Ch6 - Continuous Functions

1.

i) $f(x) = \dfrac{x^2-4}{x-2} = \dfrac{(x+2)(x-2)}{(x-2)} = x+2$

$\lim_{x\to 2} f(x) = 4$.

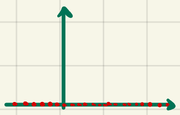
$F(x) = \begin{cases} f(x) & x \neq 2 \\ 4 & x = 2 \end{cases} \longrightarrow \forall a \ \lim_{x\to a} F(x) = F(a)$

ii) $f(x) = \dfrac{|x|}{x} = \begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases}$

$\nexists \lim_{x\to 0} f(x)$

$\lim_{x\to 0^+} f(x) = 1$
$\lim_{x\to 0^-} f(x) = -1$
$f(0)$ not defined.

iii) $f(x) = 0$ if x irrational



$\nexists \lim_{x\to a} f(x)$, for any a in the domain

$\exists \epsilon>0 \ \forall \delta>0 \ \exists x \ 0<|x-a|<\delta \wedge \neg(|f(x)-\ell|<\epsilon)$ because $f(x)$ is not defined.

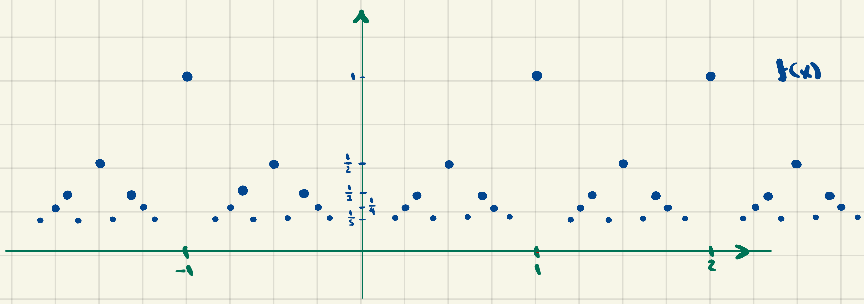
If we define $F(x) = \begin{cases} f(x) & x \text{ irrational} \\ 0 & x \text{ rational} \end{cases}$

Then $\forall a$, $\lim_{x\to a} F(x) = 0 = F(a)$, so $F(x)$ is continuous in $\mathbb{R}$.

iv) $f(x) = \dfrac{1}{q}$, $x = \dfrac{p}{q}$ rational in lowest terms

$\nexists \lim_{x\to a} f(x)$ because f is not defined at irrational numbers.

If we define $F(x)$ as 0 at irrational x and $f(x)$ at rational x, limits exist at any a and equal zero, but at rational points in the domain, F is still not continuous.

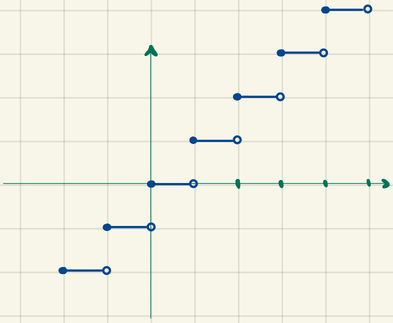


2.

From 4-17

$[x]$ = largest $z \leq x$

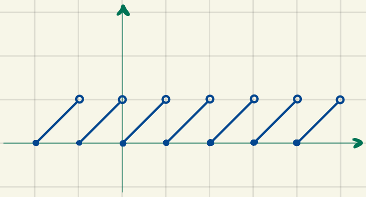
i) $f(x) = [x]$



discontinuous at integers.

continuous for $x \in \mathbb{R} \wedge x \notin \mathbb{Z}$

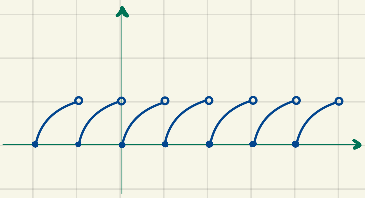
ii) $f(x) = x - [x]$



discontinuous at integers.

continuous for $x \in \mathbb{R} \wedge x \notin \mathbb{Z}$

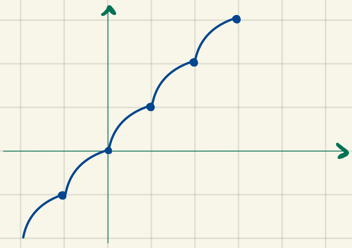
iii) $f(x) = \sqrt{x - [x]}$



discontinuous at integers.

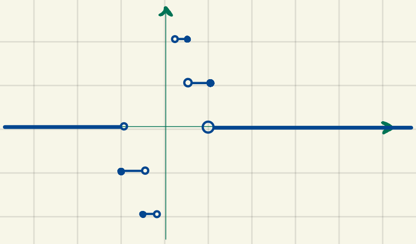
continuous for $x \in \mathbb{R} \wedge x \notin \mathbb{Z}$

iv) $f(x) = [x] + \sqrt{x - [x]}$



continuous in $\mathbb{R}$

v) $f(x) = \left[\frac{1}{x}\right]$



$(\frac{1}{2},1),(\frac{1}{3},\frac{1}{2}),\dots$

continuous in

$(1,+\infty) \cup (-\infty,-1) \cup \{x \in \mathbb{R} : \frac{1}{p} < x < \frac{1}{q},\ p,q \in \mathbb{N}-\{0\},\ p = q+1\}$

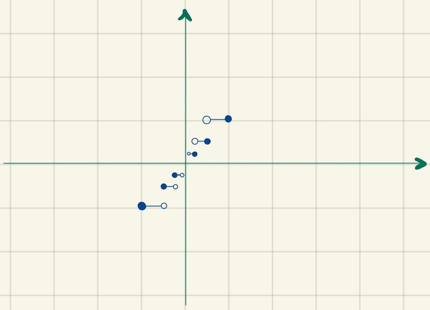
$\cup \{x \in \mathbb{R} : -\frac{1}{p} < x < -\frac{1}{q},\ p,q \in \mathbb{N}-\{0\},\ q = p+1\}$

$(-1,-\frac{1}{2}),(-\frac{1}{2},-\frac{1}{3}),\dots$

discontinuous at $x = 0$, $x = \frac{1}{p}$, $p \in \mathbb{Z}-\{0\}$

in other words continuous at $\{x \in \mathbb{R} : x \neq 0$ and $x \neq \frac{1}{p},\ p \in \mathbb{Z}-\{0\}\}$

