Ch13 - Integrals

1. $\int_0^b x^3\,dx = \frac{b^4}{4}$

Consider the interval $[0,b]$.

Let $P_n = \{t_0, \dots, t_n\}$ be the partition of $[0,b]$ into n equally spaced subintervals.

Then, in the i^{th} subinterval,

$$* \sum_{i=1}^{n} i^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

$$t_i - t_{i-1} = \frac{b}{n}$$

$$m_i = t_{i-1}^3 = \left(\frac{b(i-1)}{n}\right)^3 = \frac{b^3(i-1)^3}{n^3}$$

$$M_i = t_i^3 = \left(\frac{bi}{n}\right)^3 = \frac{b^3 i^3}{n^3}$$

$$L(f,P_n) = \sum_{i=1}^{n} t_{i-1}^3(t_i - t_{i-1}) = \sum_{i=1}^{n} \frac{b^4(i-1)^3}{n^4} = \frac{b^4}{n^4}\sum_{i=1}^{n}(i-1)^3 = \frac{b^4}{n^4}\sum_{i=0}^{n-1} i^3 = \frac{b^4}{n^4}\left[\frac{(n-1)^4}{4} + \frac{(n-1)^3}{2} + \frac{(n-1)^2}{4}\right]$$

$$= \frac{b^4}{4}\left[\frac{(n-1)^4}{n^4} + 2\frac{(n-1)^3}{n^4} + \frac{(n-1)^2}{n^4}\right]$$

$$U(f,P_n) = \sum_{i=1}^{n} t_i^3(t_i - t_{i-1}) = \sum_{i=1}^{n} \frac{b^4 i^3}{n^4} = \frac{b^4}{n^4}\sum_{i=1}^{n} i^3 = \frac{b^4}{n^4}\left[\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right] = \frac{b^4}{4}\left[1 + \frac{2}{n} + \frac{1}{n^2}\right] > \frac{b^4}{4}$$

But note that

$$\frac{b^4}{4}\left[\frac{(n-1)^4}{n^4} + 2\frac{(n-1)^3}{n^4} + \frac{(n-1)^2}{n^4}\right] = \frac{b^4}{4}\left[\left(\frac{n-1}{n}\right)^2\left(\left(\frac{n-1}{n}\right)^2 + \frac{2(n-1)}{n^2} + \frac{1}{n^2}\right)\right]$$

$$= \frac{b^4}{4}\left[\left(\frac{n-1}{n}\right)^2\left(\frac{n^2 \cancel{-2n} \cancel{+1} \cancel{+2n} \cancel{-2} \cancel{+1}}{n^2}\right)\right]$$

$$= \frac{b^4}{4}\cdot\left(\frac{n-1}{n}\right)^2 < \frac{b^4}{4}$$

Therefore

$U(f,P_n) - L(f,P_n) = \frac{b^4}{4}\cdot\frac{4}{n} = \frac{b^4}{n}$, therefore we

can make this difference as small as we want

$\to f$ integrable.

$L(f,P_n) < \frac{b^4}{4} < U(f,P_n)$ for all P_n.

But there can be only one number with this property.

The number $\int_a^b f$ has this property by definition, so

$$\frac{b^4}{4} = \sup\{L(f,P_n)\} = \inf\{U(f,P_n)\} = \int_0^b f$$

Proof

Suppose $L(f,P_n) < v_1 < U(f,P_n)$
$L(f,P_n) < v_2 < U(f,P_n)$ for all n

and $\forall \epsilon > 0\ \exists P_n$ s.t. $U(f,P_n) - L(f,P_n) < \epsilon$

Assume $v_1 \neq v_2$

Case 1: $v_1 < v_2$

Let $v_1 < c_1 < v_2$

Choose $\epsilon = \frac{v_2 - v_1}{2}$. Then,

$$U(f,P_n) < L(f,P_n) + \frac{v_2 - v_1}{2} < v_1 + \frac{v_2 - v_1}{2}$$

$$= \frac{v_1 + v_2}{2} < v_2.\ \perp.$$

Case 2, $v_2 < v_1$ is symmetric $\to \perp$.

Therefore, $v_1 = v_2$.

2. $\int_0^b x^4\,dx = \frac{b^5}{5}$

Proof

The goal is to show that $f(x) = x^4$ is integrable and that the number $\int_0^b x^4\,dx$, the integral of x^4 on $[0,b]$ is $\frac{b^5}{5}$. This is the unique number such that

$* \sum_{i=1}^{n} i^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$

$L(f,P) \le \int_0^b f \le U(f,P)$ for all P on $[0,b]$.

We start with an equally spaced partition of $[0,b]$, $P_n = \{t_0, \dots, t_n\}$ where $t_i - t_{i-1} = \frac{b}{n}$.

In each partition subinterval, $m_i = t_{i-1}^4 = \left(\frac{b(i-1)}{n}\right)^4$, $M_i = t_i^4 = \left(\frac{bi}{n}\right)^4$, $\Delta t = t_i - t_{i-1} = \frac{b}{n}$

$$L(f,P_n) = \sum_{i=1}^{n} m_i \Delta t_i = \sum t_{i-1}^4 \Delta t_i = \sum_{i=1}^{n}\left(\frac{b(i-1)}{n}\right)^4 \cdot \frac{b}{n} = \frac{b^5}{n^5}\sum_{i=1}^{n}(i-1)^4 = \frac{b^5}{n^5}\sum_{i=0}^{n-1} i^4 = \frac{b^5}{n^5}\left[\frac{(n-1)^5}{5} + \frac{(n-1)^4}{2} + \frac{(n-1)^3}{3} - \frac{n-1}{30}\right]$$

$$= \frac{b^5}{5}\cdot\frac{1}{n^5}\left[(n-1)^5 + \frac{5}{2}(n-1)^4 + \frac{5}{3}(n-1)^3 - \frac{1}{6}(n-1)\right]$$

$$= \frac{b^5}{5}\,\frac{n-1}{n^5}\left[(n-1)^4 + \frac{5}{2}(n-1)^3 + \frac{5}{3}(n-1)^2 - \frac{1}{6}\right] = \frac{b^5}{5}\,\frac{n-1}{n^5}\left[(n-1)\left((n-1)^3 + \frac{5}{2}(n-1)^2 + \frac{5}{3}(n-1)\right) - \frac{1}{6}\right]$$

$$= \frac{b^5}{5}\,\frac{n-1}{n^5}\left[(n-1)\left(\frac{6(n^3-3n^2+3n-1)+15(n^2-2n+1)+10(n-1)}{6}\right) - \frac{1}{6}\right] = \frac{b^5}{5}\,\frac{n-1}{n^5}\left[(n-1)\,\frac{6n^3-3n^2-2n-1}{6} - \frac{1}{6}\right]$$

$$= \frac{b^5}{5}\,\frac{n-1}{n^5}\cdot\frac{1}{6}\left[6n^4-3n^3-2n^2-n-6n^3+3n^2+2n+\cancel{1}-\cancel{1}\right] = \frac{b^5}{5}\,\frac{n-1}{n^5}\cdot\frac{1}{6}\left[6n^4-9n^3+n^2+n\right]$$

$$= \frac{b^5}{5}\cdot\frac{1}{6n^5}\left[6n^5-9n^4+n^3+\cancel{n^2}-6n^4+9n^3-\cancel{n^2}-n\right] = \frac{b^5}{5}\cdot\frac{1}{6n^5}\left[6n^5-15n^4+10n^3-n\right]$$

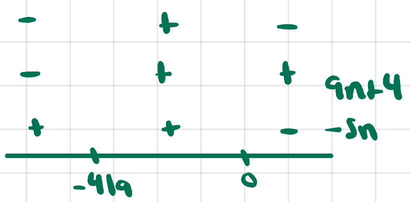
$$= \frac{b^5}{5}\,\frac{1}{6n^4}\left[6n^4-15n^3+10n^2-1\right]$$

$$\frac{1}{6n^4}\left[6n^4-15n^3+10n^2-1\right] = 1 - \frac{5}{2n} + \frac{5}{3n^2} - \frac{1}{6n^4} < 1 \rightarrow -\frac{5}{2n} + \frac{5}{3n^2} - \frac{1}{6n^4} < 0$$

$$\rightarrow 6n^4\left(-\frac{5}{2n} + \frac{5}{3n^2} - \frac{1}{6n^4}\right) < 0$$

$$\rightarrow -15n^3 + 10n^2 - 1 < 0$$

Let $g(n) = -15n^3 + 10n^2 - 1$



$g'(n) = -45n^2 + 20n < 0 \rightarrow -5n(9n+4) < 0$

$g'(n) < 0$ for $n > 0$

$g(1) = -15 + 10 - 1 = -6$

Therefore, $\frac{b^5}{5}\,\frac{1}{6n^4}\left[6n^4-15n^3+10n^2-1\right] = L(f,P) < \frac{b^5}{5}$

$U(f,P_n) = \sum_{i=1}^{n} M_i \Delta t_i = \sum_{i=1}^{n} t_i^4 \Delta t_i = \sum_{i=1}^{n} \left(\frac{bi}{n}\right)^4 \cdot \frac{b}{n} = \frac{b^5}{n^5}\sum_{i=1}^{n} i^4 = \frac{b^5}{n^5}\left[\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}\right]$

$= \frac{b^5}{5} \cdot \frac{1}{n^5}\left[n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 - \frac{n}{6}\right]$

$\frac{1}{n^5}\left[n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 - \frac{n}{6}\right] > 1$

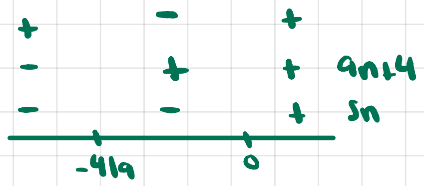
$1 + \frac{5}{2n} + \frac{5}{3n^2} - \frac{1}{6n^4} > 1$

$\rightarrow 6n^4\left(\frac{5}{2n} + \frac{5}{3n^2} - \frac{1}{6n^4}\right) > 0$

$\rightarrow 15n^3 + 10n^2 - 1 > 0$

Let $g(n) = 15n^3 + 10n^2 - 1$

$g'(n) = 45n^2 + 20n = 5n(9n+4)$



$g'(n) > 0$ for $n > 0$

$g(1) = 45 + 20 = 65 > 0$

Therefore, for any $n \in \mathbb{N}$, $\frac{1}{n^5}\left[n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 - \frac{n}{6}\right] > 1$

$\rightarrow \frac{b^5}{n^5} < U(f,P)$

Hence, we have

$L(f,P) < \frac{b^5}{5} < U(f,P)$ for any P.

$\rightarrow \int_0^b x^4 = \frac{b^5}{5}$

3. (a) $\sum_{k=1}^{n} \frac{k^p}{n^{p+1}}$ can be made as close to $\frac{1}{p+1}$ as desired, choosing n large enough

Proof

recall problem 2-7 $\sum_{i=1}^{n} k^p$ can always be written in the form $\frac{n^{p+1}}{p+1} + A_0 n^p + A_1 n^{p-1} + A_2 n^{p-2} + \cdots + A_{p-1} n$

$$\sum_{k=1}^{n} \frac{k^p}{n^{p+1}} = \frac{1}{n^{p+1}} \sum k^p = \frac{1}{n^{p+1}} \left[\frac{n^{p+1}}{p+1} + A_0 n^p + A_1 n^{p-1} + A_3 n^{p-2} + \cdots \right]$$

$$= \frac{1}{p+1} + A_1 n^{-1} + A_2 n^{-2} + \cdots + A_{p-1} n^{-p}$$

$$\sum_{k=1}^{n} \frac{k^p}{n^{p+1}} - \frac{1}{p+1} = A_1 n^{-1} + A_2 n^{-2} + \cdots + A_{p-1} n^{-p}$$

Let $A = \max(A_1, A_2, \ldots, A_{p-1})$

Then $A_1 n^{-1} + A_2 n^{-2} + \cdots + A_{p-1} n^{-p} \leq A n^{-1} + A n^{-2} + \cdots + A n^{-p} = A\left(\frac{1}{n} + \frac{1}{n^2} + \cdots + \frac{1}{n^{p-1}}\right)$

---

lemma: $\forall n, n \in \mathbb{N} \rightarrow \frac{1}{n} \geq \frac{1}{n^2}$

Proof by induction $n=1 \rightarrow 1 \geq 1$. Assume $\frac{1}{n} \geq \frac{1}{n^2}$. $\frac{1}{n+1} < 1$ since $n > 1$. $\rightarrow \frac{1}{n+1} \cdot \frac{1}{n+1} < \frac{1}{n+1}$ ■

Therefore,

$$A\left(\frac{1}{n} + \frac{1}{n^2} + \cdots + \frac{1}{n^{p-1}}\right) \leq A(p-1)\frac{1}{n}$$

For any $\epsilon > 0$, if $n > \frac{A(p-1)}{\epsilon}$ then $A\left(\frac{1}{n} + \frac{1}{n^2} + \cdots + \frac{1}{n^{p-1}}\right) < \epsilon$

That is,

$$n > \frac{A(p-1)}{\epsilon} \rightarrow \sum_{k=1}^{n} \frac{k^p}{n^{p+1}} - \frac{1}{p+1} < \epsilon$$

(b) $\int_0^b x^p dx = \frac{b^{p+1}}{p+1}$

Proof

Using the same reasoning as in problems 1 and 2, we reach

$$U(f,P_n) = \sum_{i=1}^n M_i \Delta t_i = \sum_{i=1}^n t_i \Delta t_i = \sum_{i=1}^n \left(\frac{bi}{n}\right)^p \cdot \frac{b}{n} = \frac{b^{p+1}}{n^{p+1}} \sum_{i=1}^n i^p = b^{p+1} \sum_{i=1}^n \frac{i^p}{n^{p+1}}$$

increasing n means increasing the number of subintervals in the partition.

From part a), for large enough n, $\sum_{i=1}^n \frac{i^p}{n^{p+1}}$ is as close to $\frac{1}{p+1}$ as desired.

Ie, $\forall \epsilon_1 > 0$

$$\sum_{i=1}^n \frac{i^p}{n^{p+1}} - \frac{1}{p+1} < \epsilon_1 \rightarrow b^{p+1} \sum_{i=1}^n \frac{i^p}{n^{p+1}} - \frac{b^{p+1}}{p+1} < b^{p+1} \epsilon_1$$

$\forall \epsilon > 0$, let $\epsilon_1 = \frac{\epsilon}{b^{p+1}}$. Then, $b^{p+1} \sum_{i=1}^n \frac{i^p}{n^{p+1}} - \frac{b^{p+1}}{p+1} < \epsilon$.

Therefore, we can make $U(f,P_n) = b^{p+1} \sum_{i=1}^n \frac{i^p}{n^{p+1}}$ arbitrarily close to $\frac{b^{p+1}}{p+1}$.

Similarly,

$$L(f,P_n) = \frac{b^{p+1}}{n^{p+1}} \sum_{i=1}^n (i-1)^p = \frac{b^{p+1}}{n^{p+1}} \sum_{i=1}^{n-1} i^p = \left(\frac{n-1}{n}\right)^{p+1} \cdot \frac{b^{p+1}}{(n-1)^{p+1}} \sum_{i=1}^{n-1} i^p$$

which we can also make arbitr. close to $\frac{b^{p+1}}{p+1}$.

$U(f,P) - L(f,P) = \frac{b^{p+1}}{n^{p+1}} n^p = \frac{b^{p+1}}{n}$. Ie $U(f,P) - L(f,P) < \epsilon$, $\forall \epsilon > 0$.

Therefore f integrable on $[a,b]$, and $L(f,P) \leq \int_0^b f \leq U(f,P)$, $\forall P$

Assume $\frac{b^{p+1}}{p+1} < L(f,P)$ for some partition P.

Then it is not possible to make $U(f,P)$ as close to $\frac{b^{p+1}}{p+1}$ as we want (e.g. $\epsilon = \frac{L(f,P) - \frac{b^{p+1}}{p+1}}{2}$). $\perp$.

Similarly, if we assume $\frac{b^{p+1}}{p+1} > U(f,P)$ then it's not possible to make $L(f,P)$ arbitrarily close to $\frac{b^{p+1}}{p+1}$. $\perp$.

Therefore, for all partitions P it must be that $L(f,P) \leq \frac{b^{p+1}}{p+1} \leq U(f,P)$.

But we know such a number is unique, since f integrable.

Hence $\int_0^b f = \int_0^b x^p dx = \frac{b^{p+1}}{p+1}$.

4. use partitions $P=\{t_0,\dots,t_n\}$ for which all ratios $r=\frac{t_i}{t_{i-1}}$ are equal, instead of having t_i-t_{i-1} equal.

