Ch 10 – Differentiation

1.

i) $f(x) = \sin(x + x^2)$
$f'(x) = \cos(x+x^2)(1+2x)$

ii) $f(x) = \sin x + \sin x^2$
$f'(x) = \cos x + \cos x^2 \cdot 2x$

iii) $f(x) = \sin(\cos x)$
$f'(x) = \cos(\cos x)(-\sin x)$
$= -\sin(x)\cos(\cos x)$

iv) $f(x) = \sin(\sin x)$
$f'(x) = \cos(\sin x)\cos(x)$

v) $f(x) = \sin\left(\frac{\cos x}{x}\right)$

$f'(x) = \cos\left(\frac{\cos x}{x}\right) \cdot \left[\frac{-\sin x \cdot x - \cos x}{x^2}\right]$

vi) $f(x) = \frac{\sin(\cos x)}{x}$

$f'(x) = \frac{\cos(\cos x)(-\sin x)x - \sin(\cos x)}{x^2}$

vii) $f(x) = \sin(x + \sin x)$
$f'(x) = \cos(x+\sin x)(1+\cos x)$

viii) $f(x) = \sin(\cos(\sin x))$
$f'(x) = \cos(\cos(\sin x)) \cdot (-\sin(\sin x))\cos x$

2.

i) $f(x) = \sin((x+1)^2(x+2))$

$f'(x) = \cos((x+1)^2(x+2))\left[2(x+1)(x+2) + (x+1)^2\right]$

ii) $f(x) = \sin^3(x^2+\sin x)$

$f'(x) = 3\sin^2(x^2+\sin x)\cos(x^2+\sin x)(2x+\cos x)$

iii) $f(x) = \sin^2((x+\sin x)^2)$

$f'(x) = 2\sin((x+\sin x)^2)\cos((x+\sin x)^2)\,2(x+\sin x)(1+\cos x)$

iv) $f(x) = \sin\dfrac{x^3}{\cos x^3}$

$$f'(x) = \cos\left(\frac{x^3}{\cos x^3}\right)\cdot\frac{3x^2\cos x^3 + x^3\sin x^3\cdot 3x^2}{(\cos x^3)^2}$$

v) $f(x) = \sin(x\sin x) + \sin(\sin x^2)$

$f'(x) = \cos(x\sin x)\cdot(\sin x + x\cos x) + \cos(\sin x^2)\cos(x^2)2x$

vi) $f(x) = (\cos x)^{31^2}$

$f'(x) = 31^2(\cos(x))^{31^2-1}(-\sin x)$

vii) $f(x) = \sin^2 x\,\sin x^2\,\sin^2 x^2$

$f'(x) = 2\sin x\cos x\,\sin x^2\sin^2 x^2 + \sin^2 x\cos x^2\cdot 2x\sin^2 x^2 + \sin^2 x\sin x^2\,2\sin x^2\cos x^2\,2x$

viii) $f(x) = \sin^3(\sin^2(\sin x))$

$f'(x) = 3\sin^2(\sin^2(\sin x))\cos(\sin^2(\sin x))\,2\sin(\sin x)\cos(\sin x)\cos x$

ix) $f(x) = (x+\sin^5 x)^6$

$f'(x) = 6(x+\sin^5 x)^5(1+5\sin^4 x\cos x)$

x) $f(x) = \sin(\sin(\sin(\sin(\sin x))))$

$f'(x) = \cos(\sin(\sin(\sin(\sin x))))\cos(\sin(\sin(\sin x)))\cos(\sin(\sin x))\cos(\sin x)\cos x$

xi) $f(x) = \sin((\sin^7 x^7+1)^7)$

$f'(x) = \cos((\sin^7 x^7+1)^7)\cdot 7(\sin^7 x^7+1)^6(7\sin^6 x^7\cos x^7\cdot 7x^6)$

xii) $f(x) = (((x^2+x)^3+x)^4+x)^5$

$f'(x) = 5(((x^2+x)^3+x)^4+x)^4(1+4((x^2+x)^3+x)^3(1+3(x^2+x)^2(2x+1)))$

xiii) $f(x) = \sin(x^2+\sin(x^2+\sin x^2))$

$f'(x) = \cos(x^2+\sin(x^2+\sin x^2))(2x+\cos(x^2+\sin x^2)(2x+\cos x^2\cdot 2x))$

xiv) $f(x) = \sin(6\cos(6\sin(6\cos 6x)))$

$f'(x) = \cos(6\cos(6\sin(6\cos 6x)))\cdot 6(-\sin(6\sin(6\cos 6x))\cdot 6\cos(6\cos 6x)$
$\cdot 6(-\sin 6x)\cdot 6))$

xv) $f(x) = \dfrac{\sin x^2 \sin^2 x}{1+\sin x}$

$$f'(x) = \frac{(\cos x^2 \cdot 2x \sin^2 x + \sin x^2 \cdot 2\sin x \cos x)(1+\sin x) - \sin x^2 \sin^2 x \cos x}{(1+\sin x)^2}$$

xvi) $f(x) = \dfrac{1}{x - \dfrac{2}{x+\sin x}}$

$$f'(x) = \frac{-\left(1 - \dfrac{-2(1+\cos x)}{(x+\sin x)^2}\right)}{\left(x - \dfrac{2}{x+\sin x}\right)^2}$$

xvii) $f(x) = \sin\left(\dfrac{x^3}{\sin\left(\dfrac{x^3}{\sin x}\right)}\right)$

$$f'(x) = \cos\left(\frac{x^3}{\sin\left(\frac{x^3}{\sin x}\right)}\right) \frac{3x^2 \sin\left(\frac{x^3}{\sin x}\right) - x^3 \cos\left(\frac{x^3}{\sin x}\right) \cdot \frac{3x^2 \sin x - x^3 \cos x}{\sin^2 x}}{\sin^2\left(\frac{x^3}{\sin x}\right)}$$

xviii) $f(x) = \sin\left(\dfrac{x}{x - \sin\left(\dfrac{x}{x-\sin x}\right)}\right)$

$$f'(x) = \cos\left(\frac{x}{x - \sin\left(\frac{x}{x-\sin x}\right)}\right) \frac{x - \sin\left(\frac{x}{x-\sin x}\right) - x\left(1 - \cos\left(\frac{x}{x-\sin x}\right)\frac{(x-\sin x) - x(1-\cos x)}{(x-\sin x)^2}\right)}{\left(x - \sin\left(\frac{x}{x-\sin x}\right)\right)^2}$$

3.

$f_1(x) = \tan x = \dfrac{\sin x}{\cos x}$

$f_1'(x) = \dfrac{\cos x \cos x + \sin x \sin x}{\cos^2 x} = \dfrac{1}{\cos^2 x}$

$f_2(x) = \cot x = \dfrac{\cos x}{\sin x} = \dfrac{1}{\tan(x)} = \dfrac{1}{f_1(x)}$

$f_2'(x) = -\dfrac{1}{f_1^2(x)} \cdot f_1'(x) = -\dfrac{\cos^2 x}{\sin^2 x} \cdot \dfrac{1}{\cos^2 x} = -\dfrac{1}{\sin^2 x}$

$f_3(x) = \sec x = \dfrac{1}{\cos x}$

$f_3'(x) = \dfrac{\sin x}{\cos^2 x} = \tan x \sec x$

$f_4(x) = \csc x = \dfrac{1}{\sin x}$

$f_4'(x) = \dfrac{-\cos x}{\sin^2 x} = -\cot x \csc x$

4.

$(f \circ f)(x) = f(f(x)) = g(x)$

$(f \circ f)'(x) = g'(x) = f'(f(x)) f'(x)$

we want to compute $f'(f(x))$

i) $f(x) = \dfrac{1}{1+x}$

$f'(x) = \dfrac{-1}{(1+x)^2}$

$f'(f(x)) = \dfrac{-1}{\left(\dfrac{3+x}{1+x}\right)^2}$

ii) $f(x) = \sin x$

$f'(x) = \cos x$

$f'(f(x)) = \cos(\sin x)$

iii) $f(x) = x^2$

$f'(f(x)) = 2x^2$

iv) $f(x) = 17$

$f'(f(x)) = 0$

5. $f(f'(x))$

i) $f(x) = \frac{1}{x}$

$f'(x) = -\frac{1}{x^2}$

$f(f'(x)) = \frac{1}{-\frac{1}{x^2}} = -x^2$

ii) $f(x) = x^2$

$f'(x) = 2x$

$f(f'(x)) = 4x^2$

iii) $f(x) = 17$

$f(f'(x)) = 17$

iv) $f(x) = 17x$

$f'(x) = 17$

$f(f'(x)) = 17^2$

6.

i) $f(x) = g(x + g(a))$

$f'(x) = g'(x + g(a))$

ii) $f(x) = g(x \cdot g(a))$

$f'(x) = g'(x \cdot g(a))\, g(a)$

iii) $f(x) = g(x + g(x))$

$f'(x) = g'(x + g(x))\,(1 + g'(x))$

iv) $f(x) = g(x)(x-a)$

$f'(x) = g'(x)(x-a) + g(x)$

v) $f(x) = g(a)(x-a)$

$f'(x) = g(a)$

vi) $f(x+3) = g(x^2)$

$h(x) = f(x+3) = g(x^2)$, this is $f(x)$ shifted left c units.

