Ch.5 - Limits

1. i) $\lim_{x\to 1} \frac{x^2-1}{x+1}$

$\lim_{x\to 1}(x^2-1) = 1^2-1 = 0$ δ,ϵ

$\lim_{x\to 1}(x+1) = 2$ δ,ϵ

$\lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$ Th.2 (3)

$\lim_{x\to 1} (x^2-1)\cdot\frac{1}{x+1} = 0\cdot\frac{1}{2}$ Th.2 (2)

ii) $\lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} \frac{(x-2)^3+6x^2-12x}{x-2} = \lim_{x\to 2}\left[(x-2)^2 + \frac{6x(x-2)}{x-2}\right] = \lim_{x\to 2}\left[(x-2)^2+6x\right] = 12$

$\lim_{x\to 2}(x-2) = 0$

$\lim_{x\to 2}(x^3-8) = 0$

$(x-2)^3 = x^3-3x^2\cdot 2+3x\cdot 4-8$
$= x^3-8-6x^2+12x$

iii) $\lim_{x\to 3} \frac{x^3-8}{x-2} = \frac{19}{1} = 19$ Th.2 (2)

$\lim_{x\to 3}(x-2) = 1 \neq 0$

$\lim_{x\to 3}\frac{1}{x-2} = \frac{1}{1} = 1$ Th2. (3)

iv) $\lim_{x\to y}\frac{x^n-y^n}{x-y} = \lim_{x\to y}\sum_{i=0}^{n-1} x^{n-1-i}y^i = \sum_{i=0}^{n-1} y^{n-1} = ny^{n-1}$

y is a root of x^n-y^n

$\Rightarrow x^n-y^n = (x-y)\cdot\sum_{i=0}^{n-1} x^{n-1-i}y^i$

v) $\lim_{y\to x}\frac{x^n-y^n}{x-y} = \lim_{y\to x}\frac{-1(y^n-x^n)}{-1(y-x)} = \lim_{y\to x}\frac{y^n-x^n}{y-x} = nx^{n-1}$

vi) $\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h} = \lim_{h\to 0}\frac{(\sqrt{a+h}-\sqrt{a})}{h}\frac{(\sqrt{a+h}+\sqrt{a})}{\sqrt{a+h}+\sqrt{a}} = \lim_{h\to 0}\frac{\cancel{a}+h-\cancel{a}}{\cancel{h}(\sqrt{a+h}+\sqrt{a})} = \lim_{h\to 0}\frac{1}{\sqrt{a+h}+\sqrt{a}} = \frac{1}{2\sqrt{a}}$

2. i) $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1} \frac{1-\sqrt{x}}{1^2-(\sqrt{x})^2} = \lim_{x\to 1} \frac{\cancel{1-\sqrt{x}}}{(1+\sqrt{x})\cancel{(1-\sqrt{x})}} = \lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}$

ii) $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x} = \lim_{x\to 0} \frac{1-(1-x^2)}{x(1+\sqrt{1-x^2})} = \lim_{x\to 0} \frac{x^{\cancel{2}}}{\cancel{x}(1+\sqrt{1-x^2})} = \lim_{x\to 0} \frac{x}{1+\sqrt{1-x^2}} = 0$

note

$\lim_{x\to 0} \frac{x}{1+\sqrt{1-x^2}} = \lim_{x\to 0} x \cdot \frac{1}{1+\sqrt{1-x^2}}$

Th.2 (3) $\Rightarrow \lim \frac{1}{1+\sqrt{1-x^2}} = \frac{1}{2}$

Th.2 (2) $\Rightarrow \lim_{x\to 0} x \cdot \frac{1}{1+\sqrt{1-x^2}} = 0 \cdot \frac{1}{2} = 0$

iii) $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x^2} = \lim_{x\to 0} \frac{1-(1-x^2)}{x^2(1+\sqrt{1-x^2})} = \lim_{x\to 0} \frac{\cancel{x^2}}{\cancel{x^2}(1+\sqrt{1-x^2})} = \lim_{x\to 0} \frac{1}{1+\sqrt{1-x^2}} = \frac{1}{2}$

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3. i) $f(x) = x(3-\cos x^2)$, $a = 0$

$\lim_{x\to 0} x(3-\cos x^2) = 0$ Th.2 (1), (2)

which means, by definition

$\forall \epsilon > 0, \exists \delta > 0 : |x| < \delta \Rightarrow |x(3-\cos x^2)| < \epsilon$

Let's look first at the expression $|x(3-\cos x^2)| < \epsilon$. By manipulating this expr., we may find out how close to zero x must be.

$|x(3-\cos x^2)| \leq |x||3-\cos x^2| < \epsilon$

$x^2 \geq 0 \Rightarrow 0 \leq \cos x^2 \leq 1$

$|3| - |\cos x^2| \leq |3-\cos x^2| \leq |3| + |\cos x^2|$
$3-1 \leq |3-\cos x^2| \leq 3+0$
$2 \leq |3-\cos x^2| \leq 3$

$\Rightarrow |x(3-\cos x^2)| \leq |x||3-\cos x^2| \leq 3|x| < \epsilon$

$\Rightarrow |x| < \frac{\epsilon}{3}$

$\Rightarrow \forall \epsilon > 0$, if we choose $0 < \delta < \frac{\epsilon}{3}$ then $|x(3-\cos x^2)| < \epsilon$

ii) $f(x) = x^2+5x-2$, $a=2$

$\lim_{x\to 2} f(x) = 4+10-2 = 12$ Th.2 (1), (2)

$\Rightarrow \forall \epsilon>0, \exists \delta>0 : |x-2|<\delta \Rightarrow |x^2+5x-2-(a^2+5a-2)|<\epsilon$

$\to \lim_{x\to 2} x^2 = 4 \Rightarrow \forall\epsilon>0, \exists\delta>0 : |x-2|<\delta \Rightarrow |x^2-4|<\epsilon$

$|x^2-4| = |(x+2)(x-2)| \leq |x+2||x-2|$

Assume $|x-2|<1 \Rightarrow -1<x-2<1 \Rightarrow 3<x+2<5 \Rightarrow |x+2|<5$

$|x+2||x-2| \leq 5|x-2| < \epsilon \Rightarrow |x-2| < \frac{\epsilon}{5}$

$\forall\epsilon>0, \delta < \min(1, \frac{\epsilon}{5}) \Rightarrow |x^2-4|<\epsilon$

$\to \lim_{x\to 2} 5x = 10 \Rightarrow \forall\epsilon>0 \exists\delta : |x-2|<\delta \Rightarrow |5x-10|<\epsilon$

$|5x-10| = |5(x-2)| = 5|x-2| < \epsilon \Rightarrow |x-2| < \frac{\epsilon}{5}$

$\forall\epsilon>0, \delta < \frac{\epsilon}{5} \Rightarrow |5x-10|<\epsilon$

$\to \lim_{x\to 2} (-2) = -2 \Rightarrow \forall\epsilon>0, \forall\delta>0 : |x-2|<\delta \Rightarrow |-2-(-2)| = 0 < \epsilon$

$\to \forall\epsilon>0, \delta = \min(1, \frac{\epsilon}{5}) \Rightarrow |x-2|<\delta \Rightarrow$ $|x^2-4|<\epsilon$, ie x^2 approaches 4
$|5x-10|<\epsilon$, $5x$ " 10
$|-2-(-2)| = 0<\epsilon$ -2 " -2

Choose $\epsilon = \frac{\epsilon_0}{2}$, $\forall\epsilon_0>0$

Th.2(1) $\Rightarrow |x^2+5x-(4+10)<\epsilon_0$

$\Rightarrow \forall\epsilon_0>0 \Rightarrow |x-2|<\min(1, \frac{\epsilon_0}{10}) \Rightarrow |x^2+5x-(4+10)| < \epsilon_0$

$\forall\epsilon_1$, let $\epsilon_0 = \frac{\epsilon_1}{2}$

$\Rightarrow |x-2|<\min(1, \frac{\epsilon_1}{20}) \Rightarrow |x^2+5x-(4+10)| < \frac{\epsilon_1}{2}$

$|-2-(-2|< \frac{\epsilon_1}{2}$

Th.2(1) $\Rightarrow |x^2+5x-2-(a^2+5a-2)|<\epsilon_1$

$\Rightarrow \forall\epsilon_1>0, |x-a|<\min(1, \frac{\epsilon_1}{20}) \Rightarrow |x^2+5x-2-(a^2+5a-2)|<\epsilon_1$

$\Rightarrow \lim_{x\to a}(x^2+5x-2) = a^2+5a-2$

* To determine $\lim_{x\to a}(x^2+5x-2)$ we used δ/ϵ analysis to show the limits of x^2, $5x$, and -2.

Then we used the lemmas to find the limits x^2+5x, and then x^2+5x-2.

iii) $f(x) = \frac{100}{x}$, $a = 1$

$\lim_{x \to a} 100 \cdot \frac{1}{x}$

$\forall \varepsilon > 0, \forall \delta > 0, \; |x-a| < \delta \Rightarrow |100 - 100| < \varepsilon$

$\Rightarrow \lim_{x \to a} 100 = 100$

$\forall \varepsilon > 0, \forall \delta < \varepsilon, \; |x-a| < \delta \Rightarrow |x-a| < \varepsilon$

$\Rightarrow \lim_{x \to a} x = a$

If $a \neq 0$, $\forall \varepsilon > 0$ let $\varepsilon_0 = \min\left(\frac{|a|}{2}, \frac{\varepsilon |a|^2}{2}\right)$

$\Rightarrow \forall \delta < \varepsilon_0, \; |x-a| < \delta \Rightarrow |x-a| < \varepsilon_0$

$\Rightarrow x \neq 0, \; \left|\frac{1}{x} - \frac{1}{a}\right| < \varepsilon$

$\Rightarrow \lim_{x \to a} \frac{1}{x} = \frac{1}{a}$

At this point we have

$\forall \varepsilon > 0, \forall \delta : 0 < \delta < \varepsilon, \; a \neq 0 \Rightarrow \quad |x-a| < \delta \Rightarrow |100 - 100| < \min\left(1, \frac{\varepsilon}{2(|a|+1)}\right)$

$$\left|\frac{1}{x} - \frac{1}{a}\right| < \frac{\varepsilon}{2(100+1)}$$

Lemma (2) $\Rightarrow \left|100 \cdot \frac{1}{x} - \frac{100}{a}\right| < \varepsilon$

$\Rightarrow \lim_{x \to a} \frac{100}{x} = \frac{100}{a}$, $a \neq 0$

iv) $f(x) = x^4$

$|x^2 - a^2| = |(x+a)(x-a)| \le |x+a||x-a|$

Assume $-1 < x-a < 1 \Rightarrow 2a-1 < x+a < 2a+1$

$\Rightarrow |x+a| < |2a+1| \le |2a| + |1| \le 2|a|+1$

$|x^2-a^2| < |x-a| < 2|a|+1$

$\Rightarrow |x-a| < \dfrac{\epsilon}{2|a|+1}$

$\forall \epsilon > 0,\ |x-a| < \min(1, \dfrac{\epsilon}{2|a|+1}) \Rightarrow |x^2-a^2| < \epsilon$

$|x^4 - a^4| = |(x^2)^2 - (a^2)^2| = |x^2+a^2||x^2-a^2|$

Assume $|x^2-a^2| < 1 \Rightarrow -1 < x^2-a^2 < 1 \Rightarrow 2a^2-1 < x^2+a^2 < 2a^2+1$

$\Rightarrow |x^2+a^2| < 2a^2+1$

$|x^4-a^4| < |x^2-a^2|(2a^2+1) < \epsilon$

$\forall \epsilon > 0,\ |x^2-a^2| < \min(1, \dfrac{\epsilon}{2a^2+1}) \Rightarrow |x^4-a^4| < \epsilon$

$|x-a| < \min(1, \dfrac{\min(1, \frac{\epsilon}{2a^2+1})}{2|a|+1}) \Rightarrow |x^2-a^2| < \dfrac{\epsilon}{2a^2+1}$

$\Rightarrow |x^4-a^4| < \epsilon \qquad \Rightarrow \lim\limits_{x\to a} x^4 = a^4$

$\lim\limits_{x\to a} x^2 = \lim\limits_{x\to a} x\cdot x = \lim\limits_{x\to a} x \cdot \lim\limits_{x\to a} x = a\cdot a = a^2$

Assume $\lim\limits_{x\to a} x^n = a^n$

$\lim\limits_{x\to a} x^{n+1} = \lim\limits_{x\to a} x\cdot x^n = \lim\limits_{x\to a} x \cdot \lim\limits_{x\to a} x^n = a\cdot a^n = a^{n+1}$

By induction, $\lim\limits_{x\to a} x^n = a^n$

Note

$|x-a| < \dfrac{1}{2}\dfrac{\epsilon}{(2|a|+1)(2a^2+1)} \Rightarrow |x^2-a^2| < \dfrac{1}{2}\dfrac{\epsilon}{2a^2+1}$

$\Rightarrow |x^4-a^4| < \dfrac{\epsilon}{2}$

v) $f(x) = x^4 + \dfrac{1}{x}$

$\forall \epsilon > 0,\ |x-a| < \min(1, \dfrac{\min(1, \frac{\epsilon}{2a^2+1})}{2|a|+1}) \Rightarrow |x^2-a^2| < \dfrac{\epsilon}{2a^2+1}$

$\Rightarrow |x^4-a^4| < \epsilon$

$a \neq 0, \forall \epsilon > 0$ let $\epsilon_0 = \min(\dfrac{|a|}{2}, \dfrac{\epsilon|a|^2}{2})$

$\Rightarrow \forall \delta < \epsilon_0,\ |x-a| < \delta \Rightarrow |x-a| < \epsilon_0$

$\Rightarrow x \neq 0,\ \left|\dfrac{1}{x} - \dfrac{1}{a}\right| < \epsilon$

Recall

1. $|x-x_0| < \dfrac{\epsilon}{2},\ |y-y_0| < \dfrac{\epsilon}{2} \Rightarrow |(x+y)-(x_0+y_0)| < \epsilon$

$\forall \epsilon > 0$, let $\epsilon_0 = \min\left(\min(1, \dfrac{\min(1, \frac{\epsilon}{2a^2+1})}{2|a|+1}), \min(\dfrac{|a|}{2}, \dfrac{\epsilon|a|^2}{2})\right)$

$|x-a| < \dfrac{1}{2}\epsilon_0 \Rightarrow |x^4-a^4| < \dfrac{\epsilon}{2}$

$\left|\dfrac{1}{x} - \dfrac{1}{a}\right| < \dfrac{\epsilon}{2}$

$\Rightarrow \left|(x^4+\dfrac{1}{x}) - (a^4+\dfrac{1}{a})\right| < \epsilon$

vi) $f(x) = \dfrac{x}{2-\sin^2 x}$, $a=0$

$-1 \le \sin x \le 1$

$0 \le \sin^2 x \le 1$

$-1 \le -\sin^2 x \le 0$

$1 \le 2-\sin^2 x \le 2$

$\Rightarrow \dfrac{x}{2-\sin^2 x} \le x$, $\forall x$

$\left| \dfrac{x}{2-\sin^2 x} \right| \le |x|$

$\Rightarrow \forall \varepsilon > 0,\ 0 < \delta < \varepsilon,\ |x-0| < \delta \Rightarrow \left| \dfrac{x}{2-\sin^2 x} - 0 \right| < \varepsilon$

$\Rightarrow \lim\limits_{x \to 0} \dfrac{x}{2-\sin^2 x} = 0$

vii) $f(x) = \sqrt{|x|}$, $a=0$

case 1: $\forall \varepsilon : 0 < \varepsilon < 1$

$\Rightarrow \varepsilon^2 < \varepsilon < 1$

$|x| < \varepsilon^2 \Rightarrow \sqrt{|x|} < \varepsilon < \sqrt{\varepsilon} < 1$

case 2: $1 \le \varepsilon < \infty$

$\varepsilon \le \varepsilon^2 \Rightarrow |x| < \varepsilon^2 \Rightarrow \sqrt{|x|} < \varepsilon$

viii) $f(x) = \sqrt{x}$, $a=1$

$\forall \varepsilon > 0$, we are looking for $\delta > 0$: $|x-1| < \delta \Rightarrow |\sqrt{x}-1| < \varepsilon$

the domain of f is $[0, +\infty)$, so $x \ge 0$.

consider $0 < \delta \le 1 \Rightarrow 1-\delta \le x \le \delta \Rightarrow x \ge 0$

$|\sqrt{x}-1| = \dfrac{|\sqrt{x}-1||\sqrt{x}+1|}{|\sqrt{x}+1|} = \dfrac{|x-1|}{|\sqrt{x}+1|} \le |x-1|$

because $|\sqrt{x}+1| > 1\ \forall x$

$\Rightarrow \forall \varepsilon > 0,\ \delta < \min(1, \varepsilon) \Rightarrow |x-1| < \delta \Rightarrow |\sqrt{x}-1| < |x-1| < \varepsilon$

4. $[x]$ = largest $\mathbb{Z} \leq x$

i) $f(x) = [x]$

$\forall a \in \mathbb{Z}, \lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$

example: $a = 1$

$0 < x-1 < \delta \Rightarrow 1 < x < 1+\delta$

$\delta < 1 \Rightarrow 1 < x < 1+\delta < 2 \Rightarrow f(x) = 1$ in this interval
$\Rightarrow |f(x) - 1| = 0 < \varepsilon, \forall \varepsilon > 0$

$\Rightarrow 0 < x-1 < \delta \Rightarrow |f(x)-1| < \varepsilon$

$\Rightarrow \lim_{x \to 1^+} [x] = 1$

$\delta > 0,$
$0 < 1-x < \delta \Rightarrow -1 < -x < -1+\delta \Rightarrow 1-\delta < x < 1$

$\delta < 1 \Rightarrow 0 < 1-\delta < x < 1 \Rightarrow f(x) = 0$

$\forall \varepsilon > 0, \forall \delta : 0 < \delta < 1,\ 0 < 1-x < \delta \Rightarrow |f(x) - 0| < \varepsilon$

$\Rightarrow \lim_{x \to 1^-} f(x) = 0$

general case $a \in \mathbb{Z}$

$0 < x-a < \delta \Rightarrow a < x < a+\delta$
$\delta < 1 \Rightarrow a < x < a+\delta < a+1$
$\Rightarrow 0 < x-a < \delta \Rightarrow f(x) = a$

$\forall \varepsilon > 0, 0 < \delta < 1 \Rightarrow 0 < x-a < \delta \Rightarrow |f(x) - a| = 0 < \varepsilon$

$\Rightarrow \lim_{x \to a^+} f(x) = a$

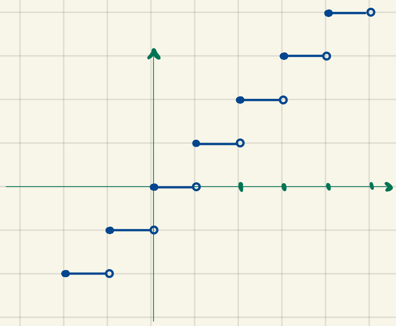
$0 < a-x < \delta, \delta > 0$
$\Rightarrow -a < -x < -a+\delta \Rightarrow a-\delta < x < a$
$\delta < 1 \Rightarrow a-1 < a-\delta < x < a$

$0 < \delta < 1,\ 0 < a-x < \delta \Rightarrow f(x) = a-1 \Rightarrow |f(x) - (a-1)| = 0 < \varepsilon$

$\Rightarrow \lim_{x \to a^-} [x] = a-1,\ a \in \mathbb{Z}$

consider $a \notin \mathbb{Z}$

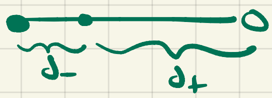
$f(a) = [a]$



let $d_- = a - [a]$
$\quad d_+ = [a+1] - a$

$|x-a| < \min(d_-, d_+)$

$\Rightarrow \lim_{x \to a} [x] = [a],\ a \notin \mathbb{Z}$



To be even more explicit

case 1: $\min(d_-, d_+) = d_-$

$|x-a| < d_- = a - [a]$

$\Rightarrow [a] - a < x - a < a - [a]$

$$\Rightarrow [a] < x < 2a - [a] = a + (a - [a]) = a + d_- < a + d_+ = [a+1]$$

$\Rightarrow [a] < x < [a+1] \Rightarrow f(x) = [a]$

therefore,

If $d_- \leq d_+$ $\forall \varepsilon > 0, \exists \delta = d_- : |x-a| < d_- \Rightarrow |f(x) - [a]| = 0 < \varepsilon$

case 2: $\min(d_-, d_+) = d_+$, ie $d_+ \leq d_-$

$|x-a| < d_+ = [a+1] - a$
$\Rightarrow a - [a+1] < x - a < [a+1] - a$
$\Rightarrow 2a - [a+1] < x < [a+1]$
$\Rightarrow a - ([a+1] - a) < x < [a+1]$
$a - d_- < a - d_+ < x < [a+1]$
$[a] < x < [a+1] \quad \Rightarrow f(x) = [a]$

