

Ch 7 - Least Upper Bounds

I.

ii) $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}$

$\sup A = 1 \in A$

$\inf A = 0 \notin A$. No least element.

iii) $A = \left\{ \frac{1}{n} : n \in \mathbb{Z} \text{ and } n \neq 0 \right\}$

$$= \left\{ \dots, -\frac{1}{3}, -\frac{1}{2}, -1, 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}$$

$\sup A = 1 \in A$

$\inf A = -1 \in A$

iv) $A = \left\{ x : x = 0 \text{ or } x = \frac{1}{n}, n \in \mathbb{N} \right\}$
 $= \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$

$\sup A = 1 \in A$

$\inf A = 0 \in A$

v) $\{x : 0 \leq x \leq \sqrt{2}, x \text{ rational}\}$

$$\forall \epsilon > 0 \exists n \in \mathbb{N} : 0 < \frac{1}{n} < \epsilon.$$

thus $\forall \epsilon > 0, \epsilon + \inf A$

$\inf A = 0 \in A$

$\sup A = \sqrt{2} \notin A$. No greatest element.

vi) $A = \{x : x^2 + x + 1 \geq 0\}$

$$\Delta = 1 - 4 = -3 \rightarrow \forall x, x^2 + x + 1 > 0$$

$A = \mathbb{R}$, unbounded.

vii) $A = \{x : x^2 + x - 1 < 0\}$

$$\Delta = 1 + 4 \cdot 5 \rightarrow x_{\text{root}} = \frac{-1 \pm \sqrt{5}}{2}$$

$$\forall x, x \in \left(\frac{-1 - \sqrt{5}}{2}, \frac{-1 + \sqrt{5}}{2} \right) \rightarrow x^2 + x - 1 < 0$$

$$A = \left\{ x : x \in \left(\frac{-1 - \sqrt{5}}{2}, \frac{-1 + \sqrt{5}}{2} \right) \right\}$$

$$\sup A = \frac{-1 + \sqrt{5}}{2} \notin A$$

$$\inf A = \frac{-1 - \sqrt{5}}{2} \notin A$$

No least or greatest element.

viii) $A = \{x : x < 0 \text{ and } x^2 + x - 1 < 0\}$

$$= \left\{ x : x \in \left(\frac{-1 - \sqrt{5}}{2}, 0 \right] \right\}$$

$\sup A = 0 \in A$ = greatest el.

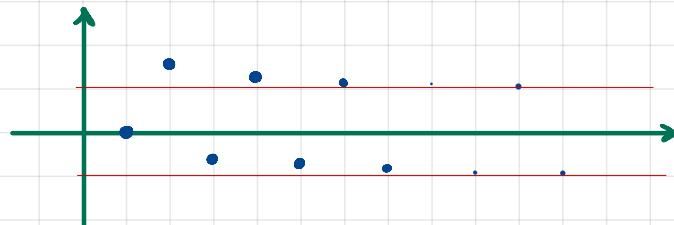
$$\inf A = \frac{-1 - \sqrt{5}}{2} \notin A$$

no least el.

ix) $A = \left\{ \frac{1}{n} + (-1)^n : n \in \mathbb{N} \right\}$

$$= \left\{ 0, \frac{1}{2} + 1, \frac{1}{3} - 1, \frac{1}{4} + 1, \dots \right\}$$

$$= \left\{ 0, \frac{3}{2}, -\frac{2}{3}, \frac{5}{4}, -\frac{4}{5}, \frac{7}{6}, -\frac{6}{7}, \dots \right\}$$



$\sup A = \frac{3}{2} \in A$ = greatest el.

$\inf A = -1 \notin A$

no least el.

2. a)

$$\begin{aligned} A + \phi &\text{ bounded below} \\ -A = \{-x : x \in A\} &\rightarrow \begin{aligned} &-A + \phi \\ &-A \text{ bounded above} \\ &-\sup(-A) = \inf(A) \end{aligned} \end{aligned}$$



Proof

Since $A + \phi$, then $\exists z \in \mathbb{R}, z \in A$. Hence, $-z \in -A$, so $-A + \phi$.

A bounded below $\rightarrow \exists z \in \mathbb{R}, \forall x \in A, z \leq x$

$\forall x \in A, -x \geq -z$

$\rightarrow -A$ bounded above.

By P13, since $-A + \phi$ is bounded above then A has least upper bound α .

i.e. $\forall x \in A, x \leq \alpha$

Therefore $\forall x \in A, -x \geq -\alpha$.

$-\alpha$ is a lower bound for A .

Is it the greatest lower bound?

Assume β is a lower bound of A and $\beta \geq -\alpha$

Assume $\beta > \alpha$.

Let x be some element in A

Then $\forall x_i \in A, -\alpha < \beta \leq x_i$

$\rightarrow \forall x_i \in A, -x_i \leq -\beta < \alpha$

Thus $\forall x_i \in -A, x_i \leq \beta < \alpha$

thus β is upper bound of $-A$.

α not sup(A)

1.

Therefore, $\beta \leq -\alpha$. i.e., $-\alpha = \inf(A)$

Therefore A has a greatest lower bound, and $\inf(A) = -\sup(-A)$ ■

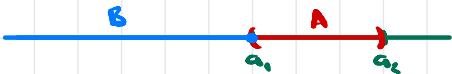
Note that this proves the "greatest lower bound property".

A is set of real numbers

$A + \phi$ $\rightarrow A$ has greatest lower bound

A bounded below

b)
 $A \neq \emptyset$ bounded below
 $B = \{x : x \text{ is lower bound of } A\}$ $\rightarrow B \neq \emptyset$
 B bounded above
 $\sup B = \inf A$



A bounded below $\rightarrow \exists j \in \mathbb{R}, \forall x, x \in A \rightarrow j \leq x$.
 Thus $j \in B$ and $B \neq \emptyset$.

Since A bounded below and $A \neq \emptyset$, A has a greatest lower bound (part a)), call it β .
 Therefore $\forall j \in \mathbb{R}$, if j is lower bound of A then $j \leq \beta$.
 Thus β is an upper bound of B .

Thus B has $\alpha = \sup(B)$.

Assume $\alpha \neq \beta$.

Case 1: $\beta < \alpha$, i.e. $\inf(A) < \sup(B)$

Then $\exists x_1, x_2 \in (\beta, \alpha) = (\inf(A), \sup(B)) \cap A, x_1 < x_2$

Then x_1 is a lower bound for A , but $x_1 > \inf(A)$. \perp

Case 2: $\alpha < \beta$, i.e. $\sup(B) < \inf(A)$

$\exists x_1, x_2 \in (\sup(B), \inf(A))$.

Then, x_1 not a lower bound because $x_1 > \sup(B)$, hence $x_1 \notin B$.

But since $x_1 < \inf(A)$ then x_1 is a lower bound of A , hence $x_1 \in B$.

\perp .

In both possible cases, \perp .

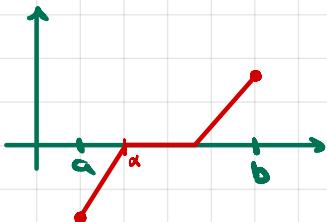
Hence $\alpha = \beta$, i.e. $\sup B = \inf A$.

3.

 f cont. on $[a, b]$ $f(a) < 0 < f(b)$

a) The proof of Theorem 7-1 used a set, $A = \{x : a \leq x \leq b : f(x) \text{ negative on } [a, x]\}$, and proved that it has a least upper bound α , for which $f(\alpha) = 0$.

However, when we think of x such that $f(x) = 0$ and $x \geq \alpha$, we can find counterexample to the claim that there is necessarily a second smallest x for which $f(x) = 0$.



$\exists x \in \mathbb{R}, x \in [a, b] \wedge f(x) = 0 \wedge [\forall y, y \in [a, b] \wedge f(y) = 0 \rightarrow y \leq x]$
ie, there is a largest x in $[a, b]$ with $f(x) = 0$.

PROOF

Let $A = \{x : a \leq x \leq b : f(x) \text{ positive in } [x, b]\}$ $b \in A, \text{ so } A \neq \emptyset$.Since f cont. on $[a, b]$ then $\lim_{x \rightarrow b^-} f(x) = f(b) > 0$ and $\lim_{x \rightarrow a^+} f(x) = f(a) < 0$.Hence, $\exists \delta > 0$ s.t. $\forall x, x \in (a, a + \delta) \rightarrow f(x) < 0$. $\rightarrow x$ is a lower bound for A .Therefore, A has a greatest lower bound, call it $\alpha = \inf A$. Then $a \leq \alpha \leq b$.Assume $f(\alpha) > 0$.Then $\exists \delta > 0, \forall x \in \mathbb{R}, \alpha - \delta < x < \alpha + \delta \rightarrow f(x) > 0$ But then $\exists x, \in (\alpha - \delta, \alpha), f(x) > 0, \forall y, y \in [x, b] \rightarrow f(y) > 0$. Hence $x \in A$, and $x < \alpha$, so α is not the greatest lower bound. \perp .Similarly, if we assume $f(\alpha) < 0$ then \perp .Hence $f(\alpha) = 0$.I.e., there is some $\alpha \in [a, b]$ such that $f(\alpha) = 0$.Additionally, since $f(\alpha) = 0$ and $\forall x, x \geq \alpha \rightarrow x \in A$ we have that $\forall x, x \geq \alpha \rightarrow f(x) > 0$.Therefore α is the largest x in $[a, b]$ s.t. $f(x) = 0$.

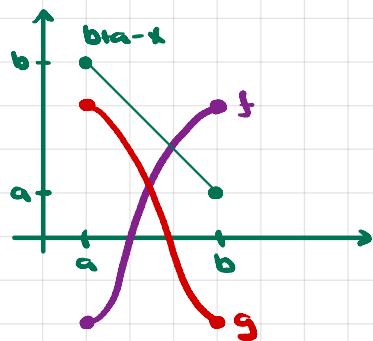
Alternative Proof

Consider the function $h(x) = b + a - x$

Then $h(a) = b$ and $h(b) = a$

Let $g(x) = f(h(x)) = f(b + a - x)$

Then $g(a) = f(b)$ and $g(b) = f(a)$.



