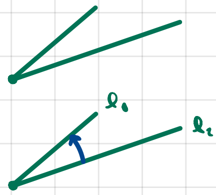
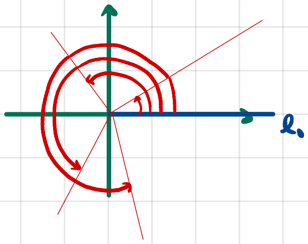
Ch 15. - The Trigonometric Functions

Elementary geometry angle: union of two half-lines w/ common initial point

Trigonometry, directed angles: pairs of half lines (ℓ_1, ℓ_2) w/ same initial point

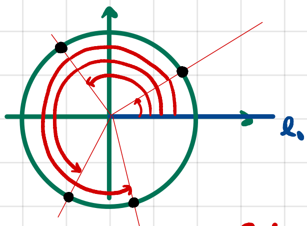


Let ℓ_1 be positive half of horiz. axis.



Then a directed angle is described completely by a second half line

But then, we can convey the same idea w/ one point on the unit circle, ie a point (x, y) w/ $x^2 + y^2 = 1$.



$\boxed{\text{sine is defined as } y\text{, cosine as } x.}$

this defines sine and cosine of a direct angle

But we want sine and cosine for each number x.

To do this we allocate an angle to every number.

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we can measure angles in degrees and define

$$\sin^\circ x = \text{sine of angle of } x \text{ degrees}$$

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or we can use radian measure.

Given any number x, choose point P on unit circle such that x is arc length.

The resulting directed angle is called angle of x radians. Define

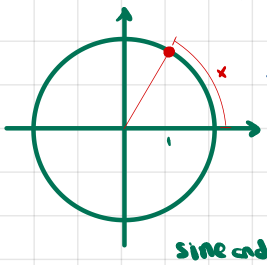
$$\sin^r x = \text{sine of angle of } x \text{ radians}$$

note

$$\sin^\circ x = \sin^r\left(\frac{2\pi}{360}x\right)$$

The def. of $\sin^r$ depends on length of a curve.

We can reformulate in terms of areas.

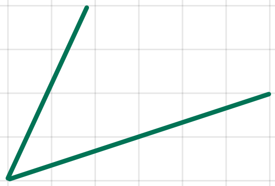


total circumf. $= 2\pi$

arc contains $\frac{x}{2\pi}$ of total circumf.

So is the area of the sector. This area is $\frac{x}{2\pi} \cdot \pi = \frac{x}{2}$

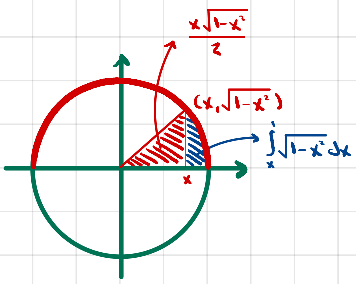
sine and cosine defined as coord. of point P which determines sector of area $\frac{x}{2}$.

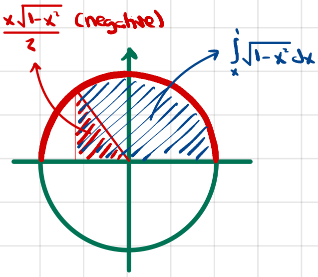


Definition $\pi = 2\cdot\int_{-1}^{1}\sqrt{1-x^2}\,dx$

Definition If $-1 \le x \le 1$ then

$$A(x) = \frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}\,dt$$





Note that

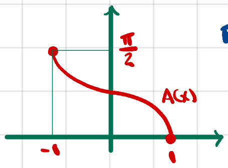
$$\left[\frac{x\sqrt{1-x^2}}{2}\right]' = \frac{1}{2}\left[\sqrt{1-x^2} + x\frac{(-2x)}{2\sqrt{1-x^2}}\right]$$ is not defined at ± 1.

$\sqrt{1-t^2}$ is cont. on $[-1,1]$, hence $F(x) = \int_1^x \sqrt{1-t^2}\,dt$ is an antiderivative on $[-1,1]$.

