

## Chapter 2 - Problems

i)  $1^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Let A be set of all  $n \in \mathbb{N}$  s.t. n satisfies i)

$$1^2 = 1 = \frac{1 \cdot 2 \cdot 3}{6} \Rightarrow 1 \text{ in } A$$

Suppose  $k \in A$

$$\text{Therefore } 1^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

$$\begin{aligned} 1^2 + \dots + k^2 + (k+1)^2 &= [k(k+1)(2k+1) + 6(k+1)^2] / 6 \\ &= (k+1)[k(2k+1) + 6(k+1)] / 6 \\ &= [(k+1)(k+2)(2(k+1)+1)] / 6 \end{aligned}$$

$\Rightarrow k+1 \in A$

$\Rightarrow$  by induction,  $A = \mathbb{N} \Rightarrow$  all  $n \in \mathbb{N}$  satisfy i)

ii)  $1^3 + \dots + n^3 = (1 + \dots + n)^2$

Let A be set of  $n \in \mathbb{N}$  s.t. ii) is true

$$1^3 = (1)^2 = 1 \Rightarrow 1 \text{ in } A$$

Suppose  $k \in A$ . Then,  $1^3 + \dots + k^3 = (1 + \dots + k)^2$

$$1^3 + \dots + k^3 + (k+1)^3 = (1 + \dots + k)^2 + (k+1)^3$$

Note that  $(x+y+z)(x+y+z) = (x+y)(x+y+z) + z(x+y+z)$

$$= (x+y)^2 + z(x+y) + z(x+y+z)$$

$$\Rightarrow (x+y)^2 = (x+y+z)^2 - z(x+y) - z(x+y+z)$$

So,

$$1^3 + \dots + k^3 + (k+1)^3 = (1 + \dots + k + k+1)^2 - (k+1)(1 + \dots + k) - (k+1)(1 + \dots + k+1) + (k+1)^3$$

$$= (1 + \dots + k + k+1)^2 - [(k+1)k(k+1)] / 2 - [(k+1)(k+1)(k+2)] / 2 + (k+1)^3$$

$$[ - (k+1)^2 (k+k+2) ] / 2 + (k+1)^3$$

$$= - (k+1)^2 (2k+2) / 2 + (k+1)^3$$

$$= - (k+1)^3 + (k+1)^3$$

$$= 0$$

$$\Rightarrow 1^3 + \dots + (k+1)^3 = (1 + \dots + k+1)^2$$

$$\Rightarrow A = \mathbb{N} \Rightarrow \text{ii) is true for all } n \in \mathbb{N}$$

$$2) \text{ i) } \sum_{i=1}^n (2i-1) = 1 + 3 + 5 + \dots + (2n-1)$$

$$\sum_{i=1}^{2n} i = \sum_{i=1}^n (2i-1) + \sum_{i=1}^n 2i$$

$$2n(2n+1)/2 - 2 \cdot n(n+1)/2$$

$$-(4n^2 + 2n - 2n^2 - 2n)/2 = 2n^2/2 = n^2$$

$$\text{ii) } \sum_{i=1}^n (2i-1)^2 = 1^2 + 3^2 + \dots + (2n-1)^2$$

$$\sum_{i=1}^{2n} i^2 = \sum_{i=1}^n (2i-1)^2 + \sum_{i=1}^n (2i)^2$$

$$\sum_{i=1}^n (2i-1)^2 = \sum_{i=1}^{2n} i^2 - \sum_{i=1}^n (2i)^2 \rightarrow 4 \sum_{i=1}^n i^2$$

$$= \frac{2n(2n+1)(2 \cdot 2n+1)}{6} - 4 \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= (2n+1)[n(4n+1) - 2n(n+1)]/3$$

$$= (2n+1)(4n^2 + n - 2n^2 - 2n)/3$$

$$= (2n+1)(2n^2 - n)/3 = n(2n+1)(2n-1)/3$$

\* Let A be set of  $n \in \mathbb{N}$  s.t.  $\sum_{i=1}^n z_i = 2 \sum_{i=1}^n i$   
 $n=1 \Rightarrow \sum_{i=1}^1 z_i = 2 = 2 \cdot \sum_{i=1}^1 i \Rightarrow 1 \text{ in A}$

assume  $k \in A \Rightarrow \sum_{i=1}^k z_i = 2 \sum_{i=1}^k i$

$$\sum_{i=1}^{k+1} z_i = \sum_{i=1}^k z_i + z(k+1)$$

$$= 2 \sum_{i=1}^k i + z(k+1)$$

$$= 2 \left( \sum_{i=1}^k i + (k+1) \right)$$

$$= 2 \sum_{i=1}^{k+1} i$$

3)  $0 \leq k \leq n$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{if } k \neq 0, n$$

$$\binom{n}{0} = \binom{n}{n} = 1 \quad (0! = 1)$$

for  $k < 0$  or  $k > n$  define  $\binom{n}{k} = 0$

a) Prove  $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

$$\frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{(n+1)!}{k!(n-k+1)!} \cdot \frac{k}{n+1} + \frac{(n+1)!}{k!(n-k+1)!} \cdot \frac{n-k+1}{n+1}$$

$$= \binom{n+1}{k} \left( \frac{k+n-k+1}{n+1} \right) = \binom{n+1}{k}$$

The other cases would involve reverse engineering the construction of  $\binom{n+1}{k}$

$$\frac{(n+1)!}{k!(n+1-k)!} \cdot \frac{k}{n+1} = \binom{n}{k-1}$$

$$\frac{(n+1)!}{k!(n+1-k)!} \cdot \frac{n+1-k}{n+1} = \binom{n}{k}$$

$$\Rightarrow \left[ \frac{(n+1)!}{k!(n+1-k)!} \right] \left( \frac{k}{n+1} + \frac{n+1-k}{n+1} \right) = \binom{n}{k-1} + \binom{n}{k}$$

With this result we have Pascal's Triangle

$$\binom{0}{0} \rightarrow \binom{1}{1} = \binom{1}{0} + \binom{1}{1}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\dots$$

$$= \begin{array}{ccccccccc} & & & 0 & & & & & \\ & & & 1 & 1 & 1 & & & \\ & & & 1 & 2 & 1 & & & \\ & & & 1 & 3 & 3 & 1 & & \\ & & & 1 & 4 & 6 & 4 & 1 & \\ & & & 1 & 5 & 10 & 10 & 5 & 1 \\ & & & & & \dots & & & \end{array}$$

3) b) use c) to prove  $\binom{n}{k}$  is always a natural number

Let A be set of  $n \in \mathbb{N}$  s.t. all  $\binom{n}{j} \in \mathbb{N}$  for  $j \in \{x \in \mathbb{N} | x \leq n, x \geq 1\} \cup \{0\}$

1 in A because  $\binom{1}{1} = 1 \in \mathbb{N}$  and  $\binom{1}{0} = 1 \in \mathbb{N}$

Suppose k in A.

$$\binom{k}{j} \in \mathbb{N}, j \in \{0, \dots, k\}$$

then,

$$\binom{k+1}{0} = 1 \in \mathbb{N}$$

$$\binom{k+1}{k} = 1 \in \mathbb{N}$$

$$\binom{k+1}{j} = \underbrace{\binom{k}{j-1} + \binom{k}{j}}_{\text{These terms both } \in \mathbb{N}}, j \in \{1, \dots, k\}$$

$$\Rightarrow \binom{k+1}{j} \in \mathbb{N}, j \in \{0, \dots, k+1\}$$

Therefore  $A = \mathbb{N} \Rightarrow \binom{n}{k} \in \mathbb{N}$  for all  $n, k \in \{x \in \mathbb{N} | x \leq n, x \geq 1\} \cup \{0\}$

c)  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Given n distinct integers, we form k-tuples.

There are  $n(n-1)(n-2)\dots(n-k+1)$  k-tuples because:

→ n possible choices for first element of the k-tuple

$n-1$  " " " second " "

(...)

$n-k+1$  " " "  $n$  " " " "

This is equal to  $\frac{n!}{k!(n-k)!}$ . Each k-tuple is unique in the order of distinct elements, but not in terms of the set of elements.

For each k-tuple, there are  $k!$  permutations of the elements, ie  $k!$  k-tuples that contain the same elements, ie represent the same set.

So, to get the number of sets of k elements from n possible elements, we divide by  $k!$ .

$$\frac{n!}{(n-k)k!} = \binom{n}{k}$$

### 3d) Binomial Theorem

$$a, b \text{ any numbers, } n \in \mathbb{N} \Rightarrow (a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n-1} a b^{n-1} + b^n$$

$$= \sum_{i=0}^n \binom{n}{i} a^{n-i} b^i$$

Using induction,

Let A be a set of all  $n \in \mathbb{N}$  for which the above equation holds.

$$(a+b)^1 = \sum_{i=0}^1 \binom{1}{i} a^{1-i} b^i = 1a^1 b^0 + 1a^0 b^1 = a+b$$

$\Rightarrow 1 \in A$

Suppose the eq. holds for  $n = k$ .

$$\begin{aligned} (a+b)^{k+1} &= (a+b)(a+b)^k \\ &= (a+b) \cdot \sum_{i=0}^k \binom{k}{i} a^{k-i} b^i = \sum_{i=0}^k \binom{k}{i} a^{k-i} b^i + \sum_{i=0}^k \binom{k}{i} a^{k-i} b^{i+1} \\ &= \binom{k}{0} a^{k+1} + \sum_{i=1}^k \binom{k}{i} a^{k-i+1} b^i + \sum_{i=1}^k \binom{k}{i-1} a^{k+1-i} b^i + \binom{k}{k} a^0 b^{k+1} \\ &= \binom{k+1}{0} a^{k+1} + \sum_{i=1}^k a^{k+1-i} b^i [\binom{k}{i} + \binom{k}{i-1}] + \binom{k+1}{k} b^{k+1} \\ &= \binom{k+1}{0} a^{k+1} b^0 + \sum_{i=1}^k \binom{k+1}{i} a^{k+1-i} b^i + \binom{k+1}{k+1} a^0 b^{k+1} \\ &= \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i \end{aligned}$$

e) ii)  $\sum_{i=0}^n \binom{n}{i} = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$

induction: A set of  $n \in \mathbb{N}$  satisfying i).

$$\sum_{i=0}^1 \binom{1}{i} = \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2^1 \Rightarrow 1 \in A$$

$$\text{Suppose } \sum_{j=0}^k \binom{k}{j} = 2^k$$

$$\begin{aligned} \sum_{j=0}^{k+1} \binom{k+1}{j} &= \binom{k+1}{k+1} + \sum_{j=0}^k \binom{k+1}{j} = \binom{k+1}{k+1} + \sum_{i=0}^k [\binom{k}{i} + \binom{k}{i-1}] \\ &= 2^k + \binom{k+1}{k+1} + \left[ \binom{k}{0} + \binom{k}{1} + \dots + \binom{k}{k-1} \right] = 2^k + \binom{k}{0} + \dots + \binom{k}{k}, \text{ because } \binom{k}{k} = \binom{k+1}{k+1} \\ &= 2^k + \sum_{i=0}^k \binom{k}{i} = 2 \cdot 2^k = 2^{k+1} \Rightarrow A = \mathbb{N} \end{aligned}$$

