Ch. 20 - Approximation by Polynomial Functions

Chapter goal: reduce computation of $f(x)$ to evaluation of polynomial that is close approximation to f.

Suppose

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

then we can express the coeff. a_i in terms of p and its various derivatives at 0

$$p(0) = a_0$$

$$p'(x) = a_1 + 2a_2 x + \cdots + a_n \cdot n x^{n-1}$$

$$p'(0) = a_1$$

$$p''(x) = 2a_2 + 3\cdot 2 a_3 x + \cdots + a_n \cdot n(n-1) x^{n-2}$$

$$p''(0) = 2a_2$$

and so on, such that in general

$$p^{(k)}(0) = k!\, a_k \quad \longrightarrow \quad a_k = \frac{p^{(k)}(0)}{k!}$$

which is true also for $k=0$ if $p^{(0)} = p$ and $0! = 1$.

If we start with a polyn. in $(x-a)$ instead of x, then we have

$$p(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$$

and by diff. and eval. at a, we reach

$$a_k = \frac{p^{(k)}(a)}{k!}$$

Suppose f is some fn s.t. $f^{(1)}(a), \ldots, f^{(n)}(a)$ exist.

Let $a_k = \dfrac{f^{(k)}(a)}{k!}$ $\quad 0 \le k \le n$

and define

$$P_{n,a}(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n$$

Then $P_{n,a}$ is called the Taylor Polynomial of degree n for f at a

Note that $P_{n,a}^{(k)}(a) = a_k \cdot k! = f^{(k)}(a)$

That is, at a, the polyn. has same k^{th} derivatives as f, for $0 \le k \le n$.

Example: Taylor Polynomial of degree n for $\sin(x)$ at 0

$f(x) = \sin x$	$f(0) = 0$	$a_0 = 0$
$f'(x) = \cos x$	$f'(0) = 1$	$a_1 = 1$
$f''(x) = -\sin x$	$f''(0) = 0$	$a_2 = 0$
$f^{(3)}(x) = -\cos x$	$f^{(3)}(0) = -1$	$a_3 = -\frac{1}{3!} = -\frac{1}{6}$

starting at $f^{(4)}$ the derivatives cycle back to the four initial $f^{(k)}$.

$$P_{n,a}(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n$$

$$a_k = \frac{f^{(k)}(a)}{k!}$$

$$P_{2n+1,0}(x) = 0 + 1\cdot(x-0) + 0 - \frac{1}{3!}(x-0)^3 + 0 + \frac{1}{5!}(x-0)^5$$
$$+ 0 - \frac{1}{7!}(x-0)^7 + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

note that we chose an odd order for the Taylor Polyn.

$P_{2n+2,0} = P_{2n+1,0}$, because the $(2n+2)^{th}$ term is zero.

Recap

Let's recap the steps of what we're doing.

For a polyn. $p(x) = \sum_{i=0}^{n} a_i x^i$, we have $p^{(k)}(x) = k!\, a_k$.

Therefore, $a_k = \dfrac{p^{(k)}(x)}{k!}$.

What if we form a polyn. but chose the coeff. such that

$$a_k = \frac{f^{(k)}(x)}{k!}$$

Note that we still have $p^{(k)}(x) = k!\, a_k$ but now this turns

into $p^{(k)}(x) = f^{(k)}(x)$.

Ie, at some specific point x, the first n derivatives coincide

between the polyn. p and the fn f.

Example: Taylor Polynomial of degree n for cos x

$f = \cos$ $\quad f(0) = 1$

$f' = -\sin$ $\quad f'(0) = 0$

$f'' = -\cos$ $\quad f''(0) = -1$

$f^{(3)} = \sin$ $\quad f^{(3)}(0) = 0$

$$P_{2n,0}(x) = \frac{f^{(0)}(0)}{0!} + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \dots + (-1)^{n+1}\frac{f^{(2n)}(0)}{(2n)!}x^{2n}$$

$$= 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + \dots + (-1)^{n+1}\frac{x^{2n}}{(2n)!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots + (-1)^{n+1}\frac{x^{2n}}{(2n)!}$$

Example: Taylor Polynomial of degree n for e^x

$f^{(k)} = e^x \quad f^{(k)}(0) = 1$

$$P_{n,0}(x) = \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^{n}\frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!}$$

Example: Taylor Polynomial of degree n for log(x)

$f = \log x$

$f' = \frac{1}{x}$

$f'' = \frac{-1}{x^2}$

$f''' = \frac{2}{x^3}$

$f^{(4)} = \frac{-3 \cdot 2}{x^4}$

$f^{(5)} = \frac{4 \cdot 3 \cdot 2}{x^5}$

$f^{(2n)} = \frac{-(2n)!}{x^{2n}}$

$f^{(2n+1)} = \frac{(2n+1)!}{x^{2n+1}}$

Let $a \neq 0$. Then

$$P_{n,a}(x) = \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k = \log(a) + \frac{1}{1! \cdot a}(x-a) - \frac{1}{2!\,a^2}(x-a)^2 + \frac{2!}{3!\,a^3}(x-a)^3 - \frac{3!}{4!\,a^4}(x-a)^4 + (-1)^{n-1}\frac{(n-1)!}{n! \cdot a^n}(x-a)^n$$

$$P_{n,1}(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots + (-1)^{n-1}\frac{(x-1)^n}{n}$$

Example: Taylor Polynomial of degree n for $\log(1+x)$

$f = \log(1+x)$ $\qquad f(0) = 0$

$f' = \dfrac{1}{1+x}$ $\qquad f'(0) = 1$

$f'' = -\dfrac{1}{(1+x)^2}$ $\qquad f''(0) = -1$

$f''' = \dfrac{2}{(1+x)^3}$ $\qquad f'''(0) = 2$

$f^{(4)} = -\dfrac{3!}{(1+x)^4}$ $\qquad f^{(4)}(0) = -3!$

$f^{(5)} = \dfrac{4!}{(1+x)^5}$ $\qquad f^{(5)}(0) = 4!$

$f^{(n)} = (-1)^{n-1}\dfrac{(n-1)!}{(1+x)^n}$ $\qquad f^{(n)}(0) = (-1)^{n-1}\cdot(n-1)!$

$$P_{n,0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-0)^k = 0 + \frac{1}{1!}x^1 + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-3!}{4!}x^4 + \frac{4!}{5!}x^5 + \dots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots + (-1)^{n-1}\frac{x^n}{n}$$

