

Ch 11 - Significance of the Derivative

Definition Let f be a fn and A a set of numbers contained in f 's domain.

A point x in A is a maximum point for f on A if $f(x) \geq f(y)$ for every y in A .

$f(x)$ is called the maximum value of f on A .

f has its maximum value on A at x .

We can define min point and min value in a similar way as above, or simply by:

x is a min point of f on A if x is a max point of $-f$ on A .

Theorem 1 Let f be any fn defined on (a,b) .

If x is a max (min) point for f on (a,b) , and f is diff at x , then $f'(x)=0$.

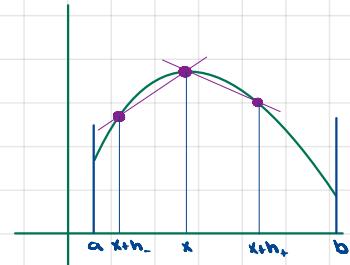
Proof

Let h be a number s.t. $x+h \in (a,b)$.

then $f(x+h) \leq f(x)$. Hence,

$$h > 0 \rightarrow \frac{f(x+h) - f(x)}{h} \leq 0 \rightarrow \lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h} \leq 0$$

$$h < 0 \rightarrow \frac{f(x+h) - f(x)}{h} \geq 0 \rightarrow \lim_{h \rightarrow 0^-} \frac{f(x+h) - f(x)}{h} \geq 0$$



since f is diff at x , these limits must be equal.

$$\rightarrow f'(x) = 0$$

Definition Let f be a fn and A a set of numbers contained in f 's domain.

A point x in A is a local maximum (minimum) point for f on A if there is some $\delta > 0$ such that x is a max (min) point for f on $A \cap (x-\delta, x+\delta)$.

Theorem 2 x local max or min for f on (a,b)
 f diff at x

$$\rightarrow f'(x) = 0$$

Proof

Direct application of Th 1. x is a max on $(a,b) \cap (x-\delta, x+\delta) = (x-\delta, x+\delta)$, for some $\delta > 0$.
By Th 1, $f'(x) = 0$.

Definition A critical point of f is a number x such that $f'(x) = 0$.

The number $f(x)$ itself is called a critical value of f .

Problem: Find max or min of f on closed interval $[a, b]$.

closed interval, continuous, therefore f has max and min values on interval

if x is a max or min point, then x must be in one of the following three classes of points

- 1) critical points of f in (a, b)
- 2) end points a and b
- 3) points in $[a, b]$ where f is not diff.

Solution

Find critical points of f in (a, b)

Find points in $[a, b]$ where f not diff

compute value of f at boundary points and endpoints

check largest and smallest values

Situations where the procedure above is not guaranteed to work

f not continuous

open interval / hole red line

In such cases, need to make other observations about the f 's.

Theorem 3 (Rolle's Theorem)

f cont. on $[a, b]$ and diff on (a, b) \rightarrow $\exists x, x \in (a, b), f'(x) = 0$
 $f(a) = f(b)$

Proof

since f is cont. on $[a, b]$, it has a max and a min value on $[a, b]$.

If the max value occurs at $x \in (a, b)$, then by Th. 1, since f diff at x , $f'(x) = 0$.

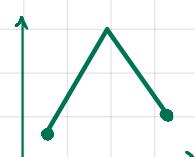
Something if the min occurs at $x \in (a, b)$.

I.e., it either the max or the min value occurs inside (a, b) , f' is zero at one of those points.

The other possible case is that both max and min occur at the endpoints.

But then the f is constant, so $f'(x) = 0$ at any $x \in [a, b]$.