(a) $t_i = ac^{i/n}$ for $c=\frac{b}{a}$

Proof

$b = t_n = rt_{n-1} = r(rt_{n-2}) = r^3 t_{n-3} = \dots = r^n t_0 = r^n a$

$b = r^n a \to r = \sqrt[n]{\frac{b}{a}}$

$\to t_i = a \cdot \frac{t_1}{a} \cdot \frac{t_2}{t_1} \cdot \dots \cdot \frac{t_i}{t_{i-1}} = a \cdot r^i = a\left(\frac{b}{a}\right)^{i/n}$

---

example: $n=3$, $a=1$, $b=2$

$r = \sqrt[3]{2} \approx 1.26$

$a = t_0 = 1$
$t_1 \approx 1.26$
$t_2 \approx 1.26 \cdot 1.26 \approx 1.58$
$t_3 \approx 1.26^3 \approx 2$

example: $n=3, a=2, b=5$

$r=\sqrt[3]{5/2}$, ie the number that when multiplied by itself three times gives the multiple $5/2$ which when multiplied by $a=2$ gives $b=5$

(b) $f(x)=x^p \to U(f,P) = a^{p+1}(1-c^{-1/n})\sum_{i=1}^{n}(c^{(p+1)/n})^i$

$$= (a^{p+1}-b^{p+1})\, c^{(p+1)/n}\, \frac{1-c^{-1/n}}{1-c^{(p+1)/n}}$$

$$= (b^{p+1}-a^{p+1})\, c^{p/n}\, \frac{1}{1+c^{1/n}+\dots+c^{p/n}}$$

Proof

recall problem 2-5 $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ if $r\neq 1$

By assumption $0<a<b$ and we partition $[a,b]$.

$$M_i = t_i^p = a^p c^{pi/n}$$

$$t_i - t_{i-1} = ac^{i/n} - ac^{(i-1)/n} = ac^{i/n}(1-c^{-1/n})$$

$$U(f,P) = \sum_{i=1}^{n} M_i(t_i-t_{i-1}) = \sum_{i=1}^{n} a^p c^{pi/n}\, ac^{i/n}(1-c^{-1/n}) = \sum_{i=1}^{n} a^{p+1} c^{i(p+1)/n}(1-c^{-1/n})$$

$$= a^{p+1}(1-c^{-1/n})\sum_{i=1}^{n}(c^{(p+1)/n})^i$$

$$= a^{p+1}(1-c^{-1/n})\, c^{\frac{p+1}{n}} \sum_{i=1}^{n}(c^{\frac{p+1}{n}})^{i-1} = a^{p+1}(1-c^{-1/n})\, c^{\frac{p+1}{n}} \sum_{i=0}^{n-1}(c^{\frac{p+1}{n}})^{i} = a^{p+1}(1-c^{-1/n})\, c^{\frac{p+1}{n}}\, \frac{1-c^{p+1}}{1-c^{\frac{p+1}{n}}}$$

$$= a^{p+1}(1-c^{p+1})\, c^{\frac{p+1}{n}}\, \frac{1-c^{-1/n}}{1-c^{\frac{p+1}{n}}} = (a^{p+1}-b^{p+1})\, c^{\frac{p+1}{n}}\, \frac{1-c^{-1/n}}{1-c^{\frac{p+1}{n}}} = (b^{p+1}-a^{p+1})\, c^{\frac{p}{n}}\, \frac{1-c^{1/n}}{1-c^{\frac{p+1}{n}}}$$

But note that $(c^{1/n})^0+(c^{1/n})^1+(c^{1/n})^2+\dots+(c^{1/n})^p = \frac{1-(c^{1/n})^{p+1}}{1-c^{1/n}}$

$$\to \frac{1-c^{1/n}}{1-c^{(p+1)/n}} = \frac{1}{1+c^{1/n}+c^{2/n}+\dots+c^{p/n}}$$

$$\to U(f,P) = (b^{p+1}-a^{p+1})\, c^{\frac{p}{n}}\, \frac{1}{1+c^{1/n}+c^{2/n}+\dots+c^{p/n}}$$

$$L(f,P) = \sum_{i=1}^{n} m_i(t_i-t_{i-1}) = \sum_{i=1}^{n} t_{i-1}^p(t_i-t_{i-1}) = \sum_{i=1}^{n} a^p c^{\frac{p}{n}(i-1)}\, ac^{i/n}(1-c^{-1/n}) = \sum_{i=1}^{n} a^{p+1} c^{\frac{i(p+1)-p}{n}}(1-c^{-1/n})$$

$$= \sum_{i=1}^{n} a^{p+1} c^{\frac{i(p+1)}{n}} c^{-\frac{p}{n}}(1-c^{-1/n}) = a^{p+1} c^{-\frac{p}{n}}(1-c^{-1/n})\sum_{i=1}^{n} c^{\frac{i(p+1)}{n}}$$

$$= c^{-\frac{p}{n}}(b^{p+1}-a^{p+1})\, c^{\frac{p}{n}}\, \frac{1}{1+c^{1/n}+c^{2/n}+\dots+c^{p/n}} = (b^{p+1}-a^{p+1})\, \frac{1}{1+c^{1/n}+c^{2/n}+\dots+c^{p/n}}$$

$$\to L(f,P) = c^{-\frac{p}{n}}\, U(f,P)$$

(c)

$$U(f,P) = (b^{p+1} - a^{p+1})\, c^{\frac{p}{n}} \frac{1}{1 + c^{1/n} + c^{2/n} + \cdots + c^{p/n}}$$

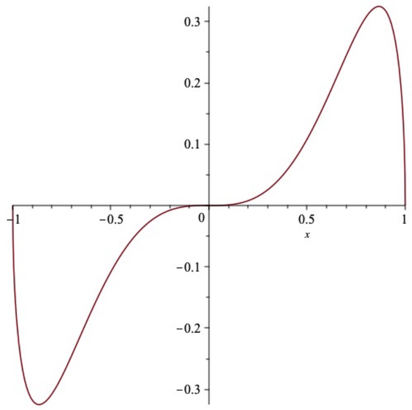
$$L(f,P) = c^{-\frac{p}{n}} U(f,P) = c^{-\frac{p}{n}} (b^{p+1} - a^{p+1})\, c^{\frac{p}{n}} \frac{1}{1 + c^{1/n} + c^{2/n} + \cdots + c^{p/n}}$$

As $n \to \infty$, every term $c^{\frac{k}{n}} \to 1$

Therefore, $U(f,P)$ and $L(f,p)$ approach $\dfrac{b^{p+1} - a^{p+1}}{1+p}$

5. (i) $\int_{-1}^{1} x^3\sqrt{1-x^2}\,dx$

Let $f(x) = x^3\sqrt{1-x^2}$



f is odd. On $[-1,0]$, the $L(f,P)$ and $U(f,P)$ are negative.

In fact, $\int_{-1}^{0} f = -\int_{0}^{1} f$, though this result wasn't addressed in the theory portion of this chapter.

Therefore $\int_{-1}^{1} f = 0$.

0, for same reason as a).

(ii) $\int_{-1}^{1}(x^5+3)\sqrt{1-x^2}\,dx = \int_{-1}^{1}\left(x^5\sqrt{1-x^2} + 3\sqrt{1-x^2}\right)dx = \cancel{\int_{-1}^{1} x^5\sqrt{1-x^2}\,dx} + 3\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{3\pi}{2}$

Let $y = \sqrt{1-x^2}$. This represents the top half of a circle of radius 1. The area is $\frac{\pi}{2}$.

6. $\int_0^x \frac{\sin t}{t+1}\,dt > 0$ for all $x>0$

Proof

Let $f(x) = \frac{\sin x}{x+1}$

For $k = 0,1,2,\dots$

$A_k = [k\cdot 2\pi,\ k\cdot 2\pi + \pi]$

$B_k = [\pi + k 2\pi,\ 2\pi + k\cdot 2\pi]$

on A_0, $\int_0^{\pi} f(x)\,dx$ is positive since $\sin(x)$ is positive in this interval.

on B_0, $\int_{\pi}^{2\pi} f(x)\,dx$ is negative. However, since $\left|\int_0^{\pi} \sin x\,dx\right| = \left|\int_{\pi}^{2\pi} \sin x\,dx\right|$, but $\frac{1}{t+1}$ is larger on B_0 than on A_0, then

$\left|\int_{\pi}^{x} f\right| < \left|\int_{\pi}^{2\pi} f\right| < \left|\int_0^{\pi} f\right|$, $x \in [0,\pi]$.

Therefore, $\int_0^x f > 0$ for $x \in [0, 2\pi]$.

The same argument can be made for $A_1 = [2\pi, 3\pi]$ and $B_1 = [3\pi, 4\pi]$ to reach the conclusion that

$\int_{2\pi}^{2\pi+x} f$ is positive and $\left|\int_{3\pi}^{x} f\right| < \left|\int_{3\pi}^{4\pi} f\right| < \left|\int_{2\pi}^{3\pi} f\right|$ for $x \in [0,\pi]$

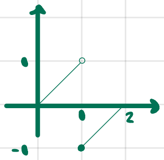
Therefore $\int_{2\pi}^{x} f > 0$ for $x \in [0, 2\pi]$.

Therefore $\int_0^x f > 0$ for $x \in [0, 4\pi]$.

Since we can make this argument for all A_k and B_k then

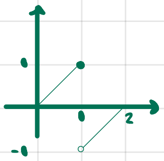
$\int_0^x f > 0$, for all $x > 0$.

7. (i) $f(x) = \begin{cases} x & 0 \le x < 1 \\ x-2 & 1 \le x \le 2 \end{cases}$



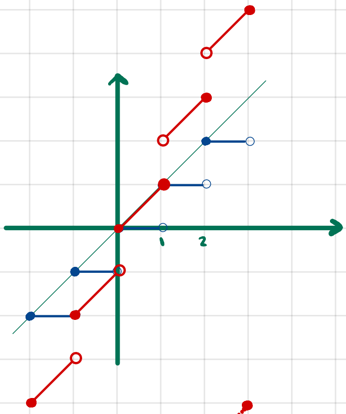
$\int_0^2 f = 0$

(ii) $f(x) = \begin{cases} x & 0 \le x \le 1 \\ x-2 & 1 < x \le 2 \end{cases}$



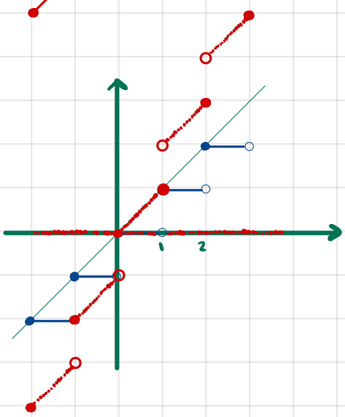
$\int_0^2 f = 0$

(iii) $f(x) = x + [x]$



$\int_0^2 f = \frac{1}{2} + \left(\frac{1}{2} + 2\right) = 3$

(iv) $f(x) = \begin{cases} x + [x] & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$



Not integrable.

$\sup\{L(f,P)\} < \inf\{U(f,P)\}$

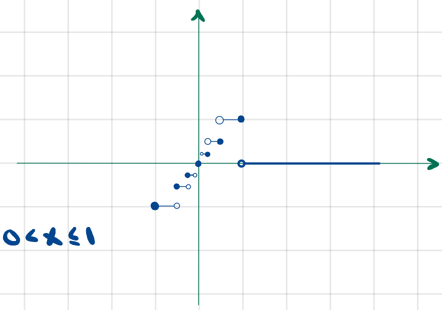
(v) $f(x) = \begin{cases} 1 & x \text{ of form } a + b\sqrt{2} \text{ for rational } a, b \\ 0 & x \text{ not of this form} \end{cases}$

$b\sqrt{2}$ irrational

$a + b\sqrt{2}$ irrational

f not integrable.

(vi) $f(x) = \begin{cases} \frac{1}{\left[\frac{1}{x}\right]} & 0 < x \le 1 \\ 0 & x = 0 \text{ or } x > 1 \end{cases}$



f integrable. $\int_0^2 f$ is an infinite sum

$1 \cdot \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3}\right) + \frac{1}{3}\left(\frac{1}{3} - \frac{1}{4}\right) + \dots$

(vii) $\frac{(2-1)\cdot 1}{2} + \frac{(1-1/2)\cdot 1}{2} + \frac{(1/2-1/4)\cdot 1}{2} + \frac{(1/4-1/8)\cdot 1}{2}$

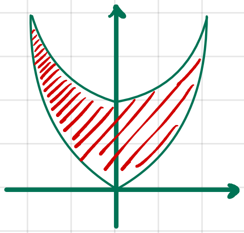
$+ (\dots)$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

$= \sum_{i=1}^{\infty} \frac{1}{2^i} = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i$

8. (i) $f(x) = x^2$ $\quad x^2 = \frac{x^2}{2} + 2 \rightarrow \frac{x^2}{2} = 2 \rightarrow x = \pm 2$

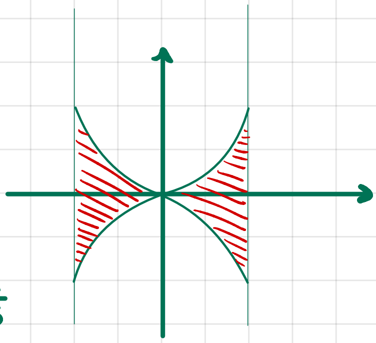
$g(x) = \frac{x^2}{2} + 2$



$$\int_{-2}^{2} x^2 dx = \frac{x^3}{3}\Big|_{-2}^{2} = \frac{8}{3} - \left(-\frac{8}{3}\right) = \frac{16}{3}$$

$$\int_{-2}^{2} \left(\frac{x^2}{2}+2\right) dx = \frac{1}{2}\frac{x^3}{3} + 2x \Big|_{-2}^{2} = \frac{8}{6} + 4 - \left(-\frac{8}{6} - 4\right) = \frac{16}{6} + 8 = \frac{8+24}{3} = \frac{32}{3}$$

$$\int_{-2}^{2} \left(\frac{x^2}{2} + 2 - x^2\right) = \frac{16}{3}$$



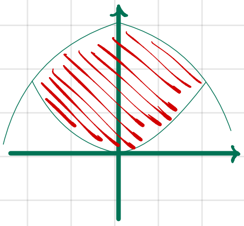
(ii) $f(x) = x^2$

$g(x) = -x^2$

$$\int_{-1}^{1} 2x^2 dx = 2\frac{x^3}{3}\Big|_{-1}^{1} = 2\left(\frac{1}{3} - \left(-\frac{1}{3}\right)\right) = \frac{4}{3}$$

(iii) $f(x) = x^2$

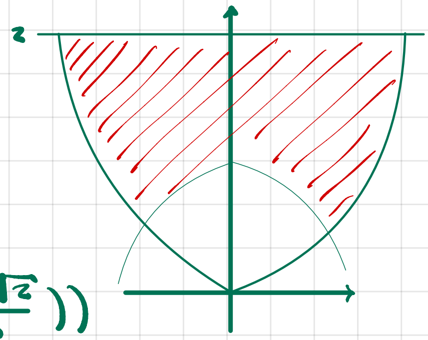
$g(x) = 1 - x^2$



$x^2 = 1 - x^2 \rightarrow x^2 = 1/2 \rightarrow x = \pm\frac{\sqrt{2}}{2}$

$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} (1 - x^2 - x^2) dx = \left(x - \frac{2}{3}x^3\right)\Big|_{-\sqrt{2}/2}^{\sqrt{2}/2} = \left(\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right) - \frac{2}{3}\left(\frac{2\sqrt{2}}{8} - \frac{-2\sqrt{2}}{8}\right)$$

$$= \frac{2\sqrt{2}}{2} - \frac{2}{3}\frac{\sqrt{2}}{2} = \frac{6\sqrt{2} - 2\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$$



(iv) $f(x) = x^2$

$g(x) = 1 - x^2$

$h(x) = 2$

$$\int_{-\sqrt{2}}^{\sqrt{2}} (2 - x^2) dx = \left(2x - \frac{x^3}{3}\right)\Big|_{-\sqrt{2}}^{\sqrt{2}} = (2\sqrt{2} - (-2\sqrt{2})) - \left(\frac{2\sqrt{2}}{3} - \left(-\frac{2\sqrt{2}}{3}\right)\right)$$

$$= 4\sqrt{2} - \frac{4\sqrt{2}}{3} = \frac{8\sqrt{2}}{3}$$

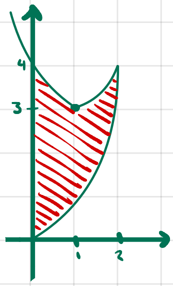
$$\frac{8\sqrt{2}}{3} - \frac{2\sqrt{2}}{3} = 2\sqrt{2}$$

(v) $f(x) = x^2$

$g(x) = x^2 - 2x + 4$

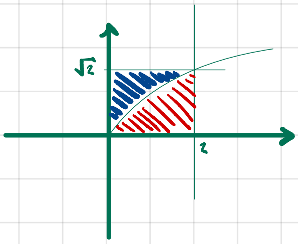
$\Delta = 4 - 4 \cdot 4 < 0 \rightarrow g(x) > 0$

$x^2 = x^2 - 2x + 4 \rightarrow 2x = 4 \rightarrow x = 2$



$\int_0^2 (-2x+4)\,dx = (4x - x^2)\Big|_0^2 = 8 - 4 = 4$

(vi) $f(x) = \sqrt{x}$



$y = \sqrt{f^{-1}(x)} \rightarrow f^{-1}(x) = y^2$

$\int_0^{\sqrt{2}} y^2\,dy = \frac{1}{3} 2\sqrt{2}$

$2 \cdot \sqrt{2} - \frac{2\sqrt{2}}{3} = \frac{4\sqrt{2}}{3}$

---

9. $\int_a^b \left(\int_c^d f(x) g(y)\, dy \right) dx$

$= \int_a^b f(x) \left(\int_c^d g(y)\, dy \right) dx$

$= \left(\int_c^d g(y)\, dy \right) \left(\int_a^b f(x)\, dx \right)$

10. $m_i' + m_i'' = \inf\{f(x_1)+g(x_2) : t_{i-1} \le x_1, x_2 \le t_i\} \le m_i$

Proof

Let $P = \{t_0, \dots, t_n\}$ be any partition of $[a,b]$.

Let $m_i = \inf\{(f+g)(x) : t_{i-1} \le x \le t_i\}$
$m_i' = \inf\{f(x) : t_{i-1} \le x \le t_i\}$
$m_i'' = \inf\{g(x) : t_{i-1} \le x \le t_i\}$

Let $A = \{f(x) : t_{i-1} \le x \le t_i\}$

$B = \{g(x) : t_{i-1} \le x \le t_i\}$

$A+B = \{f(x)+g(y) : f(x) \in A, g(y) \in B\}$

$= \{f(x)+g(y) : t_{i-1} \le x, y \le t_i\}$

Using problem 8-13,

$\inf A + \inf B = \inf(A+B)$

$m_i' + m_i'' = \inf\{f(x)+g(y) : t_{i-1} \le x, y \le t_i\}$

Now assume $m_i < m_i' + m_i''$.

Then $\exists x_1 \forall x \forall y \;\; x \in [t_{i-1}, t_i] \wedge y \in [t_{i-1}, t_i] \wedge x_1 \in [t_{i-1}, t_i]$

$\rightarrow m_i \le f(x_1) + g(x_1) < m_i' + m_i'' \le f(x) + g(y)$

But this expression must be true also for $x = x_1$ and $y = x_1$

$\rightarrow f(x_1) - g(x_1) < f(x_1) + g(x_1)$

$\perp$.

Therefore

$m_i' + m_i'' = \inf\{f(x)+g(y) : t_{i-1} \le x, y \le t_i\} \le m_i$

11. (a) every lower sum equals every upper sum

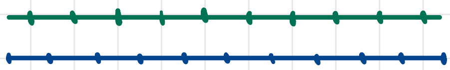
This is true for constant Fns.

Proof

Let P be some partition of $[a,b]$.

Assume $L(f,P) = U(f,P)$
In each partition subinterval, we have $m_i = M_i$, so $f(x) =$ constant in each partition subinterval.

By assumption $L(f,P') = U(f,P')$ for any other partition P'.



We can choose a partition such that the subintervals each overlap w/ two of P's subintervals, as pictured above.
Hence f is constant w/ the same value on all subintervals.

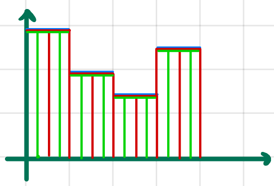
(b) some upper sum equals some (other) lower sum

Let $L(f,P_1) = L(f,P_2)$.
Let P contain P_1 and P_2.
Then

$$L(f,P_1) \le L(f,P) \le U(f,P) \le U(f,P_2) = L(f,P_1)$$

Hence $L(f,P) = U(f,P)$, so from part (a), f is constant.

