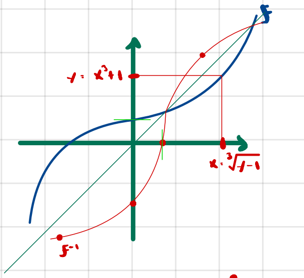
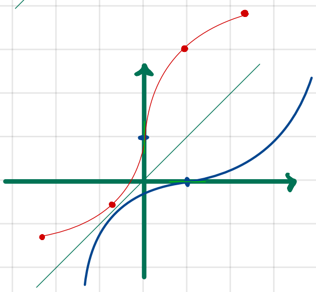
Ch. 12 - Inverse Functions

1. (i) $f(x) = x^3+1 = y$



$x^3 = y-1 \rightarrow x = \sqrt[3]{y-1}$

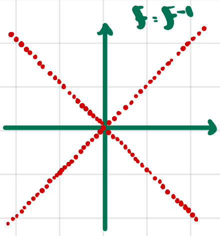
$f^{-1}(x) = \sqrt[3]{x-1}$



(ii) $f(x) = (x-1)^3 = y$

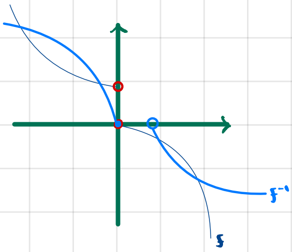
$x-1 = \sqrt[3]{y} \rightarrow x = \sqrt[3]{y}+1$

$f^{-1}(x) = 1+\sqrt[3]{x}$



(iii) $f(x) = \begin{cases} x & x \text{ rational} \\ -x & x \text{ irrational} \end{cases}$

$f(x) = f^{-1}(x)$



(iv) $f(x) = \begin{cases} -x^2 & x \geq 0 \\ 1-x^3 & x < 0 \end{cases}$

For $x \geq 0$ we have

$-x^2 = y$, so $y \leq 0$.

$x^2 = -y \rightarrow x = \sqrt{-y}$

For $x < 0$ we have

$1-x^3 = y \rightarrow x^3 = 1-y$

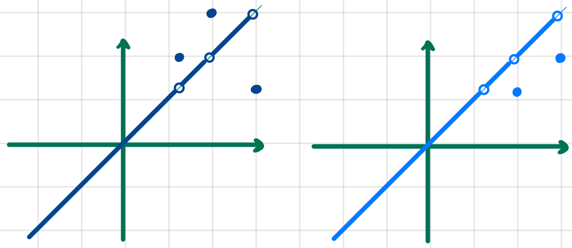
$\rightarrow x = \sqrt[3]{1-y}$

$1-y < 0 \rightarrow y > 1$

Hence we have

$f^{-1}(x) = \begin{cases} \sqrt{-x} & x \leq 0 \\ \sqrt[3]{1-x} & x > 1 \end{cases}$

(v) $f(x) = \begin{cases} x & x \neq a_1, \dots, a_n \\ a_{i+1} & x = a_i \;\; i = 1, \dots, n-1 \\ a_1 & x = a_n \end{cases}$



For $x \neq a_1, \dots, a_n$

$x = y \to f^{-1}(x) = x$

For $x = a_i$, $i = 1, \dots, n-1$

$a_{i+1} = y \to f^{-1}(x) = a_{i-1}$

For $x = a_n$

$a_1 = y \to f^{-1}(x) = a_n$

$f^{-1}(x) = \begin{cases} x & x \neq a_1, \dots, a_n \\ a_{i-1} & x = a_i, \; i = 2, \dots, n \\ a_n & x = a_1 \end{cases}$

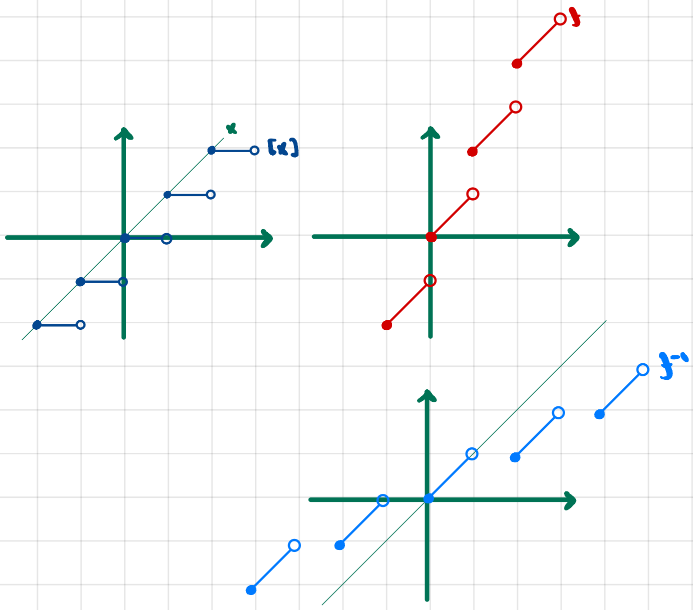
(vi) $f(x) = x + [x]$

$f(x) = x + n \quad x \in [n, n+1), n \in \mathbb{Z}$

$x + n = y \to x = y - n$

$n = 0 \to x = y$, y from 0 to 1
$n = 1 \to x = y - 1$, y from 2 to 3
$n = 2 \to x = y - 2$, y from 4 to 5

$f^{-1}(x) = x - n$, $x \in [2n, 2n+1]$ $n \in \mathbb{Z}$

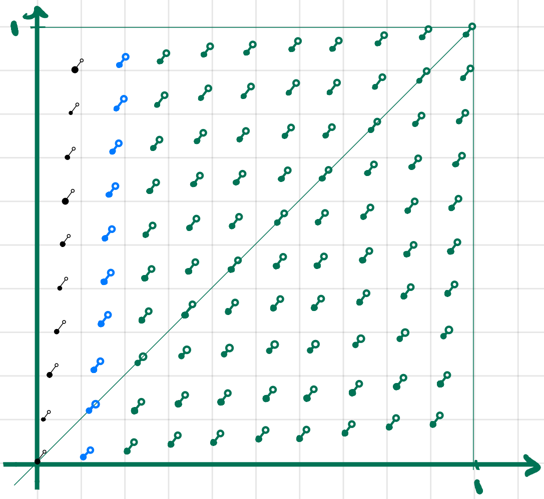


(vii) $f(0.a_1a_2a_3\ldots) = 0.a_2a_1a_3$

There are only 100 possibilities for first two decimals

0.00	0.10		0.90
0.01	0.11	(...)	0.91
0.02	0.12		0.92
...	...		...
0.09	0.19		0.99

$f^{-1} = f$.

