

Ch 7 - Three Hard Theorems

Theorem 1

f cont. on $[a,b]$ \rightarrow $\exists x \in [a,b], f(x) = 0$
 $f(a) < 0 < f(b)$ \rightarrow

Theorem 2

f cont. on $[a,b]$ \rightarrow f bounded above on $[a,b]$, i.e.
 $\exists M, \forall x \in [a,b], f(x) \leq M$

Theorem 3

f cont. on $[a,b]$ \rightarrow $\exists y \in [a,b], \forall x \in [a,b], f(y) \geq f(x)$

i.e. " f cont. on a closed interval takes its max value on that interval."

Theorem 4

f cont. on $[a,b]$ \rightarrow $\exists x \in [a,b], f(x) = c$
 $f(a) < c < f(b)$ \rightarrow

Proof

let $g = f - c$.

Then g is continuous.

$g(a) < 0 < g(b)$

By Th. 1, $\exists x \in [a,b], g(x) = 0$.

$g(x) = 0 \rightarrow f(x) = c$

Theorem 5

f cont. on $[a,b]$ \rightarrow $\exists x \in [a,b], f(x) = c$
 $f(a) > c > f(b)$ \rightarrow

Proof

$-f$ is cont. on $[a,b]$

$-f(a) < -c < -f(b)$

By Th. 4, $\exists x \in [a,b], -f(x) = -c$, i.e. $f(x) = c$.

" f cont. on two values, it takes on every value in between."

This generalization of Th. 1 is called the Intermediate Value Theorem

Theorem 6

f cont. on $[a,b]$ \rightarrow f bounded below on $[a,b]$
i.e. $\exists N \in \mathbb{R}, \forall x \in [a,b], f(x) \geq N$

Proof

$-f$ is cont. on $[a,b]$.

By Th. 2, $\exists M \in \mathbb{R}, \forall x \in [a,b], -f(x) \leq M$.

Therefore, $f(x) \geq -M$.

Therefore, $\exists N = -M, \forall x \in [a,b], f(x) \geq N$.

Th 2 and 6 imply that $-f$ cont. on $[a,b]$ is bounded on $[a,b]$.
That is, $\exists N \in \mathbb{R}, \forall x \in [a,b], |f(x)| \leq N$.

Theorem 7

f cont. on $[a,b] \rightarrow \exists y \in [a,b], \forall x \in [a,b], f(y) \leq f(x)$

" f cont. on a closed interval takes on its minimum value on that interval"

Proof

f cont. then $-f$ cont.

By Th. 3, $\exists y \in [a,b], \forall x \in [a,b], -f(y) \geq -f(x)$

Therefore, $\exists y \in [a,b], \forall x \in [a,b], f(y) \leq f(x)$.

Th 1-7 are pretty trivial consequences of Th. 1-3.

Theorem 8

Every positive number has a square root.

i.e. if $x > 0$ then $\exists z, z^2 = x$.

Proof

consider $f(x) = x^2$. Th. 8 says that for every $x \in \mathbb{R}$, f takes on the value x .

Given x , we can always find $b > 0$ such that $f(b) > x$.

For example, if $x > 1$ take $b = x$; if $x < 1$ take 1.

$f(a) < x < f(b)$.

By Th. 4, f takes on value x in $[a,b]$.

i.e., $\exists z \in \mathbb{R}, f(z) = z^2 = x$.

Theorem 8.1 Every positive number has an n^{th} root, $n \in \mathbb{N}$.

Proof

let $f(x) = x^n$. let $x > 0$.

Given x , let $b = x$ if $x > 1$ and $b = 1$ if $x < 1$.

then, $a = f(a) < x < f(b)$.

Th 4 $\rightarrow \exists z \in [a,b], f(z) = z^n = x$.

Theorem 8.2 If $n \in \mathbb{N}$ is odd then every number has an n^{th} root.

Proof

let $x \in \mathbb{R}$. If $x > 0$ then by Th. 8.1 x has an n^{th} root.

If $x < 0$ then $f(x) = f(b) < x < f(a) = 0$.

By Th. 4, $\exists z \in [b,a], f(z) = z^n = x$.

If $x = 0$ then $0^n = 0$.

In all possible cases, $\exists z \in \mathbb{R}, z^n = x$, i.e. x has an n^{th} root, n odd.

Alternatively, since $x > 0$ has n^{th} root, $x^n = x$.

If n odd then $(-x)^n = -x$.

i.e., $\exists \alpha, \beta \in \mathbb{R}, \beta < 0 \rightarrow \exists z \in \mathbb{R}, z^n = \beta$.

Therefore if n odd then $\forall x \in \mathbb{R}, \exists z \in \mathbb{R}, z^n = x$.

Theorem 8.2 is equivalent to the statement that the equation $x^n - a = 0$ has a root if n is odd.

Theorem 9

If n is odd then any equation

$$x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$$

has a root.

Proof

$$\text{Let } f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0.$$

Intuition for proof: f is sometimes positive, sometimes negative.