Th. 3 required diff at one x in (a, b) ,
o.w. Th. 3 does not apply



Theorem 4 (Mean Value Theorem)

$$f \text{ cont. on } [a, b], \text{ diff. on } (a, b) \rightarrow \exists x, x \in (a, b) \wedge f'(x) = \frac{f(b) - f(a)}{b - a}$$

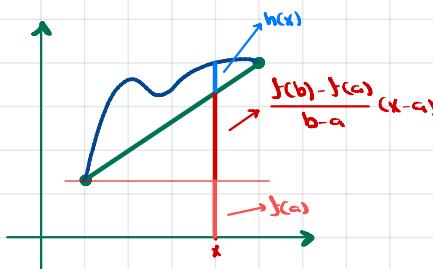
Proof

$$\text{Let } h(x) = f(x) - \frac{f(b) - f(a)}{b - a} (x - a) - f(a)$$

Then h is cont on $[a, b]$ and diff on (a, b) , and

$$h(a) = f(a)$$

$$h(b) = f(b)$$



Apply Rolle's Theorem to h , conclude that $\exists x, x \in (a, b) \wedge h'(x) = \frac{h(b) - h(a)}{b - a} = 0$

$$\text{But } h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

So the application of Rolle yields $f'(x) = \frac{f(b) - f(a)}{b - a}$

Corollary I

f defined on interval $\rightarrow f$ constant on the interval
 $f'(x) = 0$ for all x in such interval

Proof

Take any two points a and b in the interval.

$$\text{By NNT, } \exists x, x \in (a, b) \text{ and } f'(x) = \frac{f(b) - f(a)}{b - a} = 0$$

$$\rightarrow f(a) = f(b).$$

Corollary II

f, g defined on same interval $\rightarrow \exists c, f = g + c$
 $f'(x) = g'(x)$ for all x in the interval

Proof

$$(f - g)'(x) = f'(x) - g'(x) = 0$$

By corollary I, $f - g$ is constant on the interval, i.e. $f(x) - g(x) = c \rightarrow f(x) = g(x) + c$.

Definition

Category 3

$f'(x) > 0$ for all x in an interval, then f increasing on interval
 $f'(x) < 0$ " " " " " decreasing " "

Root

Consider an interval in which $f'(x) > 0$ for all x .

Let a and b be two points in this interval.

$$\text{MVT} \rightarrow \exists c \in (a,b) \wedge f'(c) = \frac{f(b)-f(a)}{b-a} > 0 \rightarrow f(b)-f(a) > 0 \rightarrow f(b) > f(a)$$

Poly nomials

prob 3-7 \rightarrow $f(x) = a_n x^n + \dots + a_1 x + a_0$ has at most n roots. (1)

We proved this result using algebra. We can now prove it using the methods of calculus.

For any two roots of t , x_1 and x_2 , we have $f(x_1) - f(x_2) = 0$.

By Rolle, there is an x between x_1 and x_2 s.t. $f'(x) = 0$.

Therefore if f has k roots then f' has at least $k-1$ roots.

Use induction to prove (ii)

$$n=1 \rightarrow f(x) = a_1 x + a_0 \rightarrow \text{one root } x = -\frac{a_0}{a_1}.$$

Assume $\deg f(x) = n = h$. $f(x) = a_n x^n + \dots + a_1 x + a_0$ has at most h roots.

Assume $g(x) = b_{n+1}x^{n+1} + b_nx^n + \dots + b_1x + b_0$ has more than $n+1$ roots. Then $g'(x) - f(x)$ has at least $n+1$ roots. \perp .

Therefore g has at most $k+1$ roots.

By induction, (1) is true for all $n \in \mathbb{N}$.

Graph Sketching

Steps

- 1) critical points
- 2) value of f' at critical points
- 3) sign of f' in regions between critical points
- 4) $f'(c) = 0$
- 5) $f(c)$ becomes large or large negative

bonus: check odd/even

if f undefined at some points, check behavior of f near such points

Theorem 5 Suppose $f'(c) = 0$. If $f''(c) > 0$, then f has local min at c ; if $f''(c) < 0$ then f has local max at c .

Proof

$$\text{By definition, } f''(c) = \lim_{h \rightarrow 0} \frac{f'(c+h) - f'(c)}{h}$$
$$= \lim_{h \rightarrow 0} \frac{f'(c+h)}$$

Assume $f''(c) > 0$.

