Ch 14 - The Fundamental Theorem of Calculus

Theorem 1 (First Fundamental Theorem of Calculus) Let f be integrable on $[a,b]$ and define F on $[a,b]$ by

$$F(x) = \int_a^x f$$

If f is cont. at $c \in [a,b]$ then F is diff at c and $F'(c) = f(c)$

(if $c = a$ or b, then $F'(c)$ is understood to be the right-hand or left-hand derivative of F)

Proof

1. Assume $c \in (a,b)$.

2. $F'(c) = \lim_{h \to 0} \frac{F(c+h)-F(c)}{h}$ Proof: by def. of derivative

3. There are two possible cases, $h>0$ and $h<0$. Proof: in the def. of $F'(c)$, we have $\forall \epsilon > 0\ \exists \delta > 0\ \forall h\ |h| < \delta$ ...

Case 1: $h>0$

Then $F(c+h)-F(c) = \int_c^{c+h} f$. Proof: $F(c+h) = \int_a^{c+h} f$, $F(c) = \int_a^c f$. $F(c+h)-F(c) = \int_a^{c+h} f - \int_a^c f = \int_a^c f + \int_c^{c+h} f - \int_a^c f = \int_c^{c+h} f$

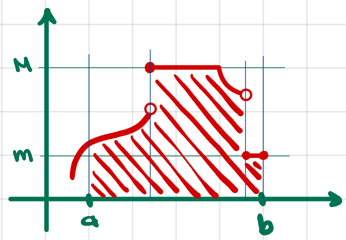
4. Define

$m_h = \inf\{f(x) : c \le x \le c+h\}$

$M_h = \sup\{f(x) : c \le x \le c+h\}$

Theorem 7 Suppose f integrable on $[a,b]$ and that $m \le f(x) \le M$ for all x in $[a,b]$.

Then, $m(b-a) \le \int_a^b f \le M(b-a)$



5. Then, $m_h h \le \int_c^{c+h} f \le M_h h$ Proof: By Th. 13-7.

6. $m_h \le \frac{F(c+h)-F(c)}{h} \le M_h$

7. Case 2: $h<0$

Then $F(c)-F(c+h) = \int_{c+h}^{c} f$

8. Define

$m_h = \inf\{f(x) : c+h \le x \le c\}$

$M_h = \sup\{f(x) : c+h \le x \le c\}$

9. Then, $m_h(-h) \le \int_{c+h}^{c} f \le M_h(-h)$ Proof: By Th. 13-7.

$$m_h \le \frac{\int_{c+h}^{c} f}{-h} \le M_h \rightarrow m_h \le \frac{F(c)-F(c+h)}{-h} \le M_h \rightarrow m_h \le \frac{F(c+h)-F(c)}{h} \le M_h$$

10. $\lim_{h\to 0} m_h = \lim_{h\to 0} M_h = f(c)$ **Proof**: $\forall \epsilon>0\ \exists \delta>0\ \forall h\ |h|<\delta \to |f(c+h)-f(c)|<\epsilon$

$\to f(c)-\epsilon \le f(c+h) \le f(c)+\epsilon$

$\to f(c)-\epsilon$ is lower bound for $\{f(x): c+h \le x \le c\}$

$\to f(c)-\epsilon \le m_h \le M_h \le f(c)+\epsilon$

$\to |m_h - f(c)| < \epsilon$

$\to \lim_{h\to 0} m_h = f(c)$

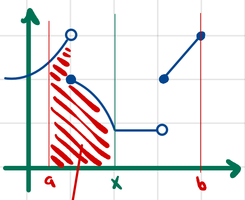
11. Take limits in the inequality $m_h \le \dfrac{F(c+h)-F(c)}{h} \le M_h$

$$\lim_{h\to 0} m_h = f(c) \le \lim \frac{F(c+h)-F(c)}{h} = F'(c) \le f(c) = \lim_{h\to 0} M_h$$

12. $F'(c) = f(c)$ **Proof**: by 11.

■

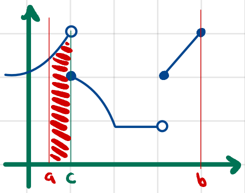
Let's go through the proof again.



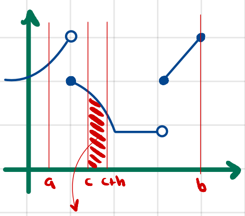
$F(x) = \int_a^x f$

Th. 8 told us that F is continuous

The First Fundamental Theorem of Calculus tells us that, in this example, since f is cont. at x, then F is diff. at x, and $F'(x) = f(x)$.



