbb{R} \wedge \forall x\ g(x) = cx$.

$c = 0 \rightarrow g(x) = 0 \rightarrow f(e^x) = 0$

let $y = e^x$. Then $y \in (0, \infty)$ and $f(y) = 0$.

$c \neq 0 \rightarrow g(x) = cx = f(e^x)$.

$f(e) = c \rightarrow f(e^x) = x f(e)$

let $y = e^x$. then $x = \log y$ and $f(y) = \log y\, f(e)$, $y > 0$.

40. $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases} \to f^{(k)}(0) = 0$ for all k

Proof

note that if $P(1/x) = 2\left(\frac{1}{x}\right)^3$ then $P(x) = 2x^3$ is polynomial.

$$f'(x) = \frac{2}{x^3} \cdot e^{-\frac{1}{x^2}}$$

$$f'(0) = \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x} = \lim_{x\to 0} \frac{1}{x e^{\frac{1}{x^2}}} = 0$$

$$f''(x) = \frac{-6}{x^4} e^{-\frac{1}{x^2}} + \frac{2}{x^3} \cdot \frac{2}{x^3} e^{-\frac{1}{x^2}} = e^{-\frac{1}{x^2}}\left(-\frac{6}{x^4} + \frac{4}{x^6}\right)$$

$$f''(0) = \lim_{x\to 0} \frac{\frac{2}{x^3} \cdot e^{-\frac{1}{x^2}}}{x} = \lim_{x\to 0} \frac{1}{x^4 e^{1/x^2}} = 0$$

$$f'''(x) = \frac{2}{x^3} \cdot e^{-\frac{1}{x^2}}\left(-\frac{6}{x^4} + \frac{4}{x^6}\right) + e^{-\frac{1}{x^2}}\left(\frac{24}{x^5} - \frac{24}{x^7}\right)$$

$$= e^{-\frac{1}{x^2}}\left(-\frac{12}{x^7} + \frac{8}{x^9}\right) + e^{-\frac{1}{x^2}}\left(\frac{24}{x^5} - \frac{24}{x^7}\right)$$

$$= e^{-\frac{1}{x^2}}\left(\frac{24}{x^5} - \frac{36}{x^7} + \frac{8}{x^9}\right)$$

Let $A = \{n : n \in \mathbb{N} \wedge f^{(n)}(x) = e^{-\frac{1}{x^2}} \cdot P(1/x) \wedge P \text{ is a polynomial}\}$

That is A is the set of $n \in \mathbb{N}$ for which the n^{th} deriv. of f has the form $e^{-\frac{1}{x^2}} P(1/x)$, where $P(1/x)$ is a polynomial.

Let's prove this by induction.

$n=1$ $\quad f'(x) = \frac{2}{x^3} \cdot e^{-\frac{1}{x^2}} = P(1/x) e^{-1/x^2}$, where $P(x) = 2x^3$.

Assume $k \in A$, ie $f^{(k)}(x) = e^{-\frac{1}{x^2}} \cdot P(1/x)$ for some polyn P.

Then $f^{(k+1)}(x) = e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3} \cdot P(1/x) + e^{-\frac{1}{x^2}} \cdot P'(1/x) \cdot \frac{(-1)}{x^2}$

$$= e^{-\frac{1}{x^2}}\left[\frac{2P(1/x)}{x^3} - \frac{P'(1/x)}{x^2}\right] = e^{-\frac{1}{x^2}}\left[Q_1(1/x) + Q_2(1/x)\right]$$

where $Q_1(1/x) = 2\left(\frac{1}{x}\right)P(1/x)$ and $Q_2(1/x) = -\left(\frac{1}{x^2}\right)P'(1/x)$ are polynomials and thus so is $Q = Q_1 + Q_2$.

$= e^{-\frac{1}{x^2}} Q(1/x)$, Q polynomial.

By induction, $A = \mathbb{N}$.

Now let $B = \{n : n \in \mathbb{N} \wedge f^{(n)}(0) = 0\}$

$n=1 \rightarrow f'(0) = \lim\limits_{x\to 0} \dfrac{e^{-\frac{1}{x^2}}}{x} = \lim\limits_{x\to 0} \dfrac{1}{xe^{\frac{1}{x^2}}} = 0 \rightarrow 1 \in B$

Assume $k \in B$, ie $f^{(k)}(0) = 0$.

