Ch. 11 Appendix - Convexity and Concavity

1.

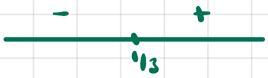
(i) $f(x) = x^3 - x^2 - 8x + 1$ on $[-2,2]$

Plan

- Find f' and f''.
- Find x s.t. $f''(x) = 0$
- Find sign of f'' at each x in domain
- Points at which f'' changes sign are inflection points

$f'(x) = 3x^2 - 2x - 8$

$f''(x) = 6x - 2 = 0 \rightarrow x = \frac{1}{3}$



f is convex in $[1/3, +\infty)$, concave in $(-\infty, 1/3]$.

$1/3$ is inflection point.

$f(1/3) = -47/27$

recall

$f'(x) = 3x^2 - 2x - 8 = 0$

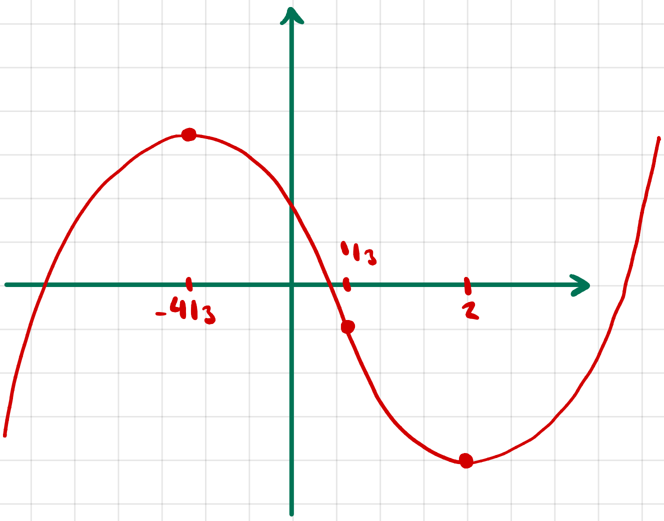
$\Delta = 4 + 96 = 100$

$x = \frac{2 \pm 10}{6}$ → 2, $-\frac{4}{3}$

$f(2) = -11$ min

$f(-4/3) = \frac{203}{27}$ max

$f(-2) = 5$

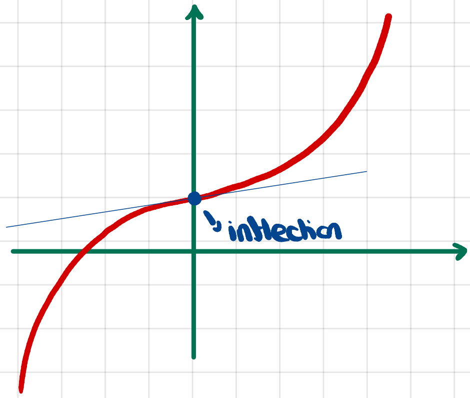


(ii) $f(x) = x^5 + x + 1$

$f'(x) = 5x^4 + 1 > 0$

$\rightarrow$ no critical points

$f''(x) = 20x^3$ $\quad$ $-$ $\;\; 0 \;\;$ $+$



(iii) $f(x) = 3x^4 - 8x^3 + 6x^2$

$f'(x) = 12x^3 - 24x^2 + 12x = 0$

$\rightarrow 12x(x^2 - 2x + 1) = 0$

$f'(x) = 12x(x-1)^2$

critical points: $0, 1$

$f(0) = 0$

$f(1) = 3 - 8 + 6 = 1$

$f''(x) = 36x^2 - 48x + 12$

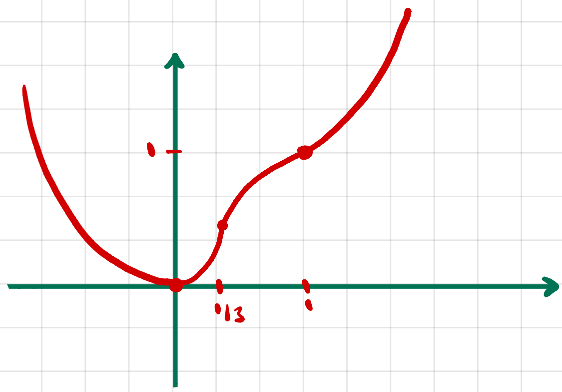
$= 12(3x^2 - 4x + 1)$

$\Delta = 16 - 12 = 4 \quad \rightarrow x = \frac{4 \pm 2}{6} \nearrow 1, \searrow 1/3$

$+$ $\quad$ $-$ $\quad$ $+$

0 $\quad$ $1/3$ $\quad$ 1

local min $\quad$ inflection $\quad$ inflection



$f(1/3) = 11/27 \approx 0.4$

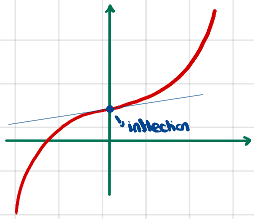
(iv) $f(x) = \dfrac{1}{x^5+x+1}$

$f'(x) = \dfrac{-1}{(x^5+x+1)^2}(5x^4+1)$

Let $g(x) = x^5+x+1$ and $h(x) = 5x^4+1$.

$h(x) \geq 0$

g(x) looks like



$g'(x) = 5x^4+1 > 0$

Therefore,

f and f' are never 0

$f' < 0$

$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$

$\lim_{x\to\infty} f'(x) = \lim_{x\to-\infty} f'(x) = 0$

$g(x_0) = 0 \to \lim_{x\to x_0^+} f(x) = +\infty$

$\lim_{x\to x_0^-} f(x) = -\infty$

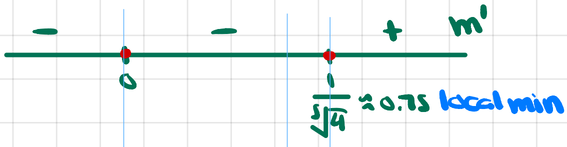
$\lim_{x\to x_0^+} f'(x) = \lim_{x\to x_0^-} f'(x) = -\infty$

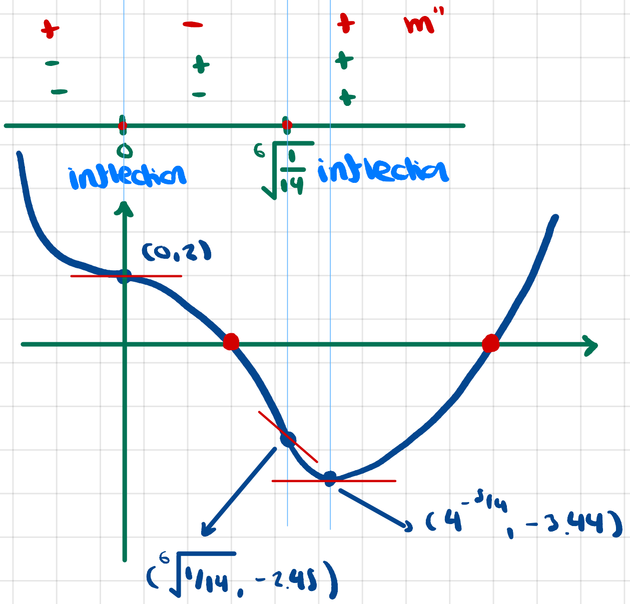
$m''(x) = 0 \to x(1680x^6 - 120) = 0$

$\to x = \sqrt[6]{\dfrac{120}{1680}} = \sqrt[6]{\dfrac{1}{14}} \approx 0.64$

$\lim_{x\to\pm\infty} m(x) = \infty$

we can determine the min of m on $\mathbb{R}$ by looking at critical points.





m'' has two roots → f'' has two roots

$$f''(x) = \frac{-20x^3(x^5+x+1)^{\cancel{2}} - (-5x^4-1)\cdot 2\cancel{(x^5+x+1)}(5x^4+1)}{(x^5+x+1)^{\cancel{4}\,3}}$$

$$= \frac{-20x^8 - 20x^4 - 20x^3 - (-10x^4-2)(5x^4+1)}{(x^5+x+1)^3}$$

$$= \frac{-20x^8 - \cancel{20x^4} - 20x^3 - (-50x^8 - \cancel{10x^4} - \cancel{10x^4} - 2)}{(x^5+x+1)^3}$$

$$= \frac{30x^8 - 20x^3 + 2}{(x^5+x+1)^3}$$

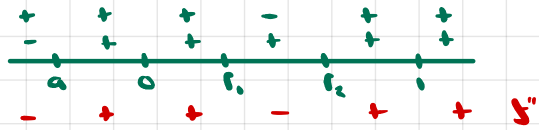
we'd like to determine the sign of f''.

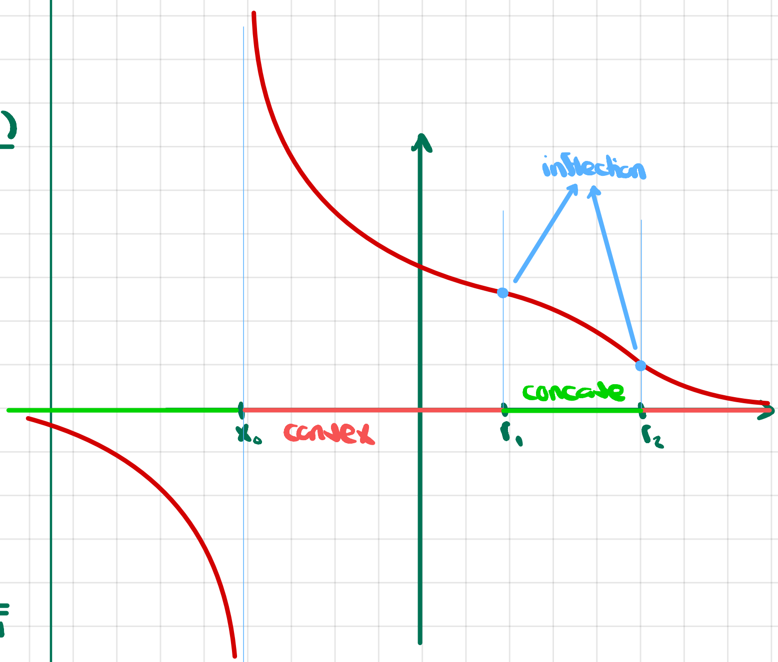
Let $m(x) = 30x^8 - 20x^3 + 2$

Then $m'(x) = 240x^7 - 60x^2 = 0 \to 60x^2(4x^5-1) = 0$ → $x = 0$, $x = \dfrac{1}{\sqrt[5]{4}}$

