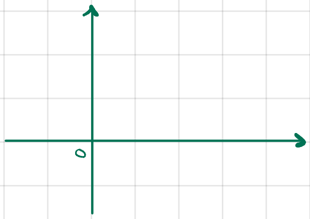
the real line: geometric picture of real numbers; prominent in thinking about proofs, but inessential to actual proofs

drawing ordered pairs requires a coordinate system



function: collection of ordered pairs

graph of fn: drawing of ordered pairs $(x, f(x))$

For our purposes, we

→ define the plane as set of all ordered pairs of real numbers

→ define straight lines as certain collections of pairs
e.g. $\{(x, cx) : x \in \mathbb{R}\}$

→ define distance between (a,b) and (c,d) as $\sqrt{(a-c)^2+(b-d)^2}$
ie we use Pythagorean Theorem in our plane

→ linear fns: $f(x) = cx+d$

→ power fns: $f(x) = x^n$

ex: $f\left(\frac{1}{n}\right) = (-1)^{n+1}$
$f\left(-\frac{1}{n}\right) = (-1)^{n+1}$
$f(x) = 1 \quad |x| \geq 1$

circle: $(x-a)^2 + (y-b)^2 = r^2$

ellipse: set of points, sum of whose distance from two "focus" points is a constant.

Two focus points $(c,0), (-c,0)$

$$[(x-c)^2+y^2]^{1/2} + [(x+c)^2+y^2]^{1/2} = 2a$$

$$[(x-c)^2+y^2]^{1/2} = 2a - [(x+c)^2+y^2]^{1/2}$$

$$(x-c)^2+y^2 = 4a^2 - 4a[(x+c)^2+y^2]^{1/2} + (x+c)^2+y^2$$
$$\cancel{x^2} - 2xc + \cancel{c^2} + \cancel{y^2} = 4a^2 + \cancel{x^2} + 2xc + \cancel{c^2} + \cancel{y^2} - 4a[(x+c)^2+y^2]^{1/2}$$
$$\cancel{4}a^2 - \cancel{4}cx = \cancel{4}a[(x+c)^2+y^2]^{1/2}$$
$$a^4 - \cancel{2a^2cx} + c^2x^2 = a^2(x^2+c^2+\cancel{2xc}+y^2)$$
$$= x^2(c^2-a^2) - a^2y^2 = a^2(c^2-a^2)$$

divide by $a^2(c^2-a^2) < 0$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad b = \sqrt{a^2-c^2}$$ is an ellipse

note

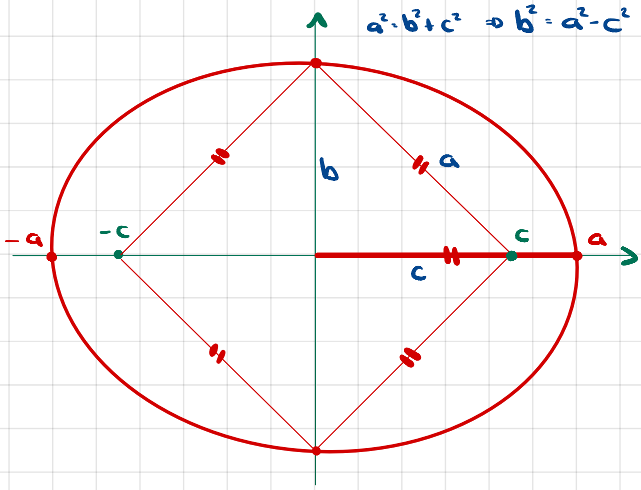
$= x^2(c^2-a^2) - a^2y^2 = a^2(c^2-a^2)$

$x^2(c^2-a^2) = a^2(y^2+c^2-a^2)$

case 1: $0 < a < c$

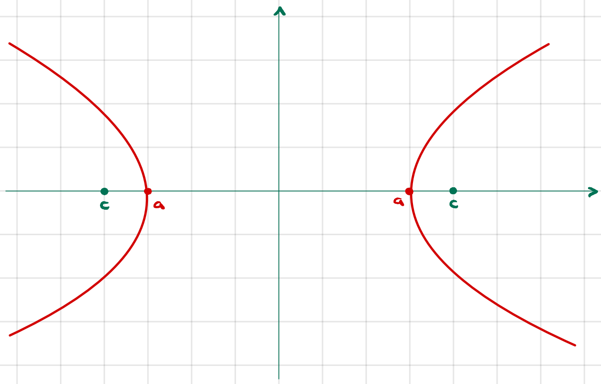
$x^2(c^2-a^2) > 0$

$a^2 = b^2 + c^2 \Rightarrow b^2 = a^2 - c^2$



hyperbola: set of points, difference of whose distance to two foci is constant.

