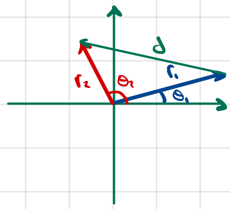
Ch.4 Appendix 3 - Polar Coordinates



1. Distance between (r_1,θ_1) and (r_2,θ_2) is $d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1-\theta_2)$

$(x_1, y_1) = (r_1\cos\theta_1, r_1\sin\theta_1)$
$(x_2, y_2) = (r_2\cos\theta_2, r_2\sin\theta_2)$

$$d = \sqrt{(r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2}$$

$$= \left[r_2^2\cos^2\theta_2 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\cos^2\theta_1 + r_2^2\sin^2\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2 + r_1^2\sin^2\theta_1 \right]^{1/2}$$

$$= \left[r_2^2 + r_1^2 - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \right]^{1/2}$$

$$= \left[r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1-\theta_2) \right]^{1/2}$$

$$= \left[r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2-\theta_1) \right]^{1/2}$$

$d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2-\theta_1)$ is the law of cosines

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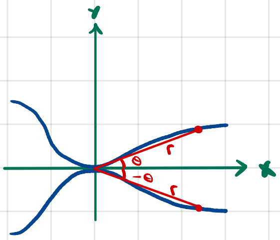
2. i) f even



$r = f(\theta)$ even

$\Rightarrow f(\theta) = f(-\theta) = r$

given a point (r,θ), the point $(r,-\theta)$ is symmetric to (r,θ) relative to the x axis

The graph of f is symmetric about x-axis.

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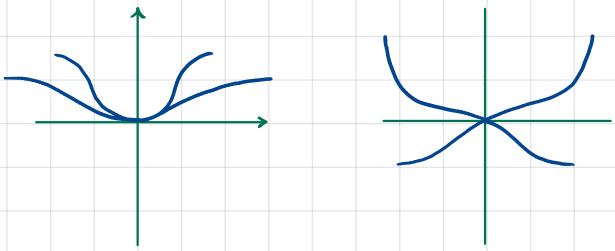
ii) f odd



$r = f(\theta)$ odd

$\Rightarrow f(\theta) = -f(-\theta)$
$f(-\theta) = -f(\theta)$

for each (r,θ) there is $(-r,-\theta)$, symmetric about y axis

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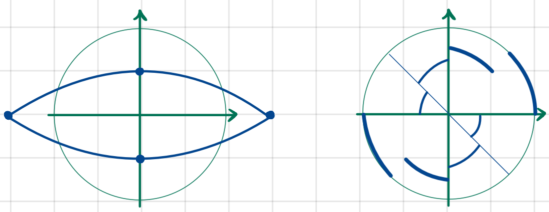
iii) $f(\theta) = f(\theta+\pi)$



$f(0) = f(\pi) = f(2\pi) = f(k\pi)$
$f(\pi/2) = f(3\pi/2) = f(\frac{\pi}{2} + k\pi)$

$\Rightarrow$ graph of f symmetric about the origin

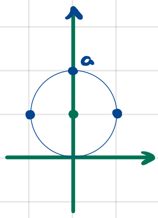
3. i) $r = a\sin\theta$

$r^2 = ar\sin\theta = ay$

$x^2 + y^2 - ay = 0$

$x^2 + y^2 - 2y\frac{a}{2} + \frac{a^2}{4} - \frac{a^2}{4} = 0$

$x^2 + (y - \frac{a}{2})^2 = (\frac{a}{2})^2$

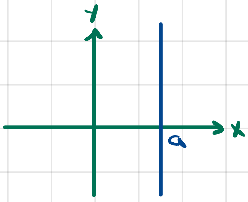


θ	r
0	0
$\pi/2$	a
π	0
$3\pi/2$	$-a$
2π	0
$\pi/4,\ 3\pi/4$	$\frac{a\sqrt{2}}{2}$
$5\pi/4,\ 7\pi/4$	$-\frac{a\sqrt{2}}{2}$

ii) $r = a\sec\theta = \frac{a}{\cos\theta}$

$1 = \frac{a}{r\cos\theta} = \frac{a}{x} \Rightarrow x = a\ ,\ a \neq 0$

$a = 0 \Rightarrow r = 0\ \forall\theta$, graph is single point.



iii) $r = \cos(2\theta)$

$\cos(2\theta) = \cos(\theta+\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