$P_{1,a}(x) = f(a) + f'(a)(x-a)$

$$\frac{f(x) - P_{1,a}(x)}{x-a} = \frac{f(x) - f(a)}{x-a} - f'(a)$$

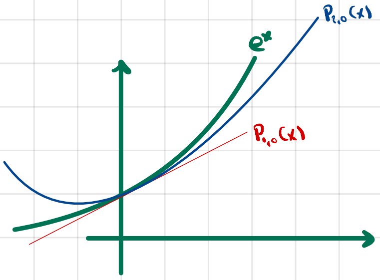
$$\lim_{x \to 0} \frac{f(x) - P_{1,a}(x)}{x-a} = \lim_{x \to 0} \frac{f(x) - f(a)}{x-a} - f'(a) = f'(a) - f'(a) = 0$$

$f(x) - P_{1,a}(x)$ becomes small relative to $x-a$.

Example: $f(x) = e^x$

$P_{1,0}(x) = 1 + x$

$P_{2,0}(x) = 1 + x + \dfrac{x^2}{2}$



$$\lim \frac{f(x) - P_{1,0}(x)}{x-0} = \lim_{x \to 0} \frac{e^x - 1 - x}{x} = \frac{0}{0} = \lim_{x \to 0} \frac{e^x - 1}{1} = 0$$

What if we compare $f(x) - P_{1,0}(x)$ to x^2?

$$\lim \frac{f(x) - P_{1,0}(x)}{(x-0)^2} = \lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \frac{0}{0} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \frac{0}{0} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}$$

$f(x) - P_{1,0}(x)$ does not become small compared to x^2.

On the other hand

$$\lim \frac{f(x) - P_{2,0}(x)}{(x-0)^2} = \lim_{x \to 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^2} = \lim_{x \to \infty} \frac{e^x - 1 - x}{2x} = \lim_{x \to \infty} \frac{e^x - 1}{2} = 0$$

Ie, relative to x^2, $f(x) - P_{2,0}(x)$ does become small.

In general, this result is true.

If $f'(a)$ and $f''(a)$ exist then

$$\lim_{x \to a} \frac{f(x) - P_{2,a}(x)}{(x-a)^2} = 0$$

in fact, the analogous result is true for the n^{th} order Taylor Polyn.

Theorem 1 Suppose $f'(a), \dots, f^{(n)}(a)$ all exist.

let $a_k = \dfrac{f^{(k)}(a)}{k!}$ $\quad 0 \le k \le n$

and define $P_{n,a}(x) = a_0 + a_1(x-a) + \dots + a_n(x-a)^n$, ie just the Taylor Polynomial of degree n for f at a as prev. defined.

Then,

$$\lim_{x \to a} \frac{f(x) - P_{n,a}(x)}{(x-a)^n} = 0$$

Proof

$$\frac{f(x) - P_{n,a}(x)}{(x-a)^n} = \frac{f(x) - \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i}{(x-a)^n} = \frac{f(x) - \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i - \frac{f^{(n)}(a)}{n!}(x-a)^n}{(x-a)^n}$$

$$= \frac{f(x) - \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i}{(x-a)^n} - \frac{f^{(n)}(a)}{n!}$$

$$= \frac{f(x) - Q(x)}{g(x)} - \frac{f^{(n)}(a)}{n!}, \quad Q(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i, \quad g(x) = (x-a)^n$$

we want to prove $\displaystyle \lim \frac{f(x) - P_{n,a}(x)}{(x-a)^n} = \lim_{x \to a} \left[\frac{f(x) - Q(x)}{g(x)} - \frac{f^{(n)}(a)}{n!} \right] = 0 \quad \to \quad \lim_{x \to a} \frac{f(x) - Q(x)}{g(x)} = \lim_{x \to a} \frac{f^{(n)}(a)}{n!}$

Note

$Q^{(k)}(a) = f^{(k)}(a) \qquad k \le n-1$

$g^{(k)}(x) = \dfrac{n!}{(n-k)!}(x-a)^{n-k}$

note that $f, f', \dots, f^{(n-1)}$ are all cont. since they are all diff.
Q is a polynomial of $n-1^{th}$ degree.
$Q, Q', \dots, Q^{(k)}$ are all cont. for any k.

Thus, $Q(a) = f(a)$ and

Thus $\displaystyle \lim_{x \to a} Q^{(k)}(x) = f^{(k)}(a)$

$$\lim_{x \to a} [f(x) - Q(x)] = f(a) - f(a) = 0$$

$$\lim_{x \to a} [f'(x) - Q'(x)] = f'(a) - f'(a) = 0$$

$(\dots)$

$$\lim_{x \to a} [f^{(n-2)}(x) - Q^{(n-2)}(x)] = 0$$

also,

$$\lim_{x\to a} g(x) = \lim_{x\to a} g'(x) = \dots = \lim_{x\to a} g^{(n-2)}(x) = 0$$

because $g(x) = (x-a)^n$

$g^{(h)}(x) = \frac{n!}{(n-h)!}(x-a)^{n-h}$

Hence,

$$\lim_{x\to a} \frac{f(x)-Q(x)}{g(x)} = \frac{0}{0} = \lim_{x\to a} \frac{f'(x)-Q'(x)}{g'(x)} = \frac{0}{0} = (\dots) \text{ apply L'Hôpital's Rule } n-2 \text{ more times}$$