Hence, on $(-1,1)$, A is diff.

$$A'(x) = \frac{1}{2}\left[\sqrt{1-x^2} + x\frac{(-2x)}{2\sqrt{1-x^2}}\right] - \sqrt{1-x^2}$$

$$= (\dots) = \frac{-1}{2\sqrt{1-x^2}} < 0$$



For each area we have

Given a number $0 \le x \le \pi$ we want to define $\cos x$ and $\sin x$ such that the point $(\cos x, \sin x)$ determines a sector of area $\frac{x}{2}$.

Definition: If $0 \le x \le \pi$ then $\cos x$ is the unique number in $[-1,1]$ such that

$$A(\cos x) = \frac{x}{2} \quad \text{and} \quad \sin x = \sqrt{1-\cos^2 x}$$

Note that $A(x) = \frac{x\sqrt{1-x^2}}{2} + \int_x^1\sqrt{1-t^2}\,dt$ is continuous, $A(1) = 0$, $A(-1) = \frac{\pi}{2}$ and so by IVT A does take on every value in between for some $y \in [-1,1]$. Also, $A' < 0$ so A is one-one.

Theorem 1 If $0 < x < \pi$ then

$$\cos' x = -\sin x$$
$$\sin' x = \cos x$$

Proof

Let $B(x) = 2A(x)$. Then $B(\cos x) = 2A(\cos x) = x$

cosine is thus the inverse of B.

$$B'(x) = 2A'(x) = \frac{-1}{\sqrt{1-x^2}}$$

T.F. since $\cos(x) = B^{-1}(x)$

$$\cos' x = B^{-1\prime}(x) = \frac{1}{B'(B^{-1}(x))} = \frac{1}{-\frac{1}{\sqrt{1-(B^{-1}(x))^2}}}$$

$$= -\sqrt{1-\cos^2 x}$$

$$= -\sin x$$

Also, $$\sin'(x) = \frac{-2\cos x(-\sin x)}{2\sqrt{1-\cos^2 x}} = \cos x$$

Recap

A directed angle can be specified by a single point on the unit circle, ie a (x,y) w/ $x^2+y^2=1$.

sine and cosine of such an angle are defined as the y and x coord.

At each point on unit circle we know sine and cosine.

If we specify the angle by a single number, sine and cosine fns can be defined.

Given the angle we know the point, given the point we know sine and cosine.

In particular, if we measure in radians we are effectively giving each point (ie directed angle) a number based on arc length.

Hence, for each arc length, a point, and a sine and cosine.

On the other hand we could just as well use sector area.

Given arc length x, the sector area is $\frac{x}{2}$. Thus instead of saying

sine x and cosine x are the coord. of the point determined by arc length x

we can say

sine x and cosine x are the coord. of the point determined by sector area $\frac{x}{2}$

The above was just background. Formalization starts here.

An area fn is defined for input being the x-coord of each point on unit circle.

This fn has certain characteristics.

cont. on $[-1,1]$
decreasing $\to$ one-one

$A(x) = \frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}\,dt$, where x is an x-coordinate.

What if the "x" above is a fn of some other number?
Out of all those fns we single one out and call it cos. It takes an input x (now arc length, domain $[0,\pi]$), and converts it to $[-1,1]$ and is such that

$$A(\cos x) = \frac{x}{2} = \frac{\cos x\sqrt{1-\cos^2 x}}{2} + \int_{\cos x}^1 \sqrt{1-t^2}\,dt = \frac{\cos x \sin x}{2} + \int_{\cos x}^1 \sqrt{1-t^2}\,dt$$

sin is then defined $\sin x = \sqrt{1-\cos^2 x}$

Given the definition we can investigate what properties these fns have.
For example, on $[0,\pi]$ we have $\sin x \geq 0$.
Since $\cos' x = -\sin x < 0$ then cos is decreasing in this interval.

