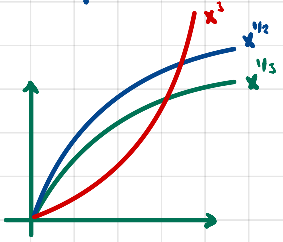
Ch 8 Appendix - Uniform Continuity

1.

a) $f(x) = x^{\alpha}$, u.c. on $[0,\infty)$ for which α?



$1/2, 1/3$

Proof that x^3 not u.c.

$$|x^3 - y^3| = |(x-y)(x^2+xy+y^2)|$$

If $|x-y|<1$ then $x^2-2xy+y^2<1$

$$x^2+xy+y^2 < 1+3xy < 4$$

$$|x^3-y^3| < 4|x-y| < \epsilon \leftrightarrow |x-y| < \frac{\epsilon}{4}$$

For some y, let $x = y + \frac{\epsilon}{8}$

Then, $|x-y| = \frac{\epsilon}{8} < \frac{\epsilon}{4}$

$$|f(y+\tfrac{\epsilon}{8}) - f(y)| = |\cancel{y^3} + 3y^2\tfrac{\epsilon}{8} + 3y\tfrac{\epsilon^2}{64} + \tfrac{\epsilon^3}{8^3} - \cancel{y^3}|$$

$$> y\frac{\epsilon^2}{64}$$

$$\frac{y\epsilon^{\cancel{2}}}{64} > \cancel{\epsilon} \rightarrow y > \frac{64}{\epsilon}$$

Therefore for a given $\epsilon>0$, if $y > \frac{64}{\epsilon}$ then we have

$x = y + \frac{\epsilon}{8}$, $|x-y| < \frac{\epsilon}{4}$ and $|f(x)-f(y)| > \epsilon$.

b) f cont. and bounded on $(0,1]$, not u.c. on $(0,1]$.

Consider $f(x) = \sin(1/x)$

Let $0<\epsilon<2$

Consider the interval $(0,\delta)$ for any $\delta>0$.

If $x = \frac{2}{\pi n}$, $n\in\mathbb{N}$ then $\sin(1/x) = 1$.

Since $\lim_{n\to\infty} \frac{2}{\pi n} = 0$, we can make $x = \frac{2}{\pi n}$ arbitrarily close to 0.

In particular, $\exists N$ s.t. $n>N \rightarrow x<\delta$.
There are therefore infinite numbers in $(0,\delta)$ of form $\frac{2\pi}{n}$, $n\in\mathbb{N}$.
For such x, $\sin(1/x) = \sin(n\frac{\pi}{2}) = 1$.
Similarly we can show there are infinite $y \in (0,\delta)$ s.t. $\sin(y) = -1$.
Hence, $|\sin x - \sin y| = 2 > \epsilon$.

Thus we've shown that $\exists\epsilon>0$ s.t. $\forall\delta>0$, $\exists x,y \in (0,1]$
$|x-y|<\delta$ and $|f(x)-f(y)| > \epsilon$.

Thus $f(x) = \sin 1/x$ is not u.c. on $(0,1]$.

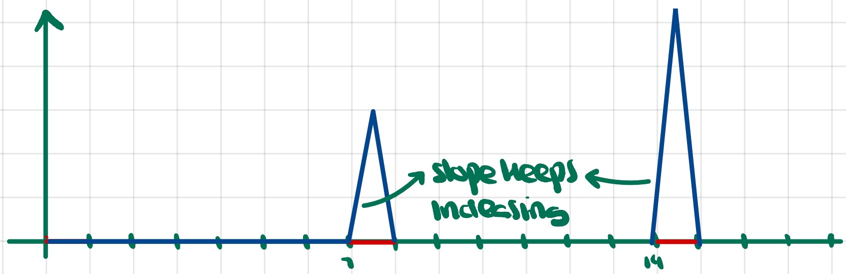
* note that $\sin 1/x$ is not c. at 0, so it's also not u.c. on $[0,1]$.

If we take any other interval $(a,1]$, $a>0$, then $\sin 1/x$ is u.c. on $(a,1]$. Just use the same $\delta>0$ used to show continuity at a, given any $\epsilon>0$.

c) f continuous, bounded on $[0,\infty)$, not u.c. on $[0,\infty)$.