Since $f^{(k)}(x) = e^{-\frac{1}{x^2}} \cdot P(1/x)$ then

$$f^{(k+1)}(x) = \lim_{h\to 0} \frac{f^{(k)}(h) - \cancel{f^{(k)}(0)}^{\,0}}{h}$$

$$= \lim_{h\to 0} \frac{e^{-\frac{1}{h^2}} P(1/h)}{h}$$

$$= \lim_{h\to\infty} \frac{h \cdot P(h)}{e^{h^2}}$$

because $h \cdot P(h)$ is polyn. and Th. 6 says $\lim\limits_{x\to\infty} \dfrac{e^x}{x^n} = \infty$.

$$= 0$$

$\rightarrow k+1 \in B$

$\rightarrow B = \mathbb{N}$

41. $f(x) = e^{-\frac{1}{x^2}} \sin(1/x) \quad x \neq 0 \quad \rightarrow f^{(n)}(0) = 0$ for all n

$f(0) = 0$

Proof

$$f'(0) = \lim_{h\to 0} \frac{e^{-\frac{1}{h^2}}\sin(1/h)}{h} = \lim_{h\to 0} e^{-\frac{1}{h^2}} \cdot \lim_{h\to 0} \frac{\sin(1/h)}{h} = 0$$

$$f'(x) = e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3}\sin(1/x) + e^{-\frac{1}{x^2}}\cos(1/x)\cdot\frac{(-1)}{x^2}$$

$$= e^{\frac{-1}{x^2}}\left(\frac{2\sin(1/x)}{x^3} - \frac{\cos(1/x)}{x^2}\right)$$

$$f''(0) = \lim_{x\to 0} \frac{e^{\frac{-1}{x^2}}\left(\frac{2\sin(1/x)}{x^3} - \frac{\cos(1/x)}{x^2}\right)}{x} = \lim_{x\to 0} e^{-\frac{1}{x^2}}\left(\frac{2\sin(1/x)}{x^4} - \frac{\cos(1/x)}{x^3}\right)$$

$$f''(x) = e^{-\frac{1}{x^2}}\left[\frac{4\sin(1/x)}{x^6} - \frac{7\sin(1/x)}{x^4} - \frac{4\cos(1/x)}{x^5} + \frac{2\cos(1/x)}{x^3}\right]$$

let $A = \{n : n\in\mathbb{N}$ and $f^{(n)}(x) = e^{-\frac{1}{x^2}}\left[\sum_{i=1}^{3n}\frac{a_i}{x^i}\sin(1/x) + \sum_{i=1}^{3n}\frac{b_i}{x^i}\cos(1/x)\right]\}$

$n=1 \rightarrow f'(x) = e^{\frac{-1}{x^2}}\left(\frac{2\sin(1/x)}{x^3} - \frac{\cos(1/x)}{x^2}\right) \rightarrow 1\in A$

Assume $n\in A$. Then

$$f^{(n)}(x) = e^{-\frac{1}{x^2}}\left[\sum_{i=1}^{3n}\frac{a_i}{x^i}\sin(1/x) + \sum_{i=1}^{3n}\frac{b_i}{x^i}\cos(1/x)\right]$$

Then

$$f^{(n+1)}(x) = \frac{2}{x^3}f^{(n)}(x) + e^{-\frac{1}{x^2}}\left[\sum_{i=1}^{3n}\left(\frac{-ia_i}{x^{i+1}}\sin(1/x) + \frac{a_i}{x^i}\cos(1/x)\frac{(-1)}{x^2}\right) + \sum_{i=1}^{3n}\left(\frac{-ia_i}{x^{i+1}}\cos(1/x) - \frac{a_i}{x^i}\sin(1/x)\frac{(-1)}{x^2}\right)\right]$$

$$= e^{-\frac{1}{x^2}}\left[\sum_{i=1}^{3n}\frac{2a_i}{x^{i+3}}\sin(1/x) + \sum_{i=1}^{3n}\frac{b_i}{x^{i+3}}\cos(1/x) + \sum_{i=1}^{3n}\frac{-ia_i}{x^{i+1}}\sin(1/x) + \sum_{i=1}^{3n}\frac{a_i}{x^{i+2}}\sin(1/x) + \sum_{i=1}^{3n}\frac{-a_i}{x^{i+2}}\cos(1/x) + \sum_{i=1}^{3n}\frac{-ia_i}{x^{i+1}}\cos(1/x)\right]$$

$$= e^{-1/x^2}\left[\sum_{i=1}^{3n+3}\frac{\alpha_i}{x^i}\sin(1/x) + \sum_{i=1}^{3n+3}\frac{\beta}{x^i}\cos(1/x)\right]$$

$A=\mathbb{N}$, ie $\forall n, n\in\mathbb{N} \rightarrow f^{(n)}(x) = e^{-\frac{1}{x^2}}\left[\sum_{i=1}^{3n}\frac{a_i}{x^i}\sin(1/x) + \sum_{i=1}^{3n}\frac{b_i}{x^i}\cos(1/x)\right]$

Now let's do another induction to prove $\forall n, n\in\mathbb{N} \rightarrow f^{(n)}(0) = 0$.

let $B = \{n : n\in\mathbb{N} \wedge f^{(n)}(0) = 0\}$

$n=1 \rightarrow f'(0) = \lim_{h\to 0}\frac{e^{-\frac{1}{h^2}}\sin(1/h)}{h} = \lim_{h\to 0} e^{-\frac{1}{h^2}} \cdot \lim_{h\to 0}\frac{\sin(1/h)}{h} = 0 \rightarrow 1\in B$

