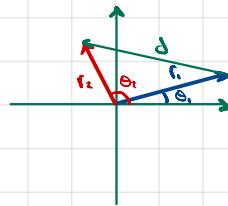


## Ch 4 Appendix 3 - Polar Coordinates

1. Distance between  $(r_1, \theta_1)$  and  $(r_2, \theta_2)$  is  $d^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)$



$$(x_1, y_1) = (r_1 \cos \theta_1, r_1 \sin \theta_1)$$

$$(x_2, y_2) = (r_2 \cos \theta_2, r_2 \sin \theta_2)$$

$$d = \sqrt{(r_2 \cos \theta_2 - r_1 \cos \theta_1)^2 + (r_2 \sin \theta_2 - r_1 \sin \theta_1)^2}$$

$$= \sqrt{r_2^2 \cos^2 \theta_2 - 2r_1 r_2 \cos \theta_1 \cos \theta_2 + r_1^2 \cos^2 \theta_1 + r_2^2 \sin^2 \theta_2 - 2r_1 r_2 \sin \theta_1 \sin \theta_2 + r_1^2 \sin^2 \theta_1}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2)}$$

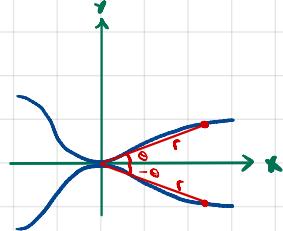
$$= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}$$

$d^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)$  is the law of cosines

2. ii } even

$$r = f(\theta) \text{ even}$$



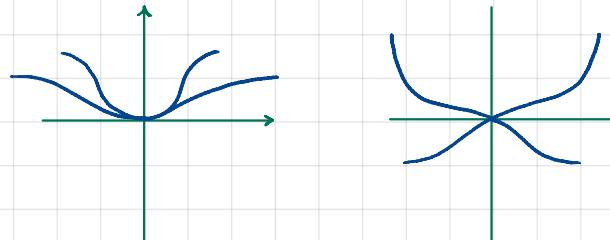
$$\Rightarrow f(\theta) = f(-\theta) = r$$

Given a point  $(r, \theta)$ , the point  $(r, -\theta)$  is symmetric to  $(r, \theta)$  relative to the x-axis.

The graph of  $f$  is symmetric about x-axis.

iii) f odd

$$r = f(\theta) \text{ odd}$$



$$\Rightarrow f(\theta) = -f(-\theta)$$

$$f(-\theta) = -f(\theta)$$

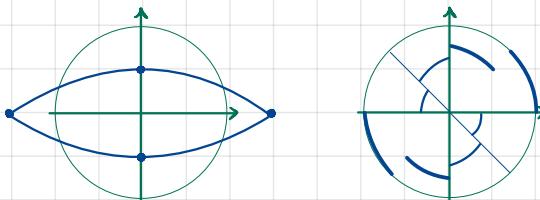
For each  $(r, \theta)$  there is  $(-r, -\theta)$ , symmetric about y-axis

iv)  $f(\theta) = f(\theta + \pi)$

$$f(\theta) = f(\pi) = f(2\pi) = f(3\pi)$$

$$f(\pi/2) = f(3\pi/2) = f(\pi + \pi/2)$$

Graph of  $f$  symmetric about the origin



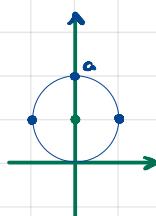
### 3. ii) $r = a \sin \theta$

$$r^2 = a \sin \theta = a y$$

$$x^2 + r^2 - 2y = 0$$

$$x^2 + y^2 - 2y + \frac{a^2}{4} + \frac{a^2}{4} - \frac{a^2}{4} = 0$$

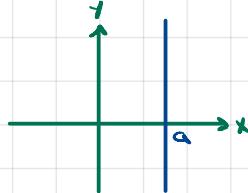
$$x^2 + (y - \frac{a}{2})^2 = (\frac{a}{2})^2$$



$\theta$	$r$
0	0
$\pi/2$	$a$
$\pi$	0
$3\pi/2$	$-a$
$2\pi$	0
$\pi/4, 3\pi/4$	$\frac{a\sqrt{2}}{2}$
$5\pi/4, 7\pi/4$	$-\frac{a\sqrt{2}}{2}$

$$\text{iii) } r = a \sec \theta = \frac{a}{\cos \theta}$$

$$1 = \frac{a}{r \cos \theta} = \frac{a}{x} \Rightarrow x = a, a \neq 0$$



$a = 0 \Rightarrow r = 0 \forall \theta$ , graph is single point.

