Ch.19 - Integration in Elementary Terms

1. (i) $\int \frac{\sqrt[5]{x^3}+\sqrt[6]{x}}{\sqrt{x}}\,dx = \int [x^{\frac{3}{5}-\frac{1}{2}}+x^{\frac{1}{6}-\frac{1}{2}}]\,dx = \int [x^{1/10}+x^{-\frac{1}{3}}]\,dx = \frac{10}{11}x^{11/10}+\frac{3x^{2/3}}{2}$

(ii) $\int \frac{dx}{\sqrt{x-1}+\sqrt{x+1}} = \int \frac{\sqrt{x-1}-\sqrt{x+1}}{x-1-x-1} = -\frac{1}{2}\left[\int\sqrt{x-1}\,dx - \int\sqrt{x+1}\,dx\right] = -\frac{1}{2}\left[(x-1)^{3/2}\cdot\frac{2}{3} - (x+1)^{3/2}\cdot\frac{2}{3}\right]$

$= \frac{1}{3}[(x+1)^{3/2}-(x-1)^{3/2}]$

(iii) $\int \frac{e^x+e^{2x}+e^{3x}}{e^{4x}}\,dx = \int [e^{-3x}+e^{-2x}+e^{-x}]\,dx = -\frac{e^{-3x}}{3}-\frac{e^{-2x}}{2}-e^{-x}$

(iv) $\int \frac{a^x}{b^x}\,dx = \int \left(\frac{a}{b}\right)^x dx = \int e^{x\log(a/b)}\,dx = \frac{e^{x\log(a/b)}}{\log(a/b)}$

(v) $\int \tan^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int\left[\frac{1}{\cos^2 x}-1\right]dx = \int \sec^2 x\,dx - \int dx$

$= \tan x - x$

(vi) $\int \frac{dx}{a^2+x^2}$ $\int \frac{1}{a^2(1+(x/a)^2)}\,dx = \frac{\arctan(x/a)}{a}$

$\arctan' x = \frac{1}{1+x^2}$

note the substit.: $f(x) = \arctan'(x)$

$g(x) = x/a$

$g'(x) = 1/a$

$= \frac{1}{a}\int \frac{1}{1+u^2}\cdot du = \frac{1}{a}\cdot\arctan(x/a)$

(vii) $\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{1}{a\sqrt{1-(x/a)^2}}\,dx = \arcsin(x/a)$

(viii) $\int \frac{dx}{1+\sin x} = \int \frac{1-\sin x}{1-\sin^2 x}\,dx = \int \frac{1-\sin x}{\cos^2 x}\,dx = \int (\sec^2 x - \tan x\cdot\sec x)\,dx = \tan x - \sec x$

(ix) $\int \frac{8x^2+6x+4}{x+1}\,dx = \int [8x-2+\frac{6}{x+1}]\,dx = 4x^2-2x+6\log(x+1)$

$$\begin{array}{r} 8x-2 \\ x+1\,\overline{)\,8x^2+6x+4} \\ \underline{8x^2+8x} \\ -2x+4 \\ \underline{-2x-2} \\ 6 \end{array}$$

(x) $\int \frac{1}{\sqrt{2x-x^2}}\,dx = \int \frac{dx}{\sqrt{-(x^2-2x+1)+1}} = \int \frac{dx}{\sqrt{1-(x-1)^2}} = \int \arcsin'(x-1)\,dx = \arcsin(x-1)$

2. (i) $\int e^x \sin(e^x)\,dx = -\cos(e^x)$

(ii) $\int x e^{-x^2} dx = \frac{1}{2}\int \frac{2x}{e^{x^2}}\,dx = \frac{1}{2}\int e^{-u}du = \frac{1}{2}(-1)e^{-u} = \frac{-e^{-x^2}}{2}$

(iii) $\int \frac{\log x}{x}\,dx = \int u\,du = \frac{(\log x)^2}{2}$

$f(x) = x$
$g(x) = \log x$
$g'(x) = 1/x$

(iv) $\int \frac{e^x dx}{e^{2x}+2e^x+1} = \int \frac{e^x dx}{(e^x+1)^2}\,dx = -(e^x+1)^{-1}$

(v) $\int e^{e^x} \cdot e^x\,dx = e^{e^x}$

Note the substit.

$f(x) = e^x$ $\quad \int e^{e^x} e^x dx = \int f(g(x))g'(x)dx = \int f(u)du = \int e^u du = e^u = e^{e^x}$
$g(x) = e^x$ $\quad u = g(x) = e^x$
$g'(x) = e^x$ $\quad du = g'(x)dx = e^x dx$

(vi) $\int \frac{x}{\sqrt{1-x^4}}\,dx = \int \frac{x}{\sqrt{1-(x^2)^2}}\,dx = \frac{1}{2}\int \arcsin'(x^2)\,dx = \frac{1}{2}\arcsin(x^2)$

$f(x) = \arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$
$g(x) = x^2$
$g'(x) = 2x$

(vii) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}}$

(viii) $\int x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int \sqrt{1-u}\,du = -\frac{1}{2}(1-u)^{3/2}\cdot\frac{2}{3} = -\frac{1}{3}(1-x^2)^{3/2}$

(ix) $\int \log(\cos x)\cdot\tan x\,dx = -\int \frac{\log(\cos x)}{\cos x}\cdot(-\sin x)\,dx = -\int \frac{\log u}{u}\,du = -\frac{[\log(u)]^2}{2} = \frac{-(\log(\cos x))^2}{2}$

$f(x) = \frac{\log x}{x}$ $\quad g(x) = \cos x$
$g'(x) = -\sin x$

(x) $\int \frac{\log(\log x)}{x\log x}\,dx = \frac{(\log(\log x))^2}{2}$

3. (i) $\int x^2e^x dx = x^2e^x - \int 2xe^x dx$
$= x^2e^x - 2xe^x + 2e^x$
$= e^x(x^2-2x+2)$

$f(x) = x^2$
$g'(x) = e^x$

$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x$
$f(x) = x$
$g'(x) = e^x$

$* \; (fg)' = f'g + fg'$

$\to f'g = (fg)' - fg'$

$\int f'g = fg - \int fg'$

(ii) $\int x^3e^{x^2}dx = \frac{1}{2}\int \underbrace{e^{x^2}\cdot 2x}_{g'} \cdot \underbrace{x^2}_{f} \cdot dx = \frac{1}{2}[x^2e^{x^2} - \int 2xe^{x^2}dx = \frac{1}{2}[x^2e^{x^2} - e^{x^2}] = \frac{e^{x^2}}{2}(x^2-1)$

$g = e^{x^2}$

(iii) $\int e^{ax}\sin(bx)dx = -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\int e^{ax}\cos(bx)$

$f = e^{ax} \quad f' = ae^{ax}$

$g' = \sin(bx) \quad g = -\frac{\cos(bx)}{b}$

$f = e^{ax} \quad f' = ae^{ax}$

$g' = \cos(bx) \quad g = \frac{\sin(bx)}{b}$

$\int e^{ax}\cos(bx) = \frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\int e^{ax}\sin(bx)dx$

Hence

$$\int e^{ax}\sin(bx)dx = -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\left[\frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\int e^{ax}\sin(bx)dx\right]$$

$$= \frac{-be^{ax}\cos(bx) + ae^{ax}\sin(bx)}{b^2} - \frac{a^2}{b^2}\int e^{ax}\sin(bx)dx$$

$$\int e^{ax}\sin(bx)dx\left[1 + \frac{a^2}{b^2}\right] = \frac{-be^{ax}\cos(bx) + ae^{ax}\sin(bx)}{b^2}$$

$$\int e^{ax}\sin(bx)dx = \frac{ae^{ax}\sin(bx) - be^{ax}\cos(bx)}{a^2+b^2}$$

(iv) $\int x^2\sin x = -x^2\cos x + 2\int x\cos x\,dx$

$f = x^2 \quad f' = 2x$

$g' = \sin x \quad g = -\cos x$

$f = x \quad f' = 1$

$g' = \cos x \quad g = \sin x$

$\int x\cos x\,dx = x\sin x - \int \sin x$
$= x\sin x + \cos x$

Hence

$\int x^2\sin x = -x^2\cos x + 2\int x\cos x\,dx = -x^2\cos x + 2x\sin x + 2\cos x$

(v) $\int (\log x)^3 dx = x\log^3 x - 3\int \log^2 x$

$f = (\log x)^3 \quad f' = 3\log^2 x \cdot \frac{1}{x}$ $\qquad f = \log^2 x \quad f' = 2\log x \cdot \frac{1}{x}$

$g' = 1 \quad g = x$ $\qquad g' = 1 \quad g = x$

$\int \log^2 x = x\log^2 x - 2\int \log x$

$= x\log^2 x - 2x\log x + 2x$

Hence

$\int (\log x)^3 dx = x\log^3 x - 3\int \log^2 x$

$= x\log^3 x - 3x\log^2 x + 6x\log x - 6x$

(vi) $\int \frac{\log(\log x)}{x} dx = \log x \cdot \log(\log x) - \int \frac{1}{x} dx = \log(x) \cdot \log(\log x) - \log(x)$

$f = \log(\log x) \quad f' = \frac{1}{x\log x}$

$g' = \frac{1}{x} \quad g = \log x$

(vii) $\int \sec^3(x) dx = \int \sec^2 x \cdot \sec x \, dx = \sec x \cdot \tan x - \int \tan^2 x \cdot \sec x \, dx$