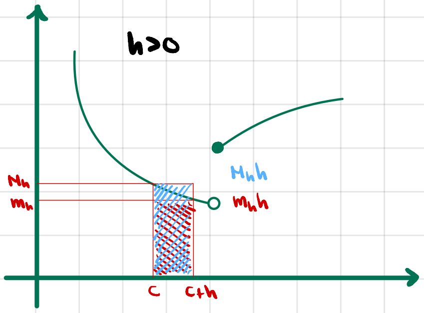
Even though F is cont. at c, it is not diff. there.

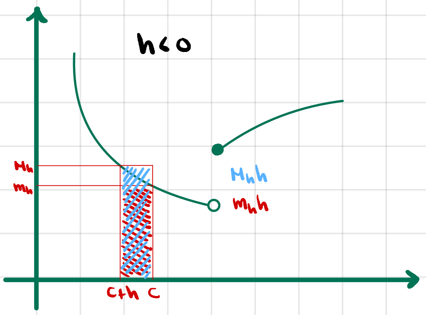


$F(c+h)-F(c)$

$\dfrac{F(c+h)-F(c)}{h}$, the change in area for a small h increment in c, is the average change in F. The derivative is the limit when $h\to 0$, the instantaneous rate of change of the area.

The Fund. Thm tells us this instant. rate of chg of f's area at c is just the value of f at c.





In both cases, since $m_h \le f \le M_h$ on $[c, c+h]$ then by Th. 13-7,

$$m_h h \le \int_c^{c+h} f \le M_h h$$

as we can see in the graphs to the left. Then

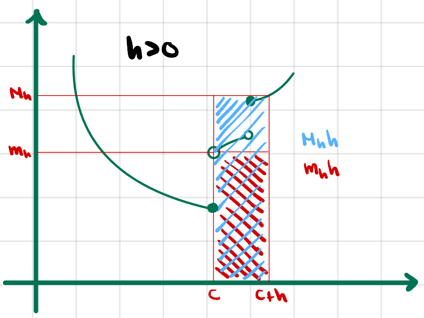
$$m_h \le \frac{\int_c^{c+h} f}{h} = \frac{F(c+h)-F(c)}{h} \le M_h$$

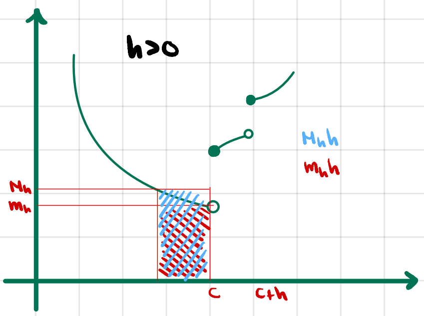
That is, the average value of f on $[c, c+h]$ is between m_h and M_h.

We can already see that if m_h and M_h converge to the same value as $h \to 0$ then $F'(c)$ will have to be that value.

Such convergence occurs if f is cont. at c, as we'll see shortly.

Note that this is true even if f is discontinuous at c. The only assumption so far is that f is integrable on $[a, b]$.





What happens when we make h very small?

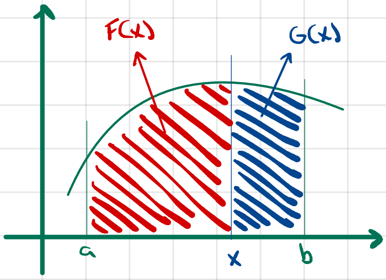
If f is cont. at c, then m_h and M_h approach $f(c)$.

But then $\dfrac{F(c+h)-F(c)}{h}$ must approach $f(c)$ as well.

Consider $G(x) = \int_x^b f = \int_a^b f - \int_a^x f$.

Then, since $\int_a^b f$ is constant, we can use the result we proved in the Fund. Thm

$G'(c) = -f(c) = -F'(c)$



As we increase x, $g(x) = \int_x^b f$ decreases and $F(x) = \int_a^x f$ increases at the same rate.

Hence if $F(x) = \int_a^x f$ and F is defined for $x<a$ then we can write

$F(x) = \int_a^x f = -\int_x^a f$

Now if $c<a$, then $F'(c) = f(c) = -(-f(c))$

A few observations

If f is cont. at c the F is diff. at c and $F'(c) = f(c)$.

If f is cont. on $[a,b]$, then F diff on $[a,b]$ and $\forall x, x \in [a,b] \to F'(x) = f(x)$

Therefore, if a fn f is cont., then it is the derivative of some other fn, namely $F(x) = \int_a^x f$.