### Alternative Proof

$$z^n = (1+1)^n$$

using Binomial Theorem:

$$= \sum_{i=0}^n \binom{n}{i}$$

$$3) \text{e) iii)} \sum_{j=0}^n (-1)^j \binom{n}{j} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + \binom{n}{n} = 0$$

induction

let A be set of  $n \in \mathbb{N}$  satisfying ii).

$$\sum_{j=0}^1 (-1)^j \binom{1}{j} = \binom{1}{0} - \binom{1}{1} = 1 - 1 = 0 \Rightarrow 1 \in A$$

$$\text{Assume } \sum_{j=0}^k (-1)^j \binom{k}{j} = 0$$

$$\begin{aligned} \text{Then } \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} &= \sum_{j=0}^{k+1} (-1)^j \left[ \binom{k}{j} + \binom{k}{j-1} \right] \\ &= \cancel{(-1)^0 \binom{k}{0}} + \sum_{j=0}^k (-1)^j \binom{k}{j} + \sum_{j=0}^{k+1} (-1)^j \binom{k}{j-1} \\ &= \cancel{(-1)^0 \binom{k}{-1}} + (-1)^0 \binom{k}{0} + \dots + (-1)^{k+1} \binom{k}{k} \\ &= (-1) \cdot \sum_{j=0}^k (-1)^j \binom{k}{j} = 0 \end{aligned}$$

$$\text{iii)} \sum_{i \text{ odd}} \binom{n}{i} = \binom{n}{1} + \binom{n}{3} + \dots = 2^{n-1}$$

induction : A set  $n \in \mathbb{N}$  satisfying iii..

$$\binom{1}{1} = 1 = 2^0, 1 \in A$$

$$\text{Assume } \sum_{i \text{ odd}} \binom{k}{i} = 2^{k-1}$$

then

$$\begin{aligned} \sum_{i \text{ odd}} \binom{k+1}{i} &= \text{if } k+1 \text{ odd } = \binom{k+1}{1} + \binom{k+1}{3} + \dots + \binom{k+1}{k+1} \\ &= \binom{k}{1} + \binom{k}{0} + \binom{k}{3} + \binom{k}{2} + \dots + \binom{k}{k} + \binom{k}{k+1} \\ &= \sum_{i=0}^k \binom{k}{i} + \cancel{\binom{k}{k+1}} = 2^k, \text{ from 3e) } \end{aligned}$$

if  $k+1$  even

$$\begin{aligned} &= \underbrace{\binom{k+1}{1} + \binom{k+1}{3} + \dots + \binom{k+1}{k-1}}_{\binom{k}{0} + \binom{k}{1}} + \underbrace{\binom{k+1}{k}}_{\binom{k}{k-1} + \binom{k}{k}} = 2^k \end{aligned}$$

Alternative Proof

$$0 = (1-1)^n = \sum_{i=0}^n \binom{n}{i} (-1)^i$$

Alternative Proof

$$\text{from ii), } \sum_{j=0}^n (-1)^j \binom{n}{j} = 0$$

$$\text{from ii), } \sum_{i \text{ odd}} (-1)^i \binom{n}{i} = 0$$

$$\text{i) - iii) } = 2 \sum_{i \text{ odd}} \binom{n}{i} = 2^n$$

$$\Rightarrow \sum_{i \text{ odd}} \binom{n}{i} = 2^{n-1}$$

$$3) \text{ e) iv)} \sum_{i \text{ even}} \binom{n}{i} = z^{n-1}$$

This results as corollary to 3eiii).

$$\sum_{i=0}^k \binom{n}{i} = \sum_{i \text{ odd}} \binom{n}{i} + \sum_{i \text{ even}} \binom{n}{i}$$

$$z^k = z^{n-1} + \sum_{i \text{ even}} \binom{n}{i}$$

$$\sum_{i \text{ even}} \binom{n}{i} = z^{n-1}$$

$$4) \text{ a) } \sum_{i=0}^k \binom{n}{i} \binom{m}{k-i} = \binom{n+m}{k}$$

$$(1+x)^n (1+x)^m = \sum_{i=0}^n \binom{n}{i} x^i \cdot \sum_{j=0}^m \binom{m}{j} x^j = \sum_{k=0}^{n+m} \binom{n+m}{k} x^k$$

The right-hand side has terms of form  $c x^k$ ,  $k=0, \dots, n+m$ .

The left-hand side has term  $(c_0 x^0 + \dots + c_n x^n)(b_0 x^0 + \dots + b_m x^m) = a_i b_j x^{i+j}$

For each  $a_i b_j x^{i+j}$ , there is a corresponding right-hand side term  $\binom{n+m}{k} x^k$

We need to determine which left-hand side summation equals each of these right-hand side terms.

For each  $k$ , we need  $i+j=k$ :

$$\sum_{i=0}^k \binom{n}{i} \binom{m}{k-i} x^k = \binom{n+m}{k} x^k$$

$$\Rightarrow \sum_{i=0}^k \binom{n}{i} \binom{m}{k-i} = \binom{n+m}{k}$$

Alternative:

$$\sum_{k=0}^{n+m} \binom{n+m}{k} x^k = \sum_{i=0}^n \binom{n}{i} x^i \cdot \sum_{j=0}^m \binom{m}{j} x^j = \sum_{i=0}^n \sum_{j=0}^m \binom{n}{i} \binom{m}{j} x^{i+j}$$

let  $k = i+j$

$$\sum_{k=0}^{n+m} \sum_{i=0}^k \binom{n}{i} \binom{m}{k-i} x^k$$

Also,

$\binom{n+m}{k}$  is the number of ways of choosing  $k$  items from  $n+m$  items.

This can also be accomplished by splitting the items into two groups, with  $n$  and  $m$  items. Then, choose  $i$  items from the group of  $n$  and  $k-i$  items from the group of  $m$ . For each  $i$ , there are  $\binom{n}{i} \binom{m}{k-i}$  ways of choosing. And we can pick 0 to  $k$  items from the  $n$  group,  $k$  to 0 items from the  $m$  group.

$$4b) \sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2 = \binom{2n}{n}$$

using 4c):

$$\binom{2n}{n} = \binom{n+n}{n} = \sum_{i=0}^n \binom{n}{i} \binom{n}{n-i}$$

$$\binom{n}{i} \cdot \frac{n!}{(n-i)! i!} = \binom{n}{n-i} \Rightarrow \binom{2n}{n} = \sum_{i=0}^n \binom{n}{i}^2$$

$$5c) 1+r+r^2+\dots+r^n = \frac{1-r^{n+1}}{1-r} \quad \text{if } r \neq 1$$

$r$  is given and  $\neq 1$ .

Let  $A$  be set of  $n \in \mathbb{N}$  s.t. the result above is true.

$$n=1 \Rightarrow 1+r = \frac{1-r^2}{1-r} = \frac{(1+r)(1-r)}{1-r} = 1-r \Rightarrow 1 \in A$$

$$\text{assume } k \in A : 1+r+\dots+r^k = \frac{1-r^{k+1}}{1-r}$$

then,

$$1+r+r^2+\dots+r^k+r^{k+1} = \frac{1-r^{k+1}}{1-r} + r^{k+1} = \frac{\cancel{1-r^k} + \cancel{r^k} - r}{1-r} = \frac{1-r^{k+2}}{1-r}$$

$\Rightarrow k+1 \in A$

By induction,  $A = \mathbb{N}$ .

$$b) S = 1+r+\dots+r^n$$

$$rS = r(1+r+\dots+r^n) = r+r^2+\dots+r^{n+1} = (1+r+\dots+r^n) + r^{n+1} - 1$$

$$rS = S + r^{n+1} - 1$$

$$S(r-1) = 1-r^{n+1} \Rightarrow S = \frac{1-r^{n+1}}{1-r}$$

6)

$$\text{ex: } \sum_{i=1}^n i^2$$

$$(k+1)^3 - k^3 = \sum_{i=0}^3 \binom{3}{i} k^i - k^3 = \cancel{\binom{3}{0} k^3} + \cancel{\binom{3}{1} k^2} + \cancel{\binom{3}{2} k} + \cancel{\binom{3}{3}} - k^3$$

$$= 3k^2 + 3k + 1$$

$$\begin{array}{ll} k=1 & \cancel{2^3 - 1^3} = 3 \cdot 1^2 + 3 \cdot 1 + 1 \\ k=2 & \cancel{3^3 - 2^3} = 3 \cdot 2^2 + 3 \cdot 2 + 1 \\ \dots & \dots \end{array}$$

$$k=n \quad (n+1)^3 - n^3 = 3 \cdot n^2 + 3n + 1$$

$$(n+1)^3 - 1^3 = 3 \cdot \sum_{i=1}^n i^2 + 3 \sum_{i=1}^n i + n \Rightarrow \sum_{i=1}^n i^2 = [(n+1)^3 - 1 - 3 \sum_{i=1}^n i - n] / 3$$

(orange bracket)

$$= (n+1)^3 - 1 - 3n(n+1)/2 - n) / 3$$

$$= (n+1)[2(n+1)^2 - 3n - 2] / 6$$

$$(k+1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k + 1$$

$$\begin{array}{ll} k=1 & \cancel{2^2 - 1^2} = 2 \cdot 1 + 1 \\ k=2 & \cancel{3^2 - 2^2} = 2 \cdot 2 + 1 \\ \dots & \dots \\ k=n & (n+1)^2 - \cancel{n^2} = 2n + 1 \end{array}$$

$$(n+1)^2 - 1^2 = 2 \cdot \sum_{i=1}^n i + n \Rightarrow \sum_{i=1}^n i = \frac{(n+1)^2 - (n+1)}{2} = \frac{n(n+1)}{2}$$

i)  $\sum_{i=1}^n i^3$ 

$$(k+1)^4 - k^4 = \cancel{\binom{4}{0} k^4} + \cancel{\binom{4}{1} k^3} + \binom{4}{2} k^2 + \binom{4}{3} k + \cancel{\binom{4}{4}} - k^4 = 4k^3 + 6k^2 + 4k + 1$$

$$(n+1)^4 - 1 = 4 \sum_{i=1}^n i^3 + 6 \sum_{i=1}^n i^2 + 4 \sum_{i=1}^n i + n$$

$$\sum_{i=1}^n i^3 = (n+1)^4 - (n+1) - 4 \sum_{i=1}^n i^2 - 6 \sum_{i=1}^n i^2 - 4 \sum_{i=1}^n i$$

ii)  $\sum_{i=1}^n i^4$ 

$$(n+1)^5 - 1 = \binom{5}{1} \sum_{i=1}^n i^4 + \binom{5}{2} \sum_{i=1}^n i^3 + \binom{5}{3} \sum_{i=1}^n i^2 + \binom{5}{4} \sum_{i=1}^n i + n$$

