

Ch. 13 Appendix - Riemann Sums

I. f, g cont. on [a, b]

P = $\{t_0, \dots, t_n\}$ partition of $[a, b]$

A = $\{x_i : x_i \in [t_{i-1}, t_i]\}$

B = $\{v_i : v_i \in [t_{i-1}, t_i]\}$

$S_n = \sum_{i=1}^n f(x_i)g(v_i)\Delta t_i$ (not a Riemann sum for $f \cdot g$)

$\rightarrow S_n$ is within ϵ of $\int_a^b fg$ if Δt_i small enough

Proof

A Riemann sum for $f \cdot g$ is $R_n = \sum_{i=1}^n f(x_i)g(x_i)\Delta t_i$.

since f and g are cont. on $[a, b]$, they are uc on $[a, b]$.

In particular, $\forall \epsilon > 0 \exists \delta > 0$ s.t. $\forall x, y, x \in [a, b] \wedge y \in [a, b] \wedge |x-y| < \delta \rightarrow |g(x)-g(y)| < \epsilon$.

Given any $\epsilon > 0$, let $\delta_1 = \frac{\epsilon}{\int_a^b f + \epsilon}$ and let $\delta_2 > 0$ s.t.

$\forall x, y, x \in [a, b] \wedge y \in [a, b] \wedge |x-y| < \delta_2 \rightarrow |g(x)-g(y)| < \epsilon$.

If we choose the partition s.t. $\Delta t_i < \delta_2$, then $|x_i - v_i| < \delta_2$, so $|g(x_i) - g(v_i)| < \epsilon$.

$S_n - R_n = \sum_{i=1}^n (g(v_i) - g(x_i))f(x_i)\Delta t_i$.

$$< \epsilon \sum_{i=1}^n f(x_i)\Delta t_i$$

But $\sum_{i=1}^n f(x_i)\Delta t_i$ is a Riemann sum for f .

By Theorem 1, we can make this sum arbitrarily close to $\int_a^b f$.

i.e. $\exists \delta_2 > 0$ s.t. if $\Delta t_i < \delta_2$, then $\left| \sum_{i=1}^n f(x_i)\Delta t_i - \int_a^b f \right| < \epsilon$.

$$\rightarrow \sum_{i=1}^n f(x_i)\Delta t_i < \int_a^b f + \epsilon$$

Therefore, if $\delta = \min(\delta_1, \delta_2)$ then $S_n - R_n < \frac{\epsilon}{\int_a^b f + \epsilon} \cdot (\int_a^b f + \epsilon) = \epsilon$

Alternative Proof

f, g cont. on $[a, b] \rightarrow \mathbb{R}$, g bounded on $[a, b]$ \wedge f, g d.c. on $[a, b]$

$\exists N$ s.t. $\forall x, y \in [a, b] \rightarrow |f(x)| \leq N$

For any $\epsilon > 0$ let $\delta_1 = \frac{\epsilon}{N(b-a)}$ and δ s.t. $|x-y| < \delta \rightarrow |g(x)-g(y)| < \epsilon$.

Let $P = \{t_0, \dots, t_n\}$ be partition of $[a, b]$.

If $\Delta t_i < \delta$ then

$$\begin{aligned}|R_n - S_n| &= \left| \sum_{i=1}^n [f(t_i)g(x_i)]\Delta t_i - \sum_{i=1}^n [f(t_i)g(v_i)]\Delta t_i \right| \\&= \left| \sum_{i=1}^n f(t_i)[g(x_i) - g(v_i)]\Delta t_i \right| \\&\leq N \cdot \frac{\epsilon}{N(b-a)} \sum_{i=1}^n \Delta t_i \\&= \epsilon\end{aligned}$$

2. f.g cont., non-neg. on $[a,b]$

For partition P consider

$\rightarrow S_n$ is within ϵ of $\int_a^b \sqrt{f(x)} dx$ if all Δt_i small enough.

$$S_n = \sum_{i=1}^n \sqrt{f(x_i) + g(w_i)} \Delta t_i$$

Proof

$$|S_n - R_n| = \left| \sum_{i=1}^n \sqrt{f(x_i) + g(w_i)} \Delta t_i - \sum_{i=1}^n \sqrt{f(x_i) + g(x_i)} \Delta t_i \right|$$

$$= \left| \sum_{i=1}^n \left(\sqrt{f(x_i) + g(x_i) + (g(w_i) - g(x_i))} - \sqrt{f(x_i) + g(x_i)} \right) \Delta t_i \right|$$

Since g is U.C. on $[a,b]$, $\forall \epsilon > 0 \exists \delta, \geq 0$ s.t. $\forall x, y, |x-y| < \delta \rightarrow |g(x) - g(y)| < \epsilon$

let $h(x) = \sqrt{x}$.