In the proof of the Fund. Thm, we assumed continuity at $c \in [a,b]$. What a and b are isn't important.

In fact,

$$\left(\int_{a_1}^x f\right)' = \left(\int_{a_2}^x f\right)' = f(x)$$

as long as f is integrable on $[a_1, b]$ and $[a_2, b]$ w/ $a_1, a_2 \le x \le b$.

Corollary f cont. on $[a,b]$
$f = g'$ for some fn g $\rightarrow \int_a^b f = g(b) - g(a)$

Proof

Let $F(x) = \int_a^x f$

1. $F'(x) = f(x) = g'(x)$ for any x in $[a,b]$

2. $\exists c,\ F = g + c$

3. What is c?

   $0 = F(a) = g(a) + c \rightarrow c = -g(a)$

   $\rightarrow F(x) = g(x) - g(a)$

4. In particular, $\int_a^b f = F(b) = g(b) - g(a)$

Let's walk through the proof

By assumption, we start with a cont. f on $[a,b]$, and we know a fn g whose derivative is f, ie $g' = f$.

Now, from the fund. thm, we already know f was the deriv. of some fn $F(x) = \int_a^x f$, on $[a,b]$, ie $F' = f$.

Thus, $F' = f = g'$.

From a problem in Ch. 11 we know that if the derivatives of two fns are the same at every point then they differ by a constant. The constant is easily found by checking the value of one of the fns at a root of the other.

But then $F(x) = g(x) - g(a) = \int_a^x f$

But then $F(b) = g(b) - g(a) = \int_a^b f$

Note that g is any fn such that $g' = f$.

Example

$f(x) = x^2$ and $g(x) = \dfrac{x^3}{3}$. Then $g'(x) = x^2 = f(x)$

Let $F(x) = \int_a^x t^2 dt$. By the corollary, $\int_a^b t^2 dt = g(b) - g(a) = \dfrac{b^3 - a^3}{3}$.

Recall that in chapter 13, we could prove f was integrable and then using arguments based on lower and upper sums we could try to find $\int_a^b f$. Now, if we know a g s.t. $g' = f$ then we can compute $\int_a^b f$ much more easily: $g(b) - g(a)$.

Example

Let $f(x) = x^n$ for $n \in \mathbb{N}$.

We know that if $g(x) = \frac{x^{n+1}}{n+1}$ then $g'(x) = f(x)$.

Hence, $\int_a^b f(x) = g(b) - g(a) = \frac{b^{n+1} - a^{n+1}}{n+1}$

Example

Let $f(x) = x^{-n}$, $n \in \mathbb{N} - \{1\}$

f is unbounded near 0, ie in any interval containing 0.

But for $a,b > 0$ or $a,b < 0$, f is continuous, thus integrable on $[a,b]$.

If $g(x) = \frac{x^{-n+1}}{-n+1}$ then $g'(x) = f(x)$

Hence $\int_a^b x^{-n} = \frac{b^{-n+1} - a^{-n+1}}{-n+1}$

In the case of $n=1$ we have $f(x) = \frac{1}{x}$. We don't yet know any fn g s.t. $g' = f$.

A few observations

The corollary says that if f is continuous then $\int_a^b f = g(b) - g(a)$.

What if f isn't continuous? f can still be integrable and thus there is a number $\int_a^b f$.

As an example, consider

$$f(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$$

Assume f is the derivative of some fn g. This contradicts Theorem 11-7, which says that a derivative cannot have jump discontinuities.

In other words, if f is the derivative of some fn, then since f exists in some interval containing 1, and

$\lim_{x \to 1^+} f = \lim_{x \to 1^-} f = \lim_{x \to 1} f = 0$ then $f(1) = \lim_{x \to 1} f = 0$, which isn't the case w/ our fn f.

Theorem 2 (Second Fundamental Theorem of Calculus)

f integrable on $[a,b]$

$f = g'$ for some g $\rightarrow \int_a^b f = g(b) - g(a)$

Proof

1. g is differentiable and hence continuous on $[a,b]$.

2. Let $P = \{t_0, \dots, t_n\}$ be any partition of $[a,b]$.

3. $\exists x_i, x_i \in [t_{i-1}, t_i] \wedge g(t_i) - g(t_{i-1}) = g'(x_i)(t_i - t_{i-1}) = f(x_i)(t_i - t_{i-1})$

Proof: By MVT applied to g on $[t_{i-1}, t_i]$.

