Ch9 - Derivatives

1.

a) $f(x) = \frac{1}{x} \longrightarrow f'(a) = -\frac{1}{a^2},\ a \neq 0$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

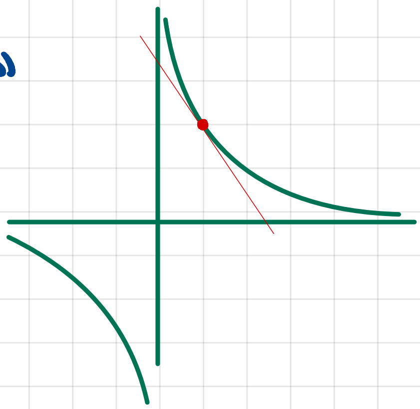
$$= \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{a} - (\cancel{a}+\cancel{h})}{\cancel{h}a(a+h)}$$

$$= \lim_{h\to 0} -\frac{1}{a(a+h)}$$

$$= -\frac{1}{a^2},\ \text{defined for } a \neq 0.$$

b)



eq. of tangent line at $(a, f(a))$

$$\frac{1}{x} = (x-a)\left(-\frac{1}{a^2}\right) + \frac{1}{a}$$

$$= -\frac{x}{a^2} + \frac{1}{a} + \frac{1}{a}$$

$$\rightarrow \frac{1}{x} + \frac{x}{a^2} = \frac{2}{a}$$

$$1 + \frac{x^2}{a^2} = \frac{2x}{a}$$

$$a^2 + x^2 = 2ax$$
$$x^2 - 2ax + a^2 = 0$$

$$\Delta = 4a^2 - 4a^2 = 0$$

There is a single solution $x = \frac{2a}{2} = a$

2.

a) $f(x) = \frac{1}{x^2} \longrightarrow f'(a) = -\frac{2}{a^3},\ a \neq 0$

$$f'(a) = \lim_{h\to 0} \frac{\frac{1}{(a+h)^2} - \frac{1}{a^2}}{h}$$

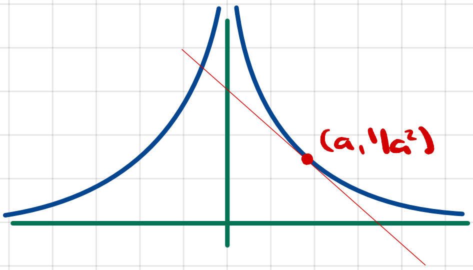
$$= \lim_{h\to 0} \frac{\cancel{a^2} - (\cancel{a^2}+2a\cancel{h}+h^{\cancel{2}})}{\cancel{h}a^2(a+h)^2}$$

$$= \lim_{h\to 0} \frac{-2a-h}{a^2(a+h)^2}$$

$$= \frac{-2a}{a^4}$$

$$= -\frac{2}{a^3},\ \text{defined for } a \neq 0$$

b)



$$\frac{1}{x^2} = -\frac{2}{a^3}(x-a) + \frac{1}{a^2}$$

$$= -\frac{2x}{a^3} + \frac{2a}{a^3} + \frac{1}{a^2}$$

$$1 = -\frac{2x^3}{a^3} + \frac{2x^2}{a^2} + \frac{x^2}{a^2}$$

$$a^3 = -2x^3 + 2ax^2 + ax^2$$

$$2x^3 - 3ax^2 + a^3 = 0$$

we know a is a solution, so $(x-a)$ is a factor.

$$\begin{array}{r|l}
 & 2x^2 - ax - a^2 \\
\hline
x-a & 2x^3 - 3ax^2 \qquad + a^3 \\
 & \underline{2x^3 - 2ax^2} \\
 & -ax^2 + a^3 \\
 & \underline{-ax^2 + a^2x} \\
 & -a^2x + a^3 \\
 & \underline{-a^2x + a^3} \\
 & 0
\end{array}$$

$$2x^3 - 3ax^2 + a^3 = (x-a)\underbrace{(2x^2 - ax - a^2)}$$

$$\Delta = a^2 - 4\cdot 2\cdot(-a^2)$$
$$= a^2 + 8a^2 = 9a^2 \longrightarrow x = \frac{a \pm 3a}{4} \begin{cases} a \\ -\frac{a}{2} \end{cases}$$

3) $f(x) = \sqrt{x} \rightarrow f'(a) = \dfrac{1}{2\sqrt{a}}, \; a > 0$

Proof

$$f'(a) = \lim_{h \to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h}$$

$$= \lim_{h \to 0} \frac{a+h-a}{(\sqrt{a+h} + \sqrt{a})h}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}}$$

$$= \frac{1}{2\sqrt{a}}, \; a > 0$$

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4. $n \in \mathbb{N}$. $S_n(x) = x^n$.

$S_1(x) = x \qquad S_1'(x) = 1$

$S_2(x) = x^2 \qquad S_2'(x) = 2x$

$S_3(x) = x^3 \qquad S_3'(x) = 3x^2$

$$S_n'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} x^{n-k} h^k - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{\binom{n}{0} x^n} + \sum_{k=1}^{n} \binom{n}{k} x^{n-k} h^k - \cancel{x^n}}{h}$$

$$= \lim_{h \to 0} \frac{\binom{n}{1} x^{n-1} h + \sum_{k=2}^{n} \binom{n}{k} x^{n-k} h^k}{h}$$

$$= n x^{n-1}$$

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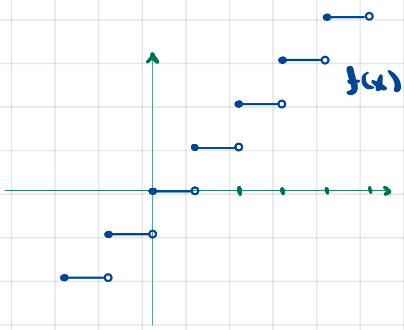
5. recall $[x]$ is the largest integer m s.t. $m \le x$.



$f(x) = [x]$

From visual inspection f is not diff. at infinitely many points.

