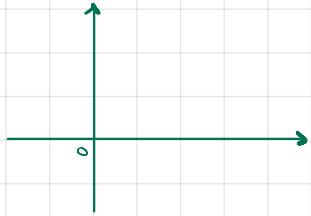


the real line: geometric picture of real numbers; prominent in thinking about proofs, but inessential to actual proofs

0

drawing ordered pairs requires a coordinate system



function: collection of ordered pairs

graph of f: drawing of ordered pairs $(x, f(x))$

For our purposes, we

- define the plane as set of all ordered pairs of real numbers
- define straight lines as certain collections of points
e.g. $\{(x, cx) : x \in \mathbb{R}\}$
- define distance between (a, b) and (c, d) as $\sqrt{(a-c)^2 + (b-d)^2}$
i.e. are like Pythagorean theorem in our plane

ellipse: set of points, sum of whose distance from two "focal" points is a constant.

Two fixed points $(c, 0), (-c, 0)$

$$[(x-c)^2 + y^2]^{1/2} + [(x+c)^2 + y^2]^{1/2} = 2a$$

$$[(x-c)^2 + y^2]^{1/2} = 2a - [(x+c)^2 + y^2]^{1/2}$$

$$(x-c)^2 + y^2 = 4a^2 - 4a[(x+c)^2 + y^2]^{1/2} + (x+c)^2 + y^2$$
 ~~$x - 2cx + c^2 + y^2 = 4a^2 + x + 2cx + c^2 + y^2 - 4a[(x+c)^2 + y^2]^{1/2}$~~
 ~~$4a^2 - 4cx = 4a[(x+c)^2 + y^2]^{1/2}$~~

$$a^4 - 3a^2cx + c^2x^2 = a^2(x^2 + c^2 + 2cx + c^2)$$

$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$

divide by $a^2(c^2 - a^2) < 0$

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad b = \sqrt{a^2 - c^2} \text{ is an ellipse}$$

• linear fun: $f(x) = cx+d$

• power fun: $f(x) = x^n$

$$\text{ex: } f\left(\frac{1}{n}\right) = (-1)^{n+1}$$

$$f\left(-\frac{1}{n}\right) = (-1)^{n+1}$$

$$f(x) = 1 \quad |x| \geq 1$$

$$\text{circle: } (x-a)^2 + (y-b)^2 = r^2$$

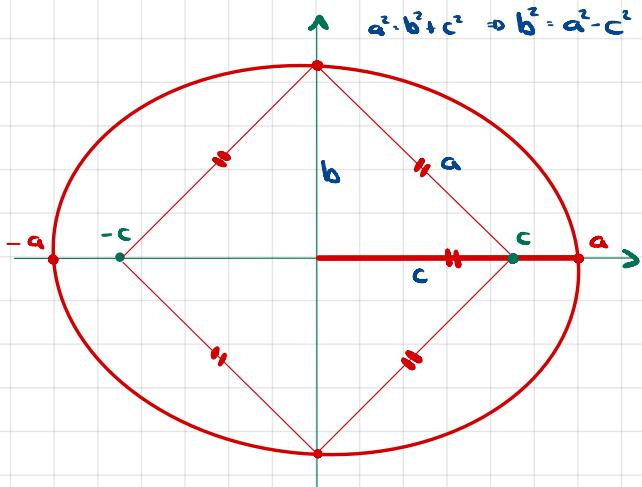
note

$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$

$$x^2(c^2 - a^2) = a^2(y^2 + c^2 - a^2)$$

case 1: $0 < a < c$

$$x^2(c^2 - a^2) > 0$$



Hyperbola: set of points, difference of whose distance to two foci is constant.

$$|(x-c)^2 + y^2 - [(x+c)^2 + y^2]| = \pm 2a$$

Consider the $+2a$ case, ie the left side of the hyperbola

$$|(x-c)^2 + y^2 - [(x+c)^2 + y^2]| = 2a$$

$$(x-c)^2 + y^2 = 2a + [(x+c)^2 + y^2]$$

$$(x-c)^2 + y^2 = 4c^2 + 4a[(x+c)^2 + y^2] + (x+c)^2 + y^2$$

$$\cancel{x^2} - 2xc + c^2 + y^2 = 4c^2 + \cancel{x^2} + 2xc + c^2 + y^2 + 4a[(x+c)^2 + y^2]$$

$-4xc - 4c^2 = 4a[(x+c)^2 + y^2]$ if we were considering the $-2a$ case, ie right side of hyperbola, the only difference here would be minus sign on the RHS term.

$$-(xc + a^2) = a[(x+c)^2 + y^2]$$

$$x^2c^2 + 3xca^2 + a^4 = a^2(x^2 + 2xc + c^2 + y^2)$$

But we square both sides so the calculations are the same.

