Ch. 17 - Planetary Motion

$\vec{c}(t) = r(t)(\cos\theta(t), \sin\theta(t))$ — assume θ is increasing; $r(t)$: sun to planet — param. curve repres. planet motion

$\vec{c}(t) = r(t)\vec{e}(\theta(t))$, $\vec{e}(t) = (\cos t, \sin t)$ — param. curve around unit circle, unit length vector

$\vec{e}'(t) = (-\sin t, \cos t)$ unit length vector

$\vec{e}(t)\cdot\vec{e}'(t) = 0$

$$\det(\vec{e}(t), \vec{e}'(t)) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

$$c'(t) = r'(t)\vec{e}(\theta(t)) + r(t)\vec{e}'(\theta(t))\cdot\theta'(t)$$

$$\det(\vec{c}(t), \vec{c}'(t)) = \det\Big(r(t)\vec{e}(\theta(t)),\ r'(t)\vec{e}(\theta(t)) + r(t)\vec{e}'(\theta(t))\cdot\theta'(t)\Big)$$

$$= \det(r(t)\vec{e}(\theta(t)), r'(t)\vec{e}(\theta(t))) + \det(r(t)\vec{e}(\theta(t)), r(t)\vec{e}'(\theta(t))\cdot\theta'(t))$$

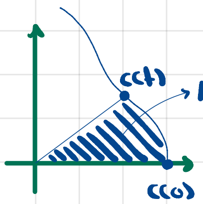
$\det(\vec{v}, \vec{w}+\vec{z}) = \det(\vec{v},\vec{w}) + \det(\vec{v},\vec{z})$

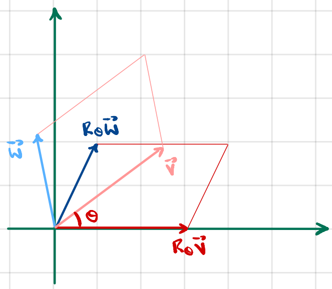
$$= r(t)r'(t)\underbrace{\cancel{\det(\vec{e}(\theta(t)), \vec{e}(\theta(t)))}}_{0} + r(t)^2\theta'(t)\underbrace{\cancel{\det(\vec{e}(\theta(t)), \vec{e}'(\theta(t)))}}_{1}$$

$a\det(\vec{v},\vec{w}) = \det(a\vec{v},\vec{w}) = \det(\vec{v},a\vec{w})$

$$\det(\vec{c}(t), \vec{c}'(t)) = r(t)^2\theta'(t)$$

$A(t)$: area swept from 0 to t. We want $A'(t)$.

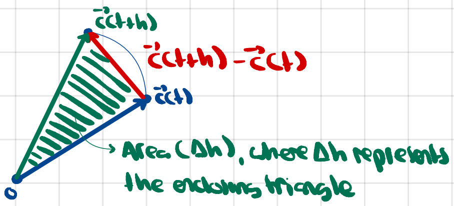




In Problem 4A1-5 we showed $\det(R_\theta(\vec{v}), R_\theta(\vec{w})) = \det(\vec{v},\vec{w})$

and $\det(\vec{v},\vec{w})$ is the area of the parallelogram formed by $\vec{v}$ and $\vec{w}$.

Therefore



$$\text{Area}(\Delta h) = \frac{1}{2}\det(\vec{c}(t), \vec{c}(t+h) - \vec{c}(t))$$

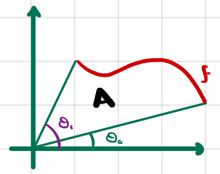
Now, $A(t+h) - A(t)$ is the area of the entire "pizza slice", approx. equal to Area(Δh).

An informal calculation shows

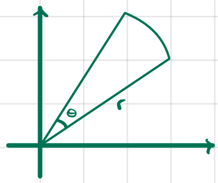
$$A'(t) = \lim_{h\to 0}\frac{A(t+h) - A(t)}{h} \approx \lim_{h\to 0}\frac{\text{Area}(\Delta h)}{h} = \frac{1}{2}\det\left(c(t), \lim_{h\to 0}\frac{\vec{c}(t+h) - c(t)}{h}\right)$$

$$= \frac{1}{2}\det(c(t), c'(t))$$

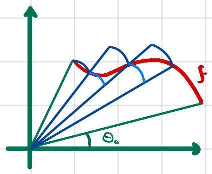
Recall that we proved in Problem 4A1-24

Given a continuous function $f(\theta)$ in polar coordinates, the area  is

$$\frac{1}{2}\int_{\theta_0}^{\theta_1} f^2(\theta)\,d\theta$$

We proved this using the fact that the area  is $\frac{r^2\theta}{2}$

and partitioning $[\theta_0, \theta_1]$ s.t. between each θ_{i-1} and θ_i we had a little wedge like the one above



we defined a $g(\theta) = \frac{f^2(\theta)}{2}$, continuous thus integr.