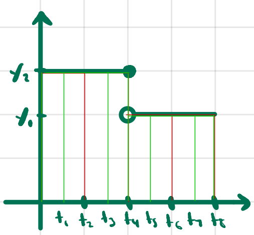
Here is an incorrect attempt at a counterexample



Assume there is some interval where f is not constant
$m_i \ne M_i$ in this interval and $m_i \Delta t_i \ne M_i \Delta t_i$.
Any partition which includes such a subinterval has $L(f,P) \ne U(f,P)$.

Therefore, if $L(f,P) = U(f,P)$ then f must be constant on each partition subinterval.

f is composed of intervals in which it is constant.



In $[t_4, t_5]$, $M_i = y_2$ and $m_i = y_1$, so $L(f,P) \ne U(f,P)$.

(c) all lower sums are equal

$L(f,P) = L(f,P')$

Assume f is not constant on $[a,b]$.
Then f takes on some minimum value m on $[a,b]$.
$\exists x, x \in [a,b] \wedge f(x) > m$

Since f is continuous, there is some interval around x in which f is close to $f(x)$ and above m.
We can choose a partition s.t. on a closed interval around x, $f > m$ so $L(f,P) > m(b-a)$.

We can also choose a partition $Q = \{a,b\}$ for which $L(f,Q) = m(b-a)$

$L(f,P) \neq L(f,Q)$

$\perp$.

T.F. f is constant on $[a,b]$.

(d) integrable fns w/ property that all lower sums are equal

12. $a<b<c<d$ $\rightarrow$ f integr. on $[b,c]$
f integrable on $[a,d]$

Proof

Recall Th.4

Theorem 4 Let $a<c<b$. Then,

f integrable on $[a,b]$ $\leftrightarrow$ f integrable on $[a,c]$ and $[c,b]$

f integrable on $[a,b]$ $\rightarrow$ $\int_a^b f = \int_a^c f + \int_c^b f$

since $a<b<d$ and f int. on $[a,d]$ then f int on $[b,d]$.

since $b<c<d$ and f int on $[b,d]$ then f int on $[b,c]$.

---

13. (a) f integrable on $[a,b]$ $\rightarrow$ $\int_a^b f \geq 0$
$\forall x, x \in [a,b] \rightarrow f(x) \geq 0$

Proof

$L(f,P) = \sum m_i \Delta t_i \geq 0$

$U(f,P) = \sum M_i \Delta t_i \geq 0$

T.F. $\sup\{L(f,P)\} \geq 0$ and $\inf\{U(f,P)\} \geq 0$ and $0 \leq \sup\{L(f,P)\} = \int_a^b f = \inf\{U(f,P)\}$

(b) f, g integ. on $[a,b]$ $\rightarrow$ $\int_a^b f \geq \int_a^b g$
$f(x) \geq g(x)$ for all x in $[a,b]$

Proof

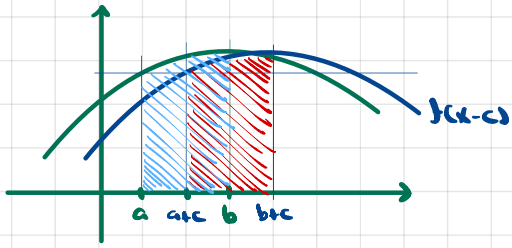
$f(x) - g(x) = f(x) + (-g(x)) \geq 0 \quad \forall x \in [a,b]$

$-g = -1 \cdot g$ is int. by Th. 6.

$f + (-g)$ int. by Th. 5.

by part a), $\int_a^b (f-g) = \int_a^b f - \int_a^b g \geq 0 \rightarrow \int_a^b f \geq \int_a^b g$

14. $\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx$



Let $g(x) = f(x-c)$

For any partition $P = \{t_0, t_1, \dots, t_n\} = \{a, t_1, \dots, t_{n-1}, b\}$ of $[a,b]$,

$P' = \{t_0+c, t_1+c, \dots, t_n+c\} = \{a+c, t_1+c, \dots, t_{n-1}+c, b+c\} = \{s_0, s_1, \dots, s_n\}$ is a partition of $[a+c, b+c]$.

And vice-versa: if P' partitions $[a+c, b+c]$ then P partitions $[a,b]$.

$\Delta s_i = s_i - s_{i-1} = t_i + c - t_{i-1} + c = t_i - t_{i-1} = \Delta t_i$

$m_{g_i} = \inf\{f(x-c) : t_i + c \le x \le t_{i-1} + c\} = \inf\{f(x-c) : t_i \le x - c \le t_{i-1}\} = \inf\{f(y) : t_i \le y \le t_{i-1}\} = m_{f_i}$

Similarly, $M_{g_i} = m_{g_i}$

$L(g, P') = \sum m_{g_i} \Delta s_i = \sum m_{f_i} \Delta t_i = L(f, P)$

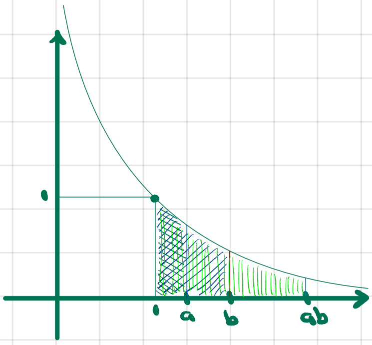
$U(g, P') = \sum M_{g_i} \Delta s_i = \sum M_{f_i} \Delta t_i = U(f, P)$

Hence, for every lower sum of a partition on $[a,b]$, there is a partition w/ the same lower sum of a partition on $[a+c, b+c]$, and vice-versa. Same for upper sums.

Therefore

$$\sup\{L(f,P)\} = \int_a^b f = \sup\{L(g,P')\} = \int_{a+c}^{b+c} g = \inf\{U(g,P')\} = \inf\{U(f,P)\}$$

15. $a,b>1 \to \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{t}\,dt = \int_1^{ab} \frac{1}{t}\,dt$



Proof

$$\int_1^{ab} \frac{1}{t}\,dt - \int_1^{b} \frac{1}{t}\,dt = \int_b^{ab} \frac{1}{t}\,dt$$

Thus, we are trying to prove $\int_1^a \frac{1}{t}\,dt = \int_b^{ab} \frac{1}{t}\,dt$.

For any partition $P = \{t_0, \dots, t_n\} = \{1, t_1, \dots, t_{n-1}, a\}$ of $[1,a]$,

$P' = \{bt_0, \dots, bt_n\} = \{b, bt_1, \dots, bt_{n-1}, ab\} = \{s_0, \dots, s_n\}$ is a partition of $[b, ab]$, and vice-versa.

$$\Delta s_i = s_i - s_{i-1} = bt_i - bt_{i-1} = b\Delta t_i$$

$$m_{P'_i} = f(s_i) = \frac{1}{bt_i} = \frac{1}{b} m_{P_i}$$

$$M_{P'_i} = f(s_{i-1}) = \frac{1}{bt_{i-1}} = \frac{1}{b} M_{P_i}$$

$$L(f,P') = \Sigma m_{P'_i} \Delta s_i = \Sigma \frac{1}{b} m_{P_i} \cdot b\Delta t_i = \Sigma m_{P_i} \Delta t_i = L(f,P)$$

$$U(f,P') = \Sigma M_{P'_i} \Delta s_i = \Sigma \frac{1}{b} M_{P_i} \cdot b\Delta t_i = \Sigma M_{P_i} \Delta t_i = U(f,P)$$

Therefore,

$$\sup\{L(f,P')\} = \int_b^{ab} \frac{1}{t}\,dt = \sup\{L(f,P)\} = \int_1^a \frac{1}{t}\,dt$$

16. $\int_{ca}^{cb} f(t)\,dt = c\int_a^b f(ct)\,dt$

Proof

Let $g(t) = f(ct)$

For $P = \{t_0, \dots, t_n\} = \{a, t_1, \dots, t_{n-1}, b\}$ partition on $[a,b]$, $P' = \{ct_0, \dots, ct_n\} = \{ca, ct_1, \dots, ct_{n-1}, cb\} = \{s_0, \dots, s_n\}$ is a partition of $[ca, cb]$.

*to be precise, this is true if $c>0$. If $c<0$, then $P' = \{cb, ct_{n-1}, \dots, ct_1, ca\}$ is a partition of $[cb, ca]$.
$= \{s_n, s_{n-1}, \dots, s_0\}$

$\Delta s_i = s_i - s_{i-1} = ct_i - ct_{i-1} = c\Delta t_i$

$\Delta s_i = -\Delta s_i = s_{i-1} - s_i = -c\Delta t_i$

$m_{P_i} = \inf\{f(ct) : t_{i-1} \le t \le t_i\} = \inf\{f(ct) : ct_{i-1} \le ct \le ct_i\} = \inf\{f(t) : ct_{i-1} \le t \le ct_i\} = m_{P'_i}$

$\inf\{f(ct) : ct_i \le ct \le ct_{i-1}\} = \inf\{f(t) : ct_i \le t \le ct_{i-1}\}$

$M_{P_i} = M_{P'_i}$ (by analogous argument)

$L(g,P) = \sum m_{P_i}\Delta t_i$

$\sum m_{P'_i}(-\Delta s_i) = \sum m_{P_i}(-c\Delta t_i) = -cL(g,P)$

$L(f,P') = \sum m_{P'_i}\Delta s_i = \sum m_{P_i} c\Delta t_i = cL(g,P)$

$U(g,P) = \sum M_{P_i}\Delta t_i$

$U(f,P') = -cU(g,P)$

$U(f,P') = \sum M_{P'_i}\Delta s_i = \sum M_{P_i} c\Delta t_i = cU(g,P)$

Hence, $-\dfrac{L(f,P')}{c} = L(g,P) \le \int_a^b f(ct)\,dt \le U(g,P) = -\dfrac{U(f,P')}{c} \rightarrow L(f,P') \le -c\int_a^b f(ct)\,dt \le U(f,P')$

$\dfrac{L(f,P')}{c} = L(g,P) \le \int_a^b f(ct)\,dt \le U(g,P) = \dfrac{U(f,P')}{c} \rightarrow L(f,P') \le c\int_a^b f(ct)\,dt \le U(f,P')$ (1)

$L(f,P') \le \int_{ca}^{cb} f(t)\,dt \le U(f,P')$ (2)

(1) and (2) are true for any partition P'. As proved previously, there is only one number satisfying inequalities such as these.

Therefore $\int_{ca}^{cb} f(t)\,dt = c\int_a^b f(ct)\,dt$

$\int_{ca}^{cb} f(t)\,dt = -c\int_a^b f(ct)\,dt$

17

$x^2+y^2=1 \to y=\pm\sqrt{1-x^2}$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \to y=\pm b\sqrt{1-\frac{x^2}{a^2}} = \pm\frac{b}{a}\sqrt{a^2-x^2}$

Let $f(x)=\sqrt{1-\frac{x^2}{a^2}}$

Then $f(ax)=\sqrt{1-x^2}$

$A_1=\int_{-1}^{1}2\sqrt{1-x^2}\,dx=2\int_{-1}^{1}f(ax)\,dx=\pi$

$A_2=\int_{-a}^{a}2b\sqrt{1-x^2/a^2}\,dx = 2b\int_{-a\cdot1}^{a\cdot1}f(x)\,dx$

By problem 16, $\int_{-a\cdot1}^{a\cdot1}f(x)\,dx=a\int_{-1}^{1}f(ax)\,dx$

Therefore, $A_2=2b\cdot a\cdot\frac{\pi}{2}=ab\pi$

---

18. We want to compute $\int_a^b x^n\,dx$

(a) $c_n=\int_0^1 x^n\,dx \to \int_0^a x^n\,dx=c_na^{n+1}$

Proof

$\int_{a\cdot0}^{a\cdot1}f(x)\,dx=a\int_0^1 f(ax)\,dx=a\int_0^1 a^nx^n\,dx=a^{n+1}\int_0^1 x^n\,dx=a^{n+1}c_n$

(b) $2^{n+1}c_na^{n+1}=2a^{n+1}\sum_{k\text{ even}}\binom{n}{k}c_k$

Proof

$\int_0^{2a}x^n\,dx=\int_{-a}^{a}(x+a)^n\,dx=\int_{-a}^{a}\sum_{i=0}^{n}\binom{n}{i}x^{n-i}a^i\,dx=\int_{-a}^{a}\binom{n}{0}x^n\,dx+\int_{-a}^{a}\binom{n}{1}x^{n-1}a+\dots+\int_{-a}^{a}\binom{n}{n}a^n\,dx$

Note that $\int_{-a}^{a}x^n\,dx=\int_{-a}^{0}x^n\,dx+\int_0^a x^n\,dx=\begin{cases}0 & \text{if } n \text{ odd}\\ 2\int_0^a x^n\,dx=2c_na^{n+1} & \text{if } n \text{ even}\end{cases}$

Therefore

$\int_{-a}^{a}(x+a)^n\,dx=\sum_{n-i\text{ even}}\binom{n}{i}\cdot2a^i\int_0^a x^{n-i}\,dx=\sum_{i\text{ even}}\binom{n}{i}2a^{n-i}\int_0^a x^i\,dx=\sum_{i\text{ even}}\binom{n}{i}2a^{n-i}c_ia^{i+1}=\sum_{i\text{ even}}\binom{n}{i}2a^{n+1}c_i$

$\int_0^{2a}x^n\,dx=c_n(2a)^{n+1}=c_n2^{n+1}a^{n+1}$, by part (a).

Therefore, $2^{n+1}a^{n+1}c_n=\sum_{i\text{ even}}\binom{n}{i}2a^{n+1}c_i$

(c) $c_n = \int_0^1 x^n dx = \frac{1}{n+1}$

Proof

We use strong induction.

$n=1 \to c_1 = \int_0^1 x\,dx = \frac{1}{2} = \frac{1}{1+1}$

Assume $c_k = \frac{1}{1+k}$ for $k<n$.

From the part (b) result,

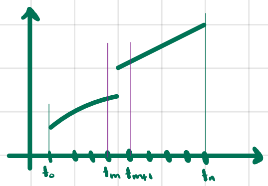
$$2^{n+1} c_n = 2 \sum_{i \text{ even}} \binom{n}{i} \frac{1}{i+1}$$
$$= \frac{2}{n+1} \sum_{i \text{ even}} \binom{n}{i} \frac{n+1}{i+1}$$
$$= \frac{2}{n+1} \sum_{i \text{ even}} \binom{n+1}{i+1}$$
$$= \frac{2}{n+1} \sum_{i \text{ odd}} \binom{n+1}{i}$$
$$= \frac{2 \cdot 2^n}{n+1}$$
$$= \frac{2^{n+1}}{n+1}$$

$\to c_n = \frac{1}{n+1}$

---

19. f bounded on $[a,b]$ $\to$ f integrable on $[a,b]$
f cont. at each point in $[a,b]$ except at $x_0 \in (a,b)$

Proof



Let $P = \{t_0, t_1, \ldots, t_m, t_{m+1}, \ldots, t_n\}$ be partition of $[a,b]$ and $x_0 \in [t_m, t_{m+1}]$

Then f is cont. on $[t_0, t_m]$ and $[t_{m+1}, t_n]$.

There is some P' of $[t_0, t_m]$ s.t. $U(f,P') - L(f,P') < \frac{\epsilon}{3}$.

and some P'' of $[t_{m+1}, t_n]$ s.t. $U(f,P'') - L(f,P'') < \frac{\epsilon}{3}$.

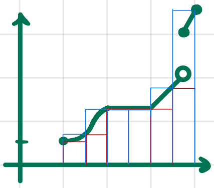
Let m s.t. $\forall x, x \in [t_m, t_{m+1}] \to f(x) \geq m$ and M s.t. $\forall x, x \in [t_m, t_{m+1}] \to f(x) \leq M$.

If $(t_{m+1} - t_m) < \frac{\epsilon}{3(M-m)}$ then if P''' contains P' and P'' we have

$$U(f,P''') - L(f,P''') < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

20. f nondecreasing on $[a,b]$. This implies f is bounded on $[a,b]$ since $f(a) \le f(x) \le f(b)$.

(a) $P = \{t_0, \dots, t_n\}$ partition of $[a,b]$.



$L(f,P) = \sum_{i=1}^{n} f(t_{i-1}) \Delta t_i$

$U(f,P) = \sum_{i=1}^{n} f(t_i) \Delta t_i$

(b) $t_i - t_{i-1} = \delta$ for each i $\rightarrow U(f,P) - L(f,P) = \delta(f(b) - f(a))$

Proof

$U(f,P) - L(f,P) = \sum (f(t_i) - f(t_{i-1})) \Delta t_i$

$= \delta(\cancel{f(t_1)} - f(t_0) + \cancel{f(t_2)} - \cancel{f(t_1)} + \cancel{f(t_3)} + \cancel{f(t_2)} + \dots + f(t_n) - \cancel{f(t_{n-1})})$

$= \delta(f(t_n) - f(t_0)) = \delta(f(b) - f(a))$

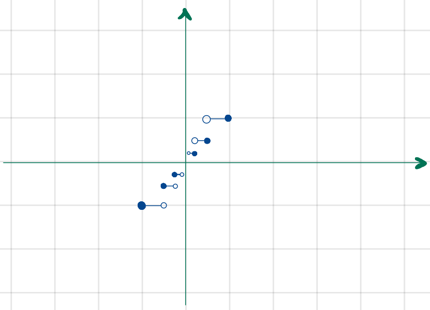
(c) f integrable

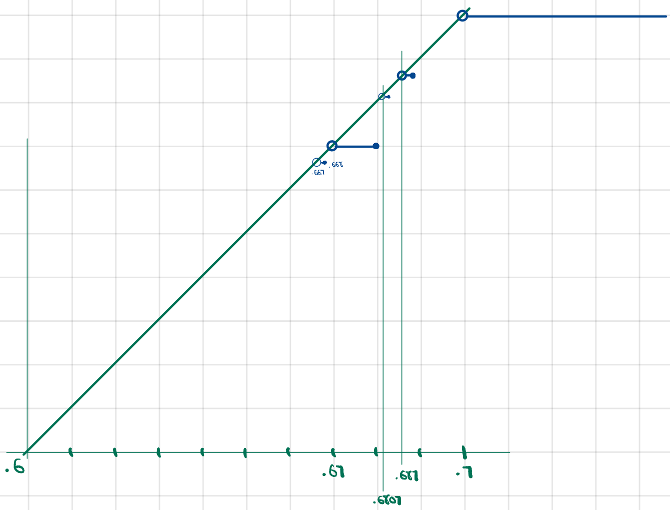
Proof

For any $\epsilon > 0$, let $\delta < \dfrac{\epsilon}{f(b) - f(a)}$. Then $U(f,P) - L(f,P) < \epsilon$.

(d) f nondecreasing on $[0,1]$, discont. at infinitely many points.

