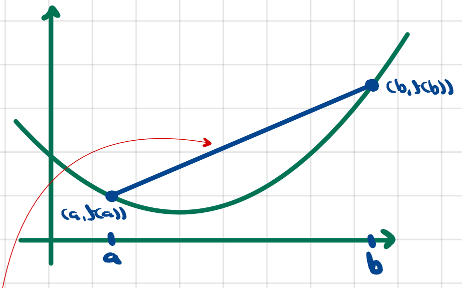
Ch. 11 Appendix - Convexity and Concavity

Definition: A fn f is convex on an interval, if for all a and b in the interval, the line segment joining $(a, f(a))$ and $(b, f(b))$ lies above the graph of f.



$$g(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a), \quad x \in [a,b]$$

This line segment lies above the graph of f at x if $g(x) > f(x)$.

$$\frac{f(b)-f(a)}{b-a}(x-a) + f(a) > f(x)$$

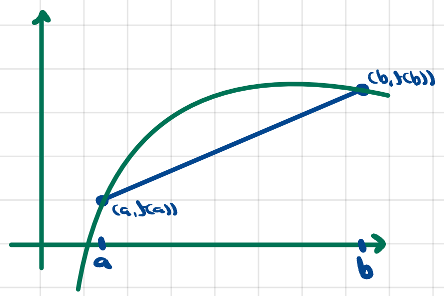
$$\rightarrow \frac{f(b)-f(a)}{b-a}(x-a) > f(x) - f(a)$$

$$\rightarrow \frac{f(b)-f(a)}{b-a} > \frac{f(x)-f(a)}{x-a}$$

Definition A fn is convex on an interval if for a, x, and b in the interval w/ $a < x < b$ we have

$$\frac{f(x)-f(a)}{x-a} < \frac{f(b)-f(a)}{b-a}$$

The definition of a concave fn is analogous



$$\frac{f(b)-f(a)}{b-a}(x-a) + f(a) < f(x)$$

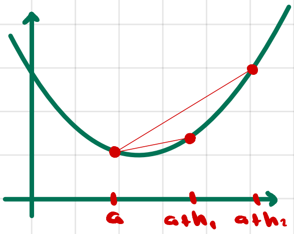
$$\frac{f(b)-f(a)}{b-a}(x-a) < f(x) - f(a)$$

$$\rightarrow \frac{f(b)-f(a)}{b-a} < \frac{f(x)-f(a)}{x-a}$$

Note that if f convex then $-f$ concave.

Theorem 1 Let f be convex. If f diff at a, then the graph of f lies above the tangent line through $(a, f(a))$, except at $(a, f(a))$.
If $a<b$ and f diff at a and b then $f'(a)<f'(b)$.

Proof



Let $0<h_1<h_2$

Since f convex, by def. of convexity on $[a, a+h_2]$

$$\frac{f(a+h_1)-f(a)}{h_1} < \frac{f(a+h_2)-f(a)}{h_2}$$

Therefore $\dfrac{f(a+h)-f(a)}{h}$ decreases as $h \to 0^+$

Therefore $f'(a) < \dfrac{f(a+h)-f(a)}{h}$ for $h>0$.

This means that for $h>0$ the secant line through $(a, f(a))$ and $(a+h, f(a+h))$ lies above the tangent line.

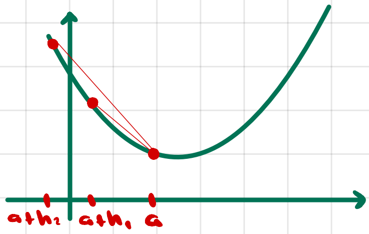
$$g(x) = \frac{f(a+h)-f(a)}{h}(x-a)+f(a)$$

$$t(x) = f'(a)(x-a)+f(a)$$

$$g(a+h) = f(a+h)-f(a)+f(a) = f(a+h)$$

$$t(a+h) = f'(a)h+f(a) < \frac{f(a+h)-f(a)}{h}h+f(a) = f(a+h)$$

ie $t(a+h) < f(a+h)$



Thus, to the right of a the graph of f lies above the tangent.

Now consider $h<0$.