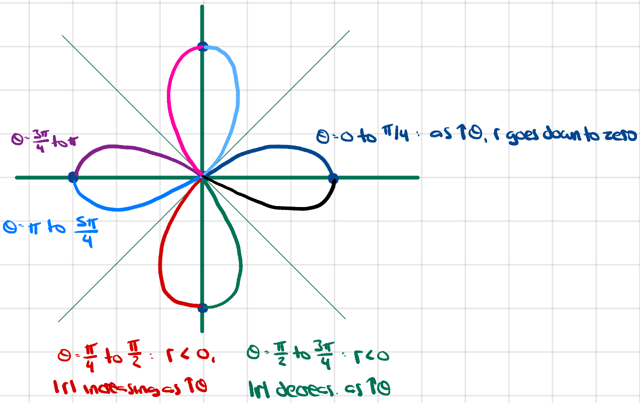
$r = 1 - 2\sin^2\theta$

$r^2 = r - 2r\sin^2\theta$

$r^3 = r^2 - 2r^2\sin^2\theta = r^2 - 2y^2 = x^2 + y^2 - 2y^2$

$r^3 = x^2 + y^2 - 2y^2 = x^2 - y^2$

$\sqrt{(x^2+y^2)^3} = x^2 - y^2$



$x = y \Rightarrow \sqrt{(2y^2)^3} = 0 \Rightarrow x = y = 0$

iv) $f(\theta) = \cos(3\theta) = \cos(2\theta + \theta)$

$= \cos(2\theta)\cos\theta - \sin 2\theta \sin\theta$

$= (\cos^2\theta - \sin^2\theta)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$

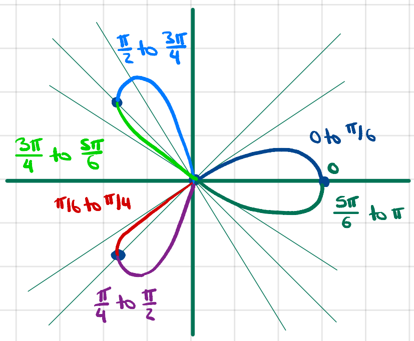
$= \cos^3\theta - 3\sin^2\theta\cos\theta$

$r^4 = r^3\cos^3\theta - 3r^2\sin^2\theta\, r\cos\theta$

$r^4 = x^3 - 3xy^2$

$(x^2+y^2)^2 = x^3 - 3xy^2$

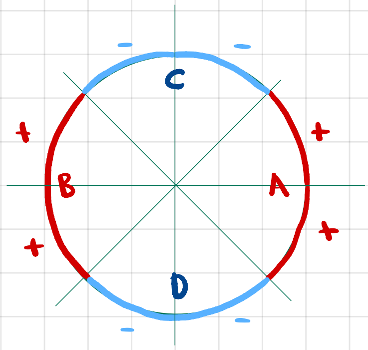
θ	$\cos 3\theta$
0	1
$\pi/2$	0
π	-1
$3\pi/2$	0
2π	1
$\pi/4$	$-\sqrt{2}/2$
$\pi/6$	0
$2\pi/6$	-1



From $\theta = 0$ to $\pi/4$, we're taking cos from $\theta = 0$ to $3\pi/4$, so $\cos(3\theta)$ from 1 to 0 at $\theta = \frac{\pi}{6}$,

Note: for $\theta \in [\pi, 2\pi]$

vi) $r = |\cos 2\theta| = f(\theta)$. Previously, $r = \cos 2\theta = g(\theta)$



$\cos(2\theta) = \cos(\theta+\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

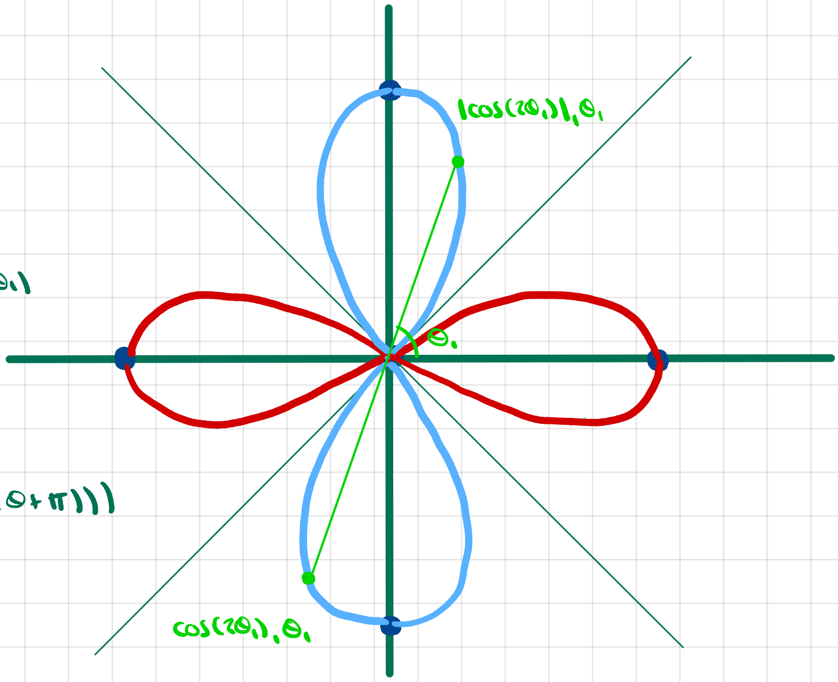
Let $A = [2\pi k - \frac{\pi}{4}, 2\pi k + \frac{\pi}{4}] = [-\frac{\pi}{4}, \frac{\pi}{4}]$

$B = [2\pi k + \frac{3\pi}{4}, 2\pi k + \frac{5\pi}{4}] = [\frac{3\pi}{4}, \frac{5\pi}{4}]$

$C = [2\pi k + \frac{\pi}{4}, 2\pi k + \frac{3\pi}{4}] = [\frac{\pi}{4}, \frac{3\pi}{4}]$

$D = [2\pi k + \frac{5\pi}{4}, 2\pi k + \frac{7\pi}{4}] = [\frac{5\pi}{4}, \frac{7\pi}{4}]$

$$r = \begin{cases} \cos(2\theta) & \theta \in A \cup B \\ -\cos(2\theta) & \theta \in C \cup D \end{cases}$$



Note that for every $\theta_1 \in C$ there is a $\theta_2 = \theta_1 + \pi \in D$.

