

Ch 7 - Three Hard Theorems

1.

ii) $f(x) = x^2$ on $(-1, 1)$

Bounded above means $\exists y \in \mathbb{R}, \forall x \in (-1, 1), f(x) \leq y$.

f taking on its max value on an interval means

there exists a value y in the interval s.t. $f(x) \leq f(y)$

for any x in the interval.

$\forall x \in (-1, 1), 0 < f(x) < 1$.

$\forall x \in (-1, 1), \exists y \in (x, 1)$.

Therefore $y > x$.

$0 \in (-1, 1)$ and $\forall x \in (-1, 1) f(x) \geq f(0)$

f is bounded. f takes on min value but not max value in $(-1, 1)$.

Note that f is continuous on $(-1, 1)$, not $[-1, 1]$.

iii) $f(x) = x^3$ on $(-1, 1)$

$\forall x \in (-1, 1), -1 < f(x) < 1$. f bounded.

f does not take on max or min in $(-1, 1)$.

Note that f is continuous on $(-1, 1)$, not $[-1, 1]$.

iv) $f(x) = x^2$ on \mathbb{R}

bounded from below, takes min value.

f cont on \mathbb{R} , which isn't a closed interval.

v) $f(x) = x^2$ on $[0, \infty)$

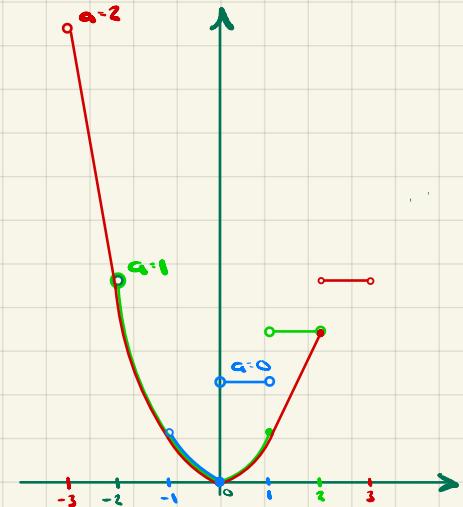
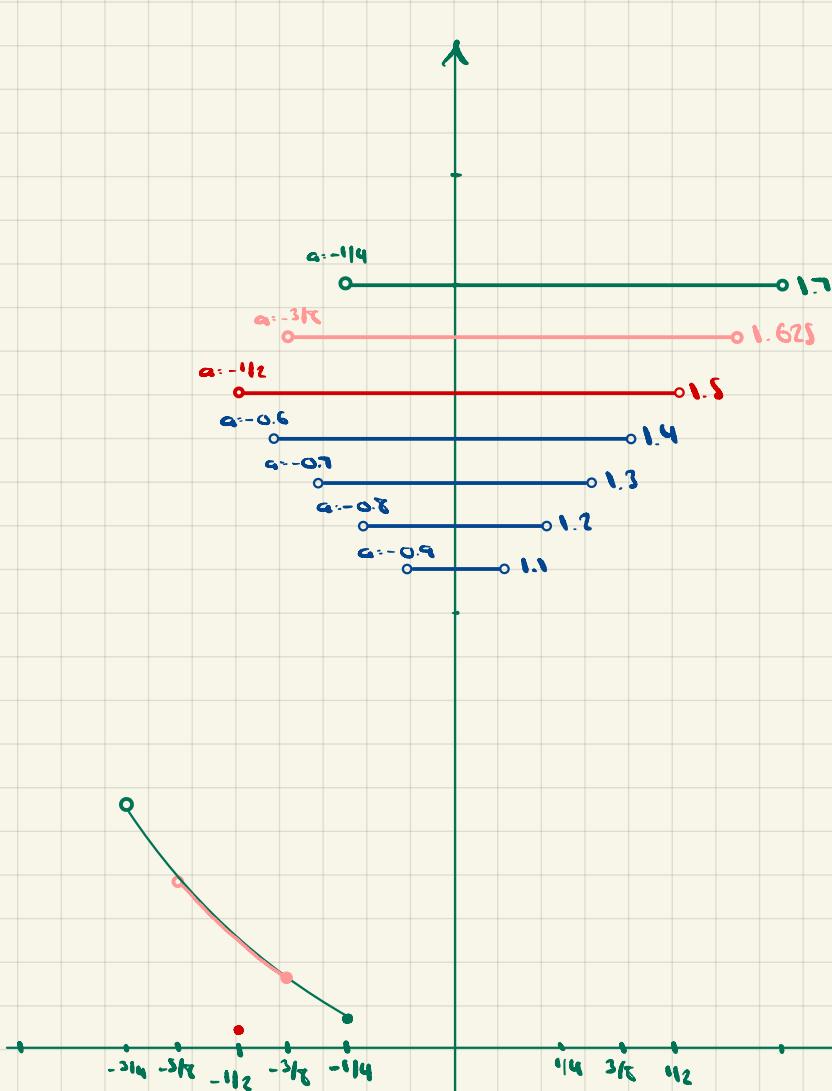
bounded from below, takes min value.

f cont. on $[0, \infty)$, which isn't a closed interval.

$$\text{v) } f(x) = \begin{cases} x^2 & x \leq a \\ a+2 & x > a \end{cases} \quad \text{on } (-a-1, a+1)$$

Assume $a > -1$. Therefore $-a-1 < a+1$.

a	interval	$f(-a-1)$	$f(a)$	$f(a+1)$
-1	(0, 0)			
0	(-1, 1)	$f(-1) = 1$	$f(0) = 0$	$f(1) = 2$
1	(-2, 2)	$f(-2) = 4$	$f(1) = 1$	$f(2) = 3$
2	(-3, 3)	$f(-3) = 9$	$f(2) = 4$	$f(3) = 4$
-1/2	(-1/2, 1/2)	$f(-1/2) = 1/4$	$f(-1/2) = 4/4$	$f(1/2) = 1.5$
-0.9	(-1, 0.1)	$f(-1) = 1.1$		$f(0.1) = 1.1$
-0.8	(-2, -0.2)	$f(-2) = 1.2$		$f(-0.2) = 1.2$
-0.7	(-3, -0.3)	$f(-3) = 1.3$		$f(-0.3) = 1.3$
-0.14	(-0.75, 0.75)	$f(-0.75) = 0.5375$	$f(0.14) = 1.116$	$f(0.75) = 1.75$
-3/8	(-3/8, 3/8)	$f(-3/8) = \frac{25}{64} = 0.3906$	$f(-3/8) = \frac{9}{64} = 0.1406$	$f(3/8) = 13/16 = 1.625$



For all a , bounded.

$$-1 \leq a \leq -\frac{1}{2} : \min = \max = a+2$$

$$-\frac{1}{2} \leq a \leq 0 : \min = a^2, \max = a+2$$

$$0 \leq a < \frac{-1+\sqrt{5}}{2} : \min = 0, \max = a+2$$

$$a > \frac{-1+\sqrt{5}}{2} : \min = 0$$

In $(a, a+1)$, $f(x) = a+2$. In $(-a-1, a]$, $f(x) = x^2$. There is no single limit value of f in $(-a-1, a]$, but if $a+2 \in (-a-1)^2$ then the max of f in $(-a-1, a+1)$ is at $a+2$ or more. This happens when $a+2 \in (a+1)^2 = a^2+2a+1 \rightarrow a^2+a-1 \geq 0$.

$$\Delta = 1+4 \cdot 1 \quad a = \frac{-1 \pm \sqrt{5}}{2}$$

$$\begin{array}{c|ccccc} & + & - & + & - \\ \hline & \frac{-1-\sqrt{5}}{2} & & \frac{-1+\sqrt{5}}{2} & \end{array}$$

If $a > \frac{-1+\sqrt{5}}{2}$, there is no max in the interval.

$$vii) f(x) = \begin{cases} x^2 & x < a \\ ax+2 & x \geq a \end{cases} \text{ on } [-a-1, a+1], a > -1.$$

Relative to vi) the difference is that even when $a+2 < (-a-1)^2$
there is a max. Therefore

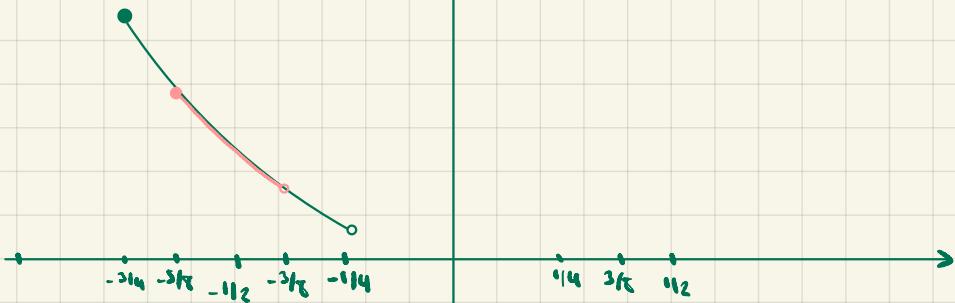
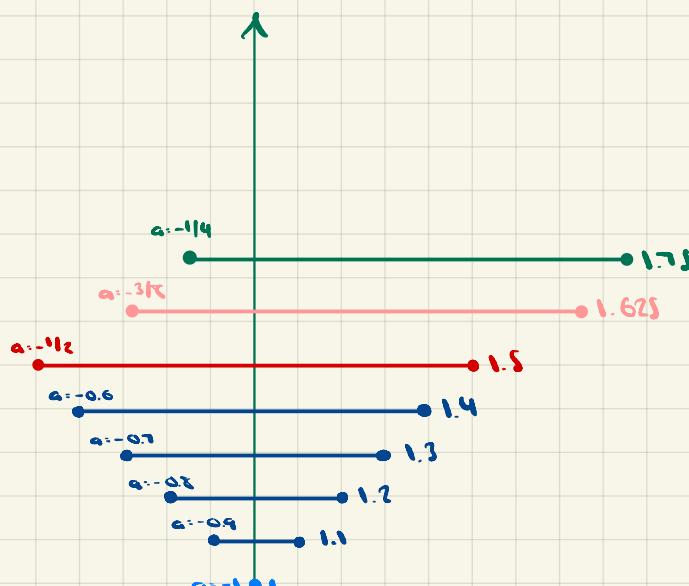
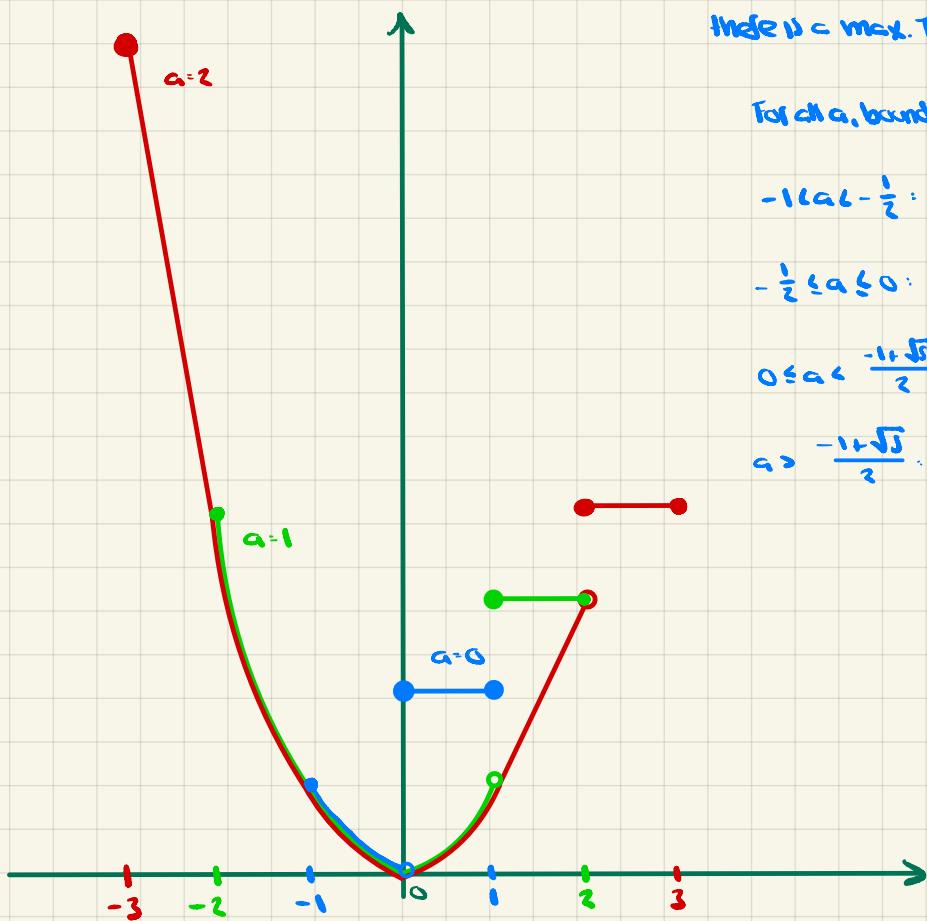
For all a , bounded.