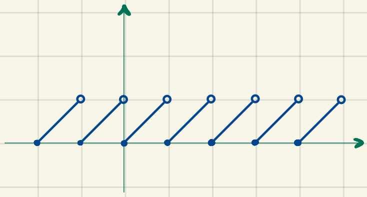
therefore

$d_+ \leq d_-, \forall \varepsilon > 0, |x-a| < d_+ \Rightarrow |f(x) - [a]| = 0 < \varepsilon$

$\Rightarrow \lim_{x \to a} [x] = [a],\ a \notin \mathbb{Z}$

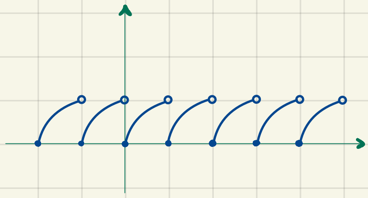
ii) $f(x) = x - [x]$

$a \in \mathbb{Z} \Rightarrow \lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$



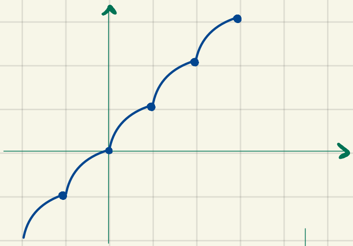
$a \notin \mathbb{Z} \Rightarrow \lim_{x \to a} (x - [x]) = a - [a]$

$\lim_{x \to a} f(x)$ exists for $x \notin \mathbb{Z}$



iii) $f(x) = \sqrt{x - [x]}$

$\lim_{x \to a} f(x)$ exists for $x \notin \mathbb{Z}$



iv) $f(x) = [x] + \sqrt{x - [x]}$

Consider $a \in \mathbb{Z}$

$\forall \delta : 0 < \delta < 1$

$0 < a - x < \delta \Rightarrow -\delta < x - a < 0 \Rightarrow 1 - \delta < x - a + 1 < 1$

$\Rightarrow 1 - \delta < \sqrt{1 - \delta} < \sqrt{x - a + 1} < 1$

$|f(x) - a| = |\cancel{a} - 1 + \sqrt{x - a + 1} - \cancel{a}|$

$< |-1 + 1 - \delta| = \delta$

Therefore, $\forall a \in \mathbb{Z}, \forall \varepsilon : 0 < \delta < \varepsilon < 1 : 0 < a - x < \delta$
$\Rightarrow |f(x) - a| < \delta < \varepsilon$

$\Rightarrow \lim_{x \to a^-} f(x) = a$

Similarly

$0 < x - a < \delta,\ 0 < \delta < 1 \Rightarrow$

$|f(x) - a| = |a + \sqrt{x - a} - a| = \sqrt{x - a} < \sqrt{\delta}$

$\forall \varepsilon > 0,\ \delta = \varepsilon^2 \Rightarrow 0 < x - a < \delta \Rightarrow |f(x) - a| < \varepsilon$

$\Rightarrow \lim_{x \to a^+} f(x) = a$

Consider $a \notin \mathbb{Z}$

Let $d_- = a - [a]$
$d_+ = [a+1] - a$

$|x - a| < \min(d_-, d_+) = \delta$

Case 1: $\min(d_-, d_+) = d_- = \delta$

$|x - a| < a - [a]$

$\Rightarrow [a] - a < x - a < a - [a]$

$\Rightarrow [a] < x < 2a - [a] = a + (a - [a])$
$= a + d_-$
$< a + d_+ = [a+1]$

$\Rightarrow [a] < x < [a+1] \Rightarrow f(x) = [a] + \sqrt{x - [a]}$

$|f(x) - [a] - \sqrt{a - [a]}| = |\sqrt{x - [a]} - \sqrt{a - [a]}|$

But, $0 < x - [a] < 2a - 2[a] = 2(a - [a])$

$\Rightarrow 0 < \sqrt{x - [a]} < \sqrt{2}\sqrt{a - [a]}$

$\Rightarrow |f(x) - [a] - \sqrt{a - [a]}| < |(\sqrt{2} - 1)\sqrt{a - [a]}| = (\sqrt{2} - 1)\delta$

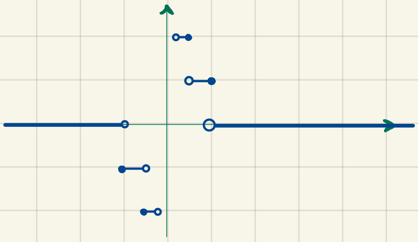
Case 2 is similar.

$\lim_{x \to a} f(x)$ exists for all $x \in \mathbb{R}$.

vi) $f(x) = \left[\frac{1}{x}\right]$

$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$ for all $x \in \mathbb{Q} : x = \frac{1}{n}, n \in \mathbb{Z}$, ie $-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \dots, -1, 1, \frac{1}{2}, \frac{1}{3}, \dots$

$\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$



consider $\forall \delta > 0 : |x - 0| = |x| < \delta$.

consider $\forall \ell \in \mathbb{R}$.

$x \in (0,1), x \in \mathbb{Q} \Rightarrow x = \frac{1}{p}, p \in \mathbb{Z}$.

$\Rightarrow f(x) = \left[\frac{1}{x}\right] = [p] = p$

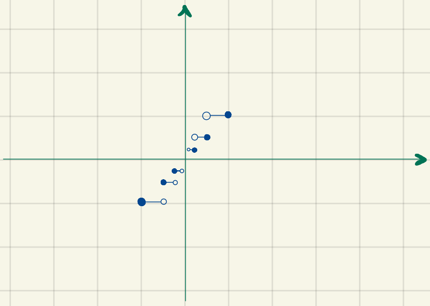
$\forall \varepsilon > 0, \delta > 0$ we can find a p: $\left|\frac{1}{p} - 0\right| < \delta \Rightarrow \frac{1}{p} < \delta \Rightarrow p > \frac{1}{\delta}$, e.g. $p = \left[\frac{1}{\delta} + 1\right]$

and such that $p - \ell > \varepsilon \Rightarrow p > \ell + \varepsilon$. For example, $\ell + \varepsilon + 1 > \ell + \varepsilon > 0$

Therefore, $p = \max\left(\left[\frac{1}{\delta} + 1\right], [\ell + \varepsilon + 1]\right) \Rightarrow |x - 0| < \delta, |f(x) - \ell| > \varepsilon$.

$\Rightarrow \lim_{x \to 0} f(x)$ does not exist.

$\lim_{x \to a} f(x)$ does exist for $\left\{a : a \neq 0 \text{ and } a \neq \frac{1}{n}, n \in \mathbb{Z}\right\}$



vii) $f(x) = \frac{1}{\left[\frac{1}{x}\right]}$

$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$ for $x \in \mathbb{Q} : x = \frac{1}{p}, p \in \mathbb{Z}\}$

At $a = 0$ we have

$$|f(x) - 0| = \left|\frac{1}{\left[\frac{1}{x}\right]}\right| < \varepsilon$$

$x > 0 \Rightarrow x < \varepsilon$

$x < 0 \Rightarrow -x < \varepsilon \Rightarrow x > -\varepsilon$ $\Rightarrow |x| < \varepsilon$

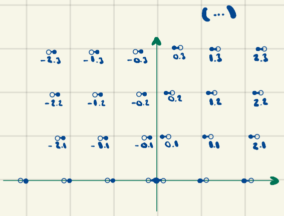
Therefore, $\forall \varepsilon > 0, |x - 0| < \varepsilon \Rightarrow |f(x) - 0| < \varepsilon$

$$\Rightarrow \lim_{x \to 0} \frac{1}{\left[\frac{1}{x}\right]} = 0$$

$\lim_{x \to a} f(x)$ exists for $a \in \left\{a : |a| \leq 1, a \neq \frac{1}{p}, p \in \mathbb{Z}\right\}$

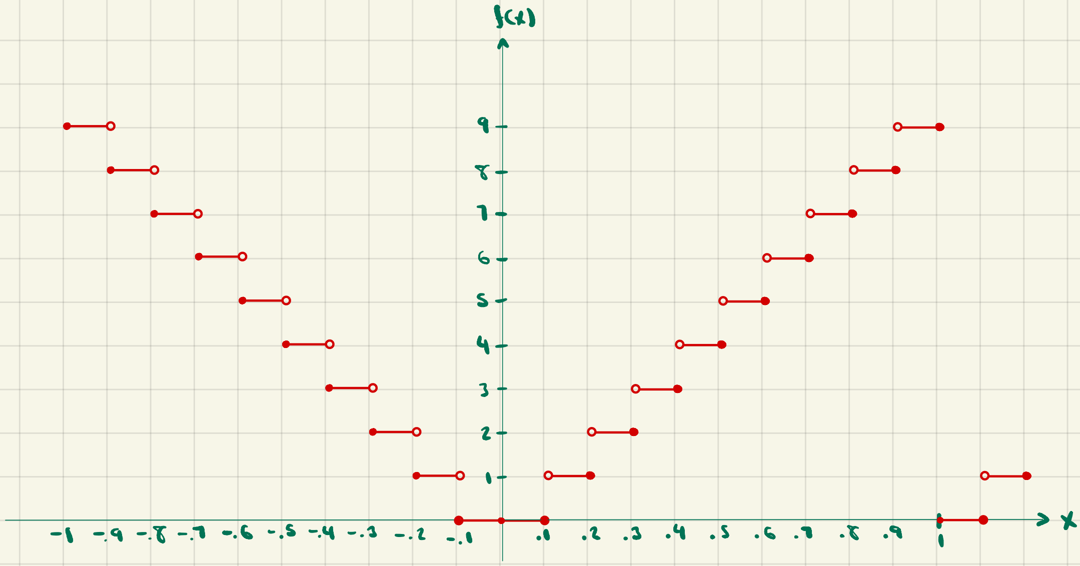
5. a)

i) $f(x)$ = 1st number of decimal expansion of x



$f(0.1) = f(0.0\bar{9}) = 0$

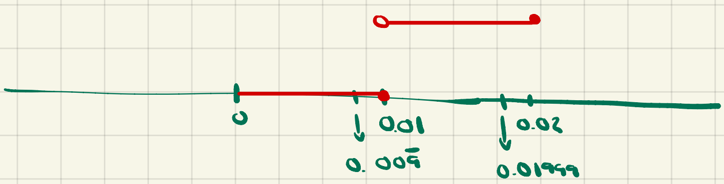
$f(0.2) = f(0.1\bar{9}) = 1$

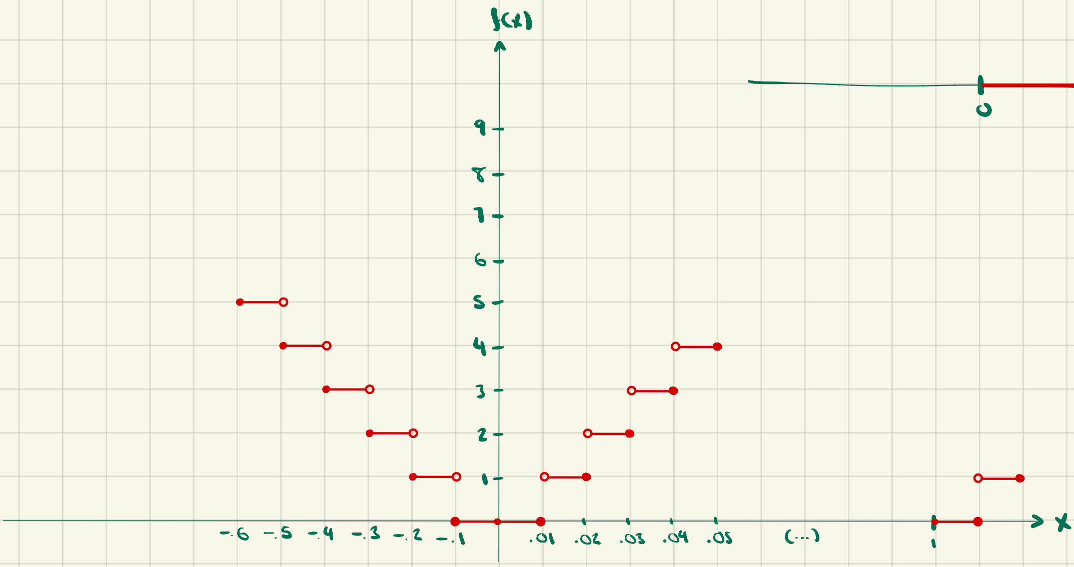


$\lim_{x \to a} f(x)$ exists for $\{a : a \neq 0.1n,\ n \neq 0,\ n \in \mathbb{Z}\}$

---

ii) $f(x)$ = 2nd number in decimal expansion of x

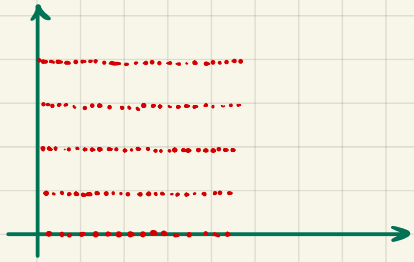




$\lim_{x \to a} f(x)$ exists for $\{a : a \neq 0.01n,\ n \neq 0,\ n \in \mathbb{Z}\}$

iii) $f(x)$ = number of 7's in decimal expansion of x if this number is finite,
0 otherwise

In any interval there are infinite numbers w/ any number of 7's, from zero to ∞.



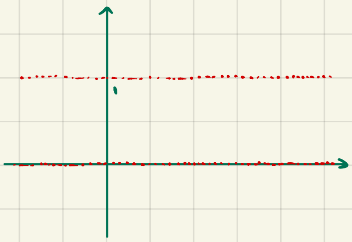
Therefore for any interval $a-\delta < x < a+\delta$, $f(x)$ equals every natural number at some x.

$\forall \epsilon > 0, \forall \delta > 0, I = (a-\delta, a+\delta), \exists x \in I : |f(x) - \ell| > \epsilon, \forall \ell \in \mathbb{R}$

$\Rightarrow \lim_{x \to a} f(x)$ does not exist for any $a \in \mathbb{R}$.

---

iv) $f(x) = 0$ if number of 7's in decimal expansion of x is finite, 1 otherwise

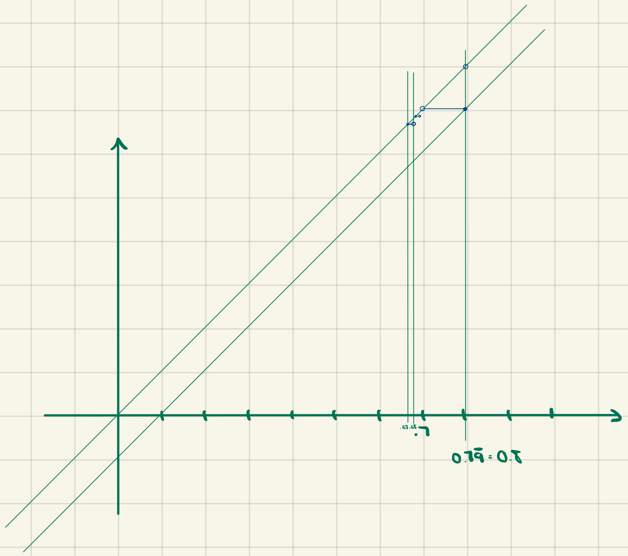


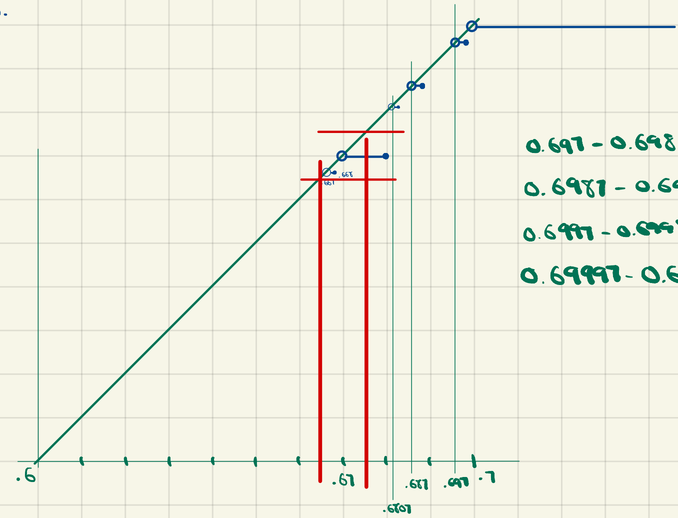
in any interval $(a-\delta, a+\delta)$, there are infinite points where f is 1, and infinite points where f is 0.

no matter what value ℓ is chosen as candidate for $\lim_{x \to a} f(x)$, $|f(x) - \ell|$ will be at least 0.5 for infinitely many points.

$\Rightarrow \lim_{x \to a} f(x)$ does not exist for any $a \in \mathbb{R}$

v) $f(x)$ = number obtained by replacing all digits in decimal expansion of x which come after first 7, if any, by 0.





0.697 - 0.698

0.6987 - 0.6988

0.6997 - 0.6998 < 0.69987 - 0.69988

0.69997 - 0.69998

$\lim_{x\to a} f(x)$ exists for all $a \in \mathbb{R}$ except if the decimal expansion of a ends in $7\bar{9}$, ie except for the points on the right (closed) side of the intervals that start on the $y = x$ line at an x with one digit 7 followed by zeros.

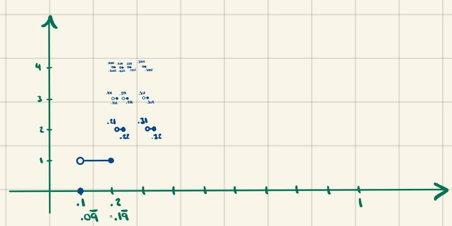
This means the limit does exist

i) for a such that a has no digit 7 in its d.e., has one digit 7 followed by zeros

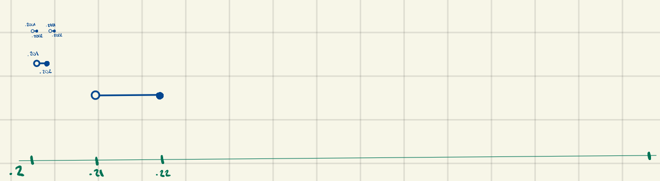
ii) a w/ at least one digit 7, followed by at least one nonzero digit, except if d.e. ends in $7\bar{9}$.

---

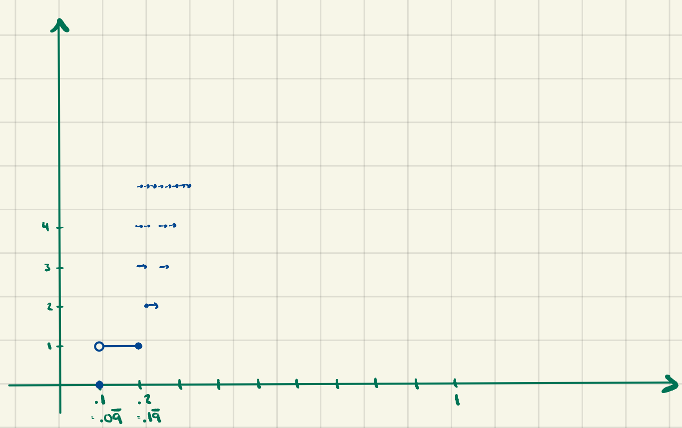
vi) $f(x) =$ 0 if 1 never appears in decimal expansion of x
n if 1 first appears in n^{th} place



The graph is composed of an infinite number of intervals of form $(a, b]$. $\forall x \in (a, b]$, $f(x)$ is constant.
However, when x approaches a from below or b from above, no matter how small an interval we can always find an x such that $f(x)$ is arbitrarily large.



$\Rightarrow \lim_{x\to a} f(x)$ exists $\forall a \in \mathbb{R}$ except if a's decimal expansion has one digit 1 followed by zeros or has no digit 1 but ends in z$\bar{0}$. Ie the limit doesn't exist on the ends of the intervals.



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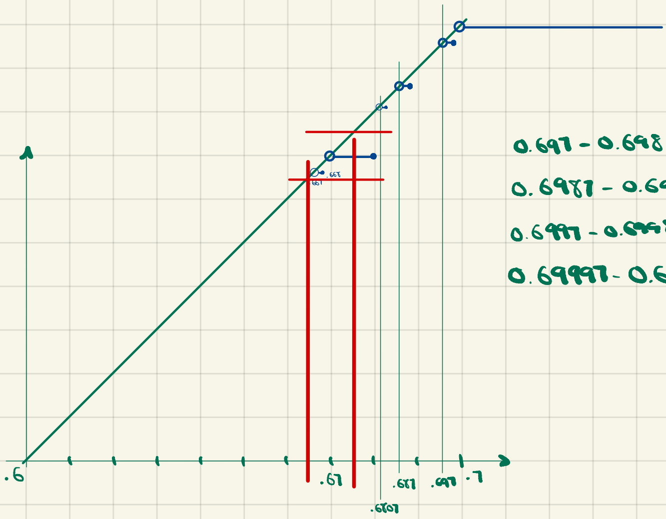
b) For all the items, nothing changes regarding existence of $\lim_{x\to a} f(x)$ when we change our use of infinite decimals ending in $\bar{9}$.

* Attempt at proving problem 5)a)v).