$f = \sec x \quad f' = \left(\frac{1}{\cos}\right)' = \frac{\sin}{\cos^2} = \sec x \cdot \tan x$

$g' = \sec^2 x \quad g = \tan x$

$\int \tan^2 x \cdot \sec x \, dx = \int (\sec^2 x - 1)\sec x \, dx = \int (\sec^3 x - \sec x) dx$

Hence,

$\int \sec^3(x) dx = \sec x \cdot \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$

$2\int \sec^3 x \, dx = \sec x \cdot \tan x + \log(\sec x + \tan x)$

$\int \sec^3 x \, dx = \frac{1}{2}\left[\sec x \cdot \tan x + \log(\sec x + \tan x)\right]$

(viii) $\int \cos(\log x) dx = x\cos(\log x) + \int \sin(\log x) dx = x\cos(\log x) + x\sin(\log x) - \int \cos(\log x) dx$

$f = \cos(\log x) \quad f' = \frac{-\sin(\log x)}{x}$ $\qquad f = \sin(\log x) \quad f' = \frac{\cos(\log x)}{x}$

$g' = 1 \quad g = x$ $\qquad g' = 1 \quad g = x$

Hence

$\int \cos(\log x) = \frac{x}{2}\left(\sin(\log x) + \cos(\log x)\right)$

(ix) $\int \sqrt{x}\log x\,dx = \frac{2}{3}\log(x)\cdot x^{3/2} - \frac{2}{3}\int \sqrt{x}\,dx = \frac{2\log(x)\cdot x^{3/2}}{3} - \frac{4}{9}x^{3/2}$

$f = \log x \quad f' = 1/x$

$g' = \sqrt{x} \quad g = \frac{2}{3}x^{3/2}$

(x) $\int x(\log x)^2\,dx = \frac{(x\log x)^2}{2} - \int x\log x\,dx = \frac{(x\log x)^2}{2} - \frac{x^2\log x}{2} + \frac{x^2}{4}$

$f = (\log x)^2 \quad f' = 2\log(x)\cdot\frac{1}{x}$

$g' = x \quad g = \frac{x^2}{2}$

$f = \log x \quad f' = 1/x$

$g' = x \quad g = x^2/2$

$\int x\log x = \frac{x^2\log x}{2} - \int \frac{x}{2}\,dx$

$= \frac{x^2\log x}{2} - \frac{x^2}{4}$

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4. (i) $\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos u}{\sqrt{1-\sin^2 u}}\,du = \int du = u = \arcsin x$

$x = \sin u$

$dx = \cos u\,du$

Altern., identity

$f(x) = \arcsin'(x)$

$g(x) = x$

$\frac{1}{\sqrt{1-x^2}} = f(g(x))\cdot g'(x)$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \arcsin'(x)\,dx = \arcsin(x)$

(ii) $\int \frac{1}{\sqrt{1+x^2}}\,dx = \int \frac{\sec^2(u)\,du}{\sec(u)} = \int \sec(u)\,du = \log(\tan u + \sec u) = \log(x + \sqrt{1+x^2})$

$x = \tan u$

$dx = \sec^2(u)\,du$

$\sqrt{1+x^2} = \sqrt{1+\tan^2 u} = \sec(u)$

(iii) $\int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\tan(u)\cdot\sec(u)}{\tan(u)}\,du = \int \sec(u) = \log(\tan u + \sec u) = \log(\sqrt{x^2-1} + x)$

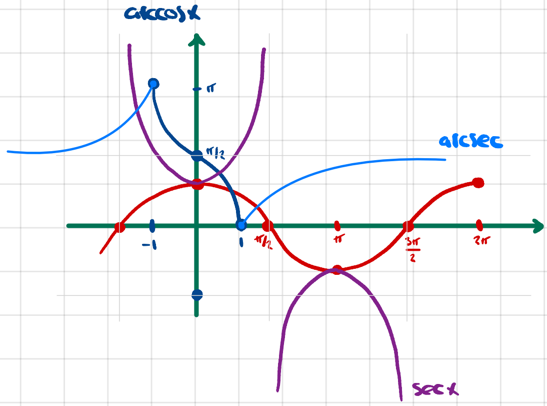
$x = \sec(u)$

$\sqrt{x^2-1} = \sqrt{\sec^2(u)-1} = \sqrt{\tan^2 u} = \tan(u)$

$dx = \tan(u)\sec(u)\,du$

(iv) $\int \frac{dx}{x\sqrt{x^2-1}}$

$f(x) = \sec(x) = \frac{1}{\cos x} \quad x \neq \frac{\pi}{2} + \pi k$



$\operatorname{arcsec}'(x) = \frac{1}{\sec'(\operatorname{arcsec} x)}$

$= \frac{1}{\tan(\operatorname{arcsec} x) \cdot x} \quad x \in (-\infty, -1] \cup [1, \infty)$

$\sec'(x) = \frac{-1}{\cos^2 x} \cdot (-\sin x) = \frac{\sin x}{\cos^2 x} = \tan(x)\sec(x) \quad x \neq \frac{\pi}{2} + \pi k$

$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} \quad x \neq \frac{\pi}{2} + k\pi$

$= \frac{1-\cos^2 x}{\cos^2 x}$

$= \sec^2 x - 1 \quad x \neq \frac{\pi}{2} + k\pi$

$\tan^2 x + 1 = \sec^2 x \quad x \neq \frac{\pi}{2} + k\pi$

$\tan^2(\operatorname{arcsec} x) = x^2 - 1 \quad x \in (-\infty, -1] \cup [1, \infty)$

$\tan(\operatorname{arcsec} x) = \pm\sqrt{x^2-1}$

In $(-\infty, -1]$, $\operatorname{arcsec} x \in (\pi/2, \pi] \to \tan(\operatorname{arcsec} x) = -\sqrt{x^2-1}$
$[1, \infty)$, $\operatorname{arcsec} x \in [0, \pi/2) \to \tan(\operatorname{arcsec} x) = \sqrt{x^2-1}$

$$\to \operatorname{arcsec}'(x) = \begin{cases} \frac{1}{x\sqrt{x^2-1}} & x \geq 1 \\ \frac{1}{-x\sqrt{x^2-1}} & x \leq -1 \end{cases}$$

$\to \operatorname{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2-1}} \quad x \in (-\infty, -1] \cup [1, \infty)$

Back to our original integral,

$\int \frac{dx}{x\sqrt{x^2-1}} = \int \operatorname{arcsec}'(x)\, dx = \operatorname{arcsec}(x)$

if we were to use substit. then

$x = \sec u \qquad dx = \sec(u)\tan(u)\, du$

$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec(u)\tan(u)\, du}{\sec(u)\tan u} = u = \operatorname{arcsec}(x)$

(v) $\int \frac{dx}{x\sqrt{1-x^2}} = \int \frac{\cancel{\cos u}\, du}{\sin u \cancel{\cos u}} = \int (\sin u)^{-1} du = \int \csc u \, du = -\log(\csc u + \cot u) = -\log\left(\frac{1}{x} + \frac{\cos(\arcsin x)}{x}\right)$

$= -\log\left[\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right]$

$x = \sin u \rightarrow u = \arcsin(x)$

$dx = \cos u \, du$

(vi) $\int \frac{dx}{x\sqrt{1+x^2}} = \int \frac{\sec^{\cancel{2}} u \, du}{\tan(u) \cdot \cancel{\sec(u)}} = \int \csc(u) du = -\log(\csc u + \cot u) = -\log\left[\frac{1}{\sin(\arctan x)} + \underbrace{\frac{\cos(\arctan x)}{\sin(\arctan x)}}_{\frac{1}{x}}\right]$

$x = \tan u \quad dx = \sec^2 u \, du$

$u = \arctan x$

$= -\log\left[\frac{1}{\sqrt{x^2+1}} + \frac{1}{x}\right]$

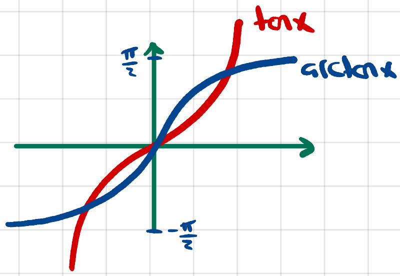
$\tan(\arctan x) = x = \frac{\sin(\arctan x)}{\cos(\arctan x)}$

$\frac{1}{\sin(\arctan x)} = \frac{1}{x\cos(\arctan x)}$

$\tan^2 x + 1 = \frac{1}{\cos^2 x} \qquad x \neq \frac{\pi}{2} + k\pi$

$x^2 + 1 = \frac{1}{\cos^2(\arctan x)}$

$\cos(\arctan x) = \frac{1}{\sqrt{x^2+1}}$ because $\arctan \in (-\frac{\pi}{2}, \frac{\pi}{2})$



(vii) $\int x^3\sqrt{1-x^2}\, dx = \int \sin^3(u) \cdot \cos(u) \cdot \cos(u) du = \int \sin(u) \cdot \sin^2(u) \cdot \cos^2(u) du$

