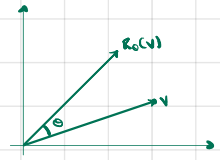
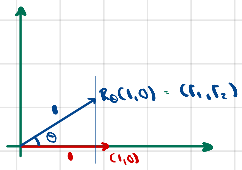
Ch4 Appendix I - Vectors

1.



a) show $R_\theta((1,0)) = (\cos\theta, \sin\theta)$

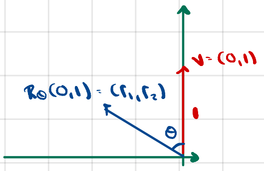
$R_\theta((0,1)) = (-\sin\theta, \cos\theta)$



$|v| = [1^2 + 0^2]^{1/2} = 1$

since we rotated only, the distance to origin stayed the same.

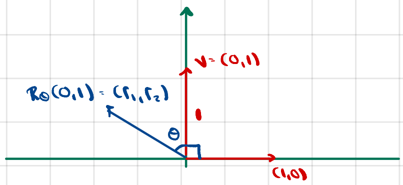
we can see that by geometry $r_1 = \cos\theta$, $r_2 = \sin\theta$



$r_1 = -\cos(90° - \theta) = -\sin\theta$

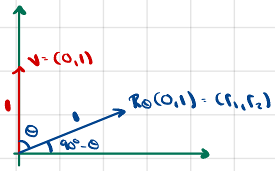
$r_2 = \sin(90° - \theta) = \cos\theta$

Alternatively, $R_\theta(0,1)$ is a rotation of $90° + \theta$ from (1,0)



$r_1 = \cos(90° + \theta) = -\sin\theta$

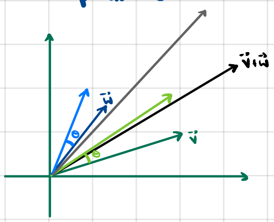
$r_2 = \sin(90° + \theta) = \cos\theta$



$r_1 = \cos(90° - \theta) = \sin\theta$

$r_2 = \sin(90° - \theta) = \cos\theta$

b) explain $R_\theta(\vec{v} + \vec{w}) = R_\theta(\vec{v}) + R_\theta(\vec{w})$



Rotation of a sum of two vectors is the same as the sum of the rotations of the two vectors.

Rotation of multiplication of vector by scalar is the same as rotation of vector, multiplied by scalar

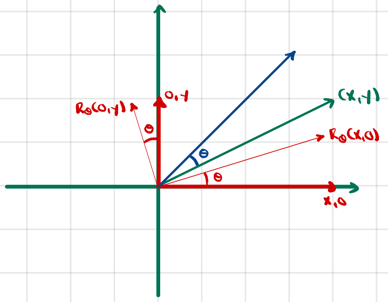
These are properties of the rotation fn or operation.

It is not true for all fns. For example, $N(\vec{v}) = \|\vec{v}\|$.

$N(\vec{v} + \vec{w}) = \|\vec{v} + \vec{w}\| = [(v_1 + w_1)^2 + (v_2 + w_2)^2]^{1/2}$

$\neq [v_1^2 + v_2^2]^{1/2} + [w_1^2 + w_2^2]^{1/2} = N(\vec{v}) + N(\vec{w})$

c) $R_\theta(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$



note that if $\vec{w} = a\vec{v}$ then $\|\vec{w}\| = \|a\vec{v}\| = (a^2v_1^2 + a_2v_2^2)^{1/2} = a\|\vec{v}\|$