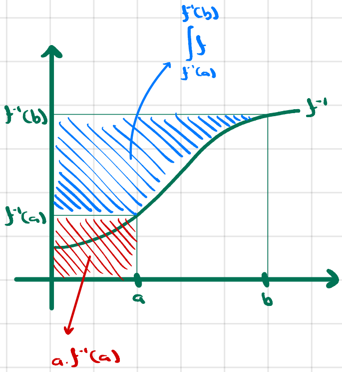
$f(x) = \dfrac{1}{\left[\frac{1}{x}\right]}$





$f(x)$ = number obtained by replacing all digits in decimal expansion of x which come after first 7, if any, by 0.

21. f increasing



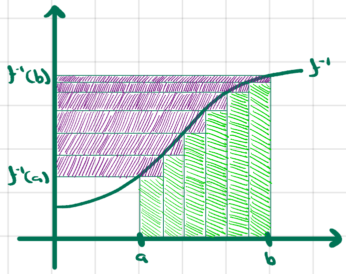
$\int_a^b f^{-1} = b \cdot f^{-1}(b) - a f^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(b)} f$ is suggested by the figure above

(a) $P = \{t_0, \dots, t_n\}$ partition of $[a,b]$

$\rightarrow L(f^{-1}, P) + U(f, P') = b f^{-1}(b) - a f^{-1}(a)$

$P' = \{f^{-1}(t_0), \dots, f^{-1}(t_n)\}$

Proof



$L(f^{-1}, P) = \sum f^{-1}(t_{i-1}) \cdot (t_i - t_{i-1})$

$U(f, P') = \sum f(f^{-1}(t_i)) \cdot (f^{-1}(t_i) - f^{-1}(t_{i-1}))$
$= \sum t_i (f^{-1}(t_i) - f^{-1}(t_{i-1}))$

$L(f^{-1}, P) + U(f, P') = \sum f^{-1}(t_{i-1}) \cdot (t_i - t_{i-1}) + \sum t_i (f^{-1}(t_i) - f^{-1}(t_{i-1}))$

$= \cancel{\sum f^{-1}(t_{i-1}) t_i} - \sum f^{-1}(t_{i-1}) t_{i-1} + \sum f^{-1}(t_i) t_i - \cancel{\sum f^{-1}(t_{i-1}) t_i}$

$= f^{-1}(\cancel{t_1}) t_1 + f^{-1}(\cancel{t_2}) t_2 + \dots + f^{-1}(t_n) t_n$
$- f^{-1}(t_0) t_0 - \cancel{f^{-1}(t_1) t_1} + \dots + \cancel{f^{-1}(t_{n-1}) t_{n-1}}$

$= f^{-1}(b) b - f^{-1}(a) a$

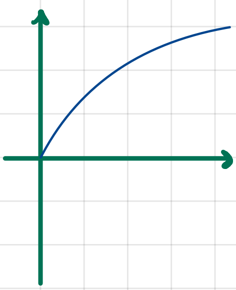
(b) $\int_a^b f^{-1} = \sup\{L(f^{-1}, P)\} = \sup\{f^{-1}(b) b - f^{-1}(a) a - U(f, P')\}$

$= f^{-1}(b) b - f^{-1}(a) a - \inf\{U(f, P')\}$

$= f^{-1}(b) b - f^{-1}(a) a - \int_{f^{-1}(a)}^{f^{-1}(b)} f$

(c) $\int_a^b \sqrt[n]{x}\, dx$ for $0 \le a < b$

$f(x) = x^n$
$f^{-1}(x) = \sqrt[n]{x}$



$\int_a^b \sqrt[n]{x}\, dx = b\sqrt[n]{b} - a\sqrt[n]{a} - \int_{\sqrt[n]{a}}^{\sqrt[n]{b}} x^n\, dx = b\sqrt[n]{b} - a\sqrt[n]{a} - \frac{1}{n+1}\left(\sqrt[n]{b^{n+1}} - \sqrt[n]{a^{n+1}}\right)$

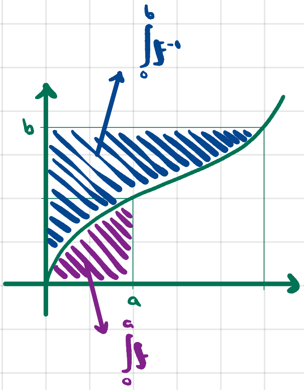
$= b\sqrt[n]{b} - a\sqrt[n]{a} - \frac{1}{n+1}\left(b\sqrt[n]{b} - a\sqrt[n]{a}\right) = \frac{n}{n+1}\left(b\sqrt[n]{b} - a\sqrt[n]{a}\right)$

22. f cont., increasing
$f(0)=0$
$a,b>0$

$\rightarrow ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx$ (Young's Inequality)

$[b = f(a) \leftrightarrow (ab = \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx)]$

Proof



$$\int_0^a f(x)dx = af(a) - \int_0^{f(a)} f^{-1}(x)dx < ab - \int_0^{f(a)} f^{-1}(x)dx$$

$$ab > \int_0^a f(x)dx + \int_0^{f(a)} f^{-1}(x)dx$$

Case 1: $b > f(a)$

f increasing $\rightarrow f^{-1}$ increas. $\rightarrow \forall x, x > f(a) \rightarrow f^{-1}(x) > f^{-1}(f(a)) = a$

$$f^{-1}(f(a))(b-f(a)) = a(b-f(a)) < \int_{f(a)}^b f^{-1}(x)dx = \int_0^b f^{-1}(x)dx - \int_0^{f(a)} f^{-1}(x)dx$$

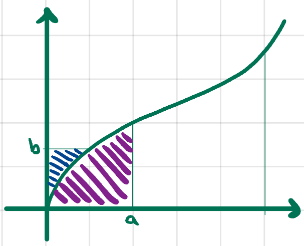
$\rightarrow ab < af(a) + \int_0^b f^{-1} - \int_0^{f(a)} f^{-1}$ (1)

In problem 21 we proved $\int_a^b f^{-1} = bf^{-1}(b) - af^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(b)} f$ which is equivalent to $\int_a^b f = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}$ (2)

For $a=0$, this is $\int_0^b f = bf(b) - \int_0^{f(b)} f^{-1}$

Therefore, from (1),

$$ab < \int_0^a f + \int_0^b f^{-1}$$



Case 2: $b < f(a)$

$$b(a - f^{-1}(b)) < \int_{f^{-1}(b)}^a f = \int_0^a f - \int_0^{f^{-1}(b)} f$$

$$ab < bf^{-1}(b) + \int_0^a f - \int_0^{f^{-1}(b)} f = \int_0^a f + \int_0^b f^{-1}$$

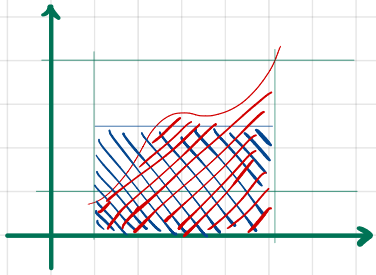
Case 3: $b = f(a)$

Starting at eq. (2), $\int_0^a f = af(a) - \int_0^{f(a)} f^{-1} = ab - \int_0^b f^{-1}$

$\rightarrow ab = \int_0^a f + \int_0^b f^{-1}$

23. (a) f integrable on $[a,b]$
$m \le f(x) \le M$ for all x in $[a,b]$ $\rightarrow \int_a^b f(x)dx = (b-a)\mu$ for some μ with $m \le \mu \le M$.

Proof



$$m(b-a) \le \frac{\int_a^b f(x)dx}{b-a}(b-a) \le M(b-a)$$

$$\mu = \frac{\int_a^b f(x)dx}{b-a}$$

(b) f cont. on $[a,b]$ $\rightarrow \int_a^b f(x)dx = (b-a)f(\xi)$ for some ξ in $[a,b]$.

Proof

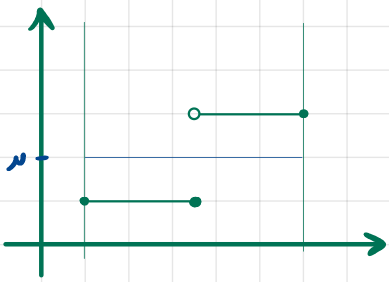
f cont on $[a,b]$ $\rightarrow$ f takes on min and max values on $[a,b]$.

Let the min be at x_1 and the max at x_2 s.t. $f(x_1) = m$ and $f(x_2) = M$.

IVT $\rightarrow$ f takes on all values between m and M on $[x_1, x_2]$, including $\mu = \frac{\int_a^b f(x)dx}{b-a}$

Ie, there is some $\xi \in [x_1, x_2]$ s.t. $f(\xi) = \mu$.

(c)



(d) f cont. on $[a,b]$ $\rightarrow$ $\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$ for some ξ in $[a,b]$ (MVT for Integrals)

g integr. and nonneg. on $[a,b]$

Proof

f cont. on $[a,b]$

$\rightarrow$ f bounded on $[a,b]$

$\rightarrow$ $m \leq f(x) \leq M$, $x \in [a,b]$

$\rightarrow$ $mg(x) \leq f(x)g(x) \leq Mg(x)$, $x \in [a,b]$.

$\rightarrow$ $m\int_a^b g \leq \int_a^b fg \leq M\int_a^b g$ (1)

Case 1: $\int_a^b g > 0$

$$\rightarrow m \leq \frac{\int_a^b fg}{\int_a^b g} \leq M$$

Since f is cont. on $[a,b]$ and $m \leq f(x) \leq M$ in this interval, by the IVT f takes on the value $\frac{\int_a^b fg}{\int_a^b g}$ at some $\xi \in [a,b]$.

Ie $f(\xi) = \int_a^b fg / \int_a^b g$

$\rightarrow$ $\int_a^b fg = f(\xi)\int_a^b g$

Case 2: $\int_a^b g = 0$

(1) $\rightarrow$ $0 \leq \int_a^b fg = 0$ $\rightarrow$ $\int_a^b fg = 0$

T.F. $\int_a^b fg = f(\xi)\int_a^b g = 0$ for any $\xi \in [a,b]$.

(e) f cont. on $[a,b]$ $\to$ $\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$ for some ξ in $[a,b]$ (MVT for Integrals)
g integr. and nonpos. on $[a,b]$

Proof

f cont. on $[a,b]$

$\to$ f bounded on $[a,b]$

$\to$ $m \le f(x) \le M$, $x \in [a,b]$ multiply by $g(x) \le 0$, switch the inequalities

$\to$ $Mg(x) \le f(x)g(x) \le mg(x)$

$\to$ $M\int_a^b g \le \int_a^b fg \le m\int_a^b g$

Case 1: $\int_a^b g = 0$

$\to \int_a^b fg = 0$ so $\forall \xi,\ \xi \in [a,b] \to \int_a^b fg = f(\xi)\int_a^b g = 0$

Case 2: $\int_a^b g < 0$

$\to m \le \dfrac{\int_a^b fg}{\int_a^b g} \le M$ divide by $\int_a^b g < 0$ and we're back to the same eq. we had changed nonneg.

since f is cont. on $[a,b]$ and $m \le f(x) \le M$ in this interval, by the IVT f takes on the value $\dfrac{\int_a^b fg}{\int_a^b g}$ at some $\xi \in [a,b]$.

Ie $f(\xi) = \int_a^b fg / \int_a^b g$

$\to \int_a^b fg = f(\xi)\int_a^b g$

(f) let $f(x) = x^3+1$, $g(x) = x$, and consider $[-1,1]$.

$f(x)g(x) = x^4 + x$

$\int_{-1}^1 (x^4+x)dx = \left(\frac{x^5}{4} + \frac{x^2}{2}\right)\Big|_{-1}^1 = \frac{1}{4} + \frac{1}{2} - \left(-\frac{1}{4} + \frac{1}{2}\right) = \frac{1}{2}$

$\int_{-1}^1 g(x)dx = 0$

T.F. $\int_{-1}^1 fg = \frac{1}{2} \ne \int_{-1}^1 g \cdot f(\xi) = 0$ for any $\xi \in [-1,1]$

Note

Recall the MVT f cont. on $[a,b]$, diff on (a,b) $\rightarrow \exists c, c \in (a,b) \wedge f'(c) = \dfrac{f(b)-f(a)}{b-a}$

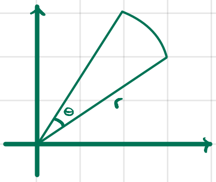
As will be proved later, $f(b)-f(a) = \int_a^b f'(x)\,dx$

Therefore, (1) says $\exists c, c \in (a,b) \wedge f'(c) = \dfrac{\int_a^b f'(x)\,dx}{b-a}$

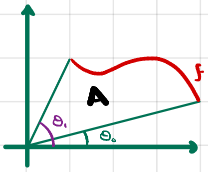
Let $g(x) = f'(x)$.

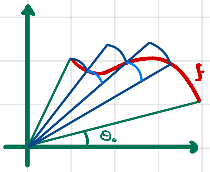
Then, $\exists \xi \in (a,b) \wedge g(\xi)(b-a) = \int_a^b g(x)\,dx$, which is the result proved in 23b.

24.



Area = $\frac{r^2\theta}{2}$





Let $g(\theta) = \frac{f^2(\theta)}{2}$. Then g is nonneg., cont. since f cont, therefore integrable.

Let $P = \{\theta_0, t_1, \dots, t_{n-1}, \theta_1\}$ be partition of $[\theta_0, \theta_1]$.

In any $[t_{i-1}, t_i]$, if $m_i = \inf\{f(\theta) : \theta \in [t_{i-1}, t_i]\}$ then $\inf\{g(\theta), \theta \in [t_{i-1}, t_i]\} = \frac{m_i^2}{2}$.

Similarly, $M_i = \sup\{f(\theta), \theta \in [t_{i-1}, t_i]\}$ and $\sup\{g(\theta), \theta \in [t_{i-1}, t_i]\} = \frac{M_i^2}{2}$

$L(g,P) = \sum \frac{m_i^2}{2}(t_i - t_{i-1})$. Each term is a small area contained in A. The sum approximates A.

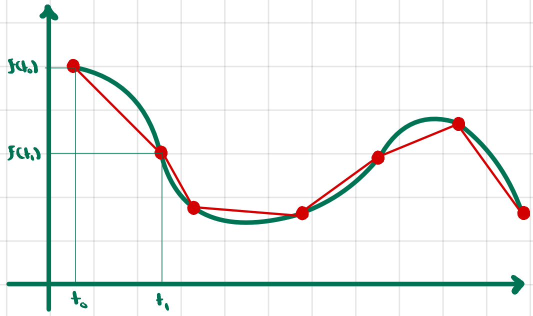
$U(g,P) = \sum \frac{M_i^2}{2}(t_i - t_{i-1})$. Each term is a small area containing A. The sum approximates A.

$\int_{\theta_0}^{\theta_1} g(\theta)d\theta = \sup\{L(g,P)\} = \inf\{U(g,P)\} = \frac{1}{2}\int_{\theta_0}^{\theta_1} f^2(\theta)d\theta = A$

25. f cont. on $[a,b]$

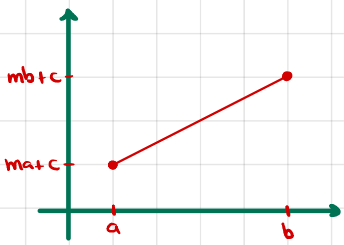
$P = \{t_0, \dots, t_n\}$ partition of $[a,b]$

$\ell(f,P) = \sum_{i=1}^{n} \sqrt{(t_i - t_{i-1})^2 + [f(t_i) - f(t_{i-1})]^2}$ = length of polygonal inscribed curve on graph of f



$d_{ab} = \sqrt{(b-a)^2 + (mb-ma)^2} = \sqrt{(b-a)^2(1+m^2)} = (b-a)\sqrt{1+m^2}$

Definition: (length of f on $[a,b]$) $\sup\{\ell(f,P)\}$, provided $\{\ell(f,P)\}$ bounded above



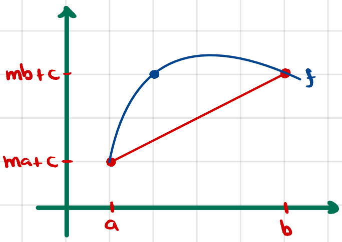
(a) f linear fn on $[a,b]$ $\to$ length of f is distance from $(a, f(a))$ to $(b, f(b))$.

Proof

Assu $f(x) = mx + c$

Then $\ell(f,P) = \sum_{i=1}^{n} \sqrt{(t_i - t_{i-1})^2 + (mt_i + \cancel{c} - mt_{i-1} - \cancel{c})^2} = \sum_{i=1}^{n} \sqrt{(t_i - t_{i-1})^2(1+m^2)} = \sqrt{1+m^2}\,(b-a)$

(b) f not linear $\to$ $\exists P = \{a, t, b\}$ of $[a,b]$ s.t. $\ell(f,P) > d_{ab}$



Proof

In problem 4-9b we proved

$$\sqrt{(x_3-x_1)^2 + (y_3-y_1)^2} \le \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} + \sqrt{(x_3-x_2)^2 + (y_3-y_2)^2}$$

Given P, consider the points $(a, f(a))$, $(t, f(t))$, and $(b, f(b))$ and apply the inequality above

$$\sqrt{(b-a)^2 + (f(b)-f(a))^2} \le \sqrt{(t-a)^2 + (f(t)-f(a))^2} + \sqrt{(b-t)^2 + (f(b)-f(t))^2}$$

$$d_{ab} \le d_{at} + d_{tb} = \ell(f,P)$$

(c) Let P be a partition $\{a, t, b\}$ as in part b). Then $d_{ab} \le d_{at} + d_{tb}$.

If we add another point to P, it is either in (a,t) or (t,b). Assume it is $c \in (a,t)$.

By part b), $d_{at} \le d_{ac} + d_{ct}$. Therefore, $d_{ab} \le d_{at} + d_{tb} \le d_{ac} + d_{ct} + d_{tb} = \ell(f,P)$

Every time we add a point, $\ell(f,P)$ increases, and d_{ab} is always smaller.

(d) f' bounded on $[a,b]$ $\rightarrow$ $L(\sqrt{1+(f')^2}, P) \leq \ell(f,P) \leq U(\sqrt{1+(f')^2}, P)$

P any partition of $[a,b]$

Proof

Let $g(x) = \sqrt{1+(f'(x))^2}$

The claim is that given a partition P, the length fn is between lower and upper sums of a specific fn, $\sqrt{1+f'^2}$.

Note that these expressions look like the one found in problem 25a.
They represent lengths of a linear fn $h(x) = m_i x + c$ between $(t_{i-1}, h(t_{i-1}))$ and $(t_i, h(t_i))$.

