

Ch 10 - Differentiation

1.

$$\text{i)} f(x) = \sin(x+x^2)$$

$$f'(x) = \cos(x+x^2)(1+2x)$$

$$\text{ii)} f(x) = \sin x + \sin x^2$$

$$f'(x) = \cos x + \cos x^2 \cdot 2x$$

$$\text{iii)} f(x) = \sin(\cos x)$$

$$f'(x) = \cos(\cos x)(-\sin x)$$

$$= -\sin x \cos(\cos x)$$

$$\text{iv)} f(x) = \sin(\sin x)$$

$$f'(x) = \cos(\sin x) \cos(x)$$

$$\text{v)} f(x) = \sin\left(\frac{\cos x}{x}\right)$$

$$f'(x) = \cos\left(\frac{\cos x}{x}\right) \cdot \left[\frac{-\sin x \cdot x - \cos x}{x^2} \right]$$

$$\text{vi)} f(x) = \frac{\sin(\cos x)}{x}$$

$$f'(x) = \frac{\cos(\cos x)(-\sin x)x - \sin(\cos x)}{x^2}$$

$$\text{vii)} f(x) = \sin(x+\sin x)$$

$$f'(x) = \cos(x+\sin x)(1+\cos x)$$

$$\text{viii)} f(x) = \sin(\cos(\sin x))$$

$$f'(x) = \cos(\cos(\sin x)) \cdot (-\sin(\sin x)) \cos x$$

2.

$$\text{i)} f(x) = \sin((x+1)^2(x+z))$$

$$f'(x) = \cos((x+1)^2(x+z)) [2(x+1)(x+z) + (x+1)^2]$$

$$\text{ii)} f(x) = \sin^3(x^2 + \sin x)$$

$$f'(x) = 3\sin^2(x^2 + \sin x) \cos(x^2 + \sin x) (2x + \cos x)$$

$$\text{iii)} f(x) = \sin^3((x+\sin x)^2)$$

$$f'(x) = 2\sin((x+\sin x)^2) \cos((x+\sin x)^2) 2(x+\sin x) (1+\cos x)$$

$$\text{iv)} f(x) = \sin \frac{x^3}{\cos x^3}$$

$$f'(x) = \cos\left(\frac{x^3}{\cos x^3}\right) \cdot \frac{3x^2 \cos x^3 + x^3 \sin x^3 \cdot 3x^2}{(\cos x^3)^2}$$

$$\text{v)} f(x) = \sin(x \sin x) + \sin(\sin x^2)$$

$$f'(x) = \cos(x \sin x) \cdot (\sin x + x \cos x) + \cos(\sin x^2) \cos(x^2) 2x$$

$$\text{vi)} f(x) = (\cos x)^{3x^2}$$

$$f'(x) = 3x^2(\cos x)^{3x^2-1} (-\sin x)$$

$$\text{vii)} f(x) = \sin^2 x \sin x^2 \sin^2 x^2$$

$$f'(x) = 2\sin x \cos x \sin x^2 \sin^2 x^2 + \sin^2 x \cos x^2 \cdot 2x \sin^2 x^2 + \sin^2 x \sin x^2 2\sin x^2 \cos x^2 2x$$

$$\text{viii)} f(x) = \sin^3(\sin^2(\sin x))$$

$$f'(x) = 3\sin^2(\sin^2(\sin x)) \cos(\sin^2(\sin x)) 2\sin(\sin x) \cos(\sin x) \cos x$$

$$\text{ix)} f(x) = (x + \sin^4 x)^6$$

$$f'(x) = 6(x + \sin^4 x)^5 (1 + 5\sin^4 x \cos x)$$

$$\text{x)} f(x) = \sin(\sin(\sin(\sin(\sin x))))$$

$$f'(x) = \cos(\sin(\sin(\sin(\sin x)))) \cos(\sin(\sin(\sin x))) \cos(\sin(\sin x)) \cos(\sin x) \cos x$$

$$\text{xii)} f(x) = \sin((\sin^3 x + 1)^7)$$

$$f'(x) = \cos((\sin^3 x + 1)^7) \cdot 7(\sin^3 x + 1)^6 (7\sin^2 x \cos x \cdot 7x^6)$$

$$\text{xiii)} f(x) = (((x^2 + x)^3 + x)^4 + x)^5$$

$$f'(x) = 5(((x^2 + x)^3 + x)^4 + x)^4 (1 + 4((x^2 + x)^3 + x)^3 (1 + 3(x^2 + x)^2 (2x + 1)))$$

$$\text{xiv)} f(x) = \sin(x^2 + \sin(x^2 + \sin x^2))$$

$$f'(x) = \cos(x^2 + \sin(x^2 + \sin x^2)) (2x + \cos(x^2 + \sin x^2)) (2x + \cos(x^2 + \sin x^2) \cdot 2x)$$

$$\text{xv)} f(x) = \sin(6\cos(6\sin(6\cos 6x)))$$

$$f'(x) = \cos(6\cos(6\sin(6\cos 6x))) \cdot 6(-\sin(6\sin(6\cos 6x)) \cdot 6\cos(6\cos 6x) \cdot 6(-\sin 6x) \cdot 6))$$

$$xvi) f(x) = \frac{\sin x^2 \sin^2 x}{1 + \sin x}$$

$$f'(x) = \frac{(\cos x^2 \cdot 2x \sin^2 x + \sin x^2 \cdot 2 \sin x \cos x)(1 + \sin x) - \sin x^2 \sin^2 x \cos x}{(1 + \sin x)^2}$$

$$xvii) f(x) = \frac{1}{x - \frac{2}{x + \sin x}}$$

$$f'(x) = \frac{-(1 - \frac{-2(1 + \cos x)}{(x + \sin x)^2})}{\left(x - \frac{2}{x + \sin x}\right)^2}$$

$$xviii) f(x) = \sin\left(\frac{x^3}{\sin\left(\frac{x^3}{\sin x}\right)}\right)$$

$$f'(x) = \cos\left(\frac{x^3}{\sin\left(\frac{x^3}{\sin x}\right)}\right) \frac{3x^2 \sin\left(\frac{x^3}{\sin x}\right) - x^3 \cos\left(\frac{x^3}{\sin x}\right) \cdot \frac{3x^2 \sin x - x^3 \cos x}{\sin^2 x}}{\sin^2\left(\frac{x^3}{\sin x}\right)}$$

$$xix) f(x) = \sin\left(\frac{x}{x - \sin\left(\frac{x}{x - \sin x}\right)}\right)$$

$$f'(x) = \cos\left(\frac{x}{x - \sin\left(\frac{x}{x - \sin x}\right)}\right) \frac{x - \sin\left(\frac{x}{x - \sin x}\right) - x(1 - \cos\left(\frac{x}{x - \sin x}\right)) \frac{(x - \sin x) - x(1 - \cos x)}{(x - \sin x)^2}}{\left(x - \sin\left(\frac{x}{x - \sin x}\right)\right)^2}$$

3.

$$f_1(x) = \tan x = \frac{\sin x}{\cos x}$$

$$f_1'(x) = \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$f_2(x) = \operatorname{cosec} x = \frac{\cos x}{\sin x} = \frac{1}{\tan x} = \frac{1}{f_1(x)}$$

$$f_2'(x) = -\frac{1}{f_1^2(x)} \cdot f_1'(x) = -\frac{\cos^2 x}{\sin^2 x} \cdot \frac{1}{\cos^2 x} = -\frac{1}{\sin^2 x}$$

$$f_3(x) = \sec x = \frac{1}{\cos x}$$

$$f_3'(x) = \frac{\sin x}{\cos^2 x} = \tan x \sec x$$

$$f_4(x) = \csc x = \frac{1}{\sin x}$$

$$f_4'(x) = \frac{-\cos x}{\sin^2 x} = -\operatorname{cosec} x \csc x$$

4.

$$(f \circ g)(x) = f(g(x)) = g(x)$$

$$(f \circ g)'(x) = g'(x) = f'(g(x))g'(x)$$

Want to compute $f'(g(x))$

$$\text{i) } f(x) = \frac{1}{1+x}$$

$$f'(x) = \frac{-1}{(1+x)^2}$$

$$f'(f(x)) = \frac{-1}{\left(\frac{1+x}{1-x}\right)^2}$$

$$\text{ii) } f(x) = \sin x$$

$$f'(x) = \cos x$$

$$f'(f(x)) = \cos(\sin x)$$

$$\text{iii) } f(x) = x^2$$

$$f'(f(x)) = 2x^2$$

$$\text{i) } f(x) = 17$$

$$f'(f(x)) = 0$$

5. $f(f'(x))$

i) $f(x) = \frac{1}{x}$

$$f'(x) = -\frac{1}{x^2}$$

$$f(f'(x)) = \frac{1}{-\frac{1}{x^2}} = -x^2$$

ii) $f(x) = x^2$

$$f'(x) = 2x$$

$$f(f'(x)) = 4x^2$$

iii) $f(x) = 17$

$$f(f'(x)) = 17$$

iv) $f(x) = 17x$

$$f'(x) = 17$$

$$f(f'(x)) = 17^2$$

6.

i) $f(x) = g(x+g(a))$

$$f'(x) = g'(x+g(a))$$

ii) $f(x) = g(x \cdot g(a))$

$$f'(x) = g'(x \cdot g(a)) \cdot g(a)$$

iii) $f(x) = g(x+g(x))$

$$f'(x) = g'(x+g(x)) \cdot (1+g'(x))$$

iv) $f(x) = g(x)(k-a)$

$$f'(x) = g'(x)(k-a) + g(x)$$

v) $f(x) = g(a)(k-x)$

$$f'(x) = g(a)$$

vi) $f(x+3) = g(x^2)$

$$h(x) = f(x+3) = g(x^2), \text{ i.e., } f(x) \text{ shifted left 3 units.}$$

$$p(x) = h(x-3) = f(x) = g((x-3)^2)$$

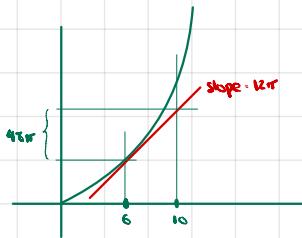
$$p'(x) = f'(x) = g'((x-3)^2) \cdot 2(x-3)$$

7.