$$-1 \leq a \leq -\frac{1}{2} : \min = \max = a+2$$

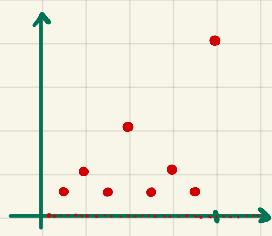
$$-\frac{1}{2} \leq a \leq 0 : \min = a^2, \max = a+2$$

$$0 \leq a \leq \frac{-1+\sqrt{5}}{2} : \min = 0, \max = a+2$$

$$a > \frac{-1+\sqrt{5}}{2} : \min = 0, \max = (-a-1)^2 \cdot (a+1)^2$$



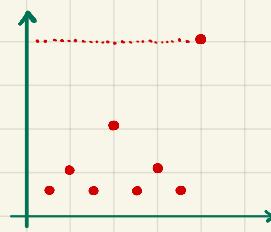
vii) $f(x) = \begin{cases} 0 & x \text{ irrational} \\ \frac{1}{\pi q} & x = \frac{p}{q} \text{ lowest terms} \end{cases} \text{ on } [0,1]$



bounded, max and min

f not cont. on $[0,1]$.

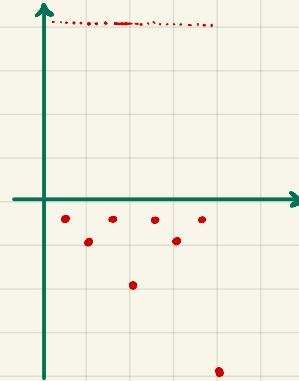
viii) $f(x) = \begin{cases} 1 & x \text{ irrational} \\ \frac{1}{\pi q} & x = \frac{p}{q} \text{ lowest terms} \end{cases} \text{ on } [0,1]$



bounded, max. 1

f not cont. on $[0,1]$.

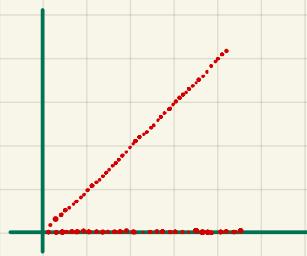
ix) $f(x) = \begin{cases} 1 & x \text{ irrational} \\ -\frac{1}{\pi q} & x = \frac{p}{q} \text{ lowest terms} \end{cases} \text{ on } [0,1]$



bounded, min, max

f not cont. on $[0,1]$.

x) $f(x) = \begin{cases} x & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases} \text{ on } [0,a]$



bounded but all a . Tches can min. 0

if a rational takes on max. a

f not cont on $[0,a]$.

xii) $f(x) = \sin^2(\cos x + \sqrt{a+x^2}) \text{ on } [0,a^2]$

bounded.

$$a+x^2 \geq 0 \rightarrow a(1+a) \geq 0$$

$$\begin{array}{ccccccc} - & & t & & t & & \\ - & & - & & + & & \\ + & -1 & - & 0 & + & & \end{array}$$

$$a \leq -1 \text{ or } a \geq 0$$

$$g(x) = \cos x + \sqrt{a+x^2} \text{ cont on } \mathbb{R}$$

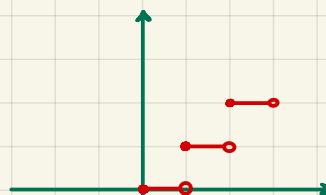
$$h(x) = \sin^2(x) \text{ cont on } \mathbb{R}$$

Th 6-2 $\rightarrow h(g(x)) \cdot f(x)$ cont on \mathbb{R} .

Th 2.6: f bounded.

Th 3.7: max, min on $[0,a^2]$.

xiii) $f(x) = [x] \text{ on } [0,a]$



bounded, max $[a]$, min 0

2.

$$\text{ii) } f(x) = x^3 - x + 3$$

x	f
-1	3
0	3
1	3
2	9
-2	-3

choose $n = -2$ then

$$f(n) = -3 < 0 < 3 = f(n+1)$$

Since f is also cont. on \mathbb{R} and in particular on $[n, n+1]$, we can use Th. 1: $\exists x \in [n, n+1], f(x) = 0$.

$$\text{iii) } f(x) = x^3 + 5x^2 + 3x + 1$$

x	f
-1	-1 + 5 - 2 + 1 + 1 = 3
0	1
1	1 + 5 + 2 + 1 + 9 = 16
-2	-32 + 8 - 16 - 4 + 1 = -32 + 80 - 3 = 45
-3	-243 + 54 - 81 + 2(-3) + 1 = -243 + 408 - 6 + 1 = 157
-4	-1024 + 8 - 256 - 8 + 1 = 249
-5	-3125 + 5 - 625 - 10 + 1 = -9

$n = -5$

$$\text{iii) } f(x) = x^3 + x + 1$$

$$f(-1) = -1 - 1 + 1 = -1$$

$$f(0) = 1$$

choose $n = -1$.

$$\text{iv) } f(x) = 4x^2 - 4x + 1$$

$$\begin{aligned} f(x) &= 4(x^2 - x + \frac{1}{4}) + 1 - 1 \\ &= 4(x - \frac{1}{2})^2 \geq 0 \end{aligned}$$

$\forall x, f(x) \geq 0$

$$f'(x) = 0$$

choose $n = 0$.

3.

$$\text{i) } \exists x, x^m + \frac{163}{1+x^2+\sin^2 x} = 119$$

Proof

$$\text{Let } g(x) = x^m + \frac{163}{1+x^2+\sin^2 x} - 119$$

$1+x^2+\sin^2 x$ is never zero.

$g(x)$ is defined and continuous for all $x \in \mathbb{R}$.

$$g(0) = 163 - 119 = 44$$

$$\lim_{x \rightarrow -\infty} g(x) = -\infty$$

$$\rightarrow \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, \forall x \in \mathbb{R}, x \geq N \Rightarrow g(x) < 0.$$

choose an $N > 0$, and an x s.t. $x \geq N \Rightarrow g(x) < 0 = 0$.

Then, $|g(x)| < 0 < |g(0)| = 44$

By Th. 1, $\exists y \in [x, 0], g(y) = 0$.

$$\text{Thus, } y^m + \frac{163}{1+y^2+\sin^2 y} - 119 = 0$$

$$\text{Thus, } y^m + \frac{163}{1+y^2+\sin^2 y} = 119.$$

$$\text{Therefore, } \exists x, x^m + \frac{163}{1+x^2+\sin^2 x} = 119$$

$$\text{ii) } \exists x \in \mathbb{R}, \sin x = x - 1$$

Proof

$$\text{Let } g(x) = \sin(x) - x + 1$$

$$f(0) = 1$$

$$f(3\pi/2) = -3\pi/2$$

Lemma: f cont. on $[a, b]$
 $f(a) > 0 > f(b) \rightarrow \exists x \in [a, b], f(x) = 0$

Since f cont. on \mathbb{R} , by the lemma above

$$\exists x \in [0, 3\pi/2], f(x) = \sin x - x + 1 = 0$$

$$\rightarrow \sin x = x - 1$$

a) in problem 3-7 we proved

$$\text{a) } \forall \text{ polyn. } f \Rightarrow \exists \text{ polyn. } g \text{ and } b \in \mathbb{R} \\ \forall a \in \mathbb{R} \quad \Rightarrow \quad \text{s.t. } f(x) = (x-a)g(x) + b \quad \forall x$$

$$\text{b) } f(a) = 0 \Rightarrow f(x) = (x-a)g(x) \text{ for some} \\ \text{polyn. } g(x)$$

c) f n degree polyn. $\Rightarrow f$ has at most n roots, i.e. numbers a with $f(a) = 0$

d) for each $n \Rightarrow \exists$ polyn of degree n
with n roots

$n-h \geq 0$ is even.

thus

\exists polyn. of degree n has at most n roots.

since $n-h \geq 0$, then if $h \geq 0$ we have $0 \leq h \leq n$.

since $n-h \leq n$, it is possible that there is an n^{th} degree polyn. with $h \leq n$ roots (though we haven't proven such a polyn. must exist if $h < n$).

let $f(x) = (x-a_1)(x-a_2)\dots(x-a_n)g(x)$ be polyn. of degree n .

$$g(x) = 1+x^{n-h}$$

then

$g(x)$ has no roots because $\forall x, 1+x^{n-h} > 0$.

But f has n roots a_1, \dots, a_n .

Therefore we have a polyn. of degree n with exactly n roots

$$f(x) = (x-a_1)\dots(x-a_n)(1+x^{n-h})$$

b) Note:

$$f(x) = (x-a)^m g(x)$$

a not root of $g \rightarrow$ a root of f has multiplicity m

Let f be poly. deg. n with k roots, including multiplicities.

Then $n-k$ is even

Proof

Let f be poly. deg. n , with k roots

We've proved that f has at most n roots, so $k \leq n$, and $n-k \geq 0$.

f can be written as $f(x) = h(x)g(x)$, where $\deg(h) = k$, h has k roots, and $\deg(g) = n-k$ and g has no roots.

↙

Results from 3-7 b

Assume $n-k$ odd. Then by Th9, $g(x)$ has a root. ⊥

Therefore $n-k$ even.

Also, $n-k$ is even \Leftrightarrow n and k both even or both odd.

e.g. $\deg(f) = n$ even, then k must be even.

$\deg(f) = n$ odd, then k must be odd.

A 3rd deg. poly. can have either one root of mult. 1 or three roots.

A 2nd deg. poly. can have 0 roots, 1 root of mult. 2, or 2 roots of mult. 1.

examples

$$f(x) = 4x^2 + 4x + 1 = 4(x+1/2)^2$$

$$\Delta = 16 - 16 = 0 \rightarrow x_{\text{root}} = -\frac{1}{2}$$

$$\deg(f) = 2 = n$$

$$k=2$$

$$n-k=0$$

$$f(x) = x(x^2 + 1)$$

$$n=3$$

$$x_{\text{root}} = 0$$

$$k=1$$

$$n-k=2, \text{even}$$

$$f(x) = x^3 + 1 = (x+1)(x^2 - x + 1)$$

$$n=3$$

$$x_{\text{root}} = -1$$

$$n-k=2, \text{even}$$

$$\begin{array}{r} x^2 - x \\ x+1 \sqrt{x^3 + 1} \\ \hline x^2 + x^2 \\ \hline -x^2 - x \\ \hline -x^2 - x \\ \hline 1 + x \end{array}$$

$$\begin{aligned} x^3 + 1 &= (x+1)(x^2 - x + 1) + (1+x) \\ &= (1+x)(x^2 - x + 1) \end{aligned}$$

$$\Delta = -3 < 0$$

no roots

$$f(x) = x(x^2 + 1) = x(x+1)(x^2 - x + 1)$$

$$n=4$$

$$x_{\text{root}} = 0$$

$$x_{\text{root}} = -1$$

$$k=2$$

$$n-k=2, \text{even}$$

$$f(x) = x^3(x+1)$$

$$n=4$$

$$x_{\text{root}} = 0$$

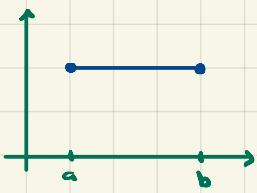
$$x_{\text{root}} = -1$$

$$k=4$$

$$n-k=0$$

5.

f cont. on $[a, b]$
 $f(x)$ always rational



Assume $f(a) \neq f(b)$.