$$\text{iv) } r = \cos(2\theta)$$

$$\cos(2\theta) = \cos(\theta + \theta) = \cos^2 \theta - \sin^2 \theta = 1 - 2\sin^2 \theta$$

$$r = 1 - 2\sin^2 \theta$$

$$r^2 = r - 2r\sin^2 \theta$$

$$r^3 = r^2 - 2r^2 \sin^2 \theta = r^2 - 2y^2 = x^2 + y^2 - 2y^2$$

$$r^3 = x^2 + y^2 - 2y^2 = x^2 - y^2$$

$$\sqrt{(x^2 + y^2)^3} = x^2 - y^2$$

$$x - y = \sqrt{(2y^2)^2} = 0 \Rightarrow x - y = 0$$

$$\text{v) } f(\theta) = \cos(3\theta) \cdot \cos(2\theta + \theta)$$

$$= \cos(2\theta) \cos \theta - \sin 2\theta \sin \theta$$

$$= (\cos^2 \theta - \sin^2 \theta) \cos \theta - (2\sin \theta \cos \theta) \sin \theta$$

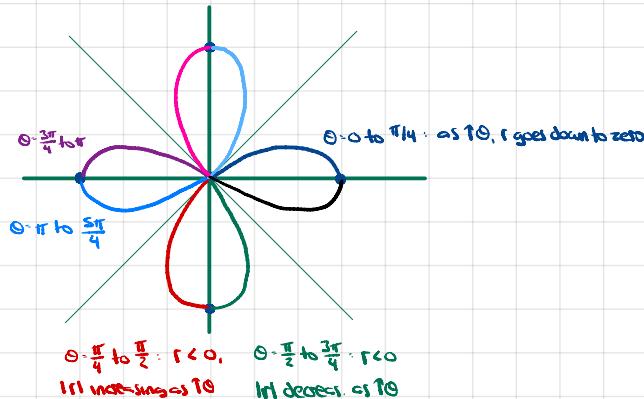
$$= \cos^3 \theta - 3\sin^2 \theta \cos \theta$$

$$r^4 = r^3 \cos^3 \theta - 3r^2 \sin^2 \theta \cos \theta$$

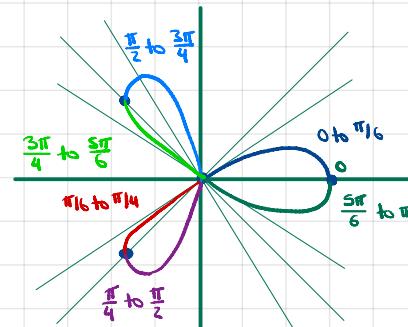
$$r^4 = x^3 - 3x^2 y^2$$

$$(x^2 + y^2)^2 = x^3 - 3x^2 y^2$$

Note: for  $\theta \in [\pi, 2\pi]$



$\theta$	$\cos 3\theta$
0	1
$\pi/2$	0
$\pi$	-1
$3\pi/2$	0
$2\pi$	1
$\pi/4$	$-\sqrt{2}/2$
$\pi/6$	0
$2\pi/3$	-1



From  $\theta = 0$  to  $\pi/4$ , we're taking  $\cos 3\theta$  from  $\theta = 0$  to  $3\pi/4$ , so  $\cos(3\theta)$  from 1 to 0 at  $\theta = \frac{\pi}{6}$ .

$$v) r = |\cos 2\theta| = f(\theta). \text{ Periodically, } r = \cos 2\theta = g(\theta)$$

$$\cos(2\theta) = \cos(\theta + \theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$$

$$\text{Let } A = [2\pi k - \frac{\pi}{4}, 2\pi k + \frac{\pi}{4}] = [-\frac{\pi}{4}, \frac{\pi}{4}]$$

$$B = [2\pi k + \frac{3\pi}{4}, 2\pi k + \frac{5\pi}{4}] = [\frac{3\pi}{4}, \frac{5\pi}{4}]$$

$$C = [2\pi k + \frac{\pi}{4}, 2\pi k + \frac{3\pi}{4}] = [\frac{\pi}{4}, \frac{3\pi}{4}]$$

$$D = [2\pi k + \frac{5\pi}{4}, 2\pi k + \frac{7\pi}{4}] = [\frac{5\pi}{4}, \frac{7\pi}{4}]$$

$$r = \begin{cases} \cos(2\theta) & \theta \in A \cup B \\ -\cos(2\theta) & \theta \in C \cup D \end{cases}$$

Note that for every  $\theta_i \in C$  there is a  $\theta_2 = \theta_1 + \pi \in D$ .

$$\text{Let } \theta_1 \in C \Rightarrow \cos(2\theta_1) < 0$$

$$\Rightarrow r_1 = f(\theta_1) = |\cos(2\theta_1)| = -\cos(2\theta_1) = -g(\theta_1)$$

For any point  $(r_1, \theta_1)$ , the point  $(-r_1, \theta)$  is the same point as  $(r_1, \theta + \pi)$  or  $(r_1, \theta - \pi)$

