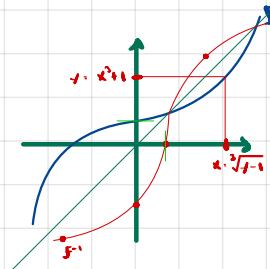


Ch. 12 - Inverse Functions

$$1. \text{ (i) } f(x) = x^3 + 1$$

$$x^3 + y - 1 \rightarrow x = \sqrt[3]{y-1}$$

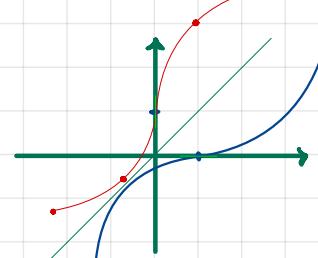
$$f^{-1}(x) = \sqrt[3]{x-1}$$



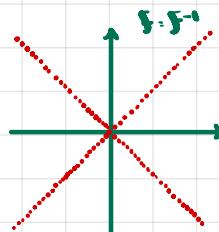
$$\text{(ii) } f(x) = (x-1)^3 + 1$$

$$x-1 = \sqrt[3]{y} \rightarrow x = \sqrt[3]{y} + 1$$

$$f^{-1}(x) = 1 + \sqrt[3]{x}$$

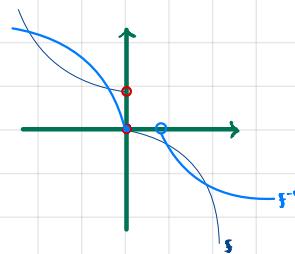


$$\text{(iii) } f(x) = \begin{cases} x & x \neq 0 \\ -x & x = 0 \end{cases}$$



$$f(x) = f^{-1}(x)$$

$$\text{(iv) } f(x) = \begin{cases} -x^2 & x \geq 0 \\ 1-x^3 & x < 0 \end{cases}$$



For $x \geq 0$ we have

$$-x^2 = y, \text{ so } y \leq 0.$$

$$x^2 = -y \rightarrow x = \sqrt{-y}$$

For $x < 0$ we have

$$1-x^3 = y \rightarrow x^3 = 1-y$$

$$\rightarrow x = \sqrt[3]{1-y}$$

$$1-y < 0 \rightarrow y > 1$$

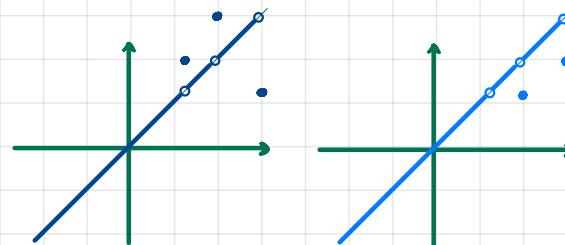
Hence we have

$$f^{-1}(x) = \begin{cases} \sqrt{-x} & x \leq 0 \\ \sqrt[3]{1-x} & x > 1 \end{cases}$$

$$(v) f(x) = \begin{cases} x & x+a_1, \dots, a_n \\ a_{i+1} & x = a_i, i=1, \dots, n-1 \\ a_n & x = a_n \end{cases}$$

For $x+a_1, \dots, a_n$

$$x+y \rightarrow f'(x) = x$$



For $x = a_i, i=1, \dots, n-1$

$$a_{i+1} - y \rightarrow f'(x) = a_{i+1}$$

For $x = a_n$

$$a_n - y \rightarrow f'(x) = a_n$$

$$f'(x) = \begin{cases} x & x+a_1, \dots, a_n \\ a_{i+1} & x = a_i, i=1, \dots, n-1 \\ a_n & x = a_n \end{cases}$$