Without loss of generality, assume $f(a) < f(b)$ and let $f(a) < c < f(b)$, c irrational.

By Th. 4 $\exists x \in [a, b], f(x) = c$. L

Therefore $f(a) = f(b)$.

Furthermore,

Assume $\exists x \in [a, b], f(x) \neq f(a) = f(b)$.

Then f cont. on $[a, x]$, and $f(a) \neq f(x)$.

But then, by the same argument above, $f(a) = f(x)$.

Therefore $f(x)$ = constant on $[a, b]$.

6.

f cont. on $[-1, 1]$

$$\forall x \quad f(x) = \sqrt{1-x^2}$$

→

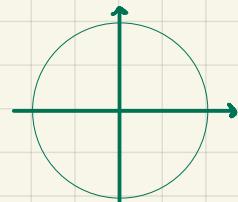
$$x^2 + (f(x))^2 = 1 \quad \forall x \in \mathbb{R}$$

$$\text{or else } \forall x \quad f(x) = -\sqrt{1-x^2}$$

PROOF

Given any x , since $a = (f(x))^2 = 1 - x^2 \geq 0$ in $[-1, 1]$, by Th. 2 we know that a has a square root.

Therefore for each $x \in [-1, 1]$, $f(x)$ is either $\sqrt{1-x^2}$ or $-\sqrt{1-x^2}$.



Let $x_1, x_2 \in (-1, 1)$. Without loss of generality assume $x_1 < x_2$.

If $f(x_1) = \sqrt{1-x_1^2}$ and $f(x_2) = -\sqrt{1-x_2^2}$ then by Th 1 $\exists x \in [x_1, x_2], f(x) = 0$.

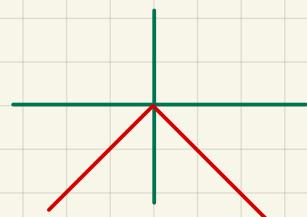
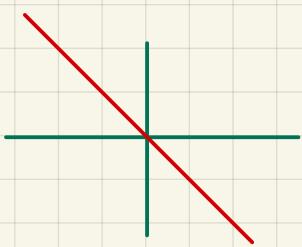
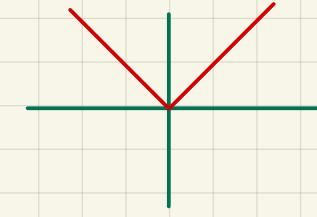
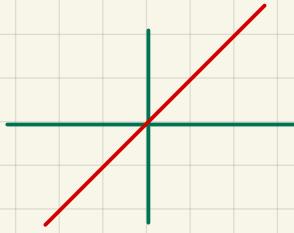
But then $x^2 + (f(x))^2 = x^2 = 1 \Leftrightarrow -1 \times x \in \mathbb{C}$. This is impossible.

The same impossibility is obtained if $f(x_1) = -\sqrt{1-x_1^2}$ and $f(x_2) = \sqrt{1-x_2^2}$.

Thus $f(x_1)$ and $f(x_2)$ have same sign, i.e. $\forall x \quad f(x) = \sqrt{1-x^2}$ or $\forall x \quad f(x) = -\sqrt{1-x^2}$

7) $\forall x \ (f(x))^2 = x^2$, f continuous.

$$\begin{aligned} f^2 &= x^2 \\ f &= \pm x \end{aligned}$$



7.

f, g cont.

$$f^2 = g^2$$

$$\forall x \ f(x) \neq 0$$

$$\forall x \ f(x) = g(x)$$

or

$$\forall x \ f(x) = -g(x)$$

Proof

Suppose $f(x_1) > 0$ and $f(x_2) < 0$, x_1, x_2 .

Then by Th. 1 $\exists x \in [x_1, x_2], f(x) = 0$.

Therefore $\forall x \ f(x) > 0$ or $\forall x \ f(x) < 0$.

Statement true for individual x.

$$\forall x \ f(x) = g(x) \quad \checkmark$$

$$\forall x \ (f(x) = g(x) \text{ or } f(x) = -g(x))$$

Case 1: $\forall x \ f(x) > 0$.

Let $x_1 \in \mathbb{R}$.

$$\text{Case 1.1: } f(x_1) = g(x_1).$$

Then $g(x_1) > 0$.

Let $x_2 \in \mathbb{R}$.

$$\text{Assume } f(x_2) = -g(x_2)$$

Then $-g(x_2) > 0$.

$$\rightarrow g(x_2) < 0.$$

Then $\exists x \in [x_1, x_2], g(x) = 0$

$$\rightarrow f(x) = 0.$$

1

$$f(x_2) = g(x_2)$$

$$\forall x \in \mathbb{R} \ f(x) = g(x)$$

$$\text{Case 1.2: } f(x_1) = -g(x_1)$$

Then $-g(x_1) > 0$

$$g(x_1) < 0$$

Let $x_2 \in \mathbb{R}$.

$$\text{Assume } f(x_2) = g(x_2).$$

Then $g(x_2) > 0$.

$$\rightarrow \exists x \in [x_1, x_2], g(x) = 0$$

$$\rightarrow f(x) = 0.$$

1

$$f(x_2) = -g(x_2)$$

$$\forall x \in \mathbb{R} \ f(x) = -g(x)$$

Therefore,

$$\forall x \in \mathbb{R} \ f(x) = -g(x) \text{ or } \forall x \in \mathbb{R} \ f(x) = g(x)$$

Case 2: $\forall x \ f(x) < 0$

Let $x_1 \in \mathbb{R}$.

$$\text{Case 2.1: } f(x_1) = g(x_1)$$

Then $g(x_1) < 0$.

Let $x_2 \in \mathbb{R}$.

$$\text{Assume } f(x_2) = -g(x_2)$$

Then $g(x_2) > 0$.

$$\exists x \in [x_1, x_2], g(x) = 0$$

$$\rightarrow f(x) = 0$$

1

$$f(x_2) = -g(x_2)$$

$$\forall x \in \mathbb{R} \ f(x) = -g(x)$$

$$\text{Case 2.2: } f(x_1) = -g(x_1)$$

Then $g(x_1) > 0$.

Let $x_2 \in \mathbb{R}$.

$$\text{Assume } f(x_2) = g(x_2)$$

$$g(x_2) > 0$$

$$\exists x \in [x_1, x_2], g(x) = 0$$

$$\rightarrow f(x) = 0$$

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$$f(x_2) = -g(x_2)$$

$$\forall x \in \mathbb{R} \ f(x) = -g(x)$$

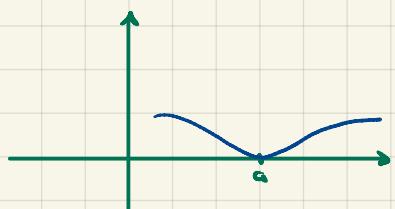
Therefore $\forall x \in \mathbb{R} \ f(x) = -g(x)$ or $\forall x \in \mathbb{R} \ f(x) = g(x)$

So thus conclude our proof by cases with:

$$\forall x \in \mathbb{R} \ f(x) = -g(x) \text{ or } \forall x \in \mathbb{R} \ f(x) = g(x)$$

9.

a)

 f cont $f(x) = 0$ only for $x=a$ $f(x) > 0$ for some $x > a$ as well
as for some $x < a$.

If f becomes negative then f must be 0 at least twice.
This contradicts an assumption, so f must be > 0 for $x \neq a$.

Proof

Let $x_1 \in \mathbb{R}$, $x_1 \neq a$ Assume $f(x_1) < 0$ Case 1: $x_1 < a$ By assumption $\exists x_2 < a$ s.t. $f(x_2) > 0$.Without loss of gen. assume $x_1 < x_2$.By Th. 1 $\exists x \in [x_1, x_2]$, $f(x) = 0$.

⊥

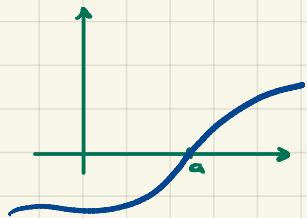
Case 2: $x_1 > a$

Analogous.

⊥

Therefore $f(x_1) > 0$.Therefore, $\forall x \in \mathbb{R} \quad x \neq a \rightarrow f(x) > 0$.

b)

 f cont $f(x) = 0$ only for $x=a$ $f(x) > 0$ for some $x > a$ $f(x) < 0$ for some $x < a$  $x < a \rightarrow f(x) < 0$ $x > a \rightarrow f(x) > 0$

Proof

Let $x_1 \in \mathbb{R}$ Assume $x_1 < a$.Assume $f(x_1) > 0$. $\exists x_2, x_2 < a$ s.t. $f(x_2) < 0$ $\exists x \in [x_1, x_2] \quad f(x) = 0$

⊥

 $f(x_1) \leq 0$ Assume $f(x_1) = 0$. ⊥ $f(x_1) < 0$ $x_1 < a \rightarrow f(x_1) < 0$ $\forall x < a \rightarrow f(x) < 0$ Similarly it we assume $x_1 > a$ we reach $\forall x > a \rightarrow f(x) > 0$.

c)

$$x^3 + x^2y + xy^2 + y^3$$

x and y not both zero

$$\begin{aligned} x^3 + x^2y + xy^2 + y^3 &= x^3 + 3x^2y + 3xy^2 + y^3 - 2x^2y - 2xy^2 \\ &= (x+y)^3 - 2xy(x+y) \\ &= (x+y)((x+y)^2 - 2xy) \\ &= (x+y)(x^2 + y^2) \end{aligned}$$

$$(x+0 \wedge y \neq 0) \vee (x \neq 0 \wedge y+0) \vee (x+0 \wedge y+0)$$

$$x+y > 0 \rightarrow y > -x$$

$$x^2 + y^2 \geq 0 \text{ for all } x, y$$

$$\text{Case 1: } x=0 \wedge y \neq 0$$

The expression becomes y^3 .

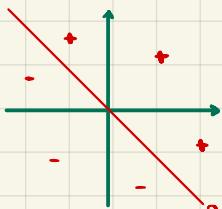


$$\text{Case 2: } x \neq 0 \wedge y=0$$

Analogous to case 1.



$$\text{Case 3: } x \neq 0 \wedge y \neq 0$$

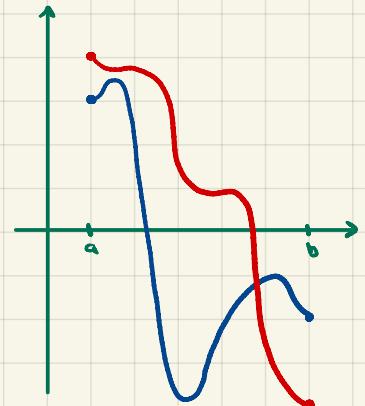
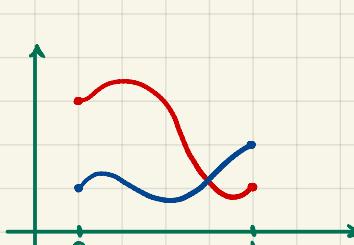


10.

f, g cont. on $[a, b]$

$$f(a) < g(a) \rightarrow f(x) = g(x) \text{ for some } x \in [a, b]$$

$$f(b) > g(b)$$



Proof

$$\text{Let } h(x) = f(x) - g(x)$$

$$h(a) = f(a) - g(a) < 0$$

$$h(b) = f(b) - g(b) > 0$$

h cont. on $[a, b]$.

$$\text{By Th. 1, } \exists x \in [a, b], h(x) = f(x) - g(x) = 0$$

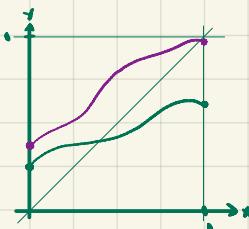
$$\rightarrow f(x) = g(x).$$

11.

f cont. on $[0, 1]$

$f(x) \in [0, 1]$ for each x

$$\rightarrow f(x) = x \text{ for some } x \in [0, 1]$$



Proof

$$\text{Let } h(x) = x.$$

Then $h(0) \leq f(0)$, $h(1) \geq f(1)$.