0.697 − 0.698

0.6987 − 0.6988

0.6997 − 0.6998 < 0.69987 − 0.69988

0.69997 − 0.69998

Case 1: a's decimal expansion contains one digit 7, followed by zeros.

Given $\forall \varepsilon > 0$, first consider $(a, a+\delta)$, $\delta > 0$: $x \in (a, a+\delta) \Rightarrow |f(x) - a| < \varepsilon$

we can easily find a δ such that $f(x) = a$ for $x \in (a, a+\delta)$.

Consider the interval $A = (a-\varepsilon, a)$, $\forall \varepsilon > 0$

- if $x \in A$ and x has no digit 7 in decimal exp., then $f(x) = x$

$\Rightarrow |f(x) - a| = |x - a| < \varepsilon$

- if $x \in A$ and x has one or more digit 7's in decimal exp, then:

a) if $a-\varepsilon$ has no digit 7's in dec. exp., then $f(a-\varepsilon) = a-\varepsilon$, $a-\varepsilon < f(x) \leq x < a$

$\Rightarrow -\varepsilon < f(x) - a < 0 < \varepsilon \Rightarrow |f(x) - a| < \varepsilon$

b) if $a-\varepsilon$ has one or more digit 7's in dec. exp, then there is some number w/o 7's between $a-\varepsilon$ and a. The reason for this is that a has one digit 7 followed by zeros. There is no number w/ one or more zeros smaller than a such that there is no number w/ no zeros between a and this number. In fact a is equivalent to a number ending in a string of 9's, with no 7's.

Choose a number between $a-\varepsilon$ and a w/ no 7's, call it n_1.

$a - \varepsilon < n_1 < a$

$a - \varepsilon < a - (a - n_1) < a$

$-\varepsilon < -(a - n_1) < 0$

$\varepsilon > a - n_1 > 0$

Call $\varepsilon_1 = a - n_1$, we are in case a) above

$\Rightarrow x \in (a-\varepsilon_1, a)$, $a-\varepsilon_1$ has no 7's in d.e., $\Rightarrow |f(x) - a| < \varepsilon_1 < \varepsilon$

6. f, g fns

$\forall \epsilon > 0, \forall x \in \mathbb{R}$

$0 < |x-2| < \sin^2\left(\frac{\epsilon^2}{9}\right) + \epsilon \Rightarrow |f(x) - 2| < \epsilon$

$0 < |x-2| < \epsilon^2 \Rightarrow |g(x) - 4| < \epsilon$

$\Rightarrow \lim_{x \to 2} f(x) = 2$

$\lim_{x \to 2} g(x) = 4$

---

i) $0 < |x-2| < \delta \Rightarrow |f(x) + g(x) - 6| < \epsilon$

$\forall \epsilon > 0$ let $\delta = \min\left(\epsilon^2, \sin^2\left(\frac{\epsilon^2}{9}\right) + \epsilon\right)$

$0 < |x-2| < \min\left(\epsilon^2/4, \sin^2(\epsilon^2/36) + \epsilon/2\right) \Rightarrow |f(x) - 2| < \epsilon/2$

$|g(x) - 4| < \epsilon/2$

Th.2 Lemma (1) $\Rightarrow |f(x) + g(x) - 6| < \epsilon$

$\Rightarrow \lim_{x \to a} (f(x) + g(x)) = 6$

---

ii) $0 < |x-2| < \delta \Rightarrow |f(x)g(x) - 8| < \epsilon$

$\forall \epsilon > 0, \quad 0 < |x-2| < \delta_1 \Rightarrow |f(x) - 2| < \min\left(1, \frac{\epsilon}{2(4+1)}\right)$

$0 < |x-2| < \delta_2 \Rightarrow |g(x) - 4| < \frac{\epsilon}{2(2+1)}$

$\Rightarrow 0 < |x-2| < \min(\delta_1, \delta_2) \Rightarrow |f(x) - 2| < \min\left(1, \frac{\epsilon}{10}\right)$

$|g(x) - 4| < \frac{\epsilon}{6}$

Th.2 Lemma (2) $\Rightarrow |f(x)g(x) - 8| < \epsilon$

$\Rightarrow \lim_{x \to 2} f(x)g(x) = 8$

---

iii)

$\forall \epsilon > 0, \exists \delta : 0 < |x-2| < \delta \Rightarrow |g(x) - 4| < \min\left(\frac{4}{2}, \frac{\epsilon \cdot 4^2}{2}\right)$

Th.2 Lemma (3) $\Rightarrow g(x) \neq 0, \left|\frac{1}{g(x)} - \frac{1}{4}\right| < \epsilon$

$\Rightarrow \lim_{x \to 2} \frac{1}{g(x)} = \frac{1}{4}$

iv) $\forall \epsilon > 0,$

$\exists \delta_1 : 0 < |x-2| < \delta_1 \Rightarrow |f(x) - 2| < \min\left(1, \frac{\epsilon}{2(\frac{1}{4}+1)}\right)$

$\exists \delta_2 : 0 < |x-2| < \delta_2 \Rightarrow g(x) \neq 0, \left|\frac{1}{g(x)} - \frac{1}{4}\right| < \frac{\epsilon}{2(2+1)}$

Th.2 Lemma (2) $\Rightarrow \left|f(x) \cdot \frac{1}{g(x)} - \frac{1}{2}\right| < \epsilon$

$\Rightarrow \lim_{x \to 2} \frac{f(x)}{g(x)} = \frac{1}{2}$

---

7. $|f(x)-\ell|<\varepsilon$ when $0<|x-a|<\delta$ $\Rightarrow$ $|f(x)-\ell|<\frac{\varepsilon}{2}$ when $0<|x-a|<\frac{\delta}{2}$

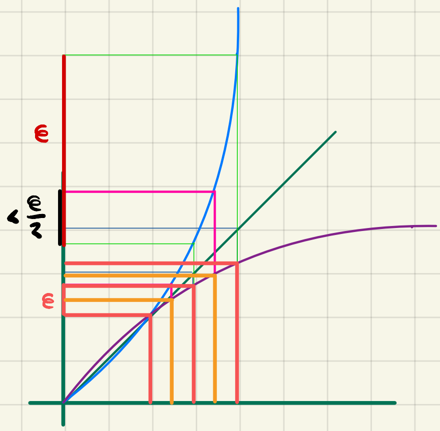
Fn f for which above statement is false

Let $f(x)=\sqrt{|x|}$

$\lim_{x\to 0}\sqrt{|x|}=0$

$|x|<\varepsilon^2 \Rightarrow \sqrt{|x|}<\varepsilon$

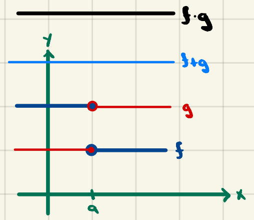
$|x|<\frac{\varepsilon^2}{2} \Rightarrow \sqrt{|x|}<\frac{\varepsilon}{\sqrt{2}}<\frac{\varepsilon}{2}$



8. a) $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ don't exist.

$\lim_{x\to a}(f(x)+g(x))$ exists.

$\lim_{x\to a} f(x)g(x)$ exists



$f(x)=\begin{cases} c_1 & x\le a \\ c_2 & x>a\end{cases}$

$g(x)=\begin{cases} c_2 & x\le a \\ c_1 & x>a\end{cases}$ $\Rightarrow$ $(f+g)(x)=c_1+c_2 \ \forall x$, $(fg)(x)=c_1c_2 \ \forall x$

Theorem 2

$\lim_{x\to a} f(x)=\ell$, $\lim_{x\to a} g(x)=m$ $\Rightarrow$ $\lim_{x\to a}(f+g)(x)=\ell+m$ (1)

$\lim_{x\to a}(f\cdot g)(x)=\ell\cdot m$ (2)

$m\neq 0 \Rightarrow \lim_{x\to a}\frac{1}{g}=\frac{1}{m}$ (3)

3. $y_0\neq 0$, $|y-y_0|<\min\left(\frac{|y_0|}{2},\frac{\varepsilon|y_0|^2}{2}\right)$

$\Rightarrow y\neq 0$, $\left|\frac{1}{y}-\frac{1}{y_0}\right|<\varepsilon$

b) $g(x)=(f(x)+g(x))+(-f(x))$

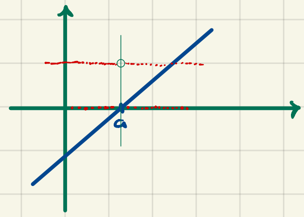
$\exists\lim_{x\to a}(f+g)(x)$, $\exists\lim_{x\to a}(-f(x))$ $\Rightarrow$ Apply Th. 2 (1)

$\Rightarrow \lim_{x\to a}(f(x)+g(x)+(-f(x)))=\lim_{x\to a} g(x)$ exists

c) $\overset{\lim f}{A}, \underset{\lim(f+g)}{B} \Rightarrow \overset{\lim g}{C}$ (proved in b))

$\neg C \Rightarrow \neg(A\wedge B) \Rightarrow (\neg A\wedge B)\,|\,(A\wedge\neg B)\,|\,(\neg A\wedge\neg B)$

$\Rightarrow A\wedge\neg B \Rightarrow \nexists\lim(f+g)$



$\forall\varepsilon: 0<\varepsilon<1$, $|x-a|<\varepsilon \Rightarrow |f(x)-0|=|x-a-0|<\varepsilon<1$

$|x-a|<\varepsilon \Rightarrow |f(x)g(x)-0|=|f(x)g(x)|\le|f(x)|<\varepsilon$

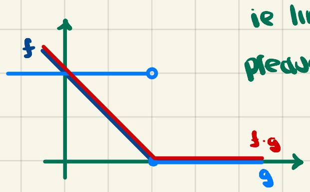
$\Rightarrow \lim_{x\to a} f(x)g(x)=0$

$\lim(f+g)$ does not exist.

d) Th. 2 (2) $\Rightarrow$ $\lim f \wedge \lim g \Rightarrow \lim f\cdot g$

By logic it is not the case that $\lim f \wedge \neg\lim g \Rightarrow \neg\lim fg$

ie $\lim g$ not existing does not preclude $\lim fg$ existing.



Altern. consider $g=\frac{f\cdot g}{f}$. $\lim_{x\to a} fg$ exists by assumption.

If $\lim_{x\to a} f$ exists and is $\neq 0$ then by Th2 (3), $\lim_{x\to a}\frac{1}{f}$ exists. Then $\lim_{x\to a}\frac{1}{f}\cdot fg$ exists by Th 2(2). If $\lim_{x\to a} f=0$, however, we can construct a counterexample, in which $\lim g$ doesn't exist but $\lim fg$ does, as shown on the left.

9. $\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h)$

$\lim_{x\to a} f(x) = \ell \Rightarrow \forall \epsilon>0, \exists\delta>0 : |x-a|<\delta \Rightarrow |f(x)-\ell|<\epsilon$

$\lim_{h\to 0} f(a+h) = \ell_2 \Rightarrow \forall\epsilon>0, \exists\delta>0 : |h-0|<\delta \Rightarrow |f(a+h)-\ell_2|<\epsilon$

Let $x = a+h$. Then $h = x-a$ and $|x-a|<\delta \Rightarrow |f(x)-\ell_2|<\epsilon$

$\Rightarrow \lim_{x\to a} f(x) = \ell_2$

$\Rightarrow \ell = \ell_2$ (because $\lim_{x\to a} f(x)$ is unique.

---

10. a) $\lim_{x\to a} f(x) = \ell \iff \lim_{x\to a}(f(x)-\ell) = 0$

$\lim_{x\to a}(-\ell) = -\ell$ because $\forall\epsilon>0, |x-a|<\epsilon \Rightarrow |-\ell-(-\ell)| = 0 < \epsilon$

Th. 2 (2) $\Rightarrow \lim_{x\to a}(f(x)+(-\ell)) = \lim_{x\to a} f(x) + \lim_{x\to a}(-\ell) = 0$

conversely

$\lim_{x\to a}(f(x)-\ell) = 0 \Rightarrow \forall\epsilon>0, \exists\delta>0 : |x-a|<\delta \Rightarrow |f(x)-\ell|<\epsilon$

$\Rightarrow \lim_{x\to a} f(x) = \ell$

b) $\lim_{x\to 0} f(x) = \lim_{x\to a} f(x-a)$

$\forall\epsilon>0, \exists\delta : |x|<\delta \Rightarrow |f(x)-\ell|<\epsilon$

$\forall\epsilon>0, \exists\delta : |x-a|<\delta \Rightarrow |f(x-a)-\ell_2|<\epsilon$

$y = x-a \Rightarrow |y|<\delta \Rightarrow |f(y)-\ell_2|<\epsilon$

$\Rightarrow \lim_{y\to 0} f(y) = \ell_2$

we know from a previous theorem that a limit at any point is unique $\Rightarrow \ell_2 = \ell$

$\Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to a} f(x-a)$

c) $\lim_{x\to 0} f(x) = \lim_{x\to 0} f(x^3)$

$\forall\epsilon>0, \exists\delta>0 : |x|<\delta \Rightarrow |f(x)-\ell|<\epsilon$

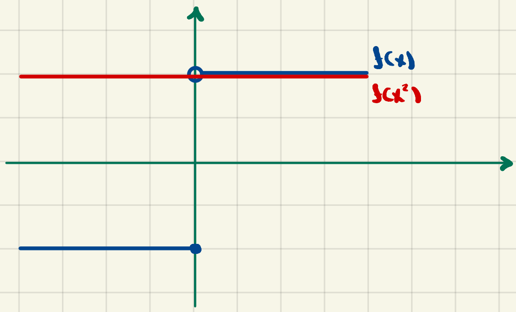
$\forall\epsilon>0, \exists\delta_1>0 : |x|<\delta_1 \Rightarrow |f(x^3)-\ell_2|<\epsilon$

$y = x^3 \Rightarrow x = \sqrt[3]{y}$

$|\sqrt[3]{y}|<\delta_1 \Rightarrow |y|<\delta_1^3 \Rightarrow |f(y)-\ell_2|<\epsilon$

$\Rightarrow \lim_{y\to 0} f(y) = \ell_2 \quad \Rightarrow \ell = \ell_2$

d) Example where $\lim_{x\to 0} f(x^2)$ exists but $\lim_{x\to 0} f(x)$ does not.



$$f(x) = \begin{cases} a & x>0 \\ -a & x\leq 0 \end{cases}$$

$f(x^2) = a \ \forall x$

11. $\left[0<|x-a|<\delta \Rightarrow f(x)=g(x)\right] \Rightarrow \lim_{x\to a} f(x) = \lim_{x\to a} g(x)$

Assume $\lim_{x\to a} f(x) = \ell$. Then

$\forall \epsilon>0$ we can choose δ' : $0<|x-a|<\delta' \Rightarrow |f(x)-\ell|<\epsilon$.

Let $\delta'<\delta$. Then

$0<|x-a|<\delta' \Rightarrow f(x)=g(x),\ |f(x)-\ell|<\epsilon \Rightarrow |g(x)-\ell|<\epsilon$

$\Rightarrow \lim_{x\to a} g(x)=\ell$

12. a) $f(x)\le g(x)\ \forall x$. Does this imply $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$?

Assume $\lim_{x\to a} f(x) = \ell > m = \lim_{x\to a} g(x)$

$\forall\epsilon>0\ \exists\delta>0 : |x-a|<\delta \Rightarrow |f(x)-\ell|<\epsilon$

Let $\epsilon = \frac{\ell-m}{2}$. Then $|x-a|<\delta \Rightarrow |f(x)-\ell| < \frac{\ell-m}{2}$

$\Rightarrow \frac{m-\ell}{2} < f(x)-\ell < \frac{\ell-m}{2}$

$\Rightarrow \frac{\ell+m}{2} < f(x) < \frac{3\ell-m}{2}$

$\ell>m \Rightarrow \frac{\ell+m}{2} > m$

$|x-a|<\delta_2 \Rightarrow |g(x)-m| < \frac{\ell-m}{2}$

$\Rightarrow \frac{m-\ell}{2} < g(x)-m < \frac{\ell-m}{2}$

$\Rightarrow \frac{3m-\ell}{2} < g(x) < \frac{\ell+m}{2}$

$\Rightarrow$

We've shown that,

$|x-a|<\min(\delta,\delta_2) \Rightarrow g(x) < \frac{\ell+m}{2} < f(x)$

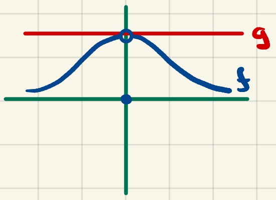
which contradicts our assumption that $f(x)\le g(x)\ \forall x$.

$\Rightarrow$ Yes, $f(x)\le g(x)\ \forall x \Rightarrow \lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$

b) Limits are a local property. Therefore, since we are investigating a limit near $x=a$, the property $f(x)\le g(x)$ need hold for x such that $0<|x-a|<\delta$, for some $\delta>0$.

The proof by contradiction in a) showed that the assumption $\ell>m$ led to a contradiction near a. No matter what happens elsewhere in the domain, if $f(x)\le g(x)$ for $x : |x-a|<\delta$ then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

c)



$f(x) < g(x)\ \forall x$ but $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x)$

13. Suppose

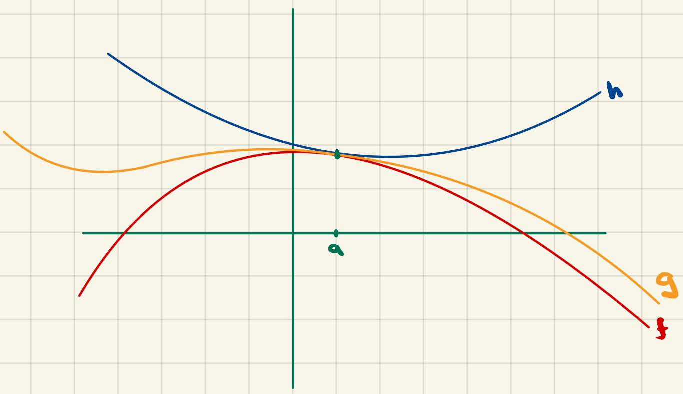
$f(x) \le g(x) \le h(x)$

$\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$

Prove

$\lim_{x\to a} g(x)$ exists

$\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = \lim_{x\to a} h(x)$



$\forall \epsilon>0, \exists \delta>0 : |x-a|<\delta \Rightarrow |f(x)-\ell|<\epsilon$
$|h(x)-\ell|<\epsilon$

$\Rightarrow \ell-\epsilon < f(x) < g(x) < h(x) < \ell+\epsilon$

$\Rightarrow |g(x)-\ell|<\epsilon$

$\Rightarrow \lim_{x\to a} g(x) = \ell$

14. a) Prove

$$\lim_{x\to 0} \frac{f(x)}{x} = \ell,\ b\neq 0 \Rightarrow \lim_{x\to 0} \frac{f(bx)}{x} = b\ell$$

$$\frac{f(bx)}{x} = \frac{b f(bx)}{bx}$$

$$\lim_{x\to 0} \frac{f(bx)}{bx}$$

$\forall \epsilon>0, \exists \delta>0 : |x|<\delta \Rightarrow \left|\frac{f(x)}{x} - \ell\right| < \epsilon$

$x = by$

$|by|<\delta \Rightarrow \left|\frac{f(by)}{by} - \ell\right| < \epsilon$

$|y| < \frac{\delta}{|b|} \Rightarrow \left|\frac{f(by)}{by} - \ell\right| < \epsilon$

$\Rightarrow \lim_{y\to 0} \frac{f(by)}{by} = \ell$

$\lim_{x\to 0} b\cdot\frac{f(by)}{by} = b\cdot\ell$ using Th.2 (2).

b) $b=0$

$$\frac{f(bx)}{x} = \frac{f(0)}{x}$$

$\lim_{x\to 0} \frac{f(0)}{x}$ is not defined because

$\lim_{x\to 0} \frac{1}{x}$ does not exist

Consider values of the form $\frac{1}{10^n}$, $n\in\mathbb{N}$.