$$(vi) f(x) = x + [x]$$

$$f(x) = x+n \quad x \in [n, n+1], n \in \mathbb{Z}$$

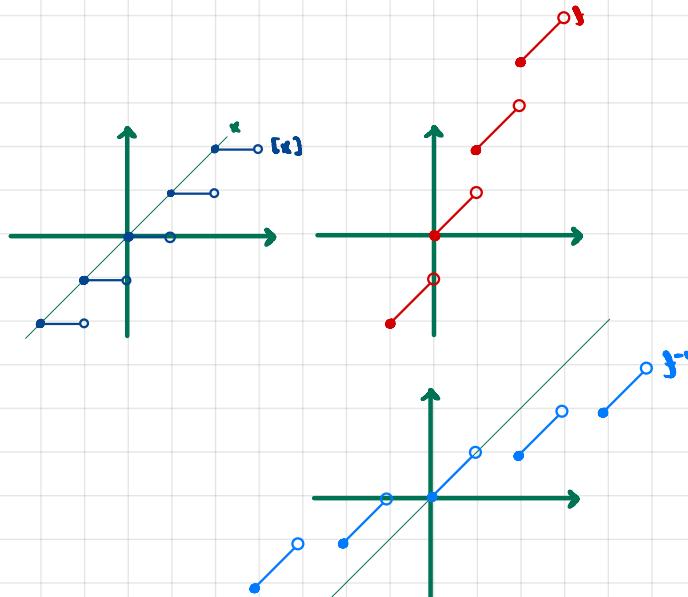
$$x+n = y \rightarrow x = y - n$$

$$n=0 \rightarrow x=y, y \text{ from } 0 \text{ to } 1$$

$$n=1 \rightarrow x=y-1, y \text{ from } 2 \text{ to } 3$$

$$n=2 \rightarrow x=y-2, y \text{ from } 4 \text{ to } 5$$

$$f'(x) = x-n, x \in [2n, 2n+1] n \in \mathbb{Z}$$

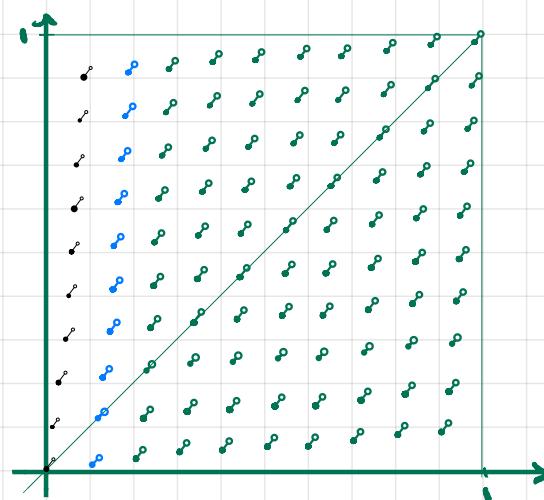


$$\text{will } f(0.9, 0.9, 0.9, \dots) = 0.9, 0.9, 0.9$$

There are only 100 possibilities for first two decimals

0.00	0.10	0.90
0.01	0.11	0.91
0.02	0.12	(...) 0.92
...
0.09	0.19	0.99

$f^{-1}(x)$

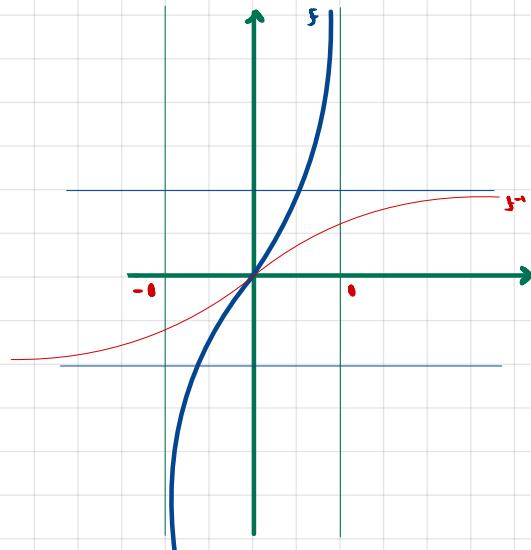


$$(viii) f(x) = \frac{x}{1-x^2} \quad -1 < x < 1$$

$$f'(x) = \frac{1-x^2-x(-2x)}{(1-x^2)^2} = \frac{1-x^2+2x^2}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2} > 0$$

$$f''(x) = \frac{2x(1-x^2) - (1+x^2) \cdot 2(-x)(-2x)}{(1-x^2)^4} = \frac{2x-2x^3+4x+4x^3}{(1-x^2)^4}$$

$$= \frac{2x^3+6x}{(1-x^2)^4} = \frac{2x(x^2+3)}{(1-x^2)^4} \rightarrow x=0 \text{ sole inflection point}$$



If y is an (inverse) fn of x , then $x = f(y)$. $y = \frac{x}{1-x^2} = f(x)$ is y as fn of x .

However, given y we can find x , so if we think of y as the indep. variable, then we can think of x as fn of y . We can go through the process of isolating all x expression from y . If we switch their names it can perhaps (or not) become clearer what variable we're trying to isolate.

$$y = f^{-1}(x) \rightarrow x = f(y) = \frac{y}{1-y^2} \longrightarrow$$

$$\rightarrow x - x \cdot y^2 = y \rightarrow x y^2 + y - x = 0$$

$$x=0 \rightarrow y=0$$

If we restrict f 's domain to $(-1, 1)$ the image is $(-\infty, \infty)$. This means the domain of f^{-1} is $(-\infty, \infty)$ and the image of f^{-1} is $(-1, 1)$.

Also, $y < 0 \Leftrightarrow x < 0$ and $y > 0 \Leftrightarrow x > 0$

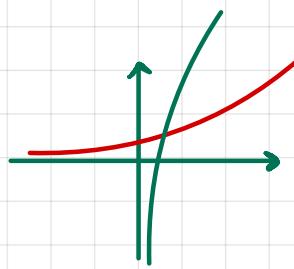
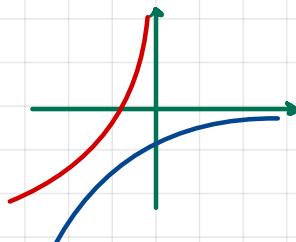
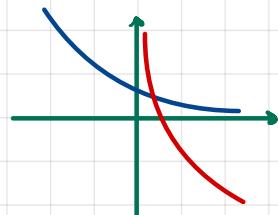
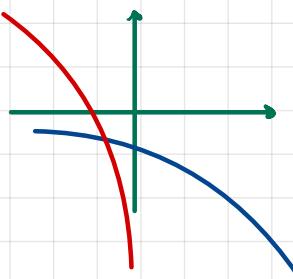
$$x \neq 0 \rightarrow 1+4x^2 > 0 \rightarrow y = \frac{-1 \pm \sqrt{1+4x^2}}{2x}. \text{ Note, however, that } y = \frac{-1 - \sqrt{1+4x^2}}{2x} < 0 \text{ if } x > 0 \text{ and}$$

> 0 if $x < 0$, so this isn't the defined sol'n. In fact, note that since $\sqrt{1+4x^2} > 1$ for $x \neq 0$ then $-1 - \sqrt{1+4x^2} < -2$ so $y < -2/2x = -1/x$

This incised sol'n represents the inverse of f for $|x| > 1$.

$$y = f^{-1}(x) = \frac{-1 + \sqrt{1+4x^2}}{2x} \text{ is the correct fn.}$$

2.

(i) f increasing, $f' > 0$ f increas. $\rightarrow f$ one-one $\rightarrow f'$ increasing fnThe image of f does not include values ≤ 0 so f' is not defined for $x \leq 0$.(ii) f increasing, $f' < 0$ f' is increasing fn, not defined for $x \geq 0$.(iii) f decreasing, $f' > 0$ f' decreases fn, undefined for $x \leq 0$ (iv) f decreases, $f' < 0$ f' decreases fn, undefined for $x \geq 0$ 

3.

Theorem f increasing $\Leftrightarrow f'$ increasing

Proof

$$x_1 = f(y_1) \longleftrightarrow y_1 = f(x_1)$$

$$x_2 = f'(y_2) \longleftrightarrow y_2 = f(x_2)$$

Let y_1 and y_2 be in f 's domain.

Then for some x_1, x_2 in f 's domain we have $y_1 = f(x_1)$ and $y_2 = f(x_2)$

Assume $y_1 < y_2$.

Then $f(x_1) < f(x_2)$ so $x_1 < x_2$.

But $x_1 = f'(y_1)$ and $x_2 = f'(y_2)$.

