Ch. 18 - The Logarithm and Exponential Functions

$f(x) = 10^x$

assumed defined for all x and to have inverse defined for positive x, $f^{-1}(x) = \log_{10} x$

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in algebra, defined only for rational x, but not irrational x

we want $f(x+y) = f(x)f(y)$ to be true.

let $f(1) = a$.

Case 1: $x \in \mathbb{N}$

$$f(x) = f(\underbrace{1+1+\cdots+1}_{x \text{ times}}) = \underbrace{f(1)\cdot f(1)\cdot \cdots \cdot f(1)}_{x \text{ times}} = a^x$$

$$\to f(n+m) = f(n)\cdot f(m) = a^n \cdot a^m = a^{n+m}$$

Case 2: $x = 0$

$0 \notin \mathbb{N}$. $f(0+n) = f(0)\cdot f(n) = a^n \cdot a^0 = a^n$

if a^0 is defined 1.

Case 3: $x = -n$, $n \in \mathbb{N} \cup \{0\}$

$f(-n+n) = f(-n)\cdot f(n) = a^{-n}\cdot a^n = a^0$

if a^{-n} defined $\frac{1}{a^n}$

Case 4: $x = \frac{1}{n}$, $n \in \mathbb{Z} - \{0\}$

$f(\sum_{i=1}^{n} 1/n) = f(1) = a = \sum f(1/n) = [f(1/n)]^n \quad \to f(1/n) = a^{1/n}$

Case 5: $x = m/n$

$$f(\sum_{i=1}^{m} 1/n) = \sum_{i=1}^{m} f(1/n) = [f(1/n)]^m = a^{m/n}$$

Thus, for all $x \in \mathbb{Q}$, $f(x) = a^x$

The idea in the definitions above is that $10^x \cdot 10^y = 10^{x+y}$ based on the idea that $10^{x+y} = 10^x 10^y$

What if x is irrational? The idea above doesn't help to define 10^x for irr. x.

Find diff. f s.t. $f(x+y) = f(x)\cdot f(y)$ $\forall x \forall y$ and $f(1) = 10$.

Assume such a fn exists

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(x)\cdot f(h) - f(x)}{h} = f(x) \lim_{h\to 0} \frac{f(h)-1}{h}$$

$f'(0) = \lim_{h\to 0} \frac{f(h)-1}{h}$. Assume this exists. Call it α. Then $f'(x) = \alpha f(x)$ for all x

This is a consequence of assuming that $f(x+y) = f(x)f(y)$

But $f^{-1}(x) = \log_{10} x$ so

↳ if f even exists

$$(f^{-1})'(x) = \log_{10}' x = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\alpha f(f^{-1}(x))} = \frac{1}{\alpha x}$$

Consider $\int_a^b x^{-1}\,dx$. The deriv. above means that if $f(x) = 10^x$ and $f^{-1}(x) = \log_{10} x$, then $(f^{-1})'(x) = \log_{10}' x = \frac{1}{f'(0)x}$

Hence, $\log_{10} 1 = 0$ so $\frac{1}{\alpha}\int_1^x \frac{1}{t}\,dt = \log_{10} x - \log_{10} 1 = \log_{10} x$

Note that $\alpha = f'(0)$ is unknown.

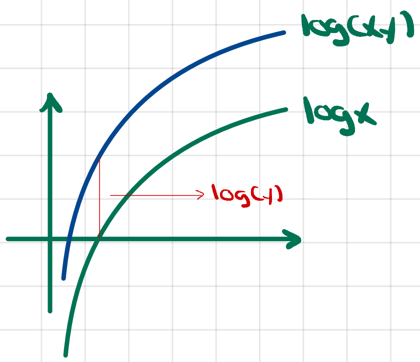
At this point, we straight up define something that we call $\log(x)$ as $\int_1^x \frac{1}{t}\,dt$, hoping that it turns out to be a logarithm of some base.