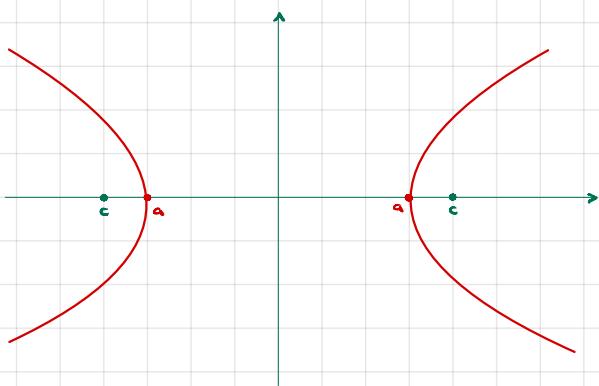
$$= a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2$$

$$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2)$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \quad \text{but } 0 > c > a \Rightarrow c^2 > a^2 \\ \Rightarrow a^2 - c^2 < 0$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, b^2 = c^2 - a^2$$



Sachinashik/Cole 11.2 Ellipses

$$\text{example 1: } 2x^2 + 9y^2 = 18$$

$$\frac{x^2}{9} + \frac{y^2}{2} = 1$$

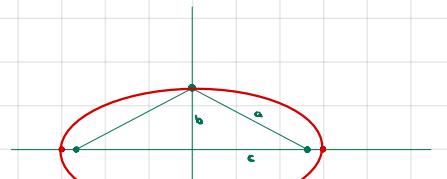
$$a^2 = 9 = b^2 + c^2 = 2 + c^2$$

$$\Rightarrow c = \sqrt{7}$$

$$\text{total string length} = 2a = 6$$

$$\text{total distance between foci} = 2\sqrt{7} \approx 5.28$$

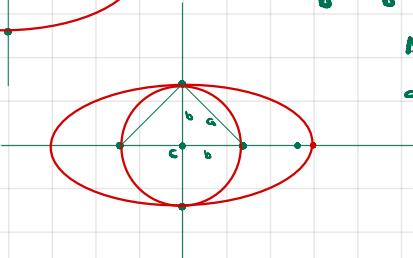
\Rightarrow not flat ellipse



simply reducing a keeping b constant reduces c because $c = \sqrt{a^2 - b^2}$
If we reduce a to b then $c = 0$.

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} = 1 \Rightarrow x^2 + y^2 = b^2$$

At this point, if we decrease c more it becomes negative, which actually just means that a starts increasing again.



$$\text{example 2: } 9x^2 + 4y^2 = 36$$

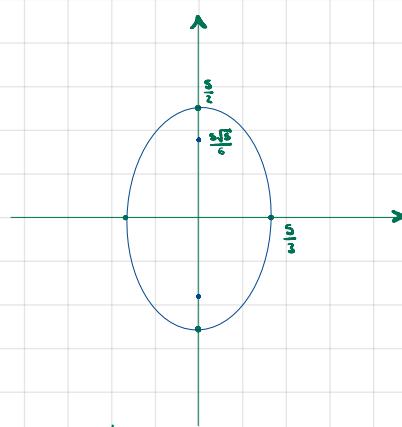
$$\frac{x^2}{36/9} + \frac{y^2}{36/4} = 1$$

$$b^2 = \frac{36}{9} = b = \frac{5}{3}$$

$$a^2 = \frac{36}{4} = a = \frac{5}{2}$$

$$c^2 = a^2 - b^2 = \frac{125}{36}$$

$$\Rightarrow c = \frac{\sqrt{125}}{6}$$



example 3

vertices $(\pm 4, 0)$

foci $(\pm 2, 0)$

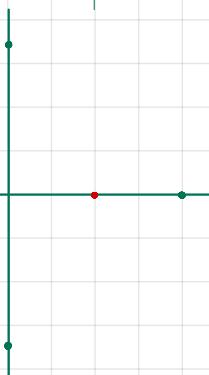
$$a = 4$$

$$c = 2$$

$$b^2 = 16 - 4 = 12$$

$$b = 2\sqrt{3} \approx 3.4$$

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$



$$\text{example 5 } 16x^2 + 9y^2 + 64x - 18y - 71 = 0$$

$$(4x)^2 + 2 \cdot 4x \cdot 8 + 64 - 64 + (3y)^2 - 2 \cdot 3y \cdot 3 + 9 - 9 - 71 = 0$$

$$(4x+8)^2 + (3y-3)^2 = 144$$

$$16(x+2)^2 + 9(y-1)^2 = 144$$

$$\frac{(x+2)^2}{9} + \frac{(y-1)^2}{16} = 1$$

$$a^2 = 16 \Rightarrow a = 4$$

$$b^2 = 9 \Rightarrow b = 3$$

$$c^2 = 16 - 9 = 7$$

$$c = \sqrt{7}$$