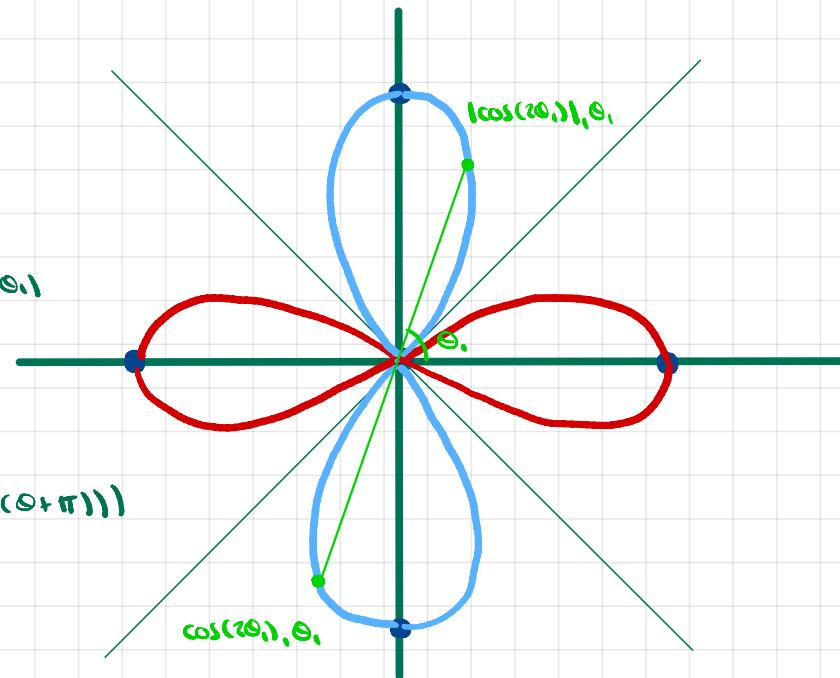
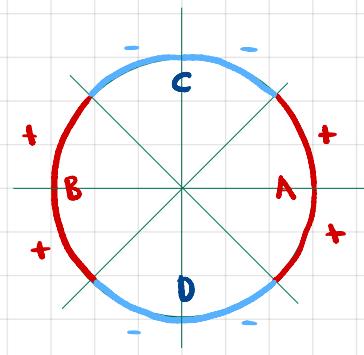
$$(r_1, \theta_1) \text{ is same point as } (r_1, g(\theta_1 + \pi)) = (r_1, \cos(2(\theta_1 + \pi)))$$

$$\begin{aligned} \Rightarrow \text{for every } \theta \in C, (r_1, f(\theta)) &= (r_1, g(\theta + \pi)) \\ &= (r_1, g(\phi)) \quad \phi \in D \end{aligned}$$

$$\text{Similarly, } \forall \theta \in D, (r_1, f(\theta)) = (r_1, g(\theta + \pi)) = (r_1, g(\phi)) \quad \phi \in C.$$

$$\text{For } \theta \in A \cup B, (r_1, f(\theta)) = (r_1, g(\theta))$$

$\Rightarrow$  The graph of  $r = f(\theta)$  is the same as that of  $r = g(\theta)$ .



$$\text{VII } r = 1 \cos 3\theta$$

$$A = [0, \frac{\pi}{6}]$$

$$B = [\frac{\pi}{2}, \frac{5\pi}{6}]$$

$$C = [\frac{\pi}{6}, \frac{\pi}{2}]$$

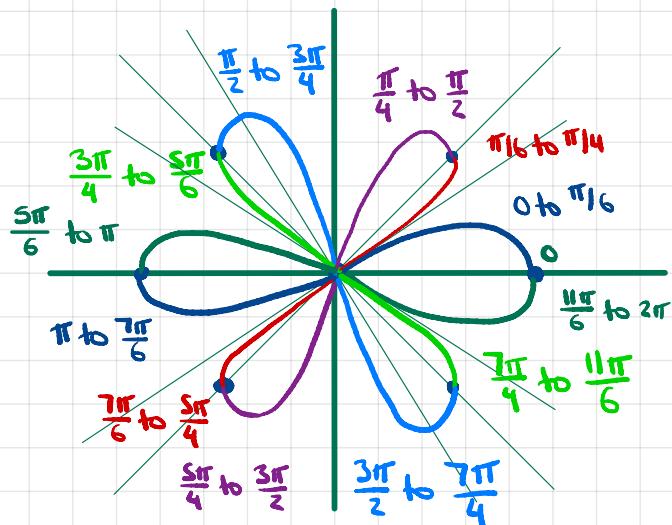
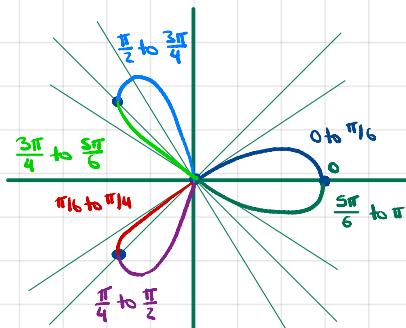
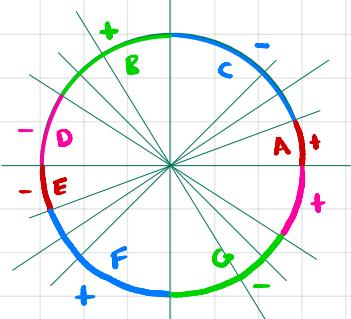
$$D = [\frac{5\pi}{6}, \pi]$$

$$r = \begin{cases} \cos 3\theta & \theta \in A \cup B \\ -\cos 3\theta & \theta \in C \cup D \end{cases}$$

For  $\theta \in A \cup B$ ,  $r = f(\theta) = g(\theta)$

For  $\theta \in C \cup D$ ,  $r = f(\theta) = -\cos 3\theta = -g(\theta)$

$\theta \in C \cup D \rightarrow$  same points as  $g(\theta)$  but cd sign of  $r$  deleted.