It's possible to show that there exists a smallest x in (a,b) s.t. $g(x)=0$.

Let x_* be that smallest x .

Since $\forall x, y \in [a,b] \wedge x < y \rightarrow h(x) > h(y)$

Then

since $\forall y \in [a,b], g(y) = 0 \rightarrow y \geq x_*$

Then $\forall y \in [a,b], g(y) = 0 \rightarrow h(y) \leq h(x_*)$

Thus

$h(x_*)$ is the least value in $[a,b]$ such that $f(h(x_*)) = 0$.

Therefore, there is a least x in $[a,b]$ s.t. $f(x) = 0$.

An easier way to see the above is using slightly different functions.

$$x = b + t(a-b) \quad t \in [0,1]$$

$$x(0) = b$$

$$x(1) = a$$

$$g(t) = f(b + t(a-b))$$

$$g(0) = f(b) > 0$$

$$g(1) = f(a) < 0$$

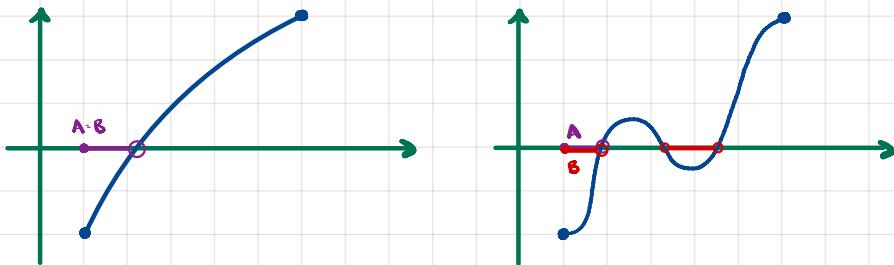
b)

Theorem 7-1

$$\begin{array}{l} f \text{ cont. on } [a,b] \\ f(a) < 0 < f(b) \end{array} \rightarrow \exists x \in [a,b], f(x)=0$$

$$A = \{x : a \leq x \leq b : f \text{ neg on } [a,x]\}$$

$$B = \{x : a \leq x \leq b : f(x) < 0\}$$



Proof

$$a \in B \text{ so } B \neq \emptyset.$$

$$b \notin B, \text{ and } \forall x, x > b \rightarrow x \notin B.$$

Therefore, b is an upper bound for B.

By P13, B has a least upper bound, $\alpha = \sup(B)$.

Assume $f(\alpha) > 0$.

Then f is positive in an open interval around α .

But then α isn't $\sup(B)$. \perp .

Assume $f(\alpha) < 0$.

Then f is neg. in an open interval around α .

But then α isn't an upper bound. \perp .

Therefore $f(\alpha) = 0$.

Ie $\exists x \in [a,b], f(x) = 0$.

Additionally, this x is s.t. $\forall y \in [\alpha, b], f(y) > 0$.

This proof locates the largest x in $[a,b]$ w/ $f(x) = 0$.

4.

a)

 f cont. on $[a, b]$ $f(a) = f(b) = 0$ $f(x_0) > 0$, for some $x_0 \in [a, b]$

$$\rightarrow \exists c, d \in \mathbb{R} \quad a \leq c < x_0 < d \leq b \quad \wedge \quad f(c) = f(d) = 0$$

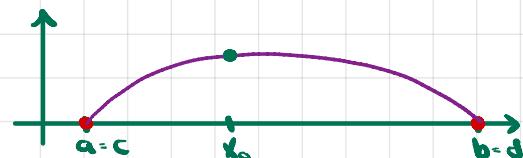
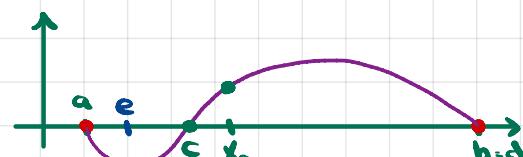
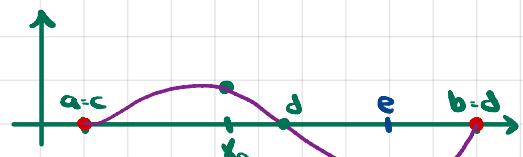
$$\wedge \forall x, x \in (c, d) \rightarrow f(x) > 0$$

Proof

Case 1: $\forall x, x \in (c, d) \rightarrow f(x) > 0$ let $c = a$ and $d = b$.

$$\text{Then } f(c) = f(d) = 0 \quad \wedge \forall x \in (c, d) \rightarrow f(x) > 0$$

$$\wedge \quad a \leq c < x_0 < d \leq b$$

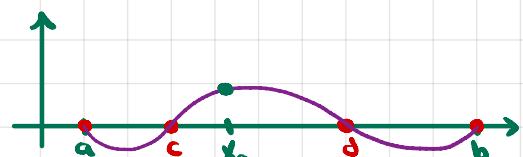
Case 2: $\exists e, e \in (a, x_0) \wedge f(e) < 0, \forall x, x \in (x_0, b) \rightarrow f(x) > 0$ By problem 3b, $\exists c, c \in (e, x_0) \wedge f(c) = 0$ $\wedge \forall x, x \in (e, x_0) \wedge f(x) > 0 \rightarrow x \leq c$, i.e. c is the least x in(e, x_0) s.t. $f(x) = 0$. Thus $\forall x, x \in (c, x_0) \rightarrow f(x) > 0$.We denote in Case 1: $\forall x, x \in (c, d) \rightarrow f(x) > 0$ Thus let $d = b$ and we have the desired result.Case 3 $\exists e, e \in (x_0, b) \wedge f(e) < 0, \forall x, x \in (a, x_0) \rightarrow f(x) > 0$

Analogous to Case 2, but using the visual proof of Theorem 7-1.

Case 4 $\exists e_1, e_2$

$$e_1 \in (a, x_0) \wedge f(e_1) < 0$$

$$e_2 \in (x_0, b) \wedge f(e_2) < 0$$

Then $\exists c_1, c_1 \in (e_1, x_0) \wedge f(c_1) = 0 \wedge \forall x, x \in (e_1, c_1) \wedge f(x) = 0 \rightarrow x \leq c_1$.

$$\exists c_2, c_2 \in (x_0, e_2) \wedge f(c_2) = 0 \wedge \forall x, x \in (x_0, c_2) \wedge f(x) = 0 \rightarrow x \geq c_2$$

Let $c = c_1$ and $d = c_2$ to obtain desired result.

b)

$$\begin{aligned} f \text{ cont. on } [a,b] & \rightarrow \exists c,d: a \leq c < d \leq b \text{ s.t.} \\ f(a) < f(b) & \quad \begin{aligned} & f(c) = f(d) \\ & \wedge f'(d) = f'(b) \\ & \wedge \forall x, x \in (c,d) \rightarrow f(a) < f(x) < f(b) \end{aligned} \end{aligned}$$

Proof

case 1: $\forall x, x \in (a,b) \rightarrow f(a) < x < f(b)$

let $c = a$ and $d = b$.

case 2: $\exists x, x \in (a,b) \wedge f(x) < f(a)$
 $\forall x, x \in (a,b) \rightarrow f(a) < f(x)$

let x_1 s.t. $x_1 \in (a,b) \wedge f(x_1) < f(a)$.

let $g(x) = f(x) - f(a)$.

Then $g(a) = 0$, $g(b) = f(b) - f(a) > 0$, and $g(x_1) = f(x_1) - f(a) < 0$.

Apply Problem 3b to g on $[x_1, b]$. obtain $x_2 \in (x_1, b)$ s.t. $g(x_2) = 0$, and x_2 is the largest x in (x_1, b) s.t. $f(x) = 0$.

$g(x_2) = 0 \rightarrow f(x_2) = f(a)$.

let $c = x_2$ and $d = b$.

case 3: $\exists x, x \in (a,b) \wedge f(x) > f(b)$
 $\forall x, x \in (a,b) \rightarrow f(x) > f(b)$

let x_1 s.t. $x_1 \in (a,b) \wedge f(x_1) > f(b)$

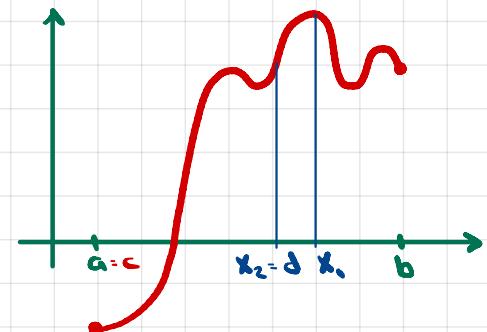
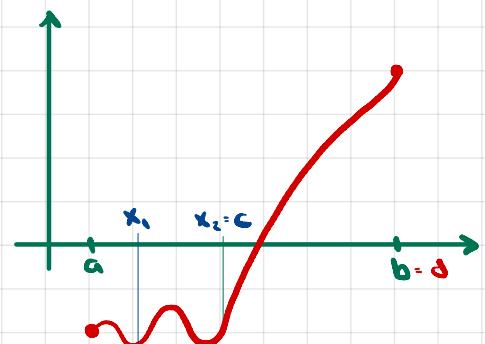
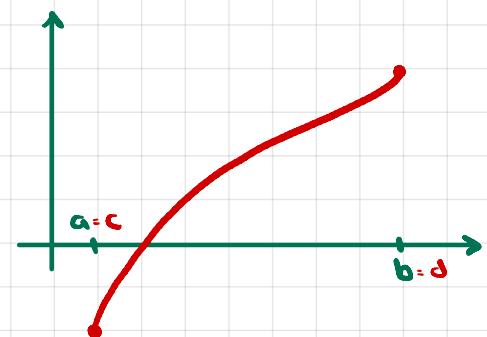
let $g(x) = f(x) - f(b)$.