$\vec{v} = (x,y) = (x,0) + (0,y)$

$R_\theta(\vec{v}) = R_\theta((x,0) + (0,y)) = R_\theta(x,0) + R_\theta(0,y)$

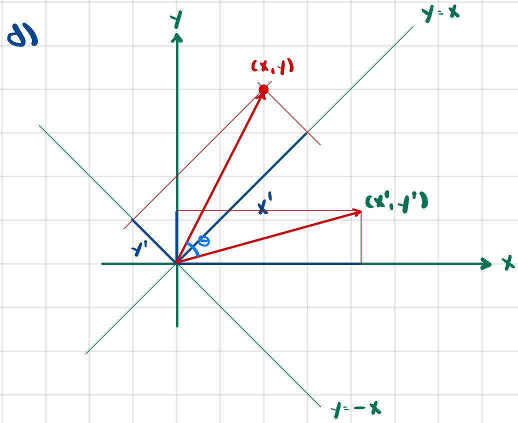
$= R_\theta(x(1,0)) + R_\theta(y(0,1))$

$= xR_\theta(1,0) + yR_\theta(0,1)$

$= x(\cos\theta, \sin\theta) + y(-\sin\theta, \cos\theta)$

$= (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$

d)



$R_\theta(x_0,y_0) = (x,y) = (x'\cos\theta - y'\sin\theta,\ x'\sin\theta + y'\cos\theta)$

$R_{-\theta}(x,y) = (x',y')$

$= (x\cos(-\theta) - y\sin(-\theta),\ x\sin(-\theta) + y\cos(-\theta))$

$= (x\cos\theta + y\sin\theta,\ -x\sin\theta + y\cos\theta)$

$\theta = 45° \Rightarrow (x',y') = \left(\frac{x+y}{\sqrt{2}},\ \frac{y-x}{\sqrt{2}}\right)$

2. $v \cdot w = v_1 \cdot w_1 + v_2 \cdot w_2$

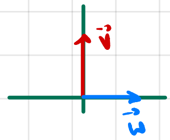
a) given $\vec{v}$, find $\vec{w}$: $\vec{v}\cdot\vec{w} = 0$

$v_1 \cdot w_1 + v_2 \cdot w_2 = 0$

$v_1 w_1 = -v_2 w_2$

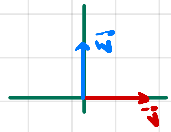
$v_1 = 0 \Rightarrow w_2 = 0$

ie $\vec{v} = (0, v_2) \Rightarrow \vec{w} = (w_1, 0)$



$v_2 = 0 \Rightarrow w_1 = 0$

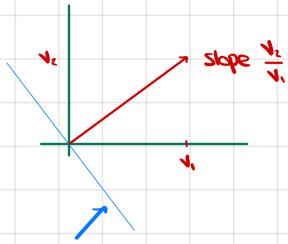
ie $\vec{v} = (v_1, 0) \Rightarrow \vec{w} = (0, w_2)$



$v_1 \neq 0,\ v_2 \neq 0$

$\Rightarrow w_1 = -\frac{v_2}{v_1} w_2$

$w_2 = -\frac{v_1}{v_2} w_1$



ie (w_2, w_1) is somewhere on this line

$\Rightarrow$ the lines on which $\vec{v}$ and $\vec{w}$ are on are perpendicular

b) $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$

commut. law for numbers

$\vec{v}\cdot\vec{w} = v_1 w_1 + v_2 w_2 = w_1 v_1 + w_2 v_2 = \vec{w}\cdot\vec{v}$

$\vec{v}\cdot(\vec{w}+\vec{z}) = \vec{v}\cdot\vec{w} + \vec{v}\cdot\vec{z}$

$v_1(w_1 + z_1) + v_2(w_2 + z_2)$

$= v_1 w_1 + v_2 w_2 + v_1 z_1 + v_2 z_2$

$= \vec{v}\cdot\vec{w} + \vec{v}\cdot\vec{z}$

$a(\vec{v}\cdot\vec{w}) = (a\vec{v})\cdot\vec{w} = v\cdot(a\vec{w})$

$a(v_1 w_1 + v_2 w_2)$

$= a v_1 w_1 + a v_2 w_2$

$= (a v_1) w_1 + (a v_2) w_2 = (a\vec{v})\cdot\vec{w}$

$= v_1 \cdot (a w_1) + v_2 \cdot (a w_2) = \vec{v}\cdot(a\vec{w})$

c) $\vec{v}\cdot\vec{v} \geq 0$

$\vec{v}\cdot\vec{v} = v_1^2 + v_2^2 \geq 0$

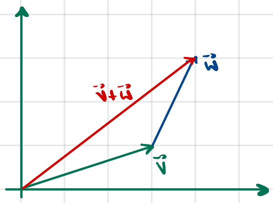
$\vec{v}\cdot\vec{v} = 0 \iff \vec{v} = 0$