Hence $f'(y_1) < f'(y_2)$

$y_1 < y_2 \rightarrow f'(y_1) < f'(y_2)$

$\forall y_1, y_2: y_1 < y_2 \rightarrow f'(y_1) < f'(y_2)$

f increasing $\rightarrow f'$ increasing

Theorem f decreasing $\rightarrow f'$ decreasing

Proof

f decreasing \rightarrow one-one $\rightarrow f'$ I.n.

Let $y_1 = f(x_1), y_2 = f(x_2)$ in f 's domain.

Assume $y_1 = f(x_1) < f(x_2) = y_2$.

Then since f decreasing, $x_1 > x_2$.

$\rightarrow f'(y_1) > f'(y_2)$

$y_1 < y_2 \rightarrow f'(y_1) > f'(y_2)$

f' decreasing

4. f, g increasing

$$\forall x \forall y x < y \rightarrow f(x) < f(y) \wedge g(x) < g(y)$$

$$\rightarrow f(x) + g(x) < f(y) + g(y)$$

$\rightarrow f+g$ increasing

Let $f(x) - g(x) = k$, both increasing.

Then $(f \cdot g)(x) = x^2$, decreasing if $x \leq 0$.

$$\forall x \forall y x < y \rightarrow g(x) < g(y) \rightarrow f(g(x)) < f(g(y))$$

Therefore, $f+g$ and $f \cdot g$ are increasing, $f \cdot g$ not necessarily.

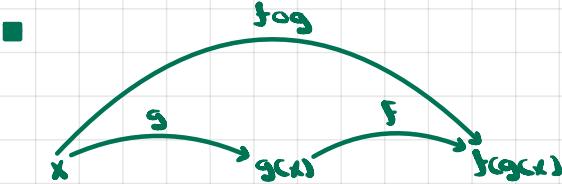
5.

(a) f, g one-one $\rightarrow f \circ g$ one-one

Proof

$$\forall x \forall y x \neq y \rightarrow g(x) \neq g(y) \rightarrow f(g(x)) \neq f(g(y))$$

$\rightarrow f \circ g$ one-one



$$f'(f(g(x))) = g(x)$$

$$g'(f'(f(g(x)))) = g'(g(x)) = x$$

Hence if $h(x) = (g' \circ f')(x)$ then

$$h(f(g(x))) = (g' \circ f') \circ (f \circ g) = I$$

and therefore

$$g' \circ f' = (f \circ g)'$$

$$a < b$$

$$c < d$$

$$\rightarrow a+c < b+c < b+d$$

$$\rightarrow a+c < b+d \checkmark$$

$$a-c < b-c \text{ but } b-c > b-d$$

we cannot establish a relationship between $a-c$ and $b-d$.

Slight, different perspective

$$\text{let } j = (f \circ g)^{-1}(x)$$

$$\text{then } x = (f \circ g)(j) = f(g(j))$$

$$\text{But then } g(j) = f'(x)$$

$$j = g'(f'(x)) = (g' \circ f')(x)$$

$$(b) g(x) = 1 + f(x)$$

$$y = g'(x) \rightarrow x = g(y) = 1 + f(y)$$

$$f(y) = x - 1$$

$$y = f'(x-1)$$

$$\rightarrow g'(x) = f'(x-1)$$

Another path is

$$h(x) = 1 + x$$

$$x = 1 + f \rightarrow y = x - 1 \rightarrow h'(y) = x - 1$$

$$g(x) = (h \circ f)(x) = h(f(x)) = 1 + f(x)$$

$$\begin{aligned} g'(x) \cdot (h \circ f)'(x) &= (f' \circ h')(x) \\ &= f'(h'(x)) \\ &= f'(x-1) \end{aligned}$$

$$6. f(x) = \frac{ax+b}{cx+d}$$
 is one-one $\Leftrightarrow ad-bc \neq 0$

Proof

$$f(x) = \frac{ax+b}{cx+d}$$
 is one-one

$$\begin{aligned} \Leftrightarrow \forall x_1 \forall x_2 x_1 \neq x_2 &\Leftrightarrow \frac{ax_1+b}{cx_1+d} \neq \frac{ax_2+b}{cx_2+d} \Leftrightarrow ax_1y_1 + adx_1 + bcy_1 + bd \neq ax_2y_1 + bcy_2 + ady_2 + bd \\ &\Leftrightarrow adx_1 + bcy_1 + bcy_2 + ady_2 \\ &\Leftrightarrow bc(y_1 - x_2) + ad(y_1 - x_2) \\ &\Leftrightarrow bc + ad \end{aligned}$$

$$\text{Let } y = f^{-1}(x).$$

$$\text{Then } x = f(y) = \frac{ay+b}{cy+d}$$

$$cyx + xd = ay + b$$

$$y(cx-a) = b - xd$$

$$y = f^{-1}(x) = \frac{b - xd}{cx - a} \quad x \neq \frac{a}{c}$$

7.

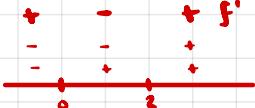
$$(i) f(x) = x^3 - 3x^2 + x^2(x-3)$$

roots at 0 and 3



$$f'(x) = 3x^2 - 6x + 3x(x-2)$$

critical points at 0 and 2



$$f''(x) = 6x - 6 = 0 \Rightarrow x=1$$



From f' we see that f is

increasing on $(-\infty, 0]$

decreasing on $[0, 2]$

increasing on $[2, +\infty)$

It is one-one on these three intervals, or subintervals of each.

$$(ii) f(x) = x^4 + x$$

$$f'(x) = 4x^3 + 1 > 0$$

f increasing on \mathbb{R} . f one-one on any interval.

$$(iii) f(x) = (1+x^2)^{-1}$$

$$f'(x) = \frac{(-1) \cdot 2x}{(1+x^2)^2} \quad \begin{array}{c} + \\ \hline - \end{array} \quad f'$$

f one-one on $(-\infty, 0]$ and $[0, +\infty)$ or subintervals of either.

$$(iv) f(x) = \frac{x+1}{x^2+1}$$

$$f'(x) = \frac{x^2+1-2x(x+1)}{(x^2+1)^2} = \frac{x^2+1-2x^2-2x}{(x^2+1)^2} = \frac{-x^2-2x+1}{(x^2+1)^2}$$



$$\Delta = 4 - 4(-1)(-1) - 8 \Rightarrow x = \frac{2 \pm 2\sqrt{2}}{-2} = -1 \pm \sqrt{2}$$

f one-one on $(-\infty, -\sqrt{2}-1], [-\sqrt{2}-1, \sqrt{2}-1], [\sqrt{2}-1, \infty)$

c. f diff

$\rightarrow g \circ f^{-1}$ satisfies $g'(x) = \frac{3}{2}g(x)^2$

$$f'(x) = \frac{1}{\sqrt{1+x^3}}$$

Proof

$g(x) = f^{-1}(x)$, that is g is simply the inverse of f .

