

Ch. 5 - Limits

$$1. \text{ i) } \lim_{x \rightarrow 1} \frac{x^2-1}{x+1}$$

$$\lim_{x \rightarrow 1} (x^2-1) = 1^2-1 = 0 \quad \text{S.E}$$

$$\lim_{x \rightarrow 1} (x+1) = 2 \quad \text{S.E}$$

$$\lim_{x \rightarrow 1} \frac{1}{x+1} = \frac{1}{2} \quad \text{Th. 2(3)}$$

$$\lim_{x \rightarrow 1} (x^2-1) \cdot \frac{1}{x+1} = 0 \cdot \frac{1}{2} \quad \text{Th. 2(2)}$$

$$ii) \lim_{x \rightarrow 2} \frac{x^3-8}{x-2} = \lim_{x \rightarrow 2} \frac{(x-2)^3 + 6x^2 - 12x}{x-2} = \lim_{x \rightarrow 2} [(x-2)^2 + \frac{6x(x-2)}{x-2}] = \lim_{x \rightarrow 2} [(x-2)^2 + 6x] \cdot 12$$

$$\lim_{x \rightarrow 2} (x-2) = 0$$

$$\lim_{x \rightarrow 2} (x^2-8) = 0$$

$$(x-2)^3 = x^3 - 3x^2 \cdot 2 + 3x \cdot 4 - 8 \\ = x^3 - 8 - 6x^2 + 12x$$

$$iii) \lim_{x \rightarrow 3} \frac{x^3-8}{x-2} = \frac{19}{1} = 19 \quad \text{Th. 2(2)}$$

$$\lim_{x \rightarrow 3} (x-2) = 1 \neq 0$$

$$\lim_{x \rightarrow 3} \frac{1}{x-2} = \frac{1}{1} = 1 \quad \text{Th. 2(3)}$$

$$iv) \lim_{x \rightarrow 1} \frac{x^n - 1^n}{x-1} = \lim_{x \rightarrow 1} \sum_{i=0}^{n-1} x^{n-1-i} \cdot 1^i = \sum_{i=0}^{n-1} 1^{n-1-i} = n \cdot 1^{n-1}$$

y is a root of $x^n - 1^n$

$$\Rightarrow x^n - 1^n = (x-1) \cdot \sum_{i=0}^{n-1} x^{n-1-i} \cdot 1^i$$

$$v) \lim_{t \rightarrow x} \frac{x^n - 1^n}{x-t} = \lim_{t \rightarrow x} \frac{-1(-1^n - x^n)}{-1(t-x)} = \lim_{t \rightarrow x} \frac{y^n - x^n}{t-x} = nx^{n-1}$$

$$vi) \lim_{h \rightarrow 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} = \lim_{h \rightarrow 0} \frac{(\sqrt{a+h} - \sqrt{a})(\sqrt{a+h} + \sqrt{a})}{h(\sqrt{a+h} + \sqrt{a})} = \lim_{h \rightarrow 0} \frac{a+h-a}{h(\sqrt{a+h} + \sqrt{a})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{a+h} + \sqrt{a}} = \frac{1}{2\sqrt{a}}$$

$$2. i) \lim_{x \rightarrow 1} \frac{1-\sqrt{x}}{1-x} = \lim_{x \rightarrow 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})} = \lim_{x \rightarrow 1} \frac{\cancel{1-\sqrt{x}}}{\cancel{(1-\sqrt{x})(1+\sqrt{x})}} = \lim_{x \rightarrow 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}$$

$$ii) \lim_{x \rightarrow 0} \frac{1-\sqrt{1-x^2}}{x} = \lim_{x \rightarrow 0} \frac{1-(1-x^2)}{x(1+\sqrt{1-x^2})} = \lim_{x \rightarrow 0} \frac{x^2}{x(1+\sqrt{1-x^2})} = \lim_{x \rightarrow 0} \frac{x}{1+\sqrt{1-x^2}} = 0$$

note

$$\lim_{x \rightarrow 0} \frac{x}{1+\sqrt{1-x^2}} = \lim_{x \rightarrow 0} x \cdot \frac{1}{1+\sqrt{1-x^2}}$$

$$\text{Th.2(3)} \Rightarrow \lim_{x \rightarrow 0} \frac{1}{1+\sqrt{1-x^2}} = \frac{1}{2}$$

$$\text{Th.2(2)} \Rightarrow \lim_{x \rightarrow 0} x \cdot \frac{1}{1+\sqrt{1-x^2}} = 0 \cdot \frac{1}{2} = 0$$

$$iii) \lim_{x \rightarrow 0} \frac{1-\sqrt{1-x^2}}{x^2} = \lim_{x \rightarrow 0} \frac{1-(1-x^2)}{x^2(1+\sqrt{1-x^2})} = \lim_{x \rightarrow 0} \frac{\cancel{x^2}}{\cancel{x^2}(1+\sqrt{1-x^2})} = \lim_{x \rightarrow 0} \frac{1}{1+\sqrt{1-x^2}} = \frac{1}{2}$$

$$3. ii) f(x) = x(3 - \cos x^2), a=0$$

$$\lim_{x \rightarrow 0} x(3 - \cos x^2) = 0 \quad \text{Th.2(i),(2)}$$

which means, by definition

$$\forall \epsilon > 0, \exists \delta > 0 : |x| < \delta \Rightarrow |x(3 - \cos x^2)| < \epsilon$$

Let's look first at the expression $|x(3 - \cos x^2)| < \epsilon$. By manipulating this expr., we may find out how close to zero x must be.

$$|x(3 - \cos x^2)| \leq |x||3 - \cos x^2| < \epsilon$$

$$x^2 \geq 0 \Rightarrow 0 \leq \cos x^2 \leq 1$$

$$|3 - \cos x^2| \leq |3 - \cos x^2| \leq |3| + |\cos x^2|$$

$$3 - 1 \leq |3 - \cos x^2| \leq 3 + 0$$

$$2 \leq |3 - \cos x^2| \leq 3$$

$$\Rightarrow |x(3 - \cos x^2)| \leq |x||3 - \cos x^2| \leq 3|x| < \epsilon$$

$$\Rightarrow |x| < \frac{\epsilon}{3}$$

$$\Rightarrow \forall \epsilon > 0, \text{ if we choose } 0 < \delta < \frac{\epsilon}{3} \text{ then } |x(3 - \cos x^2)| < \epsilon$$

$$iii) f(x) = x^2 + 5x - 2, a=2$$

$$\lim_{x \rightarrow 2} f(x) = 4 + 10 - 2 = 12 \text{ Th.2 (1), (2)}$$

$$\Rightarrow \forall \epsilon > 0, \exists \delta > 0 : |x-2| < \delta \Rightarrow |x^2 + 5x - 2 - (a^2 + 5a - 2)| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow 2} x^2 = 4 \Rightarrow \forall \epsilon > 0, \exists \delta > 0 : |x-2| < \delta \Rightarrow |x^2 - 4| < \epsilon$$

$$|x^2 - 4| = |(x+2)(x-2)| \leq |x+2||x-2|$$

$$\text{Assume } |x-2| < 1 \Rightarrow -1 < x-2 < 1 \Rightarrow 3 < x+2 < 5 \Rightarrow |x+2| < 5$$

$$|x+2||x-2| \leq 5|x-2| < \epsilon \Rightarrow |x-2| < \frac{\epsilon}{5}$$

$$\forall \epsilon > 0, \delta < \min(1, \frac{\epsilon}{5}) \Rightarrow |x^2 - 4| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow 2} 5x = 10 \Rightarrow \forall \epsilon > 0, \exists \delta : |x-2| < \delta \Rightarrow |5x-10| < \epsilon$$

$$|5x-10| = |5(x-2)| = 5|x-2| < \epsilon \Rightarrow |x-2| < \frac{\epsilon}{5}$$

$$\forall \epsilon > 0, \delta < \frac{\epsilon}{5} \Rightarrow |5x-10| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow 2} (-2) = -2 \Rightarrow \forall \epsilon > 0, \exists \delta > 0 : |x-2| < \delta \Rightarrow |-2 - (-2)| = 0 < \epsilon$$

$$\Rightarrow \forall \epsilon > 0, \delta = \min(1, \frac{\epsilon}{5}) \Rightarrow |x-2| < \delta \Rightarrow |x^2 - 4| < \epsilon, \text{ i.e. } x^2 \text{ approaches 4}$$

$$|5x-10| < \epsilon, 5x \rightarrow 10$$

$$|-2 - (-2)| = 0 < \epsilon, -2 \rightarrow -2$$

$$\text{choose } \epsilon_0 = \frac{\epsilon}{2}, \forall \epsilon_0 > 0$$

$$\text{Th.2(1)} \Rightarrow |x^2 + 5x - (4+10)| < \epsilon_0$$

$$\Rightarrow \forall \epsilon_0 > 0 \Rightarrow |x-2| < \min(1, \frac{\epsilon_0}{10}) \Rightarrow |x^2 + 5x - (4+10)| < \epsilon_0$$

$$\forall \epsilon_1, \text{ let } \epsilon_0 = \frac{\epsilon_1}{2}$$

$$\Rightarrow |x-2| < \min(1, \frac{\epsilon_1}{20}) \Rightarrow |x^2 + 5x - (4+10)| < \frac{\epsilon_1}{2}$$

$$|-2 - (-2)| < \frac{\epsilon_1}{2}$$

$$\text{Th.2(1)} \Rightarrow |x^2 + 5x - 2 - (a^2 + 5a - 2)| < \epsilon_1$$

$$\Rightarrow \forall \epsilon_1 > 0, |x-a| < \min(1, \frac{\epsilon_1}{20}) \Rightarrow |x^2 + 5x - 2 - (a^2 + 5a - 2)| < \epsilon_1$$

$$\Rightarrow \lim_{x \rightarrow a} (x^2 + 5x - 2) = a^2 + 5a - 2$$

To determine $\lim_{x \rightarrow a} (x^2 + 5x - 2)$ we used delta-epsilon analysis to

Show the limits of $x^2, 5x$, and -2 .

Then we used the lemmas to find the limits $x^2 + 5x$, and then

$$x^2 + 5x - 2.$$

$$\text{iii) } f(x) = \frac{100}{x}, a=1$$

$$\lim_{x \rightarrow a} 100 \cdot \frac{1}{x}$$

$$\forall \epsilon > 0, \forall \delta > 0, |x-a| < \delta \Rightarrow |100 - 100| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} 100 - 100$$

$$\forall \epsilon > 0, \forall \delta < \epsilon, |x-a| < \delta \Rightarrow |x-a| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} x - a$$

$$\text{If } a \neq 0, \forall \epsilon > 0 \text{ let } \epsilon' = \min\left(\frac{|a|}{2}, \frac{\epsilon|a|^2}{2}\right)$$

$$\Rightarrow \forall \delta < \epsilon', |x-a| < \delta \Rightarrow |x-a| < \epsilon.$$

$$\Rightarrow x \neq 0, \left| \frac{1}{x} - \frac{1}{a} \right| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{1}{x} = \frac{1}{a}$$

At this point we have

$$\forall \epsilon > 0, \forall \delta: 0 < \delta < \epsilon, a \neq 0 \Rightarrow |x-a| < \delta \Rightarrow |100 - 100| < \min\left(1, \frac{\epsilon}{2(|a|+1)}\right)$$
$$\left| \frac{1}{x} - \frac{1}{a} \right| < \frac{\epsilon}{2(|a|+1)}$$

$$\text{Lemma (2)} \Rightarrow |100 \cdot \frac{1}{x} - \frac{100}{a}| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{100}{x} = \frac{100}{a}, a \neq 0$$

$$\text{int } f(x) = x^4$$

$$|x^2 - a^2| = |(x+a)(x-a)| \leq |x+a||x-a|$$

Assume $-1 < x-a < 1 \Rightarrow -2a-1 < x+a < 2a+1$
 $\Rightarrow |x+a| < |2a+1| \leq |2a| + 1 \leq 2|a| + 1$

$$|x^2 - a^2| < |x-a| < 2|a| + 1$$

$$\Rightarrow |x-a| < \frac{\epsilon}{2|a|+1}$$

$$\forall \epsilon > 0, |x-a| < \min(1, \frac{\epsilon}{2|a|+1}) \Rightarrow |x^2 - a^2| < \epsilon$$

$$|x^4 - a^4| = |(x^2)^2 - (a^2)^2| = |x^2 - a^2||x^2 + a^2|$$

Assume $|x^2 - a^2| < 1 \Rightarrow -1 < x^2 - a^2 < 1 \Rightarrow 2a^2 - 1 < x^2 + a^2 < 2a^2 + 1$
 $\Rightarrow |x^2 + a^2| < 2a^2 + 1$

$$|x^4 - a^4| < |x^2 - a^2|(2a^2 + 1) < \epsilon$$

$$\forall \epsilon > 0, |x^2 - a^2| < \min(1, \frac{\epsilon}{2a^2+1}) \Rightarrow |x^4 - a^4| < \epsilon$$

$$|x-a| < \min(1, \frac{\min(1, \frac{\epsilon}{2a^2+1})}{2|a|+1}) \Rightarrow |x^2 - a^2| < \frac{\epsilon}{2a^2+1}$$

$$\Rightarrow |x^4 - a^4| < \epsilon \quad \Rightarrow \lim_{x \rightarrow a} x^4 = a^4$$

$$\lim_{x \rightarrow a} x^2 = \lim_{x \rightarrow a} x \cdot x = \lim_{x \rightarrow a} x \cdot \lim_{x \rightarrow a} x = a \cdot a = a^2$$

$$\text{Assume } \lim_{x \rightarrow a} x^k = a^k$$

$$\lim_{x \rightarrow a} x^{k+1} = \lim_{x \rightarrow a} x \cdot x^k = \lim_{x \rightarrow a} x \cdot \lim_{x \rightarrow a} x^k = a \cdot a^k = a^{k+1}$$

$$\text{By induction, } \lim_{x \rightarrow a} x^n = a^n$$

Note

$$|x-a| < \frac{1}{2} \frac{\epsilon}{(2|a|+1)(2a^2+1)} \Rightarrow |x^2 - a^2| < \frac{1}{2} \frac{\epsilon}{2a^2+1}$$

$$\Rightarrow |x^4 - a^4| < \frac{\epsilon}{2}$$

$$\text{vi) } f(x) = x^4 + \frac{1}{x}$$

$$\forall \epsilon > 0, |x-a| < \min(1, \frac{\min(1, \frac{\epsilon}{2a^2+1})}{2|a|+1}) \Rightarrow |x^2 - a^2| < \frac{\epsilon}{2a^2+1}$$

$$\Rightarrow |x^4 - a^4| < \epsilon$$

$$a \neq 0, \forall \epsilon > 0 \text{ let } \epsilon_0 = \min(\frac{|a|}{2}, \frac{\epsilon|a|^2}{2})$$

$$\Rightarrow \forall \delta < \epsilon_0, |x-a| < \delta \Rightarrow |x-a| < \epsilon_0$$

$$\Rightarrow x \neq 0, \left| \frac{1}{x} - \frac{1}{a} \right| < \epsilon$$

(Recall)

$$1. |x-x_0| < \frac{\epsilon}{2}, |f-f_0| < \frac{\epsilon}{2} \Rightarrow |(x+f) - (x_0+f_0)| < \epsilon$$

$$\forall \epsilon > 0, \text{ let } \epsilon_0 = \min(\min(1, \frac{\epsilon}{2a^2+1}), \min(\frac{|a|}{2}, \frac{\epsilon|a|^2}{2}))$$

$$|x-a| < \frac{1}{2}\epsilon_0 \Rightarrow |x^4 - a^4| < \frac{\epsilon}{2}$$

$$\left| \frac{1}{x} - \frac{1}{a} \right| < \frac{\epsilon}{2}$$

$$\Rightarrow \left| (x^4 + \frac{1}{x}) - (a^4 + \frac{1}{a}) \right| < \epsilon$$

$$\text{vii) } f(x) = \frac{x}{2 - \sin^2 x}, a=0$$

$$-1 \leq \sin x \leq 1$$

$$0 \leq \sin^2 x \leq 1$$

$$-1 \leq -\sin^2 x \leq 0$$

$$1 \leq 2 - \sin^2 x \leq 2$$

$$\Rightarrow \frac{x}{2 - \sin^2 x} \leq x, \forall x$$

$$\left| \frac{x}{2 - \sin^2 x} \right| \leq |x|$$

$$\Rightarrow \forall \epsilon > 0, \exists \delta < \epsilon, |x - 0| < \delta \Rightarrow \left| \frac{x}{2 - \sin^2 x} - 0 \right| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x}{2 - \sin^2 x} = 0$$

$$\text{viii) } f(x) = \sqrt{|x|}, a=0$$

Case 1: $\forall \epsilon > 0, \epsilon < 1$

$$\epsilon^2 < \epsilon < 1$$

$$|x| < \epsilon^2 \Rightarrow \sqrt{|x|} < \epsilon < \sqrt{\epsilon} < 1$$

Case 2: $1 \leq \epsilon < \infty$

$$\epsilon \leq \epsilon^2 \Rightarrow |x| < \epsilon^2 \Rightarrow \sqrt{|x|} < \epsilon$$

$$\text{ix) } f(x) = \sqrt{x}, a=1$$

$$\forall \epsilon > 0, \text{ we are looking for } \delta > 0 : |x - 1| < \delta \Rightarrow |\sqrt{x} - 1| < \epsilon$$

The domain of f is $[0, +\infty)$, so $x \geq 0$.

$$\text{consider } 0 < \delta \leq 1 \Rightarrow 1 - \delta \leq x \leq \delta \Rightarrow x \geq 0$$

$$|\sqrt{x} - 1| = \frac{|\sqrt{x} - 1||\sqrt{x} + 1|}{|\sqrt{x} + 1|} = \frac{|x - 1|}{|\sqrt{x} + 1|} \leq |x - 1|$$

$$\text{because } |\sqrt{x} + 1| > 1 \forall x$$

$$\Rightarrow \forall \epsilon > 0, \delta < \min(1, \epsilon) \Rightarrow |x - 1| < \delta \Rightarrow |\sqrt{x} - 1| < |x - 1| < \epsilon$$

$$4. [x] = \text{largest } z \leq x$$

$$\text{i) } f(x) = [x]$$

$$\forall a \in \mathbb{Z}, \lim_{x \rightarrow a^+} f(x) + \lim_{x \rightarrow a^-} f(x)$$

$$\text{example: } a = 1$$

$$0 < x - 1 < \delta \Rightarrow 1 < x < 1 + \delta$$

$$\begin{aligned} \delta < 1 &\Rightarrow 1 < x < 1 + \delta < 2 \Rightarrow f(x) = 1 \text{ in this interval} \\ &\Rightarrow |f(x) - 1| = 0 < \epsilon, \forall \epsilon > 0 \end{aligned}$$

$$\Rightarrow 0 < x - 1 < \delta \Rightarrow |f(x) - 1| < \epsilon$$

$$\bullet \lim_{x \rightarrow 1^-} [x] = 1$$

$$\delta > 0,$$

$$0 < 1 - x < \delta \Rightarrow -1 < -x < -1 + \delta \Rightarrow 1 - \delta < x < 1$$

$$\delta < 1 \Rightarrow 0 < 1 - \delta < x < 1 \Rightarrow f(x) = 0$$

$$\forall \epsilon > 0, \forall \delta: 0 < \delta < 1, 0 < 1 - \delta < \delta \Rightarrow |f(x) - 0| < \epsilon$$

$$\bullet \lim_{x \rightarrow 1^+} f(x) = 0$$

general case $a \in \mathbb{Z}$

$$0 < x - a < \delta \Rightarrow a < x < a + \delta$$

$$\delta < 1 \Rightarrow a < x < a + \delta < a + 1$$

$$\Rightarrow 0 < x - a < \delta \Rightarrow f(x) = a$$

$$\forall \epsilon > 0, 0 < \delta < 1 \Rightarrow 0 < x - a < \delta \Rightarrow |f(x) - a| = 0 < \epsilon$$

$$\bullet \lim_{x \rightarrow a^+} f(x) = a$$

$$0 < a - x < \delta, \delta > 0$$

$$\Rightarrow -a < -x < -a + \delta \Rightarrow a - \delta < x < a$$

$$\delta < 1 \Rightarrow a - 1 < a - \delta < x < a$$

$$0 < \delta < 1, 0 < a - x < \delta \Rightarrow f(x) = a - 1 = |f(x) - (a - 1)| = 0 < \epsilon$$

$$\bullet \lim_{x \rightarrow a^-} [x] = a - 1, a \in \mathbb{Z}$$

consider $a \notin \mathbb{Z}$

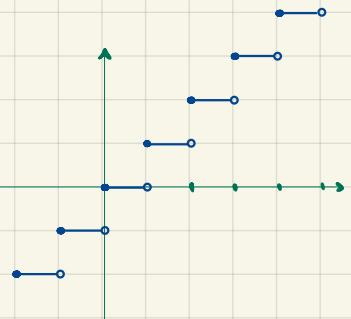
$$f(x) = [a]$$

$$\text{let } d_- = a - [a]$$

$$d_+ = [a+1] - a$$

$$|x - a| < \min(d_-, d_+)$$

$$\Rightarrow \lim_{x \rightarrow a} [x] = [a], a \notin \mathbb{Z}$$



To be even more explicit

$$\text{case 1: } \min(d_-, d_+) = d_-$$

$$|x - a| < d_- = a - [a]$$

$$\Rightarrow [a] - a < x - a < a - [a]$$

$$\begin{aligned} \Rightarrow [a] < x < [a+1] &\Rightarrow a + (a - [a]) \\ &= a + d_- \\ &< a + d_+ = [a+1] \end{aligned}$$

$$\Rightarrow [a] < x < [a+1] \Rightarrow f(x) = [a]$$

measure:

$$\text{if } d_- \leq d_+, \forall \epsilon > 0, \exists \delta = d_- : |x - a| < d_- \Rightarrow |f(x) - [a]| < \epsilon$$

$$\text{case 2: } \min(d_-, d_+) = d_+, \text{ ie } d_+ \leq d_-$$

$$|x - a| < d_+ = [a+1] - a$$

$$\Rightarrow a - [a+1] < x - a < [a+1] - a$$

$$\Rightarrow 2a - [a+1] < x < [a+1]$$

$$\Rightarrow a - ([a+1] - a) < x < [a+1]$$

$$a - d_- < a - d_+ < x < [a+1]$$

$$[a] < x < [a+1] \Rightarrow f(x) = [a]$$

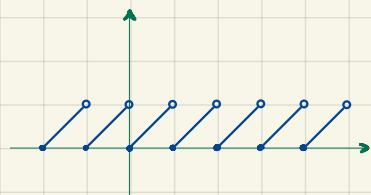
measure

$$d_+ \leq d_-, \forall \epsilon > 0, |x - a| < d_+ \Rightarrow |f(x) - [a]| < \epsilon$$

$$\bullet \lim_{x \rightarrow a} [x] = [a], a \notin \mathbb{Z}$$

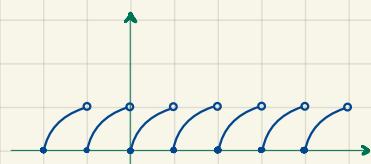
$$iii) f(x) = x - [x]$$

$$a \in \mathbb{Z} \Rightarrow \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x)$$