4.

$$\text{ii) } r = a \sin \theta$$

$$r^2 = a \sin \theta = ay$$

$$x^2 + y^2 - 2y \frac{a}{2} + \frac{a^2}{4} - \frac{a^2}{4} = 0$$

$$x^2 + (y - \frac{a}{2})^2 - (\frac{a}{2})^2 = 0$$

$$x^2 + (y - \frac{a}{2})^2 = (\frac{a}{2})^2$$

$$\text{iii) } r = a \sec \theta = \frac{a}{\cos \theta}$$

$$1 = \frac{a}{r \cos \theta} = \frac{a}{x} \Rightarrow x = a, a \neq 0$$

$a=0 \Rightarrow r=0 \forall \theta$ , graph is single point.

$$\text{iv) } r = \cos(2\theta)$$

$$\cos(2\theta) = \cos(\theta + \theta) = \cos^2 \theta - \sin^2 \theta = 1 - 2\sin^2 \theta$$

$$r = 1 - 2\sin^2 \theta$$

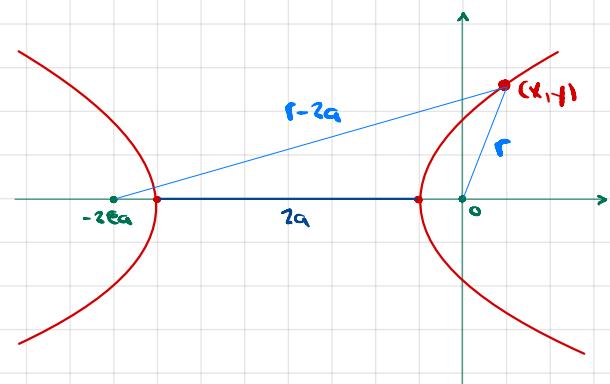
$$r^2 = r - 2r\sin^2 \theta$$

$$r^3 = r^2 - 2r^2 \sin^2 \theta = r^2 - 2y^2 = x^2 + y^2 - 2y^2$$

$$r^3 = x^2 + y^2 - 2y^2 = x^2 - y^2$$

$$\sqrt{(x^2 + y^2)^3} \cdot x^2 - y^2$$

5.



$$2\epsilon a > 2a \Rightarrow \epsilon > 1$$

distance  $(x_1, y_1)$  to focus at  $(0, 0)$  is  $(x^2 + y^2)^{1/2} - r \Rightarrow r^2 = x^2 + y^2$

distance  $(x_1, y_1)$  to focus at  $(-2\epsilon a, 0)$  is

$$s^2 = (x + 2\epsilon a)^2 + y^2 = x^2 + 4\epsilon ax + 4\epsilon^2 a^2 + y^2$$

By definition of hyperbola,  $r - s = \pm 2a$

Let's first consider the positive case, i.e. the right side of the hyperbola:

$$\Rightarrow s = r - 2a \Rightarrow s^2 = 4a^2 - 4ar + r^2$$

$$\Rightarrow 4a^2 - 4ar + r^2 = x^2 + 4\epsilon ax + 4\epsilon^2 a^2 + y^2$$

sub in expr. for  $r^2$

$$a^2 - ar = \epsilon ax + \epsilon^2 a^2$$

$$a - r = \epsilon x + \epsilon^2 a$$

$$r = a - \epsilon x - \epsilon^2 a = a(1 - \epsilon^2) - \epsilon x$$

$$= \lambda - \epsilon x = \lambda - a(1 - \epsilon^2)$$

$$r = \lambda - \epsilon r \cos \theta$$

$$r(1 - \epsilon \cos \theta) = \lambda$$

$$r = \frac{\lambda}{1 + \epsilon \cos \theta} = \frac{a(1 - \epsilon^2)}{1 + \epsilon \cos \theta}$$

Now the left side of the hyperbola:  $r - s = -2a$

$$\Rightarrow s = r + 2a \Rightarrow s^2 = r^2 + 4ar + 4a^2$$

$$\Rightarrow 4a^2 + 4ar + r^2 = x^2 + 4\epsilon ax + 4\epsilon^2 a^2 + y^2$$

$$a + r = \epsilon x + \epsilon^2 a$$

$$r = -a + \epsilon x + \epsilon^2 a = -a(1 - \epsilon^2) + \epsilon x$$

$$-\lambda + \epsilon x = -(\lambda - \epsilon x) = -(\lambda - \epsilon r \cos \theta)$$

$$r(1 - \epsilon \cos \theta) = -\lambda \Rightarrow r = -\frac{\lambda}{1 - \epsilon \cos \theta}$$

6.

$$\text{distance to } O: r^2 = x^2 + y^2$$

$$\text{distance to line } x-a: s^2 = (a-x)^2 = (a-r\cos\theta)^2 \Rightarrow s = |a - r\cos\theta|$$