$L(g,P) = \sum_{i=1}^{n} \sqrt{1+m_i^2}\, \Delta t_i$ where $m_i = \inf\{|f'(x_i)|,\ x_i \in [t_{i-1}, t_i]\}$

$U(g,P) = \sum_{i=1}^{n} \sqrt{1+M_i^2}\, \Delta t_i$ where $M_i = \sup\{|f'(x_i)|,\ x_i \in [t_{i-1}, t_i]\}$

But MVT tells us that $\exists c_i,\ c_i \in [t_{i-1}, t_i] \wedge f'(c_i) = \dfrac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}}$

By definition, $m_i \leq |f'(c_i)| \leq M_i$

Hence, $L(g,P) \leq \sum_{i=1}^{n} \sqrt{1+[f'(c_i)]^2}\, \Delta t_i \leq U(g,P)$

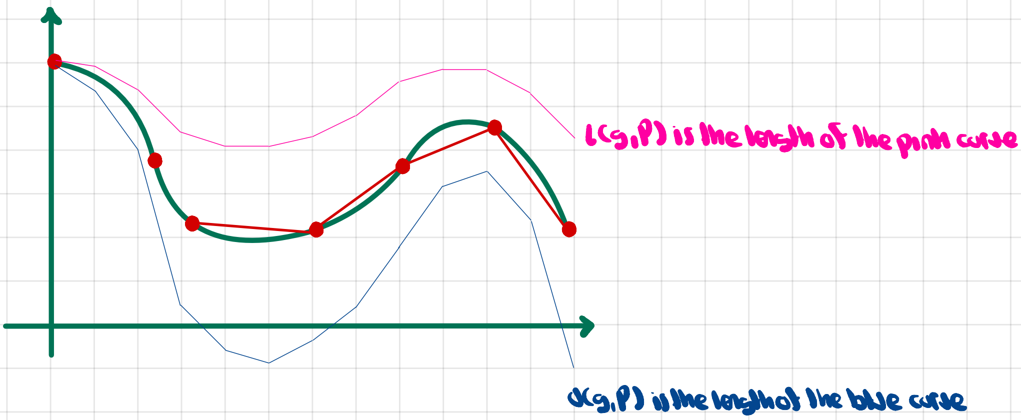
But what is $\sum_{i=1}^{n} \sqrt{1+[f'(c_i)]^2}\, \Delta t_i$?

$$\sum_{i=1}^{n} \sqrt{1+[f'(c_i)]^2}\, \Delta t_i = \sum \sqrt{(t_i - t_{i-1})^2 + (t_i - t_{i-1})^2 [f'(c_i)]^2} = \sum \sqrt{(t_i - t_{i-1})^2 + (f'(c_i)(t_i - t_{i-1}))^2}$$

$$= \sum \sqrt{(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2}$$

$$= \ell(f,P)$$

Hence, $L(g,P) \leq \ell(f,P) \leq U(g,P)$, for all P.



(e) Assume $\sup\{L(g,P)\} > \sup\{\ell(f,P)\}$

Then there is some P s.t. $L(g,P) > \sup\{\ell(f,P)\} \geq L(g,P)$. $\perp$.

Hence, $\sup\{L(g,P)\} \leq \sup\{\ell(f,P)\}$

(f) $\sup\{\ell(f,P)\} \leq \inf\{U(\sqrt{1+(f')^2}, P)\}$

Note that from e),

$$L(g,P) \leq \sup\{L(g,P)\} \leq \sup\{\ell(f,P)\}$$

If we show that

$$L(g,P) \leq \sup\{L(g,P)\} \leq \sup\{\ell(f,P)\} \leq \inf\{U(g,P)\}$$

Then, if $g = \sqrt{1+(f')^2}$ is integrable, ie if $\sup\{L(g,P)\} = \inf\{U(g,P)\}$, then

$$\sup\{L(g,P)\} = \sup\{\ell(f,P)\} = \inf\{U(g,P)\}$$

and therefore we will have shown that

$$\text{length of } f \text{ on } [a,b] = \int_a^b \sqrt{1+f'(t)^2}\, dt$$

Proof

Let P' and P'' be any two partitions of $[a,b]$.

Let P contain P' and P''. As shown in c), each additional point added to a partition Q increases $\ell(f,Q)$.

Therefore,

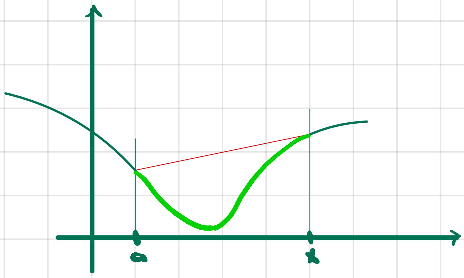
$$\ell(f,P') < \ell(f,P) \leq U(g,P) \leq U(g,P')$$

$$\ell(f,P'') < \ell(f,P) \leq U(g,P) \leq U(g,P'')$$

$\rightarrow$ $\ell(f,P') < U(g,P'')$

(g) $\mathcal{L}(x)$ is length of graph of f on $[a,x]$
$d(x)$ length of straight line segment from $(a,f(a))$ to $(x,f(x))$ $\rightarrow \left[\begin{array}{l}\sqrt{1+[f']^2} \text{ integrable on } [a,b] \\ f' \text{ cont at } a\end{array} \rightarrow \lim_{x\to a^+} \frac{\mathcal{L}(x)}{d(x)} = 1\right]$

Proof



As x approaches a, the lengths of red and light green converge.

$$\mathcal{L}(x) = \int_a^x \sqrt{1+f'(t)^2}\, dt$$

Assume $\sqrt{1+f'^2}$ integrable on $[a,b]$. Then by Th 8, $\mathcal{L}(x)$ is continuous on $[a,b]$.

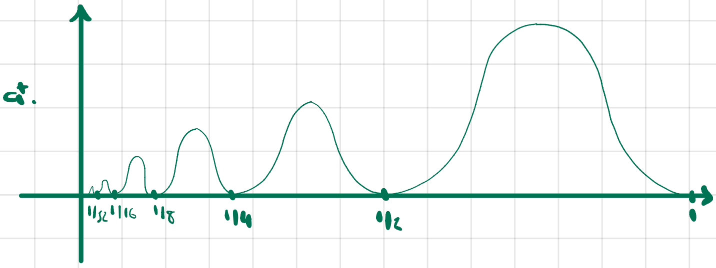
$$d(x) = \sqrt{(x-a)^2 + (f(x)-f(a))^2}$$

$$\text{MVT} \rightarrow \exists c,\ c \in (a,x) \wedge \sqrt{1+f'(c)^2} = \frac{\int_a^x \sqrt{1+f'(t)^2}\, dt}{x-a}$$

$$\lim_{x\to a^+} \frac{\mathcal{L}(x)}{d(x)} = \lim_{x\to a^+} \frac{\int_a^x \sqrt{1+f'(t)^2}\, dt}{\sqrt{(x-a)^2+(f(x)-f(a))^2}} = \lim_{x\to a^+} \frac{\sqrt{1+f'(c)^2}\,\cancel{(x-a)}}{\cancel{(x-a)}\sqrt{1+\left[\frac{f(x)-f(a)}{x-a}\right]^2}}$$

$$= \lim_{x\to a^+} \frac{\sqrt{1+f'(c)^2}}{\sqrt{1+\left[\frac{f(x)-f(a)}{x-a}\right]^2}} = \frac{\sqrt{1+f'(a)^2}}{\sqrt{1+f'(a)^2}} = 1$$

(h)



we want to compute $\frac{\mathcal{L}(x)}{d(x)}$ and take the limit as $x \to a^+$.

let $\mathcal{L}_0$ = length of f on $[1/2, 1]$.

let $\mathcal{L}(x)$ be length of f on $[0, x]$.

length of f on $[1/2^n, 1] = \mathcal{L}_0\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}\right)$

$= \mathcal{L}_0 \sum_{i=0}^{n-1}\left(\frac{1}{2}\right)^i = \mathcal{L}_0\left(\frac{1-(1/2)^n}{1-1/2}\right) = \mathcal{L}_0\left(2 - \frac{1}{2^{n-1}}\right)$

where we used the following result from 2-5

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r} \quad \text{if } r \neq 1$$

when $n \to \infty$, the length of f on $[0,1]$ is $2\mathcal{L}_0$.

$\mathcal{L}(1/2^n) = 2\mathcal{L}_0 - \mathcal{L}_0(2 - 1/2^{n-1}) = \frac{\mathcal{L}_0}{2^{n-1}}$

Thus,

$$\frac{\mathcal{L}(1/2^n)}{d(1/2^n)} = \frac{\frac{\mathcal{L}_0}{2^{n-1}}}{\frac{1}{2^n}} = 2\mathcal{L}_0$$

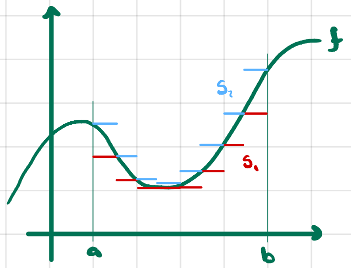
But $\mathcal{L}_0 > 0$ so $\frac{\mathcal{L}(1/2^n)}{d(1/2^n)} \neq 1$, and $\frac{\mathcal{L}(x)}{d(x)}$ can't approach 1 as $x \to a^+$.

26. fn s defined on $[a,b]$ is called a **step function** if there is partition $P = \{t_0, \dots, t_n\}$ of $[a,b]$ such that s is constant on each (t_{i-1}, t_i). Note that the values of f at t_i are arbitrary.

(a) f integrable on $[a,b]$ $\rightarrow$ $\forall \epsilon > 0$ $\exists$ step fn $s_1 \leq f$ w/ $\int_a^b f - \int_a^b s_1 < \epsilon$

$\exists$ step fn $s_2 \geq f$ w/ $\int_a^b s_2 - \int_a^b f < \epsilon$

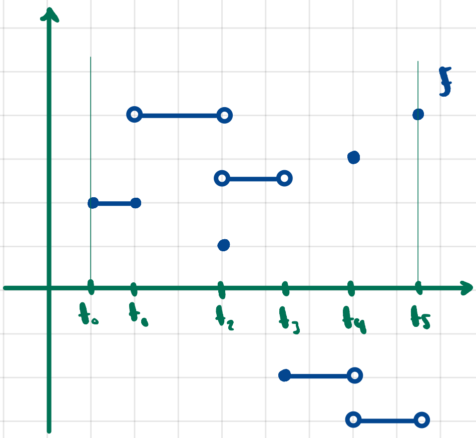
Proof



For any partition P', $L(f,P') \leq \sup\{L(f,P)\} = \int_b^a f \leq \inf\{U(f,P)\} \leq U(f,P')$

Also, $\forall \epsilon > 0$, $U(f,P') - L(f,P') < \epsilon$

Let $\epsilon > 0$ and $P' = \{t_0, \dots, t_n\}$ s.t. $U(f,P') - L(f,P') < \epsilon$



Let $s_1(x) = m_i$ if $x \in (t_{i-1}, t_i)$

$s_2(x) = M_i$ if $x \in (t_{i-1}, t_i)$

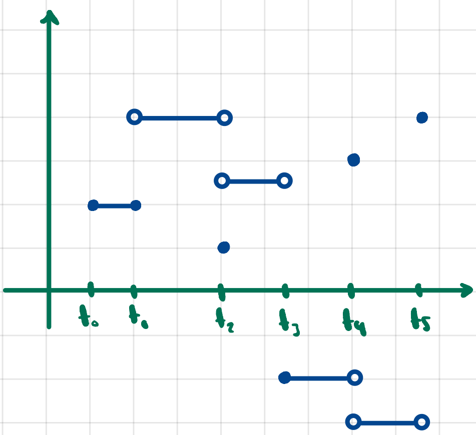
Then $L(f,P') = L(s_1,P') = U(s_1,P') = \int_a^b s_1$

$U(f,P') = L(s_2,P') = U(s_2,P') = \int_a^b s_2$

Therefore, $\int_a^b s_2 - \int_a^b s_1 < \epsilon$

But also

$L(f,P') \leq \int_a^b f \leq U(f,P')$



So

$\int_a^b s_1 \leq \int_a^b f \leq \int_a^b s_2$ $\rightarrow$ $\int_a^b f - \int_a^b s_1 < \epsilon$ and

$-\epsilon < \int_a^b s_1 - \int_a^b s_2 \leq \int_a^b f - \int_a^b s_2$

$\rightarrow \int_a^b s_2 - \int_a^b f < \epsilon$

The idea here is that we choose a P' s.t. $U(f,P') - L(f,P') < \epsilon$. Then we define s_1 and s_2 to be functions that take on the m_i and M_i values in each subinterval of P'. The integrals of these functions then correspond to the lower and upper sums of f on P', which we know differ by less than ϵ. But $\int_a^b f$ is in between, so it also differs from each of $\int_a^b s_1$ and $\int_a^b s_2$ by less than ϵ.

(b) $\forall \epsilon > 0, \exists s_1 \leq f$ and $s_2 \geq f$ s.t $\int_a^b s_2 - \int_a^b s_1 < \epsilon$ $\quad \rightarrow \quad$ f integrable

Proof

Let P' and P'' be the partitions on $[a,b]$ defining s_1 and s_2, respect.

Let $P = \{t_0, \dots, t_n\}$ be the partition containing P' and P''.

Then $m_i = \inf\{f(x) : x \in (t_{i-1}, t_i)\}$ and $M_i = \sup\{f(x) : x \in (t_{i-1}, t_i)\}$ and

$$\forall x, x \in (t_{i-1}, t_i) \rightarrow s_1(x) \leq m_i \leq M_i \leq s_2(x)$$

Also, $L(s_1, P) = U(s_1, P) = \int_a^b s_1$ and $L(s_2, P) = U(s_2, P) = \int_a^b s_2$.

Therefore,

$$\int_a^b s_1 = L(s_1, P) = \sum s_{1_i} \Delta t_i \leq L(f, P) = \sum m_i \Delta t_i \leq \sum M_i \Delta t_i = U(f, P) \leq \sum s_{2_i} \Delta t_i = U(s_2, P) = \int_a^b s_2$$

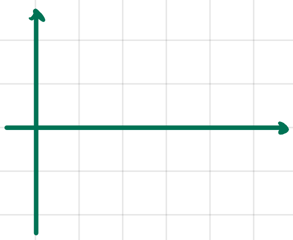
Ie, $\int_a^b s_1 \leq L(f,P) \leq U(f,P) \leq \int_a^b s_2$

But by assumption, $\int_a^b s_2 - \int_a^b s_1 < \epsilon$

Therefore,

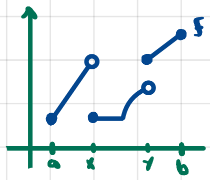
$$U(f,P) - L(f,P) < \epsilon$$

(c)



27. f integrable on $[a,b]$ $\rightarrow$ $\forall \epsilon > 0$ $\exists$ continuous g, f, h s.t. $g \leq f \leq h$ with $\int_a^b h - \int_a^b g < \epsilon$

Proof



Problem 26a tells us that there is a step fn $s \leq f$ w/ $\int_a^b f - \int_a^b s < \frac{\epsilon}{4}$

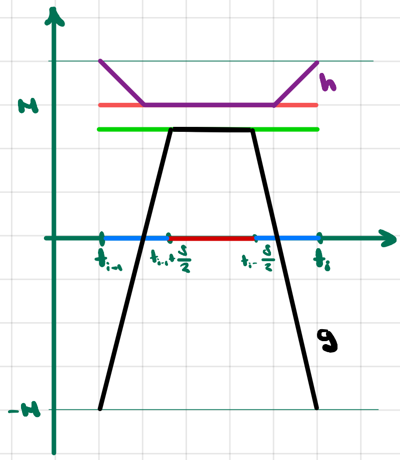
Since f is bounded there is some $M \geq 1$ s.t. $\forall x, x \in [a,b] \rightarrow |f(x)| \leq M$

Given a partition $P = \{t_0, \dots, t_n\}$, s is constant on (t_{i-1}, t_i) for $i = 1, \dots, n$.

Choose $\delta < \frac{\epsilon}{4nM}$.

Let $g = s$ on $[t_{i-1} + \frac{\delta}{2}, t_i - \frac{\delta}{2}]$

Let g be linear on $[t_i - \frac{\delta}{2}, t_i]$ and $[t_i, t_i + \frac{\delta}{2}]$ w/ $g(t_i) = -M$.



$$\int_{t_{i-1}}^{t_i} s = s_i \Delta t_i$$

$$\int_{t_{i-1}}^{t_i} g = (t_i - t_{i-1} - \delta) \cdot s_i + \int_{t_{i-1}}^{t_{i-1}+\frac{\delta}{2}} g + \int_{t_i - \frac{\delta}{2}}^{t_i} g$$

$$\leq s_i \Delta t_i - |s_i| \delta + 2|s_i| \frac{\delta}{2} + 2|s_i| \frac{\delta}{2}$$

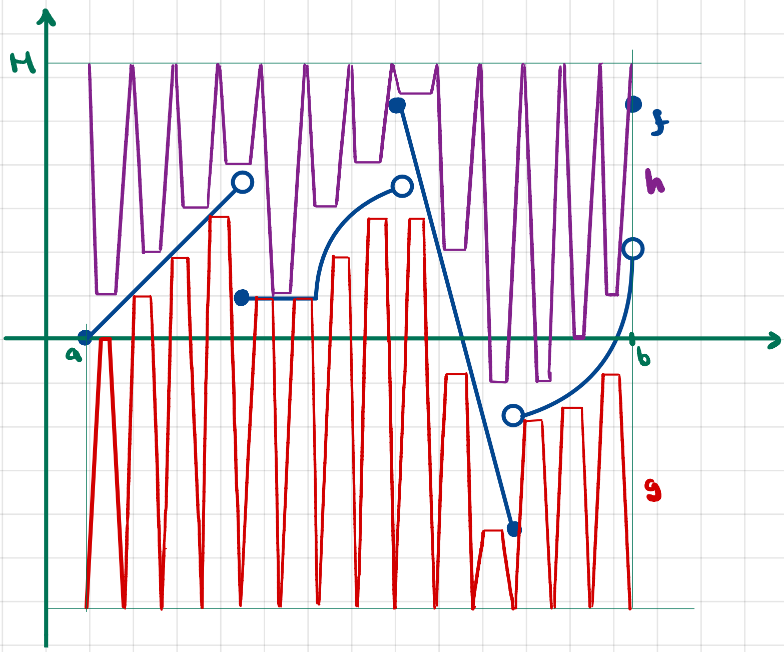
$$= s_i \Delta t_i - s_i \delta + 2|s_i| \delta = |s_i| \Delta t_i - |s_i| \delta$$

$$\int_{t_{i-1}}^{t_i} s - \int_{t_{i-1}}^{t_i} g = |s_i| \delta \leq M\delta = M \frac{\epsilon}{4nM} = \frac{\epsilon}{4n}$$

Since we have n such partition subintervals,

$$\int_a^b s - \int_a^b g = \sum s_i \delta \leq \sum M\delta = nM\delta = n \cdot \frac{\epsilon}{4n} = \frac{\epsilon}{4}$$

Therefore, $\int_a^b f - \int_a^b g \leq \frac{\epsilon}{2}$



Consider the function $-f$.