$m''(x) = 7\cdot 240x^6 - 120x$

Let a be the unique root of x^5+x+1. Then





(v) $f(x) = \frac{x+1}{x^2+1}$

$$f'(x) = \frac{x^2+1-(x+1)2x}{(x^2+1)^2} = \frac{x^2+1-2x^2-2x}{(x^2+1)^2} = \frac{-x^2-2x+1}{(x^2+1)^2}$$

$-x^2-2x+1$

$\Delta = 4-4(-1) = 8 \rightarrow x = \frac{2 \pm 2\sqrt{2}}{-2} = -1 \pm \sqrt{2}$

$-1+\sqrt{2} \approx 0.4$

$-1-\sqrt{2} \approx -2.4$

$$f(-1+\sqrt{2}) = \frac{-1+\sqrt{2}+1}{1-2\sqrt{2}+2+1} = \frac{\sqrt{2}}{4-2\sqrt{2}} = \frac{4\sqrt{2}+2\cdot 2}{16-4\cdot 2} = \frac{\sqrt{2}+1}{2} \approx 1.2071$$

$$f(-1-\sqrt{2}) = \frac{-1-\sqrt{2}+1}{1+2\sqrt{2}+2+1} = \frac{-\sqrt{2}}{4+2\sqrt{2}} = \frac{-\sqrt{2}(4-2\sqrt{2})}{16-4\cdot 2} = \frac{-4\sqrt{2}+4}{12} = \frac{1-\sqrt{2}}{3} \approx -0.13$$

critical points

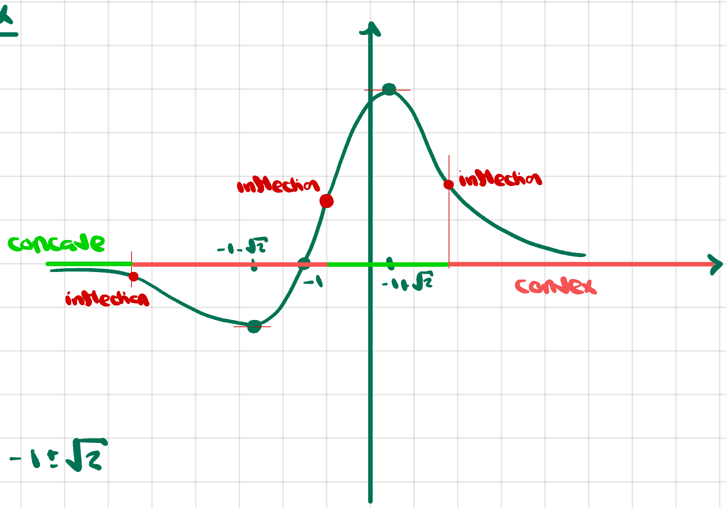
$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$

$f(x) = 0 \rightarrow x = -1$

$$f''(x) = \frac{(-2x-2)(x^2+1)^{\cancel{2}} - (-x^2-2x+1)\cdot 2\cancel{(x^2+1)}\cdot 2x}{(x^2+1)^4}$$

$$= \frac{-2x^3-2x-2x^2-2+4x^3+8x^2-4x}{(x^2+1)^4}$$

$$= \frac{2x^3+6x^2-6x-2}{(x^2+1)^3}$$



$g(x) = 2x^3+6x^2-6x-2$

$g'(x) = 6x^2+12x-6$

$\Delta = 144+4\cdot 6\cdot 6 = 288 \rightarrow x = \frac{-12 \pm 12\sqrt{2}}{12} = -1 \pm \sqrt{2}$

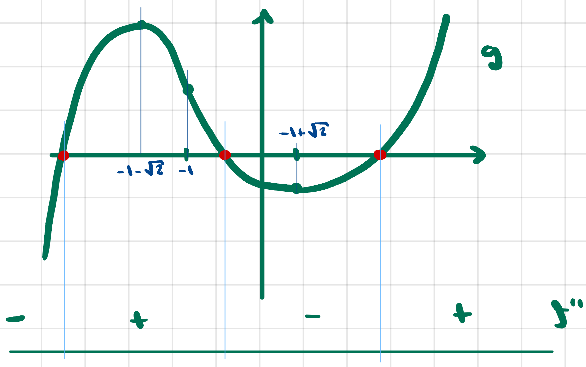
$g''(x) = 12x+12 = 0 \rightarrow x = -1$

	$-1-\sqrt{2}$		-1		$-1+\sqrt{2}$	
g''	$-$		$-$	$+$		$+$
g'	$+$		$-$	$-$		$+$

$\lim_{x\to\infty} g(x) = \infty$, $\lim_{x\to-\infty} g(x) = -\infty$

$g(-1-\sqrt{2}) \approx 19$

$g(-1) = 8$



(vi) $f(x) = \dfrac{x}{x^2-1}$

$$f'(x) = \frac{x^2-1-2x^2}{(x^2-1)^2} = \frac{-(x^2+1)}{(x^2-1)^2} \neq 0$$

f' and f aren't defined at 1 and -1.

$\lim_{x\to 1^+} f(x) = \infty$, $\lim_{x\to 1^-} f(x) = -\infty$

$\lim_{x\to -1^+} f(x) = \infty$, $\lim_{x\to -1^-} f(x) = -\infty$

$\lim_{x\to \infty} f(x) = \lim_{x\to -\infty} f(x) = 0$

$\lim_{x\to 1} f'(x) = \lim_{x\to -1} f'(x) = -\infty$

$$\lim_{x\to\infty} f'(x) = -\lim_{x\to\infty} \frac{x^2+1}{(x^2-1)^2}$$

$\lim_{x\to\infty}(x^2+1) = \lim_{x\to\infty}(x^2-1)^2 = \infty$

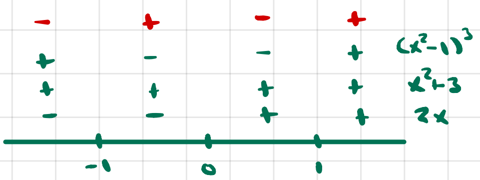
$$\lim_{x\to\infty} \frac{2x}{2(x^2-1)\cdot 2x} = 0$$

By L'Hôpital's Rule, $\lim_{x\to\infty} f'(x) = 0$

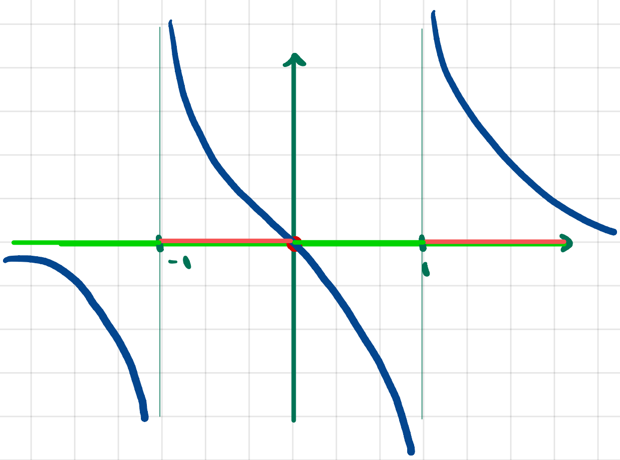
$\lim_{x\to -\infty} f'(x) = 0$

$$f''(x) = \frac{-2x(x^2-1)^2 - 2(x^2-1)2x(-x^2-1)}{(x^2-1)^4}$$

$$= \frac{2x(x^2+3)}{(x^2-1)^3}$$



$f''(x) = 0 \to x = 0$



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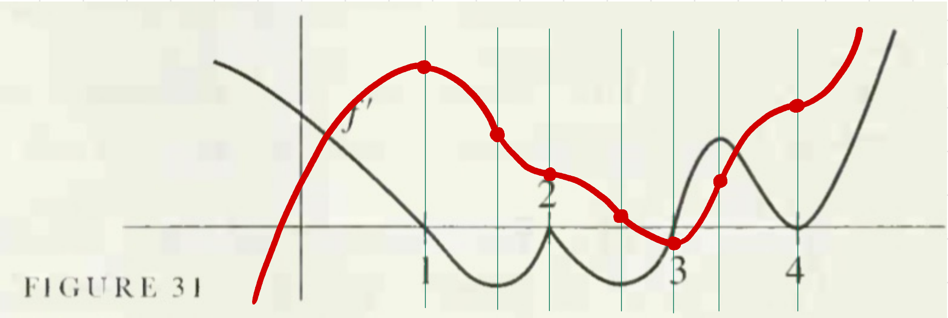
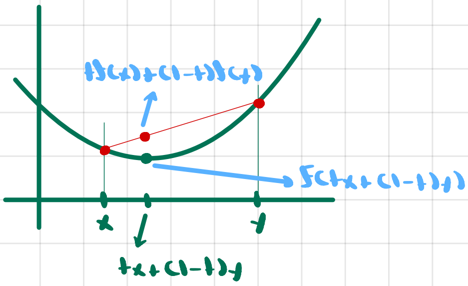


3. f convex on interval $\Leftrightarrow$ $\forall x, y$ in interval, $f(tx+(1-t)y) < tf(x)+(1-t)f(y)$, $0<t<1$

Proof



For any $x<y$ in the interval, $tx+(1-t)y \in (x,y)$ for $t \in (0,1)$.

By def. of convexity of f,

$$\frac{f(tx+(1-t)y)-f(x)}{tx+(1-t)y-x} < \frac{f(y)-f(x)}{y-x} \qquad 0<t<1$$

$$(y-x)f(tx+(1-t)y)-(y-x)f(x) < (tx+(1-t)y-x)(f(y)-f(x))$$

$$(y-x)f(tx+(1-t)y) < yf(x) - \cancel{xf(x)} + txf(y) - txf(x) + \cancel{xf(x)} + (1-t)yf(y) - (1-t)yf(x) - xf(y)$$

$$= f(x)(y-tx-y+ty) + f(y)(tx+(1-t)y-x)$$

$$= f(x)t(y-x) + f(y)(y(1-t)-x(1-t))$$

$$= f(x)t(y-x)+f(y)(1-t)(y-x)$$

$$\rightarrow f(tx+(1-t)y) < tf(x)+(1-t)f(y)$$

4.

(a) f, g convex, f increasing $\to$ $f \circ g$ convex

Proof

$h(x) = f(g(x))$

$h(ta + (1-t)b) = f(g(ta + (1-t)b))$

But $g(ta + (1-t)b) < tg(a) + (1-t)g(b)$

Since f increasing then

$$f(g(ta + (1-t)b)) < f(tg(a) + (1-t)g(b))$$
$$< tf(g(a)) + (1-t)f(g(b))$$

$$h(ta + (1-t)b) < th(a) + (1-t)h(b)$$

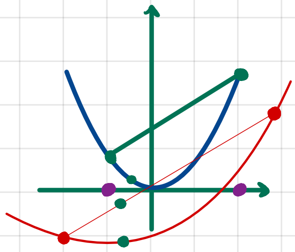
$\to h = f \circ g$ is convex.