Definition: if $x > 0$, then $\log x = \int_1^x \frac{1}{t}\,dt$

Theorem 1 if $x, y > 0$, then $\log(xy) = \log(x) + \log(y)$

Proof

$\log(x) = \int_1^x \frac{1}{t}\,dt \to \log'(x) = \frac{1}{x}$ by FTC1



let $y > 0$ be a number and let

$f(x) = \log(xy)$

Then

$f'(x) = \frac{1}{xy}\cdot y = \frac{1}{x} \quad \to f' = \log' \quad \forall x > 0$

$\to \exists c : f(x) = \log(x) + c \to \log(xy) = \log(x) + c \quad \forall x > 0$

$x = 1 \to \log(y) = c$

$\to \log(xy) = \log(x) + \log(y) \quad \forall x > 0$

Since this is true $\forall y > 0$ then

$\forall x \forall y,\ x > 0 \wedge y > 0 \to \log(xy) = \log(x) + \log(y)$

Corollary 1: $n \in \mathbb{N}, x>0 \rightarrow \log(x^n) = n\log(x)$

Proof:

$n=1 \quad \log(x) = 1\cdot\log(x)$

Assume $\log(x^k) = k\log(x)$

$\log(x^{k+1}) = \log(x x^k) = \log(x) + \log(x^k) = \log x + k\log x = (k+1)\log x$

$\blacksquare$

Corollary 2: $x,y>0$, then $\log\left(\frac{x}{y}\right) = \log x - \log y$

Proof

$\log(x) = \log\left(\frac{x}{y}\cdot y\right) = \log\left(\frac{x}{y}\right) + \log(y)$

$\rightarrow \log(x/y) = \log(x) - \log(y)$

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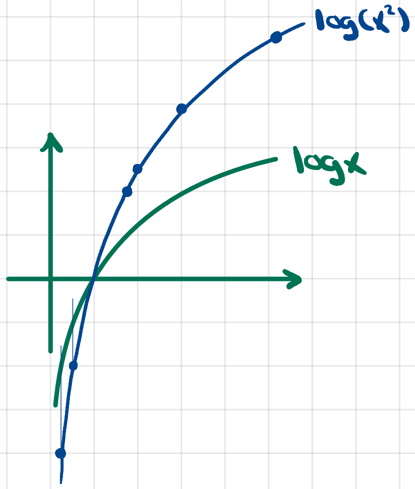
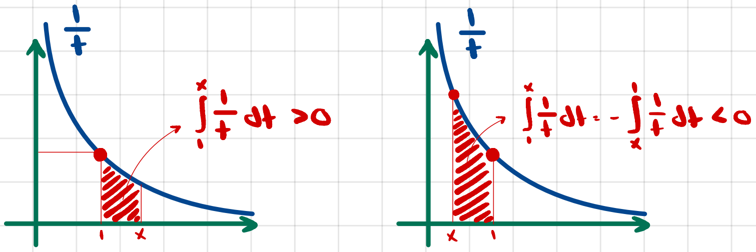
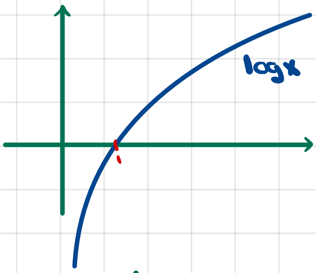


Note

Since $\log(x) = \int_1^x \frac{1}{t}\,dt$, we can intuitively see



so the graph of log x looks something like



$\log'(x) = \frac{1}{x} > 0$ for $x>0$ $\rightarrow$ $\log x$ increasing (one-one)

$\log(2^n) = n\log 2 \rightarrow \lim_{x\to\infty} \log x = \infty$

$\log\left(\frac{1}{2^n}\right) = n\log(1/2) \rightarrow \lim_{x\to 0^+} \log(x) = -\infty$

$\log x$ is cont since $\frac{1}{x}$ is integrable on (a,b), $a>0$.

IVT $\rightarrow$ log takes on all values in $\mathbb{R}$.

$\rightarrow \log^{-1}$ has domain $\mathbb{R}$

Definition: the exponential function exp, is defined as $\log^{-1}$.

