

CNT - Least Upper Bounds

Def: A set of real numbers is bounded above if there is a number $x \in \mathbb{R}$ s.t. $x \geq a$ for every $a \in A$.
Such a number is called an upper bound for A .

- This def. is for sets.

In ch. 7 we spoke of this concept for fns.

The two are closely related.

$$\text{Let } A = \{f(x) : a \leq x \leq b\}$$

Then, f bounded above on $[a,b] \Leftrightarrow A$ bounded above.

- \mathbb{R} and \mathbb{N} are not bounded above.

more accurately, "the"

Def: A number x is a least upper bound, aka supremum of A if

1) x is an upper bound of A

and 2) if y is upper bound of A then $y \geq x$.

$$x = \sup A$$

Similarly,

Set A of real numbers bounded below if $\exists x \in \mathbb{R}, \forall a \in A, x \leq a$.

Such an x is a lower bound for A .

A number x is the greatest lower bound aka infimum if

1) x is a lower bound for A

2) $\forall y, y$ lower bound for $A \rightarrow y \leq x$

$$x = \inf A$$

"most important property of real numbers"

P13: (least upper bound property)

A is set of real numbers

$$A \neq \emptyset$$

A bounded above

$\rightarrow A$ has least upper bound

Note

- If $A = \emptyset$ then $\forall y \in A, \forall x \in \mathbb{R} \rightarrow x \geq y$.

Thus every number is an upper bound of A , because there is no $x \in A$. The statement is inherently vacuously true.

But there is no least upper bound.

- P13 won't hold for \mathbb{Q} .

$$\text{e.g. } A = \{x \in \mathbb{Q} : x^2 < \sqrt{2}\}$$

A has no least upper bound.

Theorem 7-1

$$\begin{aligned} f \text{ cont. on } [a,b] \\ f(a) < c < f(b) \end{aligned} \rightarrow \exists x \in [a,b], f(x) = c$$

Proof

strategy: locate smallest $x \in [a,b]$ s.t. $f(x) = c$.

$$\text{Let } A = \{x : a \leq x \leq b : f \text{ neg. on } [a,x]\}$$

$$f(a) < c \rightarrow a \in A \rightarrow A \neq \emptyset$$

In 6-16a we proved:

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

$$\rightarrow \exists \delta > 0 \forall x, 0 < x - a < \delta \rightarrow f(x) < c$$

$$f(a) < c$$

Therefore, since f cont. on $[a,b]$ we have $\lim_{x \rightarrow a^+} f(x) = f(a) < c$, so

$$\exists \delta > 0 \forall x, a < x < a + \delta \rightarrow f(x) < c$$

thus $\forall x, a < x < a + \delta \rightarrow x \in A$.

In 6-16b we proved

$$\lim_{x \rightarrow b^-} f(x) = f(b)$$

$$\rightarrow \exists \delta > 0 \forall x, b - \delta < x < b \rightarrow f(x) > c$$

$$f(b) > c$$

Therefore since $\lim_{x \rightarrow b^-} f(x) = f(b) > c$ we have

$$\exists \delta > 0 \ b - \delta < x < b \rightarrow f(x) > c \rightarrow x \text{ upper bound of } A.$$

Thus, the assumptions of P13 are true. Therefore, according to P13, A has a least upper bound, call it α , and $a < \alpha < b$.

Let's show that $f(\alpha) = c$.

Assume $f(\alpha) < c$.

By Th. 6-3, there is an interval around α in which $f < c$.

i.e. $\exists \delta > 0, \forall x, |\alpha - x| < \delta \rightarrow f(x) < c$.

But then $\forall x, x \in (\alpha - \delta, \alpha + \delta) \rightarrow x > \alpha \wedge f(x) < c$

$\wedge (\forall y \in (\alpha, x] \rightarrow f(y) < c)$

Therefore, $x \in A$ and so x is not sup A . \perp .

Therefore $f(\alpha) \geq c$.