4. $m_i = \inf\{f(x) : t_{i-1} \le x \le t_i\}$
$M_i = \sup\{f(x) : t_{i-1} \le x \le t_i\}$

Then $m_i \le f(x_i) \le M_i$ Proof: by def. of m_i and M_i

$m_i \Delta t_i \le f(x_i) \Delta t_i \le M_i \Delta t_i$

$m_i \Delta t_i \le g(t_i) - g(t_{i-1}) \le M_i \Delta t_i$ Proof: by 3.

5. Add above eq. for all i

$\sum m_i \Delta t_i \le \sum (g(t_i) - g(t_{i-1})) \le \sum M_i \Delta t_i$

$L(f,P) \le g(b) - g(a) \le U(f,P)$ for every partition P.

6. $\int_a^b f = g(b) - g(a)$

■

Let's walk through the proof.

f is assumed integrable on $[a,b]$ and is the derivative of some fn g. Thus g is diff. on $[a,b]$, but we don't know if

If f is bounded on $[a,b]$, then $\sup\{L(f,P)\}$ and $\inf\{U(f,P)\}$ both exist, even if f is not integrable.

These numbers are also known as

lower integral of f on $[a,b]$ $= \sup\{L(f,P)\} = L\int_a^b f$

upper " " " " " $= \inf\{U(f,P)\} = U\int_a^b f$

Let's prove some theorems about $L\int_a^b f$ and $U\int_a^b f$.

Theorem $a<c<b \rightarrow L\int_a^b f = L\int_a^c f + L\int_c^b f$

Proof <u>not sure if correct</u>

Let $P = \{a=t_0, \dots, b=t_n\}$ be partition of $[a,b]$.

$L(f,P) = \sum_{i=1}^{n} m_i \Delta t_i$

Since $c \in (a,b)$, there is some t_m s.t. $c \in [t_{m-1}, t_m]$.

Let P' be P plus c. $P' = \{a = u_0 = t_0, \dots, u_{m-1} = t_{m-1}, c = u_m, u_{m+1} = t_m, \dots, u_{n+1} = t_n\}$

$L(f,P') = \sum_{i=0}^{m} m_i \Delta t_i + \sum_{i=m+1}^{n+1} m_i \Delta t_i \geq L(f,P)$

$L\int_a^c f + L\int_c^b f \geq L\int_a^b f$

$$L\int_a^b f = \underset{\text{over all partitions}}{\sup\{L(f,P)\}} \geq \underset{\text{any partition } P \text{ plus single point } c}{L(f,P')} = \sum_{i=0}^{m} m_i \Delta t_i + \sum_{i=m+1}^{n+1} m_i \Delta t_i \geq \underset{\text{any partition } P}{L(f,P)}$$

$L\int_a^b f \geq L(f,P') = L(f,P_1) + L(f,P_2) \geq L(f,P)$

Take supremum over all partitions P.

$L\int_a^b f \geq \sup\{L(f,P')\} = \sup\{L(f,P_1)\} + \sup\{L(f,P_2)\} \geq L\int_a^b f$

■

Note if f is bounded we can always build lower and upper sums. Because $L(f,P) \leq U(f,P)$ then the fact that we can compute $L(f,P)$ and $U(f,P)$ means that their sup and inf, respect., exist. Therefore, (1) is always true for bounded fns.

If f is also integrable, then $\sup\{L(f,P)\} = L\int_a^b f = U\int_a^b f = \inf\{U(f,P)\}$ and from (1) we have $\int_a^b f = \int_a^c f + \int_c^b f$

Alternative Proof

Let P_1 and P_2 be arbitrary partitions of $[a,c]$ and $[c,b]$.

Let $P = P_1 \cup P_2$ partition $[a,b]$. Then

$$L(f,P_1) + L(f,P_2) = L(f,P) \le \sup\{L(f,P)\} = L\int_a^b f$$

P_1 and P_2 can be varied independently.

Take supremum over P_1

*note: maybe easier to see if we write $L(f,P_1) \le L\int_a^b f - L(f,P_2)$, true for all P_1 and P_2. Hence the rhs is $\ge$ the sup of the left side

$$\sup\{L(f,P_1)\} + L(f,P_2) \le L\int_a^b f$$

$$L\int_a^c f + L(f,P_2) \le L\int_a^b f$$

Then take supremum over all P_2

$$L\int_a^c f + L\int_c^b f \le \int_a^b f \quad (1)$$

Now let $\epsilon > 0$. There must be a P s.t. $L(f,P) > L\int_a^b f - \epsilon$, otherwise $L\int_a^b f$ wouldn't be sup.