$\vec{v}\cdot\vec{v} = v_1^2 + v_2^2 = 0 \iff v_1^2 = -v_2^2 \iff v_1 = v_2 = 0 \Rightarrow \vec{v} = \vec{0}$

norm $\|\vec{v}\| \equiv \sqrt{\vec{v}\cdot\vec{v}}$

$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2} = \sqrt{(v_1-0)^2 + (v_2-0)^2}$, distance from $(0,0)$

d) $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ ie $\sqrt{(v_1+w_1)^2 + (v_2+w_2)^2} \leq \sqrt{v_1^2+v_2^2} + \sqrt{w_1^2+w_2^2}$



This is the triangle inequality which we proved in problem 4-9

recall

$(x_1+y_1)^2 + (x_2+y_2)^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2x_1y_1 + 2x_2y_2$

$= x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2(x_1y_1 + x_2y_2)$

$\leq x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}$

$= \left[\sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2}\right]^2$

$\Rightarrow \sqrt{(x_1+y_1)^2 + (x_2+y_2)^2} \leq \sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2}$

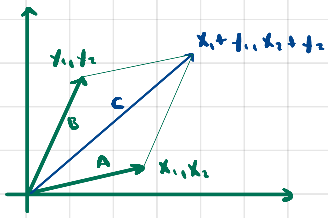
Here we used Schwarz Inequality

For equality here, either

$\vec{x} = \lambda\vec{y}$ or at least one vector is zero

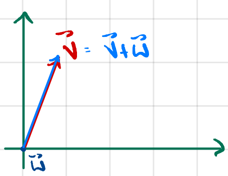
In the context of this problem, $\|\vec{v}+\vec{w}\| = \|\vec{v}\| + \|\vec{w}\|$

if either

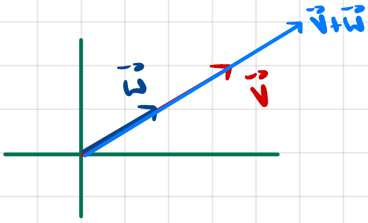


$C \leq A+B$

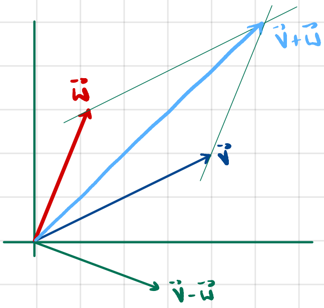
i) $\vec{v}$ and/or $\vec{w} = \vec{0}$



ii) $\vec{v} = \lambda\vec{w}$



e) $\vec{v}\cdot\vec{w} = \dfrac{\|\vec{v}+\vec{w}\|^2 - \|\vec{v}-\vec{w}\|^2}{4}$



$\|\vec{v}+\vec{w}\|^2 = (v_1+w_1)^2 + (v_2+w_2)^2$

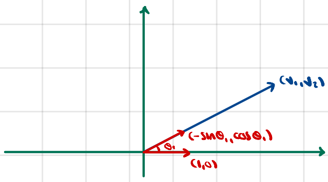
$\|\vec{v}-\vec{w}\|^2 = (v_1-w_1)^2 + (v_2-w_2)^2$

$$\|\vec{v}+\vec{w}\|^2 - \|\vec{v}-\vec{w}\|^2 = \cancel{v_1^2} + 2v_1w_1 + \cancel{w_1^2} + \cancel{v_2^2} + 2v_2w_2 + \cancel{w_2^2}$$
$$- (\cancel{v_1^2} - 2v_1w_1 + \cancel{w_1^2} + \cancel{v_2^2} - 2v_2w_2 + \cancel{w_2^2})$$