We can come up with some sketchy function like:

$$f(x) = \begin{cases} 0 & x \notin [7n, 7n+1],\ n\in\mathbb{N} \\ 7nx & x \in [7n, 7n+1/2],\ n\in\mathbb{N} \\ 7n(7n+\frac{1}{2}) - 7nx & x \in [7n+1/2, 7n+1],\ n\in\mathbb{N} \end{cases}$$



2.

a) f, g u.c. on $A \to f+g$ u.c. on A

Proof

f u.c. on A: $\forall \epsilon>0\ \exists \delta_1>0\ \forall x, \forall y,\ x \in A \wedge y \in A \wedge |x-y|<\delta_1 \to |f(x)-f(y)|< \frac{\epsilon}{2}$

g u.c. on A: $\forall \epsilon>0\ \exists \delta_2>0\ \forall x, \forall y,\ x \in A \wedge y \in A \wedge |x-y|<\delta_2 \to |g(x)-g(y)|< \frac{\epsilon}{2}$

Therefore, if $\delta = \min(\delta_1, \delta_2)$

$|x-y|<\delta \to |(f(x)+g(x))-(f(y)+g(y))| < |f(x)-f(y)|+|g(x)-g(y)|< \epsilon$

Therefore,

$\forall \epsilon>0, \exists \delta>0, \forall x, \forall y,\ x \in A \wedge y \in A \wedge |x-y|<\delta \to |(f(x)+g(x))-(f(y)+g(y))| < \epsilon.$

$f+g$ u.c. on A.

b) f, g u.c. bounded on $A \to fg$ u.c. on A

Proof

since f, g bounded in A, $\exists M$ s.t. $\forall x, x \in A \to |f(x)|< M \wedge |g(x)|< M$

since f, g u.c, choose a $\delta>0$ s.t. $\forall x, y$

$x \in A, y \in A \to \quad |f(x)-f(y)|< \frac{\epsilon}{2M}$

$|g(x)-g(y)|< \frac{\epsilon}{2M}$

Then,

$|f(x)g(x)-f(y)g(y)|$

$= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)|$

$= |f(x)(g(x)-g(y)) + g(y)(f(x)-f(y))|$

$\leq |M \frac{\epsilon}{2M} + M \frac{\epsilon}{2M}|$

$= \epsilon$

c) $f(x) = x$ and $g(x) = \sin x$ are both u.c. on $[0, +\infty)$.

$f \cdot g$ is not u.c. on $[0, \infty)$.

As x increases, the amplitude of one cycle of $f \cdot g$ grows without bound though the period stays the same. The fn grows at an ever more rapid pace at the same section of each subsequent cycle. No single $\delta > 0$ can serve to constrain f to be ϵ distance from every $f(y)$.

d) f u.c. on A
g u.c. on B $\quad \rightarrow \quad g \circ f$ u.c. on A
$\forall x,\ x \in A \rightarrow f(x) \in B$

Proof

Let $\epsilon > 0$.
Then $\exists \delta_1 > 0,\ \forall x, y,\ x \in B \wedge y \in B \wedge |x - y| < \delta_1 \rightarrow |g(x) - g(y)| < \epsilon$

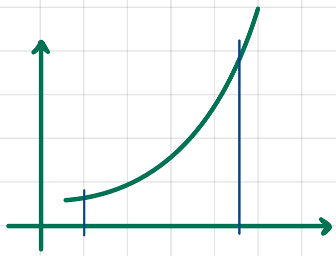
Given δ_1, $\exists \delta > 0$ s.t. $\forall x, y,\ x \in A \wedge y \in A \wedge |x - y| < \delta \rightarrow |f(x) - f(y)| < \delta_1$

Thus

$\forall x, \forall y,\ x \in A \wedge y \in B \wedge |x - y| < \delta \rightarrow |f(x) - f(y)| < \delta_1 \wedge f(x) \in B \wedge f(y) \in B$

$\rightarrow |g(f(x)) - g(f(y))| < \epsilon$

3. Theorem 1 f cont. on $[a,b]$ $\rightarrow$ f u.c. on $[a,b]$



* f is u.c on $[a,b]$ $\leftrightarrow$ f ϵ-good on $[a,b]$ for every $\epsilon>0$

let $\epsilon>0$.