Since $A(\cos 0) = 0$, $A(1) = 0$, and A one-one, we know that

$\cos(0) = 1$

Since $A(\cos\pi) = \frac{\pi}{2}$, $A(-1) = \frac{\pi}{2}$ we know $\cos(\pi) = -1$

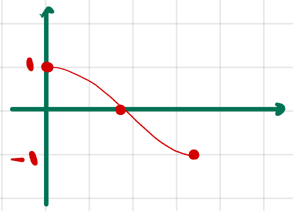
Since $\cos'(x)$ decreasing we know $\cos$ has one root in $[0,\pi]$.

$A(0) = \int_0^1 \sqrt{1-t^2}\,dt = \frac{x_r}{2} \rightarrow x_r = 2\int_0^1 \sqrt{1-t^2}\,dt = \int_{-1}^1 \sqrt{1-t^2}\,dt$ by symmetry of $\sqrt{1-t^2}$

Since this is half the circle's area we know it is $\pi/2$.

Thus $x_r = \frac{\pi}{2}$ and $\cos(\pi/2) = 0$

cos thus looks like



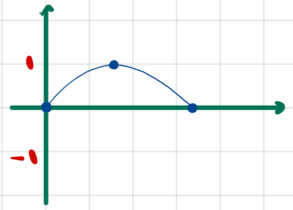
Given what we now know about the sign of cos we also know that $\sin'$ is $\begin{cases} >0 & \text{on } [0,\pi/2) \\ <0 & \text{on } (\pi/2,\pi] \end{cases}$

Also, $\sin 0 = \sqrt{1-\cos^2 0} = 0$

$\sin(\pi/2) = \sqrt{1-0} = 1$

$\sin(\pi) = \sqrt{1-1} = 0$

Thus, it looks like

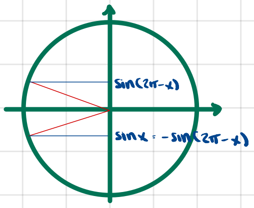


Now we want to expand the domain of cos and sin.

Seems to be by definition

$\pi \leq x \leq 2\pi \rightarrow \sin x = -\sin(2\pi - x)$

$\cos x = \cos(2\pi - x)$



Then for x outside of $[0,2\pi]$ we define

$x = 2\pi k + x'$, $k \in \mathbb{Z}$, $x' \in [0,2\pi] \rightarrow \sin x = \sin x'$

$\cos x = \cos x'$

Thus, we have

cos x defined by

$A(\cos x) = \frac{x}{2}$ $\quad 0 \le x \le \pi$ $\qquad \sin x = \sqrt{1-\cos^2 x}$ $\quad 0 \le x \le \pi$

$\cos x = \cos(2\pi - x)$ $\quad \pi \le x \le 2\pi$ $\qquad \sin x = -\sin(2\pi - x)$ $\quad \pi \le x \le 2\pi$

$\cos x = \cos x'$ $\quad x = 2\pi k + x'$ $\qquad \sin x = \sin x'$ $\quad x = 2\pi k + x'$

we check that certain properties that hold in $[0, \pi]$ still hold in the other parts of the domain.

e.g. $\sin x = \sqrt{1-\cos^2 x} \to \sin^2 x + \cos^2 x = 1$ for $0 \le x \le \pi$

in $[\pi, 2\pi]$, $\sin^2 x + \cos^2 x = \sin^2(2\pi - x) + \cos^2(2\pi - x) = 1 - \cos^2(2\pi - x) + \cos^2(2\pi - x) = 1$

Also, in $(\pi, 2\pi)$,

$$\sin'(x) = -\cos(2\pi - x)(-1) = \cos(2\pi - x) = \cos x$$

$$\cos'(x) = -\sin(2\pi - x)(-1) = -\sin x$$

and outside $[0, 2\pi]$,

$$\sin'(x) = [\sin(x')]' = \cos(x') = \cos(x)$$

$$\cos'(x) = [\cos(x')]' = -\sin(x') = -\sin(x)$$

At $x = \pi$

$$\sin'(x) = \lim_{h \to 0^+} \frac{\sin(\pi + h) - \sin(\pi)}{h} = \lim_{h \to 0^+} \frac{-\sin(2\pi - \pi - h)}{h} = \lim_{h \to 0^+} \frac{-\sin(\pi - h)}{h}$$

$$\lim_{h \to 0^+} \sin(\pi + h) = \lim_{h \to 0^+} (-\sin(2\pi - \pi - h)) = -\sin(\pi) = 0$$

$$\lim_{h \to 0^-} \sin(\pi + h) = \lim_{h \to 0^-} \sqrt{1 - \cos^2(\pi + h)} = 0$$

$\to$ sin cont at π.