(b) example of $g \circ f$ not convex

$f(x) = x^2 + 4x - 5$

$f'(x) = 2x + 4$

$f''(x) = 2$, convex

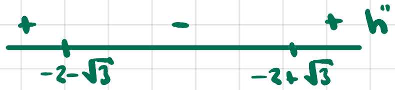


$g(x) = x^2$
$g'(x) = 2x$
$g''(x) = 2$, convex

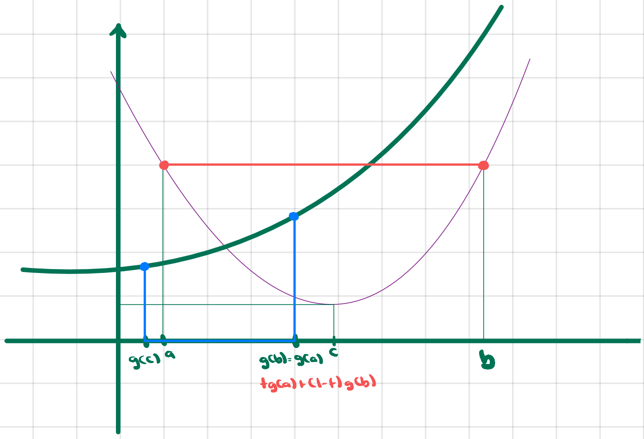
$h(x) = g(f(x)) = (x^2 + 4x - 5)^2$

$h'(x) = 4(x+5)(x-1)(x+2)$

$h''(x) = 12x^2 + 48x + 12$

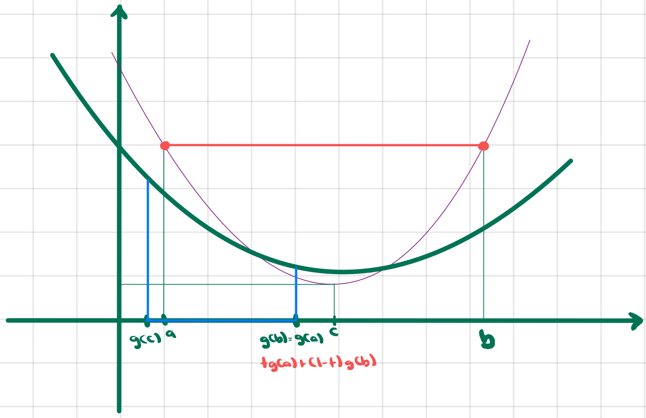


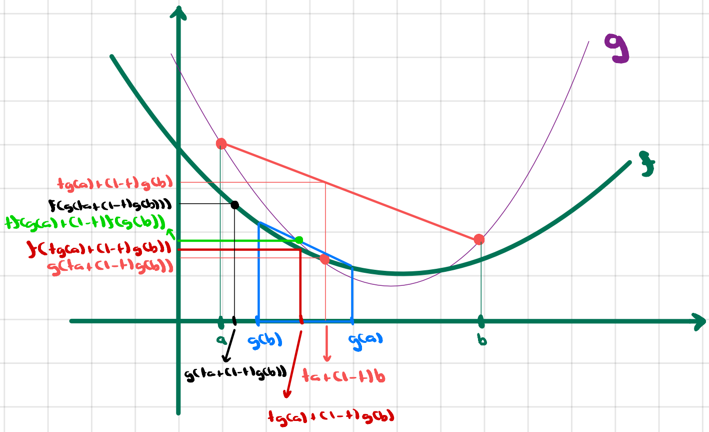
Therefore $g \circ f$ is not convex in its entire domain.



When we move from a to b as t goes from 0 to 1, the image under g is in some interval. But f is convex on this interval.
However, if we simply take the interval from g(a) to g(b), for all possible a and b, we get all possible intervals in the image of g.
f is convex on all of them.

The assumption that f increasing is necessary. Here's what can happen without it:





$tg(a) + (1-t)g(b)$ is always larger than $g(ta + (1-t)b)$, since g is convex.

Now, $f(tg(a) + (1-t)g(b)) < tf(g(a)) + (1-t)f(g(b))$, because f is convex.

But if f isn't increasing, then

$$f(g(ta + (1-t)b)) \geq f(tg(a) + (1-t)g(b))$$

Alternative Example

$f(x) = 1 + x^2 \quad x > 0$

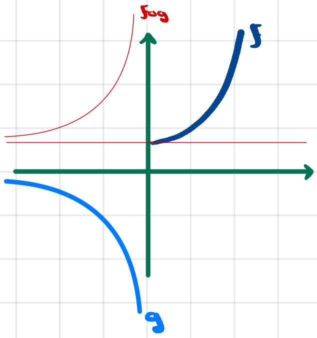
$g(x) = 1/x \quad x < 0$

$f'(x) = 2x$

$f''(x) = 2 \rightarrow$ convex

$g'(x) = -\dfrac{1}{x^2} < 0$

$g''(x) = \dfrac{2}{x^3} < 0$ for $x < 0$, concave



$h(x) = g(f(x)) = \dfrac{1}{1+x^2}$

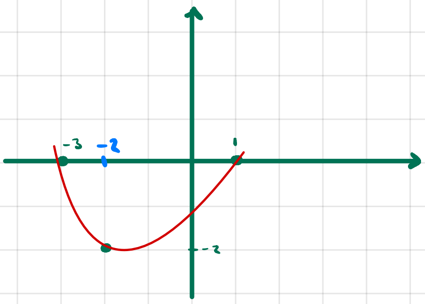
$h'(x) = \dfrac{-2x}{(1+x^2)^2}$

$h''(x) = \dfrac{-2(1+x^2)^{\cancel{2}\,1} - (-2x)\cdot 2\cancel{(1+x^2)}\cdot 2x}{(1+x^2)^{\cancel{4}\,3}}$

$= \dfrac{-2 - 2x^2 + 8x^2}{(1+x^2)^3}$

$= \dfrac{6x^2 - 2}{(1+x^2)^3}$

Therefore gof is not convex in its entire domain.



$x_{min} = -\dfrac{b}{2a} = -2$ choose an x where min occurs

$x_{r_1} = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

$x_{r_2} = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

equations for two roots

Let $a = 1$.

$-b = -4 \rightarrow b = 4$ b determined by choice of x_{min} and a.

$x_{r_1} = \dfrac{-4 + \sqrt{16 - 4c}}{2} \rightarrow 2x_{r_1} + 4 = \sqrt{16-4c}$

$x_{r_2} = \dfrac{-4 - \sqrt{16 - 4c}}{2} \rightarrow 2x_{r_2} + 4 = -\sqrt{16-4c}$

$2x_{r_1} + 4 = -2x_{r_2} - 4$

Given a and b, there is a relationship between x_{r_1} and x_{r_2}.

$2(x_{r_1} + x_{r_2}) = -8$

$x_{r_1} + x_{r_2} = -4$

e.g. $x_{r_1} = 1,\ x_{r_2} = -5$ Given the relationship we can choose x_{r_1} and x_{r_2}.

$\rightarrow 36 = 16 - 4c \rightarrow 4c = -20 \rightarrow c = -5$

And that choice determines c.

$f(x) = x^2 + 4x - 5$

$\Delta = 16 + 20 = 36 \rightarrow x_r = \dfrac{-4 \pm 6}{2} \begin{smallmatrix}\nearrow -5 \\ \searrow 1\end{smallmatrix}$

(c) f, g twice differentiable.
f, g convex $\rightarrow$ $f \circ g$ convex
f increasing

Proof

$$h(x) = f(g(x))$$

$$h'(x) = f'(g(x))g'(x)$$

$$h''(x) = \underbrace{f''(g(x))}_{>0}\underbrace{(g'(x))^2}_{>0} + \underbrace{f'(g(x))}_{>0}\underbrace{g''(x)}_{>0} > 0$$

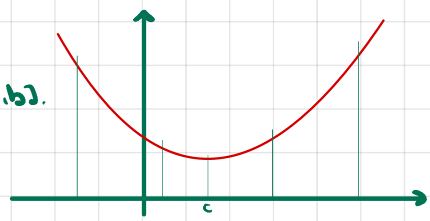
5.

(a) f diff. and convex on an interval $\rightarrow$ f increasing $\vee$ f decreasing $\vee$ $\exists c$, f increasing to right of c and decreasing to left of c.

Proof

Let $[a,b]$ be the interval.
Since f is convex, $f''(x)$ so f' is increasing.

Case 1: min at a
Then $f'(a) \geq 0$.
Then, $\forall x, x \in [a,b] \rightarrow f'(x) > 0 \rightarrow f$ increasing on $[a,b]$.



Case 2: min at b
Then $f'(b) \leq 0$.
Then, $\forall x, x \in [a,b] \rightarrow f'(x) < 0 \rightarrow f$ decreasing on $[a,b]$

Case 3: min at $c \in (a,b)$
Then $f'(c) = 0$ and $\forall x, x \in [a,c) \rightarrow f'(x) < 0$
$\forall x, x \in (c,b] \rightarrow f'(x) > 0$
$\rightarrow f$ decreasing left of c, increasing right of c.

Alternative Proof

Let $[a,b]$ be the interval.

Case 1: $f'(b) < 0$. Then $\forall x, x \in [a,b] \rightarrow f'(x) < 0$

Case 2: $f'(a) > 0$. Then $\forall x, x \in [a,b] \rightarrow f'(x) > 0$

Case 3: $f'(a) < 0$, and $\exists d, d \in (a,b) \wedge f'(d) > 0$.

Let $A = \{x : f'(x) < 0 \wedge x \in [a,b]\}$
sup A exists because $a \in A$.
Let $c = \sup A$. Then $f' < 0$ for x left of c, $f' > 0$ for x right of c.

(b) f, g convex
f increasing $\rightarrow f \circ g$ convex
f, g one-time diff.

Proof

f increasing $\rightarrow f' > 0$.

$h(x) = f(g(x))$

$h'(x) = \underbrace{f'(g(x))}_{>0} g'(x)$

By part a) since g diff and convex, either g increasing, decreasing or $\exists c$, g increasing to right of c and decreasing to left of c.

Let $x < y$

$h'(x) = f'(g(x))g'(x)$

$h'(y) = f'(g(y))g'(y)$

Th. 1 $\rightarrow$ $g'(x) < g'(y)$
$f'(x) < f'(y)$

Case 1: g increasing, ie $g' > 0$.

