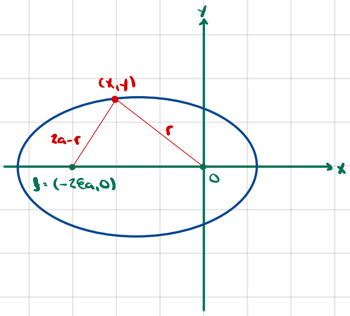
Ch 4 Appendix 2 – Polar Coordinates

Assume f fn.

graph of f in polar coord. is collection of all points P w/ polar coord. (r,θ) satisfying $r=f(\theta)$.
ie collection of all points w/ polar coord. $(f(\theta),\theta)$
ie the graph of the eq. $r=f(\theta)$

equation of ellipse in polar coord.



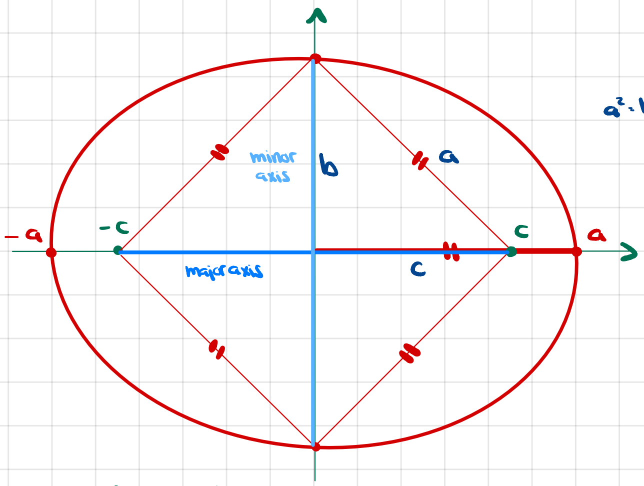
distance between foci $= 0 < 2\varepsilon a < 2a$ in ellipse
$\Rightarrow 0<\varepsilon<1$

(x,y) to O $= r = (x^2+y^2)^{1/2} \Rightarrow r^2 = x^2+y^2$ (1)

(x,y) to f $= 2a-r$ by assumption of $r_{O,(x,y)} + r_{f,(x,y)} = 2a \Rightarrow r_{f,(x,y)} = 2a-r$

$\Rightarrow (2a-r)^2 = (x+2\varepsilon a)^2 + y^2$
$4a^2 - 4ar + r^2 = x^2 + 4x\varepsilon a + 4\varepsilon^2 a^2 + y^2$ (2)

subtract (1) from (2)
$\cancel{4}a^{\cancel{2}} - \cancel{4}\cancel{a}r = \cancel{4}x\varepsilon\cancel{a} + \cancel{4}\varepsilon^2 a^{\cancel{2}}$
$a - r = x\varepsilon + \varepsilon^2 a$
$r = a - \varepsilon x - \varepsilon^2 a$
$= a(1-\varepsilon^2) - \varepsilon x$
$= \Lambda - \varepsilon x \qquad \Lambda = a(1-\varepsilon^2)$



$a^2 = b^2 + c^2 \Rightarrow b^2 = a^2 - c^2$

$r = \Lambda - \varepsilon r\cos\theta$
$r(1+\varepsilon\cos\theta) = \Lambda$

Vertices : $(a,0), (-a,0)$

$$r = \frac{\Lambda}{1+\varepsilon\cos\theta}$$

Consider the ellipse defined in ch.4, with foci at $(c,0)$ and $(-c,0)$ and equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If we move it left by c, we get the ellipse defined above, with foci at $(-2c,0)$ and $(0,0)$, assuming we choose $c = \varepsilon a$.