Theorem 2: For all x, $\exp'(x) = \exp(x)$

Proof

$$(\exp)'(x) = (\log^{-1})'(x) = \frac{1}{\log'(\log^{-1}(x))} = \frac{1}{\frac{1}{\log^{-1}x}} = \log^{-1}x = \exp(x)$$

Theorem 3 x, y any two numbers, then

$$\exp(x+y) = \exp(x)\cdot\exp(y)$$

Proof

$x' = \exp(x)$
$y' = \exp(y)$

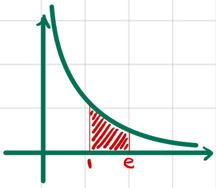
$\log x' = x$
$\log y' = y$

$x+y = \log x' + \log y' = \log x'y'$

$\exp(x+y) = x'y' = \exp(x)\cdot\exp(y)$

Definition: $e = \exp(1)$

equivalent to $1 = \log e = \int_1^e \frac{1}{t}\,dt$



From our previous analysis, we know that

$\exp(x+y) = \exp(x)\exp(y) \rightarrow \exp(x) = \exp(1+1+\cdots+1) = [\exp 1]^x = e^x$, for all rational x

What is e^x for x irrational? We define e^x for such values of x as $\exp(x) = \log^{-1}(x)$

Definition For any number x, $e^x = \exp(x) = \log^{-1}x$

We have thus found a fn $\log^{-1}(x) = e^x$, where $e = \log^{-1}(1)$.

At this point, we've defined e^x for arbitrary x. But we haven't defined a^x for arbitrary x, if $a \neq e$.

We do know $a^x = (e^{\log a})^x = ([\log^{-1}1]^{\log a})^x = e^{x\log a}$, which is defined for all x, and we use this expr. to define a^x

Definition: If $a>0$, then for any real number x,

$a^x = e^{x\log a}$

$a>0$ is required because $\log a = \int_1^a \frac{1}{t}\,dt$

Theorem 4 If $a>0$ then

(1) $(a^b)^c = a^{bc}$ for all b, c

(2) $a^1 = a$

$a^{x+y} = a^x a^y$ for all x, y

Proof

(1) $(a^b)^c = e^{c\log a^b} = e^{c\log(e^{b\log a})} = e^{cb\log a} = a^{bc}$

(2) $a^1 = e^{1\log a} = e^{\log a} = a$

$a^{x+y} = e^{(x+y)\log a} = e^{x\log a}e^{y\log a} = a^x a^y$

$a>1 \to \log a = \int_1^a t^{-1}dt > 0$

$x<y \to x\log a < y\log a \to e^{x\log a} < e^{y\log a} \to a^x < a^y$

note that $\exp(x) = e^x = \log^{-1}(x)$

$(\log^{-1})'(x) = \log^{-1}(x)$, positive because $\log' x = \frac{1}{x} > 0$.

$\to a^x$ is increasing, one-one

$a=1 \to a^x = 1^x = 1$, constant fn

$0<a<1$

$x<y \to x\log a > y\log a$, since $\log a < 0$

$\to e^{x\log a} > e^{y\log a} \to a^x > a^y$

$\to a^x$ decreasing, one-one

If $a>0$ and $a \neq 1$, or $a>1$ then a^x is one-one. Let y be any positive number

$y = a^x = e^{x\log a} = \log^{-1}(x\log a)$

$\log y = x\log a$

$x = \dfrac{\log y}{\log a}$

Thus, a^x takes on all positive values

Thus the inverse of a^x has domain $\mathbb{R}$, and is usually denoted $\log_a$.

$y = \log_a x \rightarrow x = a^y = e^{y\log a} = \log^{-1}(y\log a)$

$\log y = y\log a \rightarrow y = \dfrac{\log x}{\log a} \rightarrow \log_a x = \dfrac{\log x}{\log a}$

$f(x) = a^x = e^{x\log a} \rightarrow f'(x) = e^{x\log a}\cdot\log a = a^x\log a$

$g(x) = \log_a x = \dfrac{\log x}{\log a} \rightarrow g'(x) = \dfrac{1}{\log a}\cdot\dfrac{1}{x}$

$f(x) = g(x)^{h(x)} = e^{h(x)\log g(x)} \rightarrow f'(x) = e^{h(x)\log g(x)}\cdot\left(h'(x)\log g(x) + h(x)\cdot\dfrac{g'(x)}{g(x)}\right)$

$\log_2 10^{10^{10}}$

$= 10^{10}\cdot\log_2 10$

$= 10^{10}\cdot\dfrac{\log 10}{\log 2}$

Theorem 5 f diff. and $f'(x) = f(x)$ for all x, then there is c s.t. $f(x) = ce^x$ for all x.

i.e. fns of the form ce^x are the only ones s.t. $f = f'$ for all x.