iii)  $\sum_{i=1}^n \frac{1}{i(i+1)}$ 

$$\frac{1}{k} - \frac{1}{k+1} = \frac{k+1-k}{k(k+1)} = \frac{1}{k(k+1)}$$

$$\left. \begin{array}{l} k=1 \quad \frac{1}{1} - \frac{1}{2} = \frac{1}{1 \cdot 2} \\ k=2 \quad \frac{1}{2} - \frac{1}{3} = \frac{1}{2 \cdot 3} \\ \vdots \\ k=n \quad \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \end{array} \right\} 1 - \frac{1}{n+1} = \sum_{i=1}^n \frac{1}{i(i+1)}$$

$$S(N) \sum_{i=1}^n \frac{2i+1}{i^2(i+1)^2} = \frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \dots + \frac{2n+1}{n^2(n+1)^2}$$

$$\frac{1}{i^2} - \frac{1}{(i+1)^2} = \frac{i^2 + 2i + 1 - i^2}{i^2(i+1)^2} = \frac{2i+1}{i^2(i+1)^2}$$

$$i=1 \quad \frac{1}{1^2} - \cancel{\frac{1}{2^2}} =$$

$$i=2 \quad \cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}} =$$

$$i=n \quad \cancel{\frac{1}{n^2}} - \cancel{\frac{1}{(n+1)^2}} =$$

$$1 - \frac{1}{(n+1)^2} = \sum_{i=1}^n \frac{2i+1}{i^2(i+1)^2}$$

7)  $\sum_{k=1}^n k^p$  can always be written in form  $\frac{n^{p+1}}{p+1} + An^p + Bn^{p-1} + Cn^{p-2} + \dots$

induction

$$p=1 \quad \sum_{k=1}^n k = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$$

assume true for all  $i \in \mathbb{N}$   $i < p$

$$(kh)^{p+1} = \sum_{k=0}^{p+1} (p+1)_k h^k$$

$$(k+1)^{p+1} - k^{p+1} = (p+1)_1 h^p + \text{terms involving lower powers of } h$$

adding for  $k=1, \dots, n$

$$2^{p+1} - 1^{p+1} = (p+1)_1 \cdot 1^p + \dots$$

$$3^{p+1} - 2^{p+1} = (p+1)_2 \cdot 2^p + \dots$$

$$(n+1)^{p+1} - n^{p+1} = (p+1)_n n^p + \dots$$

$$(n+1)^{p+1} - 1 = (p+1) \sum_{k=1}^n k^p + \text{terms involving } \sum k^r \quad r < p$$

there is a term  $(p+1)_m m^p$  on the right that cancels

by assumption,  $\sum k^r$  terms can be written as expression involving  $n^i \leq p$

$$\Rightarrow \sum k^p =$$

$$p=1 \quad \sum_{k=1}^n k = 1 + 2 + \dots + n = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$p=2 \quad \sum_{k=1}^n k^2 = 1^2 + 2^2 + \dots + n^2 =$$

$$7) \sum_{k=1}^n k^p = \frac{n^{p+1}}{p+1} + A n^p + B n^{p-1} + \dots$$

$$(n+1)^2 - 1 = 2 \sum k + n \Rightarrow n^2 + 2n = 2 \sum k + n \Rightarrow \sum k = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$(n+1)^3 - 1 = 3 \sum k^2 + 3 \sum k + n \Rightarrow n^3 + 3n^2 + 3n = 3 \sum k^2 + 3 \sum k + n = 3 \sum k^2 + 3(\frac{1}{2}n^2 + \frac{1}{2}n) + n$$

$$\Rightarrow \sum k^2 = [n^3 + n^2(3 - \frac{3}{2}) + n(3 - \frac{3}{2} - 1)]/3 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$(n+1)^4 - 1 = n^4 + 4n^3 + 6n^2 + 4n = 4 \sum k^3 + 6 \sum k^2 + 4 \sum k + n = 4 \sum k^3 + 6[\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n] + 4[\frac{1}{2}n^2 + \frac{1}{2}n] + n$$

$$\sum k^3 = [n^4 + n^3[4 - \frac{6}{3}] + n^2[6 - \frac{6}{2} - \frac{4}{2}]] + n[4 - \frac{6}{6} - \frac{4}{2} - 1]/4$$

$$\sum_{k=1}^n k^p = \frac{1}{(p+1)} \cdot \left[ \underbrace{n^{p+1}}_{\text{1st term of } \sum k^{p+1}} + n^p \left[ \binom{p+1}{1} - \binom{p+1}{2} \cdot \frac{1}{(p+1)} \right] + n^{p-1} \left[ \underbrace{\binom{p+1}{2} - \binom{p+1}{3} \left[ \frac{\binom{p}{1} - \binom{p}{2} / \binom{p-1}{1}}{\binom{p}{1}} \right]}_{\text{2nd term of } \sum k^{p-1}} - \binom{p+1}{3} \cdot \underbrace{\frac{1}{(p-1)}}_{\text{1st term of } \sum k^{p-2}} \right]$$

$$+ n^{p-2} \left[ \binom{p+1}{3} - \underbrace{\text{3rd term of } \sum k^{p-1} \times \binom{p+1}{2}}_{\text{3rd term of } \sum k^{p-2} \times \binom{p+1}{3}} - \underbrace{\text{2nd term of } \sum k^{p-2} \times \binom{p+1}{3}}_{\text{2nd term of } \sum k^{p-3} \times \binom{p+1}{4}} - \text{1st term of } \sum k^{p-3} \times \binom{p+1}{4} \right]$$

8) Every natural number is either even or odd

Def

even number:  $\{2h; h \in \mathbb{Z}\}$

odd " " :  $\{2h+1; h \in \mathbb{Z}\}$

$A = \{n; n \in \mathbb{N}, n = 2h \text{ or } n = 2h+1, h \in \mathbb{Z}\}$

$1 = 2 \cdot 0 + 1 \Rightarrow 1 \in A$

assume  $n \in A$

$\Rightarrow n = 2h \text{ or } n = 2h+1, h \in \mathbb{Z}$

$n = 2h \Rightarrow n+1 = 2h+1 \Rightarrow n+1 \in A$

$n = 2h+1 \Rightarrow n+1 = 2h+2 = 2(h+1) \Rightarrow n+1 \in A$

9)  $A \subset \mathbb{N}$

$n_0 \in A \Rightarrow n \in A, \forall n \geq n_0$

$h \in A \Rightarrow h+1 \in A$

let  $B = \{n \in \mathbb{N}; n+n_0 \in A\}$

$n_0 \in A \Rightarrow n_0+1 \in A \Rightarrow 1 \in B$

assume  $k \in B \Rightarrow n_0+k \in A \Rightarrow n_0+k+1 \in A \Rightarrow k+1 \in B$

$\Rightarrow B = \mathbb{N}$

$\Rightarrow n \geq n_0 \Rightarrow n \in A$

10) well-ordering principle  $\Rightarrow$  mathem. induction

let  $A \subset \mathbb{N}, 1 \in A, h \in A \Rightarrow h+1 \in A$

let  $C = \{n \in \mathbb{N}; n \notin A\}$

Assume  $C$  is non-empty.  $\Rightarrow C$  has least element.

Let  $k$  be least element of  $C \Rightarrow k+1$  because  $1 \in A$

$\Rightarrow k-1 \in A \Rightarrow k \in A \Rightarrow k \notin C$

$\Rightarrow$  contradiction

$\Rightarrow C = \emptyset \Rightarrow A = \mathbb{N}$

II) mathem. induction  $\Rightarrow$  complete induction

$$A \subset N$$

$$1 \in A$$

$$1, \dots, n \in A \Rightarrow n+1 \in A$$

$$\text{Let } B = \{n \in N; 1, \dots, n \in A\}$$

$$1 \in A \Rightarrow 1 \in B$$

$$\begin{aligned} \text{assume } k \in B &\Rightarrow 1, \dots, k \in A \Rightarrow k+1 \in A \Rightarrow 1, \dots, k+1 \in A \Rightarrow k+1 \in B \\ &\Rightarrow B = N \Rightarrow A = N \end{aligned}$$

R(a)

i) a rational  $\Rightarrow a = \frac{m}{n}, m, n \in \mathbb{Z}$

b irrational

$$\text{assume } a+b \text{ is rational} \Rightarrow \frac{m}{n} + b = \frac{p}{q}, m, n, p, q \in \mathbb{Z} \Rightarrow b = \frac{pn - mq}{nq} \in \mathbb{Z}$$

$\Rightarrow$  contradiction

$$\Rightarrow a+b \in \mathbb{Q}'$$

ii) Now assume  $a, b \in \mathbb{Q}'$

$$\text{Take } b = 1 - a \in \mathbb{Q}'$$

$$ab = a + 1 - a = 1 \in \mathbb{Q}$$

$\Rightarrow ab, a, b \in \mathbb{Q}'$  is not necessarily irrational

b)  $a \in \mathbb{Q}, b \in \mathbb{Q}'$

$$\text{assume } ab = \frac{m}{n}, m, n \in \mathbb{Z}$$

If  $c \neq 0$ ,

$$\text{since } a = \frac{p}{q}, p, q \in \mathbb{Z} \Rightarrow b = a^{-1} \cdot \frac{m}{n} \cdot \frac{qn}{pn} \in \mathbb{Q}, \text{ contradiction}$$

$$\Rightarrow ab \in \mathbb{Q}'$$

If  $c = 0$  then  $ab = 0 \in \mathbb{Q}$

$$12) \text{ c)} \quad a = z^{\frac{1}{4}}$$

$$a^2 = z^{\frac{1}{2}} \in \mathbb{Q}'$$

$$a^4 = z \in \mathbb{Q}$$

$$\text{d)} \quad a, b \in \mathbb{Q}'$$

$$a+b = r_1$$

$$ab = r_2$$

$$r_1, r_2 \in \mathbb{Q}$$

$$a = r_1 - b$$

$$(r_1 - b)b = r_2$$

$$b^2 - r_1 b + r_2 = 0$$

$$\Delta = r_1^2 - 4r_2 \geq 0 \Rightarrow r_1^2 \geq 4r_2$$

Guess a value for  $r_1$ :  $r_1 = 0 \Rightarrow a = -b \Rightarrow -b^2 = r_2 \Rightarrow b^2 = -r_2 \Rightarrow b^2 \pm \sqrt{-r_2}$

Guess on  $r_2 \leq 0$ , say  $r_2 = -3 \Rightarrow b^2 \pm \sqrt{3}$

so,  $a = \sqrt{3}, b = -\sqrt{3}$  and  $a = -\sqrt{3}, b = \sqrt{3}$

13)

→ every  $m \in \mathbb{Z}$  can be written as either  $3n, 3n+1$  or  $3n+2$  for some  $n \in \mathbb{Z}$

Proof:

A: set of all  $m \in \mathbb{Z}$  for which  $m = 3n$  or  $m = 3n+1$  or  $m = 3n+2$

$$1 = 3 \cdot 0 + 1 \Rightarrow 1 \in A$$

assume  $k \in A$

$$k = 3n \Rightarrow k+1 = 3n+1$$

$$k = 3n+1 \Rightarrow k+1 = 3n+2$$

$$k = 3n+2 \Rightarrow k+1 = 3n+3 = 3(n+1) = 3n, n \in \mathbb{Z}$$

$$\Rightarrow k+1 = 3n \text{ or } k+1 = 3n+1 \text{ or } k+1 = 3n+2$$

→  $\forall m \in \mathbb{Z}, m = 3n$  or  $m = 3n+1$  or  $m = 3n+2$  for some  $n \in \mathbb{Z}$

→ if  $k^2$  is divisible by 3 then  $k$  is too

Proof:

$$k = 3n, n \in \mathbb{Z} \Rightarrow k^2 = 9n^2, \text{ divisible by 3}$$

$$\begin{aligned} k = 3n+1, n \in \mathbb{Z} &\Rightarrow k^2 = 9n^2 + 6n + 1 = 3(3n^2 + 2n) + 1 \\ k = 3n+2, n \in \mathbb{Z} &\Rightarrow k^2 = 9n^2 + 12n + 4 = 3(3n^2 + 4n + 1) + 1 \end{aligned} \quad \left. \right\} \text{not divisible by 3}$$

→ if  $k^2 \in \mathbb{Z}$  divisible by 3,  $k$  is of form  $3n, n \in \mathbb{Z} \Rightarrow k$  divisible by 3

Assume  $\sqrt{3}$  is rational, i.e.  $\sqrt{3} = \frac{p}{q}$ ,  $p, q \in \mathbb{N}$

$$\Rightarrow p^2 \cdot 3q^2 = p^2 \text{ divisible by } 3 \Rightarrow p \text{ divisible by } 3 \Rightarrow p = 3p', p' \in \mathbb{N}$$

$$\Rightarrow (3p')^2 \cdot 3q^2 = q^2 \cdot 3p'^2 \Rightarrow q^2 \text{ and } q \text{ divisible by } 3$$

$\Rightarrow p$  and  $q$  have common factor  $\Rightarrow$  contradiction  $\Rightarrow \sqrt{3}$  is not rational

$\sqrt{5}$

All  $k \in \mathbb{Z}$  can be written as either  $5n, 5n+1, 5n+2, 5n+3$ , or  $5n+4$  for some  $n \in \mathbb{Z}$ .

Proof of this is analogous to proof for writing all integers as  $3n, 3n+1$  or  $3n+2$ .

$$k = 5n, n \in \mathbb{Z} \Rightarrow k^2 = 25n^2$$

$$k = 5n+1, n \in \mathbb{Z} \Rightarrow k^2 = 25n^2 + 10n + 1 = 5(5n^2 + 2n) + 1$$

$$k = 5n+2, n \in \mathbb{Z} \Rightarrow k^2 = 25n^2 + 20n + 4 = 5(5n^2 + 4n) + 4$$

$$k = 5n+3, n \in \mathbb{Z} \Rightarrow k^2 = 25n^2 + 30n + 9 = 5(5n^2 + 6n + 1) + 4$$

$$k = 5n+4, n \in \mathbb{Z} \Rightarrow k^2 = 25n^2 + 40n + 16 = 5(5n^2 + 8n + 3) + 1$$

} not divisible by 5

$$\Rightarrow k^2 \text{ divisible by } 5 \Rightarrow k = 5n, n \in \mathbb{Z} \Rightarrow k \text{ divisible by } 5$$

$$\text{Similarly we can show } k^2 \text{ divisible by } 6 \Rightarrow k = 6n, n \in \mathbb{Z} \Rightarrow k \text{ divisible by } 6$$

showing that  $\sqrt{5}$  and  $\sqrt{6}$  are irrational is analogous to the proof for  $\sqrt{3}$ , by contradiction.

$\sqrt{4}$

The proofs that  $\sqrt{3}, \sqrt{5}$  and  $\sqrt{6}$  are irrational rely on the result derived above regarding divisibility of  $k^2$  and  $k$ . We can't derive that result for  $\sqrt{4}$ .

$$k = 4n \Rightarrow k^2 = 16n^2$$

$$k = 4n+1 \Rightarrow k^2 = 16n^2 + 8n + 1 = 4(4n^2 + 2n) + 1$$

$$k = 4n+2 \Rightarrow k^2 = 16n^2 + 16n + 4 = 4(4n^2 + 4n + 1)$$

$$k = 4n+3 \Rightarrow k^2 = 16n^2 + 24n + 9 = 4(4n^2 + 6n + 2) + 1$$

we see that if  $k^2$  is divisible by 4,  $k$  could be  $4n$  or  $4n+2$ , and the latter is not divisible by 4.

Assume  $\sqrt{4}$  is rational, i.e.,  $\sqrt{4} = \frac{p}{q}$ ,  $p, q \in \mathbb{N}$

$$p^2 \cdot 4q^2 \Rightarrow p \cdot 4p' \text{ or } p \cdot 4p' + 2$$

$$p \cdot 4p' \Rightarrow p \text{ div. by } 4 \Rightarrow q^2 \cdot 4p'^2 = q^2 \text{ div. by } 4 \Rightarrow q_2 \cdot 4q'_2 \text{ or } q_2 \cdot 4q'_2 + 2$$

$q_2 \cdot 4q'_2 \Rightarrow p$  and  $q$  have common factor, 4  $\Rightarrow$  contradiction

$$q_2 \cdot 4q'_2 + 2 \Rightarrow (4q'_2 + 2)^2 = 4p'^2 \Rightarrow \text{This is odd, i.e. } q_2 \cdot 4q'_2 + 2, p \cdot 4p'$$

$$p \cdot 4p' + 2 \Rightarrow (4p'^2 + 4p') = 4(p'^2 + p') + 1, \text{ non-div. by } 4 \Rightarrow q_2 + 4n, n \in \mathbb{Z} \Rightarrow \text{contradiction.}$$

with  $p$  and  $q$  non-div. by 4

$$14 \quad \sqrt{2} + \sqrt{6} \in \mathbb{Q}'$$

We've shown that  $\sqrt{6} \in \mathbb{Q}'$  in 13). Let's show that  $\sqrt{2} \in \mathbb{Q}'$ .

$\sqrt{2} \in \mathbb{Q}'$ , proof:

Any integer can be written as  $2n$  or  $2n+1$ ,  $n \in \mathbb{Z}$

Proof: Let  $A$  be set of  $n \in \mathbb{N}$  satisfying above statement.

$$1 = 2 \cdot 0 + 1 \Rightarrow 1 \in A$$

$$\text{assume } h \in A \Rightarrow h = 2n \text{ or } h = 2n+1, n \in \mathbb{N}$$

$$\Rightarrow hh = 2n^2 \text{ or } hh = 2n^2 + 2 = 2(n+1)^2 = 2m, m \in \mathbb{N}$$

$$\Rightarrow hm \in A$$

$$\Rightarrow \text{by induction, } A = \mathbb{N} \text{ so for every } h \in \mathbb{N}, h = 2n \text{ or } h = 2n+1$$

$$h^2 \text{ div by 2} \Rightarrow h \text{ div } b+2$$

Proof

$$h = 2n, n \in \mathbb{Z} \Rightarrow h^2 = 4n^2 \text{ div by 2}$$

$$h = 2n+1, n \in \mathbb{Z} \Rightarrow h^2 = 4n^2 + 4n + 1 = 4(n^2+n) + 1 \text{ not div by 2}$$

Given  $h \in \mathbb{Z}$ ,  $h^2 = 4n^2$  or  $h^2 = 4n^2 + 4n + 1$  and only  $4n^2$  is divisible by 2, and  $k$  is also div by 2.

Assume  $\sqrt{2} = \frac{p}{q}$ ,  $p, q \in \mathbb{Z}, q \neq 0$ ,  $p$  and  $q$  have no common factor, i.e.  $\sqrt{2} \in \mathbb{Q}$

$$p^2 = q^2 \cdot 2 \Rightarrow p^2 \text{ divisible by 2} \Rightarrow p \text{ div. by 2} \Rightarrow p = 2 \cdot m, m \in \mathbb{Z}$$

$$(2m)^2 = q^2 \cdot 2 \Rightarrow q^2 = 2m^2 \Rightarrow q^2 \text{ div by 2} \Rightarrow q \text{ div by 2}$$

$\Rightarrow p, q$  have common factor  $\Rightarrow$  contradiction  $\Rightarrow \sqrt{2} \notin \mathbb{Q}$

a)

From 12a) we know that  $a, b \in \mathbb{Q}'$  does not imply  $a+b \in \mathbb{Q}'$ .

$$\text{Assume } \sqrt{2} + \sqrt{6} \in \mathbb{Q} \Rightarrow \sqrt{2} + \sqrt{6} = \frac{p}{q}, p, q \in \mathbb{Z}$$

$$2 + 2\sqrt{2} + \sqrt{6} + 6 = \frac{p^2}{q^2} \Rightarrow 8 + 4\sqrt{3} = \frac{p^2}{q^2} \Rightarrow \sqrt{3} = \frac{p^2 - 8q^2}{4q^2}$$

$$p \in \mathbb{Z} \Rightarrow p \cdot p \in \mathbb{Z} \Rightarrow \frac{p^2 - 8q^2}{4q^2} \in \mathbb{Q} \Rightarrow \sqrt{3} \in \mathbb{Q} \Rightarrow \text{contradiction}$$

$$8q^2 \in \mathbb{Z}, 4q^2 \in \mathbb{Z}$$

b) Assume  $\sqrt{2} + \sqrt{3} \in \mathbb{Q} \Rightarrow \sqrt{2} + \sqrt{3} = \frac{p}{q}, p, q \in \mathbb{Z}$

$$\Rightarrow 2 + \sqrt{6} + 3 = \frac{p^2}{q^2}$$

$$\sqrt{6} = \frac{p^2 - 5q^2}{q^2} \in \mathbb{Q} \Rightarrow \text{false}$$

IS

a)  $x = p + \sqrt{q}, p, q \in \mathbb{Q} \Rightarrow x^m = a + b\sqrt{q}, a, b \in \mathbb{Q}$   
 $m \in \mathbb{N}$

$A = \text{set of } n \in \mathbb{N} \text{ s.t } x^n = a + b\sqrt{q}, a, b \in \mathbb{Q}$

$$x^1 = a + b\sqrt{q} \Rightarrow 1 \in A$$

assume  $k \in A \Rightarrow x^k = a + b\sqrt{q}, a, b \in \mathbb{Q}$

$$x^{k+1} = (p + \sqrt{q})(a + b\sqrt{q})$$

$$= pa + pb\sqrt{q} + a\sqrt{q} + bq$$

$$= \underbrace{pa + bq}_{\mathbb{Q}} + \sqrt{q} \underbrace{(pb + a)}_{\mathbb{Q}}$$

$$\Rightarrow k+1 \in A$$

$$\Rightarrow A = \mathbb{N}$$

b)  $(p - \sqrt{q})^m = a - b\sqrt{q}$

$$m=1 \quad p - \sqrt{q} = a - b\sqrt{q}$$

$$m=k \quad (p - \sqrt{q})^k = a - b\sqrt{q}$$

$$(p - \sqrt{q})^{k+1} = (p - \sqrt{q})(a - b\sqrt{q}) = pa - pb\sqrt{q} - a\sqrt{q} + bq$$

$$= \underbrace{(pa + bq)}_{\mathbb{Q}} - \sqrt{q} \underbrace{(pb + a)}_{\mathbb{Q}}$$