$g(1/10^n) = 10^n$. $\forall \delta>0, \exists n_1 \in \mathbb{N} : \frac{1}{10^{n_1}} < \delta$

$\forall n > n_1 \Rightarrow \frac{1}{10^n} < \frac{1}{10^{n_1}} \Rightarrow g(1/10^n) > g(1/10^{n_1})$

$\forall \ell>0, \exists n_2 : g(1/10^{n_2}) > \ell$ and $\forall n>n_2 \Rightarrow g(1/10^n) > \ell$

Let $\epsilon = 1$. $\forall \ell$

$\forall \delta>0, |x|<\delta \Rightarrow \exists n\in\mathbb{N} : x = \pm 1/10^n, |x|<\delta, g(x) = \pm 10^n,$

$|g(x)-\ell| > 1 \Rightarrow \lim_{x\to 0} 1/x$ does not exist.

c) $\lim\limits_{x\to 0} \frac{\sin 2x}{x}$ in terms of $\lim\limits_{x\to 0} \frac{\sin x}{x}$

$$\lim_{x\to 0} \frac{\sin x}{x} = \ell,\ b\neq 0 \Rightarrow \lim_{x\to 0} \frac{\sin(bx)}{x} = b\ell \qquad \text{(from part a))}$$

Alternatively

$$\lim_{x\to 0} \frac{\sin 2x}{x} = \lim_{x\to 0} 2\,\frac{\sin x}{x}\cos x \overset{\text{Th2(2)}}{=} 2\ell\cdot 1 = 2\ell$$

15. $\alpha = \lim\limits_{x\to 0} \frac{\sin x}{x}$

i) $$\lim_{x\to 0} \frac{\sin 3x}{x} = 3\lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \cos^2 x - \lim_{x\to 0} \frac{\sin x}{x} \cdot \cancelto{0}{\lim_{x\to 0} \sin^2 x} = 3\alpha$$

$$\sin 3x = \sin(x+2x) = \sin x\cos 2x + \sin 2x\cos x$$

$$= \sin x(\cos^2 x - \sin^2 x) + 2\sin x\cos^2 x$$

$$= 3\sin x\cos^2 x - \sin^3 x$$

ii) $$\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \lim_{x\to 0} \frac{\sin ax}{x} \cdot \frac{x}{\sin bx} = a\alpha \cdot \frac{1}{b\alpha} = \frac{a}{b}$$

iii) $$\lim_{x\to 0} \frac{\sin^2 2x}{x} = \lim \sin 2x \cdot \lim \frac{\sin 2x}{x} = 0\cdot 2\alpha = 0$$

$$\frac{(\sin 2x)^2}{x} = \frac{4\sin^2 x\cos^2 x}{x} = 4\,\frac{\sin x}{x}\cdot \sin x\cdot \cos^2 x$$

iv) $$\lim_{x\to 0} \frac{\sin^2 2x}{x^2} = \left(\lim_{x\to 0} \frac{\sin 2x}{x}\right)^2 = (2\alpha)^2 = 4\alpha^2$$

v) $$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos^2 x)} = \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos^2 x)} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \cdot \frac{1}{1+\cos^2 x} = \frac{\alpha^2}{2}$$

vi) $$\lim_{x\to 0} \frac{\tan^2 x + 2x}{x+x^2} = \lim_{x\to 0} \frac{\frac{\sin^2 x}{\cos^2 x}\cdot \frac{1}{x} + 2}{1+x} = \lim_{x\to 0} \left[\overset{\alpha}{\frac{\sin x}{x}} \cdot \overset{0}{\sin x} \cdot \overset{1}{\frac{1}{\cos^2 x}} \cdot \overset{1}{\frac{1}{1+x}} + \overset{2}{\frac{2}{1+x}} \right] = 2$$

vii) $$\lim_{x\to 0} \frac{x\sin x}{1-\cos x} = \lim_{x\to 0} \frac{x\sin x(1+\cos x)}{\sin^2 x} = \lim_{x\to 0} \frac{1+\cos x}{\frac{\sin x}{x}} = \frac{2}{\alpha}$$

viii) $$\lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0} \frac{\sin x\cosh + \sinh\cos x - \sin x}{h} = \lim_{h\to 0} \frac{\sin x(\cosh - 1) + \sinh\cos x}{h} = \sin x \cancelto{0}{\lim \frac{\cosh - 1}{h}} + \cos x \lim \frac{\sinh}{h}$$

$$\lim_{h\to 0} \frac{\cosh - 1}{h} = \lim (-1)\frac{1-\cosh}{h} = -\lim \frac{1-\cos^2 h}{h(1+\cosh)} = -\lim \frac{\sin^2 h}{h(1+\cosh)} = -\lim \frac{\sinh}{h}\cdot \frac{\sinh}{1+\cosh} = 0$$

$$\Rightarrow \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \alpha\cos x$$

ix) $\lim_{x\to 1} \frac{\sin(x^2-1)}{x-1} = \lim_{x\to 1} \frac{(x+1)\sin(x^2-1)}{x^2-1} = \lim_{x\to 1}(x+1) \cdot \underbrace{\lim_{x\to 1} \frac{\sin(x^2-1)}{x^2-1}} = 2\alpha$

*note: $\lim_{h\to 0} \frac{\sin h}{h} = \alpha$

Let $h = x^2-1$

$(|h| < \delta \Rightarrow x^2-1 < |x^2-1| < \delta \Rightarrow x^2 < 1+\delta \Rightarrow |x| < \sqrt{1+\delta}) \Rightarrow \left| \frac{\sin(x^2-1)}{x^2-1} - \ell \right| < \varepsilon$

we've shown that

$\forall \varepsilon > 0, \exists \delta_1 : |x| < \delta_2 = \sqrt{1+\delta} \Rightarrow \left| \frac{\sin(x^2-1)}{x^2-1} - \ell \right| < \varepsilon$

$\Rightarrow \lim_{x\to 0} \frac{\sin(x^2-1)}{x^2-1} = \ell$

---

x) $\lim_{x\to 0} \frac{x^2(3+\sin x)}{(x+\sin x)^2} = \lim_{x\to 0} \frac{3+\sin x}{\left(\frac{x+\sin x}{x}\right)^2} = \lim_{x\to 0} \frac{3+\sin x}{\left(1+\frac{\sin x}{x}\right)^2} = \frac{3}{(1+\alpha)^2}$

---

xi) $\lim_{x\to 1} (x^2-1)^3 \sin\left(\frac{1}{x-1}\right)^3$

Note that $\forall x \neq 1, -1 \leq \sin\left(\frac{1}{x-1}\right)^3 \leq 1$

Problem 12c $\Rightarrow -1 \leq \lim_{x\to 1} \sin\left(\frac{1}{x-1}\right)^3 \leq 1$

$f(x) = -(x^2-1)^3 \leq (x^2-1)^3 \sin\left(\frac{1}{x-1}\right)^3 \leq (x^2-1)^3 = h(x)$

$\lim_{x\to 1} f(x) = \lim_{x\to 1} h(x) = 0$

Problem 13 $\Rightarrow \lim_{x\to 1} (x^2-1)^3 \sin\left(\frac{1}{x-1}\right)^3 = 0$

16. a) $\lim_{x\to a} f(x) = l \Rightarrow \lim_{x\to a} |f|(x) = |l|$

$\forall \epsilon > 0\ \exists \delta > 0 : |x-a| < \delta \Rightarrow |f(x) - l| < \epsilon$

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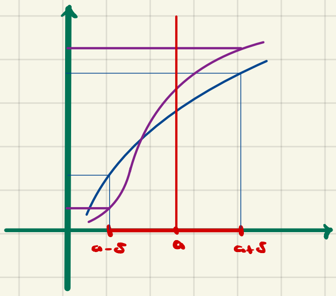
$|f(x)| - |l| \leq |f(x) - l| < \epsilon \longrightarrow \big| |f(x)| - |l| \big| \leq |f(x) - l| < \epsilon$

Therefore $\forall \epsilon > 0\ \exists \delta > 0 : |x-a| < \delta \Rightarrow \big| |f(x)| - |l| \big| < \epsilon$

$\Rightarrow \lim_{x\to a} |f(x)| = |l|$

b) $\lim_{x\to a} f(x) = l,\ \lim_{x\to a} g(x) = m \Rightarrow \lim_{x\to a} \max(f,g) = \max(l,m)$

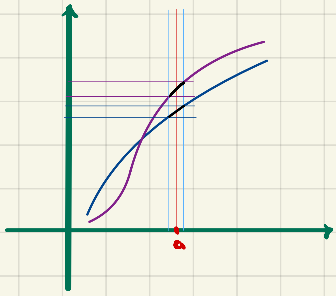


$\forall \epsilon > 0, \exists \delta_1 > 0 : |x-a| < \delta_1 \Rightarrow |f(x) - l| < \epsilon$
$|g(x) - m| < \epsilon$

Let $\epsilon = \dfrac{|l-m|}{2}$

$\exists \delta_2 > 0 : |x-a| < \delta_2 \Rightarrow l - \dfrac{|l-m|}{2} < f(x) < l + \dfrac{|l-m|}{2}$

$m - \dfrac{|l-m|}{2} < g(x) < m + \dfrac{|l-m|}{2}$



Case 1: $l > m$

$\dfrac{2l - l + m}{2} = \dfrac{l+m}{2} < f(x) < \dfrac{3l-m}{2}$

$\dfrac{2m - l + m}{2} = \dfrac{3m - l}{2} < g(x) < \dfrac{l+m}{2}$

Therefore,

$\forall x : |x-a| < \delta_2 \Rightarrow g(x) < f(x) \Rightarrow \max(f,g) = f(x) \Rightarrow |\max(f,g) - l| < \dfrac{|l-m|}{2}$

$\forall \epsilon > 0, \exists \delta = \min(\delta_1, \delta_2) : |x-a| < \delta \Rightarrow |\max(f,g) - l| < \epsilon$

$\Rightarrow \lim_{x\to a} \max(f,g)(x) = l = \max(l,m)$

Case 2: $m > l$ By symmetry, $\lim_{x\to a} \max(f,g)(x) = m = \max(l,m)$

Case 3: $m = l$

$\forall \epsilon > 0, \exists \delta > 0 : |x-a| < \delta \Rightarrow |f(x) - l| < \epsilon,\ |g(x) - l| < \epsilon \Rightarrow |\max(f,g)(x) - l| < \epsilon$

$\Rightarrow \lim_{x\to a} \max(f,g)(x) = l = m = \max(l,m)$

The proof for $\min(f,g)(x)$ is essentially the same.

Alternatively, we can use the formula

$$\max(f,g) = \frac{f+g+|f-g|}{2}$$

$$\lim_{x\to a} \max(f,g)(x) = \lim_{x\to a} \frac{f(x)+g(x)+|f(x)-g(x)|}{2} = \frac{\ell+m+|\ell-m|}{2} = \max(\ell,m)$$

$$\lim_{x\to a} |f(x)-g(x)| = |\ell-m| \quad \text{(from a))}$$

$$\min(f,g) = \frac{f+g-|f-g|}{2}$$

$$\lim_{x\to a} \min(f,g)(x) = \lim_{x\to a} \frac{f(x)+g(x)-|f(x)-g(x)|}{2} = \frac{\ell+m-|\ell-m|}{2} = \min(\ell,m)$$

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17. a) Prove $\lim_{x\to 0} \frac{1}{x}$ does not exist.

Let $\epsilon>0$. Consider some $\delta>0$.

$$|x|<\delta \Rightarrow \left|\frac{1}{x}\right| = |f(x)| > \frac{1}{\delta} \Rightarrow \frac{1}{\delta} - |\ell| < |f(x)| - |\ell| \le |f(x)-\ell|$$

$\forall \epsilon>0, \exists \delta>0: \frac{1}{\delta} - |\ell| > \epsilon$. Any $\delta: \delta < \frac{1}{|\ell|+\epsilon}$ means $|f(x)-\ell| > \frac{1}{\delta} - |\ell| > \epsilon$

Therefore we have shown that

$\forall \epsilon>0$, there is always a $\delta>0: |x|<\delta \Rightarrow |f(x)-\ell|>\epsilon$

$\Rightarrow \lim_{x\to 0} \frac{1}{x} = \ell$ is false $\forall \ell$.

b) Prove $\lim_{x\to 1} \frac{1}{x-1}$ does not exist.

Let $y = x-1$

$|x-1|<\delta \Rightarrow |y|<\delta$

If $\lim_{x\to 1} \frac{1}{x-1} = \ell$ then

$$\underbrace{\left(\forall \epsilon>0\ \exists \delta>0: |x-1|<\delta \Rightarrow |f(x)-\ell|<\epsilon\right)}_{A}$$

$$\Rightarrow \underbrace{\left(|y|<\delta \Rightarrow \left|\frac{1}{y}-\ell\right|<\epsilon\right)}_{B} \Rightarrow \underbrace{\lim_{y\to 0} \frac{1}{y} = \ell}_{C}.$$

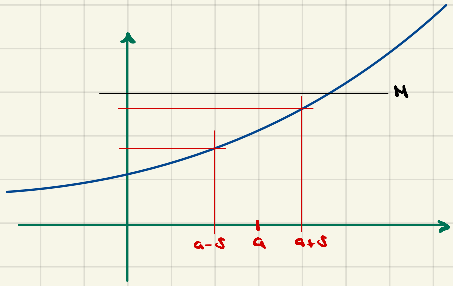
$A \Rightarrow B \Rightarrow C$

In a) we proved $\neg C$. Therefore $\neg C \Rightarrow \neg B \Rightarrow \neg A$

Ie, $\lim_{x\to 1} \frac{1}{x-1} = \ell$ is false $\forall \ell$. Therefore this limit does not exist.

18. Prove

$$\lim_{x \to a} f(x) = \ell \Rightarrow \exists \delta > 0, M : 0 < |x-a| < \delta \Rightarrow |f(x)| < M$$



In words: If x is close to a then f(x) is close to ℓ and we can find a number M larger than any f(x) in the interval that defines what "close to ℓ" means.

$\forall \epsilon > 0 \ \exists \delta > 0 : 0 < |x-a| < \delta \Rightarrow |f(x) - \ell| < \epsilon$

$\ell - \epsilon < f(x) < \ell + \epsilon$

Let $M = \max(|\ell - \epsilon|, |\ell + \epsilon|)$.

Then, $|f(x)| < M$

19. Prove

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q}' \\ 1 & x \in \mathbb{Q} \end{cases} \Rightarrow \lim_{x \to a} f(x) \text{ does not exist, } \forall a.$$

$\forall \delta > 0$ consider $x \in \mathbb{R} : |x-a| < \delta$.

$x \in \mathbb{Q}' \Rightarrow f(x) = 0 \Rightarrow |f(x) - \ell| = |\ell|$

$x \in \mathbb{Q} \Rightarrow f(x) = 1 \Rightarrow |f(x) - \ell| = |1 - \ell|$

$\Rightarrow$ value of ℓ that minimizes the maximum $|f(x) - \ell|$ is $\ell = 0.5$.

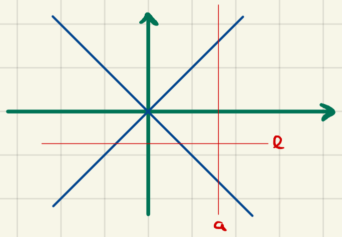
This minimum is 0.5. $\Rightarrow |f(x) - \ell| \geq 0.5$

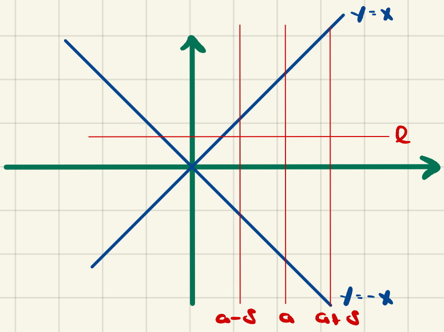
Therefore $\forall \epsilon < 0.5$, every choice of $\delta > 0$ leads to $|f(x) - \ell| \geq 0.5 > \epsilon$

$\Rightarrow \lim_{x \to a} f(x) = \ell$ is false $\forall \ell \in \mathbb{R}$.

20. Prove

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ -x & x \in \mathbb{Q}' \end{cases} \Rightarrow \lim_{x \to a} f(x) \text{ does not exist if } a \neq 0$$





$\forall \varepsilon > 0$, consider $\delta > 0$ and $x : |x - a| < \delta$

$x \in \mathbb{Q} \Rightarrow |f(x) - \ell| = |x - \ell|$

$x \in \mathbb{Q}' \Rightarrow |f(x) - \ell| = |-x - \ell| = |-(x + \ell)| = |x + \ell|$

Case 1: $a > 0, \ell > 0$

If $x \in \mathbb{R} : a - \delta < x < a + \delta,\ x \in \mathbb{Q}'$. Then $|f(x) - \ell| = |x + \ell|$

Let $x_1 \in \mathbb{Q}' : 0 < a < x_1 < a + \delta$.

Then $|f(x_1) - \ell| = |x_1 + \ell| > |a + \ell| \geq |a| = a$

Therefore in the case where $a > 0, \ell > 0, \forall \varepsilon : 0 < \varepsilon < a, \forall \delta > 0 : |x - a| < \delta$

$\Rightarrow \exists x : |f(x) - \ell| > a > \varepsilon$

$\Rightarrow \lim_{x \to a} f(x) = \ell$ is false

Case 2: $a > 0, \ell < 0$

Similar to case 1 but we look at an $x \in \mathbb{Q}$.

If $x \in \mathbb{R} : x \in (a - \delta, a + \delta),\ x \in \mathbb{Q}$ then $|f(x) - \ell| = |x - \ell| = |x + |\ell|| \geq |x|$

Let $x_1 \in \mathbb{R} : a < x_1 < a + \delta$

Then $|f(x_1) - \ell| \geq |x_1| > |a| = a$

Therefore

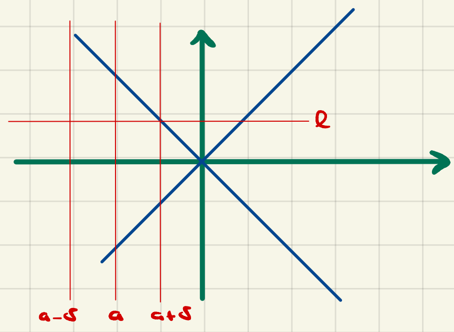
$a > 0, \ell < 0, \forall \varepsilon > 0 : 0 < \varepsilon < a, \forall \delta > 0 : |x - a| < \delta$

$\Rightarrow \exists x : |f(x) - \ell| > a > \varepsilon$

$\Rightarrow \lim_{x \to a} f(x) = \ell$ is false.

Case 3: $a<0, \ell>0$

If $x \in \mathbb{R}$: $a-\delta<x<a+\delta$, $x \in \mathbb{Q}$ then $|f(x)-\ell| = |x-\ell|$



Let $x_- \in \mathbb{Q}$: $a-\delta < x < a$

Then $|f(x_-)-\ell| = |x_- - \ell| > |a-\ell| \geq |a|$

Therefore in the case where $a<0, \ell>0$, $\forall \epsilon : 0<\epsilon<|a|$, $\forall \delta>0 : |x-a|<\delta$

$\Rightarrow \exists x : |f(x)-\ell| > |a| > \epsilon$

$\Rightarrow \lim_{x\to a} f(x) = \ell$ is false

Case 4: $a<0, \ell<0$

Analogous to case 3 but take $x_- \in \mathbb{Q}'$.

Case 5: $a=0, \ell=0$

Let $|x|<\delta$, $\delta>0$.

$x \in \mathbb{Q} \Rightarrow |f(x)-\ell| = |x| < \delta$
$x \in \mathbb{Q}' \Rightarrow |f(x)-\ell| = |x| < \delta$

Therefore, $\forall \epsilon>0$, $|x|<\epsilon \Rightarrow |f(x)|<\epsilon$

$\Rightarrow \lim_{x\to 0} f(x) = 0$

21. a) Prove $\lim_{x\to 0} g(x) = 0 \Rightarrow \lim_{x\to 0} g(x)\sin(\frac{1}{x}) = 0$

$-1 \le \sin(\frac{1}{x}) \le 1$

$-g(x) \le g(x)\sin(\frac{1}{x}) \le g(x)$, $\lim_{x\to 0}(-g(x)) = \lim_{x\to 0} g(x) \Rightarrow \lim_{x\to 0} g(x)\sin(\frac{1}{x}) = 0$

as proved in Problem 13

b) Generalization

$\lim_{x\to 0} g(x) = 0$

$\Rightarrow \lim_{x\to 0} g(x)h(x) = 0$

$|h(x)| \le M \;\; \forall x$

$-M \le h(x) \le M$

as proved in Problem 13

$-g(x)M \le g(x)h(x) \le g(x)M$, $\lim_{x\to 0}(-Mg(x)) = \lim_{x\to 0} Mg(x) = 0 \Rightarrow \lim_{x\to 0} g(x)h(x) = 0$

22. $\forall f\,[(\forall g(\nexists \lim_{x\to 0} g(x) \to \nexists \lim_{x\to 0}(f(x)+g(x)))) \leftrightarrow \exists \lim_{x\to 0} f(x)]$

contrapositive

Let f be an arbitrary fn.

Let's prove the contrapositive of $\exists \lim_{x\to 0} f(x) \to (\forall g(\nexists \lim_{x\to 0} g(x) \to \nexists \lim_{x\to 0}(f(x)+g(x))))$

Namely, $\sim(\forall g(\nexists \lim_{x\to 0} g(x) \to \nexists \lim_{x\to 0}(f(x)+g(x)))) \to \nexists \lim_{x\to 0} f(x)$

Assume $\sim(\forall g(\nexists \lim_{x\to 0} g(x) \to \nexists \lim_{x\to 0}(f(x)+g(x))))$

Then $\exists g \sim(\nexists \lim_{x\to 0} g(x) \to \nexists \lim_{x\to 0}(f(x)+g(x)))$

Then $\exists g$, $\nexists \lim_{x\to 0} g(x) \wedge \exists \lim_{x\to 0}(f(x)+g(x)))$

Assume $\lim_{x\to 0} f(x)$ exists.

In problem 8c we showed that $\exists \lim_{x\to a} f(x) \wedge \nexists \lim_{x\to a} g(x) \to \nexists \lim_{x\to 0}(f(x)+g(x))$

Therefore $\nexists \lim_{x\to 0}(f(x)+g(x))$. $\perp$.