$p(x) = h(x-3) = f(x) = g((x-3)^2)$

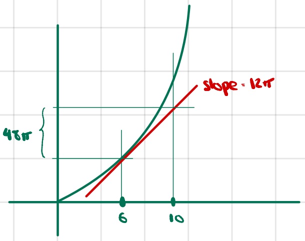
$p'(x) = f'(x) = g'((x-3)^2)\, 2(x-3)$

7.

a) $A(r(t)) = \pi r(t)^2$

$$\frac{dA(r(t))}{dt} = A'(t) = A'(r(t))\,r'(t) = 2\pi r(t)\,r'(t)$$

$$A'(6) = 2\pi r(6)\,r'(6) = 2\pi \cdot 6 \cdot 4 = 48\pi$$



b) $V(t) = \frac{4}{3}\pi r(t)^3$

$$V'(t) = 4\pi r(t)^2\,r'(t)$$
$$V'(6) = 4\pi \cdot 36 \cdot 4 = 576\pi$$

c) $A(r(t)) = \pi r(t)^2$

$$\frac{dA(r(t))}{dr} = 2\pi r(t)$$

$$\frac{dA(3)}{dr} = 2\pi \cdot 3 = 6\pi$$

$$\frac{dA(r(t))}{dt} = 2\pi r(t)\,r'(t)$$

$$\frac{dA(3)}{dt} = 2\pi \cdot 3\,r'(t_3) = 5 \rightarrow r'(t_3) = \frac{5}{6\pi}$$

$$\frac{dV(r(t))}{dt} = 4\pi r(t)^2\,r'(t)$$

$$\frac{dV(r(t_3))}{dt} = \frac{dV(3)}{dt} = 4\pi \cdot 9 \cdot \frac{5}{6\pi} = 30$$

Alltså.

$$V(t) = \frac{4}{3}\,r(t)\,A(r(t))$$

$$V'(t) = \frac{4}{3}\left(r'(t)\,A(r(t)) + r(t)\,A'(r(t))\,r'(t)\right)$$

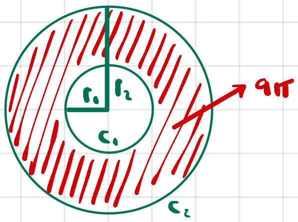
$$V'(t_3) = \frac{4}{3}\left(r'(t_3)\,A(3) + 3A'(3)\,r'(t_3)\right)$$

$$= \frac{4}{3}\left(\frac{5}{6\pi}\pi \cdot 9 + 3 \cdot 6\pi \cdot \frac{5}{6\pi}\right)$$

$$= \frac{\cancel{4}^{2}}{\cancel{3}} \cdot \frac{5}{\cancel{6}_{1}\cancel{\pi}} \cdot \cancel{9}\cancel{\pi} + \frac{4}{\cancel{3}} \cdot \cancel{18}\cancel{\pi} \cdot \frac{5}{\cancel{6}\cancel{\pi}}$$

$$= 10 + 20$$
$$= 30$$

8.



$$A_2(r_2(t)) = \pi r_2^2(t)$$

$$\frac{dA_2(r_2(t))}{dt} = \pi \cdot 2r_2(t)r_2'(t) = 10\pi$$

Circle 2's radius changes at a declining rate, to keep the rate of chg of the area of circle 2 constant.

$$r_2'(t) = \frac{5}{r_2(t)}$$

$$A_1(r_1(t)) = \pi r_1(t)^2$$

$$\frac{dA_1(r_1(t))}{dt} = \pi \cdot 2r_1(t)\,r_1'(t)$$

$$D(t) = A_2(r_2(t)) - A_1(r_1(t)) = \pi(r_2^2(t) - r_1(t)^2) = 9\pi$$

$$D'(t) = 0 = \pi(2r_2(t)\,r_2'(t) - 2r_1(t)r_1'(t))$$

$$= \pi(10 - 2r_1(t)r_1'(t))$$

$\rightarrow \; r_1(t)r_1'(t) = 5$ This is intuitively true. Since the area of 2 increases at a constant rate, then since the difference in areas stays constant, the area of 1 increases at the same rate as the area of 2.

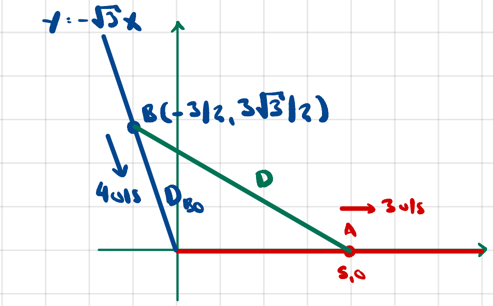
$$C_1(r_1(t)) = 2\pi r_1(t)$$

$$\frac{dC_1(r_1(t))}{dt} = 2\pi r_1'(t) = 2\pi \cdot \frac{5}{r_1(t)} = \frac{10\pi}{r_1(t)}$$

$$A_1(r_1(t)) = \cancel{\pi} r_1(t)^2 = 16\cancel{\pi} \rightarrow r_1(t) = 4$$

$$\rightarrow \frac{dC_1(r_1(t))}{dt} = 2.5\pi$$

9.



initial position of B

$OB^2 = 9 = x^2 + y^2$

$y = -\sqrt{3}x$

$9 = x^2 + 3x^2 = 4x^2$

$\to |x| = 3/2 \quad \to \quad x = -3/2$

$\to y = \frac{3\sqrt{3}}{2}$

rates of change of B's coord.

$y_B(t) = -\sqrt{3}\, x_B(t)$

$y_B'(t) = -\sqrt{3}\, x_B'(t)$

$D_{BO}(t) = \sqrt{x_B^2(t) + y_B^2(t)}$

$$D_{BO}'(t) = \frac{2x_B(t)x_B'(t) + 2y_B(t)y_B'(t)}{2\sqrt{x_B^2(t)+y_B^2(t)}}$$

$$= \frac{x_B(t)x_B'(t) + y_B(t)y_B'(t)}{\sqrt{x_B^2(t)+y_B^2(t)}}$$

B moving southeast

$$D_{BO}'(t_0) = -4 = \frac{-\frac{3}{2}x_B' + \frac{3\sqrt{3}}{2}y_B'}{\sqrt{\frac{9}{4}+\frac{27}{4}}}$$

$\to -4\cdot 3 = -\frac{3}{2}x_B' + \frac{3\sqrt{3}}{2}y_B'$

$-12 = -\frac{3}{2}x_B' + \frac{3\sqrt{3}}{2}(-\sqrt{3})x_B'$

$x_B'\left(\frac{3}{2} + \frac{9}{2}\right) = 12$

$\to x_B' = 2$

$\to y_B' = -2\sqrt{3}$

$D(t) = \sqrt{y_B(t)^2 + (x_A(t) - x_B(t))^2}$

$$D' = \frac{2y_By_B' + 2(x_A - x_B)(x_A' - x_B')}{2\sqrt{y_B^2 + (x_A - x_B)^2}}$$

$$= \frac{y_By_B' + (x_A - x_B)(x_A' - x_B')}{\sqrt{y_B^2 + (x_A - x_B)^2}}$$

$$D'(t_0) = \frac{\frac{3\sqrt{3}}{\cancel{2}}(-\cancel{2}\sqrt{3}) + (5 + 3/2)(3-2)}{\sqrt{\frac{27}{4} + (5+3/2)^2}}$$

$$= \frac{-9 + \frac{13}{2}}{7} = -\frac{5}{14} \text{ units/sec}$$

This part could have been done as follows

$D_{BO}(t) = \sqrt{x_B(t)^2 + 3x_B(t)^2}$

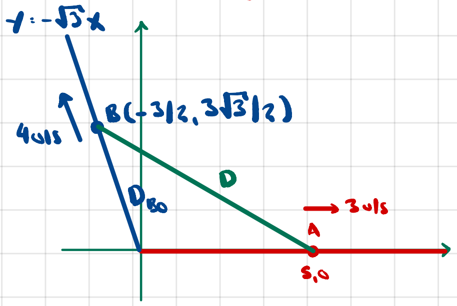
$= 2|x_B(t)| = \begin{cases} 2x_B(t) & x_B \geq 0 \\ -2x_B(t) & x_B \leq 0 \end{cases}$

since $x_B(t_0) < 0$,

$D_{BO}' = -2x_B'$

$x_B' = \frac{-4}{-2} = 2$

What if B is moving northwest?



$D_{BO}' = -2x_B' = +4 \to x_B' = -2 \to y_B' = 2\sqrt{3}$

$$D' = \frac{y_By_B' + (x_A - x_B)(x_A' - x_B')}{\sqrt{y_B^2 + (x_A - x_B)^2}} = \frac{\frac{3\sqrt{3}}{2}2\sqrt{3} + (5+3/2)(3+2)}{7}$$

$$= \frac{9 + \frac{13}{2}\cdot 5}{7} = \frac{83}{14} \text{ units/sec}$$

10. $f(x) = \begin{cases} x^2\sin(1/x) & x\neq 0 \\ 0 & x=0 \end{cases}$

h,k s.t.

$h'(x) = \sin^2(\sin(x+1))$ $\quad$ $k'(x) = f(x+1)$

$h(0) = 3$ $\quad$ $k(0) = 0$

i) $(f\circ h)'(0)$

$x\neq 0 \rightarrow f'(x) = 2x\sin(1/x) + x^2\cos(1/x)(-1/x^2)$

$2x\sin(1/x) - \cos(1/x)$

$f'(3) = 6\sin(1/3) - \cos(1/3)$

Let $g(x) = (f\circ h)(x) = f(h(x))$.

$g'(x) = (f\circ h)'(x) = f'(h(x))h'(x)$

$g'(0) = f'(h(0))h'(0) = f'(3)h'(0) = (6\sin(1/3) - \cos(1/3))\sin^2(\sin 1)$

ii) $(k\circ f)'(0)$

$f'(0) = \lim_{h\to 0} \dfrac{h^2\sin(1/h)}{h} = 0$

$k'(0) = f(1) = \sin(1)$

Let $g(x) = (k\circ f)(x) = k(f(x))$

$g'(x) = (k\circ f)'(x) = k'(f(x))f'(x)$

$g'(0) = (k\circ f)(0) = k'(f(0))f'(0) = k'(0)f'(0) = 0$

iii) $\alpha(x) = h(x^2)$. $\alpha'(x^2)$

$\alpha'(x) = h'(x^2)\cdot 2x$

$\alpha'(x^2) = h'(x^4)\cdot 2x^2$

$= \sin^2(\sin(x^4+1))\cdot 2x^2$

11. $f(x) = \begin{cases} g(x)\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

$g(0) = g'(0) = 0$

$$f'(0) = \lim_{h\to 0} \frac{g(h)\sin(1/h) - 0}{h}$$

$$= \lim_{h\to 0} \frac{g(h)}{h} \cdot \lim_{h\to 0} \sin(1/h)$$

$$= g'(0) \cdot \lim_{h\to 0} \sin(1/h)$$

$= 0$ (problem 5-21)

12. $f(x) = \frac{1}{x}$

Let $g(x)$ and $h(x) = \frac{1}{g(x)}$

$$h'(x) = \frac{-1}{g(x)^2} \cdot g'(x)$$

13.

a) $f(x) = \sqrt{1-x^2}$ $\quad -1 < x < 1$

$$f'(x) = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

b) tangent of f at $(a, \sqrt{1-a^2})$

$$g(x) = (x-a)\frac{-a}{\sqrt{1-a^2}} + f(a)$$

$$g(x) = f(x) \to \sqrt{1-x^2} = f(a) - \frac{a(x-a)}{\sqrt{1-a^2}}$$

$$\sqrt{(1-x^2)(1-a^2)} = 1 - \cancel{a} - a(x-\cancel{a})$$

$$= 1 - ax$$

$$\cancel{1} - a^2 - x^2 + \cancel{x^2a^2} = \cancel{1} - 2ax + \cancel{a^2x^2}$$

$$x^2 - 2ax + a^2 = 0$$

$$\Delta = 4a^2 - 4a^2 = 0$$

$$x = \frac{2a}{2} = a$$

14.

Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\quad b = \sqrt{a^2-c^2}$ is an ellipse

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \to y^2 = b^2 - \frac{b^2}{a^2}x^2$$

$$y(x) = \pm\sqrt{b^2 - \frac{b^2}{a^2}x^2}$$

consider $y(x) = \sqrt{b^2 - \frac{b^2}{a^2}x^2} = b\sqrt{1 - \frac{x^2}{a^2}}$

$$y'(x) = \frac{-b^2x}{a^2\sqrt{b^2 - \frac{b^2}{a^2}x^2}} = \frac{-b^2x}{a^2b\sqrt{1-\frac{x^2}{a^2}}} = \frac{-bx}{a^2\sqrt{1-\frac{x^2}{a^2}}}$$

Tangent line at $(x_0, \sqrt{b^2 - \frac{b^2}{a^2}x_0^2})$

$$g(x) = (x-x_0)\frac{(-bx_0)}{a^2\sqrt{1-\frac{x^2}{a^2}}} + b\sqrt{1 - \frac{x_0^2}{a^2}}$$

$g(x) = f(x)$

$$\to b\sqrt{1-\frac{x^2}{a^2}} = (x-x_0)\frac{(-bx_0)}{a^2\sqrt{1-\frac{x_0^2}{a^2}}} + b\sqrt{1 - \frac{x_0^2}{a^2}}$$

Let $x' = \frac{x}{a}$, $x_0' = \frac{x_0}{a}$.

$$\cancel{b}\sqrt{1-x'^2} = (x'-x_0')\frac{-\cancel{b}x_0' \cdot \cancel{a}}{\cancel{a}\sqrt{1-x_0'^2}} + \cancel{b}\sqrt{1-x_0'^2}$$

$$= \sqrt{1-x'^2} = (x'-x_0')\frac{-x_0'}{\sqrt{1-x_0'^2}} + \sqrt{1-x_0'^2}$$

This is the equation for the tangent line to $f_1(x') = \sqrt{1-x'^2}$.

From problem 13, the sol'n is $x' = x_0'$.

Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \to y^2 = b^2\left(\frac{x^2}{a^2} - 1\right) \to y = \pm b\sqrt{\frac{x^2}{a^2} - 1} \qquad \frac{x^2}{a^2} \geq 1 \to x^2 \geq a^2 \to x \geq a \text{ or } x \leq -a$$

consider $y = b\sqrt{\frac{x^2}{a^2} - 1}$, the positive portion of the hyperbola.

$$y'(x) = \frac{\frac{b \cdot \cancel{2}x}{a^2}}{\cancel{2}\sqrt{\frac{x^2}{a^2} - 1}} = \frac{bx}{a^2\sqrt{\frac{x^2}{a^2} - 1}}$$

tangent line through $\left(x_0, b\sqrt{\frac{x_0^2}{a^2} - 1}\right)$

$$g(x) = (x - x_0)\frac{bx_0}{a^2\sqrt{\frac{x_0^2}{a^2} - 1}} + b\sqrt{\frac{x_0^2}{a^2} - 1}$$

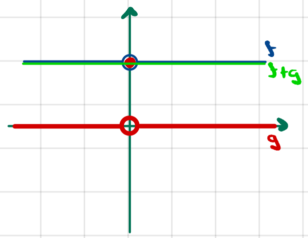
$$y(x) = g(x) \to b\sqrt{\frac{x^2}{a^2} - 1} = (x - x_0)\frac{bx_0}{a^2\sqrt{\frac{x_0^2}{a^2} - 1}} + b\sqrt{\frac{x_0^2}{a^2} - 1} \quad (1)$$

recall the eq. we reached in the ellipse case

$$b\sqrt{1 - \frac{x^2}{a^2}} = (x - x_0)\frac{(-bx_0)}{a^2\sqrt{1 - \frac{x_0^2}{a^2}}} + b\sqrt{1 - \frac{x_0^2}{a^2}} \quad (2)$$

If we square both equations they both become the same.
Thus the

15. $f+g$ diff at a



f and g both not diff. at 0.

$f \cdot g$ and f diff at a

$$(fg)'(a) = \lim_{h \to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(a+h)(g(a+h) - g(a)) + g(a)(f(a+h) - f(a))}{h}$$

$$= \lim_{h \to 0} \frac{f(a+h)(g(a+h) - g(a))}{h} + \lim_{h \to 0} \frac{g(a)(f(a+h) - f(a))}{h}$$

since $(fg)'(a)$ is defined, both the above limits exist.

see stack overflow.

$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

Therefore, $g(x)$ and $g'(x)$ exist, ie g diff at x.

16.

a) f diff at a
$f(a) \neq 0$ $\rightarrow$ $|f|$ diff at a

$$g(x) = |f(x)| = \begin{cases} f(x) & f(x) > 0 \\ -f(x) & f(x) < 0 \\ 0 & f(x) = 0 \end{cases}$$

Case 1: $f(a) > 0$

$$g'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

differentiability of f at a

Case 2: $f(a) < 0$

$$g'(a) = \lim_{h \to 0} \frac{-f(a+h) + f(a)}{h} = -\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

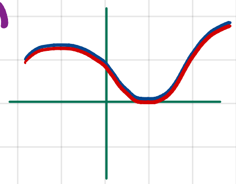
$$= -f'(a)$$

since f diff at a this limit $= f'(a)$

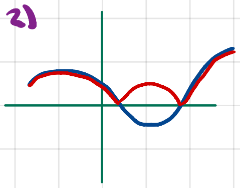
Note that in cases i) and ii), there is an interval containing a in which f is either always positive or always negative, respect.

If $f(a) = 0$ there are multiple cases to consider

1)



2)



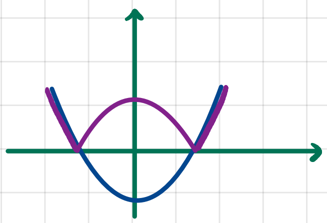
In (1) $|f| = f$ so $|f|$ is diff everywhere. Not so in 2).

Altern. Sol'n

$$g(x) = |f(x)| = \sqrt{f^2(x)}$$

$$g'(x) = \frac{2f(x)f'(x)}{2\sqrt{f^2(x)}} = f'(x)\frac{f(x)}{|f(x)|}, \quad x \neq 0.$$

b)

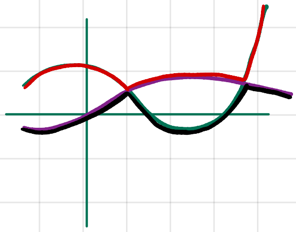


$f(x) = x^2 - 1$ $\quad$ $x^2 - 1 \geq 0 \rightarrow x^2 \geq 1$ $\quad x$

$$|f(x)| = \begin{cases} x^2 - 1 & \text{if } x \geq 1 \text{ or } x \leq -1 \\ 1 - x^2 & \text{if } -1 \leq x \leq 1 \end{cases}$$

$|f(x)|$ not diff. at ± 1.

c) f, g diff at a, $f(a) \neq g(a)$ $\rightarrow$ $\max(f,g)$, $\min(f,g)$ diff at a



Proof

$$h(x) = \max(f(x), g(x))$$

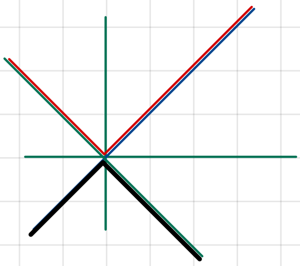
$$= \frac{f(x) + g(x) + |f(x) - g(x)|}{2}$$

$f(x) - g(x)$ is diff at a.
From a), we know that provided $f(a) - g(a) \neq 0$ ie $f(a) \neq g(a)$, $|f(x) - g(x)|$ is diff at a.
Hence $h(x)$ diff at a.

$$m(x) = \min(f(x), g(x)) = \frac{f(x) + g(x) - |f(x) - g(x)|}{2}$$

is thus also diff at a.

d) $f(a) = g(a)$



$f(x) = x$
$g(x) = -x$

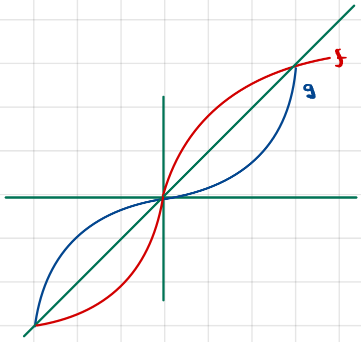
$f(0) = g(0) = 0$

$$h(x) = \frac{x - x + |x - (-x)|}{2} = |x|$$

$$m(x) = \frac{x - x - |x - (-x)|}{2} = -|x|$$

neither of h or m is diff at 0.

17. f and g fns
g takes on all values
$f \circ g$ and g diff.
f not diff.



$g(x) = x^3$, diff $\forall x$

$f(x) = \sqrt[3]{x}$

$f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$, undefined at $x=0$

$(f \circ g)(x) = x$

18.

a) $g(x) = f^2(x)$
$g'(x) = 2f(x)f'(x)$

b) $g(x) = (f'(x))^2$
$g'(x) = 2f'(x)f''(x)$

c) $\forall x, f(x) > 0$

$(f'(x))^2 = f(x) + \dfrac{1}{f^3(x)} > 0$

$2\cancel{f'(x)}f''(x) = \overset{1}{\cancel{f'(x)}} - 3(f(x))^{-4}\cancel{f'(x)}$

$f''(x) = \dfrac{1}{2} - \dfrac{3}{2f^4(x)}$

19. f, f', f'', f''' exist, ie f three times differentiable.