16

$$\text{a) } m, n \in \mathbb{N} \Rightarrow \frac{(m+2n)^2}{(m+n)^2} > 2$$

$$\frac{m^2}{n^2} < 2$$

Also

$$\frac{(m+2n)^2}{(m+n)^2} - 2 < 2 - \frac{m^2}{n^2}$$

$$\frac{(m+2n)^2}{(m+n)^2} > 2 \Leftrightarrow \frac{m^2 + 4nm + 4n^2}{m^2 + 2mn + n^2} > 2$$

$$\Rightarrow m^2 + 4\cancel{nm} + 4n^2 > 2m^2 + 4\cancel{nm} + 2n^2$$

$$m^2 < 2n^2 \Rightarrow \frac{m^2}{n^2} < 2$$

$$\frac{(m+2n)^2}{(m+n)^2} - 2 = \frac{m^2 + 4nm + 4n^2 - 2m^2 - 4mn - 2n^2}{(m+n)^2} = \frac{2n^2 - m^2}{(m+n)^2}$$

$$(m+n)^2 < n^2$$

$$\text{If } 2n^2 - m^2 > 0 \text{ then } \frac{2n^2 - m^2}{(m+n)^2} < \frac{2n^2 - m^2}{n^2} = 2 - \frac{m^2}{n^2}$$

$$\text{b) } \frac{(m+2n)^2}{(m+n)^2} < 2 \quad (\dots \text{ analogous to a}) \quad m^2 > 2n^2 \Rightarrow \frac{m^2}{n^2} > 2$$

For the second inequality:

$$\frac{(m+2n)^2}{(m+n)^2} - 2 = (\dots) \cdot \frac{2n^2 - m^2}{(m+n)^2}, \text{ but now we have } 2n^2 - m^2 < 0$$

So, if we decrease the denominator, the fraction gets more negative:

$$\frac{2n^2 - m^2}{(m+n)^2} > \frac{2n^2 - m^2}{n^2} = 2 - \frac{m^2}{n^2}$$

$$a) m, n \in \mathbb{N} \Rightarrow \frac{(m+2n)^2}{(m+n)^2} > 2$$

$$\frac{m^2}{n^2} < 2$$

Also

$$\frac{(m+2n)^2}{(m+n)^2} - 2 < 2 - \frac{m^2}{n^2}$$

### Interpretation

There is a rational number  $\frac{m^2}{n^2}$ , smaller than 2, and a rational number,  $\frac{(m+2n)^2}{(m+n)^2}$ , larger than 2, such that the second number is closer to 2 than the first, albeit from different sides of 2.

We can also say: there is a rational  $\frac{m}{n} < \sqrt{2}$ , which means there is another rational  $\frac{m+2n}{m+n} > \sqrt{2}$ ,

and the squares of these rational numbers have a relationship involving distance from the number 2.

$$b) m, n \in \mathbb{N} \Rightarrow \frac{(m+2n)^2}{(m+n)^2} < 2$$

$$\frac{m^2}{n^2} > 2$$

$$m^2 + 4mn + 4n^2 < 2m^2 + 4nm + 2n^2$$

$$2n^2 < m^2 \Rightarrow \frac{m^2}{n^2} > 2$$

Given rational  $\frac{m^2}{n^2} > 2$ , we can form another rational  $\frac{(m+2n)^2}{(m+n)^2} < 2$ . i.e., given  $m^2/n^2$  on one side of 2, we can form another rational on the other side of 2. Furthermore, these two rational numbers have the following relationship:

$$\frac{(m+2n)^2}{(m+n)^2} - 2 > 2 - \frac{m^2}{n^2}$$

Both sides are negative but the right side is more negative.  
Therefore  $m^2/n^2$  is more distant from 2 than  $(m+2n)^2/(m+n)^2$ .

$$c) \frac{m}{n} < \sqrt{2} \Rightarrow \exists \frac{m'}{n'} \text{ with } \frac{m}{n} < \frac{m'}{n'} < \sqrt{2}$$

strategy: Given  $\left(\frac{m}{n}\right)^2 < 2$ , we know there is  $\frac{(m+2n)^2}{(m+n)^2} > 2$ , with the latter closer to 2 than the former.

Given  $\frac{(m+2n)^2}{(m+n)^2} = \left(\frac{m+2n}{m+n}\right)^2 > 2$  we know there is  $\left(\frac{m_1+2n_1}{m_1+n_1}\right)^2 < 2$ , and that the latter

is closer to 2 than the former, which was closer to 2 than the original  $\left(\frac{m}{n}\right)^2$ .

$$\left(\frac{m}{n}\right)^2 < 2 \Rightarrow \frac{m}{n} < \sqrt{2}$$

$$\left(\frac{m_i}{n_i}\right)^2 = \frac{(m+2n)^2}{(m+n)^2} > 2 \Rightarrow \frac{m+2n}{m+n} > \sqrt{2}$$

$$\left(\frac{m_i}{n_i}\right)^2 > 2 \Rightarrow \frac{(m_i+2n_i)^2}{(m_i+n_i)^2} < 2$$
$$\frac{m_i+2n_i}{m_i+n_i} < \sqrt{2}$$

$$\frac{(m+2n)^2}{(m+n)^2} - 2 < 2 - \frac{m^2}{n^2} \Rightarrow 2 - \frac{(m+2n)^2}{(m+n)^2} > \frac{m^2}{n^2} - 2$$

$$\frac{(m_i+2n_i)^2}{(m_i+n_i)^2} - 2 > 2 - \frac{m_i^2}{n_i^2}$$

$$\frac{(3m+4n)^2}{(2m+3n)^2} - 2 > 2 - \frac{(m+2n)^2}{(m+n)^2} > \frac{m^2}{n^2} - 2$$

$$2 - \frac{(m_i+2n_i)^2}{(m_i+n_i)^2} < 2 - \frac{m^2}{n^2}$$

$$\frac{(m_i+2n_i)^2}{(m_i+n_i)^2} > \frac{m^2}{n^2}$$

$p \in \mathbb{N}$  is prime if we cannot write  $p = ab$ ,  $a, b \in \mathbb{N}$  unless there are  $p$  and 1.

for convenience, 1 is not prime.

We want to prove with induction that we can always apply the following procedure to factor a non-prime numbers into a product of prime numbers:

Take non-prime  $n > 1 \Rightarrow n = a \cdot b$   $a, b \in \mathbb{N}$

if  $a$  or  $b$  not prime they can be written as  $c \cdot d$   $c, d \in \mathbb{N}$

continue with this and at some point all factors are prime.

a) Let  $A = \text{set of } n \in \mathbb{N} - \{1\} \text{ that can be written as product of prime factors.}$

$2$  is prime  $\Rightarrow$  single factor prime  $\Rightarrow 2 \in A$

assume  $2, 3, \dots, k \in A$

if  $k+1$  is prime  $\Rightarrow$  single factor prime  $\Rightarrow k+1 \in A$

if  $k+1$  not prime  $\Rightarrow k+1 = ab$ ,  $a, b < k+1 \Rightarrow a, b \in A \Rightarrow k+1 \in A$

b) For  $n \in \mathbb{N}$ , prove  $\sqrt{n} \in \mathbb{Q}'$  unless  $n = m^2$ ,  $m \in \mathbb{N}$

Assume  $\sqrt{n} \in \mathbb{Q} \Rightarrow \sqrt{n} = \frac{a}{b}$ ,  $a, b \in \mathbb{N}$  since  $n \in \mathbb{N} \Rightarrow \sqrt{n} \geq 0$

$$a = b\sqrt{n}$$

$a^2 = b^2 \cdot n$ , and we know that the factorization into primes is unique to each number.

Therefore, the same factors appear on both sides. There are two of each factor in  $a^2$ , also in  $b^2$ . Therefore the same applies to  $n \Rightarrow n$  is a square.

c)  $\sqrt[n]{n} \in \mathbb{Q}'$  unless  $n = m^n$

Assume  $\sqrt[n]{n} = \frac{a}{b} \Rightarrow a = \sqrt[n]{n} \cdot b \Rightarrow a^n = nb^n$

# factors is the same on both sides.

Never  $\Rightarrow$  if a factor in  $a^n$  and  $b^n$ . whatever factor in  $n$  must also then appear  $n$  times  
 $\Rightarrow n = m^n$

d) Assume there are only  $n$  prime numbers  $p_1, \dots, p_n$ .

$p_1 \cdot p_2 \cdots p_n$  is not prime; since  $p_1$  is 2,  $p_1 \cdots p_n$  is even, so divis. by 2.

$p_1 \cdots p_n + 1$  is not divis. by any primes smaller than it. Therefore either it is prime, or there is one or more primes between  $p_n$  and  $p_1 \cdots p_n$  that factorize  $p_1 \cdots p_n + 1$ .

Either way there must be more than  $p_1, \dots, p_n$  as primes.

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$$\text{a)} \quad x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0 \Rightarrow x \in \mathbb{Q}' \text{ unless } x \in \mathbb{Z}$$

$a_{n-1}, \dots, a_0 \in \mathbb{Z}$

Assume  $x \in \mathbb{Q} \Rightarrow x = p/q, p, q \in \mathbb{Z}$

$$\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \dots + a_0 = 0$$

$$p^n + a_{n-1}p^{n-1}q + a_{n-2}p^{n-2}q^2 + \dots + a_0q^n = 0$$

For  $n=1$  and  $a_0 = -\sqrt{h}, h \in \mathbb{N}$

$$p - \sqrt{h} \cdot q = 0 \Rightarrow \frac{p}{q} = \sqrt{h}$$

Note the assumptions here:  $p, q \in \mathbb{Z} \Rightarrow p/q \in \mathbb{Q} \Rightarrow \sqrt{h} \in \mathbb{Q}$

according to 57b, we know that  $\sqrt{h}$  is actually necessarily either a natural number or an irrational number.

For the general problem,

$$p^n = -q(a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q + \dots + a_0q^{n-1})$$

$\Rightarrow p^n$  is divisible by  $q$

Assume  $q \neq \pm 1 \Rightarrow p/q \notin \mathbb{Z}$  and  $q$  is product of prime factors

$\Rightarrow p^n$  divisible by the prime factors of  $q$

$\Rightarrow p$  divisible by prime factors of  $q$

$\Rightarrow$  we can simplify  $p/q$  by dividing num. and den. by  $q$ ; thus  $p/q$  is an integer, contradicting our assumption

\* my solution posted on SE.