Also,

$$\lim_{h \to 0^+} \sin'(\pi + h) = \lim_{h \to 0^+} \cos(\pi + h) = -1$$

$$\lim_{h \to 0^-} \sin'(\pi + h) = \lim_{h \to 0^-} \cos(\pi + h) = -1$$

Then by Th. 11-7, $\sin'(\pi) = -1 = \cos(\pi)$

For $x = 2\pi$ we have s.th. similar: sin is cont at 2π and $\lim_{h \to 0} \sin'(2\pi + h) = \cos(2\pi)$, hence $\sin'(2\pi) = \cos(2\pi)$

Let's define other trig fns

$$\sec x = \frac{1}{\cos x}$$

$$\tan x = \frac{\sin x}{\cos x}$$

for $x \neq n\pi + \frac{\pi}{2}$

$$\csc x = \frac{1}{\sin x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

for $x \neq n\pi$

Theorem 2

$$\sec' x = -\cos^{-2}(x)\,(-\sin(x)) = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \cdot \sec x$$

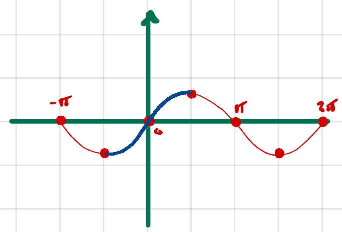
$$\tan' x = \frac{\cos x \cos x - \sin x(-\sin x)}{(\cos x)^2} = \frac{1}{\cos^2 x} = \sec^2 x$$

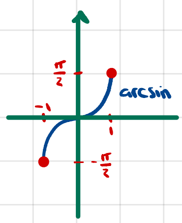
$$\csc' x = \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cdot \cot x$$

$$\cot' x = \frac{-\sin x \sin x - \cos x \cos x}{(\sin x)^2} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

Next we investigate inverse fns

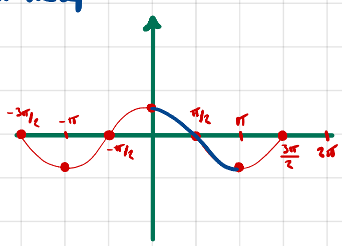
Over $\mathbb{R}$ the trig fns aren't one-one. We restrict the domains to compute the inverses

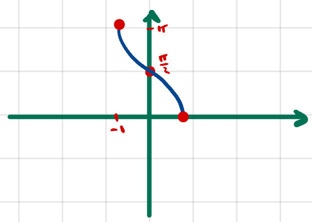




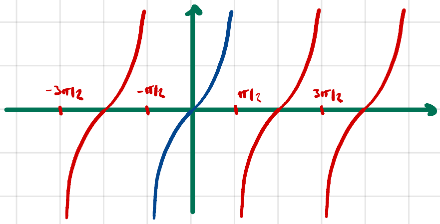
on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $\sin x$ is cnt and increasing, hence one-one, hence the inverse on $[-\pi/2, \pi/2]$ exists and is denoted arcsin.

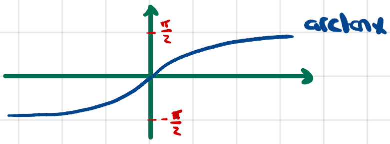
Similarly





arccos exists on $[0, \pi]$





$\tan'(0) = \frac{1}{\cos^2(0)} = 1$

Theorem 3 if $-1 < x < 1$ then

$$\arcsin' x = \frac{1}{\sqrt{1-x^2}}$$

$$\arccos' x = \frac{-1}{\sqrt{1-x^2}}$$

For all x, $\arctan' x = \frac{1}{1+x^2}$

Proof

True if $f'(f^{-1}(x)) \neq 0$. Since $\sin' = \cos$, not true when $\arcsin x = \frac{\pi}{2}$ or $-\frac{\pi}{2}$, ie $x = 1$ or -1.