$x = \sin u \quad dx = \cos u \, du \rightarrow u = \arcsin x$

$= \int \sin(u)[\cos^2 u - \cos^4 u] du$

$\sin^2(x) = \frac{1-\cos(2x)}{2}$

$= \int \cos^2(u)\sin(u) du - \int \cos^4(u)\sin(u) du$

$= \frac{\cos^3(u)}{3} - \frac{\cos^5(u)}{5}$

$x^2 + \cos^2(\arcsin x) = 1$

$= \frac{\cos^3(\arcsin x)}{3} - \frac{\cos^5(\arcsin x)}{5}$

$\cos(\arcsin x) = \sqrt{1-x^2}$

$= \frac{(1-x^2)^{3/2}}{3} - \frac{(1-x^2)^{5/2}}{5}$

(viii) $\int \sqrt{1-x^2}\,dx = \int \cos(u)\cdot\cos(u)\,du = \int \frac{1+\cos(2u)}{2}\,du = \frac{1}{2}u + \frac{\sin(2u)}{4} = \frac{1}{2}\arcsin x + \frac{\sin(2\arcsin x)}{4}$

$x = \sin u \quad dx = \cos(u)\,du$

$= \frac{1}{2}\arcsin x + \frac{x\cos(\arcsin x)}{2}$

$u = \arcsin x$

$= \frac{1}{2}\arcsin x + \frac{x\sqrt{1-x^2}}{2}$

(ix) $\int \sqrt{1+x^2}\,dx = \int \sec(u)\cdot\sec^2(u)\,du = \int \sec^3(u)\,du = \frac{1}{2}\left[\sec x\cdot\tan x + \log(\sec x + \tan x)\right]$

$x = \tan u \quad dx = \sec^2 u\,du$

(x) $\int \sqrt{x^2-1}\,dx = \int \tan^2(u)\sec(u)\,du = \int \frac{\sin^2 u}{\cos^3 u}\,du = \int \cos^{-3} u\,du - \int \cos^{-1} u\,du = \int \sec^3 u\,du - \int \sec u\,du$

$x = \sec u$

$= \frac{1}{2}\left[\sec(u)\cdot\tan(u) + \log(\sec(u)+\tan(u))\right] - \log[\sec(u)+\tan(u)]$

$dx = \sec(u)\tan(u)\,du$

$= \frac{1}{2}\left[x\cdot\tan(\operatorname{arcsec} x) + \log(x+\tan(\operatorname{arcsec} x))\right] - \log(x+\tan(\operatorname{arcsec} x))$

$u = \operatorname{arcsec}(x)$

$= \frac{1}{2}\left[|x|\sqrt{x^2-1} + \log(\right.$

$\tan(\operatorname{arcsec} x) = \pm\sqrt{x^2-1}$

In $(-\infty,-1]$, $\operatorname{arcsec} x \in (\pi/2, \pi] \rightarrow \tan(\operatorname{arcsec} x) = -\sqrt{x^2-1}$

$[1,\infty)$, $\operatorname{arcsec} x \in [0, \pi/2) \rightarrow \tan(\operatorname{arcsec} x) = \sqrt{x^2-1}$

note that $x^2-1 \geq 0 \rightarrow x^2 \geq 1 \rightarrow x \geq 1$ or $x \leq -1$

$x \leq -1 \rightarrow \operatorname{arcsec} x \in (\pi/2, \pi] \rightarrow \tan(\operatorname{arcsec} x) = -\sqrt{x^2-1} \in (-\infty, 0]$

5. (i) $\int \frac{dx}{1+\sqrt{x+1}} = \int \frac{2u}{1+u}du = \int \frac{2u+2-2}{1+u}du = \int \frac{2(1+u)-2}{1+u}du = \int 2du - \int \frac{2}{1+u}du = 2u - 2\log(1+u)$

$u = \sqrt{x+1}$

$= 2\sqrt{x+1} - 2\log(1+\sqrt{x+1})$

$du = \frac{1}{2\sqrt{x+1}}dx$

$dx = 2\sqrt{x+1}\,du = 2u\,du$

(ii) $\int \frac{dx}{1+e^x} = \int \frac{1}{1+u}\cdot\frac{1}{u}du = \int\left[\frac{A}{u} + \frac{B}{1+u}\right]du = \int\left[\frac{1}{u} - \frac{1}{1+u}\right]du = \log(u) - \log(1+u)$

$u = e^x \quad x = \log u$

$= \log(e^x) - \log(1+e^x)$

$dx = \frac{1}{u}du$

$A(1+u) + Bu = 1$

$A + u(A+B) = 1$

$A = 1$

$B = -1$

$= x - \log(1+e^x)$

Alternatively

$u = g(x) = e^x \quad \frac{du}{dx} = g'(x) = e^x$

$\underbrace{\int \frac{1}{1+e^x}\cdot\frac{1}{e^x}}_{F'(g(x))\cdot g'(x)}\cdot e^x\,dx = \underbrace{\int \frac{1}{1+u}\cdot\frac{1}{u}}_{F'(u)}du$

(iii) $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}dx = \int \frac{6u^5 du}{u^3+u^2} = 6\int \frac{u^3}{1+u}du = 6\int\left[u^2 - u + 1 - \frac{1}{1+u}\right]du$

$u = x^{1/6} \quad x = u^6$

$dx = 6u^5 du$

$$\begin{array}{r} u^2 - u + 1 \\ 1+u \,\overline{)\, u^3 \qquad\quad} \\ \underline{u^3 + u^2} \\ -u^2 \\ \underline{-u^2 - u} \\ u \\ \underline{u+1} \\ -1 \end{array}$$

$= 6\left[\frac{u^3}{3} - \frac{u^2}{2} + u - \log(1+u)\right]$

$= 6\left[\frac{\sqrt{x}}{3} - \frac{\sqrt[3]{x}}{2} + \sqrt[6]{x} - \log(1+\sqrt[6]{x})\right]$

$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\log(1+\sqrt[6]{x})$

(iv) $\int \frac{1}{\sqrt{1+e^x}}dx = \int \frac{1}{u}\cdot\frac{2u}{u^2-1}du = 2\int\frac{1}{u^2-1}du$

$u = \sqrt{1+e^x} \quad x = \log(u^2-1)$

$= 2\int\frac{1}{(u+1)(u-1)}du = 2\int\left[\frac{A}{u+1} + \frac{B}{u-1}\right]du = \int\left[\frac{1}{u-1} - \frac{1}{u+1}\right]du$

$dx = \frac{2u}{u^2-1}du$

$A(u-1) + B(u+1) = 1$

$Au - A + Bu + B = 1$

$u(A+B) + B - A = 1$

$B - A = 1 \to 2B = 1 \to B = 1/2$

$A + B = 0 \to A = -B \to A = -1/2$

$= \log(u-1) - \log(u+1)$

$= \log(\sqrt{1+e^x} - 1) - \log(\sqrt{1+e^x} + 1)$

(v) $\int \frac{dx}{2+\tan x} = \int \frac{1}{2+u}\cdot\frac{1}{1+u^2}\,du = \int\left[\frac{A}{2+u}+\frac{Bu+C}{1+u^2}\right]du = \int\left[\frac{1}{5}\frac{1}{2+u}-\frac{1}{5}\frac{(u-2)}{1+u^2}\right]du = \frac{1}{5}\int\left[\frac{1}{2+u}-\frac{u-2}{1+u^2}\right]du$

$= \frac{1}{5}\int\left[\frac{1}{2+u}-\frac{u}{1+u^2}+\frac{2}{1+u^2}\right]du$

$x = \arctan u$

$u = \tan x$

$dx = \frac{1}{1+u^2}\,du$

$A(1+u^2)+(2+u)(Bu+C)=1$

$A+Au^2+2Bu+2C+Bu^2+uC=1$

$u^2(A+B)+u(2B+C)+(A+2C)=1$

$A+2C=1 \to -B+2(-2B)=-5B=1 \to B=-\frac{1}{5}$

$A+B=0 \to A=-B \to A=\frac{1}{5}$

$2B+C=0 \to C=-2B \to C=\frac{2}{5}$

Hence,

$$\int\frac{dx}{2+\tan x} = \frac{1}{5}\left[\int\frac{1}{2+u}\,du-\frac{1}{2}\int\frac{2u}{1+u^2}\,du+2\int\frac{1}{1+u^2}\,du\right]$$

$$=\frac{1}{5}\left[\log(2+u)-\frac{1}{2}\log(1+u^2)+2\arctan(u)\right]$$

$$=\frac{1}{5}\log(2+\tan x)-\frac{1}{10}\log(\sec^2 x)+\frac{2}{5}x$$

(vi) $\int\frac{dx}{\sqrt{\sqrt{x}+1}} = \int\frac{2(u-1)}{\sqrt{u}}\,du = \int\left[2u^{1/2}-2u^{-1/2}\right]du = 2\cdot\frac{2}{3}u^{3/2}-2\cdot 2u^{1/2} = \frac{4}{3}(\sqrt{x}+1)^{3/2}-4(\sqrt{x}+1)^{1/2}$