But  $x = r\cos\theta \geq 0$  must be  $\leq a$

$$x = r\cos\theta \geq a \Rightarrow x - a \geq 0$$

$$s = x - a = \sqrt{x^2 + y^2} \geq \sqrt{x^2} = x$$

$$\Rightarrow x - a \geq x \Rightarrow -a \geq 0, \text{ but we assumed } a > 0.$$

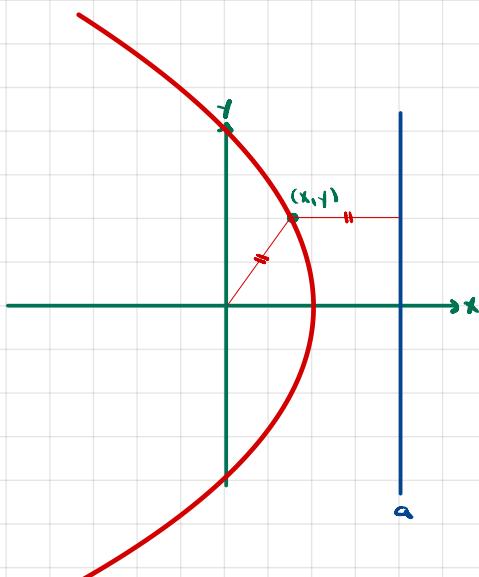
$$\Rightarrow s = a - x = a - r\cos\theta$$

$$s = r \Rightarrow r = a - r\cos\theta \Rightarrow a = r(1 + \cos\theta)$$

$$r = \frac{a}{1 + \cos\theta}$$

↓

Same form  $\frac{1}{1 + \epsilon\cos\theta}$  as hyperbola / ellipse.



7. For any  $\Lambda$  and  $\epsilon$ , consider graph of  $r = \frac{\Lambda}{1 + \epsilon\cos\theta}$  which implies  $r = \Lambda - \epsilon x$ ,  $\Lambda = a(1 - \epsilon^2)$

Show that points on the graph of these equations satisfy,

$$(1 - \epsilon^2)x^2 + y^2 = \Lambda^2 - 2\Lambda\epsilon x$$

$$\Rightarrow F = (1 - \epsilon^2) \left( \frac{2\epsilon\Lambda}{2(1 - \epsilon^2)} \right)^2 + \Lambda^2$$

$$r = \Lambda - \epsilon x$$

$F > 0, \Lambda, C > 0 \Rightarrow \text{ellipse}$

$$r^2 = x^2 + y^2 = \Lambda^2 - 2\Lambda\epsilon x + \epsilon^2 x^2$$

$$\Lambda > 0 \Rightarrow 1 - \epsilon^2 > 0 \Rightarrow -1 < \epsilon < 1 \Rightarrow F > 0$$

$$\Rightarrow x^2(1 - \epsilon^2) + y^2 = \Lambda^2 - 2\Lambda\epsilon x$$

$A < 0, C > 0, F > 0 \Rightarrow \text{hyperbola}$

$$x^2(1 - \epsilon^2) + y^2 - \Lambda^2 = 0$$

$$A < 0 \Rightarrow 1 - \epsilon^2 < 0 \Rightarrow \epsilon^2 > 1 \Rightarrow \epsilon > 1 \text{ or } \epsilon < -1$$

$$A, C < 0 \Rightarrow \epsilon^2 + 1 \Rightarrow \epsilon \neq \pm 1$$

$$F > 0 \Rightarrow (1 - \epsilon^2) \left( \frac{2\epsilon\Lambda}{2(1 - \epsilon^2)} \right)^2 + -\Lambda^2$$

$$A = 1 - \epsilon^2$$

$$\epsilon = 1 \Rightarrow F = -\Lambda^2 > 0$$

$$B = 2\epsilon\Lambda$$

$$A = 0$$

$$C = 1$$

The equation becomes

$$E = -\Lambda^2$$

$$2\epsilon\Lambda x + y^2 - \Lambda^2 = 0$$

$$Bx + y^2 + E = 0$$

parabolas

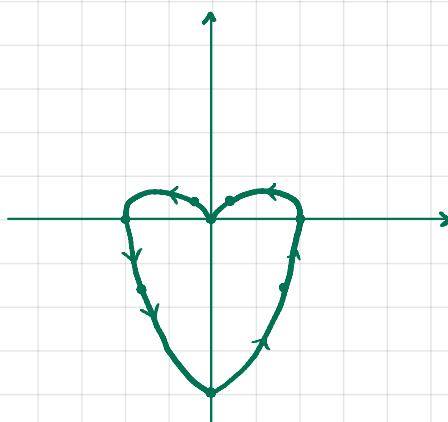
$$8. a) r = 1 - \sin\theta$$

$$r^2 = r - rs\sin\theta$$

$$x^2 + y^2 = r - y$$

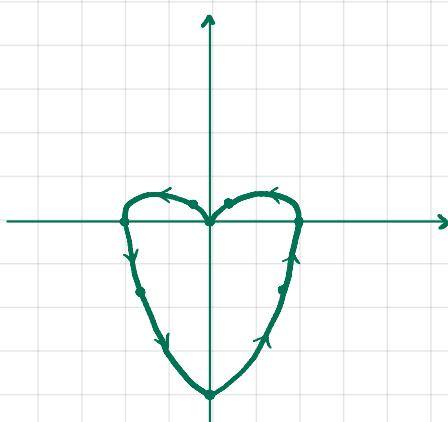
$$x^2 + y^2 = (x^2 + y^2)^{1/2} - 1$$

$\theta$	$r$
0	1
$\pi/4$	$1 - \sqrt{2}/2 \approx 0.3$
$\pi/2$	0
$3\pi/4$	$\approx 0.3$
$\pi$	1
$5\pi/4$	$\approx 1.3$
$3\pi/2$	2
$7\pi/4$	$\approx 1.3$



