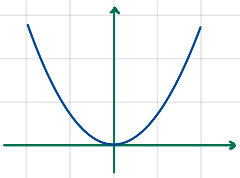
Ch 8 Appendix - Uniform continuity

Definition Function f is uniformly continuous on an interval A if for every $\epsilon>0$ there is some $\delta>0$ such that for all x and y in A,

$$|x-y|<\delta \rightarrow |f(x)-f(y)|<\epsilon$$

Examples



$f(x)=x^2$, cont. but not u.c. on $\mathbb{R}$

Proof

consider $a \in \mathbb{R}$.

Let's say you choose an $\epsilon>0$ and you have $\delta>0$ s.t.

$|x-a|<\delta \rightarrow |f(x)-f(a)|<\epsilon$ (1)

Consider $\frac{\delta}{2}$. (1) is still true if we sub $\frac{\delta}{2}$ for δ.

Assume (1) is true for all a.

Then, since $|a+\frac{\delta}{2}-a| = |\frac{\delta}{2}| < \delta$,

$|f(a+\delta/2)-f(a)|<\epsilon$

$|(a+\delta/2)^2 - a^2|<\epsilon$

$|a\delta + \frac{\delta^2}{4}|<\epsilon$

But is this really true for all a?

$|a\delta + \frac{\delta^2}{4}| > |a\delta|$

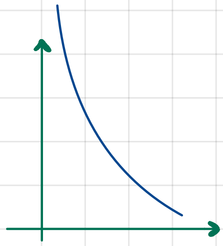
If $a > \frac{\epsilon}{\delta}$ then $|a\delta|>\epsilon$

Hence $|f(a+\delta/2)-f(a)|>\epsilon$

$\perp$

Therefore (1) is not true for all a.

$f(x)=\frac{1}{x}$, $x \in (0,1)$

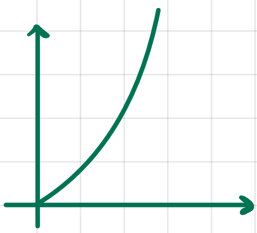


not u.c. on $(0,1)$, but c. on $(0,1)$

$f(x)=x$, u.c. on $\mathbb{R}$.

$f(x)=x^2$, $x \in [0,2]$

u.c. on $[0,2]$



$|x^2-a^2| = |(x-a)(x+a)|$

$|x-a|<1 \rightarrow a-1<x<a+1$
$\rightarrow a<x+1<a+2$
$\rightarrow |x+1|<a+2$

$|(x-a)(x+a)| \le |x-a||x+a| < (a+2)|x-a|$ if $|x-a|<1$.

$|x^2-a^2| < (a+2)|x-a| < \epsilon$

$|x-a| < \frac{\epsilon}{a+2}$

Therefore, if $|x-a|<\min(1, \epsilon/(a+2))$ then

$|x^2-a^2|<\epsilon$.

$\forall a \in [0,2]$, $|x-a|<\min(1, \epsilon/4) \rightarrow |x^2-a^2|<\epsilon$.

Ie, $f(x)=x^2$ is unif. cont. on a closed interval such as $[0,2]$.

Lemma Let $a<b<c$ and let f be continuous on $[a,c]$. Let $\epsilon>0$ and suppose the following statements are true

i) x,y both in $[a,b]$, $|x-y|<\delta_1 \to |f(x)-f(y)|<\epsilon$
ii) x,y both in $[b,c]$, $|x-y|<\delta_2 \to |f(x)-f(y)|<\epsilon$

Then, $\exists\delta>0$ s.t.

x,y both in $[a,c]$, $|x-y|<\delta \to |f(x)-f(y)|<\epsilon$

Proof

f is cont. at b so $\exists\delta_3>0$ s.t.

$$\forall x,\ |x-b|<\delta_3 \to |f(x)-f(b)|<\frac{\epsilon}{2}$$

Take two such values:

$$|x_1-b|<\delta_3 \text{ and } |x_2-b|<\delta_3 \quad \to \quad |f(x_1)-b|-|f(x_2)-b| \le |f(x_1)-f(x_2)| \le |f(x_1)-b|+|f(x_2)-b|<\epsilon$$

$$\to |f(x_1)-f(x_2)|<\epsilon$$

Let $\delta=\min(\delta_1,\delta_2,\delta_3)$

Now for any x and y in $[a,c]$ s.t. $|x-y|<\delta$ there are three cases

1) x and y in $[a,b]$. By i), $|f(x)-f(y)|<\epsilon$
2) x and y in $[b,c]$. By ii), $|f(x)-f(y)|<\epsilon$
3) x in $[a,b]$, y in $[b,c]$, or vice-versa.
   Then since $|x-b|<\delta$ and $|y-b|<\delta$, $|f(x)-f(y)|<\epsilon$

Theorem 1 f cont. on $[a,b] \to f$ u.c. on $[a,b]$

Proof

Let's introduce a new term: ϵ-good. f is ϵ-good on $[a,b]$ if there is some $\delta>0$ such that for all y and z in $[a,b]$,
$|y-z|<\delta \to |f(y)-f(z)|<\epsilon$
we want to prove that f is ϵ-good on $[a,b]$ for all $\epsilon>0$.

Let $\epsilon>0$.
Let $A=\{x : a\le x\le b \text{ and } f \text{ is } \epsilon\text{-good on } [a,x]\}$
Then $A\ne\emptyset$, since $a\in A$.
A bounded above by b.
$\to$ A has l.u.b. α
Let's prove $\alpha=b$.

Assume $\alpha<b$.
f cont at α so $\exists\delta_0>0$ s.t. $|y-\alpha|<\delta \to |f(y)-f(\alpha)|<\frac{\epsilon}{2}$
Therefore if $|z-\alpha|<\delta_0$ then $|f(y)-f(z)|<\epsilon$
Therefore f ϵ-good on $[\alpha-\delta_0,\alpha+\delta_0]$
But f also ϵ good on $[a,\alpha-\delta_0]$

By the lemma above f ϵ-good on $[a,\alpha+\delta_0]$.
So $\alpha+\delta_0$ in A. $\perp$.
Therefore $\alpha\ge b$. But $\alpha>b$ leads to $\perp$ because b is u.b.
Finally, since f cont. at b, $\exists\delta_0>0$, $b-\delta_0<y<b \to |f(y)-f(b)|<\frac{\epsilon}{2}$
So f ϵ-good on $[b-\epsilon_0,b]$. Since f ϵ-good on $[a,b-\epsilon_0]$, by the lemma, f ϵ-good on $[a,b]$.