According to the result we just proved, we can find a fn u s.t. in $[a,b]$,

$$u \le -f$$

$$\int_a^b (-f) - \int_a^b u \le \frac{\epsilon}{2}$$

But this means

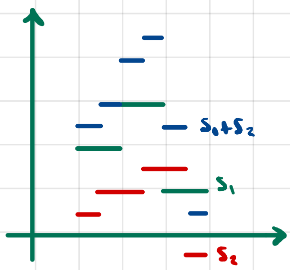
$$\int_a^b (-u) - \int_a^b f < \frac{\epsilon}{2}$$

Also, $-u \ge f$

Let $h = -u$ and we have shown that we can always find a fn h s.t. $h \ge f$ and $\int_a^b h - \int_a^b f \le \frac{\epsilon}{2}$.

---

28. (a) s_1, s_2 step fns on $[a,b]$ $\rightarrow$ $s_1 + s_2$ is step fn on $[a,b]$



Proof

By definition, there are partitions $P_1 = \{t_0, \dots, t_n\}$ and $P_2 = \{s_0, \dots, s_m\}$ s.t.

s_1 constant on (t_{i-1}, t_i) $i = 1, \dots, n$
s_2 " " (s_{i-1}, s_i) $i = 1, \dots, m$

Let P contain P_1 and P_2.

Then there are $n+m$ open subintervals in P and in each one s_1 and s_2 are constant so $s_1 + s_2$ is constant.

Hence $s_1 + s_2$ is a step fn.

(b) $\int_a^b (s_1 + s_2) = \int_a^b s_1 + \int_a^b s_2$

Proof

Let $P = \{u_0, \dots, u_{n+m}\}$ as in (a).

$$\int_a^b (s_1 + s_2) = \int_{u_0}^{u_1} (s_1+s_2) + \int_{u_1}^{u_2} (s_1+s_2) + \dots + \int_{u_{n+m-1}}^{u_{n+m}} (s_1+s_2)$$

$$= (u_1 - u_0)(s_{1_0} + s_{2_0}) + (u_2 - u_1)(s_{1_1} + s_{2_1}) + \dots (u_{n+m} - u_{n+m-1})(s_{1_{n+m-1}} - s_{2_{n+m-1}})$$

$$= (u_1 - u_0) s_{1_0} + (u_2 - u_1) s_{1_1} + \dots + (u_{n+m} - u_{n+m-1}) s_{1_{n+m-1}} + (u_1 - u_0) s_{2_0} + (u_2 - u_1) s_{2_1} + \dots + (u_{n+m} - u_{n+m-1}) s_{2_{n+m-1}}$$

$$= \int_{u_0}^{u_1} s_1 + \dots + \int_{u_{n+m-1}}^{u_{n+m}} s_1 + \int_{u_0}^{u_1} s_2 + \dots + \int_{u_{n+m-1}}^{u_{n+m}} s_2$$

$$= \int_a^b s_1 + \int_a^b s_2$$

(c) Theorem 5 f, g integrable on $[a,b]$ $\rightarrow$ $f+g$ integrable on $[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

Proof

By problem 26a, for any $\epsilon > 0$ there are step fns s_1, s_2, t_1, t_2 s.t.

$s_1 \leq f$
$s_2 \geq f$
$t_1 \leq g$
$t_2 \geq g$

and

$$\int_a^b f - \int_a^b s_1 < \epsilon/4$$

$$\int_a^b s_2 - \int_a^b f < \epsilon/4$$

$$\int_a^b g - \int_a^b t_1 < \epsilon/4$$

$$\int_a^b t_2 - \int_a^b g < \epsilon/4$$

Therefore

$$s_1 + t_1 \leq f + g \leq s_2 + t_2 \quad (1)$$

Using the result from part (b),

$$\int_a^b (s_1 + t_1) = \int_a^b s_1 + \int_a^b t_1 \leq \int_a^b f + \int_a^b g \quad (2)$$

$$\rightarrow \int_a^b (s_1 + t_1) - \int_a^b f - \int_a^b g = \left(\int_a^b s_1 - \int_a^b f\right) + \left(\int_a^b t_1 - \int_a^b g\right) > -\epsilon/2$$

$$\rightarrow 0 \leq \left(\int_a^b f + \int_a^b g\right) - \int_a^b (s_1 + t_1) < \epsilon/2 \quad (3)$$

Similarly, $\int_a^b (s_2 + t_2) = \int_a^b s_2 + \int_a^b t_2 \geq \int_a^b f + \int_a^b g \quad (4)$

$$\rightarrow \int (s_2 + t_2) - \int_a^b f - \int_a^b g = \left(\int s_2 - \int f\right) + \left(\int t_2 - \int g\right) < \epsilon/2$$

$$\rightarrow 0 \leq \int_a^b (s_2 + t_2) - \left(\int_a^b f + \int_a^b g\right) < \epsilon/2 \quad (5)$$

(3), (5) $\rightarrow$ $0 \le \int_a^b (s_2+t_2) - \int_a^b (s_1+t_1) < \epsilon$ (6)

To recap, at this point we have step fns

$s_1+t_1 \le f+g$

$s_2+t_2 \ge f+g$

and

$\int_a^b (s_2+t_2) - \int_a^b (s_1+t_1) < \epsilon$

By 26b, $f+g$ is integrable.

(2), (4) $\rightarrow$ $\int_a^b (s_1+t_1) = \int_a^b s_1 + \int_a^b t_1 \le \int_a^b f + \int_a^b g \le \int_a^b s_2 + \int_a^b t_2 = \int_a^b (s_2+t_2)$ (7)

(1) $\rightarrow$ $\int_a^b (s_1+t_1) \le \int_a^b (f+g) \le \int_a^b (s_2+t_2)$ (8)

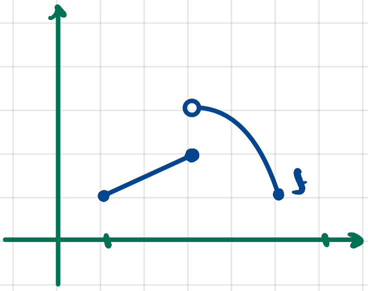
(6) $\rightarrow$ there is only one number between $\int_a^b (s_1+t_1)$ and $\int_a^b (s_2+t_2)$

Therefore $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

29. f integrable on $[a,b] \to \exists x, x \in [a,b] \wedge \int_a^x f = \int_x^b f \;\to\; \int_a^x f - \int_x^b f = 0 \;\to\; \int_a^x f + \int_x^b f - 2\int_x^b f = 0$

$$\to \frac{\int_a^b f}{2} = \int_x^b f = \int_a^x f$$

Proof



let $F(x) = \int_a^x f(t)\,dt$

Th. 8 $\to$ F cont. on $[a,b]$.

$F(a) = 0$

$F(b) = \int_a^b f(t)\,dt$

let's consider two possible cases

Case 1: $\int_a^b f \neq 0$

IVT $\to \exists x, x \in (a,b) \wedge F(x) = \dfrac{\int_a^b f(t)\,dt}{2}$

$\to \int_a^x f = \dfrac{\int_a^b f(t)\,dt}{2}$

$\to 2\int_a^x f = \int_a^x f + \int_x^b f$

$\to \int_a^x f = \int_x^b f$

Case 2: $\int_a^b f = 0$

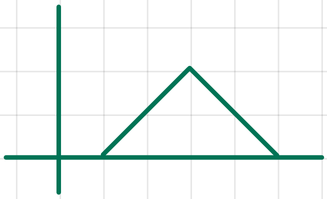
Then if $x = a$ we have

$\int_a^x f = \int_a^a f = 0 = \int_a^b f = \int_x^b f$

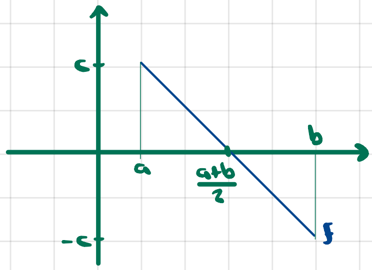
By proof by cases

$\exists x, x \in [a,b] \wedge \int_a^x f = \int_x^b f$

If we choose a fn that falls into case 2



let $f(x) = c - \dfrac{2c}{b-a}(x-a)$



let $F(x) = \int_a^x f(t)\,dt$.

Then F increasing from a to $\frac{a+b}{2}$, then decreasing from $\frac{a+b}{2}$ to b.

Assume $\exists x, x \in (a, \frac{a+b}{2}) \wedge \int_a^x f = \int_x^b f$.

we know that $\int_a^x f > 0$ (because $f > 0$ on $(a, \frac{a+b}{2})$) and

$\int_x^b f = \int_x^{(a+b)/2} f + \int_{(a+b)/2}^b f$. But $\int_{(a+b)/2}^b f < 0$ (because $f \leq 0$ on $[\frac{a+b}{2}, b]$).

Also $\left|\int_{(a+b)/2}^b f\right| \geq \int_x^{(a+b)/2} f \;\to\; -\int_{(a+b)/2}^b f \geq \int_x^{(a+b)/2} f \;\to\; \int_{(a+b)/2}^b f \leq -\int_x^{(a+b)/2} f$

$\to \int_x^{(a+b)/2} f + \int_{(a+b)/2}^b f \leq 0 \;\to\; \int_x^b f \leq 0$. But this contradicts the assumption that $\int_a^x f = \int_x^b f$ since $\int_a^x f > 0$. $\perp$.

Therefore, $\forall x, x \in (a, \frac{a+b}{2}) \to \int_a^x f \neq \int_x^b f$

Assume $\exists x, x \in [\frac{a+b}{2}, b) \wedge \int_a^x f = \int_x^b f$.

we know that $\int_b^x f < 0$ and $\int_a^x f = \int_a^{(a+b)/2} f + \int_{(a+b)/2}^x f$

But $\int_a^{(a+b)/2} f \geq \left|\int_{(a+b)/2}^x f\right| \to \int_a^{(a+b)/2} f \geq -\int_{(a+b)/2}^x f \to \int_a^{(a+b)/2} f + \int_{(a+b)/2}^x f \geq 0$

$\to \int_a^x f \geq 0$. But this contradicts our assumption that $\int_a^x f = \int_x^b f$.

$\perp$.

Therefore, $\forall x, x \in (a,b) \to \int_a^x f \neq \int_x^b f$

30. (a) $P = \{t_0, \dots, t_n\}$ part. of $[a,b]$

$U(f,P) - L(f,P) < b-a \rightarrow M_i - m_i < 1$ for some i

Proof

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)\Delta t_i < b-a$$

Assume $M_i - m_i \geq 1$ for all i.

Then

$U(f,P) - L(f,P) \geq b-a$. $\perp$.

T.F. $M_i - m_i < 1$ for some i.

(b) $\exists a_1, b_1$ w/ $a < a_1 < b_1 < b$ $\wedge$ $\sup\{f(x) : a_1 \leq x \leq b_1\} - \inf\{f(x) : a_1 \leq x \leq b_1\} < 1$

Proof

Let i s.t. $M_i - m_i < 1$.

Case 1: $i \neq 1$ $\wedge$ $i \neq n$

Let $a_1 = t_{i-1}$ and $b_1 = t_i$. Then $a < a_1 < b_1 < b$

$M_i = \sup\{f(x) : t_{i-1} \leq x \leq t_i\} = \sup\{f(x) : a_1 \leq x \leq b_1\}$

$m_i = \inf\{f(x) : t_{i-1} \leq x \leq t_i\} = \inf\{f(x) : a_1 \leq x \leq b_1\}$

Therefore, $\sup\{f(x) : a_1 \leq x \leq b_1\} - \inf\{f(x) : a_1 \leq x \leq b_1\} = M_i - m_i < 1$

Case 2: $i = 1$

Let $a_1 = \dfrac{t_1 + t_0}{2}$ and $b_1 = t_1$. Then $a < a_1 < b_1 < b$ and

$\sup\{f(x) : a_1 \leq x \leq b_1\} \leq \sup\{f(x) : t_0 \leq x \leq t_1\} = M_1$

$\inf\{f(x) : a_1 \leq x \leq b_1\} \geq \inf\{f(x) : t_0 \leq x \leq t_1\} = m_1$

T.F.

$\sup\{f(x) : a_1 \leq x \leq b_1\} - \inf\{f(x) : a_1 \leq x \leq b_1\} \leq M_1 - m_1 < 1$

Case 3: $i = n$.

Analogous to case 2: let $a_1 = t_{n-1}$ and b_1 s.t. $t_{n-1} < b_1 < t_n$.

(c) $\exists a_2, b_2$ s.t. $a_1 < a_2 < b_2 < b_1$ and $\sup\{f(x) : a_2 \le x \le b_2\} - \inf\{f(x) : a_2 \le x \le b_2\} < \frac{1}{2}$

Proof

f is integrable by assumption, so it is int. on $[a_1, b_1]$.

Choose partition P of $[a_1, b_1]$ such that $U(f,P) - L(f,P) < \frac{1}{2}(b_1 - a_1)$

Assume $\forall i, i \in \{1, \dots, n\} \to M_i - m_i \ge \frac{1}{2}$

But $U(f,P) - L(f,P) = \sum (M_i - m_i)\Delta t_i \ge \frac{1}{2}(b_1 - a_1)$.

$\perp$.

T.F. $\exists i, i \in \{1, \dots, n\} \wedge M_i - m_i < \frac{1}{2}$.

Let i s.t. $M_i - m_i < \frac{1}{2}$.

Case 1: $i \neq 1 \wedge i \neq n$.

Let $a_2 = t_{i-1}$ and $b_2 = t_i$. Then, $a_1 < a_2 < b_2 < b_1$ and by def. of M_i and m_i,

$\sup\{f(x) : a_2 \le x \le b_2\} - \inf\{f(x) : a_2 \le x \le b_2\} < \frac{1}{2}$

Case 2: $i = 1$

Let a_2 s.t. $a_1 < a_2 < t_1$ and $b_2 = t_1$. Then $a_1 < a_2 < b_2 < b_1$ and

$\sup\{f(x) : a_2 \le x \le b_2\} \le \sup\{f(x) : a_1 \le x \le b_1\} = M_1$

$\inf\{f(x) : a_2 \le x \le b_2\} \ge \inf\{f(x) : a_1 \le x \le b_1\} = m_1$

and

$\sup\{f(x) : a_2 \le x \le b_2\} - \inf\{f(x) : a_2 \le x \le b_2\} \le M_1 - m_1 < \frac{1}{2}$

Case 3: $i = n$. Analogous to case 2.

By proof by cases, we assert that

$\exists a_2, b_2$ s.t. $a_1 < a_2 < b_2 < b_1$ and $\sup\{f(x) : a_2 \le x \le b_2\} - \inf\{f(x) : a_2 \le x \le b_2\} < \frac{1}{2}$

(d) Let's use induction to show that given f integrable on $[a,b]$,

$\forall n, n \in \mathbb{N} \to \exists a_1, \dots, a_n$ and $\exists b_1, \dots, b_n$ s.t

$a_{i+1} \geq a_i \wedge b_{i+1} \geq b_i \wedge a_i \leq b_i$ and

$\sup\{f(x) : a_n \leq x \leq b_n\} - \inf\{f(x) : a_n \leq x \leq b_n\} < \frac{1}{n}$ (1)

For $n = 1$, we showed in part b that

$\exists a_1, b_1$ w/ $a < a_1 < b_1 < b \wedge \sup\{f(x) : a_1 \leq x \leq b_1\} - \inf\{f(x) : a_1 \leq x \leq b_1\} < 1$

Assume (1) is true for some n.

Choose a partition P of $[a_n, b_n]$ s.t. $U(f,P) - L(f,P) < \frac{1}{n+1}(b_n - a_n)$

Assume $\forall i, i \in \{1, \dots, k\} \to M_i - m_i \geq \frac{1}{n+1}$

But $U(f,P) - L(f,P) = \sum_{i=1}^{k}(M_i - m_i)\Delta t_i \geq \frac{1}{n+1}(b_n - a_n)$

$\perp$.

T.F. $\exists i, i \in \{1, \dots, k\} \wedge M_i - m_i < \frac{1}{n+1}$

Let i s.t. $M_i - m_i < \frac{1}{n+1}$

Case 1: $i \neq 1 \wedge i \neq k$

Let $a_{n+1} = t_{i-1}$ and $b_{n+1} = t_i$. Then, $a_n < a_{n+1} < b_{n+1} < b$ and by def. of M_i and m_i,

$\sup\{f(x) : a_{n+1} \leq x \leq b_{n+1}\} - \inf\{f(x) : a_{n+1} \leq x \leq b_{n+1}\} < \frac{1}{n+1}$

Case 2: $i = 1$

Let a_{n+1} s.t. $a_n < a_{n+1} < t_1$ and $b_{n+1} = t_1$. Then $a_n < a_{n+1} < b_{n+1} < b_n$ and

$\sup\{f(x) : a_{n+1} \leq x \leq b_{n+1}\} \leq \sup\{f(x) : a_n \leq x \leq b_n\} = M_n$

$\inf\{f(x) : a_{n+1} \leq x \leq b_{n+1}\} \geq \inf\{f(x) : a_n \leq x \leq b_n\} = m_n$

and therefore

$\sup\{f(x) : a_{n+1} \leq x \leq b_{n+1}\} - \inf\{f(x) : a_{n+1} \leq x \leq b_{n+1}\} \leq M_n - m_n < \frac{1}{n+1}$

Case 3: $i = k$ is analogous to Case 2.

Therefore, by proof by cases, $\exists a_1, \dots, a_{n+1}, b_1, \dots, b_{n+1}$ s.t. $a_{i+1} \geq a_i \wedge b_{i+1} \geq b_i \wedge a_i \leq b_i$ and

$\sup\{f(x) : a_{n+1} \leq x \leq b_{n+1}\} - \inf\{f(x) : a_{n+1} \leq x \leq b_{n+1}\} < \frac{1}{n+1}$.

By proof by induction we conclude that (1) is true.

Recall a Theorem proved in problem 8-14, the Nested Intervals Theorem

Theorem Let $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, ... such that $a_n \leq a_{n+1}$, $b_{n+1} \leq b_n$, and $a_n \leq b_n$.

Then $\exists x, \forall i, x \in I_i$.