Recall the def. of convexity for $a<x<b$:

$$\frac{f(x)-f(a)}{x-a} < \frac{f(b)-f(a)}{b-a} \iff f(x)-f(a) < \frac{x-a}{b-a}(f(b)-f(a)) \iff f(x) < f(a) + \frac{x-a}{b-a}(f(b)-f(a))$$

$$\iff -f(x) > -f(a) - \frac{x-a}{b-a}(f(b)-f(a))$$

$$\iff f(b)-f(x) > f(b)-f(a) - \frac{x-a}{b-a}(f(b)-f(a)) = (f(b)-f(a))\left(1-\frac{x-a}{b-a}\right) = (f(b)-f(a))\frac{b-x}{b-a}$$

$$\iff \frac{f(b)-f(x)}{b-x} > \frac{f(b)-f(a)}{b-a}$$

Thus, since $a+h_2 < a+h_1 < a$

$$\frac{f(a)-f(a+h_1)}{-h_1} > \frac{f(a)-f(a+h_2)}{-h_2} \iff \frac{f(a+h_1)-f(a)}{h_1} > \frac{f(a+h_2)-f(a)}{h_2}$$

Therefore the slope grows as $h \to 0^-$, and $f'(a) > \dfrac{f(a+h)-f(a)}{h}$ for any $h<0$

This means the secant through $(a+h, f(a+h))$ and $(a, f(a))$ lies above the tangent line. The calculations to show this are the same as before

$$g(x) = \frac{f(a+h)-f(a)}{h}(x-a)+f(a)$$

$$t(x) = f'(a)(x-a)+f(a)$$

$$g(a+h) = f(a+h)-f(a)+f(a) = f(a+h)$$

$$t(a+h) = f'(a)h+f(a)$$

if $f'(a) > \frac{f(a+h)-f(a)}{h} > 0$, then since $h<0$, $f'(a)h < \frac{f(a+h)-f(a)}{h}h$

if $\frac{f(a+h)-f(a)}{h} < f'(a) < 0$, then since $h<0$, $f'(a)h < \frac{f(a+h)-f(a)}{h}h$

Therefore, $t(a+h) = f'(a)h+f(a) < \frac{f(a+h)-f(a)}{h}h+f(a) = f(a+h)$

Hence, the graph of f lies above the tangent line, except at $(a, f(a))$. This is the first part of the theorem.

Now suppose $a<b$.

Then, according to the first part of the theorem

$$f'(a) < \frac{f(a+h)-f(a)}{h}$$

in particular

$$f'(a) < \frac{f(a+(b-a))-f(a)}{b-a} = \frac{f(b)-f(a)}{b-a}$$

Also

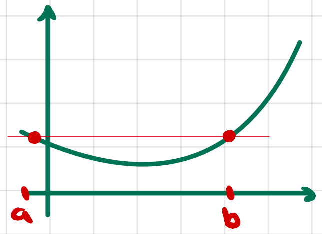
$$f'(b) > \frac{f(b+(a-b))-f(b)}{a-b} \quad (\text{since } a-b<0)$$

$$= \frac{f(b)-f(a)}{b-a}$$

Hence $f'(a) < f'(b)$

Lemma: Suppose f diff and f' increasing.

If $a<b$ and $f(a)=f(b)$, then $f(x)<f(a)=f(b)$ for $a<x<b$.



Proof

Suppose $f(x)\geq f(a)=f(b)$ for some x in (a,b).

Then the max of f on $[a,b]$ occurs at some $x_0\in(a,b)$ with $f(x_0)\geq f(a)$ and $f'(x_0)=0$.

Apply MVT to $[a,x_0]$.

$$\exists x_1,\ a<x_1<x_0 \wedge f'(x_1)=\frac{f(x_0)-f(a)}{x_0-a}\geq 0=f'(x_0)$$

$\perp$, because f' increasing by assumption.

Therefore $\forall x,\ x\in(a,b)\to f(x)<f(a)=f(b)$

Theorem 2 f diff. f' increasing $\to$ f convex

Proof

Let $a<b$.

$$\text{Let } g(x)=f(x)-\frac{f(b)-f(a)}{b-a}(x-a)$$

$$g'(x)=f'(x)-\frac{f(b)-f(a)}{b-a},$$ which is increasing.

$g(a)=g(b)=f(a)$

Apply lemma to g.

$$\forall x,\ x\in(a,b)\to g(x)<g(a)=g(b)$$

$$\to f(x)-\frac{f(b)-f(a)}{b-a}(x-a)<f(a)$$

$$\to \frac{f(x)-f(a)}{x-a}<\frac{f(b)-f(a)}{b-a}$$

Hence f convex.