vi) $f(x) = \dfrac{1}{\left[\frac{1}{x}\right]}$

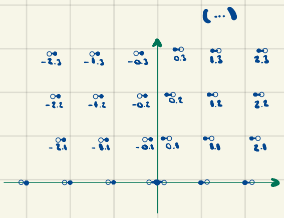


discontinuous at $x = 0$ and $x = \frac{1}{p}$, $p \in \mathbb{Z}-\{0\}$

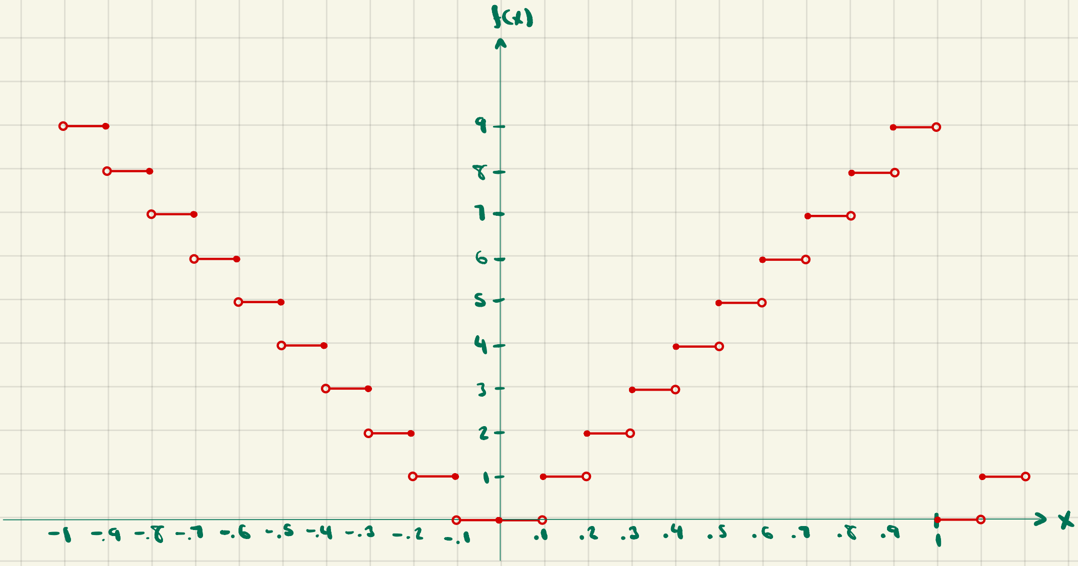
continuous at $\{x \in \mathbb{R} : 0 < |x| < 1$ and $x \neq \frac{1}{p},\ p \in \mathbb{Z}-\{0\}\}$

Fiom 4-19

i) $f(x)$ = 1st number of decimal expansion of x



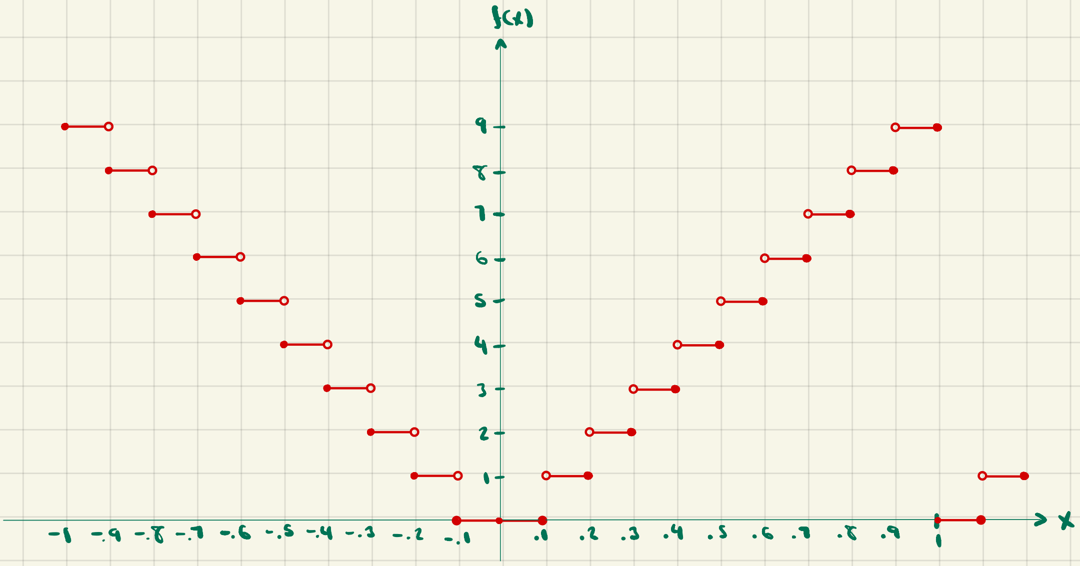
continuous at $\{x \in \mathbb{R} : x \neq 0.1n, n \in \mathbb{Z} - \{0\}\}$



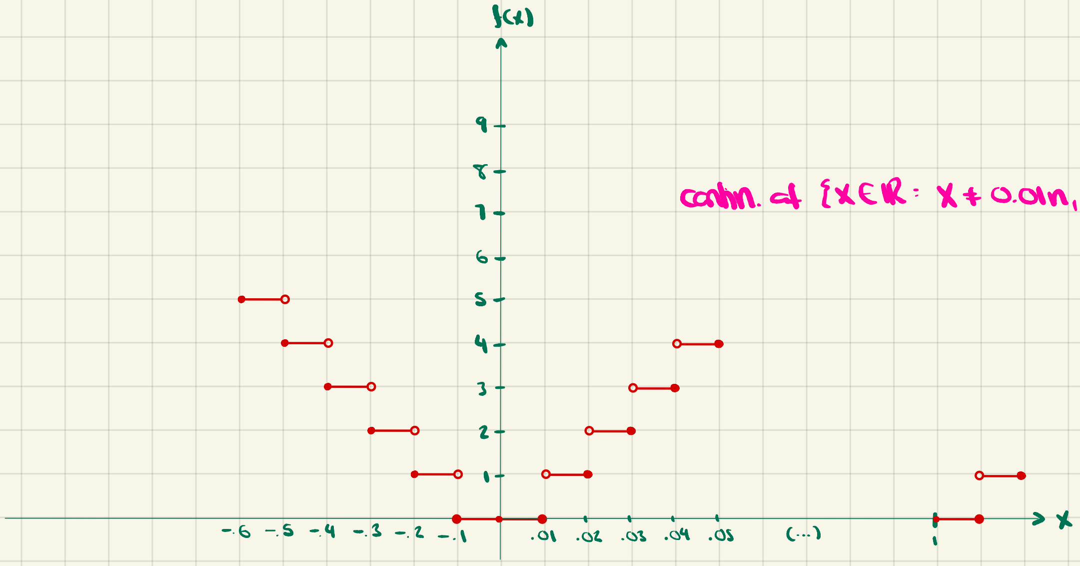
However, we are considering that, e.g., $0.0\bar{9} = 1$.

$\Rightarrow f(0.1) = f(0.0\bar{9}) = 0$

$f(0.2) = f(0.1\bar{9}) = 1$



ii) $f(x)$ = 2nd number in decimal expansion of x



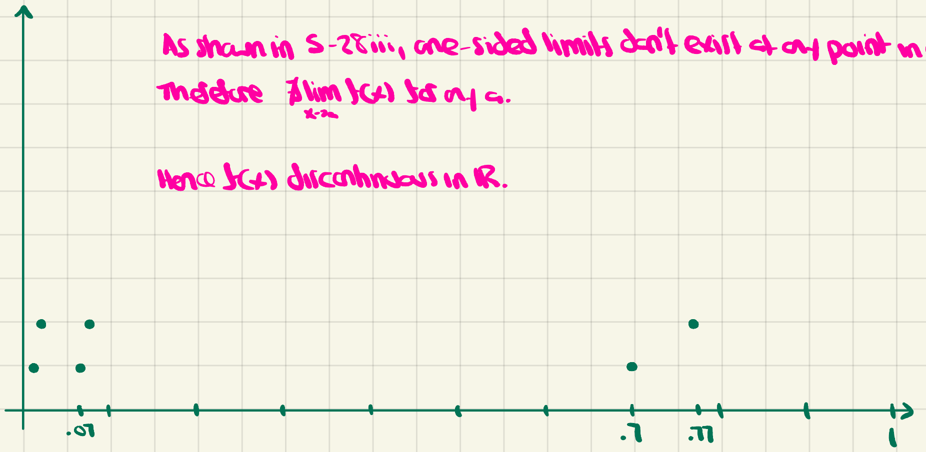
contin. at $\{x \in \mathbb{R} : x \neq 0.01n, n \in \mathbb{Z} - \{0\}\}$

iii) $f(x)$ = number of 7's in decimal expansion of x if this number is finite,
0 otherwise

Infinite 7's
irrational numbers w/ infinite 7's
rational w/ repeating decimal repres. e.g. $0.\overline{7}$, $0.\overline{74}$, $0.\overline{347}$

Finite 7's

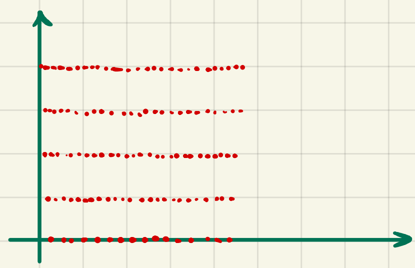
0.7	0.77
0.07	0.077
0.007	0.0077



As shown in S-28iii, one-sided limits don't exist at any point in domain.
Therefore $\nexists \lim_{x\to a} f(x)$ for any a.