$$= \lim_{x\to a} \frac{f^{(n-1)}(x) - Q^{(n-1)}(x)}{g^{(n-1)}(x)}$$

constant $= f^{(n-1)}(a)$

$$= \lim_{x\to a} \frac{f^{(n-1)}(x) - Q^{(n-1)}(x)}{n!(x-a)}$$

$$= \lim_{x\to a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{n!(x-a)}$$

$$= \frac{f^{(n)}(a)}{n!}$$

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consider the fns

$f(x) = (x-a)^n$

$g(x) = -(x-a)^n$

$f' = n(x-a)^{n-1}$

$f'' = n(n-1)(x-a)^{n-2}$

$(\dots)$

$f^{(n-1)} = n(n-1)\dots 2 \cdot 1 \cdot (x-a)$

$f^{(n)} = n!$

$g' = -n(x-a)^{n-1}$

$(\dots)$

$g^{(n-1)} = -n(n-1)\dots 2 \cdot 1\,(x-a)$

$g^{(n)} = -n!$

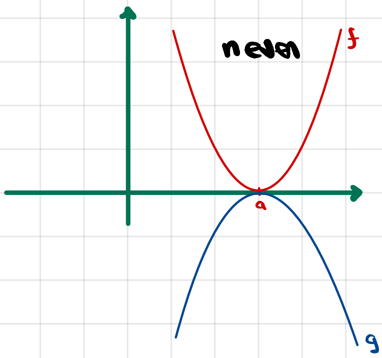
Thus, $f(a) = f'(a) = \dots = f^{(n-1)}(a) = 0$

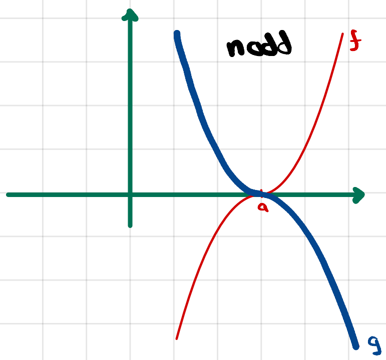
$f^{(n)}(a) \neq 0$

$g(a) = g'(a) = \dots = g^{(n-1)}(a) = 0$

$g^{(n)}(a) \neq 0$

These functions look like





The next theorem basically says that f has a local extremum in the first case, but not in the second case, and such a result is obtained based on knowledge of the n^{th} derivative when the $n-1$ first derivatives are zero.

Theorem 2 Suppose that

$f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0$
$f^{(n)}(a) \neq 0$

Then,

(1) n even, $f^{(n)} > 0 \to$ local min at a
(2) " " " $< 0 \to$ " max " "
(3) n odd $\to$ neither local max nor min at a

Proof

WLOG, assume $f(a) = 0$ (ie, replace f by $f - f(a)$)

Compute $P_{n,a}$, the n^{th} order Taylor Polyn. of f at a.

$$P_{n,a}(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Th. 1 tells us that

$$0 = \lim_{x \to a} \frac{f(x) - P_{n,a}(x)}{(x-a)^n} = \lim_{x \to a} \left[\frac{f(x)}{(x-a)^n} - \frac{f^{(n)}(a)}{n!} \right]$$

otherwise, we cannot get the difference between the difference of either term to 0.

T.F., if x suff. close to a, then the terms $\frac{f(x)}{(x-a)^n}$ and $\frac{f^{(n)}(a)}{n!}$ have the same sign

Suppose n is even. Then $(x-a)^n > 0$ for $x \neq a$. T.F. $f(x)$ has same sign as $f^{(n)}(a)$.

T.F., $f^{(n)}(a) > 0 \to f(x) > 0 = f(a)$, for such x close to a.

T.F. a is a local min.

Also, $f^{(n)}(a) < 0 \to f(x) < 0 = f(a)$, for x close to a

T.F. a is a local max

Now assume n is odd. Again, the terms $\frac{f(x)}{(x-a)^n}$ and $\frac{f^{(n)}(a)}{n!}$ must have the same sign for x suff. close to a.

$f^{(n)}(a) > 0 \to f(x)$ and $(x-a)^n$ have same sign.
$x > a \to (x-a)^n > 0 \to f(x) > 0 = f(a)$
$x < a \to (x-a)^n < 0 \to f(x) < 0 = f(a)$
$\to a$ is neither local max nor local min

$f^{(n)}(a) < 0 \to f(x)$ and $(x-a)^n$ have opposite signs
$x > a \to (x-a)^n > 0 \to f(x) < 0 = f(a)$
$x < a \to (x-a)^n < 0 \to f(x) > 0 = f(a)$
$\to a$ is neither local max nor local min

$\blacksquare$

Two functions are equal up to order n at a if

$$\lim_{x\to a}\frac{f(x)-g(x)}{(x-a)^n}=0$$

Th.1 says that $P_{n,a,f}$ and f are equal up to order n at a.

Theorem 3 Let P and Q be two polynomials in $(x-a)$, of degree $\leq n$. Suppose P and Q equal up to order n at a.

Then, $P=Q$. $\underline{e^x - x^2}$

Proof

Let $R=P-Q$.

R is polyn. of degree $\leq n$.

One way to prove our desired result is to prove that if

$$R(x)=b_0+\dots+b_n(x-a)^n$$

satisfies

$$\lim_{x\to a}\frac{R(x)}{(x-a)^n}=\lim_{x\to a}\frac{P(x)-Q(x)}{(x-a)^n}=0$$

then $R(x)=0$.

By assumption, P and Q are equal up to order n at a.

That is $\lim_{x\to a}\frac{P(x)-Q(x)}{(x-a)^k}=\lim_{x\to a}\frac{R(x)}{(x-a)^k}=0$ for $k=1,\dots,n$ (1)

$k=0\to\lim_{x\to a}R(x)=0$

But $\lim_{x\to a}R(x)=\lim_{x\to a}[b_0+\dots+b_n(x-a)^n]=b_0=0$

Thus $R(x)=b_1(x-a)+\dots+b_n(x-a)^n$

Hence $\frac{R(x)}{x-a}=b_1+b_2(x-a)+\dots+b_n(x-a)^{n-1}$

$\lim_{x\to a}\frac{R(x)}{x-a}=b_1=0$ (by (1))

By continuing in this way we reach the result that

$$b_0=b_1=\dots=b_n=0$$

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Let P_1 and P_2 be polynomials in $(x-a)$ of degree $\leq n$, each equal to f up to order n.