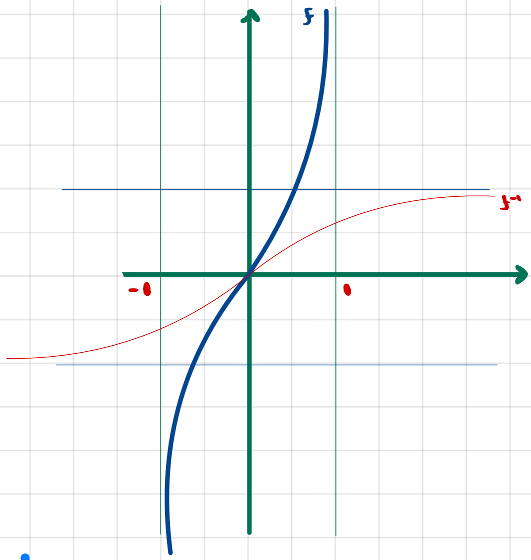


(viii) $f(x) = \dfrac{x}{1-x^2} \quad -1<x<1$

$$f'(x) = \frac{1-x^2-x(-2x)}{(1-x^2)^2} = \frac{1-x^2+2x^2}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2} > 0$$

$$f''(x) = \frac{2x(1-x^2)^{\cancel{2}} - (1+x^2)\cdot 2\cancel{(1-x^2)}(-2x)}{(1-x^2)^4} = \frac{2x-2x^3+4x+4x^3}{(1-x^2)^4}$$

$$= \frac{2x^3+6x}{(1-x)^4} = \frac{2x(x^2+3)}{(1-x)^4} \longrightarrow x=0 \text{ sole inflection point}$$



If y is an (inverse) fn of x, then x is a fn of y. $y = \dfrac{x}{1-x^2} = f(x)$ is y as fn of x. However, given y we can find x, so if we think of y as the indep. variable, then we can think of x as fn of y. We can go through the process of figuring out an expression for x. If we switch their names it can perhaps (or not) become clearer what variable we're trying to isolate.

$y = f^{-1}(x) \to x = f(y) = \dfrac{y}{1-y^2} \longrightarrow$

$\to x - xy^2 = y \to xy^2 + y - x = 0$

If we restrict f's domain to $(-1,1)$ the image is $(-\infty,\infty)$. This means the domain of f^{-1} is $(-\infty,\infty)$ and the image of f^{-1} is $(-1,1)$.
Also, $y<0 \leftrightarrow x<0$ and $y>0 \leftrightarrow x>0$

$x=0 \to y=0$

$x \neq 0 \to \Delta = 1+4x^2 > 0 \to y = \dfrac{-1 \pm \sqrt{1+4x^2}}{2x}$. Note, however, that $y = \dfrac{-1-\sqrt{1+4x^2}}{2x} < 0$ if $x>0$ and >0 if $x<0$, so this isn't the desired sol'n. In fact, note that since $\sqrt{1+4x^2} > 1$ for $x \neq 0$ then $-1-\sqrt{1+4x^2} < -2$ so $y < -2/2x = -1/x$

This incorrect sol'n represents the inverse of f for $|x|>1$.

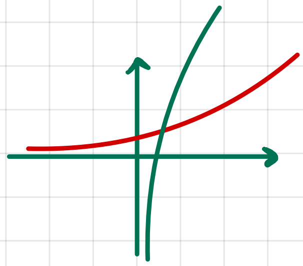
$y = f^{-1}(x) = \dfrac{-1+\sqrt{1+4x^2}}{2x}$ is the correct fn.

3.

(i) f increasing, f>0

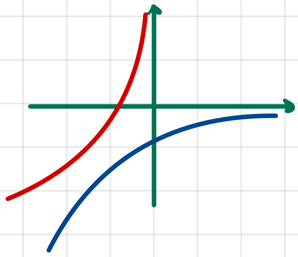
f increas. → f one-one → f^{-1} increasing fn

The image of f does not include values ≤0 so f^{-1} is not defined for x ≤0.



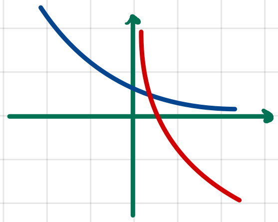
(ii) f increasing, f<0

f^{-1} is increasing fn, not defined for x ≥ 0.



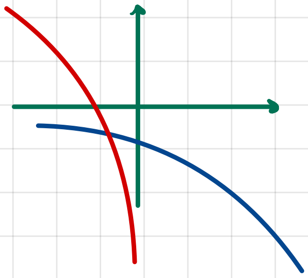
(iii) f decreasing, f>0

f^{-1} decreas. fn, undefined for x ≤0



(iv) f decreas. f<0

f^{-1} decreas. fn, undefined for x ≥0



3.

Theorem f increasing $\leftrightarrow f^{-1}$ increasing

Proof

$x_1 = f^{-1}(y_1) \longleftrightarrow y_1 = f(x_1)$

$x_2 = f^{-1}(y_2) \longleftrightarrow y_2 = f(x_2)$

Assume f increasing.

Let y_1 and y_2 be in f^{-1}'s domain.

Then for some x_1, x_2 in f's domain we have $y_1 = f(x_1)$ and $y_2 = f(x_2)$

Assume $y_1 < y_2$.
Then $f(x_1) < f(x_2)$ so $x_1 < x_2$.

But $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$.

Hence $f^{-1}(y_1) < f^{-1}(y_2)$

$y_1 < y_2 \to f^{-1}(y_1) < f^{-1}(y_2)$

$\forall y_1, y_2 \; y_1 < y_2 \to f^{-1}(y_1) < f^{-1}(y_2)$

f increasing $\to f^{-1}$ increasing

Theorem f decreasing $\to f^{-1}$ decreasing

Proof

f decreasing $\to f$ one-one $\to f^{-1}$ fn.