Therefore, $\nexists \lim_{x\to 0} f(x)$.

Therefore $\sim(\forall g(\nexists \lim_{x\to 0} g(x) \to \nexists \lim_{x\to 0}(f(x)+g(x)))) \to \nexists \lim_{x\to 0} f(x)$

So, $\exists \lim_{x\to 0} f(x) \to (\forall g(\nexists \lim_{x\to 0} g(x) \to \nexists \lim_{x\to 0}(f(x)+g(x))))$ by the law of contrapositive.

For the converse, we also prove the contrapositive of $\forall g(\nexists \lim_{x\to 0} g(x) \to \nexists \lim_{x\to 0}(f(x)+g(x))) \leftrightarrow \exists \lim_{x\to 0} f(x)$

Namely, $\nexists \lim_{x\to 0} f(x) \to \sim(\forall g(\nexists \lim_{x\to 0} g(x) \to \nexists \lim_{x\to 0}(f(x)+g(x))))$

Assume $\nexists \lim_{x \to 0} f(x)$

As we showed in Problem 8a, if $\nexists \lim_{x \to a} f(x) \wedge \nexists \lim_{x \to a} g(x)$ then $\lim_{x \to a} (f(x) + g(x))$ can exist.

For example, if $g(x) = -f(x)$ then $\nexists \lim_{x \to 0} g(x)$, $(f+g)(x) = 0$, and $\lim_{x \to 0} (f+g)(x) = 0$.

Therefore $\exists g$, $\nexists \lim_{x \to 0} g(x) \wedge \exists \lim_{x \to 0} (f(x) + g(x)))$

Therefore $\nexists \lim_{x \to 0} f(x) \to \neg(\forall g(\nexists \lim_{x \to 0} g(x) \to \nexists \lim_{x \to 0} (f(x) + g(x))))$

And by Law of Contraposition

$$\forall g(\nexists \lim_{x \to 0} g(x) \to \nexists \lim_{x \to 0} (f(x) + g(x))) \quad \to \exists \lim_{x \to 0} f(x)$$

Hence we can assert that

$$\forall g(\nexists \lim_{x \to 0} g(x) \to \nexists \lim_{x \to 0} (f(x) + g(x))) \quad \leftrightarrow \exists \lim_{x \to 0} f(x)$$

And by universal intro,

$$\forall f(\forall g(\nexists \lim_{x \to 0} g(x) \to \nexists \lim_{x \to 0} (f(x) + g(x))) \quad \leftrightarrow \exists \lim_{x \to 0} f(x))$$

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23. $\forall f\left[\left(\forall g\left(\nexists \lim_{x\to 0} g(x) \rightarrow \nexists \lim_{x\to 0}(f(x)\cdot g(x))\right)\right) \leftrightarrow \exists \lim_{x\to 0} f(x)\right]$

$\forall \epsilon>0\ \exists \delta>0\ \forall x\ |x-a|<\delta \rightarrow |f(x)-L|<\epsilon$

$\exists \epsilon>0\ \neg(\exists \delta>0\ \forall x\ |x-a|<\delta \rightarrow |f(x)-L|<\epsilon)$

$\exists \epsilon>0\ \forall \delta>0\ \exists x\ |x-a|<\delta \wedge |f(x)-L|\geq\epsilon$

a)

Let f be any fn. $\forall$Intro Ass

Assume $\exists \lim_{x\to 0} f(x) \neq 0$. Let $\lim_{x\to 0} f(x) = L_f$ $\rightarrow$Intro Ass

Let g be any fn such that $\exists \lim_{x\to 0} f(x)g(x)$ and let $L = \lim_{x\to 0} f(x)g(x)$. $\forall$Intro Ass

Since $f(x)\neq 0$ we can write $g = \frac{fg}{f}$. → this part is still too imprecise.

$\exists \lim f \neq 0$ then $\exists \lim \frac{1}{f}$.

Then $\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{1}{f(x)} f(x)\cdot g(x) = \frac{1}{L_f}\cdot L$ by Th2(2).

$\exists \lim_{x\to 0} g(x)$

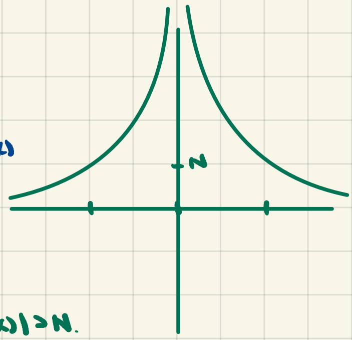
$\forall g(\exists \lim_{x\to 0} f(x)g(x) \rightarrow \exists \lim_{x\to 0} g(x))$ $\forall$Intro

By Law of Contrapositive, $\forall g(\nexists \lim_{x\to 0} g(x) \rightarrow \nexists \lim_{x\to 0}(f(x)\cdot g(x)))$

Therefore $\exists \lim_{x\to 0} f(x) \neq 0 \rightarrow \forall g(\nexists \lim_{x\to 0} g(x) \rightarrow \nexists \lim_{x\to 0}(f(x)\cdot g(x)))$ $\rightarrow$Intro

$\forall f(\exists \lim_{x\to 0} f(x) \neq 0 \rightarrow \forall g(\nexists \lim_{x\to 0} g(x) \rightarrow \nexists \lim_{x\to 0}(f(x)\cdot g(x))))$ $\forall$Intro

b) $\lim_{x\to 0} |f(x)| = \infty$



Proposition: $\lim_{x\to 0} |f(x)| = \infty \rightarrow \nexists \lim_{x\to 0} f(x)$

Assume $\lim_{x\to 0} |f(x)| = \infty$.

Then $\forall N\in\mathbb{R}\ \exists\delta>0 : \forall x\ 0<|x|<\delta \rightarrow |f(x)|>N$.

Let N_1 be one such N.

Choose a ϵ_1 such that $0<\epsilon_1\leq N_1$.

There is a $\delta_1>0$ such that $\forall x\ 0<|x|<\delta_1 \rightarrow |f(x)|>N_1$

For any $\delta<\delta_1$, it is also true that $0<|x|<\delta \rightarrow |f(x)|>N_1$.

For any $\delta>\delta_1$, we have $0<|x|<\delta_1<\delta$, so $\exists x : 0<|x|<\delta$ such that $|f(x)|>N_1$

Therefore $\forall\delta>0\ \exists x : 0<|x|<\delta \wedge |f(x)|>N_1\geq\epsilon_1$

Therefore $\nexists \lim_{x\to 0} f(x)$ $\square$

If we assume $\lim_{x\to 0}|f(x)|=\infty$, then we assume $\nexists \lim_{x\to 0} f(x)$. If we can prove $\neg\left(\forall g\left(\nexists \lim_{x\to 0} g(x) \rightarrow \nexists \lim_{x\to 0}(f(x)\cdot g(x))\right)\right)$ then we will have proved the other direction in the original biconditional.

Proposition: $\lim_{x\to 0} |f(x)| = \infty \rightarrow \lim_{x\to 0} \frac{1}{f(x)} = 0$

Assume $\lim_{x\to 0} |f(x)| = \infty$.

Then $\forall N>0\ \exists\delta>0 : \forall x\ 0<|x|<\delta \rightarrow |f(x)|>N$.

Let $\epsilon>0$.

Let $N_2 = \frac{1}{\epsilon}$.

Choose N_1 such that $0<N_2<N_1$.

Then,

$|f(x)|>N_1 \rightarrow \frac{1}{|f(x)|} < \frac{1}{N_1}$

$\forall x\ 0<|x|<\delta \rightarrow \left|\frac{1}{f(x)}\right| < \frac{1}{N_1} < \frac{1}{N_2} = \epsilon$

Therefore

$\forall\epsilon>0\ \exists\delta>0\ \forall x\ 0<|x|<\delta \rightarrow \left|\frac{1}{f(x)}\right| < \frac{1}{N_1} < \frac{1}{N}$

Therefore $\lim_{x\to 0} \frac{1}{f(x)} = 0$

Therefore $\lim_{x\to 0} |f(x)| = \infty \rightarrow \lim_{x\to 0} \frac{1}{f(x)} = 0$

Let f be any fn. ∀Intro Ass

Assume $\lim\limits_{x \to 0} |f(x)| = \infty$

Let g be any fn such that $\exists \lim\limits_{x \to 0} f(x)g(x)$

$$g = \frac{fg}{f}$$

we've shown that $\lim\limits_{x \to 0} |f(x)| = \infty \to \lim\limits_{x \to 0} \frac{1}{f(x)} = 0$

Therefore $\lim\limits_{x \to 0} \frac{1}{f(x)} = 0$.

Since $\exists \lim\limits_{x \to 0} f(x)g(x)$ by assumption, then $\lim\limits_{x \to 0} g(x) = 0 \cdot \lim fg = 0$

Therefore $\forall g \left(\exists \lim\limits_{x \to 0} f(x)g(x) \to \exists \lim\limits_{x \to 0} g(x) \right)$

By contraposition $\forall g \left(\nexists \lim g(x) \to \nexists \lim f(x)g(x) \right)$

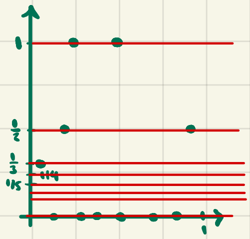
Therefore $\lim\limits_{x \to 0} |f(x)| = \infty \to \forall g \left(\nexists \lim\limits_{x \to 0} g(x) \to \nexists \lim\limits_{x \to 0} f(x)g(x) \right)$

c) $\neg\left(\exists \lim\limits_{x \to 0} f(x) \neq 0\right) \wedge \neg\left(\lim\limits_{x \to 0} |f(x)| = \infty\right) \longrightarrow \exists g.\ \nexists \lim\limits_{x \to 0} g(x) \wedge \exists \lim\limits_{x \to 0} f(x)g(x)$

$\neg\left(\exists \lim f(x) \wedge \lim f(x) \neq 0\right)$

$\nexists \lim f(x) \vee \lim f(x) = 0$

24. $n \in \mathbb{N}$



$A_1 = \{1/4, 1/2\}$
$A_2 = \{3/16, 9/10\}$
$A_3 = \{1/1000\}$

A_n = some finite set of numbers in $[0,1]$

A_n and A_m have no members in common if $m \neq n$

$$f(x) = \begin{cases} \frac{1}{n} & x \in A_n \\ 0 & \forall n \; x \notin A_n \end{cases}$$

Prove $\lim_{x \to a} f(x) = 0$ for all $a \in [0,1]$

Proof

Let a be one such $a \in [0,1]$.

Let $\epsilon > 0$.

Then $\exists n_1, n_2 \in \mathbb{N}$: $\frac{1}{n_1} < \epsilon < \frac{1}{n_2} \iff n_2 > n_1$

Let $B = \{x \in \mathbb{N} : x < n_1\}$, a finite set of natural numbers.

Hence the set $C = \{x : x \in A_n, n \in B\}$ is finite.

There is a minimum distance from the elements in C to a. Call it d_{min_C}.

Then, $x \in (a - d_{min_C}, a + d_{min_C}) \to x \notin C \to x \in A_n, n \geq n_1 \to f(x) \leq \frac{1}{n_1} < \epsilon$

Therefore $\forall \epsilon > 0 \; \exists \delta \; \forall x \; 0 < |x - a| < \delta \to f(x) < \frac{1}{n_1} < \epsilon$.

$\to \lim_{x \to a} f(x) = 0$

25. Each definition implies the standard def of limit. Hence if f applies the nonstandard defs, then the standard def is also true. The standard def is necessary for the nonstandard def.

i) $\forall \delta > 0\ \exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \varepsilon \rightarrow |f(x) - l| < \delta$

standard definition of $\lim_{x \to a} f(x) = l$.

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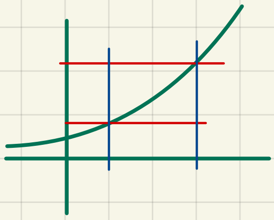
ii) $\forall \delta > 0\ \exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \varepsilon \rightarrow |f(x) - l| \leq \delta$



Let $\delta_1 > 0$.

Let $0 < \delta_2 < \delta_1$

Then $\exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \varepsilon \rightarrow |f(x) - l| \leq \delta_2 < \delta_1$

Therefore

$\forall \delta > 0\ \exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \varepsilon \rightarrow |f(x) - l| < \delta$

Therefore

$(\forall \delta > 0\ \exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \varepsilon \rightarrow |f(x) - l| \leq \delta) \rightarrow (\forall \delta > 0\ \exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \varepsilon \rightarrow |f(x) - l| < \delta)$

$(\forall \delta > 0\ \exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \varepsilon \rightarrow |f(x) - l| \leq \delta) \rightarrow \lim_{x \to a} f(x) = l$

---

iii) $\forall \delta > 0\ \exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \varepsilon \rightarrow |f(x) - l| < 5\delta$

Let $\delta_1 > 0$.

Let $\delta = \frac{\delta_1}{5}$.

Then $\exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \varepsilon \rightarrow |f(x) - l| < 5\delta = \delta_1$

$\forall \delta > 0\ \exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \varepsilon \rightarrow |f(x) - l| < \delta$

Therefore,

$(\forall \delta > 0\ \exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \varepsilon \rightarrow |f(x) - l| < \delta) \rightarrow (\forall \delta > 0\ \exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \varepsilon \rightarrow |f(x) - l| < \delta)$

$(\forall \delta > 0\ \exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \varepsilon \rightarrow |f(x) - l| < \delta) \rightarrow \lim_{x \to a} f(x) = l$

---

iv) $\forall \delta > 0\ \exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \frac{\varepsilon}{10} \rightarrow |f(x) - l| < \delta$

Let $\delta > 0$.

Let $\varepsilon > 0$ be such that $\forall x \quad 0 < |x-a| < \frac{\varepsilon}{10} \rightarrow |f(x) - l| < \delta$

Let $\varepsilon_1 = \frac{\varepsilon}{10}$.

Then $\forall x \quad 0 < |x-a| < \varepsilon_1 \rightarrow |f(x) - l| < \delta$

$\exists \varepsilon_1 > 0 : \forall x \quad 0 < |x-a| < \varepsilon_1 \rightarrow |f(x) - l| < \delta$

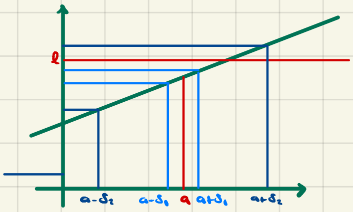
$\forall \delta > 0\ \exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \varepsilon \rightarrow |f(x) - l| < \delta$

$(\forall \delta > 0\ \exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \frac{\varepsilon}{10} \rightarrow |f(x) - l| < \delta) \rightarrow (\forall \delta > 0\ \exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \varepsilon \rightarrow |f(x) - l| < \delta)$

$(\forall \delta > 0\ \exists \varepsilon > 0 : \forall x \quad 0 < |x-a| < \frac{\varepsilon}{10} \rightarrow |f(x) - l| < \delta) \rightarrow \lim_{x \to a} f(x) = l$

26.

a) $\forall \delta > 0 \ \exists \varepsilon > 0 \quad 0 < |x-a| < \delta \to |f(x) - \ell| < \varepsilon$



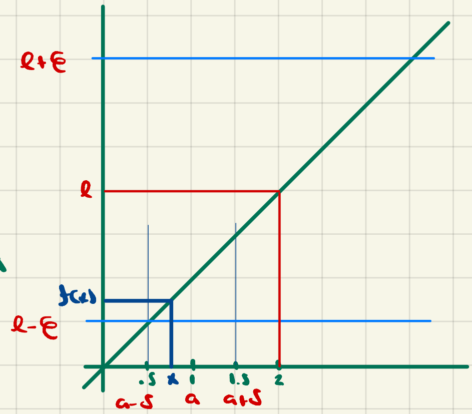
$0 < |x-a| < \delta_1 \to |f(x) - \ell| < \ell - f(a-\delta_1) = \varepsilon_1$

Here we see that $\lim_{x\to a} f(x) \neq \ell$, yet for any interval we can choose an ε to bind the difference between $f(x)$ for x in the interval and the supposed limit. In fact this definition of limit is satisfied by all numbers.

example:

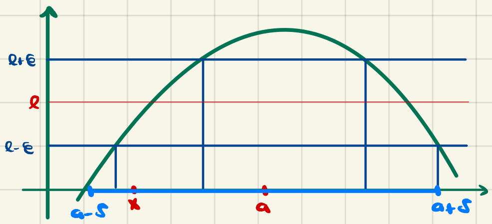
$f(x) = x$

$0 < |x-1| < 0.5 \to |f(x) - 2| < 1.5$



For any interval containing $x = a$, we can find an interval containing $f(a)$ and a number ℓ.
Given any a, this is true for any ℓ.

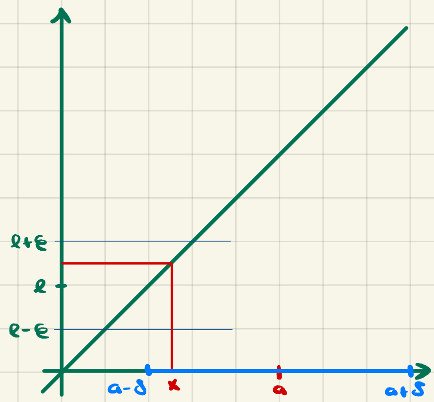
b) $\forall \varepsilon > 0 \ \exists \delta > 0 \quad |f(x) - \ell| < \varepsilon \to 0 < |x-a| < \delta$



Given an ℓ, the statement is true for any a.

example:

$f(x) = x$



For any interval containing $f(x)$ and some ℓ, we can find an interval containing x and some a.
Given ℓ this is true for all a.

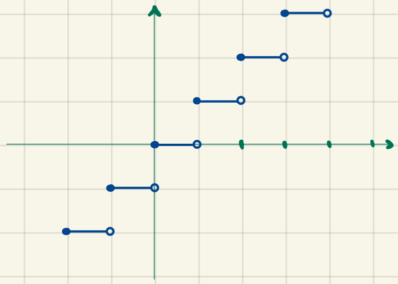
27. $[x]$ = largest $\mathbb{Z} \le x$

Limits From Above/Below

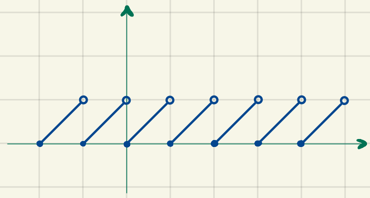
$\lim_{x \to a^+} f(x) = \ell$ or $\lim_{x \to \downarrow a} f(x) = \ell$

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < x - a < \delta \Rightarrow |f(x) - \ell| < \varepsilon$

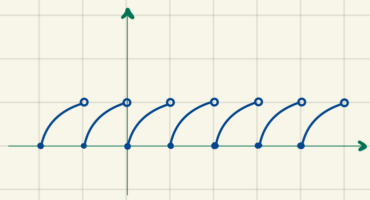
i) $f(x) = [x]$



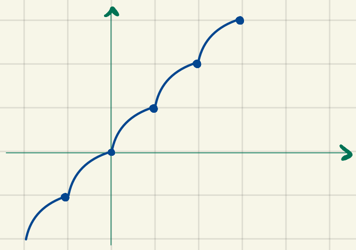
ii) $f(x) = x - [x]$



iii) $f(x) = \sqrt{x - [x]}$

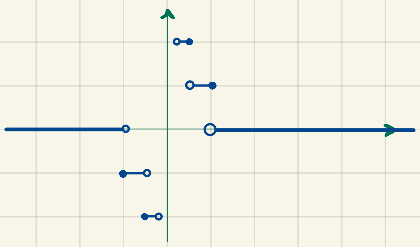


iv) $f(x) = [x] + \sqrt{x - [x]}$



For i), ii), iii), iv) $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist for all $x \in \mathbb{R}$.

v) $f(x) = \left[\frac{1}{x}\right]$

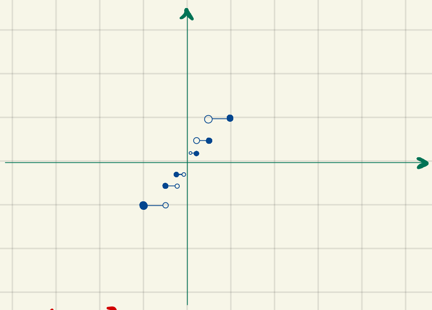


As x approaches 0^+, $f(x)$ approaches ∞.

As x approaches 0^-, $f(x)$ approaches $-\infty$.