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b) Prove  $\sqrt{6} - \sqrt{2} - \sqrt{3} \in Q'$ 

$$x = \sqrt{6} - (\sqrt{2} + \sqrt{3})$$

$$x^2 = 6 + (\sqrt{2} + \sqrt{3})^2 - 2\sqrt{6}(\sqrt{2} + \sqrt{3})$$

$$= 6 + 2 + 3 + 2\sqrt{6} - 2\sqrt{6}(\sqrt{2} + \sqrt{3})$$

$$= 11 + 2\sqrt{6}(1 - (\sqrt{2} + \sqrt{3}))$$

$$(x^2 - 11)^2 = 24(1 - (\sqrt{2} + \sqrt{3}))^2 - 24(1 + (\sqrt{2} + \sqrt{3})^2 - 2(\sqrt{2} + \sqrt{3}))$$

$$= 24(1 + 2 + 3 + 2\sqrt{6} - 2\sqrt{2} - 2\sqrt{3})$$

$$= 24(6 + 2(\sqrt{6} - \sqrt{2} - \sqrt{3}))$$

$$= 24(6 + 2x)$$

$$(x^2 - 11)^2 - 48x - 144 = 0$$

From part a we know that  $x \in \mathbb{Z}$  or  $x \in Q'$ .

Proof that  $0 < x < 1 \Rightarrow x \in Q'$

$$0 > \sqrt{6} - \sqrt{2} - \sqrt{3} > -1$$

$$\begin{array}{ll} \sqrt{6} < \sqrt{2} + \sqrt{3} & \sqrt{6} + 1 > \sqrt{2} + \sqrt{3} \\ 6 < 2 + 3 + 2\sqrt{6} & 6 + 1 + 2\sqrt{6} > 2 + 3 + 2\sqrt{6} \\ 1 < 2\sqrt{6} & 7 > 5 \end{array}$$

$$\Rightarrow \sqrt{6} - \sqrt{2} - \sqrt{3} \in ]-1, 0[ \Rightarrow x \notin \mathbb{Z} \Rightarrow x \in Q'$$

\* Proofs thus far

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i^3 = \left( \sum_{i=1}^n i \right)^2$$

$$\sum_{i=1}^n (2i-1) = n^2$$

$$\sum_{i=1}^n (2i-1)^2 = \frac{n(2n+1)(2n-1)}{3}$$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \Rightarrow \text{Pascal's Triangle}$$

$$\binom{n}{k} \text{ always } \in \mathbb{N}$$

Binomial Thm:  $(a+b)^n = \sum_{i=0}^n \binom{n}{i} a^{n-i} b^i$

$$\sum_{i=0}^n \binom{n}{i} = 2^n$$

$$\sum_{i=0}^n (-1)^i \binom{n}{i} = 0$$

$$\sum_{i=0}^n \binom{n}{i} = \sum_{i=0}^n \binom{n}{n-i} = 2^{n-1}$$

$$\sum_{i=0}^k \binom{n}{i} \binom{m}{k-i} = \binom{n+m}{k}$$

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$$

$$\sum_{i=0}^n r^i = \frac{1-r^{n+1}}{1-r} \quad r \neq 1$$

$$\sum_{k=1}^n k^p \text{ can always be written } \frac{n^{p+1}}{p+1} + An^p + Bn^{p-1} + Cn^{p-2} + \dots$$

every  $n \in \mathbb{N}$  is either even or odd

even:  $\{2k; k \in \mathbb{Z}\}$   
odd:  $\{2k+1; k \in \mathbb{Z}\}$

19 Prove  $h > -1 \Rightarrow (1+h)^n \geq 1+nh \quad \forall n$

Binomial Thm:  $(1+h)^n = \sum_{i=0}^n \binom{n}{i} h^i = 1 + \sum_{i=1}^n \binom{n}{i} h^i$

Induction

$n=1 \Rightarrow (1+h)^1 = 1+h$

assume  $n=k \Rightarrow (1+h)^k \geq 1+kh$

multiply each side by  $(1+h)$ :  $(1+h)(1+h)^k \geq (1+kh)(1+h)$  if  $1+h > 0$

$n=k+1 \Rightarrow (1+h)^{k+1} = (1+h)(1+h)^k$   
 $\geq (1+h)(1+kh) = 1+h+kh+kh^2 = 1+h(k+1)+kh^2$   
 $kh^2 \geq 0$  so  $(1+h)^{k+1} \geq 1+h(k+1)+kh^2 \geq 1+h(k+1)$

Note that this holds for  $h > -1$

If  $h=0$  then  $(1+h)^n = 1$  and  $1+nh = 1$ . The result holds trivially,  $\square$ .

$h < -1 \Rightarrow (1+h) < 0$

$(1+h)^1 = 1+h < 0$

$(1+h)^2 > 0$   
 $1+2h < 0 \Rightarrow (1+h)^2 > (1+2h)$

$(1+h)^3 < 0$      $(1+h)^3 = 1+3h+3h^2+h^3$   
 $1+3h < 0$                  $- \quad - \quad +$   
                               $- \quad + \quad +$   
 $3h^2+h^3 = h^2(3+h) \quad \underline{-3 \quad 0}$   
                               $+ \quad - \quad +$

$\Rightarrow (1+h)^3 = 1+3h+3h^2+h^3 \geq 1+3h$  if  $h \leq -3$  or  $h \geq 0$

What if we assume  $(1+h)^n \geq 1+nh$ ,  $h < -1$ ?

$(1+h)^{k+1} = (1+h)(1+h)^k$   
 $(1+h)^k (1+h) \stackrel{\text{switch sign}}{\leq} (1+kh)(1+h) = 1+h(k+1)+kh^2$

or don't obtain the relationship we're looking for.

## 20. Fibonacci seq.

$$a_1 = 1$$

$$a_2 = 1$$

$$a_n = a_{n-1} + a_{n-2} \quad n \geq 3$$

$$\text{Prove } a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Induction: A set of  $n$  fraction  $a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$  is  $n^{\text{th}}$  fibonacci number

$$n=1 \Rightarrow \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1 = a_1$$

$$n=2 \Rightarrow \frac{\left(\frac{1+2\sqrt{5}}{2}\right) - \left(\frac{1-2\sqrt{5}}{2}\right)}{4\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1 = a_2$$

$$\text{Assume } a_k = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}} \text{ for } k < n$$

$$\text{we want to show that } a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

we know that  $a_n = a_{n-1} + a_{n-2}$ , and by assumption

$$a_{n-1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}}, \quad a_{n-2} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}}{\sqrt{5}}$$

$$a = \frac{1+\sqrt{5}}{2} \quad b = \frac{1-\sqrt{5}}{2}$$

$$a_{n-1} + a_{n-2} = \frac{a^{n-1} - b^{n-1} + a^{n-2} - b^{n-2}}{\sqrt{5}} = \frac{a^{n-2}(1+a) - b^{n-2}(1+b)}{\sqrt{5}}$$

$$1+a = \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1}{2} + \frac{\sqrt{5}}{2} + \frac{5}{4} = \frac{3}{2} + \frac{\sqrt{5}}{2} = a^2$$

$$1+b = \frac{3-\sqrt{5}}{2} = \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1}{2} - \frac{\sqrt{5}}{2} + \frac{5}{4} = \frac{3}{2} - \frac{\sqrt{5}}{2} = b^2$$

$$\Rightarrow a_{n-1} + a_{n-2} = \frac{a^{n-2}a^2 - b^{n-2}b^2}{\sqrt{5}} = \frac{a^n - b^n}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Recap: we know the fib. seq. we asserted that there is a formula for the  $n^{\text{th}}$  element. we proved using induction as follows:

- verify formula valid for  $n=1, n=2$
- assume formula valid for  $k < \text{some } n$
- obtain formula for  $a_n$  based on formula assumed valid.
- since we can obtain the formula for  $a_n$  for any  $n$ , all elements of the seq. have the formula.

$$21. \text{ Schwarz Inequality} \quad \sum x_i y_i \leq \sqrt{\sum x_i^2} \sqrt{\sum y_i^2}$$

### Proof 1

$\vec{x} = \langle x_1, \dots, x_n \rangle, \vec{y} = \langle y_1, \dots, y_n \rangle$  l.i.  $\Rightarrow c_1 \vec{x} + c_2 \vec{y} = \vec{0} \Leftrightarrow c_1 = c_2 = 0$  Definition of linear indep.

$\Rightarrow \lambda \vec{x} - \vec{y} + \vec{0}$  - specific linear combination of  $c_1, c_2 \neq 0$

$\Rightarrow |\langle \lambda x_1 - y_1, \dots, \lambda x_n - y_n \rangle|^2 \geq 0$  resulting vector has length  $\geq 0$

$\Rightarrow (\lambda x_1 - y_1)^2 + \dots + (\lambda x_n - y_n)^2 \geq 0$

$$= \lambda^2 \sum_{i=1}^n x_i^2 + \sum_{i=1}^n y_i^2 - 2\lambda \sum_{i=1}^n x_i y_i \geq 0 \quad \text{What sorts of vectors does this?}$$

$$\Delta = 4 \left( \sum_{i=1}^n x_i y_i \right)^2 - 4 \sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2 \leq 0 \Rightarrow \sum x_i y_i \leq \sqrt{\sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2}$$

### Proof 2

Let  $\vec{x}$  and  $\vec{y}$  be unit vectors in  $\mathbb{R}^n$ .

$$\vec{x} = \frac{\langle x_1, \dots, x_n \rangle}{\sqrt{\sum x_i^2}} \quad \vec{y} = \frac{\langle y_1, \dots, y_n \rangle}{\sqrt{\sum y_i^2}}$$

$$(\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y}) = \|\vec{x} - \vec{y}\|^2 \geq 0$$

$$\Rightarrow \vec{x} \cdot \vec{x} - 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} \geq 0 \Rightarrow 2\vec{x} \cdot \vec{y} \leq \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y}$$

$$\frac{2 \sum x_i y_i}{\sqrt{\sum x_i^2} \sqrt{\sum y_i^2}} \leq \frac{\sum x_i^2}{\sum x_i^2} + \frac{\sum y_i^2}{\sum y_i^2} = 2$$

$$\Rightarrow \sum x_i y_i \leq \sqrt{\sum x_i^2} \sqrt{\sum y_i^2}$$

$$\begin{aligned} & (x_1^2 + x_2^2 + \dots + x_n^2)(y_1^2 + y_2^2 + \dots + y_n^2) \\ &= x_1^2 y_1^2 + x_2^2 y_2^2 + \dots + x_n^2 y_n^2 + 2x_1 x_2 y_1 y_2 + 2x_1 x_3 y_1 y_3 + 2x_1 x_4 y_1 y_4 \\ &\quad + x_2 x_3 y_2 y_3 + x_2 x_4 y_2 y_4 - 2x_1 x_2 y_1 y_2 - 2x_1 x_3 y_1 y_3 - 2x_1 x_4 y_1 y_4 \\ &\quad + x_2 x_3 y_2 y_3 + x_2 x_4 y_2 y_4 \\ &\quad + x_3 x_4 y_3 y_4 + x_3 x_5 y_3 y_5 \\ &\quad + \dots \end{aligned}$$