$= 4v_1w_1 + 4v_2w_2$

$$\Rightarrow \frac{\|\vec{v}+\vec{w}\|^2 - \|\vec{v}-\vec{w}\|^2}{4} = v_1w_1 + v_2w_2 = \vec{v}\cdot\vec{w}$$

3. a) $R_\theta(\vec{v})\cdot R_\theta(\vec{w}) = \vec{v}\cdot\vec{w}$



$R_\theta(v_1,v_2) = R_\theta(v_1,0) + R_\theta(0,v_2)$

$R_\theta(w_1,w_2) = R_\theta(w_1,0) + R_\theta(0,w_2)$

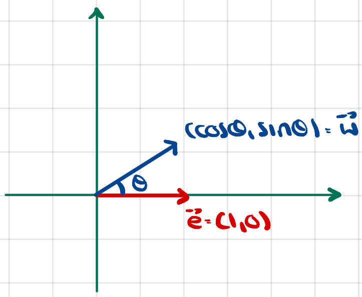
$R_\theta(\vec{v})\cdot R_\theta(\vec{w}) = R_\theta(v_1,0)\cdot R_\theta(w_1,0) + R_\theta(0,v_2)\cdot R_\theta(0,w_2)$

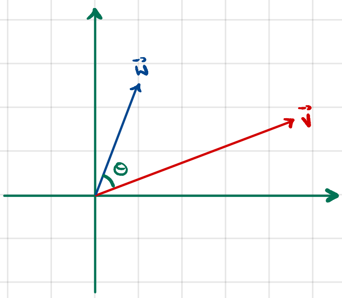
$= v_1w_1 R_\theta(1,0)\cdot R_\theta(1,0) + v_2w_2 R_\theta(0,1)\cdot R_\theta(0,1)$

$= v_1w_1 \|R_\theta(1,0)\|^2 + v_2w_2\|R_\theta(0,1)\|^2$

$= v_1w_1 + v_2w_2$

b) $\vec{e}\cdot\vec{w} = \cos\theta = \|\vec{e}\|\|\vec{w}\|\cos\theta$





Let $\hat{v} = \dfrac{\vec{v}}{\|\vec{v}\|} = \left(\dfrac{v_1}{\|\vec{v}\|}, \dfrac{v_2}{\|\vec{v}\|}\right) = (v_{1\wedge}, v_{2\wedge})$

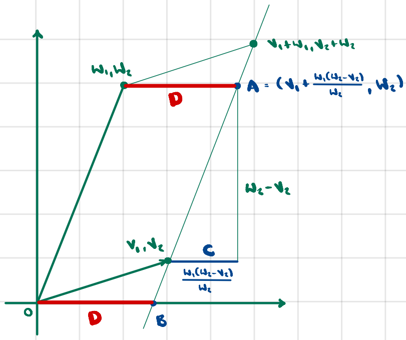
$\hat{w} = R_\theta(\hat{v}) = \dfrac{1}{\|\vec{v}\|} R_\theta(\vec{v})$

$= \dfrac{1}{\|\vec{v}\|}\left(\dfrac{v_1}{\|\vec{v}\|}\cos\theta - \dfrac{v_2}{\|\vec{v}\|}\sin\theta, \dfrac{v_1}{\|\vec{v}\|}\sin\theta + \dfrac{v_2}{\|\vec{v}\|}\cos\theta\right)$

$\hat{v}\cdot\hat{w} = \dfrac{v_1^2}{\|\vec{v}\|^2}\cos\theta - \cancel{\dfrac{v_1v_2}{\|\vec{v}\|^2}}\sin\theta + \cancel{\dfrac{v_1v_2}{\|\vec{v}\|^2}}\sin\theta + \dfrac{v_2^2}{\|\vec{v}\|^2}\cos\theta = \cos\theta$