$$\text{a) } A(r(t)) = \pi r(t)^2$$

$$\frac{dA(r(t))}{dt} = A'(t) = A'(r(t)) r'(t) = 2\pi r(t) r'(t)$$

$$A'(6) = 2\pi r(6) r'(6) = 2\pi \cdot 6 \cdot 4 = 48\pi$$



$$\text{b) } V(t) = \frac{4}{3}\pi r(t)^3$$

$$V'(t) = 4\pi r(t)^2 r'(t)$$

$$V'(6) = 4\pi \cdot 36 \cdot 4 = 576\pi$$

$$\text{c) } A(r(t)) = \pi r(t)^2$$

$$\frac{dA(r(t))}{dt} = 2\pi r(t) r'(t)$$

$$\frac{dA(3)}{dt} = 2\pi \cdot 3 = 6\pi$$

$$\frac{dA(r(t))}{dt} = 2\pi r(t) r'(t)$$

$$\frac{dA(3)}{dt} = 2\pi \cdot 3 r'(t_3) = 5 \rightarrow r'(t_3) = \frac{5}{6\pi}$$

$$\frac{dV(r(t))}{dt} = 4\pi r(t)^2 r'(t)$$

$$\frac{dV(r(t_3))}{dt} = \frac{dV(3)}{dt} = 4\pi \cdot 9 \cdot \frac{5}{6\pi} = 30$$

Allan.

$$V(t) = \frac{4}{3} r(t) A(r(t))$$

$$V'(t) = \frac{4}{3} (r'(t) A(r(t)) + r(t) A'(r(t)) r'(t))$$

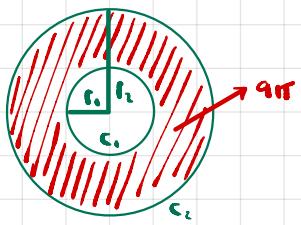
$$V'(t_3) = \frac{4}{3} (r'(t_3) A(3) + 3A'(3) r'(t_3))$$

$$= \frac{4}{3} \left(\frac{5}{6\pi} \pi \cdot 9 + 3 \cdot 6\pi \cdot \frac{5}{6\pi} \right)$$

$$= \frac{4}{3} \cdot \frac{5}{6\pi} \cdot 9\pi + \frac{4}{3} \cdot 18\pi \cdot \frac{5}{6\pi}$$

$$= 10 + 20$$

$$= 30$$



$$A_2(r_2(t)) = \pi r_2^2(t)$$

$$\frac{dA_2(r_2(t))}{dt} = \pi \cdot 2r_2(t)r_2'(t) = 10\pi$$

Circle 2's radius changes at a declining rate, to keep the rate of change of the area of circle 2 constant.

$$r_2'(t) = -\frac{5}{r_2(t)}$$

$$A_1(r_1(t)) = \pi r_1^2(t)$$

$$\frac{dA_1(r_1(t))}{dt} = \pi \cdot 2r_1(t)r_1'(t)$$

$$D(t) = A_2(r_2(t)) - A_1(r_1(t)) = \pi(r_2^2(t) - r_1^2(t)) = 9\pi$$

$$D'(t) = 0 = \pi(2r_2(t)r_2'(t) - 2r_1(t)r_1'(t))$$

$$= \pi(10 - 2r_1(t)r_1'(t))$$

$\rightarrow r_1(t)r_1'(t) = 5$ This is intuitively true, since the area of 2 increases at a constant rate, then since the difference in areas stays constant, the area of 1 increases at the same rate as the area of 2.

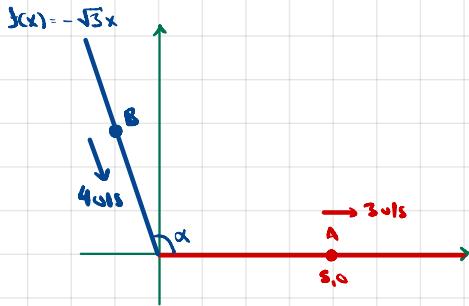
$$C_1(r_1(t)) = 2\pi r_1(t)$$

$$\frac{dC_1(r_1(t))}{dt} = 2\pi r_1'(t) = 2\pi \cdot \frac{5}{r_1(t)} = \frac{10\pi}{r_1(t)}$$

$$A_1(r_1(t)) = \cancel{\pi r_1^2(t)} = 16\cancel{\pi} \rightarrow r_1(t) = 4$$

$$\rightarrow \frac{dC_1(r_1(t))}{dt} = 2.5\pi$$

9.



B is at $(x, -\sqrt{3}x)$.

Distance from origin is $3 = \sqrt{x^2 + 3x^2} = 2x \rightarrow |x| = \frac{3}{2}$

Since $x \leq 0$ then $x = -\frac{3}{2}$

B is at $(-\frac{3}{2}, \frac{3\sqrt{3}}{2})$.

$$D_{AB} = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$$

$$x_A(t) = 5 + 3t$$

$$y_A(t) = 0$$

$$x_B(0) = -\frac{3}{2}$$

$$y_B(t) = -\sqrt{3}x_B(t) \rightarrow y_B(0) = \frac{3\sqrt{3}}{2}$$

$$\sqrt{(x_B + 3/2)^2 + (y_B - 3\sqrt{3}/2)^2} = 4$$

$$\begin{aligned} 16t^2 &= (x_B + 3/2)^2 + (y_B - 3\sqrt{3}/2)^2 \\ &= (x_B + 3/2)^2 + (-\sqrt{3}x_B - \frac{3\sqrt{3}}{2})^2 \\ &= x_B^2 + 3x_B + \frac{9}{4} + 3x_B^2 + 9x_B + \frac{27}{4} \\ &= 4x_B^2 + 12x_B + 9 \end{aligned}$$

$$\text{in general } 4x_B^2 + 12x_B + 9 - 16t^2 = 0$$

$$\Delta = 144 - 4 \cdot 4 \cdot (9 - 16t^2) = 144 - 144 + 64t^2 = 64t^2$$

$$x_B(t) = \frac{-12 \pm \sqrt{\Delta}}{8} = -\frac{3}{2} \pm t$$

$$y_B(t) = -\sqrt{3}(-\frac{3}{2} \pm t)$$

The presence of \pm reflects the fact that B might be heading northeast or southeast.

Case 1: B going northeast

$$x_A(t) = 5 + 3t$$

$$y_A(t) = 0$$

$$x_B(t) = -\frac{3}{2} + t$$

$$y_B(t) = \frac{3\sqrt{3}}{2} + \sqrt{3}t$$

$$\begin{aligned} D_{AB}(t) &= \sqrt{(5+3t + \frac{3}{2} + t)^2 + (0 - \frac{3\sqrt{3}}{2} - \sqrt{3}t)^2} \\ &= \sqrt{(\frac{13}{2} + 4t)^2 + (-\sqrt{3}t - \frac{3\sqrt{3}}{2})^2} \\ &= \sqrt{\frac{169}{4} + 52t + 16t^2 + 3t^2 + 9t + \frac{27}{4}} \\ &= \sqrt{19t^2 + 61t + 49} \end{aligned}$$

$$D_{AB}'(t) = \frac{38t + 61}{2\sqrt{19t^2 + 61t + 49}}$$

$$D_{AB}'(0) = \frac{61}{14}$$

Case 2: B heading southeast

$$x_A(t) = 5 + 3t$$

$$y_A(t) = 0$$

$$x_B(t) = -\frac{3}{2} + t$$

$$y_B(t) = \frac{3\sqrt{3}}{2} - \sqrt{3}t$$

$$\begin{aligned} D_{AB}(t) &= \sqrt{(5+3t + \frac{3}{2} - t)^2 + (0 - \frac{3\sqrt{3}}{2} + \sqrt{3}t)^2} \\ &= \sqrt{(\frac{13}{2} + 2t)^2 + (\sqrt{3}t - \frac{3\sqrt{3}}{2})^2} \\ &= \sqrt{\frac{169}{4} + 26t + 4t^2 + 3t^2 - 9t + \frac{27}{4}} \\ &= \sqrt{7t^2 + 17t + 49} \end{aligned}$$

$$D_{AB}'(t) = \frac{14t + 17}{2\sqrt{7t^2 + 17t + 49}}$$

$$D_{AB}'(0) = \frac{17}{2\sqrt{7}} =$$

Textbook Sol'n

$(x_a(t), 0)$ position of A at t.

$$x_a(t_0) = 5$$

$$x'_a(t_0) = 3$$

$$(x_b(t), -\sqrt{3}x_b(t))$$

$$D_{ba}(t) = \sqrt{x_b(t)^2 + 3x_b(t)^2} = 2x_b(t)$$

$$D'_{ba}(t) = 2x'_b(t)$$

$$x_b(t_0) = -\frac{3}{2}$$

$$x'_b(t_0) = \frac{D'_{ba}(t)}{2} = -2$$

$$D_{ab}(t) = \sqrt{(x_a(t) - x_b(t))^2 + 3x_b(t)^2}$$

$$D_{ab}(t_0) = \sqrt{(5 + 3)(-\frac{3}{2})^2 + 3(-\frac{3}{2})^2} = \sqrt{\frac{169}{4} + \frac{27}{4}} = 7$$

$$D'_{ab}(t) = \frac{2(x_a(t) - x_b(t))(x'_a(t) - x'_b(t)) + 6x_b(t)x'_b(t)}{2\sqrt{(x_a(t) - x_b(t))^2 + 3x_b(t)^2}}$$

$$D'_{ab}(t_0) = \frac{2(5 + \frac{3}{2})(3 + 2) + 6 \cdot (-\frac{3}{2})(-2)}{14} = \frac{2 \cdot \frac{13}{2} \cdot 5 + 18}{14} = \frac{83}{14}$$

$$10. f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

h, h s.t.

$$\begin{aligned} h'(x) &= \sin^2(\sin(x+1)) & h'(x) &= f(x+1) \\ h(0) &= 3 & h(0) &= 0 \end{aligned}$$

ii) $(f \circ h)'(0)$

$$x \neq 0 \rightarrow f'(x) = 2x \sin(\frac{1}{x}) + x^2 \cos(\frac{1}{x})(-\frac{1}{x^2})$$

$$2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$$

$$f'(3) = 6 \sin(\frac{1}{3}) - \cos(\frac{1}{3})$$

$$\text{let } g(x) = (f \circ h)(x) = f(h(x)).$$

$$g'(x) = (f \circ h)'(x) = f'(h(x))h'(x)$$

$$g'(0) = f'(h(0))h'(0) = f'(3)h'(0) = (6 \sin(\frac{1}{3}) - \cos(\frac{1}{3})) \sin^2(\sin 1)$$

iii) $(h \circ f)'(0)$

$$y'(0) = \lim_{h \rightarrow 0} \frac{h^2 \sin(\frac{1}{h})}{h} = 0$$

$$h'(0) = f'(1) = \sin(1)$$

$$\text{let } g(x) = (h \circ f)(x) = h(f(x))$$

$$g'(x) = (h \circ f)'(x) = h'(f(x))f'(x)$$

$$g'(0) = (h \circ f)(0) = h'(f(0))f'(0) = h'(0)f'(0) = 0$$

iv) $\alpha(x) = h(x^2), \alpha'(x^2)$

$$\alpha'(x) = h'(x^2) \cdot 2x$$

$$\alpha'(x^2) = h'(x^4) \cdot 2x^2$$

$$= \sin^2(\sin(x^4+1)) \cdot 2x^2$$

$$11. f(x) = \begin{cases} g(x) \sin(1/x) & x \neq 0 \\ 0 & x=0 \end{cases}$$

$$g(0) = g'(0) = 0$$

$$y(0) = \lim_{h \rightarrow 0} \frac{g(h) \sin(1/h) - 0}{h}$$

$$= \lim_{h \rightarrow 0} \frac{g(h)}{h} \cdot \lim_{h \rightarrow 0} \sin(1/h)$$

$$= g'(0) \cdot \lim_{h \rightarrow 0} \sin(1/h)$$

$$= 0 \quad (\text{problem 8-21})$$

$$12. f(x) = \frac{1}{x}$$

$$\text{let } g(x) \text{ and } h(x) = \frac{1}{g(x)}$$

$$h'(x) = \frac{-1}{g(x)^2} \cdot g'(x)$$

13.