Let $y_1 = f(x_1)$, $y_2 = f(x_2)$ in f^{-1}'s domain.

Assume $y_1 = f(x_1) < f(x_2) = y_2$.
Then, since f decreasing, $x_1 > x_2$.

$\to f^{-1}(y_1) > f^{-1}(y_2)$

$y_1 < y_2 \to f^{-1}(y_1) > f^{-1}(y_2)$

f^{-1} decreasing

4. f, g increasing

$\forall x \forall y \; x < y \to f(x) < f(y) \wedge g(x) < g(y)$

$\to f(x) + g(x) < f(y) + g(y)$

$\to f + g$ increasing

Let $f(x) = g(x) = x$, both increasing.

Then $(f \cdot g)(x) = x^2$, decreasing for $x \leq 0$.

$\forall x \forall y \; x < y \to g(x) < g(y) \to f(g(x)) < f(g(y))$

Therefore, $f + g$ and $f \circ g$ are increasing, $f \cdot g$ not necessarily.

* $a < b$
  $c < d$

$\to a + c < b + c < b + d$
$\to a + c < b + d$ ✓

$a - c < b - c$ but $b - c > b - d$
we cannot establish a relationship between $a - c$ and $b - d$.

5.

(a) f, g one-one $\to f \circ g$ one-one

Proof

$\forall x \forall y \; x \neq y \to g(x) \neq g(y) \to f(g(x)) \neq f(g(y))$

$\to f \circ g$ one-one

■

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[rr, bend left, "f \circ g"] & g(x) \arrow[r, "f"] & f(g(x)) \end{tikzcd}$$

$f^{-1}(f(g(x))) = g(x)$

$g^{-1}(f^{-1}(f(g(x)))) = g^{-1}(g(x)) = x$

Hence if $h(x) = (g^{-1} \circ f^{-1})(x)$ then

$h(f(g(x))) = (g^{-1} \circ f^{-1}) \circ (f \circ g) = I$

and therefore

$g^{-1} \circ f^{-1} = (f \circ g)^{-1}$

Slightly different perspective

Let $y = (f \circ g)^{-1}(x)$

Then $x = (f \circ g)(y) = f(g(y))$

But then $g(y) = f^{-1}(x)$

$y = g^{-1}(f^{-1}(x)) = (g^{-1} \circ f^{-1})(x)$

(b) $g(x) = 1 + f(x)$

$y = g^{-1}(x) \to x = g(y) = 1 + f(y)$

$f(y) = x - 1$

$y = f^{-1}(x-1)$

$\to g^{-1}(x) = f^{-1}(x-1)$

Another path is

$h(x) = 1 + x$

$x = 1 + y \to y = x - 1 \to h^{-1}(x) = x - 1$

$g(x) = (h \circ f)(x) = h(f(x)) = 1 + f(x)$

$$\begin{aligned} g^{-1}(x) = (h \circ f)^{-1}(x) &= (f^{-1} \circ h^{-1})(x) \\ &= f^{-1}(h^{-1}(x)) \\ &= f^{-1}(x-1) \end{aligned}$$

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6. $f(x) = \dfrac{ax+b}{cx+d}$ is one-one $\leftrightarrow$ $ad - bc \neq 0$

Proof

$f(x) = \dfrac{ax+b}{cx+d}$ is one-one

$\leftrightarrow \forall x \forall y \; x \neq y \leftrightarrow \dfrac{ax+b}{cx+d} \neq \dfrac{ay+b}{cy+d} \leftrightarrow a\cancel{cxy} + adx + bcy + \cancel{bd} \neq a\cancel{cxy} + bcx + ady + \cancel{bd}$

$\leftrightarrow adx + bcy \neq bcx + ady$

$\leftrightarrow bc(y-x) \neq ad(y-x)$

$\leftrightarrow bc \neq ad$

Let $y = f^{-1}(x)$.

Then $x = f(y) = \dfrac{ay+b}{cy+d}$

$cxy + xd = ay + b$

$y(cx - a) = b - xd$

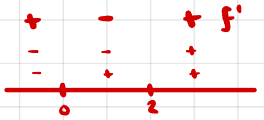
$y = f^{-1}(x) = \dfrac{b - xd}{cx - a} \quad x \neq \dfrac{a}{c}$

7.

(i) $f(x) = x^3 - 3x^2 = x^2(x-3)$

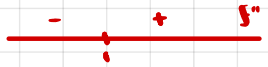
roots at 0 and 3

$f'(x) = 3x^2 - 6x = 3x(x-2)$



critical points at 0 and 2

$f''(x) = 6x - 6 = 0 \rightarrow x = 1$



From f' we see that f is

increasing on $(-\infty, 0]$

decreasing on $[0, 2]$

increasing on $[2, +\infty)$

It is one-one on these three intervals, or subintervals of each.

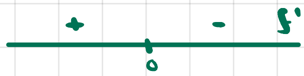
(ii) $f(x) = x^5 + x$

$f'(x) = 5x^4 + 1 > 0$

f increasing on $\mathbb{R}$. f one-one on any interval.

(iii) $f(x) = (1+x^2)^{-1}$

$f'(x) = \dfrac{(-1) \cdot 2x}{(1+x^2)^2}$

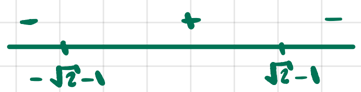


f one-one on $(-\infty, 0]$ and $[0, +\infty)$ or subintervals of either.

(iv) $f(x) = \dfrac{x+1}{x^2+1}$

$f'(x) = \dfrac{x^2+1-2x(x+1)}{(x^2+1)^2} = \dfrac{x^2+1-2x^2-2x}{(x^2+1)^2} = \dfrac{-x^2-2x+1}{(x^2+1)^2}$

$\Delta = 4 - 4(-1) \cdot 1 = 8 \rightarrow x = \dfrac{2 \pm 2\sqrt{2}}{-2} = -1 \pm \sqrt{2}$



f one-one on $(-\infty, -\sqrt{2}-1]$, $[-\sqrt{2}-1, \sqrt{2}-1]$, $[\sqrt{2}-1, \infty)$

8. f diff

$$f'(x) = \frac{1}{\sqrt{1+x^3}} \quad \rightarrow \quad g = f^{-1} \text{ satisfies } g''(x) = \frac{3}{2} g(x)^2$$

Proof

$g(x) = f^{-1}(x)$, that is g is simply the inverse of f.