Assume f not ϵ-good on $[a,b]$.
Recall the lemma used in the chapter's main text:

$a<\frac{a+b}{2}<b$, f cont. on $[a,b]$
let $\epsilon>0$ and following are true
i) x,y both in $[a,\frac{a+b}{2}]$, $|x-y|<\delta_1$ $\rightarrow$ $|f(x)-f(y)|<\epsilon$, ie f is ϵ-good on $[a,\frac{a+b}{2}]$
ii) x,y both in $[\frac{a+b}{2},c]$, $|x-y|<\delta_2$ $\rightarrow$ $|f(x)-f(y)|<\epsilon$, ie f is ϵ-good on $[\frac{a+b}{2},b]$
then,

$\exists\delta>0$ s.t. x,y both in $[a,b]$, $|x-y|<\delta$ $\rightarrow$ $|f(x)-f(y)|<\epsilon$
ie, f is ϵ-good on $[a,b]$.

Hence our assumption is the negation of the consequent. Therefore either i) or ii) must be false.

let I_1 be one of the halves in which f is not ϵ-good. Repeat the procedure on I_1: bisect and let I_2 be a half where f is not ϵ-good.

The intervals I_n are nested, thus there is some x_0 that is in all of them.

Recall that as a result of how we defined the intervals, f is not u.c. on any of the nested intervals.

But f is cont. at x_0 so we can choose $\delta>0$ s.t.

$\forall x, |x-x_0|<\delta \rightarrow |f(x)-f(x_0)|<\frac{\epsilon}{2}$

$\forall y, |y-x_0|<\delta \rightarrow |f(y)-f(x_0)|<\frac{\epsilon}{2}$

Therefore

$\forall x \forall y\ |x-x_0|<\delta$ and $|y-y_0|<\delta \rightarrow |f(y)-f(x)|<\epsilon$

f is ϵ-good on $(x_0-\delta, x_0+\delta)$.

$\perp$.

f is ϵ-good on $[a,b]$
$\forall\epsilon>0$, f is ϵ-good on $[a,b]$.

4.

Theorem 1 f cont. on $[a,b] \to f$ u.c. on $[a,b]$

Theorem 7-2 f cont. on $[a,b]$ $\to$ f bounded above on $[a,b]$, ie $\exists N, \forall x \in [a,b], f(x) \leq N$

Theorem 1 $\to$ Theorem 7-2

Intuitively, if f is u.c. on an interval, then for any x in the interval, within a certain distance from x, f stays within a certain distance from $f(x)$.

Assume Theorem 1.

Let f be cont. on $[a,b]$.

Then by Th.1, f is u.c. on $[a,b]$.

For given $\varepsilon>0$, choose $\delta>0$ s.t.

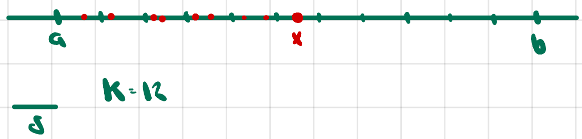
$\forall x \forall y, x \in [a,b], y \in [a,b], |x-y|<\delta \to |f(x)-f(y)|<\varepsilon$

Let $K = \dfrac{b-a}{\delta} + 1$

Then for any $x \in [a,b]$ there is a sequence

$a = a_0, a_1, a_2, \ldots, a_n = x$

with $n < K$ and $|a_{i+1} - a_i| < \delta$



Thus,

$|f(a_1) - f(a)| < \varepsilon$
$|f(a_2) - f(a_1)| < \varepsilon$
$\ldots$
$f(x) - f(a_{n-1}) < \varepsilon$

$\to |f(x) - f(a)| < K\varepsilon$
$|f(x)| \leq f(a) + K\varepsilon, \ \forall x \in [a,b]$

Therefore f is bounded on $[a,b]$.

f cont on $[a,b] \to f$ is bounded on $[a,b]$.

Th.1 $\to$ Th.7-2