$$a) f(x) = \sqrt{1-x^2} \quad -1 \leq x \leq 1$$

$$f'(x) = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

b) tangent of f at $(a, \sqrt{1-a^2})$

$$g(x) = (x-a) \frac{-a}{\sqrt{1-a^2}} + f(a)$$

$$g(x) = f(x) \rightarrow \sqrt{1-x^2} = f(a) - \frac{a(x-a)}{\sqrt{1-a^2}}$$

$$\sqrt{(1-x^2)(1-a^2)} = 1-x - a(x-a)$$

$$= 1-ax$$

~~$$1-a^2-x^2+a^2x^2 = 1-2ax+a^2x^2$$~~

$$x^2 - 2ax + a^2 = 0$$

$$\Delta = 4a^2 - 4a^2 = 0$$

$$x = \frac{2a}{2} = a$$

14.
Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad b = \sqrt{a^2-c^2} \text{ is an ellipse}$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \rightarrow y^2 = b^2 - \frac{b^2}{a^2}x^2$$

$$y(x) = \pm \sqrt{b^2 - \frac{b^2}{a^2}x^2}$$

$$\text{consider } f(x) = \sqrt{b^2 - \frac{b^2}{a^2}x^2} = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$f'(x) = \frac{-b^2x}{a^2\sqrt{b^2 - \frac{b^2}{a^2}x^2}} = \frac{-b^2x}{a^2b\sqrt{1 - \frac{x^2}{a^2}}} = \frac{-bx}{a^2\sqrt{1 - \frac{x^2}{a^2}}}$$

$$\text{Tangent line at } (x_0, \sqrt{b^2 - \frac{b^2}{a^2}x_0^2})$$

$$g(x) = (x-x_0) \frac{(-bx_0)}{a^2\sqrt{1 - \frac{x_0^2}{a^2}}} + b\sqrt{1 - \frac{x_0^2}{a^2}}$$

$$g(x) - f(x)$$

$$\rightarrow b\sqrt{1 - \frac{x^2}{a^2}} = (x-x_0) \frac{(-bx_0)}{a^2\sqrt{1 - \frac{x_0^2}{a^2}}} + b\sqrt{1 - \frac{x_0^2}{a^2}}$$

$$\text{let } x' = \frac{x}{a}, x'_0 = \frac{x_0}{a}$$

$$b\sqrt{1-x'^2} - (x'-x'_0) \frac{-bx'_0 \cdot a}{a\sqrt{1-x'^2}} + b\sqrt{1-x'^2}$$

$$\cdot \sqrt{1-x'^2} \cdot (x'-x'_0) \frac{-x'_0}{\sqrt{1-x'^2}} + \sqrt{1-x'^2}$$

This is the equation for the tangent line to $f(x') = \sqrt{1-x'^2}$.

From problem 13, the sol'n is $x' = x'_0$.

Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \rightarrow y^2 = b^2 \left(\frac{x^2}{a^2} - 1 \right) \rightarrow y = \pm b \sqrt{\frac{x^2}{a^2} - 1} \quad \frac{x^2}{a^2} \geq 1 \rightarrow x^2 \geq a^2 \rightarrow x \geq a \text{ or } x \leq -a$$

consider $y = b \sqrt{\frac{x^2}{a^2} - 1}$, the positive portion of the hyperbola.

$$f'(x) = \frac{\frac{b \cdot 1 \cdot x}{a^2}}{1 \sqrt{\frac{x^2}{a^2} - 1}} = \frac{bx}{a^2 \sqrt{\frac{x^2}{a^2} - 1}}$$

tangent line through $(x_0, b \sqrt{\frac{x_0^2}{a^2} - 1})$

$$g(x) = (x - x_0) \frac{bx_0}{a^2 \sqrt{\frac{x_0^2}{a^2} - 1}} + b \sqrt{\frac{x_0^2}{a^2} - 1}$$

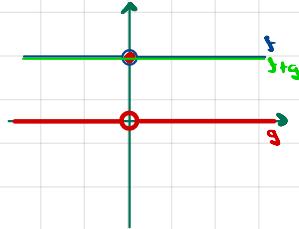
$$f(x) = g(x) \rightarrow b \sqrt{\frac{x^2}{a^2} - 1} = (x - x_0) \frac{bx_0}{a^2 \sqrt{\frac{x_0^2}{a^2} - 1}} + b \sqrt{\frac{x_0^2}{a^2} - 1} \quad (1)$$

recall the eq. we reached in the ellipse case

$$b \sqrt{1 - \frac{x^2}{a^2}} = (x - x_0) \frac{(-bx_0)}{a^2 \sqrt{1 - \frac{x_0^2}{a^2}}} + b \sqrt{1 - \frac{x_0^2}{a^2}} \quad (2)$$

If we square both equations they both become the same.
Thus the

15. $f+g$ diff at a



f and g both not diff. at 0.

$f \cdot g$ and f diff at a

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

Therefore, $g(x)$ and $g'(x)$ exist, i.e. g diff at x.

$$\begin{aligned} (fg)'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(a+h)(g(a+h) - g(a)) + g(a)(f(a+h) - f(a))}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(a+h)(g(a+h) - g(a))}{h} + \lim_{h \rightarrow 0} \frac{g(a)(f(a+h) - f(a))}{h} \end{aligned}$$

16.

a) f diff at a
 $f(a) \neq 0$ $\rightarrow |f|$ diff at a

$$g(x) = |f(x)| = \begin{cases} f(x) & f(x) > 0 \\ -f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

Case 1: $f(a) > 0$

$$g'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

differentiability of f at a

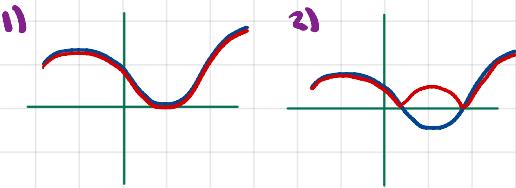
Case 2: $f(a) < 0$

$$g'(a) = \lim_{h \rightarrow 0} \frac{-f(a+h) + f(a)}{h} = -\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = -f'(a)$$

since f diff at a this limit $-f'(a)$

Note that in cases i) and ii), there is an interval containing a in which f is either strictly positive or strictly negative, respect.

If $f(a) = 0$ there are multiple cases to consider

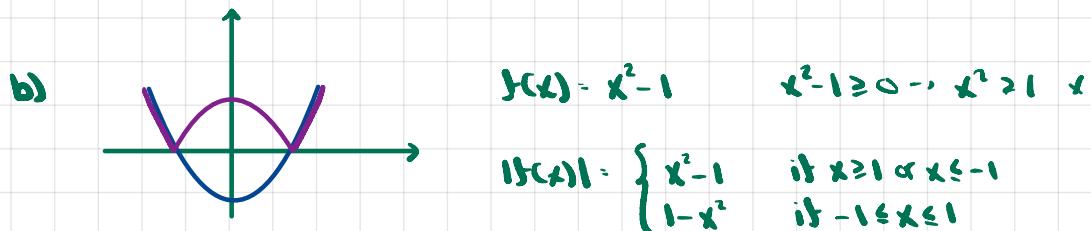


In (1) $|f| = f$ so $|f|$ is diff everywhere. Not so in (2).

Algebraic Sol'n

$$g(x) = |f(x)| = \sqrt{f^2(x)}$$

$$g'(x) = \frac{2f(x)f'(x)}{2\sqrt{f^2(x)}} = f'(x) \frac{f(x)}{|f(x)|}, x \neq 0.$$



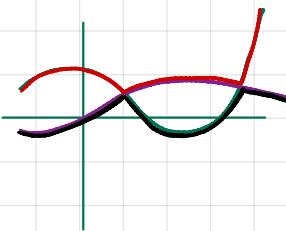
$|f(x)|$ not diff at ± 1 .

c) f, g diff at $a \rightarrow \frac{\max(f, g)}{\min(f, g)}$ diff at a

Proof

$$h(x) = \max(f(x), g(x))$$

$$\therefore \frac{f(a) + g(a) + |f(a) - g(a)|}{2}$$



$f(a) - g(a)$ is diff at a .

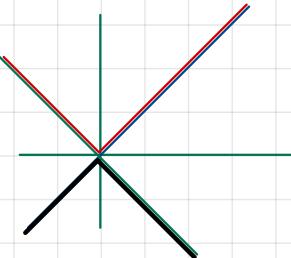
From a), we know that provided $f(a) - g(a) \neq 0$ ie $f(a) \neq g(a)$, $\frac{|f(a) - g(a)|}{2}$ is diff at a .

Hence $h(x)$ diff at a .

$$m(x) = \min(f(x), g(x)) = \frac{f(x) + g(x) - |f(x) - g(x)|}{2}$$

Thus also diff at a .

d) $f(a) = g(a)$



$$f(x) = x$$

$$g(x) = -x$$

$$f(0) = g(0) = 0$$

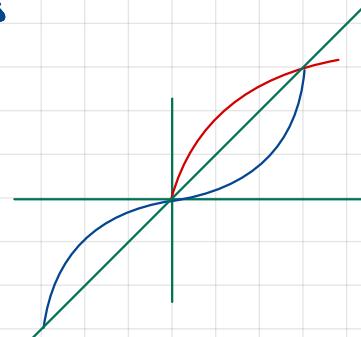
$$h(x) = \frac{x - x + |x - (-x)|}{2} = |x|$$

$$m(x) = \frac{x - x - |x - (-x)|}{2} = -|x|$$

neither of them is diff at 0.