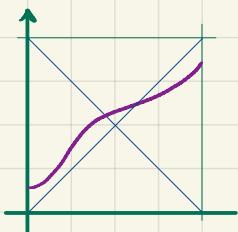
If $h(0) = f(0) = 0$ or $h(1) = f(1) = 1$ then $\exists x \in [0, 1], f(x) = x$.

If $h(0) < f(0)$ and $h(1) > f(1)$ then by problem 10 $f(x) = h(x) = x$ for some $x \in [0, 1]$.

12.

a)

f cont. on $[0,1]$
 $f(x) \in [0,1]$ for each x \rightarrow f intersects the line $y = 1 - x$ in $[0,1]$.



Proof

$$\text{Let } h(x) = 1 - x.$$

Then

$$h(0) = 1$$

$$f(0) \leq h(0)$$

$$h(1) = 0$$

$$f(1) \geq h(1)$$

Therefore, b), problem 10, $\exists x \in [0,1] f(x) = h(x) = 1 - x$.

b)

g cont. on $[0,1]$
 $g(0) = 0, g(1) = 1$ or $g(0) = 1, g(1) = 0$ \rightarrow $\exists x \in [0,1] f(x) \cdot g(x)$

Proof

$$\text{Case 1: } g(0) = 0, g(1) = 1$$

$$f(x) \geq g(x)$$

$$f(1) \leq g(1)$$

$$\text{Case 2: } g(0) = 1, g(1) = 0$$

Analogous to case 1.

$$\text{Case 1.1: } f(0) \cdot g(0) \text{ or } f(1) \cdot g(1)$$

$$\exists x \in [0,1] f(x) \cdot g(x)$$

$$\text{Case 1.2: } f(0) > g(0) \text{ and } f(1) < g(1)$$

by prob. a) $\exists x \in [0,1], f(x) \cdot g(x)$.

Therefore $\exists x \in [0,1], f(x) \cdot g(x)$.

13.

a)

$$f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x=0 \end{cases}$$

 f cont on $[-1, 1]$?

$$\lim_{x \rightarrow 0} \sin(1/x)$$

Therefore f not cont at $x=0$. f satisfies condition of INT on $[-1, 1]$

Proof

In $[-1, 1]$, the max value of f is 1, min is -1. $\frac{1}{x}$ is cont. on $(0, \infty)$.In particular, on $[2/\pi, 2/3\pi]$

$$f(2/\pi) = 1$$

$$f(2/3\pi) = -1$$

$$\rightarrow \forall c \in [-1, 1], \exists x \in [2/\pi, 2/3\pi], f(x) = c.$$

Therefore, for any two values of y in $[-1, 1]$, f takes on all values in between.

Analogously,

$$\text{Let } x_1, x_2 \in [-1, 1]$$

$$\text{Case 1: } 0 < x_1 < x_2 < 1$$

 f cont. on $[x_1, x_2]$.Therefore by Th 4 $\forall c, \exists x, f(x) < c < f(x_2)$

$$\rightarrow \exists x \in [x_1, x_2], f(x) = c.$$

$$\text{Case 2: } -1 < x_1 < x_2 < 0$$

Analogous to Case 1.

$$\text{Case 3: } x_1 < 0 < x_2$$

 f takes on all values between -1 and 1 in $[x_1, x_2]$.Therefore it takes all values in $(f(x_1), f(x_2))$ since $\forall x, f(x) \in [-1, 1]$.

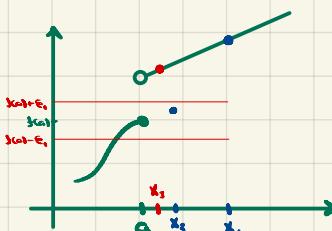
b)

$\left\{ \begin{array}{l} f \text{ satisfies cond. of INT} \\ f \text{ takes on each value only once} \end{array} \right. \rightarrow \left\{ \begin{array}{l} f \text{ continuous} \end{array} \right.$

Proof

Assume f is defined at a , but not continuous at a .Then, as proved in 6-9, for some $\epsilon > 0$, there are numbers δ sufficiently close to a s.t. $|f(x) - f(a)| < \epsilon$ or $|f(x) - f(a)| > \epsilon$.Let $\epsilon > 0$ be such as in 6-9, and δ , such that $|f(x) - f(a)| > \epsilon$, or $|f(x) - f(a)| < \epsilon$.

We need to consider two cases, but they're analogous.

Let's consider the case where $|f(x) - f(a)| > \epsilon$.By INT, $\forall c, f(a) < c < f(x_1), \exists x \in [a, x_1], f(x) = c$.Let $c \in (f(a), f(a) + \epsilon)$. Then, $f(a) < c < f(a) + \epsilon < f(x_1)$.Therefore, $\exists x_2$ s.t. $x_2 \in [a, x_1]$ and $f(x_2) = c < f(a) + \epsilon$.However, again by 6-9b, there is x_3 sufficiently close to a (e.g. $a < x_3 < x_2$) s.t. $f(x_3) > f(a) + \epsilon$.But then, $x_3 < x_2 < x_1$ and $f(x_2) < f(a) + \epsilon < f(x_3)$ and $f(x_2) < f(a) + \epsilon < f(x_3)$ By INT, f takes on value $f(a) + \epsilon$, in $[x_3, x_2]$ and in $[x_2, x_1]$, contradicting the premise that f takes on each value only once.Therefore, f continuous.

c)

- f satisfies cond. of INT
- f takes on each value only
- Switches many times

$\rightarrow f$ continuous

In 6-9 we showed that there are x arbitrarily close to a , such that $f(x) > f(a) + \epsilon$ or $f(x) < f(a) - \epsilon$.

Let n be the maximum number of times f takes on a value.

If we assume f is discontinuous at a , we can repeat the steps used in part b) $n+1$ times to reach the result that for some $\epsilon > 0$, f takes on the value $f(a) + \epsilon$ more than n times, a contradiction.

14. f cont. on $[0,1]$

$$\|f\| = \max\{ \|f\|_1, \|f\|_\infty \}$$

$$a) \forall c \quad \|fc\| = |c| \cdot \|f\|$$

let $f(x_1)$ be the max of f on $[0,1]$. we know $\exists x_1$ by Th.3.

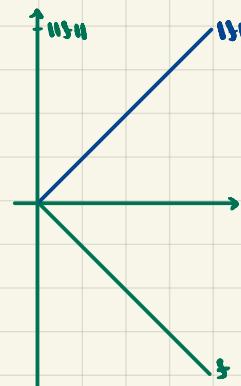
$\forall x \in [0,1], f(x) \leq f(x_1)$.

let $f(x_2)$ be the min of f on $[0,1]$. we know $\exists x_2$ by Th.7.

$\forall x \in [0,1], f(x) \geq f(x_2)$.

$$\text{Then } \|f\| = \max(\|f(x_1)\|, \|f(x_2)\|) = \frac{\|f(x_1)\| + \|f(x_2)\| + \|f(x_1) - f(x_2)\|}{2}$$

then, $\forall x \in [0,1], \|fx\| \leq \|f\|$



$$\text{Similarly, } \|fc\| = \max(\|fc(x_1)\|, \|fc(x_2)\|) = \frac{\|fc(x_1)\| + \|fc(x_2)\| + \|fc(x_1) - fc(x_2)\|}{2}$$

$$= \frac{\|c\| \|fx_1\| + \|c\| \|fx_2\| + \|c\| \|fx_1 - fx_2\|}{2}$$

$$= \frac{\|c\| \|fx_1\| + \|c\| \|fx_2\| + \|c\| (\|f(x_1)\| + \|f(x_2)\|)}{2}$$

$$= \frac{\|c\| \|fx_1\| + \|c\| \|fx_2\| + \|c\| \|f(x_1) - f(x_2)\|}{2}$$

$$= \|c\| \frac{\|f(x_1)\| + \|f(x_2)\| + \|f(x_1) - f(x_2)\|}{2}$$

$$= \|c\| \|f\|$$

$$b) \|f+g\| \leq \|f\| + \|g\|$$

Let $f(x_{\min})$ be the minimum of f on $[0,1]$.

Let $f(x_{\max})$ "maximum" " "

Then, $\forall x \in [0,1], f(x) \leq f(x_{\max}) \leq \max(\|f(x_{\min})\|, \|f(x_{\max})\|) = \|f\|$

Let $g(x_{\min})$ and $g(x_{\max})$ be defined analogously for g .

Let $f(x_{f+g_{\min}})$ and $f(x_{f+g_{\max}})$ be defined analogously for $f+g$.

Then

This proof turned out way more complicated than necessary.

$$\|f\| = \max(\|f(x_{\min})\|, \|f(x_{\max})\|) \geq \max(\|f(x_{f+g_{\min}})\|, \|f(x_{f+g_{\max}})\|)$$

$$\|g\| = \max(\|g(x_{\min})\|, \|g(x_{\max})\|) \geq \max(\|g(x_{f+g_{\min}})\|, \|g(x_{f+g_{\max}})\|)$$

$$\|f\| + \|g\| \geq \max(\|f(x_{f+g_{\min}})\|, \|f(x_{f+g_{\max}})\|) + \max(\|g(x_{f+g_{\min}})\|, \|g(x_{f+g_{\max}})\|)$$

$$= [\|f(x_{f+g_{\min}})\| + \|g(x_{f+g_{\min}})\| + \|f(x_{f+g_{\max}})\| + \|g(x_{f+g_{\max}})\| + \|f(x_{f+g_{\min}})\| - \|f(x_{f+g_{\max}})\|] / 2$$

$$\quad \|x_1 + x_2\| \geq \|x_1\| + \|x_2\| \quad \left(+ \|g(x_{f+g_{\min}})\| - \|g(x_{f+g_{\max}})\| \right) / 2$$

$$\geq [\|f(x_{f+g_{\min}}) + g(x_{f+g_{\min}})\| + \|f(x_{f+g_{\max}}) + g(x_{f+g_{\max}})\| + \|f(x_{f+g_{\min}}) + g(x_{f+g_{\min}})\| - (\|f(x_{f+g_{\max}}) + g(x_{f+g_{\max}})\|)] / 2$$

$$\geq [\|f(x_{f+g_{\min}}) + g(x_{f+g_{\min}})\| + \|f(x_{f+g_{\max}}) + g(x_{f+g_{\max}})\| + \|f(x_{f+g_{\min}}) + g(x_{f+g_{\min}})\| - (\|f(x_{f+g_{\max}}) + g(x_{f+g_{\max}})\|)] / 2$$

$$\quad \|a - b\| \geq \|a\| - \|b\| \quad \left(\frac{- (\|f(x_{f+g_{\max}}) + g(x_{f+g_{\max}})\|)}{b} \right) / 2$$

$$\geq [\|f(x_{f+g_{\min}}) + g(x_{f+g_{\min}})\| + \|f(x_{f+g_{\max}}) + g(x_{f+g_{\max}})\| + \|f(x_{f+g_{\min}}) + g(x_{f+g_{\min}})\| - \|f(x_{f+g_{\max}}) + g(x_{f+g_{\max}})\|] / 2$$