Neither $\lim_{x \to 0^+} f(x)$ nor $\lim_{x \to 0^-} f(x)$ exist.

vi) $f(x) = \dfrac{1}{\left[\frac{1}{x}\right]}$



$\lim_{x \to a^-} f(x)$ exists for $\forall x \in (-1, 1]$

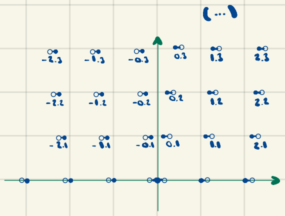
$\lim_{x \to a^+} f(x)$ exists for $\forall x \in [-1, 1)$

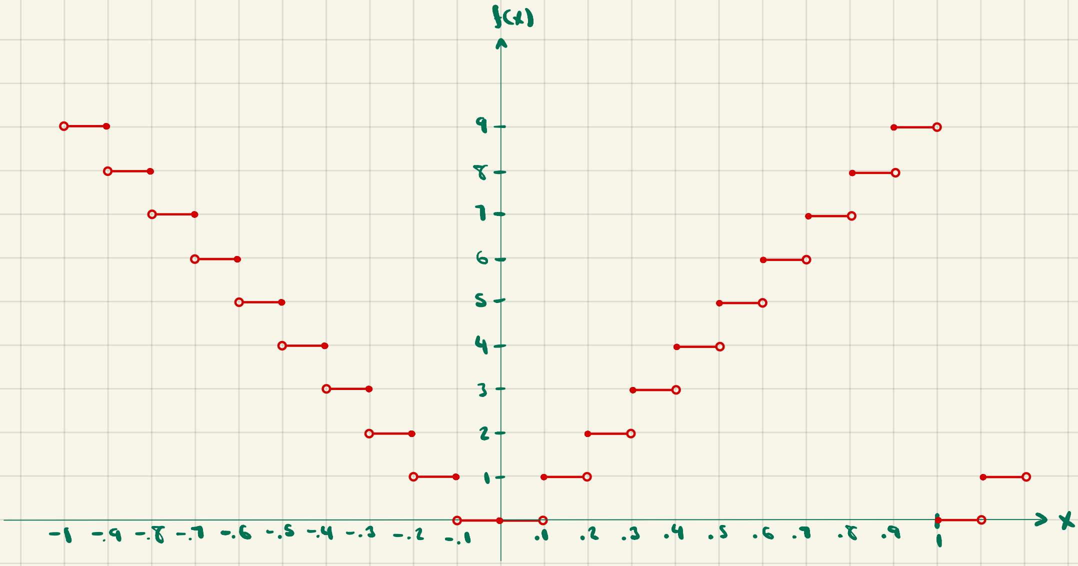
$f(x)$ isn't defined for $x > 1$ or $x < 1$.

28.

a)

i) $f(x)$ = 1st number of decimal expansion of x

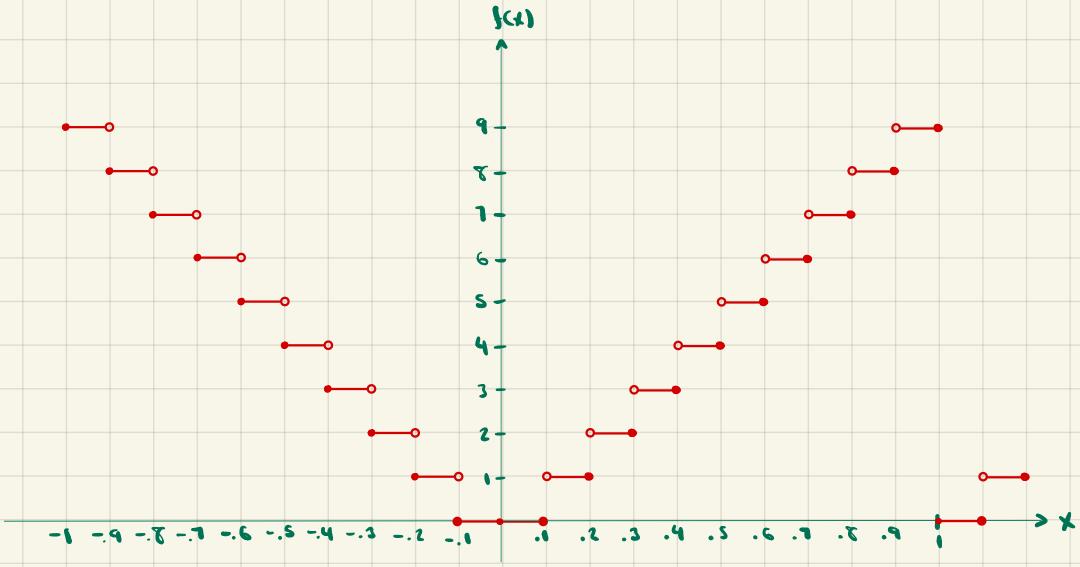




However, we are considering that, e.g., $0.0\bar{9} = 1$.

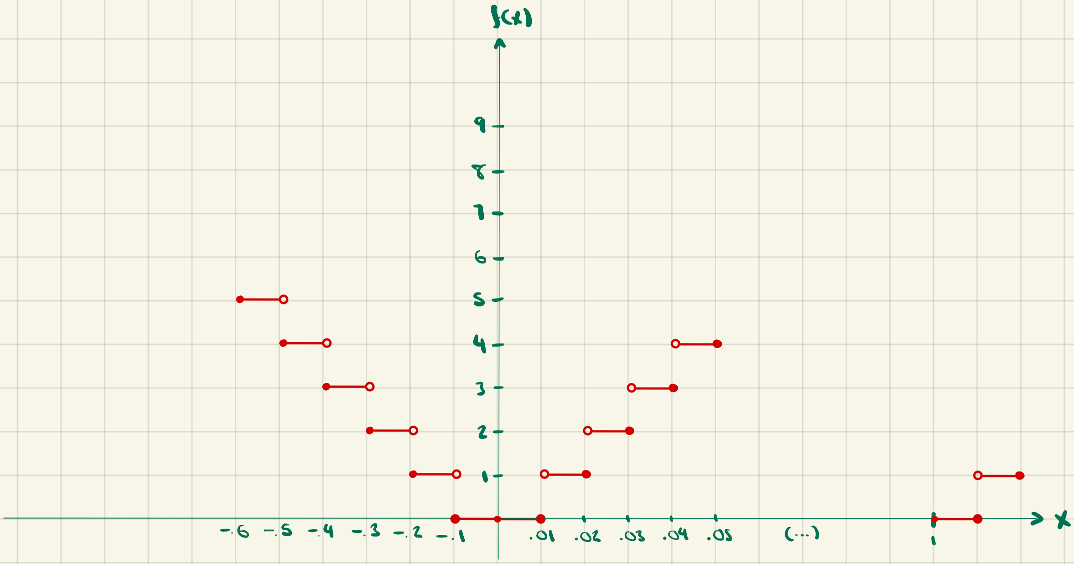
$\Rightarrow f(0.1) = f(0.0\bar{9}) = 0$

$f(0.2) = f(0.1\bar{9}) = 1$



Both $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist for all $x \in \mathbb{R}$.

ii) $f(x) = 2^{nd}$ number in decimal expansion of x



Both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist for all $x \in \mathbb{R}$.

---

iii) $f(x)$ = number of 7's in decimal expansion of x if this number is finite,
0 otherwise

Infinite 7's

irrational numbers w/ infinite 7's

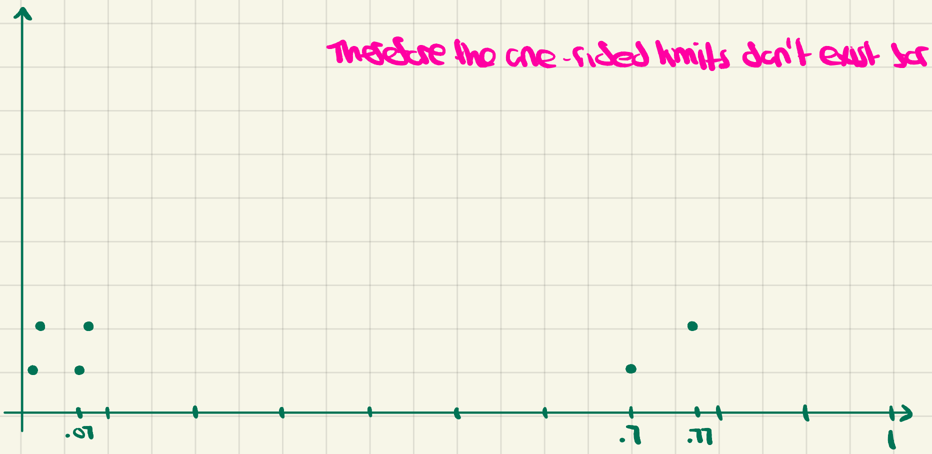
rational w/ repeating decimal repres. e.g. $0.\overline{7}$, $0.\overline{74}$, $0.\overline{347}$

For any number a, for any $\delta > 0$ consider the set $A = [a-\delta, a+\delta]$. There are infinitely many numbers with a finite number of 7s in A and an infinite number of numbers w/ infinite 7s. Hence for any $0 < \varepsilon < 1$ the definition of limit won't hold for a.

Therefore the one-sided limits don't exist for any a.

Finite 7's

0.7	0.77
0.07	0.077
0.007	0.0077

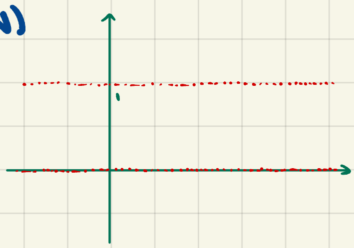


In any interval there are infinite numbers w/ any number of 7's, from zero to ∞.

$.\overline{72} \Rightarrow 10^2 \cdot .\overline{72} = 72 + .\overline{72} \Rightarrow 72 = .\overline{72}(10^2 - 1)$

$\Rightarrow .\overline{72} = \frac{72}{99}$

iv) $f(x) = 0$ if number of 7's in dec. exp. of x is finite, 1 otherwise.

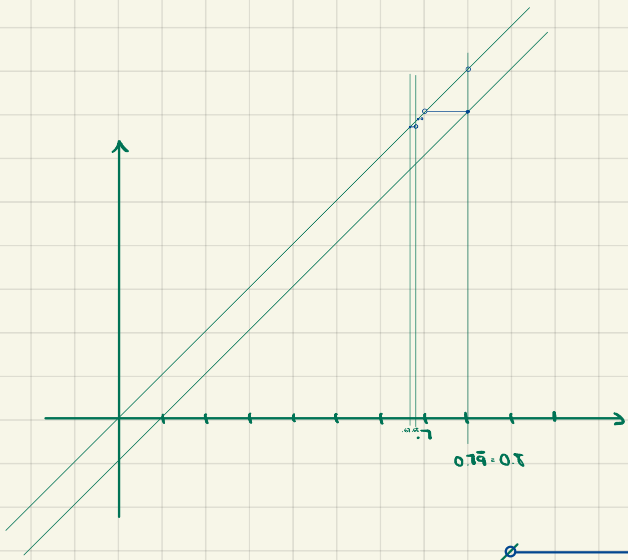


$\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ do not exist for same reason as iii).

v) $f(x)$ = number obtained by replacing all digits in decimal expansion of x which come after first 7, if any, by 0.

$f(0.77) = 0.7$
$f(0.07) = 0.07$
$f(0.\overline{72}) = 0.7$
$f(0.7\overline{9}) = 0.7$

$0.0\overline{9}$ is max subtraction.



Given any number of form m.abcd7ef...

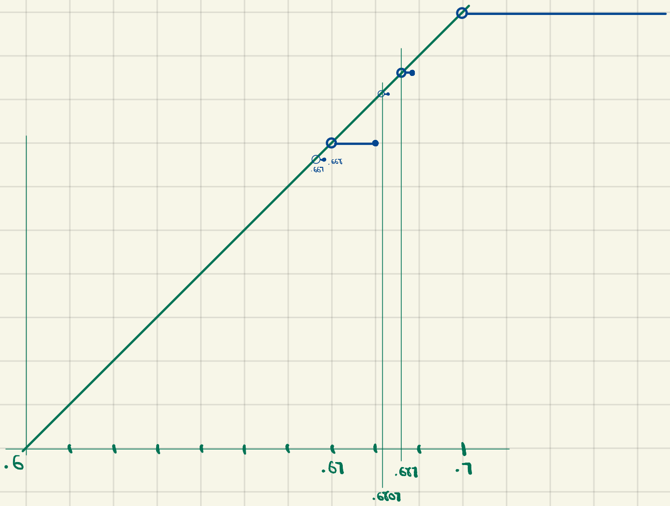
$\lim\limits_{x\to a^+} f(x)$ exists for any $x \in$ (m.abcd7, m.abcd8]

$\lim\limits_{x\to a^-} f(x)$ exists for any $x \in$ [m.abcd7, m.abcd8)

The numbers m.abcd7 and m.abcd8 are actually m.abcd6$\overline{9}$ and m.abcd7$\overline{9}$, respectively. They have left and right-sided limits, respectively.

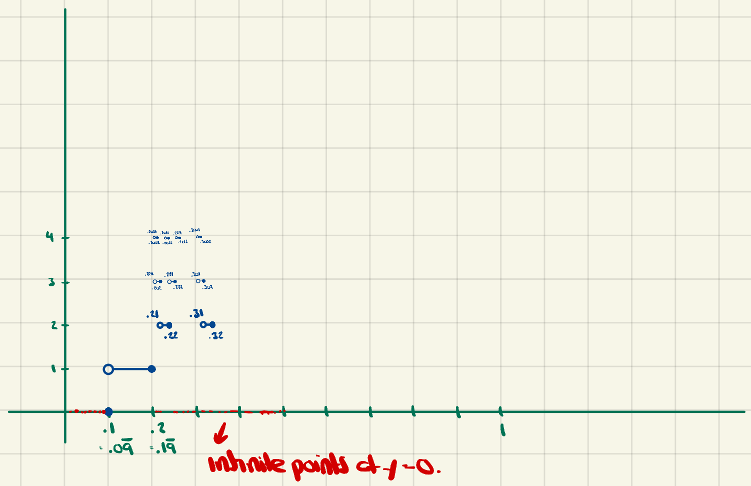
For any number a not containing any 7s in decimal expansion, both one-sided limits exist and equal a.

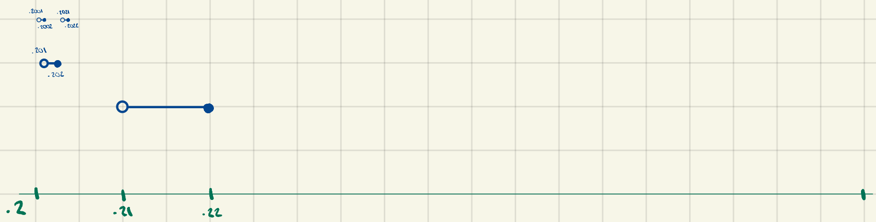
Therefore both one-sided limits exist for all x.

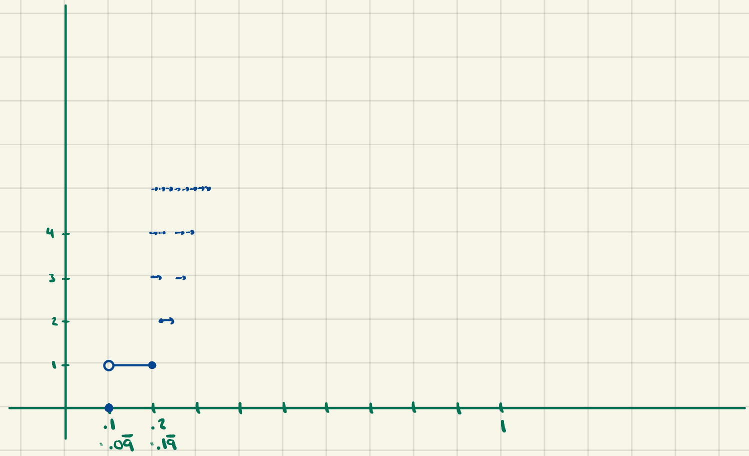


F is composed of infinite intervals like the ones above, with points on the line $y = x$ for numbers x with no 7s in them.

vi) $f(x) =$ 0 if 1 never appears in decimal expansion of x
n if 1 first appears in n^{th} place







The graph of $f(x)$ is composed of infinite intervals as above. They are of form $(m.abc1, m.abc2]$. Between two such intervals are infinite other intervals with $f(x)$ at every number from 0 to ∞.

Therefore the one-sided limits exist for every point in each interval, and the left open side has only the $\lim_{x \to a^+} f(x)$ and the right closed side has $\lim_{x \to a^-} f(x)$.

The limits do not exist for points w/ no 7s, ie the ones on the $y=0$ line.

29. Prove $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) \to \exists \lim_{x\to a} f(x)$

Assume $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \ell$

Then from the definitions of these limits:

$\forall \epsilon>0\ \exists \delta_1>0 : \forall x\ 0<x-a<\delta_1 \to |f(x)-\ell|<\epsilon$

$\forall \epsilon>0\ \exists \delta_2>0 : \forall x\ 0<a-x<\delta_2 \to |f(x)-\ell|<\epsilon$

Let $\epsilon>0$, and δ_1, δ_2 such that

$\delta_1>0 \wedge \forall x\ 0<x-a<\delta_1 \to |f(x)-\ell|<\epsilon$

$\delta_2>0 \wedge \forall x\ 0<a-x<\delta_2 \to |f(x)-\ell|<\epsilon$

Let $\delta_{min} = \min(\delta_1, \delta_2)$

Then, $\delta_{min}>0$ and $\forall x\ 0<x-a<\delta_{min} \to |f(x)-\ell|<\epsilon$ and $\forall x\ -\delta_{min}<x-a<0 \to |f(x)-\ell|<\epsilon$

That is $\delta_{min}>0 \wedge \forall x\ (((0<x-a<\delta_{min}) \vee (-\delta_{min}<x-a<0)) \to |f(x)-\ell|<\epsilon)$

Equivalently, $\delta_{min}>0 \wedge \forall x\ (0<|x-a|<\delta_{min} \to |f(x)-\ell|<\epsilon)$

Therefore, $\forall \epsilon>0\ \exists \delta>0\ \forall x\ (0<|x-a|<\delta \to |f(x)-\ell|<\epsilon)$

Therefore $\lim_{x\to a} f(x) = \ell$

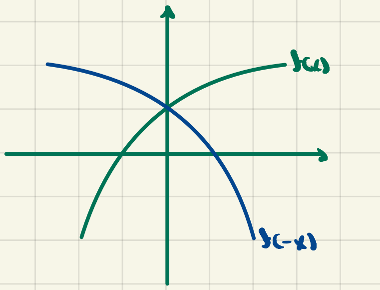
By conditional proof we've shown that $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) \to \exists \lim_{x\to a} f(x)$

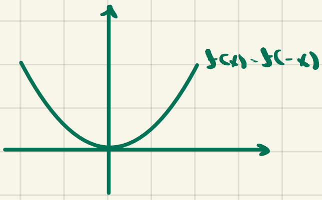
30.

i) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(-x)$

From the def of $\lim_{x\to 0^+} f(x)$ $\quad \forall \epsilon>0\ \exists \delta_1>0\ \ 0<x<\delta_1 \to |f(x)-\ell_1|<\epsilon$

From the def of $\lim_{x\to 0^-} f(-x)$ $\quad \forall \epsilon>0\ \exists \delta_2>0\ \ 0<-x<\delta_2 \to |f(-x)-\ell_2|<\epsilon$





let $\epsilon>0$ $\forall$Intro Ass

let δ_1, δ_2 such that $\delta_1>0\ \ 0<x<\delta_1 \to |f(x)-\ell_1|<\epsilon$ $\quad$ $\exists$Elim Ass

$\delta_2>0\ \ 0<-x<\delta_2 \to |f(-x)-\ell_2|<\epsilon$

let $\delta = \min(\delta_1, \delta_2)$

Then,

$\forall x\ 0<x<\delta \to |f(x)-\ell_1|<\epsilon$

$\forall x\ -\delta<x<0 \to |f(-x)-\ell_2|<\epsilon$

$0<-x<\delta$

The latter is equivalent to

$\forall x\ 0<x<\delta \to |f(x)-\ell_2|<\epsilon$

Assume $\ell_2>\ell_1$

Let $\epsilon = \frac{\ell_2-\ell_1}{2}$. Then $0<x<\delta \to |f(x)-\ell_1|<\frac{\ell_2-\ell_1}{2} \wedge |f(x)-\ell_2|<\frac{\ell_2-\ell_1}{2}$

$\frac{\ell_1-\ell_2}{2} < f(x)-\ell_1 < \frac{\ell_2-\ell_1}{2}$

$\frac{3\ell_1-\ell_2}{2} < f(x) < \frac{\ell_1+\ell_2}{2}$

Also $\frac{\ell_1-\ell_2}{2} < f(x)-\ell_2 < \frac{\ell_2-\ell_1}{2}$

$\frac{\ell_1+\ell_2}{2} < f(x) < \frac{3\ell_2-\ell_1}{2}$

Therefore, $f(x) < \frac{\ell_1+\ell_2}{2} < f(x)$. $\perp$.

Therefore $\ell_2 \le \ell_1$.

By symmetry, $\ell_2<\ell_1 \to \perp$ $\quad$ Therefore $\ell_1=\ell_2$.