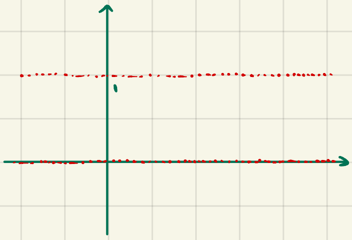
Hence $f(x)$ discontinuous in $\mathbb{R}$.

In any interval there are infinite numbers w/ any number of 7's, from zero to ∞.



$.\overline{72} \Rightarrow 10^2 \cdot .\overline{72} = 72 + .\overline{72} \Rightarrow 72 = .\overline{72}(10^2 - 1)$
$\Rightarrow .\overline{72} = \frac{72}{99}$

iv) $f(x)$ = 0 if number of 7's in dec. exp. of x is finite, 1 otherwise.

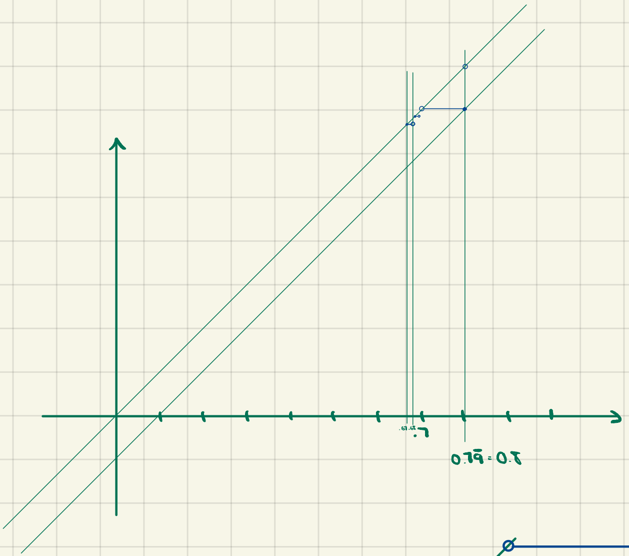


Discontinuous in $\mathbb{R}$

v) $f(x)$ = number obtained by replacing all digits in decimal expansion of x which come after first 7, if any, by 0.

$f(0.77) = 0.7$
$f(0.07) = 0.07$
$f(0.\overline{72}) = 0.7$
$f(0.7\bar{9}) = 0.7$

$0.0\bar{9}$ is max subtraction.



Given any number of form m.abcd7efg...

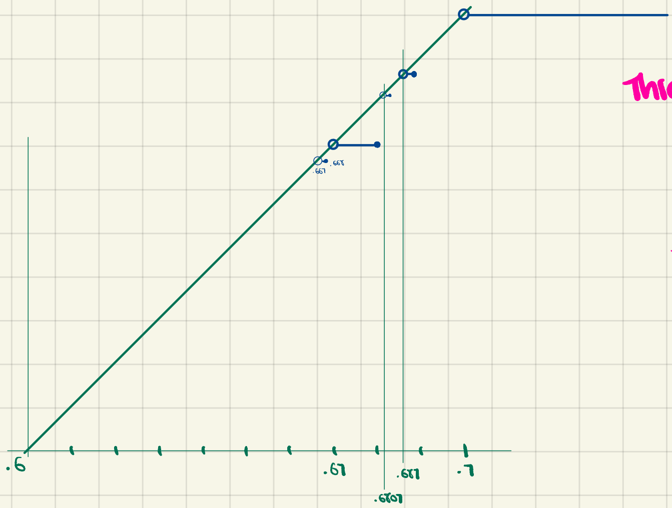
$\lim_{x \to a^+} f(x)$ exists for any $x \in$ (m.abcd7, m.abcd8]

$\lim_{x \to a^-} f(x)$ exists for any $x \in$ [m.abcd7, m.abcd8)

The numbers m.abcd7 and m.abcd8 are actually m.abcd6$\bar{9}$ and m.abcd7$\bar{9}$, respectively. They have left and right-sided limits, respectively.

For any number a not containing any 7s in decimal expansion, both one-sided limits exist and equal a.

Therefore both one-sided limits exist for all x.



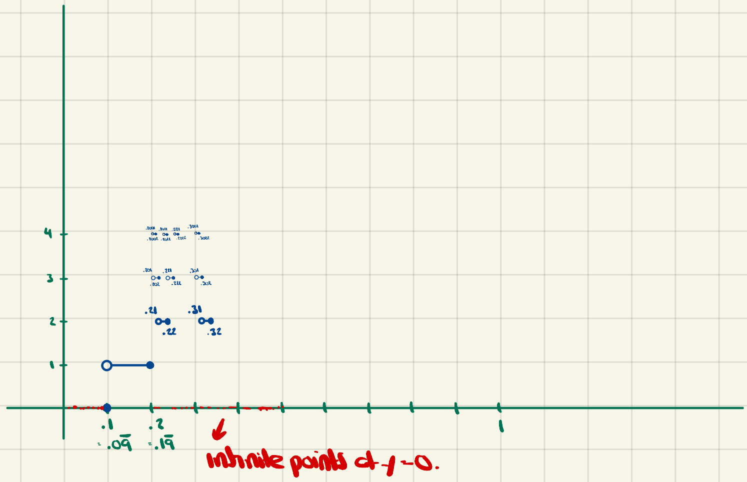
Three cases

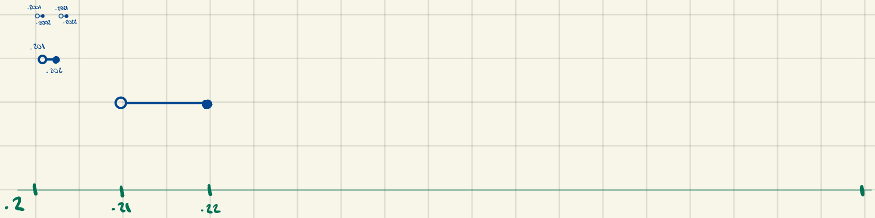
i) e.g. $x \in$ (m.abcd7, m.abcd8) : continuous

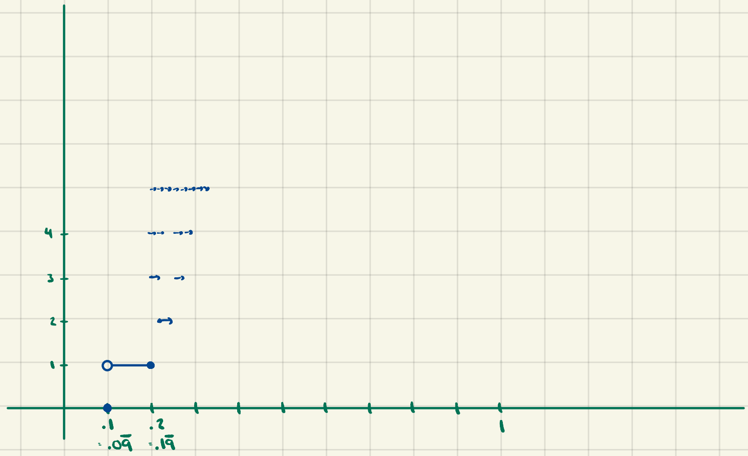
ii) x does not contain 7s in dec. expansion

F is composed of infinite intervals like the ones above, with points on the line $y = x$ for numbers x with no 7s in them.

vi) $f(x) = $ 0 if 1 never appears in decimal expansion of x
n if 1 first appears in n^{th} place







The graph of f(x) is composed of infinite intervals as above. They are of form (m.abc1, m.abc2]. Between two such intervals are infinite other intervals with f(x) at every number from 0 to ∞.