Let $\theta_1 \in C \Rightarrow \cos(2\theta_1) < 0$

$\Rightarrow r_1 = f(\theta_1) = |\cos(2\theta_1)| = -\cos(2\theta_1) = -g(\theta_1)$

For any point (r, θ), the point $(-r, \theta)$ is the same point as $(r, \theta+\pi)$ or $(r, \theta-\pi)$

(r_1, θ_1) is same point as $(r_1, g(\theta+\pi)) = (r_1, \cos(2(\theta+\pi)))$

$\Rightarrow$ For every $\theta \in C$, $(r, f(\theta)) = (r, g(\theta+\pi))$

$= (r, g(\phi))$ $\phi \in D$

Similarly, $\forall \theta \in D$, $(r, f(\theta)) = (r, g(\theta+\pi)) = (r, g(\phi))$ $\phi \in C$.

For $\theta \in A \cup B$, $(r, f(\theta)) = (r, g(\theta))$

$\Rightarrow$ The graph of $r = f(\theta)$ is the same as that of $r = g(\theta)$.

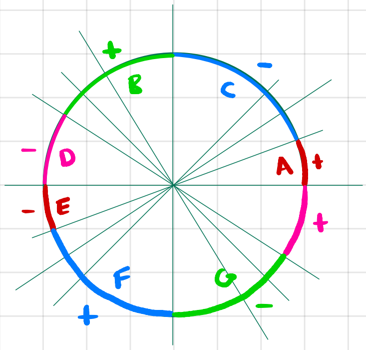
vi) $r = |\cos 3\theta|$

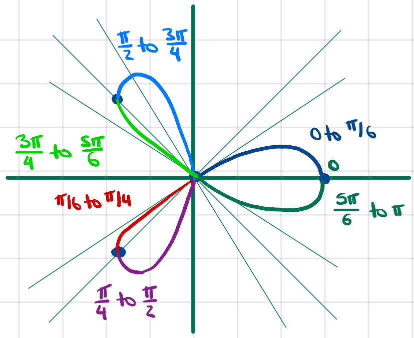
$A : [0, \frac{\pi}{6}]$

$B : [\frac{\pi}{2}, \frac{5\pi}{6}]$

$C : [\frac{\pi}{6}, \frac{\pi}{2}]$

$D : [\frac{5\pi}{6}, \pi]$



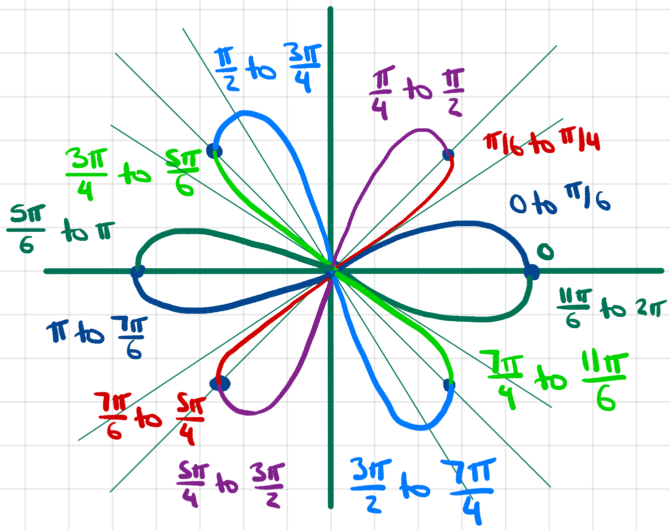


$$r = \begin{cases} \cos 3\theta & \theta \in A \cup B \\ -\cos 3\theta & \theta \in C \cup D \end{cases}$$

For $\theta \in A \cup B$, $r = f(\theta) = g(\theta)$

For $\theta \in C \cup D$ $r = f(\theta) = -\cos 3\theta = -g(\theta)$

$\theta \in C \cup D \Rightarrow$ same points as $g(\theta)$ but w/ sign of r reversed.