$f'(x) \neq 0$

Schwarzian derivative of f: $\mathcal{D}f(x) = \dfrac{f'''(x)}{f'(x)} - \dfrac{3}{2}\left(\dfrac{f''(x)}{f'(x)}\right)^2$

a) $\mathcal{D}(f \circ g) = [\mathcal{D}f \circ g] \cdot g'^2 + \mathcal{D}g$

Proof

$(f \circ g)(x) = f(g(x))$

$(f \circ g)'(x) = f'(g(x))g'(x)$

$(f \circ g)''(x) = f''(g(x))(g'(x))^2 + f'(g(x))g''(x)$

$(f \circ g)'''(x) = f'''(g(x))(g'(x))^3 + f''(g(x)) \cdot 2g'(x)g''(x) + f''(g(x))g'(x)g''(x) + f'(g(x))g'''(x)$

$$= f'''(g(x))(g'(x))^3 + 3f''(g(x))g'(x)g''(x) + f'(g(x))g'''(x)$$

$$\frac{(f \circ g)'''(x)}{(f \circ g)'(x)} = \frac{f'''(g(x))(g'(x))^3 + 3f''(g(x))g'(x)g''(x) + f'(g(x))g'''(x)}{f'(g(x))g'(x)}$$

$$\frac{(f \circ g)''(x)}{(f \circ g)'(x)} = \frac{f''(g(x))(g'(x))^2 + f'(g(x))g''(x)}{f'(g(x))g'(x)}$$

$$(\mathcal{D}f \circ g)(x) = \frac{f'''(g(x))}{f'(g(x))} - \frac{3}{2}\left(\frac{f''(g(x))}{f'(g(x))}\right)^2$$

$(\mathcal{D}f \circ g)(x) \cdot g'^2(x) + \mathcal{D}g(x)$

$$= g'^2(x)\left[\frac{f'''(g(x))}{f'(g(x))} - \frac{3}{2}\left(\frac{f''(g(x))}{f'(g(x))}\right)^2\right] + \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2$$

$\mathcal{D}(f \circ g)(x) =$

$$\frac{f'''(g(x))(g'(x))^{\cancel{3}\,2} + 3f''(g(x))\cancel{g'(x)}g''(x) + \cancel{f'(g(x))}g'''(x)}{f'(g(x))g'(x)} - \frac{3}{2}\left[\frac{f''(g(x))(g'(x))^2 + f'(g(x))g''(x)}{f'(g(x))g'(x)}\right]^2$$

$$= \frac{f'''(g(x))}{f'(g(x))}g'^2(x) + \frac{3f''(g(x))g''(x)}{f'(g(x))} + \frac{g'''(x)}{g'(x)} - \frac{3}{2}\,\frac{f''^2(g(x))\,g'^2(x)}{f'^2(g(x))}$$

$$- \frac{3f''(g(x))\cancel{(g'(x))^2}\cancel{f'(g(x))}g''(x)}{(f'(g(x))\cancel{g'(x)})^2} - \frac{3}{2}\,\frac{g''^2(x)}{g'^2(x)}$$

Therefore

$\mathcal{D}(f \circ g)(x) = (\mathcal{D}f \circ g)(x) \cdot g'(x) + \mathcal{D}g(x)$

b)

$$f(x) = \frac{ax+b}{cx+d} \qquad \to \; Df = 0 \quad \to \; D(f \circ g) = Dg$$

$$ad - bc \neq 0$$

Proof

$$f'(x) = \frac{a(cx+d) - c(ax+b)}{(cx+d)^2} = \frac{x(ac-ac) + ad - bc}{(cx+d)^2}$$

$$= \frac{ad-bc}{(cx+d)^2}$$

$$f''(x) = \frac{-(ad-bc)\cdot 2\cancel{(cx+d)}c}{(cx+d)^{\cancel{4}\,3}} = \frac{2c^2b - 2acd}{(cx+d)^3}$$

$$f'''(x) = \frac{-(2c^2b - 2acd)\cdot 3\cancel{(cx+d)^2}c}{(cx+d)^{\cancel{6}\,4}} = \frac{6ac^2d - 6c^3b}{(cx+d)^4}$$

$$Df(x) = \frac{\dfrac{6ac^2d - 6c^3b}{(cx+d)^4}}{\dfrac{ad-bc}{(cx+d)^2}} - \frac{3}{2}\left[\frac{\dfrac{2c^2b - 2acd}{(cx+d)^3}}{\dfrac{ad-bc}{(cx+d)^2}}\right]^2$$

$$= \frac{6c^2\cancel{(ad-bc)}}{(cx+d)^2\cancel{(ad-bc)}} - \frac{3}{2}\left[\frac{2c\overset{-1}{\cancel{(bc-ad)}}}{(cx+d)\cancel{(ad-bc)}}\right]^2$$

$$= \frac{6c^2}{(cx+d)^2} - \frac{3}{\cancel{2}} \; \frac{{}^{2}\cancel{4}c^2}{(cx+d)^2}$$

$$= \frac{6c^2}{(cx+d)^2} - \frac{6c^2}{(cx+d)^2}$$

$$= 0$$

20. $f^{(n)}(a)$ and $g^{(n)}(a)$ exist. $\rightarrow (f\cdot g)^{(n)}(a) = \sum_{k=0}^{n}\binom{n}{k} f^{(k)}(a)\cdot g^{(n-k)}(a)$ (Leibniz's formula)

Proof

we want to prove $\forall n\,(n \in \mathbb{N} \rightarrow [f^{(n)}(a)$ and $g^{(n)}(a)$ exist. $\rightarrow (f\cdot g)^{(n)}(a) = \sum_{k=0}^{n}\binom{n}{k} f^{(k)}(a)\cdot g^{(n-k)}(a)])$

Let $A = \{n : Q(n)\}$ where $Q(n) = (f\cdot g)^{(n)}(a) = \sum_{i=0}^{n}\binom{n}{i} f^{(i)}(a)\cdot g^{(n-i)}(a)$

ie, $A = \{n : (f\cdot g)^{(n)}(a) = \sum_{i=0}^{n}\binom{n}{i} f^{(i)}(a)\cdot g^{(n-i)}(a)\}$

we will use induction on $\mathbb{N}$.

Inductive Definition of $\mathbb{N}$

base clause: $1 \in \mathbb{N}$
inductive clause: $n \in \mathbb{N} \rightarrow n+1 \in \mathbb{N}$
final clause: $\forall n, n \in \mathbb{N} \rightarrow$ n is generated by repeated applic. of base and inductive clauses

ie, considering every set satisfying base and inductive clauses above, $\mathbb{N}$ is their intersection.

if we show that A satisfies the clauses then we can conclude $\mathbb{N} \subseteq A$.
Hence $\forall n\,(n \in \mathbb{N} \rightarrow n \in A \rightarrow Q(n))$

Base step: show that A satisfies the base clause of $\mathbb{N}$

$(f\cdot g)'(a) = f'g + fg' = \sum_{i=0}^{1}\binom{1}{i} f^{(i)}(a) g^{(n-i)}(a) \quad \rightarrow 1 \in A$

Inductive step show that A satisfies the inductive clause of $\mathbb{N}$

Assume $(f\cdot g)^{(k)}(a) = \sum_{i=0}^{k}\binom{k}{i} f^{(i)}(a) g^{(k-i)}(a)$, ie $k \in A$. Then

$$(f\cdot g)^{(k+1)}(a) = \sum_{i=0}^{k}\binom{k}{i}\left(f^{(i+1)}(a)\,g^{(k-i)}(a) + f^{(i)}(a)\,g^{(k-i+1)}(a)\right)$$

$$= \binom{k}{0} f^{(1)} g^{(k)} + \binom{k}{1} f^{(2)} g^{(k-1)} + \binom{k}{2} f^{(3)} g^{(k-2)} + \binom{k}{3} f^{(4)} g^{(k-3)} + \dots + \binom{k}{k-1} f^{(k)} g^{(1)} + \binom{k}{k} f^{(k+1)} g$$

$$\binom{k}{0} f g^{(k+1)} + \binom{k}{1} f^{(1)} g^{(k)} + \binom{k}{2} f^{(2)} g^{(k-1)} + \binom{k}{3} f^{(3)} g^{(k-2)} + \dots + \binom{k}{k-1} f^{(k-1)} g^{(2)} + \binom{k}{k} f^{(k)} g^{(1)}$$

$$= \binom{k+1}{0} f g^{(k+1)} + \binom{k+1}{1} f^{(1)} g^{(k+1-1)} + \binom{k+1}{2} f^{(2)} g^{(k+1-2)} + \binom{k+1}{3} f^{(3)} g^{(k+1-3)}$$

$$+ \dots + \binom{k+1}{k} f^{(k)} g^{(k+1-k)} + \binom{k+1}{k+1} f^{(k+1)} g$$

$$= \sum_{i=0}^{k+1}\binom{k+1}{i} f^{i} g^{k+1-i} \quad \rightarrow k+1 \in A$$

Therefore, $k \in A \rightarrow k+1 \in A$. A satisfies inductive clause of $\mathbb{N}$.

$\mathbb{N} \subseteq A$

$\forall n\,(n \in \mathbb{N} \rightarrow n \in A \rightarrow (f\cdot g)^{(n)}(a) = \sum_{i=0}^{n}\binom{n}{i} f^{(i)}(a)\cdot g^{(n-i)}(a)$

21. $f^{(n)}(g(a))$ and $g^{(n)}(a)$ exist $\to$ $(f\circ g)^{(n)}(a)$ exists.

Proof

$(f\circ g)'(x) = f'(g(x))\,g'(x)$

$(f\circ g)''(x) = f''(g(x))[g'(x)]^2 + f'(g(x))\,g''(x)$

$$(f\circ g)'''(x) = f'''(g(x))[g'(x)]^3 + f''(g(x))\cdot 2g'(x)g''(x) + f''(g(x))g'(x)g''(x) + f'(g(x))g'''(x)$$
$$= f'''(g(x))[g'(x)]^3 + 3f''(g(x))\,g''(x)g'(x) + f'(g(x))\,g'''(x)$$

Seems like $(f\circ g)^{(n)}$ is a sum of terms, each a product of some number, a certain h^{th} derivative of f ($h \le n$), and one or more derivatives of g up to n^{th} derivative.