$u=\sqrt{x}+1 \qquad \sqrt{x}=u-1$

$x=(u-1)^2$

$dx=2(u-1)\,du$

(vii) $\int\frac{4^x+1}{2^x+1}\,dx = \int\underbrace{\frac{4^x+1}{2^x+1}\cdot\frac{1}{2^x\cdot\log 2}}_{F'(g(x))}\cdot\underbrace{2^x\log 2}_{g'(x)}\,dx = \frac{1}{\log 2}\int\frac{u^2+1}{u+1}\frac{1}{u}\,du = \frac{1}{\log 2}\int\frac{u^2+1}{u(u+1)} = \frac{1}{\log 2}\int\left[1+\frac{1-u}{u^2+u}\right] = \frac{1}{\log 2}\int\left[1+\frac{1}{u}-\frac{2}{u+1}\right]du$

$u=2^x=g(x) \qquad g'(x)=\log 2\cdot 2^x=\frac{du}{dx}$

$$\begin{array}{r|l} & 1 \\ u^2+u & u^2+1 \\ & u^2+u \\ \hline & 1-u \end{array}$$

$\frac{u^2+1}{u^2+u}=1+\frac{1-u}{u^2+u}$

$\frac{1-u}{u^2+u}=\frac{1-u}{u(u+1)}=\frac{A}{u}+\frac{B}{u+1}=\frac{1}{u}-\frac{2}{u+1}$

$A(u+1)+Bu=1-u$

$Au+A+Bu=1-u$

$A+u(A+B)=1-u$

$A=1$

$A+B=-1 \to B=-2$

Hence

$$\int\frac{4^x+1}{2^x+1}\,dx = \frac{1}{\log 2}\int\left[1+\frac{1}{u}-\frac{2}{u+1}\right]du$$

$$=\frac{1}{\log 2}\left[u+\log(u)-2\log(u+1)\right]$$

$$=\frac{1}{\log 2}\left[2^x+\log 2^x-2\log(2^x+1)\right]$$

(viii) $\int e^{\sqrt{x}}\,dx = \int e^{\sqrt{x}} \cdot 2\sqrt{x} \cdot \frac{1}{2\sqrt{x}}\,dx = 2\int e^{u} \cdot u\,du = 2[ue^{u} - \int e^{u}du] = 2[ue^{u} - e^{u}] = 2e^{u}[u-1] = 2e^{\sqrt{x}}[\sqrt{x}-1]$

$f = u \quad f' = 1$

$g' = e^{u} \quad g = e^{u}$

$g(x) = \sqrt{x} \qquad g'(x) = \frac{1}{2\sqrt{x}}$

(ix) $\int \frac{\sqrt{1-x}}{1-\sqrt{x}}\,dx = \int \frac{\sqrt{1-x}}{1-\sqrt{x}} \cdot 2\sqrt{x} \cdot \frac{1}{2\sqrt{x}}\,dx = \int \frac{\sqrt{1-u^2}}{1-u} \cdot 2u\,du = 2\int \frac{\cos\theta \sin\theta}{1-\sin\theta} \cos\theta\,d\theta$

$g(x) = \sqrt{x} \qquad g'(x) = \frac{1}{2\sqrt{x}}$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$x = g^2(x)$

$= 2\int \frac{(1-\sin^2\theta)\sin\theta}{1-\sin\theta}\,d\theta$

$= 2\int (1+\sin\theta)\sin\theta\,d\theta$

$= 2\int \sin\theta\,d\theta + 2\int \sin^2\theta\,d\theta$

$= 2(-\cos\theta) + 2\int \frac{1-\cos 2\theta}{2}\,d\theta$

$= -2\cos\theta + \theta - \frac{\sin 2\theta}{2}$

$= -2\cos\theta + \theta - \sin\theta\cos\theta$

$= -2\cos(\arcsin u) + \arcsin u - u\cos(\arcsin u)$

$= -2\sqrt{1-u^2} + \arcsin(u) - u\sqrt{1-u^2}$

$= -2\sqrt{1-x} + \arcsin(\sqrt{x}) - \sqrt{x}\sqrt{1-x}$

(x) $\int \sqrt{\frac{x-1}{x+1}} \cdot \frac{1}{x^2}\,dx = \int \sqrt{\frac{\frac{1}{g(x)}-1}{\frac{1}{g(x)}+1}} \cdot (-g'(x))\,dx = -\int \sqrt{\frac{\frac{1}{u}-1}{\frac{1}{u}+1}}\,du = -\int \sqrt{\frac{1-u}{1+u}}\,du = -\int \sqrt{\frac{(1-u)^2}{1-u^2}}\,du = -\int \frac{1-u}{\sqrt{1-u^2}}\,du$

$g(x) = \frac{1}{x} \rightarrow x = \frac{1}{g(x)}$

$g'(x) = -\frac{1}{x^2}$

$u = \sin\theta$

$\theta = \arcsin(u)$

$du = \cos\theta\,d\theta$

Hence

$\int \sqrt{\frac{x-1}{x+1}} \cdot \frac{1}{x^2}\,dx = -\int \frac{1-\sin\theta}{\cos\theta}\cos\theta\,d\theta = -\int (1-\sin\theta)\,d\theta = -\theta - \cos\theta = -\arcsin(u) - \cos(\arcsin u)$

$= -\arcsin(u) - \sqrt{1-u^2}$

$= -\arcsin(1/x) - \sqrt{1-\frac{1}{x^2}}$

6. (i) $\int \frac{2x^2+7x-1}{x^3+x^2-x-1}dx = \int\left[\frac{A}{x-1}+\frac{B}{x^2+2x+1}\right]dx = \int\left[\frac{2}{x-1}+\frac{3}{(x+1)^2}\right]dx = 2\log(x-1) - 3(x+1)^{-1}$

$1^3+1^2-1-1=0$

$$\begin{array}{r|l} & x^2+2x+1 \\ \hline x-1 & x^3+x^2-x-1 \\ & \underline{x^3-x^2} \\ & 2x^2-x \\ & \underline{2x^2-2x} \\ & x-1 \\ & \underline{x-1} \\ & 0 \end{array}$$

$Ax^2+A\cdot 2x+A+Bx-B = 2x^2+7x-1$

$Ax^2+x(2A+B)+(A-B) = 2x^2+7x-1$

$A=2$

$2A+B=7 \rightarrow B=3$

$A-B=-1$

$x^3+x^2-x+1 = (x-1)(x^2+2x+1)$

$\Delta = 4-4=0$

$x=\frac{-2}{2}=-1$

(ii) $\int \frac{2x+1}{x^3-3x^2+3x-1}dx = \int\left[\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}\right]dx = \int\left[\frac{2}{(x-1)^2}+\frac{3}{(x-1)^3}\right]dx$

$= -2(x-1)^{-1} - \frac{3}{2}(x-1)^{-2}$

$$\begin{array}{r|l} & x^2-2x+1 \\ \hline x-1 & x^3-3x^2+3x-1 \\ & \underline{x^3-x^2} \\ & -2x^2+3x \\ & \underline{-2x^2+2x} \\ & x-1 \\ & \underline{x-1} \\ & 0 \end{array}$$

$A(x-1)^2+B(x-1)+C = 2x+1$

$Ax^2-2xA+A+Bx-B+C = 2x+1$

$Ax^2+x(B-2A)+(A-B+C) = 2x+1$

$A=0 \quad B=2 \quad -2+C=1 \rightarrow C=3$

$(x-1)(x^2-2x+1) = (x-1)(x-1)^2 = (x-1)^3$

$\Delta = 4-4=0$

$x=\frac{2}{2}=1$

(iii) $\int \frac{x^3+7x^2-5x+5}{(x-1)^2(x+1)^3}dx = \int\left[\frac{A}{(x-1)^2}+\frac{B}{(x+1)^3}\right]dx = \int\left[\frac{1}{(x-1)^2}+\frac{4}{(x+1)^3}\right]dx = -(x-1)^{-1}-2(x+1)^{-2}$

$A(x+1)^3+B(x-1)^2 = x^3+7x^2-5x+5$

$Ax^3+(3A+B)x^2+(3A-2B)x+A+B = x^3+7x^2-5x+5$

$\rightarrow A=1,\ B=4$

(iv) $\int \frac{2x^2+x+1}{(x+3)(x-1)^2}dx = \int\left[\frac{A}{x+3}+\frac{B}{x-1}+\frac{C}{(x-1)^2}\right]dx = \int\left[\frac{1}{x+3}+\frac{1}{x-1}+\frac{1}{(x-1)^2}\right]dx$

$A(x-1)^2+B(x+3)(x-1)+C(x+3)$

$= \log(x+3)+\log(x-1)-(x-1)^{-1}$

$\rightarrow A=B=C=1$

(v) $\int \frac{x+4}{x^2+1}dx = \frac{1}{2}\int \frac{2x}{x^2+1}dx + 4\int \frac{1}{1+x^2}dx = \frac{1}{2}\log(x^2+1) + 4\arctan(x)$

(vi) $\int \frac{x^3+x+2}{x^4+2x^2+1}dx = \int \left[\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}\right]dx = \int \left[\frac{x}{x^2+1} + \frac{2}{(x^2+1)^2}\right]dx = \frac{1}{2}\log(x^2+1) + 2\int \frac{1}{(x^2+1)^2}dx$

$(x^2)^2 + 2x^2\cdot 1 + 1$
$= (x^2+1)^2$

$\rightarrow A=1$
$B=0$
$C=0$
$D=2$

$= \frac{1}{2}\log(x^2+1) + 2\left[\frac{\arctan x}{2} + \frac{x}{2(1+x^2)}\right]$

