Ch 12 Appendix - Parametric Representation of Curves

simplest way of describing curves in the plane: path of particle moving in the plane

at each time t, two coordinates: $u(t)$, $v(t)$

$(u(t), v(t))$ is a parametric representation of a curve.

ie, the curve is represented parametrically by u and v.

all the pairs (x, y) s.t. $x = u(t)$, $y = v(t)$

the graph of f can always be described parametrically by $x = t$, $y = f(t)$

so far we are considering the curve to be composed of two functions u and v

alternatively, we can consider it as an association of a number t w/ a point in the plane, $c(t)$

from this perspective, a curve is a fn from some interval of real numbers to the plane

previously (ch. 4), we called a point in the plane a vector

a curve is thus called a **vector-valued fn**

example

$\vec{c}(t) = (\cos t, \sin t)$

we know how to add and multiply fns

$(f+g)(x) = f(x) + g(x)$
$(f \cdot g)(x) = f(x) \cdot g(x)$

we also know how to add vectors. * note that we haven't defined product of two vectors.

say we have two vector-valued functions

$c(t) = (u(t), v(t))$
$d(t) = (w(t), z(t))$

then we can define addition of vector-valued fns

$(c+d)(t) = (u(t) + w(t), v(t) + z(t))$

since we also know how to multiply a vector by a scalar, we can define multiplication of an ordinary fn with a vector-valued fn

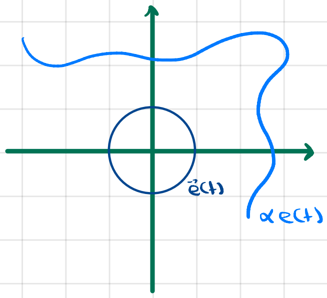
let $\alpha(t)$ be an ordinary fn

$(\alpha \cdot c)(t) = \alpha(t) \cdot c(t) = (\alpha(t) u(t), \alpha(t) v(t))$

example

curve $\alpha\vec{e}$: graph of α in polar coordinates

$$(\alpha\vec{e})(t) = (\alpha(t)\cos t, \alpha(t)\sin t)$$



$\alpha(t)$ and t are polar coordinates

$x(t) = \alpha(t)\cos t$
$y(t) = \alpha(t)\sin t$

* recall

if f is a fn, the graph of f in polar coord. is collection of all points P w/ polar coord. (r,θ) satisfying $r = f(\theta)$.

ie collection of all points w/ polar coord. $(f(\theta),\theta)$

ie the graph of the eq. $r = f(\theta)$

more generally

given any vector-valued c

define new fns r and θ by

$$c(t) = r(t)\vec{e}(\theta(t))$$

consider now the concept of limit.

Given $c(t) = (u(t), v(t))$

we define

$$\lim_{t\to a} c(t) = \lim_{t\to a}(u(t),v(t)) = (\lim_{t\to a} u(t), \lim_{t\to a} v(t))$$

given such a def., rules for limit of sum and limit of multiplication by scalar follow immed.

$$\lim_{t\to a}(c(t)+d(t)) = \lim_{t\to a}(u(t)+w(t), v(t)+z(t)) = (\lim_{t\to a}(u(t)+w(t)), \lim_{t\to a}(v(t)+z(t)))$$

$$= (\lim_{t\to a} u(t), \lim_{t\to a} v(t)) + (\lim_{t\to a} w(t), \lim_{t\to a} z(t))$$

$$= \lim_{t\to a} c(t) + \lim_{t\to a} d(t)$$

$$\lim_{t\to a} \alpha(t)c(t) = \lim_{t\to a}(\alpha(t)u(t), \alpha(t)v(t)) = (\lim_{t\to a}\alpha(t)u(t), \lim_{t\to a}\alpha(t)v(t))$$

$$= (\lim_{t\to a}\alpha(t)\lim_{t\to a}u(t), \lim_{t\to a}\alpha(t)\lim_{t\to a}v(t))$$

$$= \lim_{t\to a}\alpha(t)\cdot\lim_{t\to a}c(t)$$

consider the concept of derivative for a vector-valued fn

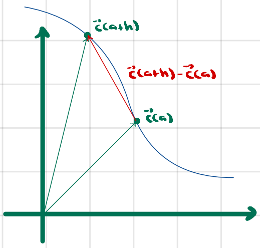
we define $c'(t) = (u'(t), v'(t))$

turns out this definition is equivalent to starting w/ the usual def. of deriv. of an ordinary function and imitating it for a vector-valued fn

$$\vec{c}'(t) = \lim_{h\to 0} \frac{\vec{c}(t+h)-\vec{c}(t)}{h} = \lim_{h\to 0} \frac{1}{h}(u(t+h)-u(t), v(t+h)-v(t))$$

$$= \left(\lim_{h\to 0} \frac{u(t+h)-u(t)}{h}, \lim_{h\to 0} \frac{v(t+h)-v(t)}{h} \right)$$

$$= (u'(t), v'(t))$$



As $h \to 0$, the red vector moves closer to the tangent line at $\vec{c}(a)$.

we define the tangent line at $\vec{c}(a)$ to be the straight line along $\vec{c}'(a)$, when $\vec{c}'(a)$ starts at $\vec{c}(a)$

The tangent line is thus the set of all points $\vec{c}(a) + s\vec{c}'(a)$

* this def does not make sense when $\vec{c}'(a) = (0,0)$.