$f(x) = m \qquad m \in \mathbb{Z}, \; m \le x < m+1$

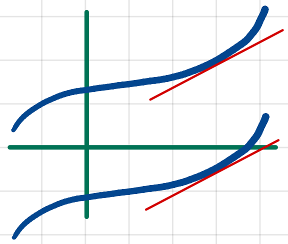
In an interval $(m, m+1)$, $m \in \mathbb{Z}$, we have

$$f'(x) = \lim_{h \to 0} \frac{m - m}{h} = 0.$$

At $x = m$, $m \in \mathbb{Z}$, f is not continuous, hence not diff.

6.

a) $g(x) = f(x) + c \rightarrow g'(x) = f'(x)$



$$g'(x) = \lim_{h \to 0} \frac{f(x+h) + c - f(x) - c}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= f'(x)$$

b) $g(x) = cf(x) \rightarrow g'(x) = cf'(x)$

$$g'(x) = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h}$$

$$= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= cf'(x)$$

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7. $f(x) = x^3$

a) $f'(x) = 3x^2$

$f'(9) = 3 \cdot 9^2 = 81$

$f'(25) = 3 \cdot 25^2 = 1725$

$f'(36) = 3 \cdot 36^2 = 3888$

b) $f'(3^2) = 3(3^2)^2 = 3 \cdot 9^2 = 81$

$f'(5^2) = f'(25)$

$f'(6^2) = f'(36)$

c) $f'(a^2) = 3a^4$

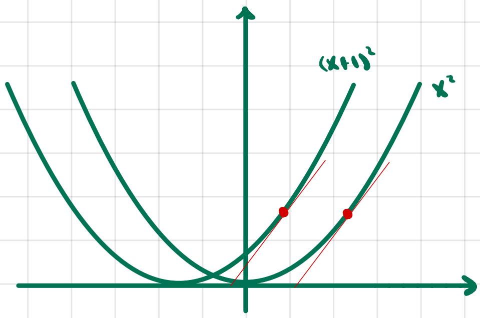
$f'(x^2) = 3x^4$

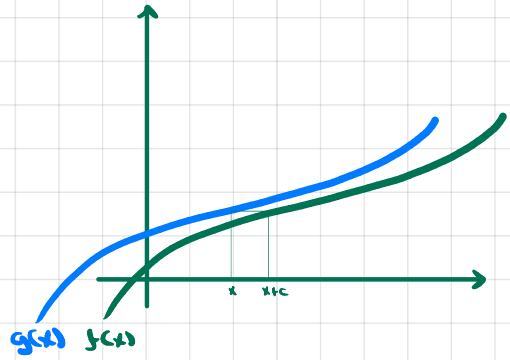
d) $g(x) = f(x^2) = x^6$

$f'(x^2) = 3x^4 \neq 6x^5 = g'(x) = \dfrac{d}{dx} f(x^2)$

8.

a) $g(x) = f(x+c) \rightarrow g'(x) = f'(x+c)$





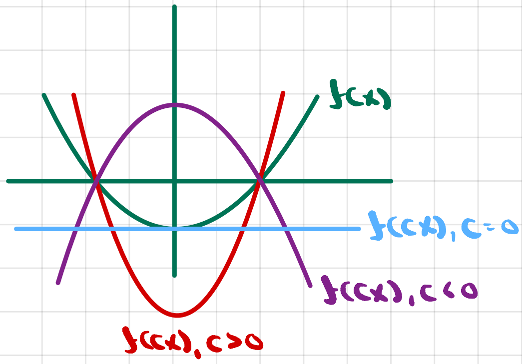
Proof

$$g'(x) = \lim_{h \to 0} \frac{f(x+c+h) - f(x+c)}{h}$$

$$= \frac{d}{dx} f(x) \Big|_{x = x+c}$$

$$= f'(x+c)$$

b) $g(x) = f(cx) \rightarrow g'(x) = cf'(cx)$



Proof

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

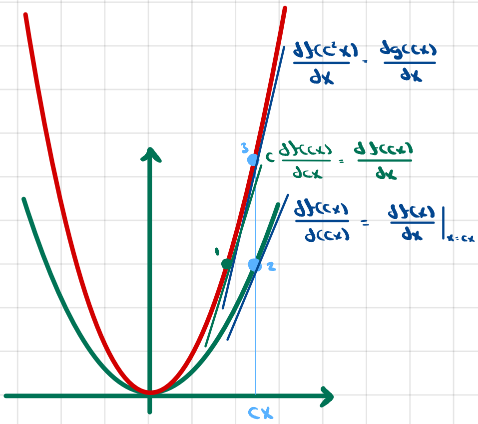
$$= \lim_{h \to 0} \frac{f(c(x+h)) - f(cx)}{h}$$

$$= \lim_{h \to 0} \frac{f(cx + ch) - f(cx)}{h}$$

$$= \lim_{h \to 0} c \cdot \frac{f(cx+ch) - f(cx)}{ch}$$

$$= c \cdot f'(cx)$$

$$= c \cdot \frac{df(cx)}{d(cx)}$$



c) f diff. and periodic, w/ period a: $f(x+a) = f(x), \forall x \rightarrow f'$ periodic

Proof

From a), $f'(x+a) = f'(x)$

9.

i) $f(x) = (x+3)^5$

$$f'(x) = \lim_{h \to 0} \frac{(x+h+3)^5 - (x+3)^5}{h}$$

$$= \lim_{h \to 0} \frac{\sum_{i=0}^{5} \binom{5}{i} (x+3)^{5-i} h^i - (x+3)^5}{h}$$

$$= \lim_{h \to 0} \frac{\sum_{i=1}^{5} \binom{5}{i} (x+3)^{5-i} h^i}{h}$$

$$= \lim_{h \to 0} \binom{5}{1}(x+3)^4 + \sum_{i=2}^{5} \binom{5}{i} (x+3)^{5-i} h^{i-1}$$

$$= 5(x+3)^4$$

$$f'(x+3) = \lim_{h \to 0} \frac{(x+3+h+3)^5 - (x+3+3)^5}{h}$$

$$= 5(x+6)^4$$

Alternatively,

Let $g(x) = x^5$.