Then $g(b) = 0$, $g(a) = f(a) - f(b) < 0$, $g(x_1) = f(x_1) - f(b) > 0$.

Apply Th. 7.1 to g on $[a, x_1]$. obtain $x_2 \in (a, x_1)$ s.t. $g(x_2) = 0$, and x_2 is the smallest x in (a, x_1) s.t. $f(x) = 0$.

$g(x_2) = 0 \rightarrow f(x_2) = f(b)$

let $c = a$ and $d = x_2$.

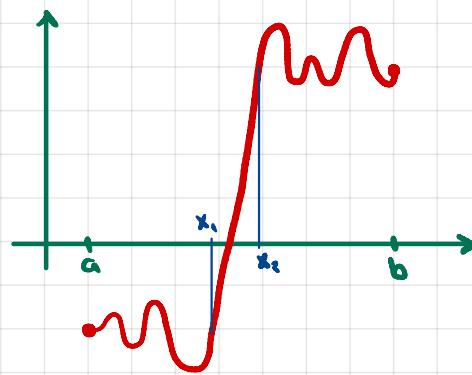


case 4 $\exists x, x \in (a,b) \wedge f(x) < f(a)$
 $\exists x, x \in (a,b) \wedge f(x) > f(b)$

Proof similar to cases 2 and 3.

Find x_1 using $g(x) = f(x) - f(a)$ and problem 3b.
" x_2 " $g(x) = f(x) - f(b)$ " theorem 7-1.

Let $c = x_1$ and $d = x_2$.



Note: each case ends up doing the same thing: c is the largest x in $[a,b]$ s.t. $f(x) = f(a)$,
 d is the smallest x in $[a,b]$ s.t. $f(x) = f(b)$.

5.

a)

$$y-x > 1 \rightarrow \exists k \in \mathbb{Z}, x < k < y$$

Proof

$$[x] \leq x < [x+1]$$

$$[x+1] \leq x+1 < [x+1]+1 = [x+2]$$

Assume $y > x+1$.Then either $y \in (x+1, [x+2])$ or $y \in ([x+2], \infty)$.Case 1: $y \in (x+1, [x+2])$ Then $x < [x+1] \leq x+1 < y < [x+2]$ Therefore $k = [x+1] \in \mathbb{Z}$ and $x < k < y$ Case 2: $y \in ([x+2], \infty)$ Then $x < [x+1] < [x+2] < y$ $k = [x+1] \in \mathbb{Z}$ and $x < k < y$ In both cases $\exists k \in \mathbb{Z}, x < k < y$.

Therefore,

$$y-x > 1 \rightarrow \exists k \in \mathbb{Z}, x < k < y$$



b)

$$x < y \rightarrow \exists r, r \in \mathbb{Q} \wedge x < r < y$$

Proof

$$y-x > 0$$

Therefore $\exists n \in \mathbb{N}, \frac{1}{n} < y-x$.

$$ny - nx > 1$$

By part a), $\exists k \in \mathbb{Z}$ s.t. $nx < k < ny$.

$$x < \frac{k}{n} < y, \frac{k}{n} \in \mathbb{Q}$$

$$\exists r \in \mathbb{Q}, x < r < y$$

■

This proof used theorem T-3, the proof of which depends on Th. T-2, which depends on the newly introduced P13.

c)

$$\begin{array}{l} r, s \in \mathbb{Q} \\ r < s \end{array} \rightarrow \exists m \in \mathbb{R} - \{\mathbb{Q}\}, r < m < s$$

Proof

$$\text{let } f(x) = x^2.$$

$$f(1) = 1 < 2 < f(\sqrt{2})$$

since f cont., $\exists z \in (1, 2)$, $f(z) = z^2 = 2$.

thus $1 < \sqrt{2} < 2$ and $\frac{1}{2} < \frac{1}{\sqrt{2}} < 1$.

since $\sqrt{2}$ irrational, so is $\frac{1}{\sqrt{2}}$.

$$\text{let } n = \frac{1}{\sqrt{2}}(s-r).$$

Then n irrational and $r < n < s$.

d)

$$x < y \rightarrow \exists n, x < n < y \text{ and } n \text{ irrational}$$

Proof

$$\exists x_1, x_2 \in \mathbb{Q} \text{ s.t. } x < x_1 < x_2 < y \quad (\text{b})$$

$$\exists n \in \mathbb{R} - \mathbb{Q} \text{ s.t. } x < n < x_2 < y \quad (\text{c})$$

6.

Let A be a set of real numbers.

A dense \rightarrow every open interval contains a point of A

\mathbb{Q} and $\mathbb{R} - \mathbb{Q}$ are both dense.

a)

A dense

f cont. $\rightarrow \forall x \in \mathbb{R} f(x) = 0$

$\forall x \in A f(x) = 0$

Proof

Assume $\exists x_0 \in \mathbb{R}, f(x_0) \neq 0$.

Then either $f(x_0) > 0$ or $f(x_0) < 0$.

Case 1: $f(x_0) > 0$

f cont, so $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

By def. this means

$\forall \epsilon > 0 \exists \delta > 0 \forall x \in \mathbb{R}, |x - x_0| < \delta \rightarrow |f(x) - f(x_0)| < \epsilon$.

Let $\epsilon = \frac{f(x_0)}{2}$. Then $\forall \delta > 0, \exists x \in (x_0 - \delta, x_0 + \delta), x \in A$.

Hence $f(x) = 0$ and $|f(x) - f(x_0)| = |f(x)| > \epsilon$.

Hence $\exists \epsilon > 0 \forall \delta > 0 \exists x, x \in (x_0 - \delta, x_0 + \delta) \wedge f(x) > \epsilon$

$\lim_{x \rightarrow x_0} f(x) \neq f(x_0)$

⊥.

Case 2 is analogous and leads to ⊥.

Therefore ⊥.

Therefore, $\forall x \in \mathbb{R} f(x) = 0$.

b)

f, g cont.

A dense

$$\rightarrow \forall x f(x) = g(x)$$

$$\forall x, x \in A \rightarrow f(x) = g(x)$$

Proof

$$\text{let } h(x) = f(x) - g(x).$$

Then f cont and $\forall x, x \in A \rightarrow h(x) = 0$.

By part a), $\forall x h(x) = 0$

Therefore $\forall x f(x) = g(x)$.

c)

f, g cont.

A dense

$$\rightarrow \forall x f(x) \geq g(x)$$

$$\forall x, x \in A \rightarrow f(x) \geq g(x)$$

Proof

$$\text{let } h(x) = f(x) - g(x)$$

Assume $\exists x_1 \in \mathbb{R}, f(x_1) < g(x_1)$

then $h(x_1) < 0$.

Since h cont., $h < 0$ in an open interval around x_1 .

Since A dense there is an x_2 in the open interval that is member of A .

But then $f(x_2) \geq g(x_2)$

$$f(x_2) - g(x_2) = h(x_2) \geq 0$$

⊥

Hence $\forall x \in \mathbb{R}, f(x) \geq g(x)$

■

If we replace \geq with $>$ the statement isn't true anymore.

Counterexample

$$f(x) = |2x|, g(x) = |x|$$

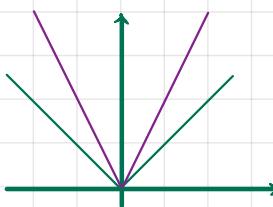
$$A = \mathbb{R} - \{0\}$$

$$0 \notin A, f(0) = g(0) = 0$$

$$\forall x \in \mathbb{R}, x \in A \rightarrow f(x) > g(x)$$

$$\forall x \in \mathbb{R}, x \notin A \rightarrow f(x) \geq g(x)$$

$$\forall x \in \mathbb{R}, f(x) \geq g(x)$$



Cont.
 $\forall x \forall y f(x+y) = f(x) + f(y)$ $\rightarrow \exists c \in \mathbb{R}, \forall x f(x) = cx$

Proof

In problem 3-16 we picked that

$$\forall x \forall y f(x+y) = f(x) + f(y) \rightarrow \exists c, \forall x \in \mathbb{Q}, f(x) = cx$$

let $g(x) = f(1)x$.

Then, f and g are continuous and $\forall x, x \in \mathbb{Q} \rightarrow f(x) = g(x)$, i.e. $f = g$ for all x in the dense set \mathbb{Q} .

By problem 6b, $f = g$ for all x .

Alternative Proof

let x_1 be an irrational number.

Assume $f(x_1) \neq cx_1$.

Case 1: $f(x_1) > cx_1$.

Let $h(x) = f(x) - cx$.

Then $h(x)$ cont. and $h(x_1) > 0$.

Hence h is positive in an open interval around x_1 .

But such an interval contains a rational number, say x_2 .

Then $h(x_2) > 0$, so $f(x_2) > cx_2$.

But since $x_2 \in \mathbb{Q}, f(x_2) = cx_2$.

1.

Case 2: $f(x_1) < cx_1$.

Analog to case 1.

1.

Therefore 1.

Therefore $f(x_1) = cx_1$.

$\forall x f(x) = cx$.

8.

$$\forall a, b, a < b \rightarrow f(a) < f(b)$$

a)

$$\exists \lim_{x \rightarrow a^-} f(x) \wedge \exists \lim_{x \rightarrow a^+} f(x)$$

Proof

$$\text{let } A = \{f(x) : x < a\}$$

Then A is bounded above by f(a).

since A ≠ ∅, then ∃ sup A = α.

Let ε > 0.

Then ∃ x_i, x_i < a ∧ f(x_i) > α - ε, because otherwise α isn't sup A.

Let δ = a - x_i.

Then, ∀ x, a - δ < x < a → x_i < x < a

→ f(x) > f(x_i)

Therefore, α - ε < f(x) ≤ α

|f(x) - α| < ε.