17. f and g inv.

g takes on all values
 f and g diff.
 f not diff.



$$g(x) = x^3, \text{ diff } \forall x$$

$$f(x) = \sqrt[3]{x}$$

$$(f \circ g)(x) = x$$

18.

a) $g(x) = f^2(x)$

$$g'(x) = 2f(x)f'(x)$$

b) $g(x) = (f'(x))^2$

$$g'(x) = 2f'(x)f''(x)$$

c) $\forall x, f'(x) > 0$

$$(f'(x))^2 = f(x) + \frac{1}{f^2(x)} > 0$$

$$2f'(x)f''(x) = \cancel{f'(x)} - 3(f'(x))^{-4} \cancel{f'(x)}$$

$$f''(x) = \frac{1}{2} - \frac{3}{2f^4(x)}$$

f, f', f'', f''' exist, i.e. three times differentiable.

$$f'(x) \neq 0$$

$$\text{Schwarzian derivative of } f: D^S(f) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left(\frac{f''(x)}{f'(x)} \right)^2$$

$$\text{a) } D(f \circ g) = [Df \circ g] \cdot g'^2 + Dg$$

Proof

$$(f \circ g)(x) = f(g(x))$$

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

$$(f \circ g)''(x) = f''(g(x))(g'(x))^2 + f'(g(x))g''(x)$$

$$\begin{aligned} (f \circ g)'''(x) &= f'''(g(x))(g'(x))^3 + f''(g(x)) \cdot 2g'(x)g''(x) + f''(g(x))g'(x)g'''(x) + f'(g(x))g'''(x) \\ &= f'''(g(x))(g'(x))^3 + 3f''(g(x))g'(x)g''(x) + f'(g(x))g'''(x) \end{aligned}$$

$$\frac{(f \circ g)'''(x)}{(f \circ g)'(x)} = \frac{f'''(g(x))(g'(x))^3 + 3f''(g(x))g'(x)g''(x) + f'(g(x))g'''(x)}{f'(g(x))g'(x)}$$

$$\frac{(f \circ g)'''(x)}{(f \circ g)'(x)} = \frac{f''(g(x))(g'(x))^2 + f'(g(x))g''(x)}{f'(g(x))g'(x)}$$

$$(Df \circ g)(x) = \frac{f''(g(x))}{f'(g(x))} - \frac{3}{2} \left(\frac{f''(g(x))}{f'(g(x))} \right)^2$$

$$(Df \circ g)(x) \cdot g'^2(x) + Dg(x)$$

$$= g'^2(x) \left[\frac{f''(g(x))}{f'(g(x))} - \frac{3}{2} \left(\frac{f''(g(x))}{f'(g(x))} \right)^2 \right] + \frac{g''(x)}{g'(x)} - \frac{3}{2} \left(\frac{g''(x)}{g'(x)} \right)^2$$

$$D(f \circ g)(x) =$$

$$\frac{f'''(g(x))(g'(x))^2 + 3f''(g(x))g'(x)g''(x) + f'(g(x))g'''(x)}{f'(g(x))g'(x)} - \frac{3}{2} \left[\frac{f''(g(x))(g'(x))^2 + f'(g(x))g''(x)}{f'(g(x))g'(x)} \right]$$

$$= \frac{f'''(g(x))}{f'(g(x))} g'^2(x) + \frac{3f''(g(x))g''(x)}{f'(g(x))} + \frac{g''(x)}{g'(x)} - \frac{3}{2} \frac{f''(g(x))g'^2(x)}{f'(g(x))}$$

$$- \frac{3f''(g(x))g'(x)f'(g(x))g''(x)}{(f'(g(x))g'(x))^2} - \frac{3}{2} \frac{g''(x)}{g'^2(x)}$$

Therefore

$$D(f \circ g)(x) = (Df \circ g)(x) \cdot g'(x) + Dg(x)$$

b)

$$f(x) = \frac{ax+b}{cx+d} \rightarrow Df = 0 \rightarrow D(f \circ g) = Dg$$

$\cancel{ad-bc \neq 0}$

Proof:

$$f'(x) = \frac{a(cx+d) - c(ax+b)}{(cx+d)^2} = \frac{x(ac-ad) + ad-bc}{(cx+d)^2}$$

$$= \frac{ad-bc}{(cx+d)^2}$$

$$f''(x) = \frac{-(ad-bc) \cdot 2(cx+d)c}{(cx+d)^4} = \frac{2c^2b - 2acd}{(cx+d)^3}$$

$$f'''(x) = \frac{-(2c^2b - 2acd) \cdot 3(cx+d)c}{(cx+d)^5} = \frac{6ac^2d - 6c^3b}{(cx+d)^4}$$

$$Df(x) = \frac{\frac{6ac^2d - 6c^3b}{(cx+d)^4}}{\frac{ad-bc}{(cx+d)^2}} - \frac{3}{2} \left[\frac{\frac{2c^2b - 2acd}{(cx+d)^3}}{\frac{ad-bc}{(cx+d)^2}} \right]^2$$

$$= \frac{6c^2(ad-bc)}{(cx+d)^2(ad-bc)} - \frac{3}{2} \left[\frac{2c(bc-ad)}{(cx+d)(ad-bc)} \right]^2$$

$$= \frac{6c^2}{(cx+d)^2} - \frac{3}{2} \cancel{\frac{2c^2}{(cx+d)^4}}$$

$$= \frac{6c^2}{(cx+d)^2} - \frac{6c^2}{(cx+d)^2}$$

$$= 0$$

$$20. f^{(n)}(a) \text{ and } g^{(n)}(a) \text{ exist.} \rightarrow (f \cdot g)^{(n)}(a) = \sum_{k=0}^n \binom{n}{k} f^{(k)}(a) \cdot g^{(n-k)}(a) \quad (\text{Leibniz's formula})$$

Proof

$$\text{we want to prove } \forall n \ (n \in \mathbb{N} \rightarrow [f^{(n)}(a) \text{ and } g^{(n)}(a) \text{ exist.} \rightarrow (f \cdot g)^{(n)}(a) = \sum_{k=0}^n \binom{n}{k} f^{(k)}(a) \cdot g^{(n-k)}(a)])$$

$$\text{let } A = \{n : Q(n)\} \text{ where } Q(n) = (f \cdot g)^{(n)}(a) = \sum_{i=0}^n \binom{n}{i} f^{(i)}(a) \cdot g^{(n-i)}(a)$$

$$\text{ie, } A = \{n : (f \cdot g)^{(n)}(a) = \sum_{i=0}^n \binom{n}{i} f^{(i)}(a) \cdot g^{(n-i)}(a)\}$$

we will use induction on \mathbb{N} .

Inductive Definition of \mathbb{N}

base clause: $1 \in \mathbb{N}$

inductive clause: $n \in \mathbb{N} \rightarrow n+1 \in \mathbb{N}$

final clause: $\forall n, n \in \mathbb{N} \rightarrow n \text{ is generated by repeated applic. of base and inductive clauses}$

ie, considering every set satisfying base and inductive clauses above, \mathbb{N} is their intersection.

if we show that A satisfies the clauses then we can conclude $\mathbb{N} \subseteq A$.

$$\text{Hence } \forall n \ (n \in \mathbb{N} \rightarrow n \in A \rightarrow Q(n))$$

Base step: show that A satisfies the base clause of \mathbb{N}

$$(f \cdot g)'(a) = f'g + fg' = \sum_{i=0}^1 \binom{i}{i} f^{(i)}(a) g^{(i-1)}(a) \rightarrow 1 \in A$$

Inductive step: show that A satisfies the inductive clause of \mathbb{N}

$$\text{Assume } (f \cdot g)^{(n)}(a) = \sum_{i=0}^n \binom{n}{i} f^{(i)}(a) g^{(n-i)}(a), \text{ ie } n \in A. \text{ Then}$$

$$\begin{aligned} & (f \cdot g)^{(n+1)}(a) = \sum_{i=0}^{n+1} \binom{n+1}{i} (f^{(i)}(a) g^{(n+1-i)}(a) + f^{(i+1)}(a) g^{(n+1-i-1)}(a)) \\ &= \binom{n}{0} f^{(n)} g^{(1)} + \binom{n}{1} f^{(n-1)} g^{(2)} + \binom{n}{2} f^{(n-2)} g^{(3)} + \binom{n}{3} f^{(n-3)} g^{(4)} + \dots + \binom{n}{n-1} f^{(1)} g^{(n)} + \binom{n}{n} f^{(0)} g^{(n+1)} \\ & \quad \binom{n}{0} f^{(n+1)} g^{(0)} + \binom{n}{1} f^{(n)} g^{(1)} + \binom{n}{2} f^{(n-1)} g^{(2)} + \binom{n}{3} f^{(n-2)} g^{(3)} + \dots + \binom{n}{n-1} f^{(1)} g^{(n)} + \binom{n}{n} f^{(0)} g^{(n+1)} \\ &= \binom{n+1}{0} f^{(n+1)} g^{(0)} + \binom{n+1}{1} f^{(n)} g^{(1)} + \binom{n+1}{2} f^{(n-1)} g^{(2)} + \binom{n+1}{3} f^{(n-2)} g^{(3)} + \dots + \binom{n+1}{n} f^{(1)} g^{(n)} + \binom{n+1}{n+1} f^{(0)} g^{(n+1)} \\ &= \sum_{i=0}^{n+1} \binom{n+1}{i} f^{(i)} g^{(n+1-i)} \rightarrow n+1 \in A \end{aligned}$$

Therefore, $n \in A \rightarrow n+1 \in A$. A satisfies inductive clause of \mathbb{N} .

$\mathbb{N} \subseteq A$

$$\forall n \ (n \in \mathbb{N} \rightarrow n \in A \rightarrow (f \cdot g)^{(n)}(a) = \sum_{i=0}^n \binom{n}{i} f^{(i)}(a) \cdot g^{(n-i)}(a))$$

21. $f^{(n)}(g(a))$ and $g^{(n)}(a)$ exist $\rightarrow (f \circ g)^{(n)}(a)$ exists.

Proof

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

$$(f \circ g)''(x) = f''(g(x)) [g'(x)]^2 + f'(g(x))g''(x)$$

$$\begin{aligned}(f \circ g)^{(n)}(x) &= f''(g(x)) [g'(x)]^2 + f'(g(x)) \cdot 2g'(x)g''(x) + f''(g(x))g'(x)g''(x) + f'(g(x))g'''(x) \\ &\quad + f''(g(x)) [g'(x)]^3 + 3f''(g(x))g''(x)g'(x) + f'(g(x))g''''(x)\end{aligned}$$

Seems like $(f \circ g)^{(n)}$ is a sum of terms, each a product of some number, a certain k^{th} derivative of f ($k \leq n$), and one or more derivatives of g up to n^{th} derivative.