$\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\hat{v}\cdot\hat{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$

4.



a) Prove $B = \left(\dfrac{v_1w_2 - w_1v_2}{w_2}, 0 \right)$

C: $\dfrac{w_2}{w_1} = \dfrac{w_2 - v_2}{C} \Rightarrow C = \dfrac{w_1(w_2-v_2)}{w_2}$

$$D:\quad D = v_1 + \frac{w_1(w_2-v_2)}{w_2} - w_1$$

$$= \frac{w_2(v_1 - \cancel{w_1}) + w_1(\cancel{w_2} - v_2)}{w_2}$$

$$= \frac{v_1w_2 - w_1v_2}{w_2}$$

$$\Rightarrow B = \left(\frac{v_1w_2 - w_1v_2}{w_2}, 0 \right)$$

height of parallelogram $= w_2$

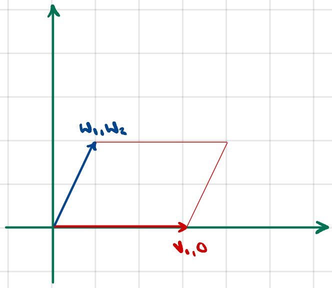
$$\text{Area} = \text{base} \times \text{height} = w_2 \cdot \frac{v_1w_2 - w_1v_2}{w_2}$$

$$= v_1w_2 - w_1v_2$$

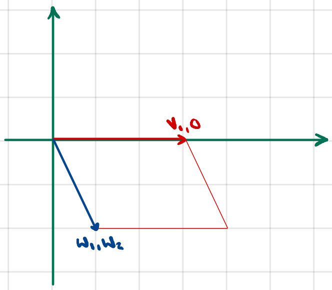
$$= \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix}$$

$$= \det(\vec{v}, \vec{w})$$

5. a)



$$\det(\vec{v}, \vec{w}) = v_1w_2 - v_2w_1$$
$$= v_1w_2 = \text{base} \times \text{height} = \text{Area}$$



$w_2 < 0 \Rightarrow \det(\vec{v}, \vec{w}) = v_1w_2 < 0$

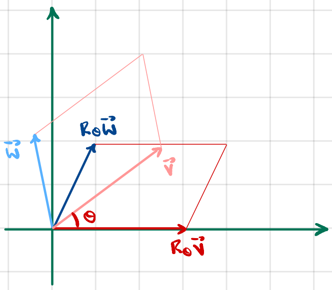
b) show $\det(R_\theta(\vec{v}), R_\theta(\vec{w})) = \det(\vec{v}, \vec{w})$

$R_\theta(\vec{v}) = (v_1\cos\theta - v_2\sin\theta,\ v_1\sin\theta + v_2\cos\theta) = (v_1\cos\theta,\ v_1\sin\theta)$

$R_\theta(\vec{w}) = (w_1\cos\theta - w_2\sin\theta,\ w_1\sin\theta + w_2\cos\theta)$

$$\det(R_\theta\vec{v}, R_\theta\vec{w}) = \cancel{v_1w_1\sin\theta\cos\theta} + v_1w_2\cos^2\theta - (\cancel{v_1w_1\sin\theta\cos\theta} - v_1w_2\sin^2\theta)$$

$$= v_1w_2(\sin^2\theta + \cos^2\theta) = v_1w_2 = \det(\vec{v}, \vec{w})$$



we can rotate any parallelogram by θ so that one of the vectors is on the x axis.
The area of the rotated parallelogram equals the area of the original, ie

$$\text{Area}_\theta = \det(R_\theta\vec{v}, R_\theta\vec{w}) = \det(\vec{v}, \vec{w}) = \text{Area}$$

If the rotation from $\vec{v}$ to $\vec{w}$ is clockwise, then when we rotate both vectors such that $\vec{v}$ is on the x axis, $\vec{w}$ is either in quadrant III or IV, both with y coord < 0 $\Rightarrow \det(\vec{v}, \vec{w}) < 0$ as per a).
If rot. from $\vec{v}$ to $\vec{w}$ counterclockwise then $\vec{w}$ is in quadrant I or II so $\det(\vec{v}, \vec{w})$ is > 0.

