Ch 11 - Significance of the Derivative

Definition Let f be a fn and A a set of numbers contained in f's domain

A point x in A is a maximum point for f on A if $f(x) \geq f(y)$ for every y in A

$f(x)$ is called the maximum value of f on A.

f has its maximum value on A at x.

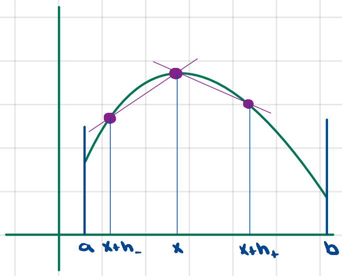
* we can define min point and min value in a similar way as above, or simply by:

x is a min point of f on A if x is a max point of $-f$ on A.

Theorem 1 Let f be any fn defined on (a,b).

If x is a max (or min) point for f on (a,b), and f is diff at x, then $f'(x) = 0$.

Proof



Let h be a number s.t. $x+h \in (a,b)$.

Then $f(x+h) \leq f(x)$. Hence,

$$h>0 \rightarrow \frac{f(x+h)-f(x)}{h} \leq 0 \rightarrow \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \leq 0$$

$$h<0 \rightarrow \frac{f(x+h)-f(x)}{h} \geq 0 \rightarrow \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} \geq 0$$

Since f is diff. at x, these limits must be equal.

$\rightarrow f'(x) = 0$

Definition Let f be a fn and A a set of numbers contained in f's domain

A point x in A is a local maximum (minimum) point for f on A if there is some $\delta > 0$ such that x is a max (min) point for f on $A \cap (x-\delta, x+\delta)$

Theorem 2 x local max or min for f on (a,b), f diff at x $\rightarrow f'(x) = 0$

Proof

Direct application of Th. 1. x is a max on $(a,b) \cap (x-\delta, x+\delta) = (x-\delta, x+\delta)$, for some $\delta > 0$.
By Th 1, $f'(x) = 0$.

Definition A critical point of fn f is a number x such that $f'(x)=0$

The number $f(x)$ itself is called a critical value of f.

Problem: find max or min of f on closed interval $[a,b]$.

closed interval, continuous, therefore f takes max and min values on interval

if x is a max or min point, then x must be in one of the following three classes of points

1) critical points of f in $[a,b]$
2) end points a and b
3) points in $[a,b]$ where f is not diff.

Solution

find critical points of f in $[a,b]$
find points in $[a,b]$ where f not diff
compute value of f at found points and endpoints
check largest and smallest values

Situations where the procedure above is not guaranteed to work

f not continuous
open interval / whole real line

in such cases, need to make other observations about the fn.

Theorem 3 (Rolle's Theorem)

f cont. on $[a,b]$ and diff on (a,b) $\rightarrow$ $\exists x,\ x \in (a,b),\ f'(x)=0$
$f(a)=f(b)$

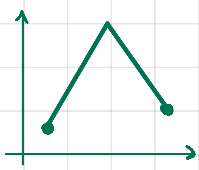
Proof

Since f is cont. on $[a,b]$, it has a max and a min value on $[a,b]$.

If the max value occurs at $x \in (a,b)$, then by Th. 1, since f diff at x, $f'(x)=0$.
Same thing if the min occurs at $x \in (a,b)$.
Ie, if either the max or the min value occurs inside (a,b), f' is zero at that or those points.

The other possible case is that both max and min occur at the endpoints.
But then the fn is constant, so $f'(x)=0$ at any $x \in [a,b]$.

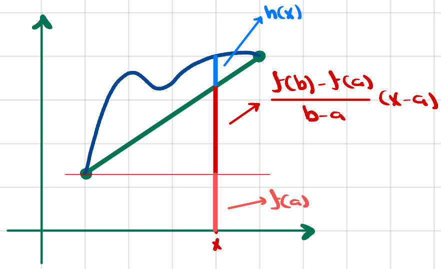
Th 3 required diff at every x in (a,b),
o.w. th. 3 does not apply



Theorem 4 (Mean Value Theorem)

f cont. on $[a,b]$, diff on (a,b) $\rightarrow \exists x, x \in (a,b) \wedge f'(x) = \dfrac{f(b)-f(a)}{b-a}$

Proof



Let $h(x) = f(x) - \dfrac{f(b)-f(a)}{b-a}(x-a) - f(a)$

Then h is cont on $[a,b]$ and diff on (a,b), and

$h(a) = f(a)$
$h(b) = f(a)$

Apply Rolle's Theorem to h, conclude that $\exists x, x \in (a,b) \wedge h'(x) = \dfrac{h(b)-h(a)}{b-a} = 0$

But $h'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$

So the application of Rolle yields $f'(x) = \dfrac{f(b)-f(a)}{b-a}$

Corollary I

f defined on interval
$f'(x) = 0$ for all x in such interval $\rightarrow$ f constant on the interval

Proof

Take any two points a and b in the interval.

By MVT, $\exists x, x \in (a,b)$ and $f'(x) = \dfrac{f(b)-f(a)}{b-a} = 0$

$\rightarrow f(a) = f(b)$.