Then $g'(x) = 5x^4$

$g(x+3) = (x+3)^5 = f(x)$

$g'(x+3) = f'(x) = 5(x+3)^4$

$h(x) = f(x+3) = g(x+6)$

$h'(x) = f'(x+3) = g'(x+6) = 5(x+6)^4$

ii) $f(x+3) = x^5$

$f(x) = f(x-3+3) = (x-3)^5$

$f'(x) = 5(x-3)^4$

$g(x) = f(x+3) = x^5$

$g'(x) = f'(x+3) = 5x^4$

Note also using the def:

$$f'(x+3) = \lim_{h \to 0} \frac{f(x+3+h) - f(x+3)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^5 - x^5}{h}$$

$$= \lim_{h \to 0} \frac{\sum_{i=1}^{5} \binom{5}{i} x^{5-i} h^i}{h}$$

$$= \lim_{h \to 0} \binom{5}{1} x^4 + \sum_{i=2}^{5} \binom{5}{i} x^{5-i} h^i$$

$$= 5x^4$$

$f'(x) = f'(x-3+3) = 5(x-3)^4$

iii) $f(x+3) = (x+5)^7$

$f'(x+3) = 7(x+5)^6$

$f'(x) = f'(x-3+3) = 7(x+2)^6$

10.

i) $f(x) = g(t+x)$

When computing $f'(x)$, t is constant.

By Ca, $f'(x) = g'(t+x)$

ii) $f(t) = g(t+x)$

Now x is constant when computing $f'(t)$

By Ca, $f'(t) = g'(t+x)$

ie $\frac{df(t)}{dt} = \frac{dg(t+x)}{dt}$

Therefore $f'(x) = g'(2x)$

11.

a)

$s'(t)$ proportional to $s(t)$

Assume $s(t) = ct^2$

Then $s'(t) = 2ct = \frac{2ct^2}{t} = \frac{2}{t} s(t)$

s' not proportional to $s(t)$.

b) $s(t) = \frac{at^2}{2}$

i) $s''(t) = a$

Proof

Using previous results that $\frac{d\, ct^2}{dt} = 2ct$ and $\frac{d\, ct}{dt} = c$
we have

$s'(t) = at$

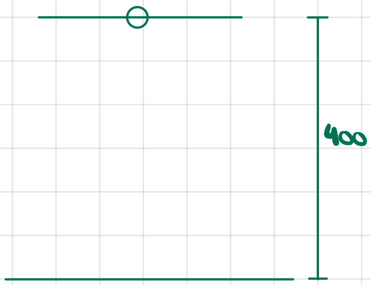
$s''(t) = a$

ii) $[s'(t)]^2 = 2a\, s(t)$

Proof

$$[s'(t)]^2 = a^2t^2 = 2a \frac{at^2}{2} = 2a\, s(t)$$

c) $a = 32$



$s(t) = \frac{32t^2}{2} = 16t^2$

$400 = 16t^2$

$t^2 = 25$

$t = 5$ sec

$s'(t) = 32t$

$s'(5) = 160$ ft/s

$s'(t) = 32t = 80$

$t = 2.5$ sec

12. $L(x)$ = speed limit x miles along road

car A: $a(t)$ position
car B: $b(t)$ "

a)

$a(t)$ is a fn of time.
$L(x)$ is function of position.

$L(a(t))$ is the speed at each time of a's trajectory.

$a'(t) = L(a(t))$

b) $b(t) = a(t-1)$

By (a), $b'(t) = a'(t-1) = L(a(t-1)) = L(b(t))$

c) $b(t) = a(t) - c$

$$b'(t) = \lim \frac{a(t+h) - c - (a(t) - c)}{h}$$

$$= \lim \frac{a(t+h) - a(t)}{h}$$

$$= a'(t) = L(a(t))$$

Ie A and B travel at same speed at all times.

$b'(t) = L(b(t)) = L(a(t) - c) = L(a(t))$

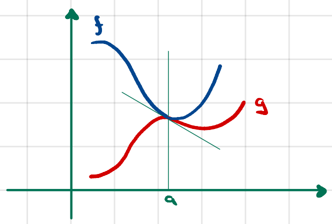
L is periodic, period c.

13. $f(a) = g(a)$

$$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0^+} \frac{g(a+h) - g(a)}{h} \quad \rightarrow h \text{ diff at } a$$

$$h(x) = \begin{cases} f(x), & x \leq a \\ g(x), & x \geq a \end{cases}$$

Proof



h diff at a $\leftrightarrow$ $\exists \lim_{m \to 0} \frac{h(a+m) - h(a)}{m}$

Problem 5-29 showed that

$\lim_{m \to 0^-} h$ exists
$\lim_{m \to 0^+} h$ exists $\rightarrow \lim_{m \to 0} h$ exists

since

$$\lim_{m \to 0^-} \frac{h(a+m) - h(a)}{m} = \lim_{m \to 0^+} \frac{f(a+m) - f(a)}{m}$$

$$\lim_{m \to 0^+} \frac{h(a+m) - h(a)}{m} = \lim_{m \to 0^+} \frac{g(a+m) - g(a)}{m}$$

Then

$$\lim_{m \to 0} \frac{h(a+m) - h(a)}{m} \text{ exists.}$$

So h is diff at a.

14.

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

$$g(h) = \frac{f(0+h) - f(0)}{h} = \begin{cases} h & h \in \mathbb{Q} \\ 0 & h \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Let $\epsilon > 0$ and $\delta = \epsilon$. Then

$\forall h, |h| < \delta = \epsilon \rightarrow (h \in \mathbb{Q}) \rightarrow (|g(h)| = |h| < \epsilon)$
$\wedge (h \in \mathbb{R} - \mathbb{Q}) \rightarrow (|g(h)| = 0 < \epsilon)$

$$\forall h, |h| < \delta = \epsilon \rightarrow \left| \frac{f(0+h) - f(0)}{h} \right| < \epsilon$$

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = 0$$

15.

a) $|f(x)| \leq x^2, \forall x \quad \rightarrow f$ diff. at 0

Let $g(x) = x^2$.
$h(x) = 0$.
$F(x) = |f(x)|$.

Then, $\forall x$, $h(x) \leq F(x) \leq g(x)$.
Therefore, $F(0) = 0$.