$$b) r = -1 - \sin\theta$$

$\theta$	$r$
0	-1
$\pi/4$	$-1 - \sqrt{2}/2 \approx -1.3$
$\pi/2$	-2
$3\pi/4$	$\approx -1.3$
$\pi$	-1
$5\pi/4$	$-1 + \frac{\sqrt{2}}{2} \approx -0.3$
$3\pi/2$	0
$7\pi/4$	$\approx -0.3$



Consider a point  $(r_1, \theta) = (-1 - \sin\theta, \theta)$  on the graph of  $r_1 = -1 - \sin\theta$ .

Now consider the point  $(r_2, \theta + \pi) = (1 - \sin(\theta + \pi), \theta + \pi)$  on the graph of  $r_2 = 1 - \sin\theta$ .

$$1 - \sin(\theta + \pi) = 1 - (-\sin\theta) = 1 + \sin\theta = -(-1 - \sin\theta)$$

$$(r_2, \theta + \pi) = (-(-1 - \sin\theta), \theta + \pi) = (-1 - \sin\theta, \theta) = (r_1, \theta)$$

Therefore every point  $(r_1, \theta)$  on  $r_1 = -1 - \sin\theta$  is the same point as  $(r_2, \theta + \pi)$ ,  $\forall \theta$ .

⇒ The graphs of  $r_1 = J_1(\theta)$  and  $r_2 = J_2(\theta)$  have the same points ⇒ same graphs.

$$c) r^2 = r - rs\sin\theta$$

$$x^2 + y^2 = r - y$$

$$x^2 + y^2 = (x^2 + y^2)^{1/2} - 1$$

$$(x^2 + y^2)^2 = x^2 + y^2 - 2y(x^2 + y^2) + y^2$$

$$(x^2 + y^2)^2 + 2y(x^2 + y^2) + y^2 = x^2 + y^2$$

$$(x^2 + y^2 + y)^2 = x^2 + y^2$$

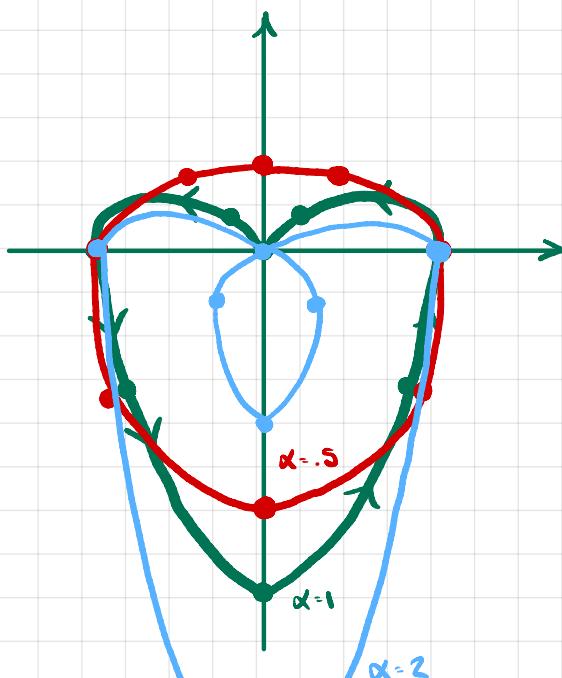
$$9. ii) r = 1 - \frac{\sin\theta}{2}$$

Note that all these graphs of form  $r = 1 - \alpha \sin\theta$  are symmetric about  $y$ -axis.

$\theta$	$1 - \sin\theta$	$1 - \frac{\sin\theta}{2}$
0	1	1
$\pi/4$	$1 - \sqrt{2}/2 \approx 0.3$	$1 - \sqrt{2}/4 \approx 0.65$
$\pi/2$	0	0.5
$3\pi/4$	2	1.5
$\pi$	$1 + \sqrt{2}/2 \approx 1.35$	