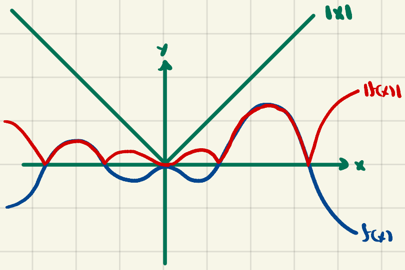
Therefore the one-sided limits exist for every point in each interval, and the left open side has only the $\lim_{x\to a^+} f(x)$ and the right closed side has $\lim_{x\to a^-} f(x)$.

The limits do not exist for points w/ no 1s, ie the ones on the y=0 line.

continuous in (m.abcd1, m.abcd2), ie in the interiors of each interval.

3.

a) $|f(x)| \le |x| \ \forall x \ \rightarrow f$ cont. at 0



$\forall x \ 0 < |f(x)| \le |x|$

$\rightarrow \ 0 < \lim_{x \to a} |f(x)| < \lim_{x \to a} |x|$

In particular, $0 < \lim_{x \to 0} |f(x)| < \lim_{x \to 0} |x| = 0$

$\rightarrow \lim_{x \to 0} |f(x)| = 0 \ \rightarrow \forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \ 0 < |x| < \delta \rightarrow ||f(x)|| < \epsilon \ \rightarrow |f(x)| < \epsilon$

$\rightarrow \lim_{x \to 0} f(x) = 0$

Also $0 < |f(0)| < |0| = 0 \ \rightarrow |f(0)| = 0 \rightarrow f(0) = 0$

Therefore $\lim_{x \to 0} f(x) = f(0) = 0$

$\rightarrow f$ cont. at $x = 0$.

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b)

$f(x) = \begin{cases} 0 & x \text{ rational} \\ x & x \text{ irrational} \end{cases}$

$\lim_{x \to 0} f(x) = f(0) = 0 \ \rightarrow f$ cont. at 0

$\nexists \lim_{x \to a} f(x)$ for $a \neq 0$. $\rightarrow f$ not cont. for $a \neq 0$.

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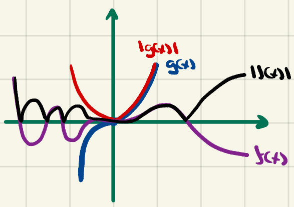
c) g cont. at 0
$g(0) = 0$ $\rightarrow f$ cont. at 0
$|f(x)| \le |g(x)|$



$\forall x \ 0 \le |f(x)| \le |g(x)| \ \rightarrow \ 0 \le \lim_{x \to 0} |f(x)| \le \lim_{x \to 0} |g(x)| = 0$

$\rightarrow \lim_{x \to 0} |f(x)| = 0$

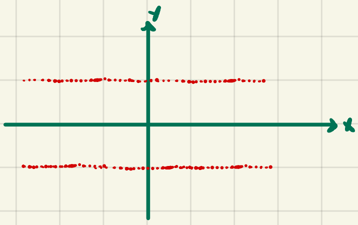
$\rightarrow \lim_{x \to 0} f(x) = 0$

Also, $0 \le |f(0)| \le |g(0)| = 0 \ \rightarrow |f(0)| = 0 \rightarrow f(0) = 0$

$\rightarrow \lim_{x \to 0} f(x) = f(0) = 0$

$\rightarrow f$ cont. at 0.

4. f continuous nowhere. $|f|$ continuous everywhere.

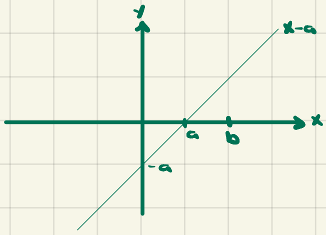


$$f(x) = \begin{cases} 1 & x \text{ rational} \\ -1 & x \text{ irrational} \end{cases} \quad \rightarrow \quad \forall a \in \mathbb{R} \ \nexists \lim_{x \to a} f(x)$$

$|f(x)| = 1 \quad x \in \mathbb{R}$

$\forall a \in \mathbb{R} \ \lim |f(x)| = 1 = f(a) \quad \rightarrow |f| \text{ cont. in } \mathbb{R}$

5. f cont. at $x = a$, not cont. at $x \neq a$.



$$f(x) = \begin{cases} 0 & x \text{ rational} \\ x-a & x \end{cases}$$

$\forall \varepsilon > 0 \quad |x-a| < \varepsilon \rightarrow |f(x)| < \varepsilon$

$\rightarrow f$ cont. at a.

But for any $b \neq a$, either b rational or b irrational

Case 1: b rational

$\rightarrow f(b) = 0$

choose $\varepsilon < |b-a|/2$

$\forall \delta > 0 \ \exists x \ |x-b| < \min(\delta, \varepsilon) \wedge x$ irrational

$\forall \delta > 0 \ \exists x \ |x-b| < \min(\delta, \varepsilon) < \delta \quad \wedge f(x) = x-a$

$(x-a) + (b-x) = b-a$

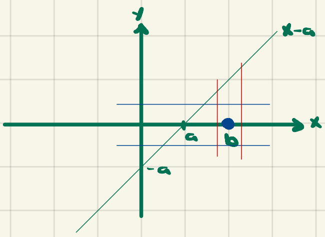
$\rightarrow x-a = (b-a) - (b-x) > (b-a) - \dfrac{|b-a|}{2}$

if $b > a$ then $x-a > \dfrac{b-a}{2} = \dfrac{|b-a|}{2}$

if $b < a$ then $x-a > \dfrac{a-b}{2} = \dfrac{|b-a|}{2}$

either way, $\forall \delta > 0 \ \exists x \ |x-b| < \delta \wedge f(x) > \dfrac{|b-a|}{2} = \varepsilon$.

$\rightarrow \lim_{x \to b} f(x) \neq 0 = f(b) \quad \rightarrow f$ not cont. at b



Case 2: b irrational

$\rightarrow f(b) = b-a$

let $\varepsilon = \dfrac{|b-a|}{2}$.

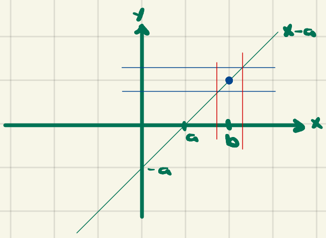
$\forall \delta > 0 \ \exists x \ |x-b| < \min(\varepsilon, \delta) < \delta \wedge x$ rational

$\forall \delta > 0 \ \exists x \ |x-b| < \min(\varepsilon, \delta) < \delta \wedge f(x) = 0$.

$\rightarrow f(x) - (b-a) = -(b-a)$

$\rightarrow |f(x) - f(b)| = |b-a| > \varepsilon$

$\rightarrow \lim_{x \to b} f(x) \neq f(b) \quad \rightarrow f$ not cont. at b.

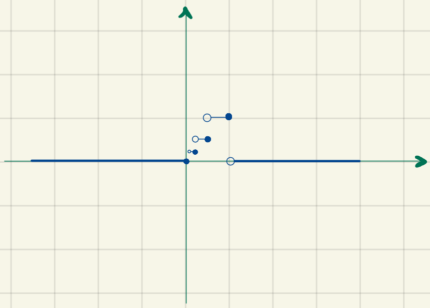


6.