Assume $h \in B$, ie $f^{(h)}(0) = 0$. Then

$$f^{(h+1)}(0) = \lim_{x\to0}\left[e^{-\frac{1}{x^2}}\left[\sum_{i=1}^{3n}\frac{a_i}{x^i}\sin(1/x) + \sum_{i=1}^{3n}\frac{b_i}{x^i}\cos(1/x)\right]\right]\Big/ x$$

Now

$$\lim_{x\to0}\left[e^{-\frac{1}{x^2}}\left[\sum_{i=1}^{3n}\frac{a_i}{x^i}\sin(1/x)\right]\Big/ x\right] = \lim_{x\to0}\left[e^{-\frac{1}{x^2}}\left[\sum_{i=1}^{3n}\frac{a_i}{x^{i+1}}\sin(1/x)\right]\right]$$

Since $-1 \leq \sin(1/x) \leq 1$ then

$$\underbrace{-\lim_{x\to0}\left[e^{-\frac{1}{x^2}}\left[\sum_{i=1}^{3n}\frac{a_i}{x^{i+1}}\right]\right]} \leq \lim_{x\to0}\left[e^{-\frac{1}{x^2}}\left[\sum_{i=1}^{3n}\frac{a_i}{x^{i+1}}\sin(1/x)\right]\right] \leq \underbrace{\lim_{x\to0}\left[e^{-\frac{1}{x^2}}\left[\sum_{i=1}^{3n}\frac{a_i}{x^{i+1}}\right]\right]}_{\text{Analogously, } = 0.}$$

$$\overset{\frac{\infty}{\infty}}{=} -\lim_{x\to\infty}\frac{\sum_{i=1}^{3n} a_i x^{i+1}}{e^{x^2}}$$

$$= -\lim_{x\to\infty}\sum_{i=1}^{3n} a_i \frac{x^{i+1}}{e^{x^2}} = 0$$

$$\to \lim_{x\to0}\left[e^{-\frac{1}{x^2}}\left[\sum_{i=1}^{3n}\frac{a_i}{x^{i+1}}\sin(1/x)\right]\right] = 0$$

$\to h \in B$

$\to B = \mathbb{N}$

42. (a) α is root of $\sum_{i=0}^{n} a_i x^i = 0$ $\rightarrow$ $y = e^{\alpha x}$ satisfies $\sum_{i=0}^{n} a_i y^{(i)} = 0$

Proof

By assumption, $\sum_{i=1}^{n} a_i \alpha^i = 0$.

Ie, if we find one solution α, we automatically find another solution $e^{\alpha x}$, for any x

$y = e^{\alpha x}$
$y' = \alpha e^{\alpha x}$
$y'' = \alpha^2 e^{\alpha x}$
$y^{(n)} = \alpha^n e^{\alpha x}$

Hence, $\sum_{i=0}^{n} a_i y^{(i)}(x) = \sum_{i=0}^{n} a_i \alpha^i e^{\alpha x} = e^{\alpha x} \sum_{i=0}^{n} a_i \alpha^i = 0$

(b) α is double root of $\sum_{i=0}^{n} a_i x^i = 0$ $\rightarrow$ $y(x) = x e^{\alpha x}$ satisfies $\sum_{i=0}^{n} a_i y^{(i)} = 0$

$y(x) = x e^{\alpha x}$
$y'(x) = e^{\alpha x} + x \alpha e^{\alpha x}$
$y''(x) = \alpha e^{\alpha x} + \alpha e^{\alpha x} + x \alpha^2 e^{\alpha x} = 2\alpha e^{\alpha x} + x\alpha^2 e^{\alpha x} = e^{\alpha x} \alpha (2 + x\alpha)$
$y'''(x) = 2\alpha^2 e^{\alpha x} + \alpha^2 e^{\alpha x} + x\alpha^3 e^{\alpha x} = 3\alpha^2 e^{\alpha x} + x\alpha^3 e^{\alpha x} = e^{\alpha x}\alpha^2 (3 + x\alpha)$
$y^{(4)}(x) = 3\alpha^3 e^{\alpha x} + \alpha^3 e^{\alpha x} + x\alpha^4 e^{\alpha x} = 4\alpha^3 e^{\alpha x} + x\alpha^4 e^{\alpha x} = e^{\alpha x}\alpha^3 (4 + x\alpha)$

$y^{(n)}(x) = n\alpha^{n-1} e^{\alpha x} + x\alpha^n e^{\alpha x} = e^{\alpha x} \alpha^{n-1} (n + x\alpha)$

Since α is a double root then

$\sum_{i=0}^{n} a_i x^i = (x-\alpha)^2 g(x)$ for some polyn. g, and α is a root of the derivative