For large $|x|$ the x^n term dominates. f is positive for large positive x and negative for large negative x .

$$\text{Recall: } f(x) = x^n \left(1 + \frac{a_{n-1}}{x} + \dots + \frac{a_0}{x^n} \right)$$

$$\text{note: } \left| \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n} \right| \leq \frac{|a_{n-1}|}{|x|} + \dots + \frac{|a_0|}{|x^n|}$$

choose a_n such that $|x|$ is larger than all the following terms:

$$\begin{aligned} & 1 \\ & 2n|a_{n-1}| \\ & 2n|a_{n-2}| \\ & \dots \\ & 2n|a_0| \end{aligned} \quad (1)$$

$$\text{Then, } |x|^n > |x|$$

$$\frac{|a_{n-1}|}{|x^n|} < \frac{|a_{n-1}|}{|x|} < \frac{|a_{n-1}|}{2n|a_{n-1}|} = \frac{1}{2n}$$

thus,

$$\left| \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n} \right| < n \cdot \frac{1}{2n} = \frac{1}{2}$$

$$-\frac{1}{2} < \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n} < \frac{1}{2}$$

$$\frac{1}{2} < 1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n} \quad (2)$$

Let $x_i > 0$ be x satisfying (1). Then

$$\frac{x_i^n}{2} < x_i^n \left(1 + \frac{a_{n-1}}{x_i} + \frac{a_{n-2}}{x_i^2} + \dots + \frac{a_0}{x_i^n} \right) = f(x_i)$$

$$\rightarrow f(x_i) > 0$$

Let x_i be another x satisfying (1), but $x_i < 0$.

Therefore $x_i^n < 0$.

Then, from (2) we have

$$\frac{x_i^n}{2} > x_i^n \left(1 + \frac{a_{n-1}}{x_i} + \frac{a_{n-2}}{x_i^2} + \dots + \frac{a_0}{x_i^n} \right) = f(x_i)$$

At this point we have

$$f(x_i) < 0 < f(x_j)$$

$$\text{By Th. 1, } \exists x \in [x_i, x_j], f(x) = 0.$$

What if n is even?

Instead of trying to solve $x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$

let's try $x^n + a_{n-1}x^{n-1} + \dots + a_0 < c$ for all possible c .

Theorem 10

n even

$$\rightarrow \exists y \in \mathbb{R}, \forall x, f(y) \leq f(x)$$

$$f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$$

Proof

Similar to proof of Th. 9, if $M = \max(1, 2n|a_{n-1}|, \dots, 2n|a_0|)$ then for all x with $|x| \geq M$ we have $\frac{1}{2} < 1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n}$ since n even, $x^n \geq 0$. So,

$$\frac{x^n}{2} < x^n \left(1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n} \right) = f(x) \text{ provided } |x| \geq M.$$

Consider $f(0)$. Let $b > 0$ s.t. $b^n \geq 2f(0)$ and $b > M$. Then, if $x \geq b$

$$f(x) > \frac{x^n}{2} \geq \frac{b^n}{2} \geq f(0)$$

Similarly, if $x \leq -b$ then $f(x) > \frac{x^n}{2} \geq \frac{(-b)^n}{2} = \frac{b^n}{2} \geq f(0)$. Thus, whether $x \geq b$ or $x \leq -b$ we have $f(x) \geq f(0)$.

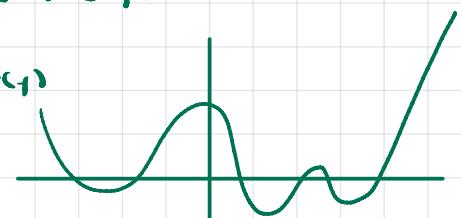
Apply Th. 7 to f on $[-b, b]$.

$$\exists y \in [-b, b], \forall x \in [-b, b], f(c_y) \leq f(x)$$

In particular $f(c_y) \leq f(0)$.

For $x \geq b$ or $x \leq -b$, $f(x) \geq f(0) \geq f(c_y)$.

Thus $\forall x \in \mathbb{R}, f(x) \geq f(0) \geq f(c_y)$



Theorem 11

Consider the eq. $x^n + a_{n-1}x^{n-1} + \dots + a_0 = c \quad (*)$

Suppose n even.

Then, $\exists m \in \mathbb{R}$ s.t. $(*)$ has sol'n for $c \geq m$ and has no sol'n for $c < m$.

Proof

Let $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$.

By Th. 10 $\exists \gamma \in \mathbb{R} \forall x f(\gamma) \leq f(x)$.

Let $m = f(\gamma)$.

If $c < m$ then there is no sol'n to $(*)$: $f(x) \geq f(\gamma) = m > c$.

If $c = m$, there is γ sol'n.

If $c > m$ then let $b \in \mathbb{R}$ s.t. $b > 1$ and $f(b) > c$.

Then $f(\gamma) = m < c < f(b)$

By Th. 4, $\exists x \in [\gamma, b], f(x) = c$.

Thus x is γ sol'n.