$$\arcsin'(x) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sin'(\arcsin x)} = \frac{1}{\cos(\arcsin x)}$$

hence this derivation is true for $x \in (-1, 1)$

But,

$$\sin^2(\arcsin x) + \cos^2(\arcsin x) = 1$$

$$x^2 + \cos^2(\arcsin x) = 1$$

note that the image of arcsin is $(-\pi/2, \pi/2)$, and cos is positive on this interval hence we take the positive square root

$$\cos(\arcsin x) = \sqrt{1-x^2}$$

Therefore, $\arcsin' x = \frac{1}{\sqrt{1-x^2}}$

$$\arccos'(x) = \frac{1}{\cos'(\arccos x)} = \frac{1}{-\sin(\arccos x)}$$

$$\sin^2(\arccos x) + x^2 = 1 \rightarrow \sin^2(\arccos x) = 1 - x^2$$

the domain of arccos is $[-1, 1]$ and the image is $[0, \pi]$. sin is positive on $[0, \pi]$.

$$\sin(\arccos x) = \sqrt{1-x^2}$$

$$\rightarrow \arccos'(x) = -\frac{1}{\sqrt{1-x^2}}$$

Alternatively, recall $B(x) = 2A(x) \rightarrow B(\cos x) = 2A(\cos x) = x \rightarrow \cos x = B^{-1}(x)$

$$\rightarrow B(x) = \arccos(x) \text{ and } B'(x) = 2A'(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\arctan'(x) = (f^{-1})'(x) = \frac{1}{\tan'(\arctan x)} = \frac{1}{\frac{1}{\cos^2(\arctan x)}} = \frac{1}{\sec^2(\arctan x)}$$

From $\sin^2 a + \cos^2 a = 1$ we obtain

$$\tan^2 a + 1 = \sec^2 a$$

T.F.

$$\arctan'(x) = \frac{1}{\tan^2(\arctan x) + 1} = \frac{1}{x^2+1}$$

Lemma: suppose f'' defined everywhere and that

$f'' + f = 0$ (1)
$f(0) = 0$
$f'(0) = 0$

Then $f = 0$

Proof

From (1), $f'f'' + ff' = 0$

$$[(f')^2 + f^2]' = 2f'f'' + 2ff' = 0$$

T.F. $[(f')^2 + f^2]$ is a constant fn.

Using (2) and (3),

$$f'(0)^2 + f(0)^2 = 0$$

so $[(f')^2 + f^2] = 0$

$\rightarrow f = 0$

Theorem 4 f'' defined everywhere
$f'' + f = 0$
$f(0) = a$
$f'(0) = b$

$\rightarrow f = b\sin + a\cos$

Proof

Let $g(x) = f(x) - b\sin x - a\cos x$
Then

$g'(x) = f'(x) - b\cos x + a\sin x$
$g''(x) = f''(x) + b\sin x + a\cos x$

T.F. the conditions in the lemma are satisfied by g

$g'' + g = f'' + f = 0$
$g(0) = f(0) - b\cdot 0 - a = 0$
$g'(0) = f'(0) - b + a\cdot 0 = 0$

Hence, $g = 0$, ie $f = b\sin + a\cos$

Theorem 5 x, y any numbers $\to$ $\sin(x+y) = \sin x \cos y + \sin y \cos x$
$\cos(x+y) = \cos x \cos y - \sin x \sin y$

Proof

Given y, define the fn

$$f(x) = \sin(x+y)$$

Then

$$f'(x) = \cos(x+y)$$
$$f''(x) = -\sin(x+y)$$

and

$$f'' + f = 0$$
$$f(0) = \sin(y)$$
$$f'(0) = \cos(y)$$

By Th.4, $f = \cos y \sin x + \sin y \cos x \quad \forall x$

Since this result is true for all y then $\forall x \forall y$ we have $\sin(x+y) = \sin x \cos y + \sin y \cos x$