6) Show $\det(\vec{v},\vec{w}+\vec{z}) = \det(\vec{v},\vec{w}) + \det(\vec{v},\vec{z})$

$\det(\vec{v}+\vec{w},\vec{z}) = \det(\vec{v},\vec{z}) + \det(\vec{w},\vec{z})$

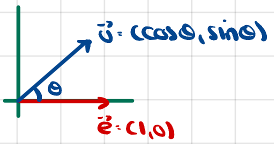
$a\det(\vec{v},\vec{w}) = \det(a\vec{v},\vec{w}) = \det(\vec{v},a\vec{w})$

$$\det(\vec{v},\vec{w}+\vec{z}) = v_1(w_2+z_2) - v_2(w_1+z_1)$$
$$= v_1w_2 - v_2w_1 + v_1z_2 - v_2z_1$$
$$= \det(\vec{v},\vec{w}) + \det(\vec{v},\vec{z})$$

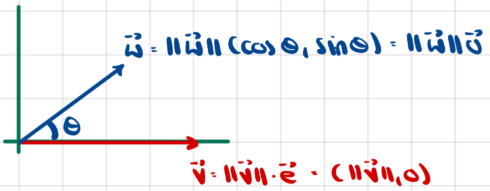
$$\det(\vec{v}+\vec{w},\vec{z}) = (v_1+w_1)z_2 - (v_2+w_2)z_1$$
$$= v_1z_2 - v_2z_1 + w_1z_2 - w_2z_1$$
$$= \det(\vec{v},\vec{z}) + \det(\vec{w},\vec{z})$$

$$a\det(\vec{v},\vec{w}) = a(v_1w_2 - v_2w_1)$$
$$= (av_1)w_2 - (av_2)w_1 = \det(a\vec{v},\vec{w})$$
$$= v_1(aw_2) - v_2(aw_1) = \det(\vec{v},a\vec{w})$$

7. Show $\det(\vec{v},\vec{w}) = \|\vec{v}\|\|\vec{w}\|\sin\theta$

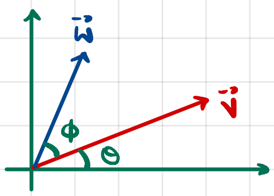


$$\det(\vec{e},\vec{u}) = e_1u_2 - e_2u_1 = \sin\theta$$



$$\det(\vec{v},\vec{w}) = \det(\|\vec{v}\|\vec{e}, \|\vec{w}\|\vec{u}) = \|\vec{v}\|\|\vec{w}\|\vec{e}\cdot\vec{u}$$

$$= \|\vec{v}\|\|\vec{w}\|\sin\theta$$



$\vec{v} = R_\theta(\vec{v}')$ for some $\vec{v}'$ along x-axis.

$\vec{w} = R_\theta(\vec{w}')$

$\|\vec{v}\| = \|\vec{v}'\|$, $\|\vec{w}\| = \|\vec{w}'\|$ because rotation doesn't change lengths, also the angle between $\vec{v}'$ and $\vec{w}'$ is maintained in rotation

$$\det(\vec{v},\vec{w}) = \det(R_\theta\vec{v}', R_\theta\vec{w}') \overset{\text{by 5b}}{=} \det(\vec{v}',\vec{w}')$$

$$\Rightarrow \det(\vec{v},\vec{w}) = \|\vec{v}'\|\|\vec{w}'\|\sin\phi = \|\vec{v}\|\|\vec{w}\|\sin\phi$$