Therefore $g'(x) = (f^{-1})'(x)$
$g''(x) = (f^{-1})''(x)$

f^{-1} is differentiable at every x in its domain because f is cont. (since it is diff.) and one-one (since f^{-1} is a function).

Theorem 5 $\rightarrow (f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))} = [1 + (f^{-1}(x))^3]^{1/2} = g'(x)$

$$g''(x) = \frac{1}{2}[1+(f^{-1}(x))^3]^{-\frac{1}{2}} \cdot 3(f^{-1}(x))^2 \cdot (f^{-1})'(x)$$

$$= \frac{3}{2} \cdot (f^{-1}(x))^2 \cdot \frac{(f^{-1})'(x)}{\sqrt{1+(f^{-1}(x))^3}}$$

But $(f^{-1})'(x) = \sqrt{1+(f^{-1}(x))^3}$

Therefore, $g''(x) = \dfrac{3}{2}(f^{-1}(x))^2$

$= \dfrac{3}{2} g(x)^2$

Theorem 5 Let f be cont. one-one fn defined on interval.
Suppose f diff at $f^{-1}(b)$, w/ derivative $f'(f^{-1}(b)) \neq 0$.
Then,
f^{-1} diff. at b and $(f^{-1})'(b) = \dfrac{1}{f'(f^{-1}(b))}$

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9. f one-one
$(f^{-1})' \neq 0 \quad \rightarrow f$ differentiable

Proof

f one-one $\rightarrow f^{-1}$ fn

f fn $\rightarrow f^{-1}$ one-one

$(f^{-1})' \neq 0$ implies f^{-1} diff. everywhere.

Let x be in f's domain.

Then since $(f^{-1})'(f^{-1}(x)) \neq 0$

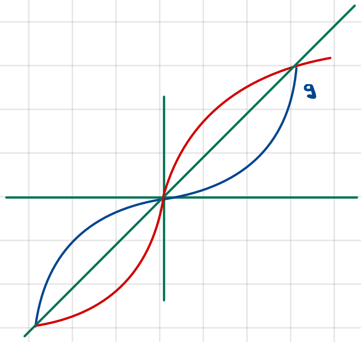
Theorem 5 says that f diff. at x and

$$f'(x) = \frac{1}{(f^{-1})'(f^{-1}(x))}$$

10.

Recall problem 10-17

17. f and g fns
g takes on all values
$f\circ g$ and g diff.
f not diff.



$g(x) = x^3$, diff $\forall x$

$f(x) = \sqrt[3]{x}$

$f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$, undefined at $x=0$

$(f\circ g)(x) = x$

In light of the current chapter, it just so happens that the fns chosen in 10-17 are inverses

$y = g(x) = x^3$

$x = g^{-1}(y) \to y = g^{-1}(y)^3 \to g^{-1}(y) = \sqrt[3]{y} = f(y).$

Since $g'(x) = 3x^2 = 0 \to x = 0$, g^{-1} is not diff at 0.

If we choose a fn g that has derivative $\neq 0$ everywhere, then if $f = g^{-1}$, f is diff. everywhere and $f\circ g = g^{-1}\circ g = I$, which is diff.

Another way to see this

$f = (f\circ g)\circ g^{-1}$

The Chain Rule says that if g^{-1} diff at a and $f\circ g$ diff at $g^{-1}(a)$ then $(f\circ g)\circ g^{-1}$ diff at a.

$f\circ g$ is diff everywhere by assumption, so the only additional condition we need is that g^{-1} diff, which happens when

$g' \neq 0$ for all a.

11. $(f^{-1})''(x)$

Let $h(x) = f^{-1}(x)$

$$h'(x) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$h''(x) = (f^{-1})''(x) = \frac{-[f'(f^{-1}(x))]'}{[f'(f^{-1}(x))]^2}$$

$$= \frac{-[f''(f^{-1}(x)) \cdot (f^{-1})'(x)]}{[f'(f^{-1}(x))]^2}$$

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12. $f'(f^{-1}(x)) \neq 0$

$\rightarrow$ $(f^{-1})^{(n)}(x)$ exists

$f^{(n)}(f^{-1}(x))$ exists

Proof

$f'(f^{-1}(x)) \neq 0$ means that for any x in f^{-1}'s domain (ie f's image), the derivative at the corresponding element $f^{-1}(x)$ in f's domain is $\neq 0$.

Now f is continuous since f' is assumed to exist, and one-one since f^{-1} is a fn.
By Th. 5, $(f^{-1})'(x)$ exists.

Let $x < y$.
Assume $f^{(k)}(x) \geq f^{(k)}(y)$.

$n = 1$. The assumptions are that $f'(f^{-1}(x))$ exists and is $\neq 0$. By Th. 5, $(f^{-1})'(x) = 1/f'(f^{-1}(x))$

Assume the statement is true for $n = k$.

Since $f^{(k)}(f^{-1}(x))$ exists so does $f^{(m)}(f^{-1}(x))$ for $1 \leq m < k$.

From $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

$$(f^{-1})''(x) = \frac{-[f''(f^{-1}(x)) \cdot (f^{-1})'(x)]}{[f'(f^{-1}(x))]^2}$$ (by problem 11)

13.

Recall

19. f, f', f'', f''' exist, ie f three times differentiable.

$f'(x) \neq 0$

Schwarzian derivative of f: $\mathcal{D}f(x) = \dfrac{f'''(x)}{f'(x)} - \dfrac{3}{2}\left(\dfrac{f''(x)}{f'(x)}\right)^2$

(a) $\mathcal{D}f(x)$ exists for all x $\rightarrow$ $\mathcal{D}f^{-1}(x)$ also exists for all x in f^{-1}'s domain

Proof

Assume $\mathcal{D}f(x)$ exists for all x.