Therefore $g'(x) = (f^{-1})'(x)$

$$g''(x) = (f^{-1})''(x)$$

f^{-1} is differentiable at every x in its domain because f is cont. (since it is diff.) and one-one (since f^{-1} is a function).

Theorem 5 $\rightarrow (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = [1 + (f^{-1}(x))^3]^{-\frac{1}{2}} = g'(x)$

$$g''(x) = \frac{1}{2} [1 + (f^{-1}(x))^3]^{-\frac{3}{2}} \cdot 3(f^{-1}(x))^2 \cdot (f^{-1})'(x)$$

$$= \frac{3}{2} \cdot (f^{-1}(x))^2 \cdot \frac{(f^{-1})'(x)}{\sqrt{1 + (f^{-1}(x))^3}}$$

But $(f^{-1})'(x) = \sqrt{1 + (f^{-1}(x))^3}$

Therefore, $g''(x) = \frac{3}{2} (f^{-1}(x))^2$
 $= \frac{3}{2} g(x)^2$

Theorem 5 Let f be cont. one-one & defined on interval.

Suppose f diff at $f^{-1}(b)$, w/ derivative $f'(f^{-1}(b)) \neq 0$.

Then,

$$f^{-1}$$
 diff at b and $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$

d. f one-one $(f^{-1})' \neq 0 \rightarrow f$ differentiable

Proof

f one-one $\rightarrow f^{-1}$ 1-1

f 1-1 $\rightarrow f^{-1}$ one-one

$(f^{-1})' \neq 0$ implies f' diff. everywhere.

Let x be in f 's domain.

Then since $(f^{-1})'(f^{-1}(x)) \neq 0$

Theorem 5 says that f diff. at x and

$$f'(x) = \frac{1}{(f^{-1})'(f^{-1}(x))}$$

Recall problem 10-17

11. f and g inv

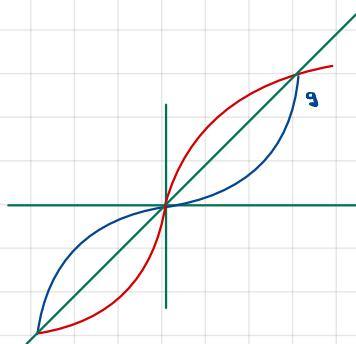
g takes on all values

fog and g diff.

f not diff.

$$g(x) = x^3, \text{ diff } \forall x$$

$$f(x) = \sqrt[3]{x}$$



$$y'(x) = \frac{1}{3\sqrt[3]{x^2}}, \text{ undefined at } x=0$$

$$(f \circ g)(x) = x$$

In light of the current chapter, it just so happens that the f's chosen in 10-17 are inverses

$$y = g(x) = x^3$$

$$x = g^{-1}(y) \rightarrow y = g^{-1}(y)^3 \rightarrow g^{-1}(y) = \sqrt[3]{y} = f(y).$$

since $g'(x) = 3x^2 = 0 \rightarrow x=0$, g' is not diff at 0.

If we choose a $\ln g$ that has derivative to 0 elsewhere, then if $f \circ g'$, f is diff. everywhere and $f \circ g \circ g' \circ g = I$, which is diff.

Another way to see this

$$f = (f \circ g) \circ g'$$

The Chain Rule says that if g' diff at a and $f \circ g$ diff at $g'(a)$ then $(f \circ g) \circ g'$ diff at a.

$f \circ g$ is diff everywhere by assumption, so the only additional condition we need is that g' diff, which happens when $g' \neq 0$ for all a.

11. $(f^{-1})''(x)$

Let $h(x) = f^{-1}(x)$

$$h'(x) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$\begin{aligned} h''(x) = (f^{-1})''(x) &= \frac{-[f'(f^{-1}(x))]'}{[f'(f^{-1}(x))]^2} \\ &= \frac{-[f''(f^{-1}(x)) \cdot (f^{-1})'(x)]}{[f'(f^{-1}(x))]^2} \end{aligned}$$

12. $f'(f^{-1}(x)) \neq 0$

$\Rightarrow (f^{-1})^{(n)}(x)$ exists

$f^{(n)}(f^{-1}(x))$ exists

Proof

$f'(f^{-1}(x)) \neq 0$ means that for any x in f' 's domain (i.e. f 's image), the derivative at the corresponding element $f^{-1}(x)$ in f 's domain is $\neq 0$.

Now f is continuous since f' is assumed to exist, and one-one since f' is a fn.
By Th.s, $(f^{-1})'(x)$ exists.

Let $x \in f$.

Assume $f^{(n)}(x) \geq f^{(m)}(y)$.

n-1. The assumptions are that $f'(f^{-1}(x))$ exists and is $\neq 0$. By Th.s, $(f^{-1})'(x) = 1/f'(f^{-1}(x))$

Assume the statement is true for n-h.

since $f^{(n)}(f^{-1}(x))$ exists so does $f^{(m)}(f^{-1}(x))$ for $1 \leq m < n$.

$$\text{From } (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$(f^{-1})''(x) = \frac{-[f''(f^{-1}(x)) \cdot (f^{-1})'(x)]}{[f'(f^{-1}(x))]^2} \quad (\text{by problem 11})$$

13.