Now add c to P to obtain P'.

$$L\int_a^b f - \epsilon < L(f,P) \le L(f,P')$$

$P' = P_1' \cup P_2'$, P_1' partition of $[a,c]$, P_2' of $[c,b]$.

$$L\int_a^b f - \epsilon < L(f,P') = L(f,P_1') + L(f,P_2') \le L\int_a^c f + L\int_c^b f$$

ie

$$L\int_a^b f - \epsilon < L\int_a^c f + L\int_c^b f$$

true for any $\epsilon > 0$.

$$\rightarrow L\int_a^b f \ge L\int_a^c f + L\int_c^b f \quad (2)$$

$$(1), (2) \rightarrow L\int_a^b f = L\int_a^c f + L\int_c^b f$$

■

Theorem $m \le f(x) \le M$ for all x in $[a,b]$ $\rightarrow$ $m(b-a) \le L\int_a^b f \le U\int_a^b f \le M(b-a)$

Proof

Let $P = \{t_0, \dots, t_n\}$ be partition of $[a,b]$.

$L(f,P) = \sum m_i \Delta t_i$

$\sum m \Delta t_i \le \sum m_i \Delta t_i \le \sum M_i \Delta t_i \le \sum M \Delta t_i$

$m(b-a) \le L(f,P) \le U(f,P) \le M(b-a)$

$m(b-a)$ is lower bound for $L(f,P)$, hence $m(b-a) \le \sup\{L(f,P)\}$

Similarly $M(b-a) \ge \inf\{U(f,P)\}$ since $M(b-a)$ is upper bound.

Hence

$$m(b-a) \le L\int_a^b f \le U\int_a^b f \le M(b-a)$$

■

Again, if f is integrable then we immediately have

$$m(b-a) \le \int_a^b f \le M(b-a)$$

Theorem 13-3 If f cont. on $[a,b]$, then f is integrable on $[a,b]$.

Proof

Define L and U on $[a,b]$

$$L(x) = L\int_a^x f$$

$$U(x) = U\int_a^x f$$

Note that these are just sums.

Let $x \in (a,b)$.

Let $h \neq 0$

Case 1: $h > 0$

$m_h = \inf\{f(t) : x \le t \le x+h\}$
$M_h = \sup\{f(t) : x \le t \le x+h\}$

$$m_h h \le L\int_x^{x+h} f \le U\int_x^{x+h} f \le M_h h$$

$$m_h h \le L(x+h) - L(x) \le U(x+h) - U(x) \le M_h h$$

$$m_h \le \frac{L(x+h)-L(x)}{h} \le \frac{U(x+h)-U(x)}{h} \le M_h$$

Case 2: $h < 0$

$m_h = \inf\{f(t) : x+h \le t \le x\}$
$M_h = \sup\{f(t) : x+h \le t \le x\}$

$m_h \le f(t) \le M_h$ for all $t \in [x+h, x]$

$$\to m_h(-h) \le L\int_{x+h}^{x} f \le U\int_{x+h}^{x} f \le M_h(-h)$$

$$m_h(-h) \le L(x) - L(x+h) \le U(x) - U(x+h) \le M_h(-h)$$

$$M_h h \le U(x+h) - U(x) \le L(x+h) - L(x) \le m_h h$$

$$m_h \le \frac{L(x+h)-L(x)}{h} \le \frac{U(x+h)-U(x)}{h} \le M_h$$

Therefore, by proof by cases,

$$m_h \le \frac{L(x+h)-L(x)}{h} \le \frac{U(x+h)-U(x)}{h} \le M_h$$

But f is continuous, so $\lim_{h\to 0} m_h = \lim_{h \to 0} M_h = f(x)$

(we proved this in the proof of Th. 1)

Therefore,

$$\lim_{h\to 0} m_h \le \lim_{h\to 0} \frac{L(x+h)-L(x)}{h}$$
$$\le \lim_{h\to 0} \frac{U(x+h)-U(x)}{h}$$
$$\le \lim_{h\to 0} M_h$$

$f(x) \le L'(x) \le U'(x) \le f(x)$

$\to L'(x) = U'(x) = f(x)$ for $x \in (a,b)$

Hence $\exists c$ s.t. $U(x) = L(x) + c$ for all $x \in [a,b]$

But $U(a) = L(a) = 0$. Hence $c = 0$.

$U(x) = L(x)$, $x \in [a,b]$

In particular,

$$U\int_a^b f = U(b) = L(b) = L\int_a^b f$$

So f is integrable on $[a,b]$.