Let $A = \{n : n \in \mathbb{N} \wedge f^{(n)}(g(a))$ and $g^{(n)}(a)$ exist $\rightarrow (f \circ g)^{(n)}(a)$ exists, and

$(f \circ g)^{(n)}(a)$ is a sum of terms of form

$$c f^{(m)}(g(a)) \cdot [g^{(m)}(a)]^{m_1} [g^{(m)}(a)]^{m_2} \cdots [g^{(m)}(a)]^{m_n}$$

where c is a number, $m, m_1, \dots, m_n \in \mathbb{N}^+$, and $k \leq n \in \mathbb{N}\}$

Consider $n=1$.

Assume $f^{(1)}(g(a))$ and $g^{(1)}(a)$ exist.

Then $(f(g(x)))' = (f \circ g)'(a) = f'(g(a)) \cdot g'(a)$, which exists.

i.e. $(f \circ g)^{(1)}(a)$ exists and is of the required form.

$\rightarrow 1 \in A$.

Assume $k \in A$. Then

$f^{(k)}(g(a))$ and $g^{(k)}(a)$ exist. $\rightarrow (f \circ g)^{(k)}(a)$ exists and has the form specified in A's def.

Assume

$f^{(k+1)}(g(a))$ and $g^{(k+1)}(a)$ exist.

Then,

$$(f \circ g)^{(k+1)}(a) = [(f \circ g)^{(k)}(a)]'$$

Every term in $(f \circ g)^{(k+1)}(a)$ is of form $c f^{(m)}(g(a)) \cdot [g^{(m)}(a)]^{m_1} [g^{(m)}(a)]^{m_2} \cdots [g^{(m)}(a)]^{m_n}$

22.

$$a) f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

$$g'_i = f$$

$$g_1(x) = \frac{a_n}{n+1} x^{n+1} + \frac{a_{n-1}}{n} x^n + \dots + a_0 x + c_1$$

$$g_2(x) = \frac{a_n}{n+1} x^{n+1} + \frac{a_{n-1}}{n} x^n + \dots + a_0 x + c_2$$

$$g'_1 = f$$

$$g'_2 = f$$

$$b) f(x) = \frac{b_2}{x^2} + \frac{b_3}{x^3} + \dots + \frac{b_m}{x^m}$$

$$g(x) = -\frac{b_2}{x} - \frac{b_3}{2x^2} - \dots - \frac{b_m}{(m-1)x^{m-1}}$$

$$\begin{aligned} g'(x) &= -(-1)b_2 x^{-2} - (-2)\frac{b_3}{2} x^{-3} - \dots - (-m+1) \frac{b_m}{m-1} x^{-(m-1)} \\ &= \frac{b_2}{x^2} + \frac{b_3}{x^3} + \dots + \frac{b_m}{x^m} \end{aligned}$$

$$\begin{aligned} c) f(x) &= a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 + \frac{b_1}{x} + \dots + \frac{b_m}{x^m} \\ &= \sum_{i=0}^n g_i(x) + \sum_{i=1}^m h_i(x), \quad g_i(x) = a_i x^i, \quad h_i(x) = \frac{b_i}{x^i} \end{aligned}$$

$$g'_i(x) = a_i i x^{i-1}, \quad i=1, \dots, n, \quad g'_0(x) = 0$$

$$h'_i(x) = -\frac{b_i}{x^{i+1}}, \quad i=1, \dots, m$$

There is no term of x^0 .

$$J(x) = \int_1^x \frac{1}{t} dt \quad x > 0$$

$$J(x_1) = \int_1^{x_1} \frac{1}{t} dt = \int_1^x \frac{1}{t} dt + \int_x^{x_1} \frac{1}{t} dt = J(x) + J(x_1)$$

$$\int_x^{x_1} \frac{1}{t} dt = \ln(x_1) - \ln(x) = \ln x_1$$

$$\int_1^x \frac{1}{t} dt = \ln x$$

23. polyn. f, degree

a) $f'(x) = 0$ for $n-1$ numbers x

$$f(x) = (x-a_1)(x-a_2) \dots (x-a_{n-1})$$

This polynomial has degree $n-1$ and $n-1$ roots.

It can be rewritten as

$$f(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \dots + b_1x + b_0$$

$$\text{As shown in problem 22, one } f(x) = \frac{b_{n-1}}{n}x^n + \dots + \frac{b_1}{2}x^2 + b_0x + c$$

is s.t. $g'(x) = f(x)$, and degree of g is n .

b) n odd, $f'(x) = 0$ for no x .

f is an n -degree polynomial, w/ n odd.
Then f' is of degree $n-1$, $n-1$ is even.

$f'(x) = x^{n-1} + S$ has no roots since $x^{n-1} = -S$ has no real sol'n.

$$f(x) = \frac{x^n}{n} + Sx + c$$

c) f' has one root only, n even.

$$f'(x) = x^{n-1} + a_0$$

$$f'(x) = 0 \rightarrow x^{n-1} = -a_0 \rightarrow x = \sqrt[n]{-a_0}$$

only one root.

$$f(x) = \frac{x^n}{n} + a_0x + c$$

d) $n-h$ odd. f' has h roots.

From 7-4b, we know that

let f be polyn. deg. n with h roots, including multiplicities.
Then $n-h$ is even

Now we have f' of degree $n-1$. $(n-1)-h$ is even.

case 1: $n-1$ even, h even.

$$f'(x) = (x-a_1) \dots (x-a_h) (x^{n-1-h} + c) \quad c > 0$$

case 2: $n-1$ odd, h odd

$$f'(x) = (x-a_1) \dots (x-a_{h-1}) (x^{n-1-h} + c)$$

24.

a) a is double root of $f \Leftrightarrow a$ is root of f and f'

Proof

Assume a is double root of f .

Then $f(x) = (x-a)^2 g(x)$, $g(x)$ polyn. and $g(a) \neq 0$.

$$f'(x) = 2(x-a)g(x) + (x-a)^2 g'(x) = 0$$

$$g(x)(x-a) (2 + (x-a)g'(x)) = 0$$

a is root of f'

a is double root of $f \rightarrow a$ is root of f and f'

Assume a is root of f and f'

$$f(x) = (x-a)g(x)$$

$$f'(x) = (x-a)g'(x) + g(x)$$

$$f'(a) = g(a) = 0 \rightarrow a \text{ is root of } g$$

$$g(x) = h(x)(x-a)$$

$$f(x) = (x-a)^2 h(x)$$

a is double root of f .

a is root of f and $f' \rightarrow a$ is double root of f

$$b) f(x) = ax^2 + bx + c \quad (a \neq 0)$$

$$f(x) = ax^2 + bx + c$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left(x^2 + 2x\frac{b}{2a} + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right)$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

If $4ac - b^2 = 0$ then $f(x) = a\left(x + \frac{b}{2a}\right)^2 \rightarrow x = -\frac{b}{2a}$ is a double root.

Also, using part a),

$$f'(x) = 2ax + b = 0 \rightarrow x = -\frac{b}{2a}.$$

If $x = -\frac{b}{2a}$ is a root of f then it is a double root of f' .

This happens if

$$f\left(-\frac{b}{2a}\right) = a\frac{b^2}{4a^2} - b\frac{b}{2a} + c$$

$$= \frac{b^2}{4a} - \frac{b^2}{2a} + c$$

$$= \frac{b^2 - 2b^2 + 4ac}{4a^2}$$

$$= 0$$

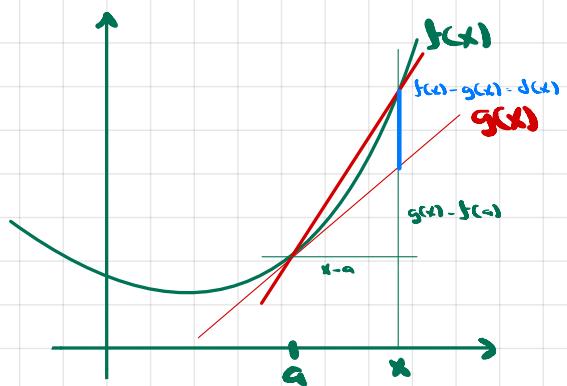
$$\rightarrow -b^2 + 4ac = 0$$

Geometrically, at $x = -\frac{b}{2a}$, the graph of f touches the x -axis, and at that point the tangent line is horizontal.

25.

f diff at a

$$d(x) = f(x) - f'(a)(x-a) - f(a)$$



We want $d'(x)$, the rate of growth of the light blue above as x changes.

$$d'(x) = f'(x) - f'(a)$$

a is the only root of $d'(x)$.

Hence, since a is also a root of $d(x)$, b, 24a) a is a double root of d .

$d(x) = (x-a)^2 h(x)$. We reached this result in q-2ab by other means.

26.

 \exists polyn. f degree $2n-1$ s.t.

a)

a_1, \dots, a_n
 b_1, \dots, b_n given numbers
 x_1, \dots, x_n distinct numbers

$$\begin{aligned}f(x_j) &= f'(x_j) = 0 \quad j \neq i \\f(x_i) &= a_i \\f'(x_i) &= b_i\end{aligned}$$

Proof

We want a $f(x)$ such that considering only points x_1, \dots, x_n , f and f' are only nonzero at a single point x_i .By problem 24, each $x_j, j \neq i$, is a double root of f.Thus f has form $\prod_{j \neq i} (x - x_j)^2 g(x)$. The $\prod_{j \neq i} (x - x_j)^2$ part has degree $2n-2$.g(x) must have degree 1, i.e. $g(x) = ax + b$.

$$f(x) = \prod_{j \neq i} (x - x_j)^2 (ax + b) = h(x)(ax + b)$$

$$f'(x) = h'(x)(ax + b) + ah(x)$$

or don't know what a and b are yet.

$$f'(x_i) = ax_i h(x_i) + bh(x_i) = a_i$$

$$f'(x_i) = a(h'(x_i)x_i + h(x_i)) + bh'(x_i) = b_i$$

$$\underbrace{\begin{bmatrix} h(x_i)x_i \\ h'(x_i)x_i + h(x_i) \end{bmatrix}}_A \begin{bmatrix} h(x_i) \\ h'(x_i) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a_i \\ b_i \end{bmatrix}$$

$$\begin{aligned}\det A &= \cancel{h(x_i)h'(x_i)x_i} - \cancel{h(x_i)h'(x_i)x_i} - \cancel{h^2(x_i)} = -h^2(x_i) \\&= -\left[\prod_{j \neq i} (x_i - x_j)^2 \right]^2 \neq 0 \text{ since } x_1, \dots, x_n \text{ are distinct.}\end{aligned}$$

Hence there is a unique solution a^*, b^* .The f we wanted is $f(x) = \prod_{j \neq i} (x - x_j)^2 (a^*x + b^*)$ b) f of degree $2n-1$

$$\forall i, f(x_i) = a_i \text{ and } f'(x_i) = b_i$$

$$\text{Let } f(x) = \sum_{i=1}^n \left[\prod_{j \neq i} (x - x_j)^2 \right] (a_i^*x + b_i^*)$$

where $\left[\prod_{j \neq i} (x - x_j)^2 \right] (a_i^*x + b_i^*)$ is the f obtained in a) for a specific i.