$\left(\sum_{i=0}^{n} a_i x^i\right)' = \sum_{i=1}^{n} i a_i x^{i-1}$ and $\sum_{i=1}^{n} i a_i \alpha^{i-1} = 0$

$$\sum_{i=0}^{n} a_i y^{(i)} = a_0 y + \sum_{i=1}^{n} a_i e^{\alpha x} \alpha^{i-1} (i + x\alpha) = a_0 x e^{\alpha x} + e^{\alpha x} \xcancel{\sum_{i=1}^{n} i a_i \alpha^{i-1}}^{\,0} + x e^{\alpha x} \sum_{i=1}^{n} a_i \alpha^i$$

$$= x e^{\alpha x} \xcancel{\sum_{i=0}^{n} a_i \alpha^i}^{\,0}$$

(c) α is root of $\sum_{i=0}^{n} a_i x^i = 0$ of order r $\rightarrow$ $y(x) = x^h e^{\alpha x}$ is solution of $\sum_{i=0}^{n} a_i y^{(i)} = 0$ for $0 \leq h \leq r-1$

Proof

$$\left(\sum_{i=0}^{n} a_i x^i\right)' = \sum_{i=0}^{n} i a_i x^{i-1} \text{ and } \sum_{i=0}^{n} i a_i \alpha^{i-1} = 0$$

$$\left(\sum_{i=0}^{n} a_i x^i\right)'' = \sum_{i=2}^{n} i(i-1) a_i x^{i-2} \text{ and } \sum_{i=2}^{n} i(i-1) a_i \alpha^{i-2} = 0$$

(...)

$$\left(\sum_{i=0}^{n} a_i x^i\right)^{(r)} = \sum_{i=r}^{n} \prod_{j=0}^{r-1} (i-j) \cdot a_i \cdot x^{i-r} \text{ and } \sum_{i=r}^{n} \prod_{j=0}^{r-1} (i-j) \cdot a_i \cdot \alpha^{i-r} = 0 = \sum_{i=r}^{n} \binom{i}{i-r} \cdot r! \, a_i \cdot \alpha^{i-r}$$

$$y(x) = x^h e^{\alpha h}$$

$$y' = h x^{h-1} e^{\alpha h} + x^h \alpha e^{\alpha h} = e^{\alpha h}(h x^{h-1} + \alpha x^h)$$

$$y'' = {}^{\alpha h}(h x^{h-1} + \alpha x^h) + e^{\alpha h}(h(h-1) x^{h-2} + \alpha h x^{h-1}) = e^{\alpha h}\left[x^{h-2} h(h-1) + x^{h-1} 2\alpha h + x^h \cdot \alpha^2\right]$$

$$y''' = \alpha e^{\alpha h}\left[x^{h-2} h(h-1) + x^{h-1} \cdot 2\alpha h + x^h \cdot \alpha^2\right] + e^{\alpha h}\left[h(h-1)(h-2) x^{h-3} + 2\alpha h(h-1) x^{h-2} + \alpha^2 h x^{h-1}\right]$$

$$= e^{\alpha h}\left[x^{h-3} \cdot h(h-1)(h-2) + x^{h-2} \cdot 3\alpha h(h-1) + x^{h-1} \cdot 3\alpha^2 h + x^h \alpha^3\right]$$

$$y^{(n)} = e^{\alpha h} \sum_{i=0}^{n} x^{h-i} \binom{n}{i} \alpha^{n-i} \binom{h}{h-i} \cdot i!$$

$$\sum_{j=0}^{n} a_j y^{(j)} = a_0 y + \sum_{j=1}^{n} a_j e^{\alpha h}\left[\sum_{i=0}^{j} x^{h-i} \binom{j}{i} \alpha^{j-i} \binom{h}{h-i} \cdot i!\right]$$

j^{th} deriv.
h constant
i cycles from 0 to j for each j

$$= a_0 x^h e^{\alpha x} + a_1 e^{\alpha h}\left[\sum_{i=0}^{1} x^{h-i} \binom{1}{i} \alpha^{1-i} \binom{h}{h-i} \cdot i!\right] + a_2 e^{\alpha h}\left[\sum_{i=0}^{2} x^{h-i} \binom{2}{i} \alpha^{2-i} \binom{h}{h-i} \cdot i!\right] + \ldots$$

$$= \left.\begin{matrix} e^{\alpha h}[a_0 x^h \\ + a_1 x^h \binom{1}{0} \alpha^1 \binom{h}{h} \cdot 0! \\ + a_2 x^h \binom{2}{0} \alpha^2 \binom{h}{h} \cdot 0! \\ + \ldots) \end{matrix}\right\} = x^h \sum_{i=0}^{n} a_i \cdot \alpha^i = 0 \text{ by assumption}$$

$$\left.\begin{matrix} + e^{\alpha h}(a_1 x^{h-1} \binom{1}{1} \alpha^{1-1} \binom{h}{h-1} 1! \\ + a_2 x^{h-1} \binom{2}{1} \alpha^{2-1} \binom{h}{h-1} \cdot 1! \\ + a_3 x^{h-1} \underbrace{\binom{3}{1}}_{i} \alpha^{3-1} \binom{h}{h-1} \cdot 1! \end{matrix}\right\} = x^{h-1} \cdot \sum_{i=1}^{n} a_i \cdot i \cdot \alpha^{i-1} = 0$$