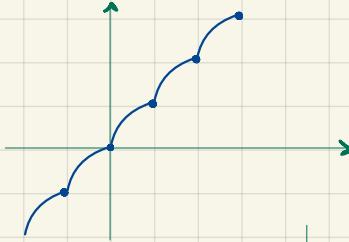


$$a \notin \mathbb{Z} \Rightarrow \lim_{x \rightarrow a} (x - [x]) = a - [a]$$

$\lim_{x \rightarrow a}$ f(x) exists for $x \notin \mathbb{Z}$



$\lim_{x \rightarrow a}$ f(x) exists for $x \notin \mathbb{Z}$



Consider $a \in \mathbb{Z}$

$$\forall \delta: 0 < \delta < 1$$

$$0 < x - a < \delta \Rightarrow -\delta < x - a < 0 + \delta$$

$$\Rightarrow 1 - \delta < \sqrt{1 - \delta} < \sqrt{x - a + 1} < 1$$

$$|f(x) - a| = |x - a + \sqrt{x - a + 1} - a|$$

$$< 1 - \delta + 1 - \delta = \delta$$

$$\text{Therefore, } \forall a \in \mathbb{Z}, \forall \epsilon: 0 < \delta < 1 \Rightarrow 0 < x - a < \delta \Rightarrow |f(x) - a| < \delta < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} f(x) = a$$

Similarly,

$$0 < x - a < \delta, 0 < \delta < 1 \Rightarrow$$

$$|f(x) - a| = |x + \sqrt{x - a} - a| = \sqrt{x - a} < \sqrt{\delta}$$

$$\forall \epsilon > 0, \delta \cdot \epsilon^2 \Rightarrow 0 < x - a < \delta \Rightarrow |f(x) - a| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} f(x) = a$$

consider $a \notin \mathbb{Z}$

$$\text{Let } d_- = a - [a]$$

$$d_+ = [a+1] - a$$

$$|x - a| < \min(d_-, d_+) = \delta$$

$$\text{Case 1: } \min(d_-, d_+) = d_- = \delta$$

$$|x - a| < a - [a]$$

$$\Rightarrow [a] - a < x - a < a - [a]$$

$$\Rightarrow [a] < x < [a] - [a] = a + (a - [a])$$

$$= a + d_-$$

$$< a + d_+ = [a+1]$$

$$\Rightarrow [a] < x < [a+1] \Rightarrow f(x) = [a] + \sqrt{x - [a]}$$

$$|f(x) - [a] - \sqrt{a - [a]}| = |\sqrt{x - [a]} - \sqrt{a - [a]}|$$

$$\text{But, } 0 < x - [a] < 2a - 2[a] - 2(a - [a])$$

$$\Rightarrow 0 < \sqrt{x - [a]} < \sqrt{2} \sqrt{a - [a]}$$

$$\Rightarrow |f(x) - [a] - \sqrt{a - [a]}| < |(\sqrt{2} - 1) \sqrt{a - [a]}| = (\sqrt{2} - 1) \delta$$

Case 2 is similar.

$\lim_{x \rightarrow a} f(x)$ exists for all $x \in \mathbb{R}$.

$$\text{vi) } f(x) = \left[\frac{1}{x} \right]$$

$\lim_{x \rightarrow a^+} f(x) + \lim_{x \rightarrow a^-} f(x)$ for all $x \in \mathbb{Q} : x = \frac{1}{n}, n \in \mathbb{Z}$, i.e. $-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \dots, -1, 1, \frac{1}{2}, \frac{1}{3}, \dots$

$$\lim_{x \rightarrow 0^+} f(x) + \lim_{x \rightarrow 0^-} f(x)$$

consider $\forall \delta > 0 : |x-0| = |x| < \delta$.

choose $\forall \ell \in \mathbb{R}$.

$$x \in (0, 1), x \in \mathbb{Q} \Rightarrow x = \frac{1}{p}, p \in \mathbb{Z}$$

$$\Rightarrow f(x) = \left[\frac{1}{x} \right] = [p] = p$$

$$\forall \epsilon > 0, \delta > 0 \text{ as defined} \Rightarrow | \frac{1}{p} - \ell | < \delta \Rightarrow \frac{1}{p} < \delta + \ell \Rightarrow p > \frac{1}{\delta + \ell}, \text{ e.g. } p = \lceil \frac{1}{\delta + \ell} + 1 \rceil$$

and note that $p - \ell > \epsilon \Rightarrow p > \ell + \epsilon$. For example, $\ell + \epsilon + 1 > \ell + \epsilon > 0$

$$\text{Therefore, } p = \max(\lceil \frac{1}{\delta + \ell} + 1 \rceil, \lceil \ell + \epsilon + 1 \rceil) \Rightarrow |x-0| < \delta, |f(x)-\ell| > \epsilon$$

$\Rightarrow \lim_{x \rightarrow 0} f(x)$ does not exist.

$\lim_{x \rightarrow a} f(x)$ does exist for $\{a : a \neq 0 \text{ and } a + \frac{1}{n}, n \in \mathbb{Z}\}$

$$\text{vii) } f(x) = \frac{1}{\left[\frac{1}{x} \right]}$$

$\lim_{x \rightarrow a^+} f(x) + \lim_{x \rightarrow a^-} f(x)$ for $x \in \mathbb{Q} : x = \frac{1}{p}, p \in \mathbb{Z}\}$

At $a=0$ we have

$$|f(x)-0| = \left| \frac{1}{\left[\frac{1}{x} \right]} \right| < \epsilon$$

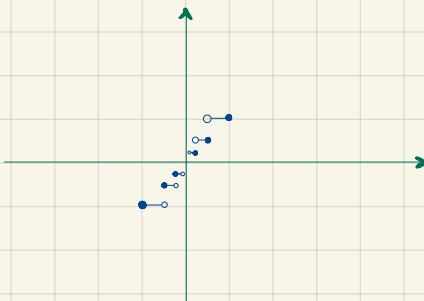
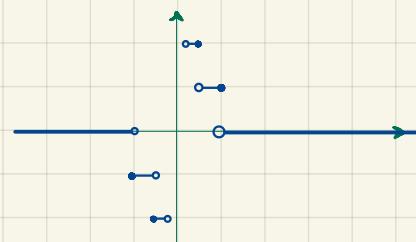
$$x > 0 \Rightarrow x < \epsilon \quad \Rightarrow |x| < \epsilon$$

$$x < 0 \Rightarrow -x < \epsilon \Rightarrow x > -\epsilon$$

$$\text{Therefore, } \forall \epsilon > 0, |x-0| < \epsilon \Rightarrow |f(x)-0| < \epsilon$$

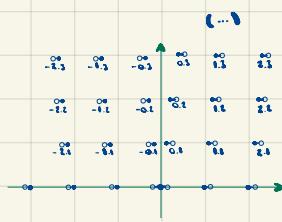
$$\Rightarrow \lim_{x \rightarrow 0} \frac{1}{\left[\frac{1}{x} \right]} = 0$$

$\lim_{x \rightarrow a} f(x)$ exists for $a \in \{a : |a| \leq 1, a + \frac{1}{p}, p \in \mathbb{Z}\}$



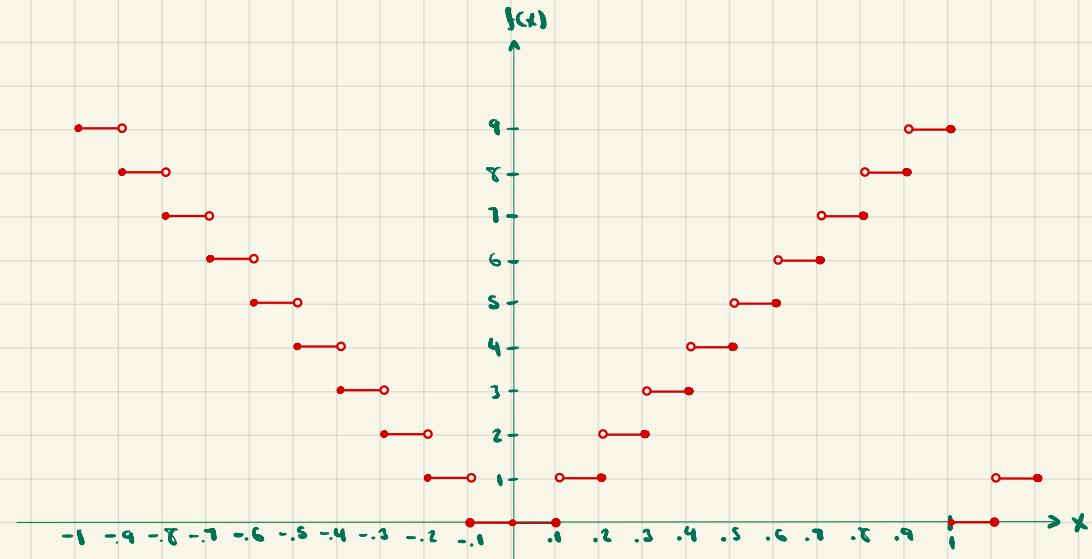
S. a)

ii) $f(x) = 1^{\text{st}} \text{ number of decimal expansion of } x$



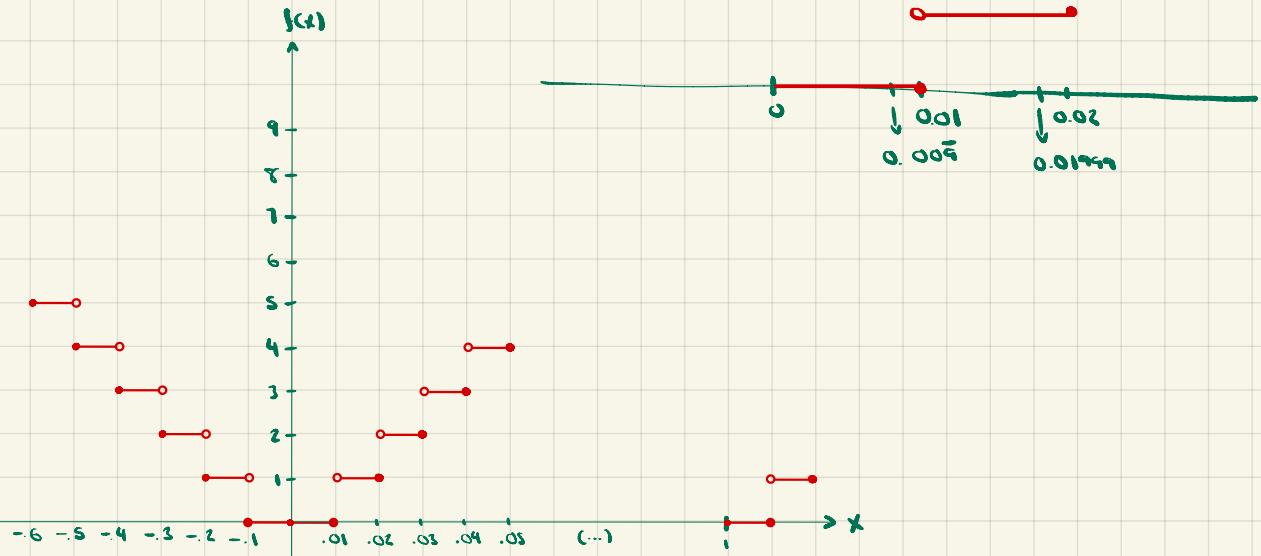
$$f(0.1) = f(0.0\bar{9}) = 0$$

$$f(0.2) = f(0.1\bar{9}) = 1$$



$\lim_{x \rightarrow a} f(x)$ exists for $\{a : a + 0.1n, n \in \mathbb{Z}\}$

iii) $f(x) = 2^{\text{nd}} \text{ number in decimal expansion of } x$



$\lim_{x \rightarrow a} f(x)$ exists for $\{a : a + 0.01n, n \in \mathbb{Z}\}$

iii) $f(x) =$ number of 7's in decimal expansion of x if this number is finite,
0 otherwise

In any interval there are infinite numbers w/
any number of 7's, from zero to ∞ .



Therefore for any interval $(a-\delta, a+\delta)$, $f(x)$ equals every natural number at some x .

$$\forall \epsilon > 0, \forall \delta > 0, I = (a-\delta, a+\delta), \exists x \in I : |f(x) - \ell| > \epsilon, \forall \ell \in \mathbb{R}$$

$\Rightarrow \lim_{x \rightarrow a} f(x)$ does not exist for any $a \in \mathbb{R}$.

iv) $f(x) = 0$ if number of 7's in decimal expansion of x is finite, 1 otherwise

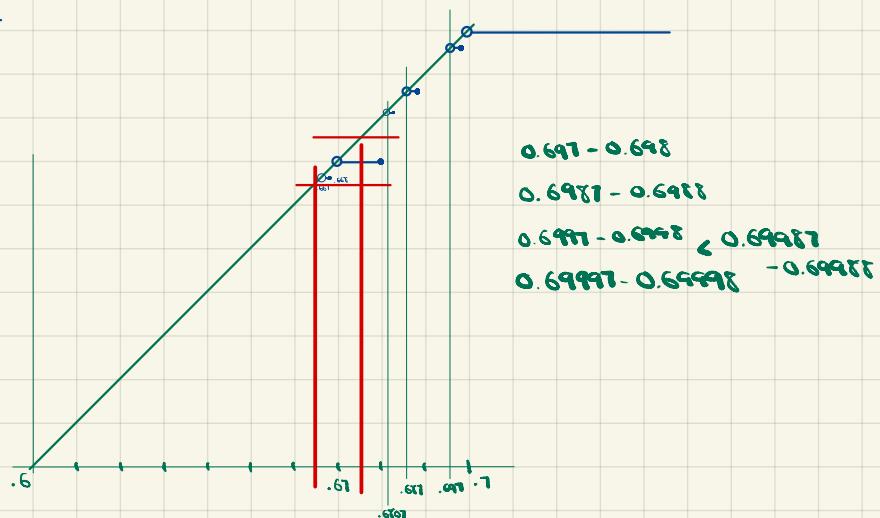
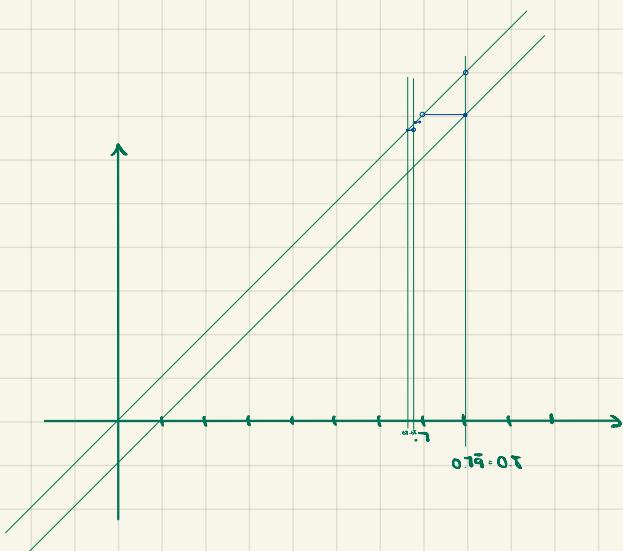


In any interval $(a-\delta, a+\delta)$, there are infinite points where f is 1, and infinite points where f is 0.

No matter what value ℓ is chosen as candidate for $\lim_{x \rightarrow a} f(x)$, $|f(x) - \ell|$ will be at least 0.5 for infinitely many points.

$\Rightarrow \lim_{x \rightarrow a} f(x)$ does not exist for any $a \in \mathbb{R}$

v) $f(x)$ - number obtained by replacing all digits in decimal expansion of x which come after first 7, i.e. by 0.



$\lim_{x \rightarrow a} f(x)$ exists for all $a \in \mathbb{R}$ except if the decimal expansion of a ends in $\bar{79}$, i.e. except for the points on the right (closed) side of the intervals that start on the $y=x$ line at or x with one digit 7 followed by zeros.

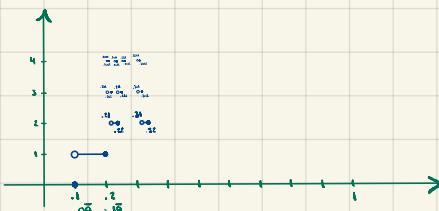
This means the limit does exist

- i) for a such that a has no digit 7 in its d.e., has one digit 7 followed by zeros
- ii) a w/ at least one digit 7, followed by at least one nonzero digit, except if d.e. ends in $\bar{79}$.

vi) $f(x)$ - 0 if 1 never appears in decimal expansion of x
n it 1 first appears in n^{th} place

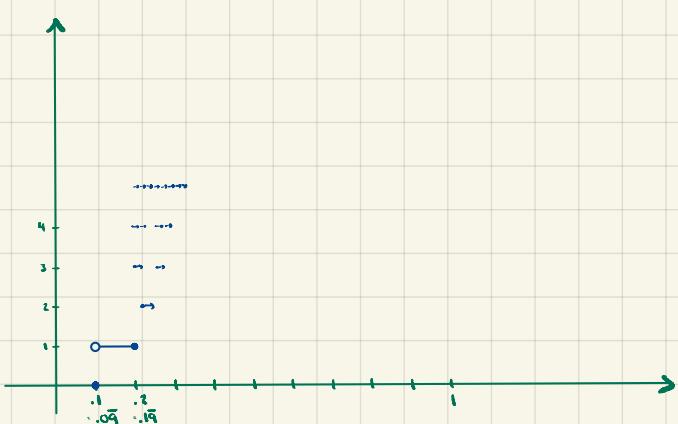
The graph is composed of an infinite number of intervals of form $(a, b]$. $\forall x \in (a, b]$, $f(x)$ is constant.

However, when x approaches a from below or b from above, no matter how small an interval we can always find an x such that $f(x)$ is arbitrarily large.



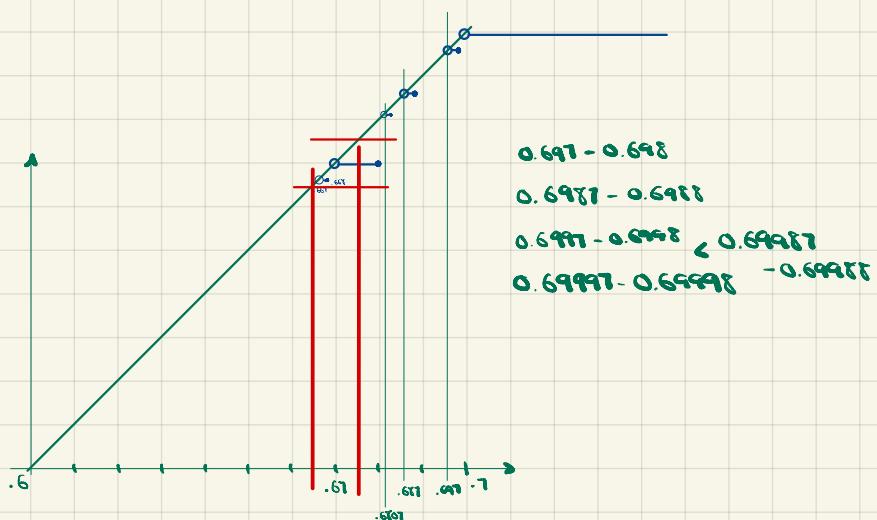
• $\lim_{x \rightarrow a} f(x)$ exists $\forall a \in \mathbb{R}$ except if a 's decimal expansion has

one digit 1 followed by zeros or has no digit 1 but ends in $\bar{0}$. I.e. the limit doesn't exist on the ends of the intervals.



- b) For all the items, nothing changes regarding existence of $\lim_{x \rightarrow a} f(x)$ when we change all use of infinite decimals ending in $\bar{9}$.

* Attempt at proving problem 5(a)(v).