From (1), we have that for any $n \in \mathbb{N}$, $\exists a_1, \ldots, a_n$ and $\exists b_1, \ldots, b_n$ s.t. $a_{i+1} \geq a_i \wedge b_{i+1} \geq b_i \wedge a_i \leq b_i$.

We can form intervals $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, ..., $I_n = [a_n, b_n]$, ... and they are nested as in the assumptions of the theorem above.

Therefore, $\exists x$ s.t. $\forall i, x \in I_i$.

(1) tells us something extra.

$$\sup\{f(x) : a_n \leq x \leq b_n\} - \inf\{f(x) : a_n \leq x \leq b_n\} < \frac{1}{n}$$

That is the range that $f(x)$ is in decreases with n.

But $\forall x, x \in [a_n, b_n] \to \inf\{f(x) : a_n \leq x \leq b_n\} \leq f(x) \leq \sup\{f(x) : a_n \leq x \leq b_n\}$

Let y s.t. $\forall i, y \in I_i$.

Then we have that $\forall n, n \in \mathbb{N}$

$$\inf\{f(x) : a_n \leq x \leq b_n\} \leq f(y) \leq \sup\{f(x) : a_n \leq x \leq b_n\}$$

$$\forall x, x \in [a_n, b_n] \to \inf\{f(x) : a_n \leq x \leq b_n\} \leq f(x) \leq \sup\{f(x) : a_n \leq x \leq b_n\}$$

$$\sup\{f(x) : a_n \leq x \leq b_n\} - \inf\{f(x) : a_n \leq x \leq b_n\} < \frac{1}{n}$$

Therefore,

$$\forall x, x \in [a_n, b_n] \to |f(y) - f(x)| < \frac{1}{n}$$

For any $\epsilon > 0$ choose n_1 s.t. $\frac{1}{n_1} < \epsilon$.

Then $\forall x, x \in [a_{n_1}, b_{n_1}] \to |f(y) - f(x)| < \epsilon$

Let $\delta = \min(y - a_n, b_n - y)$.

Then $\forall x\ |x - y| < \delta \to |f(y) - f(x)| < \epsilon \to \lim_{x \to y} f(x) = f(y)$

$\to f$ cont at y.

(e) f cont. at infinitely many points in $[a,b]$.

Proof

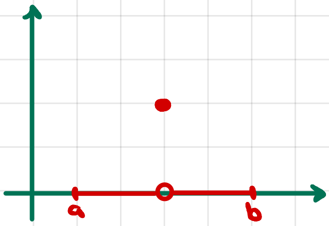
Apply (a) - (d) to any subinterval of $[a,b]$ to find a continuous point.

There are infinite subintervals $\rightarrow$ infinite points where f is cont.

---

31. f integrable on $[a,b]$

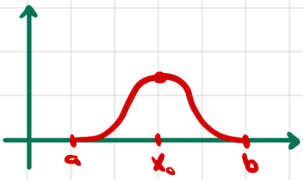
(a) $f(x) \geq 0$ for all x
$f(x) > 0$ for some x in $[a,b]$
$\int_a^b f = 0$



(b) $f(x) \geq 0$ for all x in $[a,b]$
f cont. at $x_0 \in [a,b]$
$f(x_0) > 0$

$\rightarrow \int_a^b f > 0$

Proof



Since $f(x_0) > 0$ then $\exists \delta > 0$ s.t. $\forall x, |x - x_0| < \delta \rightarrow f(x) > 0$

$$\int_a^b f = \underbrace{\int_a^{x_0-\delta} f}_{\geq 0} + \underbrace{\int_{x_0-\delta}^{x_0+\delta} f}_{>0} + \underbrace{\int_{x_0+\delta}^{b} f}_{\geq 0} > 0$$

Alternative Solution

Since f cont. at x_0, $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, |x - x_0| < \delta \rightarrow |f(x) - f(x_0)| < \varepsilon$

Let $\varepsilon = \frac{f(x_0)}{2}$. Then, $\exists \delta > 0 \ \forall x \ |x - x_0| < \delta \rightarrow \frac{f(x_0)}{2} < f(x) < \frac{3f(x_0)}{2}$

Choose a partition s.t. $x_0 \in [t_{i-1}, t_i]$ for some i and $t_i - t_{i-1} < \delta$.

Then $m_i \Delta t_i > 0$ for this particular i.

For the other subintervals it is ≥ 0.

Hence $L(f,P) > 0$ and $\int_a^b f \geq L(f,P) > 0$.

(b) f integrable on $[a,b]$
$\forall x, x \in [a,b] \to f(x) > 0$ $\to \int_a^b f > 0$

Proof

In problem 30 we showed that if f is integrable on $[a,b]$ then it is continuous at infinitely many points in $[a,b]$.

Hence, there is some $x_0 \in [a,b]$ s.t. $f(x_0) > 0$, f cont. at x_0.

By part b), $\int_a^b f > 0$.

32. (a) f cont. on $[a,b]$

$\int_a^b fg = 0$ for all cont. fns g on $[a,b]$ $\to f=0$

Proof

Assume $\neg(\forall x, x \in [a,b] \to f(x)=0)$

ie $\exists x, x \in [a,b] \wedge f(x) \neq 0$

Let x_0 s.t. $x_0 \in [a,b] \wedge f(x_0) \neq 0$.

Let $g=f$. Then $f \cdot g = f^2 \geq 0$ for $x \in [a,b]$ and $(fg)(x_0) = f^2(x_0) > 0$.

By problem 31b, $\int_a^b fg > 0$

$\perp$.

T.F. $\forall x, x \in [a,b] \to f(x)=0$

(b) f cont. on $[a,b]$

$\int_a^b fg = 0$ for those cont. fns g that satisfy $g(a)=g(b)=0$ $\to f=0$

Proof

Assume $\neg(\forall x, x \in [a,b] \to f(x)=0)$

ie $\exists x, x \in [a,b] \wedge f(x) \neq 0$

Let x_0 s.t. $x_0 \in [a,b] \wedge f(x_0) \neq 0$.

Case 1: $f(x_0) > 0$. Since f cont. then $\forall x, x \in (x_0-\delta, x_0+\delta) \to f(x) > 0$.

Let $g(x) = \begin{cases} 0 & x \in [a, x_0-\delta] \cup [x_0+\delta, b] \\ f(x) & x \in (x_0-\delta, x_0+\delta) \end{cases}$

Then $(fg)(x) = \begin{cases} 0 & x \in [a, x_0-\delta] \cup [x_0+\delta, b] \\ f^2(x) & x \in (x_0-\delta, x_0+\delta) \end{cases}$

Then $\int_a^b fg = \int_{x_0-\delta}^{x_0+\delta} f^2 > 0$. $\perp$.

Case 2: $f(x_0) < 0$. Analogous to case 1. $\perp$.

Therefore, $\forall x, x \in [a,b] \to f(x)=0$.

33. $f(x) = \begin{cases} x & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$

(a)

on $[0,1]$, $L(f,P) = 0$

(b) Let $P = \{t_0, \ldots, t_n\}$ be equally spaced partition. Then $\Delta t_i = \frac{1}{n}$.

let $g(x) = \begin{cases} t_i & x = t_i,\ i = 1, \ldots, n \\ f(x) & \text{otherwise.} \end{cases}$

Then g is the sum of f w/ n fns $p_i(x) = \begin{cases} t_i & x = t_i \\ 0 & \text{otherwise.} \end{cases}$

As we know we can change f at finitely many points w/o changing integrals of f.

Ie $\int_a^b f = \int_a^b g$.

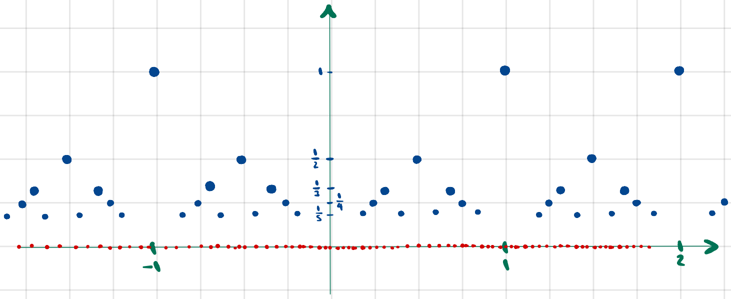
T.F. $\sup\{L(f,P)\} = \sup\{L(g,P)\} = \int_a^b f = \int_a^b g = \inf\{U(g,P)\} = \inf\{U(f,P)\}$

$U(g,P) = \sum_{i=1}^n M_i \Delta t_i = \frac{1}{n} \sum_{i=1}^n t_i = \frac{1}{n} \sum_{i=1}^n i \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2n}$

$\inf\{U(g,P)\} = \frac{1}{2}$

34. $f(x) = \begin{cases} 0 & x \text{ irrational} \\ 1/q & x = p/q \text{ in lowest terms} \end{cases}$ $\rightarrow$ f integrable on $[0,1]$ and $\int_0^1 f = 0$

Proof



Let $\epsilon > 0$. There is some $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \frac{\epsilon}{2}$. Choose one such n.

There is a finite number of rational numbers p/q such that $1/q > 1/n$.

$$1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \dots, \frac{1}{n-1}, \dots$$

Let P be a partition of $[0,1]$ s.t. the subintervals that contain the above numbers have total length $< \frac{\epsilon}{2}$.

On the other subintervals, we have $M_i \leq \frac{1}{n} < \frac{\epsilon}{2}$.

Let I_1 be the set containing all the i's from $\{1, \dots, n\}$ s.t. $[t_{i-1}, t_i]$ contains one of the rational numbers p/q s.t. $1/q > 1/n$.

Let I_2 contain the remaining i's.

Then, $U(f,P) = \sum_{i \in I_1} M_i \Delta t_i + \sum_{i \in I_2} M_i \Delta t_i$

$$\leq 1 \cdot \sum_{i \in I_1} \Delta t_i + \frac{\epsilon}{2} \sum_{i \in I_2} \Delta t_i$$

$$< 1 \cdot \frac{\epsilon}{2} + \frac{\epsilon}{2} \cdot 1$$

$$= \epsilon$$

Hence, $U(f,P) < \epsilon$, f integrable on $[0,1]$, and $\int_0^1 f = 0$.

35. f, g integrable but $g \circ f$ not integr.

Let $f(x) = \begin{cases} 0 & x \text{ irrational} \\ 1/q & x = p/q \text{ in lowest terms} \end{cases}$

$g(x) = \begin{cases} -1 & x = 0 \\ x & x \neq 0 \end{cases}$

Then

$h(x) = (g \circ f)(x) = \begin{cases} -1 & x \text{ irrational} \\ 1/q & x = p/q \text{ in lowest terms} \end{cases}$

$L(h,P) = (-1) \sum_i \Delta t_i = -1$

$U(h,P) > 0$

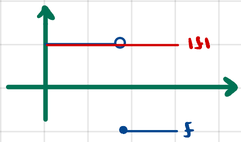
$\rightarrow U(h,P) - L(h,P) > 1 \quad \rightarrow h$ not integrable

36. f bounded on $[a,b]$
P partition of $[a,b]$

M_i', m_i' for $|f|$

(a) $M_i' - m_i' \le M_i - m_i$

Proof



$M_i' \ge M_i$
$m_i' \ge m_i$

Let $[t_{i-1}, t_i]$ be a partition subinterval.

There are three possible cases

1. $\forall x, x \in [t_{i-1}, t_i] \to f(x) \ge 0$
2. $\forall x, x \in [t_{i-1}, t_i] \to f(x) \le 0$
3. $\exists x, x \in [t_{i-1}, t_i] \wedge f(x) < 0 \quad \wedge \quad \exists y, y \in [t_{i-1}, t_i] \wedge f(y) > 0$

Case 1: $\forall x, x \in [t_{i-1}, t_i] \to f(x) \ge 0$

Then $\forall x, x \in [t_{i-1}, t_i] \to f(x) = |f(x)|$ and $M_i = M_i'$ and $m_i = m_i'$.

Hence $M_i' - m_i' = M_i - m_i$

Case 2: $\forall x, x \in [t_{i-1}, t_i] \to f(x) \le 0$

Then $\forall x, x \in [t_{i-1}, t_i] \to -f(x) = |f(x)| \to$

$m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\}$

$\to -m_i = \sup\{-f(x) : x \in [t_{i-1}, t_i]\} = \sup\{|f(x)| : x \in [t_{i-1}, t_i]\} = M_i'$

Similarly, $-M_i = m_i'$.

Hence, $M_i' - m_i' = -m_i - (-M_i) = M_i - m_i$

Case 3: $\exists x, x \in [t_{i-1}, t_i] \wedge f(x) < 0 \quad \wedge \quad \exists y, y \in [t_{i-1}, t_i] \wedge f(y) > 0$

Then $m_i < 0, M_i > 0, M_i' > 0, m_i' \ge 0$

Case 3.1: $-m_i \le M_i$

Then $M_i' = M_i$. $M_i' - m_i' \le M_i' = M_i < M_i - m_i$

Case 3.2: $-m_i > M_i$

Then $M_i' = -m_i$. $M_i' - m_i' \le M_i' = -m_i < M_i - m_i$

By proof by cases,

$M_i' - m_i' < M_i - m_i$

in case 3.

By proof by cases,

$M_i' - m_i' \le M_i - m_i$

(b) f integr. on $[a,b]$ $\rightarrow$ $|f|$ integr. on $[a,b]$

Proof

Since f integrable on $[a,b]$ then $\forall \epsilon > 0$ there is a partition P s.t.

$$U(f,P) - L(f,P) < \epsilon$$

$$\rightarrow \sum (M_i - m_i) \Delta t_i < \epsilon$$

But by part (a), $M_i' - m_i' \leq M_i - m_i$. Hence,

$$\sum (M_i' - m_i') \Delta t_i < \epsilon$$

$$\rightarrow U(|f|,P) - L(|f|,P) < \epsilon$$

$\rightarrow$ $|f|$ integr. on $[a,b]$.

(c) f, g integr. on $[a,b]$ $\rightarrow$ $\max(f,g)$ and $\min(f,g)$ integr. on $[a,b]$

Proof

$$\max(f,g)(x) = \frac{f(x) + g(x) + |f(x) - g(x)|}{2}$$

f, g int. $\rightarrow$ $f+g, f-g$ int. $\rightarrow$ $|f+g|, |f-g|$ int.

$\rightarrow$ $(f+g) + |f-g|$ int. $\rightarrow$ $\max(f,g)$ int.

Similarly

$$\min(f,g)(x) = \frac{f(x) + g(x) - |f(x) - g(x)|}{2}$$ is int.

(d) f int. on $[a,b]$ $\leftrightarrow$ $\min(f,0)$ and $\max(f,0)$ integr. on $[a,b]$.

Proof

Let $g(x) = 0$.

Assume f int. on $[a,b]$

Then, by part (c), $\max(f,g) = \max(f,0)$ and $\min(f,g) = \min(f,0)$ are both integrable.

T.F. f int. on $[a,b]$ $\rightarrow$ $\min(f,0)$ and $\max(f,0)$ integr. on $[a,b]$.

Now assume $\min(f,0)$ and $\max(f,0)$ integr. on $[a,b]$.

$$\min(f,0) = \begin{cases} 0 & f \leq 0 \\ f & f > 0 \end{cases}$$

$$\max(f,0) = \begin{cases} 0 & f \leq 0 \\ f & f \geq 0 \end{cases}$$

T.F. $f = \min(f,0) + \max(f,0)$

By Th. 5, f integr. on $[a,b]$

T.F. $\min(f,0)$ and $\max(f,0)$ integr. on $[a,b]$ $\rightarrow$ f int. on $[a,b]$

37. f int. on $[a,b]$ $\to$ $\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt$

Proof

$\sup\{L(f,P)\} = \int_a^b f = \inf\{U(f,P)\} \le \sup\{L(|f|,P)\} = \int_a^b |f| = \inf\{U(|f|,P)\}$

$\to \int_a^b f \le \int_a^b |f|$

$\left|\int_a^b f\right| \le \left|\int_a^b |f|\right| = \int_a^b |f|$

$U(fg,P) - L(fg,P) = \sum_{i=1}^{n} (M_i - m_i)\Delta t_i$

---

38. f, g integrable on $[a,b]$

$f(x), g(x) \ge 0$ for all x in $[a,b]$

P partition of $[a,b]$

M_i', m_i' of f

M_i'', m_i'' of g

M_i, m_i of fg

(a) $M_i \le M_i' M_i''$

$m_i \ge m_i' m_i''$

Proof

Let $[t_{i-1}, t_i]$ be a partition subinterval.

$\forall x, x \in [t_{i-1}, t_i] \to M_i' \ge f(x) \wedge M_i'' \ge g(x) \to f(x)g(x) \le M_i' M_i''$

$\to M_i' M_i''$ is upper bound for $\{f(x)g(x) : x \in [t_{i-1}, t_i]\}$

$\to M_i' M_i'' \ge \sup\{f(x)g(x) : x \in [t_{i-1}, t_i]\} = M_i$

Similarly,

$\forall x, x \in [t_{i-1}, t_i] \to m_i' \le f(x) \wedge m_i'' \le g(x) \to f(x)g(x) \ge m_i' m_i''$

$\to m_i' m_i''$ lower bound for $\{f(x)g(x) : x \in [t_{i-1}, t_i]\}$

$\to m_i' m_i'' \le \inf\{f(x)g(x) : x \in [t_{i-1}, t_i]\} = m_i$

(b) $U(fg,P) - L(fg,P) \le \sum_{i=1}^{n} [M_i' M_i'' - m_i' m_i''](t_i - t_{i-1})$

Proof

$U(fg,P) - L(fg,P) = \sum_{i=1}^{n} (M_i - m_i)\Delta t_i \le \sum (M_i' M_i'' - m_i' m_i'')\Delta t_i$, by part (a).