Then
$0 \leq g'(x) < g'(y)$

$g(x) < g(y)$ so $0 < f'(g(x)) < f'(g(y))$

and $h'(x) < h'(y)$

ie $(f \circ g)'$ increasing

Case 2: g decreasing, ie $g' < 0$

$g'(x) < g'(y) \leq 0$

$g(x) > g(y)$ so $0 < f'(g(x)) < f'(g(y))$

Therefore

$h'(x) = \underset{\substack{>0 \\ < f'(g(y))}}{f'(g(x))}\ \underset{\substack{<0 \\ < g'(y)}}{g'(x)} < 0$

$\rightarrow h'(x) < h'(y)$

ie $(f \circ g)'$ increasing

Case 3: $\exists c$, g increasing to right of c and decreasing to left

For any $x < y$ in $(-\infty, c]$, using same steps as case 2,

$h'(x) < h'(y)$

For any $x < y$ in $[c, +\infty)$, using same steps as in case 1,

$h'(x) < h'(y)$

For any $x \in (-\infty, c]$, $y \in (c, +\infty)$ we have $x < y$.

$h'(x) = \underbrace{f'(g(x))}_{>0}\underbrace{g'(x)}_{\leq 0} \leq 0$

$h'(y) = \underbrace{f'(g(y))}_{>0}\underbrace{g'(y)}_{>0} > 0$

$\rightarrow h'(x) < h'(y)$

c) f convex on an interval $\to$ f increasing $\vee$ f decreasing $\vee$ $\exists c$, f increasing to right of c and decreasing to left of c

Proof

Let $[a,b]$ be in the interval.

Let $x_1 \in (a,b)$.

Assume $f(a) < f(x_1)$
Let $c \in (x_1, b]$.
Assume $f(c) \le f(x_1)$

Then $\frac{f(c)-f(x_1)}{c-x_1} < \frac{f(c)-f(a)}{c-a}$

But from the def. of convexity, $\forall a, x_1, c$ $a<x_1<c$

$$\frac{f(c)-f(x_1)}{c-x_1} > \frac{f(c)-f(a)}{c-a}$$

Therefore $\perp$.

$f(c) > f(x_1)$, ie f increasing to right of x_1.

Now assume $f(x_1) > f(b)$.
Let $c \in [a, x_1)$
Assume $f(c) \le f(x_1)$.

Then $\frac{f(x_1)-f(c)}{x_1-c} > \frac{f(b)-f(c)}{b-c}$

But by convexity for any $c<x_1<b$,

$$\frac{f(x_1)-f(c)}{x_1-c} < \frac{f(b)-f(c)}{b-c}.$$

$\perp$.

T.F. $f(c) > f(x_1)$, ie f decreasing on $[a, x_1]$.

Therefore we have shown the following **lemma**:

$\forall x, x \in (a,b) \wedge f(a) < f(x)$
$\to f$ increasing right of x

$\forall x, x \in (a,b) \wedge f(x) > f(b)$
$\to f$ decreasing left of x

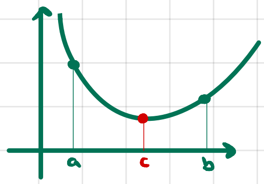
Because f is convex it is not constant.

There is some $a<b \wedge f(a) \ne f(b)$.

Case 1: $f(a) < f(b)$

Then f increasing right of b.

As shown in problem 10, since f is convex on $\mathbb{R}$ (or some open interval containing $[a,b]$) it is continuous on $[a,b]$ and hence takes on a min value therein.



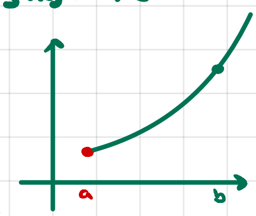
Assume the min occurs at $c \in (a,b)$
Then f decreasing left of a.

Let a' s.t. $a<a'<c$.
Then $f(a') > f(c)$ because c is the min (and $f(a') = f(c)$ would mean for all $x \in (a',c)$, $f(x) < f(a') = f(c)$, contradicting c as min).

Therefore f decreasing left of a' for all such $a<a'<c$.

T.F. f decreasing left of c.

By similar arguments, f is increasing right of c.



Now assume the min occurs at a.

Let a' s.t. $a<a'<b$.
Then $f(a') > f(a)$ and f increasing right of a'. Therefore f increasing right of a

Case 1.1 $\exists d$, $d<a \wedge f(d) > f(a)$.
The min on $[d,a]$ occurs at $c \in (d,a]$. The same reasoning used previously shows that f decreasing left of c, increasing right of c.

Case 1.2: $\forall d$ $d<a \to f(d) < f(a)$
If the min of f occurs at some $c \in (d,a)$ then, by same reasoning as before, f increasing right of c, decreasing left of c. If the min is always at d then f increasing right of d for all d, so f increasing.

Case 2: $f(a) > f(b)$ is analogous.

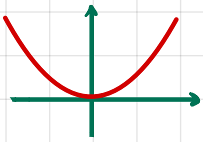
6. f twice differentiable

$x \geq 0 \to f(x) > 0$ $\quad\quad \to \exists x, x > 0 \wedge f''(x) = 0$

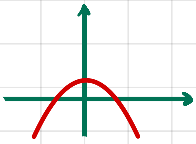
f decreasing

$f'(0) = 0$

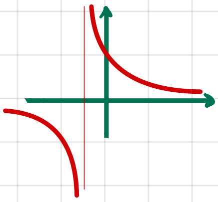
If any one of the assumptions is removed, there is a counterexample



$f(x) = x^2$, not decreasing for $x \geq 0$

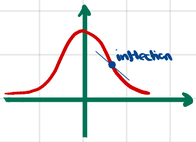


$f(x) = 1 - x^2$, $f(x) < 0$ for some $x > 0$

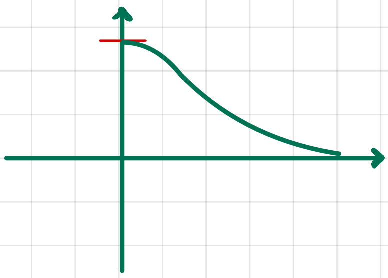


$f(x) = \frac{1}{x+1}$, $f'(0) \neq 0$

On the other hand,



$f(x) = \frac{1}{1+x^2}$ satisfies all assumptions



Proof

f decreasing for $x \geq 0$ means $f' < 0$ for $x > 0$

Let $x_0 > 0$. Then $f'(x_0) < 0$.

Suppose $\forall y, y > x_0 \to f'(y) \leq f'(x_0)$.

Then $f(y) \leq f(x_0) + f'(x_0)(y - x_0)$

Let $-\frac{f(x_0)}{f'(x_0)} < y - x_0 \to y > x_0 - \frac{f(x_0)}{f'(x_0)}$

Then $f(y) \leq 0$. $\perp$

Therefore $\exists y, y > x_0 \wedge f'(y) > f'(x_0)$.

Since $f'(0) = 0$ and $f'(x_0) < 0$ then $\exists c, c \in (0, x_0) \wedge f''(c) = \frac{f'(x_0) - f'(0)}{x_0 - 0} < 0$ (MVT)

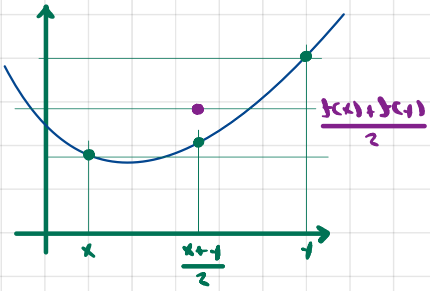
Since $f'(x_0) = \alpha < 0$ and $f'(x_1) = \beta > \alpha$, $\exists d, d \in (x_0, x_1) \wedge f''(d) = \frac{f'(x_1) - f'(x_0)}{x_1 - x_0} > 0$ (MVT)

But then $\exists x_2, x_2 \in (c, d) \wedge f''(x_2) = 0$

7.

(a) f convex $\rightarrow f\left(\frac{x+y}{2}\right) < \frac{f(x)+f(y)}{2}$

Proof



$$f \text{ convex} \rightarrow \frac{f(y)-f(x)}{y-x} > \frac{f\left(\frac{x+y}{2}\right)-f(x)}{\frac{y-x}{2}} = \frac{2\left(f\left(\frac{x+y}{2}\right)-f(x)\right)}{y-x}$$

$$\rightarrow f(y)-f(x) > 2f\left(\frac{x+y}{2}\right)-2f(x)$$

$$\rightarrow f\left(\frac{x+y}{2}\right) < \frac{f(x)+f(y)}{2}$$

(b) $f\left(\frac{x+y}{2}\right) < \frac{f(x)+f(y)}{2}$ $\rightarrow$ $f(hx+(1-h)y) < hf(x)+(1-h)f(y)$, h rational $\in [0,1]$, of form $\frac{m}{2^n}$

If the average of the function values at $\frac{x+y}{2}$ is above the value of the function at $\frac{x+y}{2}$ then the weighted average of the fn values is larger than the value of the function at the weighted x coordinates, for any weights as described above.

e.g., for $n=1$, $h = \frac{m}{2}$, but since h is in lowest terms, $m=1$.

Proof

$n=1$ $\rightarrow$ assertion true by assumption, ie $\forall x \forall y$ $f\left(\frac{x+y}{2}\right) < \frac{f(x)+f(y)}{2}$ (1)

Suppose it is true for some n, for all x, y.

Now let $h = \frac{m}{2^{n+1}}$ in lowest terms.

m must be odd otherwise it would have factor 2 in common w/ denominator.

$m-1$ and $m+1$ are even.

Hence

$h_1 = \frac{m-1}{2^{n+1}}$ and $h_2 = \frac{m+1}{2^{n+1}}$ can be expressed as $\frac{a}{2^n}$, so the assertion is true for h_1 and h_2.

But $\frac{h_1+h_2}{2} = \frac{m}{2^{n+1}} = h$

* $h = \frac{m}{2^n}$ means

$\frac{1}{2}$

$\frac{1}{4}$ $\frac{3}{4}$

$\frac{1}{8}$ $\frac{3}{8}$ $\frac{5}{8}$ $\frac{7}{8}$

$\frac{1}{16}$ $\frac{3}{16}$ $\frac{5}{16}$ $\frac{7}{16}$ $\frac{9}{16}$ $\frac{11}{16}$ $\frac{13}{16}$ $\frac{15}{16}$

Take a line segment from 0 to 1.
mark the half point: $1/2$.
now we have two segments.
mark their half points: $1/4$, $3/4$.
and so on.