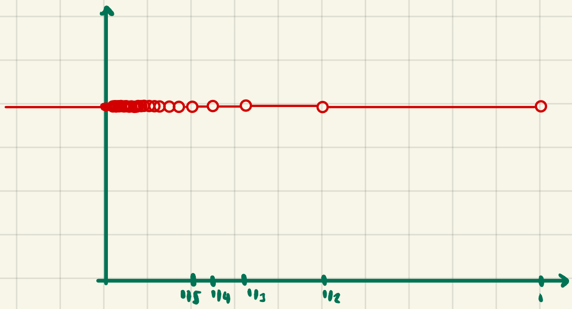
a) f discont. at $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

cont. everywhere else

$$f(x) = \begin{cases} 0 & x \le 0,\ x > 1 \\ \dfrac{1}{\left[\frac{1}{x}\right]} & 0 < x \le 1 \end{cases}$$



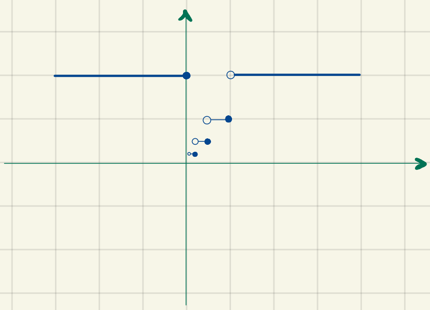
$f(x) = 1 \quad x \neq \frac{1}{n} \quad n \in \mathbb{N}$



b) f discont. at $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ and 0

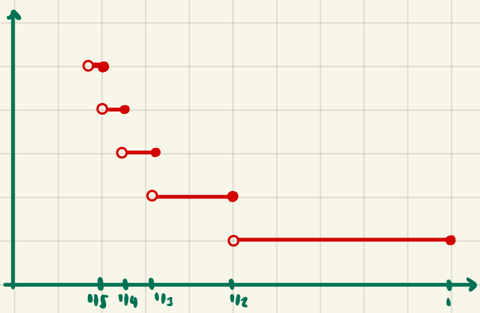
cont. everywhere else

$$f(x) = \begin{cases} 2 & x \le 0,\ x > 1 \\ \dfrac{1}{\left[\frac{1}{x}\right]} & 0 < x \le 1 \end{cases}$$



$f(x) = \left[\frac{1}{x}\right]$

f isn't defined at $x = 0$ therefore it isn't cont. there.



7. $f(x+y) = f(x) + f(y)$ → $\forall a$ f cont. at a

f cont. at 0

$f(0) = 2f(0) \rightarrow f(0) = 0$

$\lim\limits_{x \to a} f(x) - f(a)$

$\forall \epsilon > 0\ \exists \delta > 0\ \forall x\ |x-a| < \delta \rightarrow |f(x) - \ell| < \epsilon$

let $x = a+h$. Then $h = x - a$

$\forall \epsilon > 0\ \exists \delta > 0\ \forall h\ |h| < \delta \rightarrow |f(a+h) - \ell| < \epsilon$

$\leftrightarrow \lim\limits_{h \to 0} f(a+h) = \ell$

$\lim\limits_{h \to 0} f(a+h) - f(a)$

$= \lim\limits_{h \to 0} [f(a) + f(h)] - f(a)$

$= f(a) + \lim\limits_{h \to 0} f(h) - f(a)$

$= 0$

$\rightarrow \lim\limits_{h \to 0} f(a+h) = f(a)$

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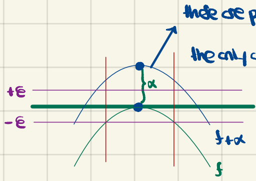
8. f cont. at a

$f(a) = 0$ → $f + \alpha$ nonzero in some open

$\alpha \neq 0$ interval containing a



there are points around $f(a) + \alpha$ that are nonzero.

the only way this wouldn't be true is if the points around

were at least α smaller than $f(a) + \alpha$. But then

f wouldn't be continuous at a.

$\lim\limits_{x \to a} f(x) = f(a) = 0 \quad \rightarrow \forall \epsilon > 0\ \exists \delta > 0\ \forall x\ |x-a| < \delta \rightarrow |f(x)| < \epsilon$

choose an ϵ such that $0 < \epsilon < \alpha$, and let $\delta > 0$ be such that $\forall x\ |x-a| < \delta \rightarrow |f(x)| < \epsilon$

Let x be one such x.

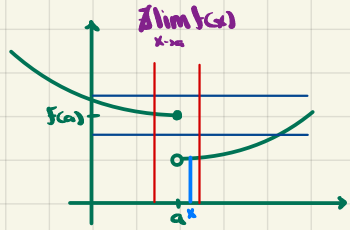
$|x-a| < \delta \rightarrow |f(x)| < \epsilon$

$a - \delta < x < a + \delta \rightarrow 0 < \alpha - \epsilon < \alpha + f(x) < \alpha + \epsilon$

Therefore $\forall x\ |x-a| < \delta \rightarrow \alpha + f(x) > 0$

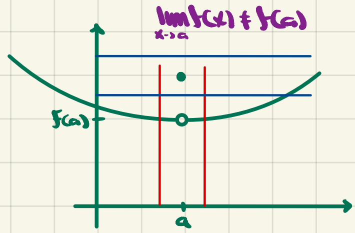
$\forall \alpha > 0\ \exists \epsilon\ 0 < \epsilon < \alpha\ \exists \delta > 0\ \forall x\ |x-a| < \delta \rightarrow \alpha + f(x) > 0$

9. a)



f defined at a, not continuous at a $\rightarrow$ $\exists \varepsilon>0\ \forall \delta>0\ \exists x\ 0<|x-a|<\delta \wedge |f(x)-f(a)|>\varepsilon$

$\lim_{x\to a} f(x) \neq f(a)$



Case 1: $\nexists \lim_{x\to a} f(x) \leftrightarrow \forall l\ \exists \varepsilon>0\ \forall \delta>0\ \exists x\ 0<|x-a|<\delta \wedge |f(x)-l|>\varepsilon$

$l=f(a) \rightarrow \exists \varepsilon>0\ \forall \delta>0\ \exists x\ 0<|x-a|<\delta \wedge |f(x)-f(a)|>\varepsilon$

Case 2: $\exists \lim_{x\to a} f(x)$ and $\lim_{x\to a} f(x) = l \neq f(a)$

$\forall \varepsilon>0\ \exists \delta>0\ \forall x\ 0<|x-a|<\delta \rightarrow |f(x)-l|<\varepsilon$

Let $\varepsilon = \frac{|l-f(a)|}{2}$. Then, $\exists \delta>0\ \forall x\ 0<|x-a|<\delta \rightarrow |f(x)-l| < \frac{|l-f(a)|}{2}$

$\rightarrow -\frac{|l-f(a)|}{2} < f(x)-l < \frac{|l-f(a)|}{2}$

Case 2.1 $l>f(a) \rightarrow \frac{f(a)-l}{2} < f(x)-l < \frac{l-f(a)}{2}$

$\rightarrow \frac{l+f(a)}{2} < f(x) < \frac{3l-f(a)}{2}$

$\rightarrow \varepsilon = \frac{l-f(a)}{2} < f(x)-f(a) < \frac{3}{2}(l-f(a))$

$\rightarrow |f(x)-f(a)| > \varepsilon$

Case 2.2 $f(a)>l \rightarrow \frac{l-f(a)}{2} < f(x)-l < \frac{f(a)-l}{2}$

$\rightarrow \frac{3l-f(a)}{2} < f(x) < \frac{f(a)+l}{2}$

$\rightarrow \frac{3(l-f(a))}{2} < f(x)-f(a) < \frac{l-f(a)}{2} < 0$

$\rightarrow \frac{f(a)-l}{2} < f(a)-f(x) < \frac{3(f(a)-l)}{2}$

$\rightarrow |f(x)-f(a)| > \frac{|l-f(a)|}{2} = \varepsilon$

Therefore,

$\exists \varepsilon>0\ \exists \delta>0\ \forall x\ 0<|x-a|<\delta \rightarrow |f(x)-f(a)|>\varepsilon$