That is

$$\lim_{x\to a}\frac{f(x)-P_1(x)}{(x-a)^n}=0$$

$$\lim_{x\to a}\frac{f(x)-P_2(x)}{(x-a)^n}=0$$

Then

$$\frac{P_1(x)-P_2(x)}{(x-a)^n}$$

$$=\frac{-f(x)+P_1(x)+f(x)-P_2(x)}{(x-a)^n}$$

$$\frac{f(x)-P_2(x)}{(x-a)^n}+\frac{P_1(x)-f(x)}{(x-a)^n}$$

$$\to\lim_{x\to a}\frac{P_1(x)-P_2(x)}{(x-a)^n}=0$$

ie P_1 and P_2 are equal up to order n.

Th.3 says that $P_1=P_2$.

Since $P_{n,a,f}$ is such a polynomial, it too equals P_1 and P_2. Ie there is only one polyn. in $(x-a)$ of degree $\leq n$ that is equal up to order n at a to f, and that polyn. is the Taylor Polyn. of degree n for f at a.

Corollary Let f be n-times diff at a.

Suppose P is polyn. in $(x-a)$ of degree $\le n$, which equals f up to order n at a.

Then,

$$P = P_{n,a,f}$$

Proof

P and $P_{n,a,f}$ equal f up to order n at a.

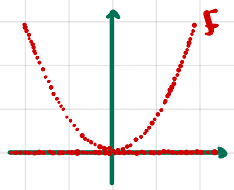
By Thm.3, $P = P_{n,a,f}$.

∎

Note: we need the assumption that f be n-times diff. at a to get the coroll. above.

Here is an example why

$$f(x) = \begin{cases} x^{n+1} & x \text{ irrational} \\ 0 & x \text{ rational} \end{cases}$$



Let $P(x) = 0$. Then at 0 we have $\lim\limits_{x \to 0} \frac{f(x) - P(x)}{x^k} = \lim \frac{x^{n+1}}{x^k} = \lim\limits_{x \to 0} x^{n+1-k}$, which equals 0 for $k = 0, 1, \dots, n$.

Ie, $P(x)$ is equal to f up to order n at 0.

$$f^{(k)}(x) = \lim_{x \to 0} \frac{f^{(k-1)}(x) - f^{(k-1)}(0)}{x}$$

$$f'(0) = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} x^n = 0$$

$f''(0) = \lim\limits_{x \to 0} \frac{f'(x)}{x}$ but $f'(x)$ is not defined. Hence, f'' and $f^{(k)}$ $k \ge 2$ are not defined

Hence, the Taylor Polynomial of any degree ≥ 1 at 0 is undefined.

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let's assume f does have n first deriv. defined.

For example, suppose

$$f(x) = \arctan(x) = \int_0^x \frac{1}{1+t^2}\,dt$$

arctan is an integral

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}$$

and we can rewrite the integral

Thus

$$\arctan(x) = \int_0^x \left[1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\right]$$

and integrate

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt$$

this portion is a pol. fn.

let

$$P(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$R(x) = (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt$$

Since $\left|\int_0^x \frac{t^{2n+2}}{1+t^2}\,dt\right| \le \left|\int_0^x t^{2n+2}\,dt\right| = \frac{|x|^{2n+3}}{2n+3}$

then

$$0 \le \lim_{x\to 0} \left|\int_0^x \frac{t^{2n+2}}{1+t^2}\,dt\right| \le \lim_{x\to 0} \frac{|x|^{2n+3}}{2n+3} = 0$$

then

$$\lim_{x\to 0} R(x) = \lim_{x\to 0} \frac{(-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt}{x^{2n+1}} = 0$$

Then,

$$\arctan(x) = P(x) + R(x)$$

$$0 = \lim_{x\to 0} \frac{(\arctan(x) - (P(x)+R(x)))}{x^{2n+1}} = \lim_{x\to 0} \frac{[\arctan(x) - P(x)]}{x^{2n+1}}$$

Thus P is equal to arctan up to order n and thus $P = P_{2n+1,0,\arctan}$

and discover the Taylor Pol fn. at 0

Ie we've found the Taylor Pol fn. of degree $2n+1$ for arctan at 0.

$$P_{2n+1,0,\arctan}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$= \sum_{i=0}^{2n+1} \frac{\arctan^{(i)}(0)}{i} x^i = \arctan(0) + \arctan'(0)x + \frac{\arctan''(0)}{2!}x^2 + \cdots + \frac{\arctan^{(2n+1)}(0)}{(2n+1)!}x^{2n+1}$$

$\to$ $\arctan(0) = 0$

$\arctan'(0) = 1$

$\arctan''(0) = 0$

$\arctan'''(0) = -\frac{1}{3} \cdot 3!$

$\arctan^{(4)}(0) = 0$

$\frac{\arctan^{(5)}(0)}{5!} = \frac{1}{5}$

(...)

$\frac{\arctan^{(k)}(0)}{k!} = 0$ k even

$\frac{\arctan^{(2l+1)}(0)}{(2l+1)!} = \frac{(-1)^l}{2l+1} \to \arctan^{(2l+1)}(0) = (-1)^l \cdot (2l)!$

Let's go back to

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{n+1}\int_0^x \frac{t^{2n+2}}{1+t^2}\,dt$$

we found that

$$\left|\int_0^x \frac{t^{2n+2}}{1+t^2}\,dt\right| \le \left|\int_0^x t^{2n+2}\,dt\right| = \frac{|x|^{2n+3}}{2n+3}$$

For $|x|<1$,

$$\frac{|x|^{2n+3}}{2n+3} \le \frac{1}{2n+3}$$

which we can make as small as we want by choosing n large enough.

Thus, using the Taylor polyn. we can compute arctan x, for $|x| \le 1$, as accurately as we want.