$$\left.\begin{matrix} + e^{\alpha h}(a_2 x^{h-2} \binom{2}{2} \alpha^{2-2} \binom{h}{h-2} 2! \\ + a_3 x^{h-2} \binom{3}{2} \alpha^{3-2} \binom{h}{h-2} \cdot 2! \\ + a_4 x^{h-2} \binom{4}{2} \alpha^{4-2} \binom{h}{h-2} \cdot 2! \\ + \ldots \end{matrix}\right\} = x^{h-2} \sum_{i=2}^{n} a_i \cdot i(i-1) \cdot \alpha^{i-2} = 0$$

$+ (\ldots)$

$= 0$

(3) $y_i(x) = x^i e^{\alpha i}$ sol'n to $\sum_{j=0}^{n} a_i y_i^{(j)} = 0$ $\to$ $c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ also a sol'n.

Proof

$$\sum_{i=1}^{n} a_i (c_1 y_1 + c_2 y_2 + \dots + c_m y_m)^{(i)} = c_1 \sum_{i=1}^{n} a_i y_1^{(i)} + \dots + c_m \sum_{i=1}^{n} a_i y_m^{(i)} = c_1 \cdot 0 + \dots + c_m \cdot 0 = 0$$

Recap

We have a polynomial equation $\sum_{i=0}^{n} a_i x^i = 0$. We want a solution for the diff. eq. $\sum_{i=1}^{n} a_i f^{(i)}(x)$.

If we find one solution α to the polynomial then we have automatically have a sol'n to the D.E.: $e^{\alpha x}$.

If α is a double root then we have an additional sol'n to the D.E.: $xe^{\alpha x}$; ie in total we have two sol'ns.

If the sol'n α has multiplicity r, then, generalizing the previous results, we automatically have r solutions for the D.E. namely, $x^h e^{\alpha x}$, for $0 \le h \le r-1$.

Finally, a linear combination of such D.E. sol'ns is another sol'n.

Thus, we have infinite sol'ns for the D.E.

43. $f'' - f = 0$ $\rightarrow f = 0$
$f'(0) = f(0) = 0$

(a) $f^2 - (f')^2 = 0$

Proof

$f'' - f = 0$ is an eq. of form $\sum_{i=0}^{2} a_i y^{(i)}$, ie a diff. eq.

$f''(0) - \cancel{f(0)}^{0} = f''(0) = 0$

Consider the polyn. eq. $x^2 - 1 = 0 \rightarrow x = \pm 1$.

By problem 42, $y(x) = e^x$ and $y(x) = e^{-x}$ are sol'ns to $f'' - f = 0$.

In fact, $c_1 e^x + c_2 e^{-x}$ are sol'ns, and represent all possible sol'ns.

Thus $f(x) = c_1 e^x + c_2 e^{-x}$ and $f'(x) = c_1 e^x - c_2 e^{-x}$.

$f(0) = c_1 + c_2 = 0 \rightarrow 2c_1 = 0 \rightarrow c_1 = 0 \rightarrow c_2 = 0$
$f'(0) = c_1 - c_2 = 0 \rightarrow c_1 = c_2$

$\rightarrow f(x) = 0$

Altern.

Start w/ $f'' - f = 0$. Then

$f'(f'' - f) = 0$

$f'f'' - f'f = 0$

$= \frac{1}{2}[2f' \cdot f'' - 2f \cdot f']$

$= \frac{1}{2}[(f')^2 - (f)^2]'$

$= 0 \quad \rightarrow (f')^2 - (f)^2 = 0$

(b) $f(x) \neq 0$ for all x in (a,b) $\rightarrow f(x) = ce^x$ or $f(x) = ce^{-x}$, for some constant c and for all x

Proof

$f'^2 = f^2 \rightarrow f = f'$ or $f = -f'$

Case 1: $f' - f = 0$

$x - 1 = 0 \rightarrow x = 1$ and by P.42, e^x is sol'n. ce^x is sol'n too

Case 2: $f' + f = 0$

$x + 1 = 0 \rightarrow x = -1$, P.42 $\rightarrow e^{-x}$ is sol'n. ce^{-x} is too.

(c) $f(x_0) \neq 0$ for $x_0 > 0$

we know $f(0) = 0$. Assume $f(x_0) \neq 0$.

Since f is diff it is continuous. There is an interval around x_0 in which either i) $f > 0$ or ii) $f < 0$, depending on the sign of $f(x_0)$.
Let's consider the first case: $f(x_0) > 0$.
The set $\{x : 0 \le x \le x_0 \wedge \forall y, y > x \rightarrow f(y) > 0\}$ has an infimum that is ≥ 0. Let's call it a. $f(a) = 0$.
that $f(x) = ce^x$ or $f(x) = ce^{-x}$ in (a, x_0).
$\lim_{x \to a} ce^{\pm x} = ce^{\pm a} \neq 0 = f(a)$, which contradicts continuity of f.