Since h is U.C. on $[a,b]$, $\forall \epsilon > 0 \exists \delta, \geq 0$ s.t. $\forall x, y, |x-y| < \delta \rightarrow |h(x) - h(y)| < \epsilon$

For any $\epsilon > 0$, let $\epsilon_i = \frac{\epsilon}{b-a}$.

let $\delta > 0$ s.t. $\forall x, y, |x-y| < \delta \rightarrow |h(x) - h(y)| < \epsilon$.

let $\delta_i > 0$ s.t. $\forall x, y, |x-y| < \delta_i \rightarrow |g(x) - g(y)| < \epsilon$

choose P s.t. $\Delta t_i < \delta_i$. Then $|v_i - x_i| < \delta_i \rightarrow |g(v_i) - g(x_i)| < \epsilon$

Notice that $f(x_i) + g(w_i) - f(x_i) - g(x_i) = \cancel{f(x_i) + g(x_i)} + (g(w_i) - g(x_i)) - \cancel{f(x_i) - g(x_i)}$
 $= g(w_i) - g(x_i) < \epsilon$

$$\rightarrow \sqrt{f(x_i) + g(x_i) + (g(w_i) - g(x_i))} - \sqrt{f(x_i) + g(x_i)} < \frac{\epsilon}{b-a}$$

$$|S_n - R_n| < \left| \frac{\epsilon}{b-a} \sum \Delta t_i \right| \cdot \epsilon$$

Alternative Proof

f, g cont. $\rightarrow f+g$ cont. $\rightarrow f+g$ bounded on $[a, b]$

$\exists M > 0$ s.t. $|f(x) + g(x)| \leq M$ on $[a, b]$

Now, $f+g$ is also u.c. on $[a, b]$, and so is g .

$$\forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } \forall x, y \in [a, b], |x - y| < \delta \rightarrow |\sqrt{x} - \sqrt{y}| < \epsilon \quad (1)$$

$$\forall \delta > 0 \exists \delta_1 > 0 \text{ s.t. } \forall i, \forall j, |x_i - y_i| < \delta_1 \rightarrow |g(x_i) - g(y_i)| < \delta \quad (2)$$

For any $\epsilon > 0$, let $\epsilon_1 = \frac{\epsilon}{b-a}$.

Let $\delta > 0$ s.t. (1) for ϵ_1 .

Let $\delta_1 > 0$ satisfying (2) for δ .

Choose the partition s.t. $\Delta t_i < \delta_1$. Then $\forall i, i=1, \dots, n \rightarrow |x_i - y_i| < \delta_1$.

$$\rightarrow |[f(x_i) + g(y_i)] - [f(x_i) + g(x_i)]| = |g(y_i) - g(x_i)| < \delta$$

$$\rightarrow \left| \sqrt{f(x_i) + g(y_i)} - \sqrt{f(x_i) + g(x_i)} \right| < \epsilon_1 = \frac{\epsilon}{b-a}$$

Hence,

$$\begin{aligned} |S_n - R_n| &= \left| \sum_{i=1}^n \sqrt{f(x_i) + g(y_i)} \Delta t_i - \sum_{i=1}^n \sqrt{f(x_i) + g(x_i)} \Delta t_i \right| \\ &\cdot \left| \sum_{i=1}^n (\sqrt{f(x_i) + g(x_i)} + (g(y_i) - g(x_i))) - \sqrt{f(x_i) + g(x_i)} \right| \Delta t_i \\ &< \frac{\epsilon}{b-a} \sum \Delta t_i = \epsilon \end{aligned}$$

let's prove a few results beforehand

let $h(x) = \sqrt{x}$.

$h(x)$ is continuous.

Proof

If $a=0$, then for any $\epsilon > 0$ choose $\delta = \epsilon^2$. Then, $|x| < \delta = \epsilon^2 \rightarrow |\sqrt{x}| < \epsilon \rightarrow \lim_{x \rightarrow 0} \sqrt{x} = 0$

If $a \in (0, \infty)$ then

$$|x-a| < \delta \rightarrow |\sqrt{x}-\sqrt{a}| |\sqrt{x}+\sqrt{a}| < \delta \rightarrow |\sqrt{x}-\sqrt{a}| < \frac{\delta}{|\sqrt{x}+\sqrt{a}|}$$

TF. if $\delta < \sqrt{a} \in \epsilon$ then we have $|x-a| < \delta \rightarrow |\sqrt{x}-\sqrt{a}| < \epsilon$.

$$\rightarrow \lim_{x \rightarrow a} \sqrt{x} = \sqrt{a}$$

$h(x)$ is cont. cont.

$$|\sqrt{x}-\sqrt{y}|^2 \leq |\sqrt{x}-\sqrt{y}| |\sqrt{x}+\sqrt{y}| = |x-y| < \epsilon^2 \rightarrow |\sqrt{x}-\sqrt{y}| < \epsilon$$