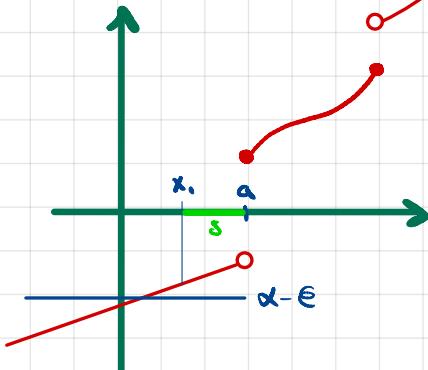
Therefore,

$$\exists \delta > 0 \forall x, x \in (a - \delta, a) \rightarrow |f(x) - \alpha| < \epsilon$$

Therefore

$$\forall \epsilon > 0 \exists \delta > 0 \forall x, x \in (a - \delta, a) \rightarrow |f(x) - \alpha| < \epsilon$$

$$\exists \lim_{x \rightarrow a^-} f(x) = \alpha$$



$$\text{let } B = \{f(x) : x > a\}$$

Then B bounded below by f(a).

Since B ≠ ∅, it has a greatest lower bound, inf B = β.

Let ε > 0

Since β = inf B, there exists an x_i ∈ B s.t.

f(x_i) < β + ε. If there were not such an x_i, then β + ε would be a lower bound greater than β, a contradiction.

Let δ = x_i - a

Then ∀ x, x ∈ (a, a + δ) → x ∈ (a, x_i)

$$\rightarrow f(x) < f(x_i) < f(x_i) < \beta + \epsilon$$

$$\rightarrow |f(x) - \beta| < \epsilon$$

Therefore,

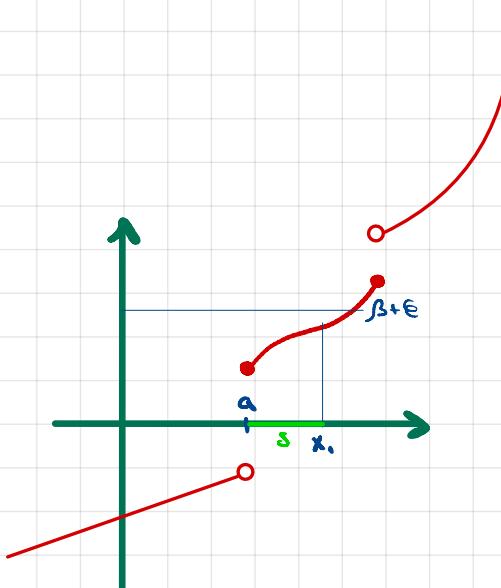
$$\exists \delta > 0 \forall x, x \in (a, a + \delta) \rightarrow |f(x) - \beta| < \epsilon$$

Hence

$$\forall \epsilon > 0 \exists \delta > 0 \forall x, x \in (a, a + \delta) \rightarrow |f(x) - \beta| < \epsilon$$

Therefore

$$\exists \lim_{x \rightarrow a^+} f(x) = \beta$$



b) f never has removable discontinuity.

Proof

Recall: $\exists \lim_{x \rightarrow a} f(x) \neq f(a) \Leftrightarrow f$ has removable discontinuity at a .

Let a be a number.

Assume $\exists \lim_{x \rightarrow a} f(x) \neq f(a)$

$$\text{Then } \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = l$$

since $l \neq f(a)$, we have two cases

Case 1: $f(a) > l$

$$\text{let } \epsilon = \frac{|f(a)-l|}{2} > 0.$$

$$\begin{aligned} \text{Then } \exists \delta > 0 \forall x \in (a, a+\delta) \rightarrow |f(x) - l| < \frac{|f(a)-l|}{2} \rightarrow l - \frac{|f(a)-l|}{2} < f(x) < l + \frac{|f(a)-l|}{2} \\ \rightarrow \frac{3l-f(a)}{2} < l < f(x) < \frac{l+f(a)}{2} < f(a). \end{aligned}$$

thus we have: $\exists x, x > a \wedge f(x) > f(a)$. \perp .

Case 2: $f(a) < l$

And to case 1. \perp .

Therefore \perp .

Therefore $\neg(\exists \lim_{x \rightarrow a} f(x) \neq f(a))$.

Therefore, $\neg(f$ has removable discontinuity at a)

Therefore $\forall a, \neg(f$ has removable discontinuity at a)

* $\neg(\exists \lim_{x \rightarrow a} f(x) \neq f(a))$

$$\Leftrightarrow \neg(\exists l \in \mathbb{R} (\lim_{x \rightarrow a} f(x) \neq l \wedge l \neq f(a)))$$

$$\Leftrightarrow \forall l \in \mathbb{R} (\lim_{x \rightarrow a} f(x) = l \wedge l = f(a))$$

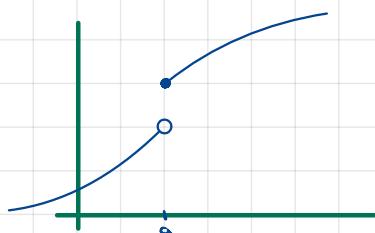
$$\Leftrightarrow \exists \lim_{x \rightarrow a} f(x) \wedge f \text{ cont at } a$$

c) f satisfies conclusion of INT. $\rightarrow f$ cont.

Proof

Assume f satisfies conclusion of INT, i.e. $\forall a, b$, if $f(a) < f(b)$ then on $[a, b]$ f takes on every value in $[f(a), f(b)]$.

Assume f is not cont. at a point a .



From a), let

$$A = \{f(x) : x < a\}$$
$$B = \{f(x) : x > a\}$$

$$\text{then } \lim_{x \rightarrow a^-} f(x) = \sup(A)$$

$$\lim_{x \rightarrow a^+} f(x) = \inf(B)$$

From b) we know the discontinuity at a isn't removable.

$$\text{Hence } \lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$$

$$\sup(A) \neq \inf(B)$$

$$\text{But then } \forall j \in (\sup(A), \inf(B)) \rightarrow \exists x, f(x) = j$$

But then f doesn't sat. INT.

⊥

Therefore $\forall a$ f cont. at a .

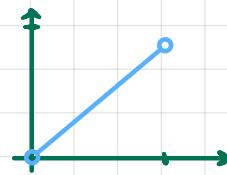
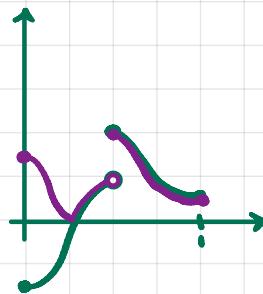
9.

 f bounded on $[0,1]$

$$\|f\| = \sup \{ |f(x)| : x \in [0,1] \}$$

$$a) \forall c, \|cf\| = |c| \cdot \|f\|$$

Proof:

Note that f is not necessarily cont. on $[0,1]$, and so neither is $\|f\|$.Therefore, there isn't necessarily the one & max or min value in $[0,1]$. E.g.

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

Let

$$m_1 = \sup \{ f(x) : x \in [0,1] \}$$

$$m_2 = \inf \{ f(x) : x \in [0,1] \}$$

$$\|f\| = \max(|m_1|, |m_2|)$$

$$\begin{aligned} \|f\| &= \max(|m_1|, |m_2|) = \frac{|cm_1| + |cm_2| + ||cm_1| - |cm_2||}{2} \\ &= \frac{|cm_1| + |cm_2| + ||cm_1| - ||cm_2||}{2} \\ &= \frac{|cm_1| + |cm_2| + ||c||(|m_1| - |m_2|)}{2} \\ &= \frac{|cm_1| + |cm_2| + |c|||m_1| - |m_2||}{2} \\ &= |c| \frac{|m_1| + |m_2| + ||m_1| - |m_2||}{2} \\ &= |c| \max(|m_1|, |m_2|) \\ &= |c| \|f\| \end{aligned}$$

$$b) \|f+g\| \leq \|f\| + \|g\|$$

Proof

Note that $\|f\| = \sup \{|f(x)| : x \in [0,1]\}$

$$\begin{aligned} &= \sup \{ |f(x)+g(x)| : x \in [0,1] \} \\ &= \|f+g\| \end{aligned}$$

That is

$$\|f\| = \|f+g\|$$

Therefore,

$$\forall x \in [0,1], |f(x)| \leq \|f\|$$

Also,

$$|f+g(x)| = |f(x)+g(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g(x)\| \quad (1)$$

since f and g are bounded on $[0,1]$, so is $f+g$. Since $\{|f(x)+g(x)| : x \in [0,1]\} \neq \emptyset$,

this set has a least upper bound: $\|f+g\| = \sup \{|f(x)+g(x)| : x \in [0,1]\}$

$$\forall x, x \in [0,1] \rightarrow |f(x)+g(x)| \leq \|f+g\|$$

Assume $\|f+g\| > \|f\| + \|g\|$

$$\text{Then } \exists x \in [0,1], \|f\| + \|g\| < |f(x)+g(x)| \leq \|f+g\|.$$

(if such an x did not exist then $\|f\| + \|g\| = \|f+g\|$). ↙

$$\text{Then } \exists x \in [0,1], |f(x)+g(x)| > \|f\| + \|g\|.$$

This contradicts (1).

⊥.