$\rightarrow \forall x, x > 0 \rightarrow f(x) = 0$

44. (a) $f''-f=0 \rightarrow \exists a,b$ s.t. $f(x)=ae^x+be^{-x}$

We want to show that there exist a and b such that $f(x)=ae^x+be^{-x}$.
First, let's check that f in facts abides by the assumption

$$f'(x)=ae^x-be^{-x}$$
$$f''(x)=ae^x+be^{-x} \quad \rightarrow f''-f=0$$

If we define

$$g(x)=ae^x+be^{-x}-f(x), \text{ for some } a,b,f(x)$$

and show that it is always zero for some a and b then we are done.

Problem 43 tells us that if we can show

$$g''-g=0$$
$$g(0)=g'(0)=0$$

then $g=0$.

$$g'(x)=ae^x-be^{-x}-f'(x)$$

$$g''(x)=ae^x+be^{-x}-f''(x)$$

$$\rightarrow g''(x)-g(x)=-f''(x)+f(x)=-(f''(x)-f(x))=0$$

so the first condition is met.

$$g(0)=a+b-f(0)=0$$
$$g'(0)=a-b-f'(0)=0 \rightarrow a=b+f'(0)$$
$$2b+f'(0)-f(0)=0 \rightarrow b=\frac{f(0)-f'(0)}{2} \rightarrow a=\frac{f(0)+f'(0)}{2}$$

Therefore, for this choice of a and b, $g=0$, so $f(x)=ae^x+be^{x}$.

(b) $f=a\sinh+b\cosh$, for some other a,b

I assume the problem means that this is another solution to $f''-f=0$.

Recall that

$$\sinh=\frac{e^x+e^{-x}}{2} \qquad \cosh=\frac{e^x-e^{-x}}{2}$$

$$a\sinh+b\cosh=e^x\left(\frac{a}{2}+\frac{b}{2}\right)+e^{-x}\left(\frac{a}{2}-\frac{b}{2}\right)$$

Now if $\frac{a}{2}+\frac{b}{2}=\frac{f(0)+f'(0)}{2}$ and $\frac{a}{2}-\frac{b}{2}=\frac{f(0)-f'(0)}{2}$ then $a\sinh+b\cosh$ is a sol'n.

$$a+b=f(0)+f'(0) \quad \rightarrow a=f(0)$$
$$a-b=f(0)-f'(0) \quad \rightarrow b=f'(0) \rightarrow f(0)\sinh+f'(0)\cosh \text{ is sol'n to } f''-f=0.$$

45. (a) $f^{(n)} = f^{(n-1)}$

If $f(x) = ce^x$ then $f^{(h)} = f$ for all $h > 0$.

Consider first the d.e. $f' = f$, true over some interval

Let $g(x)$ be some solution. Let $h(x) = g(x)e^{-x}$.

Then, $h'(x) = g'(x)e^{-x} - g(x)e^{-x} = e^{-x}(g'(x) - g(x)) = 0$

Now g is diff on the interval in question, and so is h.

Thus h is constant

$$h(x) = c = g(x)e^{-x} \to g(x) = ce^x$$

Now let $g = f^{(n-1)}$, $g' = f^{(n)}$, and $g = g'$.

Then $f^{(n-1)} = ce^x$.

Thus, $f^{(n-2)}(x) = ce^x + c_1$
$f^{(n-3)}(x) = ce^x + c_1x + c_2$
$f^{(n-4)}(x) = ce^x + \frac{c_1}{2}x^2 + c_2x + c_3$

$$f(x) = ce^x + \sum_{i=0}^{n-2} d_i x^i$$

(b) $f^{(n)} = f^{(n-2)}$

$f^{(n)} - f^{(n-2)} = 0$

$x^n - x^{n-2} = x^{n-2}(x^2 - 1) = 0$ → $x = 1$, $x = -1$, $x = 0$

Solutions: $f^{(n-2)}(x) = ce^x, ce^{-x}, c$

$$\to f(x) = ce^x + \sum_{i=0}^{n-3} \alpha_i x^i$$

46. (a) suppose there is $f \neq 0$ w/ $f' = f$, ie this diff. has a non-zero sol'n.

Let $g(x) = f(x_0+x)f(x_0-x)$ and $f(x_0) \neq 0$.

$g'(x) = f'(x_0+x)f(x_0-x) - f(x_0+x)f'(x_0-x) = 0$

$\rightarrow$ g is constant

$g(0) = f(x_0)^2 \neq 0$

$\rightarrow f(x_0+x)f(x_0-x) \neq 0$ for all x

$\rightarrow f(x) \neq 0 \ \forall x$

(b) $\exists f,\ f' = f \wedge f(0) = 1$

Assume f_1 is fn s.t. $f_1' = f_1$ and $f_1 \neq 0$.