### Proof 3

$$\sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2 = \sum_{i=1}^n x_i^2 y_i^2 + \sum_{i=1}^n \sum_{j \neq i} x_i^2 y_j^2$$

Add and subtract  $2 \sum_{i=1}^n \sum_{j \neq i} x_i y_i x_j y_j$

$$\left[ \sum_{i=1}^n x_i^2 y_i^2 + 2 \sum_{i=1}^n \sum_{j \neq i} x_i y_i x_j y_j \right] + \sum_{i=1}^n \sum_{j \neq i} x_i^2 y_j^2 - 2 \sum_{i=1}^n \sum_{j \neq i} x_i y_i x_j y_j$$

There are  $C_{n,2}$  combinations  $(i,j)$ ,  $j > i$

$$C_{n,2} = \binom{n}{2} = \frac{n!}{(n-2)!2!}$$

Take one such comb.  $(i,j)$ . We have

$$x_i^2 y_i^2 + x_j^2 y_j^2 - 2x_i y_i x_j y_j = (x_i y_j - x_j y_i)^2 \geq 0$$

$$= \left( \sum_{i=1}^n x_i y_i \right)^2 + \text{positive terms} \geq \left( \sum_{i=1}^n x_i y_i \right)^2$$

$$\Rightarrow \sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2 \geq \left( \sum_{i=1}^n x_i y_i \right)^2$$

$$\Rightarrow \sqrt{\sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2} \geq \sum_{i=1}^n x_i y_i$$

22.  $a_1, \dots, a_n \geq 0$

arithmetic mean  $A_n = \frac{a_1 + \dots + a_n}{n}$

geometric mean  $G_n = \sqrt{a_1 \dots a_n}$

satisfy  $G_n \leq A_n$

a) We have a set of  $n$  elements  $S_0 = \{a_1, \dots, a_n\}$

I)  $a_i = A_n$  for every  $i$ , then  $G_n = \sqrt{(A_n)^n} = A_n$ .

If any element is  $< A_n$  then there must be at least one other element  $> A_n$ . Assume  $a_1 < A_n$  and  $a_2 > A_n$ .

Form a new set  $S_1$  by replacing  $a_1$  w/  $\bar{a}_1 = A_n$  and  $a_2$  w/  $\bar{a}_2 = a_1 + a_2 - \bar{a}_1$ .

$$S_1 = \{\bar{a}_1, \bar{a}_2, a_3, \dots, a_n\}$$

$$\bar{A}_n = \frac{\cancel{A_n} + a_1 + a_2 - \cancel{A_n} + a_3 + \dots + a_n}{n} = A_n$$

$$G'_n = \sqrt{\bar{a}_1 \cdot \bar{a}_2 \cdot a_3 \dots a_n}$$

we can show  $\bar{a}_1 \cdot \bar{a}_2 \geq a_1 a_2$ . Therefore,  $G'_n \geq G_n$ .

The current set  $S_1$  either has all elem. equal to  $A_n$  or at least one  $< A_n$  and one  $> A_n$ . Assume  $a_3 < A_n$  and  $a_4 > A_n$ .

we apply the same steps to  $\bar{a}_3$

$$S_2 = \{\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, a_5, \dots, a_n\}$$

$$\bar{a}_3 = A_n$$

$$\bar{a}_4 = a_3 + a_4 - \bar{a}_3$$

$$\bar{A}'_n = \frac{\cancel{A_n} + a_1 + a_2 - \cancel{A_n} + \cancel{A_n} + a_3 + a_4 - \cancel{A_n} + \dots + a_n}{n} = A_n$$

$$G''_n = \sqrt{\bar{a}_1 \cdot \bar{a}_2 \cdot \bar{a}_3 \cdot \bar{a}_4 \cdot a_5 \dots a_n} \geq \sqrt{\bar{a}_1 \cdot \bar{a}_2 \cdot a_3 \dots a_n} = G'_n \geq G_n$$

\* Assume  $a_1 < A_n, a_2, \dots, a_{n-1} \leq A_n$

$$A_n = \frac{a_1 + \sum_{i=1}^{n-1} a_i}{n} \Rightarrow nA_n = \sum_{i=1}^{n-1} a_i + a_n$$

$$\text{But } \sum_{i=1}^{n-1} a_i < (n-1)A_n$$

$$\Rightarrow a_n = nA_n - \sum_{i=1}^{n-1} a_i > nA_n - (n-1)A_n = A_n \\ \Rightarrow a_n > A_n$$

\*  $\frac{\bar{a}_1 + \bar{a}_2}{2} = \frac{a_1 + a_2}{2} \Rightarrow \bar{a} = a_1 + a_2 - \bar{a}_1$

assume  $a_1 < A_n$ . Then  $a_1 > A_n$  for some  $i$ , say  $a_1 > A_n$

show  $\bar{a}_1 \cdot \bar{a}_2 \geq a_1 a_2$

sub in expr. for  $\bar{a}_1$  and  $\bar{a}_2$  into  $\bar{a}_1 \cdot \bar{a}_2 \geq a_1 a_2$

$$\bar{a}_1 \cdot \bar{a}_2 \geq a_1 a_2 \Rightarrow \bar{A}_n^2 - A_n(a_1 + a_2) + a_1 a_2 \leq 0$$

the expr. can be factored

$$\Rightarrow a_1(a_2 - A_n) + A_n(A_n - a_2) \leq 0$$

$$(A_n - a_2)(A_n + \frac{a_1(a_2 - A_n)}{A_n - a_2}) - \frac{(A_n - a_2)(A_n - a_1)}{A_n - a_2} \geq 0$$

$$a_1 = A_n \text{ or } a_2 = A_n \Rightarrow (A_n - a_2)(A_n - a_1) = 0$$

The current set  $S_1$  either has all elem. equal to  $A_n$  or at least one  $< A_n$  and one  $> A_n$ . Assume  $a_3 < A_n$  and  $a_4 > A_n$ .

we apply the same steps to  $\bar{a}_3$

$$S_2 = \{\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, a_5, \dots, a_n\}$$

$$\bar{a}_3 = A_n$$

$$\bar{a}_4 = a_3 + a_4 - \bar{a}_3$$

$$\bar{A}'_n = \frac{\cancel{A_n} + a_1 + a_2 - \cancel{A_n} + \cancel{A_n} + a_3 + a_4 - \cancel{A_n} + \dots + a_n}{n} = A_n$$

$$G''_n = \sqrt{\bar{a}_1 \cdot \bar{a}_2 \cdot \bar{a}_3 \cdot \bar{a}_4 \cdot a_5 \dots a_n} \geq \sqrt{\bar{a}_1 \cdot \bar{a}_2 \cdot a_3 \dots a_n} = G'_n \geq G_n$$

Note that at this point at least two elements,  $\bar{a}_1$  and  $\bar{a}_3$  do equal to  $A_n$ .

Each iteration replaces an elem. w/  $A_n$ , the arithm. mean stays the same, and the geom. mean increases. However, when all elem. equal  $A_n$ , then the geom. mean equals  $A_n$ .

We end up with

$$G_n < G'_n < G''_n < G'''_n < \dots < G^{(n)}_n = A_n \Rightarrow G_n < A_n$$

$$S_0 = \{a_1, \dots, a_n\}$$

$$a_1, a_2 < A_n$$

$$A_n = \frac{a_1 + a_2 + \sum_{i=3}^n a_i}{n}$$

$$nA_n = \sum_{i=1}^n a_i = a_1 + a_2$$

$$\text{i) assume } a_i = A_n \quad i=3, \dots, n$$

$$\Rightarrow a_1 + a_2 = nA_n - (n-2)A_n = 2A_n$$

which can't be true if  $a_1, a_2 < A_n$ .

$$\Rightarrow a_1 = 2A_n - a_2$$

$$\text{iii) assume } a_i = A_n \quad i=4, \dots, n$$

$$A_n = \frac{a_1 + a_2 + a_3 + \sum_{i=4}^n a_i}{n}$$

$$\Rightarrow a_1 + a_2 + a_3 = nA_n - \sum_{i=4}^n a_i = nA_n - (n-3)A_n = 3A_n$$

$$a_1 = 3A_n - a_2 - a_3$$

$a_1$	$a_2$	$a_3$
$< A_n$	$< A_n$	$> A_n$
$< A_n$	$> A_n$	depends
$> A_n$	$< A_n$	depends
$> A_n$	$> A_n$	$< A_n$

$$\text{b) Prove } G_n \leq A_n, \quad n = 2^k$$

induction on  $k$

$$k=1 \Rightarrow n=2, S_1 = \{a_1, a_2\}$$

$$\text{if } a_1 = a_2 = \frac{a_1 + a_2}{2} = A_2 \text{ then } G_2 = \sqrt{A_n^2} = A_n$$

$$\text{if } a_1 \neq A_n \text{ then } a_1 = 2A_2 - a_2$$

$$\text{Form new set } S_2 = \{\bar{a}_1, \bar{a}_2\}, \bar{a}_1 = A_2, \bar{a}_2$$

$$= a_1 + a_2 - \bar{a}_1 = \frac{a_1 + a_2}{2} = A_2$$

$$G'_2 = \sqrt{\bar{a}_1 \bar{a}_2} > \sqrt{a_1 a_2} = G_2$$

$$A'_2 = A_2$$

$$\text{since } S_2 = \{A_2, A_2\}, G'_2 = A_2 \Rightarrow G_2 < G'_2 = A_2$$

$$\text{assume } G_n \leq A_n, \quad n = 2^k$$

$$S_0 = \{a_1, \dots, a_{n-2^{k+1}}\} = \{a_1, \dots, a_{2^k}, \dots, a_{2^{k+1}}\}$$

$$G_{2^{k+1}} = \sqrt{a_1 \dots a_{2^k} \cdot a_{2^{k+1}} \dots a_{2^{k+1}}}$$

$$= [(a_1 \dots a_{2^k})(a_{2^{k+1}} \dots a_{2^{k+1}})]^{\frac{1}{2^{k+2}}}$$

$$= \sqrt[2^k]{a_1 \dots a_{2^k}} \sqrt[2^k]{a_{2^{k+1}} \dots a_{2^{k+1}}} \quad \text{This is the geometric mean of two elements}$$

$$\leq \frac{\sqrt[2^k]{a_1 \dots a_{2^k}} + \sqrt[2^k]{a_{2^{k+1}} \dots a_{2^{k+1}}}}{2} \quad (G_2 \leq A_2)$$

$$\leq \frac{\frac{\sum_{i=1}^{2^k} a_i}{2^k} + \frac{\sum_{i=1}^{2^k} a_{2^{k+1}}}{2^k}}{2} \quad (\text{using our induction assumption})$$

$$= \frac{\sum_{i=1}^{2^{k+1}} a_i}{2^{k+1}} = A_{2^{k+1}}$$

$$\Rightarrow G_{2^{k+1}} \leq A_{2^{k+1}}$$

$$\text{c) general } n, \text{ let } 2^m > n. \text{ Apply b) to } a_1, \dots, a_n, \underbrace{A_n, \dots, A_n}_{2^{m-n} \text{ times}}$$

$$G_{2^m} \leq A_{2^m} \Rightarrow [a_1 \dots a_n \cdot \underbrace{A_n}_{2^{m-n}}]^{1/2^m} \leq \frac{a_1 + \dots + a_n + (2^{m-n})A_n}{2^m}$$

$$\Rightarrow a_1 \dots a_n \cdot A_n \leq \left[ \frac{a_1 + \dots + a_n + (2^{m-n})A_n}{2^m} \right]^{2^m}$$

$$\frac{a_1 + \dots + a_n}{n} = A_n \Rightarrow a_1 + \dots + a_n = nA_n$$

$$\Rightarrow a_1 \dots a_n \cdot \underbrace{A_n}_{2^{m-n}} \leq \left[ \frac{nA_n + (2^{m-n})A_n}{2^m} \right]^{2^m} = A_n^{2^m}$$

$$\Rightarrow a_1 \dots a_n \leq A_n^n \Rightarrow (a_1 \dots a_n)^{1/n} \leq A_n$$