Therefore $\|f+g\| \leq \|f\| + \|g\|$

Assume $\exists x, x \in [0,1] \wedge \|f+g\| < |f(x)+g(x)| < \|f\| + \|g\|$

$$\text{ie } \exists x, x \in [0,1] \wedge \|f+g\| < |f(x)+g(x)| < \|f\| + \|g\|$$

$$\forall x, x \in [0,1] \vee \|f+g\| \geq (|f|+|g|)(x) \vee (|f|+|g|)(x) \geq \|f\| + \|g\|$$

Assume $x \in [0,1]$, therefore, two cases:

$$\text{Case 1: } \forall x, \|f+g\| \geq (|f|+|g|)(x)$$

$\|f+g\|$ is upper bound for $|f(x)+g(x)|$.

$$\text{But } \|f+g\| \leq \|f\| + \|g\| = \sup \{|f(x)+g(x)| : x \in [0,1]\}.$$

$$\text{Thus } \|f+g\| = \|f\| + \|g\|$$

$$\text{Case 2: } \forall x, (|f|+|g|)(x) \geq \|f\| + \|g\|$$

⊥.

$$\text{Thus } \|f+g\| = \|f\| + \|g\|$$

$$\text{Hence } \forall x, x \in [0,1] \rightarrow (\|f+g\| = \|f\| + \|g\|)$$

Attainable Proof:

$$|f(x) + g(x)| \leq |f(x)| + |g(x)|, \forall x \in [0,1]$$

$$\|f+g\| = \sup\{|f(x)| + |g(x)| : x \in [0,1]\}$$

$$\forall \epsilon > 0 \exists x_0, x_0 \in [0,1] \wedge \|f+g\| - \|f(x_0) + g(x_0)\| < \epsilon$$

$$\rightarrow \|f+g\| - |f(x_0)| - |g(x_0)| < \epsilon$$

$$\rightarrow \|f+g\| - \|f\| - \|g\| < \epsilon$$

This is true for all $\epsilon > 0$.

If $\|f+g\| > \|f\| + \|g\|$ then if we set $\epsilon = \|f+g\| - \|f\| - \|g\|$, (1) won't be true for ϵ 's larger than this chosen ϵ .

Therefore $\|f+g\| \leq \|f\| + \|g\|$.

10.

$$\alpha > 0 \rightarrow \forall k, x = k\alpha + x', k \in \mathbb{Z}, 0 \leq x' < \alpha$$

Proof

$$\text{Let } d = x - \alpha$$

Case 1: $d \leq 0$ Then $x \leq \alpha$

$$\text{choose } h=0, x'=x$$

$$\text{Then } x = h\alpha + x' = x$$

Case 2: $d > 0$ Then $x > \alpha$

$$\text{Let } A = \{y : y\alpha \leq x\}$$

 $1 \in A$ since $\alpha \leq 1$.
 $\frac{x}{\alpha}$ is an upper bound of A.
Then A has least upper bound, $\gamma = \sup A$.Assume $\gamma < \frac{x}{\alpha}$. Then $\gamma\alpha < x$ Then, by def. of $\gamma = \sup A$, $\exists y_1. (\gamma < y_1 < \frac{x}{\alpha}) \wedge y_1 \in A$. (1)

$$\text{But consider } \frac{\gamma + \frac{x}{\alpha}}{2} = \beta$$

$$\beta = \frac{\gamma + \frac{x}{\alpha}}{2} < \frac{\gamma + \frac{x}{\alpha}}{2} = \frac{\alpha\gamma + x}{2\alpha} < \frac{2x}{2\alpha} = \frac{x}{\alpha}$$

$$\text{so } \gamma < \beta < \frac{x}{\alpha}$$

$$\text{Also } \alpha\beta = \alpha \cdot \frac{\gamma + \frac{x}{\alpha}}{2} = \frac{\alpha\gamma + x}{2} < x \rightarrow \beta \in A$$

Thus $\gamma < \beta < \frac{x}{\alpha} \wedge \beta \in A$, contradicting (1)

1.

Therefore, $\gamma \geq \frac{x}{\alpha}$. However, if $\gamma > \frac{x}{\alpha}$ then γ isn't sup A. L.

$$\text{Therefore } \sup A = \gamma = \frac{x}{\alpha}$$

Let $h = \lfloor \frac{x}{\alpha} \rfloor$, which we know exists for all \mathbb{R} .

$$h \leq \frac{x}{\alpha} < h+1$$

$$h\alpha \leq x < (h+1)\alpha$$

$$0 \leq x - h\alpha < \alpha$$

Let $x' = x - h\alpha$. Then $x = h\alpha + x'$, with $h \in \mathbb{Z}$ and $0 \leq x' < \alpha$.

II.

a)

 a_1, a_2, a_3, \dots sequence of $a_n > 0$ $\rightarrow \forall \epsilon > 0 \exists n, a_n < \epsilon$

$$a_{n+1} \leq \frac{a_n}{2}$$

Proof

$$\forall n \in \mathbb{N}, a_n \leq \frac{a_1}{2^n}$$

Proof

$$A = \{n : a_{n+1} \leq \frac{a_1}{2^n}\}$$

$$1 \in A \text{ since } a_2 \leq \frac{a_1}{2}$$

Assume $k \in A$.

$$\text{Then } a_{k+1} \leq \frac{a_1}{2^k}.$$

$$a_{k+2} \leq \frac{a_{k+1}}{2} = \frac{a_1}{2^{k+1}}$$

So $k+1 \in A$.By induction we conclude that $A = \mathbb{N}$.

$$\text{Therefore } \forall n \in \mathbb{N}, a_{n+1} \leq \frac{a_1}{2^n}.$$

$$\lim_{n \rightarrow \infty} \frac{a_1}{2^n} = 0$$

$$\forall \epsilon > 0 \exists n_0 \in \mathbb{N} \text{ s.t. } 0 < a_{n+1} \leq \frac{a_1}{2^n} < \epsilon$$

Alternative Proof

$$\forall n \in \mathbb{N}, 2^n > n$$

Proof

$$A = \{n \in \mathbb{N} : 2^n > n\}$$

1 $\in A$ since $2^1 - 1 > 0$.Assume $k \in A$. Then $2^k > k$

$$2 \cdot 2^k - 2^k > 2k \geq k+1$$

A $\in \mathbb{N}$ By induction, $\forall n \in \mathbb{N}, 2^n > n$.

$$\text{Therefore } \forall n \in \mathbb{N}, \frac{1}{2^n} < \frac{1}{n} \quad \blacksquare$$

$$\text{Hence, since } a_{n+1} \leq \frac{a_1}{2^n} < \frac{a_1}{n}$$

$$\text{Then } \forall \epsilon > 0, \exists n_1 \in \mathbb{N} \text{ s.t. } \frac{1}{n_1} < \frac{\epsilon}{a_1}$$

$$\rightarrow \frac{a_1}{n_1} < \epsilon.$$

Therefore

$$a_{n+1} < \epsilon$$

b)

P is regular polygon inscribed in circle

$P' = \dots \dots \dots \dots \dots$, twice as many sides as P .

$$A(\text{circle}) - A(P') < \frac{A(\text{circle}) - A(P)}{2}$$

Proof

$$A_1 = \frac{b_1 \cdot h_1}{2}$$

$$A_2 = \frac{b_2 \cdot h_2}{2}$$

$$\sin\left(\frac{\alpha}{2}\right) = \frac{b_1}{r} \rightarrow b_1 = r \sin\left(\frac{\alpha}{2}\right) \quad (1)$$

$$\cos\left(\frac{\alpha}{2}\right) = \frac{h_1}{r} \rightarrow h_1 = r \cos\left(\frac{\alpha}{2}\right)$$

$$A_1 = \frac{r^2 \sin(\alpha/2) \cos(\alpha/2)}{2}$$

$$\sin\beta = \frac{b_1}{b_2} \rightarrow b_1 = b_2 \sin\beta \quad (2)$$

From (1) and (2)

$$r \sin(\alpha/2) = b_2 \sin\beta$$

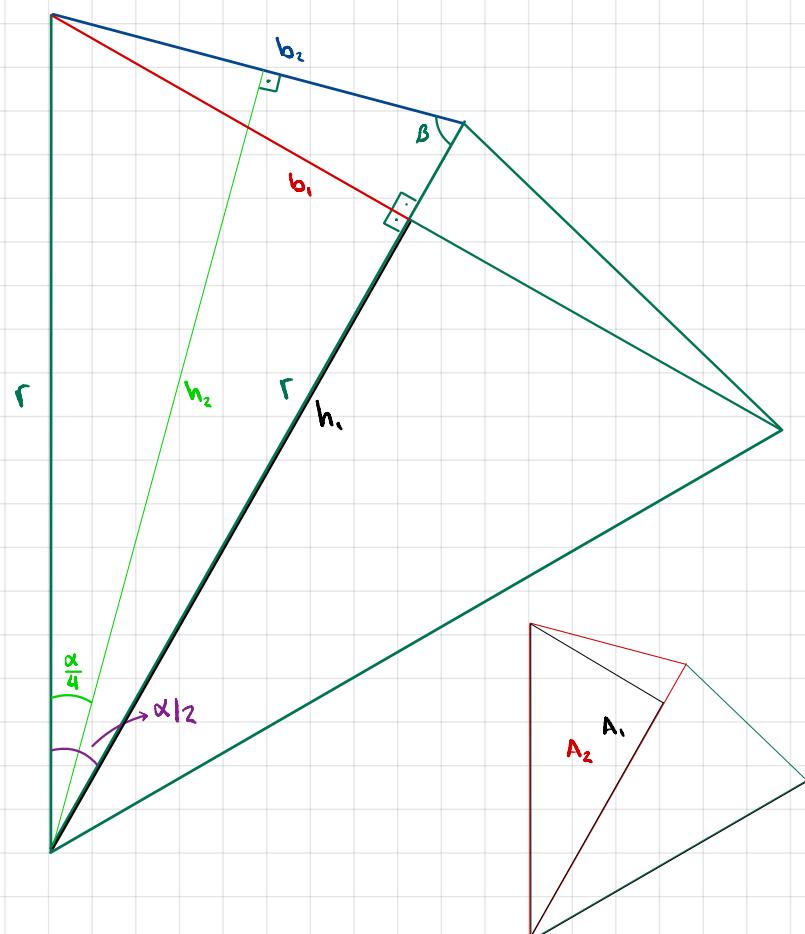
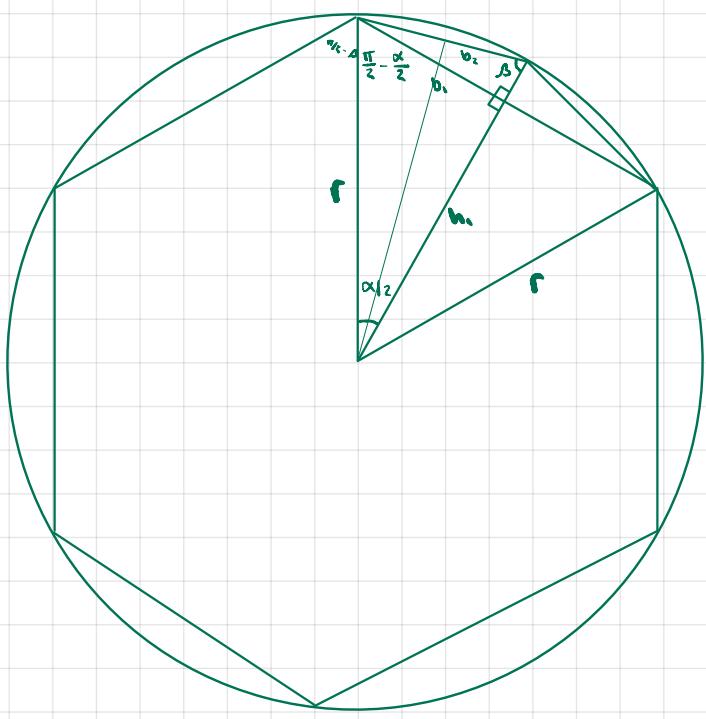
$$b_2 = \frac{r \sin(\alpha/2)}{\sin\beta} \quad (3)$$