$*$ $[\forall x, x \text{ in } f^{-1}\text{'s domain} \rightarrow f^{-1}(x) \neq 0]$

$\leftrightarrow [\forall y, y \text{ in } f\text{'s domain} \rightarrow f^{-1}(f(y)) \neq 0]$

Then for all x,

(1) f is three times diff

(2) $f'(x) \neq 0$

(3) $\mathcal{D}f(x) = \dfrac{f'''(x)}{f'(x)} - \dfrac{3}{2}\left(\dfrac{f''(x)}{f'(x)}\right)^2$

(2) $\rightarrow (f^{-1})'(x)$ exists for all x in f^{-1}'s domain and $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))} \neq 0$

Now we use problem 12

Since

$f'(f^{-1}(x)) \neq 0$ and $f''(f^{-1}(x)), f'''(f^{-1}(x))$ exist, then $(f^{-1})''(f^{-1}(x))$ and $(f^{-1})'''(f^{-1}(x))$ exist.

At this point we have

$(f^{-1})'(x) \neq 0$, for all x in f^{-1}'s domain

f^{-1} three times diff. on entire domain of f^{-1}

Therefore, the Schwarzian derivative of f^{-1} exists for all x in f^{-1}'s domain.

(b) Formula for $Df^{-1}(x)$

$(f \circ f^{-1}) = I$

$(f \circ f^{-1})(x) = x$

$D(f \circ f^{-1}) = [Df \circ f^{-1}] \cdot [(f^{-1})']^2 + Df^{-1}$

$\to Df^{-1} = \cancel{DI}^{0} - [Df \circ f^{-1}] \cdot [(f^{-1})']^2$

$$Df^{-1}(x) = -\left[Df(f^{-1}(x)) \cdot \frac{1}{f'(f^{-1}(x))^2} \right]$$

$$\to Df^{-1}(x) = \frac{Df(f^{-1}(x))}{[f'(f^{-1}(x))]^2}$$

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14. (a) $\exists f$, f differentiable $\wedge$ $f(x)^5 + f(x) + x = 0$ for all x

Proof

$f(x)^5 + f(x) + x = 0$ is an eq. for x in domain of f.

If f has an inverse, then $x = f^{-1}(y)$.
Hence

$y^5 + y + f^{-1}(y) = 0$

$f^{-1}(y) = -y^5 - y$

$(f^{-1})'(y) = -5y^4 - 1 < 0$

$\to f^{-1}$ is decreasing and hence one-one $\to f$ decreasing, one-one

f^{-1} one-one $\to f$ is a fn

f^{-1} decreasing $\to f$ decreas.

f^{-1} takes on all values, ie the image of f^{-1} is $\mathbb{R}$.
Hence, the domain of f is $\mathbb{R}$.

Since $(f^{-1})'(x) \neq 0$ for all x, f' exists for all x and equals

(b) $$f'(x) = \frac{1}{(f^{-1})'(f(x))} = \frac{1}{-5f(x)^4 - 1}$$ (Theorem 5)

(c) $5f(x)^4 f'(x) + f'(x) + 1 = 0$

$f'(x)(5f(x)^4 + 1) = -1$

$$f'(x) = \frac{1}{-5f(x)^4 - 1}$$

Slightly different

Let $g(x) = -x^5 - x$ be a fn.

g is decreasing, one-one and $\mathrm{Im}(g) = \mathbb{R}$.

Therefore

g one-one $\to f = g^{-1}$ is a fn

domain $f = \mathbb{R}$

$g(f(x)) = g(g^{-1}(x)) = x = -f(x)^5 - f(x)$

$g'(x) = -5x^4 - 1 \neq 0$ for all x, hence f' exists for all x and equals

$$f'(x) = \frac{1}{g'(f(x))} = \frac{1}{-5f(x)^4 - 1}$$

15. (a) $x^2+y^2=1$ (1)

Two functions $y=f(x)$ are defined implicitly by (1) on $x \in (-1,1)$.

$y^2 = 1-x^2$

$y = \pm\sqrt{1-x^2}$

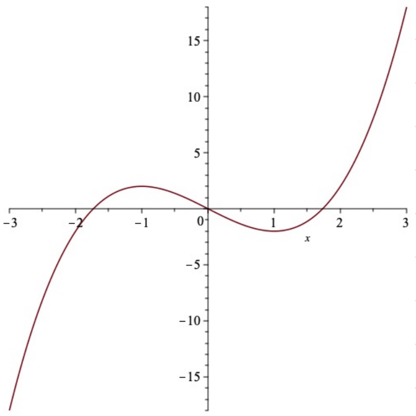
For a given x in $(0,1)$, eq. (1) is satisfied whether we plug in $\sqrt{1-x^2}$ or $-\sqrt{1-x^2}$.

(b) $x^2+[f(x)]^2=-1$ (1)

There is no function satisfying (1)

(c) $[f(x)]^3-3f(x)=x$ (1)

Consider $g(x)=x^3-3x$



If g^{-1} exists then $g(g^{-1}(x)) = x = [g^{-1}(x)]^3-3g^{-1}(x)$.

Ie g^{-1} satisfies (1). Since g isn't one-one on $\mathbb{R}$, there isn't a g^{-1} on $\mathbb{R}$.

However, since $g'(x)=3x^2-3=0 \to x=\pm1$, g has local max at -1, local min at 1, and is one-one on $(-\infty,-1)$, $(-1,1)$, and $(1,+\infty)$. Hence there is an inverse fn on each interval. Also $g'(x)\neq 0$ in these intervals.

So if we define

$g_1(x) = g(x),\ x \in (-\infty,-1)$
$g_2(x) = g(x),\ x \in (-1,1)$
$g_3(x) = g(x),\ x \in (1,+\infty)$

Then $g_i(g_i^{-1}(x)) = x = [g_i^{-1}(x)]^3-3g_i^{-1}(x)$ (2)

In order to obtain an explicit expression for $g_i^{-1}(x)$ we'd need to solve (2).
Note that since $g_i'(x)\neq 0$, g_i^{-1} is differentiable in its domain.

