Ch8 - Least Upper Bounds

Def: A set of real numbers is **bounded above** if there is a number x s.t. $x \geq a$ for every $a \in A$.
Such a number is called an **upper bound** for A.

- This def. is for sets.
  In ch.7 we spoke of this concept for fns.
  The two are closely related.

  Let $A = \{f(x) : a \leq x \leq b\}$
  Then, f bounded above on $[a,b] \leftrightarrow A$ bounded above.

- $\mathbb{R}$ and $\mathbb{N}$ are not bounded above.

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more accurately, "the"

Def: A number x is a **least upper bound**, aka **supremum** of A if

1) x is an upper bound of A
and 2) if y is upper bound of A then $y \geq x$.

$a = \sup A$

Similarly,

Set A of real numbers **bounded below** if $\exists x \in \mathbb{R}, \forall a \in A, x \leq a$.
Such an x is a **lower bound** for A.
A number x is the **greatest lower bound** aka **infimum** if
1) x is a lower bound for A
2) $\forall y$, y lower bound for $A \rightarrow y \leq x$

$x = \inf A$

"most important property of real numbers"

P13: (least upper bound property)

A is set real numbers
$A \neq \emptyset$ $\quad \rightarrow \quad$ A has least upper bound
A bounded above

Note

- if $A = \emptyset$ then $\forall y \in A, \forall x\ x \in \mathbb{R} \rightarrow x \geq y$.
  Thus every number is an upper bound of A, because there is no y in A. The statement is inherently vacuously true.
  But there is no least upper bound.

- P13 isn't true for $\mathbb{Q}$.
  e.g. $A = \{x \in \mathbb{Q} : x^2 < \sqrt{2}\}$
  A has no least upper bound.

Theorem 7-1

f cont. on $[a,b]$
$f(a) < 0 < f(b)$ $\quad \rightarrow \quad$ $\exists x \in [a,b], f(x) = 0$

Proof

strategy: locate smallest $x \in [a,b]$ s.t. $f(x) = 0$.

Let $A = \{x : a \leq x \leq b : f \text{ neg on } [a,x]\}$

$f(a) < 0 \rightarrow a \in A \rightarrow A \neq \emptyset$

In 6-16a we proved:

$\lim_{x \to a^+} f(x) = f(a)$
$f(a) < 0$ $\quad \rightarrow \quad$ $\exists \delta > 0\ \forall x\ 0 \leq x - a < \delta \rightarrow f(x) < 0$

Therefore, since f cont. on $[a,b]$ we have $\lim_{x \to a^+} f(x) = f(a) < 0$, so

$\exists \delta > 0\ \forall x\ a \leq x < a + \delta \rightarrow f(x) < 0$
Thus $\forall x\ a \leq x < a + \delta \rightarrow x \in A$.

In 6-16b we proved

$\lim_{x \to b^-} f(x) = f(b)$
$f(b) > 0$ $\quad \rightarrow \quad$ $\exists \delta\ 0 \leq b - x < \delta \rightarrow f(x) > 0$.

Therefore since $\lim_{x \to b^-} f(x) = f(b) > 0$ we have

$\exists \delta > 0\ b - \delta < x \leq b \rightarrow f(x) > 0 \rightarrow x$ upper bound of A.

Thus, the assumptions of P13 are true. Therefore, according to P13, A has a least upper bound, call it α, and $a < \alpha < b$.
Let's show that $f(\alpha) = 0$.

Assume $f(\alpha) < 0$.
By Th. 6-3, there is an interval around α in which $f < 0$.
Ie, $\exists \delta > 0, \forall x, \alpha - \delta < x < \alpha + \delta \rightarrow f(x) < 0$.
But then $\forall x, x \in (\alpha, \alpha + \delta) \rightarrow x > \alpha \wedge f(x) < 0$
$\wedge (\forall y \in [\alpha, x] \rightarrow f(y) < 0)$
Therefore, $x \in A$ and so α is not $\sup A$. $\perp$.
Therefore $f(\alpha) \geq 0$.