$$\sin\beta = \frac{h_2}{r} \rightarrow h_2 = r \sin\beta \quad (4)$$

From (3) and (4)

$$A_2 = \frac{r \sin(\alpha/2)}{\sin\beta} \cdot r \sin\beta \cdot \frac{1}{2}$$

$$A_2 = \frac{r^2 \sin(\alpha/2)}{2}$$



c)

$$\text{Let } c_n = A(\text{circle}) - A(P_{2^{n-1}})$$

where $P_{2^{n-1}}$ is a regular polygon w/ 2^{n-1} sides inscribed in the circle.

In b) we proved that $c_{n+1} < \frac{1}{2} c_n$.

Therefore by a), $\forall \epsilon > 0 \exists n \in \mathbb{N}, c_n < \epsilon$.

$\rightarrow A(\text{circle}) - A(\text{Polygon } 2^{n-1} \text{ sides}) < \epsilon$.

d) Given $\frac{A(P_{i,n})}{A(P_{i,n})} = \frac{d_i^2}{l_i^2}$

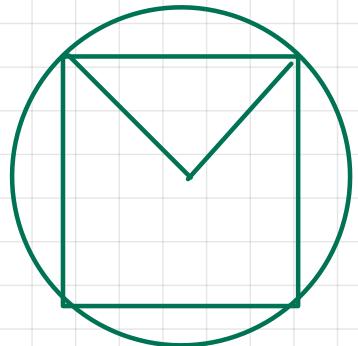
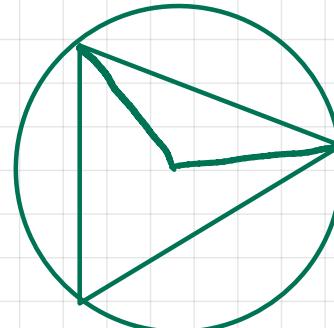
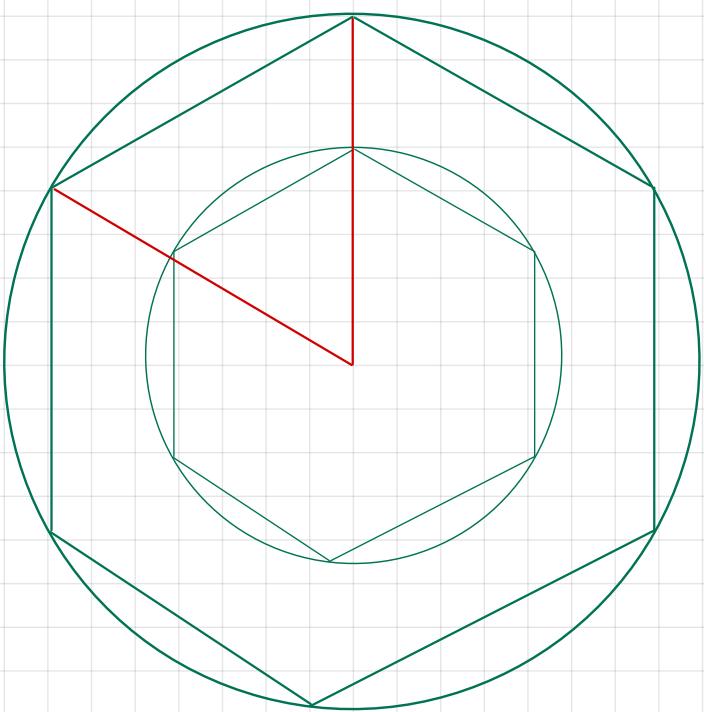
Prove $\frac{A(C_1)}{A(C_2)} = \frac{r_1^2}{r_2^2}$

Assume $\frac{A(C_1)}{A(C_2)} \neq \frac{r_1^2}{r_2^2}$

case 1: $\frac{A(C_1)}{A(C_2)} > \frac{r_1^2}{r_2^2}$

$A(C_1) - A(P_{i,n}) < \epsilon$

$A(C_1) - A(P_{i,n}) < \epsilon$



12.

A, B nonempty sets of numbers

$$\forall x \forall y, x \in A \wedge y \in B \rightarrow x \leq y$$

a) $\forall y, y \in B \rightarrow \sup A \leq y$

Proof

Every $y, y \in B$ is an upper bound for A .

Therefore, A has a supA.

Therefore $\forall y, y \in B \rightarrow \sup A \leq y$

More rigorous proof

Assume $\exists y, y \in B \wedge y < \sup A$

Then $\forall x, x \in A \rightarrow x \leq y < \sup A$

y is an upper bound for A and $y < \sup A$. \perp .

Therefore, $\forall y, y \notin B \vee y \geq \sup A$

Let $y, y \in B$.

Then $y \notin B \vee y \geq \sup A$

Case 1: $y \notin B$. \perp . Then $y \geq \sup A$.

Case 2: $y \geq \sup A$

Therefore, $y \geq \sup A$.

Therefore, $\forall y, y \in B \rightarrow y \geq \sup A$

b) $\sup A \leq \inf B$

Proof

From a), $\sup A$ is a lower bound for B .

Therefore $\exists \inf B$.

Therefore $\sup A \leq \inf B$.

13.

A, B nonempty sets of numbers, bounded above

$$\rightarrow \sup(A+B) = \sup A + \sup B$$

$$A+B = \{x+y : x \in A, y \in B\}$$

Proof

Let $x \in A, y \in B$.

Then $x \leq \sup A, y \leq \sup B$.

$$x+y \leq \sup A + \sup B$$

$$\forall x, y \in A \cap B \rightarrow x+y \leq \sup A + \sup B$$

Therefore $\sup A + \sup B$ is an upper bound for $A+B$.

Therefore $\sup(A+B)$ exists and $\sup(A+B) \leq \sup A + \sup B$

Let $\epsilon > 0$

$$\begin{aligned} \text{Let } x, y \text{ s.t. } x \in A \wedge \sup A - \frac{\epsilon}{2} \leq x \leq \sup A \\ y \in B \wedge \sup B - \frac{\epsilon}{2} \leq y \leq \sup B \end{aligned}$$

$$\text{Then } \sup A + \sup B - \epsilon \leq x+y \leq \sup(A+B)$$

$$\sup A + \sup B \leq \sup(A+B) + \epsilon$$

$$\forall \epsilon > 0 \sup A + \sup B \leq \sup(A+B) + \epsilon$$

Therefore

$$\forall \epsilon > 0 \sup(A+B) \leq \sup A + \sup B \leq \sup(A+B) + \epsilon$$

Let $\epsilon > 0$.

$$\text{Assume } \exists x, x \in A \wedge \sup A - \epsilon \leq x \leq \sup A$$

$$\forall x, x \notin A \vee (x < \sup A - \epsilon \vee x > \sup A)$$

Let $x \in A$.

Case 1: $x \notin A$. L.

Case 2: $x < \sup A - \epsilon \vee x > \sup A$

Case 2.1: $x < \sup A - \epsilon$

Case 2.2: $x > \sup A$. L

$$x < \sup A - \epsilon$$

$$x < \sup A - \epsilon$$

$$\forall x, x \in A \rightarrow x < \sup A - \epsilon$$

L

$$\exists x, x \in A \wedge \sup A - \epsilon \leq x \leq \sup A$$

$$\forall \epsilon > 0 \exists x, x \in A \wedge \sup A - \epsilon \leq x \leq \sup A$$

Similarly,

$$\forall \epsilon > 0 \exists x, x \in B \wedge \sup B - \epsilon \leq x \leq \sup B.$$

14.

a) Sequence of closed intervals

$$I_1 = [a_1, b_1], I_2 = [a_2, b_2], \dots$$

$$\rightarrow \exists x, \forall i, x \in I_i$$

$$a_n \leq a_{n+1}$$

$$b_{n+1} \leq b_n$$

$$a_n \leq b_n$$



Proof

Apply result from proof 12.

$$A = \{a_i, i \in \mathbb{N}\}$$

$$B = \{b_i, i \in \mathbb{N}\}$$

$$\text{Then } b_1 \leq b_2, \sup A \leq \inf B$$

Any $x \in [\sup A, \inf B]$ is a member of all I_i .

$$b) I_1 = (a_1, b_1), I_2 = (a_2, b_2), \dots$$

$$\text{Let } \sup A = \inf B = \alpha$$



$$a_n = \alpha - \frac{1}{n}, b_n = \alpha + \frac{1}{n}. \text{ This example shows: } x - \alpha \text{ is not always } \in I_n.$$

$$\text{Let } \sup A = \alpha, \inf B = \beta$$

Any $x \in A$ is also in I_n .It, however, α is an open endpoint.