Recall

R. f, f', f'', f''' exist, i.e. three times differentiable.

$$f'(x) \neq 0$$

$$\text{Schwarzian derivative of } f: D_f(x) = \frac{f''(x)}{f'(x)} - \frac{3}{2} \left(\frac{f''(x)}{f'(x)} \right)^2$$

(a) $D_f(x)$ exists for all $x \rightarrow f^{-1}(x)$ also exists for all x in f^{-1} 's domain

Proof

Assume $D_f(x)$ exists for all x .

$$\Leftrightarrow [\forall x, x \in f^{-1}\text{'s domain} \rightarrow f''(x) \neq 0]$$

Then for all x ,

$$\Leftrightarrow [\forall y_1, y_2 \in f^{-1}\text{'s domain} \rightarrow f^{-1}(f(y_1)) \neq 0]$$

(1) f is three times diff

(2) $f'(x) \neq 0$

$$(3) D_f(x) = \frac{f''(x)}{f'(x)} - \frac{3}{2} \left(\frac{f''(x)}{f'(x)} \right)^2$$

(2) $\rightarrow (f^{-1})'(x)$ exists for all x in f^{-1} 's domain and $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \neq 0$

Now we use problem R

Since

$f'(f^{-1}(x)) \neq 0$ and $f''(f^{-1}(x)), f'''(f^{-1}(x))$ exist, then $(f^{-1})''(f^{-1}(x))$ and $(f^{-1})'''(f^{-1}(x))$ exist.

At this point we have

$(f^{-1})'(x) \neq 0$, for all x in f^{-1} 's domain

f^{-1} three times diff. on entire domain of f^{-1}

Therefore, the Schwarzian derivative of f^{-1} exists for all x in f^{-1} 's domain.

(b) Formula for $Df^{-1}(x)$

$$(f \circ f^{-1}) = I$$

$$(f \circ f^{-1})(x) = x$$

$$D(f \circ f^{-1}) = [Df \circ Df^{-1}] \cdot [(f^{-1})']^2 + Df^{-1}$$

$$\rightarrow Df^{-1} = \cancel{Df^2} - [Df \circ f^{-1}] \cdot [(f^{-1})']^2$$

$$Df^{-1}(x) = -[Df(f^{-1}(x)) \cdot \frac{1}{f'(f^{-1}(x))^2}]$$

$$\rightarrow Df^{-1}(x) = \frac{Df(f^{-1}(x))}{[f'(f^{-1}(x))]^2}$$

14. $\forall x \in \mathbb{R}, f$ differentiable $\Rightarrow f(x^5 + f(x) + x) = 0$ for all x .

Proof

$f(x)^5 + f(x) + x = 0$ is an eq. for x in domain of f .

If f has an inverse, then $x = f^{-1}(y)$.

Hence

$$y^5 + y + f^{-1}(y) = 0$$

$$f^{-1}(y) = -y^5 - y$$

$$(f^{-1})'(y) = -5y^4 - 1 < 0$$

$\rightarrow f^{-1}$ is decreasing and have one-one $\rightarrow f$ decreasing, one-one

f one-one $\rightarrow f$ is a fn

f decreasing $\rightarrow f$ deccres.

f^{-1} takes on all values, ie the image of f^{-1} is \mathbb{R} .

Hence, the domain of f^{-1} is \mathbb{R} .

Since $(f^{-1})'(x) \neq 0$ for all x , f^{-1} exists for all x and equals

$$(b) f'(x) = \frac{1}{(f^{-1})'(f(x))} = \frac{1}{-5f(x)^4 - 1} \quad (\text{Theorem 5})$$

$$(c) 5f(x)^4 f'(x) + f'(x) + 1 = 0$$

$$f'(x)(5f(x)^4 + 1) = -1$$

$$f'(x) = \frac{1}{-5f(x)^4 - 1}$$

Slightly different

Let $g(x) = -x^5 - x$ be a fn.

g is decreasing, one-one and $\text{Im}(g) = \mathbb{R}$.

Therefore

one-one $\rightarrow f = g^{-1}$ is a fn
domain of $f = \mathbb{R}$

$$g(f(x)) = g(g^{-1}(x)) = x = -f(x)^5 - f(x)$$

$g'(x) = -5x^4 - 1 \neq 0$ for all x , hence f' exists for all x and equals

$$f'(x) = \frac{1}{g'(f(x))} = \frac{1}{-5f(x)^4 - 1}$$

$$15. (a) x^2 + y^2 = 1 \quad (1)$$

Two functions $y = f(x)$ are defined implicitly by (1) on $x \in (-1, 1)$.

$$y^2 = 1 - x^2$$

$$y = \pm \sqrt{1 - x^2}$$

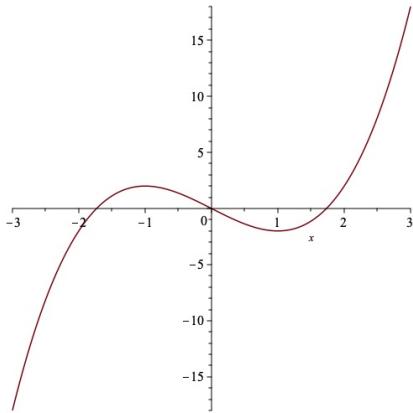
For a given x in $(0, 1)$, eq. (1) is satisfied whether we plug in $\sqrt{1 - x^2}$ or $-\sqrt{1 - x^2}$.

$$(b) x^2 + [f(x)]^2 = -1 \quad (1)$$

There is no function satisfying (1).

$$(c) [f(x)]^3 - 3x^2 = x \quad (1)$$

Consider $g(x) = x^3 - 3x$.



If g^{-1} exists then $g(g^{-1}(x)) = x = [g^{-1}(x)]^3 - 3g^{-1}(x)$.

To g^{-1} satisfies (1). Since g isn't one-one on \mathbb{R} , there isn't a g^{-1} on \mathbb{R} .

However, since $g'(x) = 3x^2 - 3 = 0 \rightarrow x = \pm 1$, g has local max at -1 , local min at 1 , and is one-one on $(-\infty, -1)$, $(-1, 1)$, and $(1, +\infty)$. Hence there is an inverse g_i on each interval. Also $g'(x) \neq 0$ in these intervals.

So if we define

$$g_1(x) = g(x), x \in (-\infty, -1)$$

$$g_2(x) = g(x), x \in (-1, 1)$$

$$g_3(x) = g(x), x \in (1, +\infty)$$

$$\text{Then } g_i(g_i^{-1}(x)) = x = [g_i^{-1}(x)]^3 - 3g_i^{-1}(x) \quad (2)$$

In order to obtain an explicit formula $g_i^{-1}(x)$ we'd need to solve (2).