Assume $f(\alpha) > c$. By Th. 6-3, $\exists \delta > 0, \forall x, |\alpha - x| < \delta \rightarrow f(x) > c$

$\rightarrow f(x) > c$, i.e. $\exists \delta > 0$ in some open interval containing α .

But then, $\forall x, x \in (\alpha - \delta, \alpha) \rightarrow f(x) > c$

Therefore, x is upper bound for A . α not sup A . \perp .

Therefore, $f(\alpha) = c$.

To prove theorems 6-2 and 6-3 we need the following theorem:

Theorem 1

f cont. at $a \rightarrow \exists \delta > 0$ s.t. f is bounded above on $(a-\delta, a+\delta)$

Proof

f cont. at a therefore $\forall \epsilon > 0 \exists \delta > 0 \forall x: |x-a| < \delta \rightarrow |f(x) - f(a)| < \epsilon$.

choose any ϵ , say $\epsilon = 1$.

then

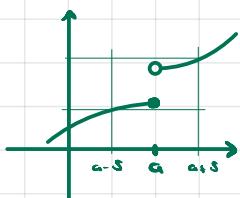
$\exists \delta > 0 \forall x \in (a-\delta, a+\delta) \rightarrow |f(x) - f(a)| < 1$

$\rightarrow |f(x) - f(a)| < 1 + |f(a)|$

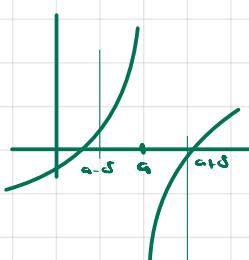
thus f is bounded on $(a-\delta, a+\delta)$.

* Note that if f not cont. at a , f may not be bounded

bounded



unbounded



• We could also prove that

f cont. on $[a, b] \rightarrow f$ bounded below on $(a-\delta, a+\delta)$

thus

f cont. on $[a, b] \rightarrow f$ bounded on $(a-\delta, a+\delta)$

the proofs are analogous to that of Th. 1.

• Also,

1) $\lim_{x \rightarrow a} f(x) = f(a) \rightarrow f$ bounded on $(a, a+\delta)$

2) $\lim_{x \rightarrow b} f(x) = f(b) \rightarrow f$ bounded on $(b-\delta, b)$

Proof of (1)

$\forall \epsilon > 0 \exists \delta > 0 \forall x: 0 < x-a < \delta \rightarrow |f(x) - f(a)| < \epsilon$.

let $\epsilon = 1$. Then $\exists \delta > 0 \forall x: a < x < a+\delta \rightarrow |f(x) - f(a)| < 1$

I.e. f bounded on $(a, a+\delta)$

Theorem 7-2

f cont. on $[a, b] \rightarrow f$ bounded above on $[a, b]$, i.e.
 $\exists M, \forall x \in [a, b], f(x) \leq M$

Proof

let $A = \{x: a \leq x \leq b \text{ and } f \text{ bounded above on } [a, x]\}$.

$a \in A$ because $x \in [a, a] \rightarrow f(x) \leq f(a)$.

so $A \neq \emptyset$.

A bounded above because $\forall x \in A, x \leq b$. b is an upper bound for A .
By P3, A has a least upper bound, $\sup A = \alpha$, and $\alpha \leq b$.

Assume $\alpha < b$.

Note that $\alpha > a$ because since f cont. on $[a, b]$ then
 $\lim_{x \rightarrow a} f(x) = f(a)$, hence f bounded on $(a, a+\delta)$. $\forall x$ in this
interval, $x \in A$, and if $a \leq x$ then $x > a$ so x not upper
bound. \perp

Th. 1 $\rightarrow \exists \delta > 0, f$ bounded on $(\alpha-\delta, \alpha+\delta)$

Since $\alpha = \sup A, \exists x_0 \in \alpha-\delta < x_0 < \alpha, x_0 \in A$

f bounded on $[a, x_0]$

$\forall x_1, x_2 \in (\alpha, \alpha+\delta) \rightarrow f$ bounded on $[x_1, x_2]$

$\rightarrow f$ bounded on $[a, x_2] \rightarrow x_2 \in A \rightarrow \alpha$ not $\sup A$
 \perp .