At this point we have that $\exists \delta>0\ (\forall x\ 0<x<\delta \to |f(x)-\ell_1|<\epsilon) \wedge (\forall x\ -\delta<x<0 \to |f(-x)-\ell_2|<\epsilon) \wedge (\ell_1=\ell_2)$

Hence, $\exists \delta>0\ (\forall x\ 0<x<\delta \to |f(x)-\ell|<\epsilon) \wedge (\forall x\ -\delta<x<0 \to |f(-x)-\ell|<\epsilon)$

$\forall$Intro $\forall \epsilon>0\ \exists \delta>0\ (\forall x\ 0<x<\delta \to |f(x)-\ell|<\epsilon) \wedge (\forall x\ -\delta<x<0 \to |f(-x)-\ell|<\epsilon)$

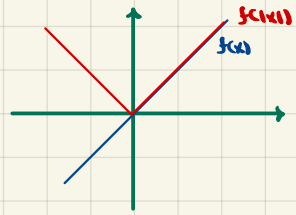
$\lim_{x\to 0^+} f(x) = \ell \quad \wedge \quad \lim_{x\to 0^-} f(-x) = \ell$

Therefore $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(-x)$

ii) $\lim_{x \to 0} f(|x|) = \lim_{x \to 0^+} f(x)$

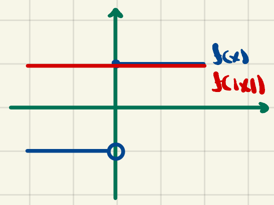
Proof

Let $\lim_{x \to 0^+} f(x) = \ell$.



$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(-x)$ by i).

$\lim_{x \to 0^-} f(-x) = \ell$ means $\forall \epsilon > 0 \ \exists \delta \ 0 < -x < \delta \to |f(-x) - \ell| < \epsilon$



For x in $-\delta < x < 0$, $|x| = -x$. Therefore, $\lim_{x \to 0^-} f(|x|) = \ell$.

Since $\lim_{x \to 0^-} f(|x|) = \ell$, if $\lim_{x \to 0} f(|x|)$ exists then $\lim_{x \to 0^+} f(|x|) = \ell$.

We can also see this by considering that $\lim_{x \to 0^+} f(|x|) = \lim_{x \to 0^+} f(x)$ since these limits consider values of x in the interval $0 < x < \delta$, in which $f(|x|) = f(x)$.

That is, for $0 \leq x < \delta$ we know that $\forall \epsilon > 0 \ \exists \delta \ \forall x \ 0 \leq x < \delta \to |f(x) - \ell| < \epsilon$.

But this is equivalent to $\forall \epsilon > 0 \ \exists \delta \ \forall x \ 0 \leq x < \delta \to |f(|x|) - \ell| < \epsilon$

That is, $\lim_{x \to 0^+} f(|x|) = \ell$.

Since $\lim_{x \to 0^-} f(|x|) = \lim_{x \to 0^+} f(|x|) = \ell$ we conclude that $\lim_{x \to 0} f(|x|) = \ell$.

iii) $\lim_{x\to 0} f(x^2) = \lim_{x\to 0^+} f(x)$

Proof

If $\lim_{x\to 0} f(x^2)$ exists then it is defined by ℓ such that

$\forall \varepsilon>0\ \exists \delta>0\ \forall x\ \ 0<|x|<\delta \rightarrow |f(x^2)-\ell|<\varepsilon$

Let $\delta_{min} = \min(1,\delta)$

Then, $\forall \varepsilon>0\ \exists \delta\ \forall x$ we have

$0<|x|<\min(1,\delta) \rightarrow |f(x^2)-\ell|<\varepsilon$

$0<x^2<x<\sqrt{x} \leq \delta_{min} \rightarrow |f(x^2)-\ell|<\varepsilon$

Since $\sqrt{x}$ is within the interval $(0,\delta_{min})$ it follows that $|f(\sqrt{x}^2)-\ell|<\varepsilon$, ie $|f(x)-\ell|<\varepsilon$.

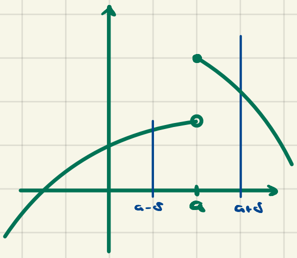
We have shown that

$\forall \varepsilon>0\ \exists \delta_1\ \forall x\ \ 0<x<\delta_1 \rightarrow |f(x)-\ell|<\varepsilon$.

This δ_1 is $\min(1,\delta)$ where δ is the delta from the limit $\lim_{x\to 0} f(x^2)$

31. $\lim_{x \to a^-} f(x) < \lim_{x \to a^+} f(x)$

Prove: $\exists \delta > 0 : \forall x \forall y \left(\begin{matrix} x < a < y \\ |x-a| < \delta \\ |y-a| < \delta \end{matrix} \rightarrow f(x) < f(y) \right)$



$\forall \varepsilon \exists \delta \forall x \; 0 < a-x < \delta \rightarrow |f(x) - \ell_-| < \varepsilon$

$\forall \varepsilon \exists \delta \forall y \; 0 < y-a < \delta \rightarrow |f(y) - \ell_+| < \varepsilon$

Let $\varepsilon = \dfrac{\ell_+ - \ell_-}{2}$

$|f(x) - \ell_-| < \dfrac{\ell_+ - \ell_-}{2} \rightarrow \dfrac{\ell_- - \ell_+}{2} < f(x) - \ell_- < \dfrac{\ell_+ - \ell_-}{2} \rightarrow \dfrac{3\ell_- - \ell_+}{2} < f(x) < \dfrac{\ell_- + \ell_+}{2}$

$|f(y) - \ell_+| < \dfrac{\ell_+ - \ell_-}{2} \rightarrow \dfrac{\ell_- - \ell_+}{2} < f(x) - \ell_+ < \dfrac{\ell_+ - \ell_-}{2} \rightarrow \dfrac{\ell_- + \ell_+}{2} < f(x) < \dfrac{3\ell_+ - \ell_-}{2}$

Hence, for $\varepsilon = \dfrac{\ell_+ - \ell_-}{2}$

$$\forall x \left(0 < a-x < \delta_1 \rightarrow \frac{3\ell_- - \ell_+}{2} < f(x) < \frac{\ell_- + \ell_+}{2} \right)$$

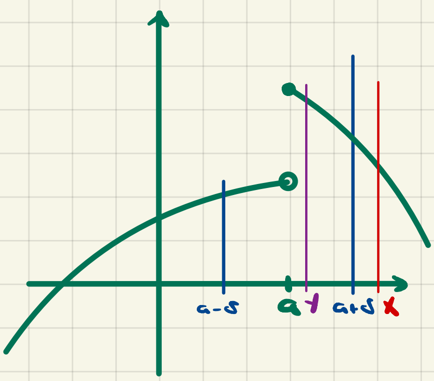
$$\forall y \left(0 < y-a < \delta_2 \rightarrow \frac{\ell_- + \ell_+}{2} < f(y) < \frac{3\ell_+ - \ell_-}{2} \right)$$

Let $\delta = \min(\delta_1, \delta_2)$. Then

$$\forall x \forall y \left(\begin{matrix} 0 < a-x < \delta \\ \\ 0 < y-a < \delta \end{matrix} \rightarrow f(x) < \frac{\ell_- + \ell_+}{2} < f(y) \right)$$

The converse is not true. Ie $\forall x \forall y \left(f(x) < f(y) \rightarrow \begin{matrix} x < a < y \\ |x-a| < \delta \\ |y-a| < \delta \end{matrix} \right)$ is false.

Here is a counterexample



we have $f(x) < f(y)$ but $\neg(x < a < y)$

32. $\lim_{x\to\infty} \dfrac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0}$, $a_n \neq 0, b_m \neq 0$, exists $\leftrightarrow m \geq n$

Assume $\exists \lim_{x\to\infty} f(x)$, $f(x) = \dfrac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0}$, $a_n \neq 0, b_m \neq 0$.

Then $\forall \epsilon > 0\ \exists N > 0$ s.t. $x > N \rightarrow \left| \dfrac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0} - \ell \right| < \epsilon$

Rewrite $f(x)$

$$\frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0} = \frac{x^n (a_n + a_{n-1}x^{-1} + \cdots a_0 x^{-n})}{x^m (b_m + b_{m-1}x^{-1} + \cdots + b_0 x^{-m})} = \frac{x^n}{x^m} \frac{(a_n + a_{n-1}x^{-1} + \cdots a_0 x^{-n})}{(b_m + b_{m-1}x^{-1} + \cdots + b_0 x^{-m})}.$$ Let $h(x) = \dfrac{x^n}{x^m}$ and $g(x) = \dfrac{(a_n + a_{n-1}x^{-1} + \cdots a_0 x^{-n})}{(b_m + b_{m-1}x^{-1} + \cdots + b_0 x^{-m})}$.

Since the following two limits exist

$\lim_{x\to\infty} (a_n + a_{n-1}x^{-1} + \cdots + a_0 x^{-n}) = a_n$ (1)

$\lim_{x\to\infty} (b_m + b_{m-1}x^{-1} + \cdots + b_0 x^{-m}) = b_m$ (2)

And since a_n and $b_m \neq 0$, we know that $\lim_{x\to\infty} g(x) = \dfrac{a_n}{b_m} \neq 0$.

From Problem 8d, given that $\lim_{x\to\infty} h(x)g(x)$ exists and $\lim_{x\to\infty} g(x)$ exists and is $\neq 0$, we know that $\lim_{x\to\infty} h(x)$ exists.

we know this because $h(x) = \dfrac{1}{g(x)} \cdot h(x)g(x)$

$\lim_{x\to\infty} g(x) \neq 0 \rightarrow \lim_{x\to\infty} \dfrac{1}{g(x)}$ exists.

$\rightarrow \lim_{x\to\infty} \dfrac{1}{g(x)} h(x)g(x)$ exists.

$\lim_{x\to\infty} h(x)g(x)$ exists by assumption

Consider $\lim_{x\to\infty} h(x)$

$\lim_{x\to\infty} \dfrac{x^n}{x^m} = \lim_{x\to\infty} \dfrac{1}{x^{m-n}}$

There are three cases to consider

case 1: $m = n \rightarrow \lim_{x\to\infty} \dfrac{1}{x^{m-n}} = 1$

case 2: $m > n \rightarrow \lim_{x\to\infty} \dfrac{1}{x^{m-n}} = 0$

case 3: $m < n \rightarrow \lim_{x\to\infty} \dfrac{1}{x^{m-n}} = \lim_{x\to\infty} x^{n-m}$, which doesn't exist. $\perp$

proof

Assume $\lim_{x\to\infty} x^{n-m} = \ell$, and $n-m > 0$

Then, $\forall \epsilon > 0\ \exists N\ \forall x\ x > N \rightarrow |x^{n-m} - \ell| < \epsilon$

Note that $\ell > 0$ because $x^{n-m} > 0$.

Let $\epsilon > 0$, and N s.t. $\forall x\ x > N \rightarrow |x^{n-m} - \ell| < \epsilon$

Let x_1 s.t. $x > N \rightarrow |x^{n-m} - \ell| < \epsilon$

Consider any x s.t. $x^{n-m} > \ell + \epsilon$ and $x > N$.

For example,

$x_1 = \ell + \epsilon + N \rightarrow x_1^{n-m} = (\ell + \epsilon + N)^{n-m} > \ell + \epsilon$

Then $|x_1^{n-m} - \ell| > |\ell + \epsilon - \ell| > |\epsilon| > \epsilon$

Therefore $\exists x : x > N \wedge |x^{n-m} - \ell| > \epsilon$

$\perp$

Therefore $n - m > 0 \rightarrow \nexists \lim_{x\to\infty} x^{n-m}$

Therefore $m \geq n$.

Also,

$m = n \rightarrow \lim_{x\to\infty} f(x) = \dfrac{a_n}{b_m}$

$m > n \rightarrow \lim_{x\to\infty} f(x) = 0$

Now assume $m \geq n$

Let $f(x) = \dfrac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0}$, $a_n \neq 0, b_m \neq 0$

Then we can rewrite $f(x)$ as $f(x) = \dfrac{x^n}{x^m} \dfrac{(a_n + a_{n-1}x^{-1} + \cdots a_0 x^{-n})}{(b_m + b_{m-1}x^{-1} + \cdots + b_0 x^{-m})}$

we know that

$$\lim_{x \to \infty} (a_n + a_{n-1}x^{-1} + \cdots + a_0 x^{-n}) = a_n$$

$$\lim_{x \to \infty} (b_m + b_{m-1}x^{-1} + \cdots + b_0 x^{-m}) = b_m$$

so $\displaystyle \lim_{x \to \infty} \frac{(a_n + a_{n-1}x^{-1} + \cdots a_0 x^{-n})}{(b_m + b_{m-1}x^{-1} + \cdots + b_0 x^{-m})} = \frac{a_n}{b_m}$

Since $m \geq n$ we have two cases:

case1: $m = n \to \displaystyle \lim_{x \to \infty} \frac{1}{x^{m-n}} = 1$

Therefore $\exists \displaystyle \lim_{x \to \infty} f(x)$

case2: $m > n \to \displaystyle \lim_{x \to \infty} \frac{1}{x^{m-n}} = 0$

Therefore $\exists \displaystyle \lim_{x \to \infty} f(x)$

Therefore $\exists \displaystyle \lim_{x \to \infty} f(x)$

33.

i) $\lim\limits_{x\to\infty} \dfrac{x+\sin^3 x}{5x+6}$

$\forall x \quad -1 < \sin^3 x < 1$

therefore $\dfrac{x-1}{5x+6} < \dfrac{x+\sin^3 x}{5x+6} < \dfrac{x+1}{5x+6}$

$$\frac{x+1}{5x+6} = \frac{x(1+\frac{1}{x})}{x(5+\frac{6}{x})} = \frac{1+\frac{1}{x}}{5+\frac{6}{x}}$$

$$\frac{x-1}{5x+6} = \frac{x(1-\frac{1}{x})}{x(5+\frac{6}{x})} = \frac{1-\frac{1}{x}}{5+\frac{6}{x}}$$

$$\lim_{x\to\infty} \frac{1+\frac{1}{x}}{5+\frac{6}{x}} = \frac{1}{5}$$

$$\lim_{x\to\infty} \frac{1-\frac{1}{x}}{5+\frac{6}{x}} = \frac{1}{5}$$

We can use the result proved in Problem 13:

Since $\dfrac{x-1}{5x+6} < \dfrac{x+\sin^3 x}{5x+6} < \dfrac{x+1}{5x+6}$

and $\lim\limits_{x\to\infty} \dfrac{1+\frac{1}{x}}{5+\frac{6}{x}} = \dfrac{1}{5}$

$\lim\limits_{x\to\infty} \dfrac{1-\frac{1}{x}}{5+\frac{6}{x}} = \dfrac{1}{5}$

then

$$\lim_{x\to\infty} \frac{x+\sin^3 x}{5x+6} = \frac{1}{5}$$

$\lim\limits_{x\to\infty} \dfrac{1}{x}$

$\forall \epsilon > 0$ choose $N = \dfrac{1}{\epsilon}$

Then, $x > N \rightarrow \dfrac{1}{x} < \dfrac{1}{N} = \epsilon$

Since $\epsilon > 0$ then $N > 0$ then $x > N \rightarrow 0 < \dfrac{1}{x} < \epsilon \rightarrow \left|\dfrac{1}{x}\right| < \epsilon$

therefore $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$

ii) $\lim\limits_{x\to\infty} \dfrac{x\sin x}{x^2+5}$

$$\frac{-x}{x^2+5} < \frac{x\sin x}{x^2+5} < \frac{x}{x^2+5}$$

$\lim\limits_{x\to\infty} \dfrac{x}{x^2+5} = \lim\limits_{x\to\infty} \dfrac{-x}{x^2+5} = 0$ (proved Problem 32)

$\rightarrow \lim\limits_{x\to\infty} \dfrac{x\sin x}{x^2+5} = 0$ (proved Problem 13)

iii) $\lim\limits_{x\to\infty} \sqrt{x^2+x} - x$

$$\sqrt{x^2+x} - x = \frac{x^2+x-x^2}{\sqrt{x^2+x}+x} = \frac{x}{x\sqrt{1+\frac{1}{x}}+x} = \frac{1}{\sqrt{1+1/x}+1}$$

$\lim\limits_{x\to\infty} \sqrt{1+1/x}$

$\forall \epsilon > 0 \quad x > \dfrac{1}{\epsilon} \rightarrow \dfrac{1}{x} < \dfrac{1}{N} \rightarrow 1+\dfrac{1}{x} - 1 < \epsilon$

$\rightarrow \lim\limits_{x\to\infty} \sqrt{1+1/x} = 1$

$\lim\limits_{x\to\infty} \left(\sqrt{1+1/x} + 1\right) = 2$

$$\lim_{x\to\infty} \frac{1}{\sqrt{1+1/x}+1} = \frac{1}{2}$$

iv) $\lim\limits_{x\to\infty} \dfrac{x^2(1+\sin^2 x)}{(x+\sin x)^2}$

$$\frac{x^2}{(x+\sin x)^2} \le \frac{x^2(1+\sin^2 x)}{(x+\sin x)^2} \le \frac{2x^2}{(x+\sin x)^2}$$

Because $x\to\infty$ then $x>0$. Therefore $(x-1)^2 < (x+\sin x)^2 < (x+1)^2$

Hence

$$\frac{x^2}{(x+1)^2} \le \frac{x^2(1+\sin^2 x)}{(x+\sin x)^2} \le \frac{2x^2}{(x-1)^2}$$

$$\lim_{x\to\infty} \frac{2x^2}{(x-1)^2} = \lim_{x\to\infty} \frac{2x^2}{x^2-2x+1} = 2 \quad \text{(Problem 32)}$$

$$\lim_{x\to\infty} \frac{x^2}{(x+1)^2} = \lim_{x\to\infty} \frac{x^2}{x^2+2x+1} = 1$$

Therefore, if $\lim\limits_{x\to\infty} \dfrac{x^2(1+\sin^2 x)}{(x+\sin x)^2}$ exists then $1 \le \lim\limits_{x\to\infty} \dfrac{x^2(1+\sin^2 x)}{(x+\sin x)^2} \le 2$ (Problem 12)

Assume the limit exists.

Then $\forall \epsilon>0\ \exists N\ x>N \to \left| \dfrac{x^2(1+\sin^2 x)}{(x+\sin x)^2} - \ell \right| < \epsilon$

choose an $\epsilon>0$, and let N_1 be such that $x>N_1 \to \left| \dfrac{x^2(1+\sin^2 x)}{(x+\sin x)^2} - \ell \right| < \epsilon$

let $x_1 > N_1$ s.t. $\sin(x_1)=0$

Then $\dfrac{x_1^2(1+\sin^2 x_1)}{(x_1+\sin x_1)^2} = 1$

$|1-\ell|<\epsilon \to -\epsilon<1-\ell<\epsilon \to -1-\epsilon<-\ell<-1+\epsilon \to 1-\epsilon<\ell<1+\epsilon$

Let $x_2>N$ s.t. $\sin(x_2)=1$

Then $\dfrac{x_2^2(1+\sin^2 x_2)}{(x_2+\sin x_2)^2} = \dfrac{2x_2^2}{(x_2+1)^2}$

$$\left|\frac{2x_2^2}{(x_2+1)^2} - \ell\right| < \epsilon \to -\frac{2x_2^2}{(x_2+1)^2} - \epsilon < -\ell < -\frac{2x_2^2}{(x_2+1)^2} + \epsilon \to \frac{2x_2^2}{(x_2+1)^2} - \epsilon < \ell < \frac{2x_2^2}{(x_2+1)^2} + \epsilon$$

$$\lim \left[\frac{2x_2^2}{(x_2+1)^2} - \epsilon\right] = 2-\epsilon$$

$$\lim \left[\frac{2x_2^2}{(x_2+1)^2} + \epsilon\right] = 2+\epsilon$$

Hence $2-\epsilon < \lim \ell < 2+\epsilon \to 2-\epsilon<\ell<2+\epsilon$

Choose $\epsilon = \frac{1}{2}$. Then we have

$1.5<\ell<2.5$

$0.5<\ell<1.5$

$\perp$

Therefore the limit doesn't exist.

