Ch.16 - π is Irrational

Let's investigate the fn $f_n(x) = \dfrac{x^n(1-x)^n}{n!}$

$0<x<1 \to f_n(x) < \dfrac{1}{n!}$

since $x^n(1-x)^n = \sum_{i=n}^{2n} c_i x^i$, $c_i \in \mathbb{Z}$ then

$f_n(x) = \dfrac{1}{n!}\sum_{i=n}^{2n} c_i x^i$, $c_i \in \mathbb{Z}$

$k<n \to f^{(k)}(x) = \dfrac{1}{n!}\sum_{i=n-k}^{2n-k} d_i x^i \to f^{(k)}(0) = 0$

$k>2n \to f^{(k)}(x) = 0$

Also,

$f_n^{(n)}(x) = \dfrac{1}{n!}\left(c_n \cdot n! + \text{terms involving } x\right) \qquad \to f_n^{(n)}(0) = c_n$

$f_n^{(n+1)}(x) = \dfrac{1}{n!}\left(c_{n+1}\cdot(n+1)! + \text{terms involving } x\right) \qquad \to f_n^{(n+1)}(0) = (n+1)\,c_{n+1}$

(...)

$f_n^{(2n)}(x) = \dfrac{1}{n!}\left((2n)!\,c_{2n}\right) \qquad \to f_n^{(2n)}(0) = \dfrac{1}{n!}\left((2n)!\,c_{2n}\right)$

$\downarrow$

All are integers.

$\to f_n^{(k)}(0)$ is an integer for all k

$f_n(x) = f_n(1-x) \quad \to f_n^{(k)}(x) = (-1)^k f_n^{(k)}(1-x)$

$f_n^{(k)}(0) = (-1)^k f_n^{(k)}(1) \in \mathbb{Z} \quad \to f_n^{(k)}(1) \in \mathbb{Z}$

Let $a>0$, $\epsilon>0$. Then, for n suffic. large, $\dfrac{a^n}{n!} < \epsilon$

Proof

$\dfrac{a}{n+1} < \dfrac{a}{2a+1} < \dfrac{a}{2a} = \dfrac{1}{2}$

Let $n \geq 2a$. Then, $\dfrac{a^{n+1}}{(n+1)!} = \dfrac{a}{n+1}\cdot\dfrac{a^n}{n!} < \dfrac{1}{2}\cdot\dfrac{a^n}{n!}$

Now let $n_0 \in \mathbb{N}$ w/ $n_0 \geq 2a$. Whatever $\dfrac{a^{n_0}}{(n_0)!}$ is, we have

$\dfrac{a^{(n_0+1)}}{(n_0+1)!} < \dfrac{1}{2}\dfrac{a^{n_0}}{n_0!}$

$\dfrac{a^{n_0+2}}{(n_0+2)!} < \dfrac{1}{2}\dfrac{a^{n_0+1}}{(n_0+1)!} < \dfrac{1}{2}\cdot\dfrac{1}{2}\dfrac{a^{n_0}}{n_0!}$

(...)

$\dfrac{a^{n_0+k}}{(n_0+k)!} < \dfrac{1}{2^k}\dfrac{a^{n_0}}{n_0!}$

We started w/ a given a and for some n, the number $a^n/n!$.
Then we took a natural number n_0 and considered $a^{n_0}/n_0!$, and then we started incrementing n_0 as: $n_0+1, n_0+2, \ldots, n_0+k$.

At some point $\dfrac{a^{n_0}}{n_0!} < 2^k\epsilon$ and when that happens, we will have a number n_0+k s.t.

$\dfrac{a^{n_0+k}}{(n_0+k)!} < \epsilon$

Theorem 1 The number π is irrational. Also, π^2 is irrational, and π being irrational follows.

Assume π^2 irrational.
Assume π rational.
Then $\pi = p/q$.
$\pi^2 = p^2/q^2$ rational.
$\perp$.
π irrational

T.F., π^2 irrat. $\to$ π irrat.

Proof

Assume π^2 is rational. Then $\pi^2 = \frac{a}{b}$ for $a, b \in \mathbb{Z}^+$

Let

$$G(x) = b^n\left[\pi^{2n} f_n(x) - \pi^{2n-2} f_n''(x) + \pi^{2n-4} f_n^{(4)}(x) - (\dots) + (-1)^n f_n^{(2n)}(x)\right]$$

Each factor $b^n\pi^{2n}, b^n\pi^{2n-2}, b^n\pi^{2n-4}, \dots, b^n\pi$ is an integer.

We can denote such factors as $b^n\pi^{2n-2k} = b^n(\pi^2)^{n-k} = b^n\left(\frac{a}{b}\right)^{n-k} = b^k a^{n-k}$, an integer.

Now,

$$G(0) = b^n \sum_{k=0}^{n} (-1)^k \pi^{2n-2k} f_n^{(2k)}(0)$$

$$G(1) = b^n \sum_{k=0}^{n} (-1)^k \pi^{2n-2k} f_n^{(2k)}(1)$$

So both $G(0)$ and $G(1)$ are integers.