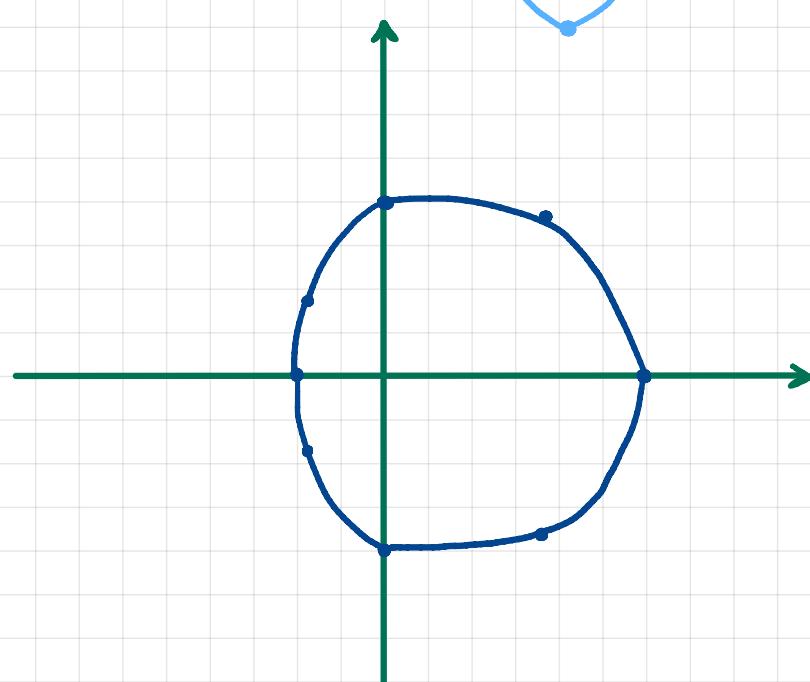
$$ii) r = 1 - 2\sin\theta$$

$\theta$	$1 - 2\sin\theta$
0	1
$\pi/4$	$1 - \sqrt{2} \approx -0.4$
$\pi/2$	-1
$3\pi/4$	3
$\pi$	$1 + \sqrt{2} \approx 2.4$
$5\pi/6$	0



$$iii) r = 2 + \cos\theta$$

$\theta$	$2 + \cos\theta$
0	3
$\pi/4$	$2 + \sqrt{2}/2 \approx 2.7$
$\pi/2$	2
$3\pi/4$	$2 - \sqrt{2}/2 \approx 1.3$
$\pi$	1
$5\pi/4$	$2 - \sqrt{2}/2 \approx 1.3$
$3\pi/2$	2
$7\pi/4$	$2 + \sqrt{2}/2 \approx 2.7$



$$\cos\theta = \sin(\theta + \frac{\pi}{2})$$

$$\Rightarrow r = 2 + \sin(\theta + \frac{\pi}{2})$$

$$= 2(1 + \frac{1}{2}\sin(\theta + \frac{\pi}{2})) = 2\underbrace{(1 - \frac{1}{2}\sin(\theta - \frac{\pi}{2}))}_{\text{looks like } f(\theta) \text{ from i) with extra term in sin.}}$$

This extra  $\pi/2$  represents a rotation counterclockwise by  $\pi/2$  rad

double the size of ii)'s graph

graph of  $r = f(\theta)$  symmetric about  $y$ -axis

The conditions for symmetry about  $y$ -axis in Cartesian coord are:

$$(x_1, y_1) \in G \Rightarrow (-x_1, y_1) \in G$$

expressed in polar coord:

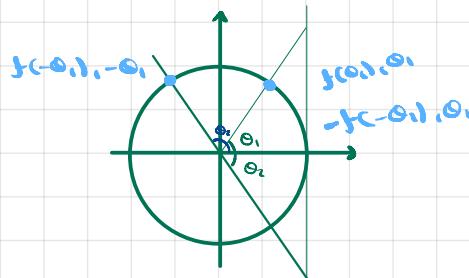
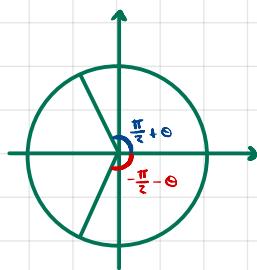
$$x = f(\theta_1) \cos \theta_1 = -f(\theta_2) \cos \theta_2$$

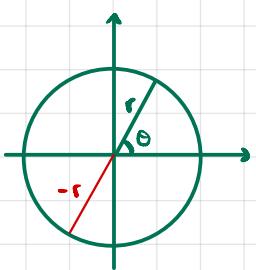
$$y = f(\theta_1) \sin \theta_1 = f(\theta_2) \sin \theta_2$$

$$\Rightarrow \tan \theta_1 = -\tan \theta_2 \Leftrightarrow \theta_1 = -\theta_2$$

$$\Rightarrow \theta_2 = -\theta_1 + 2\pi n \Rightarrow f(\theta_2) = f(-\theta_1)$$

$$\theta_2 = -\theta_1 + \pi + 2\pi n \Rightarrow f(\theta_2) =$$





$$\text{given } (r, \theta) = (\text{f}(\theta), \theta)$$

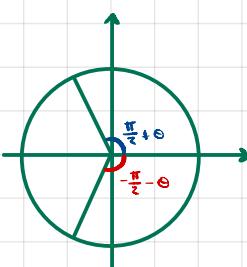
what do we know about  $(-r, \theta) = (-\text{f}(\theta), \theta) = (r, \theta + \pi) = (\text{f}(\theta), \theta + \pi)$

$$x_2 = r \cos(\theta + \pi) = -r \cos(\theta) = x_1$$

$$y_2 = -r \sin(\theta) = -y_1$$

$$\text{f}(\theta) = -\text{f}(-\theta)$$

$$\text{f}\left(\frac{\pi}{2} + \theta\right) = -\text{f}\left(-\frac{\pi}{2} - \theta\right)$$