$$\left|[(x-c)^2+y^2]^{1/2} - [(x+c)^2+y^2]^{1/2}\right| = \pm 2a$$



Consider the +2a case, ie the left side of the hyperbola

$$[(x-c)^2+y^2]^{1/2} - [(x+c)^2+y^2]^{1/2} = 2a$$

$$[(x-c)^2+y^2]^{1/2} = 2a + [(x+c)^2+y^2]^{1/2}$$

$$(x-c)^2+y^2 = 4a^2 + 4a[(x+c)^2+y^2]^{1/2} + (x+c)^2+y^2$$

$$\cancel{x^2} - 2xc + \cancel{c^2} + \cancel{y^2} = 4a^2 + \cancel{x^2} + 2xc + \cancel{c^2} + \cancel{y^2} + 4a[(x+c)^2+y^2]^{1/2}$$

$$-4xc - 4a^2 = 4a[(x+c)^2+y^2]^{1/2}$$

if we were considering the $-2a$ case, ie right side of hyperbola, the only difference here would be minus sign on the rhs term.

$$-(xc+a^2) = a[(x+c)^2+y^2]^{1/2}$$

$$x^2c^2 + \cancel{2xca^2} + a^4 = a^2(x^2+2xc+c^2+y^2)$$
$$= a^2x^2 + \cancel{2a^2xc} + a^2c^2 + a^2y^2$$

But we square both sides so the calculations are the same.

$$x^2(a^2-c^2) + a^2y^2 = a^2(a^2-c^2)$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$$

but $0 > c > a \Rightarrow c^2 > a^2$
$\Rightarrow a^2 - c^2 < 0$

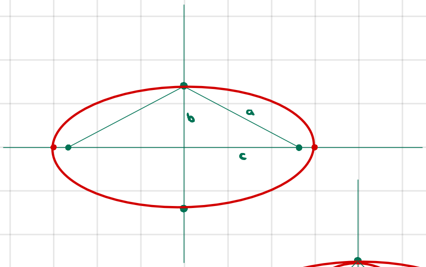
$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad b^2 = c^2 - a^2$$

Sadhaushi | Cole 11.2 Ellipses

example 1: $2x^2 + 9y^2 = 18$

$$\frac{x^2}{9} + \frac{y^2}{2} = 1$$



$a^2 = 9 = b^2 + c^2 = 2 + c^2$

$\Rightarrow c = \sqrt{7}$

total string length $= 2a = 6$

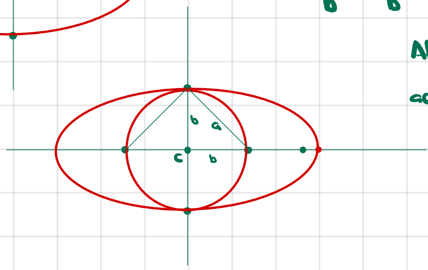
total distance between foci $= 2 \cdot \sqrt{7} \approx 5.28$

$\Rightarrow$ rel. flat ellipse

simply reducing a keeping b constant reduces c because $c = \sqrt{a^2 - b^2}$

If we reduce a to b then $c = 0$.

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} = 1 \Rightarrow x^2 + y^2 = b^2$$



At this point, if we decrease c more it becomes negative, which actually just means that a starts increasing again.

example 2: $9x^2 + 4y^2 = 25$

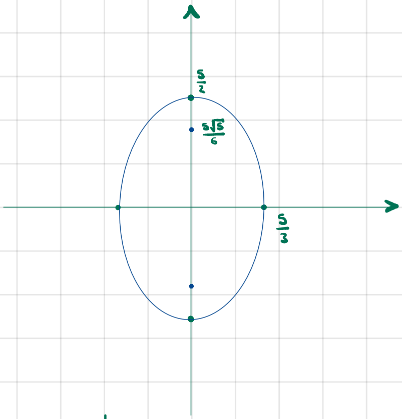
$$\frac{x^2}{25/9} + \frac{y^2}{25/4} = 1$$

$b^2 = \frac{25}{9} \Rightarrow b = \frac{5}{3}$

$a^2 = \frac{25}{4} \Rightarrow a = \frac{5}{2}$

$c^2 = a^2 - b^2 = \frac{125}{36}$

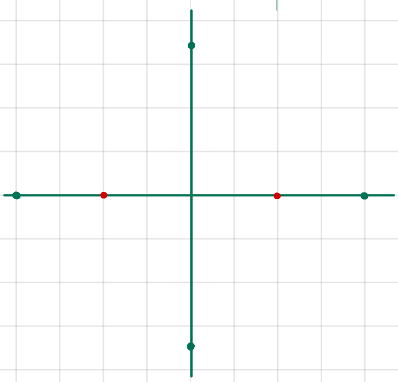
$\Rightarrow c = \frac{5\sqrt{5}}{6}$



example 3

vertices $(\pm 4, 0)$

foci $(\pm 2, 0)$



$a = 4$

$c = 2$

$b^2 = 16 - 4 = 12$

$b = 2\sqrt{3} \approx 3.4$

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

example 5 $16x^2+9y^2+64x-18y-71=0$

$(4x)^2+2\cdot4x\cdot8+64-64+(3y)^2-2\cdot3y\cdot3+9-9-71=0$

$(4x+8)^2+(3y-3)^2=144$

$16(x+2)^2+9(y-1)^2=144$

$$\frac{(x+2)^2}{9}+\frac{(y-1)^2}{16}=1$$

$a^2=16 \Rightarrow a=4$

$b^2=9 \Rightarrow b=3$

$c^2=16-9=7$

$c=\sqrt{7}$