Case 1: a 's decimal expansion contains one digit 7, followed by zeros.

Given $\epsilon > 0$, first consider $(a, a+\delta)$, $\delta > 0$: $x \in (a, a+\delta) \Rightarrow |f(x) - a| < \epsilon$

or can always find a & δ such that $f(x) - a < \epsilon$ $\forall x \in (a, a+\delta)$.

Consider the interval $A = (a-\epsilon, a)$, $\epsilon > 0$

\Rightarrow if $x \in A$ and x has no digit 7 in decimal exp., then $|f(x) - x|$

$$\Rightarrow |f(x) - a| + |x - a| < \epsilon$$

\Rightarrow if $x \in A$ and x has one or more digit 7's in decimal exp., then:

a) if $a-\epsilon$ has no digit 7's in dec. exp., then $|f(a-\epsilon) - a-\epsilon|$, $|a-\epsilon - f(a-\epsilon)| \leq \epsilon$

$$\Rightarrow -\epsilon < f(a-\epsilon) - a < 0 < \epsilon \Rightarrow |f(x) - a| < \epsilon$$

b) if $a-\epsilon$ has one or more digit 7's in dec. exp., then there is some number after 7's between $a-\epsilon$ and a . The reason for this is that a has one digit 7 followed by zeros. There is no number w/ one or more zeros smaller than a such that there is no number w/ no zeros between a and this number. In fact, a is equivalent to a number ending in a string of 9's, with no 7's.

choose a number between $a-\epsilon$ and a w/ no 7's, call it n_1 .

$$a-\epsilon < n_1 < a$$

$$a-\epsilon < a - (a-n_1) < a$$

$$-\epsilon < -(a-n_1) < 0$$

$$\epsilon > a-n_1 > 0$$

call $\epsilon_1 = a-n_1$, we are in case a) above

$$\Rightarrow x \in (a-\epsilon_1, a)$$
, $a-\epsilon_1$, has no 7's in d.e., $\Rightarrow |f(x) - a| < \epsilon_1 < \epsilon$

6. f, g fns

$$\forall \epsilon > 0, \forall x \in \mathbb{R}$$

$$0 < |x - 2| < \sin^2\left(\frac{\epsilon}{9}\right) + \epsilon \Rightarrow |f(x) - 2| < \epsilon$$

$$0 < |x - 2| < \epsilon^2 \Rightarrow |g(x) - 4| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow 2} f(x) = 2$$

$$\lim_{x \rightarrow 2} g(x) = 4$$

$$\text{ii) } 0 < |x - 2| < \delta \Rightarrow |f(x) + g(x) - 6| < \epsilon$$

$$\forall \epsilon > 0 \text{ let } \delta = \min\left(\epsilon^2, \sin^2\left(\frac{\epsilon}{9}\right) + \epsilon\right)$$

$$0 < |x - 2| < \min\left(\frac{\epsilon^2}{4}, \sin^2\left(\frac{\epsilon^2}{36}\right) + \epsilon/2\right) \Rightarrow |f(x) - 2| < \frac{\epsilon}{2}$$

$$|g(x) - 4| < \frac{\epsilon}{2}$$

$$\text{Th. 2 Lemma (i)} \Rightarrow |f(x) + g(x) - 6| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow 2} (f(x) + g(x)) = 6$$

$$\text{iii) } 0 < |x - 2| < \delta \Rightarrow |f(x)g(x) - 8| < \epsilon$$

$$\forall \epsilon > 0, \quad 0 < |x - 2| < \delta_1 \Rightarrow |f(x) - 2| < \min\left(1, \frac{\epsilon}{2(4+1)}\right)$$

$$0 < |x - 2| < \delta_2 \Rightarrow |g(x) - 4| < \frac{\epsilon}{2(3+1)}$$

$$\Rightarrow 0 < |x - 2| < \min(\delta_1, \delta_2) \Rightarrow |f(x) - 2| < \min(1, \frac{\epsilon}{10})$$

$$|g(x) - 4| < \frac{\epsilon}{6}$$

$$\text{Th. 2 Lemma (ii)} \Rightarrow |f(x)g(x) - 8| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow 2} f(x)g(x) = 8$$

iii)

$$\forall \epsilon > 0, \exists \delta: 0 < |x - 2| < \delta \Rightarrow |g(x) - 4| < \min\left(\frac{4}{2}, \frac{\epsilon \cdot 4^2}{2}\right)$$

$$\text{Th. 2 Lemma (3)} \Rightarrow g(x) \neq 0, \quad \left| \frac{1}{g(x)} - \frac{1}{4} \right| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{1}{g(x)} = \frac{1}{4}$$

$$\text{i) } \forall \epsilon > 0,$$

$$\exists \delta: 0 < |x - 2| < \delta \Rightarrow |f(x) - 2| < \min\left(1, \frac{\epsilon}{2(\frac{1}{4}+1)}\right)$$

$$\exists \delta: 0 < |x - 2| < \delta \Rightarrow g(x) \neq 0, \quad \left| \frac{1}{g(x)} - \frac{1}{4} \right| < \frac{\epsilon}{2(2+1)}$$

$$\text{Th. 2 Lemma (2)} \Rightarrow |f(x) \cdot \frac{1}{g(x)} - \frac{1}{2}| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{f(x)}{g(x)} = \frac{1}{2}$$

$$7. |f(x) - l| < \epsilon \text{ when } 0 < |x-a| < \delta \Rightarrow |f(x) - l| < \frac{\epsilon}{2} \text{ when } 0 < |x-a| < \frac{\delta}{2}$$

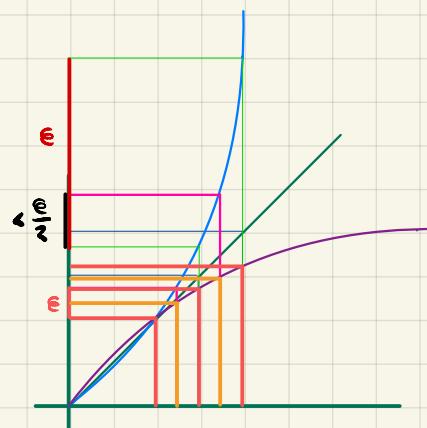
Fn 3 for which above statement is false

$$\text{Let } f(x) = \sqrt{|x|}$$

$$\lim_{x \rightarrow 0} \sqrt{|x|} = 0$$

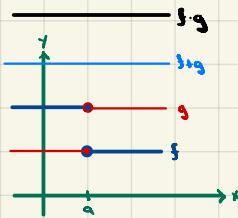
$$|x| < \epsilon^2 \Rightarrow \sqrt{|x|} < \epsilon$$

$$|x| < \frac{\epsilon^2}{2} \Rightarrow \sqrt{|x|} < \frac{\epsilon}{\sqrt{2}} < \frac{\epsilon}{2}$$



8. a) $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ don't exist.

$$\lim_{x \rightarrow a} (f(x) + g(x)) \text{ exists.}$$



Theorem 2

$$\lim_{x \rightarrow a} f(x) = l$$

$$\lim_{x \rightarrow a} (f+g)(x) = l+m \quad (1)$$

$$\lim_{x \rightarrow a} g(x) = m$$

$$\lim_{x \rightarrow a} (f \cdot g)(x) = l \cdot m \quad (2)$$

$$m \neq 0 \Rightarrow \lim_{x \rightarrow a} \frac{1}{g(x)} = \frac{1}{m} \quad (3)$$

$$f(x) = \begin{cases} c_1 & x \leq a \\ c_2 & x > a \end{cases}$$

$$\Rightarrow (f+g)(x) = c_1 + c_2 \quad \forall x$$

$$g(x) = \begin{cases} c_2 & x \leq a \\ c_1 & x > a \end{cases} \Rightarrow (f \cdot g)(x) = c_1 c_2 \quad \forall x$$

$$b) g(x) = (f(x) + g(x)) + (-f(x))$$

$$\exists \lim_{x \rightarrow a} (f+g)(x), \exists \lim_{x \rightarrow a} (-f(x)) \Rightarrow \text{Apply Th. 2 (1)}$$

$$\Rightarrow \lim_{x \rightarrow a} (f(x) + g(x) + (-f(x))) = \lim_{x \rightarrow a} g(x) \text{ exists}$$

$$\text{a) } \begin{matrix} \underset{\substack{\text{limit} \\ \text{A,B}}}{} & \underset{\substack{\text{limit} \\ \text{g}}}{} \\ A, B \rightarrow C & \\ \downarrow & \\ \lim(f+g) \end{matrix} \quad (\text{proved in b}))$$

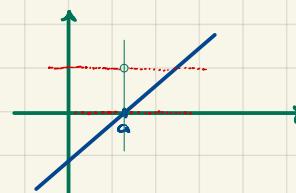
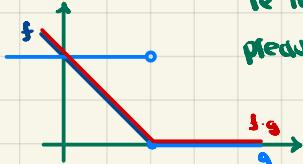
$$\neg C \Rightarrow \neg(A \wedge B) \Rightarrow (\neg A \vee \neg B) \mid (A \wedge \neg B) \mid (\neg A \wedge \neg B)$$

$$\therefore A \wedge \neg B \Rightarrow \exists \lim(f+g)$$

$\lim(f+g)$ does not exist.

$$d) \text{Th. 2 (2)} \Rightarrow \lim f + \lim g = \lim(f+g)$$

By logic it is not the case that $\lim f + \neg \lim g \Rightarrow \neg \lim(f+g)$
ie $\lim g$ not existing does not preclude $\lim(f+g)$ existing.



$$\forall \epsilon > 0 \exists \delta_1 > 0 \forall x: |x-a| < \delta_1 \Rightarrow |f(x) - l| < \epsilon$$

$$|x-a| < \delta_1 \Rightarrow |x(a+\delta_1) - a| < \delta_1 \Rightarrow |f(x(a+\delta_1)) - l| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} f(x(a+\delta_1)) = l$$

Altan. consider $g = \frac{f-g}{2}$. $\lim f-g$ exists by assumption.

If $\lim f$ exists and is $\neq 0$ then by Th2(3), $\lim \frac{1}{g}$ exists. Then $\lim \frac{f-g}{2} = \lim f - \lim g$

exists by Th2(2). If $\lim f = 0$, however, we can construct a counterexample,

in which $\lim g$ doesn't exist but $\lim(f-g)$ does, as shown on the left.

$$9. \lim_{x \rightarrow a} f(x) = \lim_{n \rightarrow \infty} f(a+n)$$

$$\lim_{x \rightarrow a} f(x) = l \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0 : |x-a| < \delta \Rightarrow |f(x) - l| < \epsilon$$

$$\lim_{n \rightarrow \infty} f(a+n) = l_2 \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0 : |a-n| < \delta \Rightarrow |f(a+n) - l_2| < \epsilon$$

Let $x = a+n$. Then $n = x-a$ and $|x-a| < \delta \Rightarrow |f(x) - l_2| < \epsilon$

$$\Rightarrow \lim_{x \rightarrow a} f(x) = l_2$$

$$\Rightarrow l = l_2 \quad (\text{because } \lim_{x \rightarrow a} f(x) \text{ is unique})$$

$$10. a) \lim_{x \rightarrow a} f(x) = l \Leftrightarrow \lim_{x \rightarrow a} (f(x) - l) = 0$$

$$\lim_{x \rightarrow a} (-l) = -l \quad \text{because } \forall \epsilon > 0, |x-a| < \delta \Rightarrow |-l - (-l)| = 0 < \epsilon$$

$$\text{Th. 2 (2)} \Rightarrow \lim_{x \rightarrow a} (f(x) + (-l)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} (-l) = 0$$

Conversely

$$\lim_{x \rightarrow a} (f(x) - l) = 0 \Rightarrow \forall \epsilon > 0, \exists \delta > 0 : |x-a| < \delta \Rightarrow |f(x) - l| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} f(x) = l$$

$$b) \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} f(x-a)$$

$$\forall \epsilon > 0, \exists \delta : |x| < \delta \Rightarrow |f(x) - l| < \epsilon$$

$$\forall \epsilon > 0, \exists \delta : |x-a| < \delta \Rightarrow |f(x-a) - l| < \epsilon$$

$$y = x-a \Rightarrow |y| < \delta \Rightarrow |f(y) - l| < \epsilon$$

$$\Rightarrow \lim_{y \rightarrow 0} f(y) = l$$

As known from a previous theorem that a limit at any point is unique $\Rightarrow l = l$

$$\Rightarrow \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} f(x-a)$$

$$c) \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} f(x^2)$$

$$\forall \epsilon > 0, \exists \delta > 0 : |x| < \delta \Rightarrow |f(x) - l| < \epsilon$$

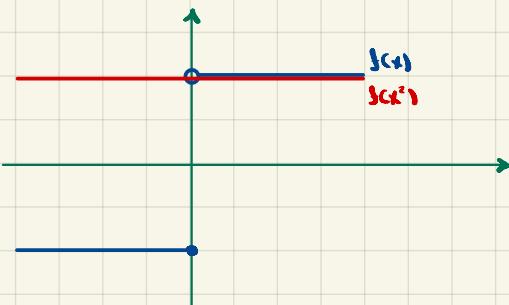
$$\forall \epsilon > 0, \exists \delta > 0 : |x| < \delta \Rightarrow |f(x^2) - l| < \epsilon$$

$$y = x^2 \Rightarrow x = \sqrt[3]{y}$$

$$|\sqrt[3]{y}| < \delta \Rightarrow |y| < \delta^3 \Rightarrow |f(y) - l| < \epsilon$$

$$\Rightarrow \lim_{y \rightarrow 0} f(y) = l \Rightarrow l = l$$

d) Example where $\lim_{x \rightarrow 0} f(x^2)$ exists but $\lim_{x \rightarrow 0} f(x)$ does not.



$$f(x) = \begin{cases} a & x > 0 \\ -a & x \leq 0 \end{cases}$$

$$f(x^2) = a \quad \forall x$$

$$\text{ii). } [0 < |x-a| < \delta \Rightarrow f(x) - g(x) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)]$$

Assume $\lim_{x \rightarrow a} f(x) = l$. Then

$$\forall \epsilon > 0 \text{ we can choose } \delta' : 0 < |x-a| < \delta' \Rightarrow |f(x)-l| < \epsilon.$$

Let $\delta' < \delta$. Then

$$0 < |x-a| < \delta' \Rightarrow f(x) - g(x), |f(x)-l| < \epsilon \Rightarrow |g(x)-l| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} g(x) = l$$

$$\text{ii. a) } f(x) \leq g(x) \quad \forall x. \text{ Does this imply } \lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)?$$

Assume $\lim_{x \rightarrow a} f(x) = l \geq m = \lim_{x \rightarrow a} g(x)$

$$\forall \epsilon > 0 \exists \delta > 0 : |x-a| < \delta \Rightarrow |f(x)-l| < \epsilon$$

$$\text{Let } \epsilon = \frac{l-m}{2}. \text{ Then } |x-a| < \delta \Rightarrow |f(x)-l| < \frac{l-m}{2}$$

$$\Rightarrow \frac{m-l}{2} < f(x)-l < \frac{l-m}{2}$$

$$\Rightarrow \frac{l+m}{2} < f(x) < \frac{3l-m}{2}$$

$$l \geq m \Rightarrow \frac{l+m}{2} > m$$

$$|x-a| < \delta_1 \Rightarrow |g(x)-m| < \frac{l-m}{2}$$

$$\Rightarrow \frac{m-l}{2} < g(x)-m < \frac{l-m}{2}$$

$$\Rightarrow \frac{3m-l}{2} < g(x) < \frac{l+m}{2}$$

We've shown that,

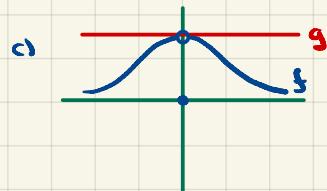
$$|x-a| < \min(\delta, \delta_1) \Rightarrow g(x) < \frac{l+m}{2} < f(x)$$

which contradicts our assumption that $f(x) \leq g(x) \forall x$.

$$\text{Yes, } f(x) \leq g(x) \forall x \Rightarrow \lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$$

b) Limits are a local property. Therefore, since we are investigating a limit near $x=a$, the property $f(x) \leq g(x)$ need hold for x such that $0 < |x-a| < \delta$, for some $\delta > 0$.

The proof by contradiction in a) showed that the assumption $l > m$ leads to a contradiction near a . No matter what happens elsewhere in the domain, if $f(x) \leq g(x)$ for $x : |x-a| < \delta$ then $\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$.



$$f(x) \leq g(x) \quad \forall x \text{ but } \lim_{x \rightarrow a} f(x) < \lim_{x \rightarrow a} g(x)$$

13. Suppose

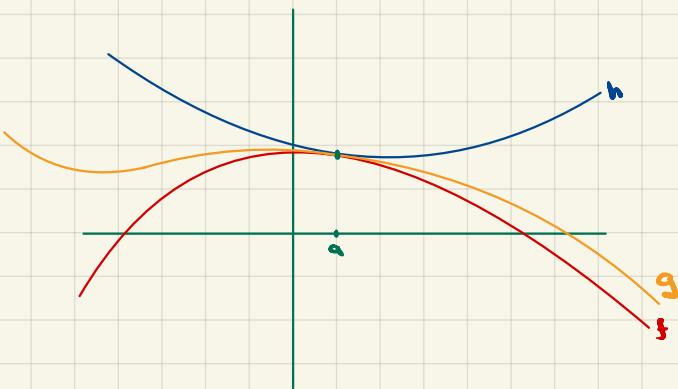
$$j(x) \leq g(x) \leq h(x)$$

$$\lim_{x \rightarrow a} j(x) = \lim_{x \rightarrow a} h(x)$$

Prove

$$\lim_{x \rightarrow a} g(x) \text{ exists}$$

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} j(x) = \lim_{x \rightarrow a} h(x)$$



$$\forall \epsilon > 0, \exists \delta > 0 : |x - a| < \delta \Rightarrow |j(x) - l| < \epsilon \\ |h(x) - l| < \epsilon$$

$$\Rightarrow l - \epsilon < j(x) < g(x) < h(x) < l + \epsilon$$

$$\Rightarrow |g(x) - l| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} g(x) = l$$

14. a) Prove

$$\lim_{x \rightarrow 0} \frac{f(x)}{x} = l, b \neq 0 \Rightarrow \lim_{x \rightarrow 0} \frac{bf(x)}{bx} = bl$$

$$\frac{f(bx)}{x} \cdot \frac{b}{b}$$

$$\lim_{x \rightarrow 0} \frac{f(bx)}{bx}$$

$$\forall \epsilon > 0, \exists \delta > 0 : |x| < \delta \Rightarrow \left| \frac{f(x)}{x} - l \right| < \epsilon$$

$$|x - by|$$

$$|bx| < \delta \Rightarrow \left| \frac{f(bx)}{bx} - l \right| < \epsilon$$

$$|y| < \frac{\delta}{|b|} \Rightarrow \left| \frac{f(bx)}{bx} - l \right| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{f(bx)}{bx} = l$$

$$\lim_{x \rightarrow 0} b \cdot \frac{f(bx)}{bx} = b \cdot l \text{ using Thm 2 (3).}$$

b) b=0

$$\frac{f(bx)}{x} \cdot \frac{f(0)}{x}$$

$\lim_{x \rightarrow 0} \frac{f(0)}{x}$ is not defined because

$\lim_{x \rightarrow 0} \frac{1}{x}$ does not exist

Consider values of the form $\frac{1}{10^n}, n \in \mathbb{N}$.

$$g(\frac{1}{10^n}) = 10^n. \forall \delta > 0, \exists n_1 \in \mathbb{N} : \frac{1}{10^{n_1}} < \delta$$

$$\forall n > n_1 \Rightarrow \frac{1}{10^n} < \frac{1}{10^{n_1}} \Rightarrow g(\frac{1}{10^n}) > g(\frac{1}{10^{n_1}})$$

$$\forall \delta > 0, \exists n_2 : g(\frac{1}{10^{n_2}}) > l \text{ and } \forall n > n_2 \Rightarrow g(\frac{1}{10^n}) > l$$

Let $\epsilon = 1, \forall \delta$

$$\forall \delta > 0, |x| < \delta \Rightarrow \exists n \in \mathbb{N} : x = \frac{1}{10^n}, |x| < \delta, g(x) = 10^n$$

$|g(x) - l| > 1 \Rightarrow \lim_{x \rightarrow 0} \frac{1}{x}$ does not exist.

$$c) \lim_{x \rightarrow 0} \frac{\sin 2x}{x} \text{ in terms of } \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = b, b \neq 0 \Rightarrow \lim_{x \rightarrow 0} \frac{\sin(bx)}{x} = b \cdot b \quad (\text{from part a})$$