27. $f(x) = (x-a)(x-b)g(x)$, $g(a) \neq 0$, $g(b) \neq 0$, a and b consecutive roots

a)
Assume $g(a)$ and $g(b)$ have different signs.
wLG, assume $g(a) < 0 < g(b)$.

Since g is polynomial, it is continuous in $[a,b]$.

INT \rightarrow g takes on every value between $g(a)$ and $g(b)$ in $[a,b]$.

$$\exists c, c \in [a,b] \wedge g(c) = 0$$

$$g(x) = (x-a)h(x)$$

$$f(x) = (x-a)(x-b)(x-c)h(x)$$

c is root of f .

a and b are not consecutive roots of f .

1.

$g(a)$ and $g(b)$ have same sign.

b)

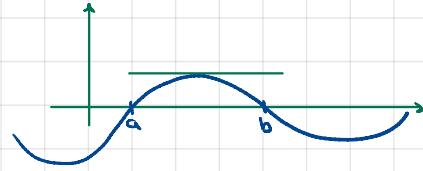
$$f(x) = (x-a)(x-b)g(x)$$

$$g(a) \neq 0$$

$$g(b) \neq 0$$

a, b consecutive roots

$$\rightarrow \exists x, a < x < b \wedge f'(x) = 0$$



Proof

$$f'(x) = (x-b)g(x) + (x-a)g(x) + (x-a)(x-b)g'(x)$$

$$f'(a) = (a-b)g(a)$$

$$f'(b) = (b-a)g(b)$$

Case 1: $g(a) > 0, g(b) > 0$

$$f'(a) < 0$$

$$f'(b) > 0$$

INT: $\exists c, c \in [a,b] \wedge f'(c) = 0$

Case 2: $g(a) < 0, g(b) < 0$

$$f'(a) > 0$$

$$f'(b) < 0$$

INT: $\exists c, c \in [a,b] \wedge f'(c) = 0$

Therefore, b is a root by case 1

$$\exists c, c \in [a,b] \wedge f'(c) = 0$$

c)

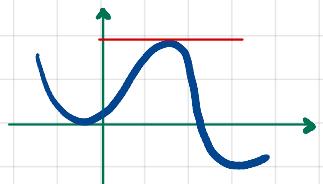
$$f(x) = (x-a)^m (x-b)^n g(x)$$

$$g(a) \neq 0$$

$$g(b) \neq 0$$

a, b consecutive roots

$$\rightarrow \exists x, a < x < b \wedge f'(x) = 0$$



Proof

$$f'(x) = m(x-a)^{m-1} (x-b)^n g(x) + n(x-a)^m (x-b)^{n-1} g(x) + (x-a)^m (x-b)^n g'(x)$$

$$\text{let } h(x) = \frac{f'(x)}{(x-a)^{m-1} (x-b)^{n-1}} = m(x-b)g(x) + n(x-a)g(x) + (x-a)(x-b)g'(x)$$

$$h(a) = m(b-a)g(a)$$

$$h(b) = n(b-a)g(b)$$

Using the same argument from a), $g(a)$ and $g(b)$ must have the same sign, otherwise $\exists c, c \in [a,b] \wedge f(c)=0$ s.t. a and b aren't consecutive, a contradiction of our assumption.

Case 1: $g(a)$ and $g(b) > 0$

$$\begin{aligned} h(a) &< 0 \\ h(b) &> 0 \end{aligned} \rightarrow \exists c, c \in [a,b] \wedge h(c)=0 \rightarrow f'(c)=0$$

Case 2: $g(a)$ and $g(b) < 0$

$$\begin{aligned} h(a) &> 0 \\ h(b) &< 0 \end{aligned} \rightarrow \exists c, c \in [a,b] \wedge h(c)=0 \rightarrow f'(c)=0$$

B) proof by cases

$$\exists c, c \in [a,b] \wedge f'(c)=0$$

28. $f(x) = xg(x)$
 g cont. at 0

$\rightarrow f$ diff at 0

Proof

$$f'(0) = \lim_{h \rightarrow 0} \frac{hg(h)}{h} = \lim_{h \rightarrow 0} g(h) = g(0)$$

29. f diff at 0 $\rightarrow f(x) = xg(x)$ for some g cont. at 0
 $f(0) = 0$ \rightarrow

Proof

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h)}{h} \quad f \text{ diff at 0 exists}$$

Let

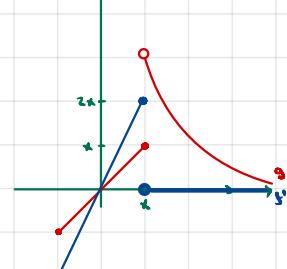
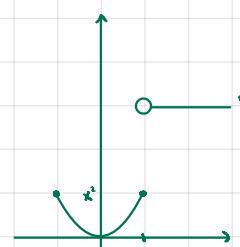
$$g(x) = \begin{cases} \frac{f(x)}{x} & x \neq 0 \\ f'(0) & x=0 \end{cases}$$

Then $g(x)$ is the slope of the line from $(0,0)$ to $(x, f(x))$.

$$f(x) = xg(x) \quad \forall x, f(0) = 0.$$

$$f'(0) = \lim_{h \rightarrow 0} g(h) = g(0), \text{ so } g \text{ cont. at 0.}$$

This result is telling us that any f that passes thru origin and is differentiable there can be written as the product of x and a g , the latter giving the slope of the line from $(0,0)$ to $(x, f(x))$, and g is cont. at 0.



$$20. \forall n \in \mathbb{N} \rightarrow [f(x) = x^{-n} \rightarrow f^{(n)}(x) = (-1)^n \frac{(n+k-1)!}{(n-1)!} x^{-n-k} = (-1)^n k! \binom{n+k-1}{k} x^{-n-k} \neq 0]$$

Proof

Let's use induction on N and n, k .

$$A = \{n \in \mathbb{N} \rightarrow [\forall h \in \mathbb{N} \rightarrow [f(x) = x^{-n} \rightarrow f^{(n)}(x) = (-1)^n \frac{(n+k-1)!}{(n-1)!} x^{-n-k} \neq 0]]\}$$

Assume $n=1$.

$$f(x) = x^{-1}$$

$$\text{Let } B_1 = \{h \in \mathbb{N} \rightarrow [f(x) = x^{-1} \rightarrow f^{(n)}(x) = (-1)^h \frac{(1+k-1)!}{(n-1)!} x^{-1-k} \neq 0]\}$$

Let $k=1$.

$$f'(x) = -x^{-2} = (-1)^1 \binom{1}{2} x^{-2}$$

$1 \in B_1$

$$\text{Assume } h \in B_1, \text{ then } f(x) = x^{-1} \rightarrow f^{(n)}(x) = (-1)^h \frac{h!}{0!} x^{-1-h} \neq 0$$

$$f^{(n)}(x) = (-1-h)(-1)^h \frac{h!}{0!} x^{-1-h-1} \neq 0$$

$$= (-1)(h+1)(-1)^h \frac{h!}{0!} x^{-1-(h+1)} \neq 0$$

$$= (-1)^{h+1} \frac{(h+1)!}{0!} x^{-1-(h+1)} \neq 0$$

$h+1 \in B_1$

$$h \in B_1 \rightarrow h+1 \in B_1$$

$$N \subseteq B_1, \text{ ie } 1 \in N \rightarrow [\forall h \in \mathbb{N} \rightarrow [f(x) = x^{-1} \rightarrow f^{(n)}(x) = (-1)^h \frac{(1+k-1)!}{(n-1)!} x^{-1-k} \neq 0]]$$

$$\text{Assume } n \in \mathbb{N} \text{ A. Then } f(x) = x^{-n} \text{ and } [\forall h \in \mathbb{N} \rightarrow [f(x) = x^{-n} \rightarrow f^{(n)}(x) = (-1)^h \frac{(n+k-1)!}{(n-1)!} x^{-n-k} \neq 0]]$$

$$\text{Let } f(x) = x^{-(n+1)}$$

$$\text{Let } B_{n+1} = \{h \in \mathbb{N} \rightarrow [f(x) = x^{-(n+1)} \rightarrow f^{(n)}(x) = (-1)^h \frac{(n+h)!}{n!} x^{-n-h-1} \neq 0\}$$

Let $k=1$.

$$f'(x) = -(n+1)x^{-n-2} = (-1)^1 \frac{(n+1)!}{n!} x^{-n-2} \rightarrow 1 \in B_{n+1}$$

$$\text{Assume } h \in B_{n+1}. \text{ Then } f(x) = x^{-(n+1)} \rightarrow f^{(n)}(x) = (-1)^h \frac{(n+h)!}{n!} x^{-n-h-1} \neq 0$$

$$f^{(n)}(x) = (-n-h-1)(-1)^h \frac{(n+h)!}{n!} x^{-n-h-2} \neq 0$$

$$= (-1)(n+h+1)(-1)^h \frac{(n+h)!}{n!} x^{-n-h-2} \neq 0$$

$$= (-1)^{h+1} \frac{(n+h+1)!}{n!} x^{-n-(h+1)-1} \neq 0$$

$h+1 \in B_{n+1}$

$$h \in B_{n+1} \rightarrow h+1 \in B_{n+1}$$

$$N \subseteq B_{n+1}, \text{ ie } \forall h, h \in \mathbb{N} \rightarrow [f(x) = x^{-(n+1)} \rightarrow f^{(n)}(x) = (-1)^h \frac{(n+h)!}{n!} x^{-n-h-1} \neq 0]$$

$n \in \mathbb{N} \rightarrow n \in \mathbb{N} \text{ A}$

$$N \subseteq \mathbb{N}, \text{ ie } \forall n \in \mathbb{N} \rightarrow [\forall h \in \mathbb{N} \rightarrow [f(x) = x^{-n} \rightarrow f^{(n)}(x) = (-1)^h \frac{(n+k-1)!}{(n-1)!} x^{-n-k} \neq 0]]$$

31. f, g differentiable

$$f(a) = g(a) = 0$$

$$\text{Let } h(x) = f(x)g(x)$$

$$\text{Assume } h(x) = x$$

$$\text{Then } h'(x) = f'(x)g(x) + f(x)g'(x) = 1$$

$$h'(0) = 0 = 1$$

⊥.

