Chapter 3 - Functions

Provisional Def: Function is a rule that assigns to each of certain real numbers, some other real number.

Polynomial Fn f is polyn. if $\forall a_0, \dots, a_n \in \mathbb{R}$ s.t. $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$

highest power: degree of f

Rational Fns $\frac{P}{q}$, P, q polynomial fns, q not always zero

Combining Fns to produce new fns

f, g fns. Define fn $f+g$ as $(f+g)(x) = f(x) + g(x)$

$\to$ domain $f+g$ = domain $f \cap$ domain g

Similarly, define $f \cdot g$, f/g, $c \cdot g$

$(f \cdot g)(x) = f(x) \cdot g(x)$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$(c \cdot g)(x) = c \cdot g(x)$ (special case of $f \cdot g$ w/ $f(x) = c$)

constant fn

Some results

$(f \cdot g) \cdot h = f \cdot (g \cdot h)$

$((f \cdot g) \cdot h)(x) = (f \cdot g)(x) \cdot h(x) = (f(x) \cdot g(x)) \cdot h(x) = f(x) \cdot g(x) \cdot h(x)$
$= f(x) \cdot (g(x) \cdot h(x)) = f \cdot (g \cdot h)(x)$

Identity fn: $I(x) = x$

$\to f(x) = \frac{x + x^2 + x\sin^2 x}{x\sin x + x\sin^2 x} = \frac{I + I \cdot I + I \cdot \sin \cdot \sin}{I \cdot \sin + I \cdot \sin \cdot \sin}$

Define composition fn $f \circ g$

$(f \circ g)(x) = f(g(x))$

domain$(f \circ g) = \{x : x \text{ in dom } g,\ g(x) \text{ in dom } f\}$

$f = I \cdot I$
$g = \sin$

$\Rightarrow$ $(f \circ g)(x) = I(\sin x) \cdot I(\sin x) = \sin^2 x$
$(g \circ f)(x) = \sin(I(x) \cdot I(x)) = \sin(x^2)$

Composition is associative $(f \circ g) \circ h = f \circ (g \circ h)$

$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x)))$
$(f \circ (g \circ h))(x) = f((g \circ h)(x)) = f(g(h(x)))$

notation

$x \longrightarrow x^2$

the function $f(x) = x^2$

Def (Function) fn is collection of pairs of numbers w/ following property: $(a,b), (a,c)$ both in the collection then $b = c$.

Def (Domain) domain of fn f is set of all a for which there is some b such that (a,b) is in f.
If a is in domain of f, by the def. of fn, there is a unique b such that (a,b) in f. This unique b is denoted $f(a)$.

Appendix - ordered Pairs

Def: (ordered Pair) $(a,b) = \{\{a\},\{a,b\}\}$

Theorem 1 $(a,b) = (c,d) \Rightarrow a = c, b = d$

proof:

$\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\}$

The two sets must contain the same elements.
The elements are themselves sets.
One element is a set with one element inside.
This one element set must be the same in both top-level sets
$\Rightarrow a = c$
$\Rightarrow \{c,b\} = \{c,d\} \Rightarrow b = d$

note

One approach is as above: define ordered pair a certain way, derive a particular property as a theorem.

Alternatively, could have simply introduced the term (a,b) and adopted the property as an axiom. The lack of definition in this case is justified by the practical fact that

i) this is the only interesting/relevant property of ordered pairs
ii) therefore there is not anything else to be done with a def. other than simply derive the property.