Let $x' = h_1 x + (1-h_1) y$

$y' = h_2 x + (1-h_2) y$

Then $\frac{x'+y'}{2} = \frac{(h_1+h_2)}{2} x + \frac{(2-h_1-h_2)}{2} y$

$= hx + (1-h)y$

Therefore

$$f(hx+(1-h)y) = f\left(\frac{x'+y'}{2}\right) < \frac{f(x')}{2} + \frac{f(y')}{2}$$

But $f(x') = f(h_1 x + (1-h_1)y) < h_1 f(x) + (1-h_1) f(y)$

$$f(hx+(1-h)y) = f\left(\frac{x'+y'}{2}\right) < \frac{f(x')}{2} + \frac{f(y')}{2}$$

$$< \frac{h_1 f(x) + (1-h_1) f(y)}{2} + \frac{h_2 f(x) + (1-h_2) f(y)}{2}$$

$$= \frac{h_1+h_2}{2} f(x) + \left(1 - \frac{h_1+h_2}{2}\right) f(y)$$

$$= h f(x) + (1-h) f(y)$$

Part b) incorrect attempt

Let $n=1$. Then $k=\frac{1}{2}$, $1-k=\frac{1}{2}$

$f\left(\frac{x+y}{2}\right) < \frac{f(x)+f(y)}{2}$ is true by assumption.

Let $n \in \mathbb{N}$ and assume $f(kx+(1-k)y) < kf(x)+(1-k)f(y)$, k rational $\in [0,1]$, of form $\frac{m}{2^n}$

$$\frac{f(y)-f(x)}{\cancel{y-x}} > \frac{f\left(\frac{m}{2^{n+1}}x + \frac{2^{n+1}-m}{2^{n+1}}y\right) - f(x)}{\frac{\cancel{(y-x)}(2^{n+1}-m)}{2^{n+1}}}$$

Here I am assuming convexity but this isn't an available assumption in this part of the problem.

$$\frac{2^{n+1}-m}{2^{n+1}}(f(y)-f(x)) > f\left(\frac{m}{2^{n+1}}x + \frac{2^{n+1}-m}{2^{n+1}}y\right) - f(x)$$

$$\rightarrow \frac{2^{n+1}-m}{2^{n+1}}f(y) + \frac{m}{2^{n+1}}f(x) > f\left(\frac{m}{2^{n+1}}x + \frac{2^{n+1}-m}{2^{n+1}}y\right)$$

$$\rightarrow f(kx+(1-k)y) < kf(x)+(1-k)f(y)$$

(c) $\forall x \forall y\ f\left(\frac{x+y}{2}\right) < \frac{f(x)+f(y)}{2}$ $\rightarrow$ f convex

f continuous

Proof

Let $0<t<1$, k of form $\frac{m}{2^n}$ between 0 and 1, and $\varepsilon>0$.

For any x and y,

$$|f(kx+(1-k)y) - f(tx+(1-t)y)| < \varepsilon$$

Because

8. $n>1$

$p_1, \dots, p_n$ positive numbers

$\sum_{i=1}^{n} p_i = 1$

(a) $\forall x_1, x_2, \dots, x_n \rightarrow \sum_{i=1}^{n} p_i x_i$ lies between smallest and largest x_i

$\sum p_i x_i$ represents a weighted average of x_i.

Let x_α be the smallest and x_β the largest x_i.

Then $x_\alpha = \sum p_i x_\alpha \leq \sum p_i x_i \leq \sum p_i x_\beta = x_\beta$

(b) $t = \sum_{i=1}^{n-1} p_i$

$\dfrac{\sum_{i=1}^{n-1} p_i x_i}{\sum_{i=1}^{n-1} p_i}$ is also an average. The weights are now $\dfrac{p_i}{\sum_{i=1}^{n-1} p_i} = q_i$

Let x_α be smallest and x_β largest among $x_1, \dots, x_{n-1}$.

Then, $x_\alpha = \sum_{i=1}^{n-1} q_i x_\alpha \leq \sum q_i x_i \leq \sum q_i x_\beta = x_\beta$

(c) f convex $\rightarrow f\left(\sum_{i=1}^{n} p_i x_i\right) < \sum_{i=1}^{n} p_i f(x_i)$ (Jensen's Inequality)

Proof

From problem 3, since f convex then

$\forall x, y$ in interval, $f(tx + (1-t)y) < tf(x) + (1-t)f(y)$, $0<t<1$

we use a proof by induction on n.

For $n=2$, $p_1 + p_2 = 1 \rightarrow p_2 = 1 - p_1$ and

$f(p_1 x_1 + p_2 x_2) < p_1 f(x_1) + p_2 f(x_2)$ by problem 3.

Assume $f\left(\sum_{i=1}^{n} p_i x_i\right) < \sum_{i=1}^{n} p_i f(x_i)$ for some n.

Then, if $p_{n+1} = (1-t)$

$$f\left(\sum_{i=1}^{n+1} p_i x_i\right) = f\left(t \cdot \frac{1}{t} \sum_{i=1}^{n} p_i x_i + (1-t) x_{n+1}\right)$$

$$< t f\left(\sum_{i=1}^{n} \frac{p_i}{t} x_i\right) + (1-t) f(x_{n+1})$$

$$< t \sum_{i=1}^{n} \frac{p_i}{t} f(x_i) + (1-t) f(x_{n+1})$$

$$= \sum_{i=1}^{n+1} p_i x_i$$

Initial Attempt

Let $x = \dfrac{\sum_{i=1}^{n-1} p_i x_i}{\sum p_i}$ and $y = x_n$.

Let $t = \sum_{i=1}^{n-1} p_i \in (0,1)$ and $(1-t) = 1 - \sum_{i=1}^{n-1} p_i$

Then

$$f(tx+(1-t)y) = f\left(\sum_{i=1}^{n-1} p_i \frac{\sum_{i=1}^{n-1} p_i x_i}{\sum_{i=1}^{n-1} p_i} + (1 - \sum_{i=1}^{n-1} p_i) x_n\right)$$

$$= f\left(\sum_{i=1}^{n-1} p_i x_i + p_n x_n\right)$$

$$= f\left(\sum_{i=1}^{n} p_i x_i\right) < \sum_{i=1}^{n-1} p_i \underbrace{f\left(\frac{\sum_{i=1}^{n-1} p_i x_i}{\sum p_i}\right)} + (1 - \sum_{i=1}^{n-1} p_i) f(x_n)$$

here we reach a recursion

If we can prove that $f\left(\dfrac{\sum_{i=1}^{n-1} p_i x_i}{\sum_{i=1}^{n-1} p_i}\right) < \sum_{i=1}^{n-1} \dfrac{p_i}{\sum_{i=1}^{n-1} p_i} f(x_i)$

then we'd have

$$f\left(\sum_{i=1}^{n} p_i x_i\right) < \cancel{\sum_{i=1}^{n-1} p_i} \cdot \sum^{n-1} \frac{p_i}{\cancel{\sum_{i=1}^{n-1} p_i}} f(x_i) + p_n f(x_n) = \sum_{i=1}^{n} p_i f(x_i)$$

9.

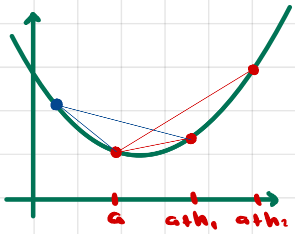
(a) Recall

Theorem 1 Let f be convex. If f diff at a, then the graph of f lies above the tangent line through $(a, f(a))$, except at $(a, f(a))$.
If $a<b$ and f diff at a and b then $f'(a)<f'(b)$.

Also

$$f'_+(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$

$$f'_-(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$$



Since f convex, by def. of convexity on $[a, a+h_2]$

$$\frac{f(a+h_1)-f(a)}{h_1} < \frac{f(a+h_2)-f(a)}{h_2}$$

Therefore $\frac{f(a+h)-f(a)}{h}$ decreases as $h\to 0^+$

Does it decrease indefinitely?

No because for $h_3<0$, because of convexity,

$$\frac{f(a)-f(a+h_3)}{h_3} < \frac{f(a+h_1)-f(a+h_3)}{h_1-h_3} < \frac{f(a+h_1)-f(a)}{h_1}$$

Therefore for any $h_3<0$, $\frac{f(a)-f(a+h_3)}{h_3}$ is a lower bound for the set $\left\{x : x = \frac{f(a+h)-f(a)}{h}, h>0\right\}$

This set has a g.l.b., ie infimum. Let's call it β.

Assume $\beta < f'_+(a)$

Then there is some $h_4>0$ s.t. $\frac{f(a+h_4)-f(a)}{h_4} < f'_+(a)$

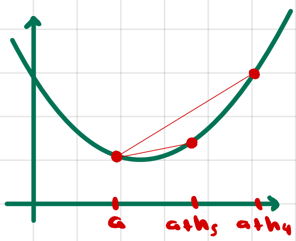
But $f'_+(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$

so there exists an interval $(a, a+\delta)$ in which $\left|\frac{f(a+h)-f(a)}{h} - f'_+(a)\right| < \epsilon$

Let $\epsilon = f'_+(a) - \frac{f(a+h_4)-f(a)}{2h}$

Then there is some h_5 such that $\left|\frac{f(a+h_5)-f(a)}{h_5} - f'(a)\right| < f'_+(a) - \frac{f(a+h_4)-f(a)}{2h}$ and $a<h_5<h_4$.

$\to -f'_+(a) + \frac{f(a+h_4)-f(a)}{2h} < \frac{f(a+h_5)-f(a)}{h_5} - f'_+(a) < f'_+(a) - \frac{f(a+h_4)-f(a)}{2h}$



$\frac{f(a+h_4)-f(a)}{2h} < \frac{f(a+h_5)-f(a)}{h_5}$, which contradicts convexity of f.

Therefore $\beta \geq f'_+(a)$

Now assume $\beta > f'_+(a)$.
Then there is some $h>0$ s.t. $\frac{f(a+h)-f(a)}{h} - f'_+(a) < \beta - f'_+(a)$, so $\frac{f(a+h)-f(a)}{h} < \beta$. $\perp$.

Therefore $\beta = f'_+(a)$.

ie $f'_+(a)$ is the infimum of $\{x : x = \frac{f(a+h)-f(a)}{h}, h>0\}$.

By an analogous proof we can show that $f'_-(a)$ is the supremum of $\{x : x = \frac{f(a+h)-f(a)}{h}, h<0\}$.

Note that for any h, $\frac{f(a+h)-f(a)}{h} \neq f'(a)$ because this violates convexity of f.