Let $f(x) = \dfrac{f_1(x)}{f_1(0)}$

Then

$$f'(x) = \frac{f_1'(x)}{f_1(0)} = \frac{f_1(x)}{f_1(0)} = f(x)$$

Also, $f(0) = 1$

47. f, g cont.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$$

we say f grows faster than g, $f \gg g$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$

f and g grow at the same rate, $f \sim g$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and is $\neq 0, \infty$

For example, for polyn. P w/ $\lim_{x\to\infty} P(x) = \infty$

$\exp \gg P$

$P \gg \log^n \quad n \in \mathbb{N}$

When is $\lim_{x\to\infty} P(x) \neq \infty$?

$P(x) =$ constant

$P(x) = -|a|x^n$, ie negative leading coeff.

(a) Given f, g w/ $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$

Let $f(x) = x$, $g(x) = x(2+\sin x)$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$

$\lim \frac{f(x)}{g(x)} = \lim \frac{1}{2+\sin x}$ doesn't exist because the expression oscillates between 1 and 1/3.

(b) $f \gg g \to f+g \sim f$

Proof

$$\lim_{x\to\infty} \frac{f(x)+g(x)}{f(x)} = \lim_{x\to\infty} \left[1 + \frac{g(x)}{f(x)}\right] = 1$$

(c) $\frac{\log f}{\log g} \geq c > 1$ for suffic. large x then $f \gg g$

Proof

$\log f \geq c \log g = \log g^c$

$f(x) \geq g(x)^{c-1}$

$\frac{f(x)}{g(x)} \geq g(x)^{c-1}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} (g(x))^{c-1} = \infty$$

(d) $f \gg g$

$F(x) = \int_0^x f$

$G(x) = \int_0^x g$

$f \gg g$ means that $\forall M > 0\ \exists N > 0$ s.t. $\forall x, x > N \to \frac{f(x)}{g(x)} > M$

$\to f(x) > Mg(x) \to \int_N^x f > \int_N^x Mg$

For any $M > 0$ let N be as above.

$$\frac{F}{G} = \frac{\int_0^N f}{\int_0^x g} + \frac{\int_N^x f}{\int_0^x g} > \frac{\int_0^N f}{\int_0^x g} + \frac{M\int_N^x g}{\int_0^x g}$$

$$\lim_{x\to\infty} \frac{M\int_N^x g}{\int_N^x g + c} = M$$

Thus

$$\lim_{x\to\infty} \frac{F(x)}{G(x)} > \lim_{x\to\infty} \left[\cancel{\frac{\int_0^N f}{\int_0^x g}}^{\,0} + \frac{M\int_N^x g}{\int_0^x g} \right] = M$$

Thus, $\forall M > 0\ \exists N > 0$ s.t. $\forall x\ x > N$

$$\to \frac{F(x)}{G(x)} > M$$

$$\to \lim_{x\to\infty} \frac{F(x)}{G(x)} = \infty$$

Alternative Proof

Let $N > 0$. Choose x_0 s.t. $f(y) \geq 2Ng(y)$ for all $y \geq x_0$.

Then

$$F(x_0 + x) = \int_0^{x_0+x} f$$

$$= \int_0^{x_0} f + \int_{x_0}^{x_0+x} f$$

$$\geq \int_0^{x_0} f + 2N \int_{x_0}^{x_0+x} g$$

$$= \int_0^{x_0} f + 2N \int_0^{x_0+x} g - 2N \int_0^{x_0} g$$

$$= \int_0^{x_0} f - 2N \int_0^{x_0} g + 2NG(x_0 + x)$$

$$= A + 2NG(x_0 + x)$$

Then

$$\frac{F(x_0+x)}{G(x_0+x)} \geq 2N + \frac{A}{G(x_0+x)}$$

That is, for any $N > 0$, $\exists x_0$ s.t. $\forall x \; x > x_0 \to \dfrac{F(x_0+x)}{G(x_0+x)} \geq 2N > N$

$$\to \lim_{x \to \infty} \frac{F(x)}{G(x)} = \infty$$

(e)

(i)

Since $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$ we have $e^x \gg x^3$.