16. (a) $[f(x)]^2 + x^2 = 1$

$2f(x)f'(x) + 2x = 0$

$f'(x) = \frac{-2x}{2f(x)} = -\frac{x}{f(x)}$

b) $f(x) = \sqrt{1-x^2} \to f'(x) = -\frac{x}{\sqrt{1-x^2}}$

$f(x) = \sqrt{1+x^2} \to f'(x) = -\frac{x}{\sqrt{1+x^2}}$

(c) $[f(x)]^3 - 3f(x) = x$

$3f(x)^2 f'(x) - 3f'(x) = 1$

$3f'(x)(f(x)^2 - 1) = 1$

$f'(x) = \frac{1}{3(f(x)^2 - 1)}$

---

17. (a) $x^3 + y^3 = 7$

$y = \sqrt[3]{7-x^3} = f(x)$. We could differentiate directly here.

Or, if we assume $y = f(x)$ we can diff. implicitly.

$3x^2 + 3y^2 y' = 0 \quad \to \quad y' = f'(x) = \frac{-x^2}{f(x)^2}$

$2x + 2yy' \cdot y' + y^2 y'' = 0$

$y'' = f''(x) = \frac{-2(x + yy'^2)}{y^2} = \frac{-2(x + f(x)f'(x)^2)}{(f(x))^2}$

(b) $f(-1) = 2$

$f'(-1) = \frac{-(-1)}{4} = -\frac{1}{4}$

$f''(-1) = \frac{-2(-1 + 2 \cdot \frac{1}{16})}{4} = \frac{2 - \frac{1}{4}}{4} = \frac{7}{16}$

18. $3x^3 + 4x^2 y - xy^2 + 2y^3 = 4$

Eq. of tangent at $(-1, 1)$?

Assume $y = f(x)$.

$9x^2 + 8xy + 4x^2 y' - y^2 - x \cdot 2yy' + 6y^2 y' = 0$

Let's already sub in $x = -1, y = 1$.

$9 - 8 + 4y' - 1 + 2y' + 6y' = 0$

$12y' = 0 \quad \to \quad f'(-1) = 0$

Tangent line: $g(x) = 1$

---

19. $y^4 + y^3 + xy = 1$

$4f^3 f' + 3f^2 f' + f + xf' = 0$

$f' = \frac{-f}{4f^3 + 3f^2 + x}$

20.

Theorem 5 Let f be cont. one-one fn defined on interval.

Suppose f diff at $f^{-1}(b)$, w/ derivative $f'(f^{-1}(b)) \neq 0$.

Then,

f^{-1} diff. at b and $\left.\frac{dx}{dy}\right|_{y=b} = \frac{1}{\left.\frac{dy}{dx}\right|_{x=f^{-1}(b)}}$

$x = y^n$ x explicitly fn of y

$\frac{1}{n} - 1 = \frac{1-n}{n}$

$y = x^{1/n}$ y explicitly fn of x

$\frac{dx^{1/n}}{dx} = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{ny^{n-1}} = \frac{1}{nx^{\frac{n-1}{n}}} = \frac{1}{nx^{1-\frac{1}{n}}} = nx^{\frac{1}{n}-1}$

diff both sides

inverse fn deriv.

21. f diff, one-one

$f' \neq 0$ $\quad\rightarrow G'(x) = f^{-1}(x)$

$f = F'$

$G(x) = x f^{-1}(x) - F(f^{-1}(x))$

Proof

$G'(x) = f^{-1}(x) + \cancel{x(f^{-1}(x))'} - \cancel{f(f^{-1}(x))(f^{-1})'(x)}$

$= f^{-1}(x)$

22. $h'(x) = \sin^2(\sin(x+1))$

$h(0) = 3$

(i) $(h^{-1})'(3)$

$h^{-1}(x) = 0$ for $x = k\pi - 1$, $k = -1, 0, 1, 2, \ldots$

$h(0) = 3 \rightarrow h^{-1}(3) = 0$

$h'(h^{-1}(3)) = h'(0) = \sin^2(\sin 1) \neq 0$

$\rightarrow (h^{-1})'(3) = \dfrac{1}{h'(h^{-1}(3))} = \dfrac{1}{\sin^2(\sin 1)}$

(ii) $\beta(x) = h(x+1)$

$(\beta^{-1})'(3)$

$\beta(-1) = h(0) = 3$

$\rightarrow \beta^{-1}(3) = -1$

$\beta'(x) = h'(x+1)$

$(\beta^{-1})'(x) = \dfrac{1}{\beta'(\beta^{-1}(x))}$

$(\beta^{-1})'(3) = \dfrac{1}{\beta'(\beta^{-1}(3))}$

$= \dfrac{1}{\beta'(-1)}$

$= \dfrac{1}{h'(0)}$

$= \dfrac{1}{\sin^2(\sin 1)}$

23. (a) An increasing and a decreasing fn intersect at most once.

Proof

Let f be increasing and g decreasing.

Assume they intersect at two points, $(x_1, f(x_1)) = (x_1, g(x_1))$ and $(x_2, f(x_2)) = (x_2, g(x_2))$, $x_1 < x_2$.

Case 1: $f(x_1) < f(x_2)$.
Then $g(x_1) < g(x_2)$. $\perp$.

Case 2: $f(x_1) > f(x_2)$. $\perp$.

Case 3: $f(x_1) = g(x_1) = f(x_2) = g(x_2)$. $\perp$.

$\perp$.

Therefore f and g intersect at most once

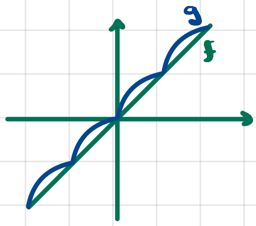
Alternatively

Assume f and g intersect at a

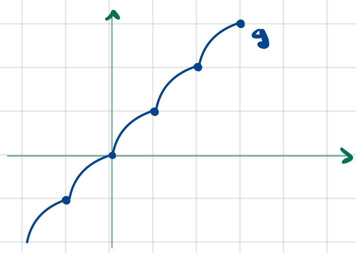
$f(a) = g(a)$

$\forall x,\ x > a \rightarrow f(x) > f(a) > g(x)$
$\quad x < a \rightarrow f(x) < f(a) < g(a)$