Let $A = \{ n : n \in \mathbb{N} \wedge f^{(n)}(g(a))$ and $g^{(n)}(a)$ exist $\to (f\circ g)^{(n)}(a)$ exists, and

$(f\circ g)^{(n)}(a)$ is a sum of terms of form

$$cf^{(h)}(g(a)).\,[g^{(1)}(a)]^{m_1}[g^{(2)}(a)]^{m_2}\cdots\cdot[g^{(n)}(a)]^{m_n}$$

where c is a number, $m_1,\dots,m_n \in \mathbb{N}^+$, and $h \le n \in \mathbb{N}.\}$

Consider $n=1$.

Assume $f^{(1)}(g(a))$ and $g^{(1)}(a)$ exist.

Then $(f(g(x)))' = (f\circ g)^{(1)}(a) = f^{(1)}(g(a))\cdot g^{(1)}(a)$, which exists.

ie $(f\circ g)^{(1)}(a)$ exists and is of the required form.

$\to 1 \in A$.

Assume $k \in A$. Then

$f^{(k)}(g(a))$
$g^{(k)}(a)$ exist. $\to$ $(f\circ g)^{(k)}(a)$ exists and has the form specified in A's def.

Assume

$f^{(k+1)}(g(a))$
$g^{(k+1)}(a)$ exist.

Then,

$$(f\circ g)^{(k+1)}(a) = [(f\circ g)^{(k)}(a)]'$$

Every term in $(f\circ g)^{(k)}(a)$ is of form $cf^{(h)}(g(a)).\,[g^{(1)}(a)]^{m_1}[g^{(2)}(a)]^{m_2}\cdots\cdot[g^{(n)}(a)]^{m_n}$

22.

a) $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

$g' = f$

$g_1(x) = \frac{a_n}{n+1} x^{n+1} + \frac{a_{n-1}}{n} x^n + \dots + a_0 x + c_1$

$g_2(x) = \frac{a_n}{n+1} x^{n+1} + \frac{a_{n-1}}{n} x^n + \dots + a_0 x + c_2$

$g_1' = f$

$g_2' = f$

b) $f(x) = \frac{b_2}{x^2} + \frac{b_3}{x^3} + \dots + \frac{b_m}{x^m}$

$g(x) = -\frac{b_2}{x} - \frac{b_3}{2x^2} - \dots - \frac{b_m}{(m-1)x^{m-1}}$

$g'(x) = -(-1) b_2 x^{-2} - (-2) \frac{b_3}{2} x^{-3} - \dots - (-(m-1)) \frac{b_m}{m-1} x^{-(m-1)-1}$

$= \frac{b_2}{x^2} + \frac{b_3}{x^3} + \dots + \frac{b_m}{x^m}$

c) $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 + \frac{b_1}{x} + \dots + \frac{b_m}{x^m}$

$= \sum_{i=0}^{n} g_i(x) + \sum_{i=1}^{m} h_i(x)$, $g_i(x) = a_i x^i$, $h_i(x) = \frac{b_i}{x^i}$

$g_i'(x) = a_i i x^{i-1}$, $i = 1, \dots, n$, $g_0'(x) = 0$

$h_i'(x) = \frac{-b_i}{x^{i+1}}$, $i = 1, \dots, m$

There is no term of x^{-1}!

$f(x) = \int_1^x \frac{1}{t} dt \qquad x > 0$

$f(xy) = \int_1^{xy} \frac{1}{t} dt = \int_1^x \frac{1}{t} dt + \int_x^{xy} \frac{1}{t} dt = f(x) + f(y)$

$\int_x^{xy} \frac{1}{t} dt = \ln(xy) - \ln(x) = \ln y$

$\int_1^y \frac{1}{t} dt = \ln y$

23. polyn. f, degree n

a) $f'(x)=0$ for $n-1$ numbers x

$f(x) = (x-a_1)(x-a_2)\cdot\cdots\cdot(x-a_{n-1})$

This polynomial has degree $n-1$ and $n-1$ roots.

It can be rewritten as

$f(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0$

As shown in problem 22, any fn $g(x) = \frac{b_{n-1}}{n}x^n + \cdots + \frac{b_1}{2}x^2 + b_0 x + c$

is s.t. $g'(x) = f(x)$, and degree of g is n.

b) n odd, $f'(x)=0$ for no x.

f is an n-degree polynomial, w/ n odd.
Then f' is of degree $n-1$, $n-1$ is even.

$f'(x) = x^{n-1} + 5$ has no roots since $x^{n-1} = -5$ has no real sol'n.

$f(x) = \frac{x^n}{n} + 5x + c$

c) f' has one root only, n even.

$f'(x) = x^{n-1} + a_0$

$f'(x) = 0 \to x^{n-1} = -a_0 \to x = \sqrt[n-1]{-a_0}$

only one root.

$f(x) = \frac{x^n}{n} + a_0 x + c$

d) $n-k$ odd. f' has k roots.

From 7-4b, we know that

Let f be polyn. deg. n with k roots, including multiplicities.
Then $n-k$ is even

Now we have f' of degree $n-1$. $(n-1)-k$ is even.

case 1: $n-1$ even, k even.

$f'(x) = (x-a_1)\cdot\cdots\cdot(x-a_k)(x^{n-1-k}+c)$ $c>0$

case 2: $n-1$ odd, k odd

$f'(x) = (x-a_1)\cdots\cdots(x-a_{k-1})(x^{n-k}+a_0)$

24.

a) a is double root of f $\leftrightarrow$ a is root of f and f'

Proof

Assume a is double root of f.

Then $f(x) = (x-a)^2 g(x)$, $g(x)$ polyn. and $g(a) \neq 0$.

$f'(x) = 2(x-a)g(x) + (x-a)^2 g'(x) = 0$

$g(x)(x-a)\left(2 + (x-a)g'(x)\right) = 0$

a is root of f'

a is double root of f $\rightarrow$ a is root of f and f'

Assume a is root of f and f'

$f(x) = (x-a)g(x)$

$f'(x) = (x-a)g'(x) + g(x)$

$f'(a) = g(a) = 0$ $\rightarrow$ a is root of g

$g(x) = h(x)(x-a)$

$f(x) = (x-a)^2 h(x)$

a is double root of f.

a is root of f and f' $\rightarrow$ a is double root of f

b) $f(x)=ax^2+bx+c \quad (a\neq 0)$

$f(x)=ax^2+bx+c$

$$=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

$$=a\left(x^2+2x\frac{b}{2a}+\frac{b^2}{4a^2}+\frac{c}{a}-\frac{b^2}{4a^2}\right)$$

$$=a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$$

If $4ac-b^2=0$ then $f(x)=a\left(x+\frac{b}{2a}\right)^2 \longrightarrow x=-\frac{b}{2a}$ is a double root.

Altern., using part a),

$f'(x)=2ax+b=0 \rightarrow x=-\frac{b}{2a}$.

If $x=-\frac{b}{2a}$ is a root of f then it is a double root of f.

This happens if

$$f\left(-\frac{b}{2a}\right)=a\frac{b^2}{4a^2}-b\frac{b}{2a}+c$$

$$=\frac{b^2}{4a}-\frac{b^2}{2a}+c$$

$$=\frac{b^2-2b^2+4ac}{4a^2}$$

$$=0$$

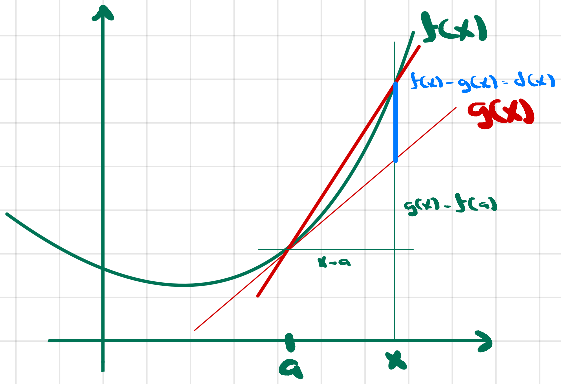
$\rightarrow -b^2+4ac=0$

Geometrically, at $-\frac{b}{2a}$, the graph of f touches the x-axis, and at that point the tangent line is horizontal.

25.

f diff at a

$d(x) = f(x) - f'(a)(x-a) - f(a)$



we want $d'(x)$, the rate of growth of the light blue distance as x changes.

$d'(x) = f'(x) - f'(a)$

a is the only root of $d'(a)$.

Hence, since a is also a root of $d(x)$, by 24a) a is a double root of d.

$d(x) = (x-a)^2 h(x)$. We reached this result in 9-20b by other means.

26.

a)

$a_1,\dots,a_n$
$b_1,\dots,b_n$ given numbers
$x_1,\dots,x_n$ distinct numbers

$\rightarrow$ $\exists$ polyn. f degree $2n-1$ s.t.

$f(x_j)=f'(x_j)=0 \quad j\neq i$

$f(x_i)=a_i$

$f'(x_i)=b_i$

Proof

we want a fn f such that considering only points $x_1,\dots,x_n$, f and f' are only nonzero at a single point x_i.

By problem 24, each x_j, $j\neq i$, is a double root of f.

Thus f has form $\left(\prod_{j\neq i}(x-x_j)^2\right)g(x)$. The $\prod_{j\neq i}(x-x_j)^2$ part has degree $2n-2$.

$g(x)$ must have degree 1, ie $g(x)=ax+b$.

$$f(x)=\prod_{j\neq i}(x-x_j)^2(ax+b)=h(x)(ax+b)$$

$$f'(x)=h'(x)(ax+b)+ah(x)$$

we don't know what a and b are yet.

$$f'(x_i)=ax_ih(x_i)+bh(x_i)=a_i$$

$$f'(x_i)=a(h'(x_i)x_i+h(x_i))+bh'(x_i)=b_i$$

$$\underbrace{\begin{bmatrix} h(x_i)x_i & h(x_i) \\ h'(x_i)x_i+h(x_i) & h'(x_i)\end{bmatrix}}_{A}\begin{bmatrix} a \\ b\end{bmatrix}=\begin{bmatrix} a_i \\ b_i\end{bmatrix}$$

$$\det A=\cancel{h(x_i)h'(x_i)x_i}-\cancel{h(x_i)h'(x_i)x_i}-h^2(x_i)=-h^2(x_i)$$

$$=-\left[\prod_{j\neq i}(x_i-x_j)^2\right]^2\neq 0 \text{ since } x_1,\dots,x_n \text{ are distinct.}$$

Hence there is a unique solution a^*, b^*.