$= \frac{1}{2}\log(x^2+1) + \arctan x + \frac{x}{1+x^2}$

$\int \frac{1}{(1+x^2)^2}dx = \int \frac{1}{(1+\tan^2(u))^2}\cdot \sec^2(u)du = \int \frac{\sec^2(u)}{\sec^4(u)}du$

$x = \tan(u)$
$dx = \sec^2 u\, du$

$= \int \frac{1}{\sec^2(u)}du$

$= \int \cos^2(u)du$

$= \int \frac{1+\cos(2u)}{2}du$

$= \frac{1}{2}u + \frac{1}{2}\cdot\frac{\sin(2u)}{2}$

$= \frac{1}{2}u + \frac{\sin(u)\cos(u)}{2}$

$= \frac{\arctan x}{2} + \frac{\sin(\arctan x)\cos(\arctan x)}{2}$

$= \frac{\arctan x}{2} + \frac{x}{2(1+x^2)}$

$\tan^2 x = \frac{\sin^2 x}{\cos^2 x}$

$x^2 = \frac{\sin^2(\arctan x)}{\cos^2(\arctan x)}$

$\sin^2(\arctan x) + \cos^2(\arctan x) = 1$

$x^2 = \frac{1-\cos^2(\arctan x)}{\cos^2(\arctan x)} = \frac{1}{\cos^2(\arctan x)} - 1$

$1+x^2 = \frac{1}{\cos^2(\arctan x)}$

$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$

$\sin^2(\arctan x) = x^2\cos^2(\arctan x) = \frac{x^2}{1+x^2}$

$\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$

(vii) $$\int \frac{3x^2+3x+1}{x^3+2x^2+2x+1}\,dx = \int \frac{3x^2+3x+1}{(x+1)(x^2+x+1)}\,dx = \int\left[\frac{A}{x+1}+\frac{Bx+C}{x^2+x+1}\right]dx = \int\left[\frac{1}{x+1}+\frac{2x}{x^2+x+1}\right]dx$$

$$\begin{array}{r|l} & x^2 \quad\quad + x + 1 \\ \hline x+1 & x^3+2x^2+2x+1 \\ & x^3+x^2 \\ \hline & \quad\; x^2+2x \\ & \quad\; x^2+x \\ \hline & \quad\quad\; x+1 \\ & \quad\quad\; x+1 \\ \hline & \quad\quad\quad 0 \end{array}$$

$$Ax^2+Ax+A+Bx^2+Bx+Cx+C$$

$$= x^2(A+B)+x(A+B+C)+(A+C) = 3x^2+3x+1$$

$$A+B=3 \rightarrow B=3-A \rightarrow B=2$$
$$A+B+C=3 \rightarrow \cancel{A}+\cancel{3}-\cancel{A}+1-A=\cancel{3} \rightarrow A=1$$
$$A+C=1 \rightarrow C=1-A \rightarrow C=0$$

Hence,

$$\int \frac{3x^2+3x+1}{x^3+2x^2+2x+1}\,dx = \log(x+1) + \int \frac{2x}{x^2+x+1}\,dx = \log(x+1) + \log(x^2+x+1) - \frac{2\sqrt{3}}{3}\cdot\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)$$

$$\int \frac{2x}{x^2+x+1}\,dx = \int \frac{2x+1-1}{x^2+x+1}\,dx = \int \frac{2x+1}{x^2+x+1}\,dx - \int \frac{1}{x^2+x+1}\,dx$$

$$= \log(x^2+x+1) - \frac{2\sqrt{3}}{3}\cdot\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)$$

$$\int \frac{1}{x^2+x+1} = \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx = \frac{4}{3}\int \frac{1}{1+\frac{4}{3}(x+\frac{1}{2})^2} = \frac{4}{3}\int \frac{1}{1+\left[\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right]^2}\,dx = \frac{4}{3}\,\frac{\sqrt{3}}{2}\int \frac{2/\sqrt{3}}{1+\left[\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right]^2}\,dx$$

$$= \frac{2\sqrt{3}}{3}\cdot\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)$$

(viii) $\int \frac{dx}{x^4+1} = \int \frac{1}{x^4+2x^2+1-2x^2}dx = \int \frac{1}{(x^2+1)^2-2x^2}dx = \int \frac{1}{(x^2+1+\sqrt{2}x)(x^2+1-\sqrt{2}x)}dx$

alg. manip. to get difference of squares, then product of quadratics

$= \int\left[\frac{Ax+B}{x^2+1+\sqrt{2}x} + \frac{Cx+D}{x^2+1-\sqrt{2}x}\right]dx = \int\left[\frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+1+\sqrt{2}x} + \frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+1-\sqrt{2}x}\right]dx = \frac{1}{8}\int\left[\frac{2\sqrt{2}x+4}{x^2+1+\sqrt{2}x} + \frac{-2\sqrt{2}x+4}{x^2+1-\sqrt{2}x}\right]dx$

partial fractions

$(Ax+B)(x^2+1-\sqrt{2}x) + (Cx+D)(x^2+1+\sqrt{2}x) = 1$

$= \frac{\sqrt{2}}{8}\int\left[\frac{2x+2\sqrt{2}}{x^2+1+\sqrt{2}x} + \frac{-2x+2\sqrt{2}}{x^2+1-\sqrt{2}x}\right]dx$

simplify

$Ax^3 + Ax - A\sqrt{2}x^2 + B + Cx^3 + Bx^2 + D - B\sqrt{2}x + Cx + C\sqrt{2}x^2 + Dx^2 + D\sqrt{2}x = x^3(A+C) + x^2(-\sqrt{2}A+B+\sqrt{2}C+D) + x(A-\sqrt{2}B+C+D\sqrt{2}) + B+D = 1$

$\left.\begin{aligned} A &= \sqrt{2}/4 \\ B &= 1/2 \\ C &= -\sqrt{2}/4 \\ D &= 1/2 \end{aligned}\right.$

Hence

$= \frac{\sqrt{2}}{8}\int \frac{2x+\sqrt{2}}{x^2+1+\sqrt{2}x}dx + \frac{\sqrt{2}}{8}\int\frac{\sqrt{2}}{x^2+1+\sqrt{2}x}dx + \frac{\sqrt{2}}{8}\int\frac{-2x+\sqrt{2}}{x^2+1-\sqrt{2}x}dx + \frac{\sqrt{2}}{8}\int\frac{\sqrt{2}}{x^2+1-\sqrt{2}x}dx$

simplify to get $g'(x)$ for first two terms

$= \frac{\sqrt{2}}{8}\int \frac{2x+\sqrt{2}}{x^2+1+\sqrt{2}x}dx + \frac{1}{4}\int\frac{1}{x^2+1+\sqrt{2}x}dx - \frac{\sqrt{2}}{8}\int\frac{2x-\sqrt{2}}{x^2+1-\sqrt{2}x}dx + \frac{1}{4}\int\frac{1}{x^2+1-\sqrt{2}x}dx$

$= \frac{\sqrt{2}}{8}\cdot\log(x^2+1+\sqrt{2}x) - \frac{\sqrt{2}}{8}\log(x^2+1-\sqrt{2}x) + \frac{1}{2}\int\frac{1}{(1+\sqrt{2}x)^2}dx + \frac{1}{2}\int\frac{1}{(1-\sqrt{2}x)^2}dx$

alg. manip. to get denominator as square

$= \frac{\sqrt{2}}{8}\cdot\log(x^2+1+\sqrt{2}x) - \frac{\sqrt{2}}{8}\log(x^2+1-\sqrt{2}x) + \frac{1}{2\sqrt{2}}\int\frac{\sqrt{2}}{(1+\sqrt{2}x)^2+1}dx + \frac{1}{2\sqrt{2}}\int\frac{\sqrt{2}}{(1-\sqrt{2}x)^2+1}dx$

then as arctan

$= \frac{\sqrt{2}}{8}\cdot\log(x^2+1+\sqrt{2}x) - \frac{\sqrt{2}}{8}\log(x^2+1-\sqrt{2}x) + \frac{\sqrt{2}}{4}\arctan(1+\sqrt{2}x) + \frac{\sqrt{2}}{4}\arctan(1-\sqrt{2}x)$

$\frac{1}{4}\int\frac{1}{x^2+1+\sqrt{2}x}dx = \frac{1}{2}\int\frac{1}{2x^2+2\sqrt{2}x+2}dx = \frac{1}{2}\int\frac{1}{(1+\sqrt{2}x)^2+1}dx$

$\frac{1}{4}\int\frac{1}{x^2+1-\sqrt{2}x}dx = \frac{1}{2}\int\frac{1}{2x^2-2\sqrt{2}x+2}dx = \frac{1}{2}\int\frac{1}{(1-\sqrt{2}x)^2+1}dx$

(ix) $\displaystyle\int \frac{2x}{(x^2+x+1)^2}\,dx = \int \frac{2x+1-1}{(x^2+x+1)^2}\,dx = \underbrace{\int \frac{2x+1}{(x^2+x+1)^2}\,dx}_{-(x^2+x+1)^{-1}} - \int \frac{1}{(x^2+x+1)^2}\,dx$

$$\int \frac{1}{(x^2+x+1)^2}\,dx = \frac{16}{9}\int \frac{1}{\left[1+\left(\frac{2}{\sqrt3}\left(x+\frac12\right)\right)^2\right]^2} = \frac{16}{9}\cdot\frac{\sqrt3}{2}\int\frac{du}{(1+u^2)^2} = \frac{8}{9}\sqrt3\left[\frac{\arctan(u)}{2} + \frac{u}{2(u^2+1)}\right]$$