Notation

If f is a fn for which $P_{n,a}(x)$ exists then we define the remainder term as

$$f(x) = P_{n,a}(x) + R_{n,a}(x)$$

$$= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n,a}(x)$$

For $n=0$

$$f(x) = f(a) + R_{0,a}(x)$$

f cont. on $[a,x]$

FTC $\rightarrow$ $f(x) = f(a) + \int_a^x f'(t)\,dt$

Ie, $R_{0,a}(x) = \int_a^x f'(t)\,dt$

* note that if $x<a$ we have

$$R_{0,a}(x) = -(f(a) - f(x)) = -\int_x^a f'(t)\,dt = \int_a^x f'(t)\,dt$$

Let

$u(t) = f'(t) \quad \rightarrow \quad u'(t) = f''(t)$

$v(t) = t - x \quad \rightarrow \quad v'(t) = 1$

Then

$$\int_a^x f'(t)\,dt = \int_a^x u(t)v'(t)\,dt = u(t)v(t)\Big|_a^x + \int_a^x f''(t)\cdot(x-t)\,dt$$

$$f(x) = f(a) + \int_a^x f'(t)\,dt = f(a) + u(x)\cancel{v(x)}^{0} - u(a)v(a) + \int_a^x f''(t)\cdot(x-t)\,dt$$

$$= f(a) + f'(a)(x-a) + \int_a^x f''(t)\cdot(x-t)\,dt$$

Thus, $R_{1,a}(x) = \int_a^x f''(t)\cdot(x-t)\,dt$

and in the case for $n=1$ we have

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x f''(t)\cdot(x-t)\,dt$$

$$= f(a) - f'(a)(a-x) - \int_x^a f''(t)(x-t)\,dt$$

Now let

$$u(t) = f''(t) \to u'(t) = f'''(t)$$

$$v(t) = \frac{-(x-t)^2}{2} \to v'(t) = x-t$$

$$\int_a^x f''(t)(x-t)\,dt = u(t)v(t)\Big|_a^x - \int_a^x f'''(t)\frac{-(x-t)^2}{2}\,dt$$

$$= u(x)\cancel{v(x)}^{0} - u(a)v(a) + \int_a^x f'''(t)\frac{(x-t)^2}{2}\,dt$$

$$= \frac{f''(a)(x-a)^2}{2} + \int_a^x \frac{f'''(t)}{2}(x-t)^2\,dt$$

Thus

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x f''(t)\cdot(x-t)\,dt$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \int_a^x \frac{f'''(t)}{2}(x-t)^2\,dt$$

$$\to R_{2,a}(x) = \int_a^x \frac{f'''(t)}{2}(x-t)^2\,dt$$

Assume that $R_{k,a}(x) = \int_a^x \frac{f^{(k+1)}}{k!}(x-t)^k\,dt$. Then

$$f(x) = \sum_{i=0}^{k} \frac{f^{(i)}(a)}{i!}(x-a)^i + \int_a^x \frac{f^{(k+1)}}{k!}(x-t)^k\,dt$$

Let $u(t) = f^{(k+1)}(t) \to u'(t) = f^{(k+2)}(t)$

$$v'(t) = \frac{(x-t)^k}{k!} \to v(t) = -\frac{(x-t)^{k+1}}{(k+1)!}$$

$$u(t)v(t)\Big|_a^x - \int_a^x f^{(k+2)}(t)\frac{-(x-t)^k}{k!}\,dt$$

$$= -u(a)v(a) + \int_a^x f^{(k+2)}(t)\frac{(x-t)^k}{k!}\,dt$$

$$= \frac{f^{(k+1)}(a)(x-a)^k}{k!} + \int_a^x f^{(k+2)}(t)\frac{(x-t)^k}{k!}\,dt$$

$$\int_a^x \frac{f^{(k+1)}}{k!}(x-t)^k\,dt = \int_a^x u(t)\cdot v'(t)\,dt = u(t)v(t)\Big|_a^x - \int_a^x f^{(k+2)}(t)\frac{(-1)(x-t)^{k+1}}{(k+1)!}\,dt = -u(a)v(a) - \int_a^x f^{(k+2)}(t)\frac{(-1)(x-t)^{k+1}}{(k+1)!}\,dt$$

$$= -f^{(k+1)}(a)\cdot\frac{(-1)(x-a)^{k+1}}{(k+1)!} + \int_a^x \frac{f^{(k+2)}(t)}{(k+1)!}(x-t)^{k+1}\,dt$$

and

$$f(x) = \sum_{i=0}^{k+1} \frac{f^{(i)}(a)}{i!}(x-a)^i + \int_a^x f^{(k+2)}(t)\frac{(x-t)^{k+1}}{(k+1)!}\,dt$$

$$\to R_{k+1,a} = \int_a^x f^{(k+2)}(t)\frac{(x-t)^{k+1}}{(k+1)!}\,dt$$

By induction, if $f^{(n+1)}$ is cont. on $[a,x]$ then $R_{n,a}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt$, the integral form of the remainder.

Can we estimate $R_{n,a}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt$?

Let m and M be min and max of $\frac{f^{(n+1)}}{n!}$ on $[a,x]$. Then

$$m\int_a^x (x-t)^n\,dt \le R_{n,a}(x) \le M\int_a^x (x-t)^n\,dt$$

$$m\,\frac{(x-a)^{n+1}}{n+1} \le R_{n,a}(x) \le M\,\frac{(x-a)^{n+1}}{n+1}$$

$\rightarrow R_{n,a}(x) = \alpha\,\frac{(x-a)^{n+1}}{n+1}$, $\alpha \in [m,M]$

But $f^{(n+1)}$ is cont., so $\frac{f^{(n+1)}}{n!}$ assumes the value α in (a,x) at some t

$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{n!}\,\frac{(x-a)^{n+1}}{n+1} = \frac{f^{(n+1)}}{(n+1)!}(x-a)^{n+1}$, the Lagrange form of the remainder

---

Lemma Suppose R is $(n+1)$-times diff. on $[a,b]$, and

$R^{(k)}(a) = 0$ for $k = 0,1,2,\dots,n$

Then, for any x in $(a,b]$ we have

$\frac{R(x)}{(x-a)^{n+1}} = \frac{R^{(n+1)}(t)}{(n+1)!}$ for some t in (a,x)

Proof

$n=0$. $\frac{R(x)}{(x-a)} = \frac{R(x)-R(a)}{x-a} = R'(t)$, the MVT.

For $n=k$, assume as inductive hypothesis that

if R is $(n+1)$-times diff on $[a,b]$ and

$R^{(k)}(a) = 0$ for $k = 0,1,2,\dots,n$

then $\frac{R(x)}{(x-a)^{k+1}} = \frac{R^{(k+1)}(t)}{(k+1)!}$

Now assume R is $(n+2)$-times diff on $[a,b]$ and $R^{(k)}(a) = 0$ for $k = 0,1,2,\dots,n+1$.