$\rightarrow \exists \varepsilon\ \forall \delta>0\ \exists x\ 0<|x-a|<\delta \wedge |f(x)-f(a)|>\varepsilon$

b) In all possible cases we proved

$\exists \varepsilon>0\ \forall \delta>0\ \exists x\ 0<|x-a|<\delta \wedge |f(x)-f(a)|>\varepsilon$

Therefore, for some ε

$|f(x)-f(a)| > \varepsilon$

$\rightarrow f(x)-f(a) > \varepsilon$ or $f(a)-f(x) > \varepsilon$

$\rightarrow f(x) > f(a)+\varepsilon$ or $f(x) < f(a)-\varepsilon$

10.

a) f cont at a $\to$ $|f|$ cont. at a

$$\lim_{x\to a} f(x) = f(a)$$

$$\forall \varepsilon>0\ \exists\delta>0\ \forall x\ |x-a|<\delta \to |f(x)-f(a)|<\varepsilon$$

$$\to |f(x)| - |f(a)| \le |f(x)-f(a)| < \varepsilon$$

From 1-12vi we know that $\forall x,y\ \ ||x|-|y|| \le |x-y|$

$$\to ||f(x)|-|f(a)|| < \varepsilon$$

Therefore

$$\forall \varepsilon>0\ \exists\delta>0\ \forall x\ |x-a|<\delta \to ||f(x)|-|f(a)|| < \varepsilon$$

$$\lim_{x\to a} |f(x)| = |f(a)|$$

$\to$ $|f|$ cont at a.

alternatively,

From problem 5-16a,

$$\lim_{x\to a} f(x) = \ell \ \longrightarrow\ \lim_{x\to a}|f(x)| = |\lim_{x\to a} f(x)| = |\ell|$$

Therefore,

$$f \text{ cont.} \leftrightarrow \lim_{x\to a} f(x) = f(a) \ \longrightarrow\ \lim_{x\to a}|f(x)| = |f(a)|$$

$\longrightarrow$ $|f|$ cont. at a

b) $\forall f$ cont. on $\mathbb{R}$ $\to$ $f = E+O$, E even and continuous
O odd and continuous

$$\forall a\ \forall\varepsilon>0\ \exists\delta>0\ \forall x\ |x-a|<\delta \to |f(x)-f(a)|<\varepsilon.$$

In problem 3-13 we showed that any fn f with domain $\mathbb{R}$ can be written $f = E+O$, with

$$E(x) = \frac{f(x)+f(-x)}{2}$$

$$O(x) = \frac{f(x)-f(-x)}{2}$$

Applying Theorem 1 we can assert that since $f(x)$ is continuous so are $E(x)$ and $O(x)$.

c) f, g continuous $\rightarrow$ $\max(f,g), \min(f,g)$ cont.

$$\max(f,g) = \frac{f+g+|f-g|}{2}$$

f and g are continuous $\rightarrow$ $\frac{f+g}{2}$ cont. by Th. 1

$g, -1$ cont $\rightarrow$ $-g$ cont. by Th. 1

$f, -g$ cont $\rightarrow$ $f-g$ cont. by Th. 1

$f-g$ cont $\rightarrow$ $|f-g|$ cont. a)

$\frac{f+g}{2}$ cont, $|f-g|$ cont. $\rightarrow$ $\frac{f+g+|f-g|}{2}$ cont. by Th. 1

$$\min(f,g) = \frac{f+g-|f-g|}{2}$$

$f, g, -|f-g|$ cont $\rightarrow$ $\min(f,g)$ cont by Th. 1

d) Every continuous fn f can be written $f = g - h$ where g and h are nonnegative and continuous.

In problem 3-15b we proved that any fn f can be written as $f = g-h$, where g and h are nonnegative, in infinitely many ways. The element of continuity is new here. However, we will now show that the same g and h are continuous if f is.

Let $g(x) = f(x) + |f(x)| + n$, $n \geq 0 \in \mathbb{R}$

For each n, $g(x)$ is a continuous fn in $\mathbb{R}$ since f, $|f|$, and n are cont. in $\mathbb{R}$.

f is thus: $f(x) = g(x) - |f(x)| - n$

Let $h(x) = |f(x)| + n \geq 0$

$h(x)$ is cont. in $\mathbb{R}$ since $|f|$ and n are cont. in $\mathbb{R}$.

Therefore we have

$f(x) = g(x) - h(x)$, $g(x), h(x) \geq 0$ and g and h cont.

---

alternatively,

Let $g(x) = \max(f,0) = \frac{f+|f|}{2}$. Then g is cont. by c) and nonneg.

Let $h(x) = -\min(f,0) = -\frac{f-|f|}{2} = \frac{|f|-f}{2}$, also cont. and nonneg.

$$g(x) - h(x) = \frac{f+|f|-|f|+f}{2} = f(x)$$

11.

Theorem 1 f, g continuous at a then

(1) $f+g$ continuous at a

(2) $f \cdot g$ continuous at a

$g(a) \neq 0$ then (3) $\frac{1}{g}$ continuous at a

Theorem 2: g cont. at a, f cont. at $g(a)$, then $f \circ g$ cont. at a.

Proof of Th 1 (iii)

Let $f(x) = \frac{1}{x}$. Note that by assumption f is cont in its domain, $\mathbb{R} - \{0\}$.

Let g be any fn such that $g(a) \neq$ and g is continuous at a.

Therefore, g cont. at a, f cont at $g(a) \neq 0$.

By Th 2, $f \circ g$ is cont. at a.

That is, $\frac{1}{g(x)}$ is cont. at a.

We have shown that

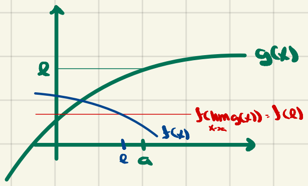
For any fn g continuous at a, if $g(a) \neq 0$ then $\frac{1}{g(x)}$ is cont. at a.

12.

a) f cont. at ℓ

$\lim\limits_{x \to a} g(x) = \ell \quad \longrightarrow \quad \lim\limits_{x \to a} f(g(x)) = f(\ell)$

When we get arbitrarily close to a, we get arbitrarily close to ℓ through g.
But since f is cont, as we get arbitrarily close to ℓ, we get arbitr. close to $f(\ell)$ through $f \circ g$.



We don't know if $g(a)$ exists or if g is cont. at a.
We can define a version of g that is cont at a:

$$G(x) = \begin{cases} g(x) & x \neq a \\ \ell & x = a \end{cases}$$

We have

G cont. at a

f cont. at $G(a)$

Therefore, by Th. 2, $f(G(x))$ cont. at a.

$\rightarrow \lim\limits_{x \to a} f(G(x)) = f(G(a)) = f(\ell)$

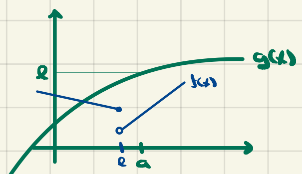
And this means

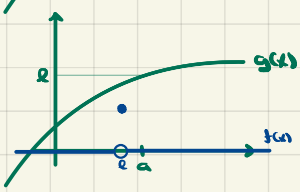
$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \; |x - a| < \delta \rightarrow |f(G(x)) - f(\ell)| < \varepsilon$

$\rightarrow \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \; 0 < |x - a| < \delta \rightarrow |f(g(x)) - f(\ell)| < \varepsilon.$

$\rightarrow \lim\limits_{x \to a} f(g(x)) = f(\ell) = f(\lim\limits_{x \to a} g(x))$

b) If f is not continuous at ℓ, then we can find an example in which $\lim\limits_{x \to a} f(g(x)) \neq f(\lim\limits_{x \to a} g(x))$





let $f(x) = \begin{cases} 0 & x \neq \ell \\ 1 & x = \ell \end{cases}$

Let $g(x) = \dfrac{\ell}{a} x$

$f(g(x)) = \begin{cases} 0 & x \neq a \\ 1 & x = a \end{cases}$

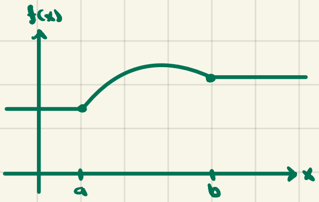
$\lim\limits_{x \to a} f(g(x)) = 0 \neq f(g(a)) = 1$

Therefore we have

f not cont. at ℓ, $\lim\limits_{x \to a} g(x) = \ell$, and $f(g(x))$ not cont. at a.