Therefore, $\alpha = b$.

At this point we know that $\forall x \in [a, b], x \in A$, because if $x \notin A$
then $\forall x_1, x_2 \in (x, b)$ could be an upper bound, \perp .

Since $\lim_{x \rightarrow b} f(x) = f(b)$ then f is bounded above on $(b-\delta, b)$.

$\exists x_0, x_0 \in (b-\delta, b)$. Then since $x_0 < b$ then $x_0 \in A$, f is
bounded on $[a, x_0]$. But f is bounded above on $[x_0, b]$.

Thus f bounded above on $[a, b], b \in A$.

Theorem 7-3

$$f \text{ cont. on } [a,b] \rightarrow \exists j \in [a,b], \forall x \in [a,b], f(x) \geq f(j)$$

Proof

f bounded on $[a,b]$, by Th. 7-2.

Therefore $A = \{f(x) : x \in [a,b]\}$ is bounded.

Since this set $\neq \emptyset$ it has a least upper bound $\alpha = \sup A$.

$$\forall x, x \in [a,b] \rightarrow f(x) \leq \alpha$$

It's to prove that $\exists j_1, j \in [a,b] \cap \alpha - f(j)$ then we have $\forall x, x \in [a,b] \rightarrow f(x) \leq f(j)$.

Assume $\forall j \in [a,b], \alpha > f(j)$

$$\text{Let } g(x) = \frac{1}{\alpha - f(x)}, x \in [a,b].$$

Since $\alpha > f(x)$ for $x \in [a,b]$, g continuous on $[a,b]$.

Let $\epsilon > 0$

Assume $\forall x \in [a,b], \alpha \geq f(x) + \epsilon$

$\exists \beta \in [f(x) + \epsilon, \alpha]$.

$$\forall x \in [a,b], \beta \geq f(x) + \epsilon > f(x)$$

Thus $\beta < \alpha$ is an upper bound for A , α is not $\sup A$.

↓

Hence it is true that $\forall \epsilon > 0 \exists x \in [a,b], \alpha - f(x) < \epsilon$.

$$\text{Therefore, } \forall \epsilon > 0 \exists x \in [a,b], g(x) = \frac{1}{\alpha - f(x)} > \frac{1}{\epsilon}$$

$\rightarrow g$ not bounded in $[a,b]$.

↓

Therefore, $\exists j \in [a,b], \alpha = f(j)$.

Archimedean property of real numbers: \mathbb{N} is not bounded.

not consequence of P1-P12

Theorem 2 \mathbb{N} is not bounded above.

Proof

Assume \mathbb{N} bounded above.

Since $\mathbb{N} \neq \emptyset$, then there is least upper bound $\alpha = \sup \mathbb{N}$.

$$\forall n \in \mathbb{N}, \alpha \geq n$$

Since $n \in \mathbb{N} \rightarrow n+1 \in \mathbb{N}$ we have

$$\forall n \in \mathbb{N}, \alpha \geq n+1$$

$$\alpha - 1 \geq n$$

Therefore $\alpha - 1$ is an upper bound for \mathbb{N} . \perp

Theorem 3 $\forall \epsilon > 0 \exists n \in \mathbb{N}, \frac{1}{n} < \epsilon$

Proof

Assume $\exists \epsilon > 0 \forall n \in \mathbb{N}, \frac{1}{n} \geq \epsilon$.

Let $\epsilon > 0$ be such a ϵ .

$$\forall n \in \mathbb{N} \quad n \leq \frac{1}{\epsilon}$$

$\frac{1}{\epsilon}$ upper bound for \mathbb{N} .

\perp

Therefore $\forall \epsilon > 0 \exists n \in \mathbb{N}, \frac{1}{n} < \epsilon$