Similarly, let $g(x) = \cos(x+y)$

$$g'(x) = -\sin(x+y)$$

$$g''(x) = -\cos(x+y)$$

$$g'' + g = 0$$

$$g(0) = \cos y$$

$$g'(0) = -\sin y$$

and by Th.4 we have

$$g = (-\sin y)\sin + \cos y \cos \quad \forall x$$

Since this is true for all y we have $\forall x \forall y$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

Recap

$\cos x$ defined by		$\sin x$ is defined by	
$A(\cos x) = \frac{x}{2}$	$0 \le x \le \pi$	$\sin x = \sqrt{1-\cos^2 x}$	$0 \le x \le \pi$
$\cos x = \cos(2\pi - x)$	$\pi \le x \le 2\pi$	$\sin x = -\sin(2\pi - x)$	$\pi \le x \le 2\pi$
$\cos x = \cos x'$	$x = 2\pi k + x'$	$\sin x = \sin x'$	$x = 2\pi k + x'$

we build the graphs based on information about $\sin', \cos', \sin'', \cos''$.

sec, csc, tan, cot are defined in terms of sin and cos, and derivatives are computed directly.

inverses are defined for specific intervals, and derivatives of the inverses are computed using the inverse function rule.

In particular, $\tan(x) \equiv \dfrac{\sin x}{\cos x}$, and $\tan^{-1}(x) = \arctan(x)$ is s.t. $\arctan'(x) = (1+x^2)^{-1}$.

Note

$$\sin x = \frac{\sin x}{\cos x}\cdot \cos x = \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \frac{\frac{\sin x}{\cos x}}{\sqrt{\frac{1}{\cos^2 x}}} = \frac{\frac{\sin x}{\cos x}}{\sqrt{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}}} = \frac{\tan x}{\sqrt{1+\tan^2 x}}$$

Note on Prob. 29. Assume we know nothing about trig. fns, and we start w/ a fn $f(x) = (1+x^2)^{-1}$.

Define $\alpha(x) = \int_0^x (1+t^2)^{-1}\,dt = \int_0^x f(t)\,dt$.

It can be shown that

α odd, increasing, one-one $\to \alpha^{-1}$ defined

$\alpha(0) = 0$

$\lim_{x\to\infty} \alpha(x) = \frac{\pi}{2} = -\lim_{x\to-\infty} \alpha(x)$, i.e. π defined as $2\lim_{x\to\infty}\alpha(x)$

$[\alpha^{-1}]'(x) = 1 + [\alpha^{-1}(x)]^2$

Consider the fn $h(x) = \dfrac{\alpha^{-1}(x)}{\sqrt{1+[\alpha^{-1}(x)]^2}}$

we can show that

$$\lim_{x\to\frac{\pi}{2}^-} h(x) = 1$$

$$\lim_{x\to-\frac{\pi}{2}^+} h(x) = -1$$

$h'(x) = \dfrac{1}{\sqrt{1+[\alpha^{-1}(x)]^2}}$, which we can write as $\dfrac{\alpha^{-1}(x)}{\alpha^{-1}(x)\sqrt{1+[\alpha^{-1}(x)]^2}}$ if $\alpha^{-1}(x) \neq 0$, which is true if $x \neq 0$.

If we keep going it will turn out that the fn h has all the properties of what we normally call sin, and α^{-1} has all the prop of what we call tan.

Right now our results are in terms of our initially defined α and α^{-1}.

If we define sin and tan as

$$\tan(x) = \alpha^{-1}(x)$$

$$\sin(x) = h(x) = \frac{\tan(x)}{\sqrt{1+\tan^2(x)}}$$

Then our results will be in terms of these fns.

E.g. $\sin'(x) = \dfrac{1}{\sqrt{1+[\alpha^{-1}(x)]^2}} = \dfrac{1}{\sqrt{1+\tan^2 x}}$

For $x \neq 0$, $\tan(x) \neq 0$ so we can rewrite $\dfrac{\tan(x)}{\tan(x)\sqrt{1+\tan^2 x}} = \dfrac{\sin x}{\tan x}$