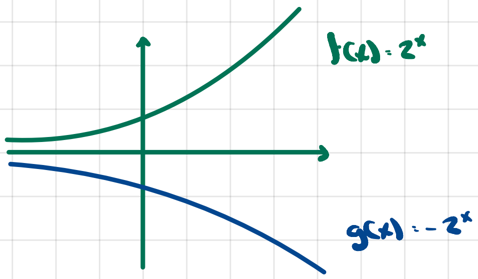
(b) f, g cont., increas.
$f(x) = g(x)$ for $x \in \mathbb{Z}$



$f(x) = x$
$g(x) = [x] + \sqrt{x - [x]}$



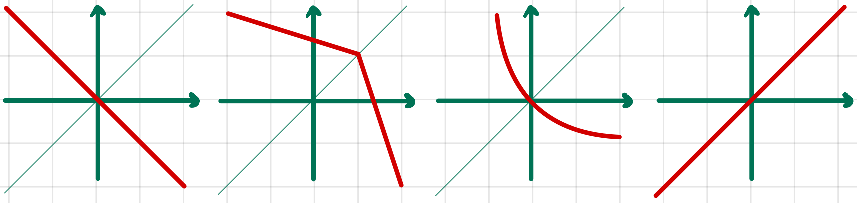
(c) f cont. increas., g cont. decreas., f and g don't intersect



24. (a) f cont on $\mathbb{R}$, $f = f^{-1}$ $\rightarrow$ $\exists x, f(x) = x$

Note

Geometrically, $f = f^{-1}$ means that these functions are symmetric relative to $y = x$.



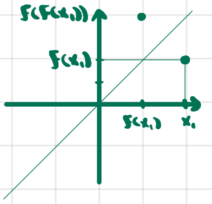
They all seem to be decreasing fns, other than $f(x) = x$.
Let's prove that they must be, unless we have $f(x) = x$.

Assume $f = f^{-1}$.

Assume f increasing.

Let $(x_1, f(x_1))$ be a point on f's graph.

If $f = f^{-1}$ then $f^{-1}(f(x_1)) = f(f(x_1)) = x_1$

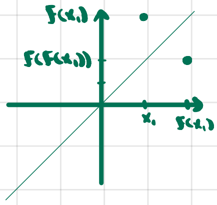


Assume $x_1 > f(x_1)$

Then $f^{-1}(f(x_1)) = f(f(x_1)) = x_1 > f(x_1)$.

We have $f(x_1) < x_1 \wedge f(f(x_1)) > f(x_1)$

$\perp$, because f increasing



Assume $x_1 < f(x_1)$

Then $f^{-1}(f(x_1)) = x_1 = f(f(x_1)) < f(x_1)$

We have $x_1 < f(x_1) \wedge f(x_1) > f(f(x_1))$

$\perp$, because f increasing

Therefore $x_1 = f(x_1)$

f increasing $\rightarrow f(x) = x$

f decreasing or $f(x) = x$

Now let's prove our main result

Proof

Assume f cont. and $f = f^{-1}$.

Then either f decreasing or $f(x) = f(x)$

Case 1: $f(x) = x$

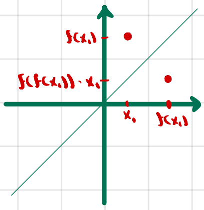
T.F. $\exists x, f(x) = x$

Case 2: f decreasing

Let $(x_1, f(x_1))$ be a point on f's graph.

Then $f^{-1}(f(x_1)) = f(f(x_1)) = x_1$

Case 2.1: $x_1 < f(x_1)$



Then

$x_1 < f(x_1) \wedge f(x_1) > f(f(x_1))$

Let $g(x) = f(x) - x$.

Then $g(x_1) = f(x_1) - x_1 > 0$

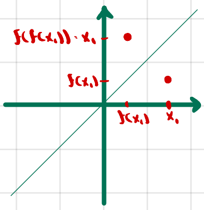
$g(f(x_1)) = f(f(x_1)) - f(x_1)$

$= x_1 - f(x_1) < 0$

Since g is cont, IVT $\rightarrow$ g is 0 for some $x \in (x_1, f(x_1))$

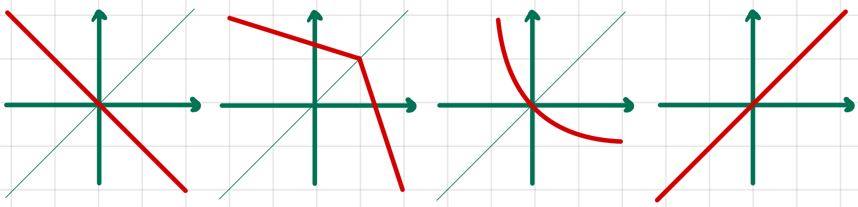
ie $f(x) = x$.

Case 2.2: $x_1 > f(x_1)$



Analogous to case 2.1.

(b)



(c)

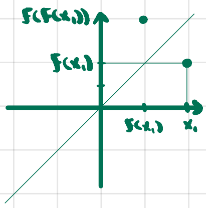
Assume $f = f^{-1}$.

Assume f increasing.

Let $(x_1, f(x_1))$ be a point on f's graph.

If $f = f^{-1}$ then $f^{-1}(f(x_1)) = f(f(x_1)) = x_1$

Assume $x_1 > f(x_1)$

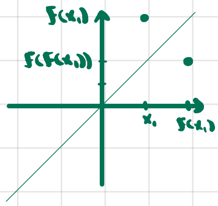


Then $f^{-1}(f(x_1)) = f(f(x_1)) = x_1 > f(x_1)$.