$a_n = \alpha, b_n = \alpha + \frac{1}{n}$. This is a counterexample. For any $x \in I_n$, there is always $a, b \in (0, \alpha)$ such that $b = \alpha + \frac{1}{n}$, because for any numbers $\frac{1}{n}$ there exist $n_i \in \mathbb{N}$ s.t. $\frac{1}{n_i} < \frac{1}{n}$.

15.

f cont. on $[a, b]$
 $f(a) < 0 < f(b)$

$$\rightarrow \text{ii)} f\left(\frac{a+b}{2}\right) = 0 \quad \text{OR}$$

iii) f has different signs at endpoints of interval $[a, \frac{a+b}{2}]$ OR

iv) f has different signs at endpoints of interval $[\frac{a+b}{2}, b]$

Proof

Since f is defined at $\frac{a+b}{2}$, there are three possible cases

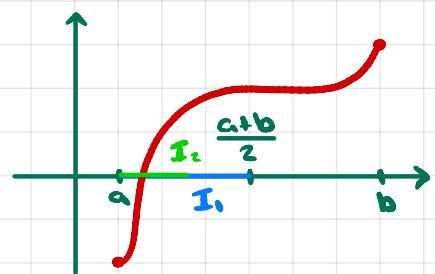
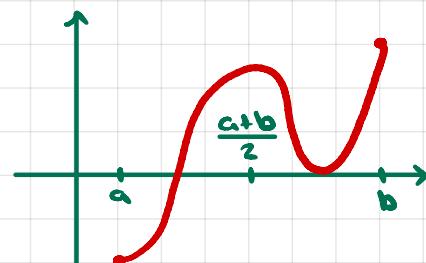
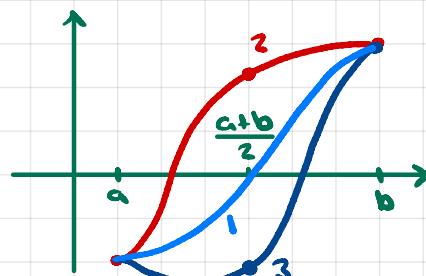
i) $f\left(\frac{a+b}{2}\right) = 0$. Therefore i).

ii) $f\left(\frac{a+b}{2}\right) > 0$. Therefore ii).

iii) $f\left(\frac{a+b}{2}\right) < 0$. Therefore iii).

Therefore i) \vee ii) \vee iii).

■



The intervals I_n are nested. Therefore there is a point c that is in all the intervals. Note that the point where f changes sign is also in every I_n , by definition.

Note that each interval I_n has length $\frac{b-a}{2^n}$.

Assume $f(c) < 0$.

Since f is continuous at c , $\exists \delta > 0$ s.t. $\forall x$, $|x - c| < \delta \Rightarrow f(x) < 0$.

Choose n such that I_n has length $\frac{b-a}{2^n} < \delta$.

Then, $\forall x \forall y$, $x \in I_n \cap I_n \Rightarrow |x - y| < \frac{b-a}{2^n} < \delta$.

In particular, $|x - c| < \delta$ so

$\forall x, x \in I_n \Rightarrow f(x) < 0$.



But then f doesn't change sign in I_n . \perp .

Therefore $f(c) \geq 0$.

Similarly we can show that assuming $f(c) > 0$ leads to \perp .

Therefore $f(c) = 0$. In conclusion, any point that is in all I_n must be a root of f .

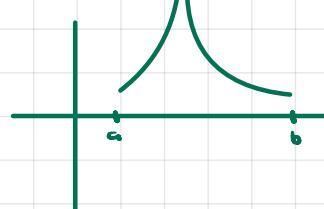
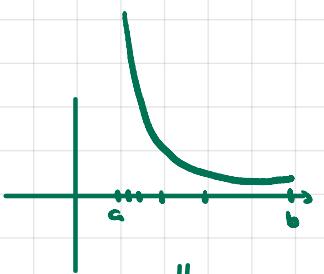
16.

f cont. on $[a,b]$, unbounded on (a,b) .

The intervals I_n are nested as described in 14a.

$\exists c, c \in [a,b]$ and c in every I_n .

f is cont at c .



The length of each bisection interval is $\frac{b-a}{2^n}$.

choose n such that $\frac{b-a}{2^n} < \delta$.

Then,

$$\forall x, |x-c| < \frac{b-a}{2^n} < \delta \rightarrow |f(x)-f(c)| < \epsilon. \\ \rightarrow f(x) < f(c) + \epsilon$$

so f is bounded in I_n . \perp .

n.

$$a) A = \{x : x < a\}$$

$$i) x \in A \wedge y < x \rightarrow y \in A$$

Proof

Since $x \in A$ then $x < a$.Since y is visible, $y < x$.

$$\rightarrow y \in A.$$

$$ii) A + \emptyset$$

Proof

$$a - 1 < a$$

$$\rightarrow a - 1 \in A$$

$$\rightarrow A + \emptyset$$

$$iii) A + R$$

Proof

$$a + b > a$$

$$\rightarrow a + b \in A$$

$$A + R$$

$$iv) x \in A \rightarrow \exists x' : x' \in A \wedge x < x'$$

Proof

$$\text{Find } x \in A, \text{ let } x' = \frac{x+a}{2}$$

Then $x < x' < a, x' \in A$.

$$b) x \in A \wedge y < x \rightarrow y \in A$$

$$A + \emptyset$$

$$A + R$$

$$x \in A \rightarrow (\exists x' \in A \wedge x < x')$$

$$\rightarrow A = \{x : x < \sup A\}$$

Proof

A + \emptyset ab a is a u.b. $\rightarrow A$ has l.u.b. β Assume $\beta < a$

$$\text{let } b = \frac{a+\beta}{2}$$

Then $\beta < b < a$

$$\rightarrow b \in A$$

 β not u.b of A . \perp Therefore β is the l.u.b of A .

iv.

a)

$$i) A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$$

Every number $a > 0$ is an almost upper bound." " " $a \leq 0$ " " " lower bound

$$ii) A = \left\{ \frac{1}{n} : n \in \mathbb{Z}, n \neq 0 \right\}$$

Any number $a > 0$ is a.u.b." " " $a < 0$ is a.l.b.

$$iii) A = \{x : x = 0 \text{ or } x = \ln n \text{ for } n \in \mathbb{N}\}$$



Same as i).

$$iv) A = \{x : 0 \leq x \leq \sqrt{2} \wedge x \in \mathbb{Q}\}$$

Any number $a \leq 0$ is a.l.b." " " $a \geq \sqrt{2}$ is a.u.b.

$$v) A = \{x : x^2 + x + 1 \geq 0\}$$

$$\Delta = 1 - 4 = -3 \rightarrow \forall x : x^2 + x + 1 \geq 0$$

A = \mathbb{R} , unbounded.

No a.l.b. nor a.u.b.

$$vi) A = \{x : x^2 + x - 1 < 0\}$$

$$\forall x, x \in \left(-\frac{1-\sqrt{5}}{2}, -\frac{1+\sqrt{5}}{2} \right) \rightarrow x^2 + x - 1 < 0$$

Any $a \leq -\frac{1-\sqrt{5}}{2}$ is a.l.b.Any $a \geq -\frac{1+\sqrt{5}}{2}$ is a.u.b.

$$\text{viii) } A = \{x : x < 0 \text{ and } x^2 + x - 1 < 0\}$$

$$= \{x : x \in \left(\frac{-1-\sqrt{5}}{2}, 0 \right]\}$$



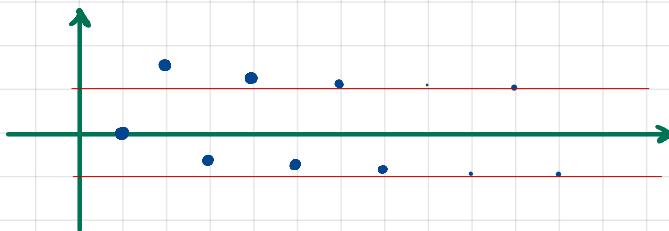
All $x \geq 0$ is a.u.b.

All $x \leq \frac{-1-\sqrt{5}}{2}$ is c.l.b.

$$\text{viii) } A = \left\{ \frac{1}{n} + (-1)^n : n \in \mathbb{N} \right\}$$

$$= \left\{ 0, \frac{1}{2} + 1, \frac{1}{3} - 1, \frac{1}{4} + 1, \dots \right\}$$

$$= \left\{ 0, 3/2, -2/3, 5/4, -4/5, 7/6, -6/7, \dots \right\}$$



Consider any element $a_n = \frac{1}{n} + (-1)^n, n \in \mathbb{N}$.

Any $\alpha > 1$ is a.u.b.

Any $\alpha \leq -1$ is c.l.b.

b) A bounded infinite set.

$\rightarrow B \neq \emptyset \wedge B \text{ bounded below}$

$$B = \{x : x \text{ is a.u.b. of } A\}$$

Proof

$A \neq \emptyset$, A bounded, therefore A has l.u.b. α

$\exists \alpha, \forall x, x \in A \rightarrow x \leq \alpha$

α is also an a.u.b. and so is any number $y > \alpha$.

There might be other $y < \alpha$ that are a.u.b.

For example,

$$\text{i) } A = \{1/n, n \in \mathbb{N}\}$$

$$\alpha = 1$$

$$B = \{x : x > 0\}$$

$B \neq \emptyset$ because $\alpha \in B$.

