

Chapter 3 - Functions

Profound Def: Function is a rule that assigns to each of certain real numbers, some other real numbers.

Polynomial Fn f is poly. if $\exists a_0, \dots, a_n \in \mathbb{R}$ s.t. $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ \rightarrow highest power: degree of f

Rational Fns $\frac{P}{Q}$, P, Q polynomial fns, Q not always zero

Combining fns to produce new fns

$f+g$ fns. Define fn $f+g$ as $(f+g)(x) = f(x) + g(x)$

\rightarrow domain $f+g = \text{domain } f \cap \text{domain } g$

Similar, define $f \cdot g$, f/g , $c \cdot g$

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

$$(c \cdot g)(x) = c \cdot g(x) \quad (\text{special case of } f \cdot g \text{ w/ } f(x) = c)$$

Some results

$$(f \cdot g) \cdot h = f \cdot (g \cdot h)$$

$$\begin{aligned} ((f \cdot g) \cdot h)(x) &= (f \cdot g)(x) \cdot h(x) = (f(x) \cdot g(x)) \cdot h(x) = f(x) \cdot g(x) \cdot h(x) \\ &= f(x) \cdot (g(x) \cdot h(x)) = f \cdot (g \cdot h)(x) \end{aligned}$$

Identity fn: $I(x) = x$

$$\rightarrow f(x) = \frac{x+x^2+x\sin^2 x}{x\sin x+x\sin^2 x} = \frac{I+I \cdot I + I \cdot \sin \cdot \sin}{I \cdot \sin + I \cdot \sin \cdot \sin}$$

Define composition fn $f \circ g$

$$(f \circ g)(x) = f(g(x))$$

$$\text{domain}(f \circ g) = \{x : x \text{ in domain } g, g(x) \text{ in domain } f\}$$

$$\begin{array}{ll} f = I \cdot I & (f \circ g)(x) = I(\sin x) \cdot I(\sin x) = \sin^2 x \\ g = \sin & (g \circ f)(x) = \sin(I(x) \cdot I(x)) = \sin(x^2) \end{array}$$

composition is associative $(f \circ g) \circ h = f \circ (g \circ h)$

$$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x)))$$

$$(f \circ (g \circ h))(x) = f((g \circ h)(x)) = f(g(h(x)))$$

notation

$$x \rightarrow x^2$$

The function $f(x) = x^2$

Def (Function) f_n is collection of pairs of numbers w/ following property: $(a,b), (c,c)$ both in the collection then $b=c$.

Def (Domain) domain of f_n is set of all a for which there is some b such that (a,b) is in f .
If a is in domain of f , b_1 the def. of f_n , there is a unique b such that (a,b) in f . This unique b is denoted $f(a)$.

Appendix - ordered Pairs

Def (ordered Pair) $(a,b) = \{\{a\}, \{a,b\}\}$

Theorem 1 $(a,b) = (c,d) \Rightarrow a=c, b=d$

proof:

$$\{\{a\}, \{a,b\}\} = \{\{c\}, \{c,d\}\}$$

The two sets must contain the same elements.

The elements are themselves sets.

One element is a set with one element inside.

This one element set must be the same in both top-level sets

$$\begin{aligned} &\Rightarrow a=c \\ &\Rightarrow \{\{a\}\} = \{\{c\}\} \Rightarrow b=d \end{aligned}$$

note

one approach is a) choose: define ordered pair as certain way, derive
a particular prop., or a theorem.

Alternatively, could have simply introduced the term (a,b) and adopted
the property as an axiom. The lack of definition in this case is justified
by the practical fact that

- i) this is the only interesting/relevant property of ordered pairs
- ii) therefore there is nothing else to be done with a def.
other than simply define the property.