4.

i) $r = a\sin\theta$

$r^2 = ar\sin\theta = ay$

$x^2+y^2-ay=0$

$$x^2+y^2-2y\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}=0$$

$$x^2+\left(y-\frac{a}{2}\right)^2=\left(\frac{a}{2}\right)^2$$

ii) $r = a\sec\theta = \dfrac{a}{\cos\theta}$

$$1=\frac{a}{r\cos\theta}=\frac{a}{x} \Rightarrow x=a \quad , a\neq 0$$

$a=0 \Rightarrow r=0\ \forall\theta$, graph is single point.

iii) $r=\cos(2\theta)$

$\cos(2\theta) = \cos(\theta+\theta) = \cos^2\theta-\sin^2\theta = 1-2\sin^2\theta$

$r = 1-2\sin^2\theta$

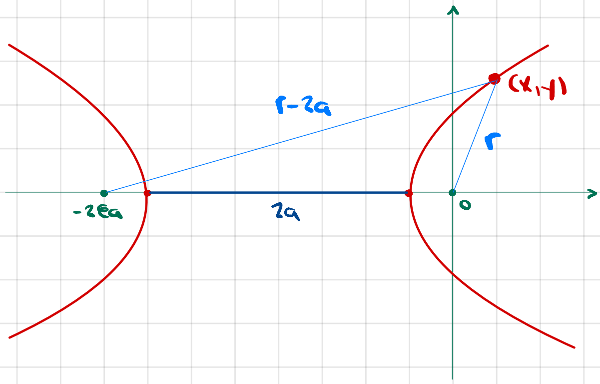
$r^2 = r-2r\sin^2\theta$

$r^3 = r^2-2r^2\sin^2\theta = r^2-2y^2 = x^2+y^2-2y^2$

$r^3 = x^2+y^2-2y^2 = x^2-y^2$

$$\sqrt{(x^2+y^2)^3} = x^2-y^2$$

5.



$2\varepsilon a > 2a \Rightarrow \varepsilon > 1$

distance (x,y) to focus at $(0,0)$ is $(x^2+y^2)^{1/2}=r \Rightarrow r^2=x^2+y^2$

distance (x,y) to focus at $(-2\varepsilon a,0)$ is

$$s^2=(x+2\varepsilon a)^2+y^2 = x^2+4\varepsilon ax+4\varepsilon^2a^2+y^2$$

By definition of hyperbola, $r-s=\pm 2a$

Let's first consider the positive case, ie the right side of the hyperbola:

$\Rightarrow s=r-2a \qquad \Rightarrow s^2=4a^2-4ar+r^2$

$\Rightarrow 4a^2-4ar+r^2 = x^2+4\varepsilon ax+4\varepsilon^2a^2+y^2$

sub in expr. for r^2

$a^2-ar = \varepsilon ax+\varepsilon^2a^2$

$a-r=\varepsilon x+\varepsilon^2 a$

$r=a-\varepsilon x-\varepsilon^2 a \quad = a(1-\varepsilon^2)-\varepsilon x$

$=\Lambda-\varepsilon x \qquad \Lambda=a(1-\varepsilon^2)$

$r=\Lambda-\varepsilon r\cos\theta$

$r(1+\varepsilon\cos\theta)=\Lambda$

$$r=\frac{\Lambda}{1+\varepsilon\cos\theta}=\frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}$$

Now the left side of the hyperbola: $r-s=-2a$

$\Rightarrow s=r+2a \qquad \Rightarrow s^2=r^2+4ar+4a^2$

$\Rightarrow 4a^2+4ar+r^2 = x^2+4\varepsilon ax+4\varepsilon^2a^2+y^2$

$a+r=\varepsilon x+\varepsilon^2 a$

$r=-a+\varepsilon x+\varepsilon^2a = -a(1-\varepsilon^2)+\varepsilon x$

$=-\Lambda+\varepsilon x = -(\Lambda-\varepsilon x) = -(\Lambda-\varepsilon r\cos\theta)$

$$r(1-\varepsilon\cos\theta)=-\Lambda \Rightarrow r=-\frac{\Lambda}{1-\varepsilon\cos\theta}$$

6.

distance to O: $r^2 = x^2 + y^2$

distance to line x = a: $s^2 = (a-x)^2 = (a - r\cos\theta)^2 \Rightarrow s = |a - r\cos\theta|$

But $x = r\cos\theta$ must be $< a$

$x = r\cos\theta > a \Rightarrow x - a > 0$

$s = x - a = \sqrt{x^2 + y^2} > \sqrt{x^2} = x$

$\Rightarrow x - a > x \Rightarrow -a > 0$, but we assumed $a > 0$.

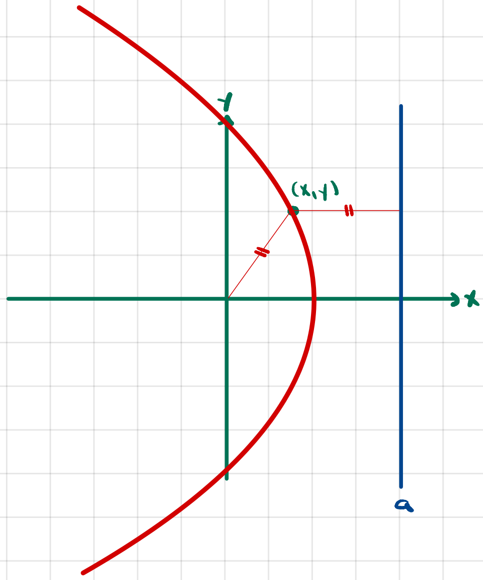
$\Rightarrow s = a - x = a - r\cos\theta$

$s = r \Rightarrow r = a - r\cos\theta \Rightarrow a = r(1+\cos\theta)$

$r = \dfrac{a}{1+\cos\theta}$

↓

same form $\dfrac{\lambda}{1+\epsilon\cos\theta}$ as a hyperbola/ellipse.