32.

a) $f(x) = \frac{1}{(x-a)^n}, x \neq a$

$$\text{Let } g(x) = f(x+a) = \frac{1}{x^n}, x \neq 0$$

By problem 30,

$$g^{(n)}(x) = f^{(n)}(x+a) = (-1)^n \frac{(n+k-1)!}{(n-1)!} x^{-n-k}, x \neq 0$$

$$g^{(n)}(x-a) = f^{(n)}(x) = (-1)^n \frac{(n+k-1)!}{(n-1)!} (x-a)^{-n-k} \quad x \neq a$$

b) $f(x) = \frac{1}{x^2-1} = \frac{1}{x+1} \cdot \frac{1}{x-1} = g(x)h(x)$

From Leibniz's formula (problem 20),

$$\begin{aligned} f^{(n)}(x) &= (g \cdot h)^{(n)}(x) = \sum_{k=0}^n \binom{n}{k} g^{(k)}(x) h^{(n-k)}(x) \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{k!}{0!} (x-1)^{-1-k} (-1)^{n-k} \frac{(n+k)!}{0!} (x+1)^{-1-k} \\ &\quad - \sum_{k=0}^n \binom{n}{k} (-1)^k k! (n+k)! (x-1)^{-1-k} \end{aligned}$$

Alternatively, write it as sum of terms and apply part a).

$$f(x) = \frac{1}{2} \left(\frac{1}{x+1} - \frac{1}{x-1} \right)$$

$$f^{(n)}(x) = \frac{1}{2} (-1)^n \frac{n!}{(n-1)!} \left[(x+1)^{-1-n} - (x-1)^{-1-n} \right]$$

$$33. f(x) = \begin{cases} x^{2n} \sin(1/x) & x \neq 0 \\ 0 & x=0 \end{cases} \rightarrow f'(0), \dots, f^{(n)}(0) \text{ exist} \\ f^{(n+1)} \text{ not continuous at } 0$$

Proof

$$f'(x) = 2nx^{2n-1} \sin(1/x) + x^{2n} \cos(1/x) (-1/x^2)$$

$$= \sin(1/x) 2nx^{2n-1} + \cos(1/x) (-x^{2n-2})$$

$$f''(x) = \cos(1/x) (-1/x^2) \cdot 2nx^{2n-1} + \cos(1/x) ((-2n-2)x^{2n-3})$$

$$- \sin(1/x) (-1/x^2) (-x^{2n-2}) + \sin(1/x) 2n(2n-1)x^{2n-2}$$

$$= \sin(1/x) \left[x^{2n-2} \frac{(2n)!}{(2n-2)!} - x^{2n-4} \right] + \cos(1/x) \left[-x^{2n-3} (2n+2n-2) \right]$$

$$= \sin(1/x) \left[x^{2n-2} \frac{(2n)!}{(2n-2)!} - x^{2n-4} \right] + \cos(1/x) \left[-x^{2n-3} 2(2n-1) \right]$$

$$f'''(x) = \cos(1/x) (-1/x^2) \left[x^{2n-2} \frac{(2n)!}{(2n-2)!} - x^{2n-4} \right] + \cos(1/x) (-2(2n-1)(2n-3)x^{2n-4})$$

$$- \sin(1/x) (-1/x^2) \left[-x^{2n-3} 2(2n-1) \right] + \sin(1/x) \left[x^{2n-3} \frac{(2n)!}{(2n-3)!} - (2n-4)x^{2n-5} \right]$$

$$= \sin(1/x) \left[x^{2n-3} \frac{(2n)!}{(2n-3)!} + x^{2n-5} (-2(2n-1)-(2n-4)) \right]$$

$$+ \cos(1/x) \left[-x^{2n-4} \left(\frac{(2n)!}{(2n-2)!} + 2(2n-1)(2n-3) \right) + x^{2n-6} \right]$$

$$= \sin(1/x) \left[x^{2n-3} \frac{(2n)!}{(2n-3)!} + x^{2n-5} (6-6n) + \cos(1/x) \left[-x^{2n-4} \left(\frac{(2n)!}{(2n-2)!} + 2(2n-3) \right) + x^{2n-6} \right] \right. \\ \left. x^{2n-4} (2-4n)(3n-3) \right]$$

Based on the calculations above we can make the following conjecture

$f^{(n)}(x)$ is composed of the following terms

$$\sin(1/x) x^{2n-h} \cdot a.$$

$$\pm \sin(1/x) x^{2n-2h} \text{ if h even}$$

$$\pm \cos(1/x) x^{2n-2h} \text{ if h odd}$$

$$\sum_{i=h}^{2n-1} (a_i x^{2n-i} \sin(1/x) + b_i x^{2n-i} \cos(1/x))$$

We check our conjecture using induction on n .

$$f'(x) = \sin(1/x) 2nx^{2n-1} + \cos(1/x)(-x^{2n-2})$$

\rightarrow conjecture true for $n=1$.

Assume it's true for n .

If it's true for $n+1$ then we should only see the following terms in $f^{(n+1)}(x)$:

$$\sin(1/x)x^{2n-(n+1)}a$$

$$\pm \sin(1/x)x^{2n-2(n+1)}$$

$$\pm \cos(1/x)x^{2n-2(n+1)}$$

$$\sum_{i=n+2}^{2(n+1)-1} (a_i x^{2n-i} \sin(1/x) + b_i x^{2n-i} \cos(1/x))$$

We check that this is the case by differentiating $f^{(n+1)}(x)$. Then $f^{(n+2)}(x)$ is composed of the terms

$$\begin{aligned} \frac{d}{dx} \sin(1/x)x^{2n-n}a &= \cos(1/x)(-1/x^2)x^{2n-n} + \sin(1/x)(2n-n)x^{2n-n-1} \\ &= \cos(1/x)x^{2n-n-2} + \sin(1/x)(2n-n)x^{2n-n-1} \\ &\quad + b_i \cos(1/x)x^{2n-(n+2)} + a_i x^{2n-(n+1)} \sin(1/x) \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} (\pm \sin(1/x)x^{2n-2n}) &= \mp \cos(1/x)(-1/x^2)x^{2n-2n} \pm \sin(1/x)(2n-2n)x^{2n-2n-1} \\ &= \mp \cos(1/x)x^{2n-2(n+1)} \pm \sin(1/x)(2n-2n)x^{2n-(2(n+1))-1} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} (\pm \cos(1/x)x^{2n-2n}) &= \pm \sin(1/x)(-1/x^2)x^{2n-2n} \pm \cos(1/x)(2n-2n)x^{2n-2n-1} \\ &= \mp \sin(1/x)x^{2n-2(n+1)} \pm \cos(1/x)(2n-2n)x^{2n-(2(n+1))-1} \end{aligned}$$

$$\frac{d}{dx} \left[\sum_{i=n+1}^{2n-1} (a_i x^{2n-i} \sin(1/x) + b_i x^{2n-i} \cos(1/x)) \right]$$

$$= \sum_{i=n+1}^{2n-1} [a_i (2n-i)x^{2n-i-1} \sin(1/x) - a_i x^{2n-i-2} \cos(1/x) + b_i (2n-i)x^{2n-i-1} \cos(1/x) + b_i x^{2n-i-2} \sin(1/x)]$$

$$= \sum_{i=n+1}^{2n-1} [\sin(1/x)(a_i (2n-i)x^{2n-(i+1)} + b_i x^{2n-(i+1)-1}) + \cos(1/x)(b_i (2n-i)x^{2n-(i+1)} - a_i x^{2n-(i+1)-1})]$$

$$= \sum_{i=n+1}^{2n-1} [\sin(1/x)(a_i (2n-i)x^{2n-(i+1)} + b_i x^{2n-(i+1)}) \cdot \frac{1}{x}]$$

$$= \sum_{i=n+1}^{2n-1} [\sin(1/x)(x^{2n-(i+1)}(a_i (2n-i) + \frac{b_i}{x}))]$$

not sure what to do w/ these terms
?? ??

$$\sum_{i=n+1}^{2(n+1)-1} [A_i \sin(1/x)x^{2n-i} + B_i \cos(1/x)x^{2n-i} + C_i \sin(1/x)x^{2n-(i+1)} + D_i \cos(1/x)x^{2n-(i+1)}]$$

Now we want to show that $f'(0), \dots, f^{(n)}(0)$ exist.

Since $f^{(n)}(x)$ is composed of the following types of terms

$$\sin(1/x)x^{2n-h}$$

$$\pm \sin(1/x)x^{2n-2h} \text{ if } h \text{ even}$$

$$\pm \cos(1/x)x^{2n-2h} \text{ if } h \text{ odd}$$

$$\sum_{i=0}^{2n-1} (a_i x^{2n-i} \sin(1/x) + b_i x^{2n-i} \cos(1/x))$$

n	x^{2n}	$f^{(n)}$
1	x^2	$f^{(1)}$
2	x^4	$f^{(2)}$
3	x^6	$f^{(3)}$

We conjecture that: every term above contains x^m , $m \geq 2$, for $k < n$, if $n \geq 2$.

Proof

$x^{2n-h}, x^{2n-h}, x^{2n-(2k+1)}, \dots, x^{2n-(2k-1)}$ appear in terms above.

For $n \geq 2$,

$$3 \leq 2(n-h)+1 = 2n-2h+1 = 2n-(2k-1) \leq 2n-h \leq 2n-1$$

■

Note that for $h \geq n$ the above is no longer true. $x^{2n-2n} = 1$.

$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h)}{h} = 0$, since all terms in f contain at least x^2 , times a bounded factor.

Assume $f^{(k)}(0) = 0$, $k < n$. Then $f^{(k+1)}(0) = \lim_{h \rightarrow 0} \frac{f^{(k)}(h)}{h} = 0$, because of

Therefore $f'(0), \dots, f^{(n)}(0)$ exist. Also $\lim_{x \rightarrow 0} f^{(k)}(x) = 0$ because of the at least x^2 factors.

$f', \dots, f^{(k)}$ are thus all continuous and diff. for $k < n$.

But $\lim_{x \rightarrow 0} f^{(n)}(x) \neq 0$, in fact since there are terms in $f^{(n)}$ that are like $\sin(1/x)$ or $\cos(1/x)$,

this limit does not exist: $f^{(n)}$ is not cont. at 0, thus not diff. at 0.

Alternative solution

Proposition

$\forall n, n \in \mathbb{N}$ h, g n times cont. diff $\rightarrow f(x) = \begin{cases} 0 \\ x^{2n}(h(x)\sin(1/x) + g(x)\cos(1/x)) \end{cases}$ is n -times diff.

