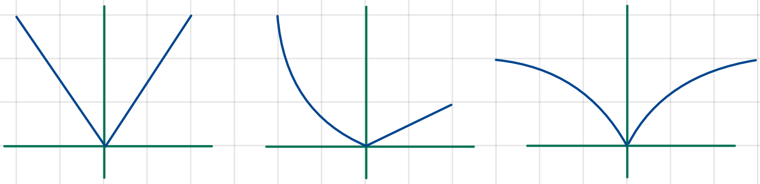
Ch 9 - Derivatives

"The most interesting and powerful results about functions will be obtained only when we restrict our attention even further, to fns which have even greater claim to be called "reasonable", which are even better behaved than most continuous fns."

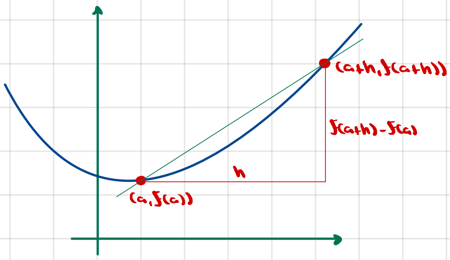
misbehaved cont. fns



The graphs are "bent" at (0,0).

We want to define tangent lines. We start w/ the notion of "secant lines."

Consider two distinct points $(a, f(a))$ and $(a+h, f(a+h))$.



The slope of the secant line is $\dfrac{f(a+h)-f(a)}{h}$

The "tangent line" at $(a, f(a))$ seems to be the limit, in some sense of the secant line as h approaches 0.
This "sense" is that of the limit of the slope of the secant.

Definition fn f is differentiable at a if

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \text{ exists.}$$

In this case the limit is denoted $f'(a)$ and is called the **derivative of f at a**.

Also, f is differentiable if f is differentiable at every point in its domain.

Definition: Tangent line to graph of f at $(a, f(a))$ is the line through $(a, f(a))$ with slope $f'(a)$.

Therefore the tangent line at $(a, f(a))$ is defined only if f diff. at a.

f' is a fn, called the derivative of f.

A few comments on notation

Leibniz notation: $\frac{df(x)}{dx}$, denotes the derivative of f.

but is ambiguous.

Does $\frac{df(x)}{dx}$ denote $f'(x)$ or f', ie is it a number or a function.

to remove ambiguity: $\left.\frac{df(x)}{dx}\right|_{x=a}$

Example

$$f(x) = \begin{cases} x^2 & x \leq 0 \\ x & x > 0 \end{cases}$$

$$\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^-} \frac{a^2+2ah+h^2-a^2}{h} = \lim_{h\to 0^-}(2a+h)$$

$$= 2a$$

$$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^+} \frac{a+h-a}{h} = \lim_{h\to 0^+} 1 = 1$$

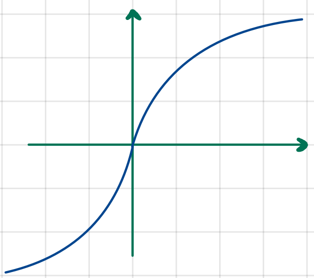
Therefore $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ does not exist.

Hence f is not differentiable at 0

It is diff. at any $x \neq 0$.

But note the the rh and lh limits do exist.

Example



$f(x) = \sqrt[3]{x}$

$$\frac{f(0+h)-f(0)}{h}$$

$$= \frac{\sqrt[3]{h}}{h} = \frac{1}{\sqrt[3]{h^2}}$$

$$f'(0) = \lim_{h\to 0} \frac{1}{\sqrt[3]{h^2}} = \infty$$

tangent line vertical.

Theorem 1: f differentiable at a $\to$ f cont. at a

Proof

$$\lim_{x\to a} f(x) = f(a) \leftrightarrow \lim_{h\to 0} f(a+h) = f(a)$$

$$\lim_{h\to 0} f(a+h) - f(a)$$

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} h$$

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \cdot \lim_{h\to 0} h$$

$$= f'(a) \cdot 0$$

$$= 0$$

Hence $\lim_{x\to a} f(x) = f(a)$, ie f cont. at a.

Definition $(f')'$, aka f'' is called the second derivative of f. If $f''(a)$ exists then f is 2-times differentiable at a.

Continuity is a feature that a fn may have.
Differentiability is a more restrictive feature.
Being twice diff. is more restrictive still.