Note that since $g_i(x) \neq 0$, g_i^{-1} is differentiable in its domain.

$$16. (a) [f(x)]^2 + x^2 = 1$$

$$2f(x)f'(x) + 2x = 0$$

$$f'(x) = \frac{-2x}{2f(x)} = -\frac{x}{f(x)}$$

$$(b) f(x) \cdot \sqrt{1-x^2} \Rightarrow f'(x) \cdot -\frac{x}{\sqrt{1-x^2}}$$

$$f(x) \cdot \sqrt{1+x^2} \Rightarrow f'(x) \cdot -\frac{x}{\sqrt{1+x^2}}$$

$$(c) [f(x)]^3 - 3f(x) = x$$

$$3f(x)^2 f'(x) - 3f'(x) = 1$$

$$3f'(x)(f(x)^2 - 1) = 1$$

$$f'(x) = \frac{1}{3(f(x)^2 - 1)}$$

$$17. (a) x^2 + y^2 = 1$$

$$y = \sqrt[3]{1-x^2} = f(x). \text{ We could differentiate directly here.}$$

Or, if we assume $y = f(x)$ we can diff. implicitly.

$$3x^2 + 3y^2 y' = 0 \Rightarrow y' \cdot f'(x) = -\frac{x^2}{f(x)^2}$$

$$2x + 2yy' \cdot y' + y^2 y'' = 0$$

$$y'' = f''(x) = \frac{-2(x+y y')}{y^2} = \frac{-2(x+f(x)f'(x))^2}{(f(x))^2}$$

$$(b) f(-1) = 2$$

$$f'(-1) = \frac{-(-1)}{4} = -\frac{1}{4}$$

$$f''(-1) = \frac{-2(-1+2 \cdot \frac{1}{16})}{4} = \frac{2 - \frac{1}{4}}{4} = \frac{7}{16}$$

$$18. 3x^2 + 4x^2 y - xy^2 + 2y^3 = 4$$

Eq. of tangent at $(-1, 1)$?

Assume $y = f(x)$.

$$9x^2 + 8x^2 y + 4x^2 y' - y^2 - x \cdot 2y y' + 6y^2 y' = 0$$

Let's already sub in $x = -1, y = 1$.

$$9 - 8 + 4y' - 1 + 2y' + 6y' = 0$$

$$12y' = 0 \Rightarrow f'(-1) = 0$$

Tangent line: $g(x) = 1$

$$19. y^4 + y^3 + 4y = 1$$

$$4y^3 y' + 3y^2 y' + y + x y' = 0$$

$$y' = \frac{-5}{4y^3 + 3y^2 + x}$$

20.

Theorem 5 Let f be cont. one-one & n defined on interval.

Suppose F diff at $F'(b)$, w/ derivative $f'(F'(b)) \neq 0$.

Then,

$$f \text{ diff. at } b \text{ and } \left| \frac{dy}{dx} \right|_{x=b} = \frac{1}{\left| \frac{dx}{dy} \right|_{y=F(b)}}$$

$x = f^n$ \times explicitly find y

$$\frac{1}{n} - 1 - \frac{1}{n-1}$$

$y = x^{\frac{1}{n}}$ \times explicitly find dx

$$\frac{dx^{\frac{1}{n}}}{dx} = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{nx^{\frac{n-1}{n}}} = \frac{1}{nx^{1-\frac{1}{n}}} = nx^{\frac{1}{n}-1}$$

diff both sides inverse in done.

21. If diff, one-one

$$f' \neq 0$$

$$f = F'$$

$$G(x) = xF'(x) - F(F'(x))$$

$$\rightarrow G(x) \cdot F'(x)$$

Proof

$$\begin{aligned} G'(x) &= F''(x) + x(F'(x))' - F(F'(x))(F'(x))'(x) \\ &= F''(x) \end{aligned}$$

$$22. h(x) = \sin^2(\sin(x+1))$$

$$h(0) = 3$$

$$(i) (h')'(3)$$

$$h'(x) = 0 \text{ for } x = k\pi - 1, k = -1, 0, 1, 2, \dots$$

$$h(0) = 3 \rightarrow h'(3) = 0$$

$$h'(h'(3)) = h'(0) = \sin^2(\sin 1) \neq 0$$

$$\rightarrow (h')'(3) = \frac{1}{h'(h'(3))} = \frac{1}{\sin^2(\sin 1)}$$

$$(ii) \beta(x) = h(x+1)$$

$$(\beta')'(3)$$

$$\beta(-1) = h(0) = 3$$

$$\rightarrow \beta'(3) = -1$$

$$\beta'(x) = h'(x+1)$$

$$(\beta')'(x) = \frac{1}{\beta'(\beta'(x))}$$

$$(\beta')'(3) = \frac{1}{\beta'(\beta'(3))}$$

$$= \frac{1}{\beta'(-1)}$$

$$= \frac{1}{h'(0)}$$

$$= \frac{1}{\sin^2(\sin 1)}$$

23. (a) An increasing and a decreasing fn intersect at most once.

Proof

Let f be increasing and g decreasing.

Assume they intersect at two points, $(x_1, f(x_1)) = (x_1, g(x_1))$ and $(x_2, f(x_2)) = (x_2, g(x_2))$, $x_1 < x_2$.

Case 1: $f(x_1) < f(x_2)$.

Then $g(x_1) > g(x_2)$. \perp .

Case 2: $f(x_1) > f(x_2)$. \perp .

Case 3: $f(x_1) = g(x_1) = f(x_2) = g(x_2)$. \perp .

\perp .

Therefore f and g intersect at most once.

Alternatively,

Assume f and g intersect at a

$$f(a) = g(a)$$

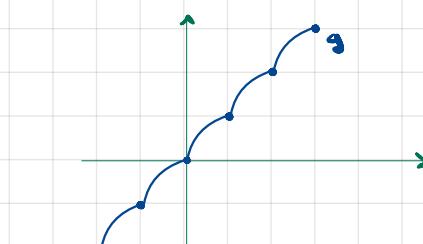
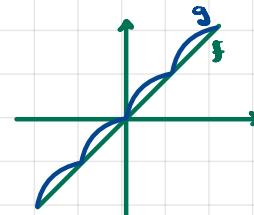
$$\begin{aligned} \forall x, x > a \rightarrow f(x) > f(a) > g(x) \\ x < a \rightarrow f(x) < f(a) < g(x) \end{aligned}$$

(b) f, g cont., increasing.