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7. For any λ and ϵ, consider graph of $r = \dfrac{\lambda}{1+\epsilon\cos\theta}$ which implies $r = \lambda - \epsilon x$, $\lambda = a(1-\epsilon^2)$

Show that points on the graph of these equations satisfy

$(1-\epsilon^2)x^2 + y^2 = \lambda^2 - 2\lambda\epsilon x$

$r = \lambda - \epsilon x$

$r^2 = x^2 + y^2 = \lambda^2 - 2\lambda\epsilon x + \epsilon^2 x^2$

$\Rightarrow x^2(1-\epsilon^2) + y^2 = \lambda^2 - 2\lambda\epsilon x$

$x^2(1-\epsilon^2) + x(2\epsilon\lambda) + y^2 - \lambda^2 = 0$

$Ax^2 + Bx + Cy^2 + E = 0$

$A, C \neq 0 \Rightarrow \epsilon^2 \neq 1 \Rightarrow \epsilon \neq \pm 1$

$= A\left(x + \frac{B}{2A}\right)^2 + y^2 = A\left(\frac{B}{2A}\right)^2 - E = F$

$A = 1 - \epsilon^2$

$B = 2\epsilon\lambda$

$C = 1$

$E = -\lambda^2$

$\Rightarrow F = (1-\epsilon^2)\left(\dfrac{2\epsilon\lambda}{2(1-\epsilon^2)}\right)^2 + \lambda^2$

$F > 0$, $A, C > 0 \Rightarrow$ ellipse

$A > 0 \Rightarrow 1 - \epsilon^2 > 0 \Rightarrow -1 < \epsilon < 1 \Rightarrow F > 0$

$A < 0$, $C > 0$, $F \neq 0 \Rightarrow$ hyperbola

$A < 0 \Rightarrow 1 - \epsilon^2 < 0 \Rightarrow \epsilon^2 > 1 \Rightarrow \epsilon > 1$ or $\epsilon < -1$

$F \neq 0 \Rightarrow (1-\epsilon^2)\left(\dfrac{2\epsilon\lambda}{2(1-\epsilon^2)}\right)^2 \neq -\lambda^2$

$\epsilon = 1 \Rightarrow F = \lambda^2 > 0$

$A = 0$

The equation becomes

$2\epsilon\lambda x + y^2 - \lambda^2 = 0$

$Bx + y^2 + E = 0$ parabolas

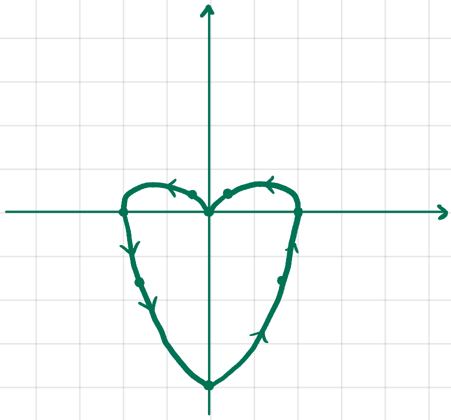
8. a) $r = 1 - \sin\theta$

$r^2 = r - r\sin\theta$

$x^2 + y^2 = r - y$

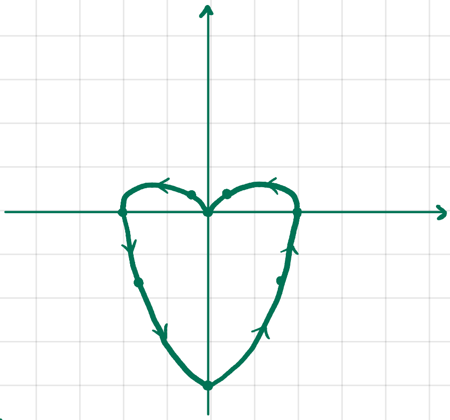
$x^2 + y^2 = (x^2+y^2)^{1/2} - y$

θ	r
0	1
$\pi/4$	$1 - \sqrt{2}/2 \approx 0.3$
$\pi/2$	0
$3\pi/4$	≈ 0.3
π	1
$5\pi/4$	≈ 1.3
$3\pi/2$	2
$7\pi/4$	≈ 1.3



b) $r = -1 - \sin\theta$

θ	r
0	-1
$\pi/4$	$-1 - \sqrt{2}/2 \approx -1.3$
$\pi/2$	-2
$3\pi/4$	≈ -1.3
π	-1
$5\pi/4$	$-1 + \frac{\sqrt{2}}{2} \approx -0.3$
$3\pi/2$	0
$7\pi/4$	≈ -0.3



Consider a point $(r_1, \theta) = (-1 - \sin\theta, \theta)$ on the graph of $r_1 = -1 - \sin\theta$.