13.

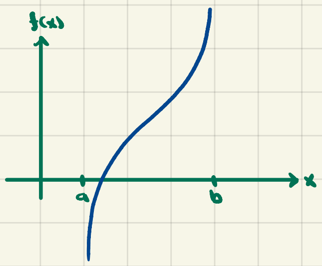
a) f cont. on $[a,b]$ $\rightarrow$ $\exists g$, g cont on $\mathbb{R}$ and $\forall x \in [a,b] \rightarrow g(x) = f(x)$



Define $g(x) = \begin{cases} f(x) & x \in [a,b] \\ a & x < a \\ b & x > b \end{cases}$

Then $g(x)$ cont on $\mathbb{R}$ and $\forall x \in [a,b] \rightarrow g(x) = f(x)$

b)



Let f be the fn depicted above: $\lim_{x \to b^-} f(x) = \infty$, $\lim_{x \to a^+} f(x) = -\infty$, f cont on (a,b).

Let g be a fn such that $\forall x$ $x \in (a,b) \rightarrow g(x) = f(x)$.

Assume g cont on $\mathbb{R}$.

$\lim_{x \to b^-} f(x) = \infty \rightarrow \lim_{x \to b^-} g(x) = \infty \rightarrow \nexists \lim_{x \to b} g(x) \rightarrow \lim_{x \to b} g(x) \neq g(b) \rightarrow g$ not cont on $\mathbb{R}$

$\perp$

Therefore g not cont on $\mathbb{R}$.

Therefore $\nexists g$ $(\forall x$ $x \in (a,b) \rightarrow g(x) = f(x)) \wedge g$ cont on $\mathbb{R}$

14.

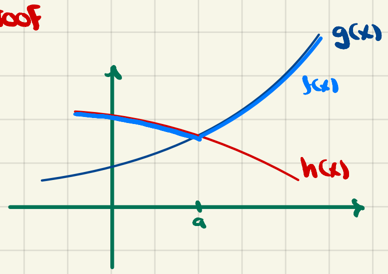
a)

g, h cont. at a

$g(a) = h(a)$ $\longrightarrow$ f cont at a

$$f(x) = \begin{cases} g(x) & x \geq a \\ h(x) & x < a \end{cases}$$

Proof



Let $\ell = g(a) = h(a)$.

$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \; |x-a| < \delta \to |g(x) - \ell| < \varepsilon$ and $|h(x) - \ell| < \varepsilon$

$|x-a| < \delta \to x \in (a-\delta, a) \cup [a, a+\delta)$

case 1: $x \in (a-\delta, a)$

$\to x < a \to f(x) = h(x)$

Therefore, $|f(x) - \ell| < \varepsilon$

case 2: $x \in [a, a+\delta)$

$\to x \geq a \to f(x) = g(x)$

Therefore, $|f(x) - \ell| < \varepsilon$

Therefore

$\forall \varepsilon > 0 \; \exists \delta > 0 \; |x-a| < \delta \to |f(x) - \ell| < \varepsilon$

Therefore $\lim_{x \to a} f(x) = g(a) = f(a)$. So f cont at a.

b)

g cont on $[a,b]$
h cont. on $[b,c]$
$g(b) = h(b)$
$f(x) = \begin{cases} g(x) & x \in [a,b] \\ h(x) & x \in [b,c] \end{cases}$

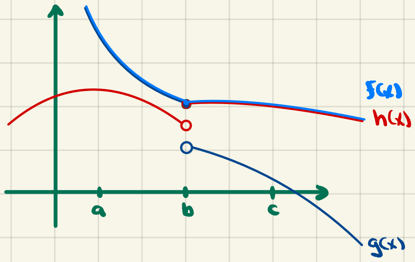
$\longrightarrow$ f cont on $[a,c]$

Proof

$\forall x \in [a,c]$, either $x \in [a,b]$ or $x \in [b,c]$.

If $x \in [a,b]$ then $f(x) = g(x)$. Since $g(x)$ cont in $[a,b]$ so is $f(x)$.

Similarly, $x \in [b,c] \to f(x) = h(x)$. Therefore $f(x)$ cont. in $[b,c]$.



Note that at this point we have

f cont on $[a,b]$ and on $[b,c]$.

f cont on $[a,b] \to \lim_{x \to b^-} f(x) = f(b)$

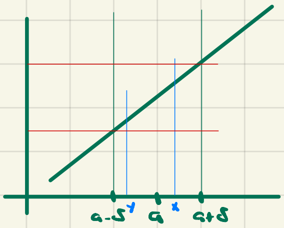
f cont on $[b,c] \to \lim_{x \to b^+} f(x) = f(b)$

Therefore $\lim_{x \to b} f(x) = f(b)$, so f cont on b.

Therefore, $\forall x \in [a,c]$, f is cont at x.

15.

f cont. at a $\rightarrow$ $\forall \epsilon>0\ \exists \delta>0\ \forall x\ \forall y\ (|x-a|<\delta$ and $|y-a|<\delta) \rightarrow |f(x)-f(y)|<\epsilon$



$\forall \epsilon>0\ \exists \delta>0\ \forall x\ |x-a|<\delta \rightarrow |f(x)-f(a)|<\epsilon$

Choose any $\epsilon>0$.

Let $\epsilon_1 = \frac{\epsilon}{2}$.

Keep $f(x)$ and $f(y)$ half as distant from $f(a)$ as you want $f(x)-f(y)$ to be. Do this by choosing the correct $\delta>0$. Now for any x and y such that $|x-a|<\delta$ and $|y-a|<\delta$, it'll be the case that $|f(x)-f(y)|<\epsilon$.

Then, $\exists \delta>0\ \forall x\ |x-a|<\delta \rightarrow |f(x)-f(a)|<\epsilon_1$

Let x and y be such that $|x-a|<\delta \wedge |y-a|<\delta$

Then $|f(x)-f(a)|<\epsilon/2$
$|f(y)-f(a)|<\epsilon/2$

Then $f(a)-\epsilon/2<f(x)<f(a)+\epsilon/2$
$f(a)-\epsilon/2<f(y)<f(a)+\epsilon/2$

$f(a)-\epsilon/2-f(y)<f(x)-f(y)<f(a)-\epsilon/2-f(y)$
$\cancel{f(a)}-\epsilon/2-(\cancel{f(a)}+\epsilon/2)<f(x)-f(y)<\cancel{f(a)}-\epsilon/2-(\cancel{f(a)}-\epsilon/2)$
$-\epsilon<f(x)-f(y)<\epsilon$

$\rightarrow |f(x)-f(y)|<\epsilon$

16. a)

$\lim_{x\to a^+} f(x) = f(a)$

$\longrightarrow \exists\delta>0\ \forall x\ 0\le x-a<\delta \to f(x)>0$

$f(a)>0$

Theorem 3

f cont. at a, $\longrightarrow$ $f(x)>0$ for all x in some interval containing a

$f(a)>0$ ie $\exists\delta>0\ \forall x\ |x-a|<\delta \to f(x)>0$

Also

$\lim_{x\to a} f(x) = f(a)$ $\longrightarrow \exists\delta>0\ \forall x\ |x-a|<\delta \to f(x)<0$