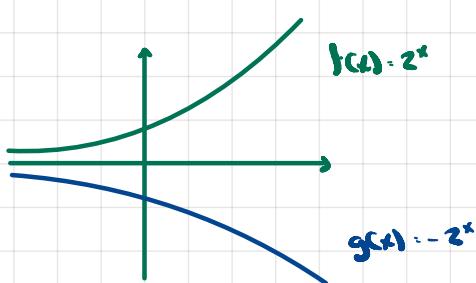
$$f(x) = g(x) \text{ for } x \in \mathbb{R}$$

$$f(x) = x$$

$$g(x) = [x] + \sqrt{x - [x]}$$



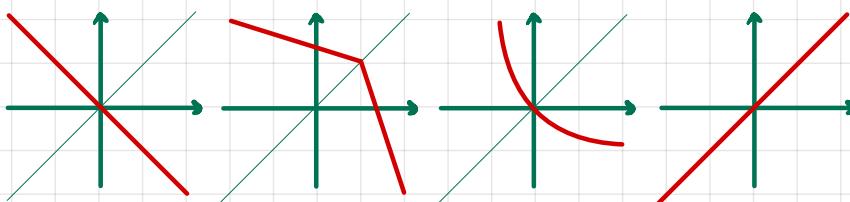
(c) f cont. increasing, g cont. decreasing, f and g don't intersect



24. (a) f is onto \mathbb{R} . $\exists x, f(x) = x$
 $f \circ f^{-1}$

Note

Geometrically, $f = f^{-1}$ means that these functions are symmetric relative to $y = x$.



The f 's all seem to be decreasing fns, other than $f(x) = x$.
 Let's prove that they must be, unless we have $f(x) = x$.

Assume $f \neq f^{-1}$.

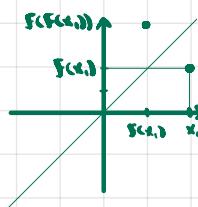
Assume f increasing.

Let $(x_1, f(x_1))$ be a point on f 's graph.

If $f = f^{-1}$ then $f^{-1}(f(x_1)) = f(f(x_1)) = x_1$.

Assume $x_1 > f(x_1)$

Then $f^{-1}(f(x_1)) = f(f(x_1)) = x_1 > f(x_1)$.

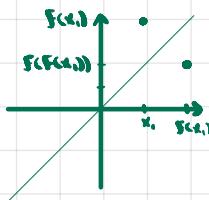


We have $f(x_1) < x_1 \wedge f(f(x_1)) > f(x_1)$

1. because f increasing

Assume $x_1 < f(x_1)$

Then $f^{-1}(f(x_1)) = x_1 < f(f(x_1)) < f(x_1)$



We have $x_1 < f(x_1) \wedge f(x_1) > f(f(x_1))$

1. because f increasing

Therefore $x_1 = f(x_1)$

f increasing $\rightarrow f(x) = x$

f decreasing or $f(x) = x$

Now let's prove our main result

Proof

Assume f cont. and $f \circ f^{-1}$.

Then either f decreasing or $f(x) = f(y)$

Case 1: $f(x) = x$

TF: $\exists x, f(x) = x$

Case 2: f decreasing

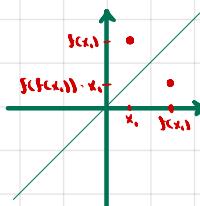
let $(x_1, f(x_1))$ be a point on f 's graph.

Then $f^{-1}(f(x_1)) = f(f(x_1)) = x_1$

Case 2.1: $x_1 < f(x_1)$

Then

$$x_1 < f(x_1) \wedge f(x_1) > f(f(x_1))$$



let $g(x) = f(x) - x$.

$$\text{Then } g(x_1) = f(x_1) - x_1 > 0$$

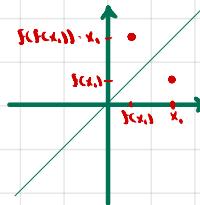
$$g(f(x_1)) = f(f(x_1)) - f(x_1)$$

$$= x_1 - f(x_1) < 0$$

Since g is cont, INT $\rightarrow g$ is 0 for some $x \in (x_1, f(x_1))$

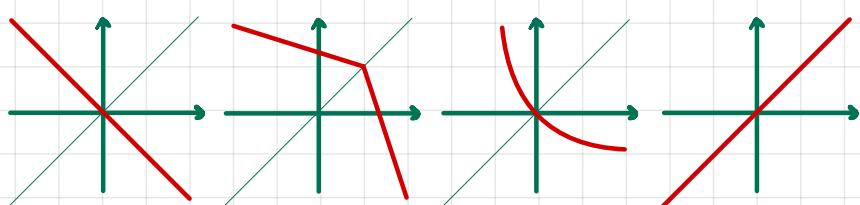
i.e. $f(x) = x$.

Case 2.2: $x_1 > f(x_1)$



Analogous to Case 2.1.

(b)



(c)

Assume $f = f^{-1}$.

Assume f increasing.

Let $(x_1, f(x_1))$ be a point on f 's graph.

If $f = f^{-1}$ then $f^{-1}(f(x_1)) = f(f(x_1)) = x_1$.

Assume $x_1 > f(x_1)$

Then $f^{-1}(f(x_1)) = f(f(x_1)) = x_1 > f(x_1)$.