For small $|h|$,

$h > 0 \rightarrow f'(c+h) > f'(c) \rightarrow f$ increasing right of c

$h < 0 \rightarrow f'(c+h) < f'(c) \rightarrow f$ decreasing left of c

$\rightarrow c$ is local minimum

Problematic Cases

- 1) f'' can be $\pm\infty$, complicated bc certain bns
- 2) $f''(c) = 0$: c could be local max, min, or neither

Theorem 6 Suppose $f''(c)$ exists. If f has local min at c , then $f''(c) \geq 0$; if f has local max at c , then $f''(c) \leq 0$.

note this is the partial converse of Th5.

Proof

Suppose f has local min at c .

If $f''(c) < 0$ then c is local max, thus f is constant, f' constant, $f'' = 0$. \perp .

Therefore $f''(c) \geq 0$.

Th5 says that if $f'(c) = 0$ and $f''(c) > 0$, then c is definitely a local min. Th5 does not say anything about the $f''(c) = 0$ possibility, in which c could be max, min, or neither.

Th6 says that if c is a local min then either $f''(c) < 0$ or $f''(c) = 0$.

Theorem 7 Suppose f is cont. at a , $f'(x)$ exists for all x in some interval containing a , except perhaps at $x=a$.
Suppose, moreover, that $\lim_{x \rightarrow a} f'(x)$ exists.

Then, $f'(a)$ exists and $f'(a) = \lim_{x \rightarrow a} f'(x)$.

Proof

$$\text{By def. } f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

For sufficiently small $h > 0$, f is cont. on $[a, a+h]$ and diff. on $(a, a+h)$, and the intermediate value theorem holds for small $h < 0$.

$$\text{N.V.T. } \Rightarrow \exists x_h, x_h \in (a, a+h) \text{ such that } \frac{f(a+h) - f(a)}{h} = f'(x_h)$$

x_h approaches a as h approaches 0, i.e. $\forall \epsilon > 0 \exists \delta, 0 < \delta < \epsilon$ s.t. $|h| < \delta \Rightarrow |x_h - a| < \delta$ and $|a+h - a| = |h| < \delta \leq \epsilon$

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} f'(x_h) = \lim_{x \rightarrow a} f'(x)$$

Proof of the last step

$$\lim_{h \rightarrow 0} f'(x_h) = l \text{ means}$$

$$\forall \epsilon > 0 \exists \delta > 0 \forall h \quad |h| < \delta \Rightarrow |f'(x_h) - l| < \epsilon$$

$$\text{Assume } |x_h - a| < \delta.$$

$$\text{Then } |x_h - a| < |a+h - a| = |h| < \delta$$

$$\text{Then } |f'(x_h) - l| < \epsilon$$

$$\text{T.F. } \forall x \quad |x - a| < \delta \Rightarrow |f'(x) - l| < \epsilon$$

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \quad |x - a| < \delta \Rightarrow |f'(x) - l| < \epsilon$$

$$\Rightarrow \lim_{x \rightarrow a} f'(x) = l$$

Theorem 7 (Cauchy Mean Value Theorem)

f, g cont. on $[a, b]$ and diff on $(a, b) \rightarrow \exists x, t \in (a, b) \wedge [f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(t)$

If $g(b) \neq g(a)$ and $g'(t)$ to be non-zero then

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(t)}{g'(t)}$$

Note

If $g(t) = k$ above, then we obtain the MVT.