Theorem 8 (Cauchy Mean Value Theorem)

f,g cont. on $[a,b]$ and diff on (a,b)

$\rightarrow \exists x, x \in (a,b) \wedge [f(b)-f(a)]g'(x) = [g(b)-g(a)]f'(x)$

if $g(b) \ne g(a)$ and $g'(x) \ne 0$ we can rewrite this as

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x)}{g'(x)}$$

Let's apply CS-MVT to

$R(x)$

$g(y) = (y-a)^{k+2}$

$\frac{R(x) - \cancel{R(a)}^{0}}{g(x) - \cancel{g(a)}^{0}} = \frac{R'(z)}{g'(z)}$, $z \in (a,x)$

$\frac{R(x)}{(x-a)^{k+2}} = \frac{R'(z)}{(k+2)(z-a)^{k+1}} = \frac{1}{k+2}\,\frac{R'(z)}{(z-a)^{k+1}}$

Since R is $(n+2)$-times diff, then R' is $(n+1)$-times diff and by the inductive h.p.,

$\frac{R'(z)}{(z-a)^{k+1}} = \frac{(R')^{(k+1)}(t)}{(k+1)!}$ for some t in (a,z)

Thus

$$\frac{R(x)}{(x-a)^{k+2}} = \frac{1}{k+2}\,\frac{(R')^{(k+1)}(t)}{(k+1)!} = \frac{R^{(k+2)}(t)}{(k+2)!}$$

∎

Theorem 4 (Taylor's Theorem)

Suppose $f', \ldots, f^{(n+1)}$ are defined on $[a, x]$ and that $R_{n,a}(x)$ is defined by

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n,a}(x) \quad (1)$$

Then

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1} \quad t \in (a, x)$$

Proof

By the def. (1), $R_{n,a}(x) = f(x) - \left[f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n\right]$

T.F. $R_{n,a}$ is $(n+1)$-times diff on $[a, x]$ and $R_{n,a}^{(k)}(a) = 0$ for $k = 0, 1, 2, \ldots, n$.

By the lemma,

$$R_{n,a}(x) = \frac{R^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1} = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}, \text{ for some } t \in (a, x).$$

■

Examples

$$f(x) = P_{n,a}(x) + R_{n,a}(x)$$

$$= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i + \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}, \ t \in (a, x)$$

$$\sin(x) = \sin(a) + \cos(a)(x-a) - \frac{\sin(a)}{2!}(x-a)^2 - \frac{\cos(a)}{3!}(x-a)^3 + \cdots + (-1)^n \frac{\sin^{(2n+1)}(a)}{(2n+1)!}(x-a)^{2n+1} + \frac{\sin^{(2n+2)}(t)}{(2n+2)!}(x-a)^{2n+2}$$

Using a Taylor Polyn. at 0, we have

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \frac{\sin^{(2n+2)}(t)}{(2n+2)!} x^{2n+2}$$

How do we estimate the remainder?

$$|\sin^{(2n+2)}(t)| \le 1 \quad \rightarrow \quad \left| \frac{\sin^{(2n+2)}(t)}{(2n+2)!} x^{2n+2} \right| \le \frac{|x|^{2n+2}}{(2n+2)!}$$

From ch. 16 we know that $\frac{x^n}{n!}$ can be made arbit. small.

Thus, by choosing a Taylor Polyn of sufficiently high order, we can compute $\sin(x)$ to any degree of accuracy.

Example: compute $\sin(2)$ with remainder $< 10^{-4}$

$$R_{n,0} \le \frac{|2|^{2n+2}}{(2n+2)!} < 10^{-4} \rightarrow n = 5$$

$$\rightarrow \sin(x) = x - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \frac{2^9}{9!} - \frac{2^{11}}{11!} + R$$

Example: e^x

$$e^x = P_{n,a}(x) + R_{n,a}(x)$$

$$= \sum_{i=0}^{n} \frac{e^x}{i!}(x-a)^i + \frac{e^t}{(n+1)!}(x-a)^{n+1}$$

$$e^x = \sum_{i=0}^{n} \frac{x^i}{i!} + \frac{e^t x^{n+1}}{(n+1)!}$$

Assume $x \geq 0$.

On $[0, x]$, max value of e^t is e^x.

Hence

$$R_{n,0} = \frac{e^t x^{n+1}}{(n+1)!} \leq \frac{e^x x^{n+1}}{(n+1)!} < \frac{4^x x^{n+1}}{(n+1)!} \quad (1)$$

Suppose $0 \leq x \leq 1$. Then

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + R_{n,0} \quad \text{and } R_{n,0} < \frac{4}{(n+1)!} \quad (\text{since } 0 \leq x \leq 1)$$

$$n = 4 \rightarrow R_{n,0} < \frac{4}{5!} < \frac{1}{10}$$

T.F. for $x = 1$ we have

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + R, \quad 0 < R < \frac{1}{10}$$

$$= 2 + \frac{17}{24} + R$$

$$\rightarrow 2 < e < 3$$

But then we can improve (1).

$$R_{n,0} = \frac{e^t x^{n+1}}{(n+1)!} \leq \frac{e^x x^{n+1}}{(n+1)!} < \frac{3^x x^{n+1}}{(n+1)!}$$

Recall some previous calculations

$f(x) = \arctan(x) = \int_0^x \frac{1}{1+t^2}\,dt$ arctan is an integral

$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \dots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}$ and we can rewrite the integrand

Thus

$$\arctan(x) = \int_0^x \left[1 - t^2 + t^4 - t^6 + \dots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} \right]$$ and integrate

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt$$

and

$$|R_{2n+1,0}(x)| = \left| (-1)^n \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt \right| \le \left| \int_0^x t^{2n+2}\,dt \right| = \frac{|x|^{2n+3}}{2n+3}$$

Let's consider $|x| \le 1$ first.

Then

$$\arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \dots + (-1)^n \frac{1}{2n+1} + R_{2n+1,0}(1) \quad \text{and} \quad |R_{2n+1,0}(1)| < \frac{1}{2n+3}$$

→ we can make the remainder arbitrarily small

However, n needs to be relatively large

example: $\frac{1}{2n+3} < 10^{-4} \to 2n+3 > 10^4 \to n > \frac{10^4 - 3}{2}$

Since $\arctan(1) = \frac{\pi}{4}$, we can use the Taylor polyn. to estimate π. However, as the ex. above shows, we need a huge computation for not even that high accuracy.