Therefore, since for $h>0$, $\frac{f(a+h)-f(a)}{h} > f'_+(a)$ and for $h<0$, $\frac{f(a+h)-f(a)}{h} < f'_-(a)$

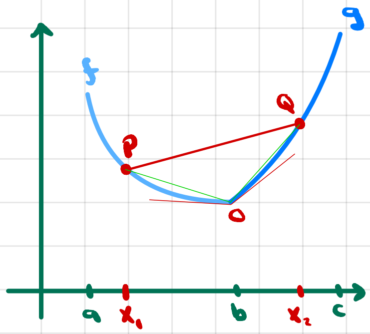
Then $f'_+(a) \geq f'_-(a)$

---

b) f convex on $[a,b]$
g convex on $[b,c]$
$f(b) = g(b)$
$f'_-(b) \leq g'_+(b)$

$\rightarrow h$ convex on $[a,c]$

$$h(x) = \begin{cases} f(x) & x \in [a,b] \\ g(x) & x \in [b,c] \end{cases}$$



Proof

Let $x_1 \in [a,b]$, $x_2 \in [b,c]$.

$$\frac{f(b)-f(x_1)}{b-x_1} < f'_-(b)$$

$$\frac{g(x_2)-f(b)}{x_2-b} > g'_+(b)$$

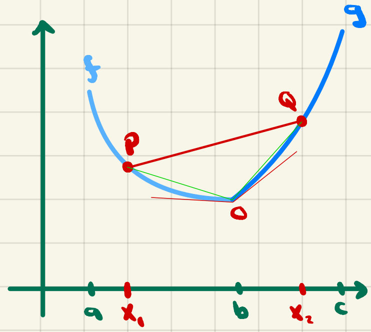
$$\rightarrow \frac{f(b)-f(x_1)}{b-x_1} < \frac{g(x_2)-f(b)}{x_2-b}$$

$$PQ: h(x) = f(x_1) + \frac{g(x_2)-f(x_1)}{x_2-x_1}(x-x_1), \quad x \in [x_1, x_2]$$

$$PO: h_2(x) = f(x_1) + \frac{f(b)-f(x_1)}{b-x_1}(x-x_1), \quad x \in [x_1, b]$$

$$\frac{g(x_2)-f(x_1)}{x_2-x_1} = \frac{g(x_2)-f(b)+f(b)-f(x_1)}{x_2-x_1} < \frac{g(x_2)-f(b)}{x_2-b} + \frac{f(b)-f(x_1)}{b-x_1}$$

Proof from Stack Exchange:



$$\frac{f(b)-f(x_1)}{b-x_1} \le \frac{g(x_2)-f(b)}{x_2-b}$$

Assume $f(b)=g(b)=0$. Then $\dfrac{-f(x_1)}{b-x_1} \le \dfrac{g(x_2)}{x_2-b}$ (1)

let $\ell(\theta) = (1-\theta)x_1 + \theta x_2 = x_1 + \theta(x_2-x_1)$

Let $\theta_b = \dfrac{b-x_1}{x_2-x_1}$. T.F. $\theta_b \in (0,1)$ and $\ell(\theta_b) = b$.

Let's show that if $\theta \in [0,\theta_b]$ then $h((1-\theta)x_1+\theta x_2) \le (1-\theta)h(x_1)+\theta h(x_2)$

Recall that $h(x) = \begin{cases} f(x) & x \in [a,b] \\ g(x) & x \in [b,c] \end{cases}$

Let $s = \theta\theta_b^{-1}$. Thus $s \in [0,1]$ and

example: $\theta_b = 1/10$, $\theta_b^{-1} = 10$
$\theta = 1/20$ $\to \theta\theta_b^{-1} = 1/2$

$$\begin{aligned} h((1-\theta)x_1+\theta x_2) &= h(x_1+\theta(x_2-x_1)) \\ &= h\left(x_1+\theta(b-x_1)\frac{(x_2-x_1)}{b-x_1}\right) \\ &= h(x_1+\theta\theta_b^{-1}(b-x_1)) \\ &= h(x_1+s(b-x_1)) \\ &= f(x_1+s(b-x_1)) \\ &= f(sb+(1-s)x_1) \\ &\le s\overset{0}{\cancel{f(b)}}+(1-s)f(x_1) \end{aligned}$$

$1-\theta\theta_b^{-1}$

we want to show $(1-s)f(x_1) \le (1-\theta)f(x_1)+\theta g(x_2)$

which is equivalent to $(\cancel{1}-\theta\theta_b^{-1})f(x_1) \le \cancel{f(x_1)} - \theta f(x_1) + \theta g(x_2)$

$$f(x_1)\theta(1-\theta_b^{-1}) \le \theta g(x_2)$$

$$(1-\theta_b^{-1})f(x_1) \le g(x_2)$$

$$\frac{b-x_2}{b-x_1}f(x_1) \le g(x_2)$$

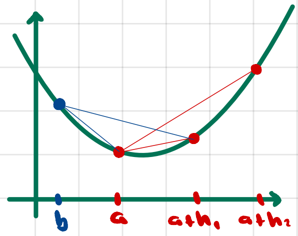
$$\frac{-f(x_1)}{b-x_1} \le \frac{g(x_2)}{x_2-b}$$

which is the same as (1).

(c)

f convex $\to$ $[\,f'_+(a) = f'_-(a) \leftrightarrow f'_+$ cont. at $a.\,]$

ie f is diff. $\leftrightarrow f'_+$ cont.



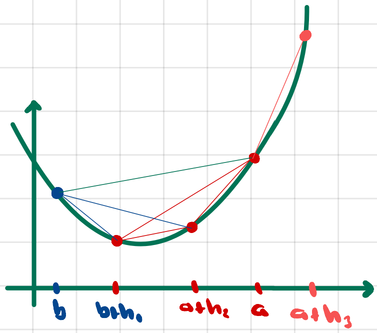
Proof

Let $b<a$.

Then

$$f'_+(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$

$$f'_-(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$$



$$f'_+(b) = \lim_{h\to 0^+} \frac{f(b+h)-f(b)}{h}$$

By convexity, given $h_1>0$ and $h_2<0$, $h_3>0$, we have

$$\frac{f(b+h_1)-f(b)}{h_1} < \frac{f(a)-f(b)}{a-b} < \frac{f(a+h_2)-f(a)}{h_2} < \frac{f(a+h_3)-f(a)}{h_3}$$

Since $f'_+(b) = \inf\{x : x = \dfrac{f(b+h)-f(b)}{h}, h>0\}$

$f'_-(a) = \sup\{x : x = \dfrac{f(a+h)-f(a)}{h}, h<0\}$

$f'_+(a) = \inf\{x : x = \dfrac{f(a+h)-f(a)}{h}, h>0\}$

Then

convexity of f assumption used to derive this

$f'_+(b) < f'_-(a) < f'_+(a)$

Assume f'_+ continuous at a.

Then $\lim_{b\to a} f'_+(b) = f'_+(a)$

This means

$\forall \epsilon>0\ \exists \delta>0\ \forall b\ 0<|b-a|<\delta \to \left| f'_+(b) - f'_+(a) \right| < \epsilon$

But $f'_+(b) - f'_+(a) < f'_-(a) - f'_+(a) < 0$

$\lim_{b\to a} (f'_+(b) - f'_+(a))$

$\to \lim_{b\to a} (f'_-(a) - f'_+(a)) = 0 \to f'_-(a) = f'_+(a)$

$\lim_{b\to a} 0 = 0$

f'_+ cont. at a $\to f'_-(a) = f'_+(a) \to f$ diff. at a.

Now assume $f'_-(a) = f'_+(a)$, ie f diff. at a.

Note: f is a (convex) fn. f'_+ is the rh deriv. fn, f'_- is the lh deriv. fn.

We want to show that f'_+ is cont. at a, ie that $\lim_{b\to a} f'_+(b) = f'_+(a)$

To do this we show first that $\lim_{b\to a^+} f'_+(b) = f'_+(a)$ and then that $\lim_{b\to a^-} f'_+(b) = f'_+(a)$.

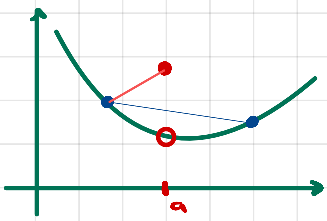
Recall that $f'_+(a)$ represents a limit: $\lim_{h\to 0^+} \dfrac{f(a+h)-f(a)}{h}$.

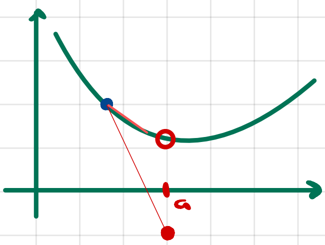
We can make $\dfrac{f(a+h)-f(a)}{h}$ as close to $f'_+(a)$ as we want.

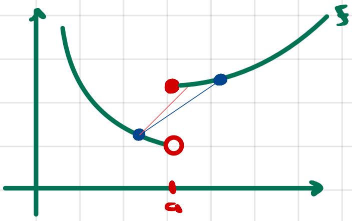
For any $\varepsilon > 0$, choose $c > a$ so that $\dfrac{f(c)-f(a)}{c-a} < f'_+(a) + \varepsilon$

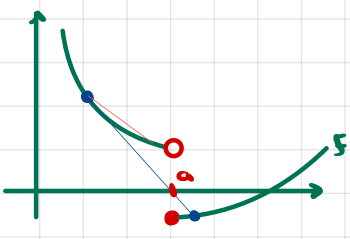
Since the rh-deriv. exists, so does the limit $\lim_{b\to a^+} f(b) = f(a)$.

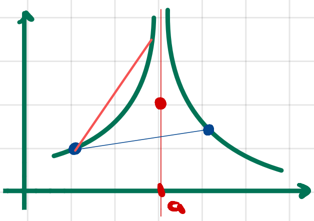
A few examples of why a function can't be convex and discontinuous at an interior point of its domain.











10.

(a) f convex on $\mathbb{R}$ or any open interval $\rightarrow f$ continuous

Proof

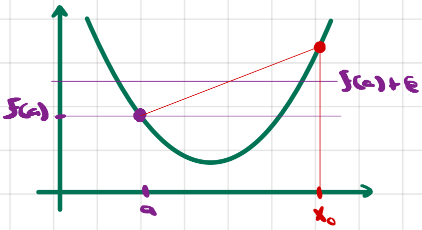
We want to show that $\lim_{x \to a} f(x) = f(a)$ for all a. We do this by showing that for any $\epsilon > 0$ there is always an interval around a in which $|f(x) - f(a)| < \epsilon$.

First we show $f(x) - f(a) < \epsilon$, then we show $f(x) - f(a) > -\epsilon$.