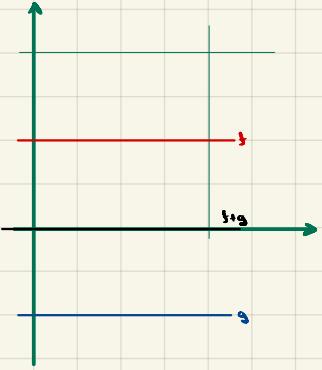
$$\geq [\|f(x_{f+g_{\min}}) + g(x_{f+g_{\min}})\| + \|f(x_{f+g_{\max}}) + g(x_{f+g_{\max}})\| + \|f(x_{f+g_{\min}}) + g(x_{f+g_{\min}})\| - \|f(x_{f+g_{\max}}) + g(x_{f+g_{\max}})\|] / 2$$

$$= [\|f(x_{f+g_{\min}}) + g(x_{f+g_{\min}})\| + \|f(x_{f+g_{\max}}) + g(x_{f+g_{\max}})\| + \|f(x_{f+g_{\min}}) + g(x_{f+g_{\min}})\| - \|f(x_{f+g_{\max}}) + g(x_{f+g_{\max}})\|] / 2$$

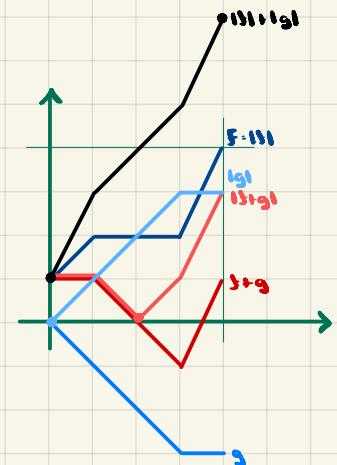
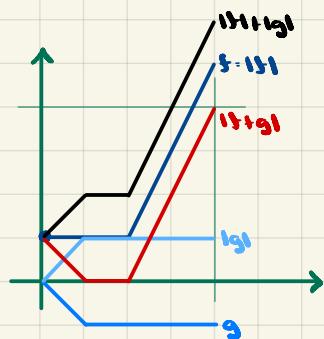
$$= \max(\|(f+g)(x_{f+g_{\min}})\|, \|(f+g)(x_{f+g_{\max}})\|)$$

$$= \|f+g\|$$

examples of $\|f+g\| \leq \|f\| + \|g\|$



$$\begin{aligned}
 c > 0 \\
 f(x) = c \\
 g(x) = -c \\
 (f+g)(x) = 0 \\
 \|f+g\| = 0 \\
 \|f\| = c \\
 \|g\| = c \\
 \|f\| + \|g\| - 2c \geq \|f+g\|
 \end{aligned}$$



Alternative proof

Let $x \in [0,1]$. Then

$$|(f+g)(x)| = |(f(x) + g(x))| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|$$

Note f and g are cont. so $|f+g|$ and $\|f+g\|$ take on max and min in $[0,1]$.

Let x_0 s.t. $\|f+g\| = |f+g|(x_0) \leq |f(x_0)| + |g(x_0)| \leq \|f\| + \|g\|$

$$c) \|h-f\| \leq \|h-g\| + \|g-f\|$$

Proof

$$\begin{aligned}
 f_i &= h-g \\
 g_i &= g-f
 \end{aligned}$$

Then

$$f_i + g_i = h-g+g-f = h-f$$

From part b),

$$\|f_i + g_i\| \leq \|f_i\| + \|g_i\|$$

$$\|h-f\| \leq \|h-g\| + \|g-f\|$$

15.

Theorem 9If $n \in \mathbb{N}$ is odd then any equation ϕ cont.

$$\lim_{x \rightarrow \infty} \frac{\phi(x)}{x^n} = \lim_{x \rightarrow -\infty} \frac{\phi(x)}{x^n} = 0$$

has a root.

a) $n \text{ odd} \rightarrow \exists x, x^n + \phi(x) = 0$

Theorem 8.2 If $n \in \mathbb{N}$ is odd then every number has an n^{th} root.**Proof**Assume n odd. (Conditional Proof Ass.)

premise: $\lim_{x \rightarrow \infty} \frac{\phi(x)}{x^n} = 0$

$$\leftrightarrow \forall \epsilon > 0 \ \exists N > 0 \ \forall x (x > N \rightarrow \left| \frac{\phi(x)}{x^n} \right| < \epsilon) \quad (\text{Def limit})$$

$$\leftrightarrow \forall \epsilon > 0 \ \exists N > 0 \ \forall x (x > N \rightarrow -\epsilon < \frac{\phi(x)}{x^n} < \epsilon)$$

choose $\epsilon = 1$. Then $\exists N > 0 \ \forall x (x > N \rightarrow -1 < \frac{\phi(x)}{x^n} < 1)$ (\exists Elim Ass. because $\exists \epsilon, \epsilon = 1$)

let N s.t. $N > 0$ and $\forall x (x > N \rightarrow -1 < \frac{\phi(x)}{x^n} < 1)$ (\exists Elim Ass.)

let x_0 , s.t. $x_0 > N > 0$ (\exists Elim Ass., because $\exists x, x > N$).Then $x_0^n > 0$ and $-x_0^n < \phi(x_0) < x_0^n$ since n odd then $-x_0^n = (-x_0)^n < \phi(x_0) < x_0^n$

$$\rightarrow (-x_0)^n < -\phi(x_0) < x_0^n$$

let $f(x) = x^n$. Then $f(-x_0) < -\phi(x_0) < f(x_0)$ since f cont in \mathbb{R} , $\exists z \in [-x_0, x_0], f(z) = z^n = -\phi(x_0)$

$$\rightarrow x_0^n + \phi(x_0) = 0$$

Therefore, $\exists x, x^n + \phi(x) = 0$. (\exists Elim)Therefore, $\exists x, x^n + \phi(x) = 0$. (\exists Elim)Therefore, $\exists x, x^n + \phi(x) = 0$. (\exists Elim)Therefore n odd $\rightarrow \exists x, x^n + \phi(x) = 0$.

Archimedean Property

$$\lim_{x \rightarrow \infty} \frac{\phi(x)}{x^n} = 0$$

$$\Leftrightarrow \forall \epsilon > 0 \exists N > 0 \forall x (x > N \rightarrow \left| \frac{\phi(x)}{x^n} \right| < \epsilon)$$

$$\text{Let } \epsilon = \frac{1}{2}$$

$$\text{Let } N \text{ s.t. } \forall x (x > N \rightarrow \left| \frac{\phi(x)}{x^n} \right| < \frac{1}{2})$$

Let b s.t. $b > N$

$$\text{Then } \left| \frac{\phi(b)}{b^n} \right| < \frac{1}{2}$$

$$-\frac{1}{2} < \frac{\phi(b)}{b^n} < \frac{1}{2}$$

$$\frac{1}{2} < 1 + \frac{\phi(b)}{b^n}$$

$$0 < \frac{b^n}{2} < b^n \left(1 + \frac{\phi(b)}{b^n}\right) = b^n + \phi(b)$$

Similarly

$$\lim_{x \rightarrow -\infty} \frac{\phi(x)}{x^n} = 0 \Leftrightarrow \forall \epsilon > 0 \exists N_2 < 0 \forall x (x < N_2 \rightarrow \left| \frac{\phi(x)}{x^n} \right| < \epsilon)$$

$$\text{Let } \epsilon = \frac{1}{2}$$

$$\text{Let } N_2 < 0 \text{ s.t. } \forall x (x < N_2 \rightarrow \left| \frac{\phi(x)}{x^n} \right| < \frac{1}{2})$$

Let a s.t. $a < N_2$.

$$\text{Then } \left| \frac{\phi(a)}{a^n} \right| < \frac{1}{2}$$

$$-\frac{1}{2} < \frac{\phi(a)}{a^n} < \frac{1}{2}$$

Since $n \geq 1$, $a^n < 0$.

$$\frac{1}{2} < 1 + \frac{\phi(a)}{a^n} < \frac{3}{2}$$

$$0 > \frac{a^n}{2} > a^n \left(1 + \frac{\phi(a)}{a^n}\right) > \frac{2a^n}{2}$$

$$\text{i.e., } a^n \left(1 + \frac{\phi(a)}{a^n}\right) - a^n + \phi(a) < 0$$

Thus we have

$$a^n + \phi(a) - f(a) < 0 < f(b) = b^n + \phi(b)$$

$$\exists x \in [a, b], f(x) = x^n + \phi(x) = 0.$$

b)

 ϕ cont.

$$\lim_{x \rightarrow \infty} \frac{\phi(x)}{x^n} = \lim_{x \rightarrow -\infty} \frac{\phi(x)}{x^n} = 0$$

n even

$$\forall x \exists y, y^n + \phi(y) \leq x^n + \phi(x)$$

Proof

$$\lim_{x \rightarrow \infty} \frac{\phi(x)}{x^n} = 0 \iff \forall \epsilon > 0 \exists N_1 > 0 \forall x (x > N_1 \rightarrow \left| \frac{\phi(x)}{x^n} \right| < \epsilon)$$

$$\lim_{x \rightarrow -\infty} \frac{\phi(x)}{x^n} = 0 \iff \forall \epsilon > 0 \exists N_2 < 0 \forall x (x < N_2 \rightarrow \left| \frac{\phi(x)}{x^n} \right| < \epsilon)$$

Let $N_{\max} = \max(N_1, |N_2|)$. Then $\forall \epsilon > 0 \exists N > 0 \forall x (|x| > N \rightarrow \left| \frac{\phi(x)}{x^n} \right| < \epsilon)$

Let $\epsilon = \frac{1}{2}$.

$$\text{Let } N > 0 \text{ s.t. } \forall x (|x| > N \rightarrow \left| \frac{\phi(x)}{x^n} \right| < \frac{1}{2})$$

Let $N_1 > 2\phi(0)$.Let $N = \max(N_1, N_2)$.

Then $N > 2\phi(0)$ and $\forall x (|x| > N \rightarrow \left| \frac{\phi(x)}{x^n} \right| < \frac{1}{2})$

Let $x, s.t. |x|, l > N$. Then $-\frac{1}{2} < \frac{\phi(x)}{x^n} < \frac{1}{2}$.

$$\frac{x_i^n}{2} < x_i^n + \phi(x_i) < \frac{3x_i^n}{2}$$

But $x_i^n > N > 2\phi(0)$ so $\frac{x_i^n}{2} > \phi(0)$.

Therefore

$$\phi(0) < x_i^n + \phi(x_i)$$

Therefore $\forall x x > N \text{ or } x < -N \rightarrow x^n + \phi(x) > \phi(0)$.

But $0 \in [-N, N]$, and $y(x) = x^n + \phi(x)$ is cont. on $[-N, N]$.

Therefore it has on its minimum value on that interval, say at x_2 .

Therefore $\forall x \in [-N, N] x^n + \phi(x) \geq x_2^n + \phi(x_2)$.