We have $f(x_1) < x_1 \wedge f(f(x_1)) > f(x_1)$

$\perp$, because f increasing

Assume $x_1 < f(x_1)$



Then $f^{-1}(f(x_1)) = x_1 = f(f(x_1)) < f(x_1)$

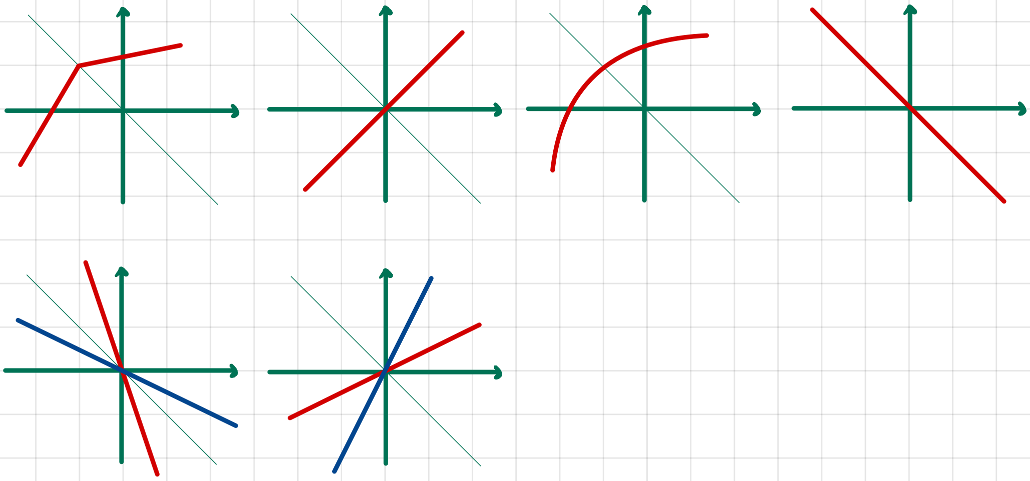
We have $x_1 < f(x_1) \wedge f(x_1) > f(f(x_1))$

$\perp$, because f increasing

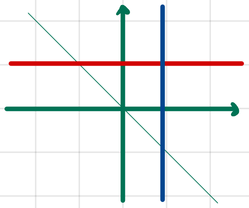
Therefore $x_1 = f(x_1)$

f increasing $\rightarrow f(x) = x$

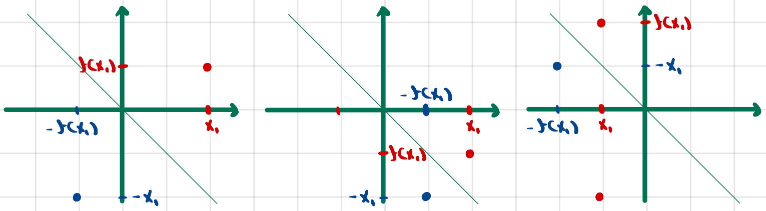
25.



Examples that don't work



From the graphs, it seems that f must be one-one



f one-one $\rightarrow$ f reflected through antidiagonal is a fn.

Proof

Let $(x_1, f(x_1))$ be a point.
Since f is one-one, for any other $(x, f(x))$ we have $x \neq x_1 \leftrightarrow f(x) \neq f(x_1)$ (1)

Consider the fn represented by points $(-f(x_1), -x_1)$, $x_1 \in$ domain of f.

The domain of this fn is the image of f.

It is in fact a fn because of (1): $f(x) = f(x_1) \rightarrow x = x_1$.

It is one-one also because of (1): $f(x) \neq f(x_1) \rightarrow x \neq x_1$.

26.

(a) f non-decreasing, not increasing $\to$ f constant on some interval

Proof

f not increasing means $\exists x, y$ in domain, $x < y \wedge f(x) \geq f(y)$ (1)

f non-decreasing means $\forall x, y$ $x < y \to f(x) \leq f(y)$ (2)

Therefore, $\exists x, y$ in domain, $x < y \wedge f(x) = f(y)$.

let x, y be such that they satisfy $x < y \wedge f(x) = f(y)$

consider the interval (x, y).

For any $x_1 \in (x, y)$, if $f(x_1) > f(x) = f(y)$ then f decreasing between x_1 and y. $\perp$.
if $f(x_1) < f(x) = f(y)$ " " decreasing " x " x_1. $\perp$.

T.F. $\forall x_1 \in (x, y) \to f(x_1) = f(x) = f(y)$

$\to$ f constant on (x, y).

(b) f diff. and non-decreasing $\to$ $f'(x) \geq 0$ for all x.

Proof

f non-decreasing means $\forall x, y$ $x < y \to f(x) \leq f(y)$

$\forall a$, $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a} = \lim_{x \to a^+} \dfrac{f(x) - f(a)}{x - a}$

But $\dfrac{f(x) - f(a)}{x - a} \geq 0$ for $x \to a^+$.

Therefore $f'(a) \geq 0$.

(c) $f'(x) \geq 0$ for all x $\to$ f is nondecreasing

Proof

let x, y in f's domain, $x < y$.

MVT $\to$ $\exists c, c \in (x, y) \wedge \dfrac{f(y) - f(x)}{y - x} = f'(c) \geq 0$

$\to f(y) \geq f(x)$

T.F. $\forall x, y$ in domain, $x < y \to f(x) \leq f(y)$

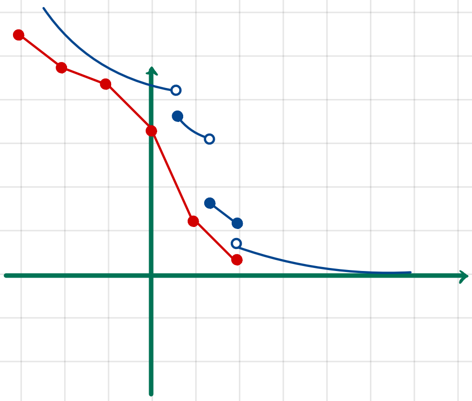
$\to f$ nondecreasing.

27.

(a) $f(x) > 0$ for all x
f decreasing
$\to$ $\exists g$, g continuous and decreasing s.t. $0 < g(x) \leq f(x)$ for all x

Proof

In the interval $[n, n+1]$, let g be the linear fn such that $g(n) = f(n+1)$ and $g(n+1) = f(n+2)$.



Let $x \in [n, n+1]$.

$g(n) = f(n+1)$

$\forall x, x \in [n, n+1] \to f(x) > f(n+1) = g(n) \geq g(x)$

(b) $\lim_{x \to \infty} \frac{g(x)}{f(x)} = 0$

On $[n, n+1]$, let g be the linear fn such that $g(n) = \frac{f(n+1)}{n+1}$ and $g(n+1) = \frac{f(n+2)}{n+2}$

Then since $\lim_{x \to \infty} f(x)$ must be some number ≥ 0, and $\lim_{n \to \infty} (n+1) = \infty$, we have $\lim_{x \to \infty} \frac{g(x)}{f(x)} = 0$