However, since $|R_{2n+1,0}(x)| < \frac{|x|^{2n+3}}{2n+3}$, if we reduce the x we get a big reduction in this upper bound.

let's now consider $\log(1+x)$

$$P_{n,0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-0)^k = 0 + \frac{1}{1!}x^1 + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-3!}{4!}x^4 + \frac{4!}{5!}x^5 + \dots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots + (-1)^{n-1}\frac{x^n}{n}$$

$$R_{n,0}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-0)^{n+1} = \frac{(-1)^n n!}{(1+t)^{n+1}} \cdot \frac{1}{(n+1)!} \cdot x^{n+1} \quad t \in (0,x)$$

$$= \frac{(-1)^n x^{n+1}}{(n+1)(1+t)^{n+1}}$$

and

$$|R_{n,0}(x)| \leq \frac{x^{n+1}}{n+1} \quad (\text{for } x \geq 0)$$

$$f^{(k)}(0) = (-1)^{k-1}(k-1)!$$

$$f^{(k)} = (-1)^{k-1}\frac{(k-1)!}{(1+x)^k} \qquad f^{(k)}(0) = (-1)^{k-1}(k-1)!$$

$$f^{(k+1)} = (-1)^k \frac{k!}{(1+x)^{k+1}}$$

spivak actually does the following

$$\frac{1}{1+x} = \sum_{i=0}^{n-1}(-1)^i x^i + (-1)^n \frac{x^n}{1+x}$$

$$\int_0^x \frac{1}{1+t}\,dt = \log(1+x) = \int_0^x \sum_{i=0}^{n-1}(-1)^i t^i\,dt + \int_0^x (-1)^n \frac{u^n}{1+u}\,du$$

$$= \sum_{i=1}^{n}(-1)^i \frac{x^i}{i} + \int_0^x (-1)^n \frac{u^n}{1+u}\,du$$

integral form of the remainder

$$\begin{array}{r|l} & 1 - x + x^2 - x^3 \\ \hline 1+x & 1 \\ & \underline{1+x} \\ & -x \\ & \underline{-x - x^2} \\ & x^2 \\ & \underline{x^2 + x^3} \\ & -x^3 \\ & \underline{-x^3 - x^4} \\ & x^4 \end{array}$$

let $f(x) = \dfrac{x^{n+1}}{n+1}$

For $x>0$ we have

$$f(0) = 0 < \int_0^x \frac{u^n}{1+u}\,du < \int_0^x u^n\,du = \frac{x^{n+1}}{n+1} = f(x)$$

$$\text{IVT} \to \exists t, t \in (0,x) \wedge f(t) = \frac{t^{n+1}}{n+1} = \int_0^x \frac{u^{n+1}}{1+u}\,du$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \frac{x^4}{1+x}$$

$$\frac{1}{1+x} = \sum_{i=0}^{n}(-1)^i x^i + (-1)^{n+1}\frac{x^{n+1}}{1+x}$$

Thus,

$$\int_0^x \frac{1}{1+t}\,dt = \sum_{i=1}^{n}(-1)^i \frac{x^i}{i} + (-1)^{n+1} \cdot \frac{t^{n+1}}{n+1}, \quad t \in (0,x)$$

Note that

$R_{n,0}(x) = (-1)^{n+1}\dfrac{t^{n+1}}{n+1}$ $t \in (0,x)$, and $|R_{n,0}(x)| < \dfrac{x^{n+1}}{n+1}$. This is an upper bound we can actually compute the magnitude of because we know x and n (as opposed to t).

For $0 \leq x \leq 1$, this remainder decreases with n.

Now let's consider $|x|>1$.

we found the following for the remainders of arctan and $\log(1+x)$

$$|R_{2n+1,0}(x)| < \frac{|x|^{2n+3}}{2n+3} \quad \text{for arctan}$$

$$|R_{n,0}(x)| < \frac{x^{n+1}}{n+1} \quad x>0 \text{ for } \log(1+x)$$

$$< \frac{x^{n+1}}{(1+x)(n+1)} \quad -1<x\leq 0$$

which grows as n becomes large.

Do the remainders actually grow or is this just bad estimates?

$t \in [0,x]$, or $[x,0]$ if $x<0$, then

$$1+t^2 \leq 1+x^2 \leq 2x^2 \quad |x|\geq 1$$

hence,

$$\left|\int_0^x \frac{t^{2n+2}}{1+t^2}\,dt\right| \geq \frac{1}{2x^2}\left|\int_0^x t^{2n+2}\,dt\right| = \frac{1}{2x^2}\cdot\left|\frac{x^{2n+3}}{2n+3}\right| = \frac{|x|^{2n+1}}{4n+6}$$, which also grows with n.

Thus, for arctan, $R(x) = (-1)^{n+1}\int_0^x \frac{t^{2n+2}}{1+t^2}\,dt$, and $|R(x)| \geq \frac{|x|^{2n+1}}{4n+6}$, so the remainder does blow up.

For $\log(1+x)$, $x>0$ and $t\in[0,x]$ we have

$$1+t \leq 1+x \leq 2x, \quad x\geq 1.$$

$$|R_{n,0}(x)| = \int_0^x \frac{t^n}{n+1}\,dt \geq \frac{1}{2x}\int_0^x t^n\,dt = \frac{1}{2x}\frac{x^{n+1}}{n+1} = \frac{x^n}{2n+2}$$, which represents a lower bound that blows up w/ n.

Thus the Taylor polynomials for arctan(x) and $\log(1+x)$ don't allow us to compute the value of these fns to arbitrary accuracy.