Also, $\lim \frac{e^x}{x+e^{-5x}} = \lim \frac{e^x}{\frac{xe^{5x}+1}{e^{5x}}} = \lim \frac{e^{6x}}{1+xe^{5x}} = \frac{\infty}{\infty} = \lim \frac{6e^{6x}}{e^{5x}+5xe^{5x}} = \lim \frac{6e^{6x}}{e^{5x}(1+5x)}$

$= \lim \frac{e^x}{1+5x} = \infty \to e^x \gg x+e^{-5x}$

Since $\lim_{x\to\infty} \frac{x}{(\log x)^n} = \infty$ we have $x^3 \gg \log(4x)$
$x+e^{-5x} \gg \log(4x)$

$\lim_{x\to\infty} \frac{e^{x\log x}}{e^x} = \lim_{x\to\infty} e^{x(\log x - 1)} = \infty \quad e^{x\log x} \gg e^x$

$\lim_{x\to\infty} \frac{x^3+3\log x}{x^3\log x} = \lim_{x\to\infty} \left[\frac{1}{\log x} + \frac{3}{x^3}\right] = 0 \to x^3\log x \gg x^3+\log x^3$

$\lim \frac{(\log x)^x}{e^x} = \lim \frac{e^{\log(\log x)\cdot x}}{e^x} = \lim_{x\to\infty} e^{x(\log(\log x)-1)} = \infty \to (\log x)^x \gg e^x$

$\lim \frac{x^x}{(\log x)^x} = \lim e^{\log \frac{x}{\log x} \cdot x} = \infty \to x^x \gg (\log x)^x$

$x^x \gg (\log x)^x \gg e^x \gg x^3\log x \gg x^3+\log(x^3) \gg x^3 \gg x+e^{-5x} \gg \log(4x)$

(ii)

$\log(x^x) = \log(e^{x\log x}) = x\log(x)$

$x\log^2 x \gg x\log(x)$

$\frac{x^{\log x}}{x\log^2 x} = \frac{e^{(\log x)^2}}{x\log^2 x} = \frac{e^{y^2}}{e^y y^2} = \frac{(e^y)^{y-1}}{y^2} = \infty \rightarrow x^{\log x} \gg x\log^2 x$

$\frac{e^{(\log x)^2}}{e^{5x}} = e^{(\log x)^2 - 5x} = e^{\log x(\log x - \frac{5x}{\log x})} = 0 \rightarrow e^{5x} \gg x^{\log x}$

$\frac{(\log x)^x}{e^{5x}} = \frac{e^{\log(\log x)\cdot x}}{e^{5x}} = e^{x(\log(\log x) - 5)} = \infty \rightarrow (\log x)^x \gg e^{5x}$

From part a), $x^x \gg (\log x)^x$

$\frac{e^{x^2}}{x^x} = \frac{e^{x^2}}{e^{x\log x}} = e^{x^2 - x\log x} = e^{x^2(1 - \frac{\log x}{x})} = \infty \rightarrow e^{x^2} \gg x^x$

$e^{x^2} \gg x^x \gg (\log x)^x \gg e^{5x} \gg x^{\log x} \gg x\log^2 x \gg x\log x$

(iii) From ii), $e^{x^2} \gg x^x \gg e^x$ $\quad y = \log x \quad x = e^y$

$\frac{x^x}{(\log x)^{2x}} = \frac{e^{x\log x}}{e^{2x\log(\log x)}} = e^{x\log x - 2x\log(\log x)} = e^{x(\log x - 2\log(\log x))} = e^{e^y(y - 2\log y)} = e^{\frac{e^y}{y}(1 - 2\frac{\log y}{y})} = \infty$

$\rightarrow x^x \gg (\log x)^{2x}$

$\frac{(\log x)^{2x}}{e^x} = e^{2x\log(\log x) - x} = e^{x(2\log(\log x) - 1)} = \infty$

$\rightarrow (\log x)^{2x} \gg e^x$

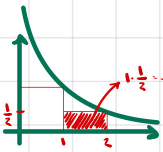
$\frac{e^x}{2^x} = \frac{e^x}{e^{x\log 2}} = e^{x(1-\log 2)} = \infty$ since $1 - \log 2 > 0$. $\rightarrow$ $\log e = 1$

$\log 1 = 0$

and a geometric argument showed $2 < e < 4$

plus log is increasing

$\rightarrow e^x \gg 2^x$

$\frac{e^{x/2}}{2^x} = \frac{e^{x/2}}{e^{x\log 2}} = e^{\frac{x}{2}(1 - 2\log 2)} = 0$ since $1 - 2\log 2 < 0$ $\rightarrow$ we know this because 

$1 \cdot \frac{1}{2} = \frac{1}{2} \rightarrow \log 2 = \int_1^2 t^{-1}\,dt > \frac{1}{2}$, hence $2\log 2 > 1$

$\rightarrow 2^x \gg e^{x/2}$

$\frac{e^{x/2}}{x^e} = \frac{e^{x/2}}{e^{e\log x}} = e^{\frac{x}{2} - e\log x} = e^{\frac{e^y}{2} - ey} = e^{\frac{e^y}{y}(\frac{y}{2} - e)} = \infty$

$\rightarrow e^{x/2} \gg x^e$

$e^{x^2} \gg x^x \gg (\log x)^{2x} \gg e^x \gg 2^x \gg e^{x/2} \gg x^e$

48. g_1, g_2, g_3 cont. Fns $\to$ there is cont. f such that $f \gg g_i$

Proof

Let $f = g_1 g_2 g_3$

$\lim \frac{f}{g_1} = \lim g_2 g_3 =$