Now consider the point $(r_2, \theta + \pi) = (1 - \sin(\theta + \pi), \theta + \pi)$ on the graph of $r_2 = 1 - \sin\theta$.

$$1 - \sin(\theta + \pi) = 1 - (-\sin\theta) = 1 + \sin\theta = -(-1 - \sin\theta)$$

$$(r_2, \theta + \pi) = (-(-1 - \sin\theta), \theta + \pi) = (-1 - \sin\theta, \theta) = (r_1, \theta)$$

Therefore every point (r_1, θ) on $r_1 = -1 - \sin\theta$ is the same point as $(r_2, \theta + \pi)$, $\forall \theta$.

$\Rightarrow$ The graphs of $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ have the same points $\Rightarrow$ same graphs.

c) $r^2 = r - r\sin\theta$

$x^2 + y^2 = r - y$

$x^2 + y^2 = (x^2 + y^2)^{1/2} - y$

$(x^2 + y^2)^2 = x^2 + y^2 - 2y(x^2 + y^2) + y^2$

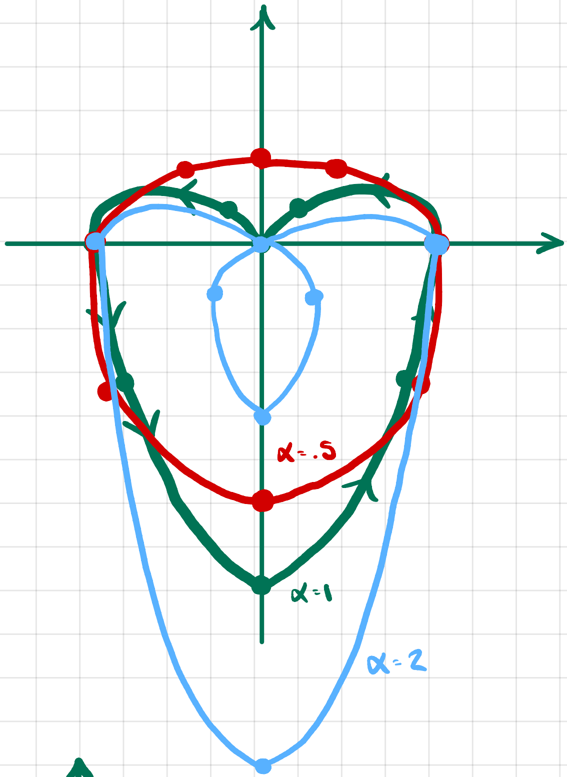
$(x^2 + y^2)^2 + 2y(x^2 + y^2) + y^2 = x^2 + y^2$

$(x^2 + y^2 + y)^2 = x^2 + y^2$

9. i) $r = 1 - \frac{\sin\theta}{2}$

note that all these graphs of form $r = 1 - \alpha \sin\theta$ are symmetric about y-axis

θ	$1 - \sin\theta$	$1 - \frac{\sin\theta}{2}$
0	1	1
$\pi/4$	$1 - \sqrt{2}/2 \approx 0.3$	$1 - \sqrt{2}/4 \approx 0.65$
$\pi/2$	0	0.5
$3\pi/2$	2	1.5
$7\pi/4$	≈ 1.3	$1 + \sqrt{2}/4 \approx 1.35$

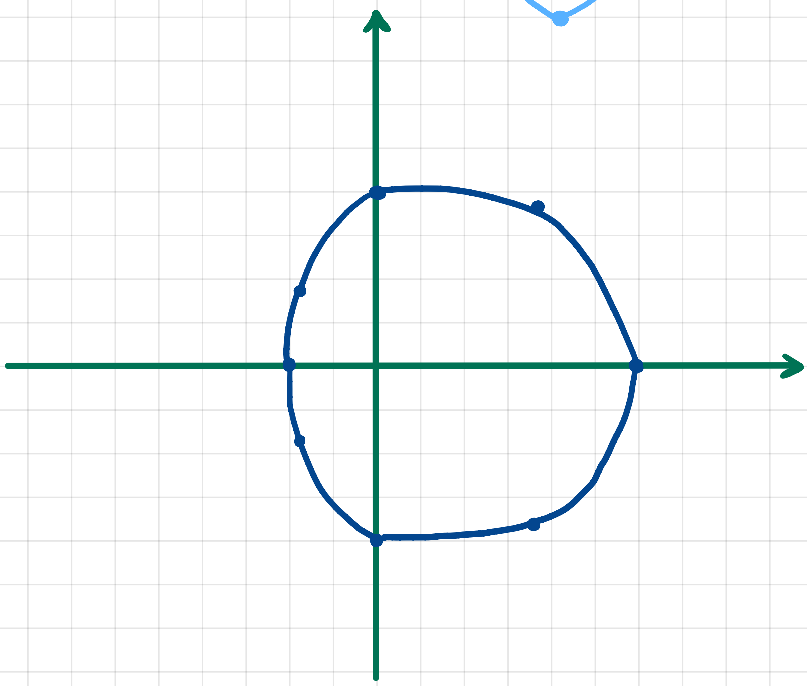


ii) $r = 1 - 2\sin\theta$

θ	$1 - 2\sin\theta$
0	1
$\pi/4$	$1 - \sqrt{2} \approx -.4$
$\pi/2$	-1
$3\pi/2$	3
$7\pi/4$	$1 + \sqrt{2} \approx 2.4$
$\frac{\pi}{6}$	0

iii) $r = 2 + \cos\theta$

θ	$2 + \cos\theta$
0	3
$\pi/4$	$2 + \sqrt{2}/2 \approx 2.7$
$\pi/2$	2
$3\pi/4$	$2 - \sqrt{2}/2 \approx 1.3$
π	1
$5\pi/4$	$2 - \sqrt{2}/2 \approx 1.3$
$3\pi/2$	2
$7\pi/4$	≈ 2.7