Proof

Let $g(x) = \dfrac{f(x)}{e^x}$ (recall that $e^x \neq 0$ for all x)

Then

$$g'(x) = \frac{e^x f(x) - f'(x)e^x}{e^{2x}} = \frac{f(x) - f'(x)}{e^x} = 0$$

$\rightarrow$ $g(x)$ is constant

$\exists c$ s.t. $g(x) = \dfrac{f(x)}{e^x} = c \rightarrow f(x) = ce^x$

Theorem 6 $n \in \mathbb{N} \rightarrow \lim\limits_{x \to \infty} \dfrac{e^x}{x^n} = \infty$

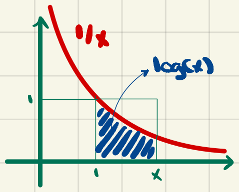
Proof

Step 1: $e^x > x$ for all x $\rightarrow \lim\limits_{x \to \infty} e^x = \infty$

if we prove $x > \log x$ for $x > 0$, then $\log^{-1} x = e^x > x$

$x < 1 \rightarrow \log x < 0 < x < 1$

$x > 1 \rightarrow$



$x - 1$ is an upper limit for $\int_1^x t^{-1}\,dt$ so $\log x < x - 1 < x$

Step 2 $\lim\limits_{x \to \infty} \dfrac{e^x}{x} = \infty$

$$\frac{e^x}{x} = \frac{e^{x/2} \cdot e^{x/2}}{\frac{x}{2} \cdot 2} = \frac{1}{2} \underbrace{\frac{e^{x/2}}{x/2}}_{>1 \text{ by step 1}} \cdot \overbrace{e^{x/2}}^{\lim\limits_{x \to \infty} e^{x/2} = \infty}$$

$\rightarrow \lim\limits_{x \to \infty} e^x/x = \infty$

Step 3 $\lim\limits_{x \to \infty} \dfrac{e^x}{x^n} = \infty$

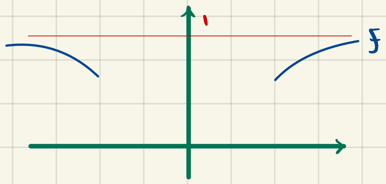
$$\frac{e^x}{x^n} = \frac{(e^{x/n})^n}{(\frac{x}{n})^n \cdot n^n} = \frac{1}{n^n} \underbrace{\left(\frac{e^{x/n}}{x/n}\right)^n}_{\text{becomes arbitrarily large}}$$

Let's investigate the fn

$f(x) = e^{-\frac{1}{x^2}} \quad x \neq 0$

$f'(x) = e^{-\frac{1}{x^2}}(-1)(-2)x^{-3} = \dfrac{2e^{-\frac{1}{x^2}}}{x^3}$

$-$ $\quad 0 \quad$ $+$ $\quad f'$



$|x|$ large $\to -\frac{1}{x^2}$ small $\to e^{-\frac{1}{x^2}}$ close to 1

x very small $\to \frac{1}{x^2}$ large $\to e^{-\frac{1}{x^2}} = \dfrac{1}{e^{\frac{1}{x^2}}} \approx 0$

$\to \lim\limits_{x\to 0} e^{-\frac{1}{x^2}} = 0$

Proof: If $\delta > 0$ then $|x| < \delta \to -\delta < x < \delta \to 0 < x^2 < \delta^2 \to 0 < \frac{1}{\delta^2} < \frac{1}{x^2} \to 1 < e^{1/\delta^2} < e^{1/x^2}$

$\to 0 < e^{-1/x^2} < e^{-1/\delta^2} < 1$

For any ϵ s.t. $0 < \epsilon < 1$, choose δ s.t. $\epsilon = \dfrac{1}{e^{1/\delta^2}}$.

This means $e^{1/\delta^2} = \epsilon^{-1} \to \frac{1}{\delta^2} = -\log \epsilon \to \delta = \dfrac{1}{\sqrt{-\log \epsilon}}$.