Assume $f(\alpha) > 0$. By Th. 6-3, $\exists \delta > 0, \forall x\ \alpha - \delta < x < \alpha + \delta$
$\rightarrow f(x) > 0$, ie $f > 0$ in some open interval containing α.
But then, $\forall x, x \in (\alpha - \delta, \alpha) \rightarrow f(x) > 0$
Therefore, x is upper bound for A. α not $\sup A$. $\perp$.
Therefore, $f(\alpha) = 0$. ∎

To prove theorems 6-2 and 6-3 we need the following theorem:

Theorem 1

f cont. at a $\rightarrow \exists \delta>0$ s.t. f is bounded above on $(a-\delta, a+\delta)$

Proof

f cont at a therefore $\forall \varepsilon>0\ \exists \delta>0\ \forall x\ |x-a|<\delta \rightarrow |f(x)-f(a)|<\varepsilon$.

Choose any ε, say $\varepsilon=1$.

Then

$\exists \delta>0\ \forall x\ x \in (a-\delta, a+\delta) \rightarrow |f(x)-f(a)|<1$

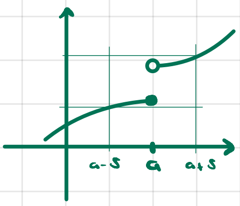
$\rightarrow f(x)-f(a)<1$

$\rightarrow f(x) < 1+f(a)$

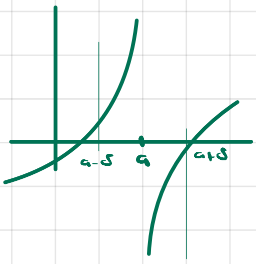
Thus f is bounded on $(a-\delta, a+\delta)$.

* Note that if f not cont. it may or may not be bounded

bounded



unbounded



- we could also prove that

f cont on $[a,b] \rightarrow f$ bounded below on $(a-\delta, a+\delta)$

thus

f cont on $[a,b] \rightarrow f$ bounded on $(a-\delta, a+\delta)$

the proofs are analogous to that of Th. 1.

- Also,

1) $\lim_{x\to a^+} f(x)=f(a) \rightarrow f$ bounded on $(a, a+\delta)$

2) $\lim_{x\to b^-} f(x)=f(b) \rightarrow f$ bounded on $(b-\delta, b)$

Proof of (1)

$\forall \varepsilon>0\ \exists \delta>0\ \forall x\ 0<x-a<\delta \rightarrow |f(x)-f(a)|<\varepsilon$.

Let $\varepsilon=1$. Then $\exists \delta>0\ \forall x\ a<x<a+\delta \rightarrow f(x)-f(a)<1$

Ie f bounded on $(a, a+\delta)$

Theorem 7-2

f cont. on $[a,b] \rightarrow f$ bounded above on $[a,b]$, ie $\exists N, \forall x \in [a,b], f(x) \leq N$

Proof

Let $A=\{x: a\leq x\leq b$ and f bounded above on $[a,x]\}$.

$a \in A$ because $x\in[a,a] \rightarrow f(x)\leq f(a)$.
So $A \neq \emptyset$.
A bounded above because $\forall x \in A, x \leq b$. b is an upper bound for A.
By P13, A has a least upper bound, $\sup A = \alpha$, and $\alpha \leq b$.

Assume $\alpha < b$.
Note that $\alpha > a$ because since f cont on $[a,b]$ then $\lim_{x\to a^+} f(x)=f(a)$, hence f bounded on $(a, a+\delta)$. $\forall x$ in this interval, $x \in A$, and if $\alpha \leq a$ then $x > \alpha$ so α not upper bound. $\perp$

Th1 $\rightarrow \exists \delta>0$, f bounded on $(\alpha-\delta, \alpha+\delta)$
Since $\alpha=\sup A$, $\exists x_0 \in \alpha-\delta < x_0 < \alpha$, $x_0 \in A$
f bounded on $[a, x_0]$
$\forall x_1, x_1 \in (\alpha, \alpha+\delta) \rightarrow f$ bounded on $[x_0, x_1]$
$\rightarrow f$ bounded on $[a, x_1] \rightarrow x_1 \in A \rightarrow \alpha$ not $\sup A$
$\perp$.