$x=0$
 $x \neq 0$

$(g(0) \neq 0 \text{ or } h(0) \neq 0) \rightarrow f^{(n)}$ discnt. at 0

Proof induction on n

$n=1$

$$f(x) = \begin{cases} 0 \\ x^2(h(x)\sin(1/x) + g(x)\cos(1/x)) \end{cases}$$

$$f'(0) = \lim_{m \rightarrow 0} \frac{m^2(h(m)\sin(1/m) + g(m)\cos(1/m))}{m}$$

$$= \lim_{m \rightarrow 0} m(h(m)\sin(1/m) + g(m)\cos(1/m))$$

$$= 0$$

$$f'(x) = 2x(h(x)\sin(1/x) + g(x)\cos(1/x))$$

$$+ x^2(h'(x)\sin(1/x) - h(x)\cos(1/x)) + g'(x)\cos(1/x) + g(x)\sin(1/x)$$

$$= 2x(h(x)\sin(1/x) + g(x)\cos(1/x)) + x^2h'(x)\sin(1/x) - h(x)\cos(1/x) + x^2g'(x)\cos(1/x) + g(x)\sin(1/x)$$

$\lim_{x \rightarrow 0} f'(x)$ doesn't exist.

Assume that f for some n

h, g n times cont. diff $\rightarrow f(x) = \begin{cases} 0 \\ x^{2n}(h(x)\sin(1/x) + g(x)\cos(1/x)) \end{cases}$ is n -times diff.

$x=0$
 $x \neq 0$

$(g(0) \neq 0 \text{ or } h(0) \neq 0) \rightarrow f^{(n)}$ discnt. at 0

Let $h(x), g(x)$ be n times diff.

Then if $f'(x) = \begin{cases} 0 \\ x^{2n}(h(x)\sin(1/x) + g(x)\cos(1/x)) \end{cases}$

$x=0$
 $x \neq 0$

By applying the inductive hypothesis

$f'(x)$ is n -times diff, so f is $n+1$ -times diff; $f^{(n+1)}$ is discnt. at 0 if $g(0) \neq 0$ or $h(0) \neq 0$.

34.

$$\forall n \in \mathbb{N}, f(x) = \begin{cases} x^{2n+1} \sin(\pi x) & x \neq 0 \\ 0 & x=0 \end{cases} \rightarrow \begin{array}{l} f \text{ n-times diff. at } 0 \\ f \text{ cont. at } 0 \\ f \text{ not diff. at } 0. \end{array}$$

Proof

n=1

$$f(x) = \begin{cases} x^3 \sin(\pi x) & x \neq 0 \\ 0 & x=0 \end{cases}$$

$$f'(x) = 3x^2 \sin(\pi x) + x^3 \cos(\pi x)(-\pi x^2) \quad x \neq 0$$

$$\lim_{h \rightarrow 0} \frac{h^3 \sin(\pi h)}{h} = 0 \quad x=0$$

$$\lim_{x \rightarrow 0} f'(x) = 0 \rightarrow f'(x) = f' \text{ cont. at } 0$$

$$f''(0) = \lim_{h \rightarrow 0} \frac{3h^2 \sin(\pi h) - h \cos(\pi h)}{h}$$

this limit does not exist because $\lim_{h \rightarrow 0} \cos(\pi h)$ doesn't exist.

$$\rightarrow f''(0) = f'' \text{ not diff.}$$

n Assume that

$$f(x) = \begin{cases} x^{2n+1} \sin(\pi x) & x \neq 0 \\ 0 & x=0 \end{cases} \rightarrow \begin{array}{l} f \text{ n-times diff. at } 0 \\ f \text{ cont. at } 0 \\ f \text{ not diff. at } 0. \end{array}$$

$$\text{let } g(x) = \begin{cases} x^{2(n+1)+1} \sin(\pi x) & x \neq 0 \\ 0 & x=0 \end{cases}$$

$$\text{Then } g'(x) = (2n+3)x^{2(n+1)} \sin(\pi x) - x^{2(n+1)-1} \cos(\pi x)$$

$$= \underline{(2n+3)x^{2(n+1)} \sin(\pi x)} - \underline{x^{2n+1} \cos(\pi x)}$$

apply problem 33

$\rightarrow n+1$ time diff
 $(n+1)^{\text{th}}$ diff not cont at 0

can't quite apply inductive hypothesis
 but the $\cos(\pi x)$ case is analog
 if we could apply i.h.

n times diff at 0
 n^{th} deriv. cont at 0, not diff at 0

see next page for alternative and more general sol'n.

$$\forall n, n \in \mathbb{N}, f(x) = \begin{cases} h(x)x^{2n+1}\sin(1/x) + g(x)x^{2n+1}\cos(1/x) & x \neq 0 \\ 0 & x=0 \end{cases} \rightarrow \begin{array}{l} f \text{ n-times diff at } 0 \\ f \text{ cont. at } 0 \\ f^{(n)} \text{ not diff. at } 0. \end{array}$$

h, g n-times diff

Proof

$n=1$

$$f(x) = \begin{cases} h(x)x^3\sin(1/x) + g(x)x^3\cos(1/x) & x \neq 0 \\ 0 & x=0 \end{cases}$$

$$= \begin{cases} x^3(h(x)\sin(1/x) + g(x)\cos(1/x)) & x \neq 0 \\ 0 & x=0 \end{cases}$$

$$f'(x) = 3x^2(h(x)\sin(1/x) + g(x)\cos(1/x)) + x^3[h(x)\sin(1/x) - \frac{h(x)\cos(1/x)}{x^2}] + g'(x)\cos(1/x) + \frac{g(x)\sin(1/x)}{x^2}$$

$$= 3x^2(h(x)\sin(1/x) + g(x)\cos(1/x)) + x^3h(x)\sin(1/x) - xh(x)\cos(1/x)$$

$$\lim_{x \rightarrow 0} f'(x) = 0 \quad + x^3g'(x)\cos(1/x) + xg(x)\sin(1/x)$$

$$f'(0) = \lim_{m \rightarrow 0} \frac{h(m)m^3\sin(1/m) + g(m)m^3\cos(1/m)}{m} = 0$$

$\rightarrow f^{(n)}(0) = f'(0)$ exists, f' cont. at 0.

$$f''(0) = \lim_{m \rightarrow 0} \frac{[3m^2(h(m)\sin(1/m) + g(m)\cos(1/m)) + m^3h'(m)\sin(1/m) - mh(m)\cos(1/m) + m^3g'(m)\cos(1/m) + mg(m)\sin(1/m)]}{m}$$

Blue terms go to zero, but red terms depend on $\lim_{m \rightarrow 0} h(m)$ and $\lim_{m \rightarrow 0} g(m)$. If these are $\neq 0$ then

$f''(0)$ does not exist.

n

$$\text{Assume } f(x) = \begin{cases} h(x)x^{2n+1}\sin(1/x) + g(x)x^{2n+1}\cos(1/x) & x \neq 0 \\ 0 & x=0 \end{cases}$$

$\rightarrow \begin{array}{l} f \text{ n-times diff at } 0 \\ f \text{ cont. at } 0 \\ f^{(n)} \text{ not diff. at } 0. \end{array}$

$$\text{let } i(x) = \begin{cases} h(x)x^{2n+1+1}\sin(1/x) + g(x)x^{2n+1+1}\cos(1/x) & x \neq 0 \\ 0 & x=0 \end{cases}$$

$$= \begin{cases} x^{2n+2+1}(h(x)\sin(1/x) + g(x)\cos(1/x)) & x \neq 0 \\ 0 & x=0 \end{cases}$$

h, g $(n+1)$ -times diff.

$$i'(0) = \lim_{m \rightarrow 0} \frac{m^{2(n+1)}(h(m)\sin(1/m) + g(m)\cos(1/m))}{m}$$

$$= \lim_{m \rightarrow 0} (m^{2(n+1)}(h(m)\sin(1/m) + g(m)\cos(1/m))) = 0$$

$$i'(x) = \frac{(2n+3)x^{2(n+1)}(h(x)\sin(1/x) + g(x)\cos(1/x)) + x^{2(n+1)}(h'(x)\sin(1/x) - h(x)\cos(1/x) + g'(x)\cos(1/x) + g(x)\sin(1/x))}{x^2} \quad x \neq 0$$

apply problem 33
 g' is $(n+1)$ times diff
 $g^{(n+1)}$ not cont. at 0

Then

$$i'(x) = \begin{cases} 0 & \text{if } x=0 \\ x^{2n+1} \sin(1/x) ((2n+3)h(x) + x^2 h'(x) + g(x)) \\ + x^{2n+1} \cos(1/x) ((2n+3)g(x) + x^2 g'(x) - h(x)), & x \neq 0 \end{cases}$$

$$= \begin{cases} x^{2n+1} \sin(1/x) h_1(x) + x^{2n+1} \cos(1/x) h_2(x) & x \neq 0 \\ 0 & x=0 \end{cases}$$

$h_1(x)$ and $h_2(x)$ are n times diff.

Apply inductive hypothesis to i'

$$\rightarrow i' \text{ } n \text{ times diff at 0} \rightarrow i \text{ } (n+1) \text{ times diff at 0}$$

$i^{(n+1)}$ cont at 0
 $i^{(n+1)}$ not diff at 0

Chain Rule, Leibniz notation

$$\frac{d}{dx}(g(f(x))) = \frac{d}{dy}(g(y)) \Big|_{y=f(x)} \cdot \frac{dy}{dx}$$

Allan.

$$y = g(x)$$

$$z = f(y)$$

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$$

↑↑↑↑↑
reflects z's in this eq refers
to different functions

i) $z = \sin y, y = x + x^2$

$$z(y) = \sin y$$

$$y(x) = x + x^2$$

$$z(y(x)) = \sin(x + x^2)$$

$$\begin{aligned}\frac{dz}{dx} &= \cos(y) \Big|_{y=x+x^2} (1+2x) \\ &= \cos(x+x^2) \cdot (1+2x)\end{aligned}$$

ii) $z = \sin y, y = \cos x$

$$\begin{aligned}\frac{dz}{dx} &= \cos(y) \Big|_{y=\cos x} (-\sin x) \\ &= \cos(\cos x) \cdot (-\sin x)\end{aligned}$$

iii) $z = \sin u, u = \sin x$

$$\begin{aligned}\frac{dz}{dx} &= \cos(u) \Big|_{u=\sin x} \cos x \\ &= \cos(\sin x) \cdot \cos(x)\end{aligned}$$

iv) $z = \sin v, v = \cos u, u = \sin x$

$$\begin{aligned}z(v) &= \sin v \\ v(u) &= \cos u \rightarrow w(u) = z(v(u)) = \sin(\cos(u)) \\ u(x) &= \sin x\end{aligned}$$

$$\frac{dz}{dx} = \frac{du}{dx} \cdot \frac{dv}{du} = \left(\frac{dz}{dv} \frac{dv}{du} \right) \frac{du}{dx}$$

$$= \left(\cos v \Big|_{v=\cos u} \cdot (-\sin u) \right) \Big|_{u=\sin x} \cdot \cos x = \cos(\cos(\sin x)) \cdot (-\sin(\sin x)) \cdot \cos x$$