$\cos\theta = \sin(\theta + \frac{\pi}{2})$

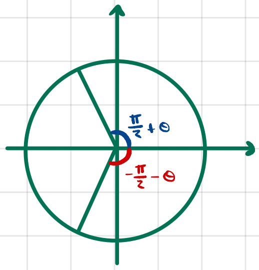
$\Rightarrow r = 2 + \sin(\theta + \frac{\pi}{2})$

$= 2(1 + \frac{1}{2}\sin(\theta + \frac{\pi}{2})) = \underbrace{2(1 - \frac{1}{2}\sin(\theta - \frac{\pi}{2}))}$

looks like $f(\theta)$ from i) with extra term in sin.
this extra $\pi/2$ represents a rotation counterclockwise by $\pi/2$ rad

double the size of i)'s graph

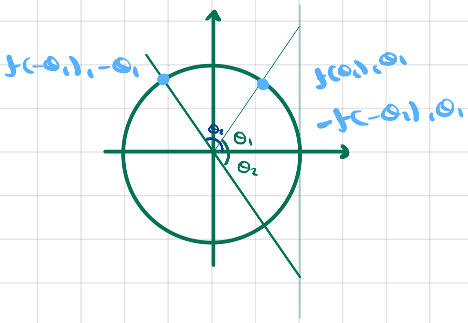
graph of $r = f(\theta)$ symmetric about y-axis



The conditions for symmetry about y axis in cartesian coord are:

$(x, y) \in G \Rightarrow (-x, y) \in G$

expressed in polar coord:



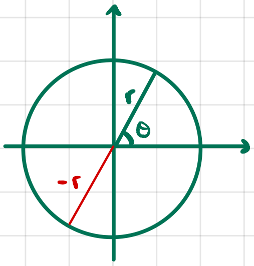
$x = f(\theta_1)\cos\theta_1 = -f(\theta_2)\cos\theta_2$

$y = f(\theta_1)\sin\theta_1 = f(\theta_2)\sin\theta_2$

$\Rightarrow \tan\theta_1 = -\tan\theta_2 \iff \theta_1 = -\theta_2$

$\Rightarrow \theta_2 = -\theta_1 + 2\pi n \Rightarrow f(\theta_2) = f(-\theta_1)$

$\theta_2 = -\theta_1 + \pi + 2\pi n \Rightarrow f(\theta_2) =$



given $(r,\theta) = (f(\theta),\theta)$

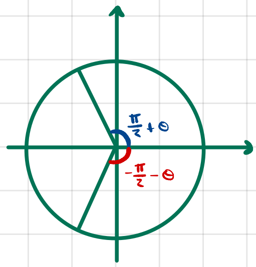
what do we know about $(-r,\theta) = (-f(\theta),\theta) = (r, \theta+\pi) = (f(\theta), \theta+\pi)$

$x_2 = r\cos(\theta+\pi) = -r\cos(\theta) = x_1$

$y_2 = -y_1$

$f(\theta) = -f(-\theta)$

$f(\frac{\pi}{2}+\theta) = -f(-\frac{\pi}{2}-\theta)$



I want to prove that f odd $\Rightarrow$ for every point $(f(\frac{\pi}{2}+\theta), \frac{\pi}{2}+\theta)$ there is a point $(f(\frac{\pi}{2}+\theta), \frac{\pi}{2}-\theta)$.

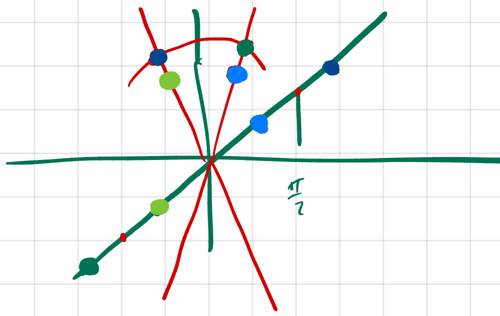
Consider the polar point $(r_1,\theta_1) = (f(\frac{\pi}{2}+\theta), \frac{\pi}{2}+\theta)$

f odd $\Rightarrow (r_1,\theta_1) = (-f(-\frac{\pi}{2}-\theta), \frac{\pi}{2}+\theta)$

$= (f(-\frac{\pi}{2}-\theta), -\frac{\pi}{2}+\theta)$

$\Rightarrow (f(-\frac{\pi}{2}-\theta), -\frac{\pi}{2}+\theta)$ is on the graph.