The f we wanted is $f(x)=\prod_{j\neq i}(x-x_j)^2\,(a^*x+b^*)$

---

b) f of degree $2n-1$

$\forall i$ $f(x_i)=a_i$ and $f'(x_i)=b_i$

Let $f(x)=\sum_{i=1}^{n}\left[\prod_{j\neq i}(x-x_j)^2\right](a_i^*x+b_i^*)$

where $\left[\prod_{j\neq i}(x-x_j)^2\right](a_i^*x+b_i^*)$ is the fn obtained in a) for a specific i.

27. $f(x) = (x-a)(x-b)g(x)$, $g(a) \neq 0$, $g(b) \neq 0$, a and b consecutive roots

a)

Assume $g(a)$ and $g(b)$ have different signs.
WLOG, assume $g(a) < 0 < g(b)$.

Since g is polynomial, it is continuous in $[a,b]$.

IVT $\rightarrow$ g takes on every value between $g(a)$ and $g(b)$ in $[a,b]$.

$\exists c,\ c \in [a,b] \wedge g(c) = 0$

$g(x) = (x-c)h(x)$

$f(x) = (x-a)(x-b)(x-c)h(x)$

c is root of f.

a and b are not consecutive roots of f.

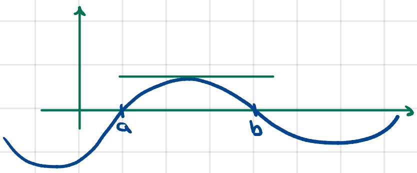
$\perp$.

$g(a)$ and $g(b)$ have same sign.

b)

$f(x) = (x-a)(x-b)g(x)$
$g(a) \neq 0$
$g(b) \neq 0$
a, b consecutive roots

$\longrightarrow$ $\exists x,\ a < x < b \wedge f'(x) = 0$



Proof

$f'(x) = (x-b)g(x) + (x-a)g(x) + (x-a)(x-b)g'(x)$

$f'(a) = (a-b)g(a)$
$f'(b) = (b-a)g(b)$

Case 1: $g(a) > 0,\ g(b) > 0$

$f'(a) < 0$
$f'(b) > 0$
IVT: $\exists c,\ c \in [a,b] \wedge f'(c) = 0$

Case 2: $g(a) < 0,\ g(b) < 0$

$f'(a) > 0$
$f'(b) < 0$
IVT: $\exists c,\ c \in [a,b] \wedge f'(c) = 0$

Therefore, by proof by cases

$\exists c,\ c \in [a,b] \wedge f'(c) = 0$

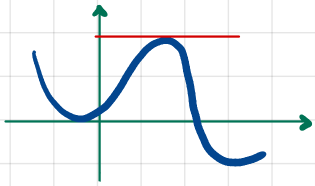
c)

$f(x) = (x-a)^m (x-b)^n g(x)$

$g(a) \neq 0$ $\longrightarrow$ $\exists x,\ a<x<b \wedge f'(x)=0$

$g(b) \neq 0$

a, b consecutive roots



Proof

$$f'(x) = m(x-a)^{m-1}(x-b)^n g(x) + n(x-a)^m (x-b)^{n-1} g(x) + (x-a)^m (x-b)^n g'(x)$$

Let $h(x) = \dfrac{f'(x)}{(x-a)^{m-1}(x-b)^{n-1}} = m(x-b)g(x) + n(x-a)g(x) + (x-a)(x-b)g'(x)$

$h(a) = m(a-b)g(a)$

$h(b) = n(b-a)g(b)$

Using the same argument from a), $g(a)$ and $g(b)$ must have the same sign, otherwise $\exists c,\ c \in [a,b] \wedge f(c)=0$ s.t. a and b aren't consecutive, a contradiction w/ our assumptions.

Case 1: $g(a)$ and $g(b) > 0$

$h(a) < 0$
$h(b) > 0$ $\rightarrow \exists c,\ c \in [a,b] \wedge h(c)=0 \rightarrow f'(c)=0$

Case 2: $g(a)$ and $g(b) < 0$

$h(a) > 0$
$h(b) < 0$ $\rightarrow \exists c,\ c \in [a,b] \wedge h(c)=0 \rightarrow f'(c)=0$

By proof by cases

$\exists c,\ c \in [a,b] \wedge f'(c)=0$

28. $f(x) = xg(x)$, g cont. at 0 $\rightarrow$ f diff at 0

Proof

$$f'(0) = \lim_{h \to 0} \frac{hg(h)}{h} = \lim_{h \to 0} g(h) = g(0)$$

29. f diff at 0, $f(0) = 0$ $\rightarrow$ $f(x) = xg(x)$ for some g cont. at 0

Proof

$$f'(0) = \lim_{h \to 0} \frac{f(h)}{h} \quad \text{exists}$$

f diff at 0

Let

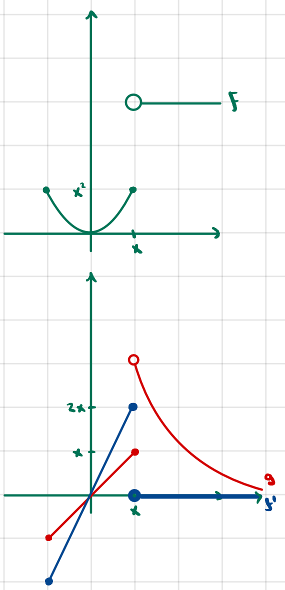
$$g(x) = \begin{cases} \dfrac{f(x)}{x} & x \neq 0 \\ f'(0) & x = 0 \end{cases}$$

Then $g(x)$ is the slope of the line from $(0,0)$ to $(x, f(x))$.

$f(x) = xg(x)$ $\forall x$, $f(0) = 0$.

$f'(0) = \lim_{h \to 0} g(h) = g(0)$, so g cont. at 0.

This result is telling us that any fn that passes thru origin and is differentiable there can be written as the product of x and a fn g, the latter giving the slope of the line from $(0,0)$ to $(x, f(x))$, and g is cont. at 0.



30. $\forall n\,(n\in\mathbb{N} \rightarrow (f(x)=x^{-n} \rightarrow f^{(k)}(x)=(-1)^k \frac{(n+k-1)!}{(n-1)!}x^{-n-k} = (-1)^k k! \binom{n+k-1}{k} x^{-n-k} \quad x\neq 0)$

Proof

Let's use induction on n and on k.

$A=\{n : n\in\mathbb{N} \rightarrow [\forall k : k\in\mathbb{N} \rightarrow [f(x)=x^{-n} \rightarrow f^{(k)}(x)=(-1)^k \frac{(n+k-1)!}{(n-1)!}x^{-n-k} \quad x\neq 0]]\}$

Assume $n=1$.

$f(x)=x^{-1}$

Let $B_1=\{k : k\in\mathbb{N} \rightarrow [f(x)=x^{-1} \rightarrow f^{(k)}(x)=(-1)^k \frac{(1+k-1)!}{(n-1)!}x^{-1-k} \quad x\neq 0]\}$

Let $k=1$.

$f'(x)=-x^{-2}=(-1)^1\binom{1}{0}x^{-2}$

$1\in B_1$

Assume $k\in B_1$. Then $f(x)=x^{-1} \rightarrow f^{(k)}(x)=(-1)^k \frac{k!}{0!}x^{-1-k} \quad x\neq 0$

$f^{(k+1)}(x)=(-1-k)(-1)^k \frac{k!}{0!}x^{-1-k-1} \quad x\neq 0$

$=(-1)(k+1)(-1)^k \frac{k!}{0!}x^{-1-(k+1)} \quad x\neq 0$

$=(-1)^{k+1} \frac{(k+1)!}{0!}x^{-1-(k+1)} \quad x\neq 0$

$k+1\in B_1$

$k\in B_1 \rightarrow k+1\in B_1$

$\mathbb{N}\subseteq B_1$, ie $1\in\mathbb{N}\rightarrow [\forall k\in\mathbb{N} \rightarrow [f(x)=x^{-1} \rightarrow f^{(k)}(x)=(-1)^k \frac{(1+k-1)!}{(n-1)!}x^{-1-k} \quad x\neq 0]]$

$1\in A$

Assume $n\in A$. Then $f(x)=x^{-n}$ and $[\forall k : k\in\mathbb{N} \rightarrow [f(x)=x^{-n} \rightarrow f^{(k)}(x)=(-1)^k \frac{(n+k-1)!}{(n-1)!}x^{-n-k} \quad x\neq 0]]$

Let $f(x)=x^{-(n+1)}$

Let $B_{n+1}=\{k : k\in\mathbb{N} \rightarrow [f(x)=x^{-(n+1)} \rightarrow f^{(k)}(x)=(-1)^k \frac{(n+k)!}{n!}x^{-n-k-1} \quad x\neq 0$

Let $k=1$.

$f'(x)=-(n+1)x^{-n-2}=(-1)^1 \frac{(n+1)!}{n!}x^{-n-2} \rightarrow 1\in B_{n+1}$

Assume $k\in B_{n+1}$. Then $f(x)=x^{-(n+1)} \rightarrow f^{(k)}(x)=(-1)^k \frac{(n+k)!}{n!}x^{-n-k-1} \quad x\neq 0$

$f^{(k+1)}(x)=(-n-k-1)(-1)^k \frac{(n+k)!}{n!}x^{-n-k-2} \quad x\neq 0$

$=(-1)(n+k+1)(-1)^k \frac{(n+k)!}{n!}x^{-n-k-2} \quad x\neq 0$

$=(-1)^{k+1} \frac{(n+k+1)!}{n!}x^{-n-(k+1)-1} \quad x\neq 0$

$k+1\in B_{n+1}$

$k\in B_{n+1} \rightarrow k+1\in B_{n+1}$

$\mathbb{N}\subseteq B_{n+1}$, ie $\forall k, k\in\mathbb{N} \rightarrow [f(x)=x^{-(n+1)} \rightarrow f^{(k)}(x)=(-1)^k \frac{(n+k)!}{n!}x^{-n-k-1} \quad x\neq 0$

$n+1\in A$

$n\in A \rightarrow n+1\in A$

$\mathbb{N}\subseteq A$, ie $\forall n\,(n\in\mathbb{N} \rightarrow [\forall k : k\in\mathbb{N} \rightarrow [f(x)=x^{-n} \rightarrow f^{(k)}(x)=(-1)^k \frac{(n+k-1)!}{(n-1)!}x^{-n-k} \quad x\neq 0]]$

31. f, g differentiable
$f(0) = g(0) = 0$

Let $h(x) = f(x)g(x)$

Assume $h(x) = x$

Then $h'(x) = f'(x)g(x) + f(x)g'(x) = 1$

$h'(0) = 0 = 1$

$\perp$.