If we apply MVT to f and g separately,

$$f'(t) = \frac{f(b) - f(a)}{b - a}$$

$$g'(t) = \frac{g(b) - g(a)}{b - a}$$

$$\frac{f'(x)}{g'(t)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof

$$\text{Let } h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$$

Then h is cont. on $[a, b]$, diff on (a, b) and

$$\begin{aligned} h(a) &= f(a)[g(b) - g(a)] - g(a)[f(b) - f(a)] \\ &= f(a)g(b) - g(a)f(b) = h(b) \end{aligned}$$

Apply Rolle's Theorem

$$\exists x, t \in (a, b) \wedge h'(x) = 0$$

$$\rightarrow f'(x)[g(b) - g(a)] - g'(x)[f(b) - f(a)] = 0$$

$$\rightarrow \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(x)}$$

Theorem 9 (L'Hôpital's Rule)

$$\lim_{x \rightarrow a} f(x) = 0 \text{ and } \lim_{x \rightarrow a} g(x) = 0$$

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \text{ exists.}$$

$$\rightarrow \lim_{x \rightarrow a} \frac{f(x)}{g(x)} \text{ exists}$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Proof

The hypothesis that $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists contains two implicit assumptions

(1) $\exists \delta > 0, \forall x, x \in (a-\delta, a) \cup (a, a+\delta) \rightarrow f'(x) \text{ and } g'(x) \text{ exist.}$

(2) in $(a-\delta, a+\delta)$, $g'(x) \neq 0$ except perhaps at $x=a$

Remember that $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = l$ means $\forall \epsilon > 0 \exists \delta > 0 \forall x, 0 < |x-a| < \delta \rightarrow \left| \frac{f'(x)}{g'(x)} - l \right| < \epsilon$

T.F. $\frac{f'(x)}{g'(x)}$ is defined in $(a-\delta) \cup (a+\delta)$, so $f'(x)$ and $g'(x)$ both exist and $g'(x) \neq 0$.

f and g , however, are not assumed to be defined at a . If we define $f(a) = g(a) = 0$ then f and g are cont. at a .

Then the assumptions of MVT and Cauchy MVT are met by f and g on $[a, x]$, where $a < x < a+\delta$.

$$\text{MVT} \rightarrow \exists c, c \in (a, x) \wedge g'(c) = \frac{g(x) - g(a)}{x - a}$$

If $g'(x) = 0$ then $g'(c) = 0$, contradicting (2). T.F. $g'(x) \neq 0$

Cauchy MVT $\rightarrow \exists d_x, d_x \in (a, x)$ s.t.

$$[f(x) - f(a)] g'(d_x) = [g(x) - g(a)] f'(d_x)$$

$$\rightarrow \frac{f(x)}{g(x)} = \frac{f'(d_x)}{g'(d_x)} \quad (1) \quad *At this point we consider a and d_x s.t.$$

$a < d_x < x < a+\delta$, but we want to know what d_x approaches a as x approaches a , i.e. happens to the fraction in (1) all around a , so we need to go through the same

$$|d_x - a| < |x - a|$$

$$\text{so } 0 < |x - a| < \delta \rightarrow 0 < |d_x - a| < \delta$$

calculation for
 $a-\delta < x < a+\delta$

consider $[x, a] \cup [a-\delta, x]$.

Applic. of MVT stays the same.

Cauchy MVT becomes

$\exists d_x, d_x \in (x, a)$ s.t.

$$[(x) - f(a)] g(d_x) = [g(x) - g(a)] f'(d_x)$$

and since the $-$ sign is on both sides we have the same result as before, namely,

$$\frac{f(x)}{g(x)} = \frac{f'(d_x)}{g'(d_x)}$$

Take the limit as $x \rightarrow a$ on both sides of (1)

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(d_x)}{g'(d_x)}$$

But $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists by assumption, and it means

$$\forall \epsilon > 0 \exists \delta > 0 \forall x, |x - a| < \delta \rightarrow \left| \frac{f'(x)}{g'(x)} - l \right| < \epsilon$$

$$\text{But } |x - a| < \delta \rightarrow |d_x - a| < \delta \rightarrow \left| \frac{f'(d_x)}{g'(d_x)} - l \right| < \epsilon$$

T.F.

$$\forall \epsilon > 0 \exists \delta > 0 \forall x, |x - a| < \delta \rightarrow \left| \frac{f'(x)}{g'(x)} - l \right| < \epsilon \rightarrow \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(d_x)}{g'(d_x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$