In particular, $0 + \phi(0) = \phi(0) \geq x_2^n + \phi(x_2)$

Therefore $\forall x \in \mathbb{R} x^n + \phi(x) \geq \phi(0) \geq x_2^n + \phi(x_2)$

16.

a)

 f cont on (a, b)

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow b^-} f(x) = \infty$$

 f has a minimum on $[a, b]$

Proof

Let $x_1 \in (a, b)$ Let $M = f(x_1)$ Since $\lim_{x \rightarrow a^+} f(x) = \infty$, $\exists \delta_1 > 0 \forall x, 0 < x - a < \delta_1 \rightarrow f(x) > M$ Since $\lim_{x \rightarrow b^-} f(x) = \infty$, $\exists \delta_2 > 0 \forall x, 0 < b - x < \delta_2 \rightarrow f(x) > M$ Let $\delta = \min(\delta_1, \delta_2)$. Then $\forall x, a + \delta < x < a + \delta \text{ or } b - \delta < x < b \rightarrow f(x) > M$

Consider the following intervals

$(a, a + \delta)$

$[a + \delta, b - \delta]$

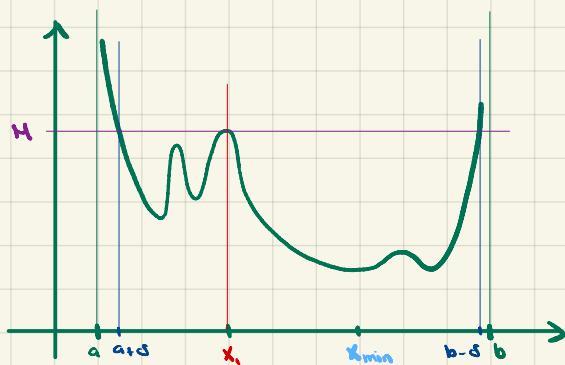
$(b - \delta, b)$

Their union is (a, b) . Since f is cont. on $[a + \delta, b - \delta]$,

$\exists x_{\min} \in [a + \delta, b - \delta] \quad \forall x \in [a + \delta, b - \delta] \quad f(x) \geq f(x_{\min})$

$\text{But } x_1 \in [a + \delta, b - \delta], \text{ so } \forall x, x \in (a, a + \delta) \text{ or } x \in (b - \delta, b) \rightarrow f(x) > f(x_1) \geq f(x_{\min})$

Therefore $\forall x \in (a, b) \quad f(x) \geq f(x_{\min})$



b)

f cont on $(-\infty, \infty)$

\rightarrow f has a minimum on all of (c, b)

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = \infty$$

Proof

$$\forall M > 0 \exists N > 0 \forall x > N \rightarrow f(x) > M$$

$$\forall M < 0 \exists N < 0 \forall x < N \rightarrow f(x) < M$$

Let $x_0 \in \mathbb{R}$. Let $M = f(x_0)$.

$$\exists N > 0 \forall x |x| > N \rightarrow f(x) > M$$

Consider the intervals

$$(-\infty, -N), [-N, N], (N, \infty)$$

There is an $x_{\min} \in [-N, N]$ s.t. $\forall x \in [-N, N], f(x) \geq f(x_{\min})$.

But $x_0 \in [-N, N]$, so $\forall x, x \in (-\infty, -N)$ or $x \in (N, \infty) \rightarrow x > f(x_0) \geq f(x_{\min})$.

Therefore,

$$\forall x \in \mathbb{R}, f(x) \geq f(x_{\min})$$

17.

$$\{ \text{any polynomial} \} \rightarrow \exists f, \forall x |f(x)| \leq |f(x)|$$

Proof

Polynomial Fn f is polyn if $\forall a_0, \dots, a_n \in \mathbb{R}$ s.t. $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ $\xrightarrow{\text{highest power: degree of } f}$

since f polynomial then f cont in \mathbb{R} .

by 6-10a, f cont at $a \rightarrow f$ cont at a

This f cont in \mathbb{R} .

$$\text{Note that } \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} a_n x^n$$

$$\text{In 5-16a we proved that: } \lim_{x \rightarrow \infty} f(x) = L \rightarrow \lim_{x \rightarrow \infty} |f(x)| = |L|$$

Therefore

$$\lim_{x \rightarrow \infty} |f(x)| = \lim_{x \rightarrow \infty} |f(x)|$$

$$\lim_{x \rightarrow -\infty} |f(x)| = \lim_{x \rightarrow -\infty} |f(x)|$$

Since $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = \pm \infty$ then

$$\lim_{x \rightarrow \infty} |f(x)| = \lim_{x \rightarrow -\infty} |f(x)| = \infty$$

we thus have the assumption made in 16b:

$|f|$ cont on \mathbb{R}

$$\lim_{x \rightarrow \infty} |f(x)| = \lim_{x \rightarrow -\infty} |f(x)| = \infty$$

Therefore $|f|$ has a minimum on $(-\infty, \infty)$.

$$\exists c \in \mathbb{R}, \forall x |f(x)| \leq |f(c)|$$

→ Proof

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \\ &= a_n x^n \left(1 + \frac{a_{n-1}}{a_n} \cdot \frac{1}{x} + \frac{a_{n-2}}{a_n} \frac{1}{x^2} + \dots + \frac{a_0}{x^n}\right) \end{aligned}$$

$$\lim_{x \rightarrow \infty} f(x) = \left(\lim_{x \rightarrow \infty} a_n x^n\right) \left(\lim_{x \rightarrow \infty} \left(1 + \frac{a_{n-1}}{a_n} \cdot \frac{1}{x} + \frac{a_{n-2}}{a_n} \frac{1}{x^2} + \dots + \frac{a_0}{x^n}\right)\right)$$

$$= \lim_{x \rightarrow \infty} a_n x^n$$

$$\begin{aligned} &\# \lim_{x \rightarrow \infty} f(x) = \infty \\ &\lim_{x \rightarrow \infty} g(x) = c > 0 \quad \rightarrow \quad \lim_{x \rightarrow \infty} f(x)g(x) = \infty \end{aligned}$$

$$\begin{aligned} \forall M_1 > 0 \exists N_1 > 0 \forall x > N_1 \rightarrow |f(x)| > M_1, \\ \forall \epsilon > 0 \exists N_2 > 0 \forall x > N_2 \rightarrow |g(x) - c| < \epsilon \end{aligned}$$

$$\forall N > 0 \text{ let } N_1 > 0 \text{ ad } 0 < \epsilon < c \text{ such that } N_1(c - \epsilon) = N.$$

$$\text{let } N = \max(N_1, N_2)$$

Then

$$\forall x > N \rightarrow |f(x)g(x)| > N_1(c - \epsilon) = N > 0$$

That is, we've shown

$$\forall N > 0 \exists N > 0 \forall x > N \rightarrow |f(x)g(x)| > N$$

$$\lim_{x \rightarrow \infty} f(x)g(x) = \infty$$

Alternative Proof

$$J(x) = b_n x^n + b_{n-1} x^{n-1} + \dots + b_0$$

$$= b_n x^n \left(1 + \frac{b_{n-1}}{b_n} \cdot \frac{1}{x} + \frac{b_{n-2}}{b_n} \cdot \frac{1}{x^2} + \dots + \frac{b_0}{b_n} \cdot \frac{1}{x^n} \right)$$

$$= b_n x^n \left(1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n} \right)$$

note: $\left| \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n} \right| \leq \frac{|a_{n-1}|}{|x|} + \dots + \frac{|a_0|}{|x^n|}$ here we shall take the same steps used to prove Theorem 9.

$$\text{let } M = \max(1, 2|a_{n-1}|, \dots, 2|a_0|)$$

Then $\forall x$ s.t. $|x| > M$

$$|x|^n > |x|$$

$$\frac{|a_{n-1}|}{|x|^n} < \frac{|a_{n-1}|}{|x|} < \frac{|a_{n-1}|}{2|a_{n-1}|} = \frac{1}{2}$$

Thus,

$$\left| \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n} \right| < n \cdot \frac{1}{2} = \frac{1}{2}$$

$$-\frac{1}{2} < \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n} < \frac{1}{2}$$

$$\frac{1}{2} < 1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n}$$

$$\frac{bx^n}{2} < bx^n \left(1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n} \right) = J(x)$$

Therefore

$$|J(x)| - \left| bx^n \left(1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n} \right) \right| > \left| \frac{bx^n}{2} \right|$$

$|x| > M$ that thus $|J(x)| > |J(x)|$, i.e. $|J(x)|$ less than $-\frac{1}{2}$ of the highest order term $b_n x^n$.

choose some x , such that $|x| > M$ and also $|x^n| \geq \frac{2|a_0|}{|b|}$

Then, $\forall x$, $|x| > x_1 \rightarrow |J(x)| > |a_0|$

Thus, the minimum on $[-x_1, x_1]$ is the minimum on \mathbb{R} .

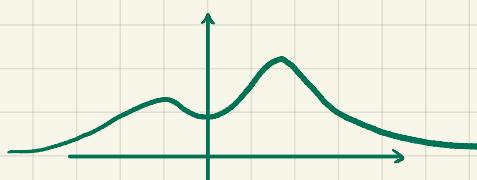
18.

f cont.

$$\forall x \ f(x) > 0$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow \infty} f(x) = 0$$

$$\rightarrow \exists \forall x \ f(x) \geq f(x)$$



Proof

Def of limit at ∞

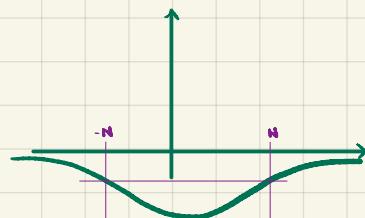
$$\forall \epsilon > 0 \ \exists N > 0 \ \forall |x| > N \rightarrow |f(x)| < \epsilon.$$

Let $x_1 \in \mathbb{R}$ and $f(x_1) > 0$.Let $\epsilon \cdot \frac{f(x_1)}{2} < f(x_1)$. Thensince $\forall x \ f(x) > 0$

$$\exists N > 0 \ \forall |x| > N \rightarrow 0 < f(x) < \frac{f(x_1)}{2} < f(x_1)$$

since f cont.

$$\text{Also } \exists j \in [-N, N], \forall x \in [-N, N], f(x) \leq f(j)$$

In particular $x_1 \in [-N, N]$, thus $f(j) \geq f(x_1)$.Hence $\forall x \in \mathbb{R} \ f(j) \geq f(x)$.Note that if $f(x)$ could be neg. we could have the situation

in which there is no max.

19.

a)

$$\begin{aligned} f \text{ cont. on } [a,b] \\ x \in \mathbb{R} \end{aligned} \rightarrow \exists y \in [a,b], \forall z \in [a,b], D((z, f(z)), (x, 0)) \geq D((y, f(y)), (x, 0))$$

ie there is a point on graph of f in $[a,b]$ that is closest to $(x,0)$.

Proof

Distance between two points $(x,0)$ and $(z, f(z))$

$$D(z) = \sqrt{(x-z)^2 + f^2(z)}, z \in [a,b]$$

 $D(z)$ cont. in $[a,b]$

Proof

$$(x-z)^2 \text{ and } f^2(z) \text{ cont. in } [a,b] \rightarrow (x-z)^2 + f^2(z) \text{ cont. in } [a,b]$$

$$\text{Therefore } \lim_{z \rightarrow y} [(x-z)^2 + f^2(z)] = (x-y)^2 + f^2(y) \geq 0, \forall y \in [a,b]$$

$$g(y) = \sqrt{x} \text{ cont. in } [0, \infty)$$

$$\text{Therefore, } \forall y \in [a,b], \lim_{z \rightarrow y} [(x-z)^2 + f^2(z)] = 0 \geq 0, \text{ and } g \text{ cont. at } 0.$$

$$\text{Therefore by 6-12a, } \lim_{z \rightarrow y} g(\sqrt{(x-z)^2 + f^2(z)}) = g(\lim_{z \rightarrow y} \sqrt{(x-z)^2 + f^2(z)}) = \sqrt{(x-y)^2 + f^2(y)}$$

$$\text{i.e., } \lim_{z \rightarrow y} D(z) = D(y).$$

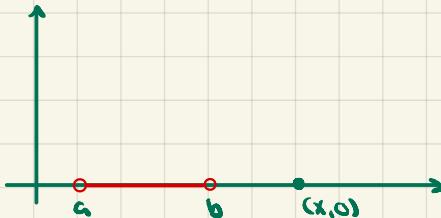
Therefore $D(z)$ cont. in $[a,b]$.Therefore, D takes a minimum value in $[a,b]$.