We have $f(x_1) < x_1 \wedge f(f(x_1)) > f(x_1)$

L, because f increasing

Assume $x_1 < f(x_1)$

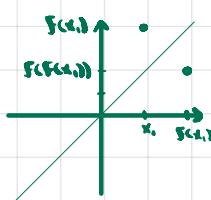
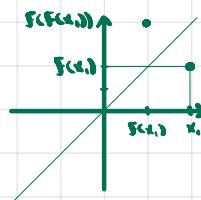
Then $f^{-1}(f(x_1)) = x_1 < f(f(x_1)) < f(x_1)$

We have $x_1 < f(x_1) \wedge f(x_1) > f(f(x_1))$

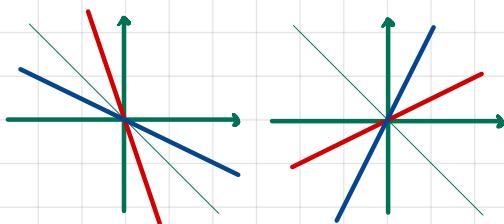
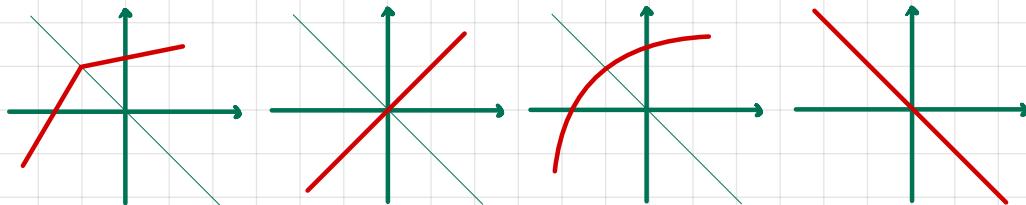
L, because f increasing

Therefore $x_1 = f(x_1)$

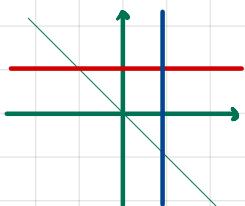
f increasing $\rightarrow f(x) = x$



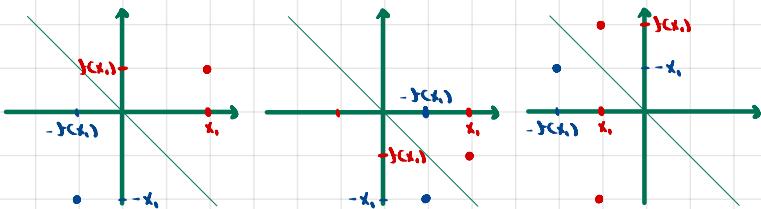
25.



Examples that don't work



From the graphs, it seems that f must be one-one



f one-one $\rightarrow f$ reflected through antidiagonal is a fn.

Proof

Let $(x_1, f(x_1))$ be a point.

Since f is one-one, for any other $(x, f(x))$ we have $x \neq x_1 \Leftrightarrow f(x) \neq f(x_1)$ (1)

Consider the fn represented by points $(-f(x_1), -x_1)$, $x_1 \in \text{domain of } f$.

The domain of this fn is the range of f .

It is in fact a fn because of (1): $f(x) = f(x_1) \rightarrow x = x_1$.

It is one-one also because of (1): $f(x) \neq f(x_1) \rightarrow x \neq x_1$.

26.

(a) f non-decreasing, not increasing $\rightarrow f$ constant on some interval

Proof

f not increasing means $\exists x, y$ in domain, $x < y \wedge f(x) \geq f(y)$ (1)

f non-decreasing means $\forall x, y \in \text{domain}, x < y \rightarrow f(x) \leq f(y)$ (2)

Therefore, $\exists x, y$ in domain, $x < y \wedge f(x) = f(y)$.

let x, y be such that $f(x) = f(y)$, $x < y \wedge f(x) = f(y)$

consider the interval (x, y) .

For any $x, z \in (x, y)$, if $f(z) > f(x) = f(y)$ then f decreasing between x and y . \perp .

if $f(z) < f(x) = f(y)$ " " decreasing " " $x = z \perp$.

T.F. $\forall x, z \in (x, y) \rightarrow f(z) = f(x) = f(y)$

$\rightarrow f$ constant on (x, y) .

(b) f diff. and non-decreasing $\rightarrow f'(a) \geq 0$ for all a .

Proof

f non-decreasing means $\forall x, y \in \text{domain}, x < y \rightarrow f(x) \leq f(y)$

$$\forall a, f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a}$$

But $\frac{f(x) - f(a)}{x - a} \geq 0$ for $x \rightarrow a^+$.

Therefore $f'(a) \geq 0$.

(c) $f'(x) \geq 0$ for all $x \rightarrow f$ is nondecreasing

Proof

let x, y in f 's domain, $x < y$.

$$\text{NNT} \rightarrow \exists c, c \in (x, y) \wedge \frac{f(y) - f(x)}{y - x} = f'(c) \geq 0$$

$$\rightarrow f(y) \geq f(x)$$

T.F. $\forall x, y$ in domain, $x < y \rightarrow f(x) \leq f(y)$

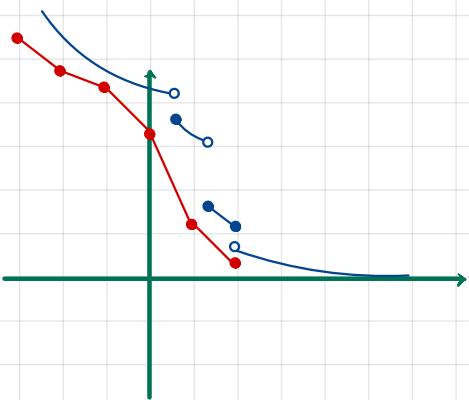
$\rightarrow f$ nondecreasing.

27.

(a) $f(x) > 0$ for all $x \rightarrow \exists g$, g continuous and decreasing s.t. $0 < g(x) \leq f(x)$ for all x
 ↓ decreasing

Proof

In the interval $[n, n+1]$, let g be the linear function that $g(n) = f(n+1)$ and $g(n+1) = f(n+2)$.



Let $x \in [n, n+1]$.

$$g(n) = f(n+1)$$

$$\forall x, x \in [n, n+1] \rightarrow f(x) > f(n+1) = g(n) \geq g(x)$$

$$(b) \lim_{x \rightarrow \infty} \frac{g(x)}{f(x)} = 0$$

on $[n, n+1]$, let g be the linear function that $g(n) = \frac{f(n+1)}{n+1}$ and $g(n+1) = \frac{f(n+2)}{n+2}$

That since $\lim_{x \rightarrow \infty} f(x)$ must be some number ∞ , and $\lim_{n \rightarrow \infty} (n+1) = \infty$, we have $\lim_{x \rightarrow \infty} \frac{g(x)}{f(x)} = 0$