

Ch.4 - Graphs

i) $|x-3| < 1$

ii) $|x-3| \leq 1$

iii) $|x-a| < \epsilon$

iv) $|x^2-1| < \frac{1}{2}$

$$\sqrt{(x^2-1)^2} < \frac{1}{2}$$

$$-\frac{1}{2} < x^2-1 < \frac{1}{2}$$

$$\frac{1}{2} < x^2 < \frac{3}{2}$$

$$\frac{\sqrt{2}}{2} < x < \frac{\sqrt{6}}{2} \cup -\frac{\sqrt{6}}{2} < x < -\frac{\sqrt{2}}{2}$$

v) $\frac{1}{1+x^2} \geq \frac{1}{5}$

$$5 \geq 1+x^2$$

$$x^2 \leq 4$$

$$-2 \leq x \leq 2$$

vi) $\frac{1}{1+x^2} \leq a$

case 1: $a > 0 \Rightarrow 1+x^2 \geq \frac{1}{a} \Rightarrow x^2 \geq \frac{1-a}{a}$

$\Rightarrow \sin x^2 \geq 0, 1-a \geq 0 \Rightarrow a \leq 1 \Rightarrow 0 < a \leq 1$

$\Rightarrow x \geq \sqrt{\frac{1-a}{a}}, x \leq -\sqrt{\frac{1-a}{a}}, 0 < a \leq 1$

case 2: $a \leq 0 \Rightarrow 0 < \frac{1}{1+x^2} \leq a \leq 0$, false

\Rightarrow there are no solutions if $a < 0$

vii) $x^2+1 \geq 2$

$$x^2 \geq 1 \Rightarrow x \geq 1 \cup x \leq -1$$

viii) $(x+1)(x-1)(x-2) > 0$

2. a) consider $[a, b], b > 0$

Prove $x \in [a, b] \Rightarrow x = tb$, for some $t \in [0, 1]$

$$0 \leq x \leq b$$

$$0 \leq \frac{x}{b} \leq 1$$

$$x = \frac{x}{b} \cdot b$$

$$t = \frac{x}{b} \Rightarrow x = tb$$

t is a ratio reflecting the proportion of b that is x .

The midpoint of $[a, b]$ is $x = 0.5b$

b) $x \in [a, b] \Rightarrow x = (1-t)a + tb$, for some $t \in [0, 1]$

$$a \leq x \leq b \Rightarrow 0 \leq x-a \leq b-a$$

$$\Rightarrow b \geq a, x-a = t(b-a), t = \frac{x-a}{b-a}$$

$$\Rightarrow x = a + t(b-a)$$

$$\text{midpoint of } [a, b]: x = a + 0.5(b-a) = \frac{a+b}{2}$$

$$\text{1/3 point: } x = a + \frac{1}{3}(b-a) = \frac{2}{3}a + \frac{1}{3}b$$

$$\text{c) } 0 \leq t \leq 1 \Rightarrow (1-t)a + tb \in [a, b]$$

$$(1-t)a + tb = a + t(b-a)$$

$$b \geq a, t \geq 0 \Rightarrow a \leq a + t(b-a)$$

$$t \leq 1 \Rightarrow a + t(b-a) \leq b$$

$$\Rightarrow a \leq a + t(b-a) \leq b$$

$$\Rightarrow a + t(b-a) - (1-t)a + tb \in [a, b]$$

d) Prove points of open interval (a, b) are of form $(1-t)a + tb$ for $0 < t < 1$

$$x \in (a, b) \Rightarrow 0 < x < b \Rightarrow 0 < \frac{x}{b} < 1$$

$$\Rightarrow x = tb, t = \frac{x}{b}$$

$$x \in (a, b) \Rightarrow a < x < b \Rightarrow a < x-a < b-a$$

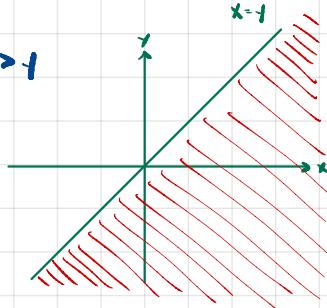
$$\Rightarrow 0 < \frac{x-a}{b-a} < 1$$

$$\Rightarrow x-a = t(b-a), t = \frac{x-a}{b-a}$$

$$\Rightarrow x = a + t(b-a), 0 < t < 1$$

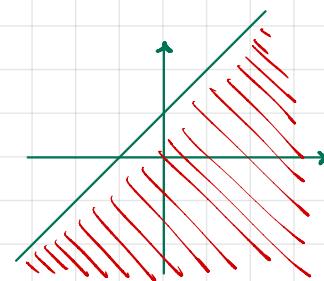
3.