To

$$\forall x \in \mathbb{R}, \exists z \in [a,b] \forall y \in [a,b] D(z) \leq D(y)$$

b)

$$\begin{aligned} f \text{ cont. on } (a,b) \\ x \in \mathbb{R} \end{aligned}$$



$$D(z) = \sqrt{(x-z)^2} = |z-x| \text{ cont. in } (a,b).$$

There is no minimum value of $D(z)$ in (a,b) .

$$\forall z \in [a,b], \exists z' \in (a,b) \text{ and } D(z') < D(z).$$

c)

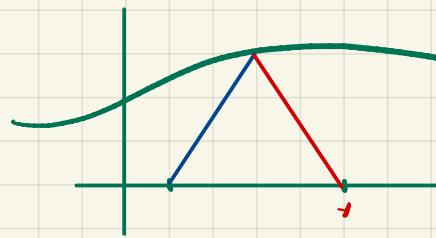
f cont. on \mathbb{R} .
 $x \in \mathbb{R}$.
 $\exists y \in \mathbb{R}, \forall z \in \mathbb{R}, D((z, f(z)), (x, 0)) \geq D((y, f(y)), (x, 0))$
 ie there is a point $(y, f(y))$ on graph of f in \mathbb{R} that is closest to $(x, 0)$.

Proof

$$D(z) = \sqrt{(x-z)^2 + f^2(z)}, z \in \mathbb{R}$$

 $D(z)$ cont. in \mathbb{R} .

$$\lim_{z \rightarrow \infty} D(z) = \lim_{z \rightarrow \infty} D(z) = \infty$$

By T-16b, D has a minimum on $(-\infty, \infty) = \mathbb{R}$.

d)

 f cont. on \mathbb{R} . $x \in \mathbb{R}$

$$g(x) = \min_z \sqrt{(x-z)^2 + f^2(z)}$$

$$\rightarrow g(y) \leq g(x) + |x-y|$$

 g cont.

Proof

Let $x_i \in \mathbb{R}$, we know that $\exists z_{x_i, \min} \in \mathbb{R}, \forall y_i \in \mathbb{R}, D_{x_i}(z_{x_i, \min}) \leq D_{x_i}(y_i)$, where

$$D_{x_i}(z) = \sqrt{(x_i - z)^2 + f^2(z)}.$$

$$\text{Then } g(x_i) = D_{x_i}(z_{x_i, \min})$$

 $\forall y_i \in \mathbb{R}, \exists z_{x_i, \min} \in \mathbb{R}, \forall y_i \in \mathbb{R}, D_{x_i}(z_{x_i, \min}) \leq D_{x_i}(y_i)$, where $D_{x_i}(z) = \sqrt{(y_i - z)^2 + f^2(z)}$.

$$\text{Then } g(y_i) = D_{x_i}(z_{x_i, \min})$$

Therefore since $z_{x_i, \min} \in \mathbb{R}$, we have $g(y_i) = D_{x_i}(z_{x_i, \min}) \leq D_{x_i}(z_{x_i, \min}) \quad (1)$ Consider the points $(x_i, 0), (y_i, 0)$, and $(z_{x_i, \min}, f(z_{x_i, \min}))$.

By the triangle inequality, we have:

$$\sqrt{(z_{x_i, \min} - y_i)^2 + f^2(z_{x_i, \min})} \leq \sqrt{(z_{x_i, \min} - x_i)^2 + f^2(z_{x_i, \min})} + \sqrt{(x_i - y_i)^2}$$

$$D_{x_i}(z_{x_i, \min}) \leq D_{x_i}(z_{x_i, \min}) + |x_i - y_i| = g(x_i) + |x_i - y_i| \quad (2)$$

by (1) and (2), $g(y_i) \leq g(x_i) + |x_i - y_i|$.Hence $\forall y \forall x, g(y) \leq g(x) + |x - y|$.Ass continuity of g

$$\forall \forall x, g(y) - g(x) \leq |x - y|$$

Therefore we have

$$g(y) - g(x) \leq |x - y| \rightarrow -|x - y| \leq g(x) - g(y)$$

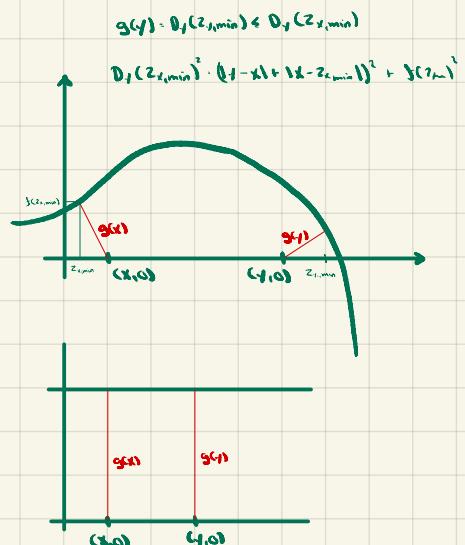
$$g(x) - g(y) \leq |x - y|$$

$$\text{Hence } |g(y) - g(x)| \leq |x - y|$$

$$\forall \epsilon > 0 \exists \delta - \epsilon \\ |x - a| < \delta \rightarrow |g(x) - g(a)| < \epsilon$$

Therefore

$$\lim_{x \rightarrow a} g(x) = g(a)$$

 $\rightarrow g$ cont. on \mathbb{R} .

e)

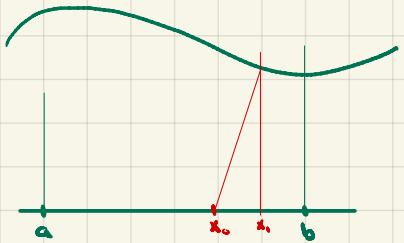
$\exists x_0, x_1$ s.t.

$$x_0, x_1 \in [a, b]$$

$$\rightarrow D((x_0, 0), (x_1, f(x_1))) \leq D((x'_0, 0), (x'_1, f(x'_1)))$$

$$\forall x'_0, x'_1 \in [a, b]$$

in other words, there is a minimum distance from $[a, b]$ on the x -axis to the graph of f .



Proof

In d) we showed that $g(x) = \min_z D_x(z)$ is continuous in \mathbb{R} .

Therefore it is cont. in $[a, b]$. Hence it takes a min in $[a, b]$.

20.

a)

 f cont. on $[0,1]$

$$f(0) = f(1)$$

 $\forall n \in \mathbb{N}$

$$\exists x, f(x) = f(x + \frac{1}{n})$$

Proof

 $\forall n \in \mathbb{N}$.

$$\text{let } g(x) = f(x) - f(x + \frac{1}{n})$$

 $\text{Assume } \forall x \in \mathbb{R}, g(x) \neq 0$ $\text{Then either } \forall x, f(x) > f(x + \frac{1}{n}) \text{ or } \forall x, f(x) < f(x + \frac{1}{n})$ Case 1: $\forall x, f(x) > f(x + \frac{1}{n})$

$$f(0) > f(1/n) > f(2/n) > \dots > f(1)$$

⊥

Case 2: $\forall x, f(x) < f(x + \frac{1}{n})$

$$f(0) < f(1/n) < f(2/n) < \dots < f(1)$$

⊥

Therefore ⊥.

Therefore $\exists x, g(x) = 0$.I.e. $\exists x, f(x) = f(x + \frac{1}{n})$ Therefore $\forall n \in \mathbb{N} \exists x \in \mathbb{R} f(x) = f(x + \frac{1}{n})$

b)

$$0 < a < 1$$

$$\forall n \in \mathbb{N}, a \neq \frac{1}{n}$$

 f cont. on $[0,1]$, $f(0) = f(1)$, but does not satisfy $f(x) = f(x + a)$ for any x .

$$\frac{1}{n+1} < a < \frac{1}{n} \quad (1)$$

Define f on $[0,a]$, subject only to following conditions

$$f(0) = 0$$

$$f(a) > 0$$

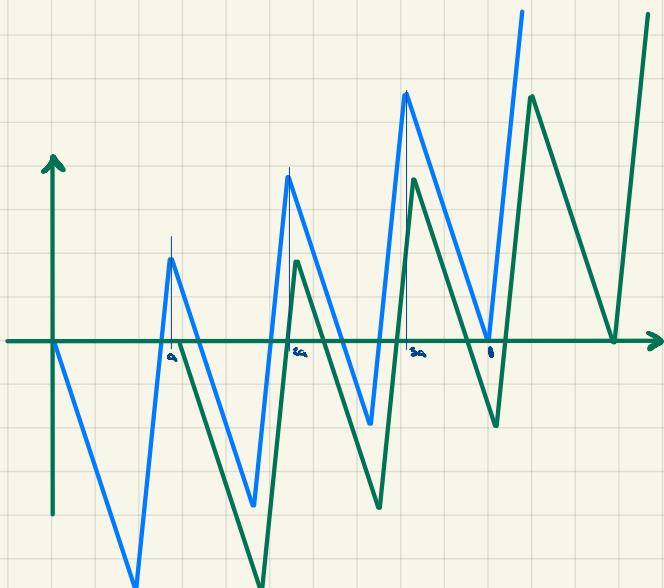
$$f(1-na) = -nf(a)$$

Because of (1),

$$\frac{n}{n+1} < na < 1$$

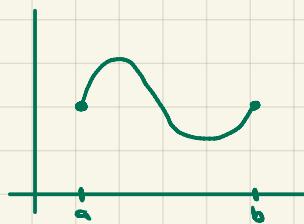
$$1 < 1-na < \frac{n}{n+1}$$

$$0 < z < 1-na < \frac{1}{n+1} < a$$

we see that the numbers 0, 1-na, and a are distinct. Therefore we can define f as above.Then define f on $[ka, (k+1)a]$ by $f(ka+x) = f(x) + ka$.

21.

a) Prove:

 $\exists f$ s.t. f defined on $[a,b]$ cont., and f takes on every value twice.

Hint:

Given f continuous on $[a,b]$, if f takes each value exactly twice thenif $a < b$ and $f(a) = f(b)$ then either

1) $\forall x \in (a,b) f(x) > f(a)$ and $\forall x, x < a \text{ or } x > b \rightarrow f(x) < f(a)$

or

2) $\forall x \in (a,b) f(x) < f(a)$ and $\forall x, x < a \text{ or } x > b \rightarrow f(x) > f(a)$

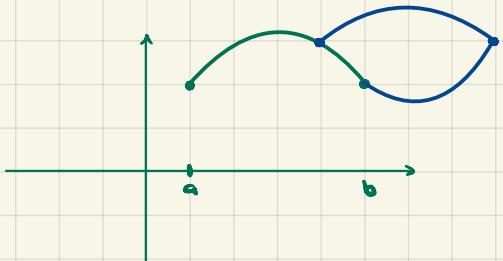
Proof of Hint

 $f(a) = f(b)$, so f takes on the value $f(a)$ twice.If for some $x,y \in (a,b)$ we have $f(x) < f(a) < f(y)$ or $f(y) < f(a) < f(x)$, then $\exists z \in (a,b)$ s.t. $f(z) = f(a)$. But then f takes on value $f(a)$ more than once.Therefore $\forall x \in [a,b]$ either $f(x) > f(a)$ or $f(x) < f(a)$.

Note also that

 $f(x) > f(a)$ in (a,b) then $f(x) < f(a)$ for $x < a$ or $x > b$,
and $f(x) < f(a)$ in (a,b) then $f(x) > f(a)$ for $x < a$ or $x > b$

Main Proof

Assume f takes on every value exactly twice.let $f(a) = f(b)$.Case 1: $\forall x \in (a,b) f(x) > f(a)$ Then f has a max in \mathbb{R} . Therefore f doesn't take on values larger than the max. \perp Case 2: $\forall x \in (a,b) f(x) < f(a)$ Then f has a min in \mathbb{R} . Therefore f doesn't take on values smaller than the min. \perp Therefore f does not take on every value twice.

b) $\exists f$ s.t. f cont. and takes on each value either 0 or 2 times.