Attention,

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{x} = \lim_{x \rightarrow 0} 2 \frac{\sin x}{x} \cos x \stackrel{\text{Th2(c)}}{\downarrow} = 2b \cdot 1 = 2b$$

$$15. \alpha - \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

$$ii) \lim_{x \rightarrow 0} \frac{\sin 3x}{x} = 3 \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \cos^3 x = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \sin^3 x = 3\alpha$$

$$\sin 3x = \sin(x+2x) = \sin x \cos 2x + \sin 2x \cos x$$

$$= \sin x (\cos^2 x - \sin^2 x) + 2 \sin x \cos^2 x$$

$$= 3 \sin x \cos^2 x - \sin^3 x$$

$$iii) \lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} \cdot \lim_{x \rightarrow 0} \frac{\sin cx}{x} \cdot \frac{x}{\sin bx} = a \alpha \cdot \frac{1}{b \alpha} = \frac{a}{b}$$

$$iv) \lim_{x \rightarrow 0} \frac{\sin^2 2x}{x} = \lim_{x \rightarrow 0} \sin 2x \cdot \lim_{x \rightarrow 0} \frac{\sin 2x}{x} = 0 \cdot 2\alpha = 0$$

$$\frac{(\sin 2x)^2}{x} = \frac{4 \sin^2 x \cos^2 x}{x} = 4 \frac{\sin x}{x} \cdot \sin x \cdot \cos^2 x$$

$$v) \lim_{x \rightarrow 0} \frac{\sin^2 2x}{x^2} = \left(\lim_{x \rightarrow 0} \frac{\sin 2x}{x} \right)^2 = (2\alpha)^2 = 4\alpha^2$$

$$vi) \lim_{x \rightarrow 0} \frac{1-\cos x}{x^2} = \lim_{x \rightarrow 0} \frac{1-\cos^2 x}{x^2(1+\cos^2 x)} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2(1+\cos^2 x)} = \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 \cdot \frac{1}{1+\cos^2 x} = \frac{\alpha^2}{2}$$

$$vii) \lim_{x \rightarrow 0} \frac{\tan^2 x + 2x}{x+x^2} = \lim_{x \rightarrow 0} \frac{\frac{\sin^2 x}{\cos^2 x} \cdot \frac{1}{x} + 2}{1+x} = \lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \stackrel{0}{\sin x} \frac{1}{\cos^2 x} \cdot \frac{1}{1+x} + \frac{2}{1+x} \right] = 2$$

$$viii) \lim_{x \rightarrow 0} \frac{x \sin x}{1-\cos x} = \lim_{x \rightarrow 0} \frac{x \sin x (1+\cos x)}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{1+\cos x}{\frac{\sin x}{x}} = \frac{2}{\alpha}$$

$$ix) \lim_{h \rightarrow 0} \frac{\sin(e^{ih}) - \sin x}{h} = \lim_{h \rightarrow 0} \frac{\sin x \cosh h + \sinh \cos x - \sin x}{h} = \lim_{h \rightarrow 0} \frac{\sin x (\cosh h - 1) + \sinh \cos x}{h} = \sin x \lim_{h \rightarrow 0} \frac{\cosh h - 1}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sinh}{h}$$

$$\lim_{h \rightarrow 0} \frac{\cosh h - 1}{h} = \lim_{h \rightarrow 0} (-1) \frac{1-\cosh h}{h} = -\lim_{h \rightarrow 0} \frac{1-\cosh h}{h(1+\cosh h)} = -\lim_{h \rightarrow 0} \frac{\sinh h}{h(1+\cosh h)} = -\lim_{h \rightarrow 0} \frac{\sinh h}{h} \cdot \frac{\cosh h}{1+\cosh h} = 0$$

$$x) \lim_{h \rightarrow 0} \frac{\sin(e^{ih}) - \sin x}{h} = x \cos x$$

$$\text{ix)} \lim_{x \rightarrow 1} \frac{\sin(x^2-1)}{x-1} = \lim_{x \rightarrow 1} \frac{(x+1)\sin(x^2-1)}{x^2-1} = \lim_{x \rightarrow 1} (x+1) \cdot \underbrace{\lim_{x \rightarrow 1} \frac{\sin(x^2-1)}{x^2-1}}_{=2\alpha} = 2\alpha$$

*note: $\lim_{h \rightarrow 0} \frac{\sinh}{h} = \alpha$

Let $h = x^2 - 1$

$$(|h| < \delta \Rightarrow x^2 - 1 < |x^2 - 1| < \delta \Rightarrow x^2 < 1 + \delta \Rightarrow |x| < \sqrt{1+\delta}) \Rightarrow \left| \frac{\sin(x^2-1)}{x^2-1} - \alpha \right| < \epsilon$$

we can show that

$$\forall \epsilon > 0, \exists \delta_1: |x| < \delta_1 = \sqrt{1+\delta} \Rightarrow \left| \frac{\sin(x^2-1)}{x^2-1} - \alpha \right| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin(x^2-1)}{x^2-1} = \alpha$$

$$\text{x)} \lim_{x \rightarrow 0} \frac{x^2(3+\sin x)}{(x+\sin x)^2} = \lim_{x \rightarrow 0} \frac{3+\sin x}{\left(\frac{x+\sin x}{x}\right)^2} = \lim_{x \rightarrow 0} \frac{3+\sin x}{\left(1+\frac{\sin x}{x}\right)^2} = \frac{3}{(1+\alpha)^2}$$

$$\text{xii)} \lim_{x \rightarrow 1} (x^2-1)^3 \sin\left(\frac{1}{x-1}\right)^3$$

$$\text{Note that } \forall x \neq 1, -1 \leq \sin\left(\frac{1}{x-1}\right)^3 \leq 1$$

$$\text{Problem 12c} \Rightarrow -1 \leq \lim_{x \rightarrow 1} \sin\left(\frac{1}{x-1}\right)^3 \leq 1$$

$$f(x) = -(x^2-1)^3 \leq (x^2-1)^3 \sin\left(\frac{1}{x-1}\right)^3 \leq (x^2-1)^3 = h(x)$$

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} h(x) = 0$$

$$\text{Problem 13} \Rightarrow \lim_{x \rightarrow 1} (x^2-1)^3 \sin\left(\frac{1}{x-1}\right)^3 = 0$$

$$16. a) \lim_{x \rightarrow a} f(x) = l \Rightarrow \lim_{x \rightarrow a} |f(x) - l| = 0$$

$$\forall \epsilon > 0 \exists \delta > 0 : |x - a| < \delta \Rightarrow |f(x) - l| < \epsilon$$

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$$|f(x) - l| \leq |f(x) - l| + |l - f(x)| < \epsilon \rightarrow |f(x) - l| + |l - f(x)| < 2\epsilon$$

Therefore $\forall \epsilon > 0 \exists \delta > 0 : |x - a| < \delta \Rightarrow |f(x) - l| < \epsilon$

$$\Rightarrow \lim_{x \rightarrow a} |f(x)| = l$$

$$b) \lim_{x \rightarrow a} f(x) = l, \lim_{x \rightarrow a} g(x) = m \Rightarrow \lim_{x \rightarrow a} \max(f, g) = \max(l, m)$$

$$\forall \epsilon > 0, \exists \delta_1, \delta_2 > 0 : |x - a| < \delta_1 \Rightarrow |f(x) - l| < \epsilon$$

$$|g(x) - m| < \epsilon$$

$$\text{let } \epsilon' = \frac{|l-m|}{2}$$

$$\exists \delta_2 > 0 : |x - a| < \delta_2 \Rightarrow l - \frac{|l-m|}{2} < f(x) < l + \frac{|l-m|}{2}$$

$$m - \frac{|l-m|}{2} < g(x) < m + \frac{|l-m|}{2}$$

Case 1: $l > m$

$$\frac{2l-l+m}{2} = \frac{l+m}{2} < f(x) < \frac{3l-m}{2}$$

$$\frac{2m-l+m}{2} = \frac{3m-l}{2} < g(x) < \frac{l+m}{2}$$

Therefore,

$$\forall x : |x - a| < \delta_2 \Rightarrow g(x) < f(x) \Rightarrow \max(f, g) = f(x) \Rightarrow |\max(f, g) - l| < \frac{|l-m|}{2}$$

$$\forall \epsilon > 0, \exists \delta = \min(\delta_1, \delta_2) : |x - a| < \delta \Rightarrow |\max(f, g) - l| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} \max(f, g)(x) = l = \max(l, m)$$

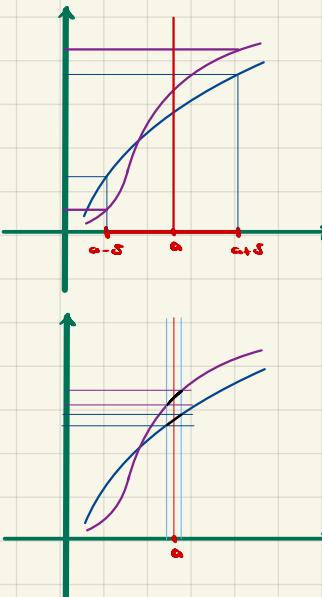
Case 2: $m > l$ By symmetry, $\lim_{x \rightarrow a} \max(f, g)(x) = m = \max(l, m)$

Case 3: $m = l$

$$\forall \epsilon > 0, \exists \delta > 0 : |x - a| < \delta \Rightarrow |f(x) - l| < \epsilon, |g(x) - l| < \epsilon \Rightarrow |\max(f, g)(x) - l| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} \max(f, g)(x) = l = m = \max(l, m)$$

The proof for $\min(f, g)(x)$ is essentially the same.



Alternatively, we can use the formula

$$\max(f, g) = \frac{f + g + |f - g|}{2}$$

$$\lim_{x \rightarrow a} \max(f, g)(x) = \lim_{x \rightarrow a} \frac{f(x) + g(x) + |f(x) - g(x)|}{2} = \frac{l + m + |l - m|}{2} = \max(l, m)$$

$$\lim_{x \rightarrow a} |f(x) - g(x)| = |l - m| \quad (\text{from a})$$

$$\min(f, g) = \frac{f + g - |f - g|}{2}$$

$$\lim_{x \rightarrow a} \min(f, g)(x) = \lim_{x \rightarrow a} \frac{f(x) + g(x) - |f(x) - g(x)|}{2} = \frac{l + m - |l - m|}{2} = \min(l, m)$$

17. a) Prove $\lim_{x \rightarrow 0} \frac{1}{x}$ does not exist.

Let $\epsilon > 0$. Consider some $\delta > 0$.

$$|x| < \delta \Rightarrow \left| \frac{1}{x} \right| - l > \frac{1}{\delta} \Rightarrow \frac{1}{\delta} - |l| < |f(x)| - |l| \leq |f(x) - l|$$

$$\forall \epsilon > 0, \exists \delta > 0 : \frac{1}{\delta} - |l| > \epsilon. \text{ Any } \delta : \delta < \frac{1}{|l| + \epsilon} \text{ means } |f(x) - l| > \frac{1}{\delta} - |l| > \epsilon$$

Therefore we have shown that

$$\forall \epsilon > 0, \text{ there is always } \delta > 0 : |x| < \delta \Rightarrow |f(x) - l| > \epsilon$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1}{x} = l \text{ is false } \forall l.$$

b) Prove $\lim_{x \rightarrow 1} \frac{1}{x-1}$ does not exist.

Let $f = x-1$

$$|x-1| < \delta \Rightarrow |f| < \delta$$

$$\text{If } \lim_{x \rightarrow 1} \frac{1}{x-1} = l \text{ then}$$

$$(\forall \epsilon > 0 \exists \delta > 0 : |x-1| < \delta \Rightarrow |f(x) - l| < \epsilon)$$

$$\Rightarrow (|f| < \delta \Rightarrow |\frac{1}{f} - l| < \epsilon) \Rightarrow \lim_{f \rightarrow 0} \frac{1}{f} = l$$

$$A \circ B \circ C$$

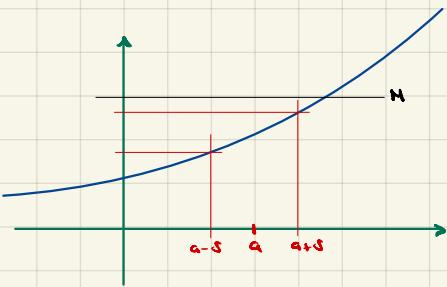
In a) we proved $\neg C$. Therefore $\neg C \circ \neg B \circ \neg A$

i.e., $\lim_{x \rightarrow 1} \frac{1}{x-1} = l$ is false $\forall l$. Therefore the limit does not exist.

18. Prove

$$\lim_{x \rightarrow a} f(x) = L \Rightarrow \exists \delta > 0, N : 0 < |x - a| < \delta \Rightarrow |f(x) - L| < N$$

means: If x is close to a then $f(x)$ is close to L and we can find a number N less than $|f(x)|$ in the interval that defines what "close to L " means.



$$\forall \epsilon > 0 \exists \delta > 0 : 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$

$$L - \epsilon < f(x) < L + \epsilon$$

$$\text{Let } N = \max(|L - \epsilon|, |L + \epsilon|).$$

$$\text{Then, } |f(x)| < N$$

19. Prove

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q}' \\ 1 & x \in \mathbb{Q} \end{cases} \Rightarrow \lim_{x \rightarrow a} f(x) \text{ does not exist, } \forall a.$$

$\forall \delta > 0$ consider $x \in \mathbb{R} : |x - a| < \delta$.

$$x \in \mathbb{Q}' \Rightarrow f(x) = 0 \Rightarrow |f(x) - L| = |0|$$

$$x \in \mathbb{Q} \Rightarrow f(x) = 1 \Rightarrow |f(x) - L| = |1 - L|$$

\Rightarrow value of L that minimizes the maximum $|f(x) - L|$ is $L = 0.5$.

This minimum is 0.5. $\Rightarrow |f(x) - L| \geq 0.5$

Therefore $\forall \epsilon < 0.5$, every choice of $\delta > 0$ leads to $|f(x) - L| \geq 0.5 > \epsilon$

$\Rightarrow \lim_{x \rightarrow a} f(x) = L$ is false $\forall L \in \mathbb{R}$.

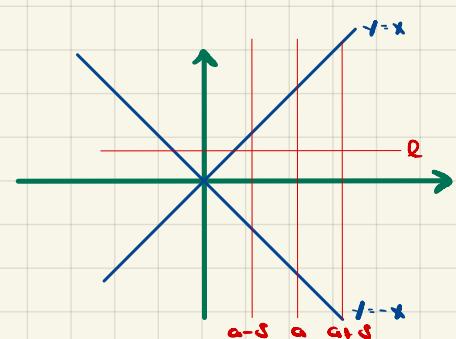
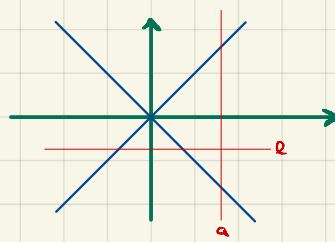
20. Proof

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ -x & x \in \mathbb{Q}' \end{cases} \Rightarrow \lim_{x \rightarrow a} f(x) \text{ does not exist if } a \neq 0$$

$\forall \epsilon > 0$, consider $\delta > 0$ and $x : |x - a| < \delta$

$$x \in \mathbb{Q} \Rightarrow |f(x) - \ell| = |x - \ell|$$

$$x \in \mathbb{Q}' \Rightarrow |f(x) - \ell| = |-x - \ell| = |-(x + \ell)| = |x + \ell|$$



Case 1: $a > 0, \ell > 0$

If $x \in \mathbb{R} : a - \delta < x < a + \delta, x \in \mathbb{Q}$. Then $|f(x) - \ell| = |x - \ell|$

Let $x_+ \in \mathbb{Q}' : 0 < x_+ < a + \delta$.

$$\text{Then } |f(x_+) - \ell| = |x_+ - \ell| > |a + \ell| \geq |a| = a$$

Therefore in the case where $a > 0, \ell > 0, \forall \epsilon > 0 \exists \delta < a, \forall \delta > 0 : |x - a| < \delta$

$$\Rightarrow \exists x : |f(x) - \ell| > a > \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} f(x) - \ell \text{ is false}$$

Case 2: $a > 0, \ell < 0$

Similar to case 1 but we look at $x \in \mathbb{Q}'$.

If $x \in \mathbb{R} : x \in (a - \delta, a + \delta), x \in \mathbb{Q}$ then $|f(x) - \ell| = |x - \ell| = |x + |\ell|| \geq |x|$

Let $x_+ \in \mathbb{R} : a < x_+ < a + \delta$

$$\text{Then } |f(x_+) - \ell| \geq |x_+| > |a| = a$$

Therefore

$a > 0, \ell < 0, \forall \epsilon > 0 : 0 < \delta < a, \forall \delta > 0 : |x - a| < \delta$

$$\Rightarrow \exists x : |f(x) - \ell| > a > \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} f(x) - \ell \text{ is false.}$$

Case 3: $a > 0, \ell > 0$

If $x \in \mathbb{R}$: $a - \delta < x < a$, $x \in \mathbb{Q}$ then $|f(x) - \ell| = |x - \ell|$

Let $x \in \mathbb{Q}$: $a - \delta < x < a$

Then $|f(x) - \ell| = |x - \ell| > |a - \ell| \geq |\ell|$

Therefore in the case where $a > 0, \ell > 0, \forall \epsilon > 0 \exists \delta > 0 : |x - a| < \delta$

$\Rightarrow \exists x : |f(x) - \ell| > |\ell| > \epsilon$

$\Rightarrow \lim_{x \rightarrow a} f(x) \neq \ell$

Case 4: $a < 0, \ell < 0$

Analogous to Case 3 but take $x \in \mathbb{Q}'$.

Case 5: $a = 0, \ell = 0$

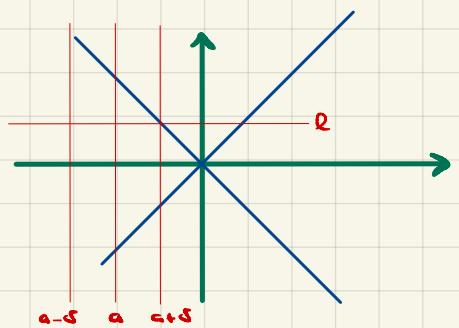
Let $|x| < \delta, \delta > 0$.

$x \in \mathbb{Q} \Rightarrow |f(x) - \ell| = |x| < \delta$

$x \in \mathbb{Q}' \Rightarrow |f(x) - \ell| = |x| < \delta$

Therefore, $\forall \epsilon > 0, \exists \delta < \epsilon \Rightarrow |f(x)| < \epsilon$

$\Rightarrow \lim_{x \rightarrow 0} f(x) = 0$



$$21. \text{ a) Prove } \lim_{x \rightarrow 0} g(x) = 0 \Rightarrow \lim_{x \rightarrow 0} g(x) \sin\left(\frac{1}{x}\right) = 0$$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$-g(x) \leq g(x) \sin\left(\frac{1}{x}\right) \leq g(x), \lim_{x \rightarrow 0} (-g(x)) = \lim_{x \rightarrow 0} g(x) \Rightarrow \lim_{x \rightarrow 0} g(x) \sin\left(\frac{1}{x}\right) = 0$$

as proved in Problem 13

b) Generalization

$$\lim_{x \rightarrow 0} g(x) = 0 \Rightarrow \lim_{x \rightarrow 0} g(x)h(x) = 0$$

$$|h(x)| \leq M \quad \forall x$$

$$-M \leq h(x) \leq M$$

$$-g(x)M \leq g(x)h(x) \leq g(x)M, \lim_{x \rightarrow 0} (-Mg(x)) = \lim_{x \rightarrow 0} Mg(x) = 0 \Rightarrow \lim_{x \rightarrow 0} g(x)h(x) = 0$$

as proved in Problem 13

$$22. \text{ vs } \left[\left(\forall g \left(\nexists \lim_{x \rightarrow 0} g(x) \rightarrow \nexists \lim_{x \rightarrow 0} (f(x) + g(x)) \right) \right) \leftrightarrow \exists \lim_{x \rightarrow 0} f(x) \right]$$

contrapositive

Let f be a function.

Let's prove the contrapositive of $\exists \lim_{x \rightarrow 0} f(x) \rightarrow (\forall g \left(\nexists \lim_{x \rightarrow 0} g(x) \rightarrow \nexists \lim_{x \rightarrow 0} (f(x) + g(x)) \right))$

$$\text{Namely, } \neg(\forall g \left(\nexists \lim_{x \rightarrow 0} g(x) \rightarrow \nexists \lim_{x \rightarrow 0} (f(x) + g(x)) \right)) \rightarrow \exists \lim_{x \rightarrow 0} f(x)$$

$$\text{Assume } \neg(\forall g \left(\nexists \lim_{x \rightarrow 0} g(x) \rightarrow \nexists \lim_{x \rightarrow 0} (f(x) + g(x)) \right))$$

$$\text{Then } \exists g, \nexists \lim_{x \rightarrow 0} g(x) \wedge \exists \lim_{x \rightarrow 0} (f(x) + g(x))$$

$$\text{Then } \exists g, \nexists \lim_{x \rightarrow 0} g(x) \wedge \exists \lim_{x \rightarrow 0} (f(x) + g(x))$$

Assume $\lim_{x \rightarrow 0} f(x)$ exists.

In problem 8c we showed that $\exists \lim_{x \rightarrow 0} f(x) \wedge \nexists \lim_{x \rightarrow 0} g(x) \rightarrow \nexists \lim_{x \rightarrow 0} (f(x) + g(x))$

Therefore $\nexists \lim_{x \rightarrow 0} (f(x) + g(x))$. \perp .

Therefore, $\lim_{x \rightarrow 0} f(x)$.

$$\text{Therefore } \neg(\forall g \left(\nexists \lim_{x \rightarrow 0} g(x) \rightarrow \nexists \lim_{x \rightarrow 0} (f(x) + g(x)) \right)) \rightarrow \exists \lim_{x \rightarrow 0} f(x)$$

So, $\exists \lim_{x \rightarrow 0} f(x) \rightarrow (\forall g \left(\nexists \lim_{x \rightarrow 0} g(x) \rightarrow \nexists \lim_{x \rightarrow 0} (f(x) + g(x)) \right))$ by the law of contrapositive.