Also $\lim_{x \to 0} h(x) = \lim_{x \to 0} F(x) = \lim_{x \to 0} g(x)$ (Problem 5-13)

$$F'(0) = \lim_{m \to 0} \frac{|f(m)| - |f(0)|}{m}$$

$$= \lim_{m \to 0} \frac{|f(m)|}{m}$$

But

$$0 \leq |f(m)| \leq g(m) = |g(m)| = m^2$$

Case 1: $m > 0$

$$0 \leq \frac{|f(m)|}{m} \leq \frac{m^2}{m}$$

$$0 \leq \frac{|f(m)|}{m} \leq m, \quad \lim_{m \to 0} 0 = \lim_{m \to 0} m = 0$$

By 5-13, $\lim_{m \to 0} \frac{|f(m)|}{m}$ exists and equals 0.

$\rightarrow F'(0) = 0$

Case 2: $m < 0$

$$m = \frac{m^2}{m} \leq \frac{|f(m)|}{m} \leq 0$$

$$\lim_{m \to 0} m = \lim_{m \to 0} 0$$

By 5-13, $\lim_{m \to 0} \frac{|f(m)|}{m} = 0$.

$\rightarrow F'(0) = 0$

In both possible cases, $F'(0)$ exists and equals 0.

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b) $|f(x)| \leq |g(x)|, \forall x$

$\lim_{x \to 0} \frac{g(x)}{x} = 0 \quad \rightarrow f$ diff at 0.

Proof

$\forall x$, $h(x) = 0 \leq |f(x)| \leq |g(x)|$

Then $|f(0)| = 0$ and by 5-13, $\lim_{x \to 0} |f(x)| = 0$

$$f'(0) = \lim_{m \to 0} \frac{|f(m)| - f(0)}{m}$$

$$= \lim_{m \to 0} \frac{|f(m)|}{m}$$

But $0 \leq |f(m)| \leq g(m) = |g(m)|$

As in a) consider two cases.

Case 1: $m > 0$

$$0 \leq \frac{|f(m)|}{m} \leq \frac{|g(m)|}{m}$$

$$\lim_{m \to 0} 0 = \lim_{m \to 0} \frac{|g(m)|}{m} = 0$$

By 5-13, $\lim_{m \to 0} \frac{|f(m)|}{m}$ exists and equals 0.

$\rightarrow F'(0) = 0$

Case 2: $m < 0$

$$\frac{|g(m)|}{m} \leq \frac{|f(m)|}{m} \leq 0$$

$$\lim_{m \to 0} 0 = \lim_{m \to 0} \frac{|g(m)|}{m} = 0$$

By 5-13, $\lim_{m \to 0} \frac{|f(m)|}{m}$ exists and equals 0.

$\rightarrow F'(0) = 0$

$\lim_{x\to0} \frac{g(x)}{x} = 0 \to \lim_{x\to0} g(x) = 0$

Proof

$\forall \epsilon>0\ \exists \delta>0\ \forall x\ |x|<\delta \to \left|\frac{g(x)}{x}\right| < \epsilon$

$\to |g(x)| < |x|\epsilon < \delta\epsilon$

Therefore if $\delta<1$ then $|g(x)|<\epsilon$.

Ie,

$\forall \epsilon>0\ \exists\delta>0\ \forall x\ |x|<\delta \to |g(x)|<\epsilon$

$\lim_{x\to0} g(x) = 0$.

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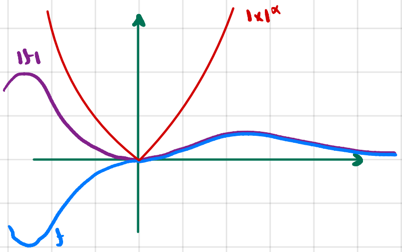
16. $\alpha>1$

$|f(x)| \le |x|^\alpha$ $\to$ f diff at 0

Proof



$0 \le |f(x)| \le |x|^\alpha \to |f(0)| = f(0) = 0$

$\to \lim_{x\to0} |f(x)| = 0$

$\lim_{x\to0} \frac{f(x)-f(0)}{x} = \lim_{x\to0} \frac{f(x)}{x} = f'(0)$

$-|x|^\alpha \le f(x) \le |x|^\alpha$

Case 1: $x>0$

$-x^{\alpha-1} \le \frac{f(x)}{x} \le x^{\alpha-1}$

$\lim_{x\to0}(-x^{\alpha-1}) = \lim x^{\alpha-1} = 0$

$\lim_{x\to0} \frac{f(x)}{x} = 0$ (5-13)

Case 2: $x<0$

$-(-x)^{\alpha-1} \le \frac{f(x)}{x} \le -x^{\alpha-1}$

$-\lim_{x\to0}(-x)^{\alpha-1} = -\lim x^{\alpha-1} = 0$

$\to \lim_{x\to0} \frac{f(x)}{x} = 0$

In both possible cases, $f'(0)=0$.

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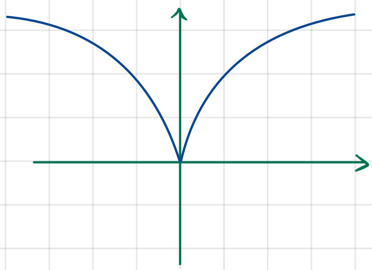
17. $0<\beta<1$

$|f(x)| \ge |x|^\beta$ $\to$ f not diff at 0

$f(0)=0$



Proof

$f(x) = |x|^\beta$

$0 < |x|^\beta < \epsilon \leftrightarrow |x| < \epsilon^{1/\beta}$

$\forall\epsilon>0\ \exists\delta>0,\ \forall x\ |x|<\epsilon^{1/\beta} = \delta \to \left||x|^\beta - 0\right| < \epsilon$

$\to \lim_{x\to0} f(x) = 0$

$\lim_{x\to0} \frac{f(x)-f(0)}{x} = \lim_{x\to0} \frac{|x|^\beta}{x}$

$\lim_{x\to0^+} \frac{|x|^\beta}{x} = \lim_{x\to0^+} x^{\beta-1}$

$0<\beta<1 \to \lim_{x\to0^+} x^{\beta-1} = \lim_{x\to0^+} \frac{1}{x^{|\beta-1|}} = \infty$

Therefore $\nexists \lim_{x\to0^+} f(x)$, so f not diff. at 0.