$\left(x^2+2x\cdot\frac12+\frac14+\frac34\right)^2$

$= \left[\left(x+\frac12\right)^2+\frac34\right]^2$

$= \left[\frac34\left(\frac43\left(x+\frac12\right)^2+1\right)\right]^2$

$= \frac{9}{16}\left[1+\left(\frac{2}{\sqrt3}\left(x+\frac12\right)\right)^2\right]^2$

$u = \frac{2}{\sqrt3}\left(x+\frac12\right)$

$du = \frac{2}{\sqrt3}\,dx$

$dx = \frac{\sqrt3}{2}\,du$

Hence

$$\int \frac{2x}{(x^2+x+1)^2}\,dx = \frac{4\sqrt3}{9}\left[\arctan\left(\frac{2}{\sqrt3}\left(x+\frac12\right)\right) + \frac{\frac{2}{\sqrt3}\left(x+\frac12\right)}{2\left(\frac43\left(x+\frac12\right)^2+1\right)}\right]$$

$$\int\frac{du}{(1+u^2)^2} = \int\left[\frac{1}{1+u^2} - \frac{u^2}{(1+u^2)^2}\right]du = \arctan(u) - \frac{\arctan(u)}{2} + \frac{u}{2(u^2+1)} = \frac{\arctan(u)}{2} + \frac{u}{2(u^2+1)}$$

$$\int\frac{u^2}{(1+u^2)^2}\,du = \int\frac{\tan^2\theta}{\sec^{\cancel{2}4}\theta}\cancel{\sec^2\theta}\,d\theta = \int\frac{\sin^2\theta}{\cancel{\cos^2\theta}}\cdot\cancel{\cos^2\theta}\,\cos^2\theta\,d\theta = \int\sin^2\theta\,d\theta = \frac{\theta}{2} - \frac{\sin\theta\cos\theta}{2}$$

$u = \tan\theta \qquad du = \sec^2\theta\,d\theta$

$\theta = \arctan u$

$$= \frac{\arctan(u)}{2} - \frac{\frac{u}{\sqrt{u^2+1}}\cdot\frac{1}{\sqrt{u^2+1}}}{2}$$

$$= \frac{\arctan(u)}{2} - \frac{u}{2(u^2+1)}$$

$$\int\cos^2x\,dx = \frac{x}{2} + \frac{\sin(x)\cos(x)}{2}$$

$$\int\cos^4x\,dx = \frac{3x}{8} + \frac{3\sin x\cos x}{8} + \frac{\sin x\cos^3x}{4}$$

$$(x)\ \int \frac{3x}{(x^2+x+1)^3}\,dx = \int \frac{\frac{3}{2}(2x+1)-\frac{3}{2}}{(x^2+x+1)^3}\,dx = \frac{3}{2}\int \frac{2x+1}{(x^2+x+1)^3}\,dx - \frac{3}{2}\int \frac{1}{(x^2+x+1)^3}\,dx$$

$$= \frac{3}{2}\,\frac{(x^2+x+1)^{-2}}{-2} - \frac{3}{2}\left[\frac{3}{8}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{3\left(\frac{2x+1}{\sqrt{3}}\right)^3 + 5\,\frac{2x+1}{\sqrt{3}}}{8\left[\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right]^2}\right]$$

$$\int \frac{1}{(x^2+x+1)^3}\,dx = \left[\frac{4}{3}\right]^3 \int \frac{1}{\left[1+\left(\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right)^2\right]^3}\,dx = \left[\frac{4}{3}\right]^3 \int \frac{\sqrt{3}/2}{(1+u^2)^3}\,du$$

$$u = \frac{2x}{\sqrt{3}} + \frac{1}{\sqrt{3}}$$

$$du = \frac{2}{\sqrt{3}}\,dx$$

$$= \frac{32}{9\sqrt{3}}\left[\frac{\arctan(u)}{2} + \frac{u}{2(u^2+1)} - \frac{\arctan(u)}{8} - \frac{u^3-u}{8(u^2+1)^2}\right]$$

$$= \frac{32}{9\sqrt{3}}\left[\frac{3\arctan(u)}{8} + \frac{u}{2(u^2+1)} - \frac{u^3-u}{8(u^2+1)^2}\right]$$

$$= \frac{32}{9\sqrt{3}}\left[\frac{3\arctan(u)}{8} + \frac{3u^3+5u}{8(u^2+1)^2}\right]$$

$$\left[x^2+2\cdot\frac{1}{2}\cdot x+\frac{1}{4}+\frac{3}{4}\right]^3$$

$$= \left[(x+\frac{1}{2})^2+\frac{3}{4}\right]^3$$

$$= \left[\frac{3}{4}\left(1+\frac{4}{3}(x+\frac{1}{2})^2\right)\right]^3$$

$$= \left(\frac{3}{4}\right)^3\left[1+\frac{4}{3}(x+\frac{1}{2})^2\right]^3$$

$$= \left(\frac{3}{4}\right)^3\left[1+\left(\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right)^2\right]^3$$

$$\int \frac{1}{(1+u^2)^3}\,du = \int\left[\frac{1}{(1+u^2)^2} - \frac{u^2}{(1+u^2)^3}\right]du = \frac{\arctan(u)}{2} + \frac{u}{2(u^2+1)} - \frac{\arctan(u)}{8} - \frac{u^3-u}{8(u^2+1)^2}$$

$$\int \frac{u^2}{(1+u^2)^3}\,du = \int \frac{\tan^2\theta}{\sec^{\cancel{6}\,4}\theta}\cdot\cancel{\sec^2\theta}\,d\theta = \int \frac{\sin^2\theta}{\cancel{\cos^2\theta}}\cdot\cos^{\cancel{4}\,2}\theta\,d\theta = \int \sin^2\theta\cos^2\theta\,d\theta = \frac{\theta}{8} + \frac{\sin\theta\cos\theta}{8} - \frac{\sin\theta\cos^3\theta}{4}$$

$u = \tan\theta$

$du = \sec^2\theta\,d\theta$

$\theta = \arctan(u)$

$$= \frac{\arctan(u)}{8} + \frac{\sin(\arctan(u))\cos(\arctan(u))}{8} - \frac{\sin(\arctan(u))\cos^3(\arctan(u))}{4}$$

$$= \frac{\arctan(u)}{8} + \frac{\frac{u}{\sqrt{u^2+1}}\cdot\frac{1}{\sqrt{u^2+1}}}{8} - \frac{\frac{u}{\sqrt{u^2+1}}\left[\frac{1}{\sqrt{u^2+1}}\right]^3}{4}$$

$$= \frac{\arctan(u)}{8} + \frac{u}{8(u^2+1)} - \frac{u}{4(u^2+1)^2}$$

$$= \frac{\arctan(u)}{8} + \frac{u^3-u}{8(u^2+1)^2}$$

7. $$\int \frac{dx}{\sqrt{x^n - x^2}} = \int \frac{dx}{x\sqrt{x^{n-2}-1}} = \int \frac{1}{u^{1/(n-2)}\sqrt{u-1}} \cdot \frac{1}{n-2} u^{\frac{3-n}{n-2}} du = \frac{1}{n-2}\int \frac{du}{u\sqrt{u-1}} = \frac{2\arctan(\sqrt{u-1})}{n-2}$$

$$= \frac{2}{n-2}\arctan\left(\sqrt{x^{n-2}-1}\right)$$

$$\sqrt{x^n - x^2}$$
$$= \sqrt{x^2(x^{n-2}-1)}$$
$$= x\sqrt{x^{n-2}-1}$$

$$u = x^{n-2} \rightarrow x = u^{1/(n-2)}$$
$$dx = \frac{1}{n-2} u^{\frac{3-n}{n-2}} du$$

$$\int \frac{du}{u\sqrt{u-1}} = \int \frac{2v\,dv}{v(v^2+1)} = \int \frac{2\,dv}{1+v^2} = 2\arctan(v) = 2\arctan(\sqrt{u-1})$$

$$v = \sqrt{u-1} \rightarrow v^2+1 = u$$
$$dv = \frac{1}{2\sqrt{u-1}} du$$
$$du = 2v\,dv$$