Since A is bounded below, $\exists \beta, \forall x, x \in A \rightarrow x \geq \beta$.

Let $x \in B$.

Assume $x < \beta$.

Then, because x is a.u.b. of A, there are finite numbers of elements of A larger than x . But all elements of A are less than β , thus larger than x , and A has infinite elements. \perp .

Therefore $x \geq \beta$

$$\forall x, x \in B \rightarrow x \geq \beta$$

Thus, B is bounded below.

Therefore $B \neq \emptyset$ and B is bounded below ■

$$\text{c) } B \neq \emptyset \wedge B \text{ bounded below} \rightarrow \exists \inf B$$

$\inf B$ is the g.l.b. of the a.u.b. of A.

This will called the limit superior of A, denoted $\limsup A$ or $\limsup A$.

$$\text{i) } A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$$



$$\limsup A = 0$$

$$\text{ii) } A = \left\{ \frac{1}{n} : n \in \mathbb{Z}, n \neq 0 \right\}$$



$$\limsup A = 0$$

$$\text{iii) } A = \{x : x = 0 \text{ or } x = 1/n \text{ for } n \in \mathbb{N}\}$$



$$\limsup A = 0$$

$$\text{iv) } A = \{x : 0 \leq x \leq \sqrt{2} \text{ and } x \in \mathbb{Q}\}$$



$$\limsup A = \sqrt{2}$$

$$\text{v) } A = \{x : x^2 + x - 1 \geq 0\} = \mathbb{R}, \text{ has no limit superior.}$$

$$\text{vi) } A = \{x : x^2 + x - 1 < 0\}$$



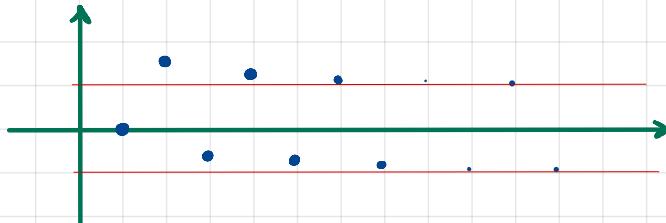
$$\limsup A = \frac{-1+\sqrt{5}}{2}$$

$$\text{viii) } A = \{x : x < 0 \text{ and } x^2 + x - 1 < 0\}$$



$$\limsup A = 0$$

$$\text{viii) } A = \left\{ \frac{1}{n} + (-1)^n : n \in \mathbb{N} \right\}$$



$$\limsup A = 1$$

d) $\underline{\lim} A$ is the limit inferior of A and is $\sup C$, where $C = \{x : x \text{ is a.l.b. of } A\}$.

$$\text{i) } A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$$



$$\liminf A = 0$$

$$\text{ii) } A = \left\{ \frac{1}{n} : n \in \mathbb{Z}, n \neq 0 \right\}$$



$$\liminf A = 0$$

$$\text{iii) } A = \{x : x = 0 \text{ or } x = \frac{1}{n} \text{ for } n \in \mathbb{N}\}$$



$$\liminf A = 0$$

$$\text{iv) } A = \{x : 0 \leq x \leq \sqrt{2} \text{ and } x \in \mathbb{Q}\}$$



$$\liminf A = 0$$

$$\text{v) } A = \{x : x^2 + x + 1 \geq 0\} = \mathbb{R}, \text{ has no limit inferior}$$

$$\text{vi) } A = \{x : x^2 + x - 1 < 0\}$$



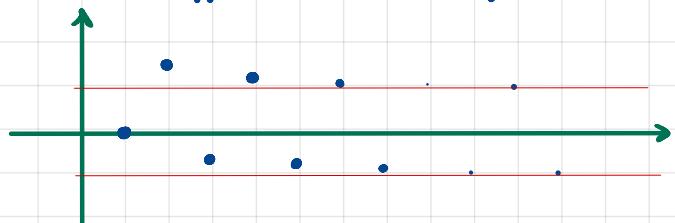
$$\liminf A = \frac{-1-\sqrt{5}}{2}$$

$$\text{viii) } A = \{x : x < 0 \text{ and } x^2 + x - 1 < 0\}$$



$$\liminf A = \frac{-1-\sqrt{5}}{2}$$

$$\text{viii) } A = \left\{ \frac{1}{n} + (-1)^n : n \in \mathbb{N} \right\}$$



$$\liminf A = -1$$

19. Bounded infinite set.

a) $\liminf A \leq \limsup A$

Proof



Assume $\liminf A > \limsup A$

From the definitions of limit inferior and limit superior,

there is a finite number of elements of A that are smaller than $\liminf A$, and a finite number larger than $\limsup A$.

Let α s.t. $\liminf A \leq \alpha \leq \limsup A$

Then there is finite numbers of elements of A smaller than α , but also larger than α . Hence A is finite. \perp .

Hence, $\liminf A \leq \limsup A$

b) $\liminf A \leq \sup A$



Proof

Intuitively, all elements of A are $\leq \sup A$, but a finite number are larger than $\liminf A$.

Assume $\liminf A > \sup A$

There must be infinite points $a \in A$ s.t. $\sup A < a < \liminf A$, otherwise $\sup A$ would be a.u.b. and thus $\liminf A$ wouldn't exactly be the limit superior of A.

But then $\sup A$ isn't the supremum of A. \perp .

Hence $\liminf A \leq \sup A$

c) $\liminf A < \sup A \rightarrow A$ contains a largest element

Intuitively, there is finite number of elements of A between $\liminf A$ and $\sup A$. one of them is largest among this finite set.

But then it is also the largest in A since all other members of A are smaller than $\liminf A$.

Proof

Let $B = \{x : x \in A \wedge x > \liminf A\}$. Then $\forall x, x \in B \rightarrow x < \sup A$, i.e. $\sup A - \sup B$.

$B \neq \emptyset$ because otherwise $\liminf A$ would be $\sup A$, a \perp .

Thus because $B \neq \emptyset$ and B is finite, one of its elements is largest. Call it α . $\forall x, x \in B \rightarrow x \leq \alpha$.

But

$$\forall x, x \in B \rightarrow x \in A \wedge x \geq \liminf A$$

$$\forall x, x \in A \rightarrow x \in B \vee x \leq \liminf A$$

Therefore

$$\forall x, x \in A \rightarrow \liminf A < x \leq \alpha$$

$$\text{Hence } \forall x, x \in A \rightarrow x \leq \alpha.$$

d)

i) $\liminf A \geq \inf A$

Intuitively, all elements of A are larger than or equal to $\inf A$, but a finite number of them are smaller than $\liminf A$.

Proof

Assume $\liminf A < \inf A$.

There must be infinite points a s.t. $a \in A$ and $\liminf A < a < \inf A$, otherwise $\inf A$ would be a.u.b. and limit inferior.

But then $\inf A$ isn't a lower bound. \perp .

Hence $\liminf A \geq \inf A$.

ii) $\underline{\lim} A > \inf A \rightarrow A$ contains smallest element

Intuitively, there is a finite number of elements of A between $\inf A$ and $\underline{\lim} A$. There is a smallest among them. Since all such elements are $< \underline{\lim} A$, and all the rest of the (infinite) elements are $> \underline{\lim} A$, the smallest in the finite set is the smallest of entire A .

Proof

$$\text{Let } B = \{x : x \in A \wedge x < \underline{\lim} A\}$$

Assume $B = \emptyset$.

Then $\forall x, x \in A \rightarrow x \geq \underline{\lim} A$

Then $\underline{\lim} A$ is Lb. and $\underline{\lim} A > \inf A$. \perp

Thus $B \neq \emptyset$.

Since B is finite, it contains a smallest element α

$$\forall x, x \in B \rightarrow x \geq \alpha.$$

But it is also true that

$$\forall x, x \in B \rightarrow x \in A \wedge x \geq \underline{\lim} A.$$

$$\begin{aligned} \forall x, x \in A \rightarrow \inf A \leq x \leq \underline{\lim} A \vee x \geq \underline{\lim} A \\ \rightarrow x \in B \vee x \geq \underline{\lim} A \end{aligned}$$

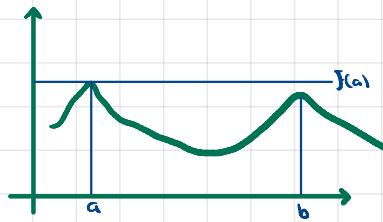
Hence

$$\forall x, x \in A \rightarrow \alpha \leq x \leq \underline{\lim} A$$

$$\forall x, x \in A \rightarrow x \geq \alpha.$$

20. f cont. on \mathbb{R} .

point x is shadow point of f if there is $y > x$ s.t. $f(y) > f(x)$



a)

Assume $f(a) > f(b)$

Consider the closed interval $I = [a, b]$.

I is bounded and nonempty.

Since f cont. on I , it takes on a max value in I , say at α .

Assume $\alpha \neq a$.

Then $\alpha > a$ and $f(\alpha) \geq f(a)$.

Assume $f(\alpha) > f(a)$

Then a is shadow point.

1.

Therefore $f(\alpha) = f(a) > f(b)$

Since b is not a shadow point, $\forall y, y > b \rightarrow f(y) \leq f(b)$.

Therefore $\forall y, y > \alpha \rightarrow f(y) \leq f(\alpha)$

But then α is not a shadow point.

1.

Therefore, $\alpha = a$.

b) Since f cont. on $[a, b]$ it takes on all values in $[f(a), f(b)]$.

Therefore, $\exists x, x \in [a, b] \wedge f(x) \in (f(a), f(b))$

$\rightarrow f(x) > f(b)$

Some such point is not a shadow point.

But by assumption all points in (a, b) are shadow points.

1.

Therefore $f(a) \leq f(b)$, by contradiction.

But $f(a) = f(b)$ because a & b are shadow points.