32.

a) $f(x) = \dfrac{1}{(x-a)^n}$, $x \neq a$

Let $g(x) = f(x+a) = \dfrac{1}{x^n}$, $x \neq 0$

By problem 30,

$$g^{(k)}(x) = f^{(k)}(x+a) = (-1)^k \frac{(n+k-1)!}{(n-1)!} x^{-n-k}, \quad x \neq 0$$

$$g^{(k)}(x-a) = f^{(k)}(x) = (-1)^k \frac{(n+k-1)!}{(n-1)!} (x-a)^{-n-k} \quad x \neq a$$

b) $f(x) = \dfrac{1}{x^2-1} = \dfrac{1}{x+1} \cdot \dfrac{1}{x-1} = g(x)h(x)$

From Leibniz's formula (problem 20):

$$f^{(n)}(x) = (g \cdot h)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} g^{(k)}(x)\, h^{(n-k)}(x)$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{k!}{0!} (x-1)^{-1-k} (-1)^{n-k} \frac{(n+k)!}{0!} (x+1)^{-1-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^n k!\,(n+k)!\,(x^2-1)^{-1-k}$$

Alternatively, write f as sum of terms and apply part a).

$$f(x) = \frac{1}{2}\left(\frac{1}{x+1} - \frac{1}{x-1} \right)$$

$$f^{(n)}(x) = \frac{1}{2}(-1)^n \frac{n!}{(n-1)!} \left[(x+1)^{-1-n} - (x-1)^{-1-n} \right]$$

33. $f(x) = \begin{cases} x^{2n}\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ $\to$ $f'(0), \dots, f^{(n)}(0)$ exist, $f^{(n)}$ not continuous at 0

Proof

$$f'(x) = 2n x^{2n-1}\sin(1/x) + x^{2n}\cos(1/x)(-1/x^2)$$

$$= \sin(1/x)\,2n x^{2n-1} + \cos(1/x)(-x^{2n-2})$$

$$f''(x) = \cos(1/x)(-1/x^2)\cdot 2n x^{2n-1} + \cos(1/x)(-(2n-2)x^{2n-3})$$

$$- \sin(1/x)(-1/x^2)(-x^{2n-2}) + \sin(1/x)\,2n(2n-1)x^{2n-2}$$

$$= \sin(1/x)\left[x^{2n-2}\frac{(2n)!}{(2n-2)!} - x^{2n-4}\right] + \cos(1/x)\left[-x^{2n-3}(2n+2n-2)\right]$$

$$= \sin(1/x)\left[x^{2n-2}\frac{(2n)!}{(2n-2)!} - x^{2n-4}\right] + \cos(1/x)\left[-x^{2n-3}\,2(2n-1)\right]$$

$$f'''(x) = \cos(1/x)(-1/x^2)\left[x^{2n-2}\frac{(2n)!}{(2n-2)!} - x^{2n-4}\right] + \cos(1/x)\left(-2(2n-1)(2n-3)x^{2n-4}\right)$$

$$- \sin(1/x)(-1/x^2)\left[-x^{2n-3}\,2(2n-1)\right] + \sin(1/x)\left[x^{2n-3}\frac{(2n)!}{(2n-3)!} - (2n-4)x^{2n-5}\right]$$

$$= \sin(1/x)\left[x^{2n-3}\frac{(2n)!}{(2n-3)!} + x^{2n-5}\left(-2(2n-1)-(2n-4)\right)\right.$$

$$\left. + \cos(1/x)\left[-x^{2n-4}\left(\frac{(2n)!}{(2n-2)!} + 2(2n-1)(2n-3)\right) + x^{2n-6}\right]\right.$$

$$= \sin(1/x)\left[x^{2n-3}\frac{(2n)!}{(2n-3)!}\right. + x^{2n-5}(6-6n) + \cos(1/x)\left[-x^{2n-4}\left(\frac{(2n)!}{(2n-2)!} + 2(2n-3)\right) + x^{2n-6}\right]$$

$$x^{2n-4}(2-4n)(3n-3)$$

Based on the calculations above we can make the following conjecture

$f^{(k)}(x)$ is composed of the following terms

$\sin(1/x)\,x^{2n-k}\cdot a.$

$\pm \sin(1/x)\,x^{2n-2k}$ if k even

$\pm \cos(1/x)\,x^{2n-2k}$ if k odd

$$\sum_{i=k+1}^{2k-1}\left(a_i x^{2n-i}\sin(1/x) + b_i x^{2n-i}\cos(1/x)\right)$$

we check our conjecture using induction on k

$$f'(x) = \sin(1/x)\, 2n x^{2n-1} + \cos(1/x)(-x^{2n-2})$$

$\to$ conjecture true for $k=1$.

$\sin(1/x)\, x^{2n-(k+1)} \quad a$

Assume it's true for k.

$\pm \sin(1/x)\, x^{2n-2(k+1)}$ if k even

If it's true for $k+1$ then we should only see the following terms in $f^{(k+1)}(x)$:

$\pm \cos(1/x)\, x^{2n-2(k+1)}$ if k odd

$$\sum_{i=k+2}^{2(k+1)-1} \left(a_i x^{2n-i} \sin(1/x) + b_i x^{2n-i} \cos(1/x) \right)$$

we check that this is the case by differentiating $f^{(k)}(x)$. Then $f^{(k+1)}(x)$ is composed of the terms

$$\frac{d}{dx} \sin(1/x)\, x^{2n-k} \cdot a = \cos(1/x)(-1/x^2)\, x^{2n-k} + \sin(1/x)(2n-k)\, x^{2n-k-1}$$

$$= \cos(1/x)\, x^{2n-k-2} + \sin(1/x)(2n-k)\, x^{2n-k-1}$$

$$= b_2 \cos(1/x)\, x^{2n-(k+2)} + a_1 x^{2n-(k+1)} \sin(1/x)$$

$$\frac{d}{dx}\left(\pm \sin(1/x)\, x^{2n-2k}\right) = \pm \cos(1/x)(-1/x^2)\, x^{2n-2k} \pm \sin(1/x)(2n-2k)\, x^{2n-2k-1}$$

$$= \mp \cos(1/x)\, x^{2n-2(k+1)} \pm \sin(1/x)(2n-2k)\, x^{2n-(2(k+1)-1)}$$

$$\frac{d}{dx}\left(\pm \cos(1/x)\, x^{2n-2k}\right) = \pm \sin(1/x)(-1/x^2)\, x^{2n-2k} \pm \cos(1/x)(2n-2k)\, x^{2n-2k-1}$$

$$= \mp \sin(1/x)\, x^{2n-2(k+1)} \pm \cos(1/x)(2n-2k)\, x^{2n-(2(k+1)-1)}$$

$$\frac{d}{dx}\left[\sum_{i=k+1}^{2k-1} \left(a_i x^{2n-i} \sin(1/x) + b_i x^{2n-i} \cos(1/x) \right) \right]$$

$$= \sum_{i=k+1}^{2k-1} \left[a_i (2n-i)\, x^{2n-i-1} \sin(1/x) - a_i x^{2n-i-2} \cos(1/x) + b_i (2n-i)\, x^{2n-i-1} \cos(1/x) + b_i x^{2n-i-2} \sin(1/x) \right]$$

$$= \sum_{i=k+1}^{2k-1} \left[\sin(1/x)\left(a_i (2n-i)\, x^{2n-(i+1)} + b_i x^{2n-(i+1)-1} \right) + \cos(1/x)\left(b_i (2n-i)\, x^{2n-(i+1)} - a_i x^{2n-(i+1)-1} \right) \right]$$

$$= \sum_{i=k+1}^{2k-1} \left[\sin(1/x)\left(a_i (2n-i)\, x^{2n-(i+1)} + b_i x^{2n-(i+1)} \cdot \frac{1}{x} \right) \right.$$

$$= \sum_{i=k+1}^{2k-1} \left[\sin(1/x)\left(x^{2n-(i+1)} \left(a_i (2n-i) + \frac{b_i}{x} \right) \right) \right.$$

not sure what to do w/ these terms

?? ??

$$\sum_{i=k+1}^{2(k+1)-1} \left[A_i \sin(1/x)\, x^{2n-i} + B_i \cos(1/x)\, x^{2n-i} + c_i \sin(1/x)\, x^{2n-(i+1)} + d_i \cos(1/x)\, x^{2n-(i+1)} \right]$$

Now we want to show that $f'(0), \dots, f^{(n)}(0)$ exist.

n	x^{2n}	$f^{(n)}$
1	x^2	$f^{(1)}$
2	x^4	$f^{(2)}$
3	x^6	$f^{(3)}$

Since $f^{(k)}(x)$ is composed of the following types of terms

$\sin(1/x)\, x^{2n-k} \cdot a.$

$\pm \sin(1/x) x^{2n-2k}$ if k even

$\pm \cos(1/x) x^{2n-2k}$ if k odd

$$\sum_{i=k+1}^{2k-1} \left(a_i x^{2n-i} \sin(1/x) + b_i x^{2n-i} \cos(1/x) \right)$$

We conjecture that: every term above contains x^m, $m \geq 2$, for $k < n$, if $n \geq 2$.

Proof

$x^{2n-k}, x^{2n-k}, x^{2n-(k+1)}, \dots, x^{2n-(2k-1)}$ appear in terms above.

For $n \geq 2$,

$$3 \leq 2(n-k)+1 = 2n-2k+1 = 2n-(2k-1) \leq 2n-k \leq 2n-1$$

■

Note that for $k \geq n$ the above is no longer true. $x^{2n-2n} = 1$.

$f'(0) = \lim_{h \to 0} \frac{f(0+h)}{h} = 0$, since all terms in f contain at least x^2, times a bounded factor.

Assume $f^{(k)}(0) = 0$, $k < n$. Then $f^{(k+1)}(0) = \lim_{h \to 0} \frac{f^{(k)}(h)}{h} = 0$, because of

Therefore $f'(0), \dots, f^{(n)}(0)$ exist. Also $\lim_{x \to 0} f^{(k)}(x) = 0$ because of the at least x^2 factors.

$f', \dots, f^{(k)}$ are thus all continuous and diff. for $k < n$.