Therefore, $\alpha = b$.

At this point we know that $\forall x \in [a,b), x \in A$, because if $x \notin A$ then $\forall x_1, x_1 \in (x,b)$ would be an upper bound, a $\perp$.

Since $\lim_{x\to b^-} f(x)=f(b)$ then f is bounded above on $(b-\delta, b]$. $\exists x_0$, $x_0 \in (b-\delta, b)$. Thus since $x_0<b$ then x_0 in A, ie f bounded on $[a, x_0]$. But f also bounded above on $[x_0, b]$. Thus f bounded above on $[a,b]$, $b \in A$.

Theorem 7-3

f cont. on $[a,b]$ $\rightarrow$ $\exists y \in [a,b], \forall x \in [a,b], f(y) \geq f(x)$

Proof

f is bounded on $[a,b]$, by Th. 7-2.

Therefore $A = \{f(x) : x \in [a,b]\}$ is bounded.

Since this set $\neq \phi$ it has a least upper bound $\alpha = \sup A$.

$\forall x, x \in [a,b] \rightarrow f(x) \leq \alpha$

If we prove that $\exists y, y \in [a,b] \wedge \alpha = f(y)$ then we'd have $\forall x, x \in [a,b] \rightarrow f(x) \leq f(y)$.

Assume $\forall y \in [a,b], \alpha \neq f(y)$

Let $g(x) = \dfrac{1}{\alpha - f(x)}$, $x \in [a,b]$.

Since $\alpha \neq f(x)$ for $x \in [a,b]$, g is continuous on $[a,b]$.

Let $\epsilon > 0$

Assume $\forall x \in [a,b], \alpha \geq f(x) + \epsilon$

$\exists \beta \in [f(x) + \epsilon, \alpha)$.

$\forall x \in [a,b], \beta \geq f(x) + \epsilon > f(x)$

Thus $\beta < \alpha$ is an upper bound for A, α is not $\sup A$.

$\perp$

Hence it is true that $\forall \epsilon > 0 \ \exists x \in [a,b], \alpha - f(x) < \epsilon$.

Therefore, $\forall \epsilon > 0 \ \exists x \in [a,b], \ g(x) = \dfrac{1}{\alpha - f(x)} > \dfrac{1}{\epsilon}$

-> g not bounded in $[a,b]$.

$\perp$

Therefore, $\exists y \in [a,b], \alpha = f(y)$.

Archimedean property of real numbers: $\mathbb{N}$ is not bounded.

not consequence of P1-P12

Theorem 2 $\mathbb{N}$ is not bounded above.

Proof

Assume $\mathbb{N}$ bounded above.

Since $\mathbb{N} \neq \emptyset$, then there is least upper bound $\alpha = \sup \mathbb{N}$.

$\forall n \in \mathbb{N},\ \alpha \geq n$

Since $n \in \mathbb{N} \to n+1 \in \mathbb{N}$ we have

$\forall n \in \mathbb{N},\ \alpha \geq n+1$
$\alpha - 1 \geq n$

Therefore $\alpha - 1$ is an upper bound for $\mathbb{N}$. $\perp$

Theorem 3 $\forall \varepsilon > 0\ \exists n \in \mathbb{N},\ \frac{1}{n} < \varepsilon$

Proof

Assume $\exists \varepsilon > 0\ \forall n \in \mathbb{N},\ \frac{1}{n} \geq \varepsilon$.

Let $\varepsilon > 0$ be such an ε.

$\forall n \in \mathbb{N}\quad n \leq \frac{1}{\varepsilon}$

$\frac{1}{\varepsilon}$ upper bound for $\mathbb{N}$.

$\perp$

Therefore $\forall \varepsilon > 0\ \exists n \in \mathbb{N},\ \frac{1}{n} < \varepsilon$