Note, however, that

$$\log_{10} x = \log_{10}(10^n \cdot y) \quad \text{where } 0<y\leq 1$$

$$= n + \underbrace{\log_{10}(y)}_{\text{this we know by assumption}}$$

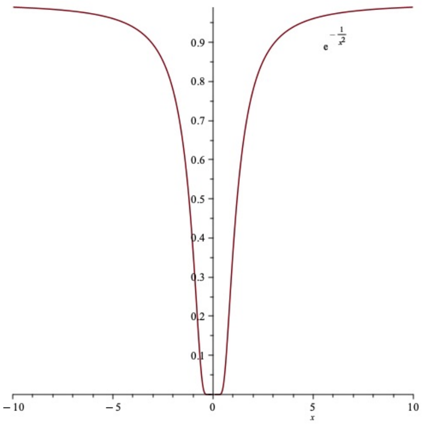
Also

$$\log x = \frac{\log_{10} x}{\log_{10} e} \quad \text{and} \quad \log_{10} e = \log_{10}\left(10\cdot\frac{e}{10}\right) = 1 + \underbrace{\log_{10}(e/10)}_{\text{this we know since } 0<e/10<1}$$

Hence if we know $\log_b x$ for $|x|<1$ then we know it for all x

Consider the fn

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



$$f'(0) = \lim_{h \to 0} \frac{e^{-1/h^2}}{h} = 0$$

$$f'(x) = \frac{2}{x} e^{-1/x^2}$$

$$f''(0) = \lim_{h \to 0} \frac{\frac{2}{h} e^{-1/h^2}}{h} = \lim_{h \to 0} \frac{2e^{-1/h^2}}{h^2} = 0$$

...

$$f^{(n)}(0) = 0$$

and $P_{n,0}(x) = 0$

→ remainder term is always $f(x)$

Theorem 5 e is irrational

Proof

$e = e^1 = \sum_{i=0}^{n} \frac{1}{i!} + R_n$ where $0 < R_n < \frac{3}{(n+1)!}$

* we are using Taylor's Theorem to show e is irrational.

Suppose e is rational, ie $e = \frac{a}{b}$, a and b positive integers.

Let $n > b$ and $n > 3$. Then

$$\frac{a}{b} = 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!} + R_n$$

$$\underbrace{\frac{n!\,a}{b}}_{\text{integer}} = \underbrace{n! + n! + \frac{n!}{2!} + \dots + 1}_{\text{integer terms}} + n!\,R_n$$

→ $n!\,R_n$ must be integer

But $0 < R_n < \frac{3}{(n+1)!}$ → $0 < n!\,R_n < \frac{3}{n+1} < \underbrace{\frac{3}{4}}_{\text{when } n=1,\ \max} < 1$

$\perp$, since $n!\,R_n$ is integer.

■

let's see

In Ch. 15 we proved that

$$\left.\begin{array}{l} f''+f=0 \\ f(0)=0 \\ f'(0)=0 \end{array}\right. \quad \rightarrow \quad f=0$$

Let's prove this now using Taylor's Theorem
we have

$$f''=-f$$
$$f'''=(f'')'=-f'$$
$$f^{(4)}=(f''')'=-f''=f$$
$$f^{(5)}=f'$$

All $f^{(k)}$ exist, and they are each one of $f, f', -f, -f'$
Also, $f(0)=f'(0)=0$ so $f^{(k)}(0)=0$

Taylor's Theorem $\rightarrow$ $f(x)=\dfrac{f^{(n+1)}(t)}{(n+1)!}(x-a)^n$, $t \in [0,x]$, for any n.

$f^{(n+1)}$ is cont. since $f^{(n+2)}$ exists

T.F. $\exists M$ s.t. $|f^{(n+1)}(t)| \leq M$ for $0 \leq t \leq x$ and all n

ie $f^{(n+1)}$ is bounded on $[0,x]$.

$\rightarrow$ $|f(x)| \leq \dfrac{M|x|^{n+1}}{(n+1)!}$ but this can be made arbitrarily small.

Hence $|f(x)| < \epsilon$ for any $\epsilon > 0$ and $f(x)=0$.

Recap

We defined the Taylor Polyn.

Then we showed that if a fn has the first n derivatives then $\lim_{x \to a} \frac{f(x) - P_{n,a}(x)}{(x-a)^n} = 0$

This led to a result about local minima and maxima when the first n-1 deriv. are 0 at a point but the n^{th} isn't.

Then we defined the concept of two fns being equal up to order n at a, and showed that there is only one polyn. in $(x-a)$ that is equal to f up to order n at a, and that polyn. is $P_{n,a,f}$.

Then we defined the remainder term as the difference between a function and $P_{n,a}$, and showed that for any function that has (n+1) deriv. defined on an interval $[a,x]$, the remainder can be expressed in terms of $f^{(n+1)}$, and furthermore is decreasing in n.

T.F. for some fixed x, if we increase the degree of the Taylor Polyn., the remainder decreases.

With the formula for the remainder (which is the result of Taylor's Theorem), we can write functions as the sum of a polyn. and this remainder.

	at 0
$f(x) = \sin x$	0
$f' = \cos$	1
$f'' = -\sin$	0
$f''' = -\cos$	-1
$f^{(4)} = \sin$	0

$P_{0,0}(x) = 0$ $\qquad \sin(x) = 0 + \cos(t)\cdot x \quad t \in (0,x)$

$P_{1,0}(x) = 0 + x = x$ $\qquad \sin(x) = x + \frac{(-\sin(t))}{2!}\cdot x^2 \quad t \in (0,x)$

$P_{2,0}(x) = 0 + x + 0$ $\qquad \sin(x) = x + \frac{(-\cos(t))}{3!}x^3 \quad t \in (0,x)$

$P_{3,0}(x) = 0 + x + 0 - \frac{x^3}{3!}$ $\qquad \sin(x) = x - \frac{x^3}{3!} + \frac{\sin(t)}{4!}x^4 \quad t \in (0,x)$

$P_{n,0}(x) = \sum_{i=0}^{n}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$ $\qquad \sin(x) = \sum_{i=0}^{n}(-1)^n\frac{x^{2n+1}}{(2n+1)!} + \frac{\sin^{(n+1)}(t)}{(2n+2)!}x^{2n+2}$

$(2n+1)^{th}$ order Taylor polyn. involves the $(2n+1)^{th}$ derivative
$2n+1$ is always odd.
The $(2n+2)^{th}$ order Taylor polyn. is the same, but involves the $(2n+2)^{th}$ order polyn., so the remainder involves the $(2n+3)^{th}$ derivative.