Then, $0 < e^{-1/x^2} < \epsilon$

Ie, for any $\epsilon > 0$,

if $\epsilon \geq 1$ then simply choose $\delta = \dfrac{1}{\sqrt{-\log \alpha}}$ for $0 < \alpha < 1$

then $0 < e^{-1/x^2} < \alpha < 1 < \epsilon$

if $0 < \epsilon < 1$ then choose $\delta = \dfrac{1}{\sqrt{-\log \epsilon}}$ and this results in $0 < e^{-1/x^2} < \epsilon$

Thus, we've shown

$\forall \epsilon > 0 \ \exists \delta > 0$ s.t. $\forall x \ |x| < \delta \to |e^{-1/x^2}| < \epsilon \to \lim\limits_{x \to 0} e^{-1/x^2} = 0$

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f is not defined at 0 but since $\lim\limits_{x\to 0} f(x) = 0$ then if we define

$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$

then f is cont. at 0.

Also,

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{e^{-\frac{1}{h^2}} - 0}{h} = \lim_{h\to 0} \frac{1}{e^{1/h^2}\cdot h} = \lim_{h\to 0} \frac{1/h}{e^{1/h^2}}$$

$$\lim_{h\to 0^+} \frac{1/h}{e^{1/h^2}} = \lim_{x\to\infty} \frac{x}{e^{x^2}}$$

$$\lim_{h\to 0^-} \frac{1/h}{e^{1/h^2}} = -\lim_{x\to\infty} \frac{x}{e^{x^2}}$$

From Th 6 we know that $\lim\limits_{x\to\infty} \frac{e^x}{x^n} = \infty$.

Therefore $\lim\limits_{x\to\infty} \frac{e^{x^2}}{x^n} = \infty$ and $\lim\limits_{x\to\infty} \frac{x^n}{e^x} = 0$

Thus

$$\lim_{h\to 0^+} \frac{1/h}{e^{1/h^2}} = \lim_{h\to 0^-} \frac{1/h}{e^{1/h^2}} = 0$$

and

$$f'(0) = 0$$

$$x \neq 0 \to f'(x) = e^{-1/x^2} \cdot \frac{2}{x^3}$$

Thus

$$f'(x) = \begin{cases} e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

What about f''?

$$f''(0) = \lim_{h\to 0} \frac{f'(h)-f'(0)}{h} = \lim_{h\to 0} \frac{e^{-\frac{1}{h^2}} \cdot \frac{2}{h^3}}{h} = \lim_{h\to 0} \frac{2}{h^4 e^{1/h^2}} = \lim_{h\to 0} \frac{2\cdot(1/h^4)}{e^{(1/h^2)}}$$

$$= \lim_{x\to\infty} \frac{2x^4}{e^{x^2}} = 0$$

$$f''(x) = \begin{cases} e^{-1/x^2} \cdot \frac{-6}{x^4} + e^{-1/x^2} \cdot \frac{4}{x^6}, & x \neq 0 \\ 0 & x = 0 \end{cases}$$

This process can be continued indefinitely.

$$f^{(n)}(0) = 0.$$

Recap

Given a number $a>0$, it is possible to define a fn $f(x)=a^x$ on $\mathbb{Q}$ s.t. $f(1)=a^1=a$ and

$$\forall x,y \in \mathbb{Q} \to f(x+y) = f(x)\cdot f(y) = a^{x+y} \quad \text{(A)}$$

Proof

$x \in \mathbb{Q} \to x$ is either

1) a natural number $1,2,3,\dots$
2) the number 0
3) $-n$ for $n \in \mathbb{N}$
4) $1/n$, $n \in \mathbb{Z}-\{0\}$
5) m/n, $m \in \mathbb{Z}$, $n \in \mathbb{Z}-\{0\}$

Let's define f as follows

defined previously for $n \in \mathbb{N}$

1) $n \in \mathbb{N}$, $f(n) = a^n = \underbrace{a\cdot a\cdot a \cdots a}_{n \text{ times}}$, by definition of a^n for $n \in \mathbb{N}$.

then, if $m \in \mathbb{N}$, $f(n+m) = a^{n+m} = \underbrace{a\cdot a \cdot \dots \cdot a}_{n+m \text{ times}} = a^n \cdot a^m$