34. $\lim_{x\to 0^+} f(1/x) = \lim_{x\to\infty} f(x)$

Proof

If $\lim_{x\to\infty} f(x)$ exists and equals ℓ then

$\forall \epsilon>0\ \exists N\ \forall x\ x>N \to |f(x)-\ell|<\epsilon$ (1)

Let $y = \frac{1}{x}$

Then (1) is equivalent to

$\forall\epsilon>0\ \exists N\ \forall y\ y<\frac{1}{N} \to |f(1/y)-\ell|<\epsilon$

Let $\delta = \frac{1}{N}$. Then

$\forall\epsilon>0\ \exists\delta\ \forall y\ y<\delta \to |f(1/y)-\ell|<\epsilon$

Therefore $\lim_{y\to 0^+} f(1/y) = \ell$

35. $\alpha = \lim_{x\to 0} \frac{\sin x}{x}$

i) $\lim_{x\to\infty} \frac{\sin x}{x} = \lim_{x\to\infty} \frac{1}{x}\sin x$

$\forall x>0 \quad -\frac{1}{x} < \frac{1}{x}\sin x < \frac{1}{x}$

$\lim_{x\to\infty}\left(-\frac{1}{x}\right) = \lim_{x\to\infty}\frac{1}{x} = 0 \to \lim_{x\to\infty}\frac{\sin x}{x} = 0$

ii) $\lim_{x\to\infty} x\sin\frac{1}{x}$

$\forall\epsilon>0\ \exists N>0\ \forall x\ x>N \to |x\sin\frac{1}{x}-\ell|<\epsilon$

let $y = \frac{1}{x}$

$\forall\epsilon>0\ \exists N>0\ \forall y\ 0<y<\frac{1}{N} \to \left|\frac{\sin y}{y}-\ell\right|<\epsilon$

$\lim_{y\to 0} \frac{\sin y}{y} = \ell = \alpha$

Alternatively, note that $\lim_{x\to\infty} x\sin(1/x) = \lim_{x\to\infty}\frac{\sin(1/x)}{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f(1/x) = \lim_{x\to 0^+}\frac{\sin x}{x} = \alpha$

36. $\lim_{x \to -\infty} f(x) = \ell$

a) $\lim_{x \to -\infty} \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0}$

Proof

$\forall \varepsilon > 0\ \exists N < 0\ \forall x\ x < N \to |f(x) - \ell| < \varepsilon$

Let $f(x) = \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0} = \frac{x^n}{x^m} \frac{(a_n + a_{n-1} x^{-1} + \cdots a_0 x^{-n})}{(b_m + b_{m-1} x^{-1} + \cdots + b_0 x^{-m})}$

$\lim \frac{1}{x^{m-n}}$

There are three cases to consider

case 1: $m = n \to \lim_{x \to -\infty} \frac{1}{x^{m-n}} = 1$

case 2: $m > n \to \lim_{x \to -\infty} \frac{1}{x^{m-n}} = 0$

$x < N < 0 \to \frac{1}{N} < \frac{1}{x} < 0$

if $m-n$ even then $0 < \frac{1}{x^{m-n}} < \frac{1}{N^{m-n}}$

if $N = -\frac{1}{\varepsilon^{1/(m-n)}}$ then we have $x < N < 0 \to 0 < \frac{1}{x^{m-n}} < \varepsilon \to \left|\frac{1}{x^{m-n}}\right| < \varepsilon$

$\forall \varepsilon > 0\ \exists N < 0\ \forall x\ x < N < 0 \to \left|\frac{1}{x^{m-n}}\right| < \varepsilon$

Therefore $m-n$ even $\to \lim_{x \to -\infty} \frac{1}{x^{m-n}} = 0$

if $m-n$ odd then $\frac{1}{N^{m-n}} < \frac{1}{x^{m-n}} < 0$

if $N = -\frac{1}{\varepsilon^{1/(m-n)}}$ then $-\varepsilon < \frac{1}{x^{m-n}} < 0 \to \left|\frac{1}{x^{m-n}}\right| < \varepsilon$

therefore $\forall \varepsilon > 0\ \exists N < 0\ x < N < 0 \to \left|\frac{1}{x^{m-n}}\right| < \varepsilon$

Therefore $m-n$ odd $\to \lim_{x \to -\infty} \frac{1}{x^{m-n}} = 0$

case 3: $m < n \to \lim_{x \to -\infty} \frac{1}{x^{m-n}} = \lim_{x \to \infty} x^{n-m}$, which doesn't exist. $\perp$

Note that each term in $a_n + a_{n-1} x^{-1} + \cdots + a_0 x^{-n}$ has "$m > n$" so the limits of these terms are zero.

Hence $\lim_{x \to -\infty} \frac{x^n}{x^m} \frac{(a_n + a_{n-1} x^{-1} + \cdots a_0 x^{-n})}{(b_m + b_{m-1} x^{-1} + \cdots + b_0 x^{-m})}$ is $\frac{a_n}{b_m} \cdot \lim_{x \to -\infty} \frac{1}{x^{m-n}}$

$m > n \to \lim_{x \to -\infty} f(x) = 0$

(Same answers as for $\lim_{x \to \infty} f(x)$)

$m = n \to \lim_{x \to -\infty} f(x) = \frac{a_n}{b_m}$

b) $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(-x)$

Proof

$\forall \varepsilon>0\ \exists N>0\ \forall x\ 0<N<x \to |f(x)-\ell|<\varepsilon$

Let $y=-x$

$\forall \varepsilon>0\ \exists N>0\ \forall y\ y<-N<0 \to |f(-y)-\ell|<\varepsilon$

$\forall \varepsilon>0\ \exists N<0\ \forall y\ y<N<0 \to |f(-y)-\ell|<\varepsilon$

$\lim_{x\to-\infty} f(-x) = \lim_{x\to\infty} f(x) = \ell$

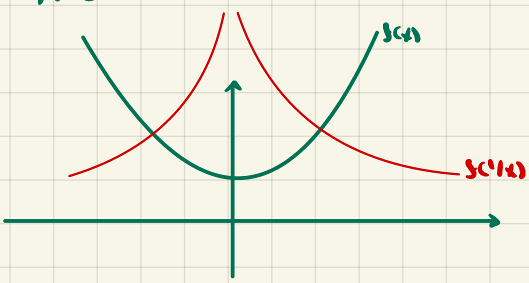
Interpretation of the substitution

For any x s.t. $0<N<x$ there is an associated $y=-x$ by our definition.
Hence it is true that $0<N<-y \to |f(-y)-\ell|<\varepsilon$.
Equivalent to $y<-N<0 \to |f(-y)-\ell|<\varepsilon$.
For any y s.t. $y<-N<0$ we know there is an x s.t. $0<N<x$,
So then $|f(x)-\ell|<\varepsilon$. But then $|f(-y)-\ell|<\varepsilon$.

$y<-N<0 \to 0<N<x \to |f(x)-\ell|<\varepsilon \to |f(-y)-\ell|<\varepsilon$

$0<-x<\delta$

$-\delta<x<0$



c) $\lim_{x\to0^-} f\left(\frac{1}{x}\right) = \lim_{x\to-\infty} f(x)$

Proof

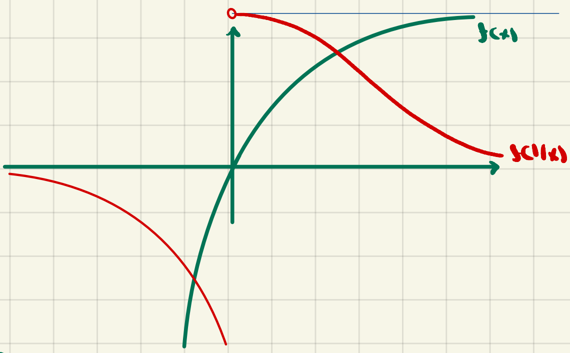
$\forall \varepsilon>0\ \exists N<0\ \forall x\ x<N<0 \to |f(x)-\ell|<\varepsilon$

$x = \frac{1}{y}$

$\frac{1}{y}<N<0 \to |f(1/y)-\ell|<\varepsilon$

$\frac{1}{N}<y<0 \to |f(1/y)-\ell|<\varepsilon$

$0<-y<-\frac{1}{N} \to |f(1/y)-\ell|<\varepsilon$



Let $\delta=-\frac{1}{N}>0$.
Then $\forall \varepsilon>0\ \exists \delta>0\ \forall y\ 0<-y<\delta \to |f(1/y)-\ell|<\varepsilon$

$\lim_{x\to0^-} f(1/x) = \ell = \lim_{x\to-\infty} f(x)$

37. $\lim_{x\to a} f(x) = \infty \leftrightarrow \forall N\ \exists\delta>0\ \ 0<|x-a|<\delta \to f(x)>N$

a) $\lim_{x\to 3} \frac{1}{(x-3)^2} = \infty$

Proof

Let N be any number.

Case 1: $N>0$

$$\frac{1}{(x-3)^2} > N > 0 \leftrightarrow 0<(x-3)^2 < \frac{1}{N} \leftrightarrow 0< |x-3| < (1/N)^{1/2}$$

Therefore $\exists\delta>0\ \ 0<|x-3|<\delta \to \frac{1}{(x-3)^2} > N$

Case 2: $N\leq 0$

$$\frac{1}{(x-3)^2} > 0 > N$$

Therefore $\forall\delta>0\ \ 0<|x-3|<\delta \to \frac{1}{(x-3)^2} > N$

Therefore $\forall N\ \exists\delta>0\ \ 0<|x-3|<\delta \to \frac{1}{(x-3)^2} > N$

Therefore $\lim_{x\to 3} \frac{1}{(x-3)^2} = \infty$

b) $\forall x\ \ f(x) > \epsilon > 0$, $\lim_{x\to a} g(x) = 0$ $\to$ $\lim_{x\to a} \frac{f(x)}{|g(x)|} = \infty$

Proof

$\lim_{x\to a} g(x) = 0$ means

$\forall\epsilon_g\ \exists\delta\ \forall x\ \ 0<|x-a|<\delta \to |g(x)|<\epsilon_g$

$|g(x)| < \epsilon_g \to \frac{1}{|g(x)|} > \frac{1}{\epsilon_g}$

$\to \left|\frac{f(x)}{|g(x)|}\right| = \frac{f(x)}{|g(x)|} > \frac{f(x)}{\epsilon_g} > \frac{\epsilon}{\epsilon_g}$

For any N, $\frac{\epsilon}{\epsilon_g} > N \leftrightarrow \epsilon_g < \frac{\epsilon}{N}$

Therefore if $\forall N$ if we choose $\epsilon_g < \frac{\epsilon}{N}$ then $\exists\delta\ \forall x\ \ 0<|x-a|<\delta \to \left|\frac{f(x)}{|g(x)|}\right| > N$

Therefore $\forall N\ \exists\delta\ \forall x\ \ 0<|x-a|<\delta \to \left|\frac{f(x)}{|g(x)|}\right| > N$

So $\lim_{x\to\infty} \frac{f(x)}{|g(x)|} = \infty$

38.

a)

$\lim_{x\to a^+} f(x) = \infty \iff \forall N\ \exists \delta>0\ \forall x\ 0<x-a<\delta \to f(x)>N$

$\lim_{x\to a^-} f(x) = \infty \iff \forall N\ \exists \delta>0\ \forall x\ 0<a-x<\delta \to f(x)>N$

b) $\lim_{x\to 0^+} \frac{1}{x} = \infty$

Proof

$\forall \delta>0 \quad 0<x<\delta \to 0<\frac{1}{\delta}<\frac{1}{x}$

$\forall N>0$ let $\frac{1}{\delta}=N$. Then $\delta = \frac{1}{N}$

$\forall N>0\ \exists \delta>0\ 0<x<\delta \to \frac{1}{x}>N$

Also $\forall N\leq 0\ \exists \delta>0\ 0<x<\delta \to \frac{1}{x}>N$

So $\forall N\ \exists\delta>0\ 0<x<\delta \to \frac{1}{x}>N$

Therefore $\lim_{x\to 0^+}\frac{1}{x} = \infty$

c) $\lim_{x\to 0^+} f(x) = \infty \iff \lim_{x\to\infty} f\left(\frac{1}{x}\right) = \infty$

Proof

Assume $\lim_{x\to 0^+} f(x) = \infty$.

Then $\forall N\ \exists\delta>0\ 0<x<\delta \to f(x)>N$

Let $x=\frac{1}{y}$. Then

$0<\frac{1}{y}<\delta \to 0<\frac{1}{\delta}<y \to f(1/y)>N$

$\forall N\ \exists M\ y>M>0 \to f(1/y)>N$

$\lim_{x\to\infty} f(1/x) = \infty$

Assume $\lim_{x\to\infty} f\left(\frac{1}{x}\right) = \infty$

$\forall N\ \exists M\ \forall x\quad x>M>0 \to f(1/x)>N$

Let $y=\frac{1}{x}$. Then

$\forall N\ \exists M\ \forall y\ \frac{1}{y}>M>0 \to f(y)>N$

$\frac{1}{y}>M>0 \to 0<y<\frac{1}{M}$

$\forall N\ \exists\delta\ \forall x\ 0<x<\delta \to f(x)>N$

$\lim_{x\to 0^+} f(x) = \infty$

39.

i) $\lim_{x\to\infty} \frac{x^3+4x-7}{7x^2-x+1}$

Problem 32: exponent 2 < exponent 3

$\to \lim_{x\to\infty} \frac{x^3+4x-7}{7x^2-x+1} = \infty$

ii) $\lim_{x\to\infty} x(1+\sin^2 x)$

$1 < 1+\sin^2 x < 2$. $\nexists \lim_{x\to\infty}(1+\sin^2 x)$

$\nexists \lim_{x\to\infty} x$

$\lim_{x\to\infty} x < \lim_{x\to\infty} x(1+\sin^2 x) < \lim_{x\to\infty} 2x$

$\to \infty < \lim_{x\to\infty} x(1+\sin^2 x) < \infty$

$\to \lim_{x\to\infty} x(1+\sin^2 x) = \infty$

iii) $\lim_{x\to\infty} x\sin^2 x$

$0 \le x\sin^2 x \le x$

Does not exist.

iv) $\lim_{x\to\infty} x^2\sin(1/x) = \lim_{x\to\infty}[x\cdot(x\sin(1/x))]$

From Problem 35, $\lim_{x\to\infty} x\sin(1/x) = \lim_{x\to 0}\frac{\sin(x)}{x} = \alpha$

Assume $\lim_{x\to\infty} x^2\sin(1/x) = \ell$.

Then $\forall \epsilon>0\ \exists N\ \forall x\ x>N \to |x\cdot x\sin(1/x) - \ell| < \epsilon$

$\ell - \epsilon < x\cdot x\sin(1/x) < \ell+\epsilon$

v) $\lim_{x\to\infty} \sqrt{x^2+2x} - x$

$\sqrt{x^2+2x}-x = \frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x} = \frac{2x}{x\sqrt{1+2/x}+x} = \frac{2}{\sqrt{1+2/x}+1}$

$\lim_{x\to\infty} \sqrt{1+2/x} = 1$

$\lim_{x\to\infty} \sqrt{x^2+2x} - x = 2$

vi) $\lim_{x\to\infty} x(\sqrt{x+2}-\sqrt{x})$

$x(\sqrt{x+2}-\sqrt{x}) = \frac{x(\cancel{x}+2-\cancel{x})}{\sqrt{x+2}+\sqrt{x}} = \frac{2x}{\sqrt{x+2}+\sqrt{x}} = \frac{2x}{x\sqrt{\frac{1}{x}+\frac{2}{x^2}}+x\sqrt{\frac{1}{x}}}$

$= \frac{2}{\sqrt{\frac{1}{x}+\frac{2}{x^2}}+\sqrt{\frac{1}{x}}}$

$\lim_{x\to\infty}\sqrt{1/x} = \lim\sqrt{\frac{1}{x}+\frac{2}{x^2}} = 0$

$\lim_{x\to\infty} \frac{1}{x} = 0 \leftrightarrow \forall\epsilon>0\ \forall x\ x>2\epsilon^2 \to \frac{1}{x} < \frac{1}{2\epsilon^2}$

$\lim_{x\to\infty}\sqrt{1/x} = 0 \leftrightarrow \forall\epsilon>0\ \forall x\ x>\epsilon^2 \to \sqrt{1/x} < \frac{1}{\epsilon}$

$\lim_{x\to\infty} \frac{2}{x^2} = 0 \leftrightarrow \forall\epsilon\ \forall x\ x>2\epsilon \to \frac{2}{x^2} < \frac{1}{2\epsilon^2}$

$\lim_{x\to\infty}\sqrt{\frac{1}{x}+\frac{2}{x^2}} = 0 \leftrightarrow \forall\epsilon\ \forall x\ x>\max(2\epsilon^2, 2\epsilon) \to \frac{1}{x}+\frac{2}{x^2} < \frac{1}{\epsilon^2}$

$\to \sqrt{\frac{1}{x}+\frac{2}{x^2}} < \frac{1}{\epsilon}$

Therefore $\forall\epsilon>0\ \forall x$

$x>\max(2\epsilon^2, 2\epsilon, \epsilon^2) \to \sqrt{\frac{1}{x}+\frac{2}{x^2}}+\sqrt{\frac{1}{x}} < \frac{2}{\epsilon}$

$\to \frac{2}{\sqrt{\frac{1}{x}+\frac{2}{x^2}}+\sqrt{\frac{1}{x}}} > \epsilon$

That is $\forall N>0\ \exists N>0\ \forall x\ x>N \to \frac{2}{\sqrt{\frac{1}{x}+\frac{2}{x^2}}+\sqrt{\frac{1}{x}}} > N$

Hence $\lim_{x\to\infty} x(\sqrt{x+2}-\sqrt{x}) = \infty$

viii) $\lim_{x\to\infty} \frac{\sqrt{|x|}}{x}$

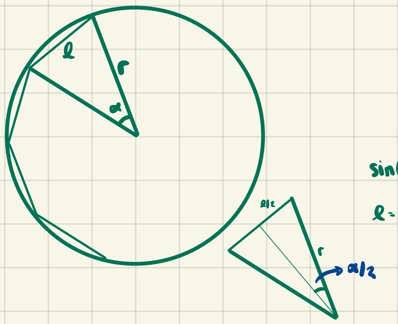
$\forall N>0\ \forall x>N \to |x|=x \to \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{N}} \leftrightarrow \sqrt{x} > \sqrt{N} \leftrightarrow x > N$

$\forall \varepsilon>0\ \forall x > \frac{1}{\varepsilon^2} \to \frac{\sqrt{|x|}}{x} = \frac{1}{\sqrt{x}} < \varepsilon$

$\lim_{x\to\infty} \frac{\sqrt{|x|}}{x} = 0$

40

a)



$\sin(\frac{\alpha}{2}) = \frac{l}{2r}$

$l = 2r\sin(\alpha/2)$

Perimeter $= nl$

$= n \cdot 2r\sin(\alpha/2)$

$= 2nr\sin(\pi/n)$

$n = \frac{2\pi}{\alpha} \to \alpha = \frac{2\pi}{n}$

b) $\lim_{n\to\infty} P(n) = \lim_{n\to\infty} 2nr\sin(\pi/n) = 2r\lim_{n\to\infty} n\sin(\pi/n)$

$= 2\pi r \lim_{n\to\infty} \frac{\sin(\pi/n)}{\pi/n} = 2\pi r\alpha$ (as per Problem 35)

c) $\lim_{x\to\infty} \frac{\sin x}{x} = 1$

41.

a) $c>1$, $f(n) = c^{1/n} = \sqrt[n]{c}$

Bernoulli's Inequality says $\forall \varepsilon>0\ \forall n>0\ 1 < 1+n\varepsilon \leq (1+\varepsilon)^n$

$\lim_{n\to\infty}(1+n\varepsilon) = \infty \to \lim_{n\to\infty}(1+\varepsilon)^n = \infty$

That is $\forall \varepsilon>0\ \forall M>0\ \exists N>0\ \forall n\ n>N \to (1+\varepsilon)^n > M$

Therefore, $\forall \varepsilon>0\ \exists N>0\ \forall n\ n>N$

$\to (1+\varepsilon)^n > c \to c^{1/n} - 1 < \varepsilon$

Alternatively,

Let $c>1$

Assume $c^{1/n} > 1+\varepsilon$, for all $n\in\mathbb{N}$

Then $c > (1+\varepsilon)^n > 1+n\varepsilon$

$\to n < \frac{c-1}{\varepsilon}$. $\perp$ because we can choose $n_1 = [\frac{c-1}{\varepsilon}]+1$,

and $\forall n\in\mathbb{N}, n>n_1 \to n > \frac{c-1}{\varepsilon}$.

Therefore $\forall \varepsilon>0\ \exists N\ \forall n, n>N \to 1 < c^{1/n} < 1+\varepsilon$

b) Let $c>0$.

Case 1: $c>1$. By part a), $c^{1/n}$ approaches 1.

Case 2: $c=1$. $c^{1/n}=1$.

Case 3: $0<c<1$

Then $\frac{1}{c} > 1$. By part a), $(\frac{1}{c})^{1/n}$ approaches 1.

$\forall \varepsilon>0\ \exists N\ \forall n, n>N \to 1 < \frac{1}{c^{1/n}} < 1+\varepsilon$

$c^{1/n} = \frac{1}{\frac{1}{c^{1/n}}} \to 1$ by Theorem 2.

42.

$\delta = \min\left(\sqrt{a^2+\epsilon}-a,\ a-\sqrt{a^2-\epsilon}\right)$

$|x-a|<\delta \rightarrow a-\delta<x<a+\delta$

if $\delta = \sqrt{a^2+\epsilon}-a$ then $2a-\sqrt{a^2+\epsilon}<x<\sqrt{a^2+\epsilon}$

if $\delta = a-\sqrt{a^2-\epsilon}$ then $\sqrt{a^2-\epsilon}<x<2a-\sqrt{a^2-\epsilon}$

For $f(x)=x^2$ specifically,

$$\sqrt{a^2+\epsilon}-a-\left(a-\sqrt{a^2-\epsilon}\right)$$
$$\sqrt{a^2+\epsilon}+\sqrt{a^2-\epsilon}-2a<0$$

Therefore $\delta = \sqrt{a^2+\epsilon}-a$, so $2a-\sqrt{a^2+\epsilon}<x<\sqrt{a^2+\epsilon}$

this term was omitted