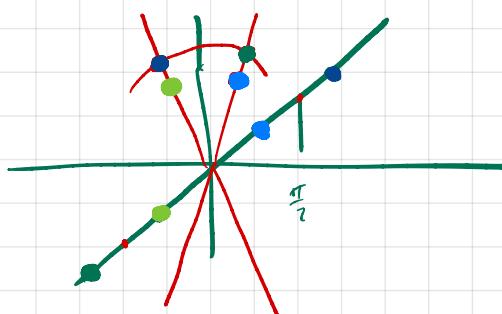
I want to prove that  $\text{f}$  odd  $\Rightarrow$  for every point  $(\text{f}\left(\frac{\pi}{2} + \theta\right), \frac{\pi}{2} + \theta)$  there is a point  $(\text{f}\left(\frac{\pi}{2} + \theta\right), \frac{\pi}{2} - \theta)$ .

Consider the polar point  $(r_1, \theta_1) = (\text{f}\left(\frac{\pi}{2} + \theta\right), \frac{\pi}{2} + \theta)$

$$\begin{aligned} \text{f odd} &\Rightarrow (r_1, \theta_1) = (-\text{f}\left(-\frac{\pi}{2} - \theta\right), \frac{\pi}{2} + \theta) \\ &= (\text{f}\left(-\frac{\pi}{2} - \theta\right), -\frac{\pi}{2} + \theta) \end{aligned}$$

$\Rightarrow (\text{f}\left(-\frac{\pi}{2} - \theta\right), -\frac{\pi}{2} + \theta)$  is on the graph.

But so is  $(\text{f}\left(-\frac{\pi}{2} - \theta\right), -\frac{\pi}{2} - \theta)$



Consider polar point  $(r_1, \theta_1) = (\text{f}\left(-\frac{\pi}{2} - \theta\right), -\frac{\pi}{2} - \theta)$

$$\begin{aligned} \text{f odd} &\Rightarrow (r_1, \theta_1) = (-\text{f}\left(\frac{\pi}{2} + \theta\right), -\frac{\pi}{2} - \theta) \\ &= (\text{f}\left(\frac{\pi}{2} + \theta\right), \frac{\pi}{2} - \theta) \end{aligned}$$

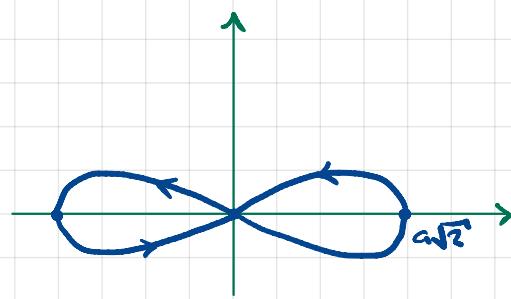
$\Rightarrow (\text{f}\left(\frac{\pi}{2} + \theta\right), \frac{\pi}{2} - \theta)$  is on the graph.

But so is  $(\text{f}\left(\frac{\pi}{2} + \theta\right), \frac{\pi}{2} + \theta)$

10. a)  $r^2 \cdot 2a^2 \cos(2\theta)$

$$r^2 \geq 0 \Rightarrow \cos 2\theta \geq 0 \Rightarrow \theta \in [-\pi/4, \pi/4] \cup [3\pi/4, 5\pi/4]$$

$\theta$	$2a^2 \cos(2\theta)$	$r$
$-\pi/4$	0	0
0	$2a^2$	$a\sqrt{2}$
$\pi/4$	0	0
$3\pi/4$	0	0
$\pi$	$2a^2$	
$5\pi/4$	0	
$3\pi/2$		



Note that  $f(\theta) = 2a^2 \cos(2\theta)$  is even.

$$f(-\theta) = 2a^2 \cos(-2\theta) = 2a^2 \cos(2\theta) = f(\theta)$$

b)  $r^2 = 2a^2 \cos(2\theta)$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$r^4 = 2a^2(r^2 \cos^2 \theta - r^2 \sin^2 \theta)$$

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$$

c)

$$d_1 = \sqrt{(x+a)^2 + y^2}$$

$$d_2 = \sqrt{(x-a)^2 + y^2}$$

$$d_1 d_2 = a^2 \Rightarrow \sqrt{[(x+a)^2 + y^2][(x-a)^2 + y^2]}$$

$$= \sqrt{x^4 + 2x^2y^2 + y^4 + a^4 - 2a^2x^2 + 2a^2y^2}$$

$$= \sqrt{(x^2 + y^2)^2 - 2a^2(x^2 - y^2) + a^4}$$

$$\Rightarrow \cancel{d_1^4} = (x^2 + y^2)^2 - 2a^2(x^2 - y^2) + \cancel{d_1^4}$$

$$(x^2 + y^2)^2 - 2a^2(x^2 - y^2)$$

$$d) \{(x, y) : d_1 d_2 = b, b > a^2\}$$