$\to$ (A) holds

new definition

2) define $a^0 = 1$. Then

$$a = f(1) = a\cdot a^0 = f(1)\cdot f(0) = f(1+0)$$

$\to$ (A) holds

new definition

3) define $a^{-n} = \dfrac{1}{a^n}$, $n \in \mathbb{N}$. Then,

$$1 = f(0) = f(n+(-n)) = a^n \cdot \frac{1}{a^n} = f(n)\,f(-n)$$

$\to$ (A) holds

new definition

4) define $a^{1/n} = \sqrt[n]{a}$, $n \in \mathbb{N}-\{0\}$

$$f(1) = f\Big(\sum_{i=1}^{n} 1/n\Big) = a = \underbrace{a^{1/n}\cdot a^{1/n} \cdot \dots \cdot a^{1/n}}_{n \text{ times}} = \prod_{i=1}^{n} f(1/n)$$

$\to$ (A) holds.

new definition

5) define $a^{m/n} = \sqrt[m]{a^n}$, $n \in \mathbb{N}-\{0\}$, $M \in \mathbb{N}$. Then,

$$f(m/n) = f\Big(\sum_{i=1}^{m} 1/n\Big) = a^{m/n} = \underbrace{a^{1/n}\cdot a^{1/n} \cdot \dots \cdot a^{1/n}}_{m \text{ times}} = \prod_{i=1}^{m} f(1/n)$$

$\to$ (A) holds.

Thus, for all numbers $x \in \mathbb{Q}$ we have $f(x) = a^x$ and $\forall y \in \mathbb{Q}$, $f(x+y) = f(x)f(y)$. To achieve this all we did was define the symbol a^x when x is either $0, -n, 1/n$, or m/n.

On the previous page, we made some new definitions and showed that the result was (A) holding for all $\mathbb{Q}$.

Thus, we proved the existence of a fn s.t. (A) holds.

Now assume (A) holds, ie $\forall x,y \in \mathbb{Q} \rightarrow f(x+y) = f(x) \cdot f(y)$ (A), and $f(1) = a$.

1) $n \in \mathbb{N}$. $f(n) = f(\sum_{i=1}^{n} 1) = \prod_{i=1}^{n} f(1) = \prod_{i=1}^{n} a = a^n$

$\rightarrow f(n) = a^n$

2) $x = 0$. $f(1) = f(1) \cdot f(0) = a \cdot f(0) = a \quad \rightarrow f(0) = 1 = a^0$

3) $x = -n, n \in \mathbb{N}$. $1 = f(0) = f(-n+n) = f(n) \cdot f(-n) \rightarrow f(-n) = \dfrac{1}{f(n)} = \dfrac{1}{a^n} = a^{-n}$

4) $x = 1/n$, $n \in \mathbb{Z} - \{0\}$.

$f(1) = f(\sum_{i=1}^{n} 1/n) = \prod_{i=1}^{n} f(1/n) = (f(1/n))^n \rightarrow f(1/n) = a^{1/n}$

5) $x = m/n$, $m \in \mathbb{Z}, n \in \mathbb{Z} - \{0\}$

$f(m/n) = f(\sum_{i=1}^{m} 1/n) = \prod_{i=1}^{m} f(1/n) = [f(1/n)]^m = [a^{1/n}]^m = a^{m/n}$

$\rightarrow f(m/n) = a^{m/n}$

Seems we have proved that if (A) holds then $f(x) = a^x$, ie the fn satisfying (A) is unique.

But then we set out to find an f that satisfies $f(x+y) = f(x)f(y)$ for all x,y, not only the rational ones.

Assuming $f(1) = a$, we know that for rational x, f is the fn a^x. We don't know what a^x means for x irrational, but if we figure out what $f(x)$ is we can set a^x to $f(x)$ for such x.

We assumed f exists and investigated its derivative, and obtained $f'(x) = \alpha f(x)$ for all x, where $\alpha = f'(0)$.

Assuming f has an inverse f^{-1}, using the inverse fn theorem, we reached $(f^{-1})'(x) = \dfrac{1}{\alpha f(x)}$.

Now, for rational x this inverse was called $\log_a x$.

If we still call it that then $\log_a' x = \dfrac{1}{\alpha f(x)}$

But this tells us a lot about $\log_a x$. FTC1 $\rightarrow \log_a x = \dfrac{1}{\alpha} \int_1^x \dfrac{1}{t} dt$, ie $\log_a x$ is proportional to an integral we haven't been able to solve yet, $\int_1^x \frac{1}{t} dt$.