Note that the limit from below also doesn't exist

$\lim_{x\to0^-} \frac{|x|^\beta}{x} = \lim_{x\to0^-} \frac{(-x)^\beta}{x}$

$0<\beta<1 \to \lim_{x\to0^-} \frac{|x|^\beta}{x} = -\lim_{x\to0^-} x^{\beta-1} = -\lim_{x\to0^-} \frac{1}{x^{|\beta-1|}} = \infty$

18.

$$f(x) = \begin{cases} 0 & x \in \mathbb{R}-\mathbb{Q} \\ 1/q & x = p/q \text{ in lowest terms} \end{cases} \quad \rightarrow \quad f \text{ not diff at } a, \forall a$$

Proving the statement above involves proving that $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ does not exist for any a.

If we prove this for all $a \in \mathbb{R}-\mathbb{Q}$, then we will have proven it for all a, because $\mathbb{R}-\mathbb{Q}$ and $\mathbb{Q}$ are both dense. Any interval around a rational a contains irrational numbers. Any interval around one of the latter contains x s.t. $|f(x)-f(a)| > \epsilon$.

Proof

Let $a = m.a_1a_2a_3\ldots$ be decimal expansion of $a \in \mathbb{R}-\mathbb{Q}$

Consider $\frac{f(a+h)-f(a)}{h} = \frac{f(a+h)}{h}$

Let $h = -0.000\ldots00a_{n+1}a_{n+2} \in \mathbb{R}-\mathbb{Q}$

Then $a+h = a.a_1a_2a_3\ldots a_n000\ldots = a + \frac{a_1a_2\ldots a_n}{10^n} = \frac{a10^n + a_1a_2\ldots a_n}{10^n} \in \mathbb{Q}$

$f(a+h) = \frac{1}{10^n}$

$\frac{f(a+h)}{h} = \frac{1}{10^n h} \rightarrow \infty$

For any $\delta > 0$, there is some h of the form above such that $|h| < \delta$ and $\frac{f(a+h)}{h} > \epsilon$.

Thus, $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $|h| < \delta \wedge \frac{f(a+h)-f(a)}{h} > \epsilon$

Thus, for $a \in \mathbb{R}-\mathbb{Q}$, $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ does not exist.

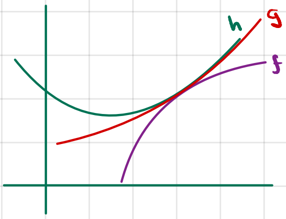
For $a \in \mathbb{Q}$, $\forall \delta > 0$, $\forall \epsilon > 0$

$\exists h \in \mathbb{R}-\mathbb{Q}$ s.t. $|h| < \delta$
h has the form shown above
$a+h \in \mathbb{Q}$
$f(a+h) > \epsilon$

19.

a) $f(a) = g(a) = h(a)$
$f(x) \le g(x) \le h(x), \forall x$ $\rightarrow$ g diff at a
$f'(a) = h'(a)$ $\quad$ $f'(a) = g'(a) = h'(a)$

Proof



$$f(a+m) \le g(a+m) \le h(a+m)$$

$$f(a+m) - f(a) \le g(a+m) - g(a) \le h(a+m) - h(a)$$

case 1: $m > 0$

$$\frac{f(a+m)-f(a)}{m} \le \frac{g(a+m)-g(a)}{m} \le \frac{h(a+m)-h(a)}{m}$$

$$\lim_{m\to 0^+} \frac{f(a+m)-f(a)}{m} = \lim_{m\to 0^+} \frac{h(a+m)-h(a)}{m} = l$$

$$\rightarrow \lim_{m\to 0^+} \frac{g(a+m)-g(a)}{m} = l$$

case 2: $m < 0$

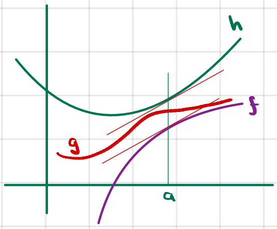
$$\frac{h(a+m)-h(a)}{m} \le \frac{g(a+m)-g(a)}{m} \le \frac{f(a+m)-f(a)}{m}$$

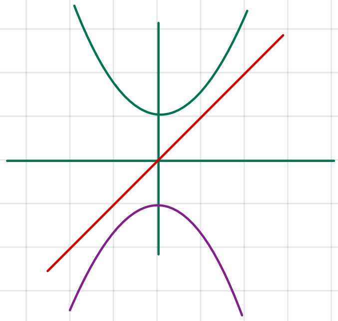
$$\lim_{m\to 0^-} \frac{f(a+m)-f(a)}{m} = \lim_{m\to 0^-} \frac{h(a+m)-h(a)}{m} = l$$

$$\rightarrow \lim_{m\to 0^-} \frac{g(a+m)-g(a)}{m} = l$$

Therefore $\lim_{m\to 0} \frac{g(a+m)-g(a)}{m} = l$, so $g'(a) = f'(a) = h'(a)$.

b) $f(x) \le g(x) \le h(x), \forall x$
$f'(a) = h'(a)$





$f(x) = -1 - x^2$
$h(x) = 1 + x^2$
$g(x) = x$

$g(x) \ge f(x) \rightarrow x \ge -1 - x^2$
$x^2 + x + 1 \ge 0$
$\Delta = 1 - 4 = -3$
$\rightarrow g(x) \ge f(x)\ \forall x$

$g(x) \le h(x) \rightarrow x \le 1 + x^2$
$x^2 - x + 1 \ge 0$
$\Delta = 1 - 4 = -3$
$\rightarrow g(x) \le h(x)\ \forall x$

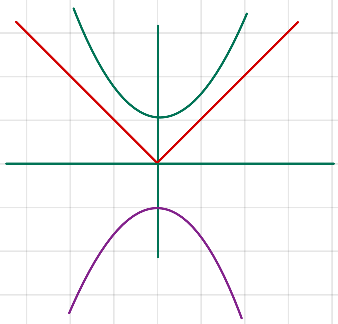
T.F. $f(x) \le g(x) \le h(x)\ \forall x$

Consider $x = 0$.

$f'(x) = -2x$
$h'(x) = 2x$
$g'(x) = 1$
$f'(0) = h'(0) = 0 \ne g'(0)$.

Thus we have a counterexample.

Here is another:



In this case g is not diff at 0 as well.