Corollary II

f, g defined on same interval
$f'(x) = g'(x)$ for all x in the interval $\rightarrow \exists c, f = g + c$

Proof

$(f-g)'(x) = f'(x) - g'(x) = 0$

By corollary I, $f-g$ is constant on the interval, ie $f(x) - g(x) = c \rightarrow f(x) = g(x) + c$.

Definition

A fn is increasing on an interval if $\forall a \forall b$, a and b in interval and $a < b \to f(a) < f(b)$
" " " decreasing " " " " " " " " " $\to f(a) > f(b)$

Corollary 3

$f'(x) > 0$ for all x in an interval, then f increasing on interval
$f'(x) < 0$ " " " " " " " decreasing " "

Proof

Consider an interval in which $f'(x) > 0$ for all x.
Let a and b be two points in this interval.

MVT $\to \exists x \in (a,b) \wedge f'(x) = \dfrac{f(b)-f(a)}{b-a} > 0 \to f(b)-f(a) > 0 \to f(b) > f(a)$

Polynomials

prob 3-7 $\to f(x) = a_n x^n + \dots + a_1 x + a_0$ has at most n roots. (1)

we proved this result using algebra. we can now prove it using the theorems of calculus.

For any two roots of f, x_1 and x_2, we have $f(x_1) = f(x_2) = 0$.

By Rolle, there is an x between x_1 and x_2 s.t. $f'(x) = 0$.

Therefore if f has k roots then f' has at least k-1 roots.

Use induction to prove (1)

$n=1 \to f(x) = a_1 x + a_0 \to$ one root $x = \dfrac{-a_0}{a_1}$.

Assume true for $n=k$. $f(x) = a_k x^k + \dots + a_1 x + a_0$ has at most k roots.

Assume $g(x) = b_{k+1} x^{k+1} + b_k x^k + \dots + b_1 x + b_0$ has more than k+1 roots.
Then $g'(x) = f(x)$ has at least k+1 roots. $\perp$.

Therefore g has at most k+1 roots.

By induction, (1) is true for all $n \in \mathbb{N}$.

Graph Sketching

Steps

1) critical points
2) value of f at critical points
3) sign of f' in regions between critical points
4) $f(x) = 0$
5) f as x becomes large or large negative

bonus: check odd/even

if f undefined at some points, check behavior of f near such points

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Theorem 5 Suppose $f'(a) = 0$. If $f''(a) > 0$, then f has local min at a; if $f''(a) < 0$ then f has local max at a.

Proof

By definition, $f''(a) = \lim_{h \to 0} \dfrac{f'(a+h) - f'(a)}{h}$

$= \lim_{h \to 0} \dfrac{f'(a+h)}{h}$

Assume $f''(a) > 0$.
Then for small $|h|$,
$h > 0 \to f'(a+h) > 0 \to f$ increasing right of a
$h < 0 \to f'(a+h) < 0 \to f$ decreasing left of a $\to a$ is local minimum

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Problematic cases

1) f'' can be very complicated for certain fns
2) $f''(a) = 0$: a could be local max, min, or neither

Theorem 6 Suppose $f''(a)$ exists. If f has local min at a, then $f''(a) \geq 0$; if f has local max at a, then $f''(a) \leq 0$.

note this is the partial converse of Th 5.

Proof

Suppose f has local min at a.
If $f''(a) < 0$ then a is local max, thus f is constant, f' constant, $f'' = 0$. $\perp$.
Therefore $f''(a) \geq 0$.

Th 5 says that if $f'(a) = 0$ and $f''(a) > 0$, then a is definitely a local min. Th 5 does not say anything about the $f''(a) = 0$ possibility, in which a could be max, min, or neither.
Th 6 says that if a is a local min then either $f''(a) < 0$ or $f''(a) = 0$.

Theorem 7 Suppose f cont. at a, $f'(x)$ exists for all x in some interval containing a, except perhaps at $x=a$. Suppose, moreover, that $\lim_{x\to a} f'(x)$ exists.

Then, $f'(a)$ exists and $f'(a) = \lim_{x\to a} f'(x)$.

Proof

By def, $f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

For sufficiently small $h>0$, f is cont. on $[a,a+h]$ and diff on $(a,a+h)$, and a similar assertion holds for small $h<0$.

MVT $\to \exists \alpha_h,\ \alpha_h \in (a,a+h)$ such that $\dfrac{f(a+h)-f(a)}{h} = f'(\alpha_h)$

α_h approaches a as h approaches 0, ie $\forall \varepsilon>0\ \exists \delta,\ 0<\delta<\varepsilon$ s.t. $|h|<\delta \to |\alpha_h - a| < |a+h-a| = |h| < \delta < \varepsilon$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} f'(\alpha_h) = \lim_{x\to a} f'(x)$$

Proof of the last step

$\lim_{h\to 0} f'(\alpha_h) = \ell$ means

$\forall \varepsilon>0\ \exists \delta>0\ \forall h\ |h|<\delta \to |f'(\alpha_h)-\ell| < \varepsilon$

Assume $|\alpha_h - a| < \delta$.