For the converse, we also prove the contrapositive of $\forall g \left(\nexists \lim_{x \rightarrow 0} g(x) \rightarrow \nexists \lim_{x \rightarrow 0} (f(x) + g(x)) \right) \leftrightarrow \exists \lim_{x \rightarrow 0} f(x)$

$$\text{Namely, } \nexists \lim_{x \rightarrow 0} f(x) \rightarrow \neg(\forall g \left(\nexists \lim_{x \rightarrow 0} g(x) \rightarrow \nexists \lim_{x \rightarrow 0} (f(x) + g(x)) \right))$$

Assume $\exists \lim_{x \rightarrow 0} f(x)$

As we showed in Problem 8a, if $\exists \lim_{x \rightarrow 0} f(x) = \exists \lim_{x \rightarrow 0} g(x)$ then $\lim_{x \rightarrow 0} (f(x) + g(x))$ can exist.

For example, if $g(x) = -f(x)$ then $\exists \lim_{x \rightarrow 0} g(x)$, $(f+g)(x) = 0$, and $\lim_{x \rightarrow 0} (f+g)(x) = 0$.

Therefore $\exists g, \exists \lim_{x \rightarrow 0} g(x) \wedge \exists \lim_{x \rightarrow 0} (f(x) + g(x))$

Therefore $\exists \lim_{x \rightarrow 0} f(x) \rightarrow \neg (\forall g (\exists \lim_{x \rightarrow 0} g(x) \rightarrow \exists \lim_{x \rightarrow 0} (f(x) + g(x))))$

And by Law of Contraposition

$\forall g (\exists \lim_{x \rightarrow 0} g(x) \rightarrow \exists \lim_{x \rightarrow 0} (f(x) + g(x))) \rightarrow \exists \lim_{x \rightarrow 0} f(x)$

Hence we can assert that

$\forall g (\exists \lim_{x \rightarrow 0} g(x) \rightarrow \exists \lim_{x \rightarrow 0} (f(x) + g(x))) \leftrightarrow \exists \lim_{x \rightarrow 0} f(x)$

And by universal intro,

$\forall f (\forall g (\exists \lim_{x \rightarrow 0} g(x) \rightarrow \exists \lim_{x \rightarrow 0} (f(x) + g(x))) \leftrightarrow \exists \lim_{x \rightarrow 0} f(x))$

$$23. \forall g (\lim_{x \rightarrow 0} g(x) \rightarrow \lim_{x \rightarrow 0} (f(x) \cdot g(x))) \leftrightarrow \lim_{x \rightarrow 0} f(x)$$

a)

Let f be any fn. **Vintro Ass**

Assume $\lim_{x \rightarrow 0} f(x) \neq 0$. Let $\lim_{x \rightarrow 0} f(x) = b$. \rightarrow **Intro Ass**

$$\begin{aligned} \forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \ |x - a| < \delta \rightarrow |f(x) - b| < \epsilon \\ \exists \delta > 0 \ \forall \epsilon > 0 \ \forall x \ |x - a| < \delta \rightarrow |f(x) - b| < \epsilon \\ \exists \delta > 0 \ \forall \epsilon > 0 \ \exists x \ |x - a| < \delta \wedge |f(x) - b| < \epsilon \end{aligned}$$

Let g be any fn such that $\lim_{x \rightarrow 0} g(x) \neq 0$ and let $L = \lim_{x \rightarrow 0} g(x)$. **Vintro Ass**

choose $f(x) \neq 0$ we can write $g = \frac{f}{f}$. \rightarrow this part is still too imprecise.

$\exists \delta > 0$ then $\exists n \frac{1}{\delta}$.

$$\text{Then } \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} \frac{1}{f(x)} \cdot f(x) \cdot g(x) = \frac{1}{b} \cdot L \text{ by Th2(2).}$$

$\lim_{x \rightarrow 0} g(x)$

$$\forall g (\lim_{x \rightarrow 0} f(x) \neq 0 \rightarrow \lim_{x \rightarrow 0} g(x)) \text{ Vintro}$$

By law of contrapositive, $\forall g (\lim_{x \rightarrow 0} g(x) \rightarrow \lim_{x \rightarrow 0} (f(x) \cdot g(x)))$

Therefore $\lim_{x \rightarrow 0} f(x) \neq 0 \rightarrow \forall g (\lim_{x \rightarrow 0} g(x) \rightarrow \lim_{x \rightarrow 0} (f(x) \cdot g(x))) \rightarrow$ **Intro**

$$\forall \lim_{x \rightarrow 0} f(x) \neq 0 \rightarrow \forall g (\lim_{x \rightarrow 0} g(x) \rightarrow \lim_{x \rightarrow 0} (f(x) \cdot g(x))) \text{ Vintro}$$

$$b) \lim_{x \rightarrow 0} |f(x)| = \infty$$

$$\text{Proposition: } \lim_{x \rightarrow 0} |f(x)| = \infty \rightarrow \lim_{x \rightarrow 0} f(x)$$

$$\text{Assume } \lim_{x \rightarrow 0} |f(x)| = \infty.$$

$$\text{Then } \forall N \in \mathbb{R} \ \exists \delta > 0 : \forall x \ 0 < |x| < \delta \rightarrow |f(x)| > N.$$

$$\text{Let } N_1 \text{ be one such } N.$$

$$\text{choose } \epsilon, \text{ such that } 0 < \epsilon < N_1.$$

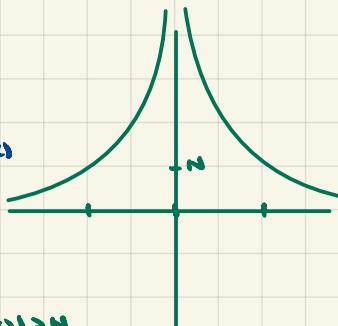
$$\text{Then } \exists \delta > 0 \text{ such that } \forall x \ 0 < |x| < \delta \rightarrow |f(x)| > N_1.$$

$$\text{For any } \delta < \delta_1 \text{ it is also true that } 0 < |x| < \delta \rightarrow |f(x)| > N_1.$$

$$\text{For any } \delta > \delta_1 \text{ we have } 0 < |x| < \delta < \delta_1 \text{ so } \exists x : 0 < |x| < \delta \text{ such that } |f(x)| > N_1.$$

$$\text{Therefore } \forall \delta > 0 \ \exists x : 0 < |x| < \delta \rightarrow |f(x)| > N_1 \geq \epsilon.$$

$$\text{Therefore } \lim_{x \rightarrow 0} f(x) \quad \square$$



$$\text{Proposition: } \lim_{x \rightarrow 0} |f(x)| = \infty \rightarrow \lim_{x \rightarrow 0} \frac{1}{|f(x)|} = 0$$

$$\text{Assume } \lim_{x \rightarrow 0} |f(x)| = \infty.$$

$$\text{Then } \forall N > 0 \ \exists \delta > 0 : \forall x \ 0 < |x| < \delta \rightarrow |f(x)| > N.$$

$$\text{Let } \epsilon > 0.$$

$$\text{Let } N_2 = \frac{1}{\epsilon}.$$

$$\text{choose } N_1 \text{ such that } 0 < N_1 < N_2.$$

$$\text{Then,}$$

$$|f(x)| > N_1 \rightarrow \frac{1}{|f(x)|} < \frac{1}{N_1}.$$

$$\forall x \ 0 < |x| < \delta \rightarrow \left| \frac{1}{|f(x)|} \right| < \frac{1}{N_1} < \frac{1}{N_2} = \epsilon$$

Therefore

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \ 0 < |x| < \delta \rightarrow \left| \frac{1}{|f(x)|} \right| < \frac{1}{N_1} < \frac{1}{N_2} = \epsilon$$

$$\text{Therefore } \lim_{x \rightarrow 0} \frac{1}{|f(x)|} = 0$$

$$\text{Therefore } \lim_{x \rightarrow 0} |f(x)| = \infty \rightarrow \lim_{x \rightarrow 0} \frac{1}{|f(x)|} = 0$$

If we assume $\lim_{x \rightarrow 0} |f(x)| = \infty$, then we assume $\lim_{x \rightarrow 0} f(x)$. If we can prove

$\neg (\forall g (\lim_{x \rightarrow 0} g(x) \rightarrow \lim_{x \rightarrow 0} (f(x) \cdot g(x))))$ then we will have proved the

other direction in the original biconditional.

Let f be any fn. VITALLY

Assume $\lim_{x \rightarrow 0} |f(x)| = \infty$

Let g be any fn such that $\lim_{x \rightarrow 0} f(x)g(x)$

$$g = \frac{f}{f}$$

we've shown that $\lim_{x \rightarrow 0} |f(x)| = \infty \rightarrow \lim_{x \rightarrow 0} \frac{1}{|f(x)|} = 0$

therefore $\lim_{x \rightarrow 0} \frac{1}{|f(x)|} = 0$.

Since $\lim_{x \rightarrow 0} f(x)g(x)$ by assumption, then $\lim_{x \rightarrow 0} g(x) = 0 \cdot \lim_{x \rightarrow 0} f(x) = 0$

Therefore $\forall g (\lim_{x \rightarrow 0} f(x)g(x) \rightarrow \lim_{x \rightarrow 0} g(x))$

By contraposition $\forall g (\lim_{x \rightarrow 0} g(x) \rightarrow \lim_{x \rightarrow 0} f(x)g(x))$

c) $\neg(\lim_{x \rightarrow 0} f(x) \neq 0) \wedge \neg(\lim_{x \rightarrow 0} |f(x)| = \infty) \rightarrow \exists g, \lim_{x \rightarrow 0} g(x) \sim \lim_{x \rightarrow 0} f(x)g(x)$

$\neg(\lim_{x \rightarrow 0} f(x) \sim \lim_{x \rightarrow 0} f(x) + 0)$

$\lim_{x \rightarrow 0} f(x) \vee \lim_{x \rightarrow 0} f(x) = 0$

24. $n \in \mathbb{N}$

$A_n = \text{some finite set of numbers in } [0, 1]$

A_n and A_m have no members in common if $m \neq n$

$$f(x) = \begin{cases} \frac{1}{n} & x \in A_n \\ 0 & \forall n \quad x \notin A_n \end{cases}$$

Prove $\lim_{x \rightarrow a} f(x) = 0$ for all $a \in [0, 1]$

Proof

Let a be one such $a \in [0, 1]$.

Let $\epsilon > 0$.

$$\text{Then } \exists n_1, n_2 \in \mathbb{N} : \frac{1}{n_1} < \epsilon < \frac{1}{n_2} \Leftrightarrow n_2 > n_1.$$

Let $B = \{x \in \mathbb{N} : x < n_1\}$, a finite set of natural numbers.

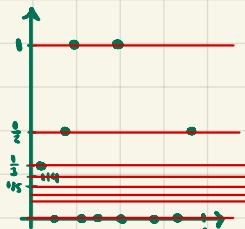
Hence the set $C = \{x : x \in A_n, n \in B\}$ is finite.

There is a minimum distance from the elements in C to a . Call it d_{\min} .

$$\text{Then, } x \in (a - d_{\min}, a + d_{\min}) \rightarrow x \notin C \rightarrow x \notin A_n, n \geq n_1 \rightarrow f(x) \leq \frac{1}{n_1} < \epsilon$$

Therefore $\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \quad 0 < |x - a| < \delta \rightarrow f(x) < \frac{1}{n_1} < \epsilon$.

$$\rightarrow \lim_{x \rightarrow a} f(x) = 0$$



$$A_1 = \{1/4, 1/2\}$$

$$A_2 = \{1/6, 1/3\}$$

$$A_3 = \{1/10, 1/5\}$$

25. Each definition implies the standard def of limit. If we apply the nonstandard def, then the standard def is necessary for the nonstandard def.

$$\text{i) } \forall \delta > 0 \exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \epsilon \rightarrow |f(x) - l| < \delta$$

standard definition of $\lim_{x \rightarrow a} f(x) = l$.

$$\text{iii) } \forall \delta > 0 \exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \epsilon \rightarrow |f(x) - l| < \delta$$

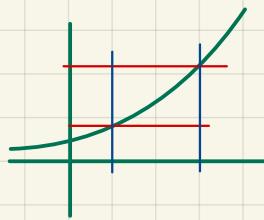
Let $\delta_1 > 0$.

Let $0 < \delta_2 < \delta_1$

Then $\exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \epsilon \rightarrow |f(x) - l| \leq \delta_1 < \delta_2$

Therefore

$$\forall \delta > 0 \exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \epsilon \rightarrow |f(x) - l| < \delta$$



Therefore

$$(\forall \delta > 0 \exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \epsilon \rightarrow |f(x) - l| < \delta) \rightarrow (\forall \delta > 0 \exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \epsilon \rightarrow |f(x) - l| < \delta)$$

$$(\forall \delta > 0 \exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \epsilon \rightarrow |f(x) - l| < \delta) \rightarrow \lim_{x \rightarrow a} f(x) = l$$

$$\text{iii) } \forall \delta > 0 \exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \epsilon \rightarrow |f(x) - l| < \delta$$

Let $\delta > 0$.

Let $\delta = \frac{\delta}{5}$.

Then $\exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \epsilon \rightarrow |f(x) - l| < \delta$.

$$\forall \delta > 0 \exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \epsilon \rightarrow |f(x) - l| < \delta$$

Therefore,

$$(\forall \delta > 0 \exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \epsilon \rightarrow |f(x) - l| < \delta) \rightarrow (\forall \delta > 0 \exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \epsilon \rightarrow |f(x) - l| < \delta)$$

$$(\forall \delta > 0 \exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \epsilon \rightarrow |f(x) - l| < \delta) \rightarrow \lim_{x \rightarrow a} f(x) = l$$

$$\text{iv) } \forall \delta > 0 \exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \frac{\epsilon}{10} \rightarrow |f(x) - l| < \delta$$

Let $\delta > 0$.

Let $\epsilon > 0$ be such that $\forall x \quad 0 < |x - a| < \frac{\epsilon}{10} \rightarrow |f(x) - l| < \delta$

Let $\epsilon_1 = \frac{\epsilon}{10}$.

Then $\forall x \quad 0 < |x - a| < \epsilon_1 \rightarrow |f(x) - l| < \delta$

$$\exists \epsilon_1 > 0 : \forall x \quad 0 < |x - a| < \epsilon_1 \rightarrow |f(x) - l| < \delta$$

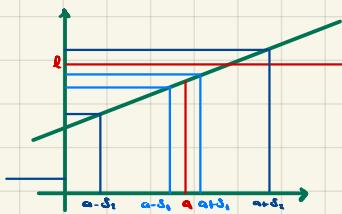
$$\forall \delta > 0 \exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \epsilon \rightarrow |f(x) - l| < \delta$$

$$(\forall \delta > 0 \exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \frac{\epsilon}{10} \rightarrow |f(x) - l| < \delta) \rightarrow (\forall \delta > 0 \exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \epsilon \rightarrow |f(x) - l| < \delta)$$

$$(\forall \delta > 0 \exists \epsilon > 0 : \forall x \quad 0 < |x - a| < \frac{\epsilon}{10} \rightarrow |f(x) - l| < \delta) \rightarrow \lim_{x \rightarrow a} f(x) = l$$

26.

$$\text{a) } \forall \delta > 0 \exists \epsilon > 0 \quad 0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon$$



$$0 < |x - a| < \delta \rightarrow |f(x) - L| < L - f(a - \delta) = \epsilon.$$

Here we see that $\lim_{x \rightarrow a} f(x) = L$, for for any interval we can choose an ϵ to bind the difference between $f(x)$ for x in the interval and the supposed limit. In fact this definition of limit is satisfied by all numbers.

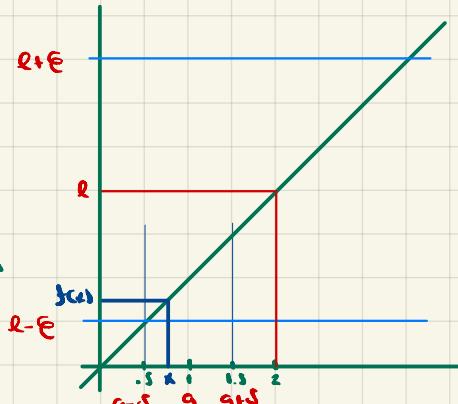
example:

$$f(x) = x$$

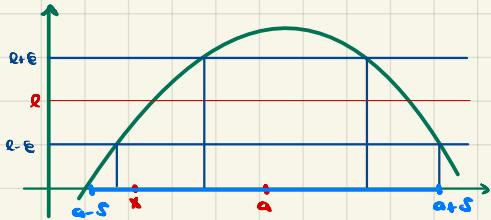
$$0 < |x - 1| < 0.5 \rightarrow |f(x) - 1| < 1.5$$

For any interval containing $x = a$, we can find an interval containing $f(a)$ and a number L .

Given any a , this is true for any L .



$$\text{b) } \forall \epsilon > 0 \exists \delta > 0 \quad |f(x) - L| < \epsilon \rightarrow 0 < |x - a| < \delta$$



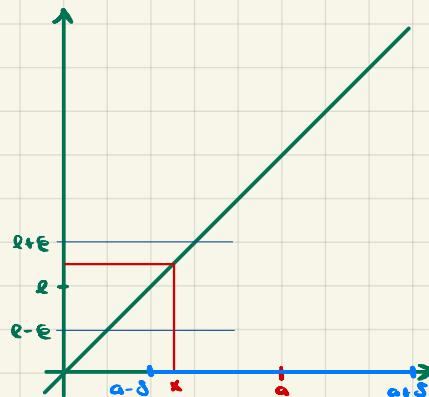
Given any L , the statement is true for any a .

example:

$$f(x) = x$$

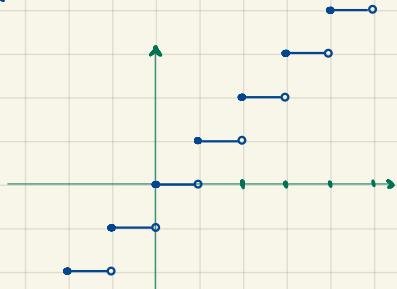
For any interval containing $f(a)$ and some L , we can find an interval containing $x = a$ and some a .

Given L this is true for all a .



21. $[x] = \text{largest } z \leq x$

i) $f(x) = [x]$

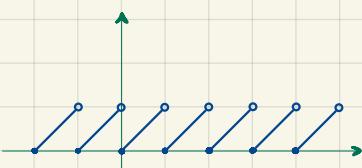


Limits From Above/Below

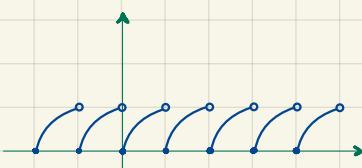
$$\lim_{x \rightarrow a^+} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) = L$$

$$\forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$

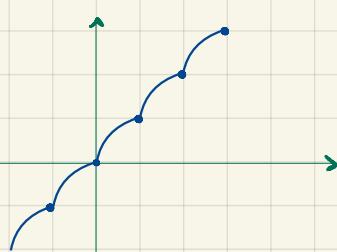
ii) $f(x) = x - [x]$



iii) $f(x) = \sqrt{x - [x]}$

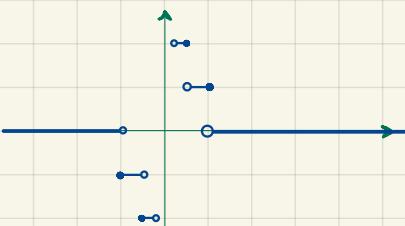


iv) $f(x) = [x] + \sqrt{x - [x]}$



For i), ii), iii), iv) $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow a^-} f(x)$ exist for all $x \in \mathbb{R}$.

v) $f(x) = \left[\frac{1}{x} \right]$

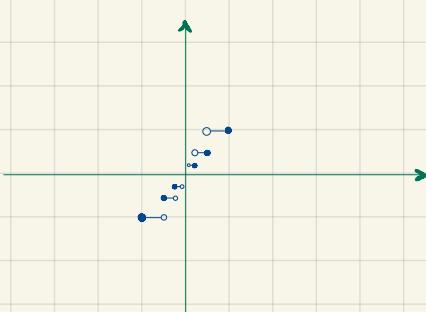


As x approaches 0^+ , $f(x)$ approaches ∞ .

As x approaches 0^- , $f(x)$ approaches $-\infty$.

Neither $\lim_{x \rightarrow 0^+} f(x)$ nor $\lim_{x \rightarrow 0^-} f(x)$ exist.

vi) $f(x) = \frac{1}{\left[\frac{1}{x} \right]}$

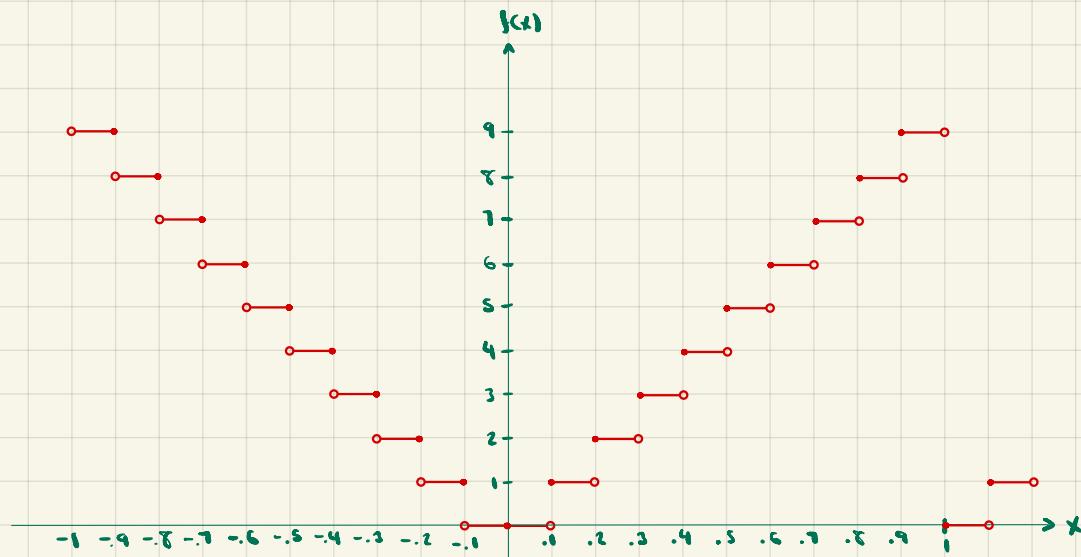
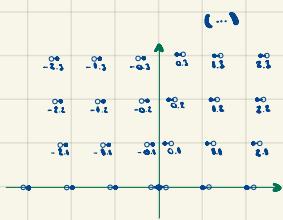


$\lim_{x \rightarrow a^+} f(x)$ exists for $\forall x \in (-1, 1]$

$\lim_{x \rightarrow a^-} f(x)$ exists for $\forall x \in [-1, 1)$

$f(x)$ is not defined for $x > 1$ or $x < -1$.