Let $x_0 > a$.

$$g(x) = \cancel{f(a)} + \frac{f(x_0) - f(a)}{x_0 - a}(x - a) < \cancel{f(a)} + \epsilon$$

$\rightarrow (f(x_0) - f(a))(x - a) < \epsilon(x_0 - a)$

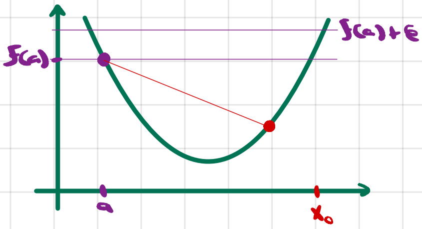


Case 1: $f(x_0) > f(a)$

$$\rightarrow x < a + \epsilon \frac{x_0 - a}{f(x_0) - f(a)}$$

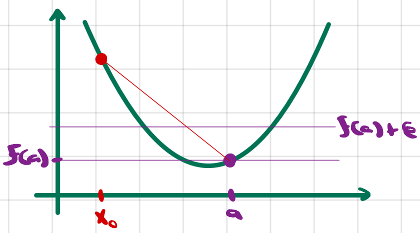
Since $g(a) = f(a)$ and $\forall x, x \in (a, x_0) \rightarrow f(x) < g(x)$, then for $x \in \left(a, \min\left(x_0, a + \epsilon \frac{x_0 - a}{f(x_0) - f(a)}\right)\right)$ we have

$f(x) < f(a) + \epsilon$.



Case 2: $f(x_0) < f(a) < f(a) + \epsilon$

Now let $x_0 < a$.



$$g(x) = \cancel{f(a)} + \frac{f(x_0) - f(a)}{x_0 - a}(x - a) < \cancel{f(a)} + \epsilon$$

Case 1: $f(x_0) > f(a)$

$\rightarrow (f(x_0) - f(a))(x - a) > \epsilon(x_0 - a)$

$x > a + \epsilon \frac{x_0 - a}{f(x_0) - f(a)}$, note that this is some number smaller than a.

Since $g(a) = f(a)$, and $\forall x\ x \in (x_0, a) \rightarrow f(x) < g(x)$, then for $x \in \left(\max\left(x_0, a + \epsilon \frac{x_0 - a}{f(x_0) - f(a)}\right), a\right)$ we have

$f(x) < g(x) < f(a) + \epsilon$

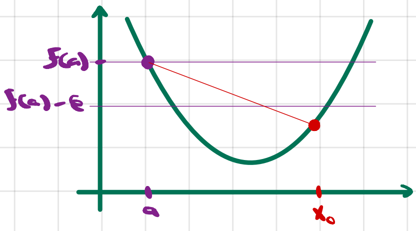
Case 2: $f(x_0) < f(a) < f(a) + \epsilon$

Therefore, for any $\epsilon > 0$, there is an interval around a such that $f(x) < f(a) + \epsilon$.

Now let's consider points where $f(x) > f(a) - \epsilon$.

Let $x_0 > a$.

Case 1: $f(x_0) > f(a) > f(a) - \epsilon$

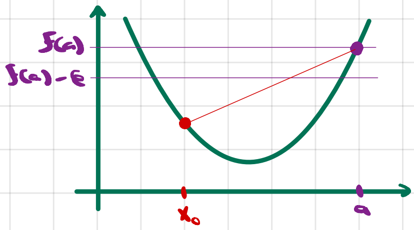


Case 2: $f(x_0) < f(a)$

$$g(x) = \cancel{f(a)} + \frac{f(x_0) - f(a)}{x_0 - a}(x - a) > \cancel{f(a)} - \epsilon$$

$$\rightarrow x < a - \epsilon \frac{x_0 - a}{f(x_0) - f(a)} = \alpha$$

$\forall x, x \in (a, \min(x_0, \alpha)) \rightarrow f(x) > f(a) - \epsilon$.



Let $x_0 < a$

Case 1: $f(x_0) > f(a) > f(a) - \epsilon$

Case 2: $f(x_0) < f(a)$

$$g(x) = \cancel{f(a)} + \frac{f(x_0) - f(a)}{x_0 - a}(x - a) > \cancel{f(a)} - \epsilon$$

$$(f(x_0) - f(a))(x - a) > -\epsilon(x_0 - a)$$

$$x > a - \epsilon \frac{x_0 - a}{f(x_0) - f(a)} = \beta$$

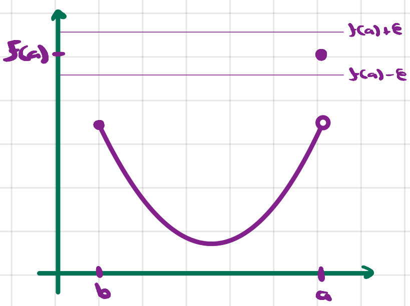
$\forall x, x \in (\max(x_0, \beta), a) \rightarrow f(x) > f(a) - \epsilon$

Therefore, for any $\epsilon > 0$ there is an interval around a where $|f(x) - f(a)| < \epsilon$.

---

b) Example of convex fn on closed interval that is not continuous.

The proof above considered a to be some interior point of an open interval.
we can't apply the proof to endpoints of closed intervals. The following fn is convex on $[b, a]$ but discont. at a.



11. f is weakly convex on an interval if for $a<b<c$ in this interval we have

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a}$$

(a) f weakly convex $\rightarrow$ $\left[f \text{ convex} \leftrightarrow \text{graph of } f \text{ contains no straight line segments.} \right]$

Proof

Assume f convex.

Assume graph of f contains straight line segment.

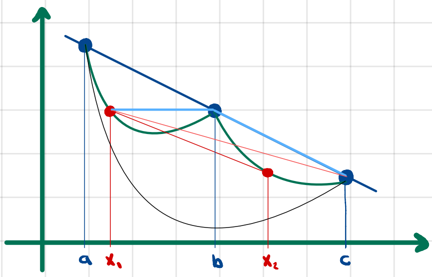
Let $a<b<c$ be on the line segment portion of f's graph.

$$\frac{f(b)-f(a)}{b-a} = \frac{f(c)-f(a)}{c-a}$$

$\perp$, because if f (strictly) convex then $\dfrac{f(b)-f(a)}{b-a} < \dfrac{f(c)-f(a)}{c-a}$

Therefore f's graph has no straight line segments.

f convex $\rightarrow$ graph of f contains no straight line segments.



Assume graph of f contains no straight line segments.

For any $a<b<c$, we know that $\dfrac{f(b)-f(a)}{b-a} \leq \dfrac{f(c)-f(a)}{c-a}$

Assume $\dfrac{f(b)-f(a)}{b-a} = \dfrac{f(c)-f(a)}{c-a} = \alpha$

Let $x_2 \in (b,c)$. Then $\dfrac{f(x_2)-f(b)}{x_2-b} \leq \alpha$ (1)

Let $x_1 \in (a,b)$

Assume $\dfrac{f(x_1)-f(a)}{x_1-a} < \alpha$ (2)

(1) $\rightarrow$ $f(x_1)-f(a) < \dfrac{x_1-a}{b-a}(f(b)-f(a))$

$f(x_1) < f(a) + \dfrac{x_1-a}{b-a}(f(b)-f(a))$

$f(b)-f(x_1) > f(b)-f(a) - \dfrac{x_1-a}{b-a}(f(b)-f(a)) = (f(b)-f(a))\left[1-\dfrac{x_1-a}{b-a}\right] = (f(b)-f(a))\dfrac{b-x_1}{b-a}$

$\rightarrow \dfrac{f(b)-f(x_1)}{b-x_1} > \dfrac{f(b)-f(a)}{b-a}$

$\rightarrow \dfrac{f(b)-f(x_1)}{b-x_1} > \alpha$

Then, $\alpha < \dfrac{f(b)-f(x_1)}{b-x_1} \leq \dfrac{f(x_2)-f(x_1)}{x_2-x_1} \leq \dfrac{f(x_2)-f(b)}{x_2-b}$, $\perp$ w/ (1).

Therefore, $\forall x, x \in (a,b) \rightarrow \dfrac{f(x_1)-f(a)}{x_1-a} = \alpha$

But then the graph of f contains a straight line segment.

$\perp$.

Therefore $\dfrac{f(b)-f(a)}{b-a} < \dfrac{f(c)-f(a)}{c-a}$

That is, f is convex.

(b)

Theorem 1 Let f be weakly convex. If f diff at a, then the graph of f lies above or on the tangent line through $(a, f(a))$, except at $(a, f(a))$. If $a<b$ and f diff at a and b then $f'(a) \leq f'(b)$.

Proof

Let $0<h_1<h_2$

Since f weakly convex, by def. of weak convexity on $[a, a+h_2]$

$$\frac{f(a+h_1)-f(a)}{h_1} \leq \frac{f(a+h_2)-f(a)}{h_2}$$

Therefore $\dfrac{f(a+h)-f(a)}{h}$ decreases or stays constant as $h \to 0^+$

Therefore $f'(a) \leq \dfrac{f(a+h)-f(a)}{h}$ for $h>0$.

This means that for $h>0$ the secant line through $(a, f(a))$ and $(a+h, f(a+h))$ lies above or on the tangent line.

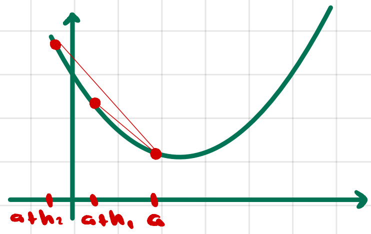
$$g(x) = \frac{f(a+h)-f(a)}{h}(x-a)+f(a)$$

$$t(x) = f'(a)(x-a)+f(a)$$

$$g(a+h) = f(a+h)-f(a)+f(a) = f(a+h)$$

$$t(a+h) = f'(a)h+f(a) \leq \frac{f(a+h)-f(a)}{h}h+f(a) = f(a+h)$$

ie $t(a+h) \leq f(a+h)$



Thus, to the right of a the graph of f lies above or on the tangent.

Now consider $h<0$.

Since $a+h_2<a+h_1<a$, $\dfrac{f(a)-f(a+h_1)}{-h_1} \geq \dfrac{f(a)-f(a+h_2)}{-h_2} \iff \dfrac{f(a+h_1)-f(a)}{h_1} \geq \dfrac{f(a+h_2)-f(a)}{h_2}$

Therefore the slope grows or stays constant as $h \to 0^-$, and $f'(a) \geq \dfrac{f(a+h)-f(a)}{h}$ for any $h<0$

This means the secant through $(a+h, f(a+h))$ and $(a, f(a))$ lies above or on the tangent line.
The calculations to show this are the same as before

$$g(x) = \frac{f(a+h)-f(a)}{h}(x-a)+f(a) \to g(a+h) = f(a+h)-f(a)+f(a) = f(a+h)$$

$$t(x) = f'(a)(x-a)+f(a) \to t(a+h) = f'(a)h+f(a)$$

if $f'(a) \geq \dfrac{f(a+h)-f(a)}{h} \geq 0$, then since $h<0$, $f'(a)h \leq \dfrac{f(a+h)-f(a)}{h}h$

if $\dfrac{f(a+h)-f(a)}{h} \leq f'(a) < 0$, then since $h<0$, $f'(a)h \leq \dfrac{f(a+h)-f(a)}{h}h$

Therefore, $t(a+h) = f'(a)h+f(a) \leq \dfrac{f(a+h)-f(a)}{h}h+f(a) = f(a+h)$

Hence, the graph of f lies above or on the tangent line, except at $(a, f(a))$. This is the first part of the theorem.