Then $|\alpha_h - a| < |a+h-a| = |h| < \delta$

Then $|f'(\alpha_h) - \ell| < \varepsilon$

T.F. $\forall x\ |x-a|<\delta \to |f'(x)-\ell| < \varepsilon$

$\forall \varepsilon>0\ \exists \delta>0\ \forall x\ |x-a|<\delta \to |f'(x)-\ell| < \varepsilon$

$\to \lim_{x\to a} f'(x) = \ell$

Theorem 8 (Cauchy Mean Value Theorem)

f, g cont. on $[a,b]$ and diff on (a,b) $\to \exists x, x \in (a,b) \wedge [f(b)-f(a)]g'(x) = [g(b)-g(a)]f'(x)$

if $g(b) \neq g(a)$ and $g'(x) \neq 0$ we can rewrite this as

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x)}{g'(x)}$$

Note

if $g(x) = x$ above, then we obtain the MVT.

If we apply MVT to f and g separately

$$f'(x) = \frac{f(b)-f(a)}{b-a}$$

$$g'(y) = \frac{g(b)-g(a)}{b-a}$$

$$\frac{f'(x)}{g'(y)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof

Let $h(x) = f(x)[g(b)-g(a)] - g(x)[f(b)-f(a)]$

Then h is cont. on $[a,b]$, diff on (a,b) and

$h(a) = f(a)[g(b)-\cancel{g(a)}] - g(a)[f(b)-\cancel{f(a)}]$

$= f(a)g(b) - g(a)f(b) = h(b)$

Apply Rolle's Theorem

$\exists x, x \in (a,b) \wedge h'(x) = 0$

$\to f'(x)[g(b)-g(a)] - g'(x)[f(b)-f(a)] = 0$

$\to \dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(x)}{g'(x)}$

Theorem 9 (L'Hôpital's Rule)

$\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

$\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists.

$\rightarrow$ $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Proof

The hypothesis that $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists contains two implicit assumptions

(1) $\exists \delta > 0, \forall x, x \in (a-\delta, a) \cup (a, a+\delta) \rightarrow f'(x)$ and $g'(x)$ exist.

(2) in $(a-\delta, a+\delta)$, $g'(x) \neq 0$ except perhaps at $x = a$

Remember that $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \ell$ means $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x\ 0 < |x-a| < \delta \rightarrow \left| \frac{f'(x)}{g'(x)} - \ell \right| < \varepsilon$

T.F. $\frac{f'(x)}{g'(x)}$ is defined in $(a-\delta) \cup (a+\delta)$, so $f'(x)$ and $g'(x)$ both exist and $g'(x) \neq 0$.

f and g, however, are not assumed to be defined at a. If we define $f(a) = g(a) = 0$ then f and g are cont. at a.

Then the assumptions of MVT and Cauchy MVT are met by f and g on $[a, x]$, where $a < x < a+\delta$.

MVT $\rightarrow \exists c, c \in (a, x) \wedge g'(c) = \frac{g(x) - g(a)}{x-a}$

If $g(x) = 0$ then $g'(c) = 0$, contradicting (2). T.F. $g(x) \neq 0$

Cauchy MVT $\rightarrow \exists \alpha_x, \alpha_x \in (a, x)$ s.t.

$[f(x) - \underbrace{f(a)}_{0}]\, g'(\alpha_x) = [g(x) - \underbrace{g(a)}_{0}]\, f'(\alpha_x)$

$\rightarrow \frac{f(x)}{g(x)} = \frac{f'(\alpha_x)}{g'(\alpha_x)}$

α_x approaches a as x approaches a.

Since $\lim_{y\to a} \frac{f'(y)}{g'(y)}$ exists by assumption, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(\alpha_x)}{g'(\alpha_x)} = \lim_{y\to a} \frac{f'(y)}{g'(y)}$

Note

$|\alpha_x - a| < |x-a|$ so $0 < |x-a| < \delta \rightarrow 0 < |\alpha_x - a| < \delta$

$\lim_{x\to a} \frac{f'(\alpha_x)}{g'(\alpha_x)} = \ell$ means $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x,\ 0 < |x-a| < \delta \rightarrow \left| \frac{f'(\alpha_x)}{g'(\alpha_x)} - \ell \right| < \varepsilon$

Assume $0 < |\alpha_x - a| < \delta$

Then $\left| \frac{f'(\alpha_x)}{g'(\alpha_x)} - \ell \right| < \varepsilon$. Therefore $0 < |\alpha_x - a| < \delta \rightarrow \left| \frac{f'(\alpha_x)}{g'(\alpha_x)} - \ell \right| < \varepsilon$

T.F. $\forall y,\ 0 < |y-a| < \delta \rightarrow \left| \frac{f'(y)}{g'(y)} - \ell \right| < \varepsilon$, Ie $\lim_{y\to a} \frac{f'(y)}{g'(y)}$

$\blacksquare$