$f(a)<0$

Proof

$\lim_{x\to a^+} f(x) = f(a) \iff \forall\epsilon>0\ \exists\delta>0\ \forall x\ 0\le x-a<\delta \to |f(x)-f(a)|<\epsilon$

choose $\epsilon = \frac{f(a)}{2}$. Then $\exists\delta>0\ \forall x\ 0\le x-a<\delta \to f(a)-\frac{f(a)}{2} < f(x) < f(a)+\frac{f(a)}{2}$

$\to 0 < \frac{f(a)}{2} < f(x) < \frac{3f(a)}{2}$

$\to f(x)>0$

Therefore, $\exists\delta>0\ \forall x\ 0\le x-a<\delta \to f(x)>0$

$\lim_{x\to a^+} f(x) = f(a)$

$\longrightarrow \exists\delta>0\ \forall x\ 0\le x-a<\delta \to f(x)<0$

$f(a)<0$

Proof

Analogous proof. Note that we reach the following step

$\frac{f(a)}{2} < f(x) < \frac{3f(a)}{2} < 0$

$\to f(x)<0$

Therefore $\exists\delta>0\ 0\le x-a<\delta \to f(x)<0$

---

b)

$\lim_{x\to b^-} f(x) = f(b)$

$\longrightarrow \exists\delta\ 0\le b-x<\delta \to f(x)>0.$

$f(b)>0$

Proof

$\lim_{x\to b^-} f(x) \iff \forall\epsilon>0\ \exists\delta>0\ \forall x\ 0\le b-x<\delta \to |f(x)-f(b)|<\epsilon$

choose $\epsilon = \frac{f(b)}{2}$. Then $\forall x\ 0\le b-x<\delta \to 0 < \frac{f(b)}{2} < f(x) < \frac{3f(b)}{2}$

Therefore $\exists\delta\ 0\le b-x<\delta \to f(x)>0.$

17. $\exists \lim_{x\to a} f(x) \neq f(a) \leftrightarrow f$ has removable discontinuity at a

a)

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Note that if $h(x) = {}^1/_x$ and $g(x) = \sin(x)$ then $f(x) = g(h(x))$.

$g(x)$ is continuous in $\mathbb{R}$. $h(x)$ is continuous in $\mathbb{R} - \{0\}$.
we cannot apply Th 2 to say $f(x)$ cont. at 0, because $h(x)$ not cont. at 0.

$\lim_{x\to 0} f(x) = \lim_{x\to 0} \sin({}^1/_x)$

Also, $\nexists \lim_{x\to 0} h(x)$.

$f(x)$ is continuous $\forall x \in \mathbb{R} - \{0\}$, but $\nexists \lim_{x\to 0} f(x)$. Therefore there is no removable discontinuity at 0.

If we now let $f(x) = \begin{cases} x\sin({}^1/_x) & x \neq 0 \\ 1 & x = 0 \end{cases}$

Now $\lim_{x\to 0} f(x) = 0 \longrightarrow$ Proof

Therefore $\exists \lim_{x\to 0} f(x) \neq f(0)$

$\to$ f has removable discontinuity, ie if we redefine $f(0)$ to 0, then f is continuous on $\mathbb{R}$.

Proof

$-1 < \sin({}^1/_x) < 1$

$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \ |x| < \delta \to -\delta < x < \delta$

$|x| < \delta \to -\delta < x < \delta$

$\forall x \quad 0 \leq \sin({}^1/_x) \leq 1 \quad$ or $\quad -1 \leq \sin({}^1/_x) < 0$

case 1: $0 \leq \sin({}^1/_x) \leq 1$

$\to \quad 0 < -\delta\sin({}^1/_x) < x\sin({}^1/_x) < \delta\sin({}^1/_x) < \delta$

case 2: $-1 \leq \sin({}^1/_x) < 0$

$\to \quad -\delta\sin({}^1/_x) > x\sin({}^1/_x) > \delta\sin({}^1/_x)$

$\to \quad -\delta < \delta\sin({}^1/_x) < x\sin({}^1/_x) < -\delta\sin({}^1/_x) < 0$

Therefore $-\delta < x\sin({}^1/_x) < \delta$

we have proved: $\forall \epsilon > 0 \ \forall \delta \ \delta < \epsilon$

$\to \left(\forall x \ |x| < \delta \to |x\sin({}^1/_x)| < \delta < \epsilon\right)$

Therefore $\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \ |x| < \delta \to |x\sin({}^1/_x)| < \epsilon$.

$\to \lim_{x\to 0} x\sin({}^1/_x) = 0$

b)

f has removable discontinuity at a

$\lim_{x \to a} f(x) \neq f(a)$

$\forall x \; x \neq a \to g(x) = f(x)$ $\longrightarrow$ g cont. at a

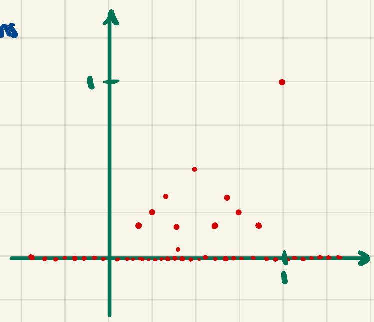
$g(a) = \lim_{x \to a} f(x)$

Proof

$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = g(a)$

Therefore g cont. at a.

c) $f(x) = \begin{cases} 0 & x \text{ irrational} \\ \frac{1}{q} & x = \frac{p}{q} \text{ in lowest terms} \end{cases}$



g defined by $g(x) = \lim_{y \to x} f(y)$

$g(x)$ is the limit of $f(x)$ at x.

This limit equals 0 for all x.

Therefore, $g(x) = 0$.

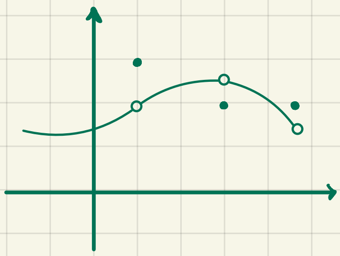
d)

every point of discontinuity of f is a removable discontinuity.

therefore $\exists \lim_{y \to x} f(y)$ for all x, but f may be discontinuous at a finite or infinite number of these points. → g continuous

$g(x) = \lim_{y \to x} f(y)$

Proof

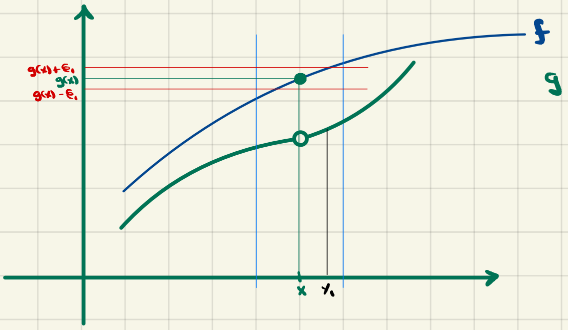


For every $x \in \mathbb{R}$, either f is continuous at x or discontinuous at x.

If f continuous at x then $\lim_{y \to x} f(y) = f(x) = g(x)$.

$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall y \; |y - x| < \delta \to |f(y) - f(x)| < \varepsilon$

Assume g not cont. at x. That is assume $\lim_{y \to x} g(y) \neq g(x)$



Then $\exists \varepsilon > 0 \; \forall \delta > 0 \; \exists y \; 0 < |y - x| < \delta \to |g(y) - g(x)| > \varepsilon$

Let ε_1 be one such ε, δ_1 one such δ, and y_1 one such y.

$0 < |y_1 - x| < \delta_1 \to |g(y_1) - g(x)| > \varepsilon_1$

Then $g(y_1) = \lim_{x \to y_1} f(x)$.

Therefore $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall y \; |y - y_1| < \delta \to |f(y) - g(y_1)| < \varepsilon$.

It is either the case that g(

f defined at a, not continuous at a → $\exists \varepsilon > 0 \; \forall \delta > 0 \; \exists x \; 0 < |x - a| < \delta \to |f(x) - f(a)| > \varepsilon$