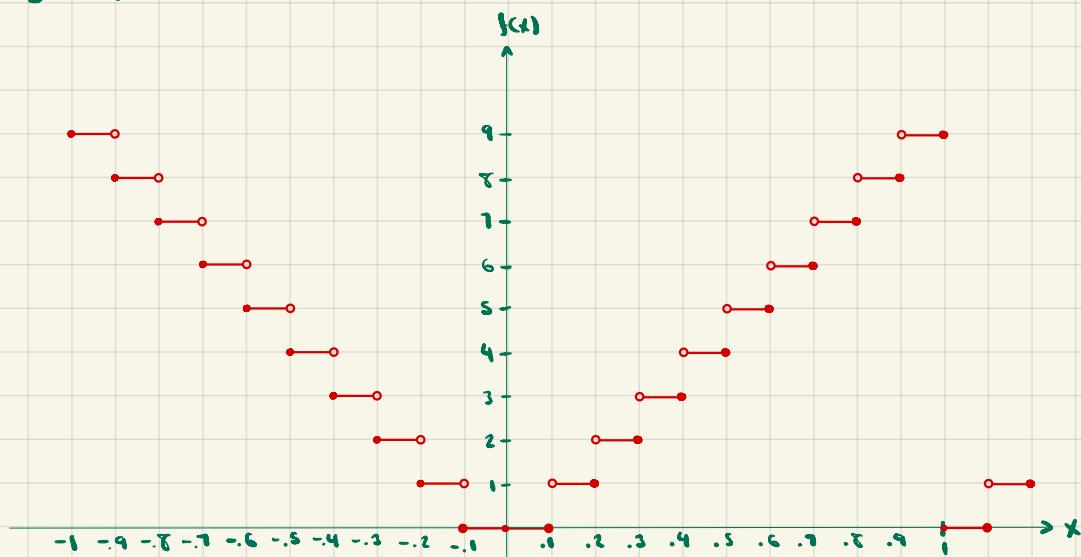
a)

ii) $f(x) = 1^{\text{st}} \text{ number of decimal expansion of } x$ 

However, we are considering that, e.g., $0.\bar{0} = 1$.

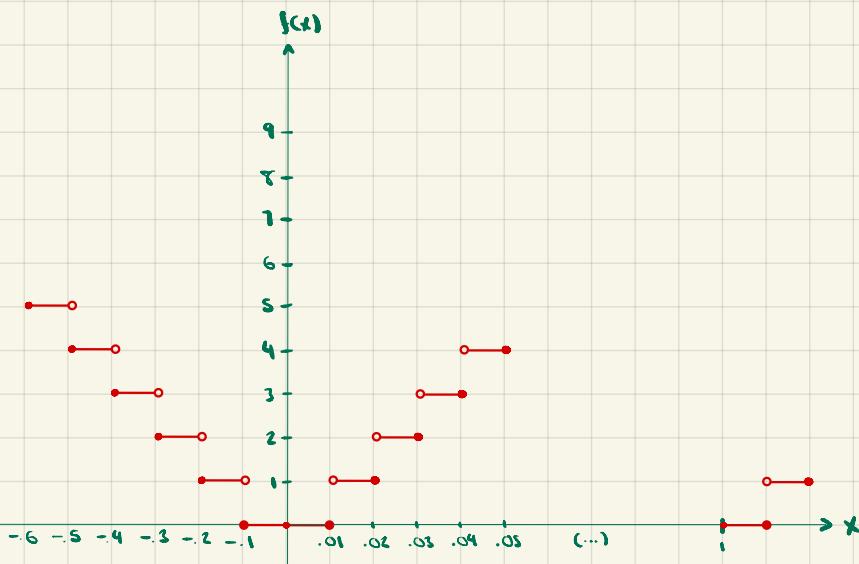
$$\Rightarrow f(0.1) = f(0.\bar{0}) = 0$$

$$f(0.2) = f(0.1\bar{9}) = 1$$



Both $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow a^-} f(x)$ exist for all $x \in \mathbb{R}$.

ii) $f(x) = 2^{\text{nd}} \text{ number in decimal expansion of } x$



Both $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ exist for all $x \in \mathbb{R}$.

iii) $f(x) = \begin{cases} \text{number of 7's in decimal expansion of } x \text{ if this number is finite,} \\ 0 \text{ otherwise.} \end{cases}$

Infinite 7's

Rational numbers w/ infinite 7's

Example of repeating decimal repes. e.g. $0.\overline{7}, 0.\overline{7}\overline{4}, 0.\overline{2}\overline{4}\overline{7}$

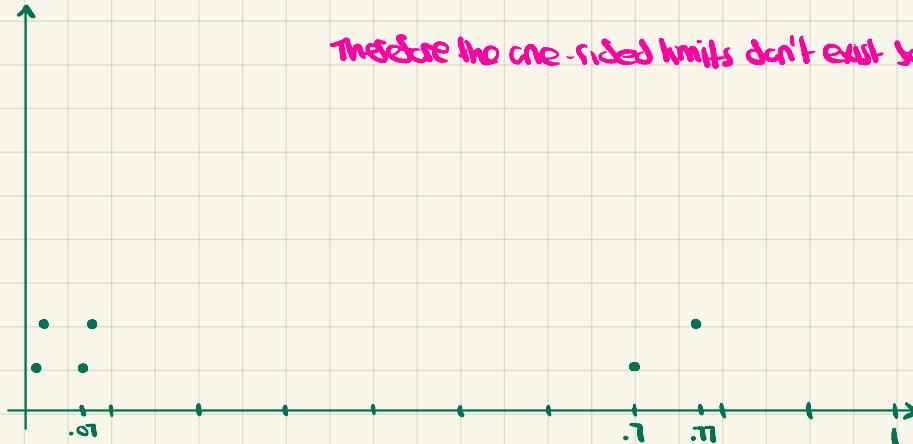
For any number a , for any $\delta > 0$ consider the set $A = [a-\delta, a+\delta]$.

There are infinitely many numbers with a finite number of 7's in A and an infinite number of numbers w/ infinite 7's.

Hence for any $a \in \mathbb{R}$ the definition of limit will fail for a .

Finite 7's

0.7	0.77
0.07	0.077
0.007	0.0077



Therefore the one-sided limits don't exist for any a .

In any interval there are infinite numbers w/ any number of 7's, from zero to infinity.



$$0.\overline{72} \Rightarrow 10^2 \cdot 0.\overline{72} = 72 + 0.\overline{72} = 72 = 0.\overline{72}(10^2 - 1)$$

$$= 0.\overline{72} \cdot \frac{72}{99}$$

iv)

$\lim_{x \rightarrow a^+} f(x) = 0$ if number of 7's in dec. exp. of x is finite, 1 otherwise.



$\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ do not exist for some reason (d, iii).

v) $f(x)$ - number obtained by replacing all digits in decimal expansion of x which come after first 7, if any, by 0.

$$f(0.77) = 0.7$$

$$f(0.07) = 0.07$$

$$f(0.\overline{72}) = 0.7$$

$$f(0.7\overline{9}) = 0.7$$

$0.0\overline{9}$ is max subtraction.

Given any number of form $m.\overline{abcd7e\dots}$

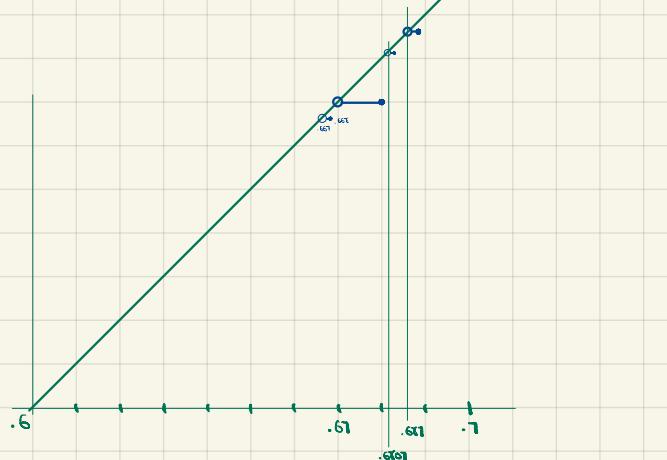
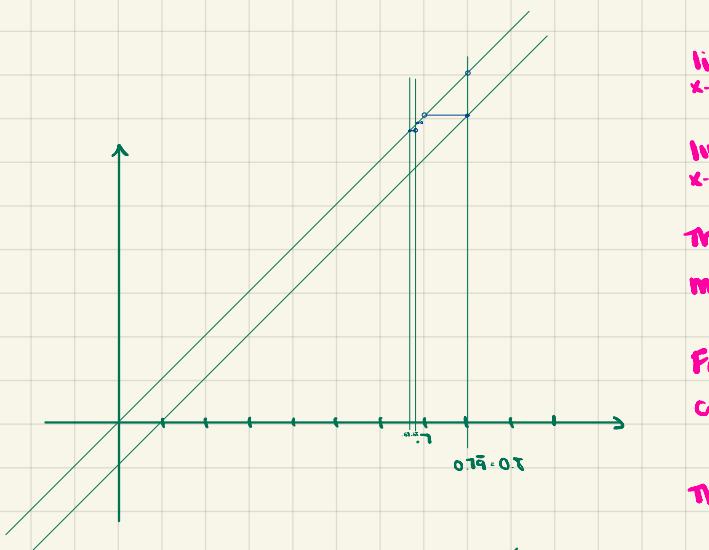
$\lim_{x \rightarrow a^+} f(x)$ exists for any $x \in (m.\overline{abcd7}, m.\overline{abcd8}]$

$\lim_{x \rightarrow a^-} f(x)$ exists for any $x \in [m.\overline{abcd7}, m.\overline{abcd8})$

The numbers $m.\overline{abcd7}$ and $m.\overline{abcd8}$ are called $m.\overline{abcd6}\overline{9}$ and $m.\overline{abcd7}\overline{9}$, respectively. They have left and right-sided limits, respectively.

For any number a not containing any 7's in decimal expansion, both one-sided limits exist and equal a .

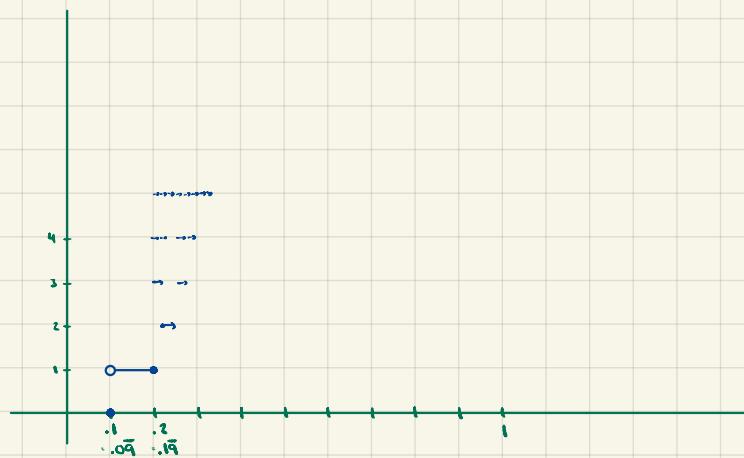
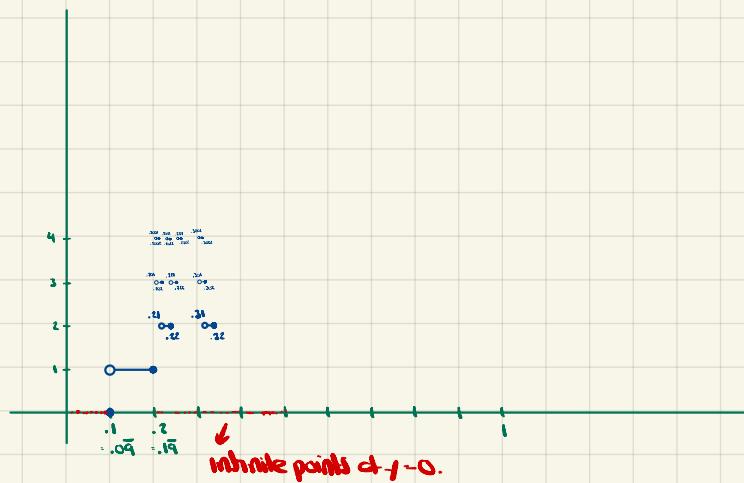
Therefore both one-sided limits exist for all x .



F is composed of infinite intervals like the ones above, with points on the line $y = 1$ for numbers x with no 7's in them.

vii) $f(x) =$

- o if 1 never appears in decimal expansion of x
- n if 1 first appears in n^{th} place



The graph of $f(x)$ is composed of infinite intervals as above. They are of form $(m, abcl, m, abcl_2]$. Between two such intervals are infinite line intervals with $f(x) = 0$ for every number from 0 to ∞ .

Indicate the one-sided limits exist for every point in each interval, and the left open side has only the $\lim_{x \rightarrow a^-} f(x)$ and the right closed side has $\lim_{x \rightarrow a^+} f(x)$.

The limits do not exist for points w/ no 1s, i.e. the ones on the $y=0$ line.

$$29. \text{ Prove } \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) \rightarrow \exists \lim_{x \rightarrow a} f(x)$$

$$\text{Assume } \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$$

Then from the definitions of these limits:

$$\forall \epsilon > 0 \exists \delta_1 > 0 : \forall x \quad 0 < x - a < \delta_1 \rightarrow |f(x) - L| < \epsilon$$

$$\forall \epsilon > 0 \exists \delta_2 > 0 : \forall x \quad 0 < a - x < \delta_2 \rightarrow |f(x) - L| < \epsilon$$

Let $\epsilon > 0$, and δ_1, δ_2 such that

$$\delta_1 > 0 \wedge \forall x \quad 0 < x - a < \delta_1 \rightarrow |f(x) - L| < \epsilon$$

$$\delta_2 > 0 \wedge \forall x \quad 0 < a - x < \delta_2 \rightarrow |f(x) - L| < \epsilon$$

$$\text{let } \delta_{\min} = \min(\delta_1, \delta_2)$$

$$\text{Then, } \delta_{\min} > 0 \text{ and } \forall x \quad 0 < x - a < \delta_{\min} \rightarrow |f(x) - L| < \epsilon \text{ and } \forall x \quad -\delta_{\min} < x - a < 0 \rightarrow |f(x) - L| < \epsilon$$

$$\text{That is } \delta_{\min} > 0 \wedge \forall x \quad ((0 < x - a < \delta_{\min}) \vee (-\delta_{\min} < x - a < 0)) \rightarrow |f(x) - L| < \epsilon$$

$$\text{Equivalently, } \delta_{\min} > 0 \wedge \forall x \quad (0 < |x - a| < \delta_{\min} \rightarrow |f(x) - L| < \epsilon)$$

$$\text{Therefore, } \forall \epsilon > 0 \exists \delta > 0 \quad \forall x \quad (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)$$

$$\text{Therefore } \lim_{x \rightarrow a} f(x) = L$$

$$\text{By conditional proof we've shown that } \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) \rightarrow \exists \lim_{x \rightarrow a} f(x)$$

30.

$$\text{ii) } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(-x)$$

From the def of $\lim_{x \rightarrow 0^+} f(x)$ $\forall \epsilon > 0 \exists \delta_1 > 0 \quad 0 < x < \delta_1 \rightarrow |f(x) - l_1| < \epsilon$

From the def of $\lim_{x \rightarrow 0^-} f(-x)$ $\forall \epsilon > 0 \exists \delta_2 > 0 \quad 0 < -x < \delta_2 \rightarrow |f(-x) - l_2| < \epsilon$

let $\epsilon > 0$ **Vi intro Ass**

let δ_1, δ_2 such that $\delta_1 > 0 \quad 0 < x < \delta_1 \rightarrow |f(x) - l_1| < \epsilon$
 $\delta_2 > 0 \quad 0 < -x < \delta_2 \rightarrow |f(-x) - l_2| < \epsilon$ **je ihm Ass**

let $\delta = \min(\delta_1, \delta_2)$

Then,

$$\forall x \quad 0 < x < \delta \rightarrow |f(x) - l_1| < \epsilon$$

$$\forall x \quad -\delta < x < 0 \rightarrow |f(-x) - l_2| < \epsilon$$

The latter is equivalent to

$$\forall x \quad 0 < -x < \delta \rightarrow |f(-x) - l_2| < \epsilon$$

Assume $l_2 > l_1$.

$$\text{let } \epsilon = \frac{l_2 - l_1}{2}. \text{ Then } 0 < x < \delta \rightarrow |f(x) - l_1| < \frac{l_2 - l_1}{2} \wedge |f(x) - l_2| < \frac{l_2 - l_1}{2}$$

$$\frac{l_1 - l_2}{2} < f(x) - l_1 < \frac{l_2 - l_1}{2}$$

$$\frac{3l_1 - l_2}{2} < f(x) < \frac{l_1 + l_2}{2}$$

$$\text{Also } \frac{l_1 - l_2}{2} < f(x) - l_2 < \frac{l_2 - l_1}{2}$$

$$\frac{l_1 + l_2}{2} < f(x) < \frac{3l_2 - l_1}{2}$$

$$\text{Therefore, } f(x) < \frac{l_1 + l_2}{2} < f(x). \perp.$$

Therefore $l_2 \leq l_1$.

By symmetry, $l_1 < l_2 \rightarrow \perp$ Therefore $l_1 = l_2$.

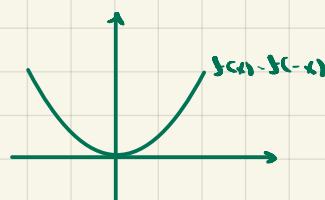
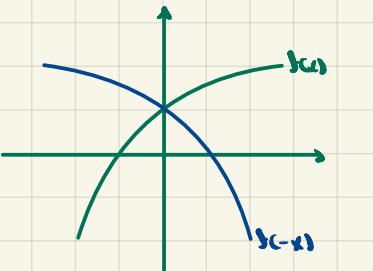
At this point we have that $\exists \delta > 0 (\forall x \quad 0 < x < \delta \rightarrow |f(x) - l_1| < \epsilon) \wedge (\forall x \quad -\delta < x < 0 \rightarrow |f(-x) - l_1| < \epsilon) \wedge (l_1 = l_2)$

Hence, $\exists \delta > 0 (\forall x \quad 0 < x < \delta \rightarrow |f(x) - l_1| < \epsilon) \wedge (\forall x \quad -\delta < x < 0 \rightarrow |f(-x) - l_1| < \epsilon)$

Vi intro $\forall \epsilon > 0 \exists \delta > 0 (\forall x \quad 0 < x < \delta \rightarrow |f(x) - l_1| < \epsilon) \wedge (\forall x \quad -\delta < x < 0 \rightarrow |f(-x) - l_1| < \epsilon)$

$$\lim_{x \rightarrow 0^+} f(x) = l_1 \wedge \lim_{x \rightarrow 0^-} f(-x) = l_1$$

$$\text{Therefore } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(-x)$$



$$\text{ii) } \lim_{x \rightarrow 0^+} f(|x|) = \lim_{x \rightarrow 0^+} f(x)$$

Proof

Let $\lim_{x \rightarrow 0^+} f(x) = l$.

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} f(-x) \text{ by ii.}$$

$$\lim_{x \rightarrow 0^+} f(-x) = l \text{ means } \forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } |f(-x) - l| < \epsilon$$

For $x \in (-\delta, 0)$, $|x| = -x$. Therefore, $\lim_{x \rightarrow 0^+} f(|x|) = l$.

Since $\lim_{x \rightarrow 0^+} f(|x|) = l$, if $\lim_{x \rightarrow 0^+} f(x)$ exists then $\lim_{x \rightarrow 0^+} f(x) = l$.

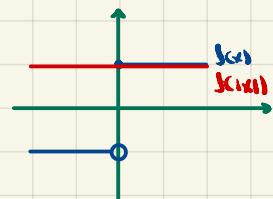
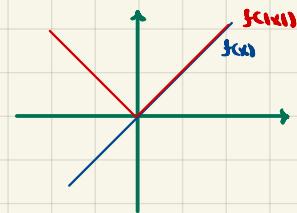
We can also see this by considering that $\lim_{x \rightarrow 0^+} f(|x|) = \lim_{x \rightarrow 0^+} f(x)$ since these limits consider values of x in the interval $0 < x < \delta$, in which $f(|x|) = f(x)$.