Now suppose $a<b$.

Then, according to the first part of the theorem

$$f'(a) \leq \frac{f(a+h)-f(a)}{h}$$

in particular

$$f'(a) \leq \frac{f(a+(b-a))-f(a)}{b-a} = \frac{f(b)-f(a)}{b-a}$$

Also

$$f'(b) \geq \frac{f(b+(a-b))-f(b)}{a-b} \quad (\text{since } a-b<0)$$

$$= \frac{f(b)-f(a)}{b-a}$$

Hence $f'(a) \leq f'(b)$

Lemma: suppose f diff and f' constant or increasing.

If $a<b$ and $f(a)=f(b)$, then $f(x) \leq f(a)=f(b)$ for $a<x<b$.

Proof

Suppose $f(x)>f(a)=f(b)$ for some x in (a,b).

Then the max of f on $[a,b]$ occurs at some $x_0 \in (a,b)$ with $f(x_0)>f(a)$ and $f'(x_0)=0$.

Apply MVT to $[a,x_0]$.

$$\exists x_1,\ a<x_1<x_0 \wedge f'(x_1)=\frac{f(x_0)-f(a)}{x_0-a}>0=f'(x_0)$$

$\perp$, because f' increasing by assumption.

Therefore $\forall x,\ x\in(a,b) \to f(x) \leq f(a)=f(b)$

Theorem 2 f diff. $\rightarrow$ f weakly convex
f' increasing or constant

Proof

Let $a<b$.

Let $g(x) = f(x) - \dfrac{f(b)-f(a)}{b-a}(x-a)$

$g'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$, which is increasing.

$g(a) = g(b) = f(a)$

Apply lemma to g.

$\forall x,\ x \in (a,b) \rightarrow g(x) \leq g(a) = g(b)$

$\rightarrow f(x) - \dfrac{f(b)-f(a)}{b-a}(x-a) \leq f(a)$

$\rightarrow \dfrac{f(x)-f(a)}{x-a} \leq \dfrac{f(b)-f(a)}{b-a}$

Hence f weakly convex.

Theorem 3 f diff

f lies above or on each tangent line except at point of contact $\rightarrow$ f weakly convex

Proof

Let $a < b$.

Tangent line at $(a, f(a))$: $g(x) = f'(a)(x-a) + f(a)$

$(b, f(b))$ lies above or on the tangent line: $f(b) \geq f'(a)(b-a) + f(a)$

$f(b) - f(a) \geq f'(a)(x-a)$ (1)

Tangent line at $(b, f(b))$: $h(x) = f'(b)(x-b) + f(b)$

$(a, f(a))$ lies above or on: $f(a) \geq f'(b)(a-b) + f(b)$

$f(b) - f(a) \leq f'(b)(b-a)$ (2)

Hence

$f'(a)(x-a) \leq f'(b)(x-a)$

$\rightarrow f'(a) \leq f'(b)$

By Th. 2, f is weakly convex.

Theorem 4 f diff on an interval and intersects each of its tangent lines just once $\rightarrow$ f is either convex or concave on that interval

Proof

Two parts.

(1) no straight line can intersect the graph of f in three different points.

Assume some straight line did intersect graph of f at $(a, f(a))$, $(b, f(b))$, and $(c, f(c))$, w/ $a<b<c$.

Then we'd have

$$\frac{f(b)-f(a)}{b-a} = \frac{f(c)-f(b)}{c-b}$$

Let $g(x) = \dfrac{f(x)-f(a)}{x-a}$ $\quad x \in [b, c]$

Then $g(b) = g(c)$

Rolle's Thm $\rightarrow \exists x,\ x \in (b,c) \wedge g'(x) = 0$

$$g'(x) = \frac{(x-a)f'(x) - (f(x)-f(a))}{(x-a)^2} = 0$$

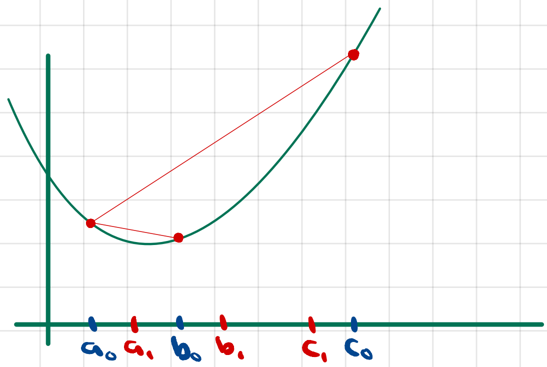
$$\rightarrow f'(x) = \frac{f(x)-f(a)}{x-a}$$

ie the tangent has slope equal to the slope between $(a, f(a))$ and $(x, f(x))$. Thus the tangent passes through $(x, f(x))$ and $(a, f(a))$, contradicting our assumption that f intersects each tangent line only once.

---

(2)

Suppose $a_0 < b_0 < c_0$ and $a_1 < b_1 < c_1$ are points in the interval.



Let
$x_t = (1-t)a_0 + ta_1$
$y_t = (1-t)b_0 + tb_1$ $\quad 0 \le t \le 1$
$z_t = (1-t)c_0 + tc_1$

Then $x_0 = a_0$, $x_1 = a_1$, x_t points lie between a_0 and a_1.
Analogous statements for y_t and z_t.

Moreover, $x_t < y_t < z_t \quad 0 \le t \le 1$

$$\text{Let } g(t) = \frac{f(y_t)-f(x_t)}{y_t - x_t} - \frac{f(z_t)-f(x_t)}{z_t - x_t} \quad 0 \le t \le 1$$

Assume $g(t) = 0$
$\rightarrow f(y_t) - f(x_t) = f(z_t) - f(x_t) = 0$
$\rightarrow f(x_t) = f(y_t) = f(z_t)$
$\rightarrow$ a straight line $y = f(x_t)$ intersects f in three places.
$\perp$ with part (1).

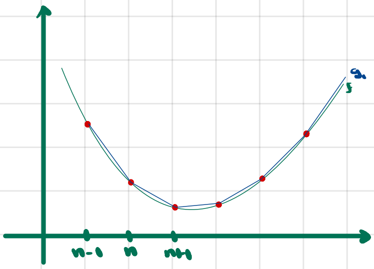
Hence $g(t) \neq 0$ for all $t \in [0,1]$. Ie, $\dfrac{f(y_t)-f(x_t)}{y_t - x_t} \neq \dfrac{f(z_t)-f(x_t)}{z_t - x_t}$.

Therefore either $g(t) > 0$ or $g(t) < 0$, ie either f convex or concave.

12. Two convex fns f and g s.t.

$$f(x) = g(x) \iff x \text{ integer}$$

Proof



f convex, g_1 weakly convex.

Can we make g_1 convex?

The straight line portions of g_1 must be replaced w/ convex curves.

For each integer n, consider $[n-1, n]$ and $[n, n+1]$.

From problem 9b, if we define a convex h_1 on $[n-1,n]$ and a convex h_2 on $[n,n+1]$ such that $h_1(n) = h_2(n)$ and $h_{1-}'(n) \leq h_{2+}'(n)$ then the

fn $g(x) = \begin{cases} h_1(x) & x \in [n-1,n] \\ h_2(x) & x \in [n,n+1] \end{cases}$ is convex on $[n-1,n+1]$.

Here is a possible way to do this:

For $x \in [n-1,n]$, $g_1(x) = f(n-1) + (f(n)-f(n-1))(x-(n-1))$

$$g_1'(x) = f(n) - f(n-1)$$

$$\text{Let } g(x) = \frac{f(x) + g_1(x)}{2}$$

$$\text{Then } g'(x) = \frac{f'(x) + g_1'(x)}{2} = \frac{f'(x) + f(n) - f(n-1)}{2}$$

For $x \in [n, n+1]$, $g_1(x) = f(n) + (f(n+1) - f(n))(x-n)$

$$g_1'(x) = f(n+1) - f(n)$$

$$g'(x) = \frac{f'(x) + f(n+1) - f(n)}{2}$$

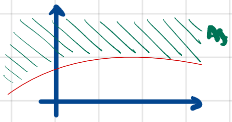
Because f is convex, $f(n+1) - f(n) > f(n) - f(n-1)$

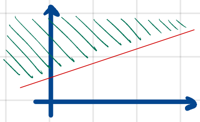
$$\text{Hence } g_-'(n) = \frac{f'(n) + f(n) - f(n-1)}{2} < \frac{f'(n) + f(n+1) - f(n)}{2} = g_+'(n)$$

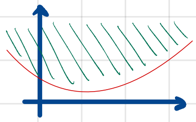
13. A : set of points in the plane.

A contains line segment joining any two points in A $\rightarrow$ A convex

$A_f = \{(x,y) : y \geq f(x)\}$



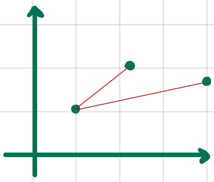




A_f convex $\leftrightarrow$ f is weakly convex

Proof

Assume A_f is convex.



Assume f is not weakly convex.

Then $\exists a, x, b$ in f's domain such that

$$\frac{f(x)-f(a)}{x-a} > \frac{f(b)-f(a)}{b-a}$$

$$f(x) > f(a) + \frac{x-a}{b-a}(f(b)-f(a))$$

The segment joining $(a, f(a))$ and $(b, f(b))$ is

$$g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a), \quad x \in [a,b]$$

Therefore A_f contains $(a, f(a))$ and $(b, f(b))$ but not $(x, g(x))$ because $y \geq f(x) > g(x)$

$\rightarrow$ A not convex.

$\perp$.

$\rightarrow$ f weakly convex.

Assume f weakly convex.

Then $\forall a, x, b$ in f's domain, $\dfrac{f(x)-f(a)}{x-a} \leq \dfrac{f(b)-f(a)}{b-a}$

$\rightarrow f(x) \leq f(a) + (x-a)\dfrac{f(b)-f(a)}{b-a} = g(x), \quad x \in [a,b]$

ie f lies below or at the line segment connecting $(a, f(a))$ and $(b, f(b))$.

Therefore the line segment $\in A_f$.

$\rightarrow$ A_f convex