Proof:

Assume for proof by contrd. that $\exists f$ s.t. f cont. and takes on each value either 0 or 2 times.

Let $x_1 \in \mathbb{R}$. Then there is a value $f(x_1)$, and so $\exists x_2 \in \mathbb{R}$ s.t. $f(x_2) = f(x_1)$.

Since f cont. in \mathbb{R} it is cont. in $[x_1, x_2]$.

As shown in a), either $\forall x, x \in [x_1, x_2] \rightarrow f(x) > f(x_1)$ or $\forall x, x \in [x_1, x_2] \rightarrow f(x) < f(x_1)$.

Case 1: $\forall x, x \in (x_1, x_2) \rightarrow f(x) > f(x_1)$

Then $\forall x, x < x_1$ or $x > x_2 \rightarrow f(x) < f(x_1)$

Also, $\exists x_{\max} \in [x_1, x_2]$ s.t. $\forall x \in [x_1, x_2], f(x) \leq f(x_{\max})$

Assume $\exists x_{\max} \in [x_1, x_2]$ s.t. $f(x_{\max}) = f(x_{\min})$.

Then either: $\forall x, x \in [x_{\max}, x_{\min}] \rightarrow f(x) > f(x_{\max})$

or $\forall x, x \in [x_{\max}, x_{\min}] \rightarrow f(x) < f(x_{\max})$

Either case leads to \perp .

Thus $\exists x_{\max} \in [x_1, x_2]$ s.t. $f(x_{\max}) = f(x_{\min})$.

That is, there is only one value x_{\max} s.t. $\forall x \in [x_1, x_2], f(x) \leq f(x_{\max})$.

In fact, $\forall x \in \mathbb{R}, f(x) \leq f(x_{\max})$.

Thus f takes on value $f(x_{\max})$ only once.

\perp .

Case 2: $\forall x, x \in (x_1, x_2) \rightarrow f(x) < f(x_1)$

Analogous to case 1 using min. in $[x_1, x_2]$

\perp

Therefore \perp in both possible cases

Therefore, $\exists f$ s.t. f cont. and takes on each value either 0 or 2 times.

Then $\forall x \in \mathbb{R}$ s.t. $x < a$ or $x > b, f(x) \neq f(a)$.

Note that if two values $x_1, x_2 \in \mathbb{R}$ are both smaller than a , then $f(x_1) \neq f(x_2)$. Same result is true if x_1, x_2 are both larger than b .

Thus, for $x_1, x_2 \in \mathbb{R}$ s.t. $f(x_1) = f(x_2) \neq f(a) = f(b)$, it must be that $x_1 < a$ and $x_2 > b$ or $x_1 < a$ and $x_2 > b$.

Proof

Assume $x_1, x_2 \in \mathbb{R}$ s.t. $x_1 < x_2 < a$ and $f(x_1) = f(x_2)$.

Case 1.1 $\forall x \in (x_1, x_2) f(x) > f(x_1)$

$\forall x \in (x_2, a], f$ takes on all values in $[f(x_2), f(a)]$.

Let $x_3 \in (x_1, x_2), f(x_3) > f(x_1)$

f takes on value $f(x_3)$ twice in (x_1, x_2) and on additional time in $(x_2, a]$. \perp

Case 1.2 $\forall x \in (x_1, x_2) f(x) < f(x_1)$

Then $\forall x, x > x_2 \text{ or } x < x_1 \rightarrow f(x) > f(x_1)$

Let m be the max of f on $[x_1, x_2]$.

Then $\forall x, f(x) \geq m$.

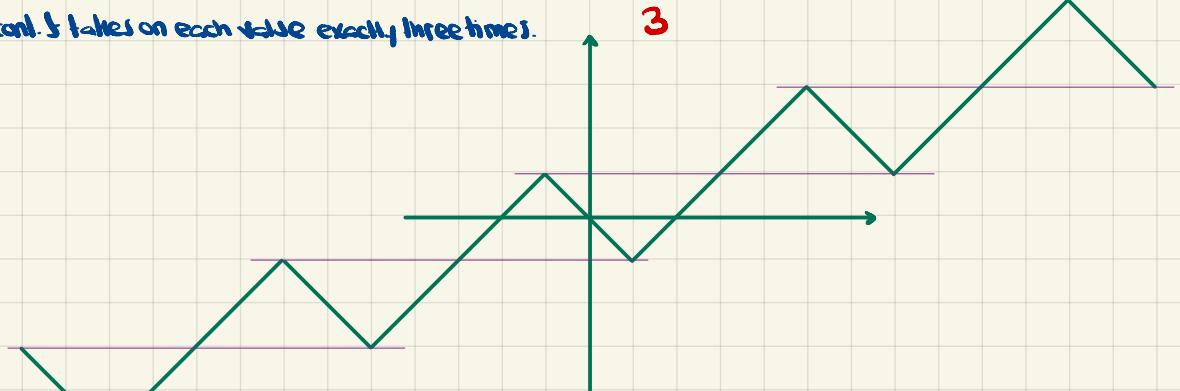
i.e. f doesn't take on values smaller than m .

\perp

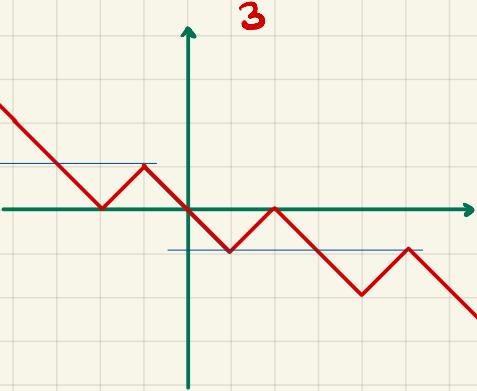
Hence, $x_1 < x_2 < a \rightarrow f(x_1) \neq f(x_2)$.

c) cont. It takes on each value exactly three times.

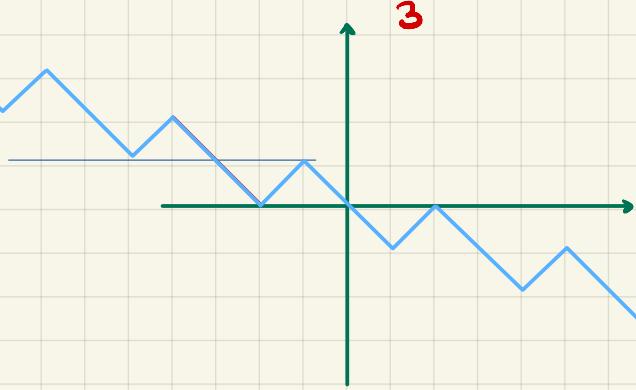
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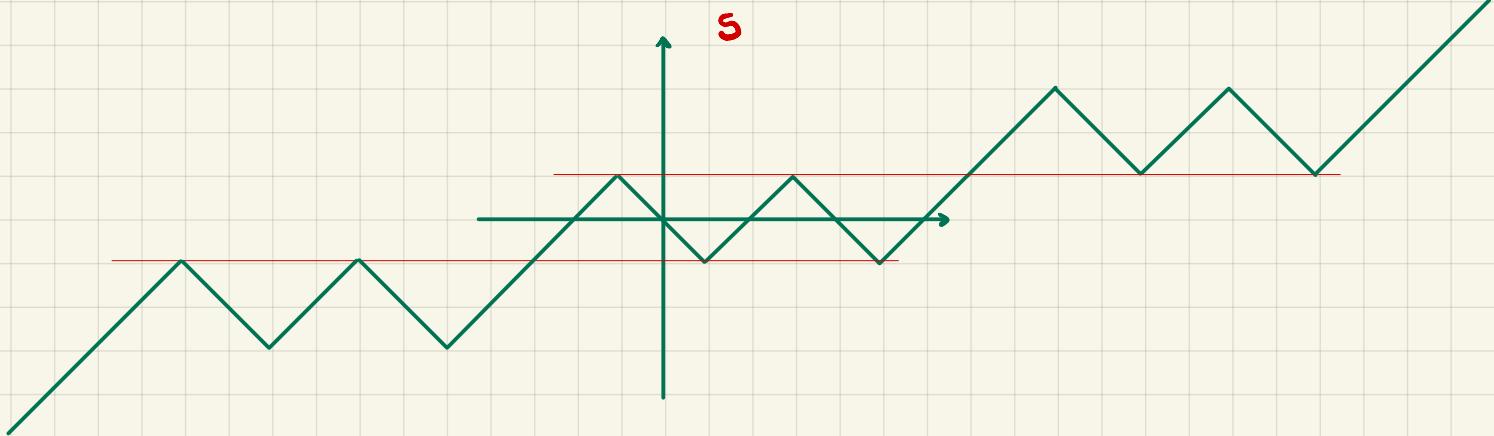
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3



5



d) n even $\rightarrow \exists f$ s.t. f takes on each value n times.

Assume n is even.

In a) we proved the case $n=2$.

Let $n=4$. Let x_1, x_2, x_3, x_4 s.t. $f(x_1)=f(x_2)=f(x_3)=f(x_4)$

Note that $\forall x \in \mathbb{R}, x+x_1 = x+x_2 = x+x_3 = x+x_4 \rightarrow f(x) \neq f(x_i)$.

Let y_1, y_2, y_3, y_4 s.t.

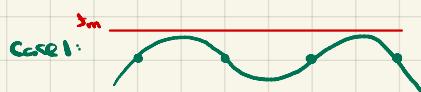
Assume $f(y_1) < f(x_i) < f(y_2)$

Then $\exists y \in (y_1, y_2), f(y) = f(x_i)$. L.

* Or also do it in L if we assume $f(y_1) > f(x_i) > f(y_2)$.

Therefore $\forall y_1, y_2 \in [x_i, x_{i+1}] \rightarrow f(y_1) > f(y_2)$ or $\forall y_1, y_2 \in [x_i, x_{i+1}] \rightarrow f(y_1) < f(y_2)$

For each of $(x_i, x_{i+1}), (x_2, x_3), (x_3, x_4)$, if f takes on n values in an interval it takes on the same value in that interval except at maximum or minimum. Therefore, relative to $f(x_i)$, the sign of f can be the same in at most two of such intervals. Hence we have two possible cases



Note that $x < x_i$ or $x > x_4 \rightarrow f(x) \neq f(x_i)$.

Therefore f doesn't take on values above $f(x_i)$. L.



Note that $x < x_i$ or $x > x_4 \rightarrow f(x) \neq f(x_i)$.

Therefore, f doesn't take on values below $f(x_i)$. L.

n even $\rightarrow \exists f$ s.t. f takes on every value n times.

general case

$$x_1 \quad x_2 \quad x_3 \quad \dots \quad x_n$$

$$f(x_1) = f(x_2) = \dots = f(x_n)$$

for any intervals (x_i, x_j) , $i, j \in \{1, 2, \dots, n\}$, $\forall x, x \in (x_i, x_j) \rightarrow f(x) > f(x_i)$ or $\forall x, x \in (x_i, x_j) \rightarrow f(x) < f(x_i)$

for any value f taken on in (x_i, x_j) it takes it on twice, except maximum or minimum (and there must be max or min because $\forall x, x \in (x_i, x_j) \rightarrow f(x) \neq f(x_i) = f(x_j)$, so either the max on $[x_i, x_j]$ is the max on (x_i, x_j) or the min on (x_i, x_j) is the min on $[x_i, x_j]$).
Therefore $\frac{n}{2}$ intervals have $f > f(x_i)$, $n-1-\frac{n}{2}$ have $f < f(x_i)$, or vice-versa.

$\forall x \in x_i$ or $x > x_i \rightarrow f(x) < f(x_i)$ in the first case, $f(x) > f(x_i)$ in the second.

In the first case, f is bounded above, in the second case it's bounded below. L both cases.

n even $\rightarrow \exists f$ s.t. f takes on every value n times.