That is, for $0 < x < \delta$ we know that $\forall \epsilon > 0 \exists \delta' > 0 \text{ s.t. } |f(x) - l| < \epsilon$.

But this is equivalent to $\forall \epsilon > 0 \exists \delta' > 0 \text{ s.t. } |f(|x|) - l| < \epsilon$

That is, $\lim_{x \rightarrow 0^+} f(|x|) = l$.

Since $\lim_{x \rightarrow 0^+} f(|x|) = \lim_{x \rightarrow 0^+} f(x) = l$ we conclude that $\lim_{x \rightarrow 0^+} f(x) = l$.



$$\text{iii) } \lim_{x \rightarrow 0} f(x^2) = \lim_{x \rightarrow 0^+} f(x)$$

Proof

If $\lim_{x \rightarrow 0} f(x)$ exists then it is defined by L such that

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \quad 0 < |x| < \delta \rightarrow |f(x) - L| < \epsilon$$

$$\text{let } \delta_{\min} = \min(1, \delta)$$

Then, $\forall \epsilon > 0 \exists \delta \forall x$ we have

$$0 < |x| < \min(1, \delta) \rightarrow |f(x) - L| < \epsilon$$

$$0 < x^2 < x < \sqrt{x} \leq \delta_{\min} \rightarrow |f(x^2) - L| < \epsilon$$

Since \sqrt{x} is within the interval $(0, \delta_{\min})$ it follows that $|f(\sqrt{x}) - L| < \epsilon$, ie $|f(x) - L| < \epsilon$.

We have shown that

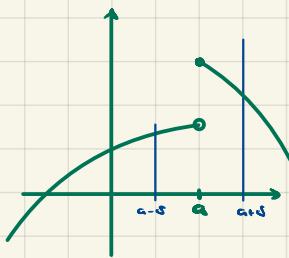
$$\forall \epsilon > 0 \exists \delta \forall x \quad 0 < |x| < \delta \rightarrow |f(x) - L| < \epsilon.$$

This δ_1 is $\min(1, \delta)$ where δ is the delta from the limit $\lim_{x \rightarrow 0} f(x^2)$

$$31. \lim_{x \rightarrow a} f(x) < \lim_{x \rightarrow a} g(x)$$

Prove: $\exists \delta > 0 : \forall x, y$

$$\begin{cases} x < a < y \\ |x-a| < \delta \\ |y-a| < \delta \end{cases} \rightarrow f(x) < g(y)$$



$$\forall \epsilon \exists \delta_1 \forall x (a-\delta < x < a) \rightarrow |f(x)-l_-| < \epsilon$$

$$\forall \epsilon \exists \delta_2 \forall y (a < y < a+\delta) \rightarrow |g(y)-l_+| < \epsilon$$

$$\text{Let } \delta = \frac{l_+ - l_-}{2}$$

$$|f(x)-l_-| < \frac{l_+-l_-}{2} \rightarrow \frac{l_--l_+}{2} < f(x)-l_- < \frac{l_+-l_-}{2} \rightarrow \frac{3l_--l_+}{2} < f(x) < \frac{l_++l_-}{2}$$

$$|g(y)-l_+| < \frac{l_+-l_-}{2} \rightarrow \frac{l_--l_+}{2} < g(y)-l_+ < \frac{l_+-l_-}{2} \rightarrow \frac{l_+-l_+}{2} < g(y) < \frac{3l_+-l_-}{2}$$

$$\text{Hence, for } \delta = \frac{l_+-l_-}{2}$$

$$\forall x \left(0 < a-x < \delta_1 \rightarrow \frac{3l_--l_+}{2} < f(x) < \frac{l_++l_-}{2} \right)$$

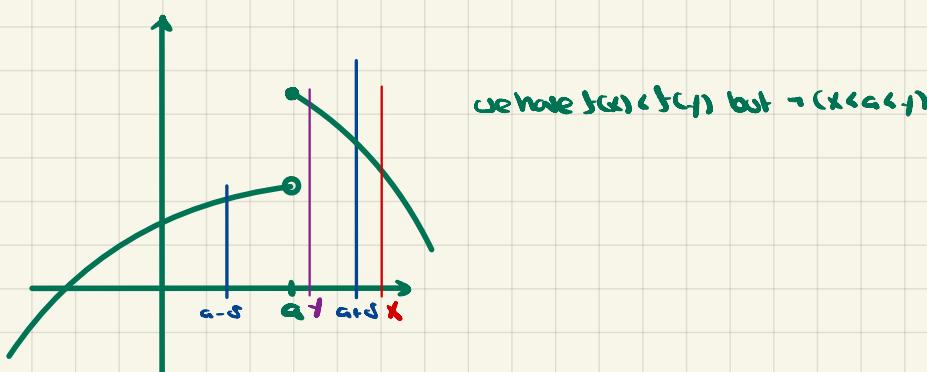
$$\forall y \left(0 < y-a < \delta_2 \rightarrow \frac{l_+-l_+}{2} < g(y) < \frac{3l_+-l_-}{2} \right)$$

Let $\delta = \min(\delta_1, \delta_2)$. Then

$$\forall x, y \left(\begin{array}{l} 0 < a-x < \delta \\ 0 < y-a < \delta \end{array} \rightarrow f(x) < \frac{l_++l_-}{2} < g(y) \right)$$

The converse is not true. I.e. $\forall x, y \left(f(x) < g(y) \rightarrow \begin{array}{l} x < a < y \\ |x-a| < \delta \\ |y-a| < \delta \end{array} \right)$ is false.

Here is a counterexample:



$$32. \lim_{x \rightarrow \infty} \frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0}, a_n \neq 0, b_m \neq 0, \text{ exists } \Leftrightarrow m \geq n$$

$$\text{Assume } \exists \lim_{x \rightarrow \infty} f(x), f(x) = \frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0}, a_n \neq 0, b_m \neq 0.$$

$$\text{Then } \forall \epsilon > 0 \exists N > 0 \text{ s.t. } x > N \Rightarrow \left| \frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0} - L \right| < \epsilon$$

Rewrite $f(x)$

$$\frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0} \cdot \frac{x^m (a_0 + a_1 x^{-1} + \dots + a_{n-m} x^{-(n-m)})}{x^m (b_0 + b_1 x^{-1} + \dots + b_{m-n} x^{-(m-n)})} = \frac{x^n}{x^m} \cdot \frac{(a_0 + a_1 x^{-1} + \dots + a_{n-m} x^{-(n-m)})}{(b_0 + b_1 x^{-1} + \dots + b_{m-n} x^{-(m-n)})}. \text{ Let } h(x) = \frac{x^n}{x^m} \text{ and } g(x) = \frac{(a_0 + a_1 x^{-1} + \dots + a_{n-m} x^{-(n-m)})}{(b_0 + b_1 x^{-1} + \dots + b_{m-n} x^{-(m-n)})}.$$

Since the following two limits exist

$$\lim_{x \rightarrow \infty} (a_0 + a_1 x^{-1} + \dots + a_{n-m} x^{-(n-m)}) \cdot a_n \quad (1)$$

$$\lim_{x \rightarrow \infty} (b_0 + b_1 x^{-1} + \dots + b_{m-n} x^{-(m-n)}) \cdot b_m \quad (2)$$

$$\text{And since } a_n \text{ and } b_m \neq 0, \text{ we know that } \lim_{x \rightarrow \infty} g(x) = \frac{a_n}{b_m} \neq 0.$$

From Problem 3d, given that $\lim_{x \rightarrow \infty} h(x)g(x)$ exists and $\lim_{x \rightarrow \infty} g(x)$ exists and it $\neq 0$, we know that $\lim_{x \rightarrow \infty} h(x)$ exists.

We know this because $h(x) \cdot \frac{1}{g(x)} \cdot h(x)g(x)$

$$\lim_{x \rightarrow \infty} g(x) \neq 0 \rightarrow \lim_{x \rightarrow \infty} \frac{1}{g(x)} \text{ exists.} \\ \rightarrow \lim_{x \rightarrow \infty} \frac{1}{g(x)} h(x)g(x) \text{ exists.}$$

$\lim_{x \rightarrow \infty} h(x)g(x)$ exists by assumption

Consider $\lim_{x \rightarrow \infty} h(x)$

$$\lim_{x \rightarrow \infty} \frac{x^n}{x^m} = \lim_{x \rightarrow \infty} \frac{1}{x^{m-n}}$$

There are three cases to consider

$$\text{Case 1: } m=n \rightarrow \lim_{x \rightarrow \infty} \frac{1}{x^{m-n}} = 1$$

$$\text{Case 2: } m > n \rightarrow \lim_{x \rightarrow \infty} \frac{1}{x^{m-n}} = 0$$

$$\text{Case 3: } m < n \rightarrow \lim_{x \rightarrow \infty} \frac{1}{x^{m-n}} = \lim_{x \rightarrow \infty} x^{n-m}, \text{ which doesn't exist. } \perp$$

Therefore $m \geq n$.

Also,

$$m=n \rightarrow \lim_{x \rightarrow \infty} f(x) = \frac{a_n}{b_m}$$

$$m > n \rightarrow \lim_{x \rightarrow \infty} f(x) = 0$$

proof

$$\text{Assume } \lim_{x \rightarrow \infty} x^{n-m} = L, \text{ and } n-m > 0$$

$$\text{Then, } \forall \epsilon > 0 \exists N \text{ s.t. } \forall x > N \rightarrow |x^{n-m} - L| < \epsilon$$

Note that $L > 0$ because $x^{n-m} > 0$.

$$\text{Let } \epsilon' > 0, \text{ and } N \text{ s.t. } \forall x > N \rightarrow |x^{n-m} - L| < \epsilon'$$

$$\text{Let } x, \text{ s.t. } x > N \rightarrow |x^{n-m} - L| < \epsilon'$$

$$\text{Consider any } x \text{ s.t. } x^{n-m} > L + \epsilon' \text{ and } x > N.$$

For example,

$$x_i = L + \epsilon' + N \rightarrow x_i^{n-m} = (L + \epsilon' + N)^{n-m} > L + \epsilon' \\ \text{Then } |x_i^{n-m} - L| > |L + \epsilon' - L| > |\epsilon'| > \epsilon'$$

$$\text{Therefore } \exists x: x > N \wedge |x^{n-m} - L| > \epsilon$$

\perp

$$\text{Therefore } n-m > 0 \rightarrow \lim_{x \rightarrow \infty} x^{n-m}$$

Now assume $m \geq n$

Let $f(x) = \frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0}$, $a_n \neq 0, b_m \neq 0$

Then we can rewrite $f(x)$ as $f(x) \cdot \frac{x^n}{x^n} \frac{(a_n + a_{n-1}x^{-1} + \dots + a_0x^{-n})}{(b_m + b_{m-1}x^{-1} + \dots + b_0x^{-m})}$

we know that

$$\lim_{x \rightarrow \infty} (a_n + a_{n-1}x^{-1} + \dots + a_0x^{-n}) = a_n$$

$$\lim_{x \rightarrow \infty} (b_m + b_{m-1}x^{-1} + \dots + b_0x^{-m}) = b_m$$

$$\text{so } \lim_{x \rightarrow \infty} \frac{(a_n + a_{n-1}x^{-1} + \dots + a_0x^{-n})}{(b_m + b_{m-1}x^{-1} + \dots + b_0x^{-m})} = \frac{a_n}{b_m}$$

Since $m \geq n$ we have two cases:

Case 1: $m = n \rightarrow \lim_{x \rightarrow \infty} \frac{1}{x^{m-n}} = 1$

Therefore $\lim_{x \rightarrow \infty} f(x)$

Case 2: $m > n \rightarrow \lim_{x \rightarrow \infty} \frac{1}{x^{m-n}} = 0$

Therefore $\lim_{x \rightarrow \infty} f(x)$

Therefore $\lim_{x \rightarrow \infty} f(x)$

33.

$$\text{i) } \lim_{x \rightarrow \infty} \frac{x + \sin^3 x}{5x+6}$$

$\forall x \quad -1 < \sin^3 x < 1$

$$\text{therefore } \frac{x-1}{5x+6} < \frac{x + \sin^3 x}{5x+6} < \frac{x+1}{5x+6}$$

$$\frac{x+1}{5x+6} = \frac{x(1 + \frac{1}{x})}{x(5 + \frac{6}{x})} = \frac{1 + \frac{1}{x}}{5 + \frac{6}{x}}$$

$$\frac{x-1}{5x+6} = \frac{x(1 - \frac{1}{x})}{x(5 + \frac{6}{x})} = \frac{1 - \frac{1}{x}}{5 + \frac{6}{x}}$$

$$\lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x}}{5 + \frac{6}{x}} = \frac{1}{5}$$

$$\lim_{x \rightarrow \infty} \frac{1 - \frac{1}{x}}{5 + \frac{6}{x}} = \frac{1}{5}$$

We can use the result proved in Problem 13:

$$\text{since } \frac{x-1}{5x+6} < \frac{x + \sin^3 x}{5x+6} < \frac{x+1}{5x+6}$$

$$\text{and } \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x}}{5 + \frac{6}{x}} = \frac{1}{5}$$

$$\lim_{x \rightarrow \infty} \frac{1 - \frac{1}{x}}{5 + \frac{6}{x}} = \frac{1}{5}$$

then

$$\lim_{x \rightarrow \infty} \frac{x + \sin^3 x}{5x+6} = \frac{1}{5}$$

$$\lim_{x \rightarrow \infty} \frac{1}{x}$$

$$\forall \epsilon > 0 \text{ choose } N = \frac{1}{\epsilon}$$

$$\text{Then, } x > N \Rightarrow \frac{1}{x} < \frac{1}{N} = \epsilon$$

$$\text{since } \epsilon > 0 \text{ then } N > 0 \text{ then } x > N \Rightarrow 0 < \frac{1}{x} < \epsilon \rightarrow \left| \frac{1}{x} \right| < \epsilon$$

$$\text{Therefore } \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$\text{ii) } \lim_{x \rightarrow \infty} \frac{x \sin x}{x^2 + 5}$$

$$\frac{-x}{x^2 + 5} < \frac{x \sin x}{x^2 + 5} < \frac{x}{x^2 + 5}$$

$$\lim_{x \rightarrow \infty} \frac{x}{x^2 + 5} = \lim_{x \rightarrow \infty} \frac{-x}{x^2 + 5} = 0 \quad (\text{proved Problem 32})$$

$$\rightarrow \lim_{x \rightarrow \infty} \frac{x \sin x}{x^2 + 5} = 0 \quad (\text{proved Problem 13})$$

$$\text{iii) } \lim_{x \rightarrow \infty} \sqrt{x^2 + x} - x$$

$$\sqrt{x^2 + x} - x = \frac{x^2 + x - x^2}{\sqrt{x^2 + x} + x} = \frac{x}{x\sqrt{1 + \frac{1}{x}} + x} = \frac{1}{\sqrt{1 + \frac{1}{x}} + 1}$$

$$\lim_{x \rightarrow \infty} \sqrt{1 + \frac{1}{x}}$$

$$\forall \epsilon > 0 \quad x > \frac{1}{\epsilon} \Rightarrow \frac{1}{x} < \frac{1}{N} \Rightarrow 1 + \frac{1}{x} - 1 < \epsilon$$

$$\rightarrow \lim_{x \rightarrow \infty} \sqrt{1 + \frac{1}{x}} - 1$$

$$\lim_{x \rightarrow \infty} (\sqrt{1 + \frac{1}{x}} + 1) - 2$$

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{x}} + 1} = \frac{1}{2}$$

$$n) \lim_{x \rightarrow \infty} \frac{x^2(1+\sin^2 x)}{(x+\sin x)^2}$$

$$\frac{x^2}{(x+\sin x)^2} \leq \frac{x^2(1+\sin^2 x)}{(x+\sin x)^2} \leq \frac{2x^2}{(x+1)^2}$$

Because $x \rightarrow \infty$ then $x > 0$. Therefore $(x-1)^2 < (x+\sin x)^2 < (x+1)^2$

$$\text{Hence } \frac{x^2}{(x+1)^2} \leq \frac{x^2(1+\sin^2 x)}{(x+\sin x)^2} \leq \frac{2x^2}{(x-1)^2}$$

$$\lim_{x \rightarrow \infty} \frac{2x^2}{(x-1)^2} = \lim_{x \rightarrow \infty} \frac{2x^2}{x^2 - 2x + 1} = 2 \quad (\text{Problem 32})$$

$$\lim_{x \rightarrow \infty} \frac{x^2}{(x+1)^2} = \lim_{x \rightarrow \infty} \frac{x^2}{x^2 + 2x + 1} = 1$$

Therefore, if $\lim_{x \rightarrow \infty} \frac{x^2(1+\sin^2 x)}{(x+\sin x)^2}$ exists then $1 \leq \lim_{x \rightarrow \infty} \frac{x^2(1+\sin^2 x)}{(x+\sin x)^2} \leq 2 \quad (\text{Problem 12})$

Assume the limit exists.

$$\text{Then } \forall \epsilon > 0 \exists N \in \mathbb{N} \rightarrow \left| \frac{x^2(1+\sin^2 x)}{(x+\sin x)^2} - L \right| < \epsilon$$

$$\text{choose an } \epsilon > 0, \text{ and let } N_1 \text{ be such that } x > N_1 \rightarrow \left| \frac{x^2(1+\sin^2 x)}{(x+\sin x)^2} - L \right| < \epsilon$$

Let $x_1 > N_1$ s.t. $\sin(x_1) = 0$

$$\text{Then } \frac{x_1^2(1+\sin^2 x_1)}{(x_1+\sin x_1)^2} = 1$$

$$|1-L| < \epsilon \rightarrow -\epsilon < 1-L < \epsilon \rightarrow -1-\epsilon < -L < -1+\epsilon \rightarrow 1-\epsilon < L < 1+\epsilon$$

Let $x_2 > N$ s.t. $\sin(x_2) = 1$

$$\text{Then } \frac{x_2^2(1+\sin^2 x_2)}{(x_2+\sin x_2)^2} = \frac{2x_2^2}{(x_2+1)^2}$$

$$\left| \frac{2x_2^2}{(x_2+1)^2} - L \right| < \epsilon \rightarrow -\frac{2x_2^2}{(x_2+1)^2} - \epsilon < -L < -\frac{2x_2^2}{(x_2+1)^2} + \epsilon \rightarrow \frac{2x_2^2}{(x_2+1)^2} - \epsilon < L < \frac{2x_2^2}{(x_2+1)^2} + \epsilon$$

$$\lim \left[\frac{2x_2^2}{(x_2+1)^2} - \epsilon \right] = 2 - \epsilon$$

$$\lim \left[\frac{2x_2^2}{(x_2+1)^2} + \epsilon \right] = 2 + \epsilon$$

$$\text{Hence } 2 - \epsilon < L < 2 + \epsilon \rightarrow 2 - \epsilon < L < 2 + \epsilon$$

$$\text{choose } \epsilon = \frac{1}{2}. \text{ Then we have } 1.5 < L < 2.5$$

$$0.5 < L < 1.5$$

\perp

Therefore the limit doesn't exist.

$$34. \lim_{x \rightarrow 0} f(1/x) = \lim_{x \rightarrow \infty} f(x)$$

Proof

If $\lim_{x \rightarrow \infty} f(x)$ exists and equals l , then

$$\forall \epsilon > 0 \exists N \forall x > N \rightarrow |f(x) - l| < \epsilon \quad (1)$$

$$\text{Let } f_1(x) = \frac{1}{x}$$

Then (1) is equivalent to

$$\forall \epsilon > 0 \exists N \forall y > \frac{1}{N} \rightarrow |f_1(y) - l| < \epsilon$$

$$\text{Let } \delta = \frac{1}{N}. \text{ Then}$$

$$\forall \epsilon > 0 \exists \delta > 0 \forall y > \delta \rightarrow |f_1(y) - l| < \epsilon$$

$$\text{Therefore } \lim_{y \rightarrow 0^+} f_1(y) = l$$

$$35. \alpha = \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

$$\text{i)} \lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{1}{x} \sin x$$

$$\forall x > 0 \quad -\frac{1}{x} < \frac{1}{x} \sin x < \frac{1}{x}$$

$$\lim_{x \rightarrow 0^+} \left(-\frac{1}{x}\right) = \lim_{x \rightarrow 0^+} \frac{1}{x} = 0 \rightarrow \lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 0$$

$$\text{ii)} \lim_{x \rightarrow 0} x \sin \frac{1}{x}$$

$$\forall \epsilon > 0 \exists N > 0 \forall x > N \rightarrow |x \sin \frac{1}{x} - l| < \epsilon$$

$$\text{let } f_1(x) = \frac{1}{x}$$

$$\forall \epsilon > 0 \exists N > 0 \forall y > \frac{1}{N} \rightarrow \left|\frac{\sin y}{y} - l\right| < \epsilon$$

$$\lim_{y \rightarrow 0^+} \frac{\sin y}{y} = l = \alpha$$

$$\text{Alternatively, note that } \lim_{x \rightarrow 0} x \sin(1/x) = \lim_{x \rightarrow 0} \frac{\sin(1/x)}{1/x} = \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} f(1/x) = \lim_{x \rightarrow 0} \frac{\sin x}{x} = \alpha$$

36. $\lim_{x \rightarrow \infty} f(x) = L$

a) $\lim_{x \rightarrow \infty} \frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0}$

Proof

$\forall \epsilon > 0 \exists N < 0 \forall x > N \rightarrow |f(x) - L| < \epsilon$

Let $f(x) = \frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0} = \frac{x^n}{x^m} \frac{(a_n + a_{n-1}x^{-1} + \dots + a_0 x^{-n})}{(b_m + b_{m-1}x^{-1} + \dots + b_0 x^{-m})}$

$\lim_{x \rightarrow \infty} \frac{1}{x^{m-n}}$

There are three cases to consider

Case 1: $m=n \rightarrow \lim_{x \rightarrow \infty} \frac{1}{x^{m-n}} = 1$

Case 2: $m > n \rightarrow \lim_{x \rightarrow \infty} \frac{1}{x^{m-n}} = 0$

$x < N < 0 \rightarrow \frac{1}{N} < \frac{1}{x} < 0$

if $m-n$ even then $0 < \frac{1}{x^{m-n}} < \frac{1}{N^{m-n}}$

if $N = -\frac{1}{\epsilon^{1/(m-n)}}$ then we have $x < N < 0 \rightarrow 0 < \frac{1}{x^{m-n}} < \epsilon \rightarrow \left| \frac{1}{x^{m-n}} \right| < \epsilon$

$\forall \epsilon > 0 \exists N < 0 \forall x < N < 0 \rightarrow \left| \frac{1}{x^{m-n}} \right| < \epsilon$

Therefore $m-n$ even $\rightarrow \lim_{x \rightarrow \infty} \frac{1}{x^{m-n}} = 0$

if $m-n$ odd then $\frac{1}{N^{m-n}} < \frac{1}{x^{m-n}} < 0$

if $N = -\frac{1}{\epsilon^{1/(m-n)}}$ then $-\epsilon < \frac{1}{x^{m-n}} < 0 \rightarrow \left| \frac{1}{x^{m-n}} \right| < \epsilon$

Therefore $\forall \epsilon > 0 \exists N < 0 \forall x < N < 0 \rightarrow \left| \frac{1}{x^{m-n}} \right| < \epsilon$

Therefore $m-n$ odd $\rightarrow \lim_{x \rightarrow \infty} \frac{1}{x^{m-n}} = 0$

Case 3: $m < n \rightarrow \lim_{x \rightarrow \infty} \frac{1}{x^{m-n}} = \lim_{x \rightarrow \infty} x^{n-m}$, which doesn't exist. \perp

Note that each term in $a_n + a_{n-1}x^{-1} + \dots + a_0 x^{-n}$ has "m>n" so the limits of their terms are zero.