From this definition of derivative we can derive formulas for derivatives of sum of vector-valued fns and multiplic. of vv-fn by scalar fn.

$$(\vec{c}+\vec{d})'(t) = (u(t)+w(t), v(t)+z(t))' = ((u(t)+w(t))', (v(t)+z(t))') = (u'(t)+w'(t), v'(t)+z'(t))$$

$$= (u'(t), v'(t)) + (w'(t)+z'(t))$$

$$= \vec{c}'(t) + \vec{d}'(t)$$

$$(\alpha c)'(t) = (\alpha(t)u(t), \alpha(t)v(t))' = ((\alpha(t)u(t))', (\alpha(t)v(t))') = (\alpha'u+\alpha u', \alpha'v+\alpha v') = \alpha'(u,v) + \alpha(u',v')$$

$$= \alpha(t)c'(t) + \alpha'(t)c(t)$$

Alternatively, we can derive the same results starting w/ the derivative as a limit

$$(\vec{c}+\vec{d})'(t) = \lim_{h\to 0} \frac{1}{h}\left[(\vec{c}+\vec{d})(t+h) - (\vec{c}+\vec{d})(t)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[(u(t+h)+w(t+h), v(t+h)+z(t+h)) - (u(t)+w(t), v(t)+z(t))\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[(u(t+h)-u(t), v(t+h)-v(t)) - (w(t+h)-w(t), z(t+h)-z(t))\right]$$

$$= \lim_{h\to 0} \frac{1}{h}(u(t+h)-u(t), v(t+h)-v(t)) - \lim_{h\to 0} \frac{1}{h}(w(t+h)-w(t), z(t+h)-z(t))$$

$$= \left(\lim_{h\to 0} \frac{u(t+h)-u(t)}{h}, \lim_{h\to 0} \frac{v(t+h)-v(t)}{h}\right) - \left(\lim_{h\to 0} \frac{w(t+h)-w(t)}{h}, \lim_{h\to 0} \frac{z(t+h)-z(t)}{h}\right)$$

$$= (u'(t), v'(t)) + (w'(t), z'(t))$$

$$= c'(t) + d'(t)$$

$$(\alpha\vec{c})'(t) = \lim_{h\to 0} \frac{\alpha(t+h)\vec{c}(t+h) - \alpha(t)\vec{c}(t)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\alpha(t+h)\vec{c}(t+h) + \alpha(t+h)\vec{c}(t) - \alpha(t+h)\vec{c}(t) - \alpha(t)\vec{c}(t)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\alpha(t+h)(\vec{c}(t+h) - \vec{c}(t)) + \vec{c}(t)(\alpha(t+h) - \alpha(t))\right]$$

$$= \lim_{h\to 0} \alpha(t+h) \cdot \lim_{h\to 0} \frac{\vec{c}(t+h) - \vec{c}(t)}{h} + \lim_{h\to 0} \vec{c}(t) \cdot \lim_{h\to 0} \frac{\alpha(t+h) - \alpha(t)}{h}$$

$$= \alpha(t)\, c'(t) + \alpha'(t)\, c(t)$$

Let's consider now the derivative of a composition of a vector-valued fn and a scalar fn

$$\vec{d}(t) = \vec{c}(p(t)) = (\vec{c} \circ p)(t)$$

p corresponds to a reparametrization of c

d passes through the same points as $\vec{c}(t)$ but at different times

$$\vec{d}(t) = (u(p(t)), v(p(t))) = ((u\circ p)(t), (v\circ p)(t))$$

$$\vec{d}'(t) = (u'(p(t))p'(t), v'(p(t))p'(t))$$

$$= p'(t)\,(u'(p(t)), v'(p(t)))$$

$$= \vec{c}'(p(t))\,p'(t)$$

Consider the situation $p(a) = a$.

Then $\vec{d}(a) = \vec{c}(a)$, ie at time a the two particles are at the same position.

$\vec{d}'(a) = c'(a)\,p'(a)$. The tangent vectors are multiples of one another.

If $p'(a) = 0$, the tangent line for d is undefined.