But so is $(f(-\frac{\pi}{2}-\theta), -\frac{\pi}{2}-\theta)$



Consider polar point $(r_1,\theta_1) = (f(-\frac{\pi}{2}-\theta), -\frac{\pi}{2}-\theta)$

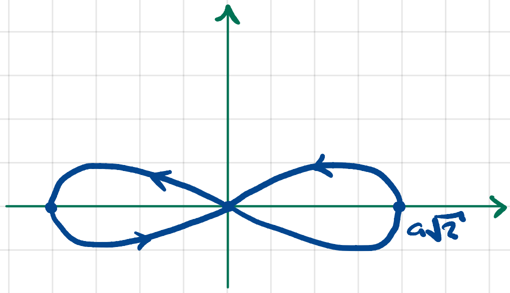
f odd $\Rightarrow (r_1,\theta_1) = (-f(\frac{\pi}{2}+\theta), -\frac{\pi}{2}-\theta)$

$= (f(\frac{\pi}{2}+\theta), \frac{\pi}{2}-\theta)$

$\Rightarrow (f(\frac{\pi}{2}+\theta), \frac{\pi}{2}-\theta)$ is on the graph.

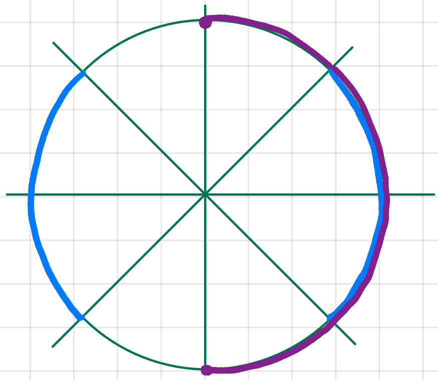
But so is $(f(\frac{\pi}{2}+\theta), \frac{\pi}{2}+\theta)$

10. a) $r^2 = 2a^2\cos(2\theta)$



$r^2 \geq 0 \Rightarrow \cos 2\theta \geq 0 \Rightarrow \theta \in [-\pi/4, \pi/4] \cup [3\pi/4, 5\pi/4]$

θ	$2a^2\cos(2\theta)$	r
$-\pi/4$	0	0
0	$2a^2$	$a\sqrt{2}$
$\pi/4$	0	0
$3\pi/4$	0	
π	$2a^2$	
$5\pi/4$	0	
$3\pi/2$		



note that $f(\theta) = 2a^2\cos(2\theta)$ is even

$f(-\theta) = 2a^2\cos(-2\theta) = 2a^2\cos(2\theta) = f(\theta)$

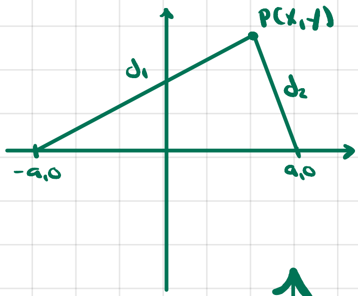
b) $r^2 = 2a^2\cos(2\theta)$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$r^4 = 2a^2(r^2\cos^2\theta - r^2\sin^2\theta)$

$(x^2+y^2)^2 = 2a^2(x^2-y^2)$

c)



$d_1 = \sqrt{(x+a)^2 + y^2}$

$d_2 = \sqrt{(x-a)^2 + y^2}$

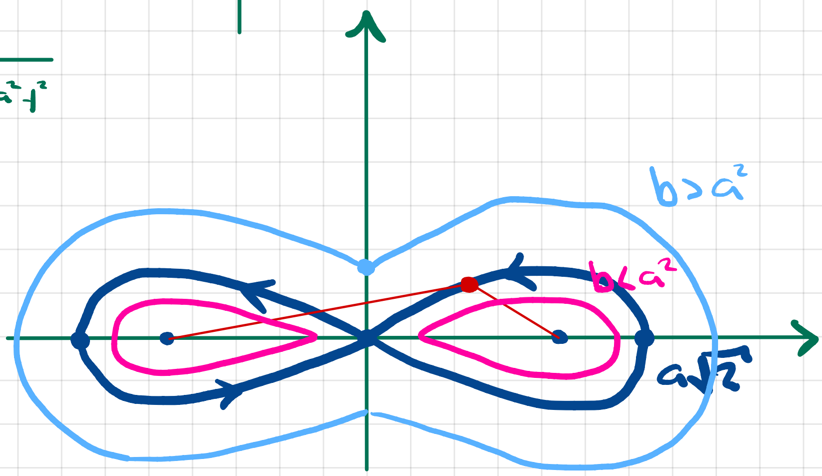
$d_1 d_2 = a^2 \Rightarrow \sqrt{[(x+a)^2 + y^2][(x-a)^2 + y^2]}$

$= \sqrt{x^4 + 2x^2y^2 + y^4 + a^4 - 2a^2x^2 + 2a^2y^2}$

$= \sqrt{(x^2+y^2)^2 - 2a^2(x^2-y^2) + a^4}$

$\Rightarrow \cancel{a^4} = (x^2+y^2)^2 - 2a^2(x^2-y^2) + \cancel{a^4}$

$(x^2+y^2)^2 = 2a^2(x^2-y^2)$



d) $\{(x,y) : d_1 d_2 = b,\ b > a^2\}$