(c) $U(fg,P) - L(fg,P) \le M\left\{ \sum_{i=1}^{n} (M_i' - m_i')\Delta t_i + \sum_{i=1}^{n} (M_i'' - m_i'')\Delta t_i \right\}$

Proof

f, g int. $\to f, g$ bounded $\to \exists M$ s.t. $|f| \le M$ and $|g| \le M$

$\to M_i' \le M \qquad m_i' \ge -M$
$\quad M_i'' \le M \qquad m_i'' \ge -M$

From the result proved in part (b), we have

$$\begin{aligned} U(fg,P) - L(fg,P) &\le \sum_{i=1}^{n} [M_i' M_i'' - m_i' m_i''](t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} [M_i' M_i'' - M_i'' m_i' + M_i'' m_i' - m_i' m_i'']\Delta t_i \\ &= \sum_{i=1}^{n} [M_i''(M_i' - m_i') + m_i'(M_i'' - m_i'')]\Delta t_i \\ &\le M \sum_{i=1}^{n} [(M_i' - m_i') + (M_i'' - m_i'')]\Delta t_i \end{aligned}$$

(d) fg integrable

Choose P_1 s.t. $U(f,P_1) - L(f,P_1) = \sum_{i=1}^{n} (M_i' - m_i')\Delta t_i < \frac{\epsilon}{2M}$

and P_2 s.t. $U(g,P_2) - L(g,P_2) = \sum_{i=1}^{n} (M_i'' - m_i'')\Delta t_i < \frac{\epsilon}{2M}$

Let P_3 contain P_1 and P_2. Then

$$U(f,P_3) - L(f,P_3) = \sum_{i=1}^{n} (M_i''' - m_i''')\Delta t_i < \frac{\epsilon}{2M}$$

$$U(g,P_3) - L(g,P_3) = \sum_{i=1}^{n} (M_i^{(4)} - m_i^{(4)})\Delta t_i < \frac{\epsilon}{2M}$$

and from part (c),

$$U(fg,P_3) - L(fg,P_3) \le M\left[(U(f,P_3) - L(f,P_3)) + (U(g,P_3) - L(g,P_3)) \right]$$

$$< M \cdot \frac{2\epsilon}{2M} = \epsilon$$

$\to fg$ integrable

Recap of this problem so far

We found a relationship between M_i and the pair M_i' and M_i'', and the same for m_i w/ the pair m_i' and m_i''.

We found an initial expr. for $U(fg,P)-L(fg,P)$ and then manipulated it to obtain

$$U(fg,P)-L(fg,P) = M\left[(U(f,P)-L(f,P))+(U(g,P)-L(g,P))\right]$$

But then $U(fg,P)-L(fg,P)$ is a fn of things we can make as small as we want.

Therefore we can make $U(fg,P)-L(fg,P)$ as small as we want, and therefore we've shown fg integrable from f and g

being integrable.

The assumption $f(x),g(x)\geq 0$ was used to prove that $M_i \leq M_i'M_i''$ and $m_i \geq m_i'm_i''$.

Consider the case $f(x)\leq 0$ and $g(x)\leq 0$, for all x in $[a,b]$.

$$\forall x,\ x\in[t_{i-1},t_i] \to f(x)\leq M_i'\leq 0 \wedge g(x)\leq M_i''\leq 0 \to f(x)g(x)\leq M_i'M_i''$$

This is exactly what we had in the $f,g\geq 0$ case. Therefore, $M_i\leq M_i'M_i''$, and similarly $m_i\geq m_i'm_i''$.

By the same steps as used in b), c), and d) we conclude that fg integrable.

Now consider the case $f(x)\geq 0$ and $g(x)\leq 0$, for all x in $[a,b]$.

$$\forall x,\ x\in[t_{i-1},t_i] \to 0\leq m_i'\leq f(x)\leq M_i' \wedge m_i''\leq g(x)\leq M_i''\leq 0 \to m_i''M_i'\leq f(x)g(x)\leq m_i'M_i''$$

Therefore, $m_i''M_i'\leq m_i$ and $m_i'M_i''\geq M_i$

$$U(fg,P)-L(fg,P) = \sum_{i=1}^{n}(M_i-m_i)\Delta t_i \leq \sum(m_i'M_i''-m_i''M_i')\Delta t_i$$

$$= \sum_{i=1}^{n}(m_i'M_i''-M_i''M_i'+M_i''M_i'-m_i''M_i')\Delta t_i$$

$$= \sum_{i=1}^{n}\left(M_i'(M_i''-m_i'')-M_i''(M_i'-m_i')\right)\Delta t_i$$

f,g bounded $\to \exists M$ s.t. $|f(x)|<M$, $|g(x)|<M$.

$$\leq M\left[(U(g,P)-L(g,P))-(U(f,P)-L(f,P))\right]$$

Choose P s.t. $U(g,P)-L(g,P)<\frac{\epsilon}{2M}$ and $U(f,P)-L(f,P)<\frac{\epsilon}{2M} \to -(U(f,P)-L(f,P))>-\frac{\epsilon}{2M}$

$$\leq M\left[\frac{\epsilon}{2M}-\left(-\frac{\epsilon}{2M}\right)\right]=\epsilon$$

Hence fg is integrable.

At this point we know that fg integrable on $[a,b]$ in the specific cases in which f and g are ≥ 0 in $[a,b]$, ≤ 0 in $[a,b]$, or one is ≥ 0 in $[a,b]$ and the other is ≤ 0 in $[a,b]$.

But $f(x) = \max(f(x),0) + \min(f(x),0)$

and $g(x) = \max(g(x),0) + \min(g(x),0)$

where $\max(f,0)$ and $\max(g,0)$ are ≥ 0 and $\min(f,0)$ and $\min(g,0)$ are ≤ 0.

$$f(x)g(x) = \max(f(x),0)\max(g(x),0) + \max(f(x),0)\min(g(x),0)$$
$$+ \min(f(x),0)\max(g(x),0) + \min(f(x),0)\min(g(x),0)$$

Each fn in the sum is formed by fns which fit the specific cases we considered previously, and hence each fn in the sum is integrable. Hence fg is integrable.

Note that fg has no restrictions on the signs of f and g.

Therefore we've proved that

f, g integrable on $[a,b]$ $\rightarrow$ fg integrable on $[a,b]$

39. f, g integrable on $[a,b]$

Cauchy-Schwarz Inequality: $\left(\int_a^b fg\right)^2 \le \left(\int_a^b f^2\right)\left(\int_a^b g^2\right)$

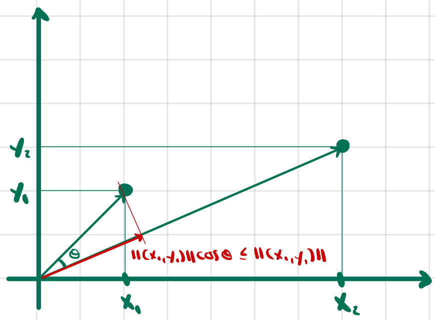
(a) Schwarz Ineq. is special case of Cauchy-Schwarz Ineq.

Proof

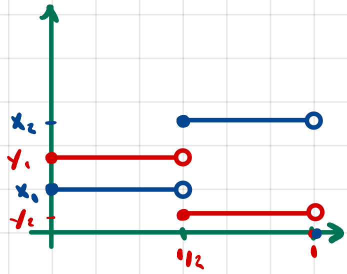
Recall the Schwarz Inequalities seen in prev. chapters

1-19: $x_1y_1 + x_2y_2 \le \sqrt{x_1^2 + x_2^2}\,\sqrt{y_1^2 + y_2^2}$

2-21: $\sum_{i=1}^{n} x_i y_i \le \sqrt{\sum_{i=1}^{n} x_i^2}\,\sqrt{\sum_{i=1}^{n} y_i^2}$



$$f(x) = \begin{cases} x_1 & 0 \le x < \frac{1}{2} \\ x_2 & \frac{1}{2} \le x < 1 \\ 0 & x = 1 \end{cases}$$



$$g(x) = \begin{cases} y_1 & 0 \le x < \frac{1}{2} \\ y_2 & \frac{1}{2} \le x < 1 \\ 0 & x = 1 \end{cases}$$

$$\int_0^1 fg = \int_0^{1/2} x_1y_1\,dx + \int_{1/2}^1 x_2y_2 = \frac{x_1y_1 + x_2y_2}{2} = \frac{\sum_{i=1}^{2} x_iy_i}{2}$$

$$\left(\int_0^1 fg\right)^2 = \frac{\left(\sum_{i=1}^{2} x_iy_i\right)^2}{4}$$

$$\int_0^1 f^2 = \int_0^{1/2} x_1^2\,dx + \int_{1/2}^1 x_2^2\,dx = \frac{x_1^2 + x_2^2}{2} = \frac{\sum_{i=1}^{2} x_i^2}{2}$$

$$\int_0^1 g^2 = \frac{y_1^2 + y_2^2}{2} = \frac{\sum_{i=1}^{2} y_i^2}{2}$$

Therefore,

$$\left(\int_0^1 fg\right)^2 \le \left(\int_0^1 f^2\right)\left(\int_0^1 g^2\right) \longrightarrow \frac{\left(\sum_{i=1}^{2} x_iy_i\right)^2}{4} \le \frac{\sum_{i=1}^{2} x_i^2}{2}\,\frac{\sum_{i=1}^{2} y_i^2}{2} \longrightarrow \left(\sum_{i=1}^{2} x_iy_i\right)^2 \le \sum_{i=1}^{2} x_i^2 \sum_{i=1}^{2} y_i^2$$

More generally, let

$$f(x) = \begin{cases} x_i & \frac{i-1}{n} < x < \frac{i}{n} \\ 0 & 1 \end{cases}$$

$$g(x) = \begin{cases} y_i & \frac{i-1}{n} < x < \frac{i}{n} \\ 0 & 1 \end{cases}$$

Then $\left(\int_0^1 fg\right)^2 = \frac{\left(\sum x_iy_i\right)^2}{n^2} \le \left(\int_0^1 f^2\right)\left(\int_0^1 g^2\right) = \frac{\sum x_i^2}{n} \cdot \frac{\sum y_i^2}{n} \longrightarrow \longrightarrow \left(\sum_{i=1}^{n} x_iy_i\right)^2 \le \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2$

(b) Let's prove $\left(\int_a^b fg\right)^2 \leq \left(\int_a^b f^2\right)\left(\int_a^b g^2\right)$ in three different ways.

Proof 1

i) Assume there is some number $\lambda \geq 0$ s.t. $f = \lambda g$ for all x in $[a,b]$

$$\int_a^b fg = \int_a^b \lambda g g = \lambda \int_a^b g^2$$

$$\left(\int_a^b fg\right)^2 = \lambda^2 \left(\int_a^b g^2\right)^2$$

$$\int_a^b f^2 = \lambda^2 \int_a^b g^2$$

$$\left(\int_a^b f^2\right)\left(\int_a^b g^2\right) = \left(\lambda^2 \int_a^b g^2\right)\left(\int_a^b g^2\right) = \lambda^2 \left(\int_a^b g^2\right)^2$$

Therefore, $\left(\int_a^b fg\right)^2 = \left(\int_a^b f^2\right)\left(\int_a^b g^2\right)$

---

ii) Now assume $f = g = 0$. Then $\left(\int_a^b fg\right) = 0 = \left(\int_a^b f^2\right)\left(\int_a^b g^2\right)$

---

iii) Now assume $\nexists \lambda$ s.t. $f = \lambda g$ for all x in $[a,b]$.

we know this is ≥ 0 because $(f-\lambda g)^2 \geq 0$.

Then, $\int_a^b (f-\lambda g)^2 = \int_a^b (f^2 - 2\lambda fg + \lambda^2 g^2) = \lambda^2 \int_a^b g^2 - 2\lambda \int_a^b fg + \int_a^b f^2 \geq 0$

There are two possible cases

Case 1: $\lambda^2 \int_a^b g^2 - 2\lambda \int_a^b fg + \int_a^b f^2 > 0$

$\Delta = 4\left(\int_a^b fg\right)^2 - 4\int_a^b g^2 \int_a^b f^2 < 0 \quad \to \quad \left(\int_a^b fg\right)^2 < \int_a^b f^2 \int_a^b g^2$

Case 2: $\lambda^2 \int_a^b g^2 - 2\lambda \int_a^b fg + \int_a^b f^2 = 0$

$\Delta = 0 \leftrightarrow$ one solution $\leftrightarrow \int_a^b (f-\lambda g)^2 = 0 \leftrightarrow \left(\int_a^b fg\right)^2 = \int_a^b f^2 \int_a^b g^2$

$$\lambda = \frac{2\int_a^b fg}{2\int_a^b g^2} = \frac{\int_a^b fg}{\int_a^b g^2}$$

By proof by cases, $\left(\int_a^b fg\right)^2 \leq \int_a^b f^2 \int_a^b g^2$.

Proof 2

Previously (2-21), we used the fact that the difference between two vectors is a vector w/ non-negative length to derive

$\sum x_i y_i \leq \sqrt{\sum x_i^2}\sqrt{\sum y_i^2}$. Each individual vector coordinate obeyed $2x_i y_i \leq x_i^2 + y_i^2$.

$$(x-y)^2 \geq 0 \rightarrow x^2+y^2-2xy \geq 0 \rightarrow 2xy \leq x^2+y^2$$

let $x = \dfrac{f(x)}{\sqrt{\int_a^b f^2}}$ and $y = \dfrac{g(x)}{\sqrt{\int_a^b g^2}}$

Then $$\frac{2f(x)g(x)}{\sqrt{\int_a^b f^2 \int_a^b g^2}} \leq \frac{f^2(x)}{\int_a^b f^2} + \frac{g^2(x)}{\int_a^b g^2}$$

Integrate both sides

$$\frac{2\int_a^b fg}{\sqrt{\int_a^b f^2 \int_a^b g^2}} \leq \frac{\int_a^b f^2}{\int_a^b f^2} + \frac{\int_a^b g^2}{\int_a^b g^2} = 2 \quad \rightarrow \int_a^b fg \leq \sqrt{\int_a^b f^2 \int_a^b g^2}$$

Proof 3

$$\int_a^b \left[\int_a^b (f(x)g(y) - f(y)g(x))^2 dx\right] dy = \int_a^b \left[\int_a^b (f(x)^2 g(y)^2 - 2f(x)g(x)f(y)g(y) + f(y)^2 g(x)^2)\, dx\right] dy$$

$$= \int_a^b \left[g(y)^2 \int_a^b f(x)^2 dx - 2f(y)g(y)\int_a^b f(x)g(x)\,dx + f(y)^2 \int_a^b g(x)^2 dx \right] dy$$

$$= \int_a^b g(y)^2 \int_a^b f(x)^2 + \int_a^b g(y)^2 dy \int_a^b f(x)^2 dx - 2\int_a^b fg \int_a^b fg$$

$$= \int_a^b g^2 \int_a^b f^2 + \int_a^b g^2 \int_a^b f^2 - 2\left(\int_a^b fg\right)^2 = 2\int_a^b g^2 \int_a^b f^2 - 2\left(\int_a^b fg\right)^2$$

Therefore, we have shown

$$\int_a^b \left[\int_a^b (f(x)g(y) - f(y)g(x))^2 dx\right] dy = 2\int_a^b g^2 \int_a^b f^2 - 2\left(\int_a^b fg\right)^2$$

$$\rightarrow \left(\int_a^b fg\right)^2 = \int_a^b g^2 \int_a^b f^2 - \frac{1}{2}\int_a^b \left[\int_a^b (f(x)g(y) - f(y)g(x))^2 dx\right] dy \leq \int_a^b f^2 \int_a^b g^2$$

Ie $$\int_a^b fg \leq \sqrt{\int_a^b f^2 \int_a^b g^2}$$

(c)

Assume f and g are continuous and $\left(\int_a^b fg\right)^2 = \int_a^b f^2 \int_a^b g^2$

Then $\int_a^b (f-\lambda g)^2 = 0$.

Assume $f(y) \neq \lambda g(y)$ at some $\in [a,b]$.

Then $\exists \delta > 0$ s.t. $x \in (y-\delta, y+\delta) \to f(x) \neq \lambda g(x)$

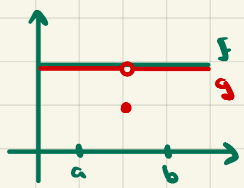
But then $\int_{y-\delta}^{y+\delta} (f-\lambda g)^2 > 0$

Hence $\int_a^b (f-\lambda g) > 0$. $\perp$.

T.F. $\forall x, x \in [a,b] \to f(x) = \lambda g(x)$

If, on the other hand, at least one of f or g is discontinuous on [a,b], then the above argument isn't valid.

Here is a counterexample:



(d) $\left(\int_0^1 f\right)^2 \leq \left(\int_0^1 f^2\right)$

Proof

Let $g(x)=1$. By Cauchy-Schwarz,

$$\left(\int_a^b fg\right)^2 \leq \int_a^b f^2 \int_a^b g^2 \to \left(\int_0^1 f\right)^2 \leq \int_0^1 f^2 \int_0^1 1\,dx = \int_0^1 f^2$$

If 0 and 1 are replaced by a and b,

$$\left(\int_a^b f\right)^2 \leq \int_a^b f^2 \cdot (b-a)$$

40. f int. on $[0,x]$, for all $x>0$.

$$\lim_{x\to\infty} f(x) = a \rightarrow \lim_{x\to\infty} \frac{1}{x}\int_0^x f(t)\,dt = a$$

Proof

$\lim_{x\to\infty} f(x) = a$ means $\forall \epsilon>0\ \exists N\ \forall x\ x>N \rightarrow |f(x)-a|<\epsilon$

$\rightarrow a-\epsilon < f(x) < a+\epsilon$

Let P be a partition of $[N, N+M]$.

$L(f,P) = \sum m_i \Delta t_i$

$\sum (a+\epsilon)\Delta t_i = M(a+\epsilon)$ is an upper bound.

$U(f,P) = \sum M_i \Delta t_i$

$\sum (a-\epsilon)\Delta t_i = M(a-\epsilon)$ is a lower bound.

Therefore $L(f,P) < M(a+\epsilon)$ and $M(a-\epsilon) < U(f,P)$.

Since $L(f,P) \le \int_N^{N+M} f \le U(f,P)$, it must be that

$$M(a-\epsilon) \le \int_N^{N+M} f \le M(a+\epsilon)$$

$$\left|\int_N^{N+M} f - Ma\right| < \epsilon M$$

$$\left|\frac{\int_N^{N+M} f}{N+M} - \frac{Ma}{N+M}\right| < \frac{\epsilon M}{N+M}$$

$\rightarrow 0$ as $M\to\infty$ $\qquad$ $\rightarrow a$ as $M\to\infty$

$$\frac{Ma}{N+M} - \frac{M\epsilon}{N+M} < \frac{\int_N^{N+M} f}{N+M} < \frac{Ma}{N+M} + \frac{M\epsilon}{N+M}$$

Choose M s.t. $\left|\frac{Ma}{N+M} - a\right| < \epsilon$

Then

$$\frac{Ma}{N+M} < a+\epsilon, \quad \frac{M\epsilon}{N+M} < \epsilon$$

and from (1)

$$a-\epsilon-\epsilon < \frac{\int_N^{N+M} f}{N+M} < a+\epsilon+\epsilon$$

$$-2\epsilon < \frac{\int_N^{N+M} f}{N+M} - a < 2\epsilon$$

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x f(t)\,dt$$

$$= \lim_{x\to\infty}\left[\frac{1}{x}\int_0^N f(t)\,dt + \frac{1}{x}\int_N^x f(t)\,dt\right]$$

$$= \lim_{M\to\infty} \frac{1}{N+M}\int_0^N f(t)\,dt + \frac{1}{N+M}\int_N^{N+M} f(t)\,dt$$

$= 0 + a$

$= a$