20. f polynomial fn

a) $f(x)=x^4$

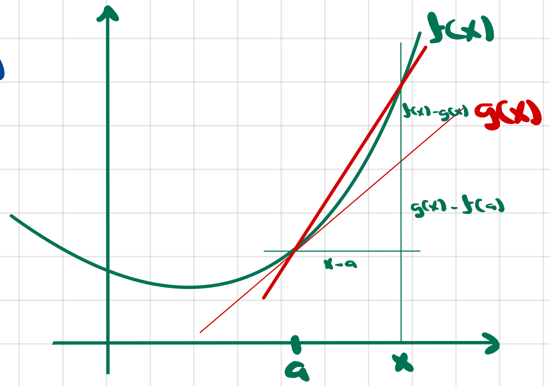
$g(x)=f(a)+(x-a)f'(a)$
$\quad = a^4+(x-a)\cdot 4a^3$

$d(x)=f(x)-g(x)$
$\quad = x^4-a^4-(x-a)4a^3$
$\quad = (x-a)(x^3+ax^2+a^2x+a^3)-(x-a)4a^3$
$\quad = (x-a)(x^3+ax^2+a^2x-3a^3)$
$\quad = (x-a)^2(x^2+2ax+3a^2)$

$$\begin{array}{r|l} & x^3+ax^2+a^2x+a^3 \\ \hline x-a & x^4 \qquad\qquad -a^4 \\ & x^4-ax^3-a^4 \\ & ax^3 \qquad -a^4 \\ & ax^3-a^2x^2 \\ & a^2x^2-a^4 \\ & a^2x^2-a^3x \\ & a^3x-a^4 \\ & a^3x-a^4 \\ & 0 \end{array}$$

$$\begin{array}{r|l} & x^2+2ax+3a^2 \\ \hline x-a & x^3+ax^2+a^2x-3a^3 \\ & x^3-ax^2 \\ & 2ax^2+a^2x-3a^3 \\ & 2ax^2-2a^2x \\ & 3a^2x-3a^3 \\ & 3a^2x-3a^3 \\ & 0 \end{array}$$

b)



$d(x)=f(x)-g(x)=f(x)-f(a)-f'(a)(x-a)$

$\dfrac{d(x)}{x-a}=\dfrac{f(x)-f(a)}{x-a}-f'(a)$

If we define $m(x)=f(x)-f(a)$, then since $m(a)=0$, we can write $m(x)=f(x)-f(a)=(x-a)n(x)$, for some polyn. $n(x)$. Thus $n(x)=\dfrac{f(x)-f(a)}{x-a}$, ie $f(x)-f(a)$ is divisible by $x-a$

we defined $d(x)=f(x)-g(x)=f(x)-f(a)-f'(a)(x-a)$

$d(a)=0$. Hence, by 3-7b, $d(x)=(x-a)h(x)$, for some polyn. $h(x)$.

Thus $h(x)=\dfrac{d(x)}{x-a}$, $x\neq a$.

$h(x)=\dfrac{f(x)-f(a)}{x-a}-f'(a)$, the difference between the slope of the secant through $(a,f(a))$ and $(x,f(x))$

and the slope (derivative) of f at a.

$\lim\limits_{x\to a} h(x)=\lim\limits_{x\to a}\left[\dfrac{f(x)-f(a)}{x-a}-f'(a)\right]=f'(a)-f'(a)=0$

As x approaches a, the secant slope approaches the tangent slope.

At a, the slopes coincide, h(x) is zero, and so it can be written $h(x)=(x-a)p(x)$.

Thus $d(x)=(x-a)^2p(x)=f(x)-f(a)-f'(a)(x-a)$

clearly then, $f(x)-f(a)$ is div. by $(x-a)$.

21.

a) $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

Proof

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

means

$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \; |x-a| < \delta \rightarrow |f(x) - f'(a)| < \varepsilon$

Let $h = x - a$

$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \; |h| < \delta \rightarrow |f(a+h) - f'(a)| < \varepsilon$

$\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

b) Derivatives are local property

$\forall x, \, x \in (m_1, m_2), \, a \in (m_1, m_2) \rightarrow f(x) = g(x) \quad \rightarrow f'(a) = g'(a)$

Proof

Assume $g'(a)$ exists.
Then

$\forall \varepsilon > 0 \, \exists \delta_1 > 0 \, \forall x \; |x-a| < \delta_1 \rightarrow \left| \dfrac{g(x) - g(a)}{x - a} - g'(a) \right| < \varepsilon$

Since $f(x) = g(x)$ in an open interval around a

Let $\delta = \min(\delta_1, |m_2 - a|, |a - m_1|)$.

Then,

$\forall \varepsilon > 0 \; \forall x \; |x-a| < \delta \rightarrow \left| \dfrac{f(x) - f(a)}{x - a} - g'(a) \right| < \varepsilon$

Thus $f'(a) = g'(a)$.

22.

a) f diff. at x $\to$ $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}$

Proof

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x-h)+f(x-h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{h} + \lim_{h\to 0} \frac{f(x-h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{h} - \lim_{h\to 0} \frac{f(x-h)-f(x)}{-h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{h} - f'(x)$$

Thus

$$2f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}$$

Note

$\lim_{h\to 0} f(x+h) = \ell$ means

$|h|<\delta \to |f(x+h)-\ell|<\epsilon$

in particular this is true for $-h$

$|-h|<\delta \to |f(x-h)-\ell|<\epsilon$

which means

$$\lim_{h\to 0} f(x-h) = \lim_{h\to 0} f(x+h) = \ell$$

The important detail is that

$\forall h,\ |-h| = |h|$

Thus

$|-h|<\delta \leftrightarrow |h|<\delta$

Thus

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\lim_{-h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\lim_{h\to 0} \frac{f(x-h)-f(x)}{h}$$

$$\lim_{-h\to 0} \frac{f(x-h)-f(x)}{h}$$

are all equal to $f'(x)$

b) $f'(x) = \lim\limits_{h,k \to 0^+} \dfrac{f(x+h) - f(x-k)}{h+k}$

Proof

$$\frac{f(x+h) - f(x-k)}{h+k} = \frac{h}{h+k} \cdot \frac{f(x+h) - f(x)}{h} + \frac{k}{h+k} \frac{f(x+k) - f(x)}{k}$$