Conversely, given the ellipse $r = \frac{\Lambda}{1+\varepsilon\cos\theta}$, $\Lambda = a(1-\varepsilon^2)$

if we want the eq. in Cartesian coord centered on the ellipse's midpoint, we have

$$a = \frac{\Lambda}{1-\varepsilon^2}$$

$$c = \varepsilon a$$

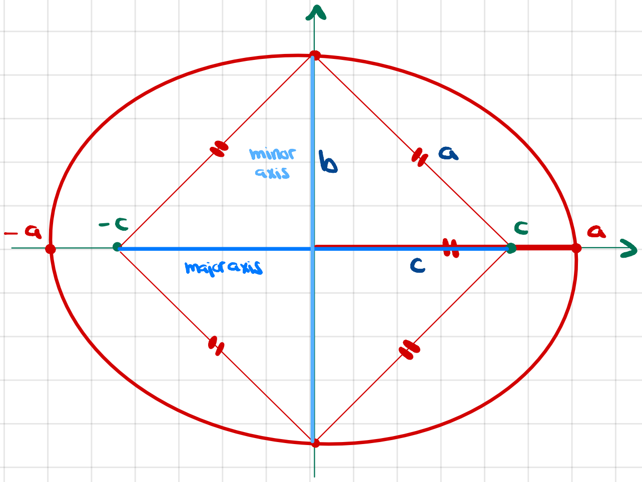
$$b = \sqrt{a^2-c^2} = \sqrt{a^2-\varepsilon^2 a^2} = a\sqrt{1-\varepsilon^2} = \frac{\Lambda}{\sqrt{1-\varepsilon^2}}$$

we thus obtain lengths of major and minor axes, a and b, immediately from Λ and ε.

$$\epsilon = \frac{c}{a} = \frac{\sqrt{a^2-b^2}}{a} = \sqrt{1-\left(\frac{b}{a}\right)^2}$$

is the **eccentricity** of the ellipse, determines "shape" (ratio of major and minor axes)

$$= \frac{\text{distance center to focus}}{\text{distance center to vertex}}$$

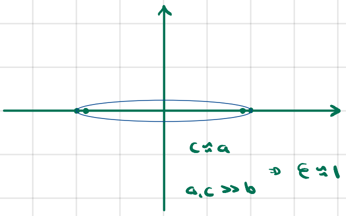


Fix a, vary b.

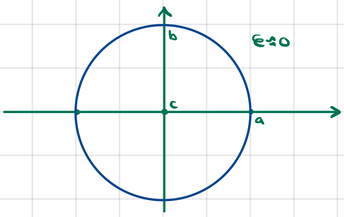
$0 < \sqrt{a^2-b^2} < a \Rightarrow 0 < \frac{\sqrt{a^2-b^2}}{a} < 1$

$\Rightarrow 0 < \epsilon < 1$

c close to a $\Rightarrow b^2 = a^2 - c^2 \approx 0$
$\Rightarrow b \approx 0$
$\Rightarrow \epsilon \approx 1$
very flat ellipse



$b \approx a \Rightarrow c \approx 0 \Rightarrow \epsilon \approx 0$

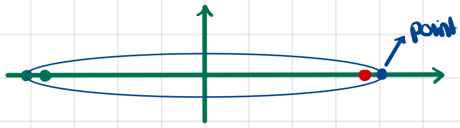


Λ determines "size": $\uparrow \Lambda$ means $\uparrow r$ for each θ

example Halley's comet has elliptical path around sun with sun at one focus point.

$\epsilon = 0.967 = \frac{c}{a} = \frac{\sqrt{a^2-b^2}}{a} = \sqrt{1-\frac{b^2}{a^2}}$

$c = 0.967a$



min distance = $a - c = 0.587$ AU

$a - 0.967a = 0.587 \Rightarrow a = \frac{0.587}{0.033} = 17.78$ AU

max distance = $a + c = a(1.967) = 34.98$ AU $= 3.25 \times 10^9$ miles

$\min (x-c)^2 + y^2$ s.t. $\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$

$g(x) = \frac{1}{a^2}x^2 + \frac{1}{a^2-c^2}y^2 - 1 = 0$

$\nabla f = (2(x-c), 2y)$

$\nabla g = \left(\frac{2x}{a^2}, \frac{2y}{a^2-c^2}\right)$

$\nabla f = \lambda \nabla g = \quad 2(x-c) = \frac{\lambda 2x}{a^2}$

$2y = \frac{2y}{a^2-c^2}$

$\Rightarrow 2y(a^2-c^2) = 2y \Rightarrow y = 0$

$y = 0 \Rightarrow \frac{x^2}{a^2} = 1 \Rightarrow x = \pm a$

$\Rightarrow \cancel{2}(a-c) = \frac{\lambda \cdot \cancel{2}\cancel{a}}{a^{\cancel{2}}}$

$\Rightarrow \lambda = a(a-c) > 0$