### 23. recursive def. of $a^n$

$$a^1 = a$$

$$a^{n+1} = a^n \cdot a$$

prove

$$a^{n+m} = a^n \cdot a^m$$

$$(a^n)^m = a^{nm}$$

$$\text{i) } A = \{m \in \mathbb{N} \mid a^{n+m} = a^n \cdot a^m\}$$

$$m=1 \Rightarrow a^{n+1} = a^n \cdot a \text{ by definition. } \Rightarrow 1 \in A$$

$$\text{assume } k \in A \Rightarrow a^{nk} = a^n \cdot a^k$$

$$\text{by def. } a^{n+k+1} = a^{nk} \cdot a = a^n \cdot a^k \cdot a = a^n a^{k+1}$$

$$\Rightarrow k+1 \in A$$

$$\text{ii) } A = \{m \in \mathbb{N} \mid (a^n)^m = a^{nm}\}$$

$$m=1 \Rightarrow (a^n)^1 = a^n = a^{n \cdot 1} \Rightarrow 1 \in A$$

$$\text{assume } k \in A \Rightarrow (a^n)^k = a^{nk}$$

multiply both sides by  $a^n$

$$(a^n)^k \cdot a^n = a^{nk} \cdot a^n$$

by definition on each side

$$(a^n)^{k+1} = a^{nk+n} = a^{n(k+1)} \Rightarrow k+1 \in A$$

### 24.

P1 a,b,c any numbers, then  $a + (b+c) = (a+b)+c$  (associative law for addition)

P2 a,b any numbers, then  $a+b = b+a$  (commutative law for addition)

recursive def. of multiplication

$$\text{A } 1 \cdot b = b$$

$$\text{B } (a+1) \cdot b = a \cdot b + b$$

$$\text{i) Prove } a(b+c) = a \cdot b + a \cdot c$$

$$\text{let } A = \{a \in \mathbb{N} \mid a(b+c) = ab+ac\}$$

$$a \cdot 1 \stackrel{\text{A}}{=} 1 \cdot (b+c) \stackrel{\text{A}}{=} b+c = 1 \cdot b + 1 \cdot c \Rightarrow 1 \in A$$

$$\text{assume } k \in A \Rightarrow k(b+c) = kb+kc$$

$(k+1)(b+c)$  by definition of multip. is

B induction assumption

$$k(b+c) + (b+c) = kb+kc + b+c$$

$$= (kb+b) + (kc+c) \quad \text{P1}$$

$$= (k+1)b + (k+1)c \quad \text{B}$$

$$\text{ii) Prove } a \cdot 1 = a$$

$$\text{let } A = \{a \in \mathbb{N} \mid a \cdot 1 = a\}$$

$$a \cdot 1 \stackrel{\text{def.}}{=} a \cdot 1 + 1 \cdot 1 - 1 \quad A \Rightarrow 1 \in A$$

$$\text{assume } k \in A \Rightarrow k \cdot 1 = k$$

$$\begin{aligned} (k+1) \cdot 1 &= k \cdot 1 + 1 \cdot 1 \\ &= k+1 \quad \text{Ind-ass.} \end{aligned}$$

$$\text{iii) } a \cdot b = b \cdot a$$

$$A = \{a \in \mathbb{N} \mid a \cdot b = b \cdot a\}$$

$$a \cdot 1 \stackrel{\text{def.}}{=} 1 \cdot a = a \cdot 1, \text{ proved in ii) } \Rightarrow 1 \in A$$

$$\text{assume } k \in A \Rightarrow kb = bk$$

$$\begin{aligned} (k+1)b &\stackrel{\text{B, Ind-ass.}}{=} kb + b = bk + b = bk + b \cdot 1 = b(k+1) \\ &\Rightarrow k+1 \in A \end{aligned}$$

25.

a) set A of real numbers is inductive if

- (1)  $1 \in A$
- (2)  $k \in A \Rightarrow k+1 \in A$

i) Prove  $\mathbb{N}$  is inductive

$1 \in \mathbb{N}$

assume  $k \in \mathbb{N}$ . Then  $k+1 \in \mathbb{N}$

ii) Prove  $A = \{k \in \mathbb{N} \mid k > 0\}$  inductive

$1 \in A$

$k \in A \Rightarrow k > 0 \Rightarrow k+1 > 0 \Rightarrow k+1 \in A$

$\Rightarrow A$  inductive

iii) Prove  $A = \{k \in \mathbb{N} \mid k > 0, k + \frac{1}{2}\}$  inductive

$1 > 0, 1 + \frac{1}{2} = 1 \in A$

$k \in A \Rightarrow k > 0, k + \frac{1}{2} \Rightarrow k+1 > 1 \Rightarrow k+1 \in A$

$\Rightarrow A$  inductive

iv) Prove  $A = \{k \in \mathbb{N} \mid k > 0, k + 5\}$  inductive

$4 \in A \Rightarrow 4+1=5 \notin A \Rightarrow A$  not inductive

v) Prove  $A, B$  inductive  $\Rightarrow C = \{k \in \mathbb{N} \mid k \in A, k \in B\}$  inductive

$1 \in A, 1 \in B \Rightarrow 1 \in C$

$k \in A, k \in B \Rightarrow k+1 \in A, k+1 \in B \Rightarrow k+1 \in C$

$\Rightarrow C$  inductive

b) 1 is in every inductive set. Because  $k \in A \Rightarrow k+1 \in A, 2, 3, 4, \dots$

are all in every induct. set. So  $\mathbb{N}$  is in every inductive set.

vi) Prove  $1 \in \mathbb{N}$

By def. of inductive set, 1 is in every induct. set.  $\Rightarrow 1 \in \mathbb{N}$

vii)  $k \in \mathbb{N} \Rightarrow k \in$  every inductive set

$\Rightarrow k+1 \in$  every inductive set, by def. of induct. set.

## 26. 3 spindles

$n$  concentric rings of decreasing diameter

$$\text{ops.2 } S_1 \rightarrow A$$

$$S_2 \rightarrow B$$

$$S_1 \rightarrow B \Rightarrow S_2, \text{cn } B$$

in three moves you change location of a two-item stack.  $= 2^2 - 1$

$$S_3 \rightarrow A$$

$$\text{ops.2}(S_1, S_2) \Rightarrow S_3, S_1, \text{cn } A$$

one move to lay next layer, 3 moves to change loc of 3-item stack.

$$S_4 \rightarrow B$$

$$\text{ops.2}(S_1, S_2) \Rightarrow S_4, S_1, \text{cn } C$$

$$\text{ops.3} \left\{ \begin{array}{l} S_3 \rightarrow B \\ \text{ops.2}(S_1, S_2) \Rightarrow S_4, S_1, \text{cn } B \end{array} \right.$$

ops.3 moves  $\approx$  3-item stack, takes 7 moves.  $= 2^3 - 1$

ops.4 moves 4-item stack.

$\Rightarrow$  ops.3 to move 3-item stack

$\Rightarrow$  move largest piece

$\Rightarrow$  ops.3 to place 3-item stack back on top

15 moves  $= 2^4 - 1$

**Proposition:** a stack of  $n$  rings  $r_1, r_2, \dots, r_n$  with  $\text{size}(r_i) < \text{size}(r_j) \Leftrightarrow i < j$ , cn  $\Rightarrow$  spindle, can be moved to another spindle in  $2^n - 1$  moves.

$$A = \{n \in \mathbb{N} \mid \text{proposition is true}\}$$

$n=1 \Rightarrow$  move the sole ring  $r_1$  to the other spindle.

$\Rightarrow 1$  move necessary  $= 2^1 - 1$

$\Rightarrow 1 \in A$

assume  $k \in A \Rightarrow 2^{k-1}$  moves to move  $\approx k$ -ring stack

consider a  $(k+1)$  ring stack.

It takes  $2^k - 1$  moves to move the topmost  $k$  rings.

An additional move to move the bottom ring.

Again  $2^k - 1$  to move the  $k$  rings onto the bottom ring.

Total moves:  $2^k - 1 + 1 + 2^k - 1 = 2^{k+1} - 1$

$\Rightarrow k+1 \in A$

**Proposition:**  $2^n - 1$  is the minimum number of moves to move  $n$  rings stack to another spindle.

$$A = \{n \in \mathbb{N} \mid \text{proposition is true}\}$$

$n=1 \Rightarrow 2^1 - 1 = 1$  min moves to move one ring.

$\Rightarrow 1 \in A$

assume  $n-k \in A \Rightarrow 2^{n-k} - 1$  min moves to move  $k$ -ring stack.

if we have a  $(k+1)$ -ring stack, it takes  $2^{k+1} - 1$  to move the top  $k$ , by assumption the min number of moves. Now there is a one ring stack. Min moves to move it is 1. Finally, min to move  $k$  rings is another  $2^{k+1} - 1$ . Total  $2^{k+1} - 1$ .

27. Proposition: For  $n$  professors addressed by prof X at meeting 0,  
all  $n$  professors resign at the  $n^{\text{th}}$  meeting.

$$A = \{n \in \mathbb{N} \mid \text{prop. true for } n\}$$

Denote meeting  $X$  by  $m_X$ .

$n=2$

at  $m_1$ , each professor thinks

- i) the other professor did not find a mistake in my code
- ii) therefore the other prof has concluded that I found a mistake in his code
- iii) therefore the other prof. will resign

no one resigns at  $m_1$ .

at  $m_2$ , each prof. thinks

- i) if the other prof. did not resign, then he must have thought I would resign because he found a mistake in my code.
- ii) Because a mistake in my code was found, I will resign.

both resign at  $m_2$ .

$n=3$

$m_1$ : each prof. thinks

- i) the other two prof. represent the  $n=2$  case and will resign at  $m_2$ .

no one resigns at  $m_1$  or  $m_2$ .

$m_2$ : each prof. thinks

- i) the other two prof.