Theorem 3 f diff
f lies above each tangent line except at point of contact $\to f$ convex

Proof

Let $a<b$.

Tangent line at $(a, f(a))$: $g(x) = f'(a)(x-a) + f(a)$

$(b, f(b))$ lies above the tangent line: $f(b) > f'(a)(b-a) + f(a)$

$f(b) - f(a) > f'(a)(x-a)$ (1)

Tangent line at $(b, f(b))$: $h(x) = f'(b)(x-b) + f(b)$

$(a, f(a))$ lies above: $f(a) > f'(b)(a-b) + f(b)$

$f(b) - f(a) < f'(b)(b-a)$ (2)

Hence

$f'(a)(x-a) < f'(b)(x-a)$

$\to f'(a) < f'(b)$

By Th. 2, f is convex.

Inflection Point

A number a is an inflection point of f if the tangent line to the graph of f at $(a, f(a))$ crosses the graph.

f'' has different signs to the left and to the right of a.

$x=0$ on $f(x) = x^4$ is not an inflection point.

Theorem 4 f diff on an interval and intersects each of its tangent lines just once $\to$ f is either convex or concave on that interval

Proof

Two parts.

(1) no straight line can intersect the graph of f in three different points.

Assume some straight line did intersect graph of f at $(a, f(a))$, $(b, f(b))$, and $(c, f(c))$, w/ $a<b<c$.

Then we'd have

$$\frac{f(b)-f(a)}{b-a} = \frac{f(c)-f(b)}{c-b}$$

Let $g(x) = \dfrac{f(x)-f(a)}{x-a}$ $\quad x \in [b,c]$

Then $g(b) = g(c)$

Rolle's Thm $\to \exists x,\ x \in (b,c) \wedge g'(x) = 0$

$$g'(x) = \frac{(x-a)f'(x) - (f(x)-f(a))}{(x-a)^2} = 0$$

$$\to f'(x) = \frac{f(x)-f(a)}{x-a}$$

ie the tangent has slope equal to the slope between $(a, f(a))$ and $(x, f(x))$. Thus the tangent passes through $(x, f(x))$ and $(a, f(a))$, contradicting our assumption that f intersects each tangent line only once.

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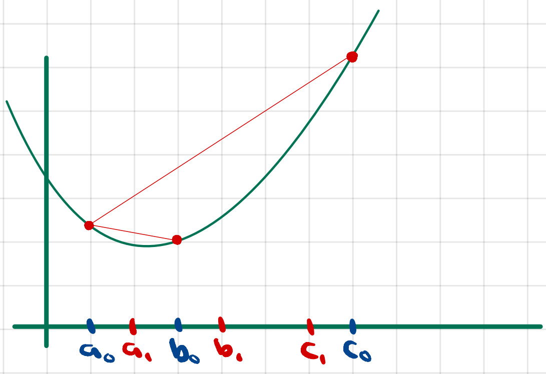
(2)

Suppose $a_0 < b_0 < c_0$ and $a_1 < b_1 < c_1$ are points in the interval.



Let
$$x_t = (1-t)a_0 + ta_1$$
$$y_t = (1-t)b_0 + tb_1 \qquad 0 \le t \le 1$$
$$z_t = (1-t)c_0 + tc_1$$

Then $x_0 = a_0$, $x_1 = a_1$, x_t points lie between a_0 and a_1.

Analogous statements for y_t and z_t.

Moreover, $x_t < y_t < z_t \quad 0 \le t \le 1$

$$\text{Let } g(t) = \frac{f(y_t)-f(x_t)}{y_t - x_t} - \frac{f(z_t)-f(x_t)}{z_t - x_t} \qquad 0 \le t \le 1$$

Assume $g(t) = 0$

$\to f(y_t) - f(x_t) = f(z_t) - f(x_t) = 0$

$\to f(x_t) = f(y_t) = f(z_t)$

$\to$ a straight line $y = f(x_t)$ intersects f in three places.

$\perp$ with part (1).

Hence $g(t) \ne 0$ for all $t \in [0,1]$. Ie, $\dfrac{f(y_t)-f(x_t)}{y_t - x_t} \ne \dfrac{f(z_t)-f(x_t)}{z_t - x_t}$.

Therefore either $g(1) > 0$ or $g(1) < 0$, ie either f convex or concave.