Alternative Solution

$$1 - \frac{1}{n-2} = \frac{n-2-1}{n-2}$$

$$g(x)^2 = x^{n-2} - 1 \rightarrow x = [1+g(x)^2]^{1/(n-2)}$$

$$2g(x)g'(x) = (n-2)x^{n-3} = (n-2)\frac{x^{n-2}}{x} = (n-2)\frac{(g(x)^2+1)}{x} = (n-2)(g(x)^2+1)^{\frac{n-3}{n-2}}$$

$$g'(x) = \frac{(n-2)(g(x)^2+1)^{\frac{n-3}{n-2}}}{2g(x)}$$

$$\int \frac{dx}{x\sqrt{x^{n-2}-1}} = \int \frac{1}{x\sqrt{x^{n-2}-1}} \cdot \frac{2g(x)}{(n-2)(g(x)^2+1)^{\frac{n-3}{n-2}}} \frac{(n-2)(g(x)^2+1)^{\frac{n-3}{n-2}}}{2g(x)} dx$$

$$= \int \frac{1}{x\cancel{g(x)}} \cdot \frac{2\cancel{g(x)}}{(n-2)(g(x)^2+1)^{\frac{n-3}{n-2}}} \frac{(n-2)(g(x)^2+1)^{\frac{n-3}{n-2}}}{2g(x)} dx$$

$$= \frac{2}{n-2}\int \frac{1}{x(g(x)^2+1)^{\frac{n-3}{n-2}}} \cdot \frac{(n-2)(g(x)^2+1)^{\frac{n-3}{n-2}}}{2g(x)} dx$$

$$= \frac{2}{n-2}\int \frac{1}{(1+g(x)^2)} \frac{(n-2)(g(x)^2+1)^{\frac{n-3}{n-2}}}{2g(x)} dx$$

$$= \frac{2}{n-2}\int \frac{1}{1+u^2} du$$

$$= \frac{2}{n-2}\arctan(u)$$

$$= \frac{2}{n-2}\arctan\left(\sqrt{x^{n-2}-1}\right)$$

Another Alternative Solution

$$\int \frac{dx}{\sqrt{x^n - x^2}} = \int \frac{dx}{x\sqrt{x^{n-2}-1}} = \int \frac{1}{g(x)^{\frac{1}{\alpha}}\sqrt{g(x)^{\frac{n-2}{\alpha}}-1}} \cdot \frac{1}{\alpha g(x)^{\frac{\alpha-1}{\alpha}}} \alpha g(x)^{\frac{\alpha-1}{\alpha}} dx$$

$g(x) = x^\alpha$

$g'(x) = \alpha x^{\alpha-1}$

$= \alpha \frac{x^\alpha}{x}$

$= \alpha \frac{g(x)}{x}$

$= \alpha g(x)^{\frac{\alpha-1}{\alpha}}$

$$= \frac{1}{\alpha}\int \frac{1}{g(x)} \cdot \frac{1}{\sqrt{g(x)^{\frac{n-2}{\alpha}}-1}} \alpha g(x)^{\frac{\alpha-1}{\alpha}} dx$$

$$= \frac{1}{\alpha}\int \frac{1}{u\sqrt{u^{\frac{n-2}{\alpha}}-1}} du$$

if $\alpha = \frac{2}{n-2}$ then $= \frac{n-2}{2}\int \frac{du}{u\sqrt{u^2-1}} = \frac{n-2}{2}\operatorname{arcsec}(u) = \frac{n-2}{2}\operatorname{arcsec}(x^{\frac{2}{n-2}})$

altern. $= \frac{1}{\alpha}\int \frac{1}{\sqrt{u^{2+\frac{n-2}{\alpha}}-u^2}} du = \frac{2}{2-n}\int \frac{1}{\sqrt{1-u^2}} du = \frac{2}{2-n}\arcsin(u) = \frac{2}{2-n}\arcsin(x^{\frac{2-n}{2}})$

$2 + \frac{n-2}{\alpha} = 0 \rightarrow n-2 = -2\alpha \rightarrow \alpha = \frac{2-n}{2}$

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8. (i) $\int \frac{\arctan(x)}{1+x^2}dx = \int \arctan(x)\cdot\arctan'(x)\,dx = \int g(x)\cdot g'(x)\,dx = \int F'(g(x))g'(x)\,dx$

$g(x) = \arctan(x)$

$= \int u\,du = \frac{u^2}{2} = \frac{\arctan^2(x)}{2}$

$g'(x) = \frac{1}{1+x^2}$

(ii) $\int \frac{x\arctan(x)}{(1+x^2)^2}dx = \int \frac{x\arctan x}{1+x^2}\cdot\frac{1}{1+x^2}dx = \int \frac{\tan(g(x))\,g(x)}{1+\tan^2(g(x))}\cdot g'(x)\,dx = \int \sin(g(x))\cos(g(x))\,g(x)\,g'(x)\,dx$

$g(x) = \arctan(x)$

$x = \tan(g(x))$

$$= \int \sin(u)\cos(u)\cdot u\,du = \frac{u\sin^2(u)}{2} - \frac{1}{2}\int \sin^2(u)\,du = \frac{u\sin^2(u)}{2} - \frac{1}{2}\left[\frac{u}{2} - \frac{\sin u\cos u}{2}\right]$$

$f = u \quad f' = 1$

$g' = \sin u\cos u \quad g = \frac{\sin^2 u}{2}$

$$= \frac{u\sin^2(u)}{2} - \frac{u}{4} + \frac{\sin u\cos u}{4}$$

$$= \frac{\arctan(x)}{2}\cdot\frac{x^2}{x^2+1} - \frac{\arctan(x)}{4} + \frac{x}{4(x^2+1)}$$

$$= \frac{\arctan x}{4}\left[\frac{2x^2}{x^2+1} - 1\right] + \frac{x}{4(x^2+1)}$$

$$= \frac{\arctan(x)}{4}\cdot\frac{(x^2-1)}{(x^2+1)} + \frac{x}{4(x^2+1)}$$

(iii) $\int \log\sqrt{1+x^2}\,dx = x\log\sqrt{1+x^2} - \int \frac{x^2}{1+x^2}\,dx = x\log\sqrt{1+x^2} - x + \arctan(x)$

$f = \log(1+x^2)^{1/2} \quad f' = \frac{1}{\sqrt{1+x^2}}\cdot\frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}} = \frac{x}{1+x^2}$

$g' = 1 \qquad g = x$

$$\int \frac{x^2}{1+x^2}\,dx = \int \frac{1+x^2-1}{1+x^2}\,dx = \int\left[1 - \frac{1}{1+x^2}\right]dx = x - \arctan(x)$$

(iv) $\int x\log\sqrt{1+x^2}\,dx = \frac{x^2\log\sqrt{1+x^2}}{2} - \frac{1}{2}\int\frac{x^3}{1+x^2}\,dx = \frac{x^2\log\sqrt{1+x^2}}{2} - \frac{x^2}{4} + \frac{\log(1+x^2)}{4}$

$f = \log(1+x^2)^{1/2} \quad f' = \frac{1}{\sqrt{1+x^2}}\cdot\frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}} = \frac{x}{1+x^2}$

$g' = x \qquad g = \frac{x^2}{2}$

$$\int\frac{x^3}{1+x^2}\,dx = \int\left[x - \frac{x}{1+x^2}\right]dx = \frac{x^2}{2} - \frac{\log(1+x^2)}{2}$$

$$\begin{array}{r|l} & x \\ 1+x^2 & x^3 \\ & x^3+x \\ \hline & -x \end{array}$$

(v) $\int\frac{x^2-1}{x^2+1}\cdot\frac{1}{\sqrt{1+x^4}}\,dx = \int\frac{x^2-1}{x^2+1}\cdot\frac{1}{\sqrt{1+x^4}}\cdot\frac{(x^2+1)^2}{4x}\cdot\frac{4x}{(x^2+1)^2}\,dx = \int u\cdot\frac{1}{\sqrt{1+\frac{(u+1)^2}{(1-u)^2}}}\cdot\frac{\left(\frac{2}{1-u}\right)^2}{4\sqrt{\frac{u+1}{1-u}}}\,du$

$u = g(x) = \frac{x^2-1}{x^2+1} \qquad g'(x) = \frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2}$

$ux^2+u = x^2-1 \qquad = \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}$

$x^2(1-u) = u+1 \qquad = \frac{4x}{(x^2+1)^2} = \frac{du}{dx}$

$x = \sqrt{\frac{u+1}{1-u}}$

$$= \int u\cdot\frac{1}{\sqrt{1+\frac{(u+1)^2}{(1-u)^2}}}\cdot\frac{\cancel{4}}{(1-u)^2}\cdot\frac{1}{\cancel{4}}\sqrt{\frac{1-u}{1+u}}$$

$$= \int u\cdot\frac{1}{\sqrt{1+\frac{(u+1)^2}{(1-u)^2}}}\cdot\sqrt{\frac{1-u}{1+u}}\cdot\frac{1}{(1-u)^2}\,du$$

$$= \int\frac{u}{(1-u)^{\cancel{2}}}\cdot\sqrt{\frac{\cancel{(1-u)^2}}{2u^2+2}}\cdot\sqrt{\frac{1-u}{1+u}}\,du \quad \int\frac{u}{1-u}\cdot\frac{1}{\sqrt{2+2u^2}}\cdot\sqrt{\frac{1-u}{1+u}}\,du = \frac{1}{\sqrt{2}}\int\frac{u}{\sqrt{1-u}}\cdot\frac{1}{\sqrt{1+u^2}}\cdot\frac{1}{\sqrt{1+u}}\,du$$

$$= \frac{1}{\sqrt{2}}\int\frac{u}{\sqrt{1+u^2}\sqrt{1-u^2}}\,du = \frac{1}{\sqrt{2}}\cdot\frac{1}{2}\cdot\arcsin(u^2) = \frac{\sqrt{2}}{4}\cdot\arcsin\left(\left(\frac{x^2-1}{x^2+1}\right)^2\right)$$

$v = h(u) = u^2$

$h'(u) = 2u$

$$\int\frac{u}{\sqrt{1+u^2}\sqrt{1-u^2}}\,du = \int\frac{\cancel{u}}{\sqrt{1+u^2}\sqrt{1-u^2}}\cdot\frac{1}{2\cancel{u}}\cdot 2u\,du = \frac{1}{2}\int\frac{dv}{\sqrt{1-v^2}} = \frac{1}{2}\arcsin(v) = \frac{1}{2}\arcsin(u^2)$$

