Ch 7 - Three Hard Theorems

Theorem 1

f cont. on $[a,b]$, $f(a) < 0 < f(b)$ $\rightarrow$ $\exists x \in [a,b], f(x) = 0$

Theorem 2

f cont. on $[a,b]$ $\rightarrow$ f bounded above on $[a,b]$, ie

$\exists N, \forall x \in [a,b], f(x) \leq N$

Theorem 3

f cont. on $[a,b]$ $\rightarrow$ $\exists y \in [a,b], \forall x \in [a,b], f(y) \geq f(x)$

ie "a cont. fn on a closed interval takes its max value on that interval."

Theorem 4

f cont. on $[a,b]$, $f(a) < c < f(b)$ $\rightarrow$ $\exists x \in [a,b], f(x) = c$

Proof

Let $g = f - c$.

Then g is continuous.

$g(a) < 0 < g(b)$

By Th. 1, $\exists x \in [a,b], g(x) = 0$.

$g(x) = 0 \rightarrow f(x) = c$

Theorem 5

f cont. on $[a,b]$, $f(a) > c > f(b)$ $\rightarrow$ $\exists x \in [a,b], f(x) = c$

Proof

$-f$ is cont. on $[a,b]$

$-f(a) < -c < -f(b)$

By Th 4., $\exists x \in [a,b], -f(x) = -c$, ie $f(x) = c$.

"If a cont. fn takes on two values, it takes on every value in between."

This generalization of Th. 1 is called the Intermediate Value Theorem

Theorem 6

f cont. on $[a,b]$ $\rightarrow$ f bounded below on $[a,b]$

ie. $\exists N \in \mathbb{R}, \forall x \in [a,b], f(x) \geq N$

Proof

$-f$ is cont. on $[a,b]$.

By Th. 2, $\exists N \in \mathbb{R}, \forall x \in [a,b], -f(x) \leq N$.

Therefore, $f(x) \geq -N$.

Therefore, $\exists N' = -N, \forall x \in [a,b], f(x) \geq N'$.

Th 2 and 6 imply that a fn cont. on $[a,b]$ is bounded on $[a,b]$.

That is, $\exists N \in \mathbb{R}, \forall x \in [a,b], |f(x)| \leq N$.

Theorem 7

f cont. on $[a,b]$ $\rightarrow$ $\exists y \in [a,b], \forall x \in [a,b], f(y) \leq f(x)$

"a cont. fn on a closed interval takes on its minimum value on that interval"

Proof

f cont. then $-f$ cont.

By Th. 3 $\exists y \in [a,b], \forall x \in [a,b], -f(y) \geq -f(x)$

Therefore $\exists y \in [a,b], \forall x \in [a,b], f(y) \leq f(x)$.

Th. 1-7 are pretty trivial consequences of Th. 1-3.

Theorem 8

Every positive number has a square root.

Ie, if $\alpha > 0$ then $\exists x, x^2 = \alpha$.

Proof

Consider $f(x) = x^2$. Th 8 says that for every $\alpha \in \mathbb{R}$, f takes on the value α.

Given α, we can always find $b > 0$ such that $f(b) > \alpha$.

For example, if $\alpha > 1$ take $b = \alpha$; if $\alpha < 1$ take 1.

$f(0) < \alpha < f(b)$.

By Th. 4, f takes on value α in $[0,b]$.

Ie, $\exists x \in \mathbb{R}, f(x) = x^2 = \alpha$.

Theorem 8.1

Every positive number has an n^{th} root, $n \in \mathbb{N}$.

Proof

Let $f(x) = x^n$. Let $\alpha > 0$.

Given α, let $b = \alpha$ if $\alpha > 1$ and $b = 1$ if $\alpha < 1$.

Then, $0 = f(0) < \alpha < f(b)$.

Th 4 $\rightarrow$ $\exists x \in [0,b], f(x) = x^n = \alpha$.

Theorem 8.2

If $n \in \mathbb{N}$ is odd then every number has an n^{th} root.

Proof

Let $\alpha \in \mathbb{R}$. If $\alpha > 0$ then by Th. 8.1 α has an n^{th} root.

If $\alpha < 0$ then $f(\alpha) = f(b) < \alpha < f(0) = 0$.

By Th. 4, $\exists x \in [b,0], f(x) = x^n = \alpha$

If $\alpha = 0$ then $0^n = \alpha$.

In all possible cases, $\exists x \in \mathbb{R}, x^n = \alpha$, ie α has an n^{th} root, n odd.

Alternatively, since $\alpha > 0$ has n^{th} root, $x^n = \alpha$.

If n odd then $(-x)^n = -\alpha$.

Ie, for all $\beta \in \mathbb{R}, \beta < 0$ $\rightarrow$ $\exists x \in \mathbb{R}, x^n = \beta$.

Therefore if n odd then $\forall \alpha \in \mathbb{R}, \exists x \in \mathbb{R}, x^n = \alpha$.