Let $\varepsilon > 0$. There is $\delta > 0$ s.t. for $0 < h < \delta$ and $0 < k < \delta$

$$-\varepsilon < \frac{f(x+h) - f(x)}{h} - f'(x) < \varepsilon \qquad (1)$$

$$-\varepsilon < \frac{f(x) - f(x-k)}{k} - f'(x) < \varepsilon \qquad (2)$$

The latter follows because

$$|h| < \delta \to \left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| < \varepsilon$$

$$|-k| = |k| < \delta \to \left| \frac{f(x-k) - f(x)}{-k} - f'(x) \right| < \varepsilon$$

$$\to \left| \frac{f(x) - f(x-k)}{-k} - f'(x) \right| < \varepsilon$$

Back to (1) and (2).

$$-\varepsilon \frac{h}{h+k} < \frac{h}{h+k}\left(\frac{f(x+h) - f(x)}{h} - f'(x) \right) < \frac{h}{h+k} \varepsilon$$

$$-\varepsilon \frac{k}{h+k} < \frac{k}{h+k}\left(\frac{f(x) - f(x-k)}{k} - f'(x) \right) < \frac{k}{h+k} \varepsilon$$

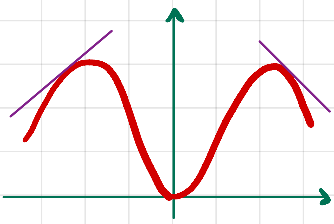
$$\to -\varepsilon < \frac{f(x+h) - f(x-k)}{h+k} - f'(x) < \varepsilon$$

$$\to \lim_{h,k \to 0^+} \frac{f(x+h) - f(x-k)}{h+k} = f'(x)$$

23.

f even $\rightarrow$ $f'(x) = -f'(-x)$
ie the derivative is odd

Proof



Let $g(x) = f(-x)$

$g'(x) = -f'(-x)$, by problem 8b).

Since f is even,

$f(x) = f(-x) = g(x)$

Therefore,

$f'(x) = -f'(-x) = g'(x)$

24.

f odd $\rightarrow$ $f'(x) = f'(-x)$
ie the derivative is even

Proof

Since f odd, then $f(x) = -f(-x)$

$f'(x) = -1 \cdot (-1) \cdot f'(-x)$
$= f'(-x)$

(Problems 8b) and 6b))

25.

If f even then $f(x) = f(-x)$

$f'(x) = -f'(-x)$, odd.

Therefore by 24), f'' is even, f''' is odd, ...

Therefore,

if f even then

n odd $\rightarrow$ $f^{(n)}$ odd
n even $\rightarrow$ $f^{(n)}$ even

if f odd then

n odd $\rightarrow$ $f^{(n)}$ even
n even $\rightarrow$ $f^{(n)}$ odd

26.

i) $f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{\sum_{i=1}^{3} \binom{3}{i} x^{3-i} h^i}{h} = 3x^2$$

$$f''(x) = \lim_{h\to 0} \frac{3(x+h)^2 - 3x^2}{h} = \lim_{h\to 0} \frac{6xh + 3h^2}{h} = 6x$$

ii) $f(x) = x^5$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^5 - x^5}{h} = \lim_{h\to 0} \frac{\sum_{i=1}^{5} \binom{5}{i} x^{5-i} h^i}{h} = 5x^4$$

$$f''(x) = \lim_{h\to 0} \frac{5(x+h)^4 - 5x^4}{h} = \lim_{h\to 0} \frac{5\sum_{i=1}^{4} \binom{4}{i} x^{4-i} h^i}{h} = 20x^3$$

iii) $f'(x) = x^4$

$$f''(x) = \lim \frac{(x+h)^4 - x^4}{h} = 4x^3$$

iv) $f(x+3) = x^5$

$f'(x) = f'(x+3)$

Hence, $f'(x+3) = 5x^4$

$f''(x+3) = f''(x+3) = 20x^3$

22.

$S_n(x) = x^n \quad 0 \leq k \leq n \quad \rightarrow \quad S_n^{(k)}(x) = \frac{n!}{(n-k)!} x^{n-k}$

$= k! \binom{n}{k} x^{n-k}$

Proof

Let's use induction.

Let $A = \{ k : S_n^{(k)} = \frac{n!}{(n-k)!} x^{n-k} \}$

$$S_n'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\sum_{i=1}^{n} \binom{n}{i} x^{n-i} h^i}{h}$$

$$= \lim_{h \to 0} \left(\binom{n}{1} x^{n-1} + \sum_{i=2}^{n} \binom{n}{i} x^{n-i} h^{i-1} \right)$$

$$= \binom{n}{1} x^{n-1}$$

$\rightarrow 1 \in A$

Assume $S_n^{(k)}(x) = \frac{n!}{(n-k)!} x^{n-k}$

$$S_n^{(k+1)}(x) = \lim_{h \to 0} \frac{\frac{n!}{(n-k)!} (x+h)^{n-k} - \frac{n!}{(n-k)!} x^{n-k}}{h}$$

$$= \frac{n!}{(n-k)!} \lim_{h \to 0} \frac{(x+h)^{n-k} - x^{n-k}}{h}$$

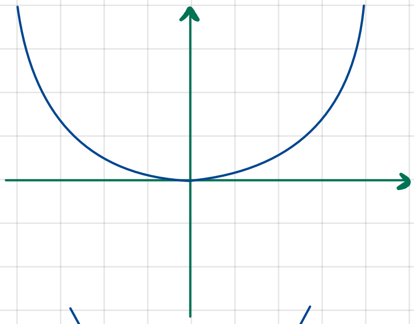
$$= \frac{n!}{(n-k)!} (n-k) x^{n-k-1}$$

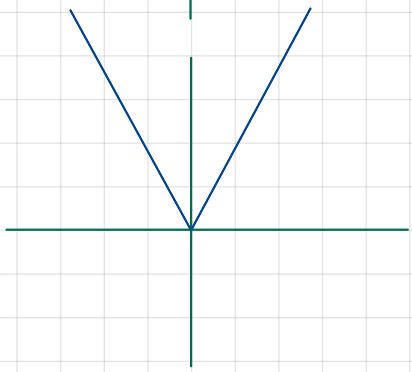
$$= \frac{n!}{(n-k-1)!} x^{n-k-1}$$

$\rightarrow A = \mathbb{N}$, ie $\forall k \in \mathbb{N}$, $S_n^{(k)}(x) = \frac{n!}{(n-k)!} x^{n-k}$