By def. of integral,

sum of small wedges, approx. A

$$\underbrace{\int_{\theta_0}^{\theta_1} g(\theta)\,d\theta}_{A} = \sup\{L(g,P)\} = \sup\left\{\sum \frac{m_i^2}{2}(t_i - t_{i-1})\right\} = \inf\{U(g,P)\} = \inf\left\{\sum \frac{M_i^2}{2}(t_i - t_{i-1})\right\}$$

sum of small wedges, approx. A

$\vec{c}(t)$ is parametrized by t.

we can transform it into a param. by angular polar coord. ϕ.

Then we'd have a fn $\rho(\phi)$ repres. the curve in polar coord.
(the param. would be $(\rho(\phi)\cos(\phi), \rho(\phi), \sin(\phi))$)

The area $A(t)$ would thus be

$$A(t) = \frac{1}{2}\int_{\theta(0)}^{\theta(t)} \rho(\phi)^2\,d\phi \qquad (1)$$

Note

$r = \rho(\phi)$ r as function of angle

but ϕ is fn of time. $\phi = \theta(t)$

$r(t) = \rho(\theta(t))$

but t is a fn of angle, $t = \theta^{-1}(\phi)$

T.F. $r(\theta^{-1}(\phi)) = \rho(\phi)$

Differentiate (1)

$$A'(t) = \frac{1}{2}\theta'(t)\rho^2(\theta(t))$$

$$= \frac{1}{2} r(t)^2 \theta'(t)$$ because $\rho(\theta(t)) = r(t)$; ie $A' = \frac{1}{2} r^2 \theta'$

but we proved before that $\det(c(t), c'(t)) = r^2(t)\theta'(t)$

Thus, $A'(t) = \frac{1}{2}\det(c(t), c'(t)) = \frac{1}{2} r^2(t)\theta'(t)$

Consider Kepler's Second Law

Equal areas are swept out by the radius vector in equal times.
(ie, the area swept out in time t is proportional to t)

This is equiv. to A' is constant, ie $A'' = 0$.

$$A''(t) = \frac{1}{2}\left[\det(\cancel{c'(t), c'(t)}) + \det(c(t), c''(t))\right]$$
$$= \frac{1}{2}\det(c(t), c''(t))$$

$* \det(c,d)(t) = \det(c(t), d(t))$

$[\det(c,d)]'(t) = \det(c',d)(t) + \det(c,d')(t)$

T.F. Kepler's 2^{nd} Law is equiv. to $\det(c(t), c''(t)) = 0$

Theorem 1 Kepler's Second Law true $\leftrightarrow$ the force is central (always points along c(t))
each planetary path $c(t) = r(t)\vec{e}(\theta(t))$ satisfies

$$r^2\theta' = \det(c, c') = \text{constant}$$

Proof

According Newton's first law, $\vec{F} = m \cdot \vec{a}$. Hence, $\vec{a}$ and $\vec{F}$ have same direction.

T.F. $\vec{c}''(t)$ points in direction of $\vec{c}(t)$, since $\vec{F}$ points along $\vec{c}(t)$.

But then $\det(\vec{c}, \vec{c}'') = 0$

Prev. we showed

$$A''(t) = \frac{1}{2}\det(c(t), c''(t))$$

T.F. we have $A'' = 0$ which is equiv. to Kepler's 2^{nd} law.

Also, since

$$A'(t) = \frac{1}{2}\det(c(t), c'(t)) = \frac{1}{2} r^2(t)\theta'(t)$$

So, $A'' = 0$ means $[r^2\theta']' = 0 \rightarrow r^2\theta' = \text{constant}$.

Theorem 2: If

sun's gravitational force is a central force that satisfies inverse square law,

then

the path of any body in it will be a conic section having the sun at one focus (more precisely, either an ellipse, parabola, or one branch of a hyperbola).

Proof

Th.1 $\to r^2\theta' = \det(c, c') = M$, for some constant M

Inverse square law hypothesis: $c''(t) = -\dfrac{H}{r(t)^2}\vec{e}(\theta(t))$ for some constant H

$$\frac{c''}{\theta'} = -\frac{H}{r^2\theta'}\vec{e}$$

$$\to \frac{c''(t)}{\theta'(t)} = -\frac{H}{M}\vec{e}(\theta(t)) \quad (1)$$

Note that

$$[c'\circ\theta^{-1}]'(\theta(t)) = [c'(\theta^{-1}(\theta(t)))]' = c''(t)\cdot(\theta^{-1})'(\theta(t)) = c''(t)\cdot\frac{1}{\theta'(\theta^{-1}(\theta(t)))} = \frac{c''(t)}{\theta'(t)}$$

which is the left-hand side of (1)

now a trick let $D = c'\circ\theta^{-1}$. Ie $D(\theta(t)) = c'(\theta^{-1}(\theta(t)))$

Then $D'(\theta(t)) = \dfrac{c''(t)}{\theta'(t)}$ and

$$D'(\theta(t)) = -\frac{H}{M}\vec{e}(\theta(t)) = -\frac{H}{M}(\cos\theta(t), \sin\theta(t))$$

more simply

$D'(u) = -\dfrac{H}{M}(\cos u, \sin u)$ for all u of the form $u = \theta(t)$ for some t.

$$\to D(u) = \left(-\frac{H}{M}\sin u + A, \frac{H}{M}\cos u + B\right)$$