Theorem 8.2 is equivalent to the statement that the equation $x^n - \alpha = 0$ has a root if n is odd.

Theorem 9

If n is odd then any equation

$$x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$$

has a root.

Proof

Let $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$

Intuition for proof: f is sometimes positive, sometimes negative. For large $|x|$ the x^n term dominates: f is positive for large positive x and negative for large negative x.

Rewrite f: $f(x) = x^n\left(1 + \frac{a_{n-1}}{x} + \dots + \frac{a_0}{x^n}\right)$

note: $\left|\frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n}\right| \le \frac{|a_{n-1}|}{|x|} + \dots + \frac{|a_0|}{|x^n|}$

choose an x such that $|x|$ is larger than all the following terms:

1
$2n|a_{n-1}|$
$2n|a_{n-2}|$ (1)
$\dots$
$2n|a_0|$

Then,

$|x|^k > |x|$

$$\frac{|a_{n-k}|}{|x^k|} < \frac{|a_{n-k}|}{|x|} < \frac{|a_{n-k}|}{2n|a_{n-k}|} = \frac{1}{2n}$$

Thus,

$$\left|\frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n}\right| < n \cdot \frac{1}{2n} = \frac{1}{2}$$

$$-\frac{1}{2} < \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n} < \frac{1}{2}$$

$$\frac{1}{2} < 1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n} \qquad (2)$$

Let $x_1 > 0$ be x satisfying (1). Then

$$\frac{x_1^n}{2} < x_1^n\left(1 + \frac{a_{n-1}}{x_1} + \frac{a_{n-2}}{x_1^2} + \dots + \frac{a_0}{x_1^n}\right) = f(x_1)$$

$\to f(x_1) > 0$

Let x_2 be another x satisfying (1), but $x_2 < 0$.
Therefore $x_2^n < 0$.
Then, from (2) we have:

$$\frac{x_2^n}{2} > x_2^n\left(1 + \frac{a_{n-1}}{x_2} + \frac{a_{n-2}}{x_2^2} + \dots + \frac{a_0}{x_2^n}\right) = f(x_2)$$

At this point we have

$$f(x_2) < 0 < f(x_1)$$

By Th. 1, $\exists x \in [x_2, x_1], f(x) = 0$.

What if n is even?

instead of trying to solve $x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$
let's try $x^n + a_{n-1}x^{n-1} + \dots + a_0 = c$ for all possible c.

Theorem 10

n even
$f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$ $\to \exists y \in \mathbb{R}, \forall x\ f(y) \le f(x)$

Proof

Similar to proof of Th. 9, if $M = \max(1, 2n|a_{n-1}|, \dots, 2n|a_0|)$ then for all x with $|x| \ge M$ we have $\frac{1}{2} < 1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n}$
Since n even, $x^n \ge 0$. So,

$$\frac{x^n}{2} < x^n\left(1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n}\right) = f(x) \quad \text{provided } |x| \ge M.$$

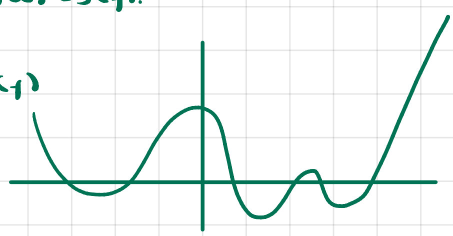
Consider $f(0)$. Let $b > 0$ s.t. $b^n \ge 2f(0)$ and $b > M$. Then, if $x \ge b$

$$f(x) > \frac{x^n}{2} \ge \frac{b^n}{2} \ge f(0)$$

Similarly, if $x \le -b$ then $f(x) > \frac{x^n}{2} \ge \frac{(-b)^n}{2} = \frac{b^n}{2} \ge f(0)$
Thus, whether $x \ge b$ or $x \le -b$ we have $f(x) \ge f(0)$.

Apply Th. 7 to f on $[-b, b]$.
$\exists y \in [-b, b], \forall x \in [-b, b], f(y) \le f(x)$
In particular $f(y) \le f(0)$
For $x \ge b$ or $x \le -b$, $f(x) \ge f(0) \ge f(y)$.

Thus $\forall x \in \mathbb{R}\ f(x) \ge f(0) \ge f(y)$



Theorem 11

Consider the eq. $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = c$ $(*)$

Suppose n even.

Then, $\exists m \in \mathbb{R}$ s.t. $(*)$ has sol'n for $c \geq m$ and has no sol'n for $c < m$.

Proof

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$.

By Th. 10 $\exists y \in \mathbb{R}$ $\forall x$ $f(y) \leq f(x)$.

Let $m = f(y)$.

If $c < m$ then there is no sol'n to $(*)$: $f(x) \geq f(y) = m > c$.
If $c = m$, there is a sol'n.

If $c > m$ then let $b \in \mathbb{R}$ s.t. $b > y$ and $f(b) > c$.

Then $f(y) = m < c < f(b)$

By Th. 4, $\exists x \in [y, b]$, $f(x) = c$.
Thus x is a sol'n.