But $\lim_{x \to 0} f^{(n)}(x) \neq 0$, in fact since there are terms in $f^{(n)}$ that are like $\sin(1/x)$ or $\cos(1/x)$,

this limit does not exist: $f^{(n)}$ is not cont. at 0, thus not diff at 0.

Alternative solution

Proposition

$\forall n, n \in \mathbb{N}$ h, g n times cont. diff $\rightarrow$ $f(x) = \begin{cases} 0 & x = 0 \\ x^{2n}(h(x)\sin(1/x) + g(x)\cos(1/x)) & x \neq 0 \end{cases}$ is n-times diff.

$(g(0) \neq 0$ or $h(0) \neq 0) \rightarrow f^{(n)}$ discont. at 0

Proof induction on n

$n = 1$

$$f(x) = \begin{cases} 0 \\ x^2(h(x)\sin(1/x) + g(x)\cos(1/x)) \end{cases}$$

$$f'(0) = \lim_{m \to 0} \frac{m^2(h(m)\sin(1/m) + g(m)\cos(1/m))}{m}$$

$$= \lim_{m \to 0} m(h(m)\sin(1/m) + g(m)\cos(1/m))$$

$$= 0$$

$$f'(x) = 2x(h(x)\sin(1/x) + g(x)\cos(1/x)) + x^2\left(h'(x)\sin(1/x) - \frac{h(x)\cos(1/x)}{x^2} + g'(x)\cos(1/x) + \frac{g(x)\sin(1/x)}{x^2}\right)$$

$$= 2x(h(x)\sin(1/x) + g(x)\cos(1/x)) + x^2 h'(x)\sin(1/x) - h(x)\cos(1/x) + x^2 g'(x)\cos(1/x) + g(x)\sin(1/x)$$

$\lim_{x \to 0} f'(x)$ doesn't exist.

Assume that for some n

h, g n times cont. diff $\rightarrow$ $f(x) = \begin{cases} 0 & x = 0 \\ x^{2n}(h(x)\sin(1/x) + g(x)\cos(1/x)) & x \neq 0 \end{cases}$ is n-times diff.

$(g(0) \neq 0$ or $h(0) \neq 0) \rightarrow f^{(n)}$ discont. at 0

Let $h(x), g(x)$ be n times diff.

Then if $f'(x) = \begin{cases} 0 & x = 0 \\ x^{2n}(h(x)\sin(1/x) + g(x)\cos(1/x)) & x \neq 0 \end{cases}$

By applying the inductive hypothesis

$f'(x)$ is n-times diff, so f is n+1-times diff; $f^{(n+1)}$ is discont. at 0 if $g(0) \neq 0$ or $h(0) \neq 0$.

34.

$\forall n, n \in \mathbb{N}, \ f(x) = \begin{cases} x^{2n+1}\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases} \rightarrow$ f n-times diff at 0, $f^{(n)}$ cont. at 0, $f^{(n)}$ not diff. at 0.

Proof

n=1

$$f(x) = \begin{cases} x^3\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f'(x) = 3x^2\sin(1/x) + x^3\cos(1/x)(-1/x^2) \quad x \neq 0$$

$$\lim_{h\to 0} \frac{h^3\sin(1/h)}{h} = 0 \quad x = 0$$

$\lim_{x\to 0} f'(x) = 0 \rightarrow f^{(n)} = f'$ cont. at 0

$$f''(0) = \lim_{h\to 0} \frac{3h^2\sin(1/h) - h\cos(1/h)}{h}$$

this limit does not exist because $\lim_{h\to 0}\cos(1/h)$ doesn't exist.

$\rightarrow f^{(n)} = f''$ not diff.

n Assume that

$f(x) = \begin{cases} x^{2n+1}\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases} \rightarrow$ f n-times diff at 0, $f^{(n)}$ cont. at 0, $f^{(n)}$ not diff. at 0.

Let $g(x) = \begin{cases} x^{2(n+1)+1}\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

Then $g'(x) = (2n+3)x^{2(n+1)}\sin(1/x) - x^{2(n+1)-1}\cos(1/x)$

$$= \underline{(2n+3)x^{2(n+1)}\sin(1/x)} - \underline{x^{2n+1}\cos(1/x)}$$

apply problem 33
$\rightarrow$ n+1 times diff
$(n+1)^{th}$ deriv. not cont at 0

can't quite apply inductive hypothesis but the $\cos(1/x)$ case is analogous
if we could apply i.h.
n times diff at 0
n^{th} deriv. cont at 0, not diff at 0

See next page for alternative and more general sol'n.

$\forall n, n \in \mathbb{N}$, $f(x) = \begin{cases} h(x)x^{2n+1}\sin(1/x) + g(x)x^{2n+1}\cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ $\rightarrow$ f n-times diff at 0, $f^{(n)}$ cont. at 0, $f^{(n)}$ not diff. at 0.

h, g n times diff

Proof

$n=1$

$$f(x) = \begin{cases} h(x)x^3\sin(1/x) + g(x)x^3\cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$= \begin{cases} x^3(h(x)\sin(1/x) + g(x)\cos(1/x)) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f'(x) = 3x^2(h(x)\sin(1/x) + g(x)\cos(1/x)) + x^3\left[h'(x)\sin(1/x) - \frac{h(x)\cos(1/x)}{x^2} + g'(x)\cos(1/x) + \frac{g(x)\sin(1/x)}{x^2}\right]$$

$$= 3x^2(h(x)\sin(1/x) + g(x)\cos(1/x)) + x^3h'(x)\sin(1/x) - xh(x)\cos(1/x) + x^3g'(x)\cos(1/x) + xg(x)\sin(1/x)$$

$\lim_{x\to 0} f'(x) = 0$

$$f'(0) = \lim_{m\to 0} \frac{h(m)m^3\sin(1/m) + g(m)m^3\cos(1/m)}{m} = 0$$

$\rightarrow f^{(n)}(0) = f'(0)$ exists, f' cont. at 0.

$$f''(0) = \lim_{m\to 0} \frac{\left[3m^2(h(m)\sin(1/m) + g(m)\cos(1/m)) + m^3h'(m)\sin(1/m) - mh(m)\cos(1/m) + m^3g'(m)\cos(1/m) + mg(m)\sin(1/m)\right]}{m}$$

Blue terms go to zero, but red terms depend on $\lim_{m\to 0} h(m)$ and $\lim_{m\to 0} g(m)$. If these are $\neq 0$ then $f''(0)$ does not exist.

n

Assume $f(x) = \begin{cases} h(x)x^{2n+1}\sin(1/x) + g(x)x^{2n+1}\cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ $\rightarrow$ f n-times diff at 0, $f^{(n)}$ cont. at 0, $f^{(n)}$ not diff. at 0.

Let $i(x) = \begin{cases} h(x)x^{2(n+1)+1}\sin(1/x) + g(x)x^{2(n+1)+1}\cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

$$= \begin{cases} x^{2(n+1)+1}(h(x)\sin(1/x) + g(x)\cos(1/x)) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

h, g $(n+1)$-times diff.

$$i'(0) = \lim_{m\to 0} \frac{m^{2(n+1)+1}(h(m)\sin(1/m) + g(m)\cos(1/m))}{m}$$

$$= \lim_{m\to 0}\left(m^{2(n+1)}(h(m)\sin(1/m)+g(m)\cos(1/m))\right) = 0$$

$$i'(x) = \underline{(2n+3)x^{2(n+1)}(h(x)\sin(1/x) + g(x)\cos(1/x))} + x^{2(n+1)+1}\left(h'(x)\sin(1/x) - \frac{h(x)\cos(1/x)}{x^2}\right.$$

$$\left. + g'(x)\cos(1/x) + \frac{g(x)\sin(1/x)}{x^2}\right) \quad x \neq 0$$

apply problem 33

g' is $(n+1)$ times diff

$g^{(n+1)}$ not cont. at 0

Thus

$$i'(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^{2n+1}\sin(1/x)\left((2n+3)h(x) + x^2h'(x) + g(x)\right) \\ \quad + x^{2n+1}\cos(1/x)\left((2n+3)g(x) + x^2g'(x) - h(x)\right), & x \neq 0 \end{cases}$$

$$= \begin{cases} x^{2n+1}\sin(1/x)h_1(x) + x^{2n+1}\cos(1/x)h_2(x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$h_1(x)$ and $h_2(x)$ are n times diff.

Apply inductive hypothesis to i'

$\to$ i' n times diff at 0 $\to$ i $n+1$ times diff at 0

$i^{(n+1)}$ cont at 0

$i^{(n+1)}$ not diff at 0

35.

Chain Rule, Leibnizian notation

$$\frac{d f(g(x))}{dx} = \frac{d f(y)}{dy}\bigg|_{y=g(x)} \cdot \frac{d g(x)}{dx}$$

Alton.

$y = g(x)$

$z = f(y)$

$$\underset{f\circ g}{\frac{dz}{dx}} = \underset{f}{\frac{dz}{dy}} \frac{dy}{dx}$$

ie the two z's in this eq refer to different functions

i) $z = \sin y$, $y = x + x^2$

$z(y) = \sin y$

$y(x) = x + x^2$

$z(y(x)) = \sin(x + x^2)$

$$\frac{dz}{dx} = \cos(y)\Big|_{y=x+x^2} (1+2x)$$

$$= \cos(x+x^2)\cdot(1+2x)$$

ii) $z = \sin y$, $y = \cos x$

$$\frac{dz}{dx} = \cos(y)\Big|_{y=\cos x} (-\sin x)$$

$$= \cos(\cos x)\cdot(-\sin x)$$

iii) $z = \sin u$, $u = \sin x$

$$\frac{dz}{dx} = \cos(u)\Big|_{u=\sin x} \cos x$$

$$= \cos(\sin x)\cdot\cos(x)$$

iv) $z = \sin v$, $v = \cos u$, $u = \sin x$

$z(v) = \sin v$

$v(u) = \cos u \quad \rightarrow \quad w(u) = z(v(u)) = \sin(\cos(u))$

$u(x) = \sin x$

$$\frac{dz}{dx} = \frac{dw}{du}\cdot\frac{du}{dx} = \left(\frac{dz}{dv}\frac{dv}{du}\right)\frac{du}{dx}$$

$$= \left(\cos v\Big|_{v=\cos u}\cdot(-\sin u)\right)\Big|_{u=\sin x}\cdot\cos x = \cos(\cos(\sin x))\cdot(-\sin(\sin x))\cdot\cos x$$