(vi) $\int \arcsin(\sqrt{x})\,dx = \int \arcsin(u)\cdot 2u\,du = u^2\arcsin(u) - \int \frac{u^2}{\sqrt{1-u^2}}\,du = u^2\arcsin(u) - \frac{\arcsin(u)}{2} + \frac{u\sqrt{1-u^2}}{2}$

$= x\arcsin(\sqrt{x}) - \frac{\arcsin(\sqrt{x})}{2} + \frac{\sqrt{x}\sqrt{1-x}}{2}$

$u = \sqrt{x}$ $\quad f = \arcsin(u) \to f' = \frac{1}{\sqrt{1-u^2}}$

$x = u^2 \to dx = 2u\,du$ $\quad g' = 2u \quad g = u^2$

$\int \frac{u^2}{\sqrt{1-u^2}}\,du = \int \frac{\sin^2\theta}{\cos\theta}\cdot\cos\theta\,d\theta = \int \sin^2\theta\,d\theta = \frac{\theta}{2} - \frac{\sin(\theta)\cos(\theta)}{2}$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$= \frac{\arcsin(u)}{2} - \frac{u\cdot\cos(\arcsin(u))}{2}$

$= \frac{\arcsin(u)}{2} - \frac{u\sqrt{1-u^2}}{2}$

(vii) $\int \frac{x}{1+\sin x}\,dx = x(\tan x + \sec x) - \int(\tan x + \sec x)\,dx = x(\tan x + \sec x) + \log(\cos x) + \log(\tan x + \sec x)$

$f = x \quad f' = 1$

$g' = \frac{1}{1+\sin x}$

$g = \int \frac{dx}{1+\sin x} = \int \frac{1-\sin x}{1-\sin^2 x}\,dx = \int \frac{1-\sin x}{\cos^2 x}\,dx = \int(\sec^2 x - \tan x\cdot\sec x)\,dx = \tan x - \sec x$

$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{\sin x}{\cos x}\cdot\frac{du}{-\sin x} = -\int u^{-1}\,du = -\log(\cos x)$

$u = \cos x$

$du = -\sin x\,dx$

(viii) $\int e^{\sin x}\cdot\frac{x\cos^3 x - \sin x}{\cos^2 x}\,dx = \int e^{\sin x}\cdot x\cos x\,dx - \int \frac{e^{\sin x}\sin x}{\cos^2 x}$

$\int e^{\sin x}\cdot x\cos x\,dx = xe^{\sin x} - \int e^{\sin x}\,dx$

$= xe^{\sin x} - \cancel{\int e^{\sin x}\,dx} - e^{\sin x}\sec x + \cancel{\int e^{\sin x}\,dx}$

$f = x \quad f' = 1$

$= xe^{\sin x} - e^{\sin x}\sec x$

$g' = e^{\sin x}\cos x \quad g = e^{\sin x}$

$\int \frac{e^{\sin x}\sin x}{\cos^2 x}\,dx = \int e^{\sin x}\cdot\tan(x)\cdot\sec(x)\,dx = e^{\sin x}\sec x - \int e^{\sin x}\cos x\sec x = e^{\sin x}\sec x - \int e^{\sin x}\,dx$

$f = e^{\sin x} \quad f' = e^{\sin x}\cdot\cos x$

$g' = \sec x\tan x \to g = \sec x$

c(x) $\int \sqrt{\tan x}\, dx$

$$\int \sqrt{\tan x}\, dx = \int \sqrt{\tan x}\, \frac{2\sqrt{\tan x}}{\sec^2 x} du = 2\int \frac{\tan x}{\sec^2 x} du = 2\int \frac{u^2}{1+u^4} du$$

$$u = \sqrt{\tan x} \to u^2 = \tan(x) = \frac{\sin x}{\cos x} \to \sec^2 x = \frac{u^4}{\sin^2 x} = \frac{u^4}{\frac{u^4}{1+u^4}} = 1+u^4$$

$$du = \frac{\sec^2 x}{2\sqrt{\tan x}} dx$$

$$dx = \frac{2\sqrt{\tan x}}{\sec^2 x} du$$

$$x = \arctan(u^2)$$

$$\sin(\arctan x) = \frac{x}{\sqrt{x^2+1}} \to \sin(\arctan u^2) = \frac{u^2}{\sqrt{u^4+1}} \to \sin^2(\arctan u^2) = \frac{u^4}{1+u^4}$$

$$\int \frac{u^2}{1+u^4} du = \int \frac{u^2}{(u^2+\sqrt{2}u+1)(u^2-\sqrt{2}u+1)} du = \int\left[\frac{Au+B}{u^2+\sqrt{2}u+1} + \frac{Cu+D}{u^2-\sqrt{2}u+1}\right] du$$

$$= \int\left[\frac{-u/2\sqrt{2}}{u^2+\sqrt{2}u+1} + \frac{u/2\sqrt{2}}{u^2-\sqrt{2}u+1}\right] du$$

$$= \frac{1}{4}\int\left[\frac{-u\sqrt{2}}{u^2+\sqrt{2}u+1} + \frac{u\sqrt{2}}{u^2-\sqrt{2}u+1}\right] du$$

$$= -\frac{\sqrt{2}}{8}\int\left[\frac{2u+\sqrt{2}-\sqrt{2}}{u^2+\sqrt{2}u+1} + \frac{2u-\sqrt{2}+\sqrt{2}}{u^2-\sqrt{2}u+1}\right] du$$

$$= -\frac{\sqrt{2}}{8}\int \frac{2u+\sqrt{2}}{u^2+\sqrt{2}u+1} + \frac{1}{4}\int \frac{1}{u^2+\sqrt{2}u+1} du$$

$$-\frac{\sqrt{2}}{8}\int \frac{2u-\sqrt{2}}{u^2-\sqrt{2}u+1} - \frac{1}{4}\int \frac{1}{u^2-\sqrt{2}u+1} du$$

$$= -\frac{\sqrt{2}}{8}\log(u^2+\sqrt{2}u+1)$$

$$\begin{aligned} &Au^3 - A\sqrt{2}u^2 + Au \quad +B+D \\ &+Cu^3 + Bu^2 \quad -B\sqrt{2}u \\ &+C\sqrt{2}u^2 + Cu \\ &+Du^2 \quad +D\sqrt{2}u \end{aligned}$$

$$u^3[A+C] + u^2[B+D+C\sqrt{2}-A\sqrt{2}] + u[A-B\sqrt{2}+C+D\sqrt{2}] + (B+D)$$

$$A+C=0 \to A=-C$$

$$B+D=0 \to B=-D \to B=D=0$$

$$\cancel{B}+\cancel{D}+C\sqrt{2}-A\sqrt{2}=1 \to 2C\sqrt{2}=1 \to C=\frac{1}{2\sqrt{2}} \to A=-\frac{1}{2\sqrt{2}}$$

$$\cancel{A}-B\sqrt{2}+\cancel{C}+D\sqrt{2}=0 \to \sqrt{2}(D-B)=0 \to D=B$$

(ii) $\displaystyle\int \frac{1+\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin^2 x}\,dx + \int \frac{\cos x}{\sin^2 x}\,dx$

$$\int \frac{1}{\sin^2 x}\,dx = \int \frac{1}{\cancel{\sin^2 x}} \cdot \frac{\sin^{\cancel{3}} x}{-2\cos x} \cdot \frac{-2\cos x}{\sin^3 x}\,dx = -\frac{1}{2}\int \frac{\sin x}{\cos x}\,\frac{-2\cos x}{\sin^3 x}\,dx = -\frac{1}{2}\int \tan\left[\arcsin\left(1/\sqrt{g(x)}\right)\right] g'(x)\,dx$$

$u = g(x) = \sin^{-2}(x)$ $\quad g'(x) = -2\sin^{-3}(x)\cos(x) = \dfrac{-2\cos x}{\sin^3 x}$

$\sin^2 x = \dfrac{1}{g(x)} \rightarrow \sin x = \dfrac{1}{\sqrt{g(x)}} \rightarrow x = \arcsin\left(1/\sqrt{g(x)}\right)$

$$= -\frac{1}{2}\int \tan\left(\arcsin\left(1/\sqrt{u}\right)\right)du$$

$$= -\frac{1}{2}\int \frac{\frac{1}{\sqrt{u}}}{\sqrt{1-\frac{1}{u}}} = -\frac{1}{2}\int \frac{1}{\sqrt{u}} \cdot \sqrt{\frac{u}{u-1}}\,du$$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{u-1}}\,du$$

$$= -\frac{1}{2}\,\frac{(u-1)^{1/2}}{1/2} = -(u-1)^{1/2}$$

$$= -\left(\frac{1}{\sin^2 x} - 1\right)^{1/2}$$

$$= -(\csc^2 x - 1)^{1/2}$$

$$= -(\cot^2 x)^{1/2} = -\cot(x)$$

$$\int \frac{\cos x}{\sin^2 x}\,dx = -(\sin x)^{-1}$$

Hence

$$\int \frac{1+\cos x}{\sin^2 x}\,dx = -\cot(x) - (\sin x)^{-1}$$