Differentiate twice.

$$G'(x) = b^n \sum_{k=0}^{n} (-1)^k \pi^{2n-2k} f_n^{(2k+1)}(x)$$

$$G''(x) = b^n \sum_{k=0}^{n} (-1)^k \pi^{2n-2k} f_n^{(2k+2)}(x)$$ note the last term, $b^n(-1)^n f_n^{(2n+2)}(x)$, is zero

$$G''(x) + \pi^2 G(x) = \cancel{b^n\pi^{2n} f_n''(x)} - \cancel{b^n\pi^{2n-2} f_n^{(4)}(x)} + \dots + \cancel{(-1)^{n-1}\pi^2 f_n^{(2n)}(x)}$$
$$+ b^n\pi^{2n+2} f_n(x) - \cancel{b^n\pi^{2n} f_n''(x)} + \cancel{\pi^{2n-2} f_n^{(4)}(x)} - \dots + \cancel{(-1)^n \pi^2 f_n^{(2n)}(x)}$$
$$= b^n\pi^{2n+2} f_n(x)$$
$$= \frac{a^n}{\pi^{2n}}\pi^{2n+2} f_n(x)$$
$$= a^n \pi^2 f_n(x)$$

Let

$$H(x) = G'(x)\sin(\pi x) - \pi G(x)\cos(\pi x)$$

Then,

$$H'(x) = \cancel{\pi G'(x)\cos(x)} + G''(x)\sin(\pi x) + \pi^2 G(x)\sin(\pi x) - \cancel{\pi G'(x)\cos(\pi x)}$$

$$= \sin(\pi x)\left(G''(x) + \pi^2 G(x)\right)$$

$$= \sin(\pi x)\,\pi^2 a^n f_n(x)$$

$$\text{FTC2} \rightarrow \int_0^1 H'(x)\,dx = \pi^2 \int_0^1 a^n f_n(x)\sin(\pi x)\,dx$$

$$= H(1) - H(0)$$

$$= \cancelto{0}{G'(1)\sin(\pi)} - \pi G(1)\cos(\pi) - \left[\cancelto{0}{G'(0)\sin(0)} - \pi G(0)\cos(0)\right]$$

$$= \pi\left(G(1) + G(0)\right)$$

Thus

$$\pi \int_0^1 a^n f_n(x)\sin(\pi x)\,dx = G(1) - G(0) \text{ is an integer.}$$

As we've seen previously,

$$0 < f_n(x) < \frac{1}{n!} \quad \text{for } 0<x<1$$

so

$$0 < \pi a^n f_n(x)\sin(\pi x) < \frac{\pi a^n}{n!} \quad 0<x<1$$

Integrate from 0 to 1

$$0 < \pi \int_0^1 a_n \sin(\pi x) f_n(x)\,dx < \frac{\pi a^n}{n!}(1-0)$$

The reasoning to reach this point is true for any n.

As we showed previously for n sufficiently large, $\frac{a^n}{n!} < \epsilon$, for $\epsilon > 0$. Let $\epsilon = \frac{1}{\pi}$. Then $\frac{\pi a^n}{n!} < 1$

and so

$$0 < \pi \int_0^1 a_n \sin(\pi x) f_n(x)\,dx < \frac{\pi a^n}{n!} < 1$$

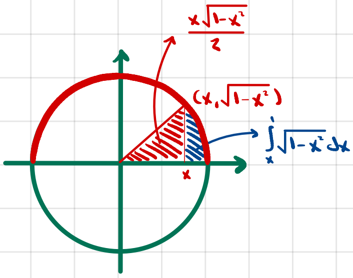
But $\pi \int_0^1 a_n \sin(\pi x) f_n(x)\,dx$ is an integer. $\perp$.

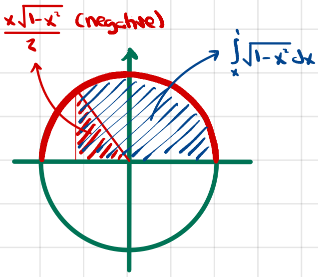
Hence, by proof by contradiction, π^2 is irrational.

Note

In ch. 15 we defined

Definition $\pi = 2 \cdot \int_{-1}^{1} \sqrt{1-x^2}\,dx$





Definition If $-1 \le x \le 1$ then

$$A(x) = \frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}\,dt$$

and then cos and sin were defined

$$A(\cos x) = \frac{x}{2} \quad x \in [0, \pi]$$

$$\sin(x) = \sqrt{1-\cos^2 x}$$

These definitions led to

$$\sin(\pi) = \sin(0) = 0$$

$$\cos(\pi) = -1, \quad \cos(0) = 1$$

which was used in our proof of Th. 1.

We also used the fact that

$$\sin' = \cos$$

$$\cos' = -\sin$$