28.

a) $f(x) = |x|^3$





$$f(x) = \begin{cases} x^3 & x \geq 0 \\ (-x)^3 & x < 0 \end{cases}$$

$x > 0 \rightarrow f'(x) = 3x^2$

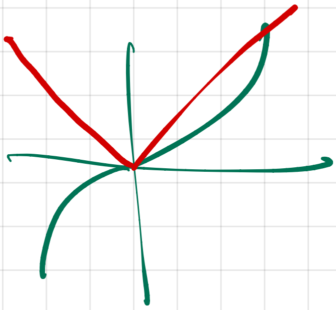
$f''(x) = 6x$

$f'''(x) = 6$

$x < 0 \rightarrow f'(x) = -3x^2$

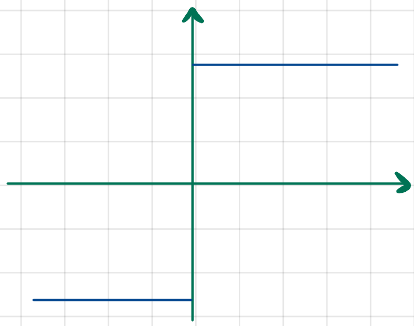
$f''(x) = -6x$

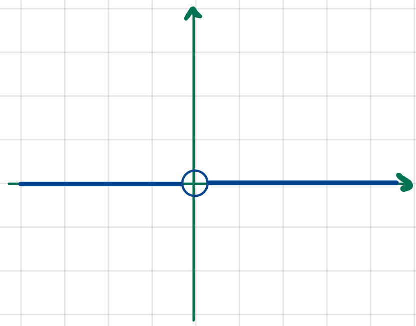
$f'''(x) = -6$



Therefore $f'''(0)$ does not exist.

b)





$$f(x) = \begin{cases} x^4 & x \geq 0 \\ -x^4 & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 4x^3 & x \geq 0 \\ -4x^3 & x < 0 \end{cases}$$

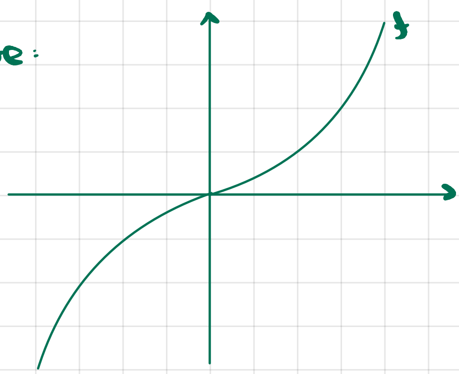
$$f''(x) = \begin{cases} 12x^2 & x \geq 0 \\ -12x^2 & x < 0 \end{cases}$$

$$f'''(x) = \begin{cases} 24x & x \geq 0 \\ -24x & x < 0 \end{cases}$$

$$f^{(4)}(x) = \begin{cases} 24 & x \geq 0 \\ -24 & x < 0 \end{cases}$$

Therefore $f^{(4)}(x)$ does not exist at $x = 0$.

Note:



f is odd : $f(x) = -f(-x)$

f' is even: $f'(x) = f'(-x)$

$4x^3 = -4(-x^3) = 4x^3$

f'' is odd : $12x^2 = -(-12(-x)^2) = 12x^2$

29. $f(x) = \begin{cases} x^n & x \geq 0 \\ 0 & x \leq 0 \end{cases} \rightarrow$ $f^{(n-1)}(x)$ exists, $f^{(n)}(x)$ does not

Proof

From problem 27,

$$f^{(n-1)}(x) = \frac{n!}{(n-k)!}x$$

$$f^{(n-1)}(0) = 0$$

Therefore $\lim_{h \to 0^+} \frac{f^{(n-1)}(h) - f^{(n-1)}(0)}{h} = \frac{n!}{(n-k)!}$

$$\lim_{h \to 0^-} \frac{f^{(n-1)}(h) - f^{(n-1)}(0)}{h} = 0$$

$f^{(n)}(0)$ does not exist.

30.

i) $\frac{dx^n}{dx} = nx^{n-1}$

$f'(x) = nx^{n-1}$ if $f(x) = x^n$

ii) $\frac{dz}{dy} = -\frac{1}{y^2}$, if $z = \frac{1}{y}$

if $f(x) = \frac{1}{x}$ then $f'(x) = \frac{1}{x^2}$

iii) $\frac{d[f(x)+c]}{dx} = \frac{df(x)}{dx}$

$g(x) = f(x) + c$
$g'(x) = f'(x)$

iv) $\frac{d(cf(x))}{dx} = c\frac{df(x)}{dx}$

$g(x) = cf(x)$
$g'(x) = cf'(x)$

v) $\frac{dz}{dx} = \frac{dy}{dx}$ if $z = y + c$

$g(y) = y$
$h(y) = g(y) + c$
$h'(y) = g'(y)$

vi) $\left.\frac{dx^3}{dx}\right|_{x=a^2} = 3a^4$

$f(x) = x^3$
$f'(x) = 3x^2$
$f'(a^2) = 3a^4$

vii) $\left.\frac{df(x+a)}{dx}\right|_{x=b} = \left.\frac{df(x)}{dx}\right|_{x=b+a}$

$g(x) = f(x+a)$
$g'(x) = f'(x+a)$
$g'(b) = f'(b+a)$

viii) $\left.\frac{df(cx)}{dx}\right|_{x=b} = c\left.\frac{df(x)}{dx}\right|_{x=cb}$

$g(x) = f(cx)$
$g'(x) = cf'(cx)$
$g'(b) = cf'(cb)$

ix) $\frac{df(cx)}{dx} = c\left.\frac{df(y)}{dy}\right|_{y=cx}$

$g(x) = f(cx)$
$g'(x) = cf'(cx) = c\frac{df(cx)}{dcx} = c\left.\frac{df(y)}{dy}\right|_{y=cx}$

x) $\frac{d^k x^n}{dx^k} = k!\binom{n}{k}x^{n-k}$

$f(x) = x^n$

$f^{(k)}(x) = k!\binom{n}{k}x^{n-k}$