Could it be that for some base a we have $\alpha = f'(0) = 1$? Let's call that base e. Then we'd have $\log_e x = \int_1^x \frac{1}{x} dt$,

$\log_e' x = \dfrac{1}{x} > 0$ for $x > 0$, and immediately some other properties: $\log_e xy = \log_e x + \log_e y$,

$\log_e \dfrac{x}{y} = \log_e x - \log_e y$, and $\log_e (x^n) = n \log_e x$.

Since $\log_e' x = \frac{1}{x} > 0$, this fn has a defined inverse (on $\mathbb{R}$ since can show that $\log_e x$ is unbounded).

What can we say about $\log_e^{-1}$?

Using the inverse fn law

$$(\log_e^{-1})'(x) = \frac{1}{\log_e'(\log_e^{-1}(x))} = \frac{1}{\frac{1}{\log_e^{-1}(x)}} = \log_e^{-1}(x)$$

That is,

$$(\log_e^{-1})'(x) = \log_e^{-1}(x)$$

If we give this fn a name, $\exp(x) = \log_e^{-1}(x)$ then we have $\exp'(x) = \exp(x)$

Another property of $\log_e^{-1}(x) = \exp(x)$ is

$$\log_e^{-1}(x+y) = \log_e^{-1}(x) \cdot \log_e^{-1}(y)$$

Okay, at this point, this fn $\log_e^{-1}$ has the property $f(x+y) = f(x) \cdot f(y)$, for all x, y.

What do we know about such fns?

At rational x they equal a^x, where $a = f(1)$.

Well, $\exp(1) = \log_e^{-1}(1) = e$ because $\log_e e = 1$.

Note that we conjectured that this base number e exists but we don't know what it is exactly.

In fact, we define $\exp(1) = \log_e^{-1}(1) = e$. Thus $\log_e^{-1}(x) = [\log_e^{-1}(1)]^x$ for $x \in \mathbb{Q}$.

Let me just recap w/o the e subscript.

We defined a fn $\log x = \int_1^x \frac{1}{t} dt$.

Its derivative is $\frac{1}{x} > 0$ so $\log x$ is increasing, and it is thus one-one.

The inverse exists on $\mathbb{R}$ and $(\log^{-1})'(x) = \log^{-1} x$

Also $\log^{-1}(x+y) = \log^{-1} x \cdot \log^{-1} y$

Thus we know that $\log^{-1}(x) = (\log^{-1}(1))^x$, ie $\exp(x) = (\exp(1))^x$, for rational x.

Again, we started with $\log x = \int_1^x t^{-1} dt$, defined on $\mathbb{R}^+$, image $\mathbb{R}$. $\log^{-1}(x)$ is a fn defined on $\mathbb{R}$ and for $x \in \mathbb{Q}$,

$\log^{-1}(x) = e^x = (\log^{-1}(1))^x$. What we do now is define e^x for $x \in \mathbb{R} \setminus \mathbb{Q}$ as $\log^{-1}(x)$.

And we found the fn we were looking: $\log^{-1} x$, diff, $\log^{-1}(x+y) = \log^{-1}(x) \log^{-1}(y)$, and for all x not just rationals.

Also, $\log^{-1}(x) = e^x$, for all numbers x.

For rational x this is a consequence of the property $f(x+y) = f(x)f(y)$.

For irrational x, it is a definition.

At this point we don't know what a^x is for x irrational (recall that for rational x, we have by def. that

$$a^n = \prod_{i=1}^{n} a \qquad n \in \mathbb{N}$$

$$a^0 = 1$$

$$a^{-n} = \frac{1}{a^n} \qquad n \in \mathbb{N}$$

$$a^{1/n} = \sqrt[n]{a} \qquad n \in \mathbb{Z} - \{0\}$$

$$a^{m/n} = (\sqrt[n]{a})^m \quad n \in \mathbb{Z} - \{0\}, m \in \mathbb{Z}$$

But now we have

$$a^x = (e^{\log a})^x = (\log^{-1}(\log a)) = e^{x \log a},$$ defined everywhere.

T.F. we define, for $a > 0$ and all x

$$a^x = e^{x \log a}$$

$$e = \log^{-1}(1)$$

$$e^x = \log^{-1}(x)$$

$$e^{\log a} = \log^{-1}(\log(a)) = a$$