Here $\lim_{x \rightarrow \infty} \frac{x^n}{x^m} \frac{(a_n + a_{n-1}x^{-1} + \dots + a_0 x^{-n})}{(b_m + b_{m-1}x^{-1} + \dots + b_0 x^{-m})}$ is $\frac{a_n}{b_m} \cdot \lim_{x \rightarrow \infty} \frac{1}{x^{m-n}}$

$m > n \rightarrow \lim_{x \rightarrow \infty} f(x) = 0$

(Same answer as for $\lim_{x \rightarrow \infty} f(x)$)

$m < n \rightarrow \lim_{x \rightarrow \infty} f(x) = \frac{a_n}{b_m}$

$$b) \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(-x)$$

Proof

$$\forall \epsilon > 0 \exists N > 0 \forall x > N \rightarrow |f(x) - l| < \epsilon$$

$$\text{Let } f(-x)$$

$$\forall \epsilon > 0 \exists N' > 0 \forall y < -N' < 0 \rightarrow |f(-y) - l| < \epsilon$$

$$\forall \epsilon > 0 \exists N' > 0 \forall y < -N' < 0 \rightarrow |f(-y) - l| < \epsilon$$

$$\lim_{x \rightarrow -\infty} f(-x) = \lim_{x \rightarrow \infty} f(x) = l$$

Interpretation of the substitution

For any $x \in \mathbb{R}$, $x \neq 0$ there is an associated $y = -x$ by definition.

Hence it is true that $x \neq 0 \rightarrow |f(-y) - l| < \epsilon$.

Equivalent to $y < -N' < 0 \rightarrow |f(-y) - l| < \epsilon$.

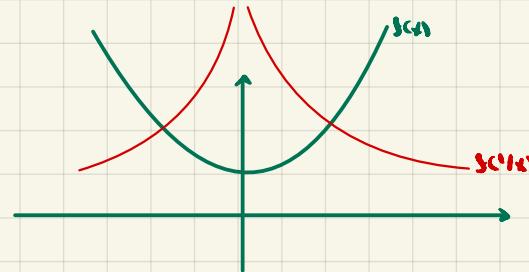
For any $y \in \mathbb{R}$, $y \neq 0$ we know there is $x \in \mathbb{R}$, $x \neq 0$,

so then $|f(x) - l| < \epsilon$. But then $|f(-y) - l| < \epsilon$.

$$y < -N' < 0 \rightarrow x \neq 0 \rightarrow |f(x) - l| < \epsilon \rightarrow |f(-y) - l| < \epsilon$$

$$c) \lim_{x \rightarrow 0} f(\frac{1}{x}) = \lim_{x \rightarrow \infty} f(x)$$

$$\begin{aligned} &x \in \mathbb{R} \\ &-N < x < 0 \end{aligned}$$



Proof

$$\forall \epsilon > 0 \exists N < 0 \forall x < N \rightarrow |f(x) - l| < \epsilon$$

$$x = \frac{1}{y}$$

$$\frac{1}{y} < N < 0 \rightarrow |f(\frac{1}{y}) - l| < \epsilon$$

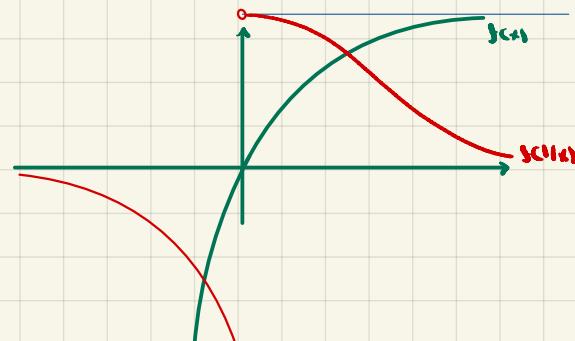
$$\frac{1}{N} < y < 0 \rightarrow |f(\frac{1}{y}) - l| < \epsilon$$

$$0 < -y < -\frac{1}{N} \rightarrow |f(-y) - l| < \epsilon$$

$$\text{Let } \delta = -\frac{1}{N} > 0.$$

$$\text{Then } \forall \epsilon > 0 \exists \delta > 0, 0 < -y < \delta \rightarrow |f(-y) - l| < \epsilon$$

$$\lim_{x \rightarrow 0^+} f(\frac{1}{x}) = l = \lim_{x \rightarrow \infty} f(x)$$



$$31. \lim_{x \rightarrow a} f(x) = \infty \Leftrightarrow \forall N \exists \delta > 0 \quad 0 < |x-a| < \delta \rightarrow f(x) > N$$

$$\text{a)} \lim_{x \rightarrow 3} \frac{1}{(x-3)^2} = \infty$$

Proof

Let N be any number.

Case 1: $N > 0$

$$\frac{1}{(x-3)^2} > N > 0 \Leftrightarrow 0 < (x-3)^2 < \frac{1}{N} \Leftrightarrow 0 < |x-3| < (\frac{1}{N})^{1/2}$$

$$\text{Therefore } \exists \delta > 0 \quad 0 < |x-3| < \delta \rightarrow \frac{1}{(x-3)^2} > N$$

Case 2: $N \leq 0$

$$\frac{1}{(x-3)^2} > 0 > N$$

$$\text{Therefore } \forall \delta > 0 \quad 0 < |x-3| < \delta \rightarrow \frac{1}{(x-3)^2} > N$$

$$\text{Therefore } \forall N \exists \delta > 0 \quad 0 < |x-3| < \delta \rightarrow \frac{1}{(x-3)^2} > N$$

$$\text{Therefore } \lim_{x \rightarrow 3} \frac{1}{(x-3)^2} = \infty$$

$$\text{b) } \forall x \quad f(x) > \epsilon > 0$$

$$\lim_{x \rightarrow a} g(x) = 0 \rightarrow \lim_{x \rightarrow a} \frac{f(x)}{|g(x)|} = \infty$$

Proof

$\lim_{x \rightarrow a} g(x) = 0$ means

$$\forall \epsilon_g \exists \delta \forall x \quad 0 < |x-a| < \delta \rightarrow |g(x)| < \epsilon_g$$

$$|g(x)| < \epsilon_g \rightarrow \frac{1}{|g(x)|} > \frac{1}{\epsilon_g}$$

$$\rightarrow \left| \frac{f(x)}{|g(x)|} \right| = \frac{|f(x)|}{|g(x)|} > \frac{|f(x)|}{\epsilon_g} > \frac{\epsilon}{\epsilon_g}$$

$$\text{For any } N, \quad \frac{\epsilon}{\epsilon_g} > N \Leftrightarrow \epsilon_g < \frac{\epsilon}{N}$$

$$\text{Therefore if } \forall N \text{ it decrease } \epsilon_g < \frac{\epsilon}{N} \text{ then } \exists \delta \forall x \quad 0 < |x-a| < \delta \rightarrow \left| \frac{f(x)}{|g(x)|} \right| > N$$

$$\text{Therefore } \forall N \exists \delta \forall x \quad 0 < |x-a| < \delta \rightarrow \left| \frac{f(x)}{|g(x)|} \right| > N$$

$$\text{So } \lim_{x \rightarrow a} \frac{f(x)}{|g(x)|} = \infty$$

38.

a)

$$\lim_{x \rightarrow a^+} f(x) = \infty \Leftrightarrow \forall N \exists \delta > 0 \forall x \quad 0 < x - a < \delta \rightarrow f(x) > N$$

$$\lim_{x \rightarrow a^-} f(x) = \infty \Leftrightarrow \forall N \exists \delta > 0 \forall x \quad a < x < \delta \rightarrow f(x) > N$$

b) $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$

Proof

$$\forall \delta > 0 \quad 0 < x < \delta \rightarrow 0 < \frac{1}{\delta} < \frac{1}{x}$$

$$\forall N > 0 \text{ let } \frac{1}{\delta} = N. \text{ Then } \delta = \frac{1}{N}$$

$$\forall N > 0 \exists \delta > 0 \quad 0 < x < \delta \rightarrow \frac{1}{x} > N$$

$$\text{Also } \forall N \exists \delta > 0 \quad 0 < x < \delta \rightarrow \frac{1}{x} > N$$

$$\text{So } \forall N \exists \delta > 0 \quad 0 < x < \delta \rightarrow \frac{1}{x} > N$$

Therefore $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$

c) $\lim_{x \rightarrow 0^+} f(x) = \infty \Leftrightarrow \lim_{x \rightarrow 0^+} f(\frac{1}{x}) = \infty$

Proof

Assume $\lim_{x \rightarrow 0^+} f(x) = \infty$.

$$\text{Then } \forall N \exists \delta > 0 \quad 0 < x < \delta \rightarrow f(x) > N$$

$$\text{Let } x = \frac{1}{y}. \text{ Then}$$

$$0 < \frac{1}{y} < \delta \rightarrow 0 < \frac{1}{\delta} < y \rightarrow f(1/y) > N$$

$$\forall N \exists M \quad y > M > 0 \rightarrow f(1/y) > N$$

$$\lim_{x \rightarrow 0^+} f(1/x) = \infty$$

Assume $\lim_{x \rightarrow 0^+} f(\frac{1}{x}) = \infty$

$$\forall N \exists M \quad x > M > 0 \rightarrow f(1/x) > N$$

$$\text{Let } y = \frac{1}{x}. \text{ Then}$$

$$\frac{1}{y} > M > 0 \rightarrow 0 < y < \frac{1}{M}$$

$$\forall N \exists M \quad x > M > 0 \rightarrow f(x) > N$$

$$\lim_{x \rightarrow 0^+} f(x) = \infty$$

39.

$$\text{i)} \lim_{x \rightarrow \infty} \frac{x^3 + 4x - 7}{7x^2 - x + 1}$$

Problem 32: exponent 2 < exponent 3

$$\rightarrow \lim_{x \rightarrow \infty} \frac{x^3 + 4x - 7}{7x^2 - x + 1} = \infty$$

$$\text{ii)} \lim_{x \rightarrow \infty} x(1 + \sin^2 x)$$

$$1 < 1 + \sin^2 x < 2. \quad \lim_{x \rightarrow \infty} (1 + \sin^2 x) = 2$$

$$\cancel{\lim_{x \rightarrow \infty} x}$$

$$\lim_{x \rightarrow \infty} x < \lim_{x \rightarrow \infty} x(1 + \sin^2 x) < \lim_{x \rightarrow \infty} 2x$$

$$\rightarrow \infty < \lim_{x \rightarrow \infty} x(1 + \sin^2 x) < \infty$$

$$\rightarrow \lim_{x \rightarrow \infty} x(1 + \sin^2 x) = \infty$$

$$\text{iii)} \lim_{x \rightarrow \infty} x \sin^2 x$$

$$0 \leq x \sin^2 x \leq x$$

Does not exist.

$$\text{iv)} \lim_{x \rightarrow \infty} x^2 \sin(\frac{1}{x}) = \lim_{x \rightarrow \infty} [x \cdot (\sin(\frac{1}{x}))]$$

$$\text{From Problem 35, } \lim_{x \rightarrow \infty} x \sin(\frac{1}{x}) = \lim_{x \rightarrow \infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} = 0$$

$$\text{Assume } \lim_{x \rightarrow \infty} x^2 \sin(\frac{1}{x}) = R.$$

$$\text{Then } \forall \epsilon > 0 \exists N \forall x \ x > N \rightarrow |x \cdot x \sin(\frac{1}{x}) - R| < \epsilon$$

$$R - \epsilon < x \cdot x \sin(\frac{1}{x}) < R + \epsilon$$

$$\text{v)} \lim_{x \rightarrow \infty} \sqrt{x^2 + 2x} - x$$

$$\sqrt{x^2 + 2x} - x = \frac{x^2 + 2x - x^2}{\sqrt{x^2 + 2x} + x} = \frac{2x}{x\sqrt{1 + \frac{2}{x}}} = \frac{2}{\sqrt{1 + \frac{2}{x}}} \rightarrow \frac{2}{\sqrt{1 + 0}} = 2$$

$$\lim_{x \rightarrow \infty} \sqrt{1 + \frac{2}{x}} = 1$$

$$\lim_{x \rightarrow \infty} \sqrt{x^2 + 2x} - x = 2$$

$$\text{vi)} \lim_{x \rightarrow \infty} x(\sqrt{x+2} - \sqrt{x})$$

$$x(\sqrt{x+2} - \sqrt{x}) = \frac{x(\sqrt{x+2} - \sqrt{x})}{\sqrt{x+2} + \sqrt{x}} = \frac{2x}{\sqrt{x+2} + \sqrt{x}} = \frac{2x}{\sqrt{\frac{1}{x} + \frac{2}{x^2}} + \sqrt{\frac{1}{x}}}$$

$$\lim_{x \rightarrow \infty} \sqrt{1/x} = \lim_{x \rightarrow \infty} \sqrt{\frac{1}{x} + \frac{2}{x^2}} = 0$$

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0 \Leftrightarrow \forall \epsilon > 0 \ \forall x \ x > 2\epsilon^2 \rightarrow \frac{1}{x} < \frac{1}{2\epsilon^2}$$

$$\lim_{x \rightarrow \infty} \sqrt{1/x} = 0 \Leftrightarrow \forall \epsilon > 0 \ \forall x \ x > \epsilon^2 \rightarrow \sqrt{1/x} < \frac{1}{\epsilon}$$

$$\lim_{x \rightarrow \infty} \frac{2}{x^2} = 0 \Leftrightarrow \forall \epsilon > 0 \ \forall x \ x > 2\epsilon \rightarrow \frac{2}{x^2} < \frac{1}{2\epsilon^2}$$

$$\lim_{x \rightarrow \infty} \sqrt{\frac{1}{x} + \frac{2}{x^2}} = 0 \Leftrightarrow \forall \epsilon > 0 \ \forall x \ x > \max(2\epsilon^2, 2\epsilon) \rightarrow \frac{1}{x} + \frac{2}{x^2} < \frac{1}{\epsilon^2}$$

$$\rightarrow \sqrt{\frac{1}{x} + \frac{2}{x^2}} < \frac{1}{\epsilon}$$

Therefore $\forall \epsilon > 0 \ \forall x$

$$x > \max(2\epsilon^2, 2\epsilon, \epsilon^2) \rightarrow \sqrt{\frac{1}{x} + \frac{2}{x^2}} + \sqrt{\frac{1}{x}} < \frac{2}{\epsilon}$$

$$\rightarrow \frac{2}{\sqrt{\frac{1}{x} + \frac{2}{x^2}} + \sqrt{\frac{1}{x}}} > \epsilon$$

$$\text{That is } \forall N > 0 \ \exists N > 0 \ \forall x \ x > N \rightarrow \frac{2}{\sqrt{\frac{1}{x} + \frac{2}{x^2}} + \sqrt{\frac{1}{x}}} > N$$

$$\text{Hence } \lim_{x \rightarrow \infty} x(\sqrt{x+2} - \sqrt{x}) = \infty$$

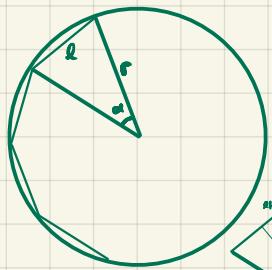
$$\text{vii) } \lim_{x \rightarrow \infty} \frac{\sqrt{|x|}}{x}$$

$$\forall N > 0 \quad \forall x > N \Rightarrow |x| = x \Rightarrow \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{N}} \Leftrightarrow \sqrt{x} > \sqrt{N} \Leftrightarrow x > N$$

$$\forall \epsilon > 0 \quad \forall x > \frac{1}{\epsilon^2} \Rightarrow \frac{\sqrt{|x|}}{x} = \frac{1}{\sqrt{x}} < \epsilon$$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{|x|}}{x} = 0$$

40



$$\sin(\alpha/2) = \frac{l}{2r}$$

$$l = 2r \sin(\alpha/2)$$

$$n = \frac{2\pi}{\alpha} \Rightarrow \alpha = \frac{2\pi}{n}$$

$$\text{Perimeter} = nl$$

$$= n \cdot 2r \sin(\alpha/2)$$

$$= 2nr \sin(\pi/n)$$

b)

$$\lim_{n \rightarrow \infty} P(n) = \lim_{n \rightarrow \infty} 2nr \sin(\pi/n) = 2r \lim_{n \rightarrow \infty} n \sin(\pi/n) = 2r \pi \lim_{n \rightarrow \infty} \frac{\sin(\pi/n)}{\pi/n} = 2r\pi\alpha \quad (\Rightarrow \text{per Problem 35})$$

$$\text{c) } \lim_{x \rightarrow \infty} \frac{\sin x}{x} = 1$$

41.

$$\text{a) } c > 1, \quad f(n) = c^n \cdot \sqrt[n]{c}$$

$$\lim_{n \rightarrow \infty} c^n =$$

Bernoulli's Inequality says $\forall \epsilon > 0 \quad \forall n > 0 \quad 1 + n\epsilon \leq (1+\epsilon)^n$

$$\lim_{n \rightarrow \infty} (1+n\epsilon) = \infty \rightarrow \lim_{n \rightarrow \infty} (1+\epsilon)^n = \infty$$

That is $\forall \epsilon > 0 \quad \exists N > 0 \quad \forall n > N \rightarrow (1+\epsilon)^n > N$

Therefore, $\forall \epsilon > 0 \quad \exists N > 0 \quad \forall n > N \rightarrow (1+\epsilon)^n > c \rightarrow c^n - 1 < \epsilon$

Alternatively, by contradiction:

Assume $c^n > 1 + \epsilon, \quad \forall n \in \mathbb{N}$

Then $c > (1+\epsilon)^{1/n} > 1 + \epsilon$

$$\rightarrow n < \frac{c-1}{\epsilon} < 0. \quad \text{I because } n \in \mathbb{N}.$$

Therefore, $1 < c^n < 1 + \epsilon$

$$\forall \epsilon > 0 \quad \forall n \in \mathbb{N} > 0 \quad 0 < c^n - 1 < \epsilon \rightarrow 0 < c^n - 1 < \epsilon$$

$$\text{Therefore } \lim_{n \rightarrow \infty} c^n = 1$$

42.

$$\delta = \min(\sqrt{a^2 + \epsilon} - a, a - \sqrt{a^2 - \epsilon})$$

$$|x-a| < \delta \rightarrow a-\delta < x < a+\delta$$

$$\text{if } \delta = \sqrt{a^2 + \epsilon} - a \text{ then } 2a - \sqrt{a^2 + \epsilon} < x < \sqrt{a^2 + \epsilon}$$

$$\text{if } \delta = a - \sqrt{a^2 - \epsilon} \text{ then } \sqrt{a^2 - \epsilon} < x < 2a - \sqrt{a^2 - \epsilon}$$

For $f(x) = x^2$ specifically,

$$\begin{aligned}\sqrt{a^2 + \epsilon} - a &= (a - \sqrt{a^2 - \epsilon}) \\ \sqrt{a^2 + \epsilon} + \sqrt{a^2 - \epsilon} - 2a &< 0\end{aligned}$$

$$\text{Therefore } \delta = \sqrt{a^2 + \epsilon} - a, \text{ so } 2a - \sqrt{a^2 + \epsilon} < x < \sqrt{a^2 + \epsilon}$$

thus terms omitted