$$\text{i)} x > y$$

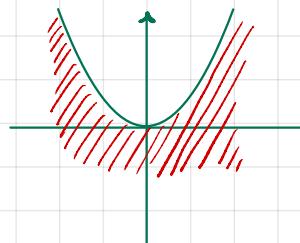


$$\text{ii)} x+a > y+b$$

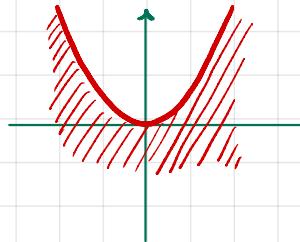
$$y < x + (a-b)$$



$$\text{iii)} y < x^2$$



$$\text{iv)} y < x^2$$



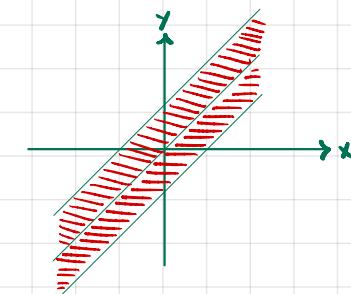
$$\text{v)} |x-y| < 1$$

$$y > x \Rightarrow y - x < 1$$

$$\Rightarrow y < x+1$$

$$y < x \Rightarrow x - y < 1$$

$$\Rightarrow y > x-1$$



$$\text{vi)} |x+y| < 1$$

$$x+y \geq 0 \Rightarrow y \geq -x$$

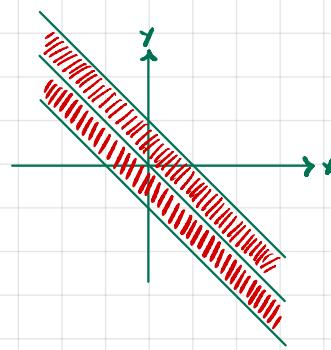
$$\Rightarrow x+y < 1 \Rightarrow y < 1-x$$

$$\Rightarrow -x < y < 1-x$$

$$x+y < 0 \Rightarrow y < -x$$

$$\Rightarrow -(x+y) < 1$$

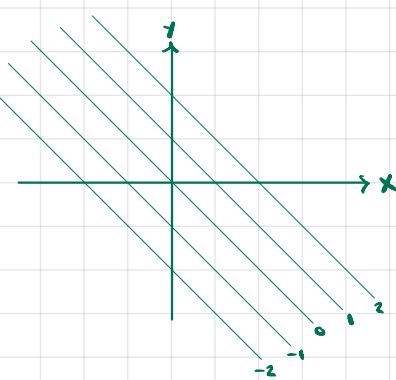
$$\Rightarrow y > -1-x$$



$$vii) x+y \in \mathbb{Z}$$

$$x+y = i, \forall i \in \mathbb{Z}$$

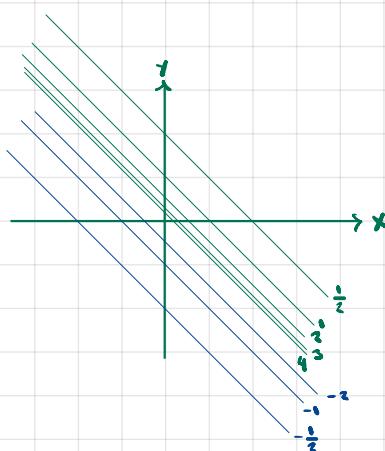
$$i = i - x, \forall i \in \mathbb{Z}$$



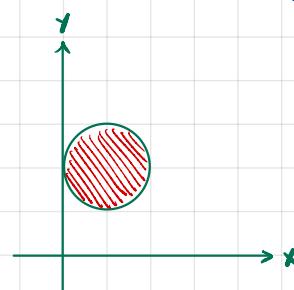
$$viii) \frac{1}{x+y} \in \mathbb{Z}$$

$$\frac{1}{x+y} = i, i \in \mathbb{Z}, i \neq 0$$

$$x+y = \frac{1}{i} = i - \frac{1}{i} - x$$



$$ix) (x-1)^2 + (y-2)^2 < 1$$



$$x) x^2 < y < x^4$$

$$x^2 < y < x^4 \Rightarrow x < x^2 \cup x > -x^2$$

$$x < x^2 \Rightarrow x(1-x) < 0$$

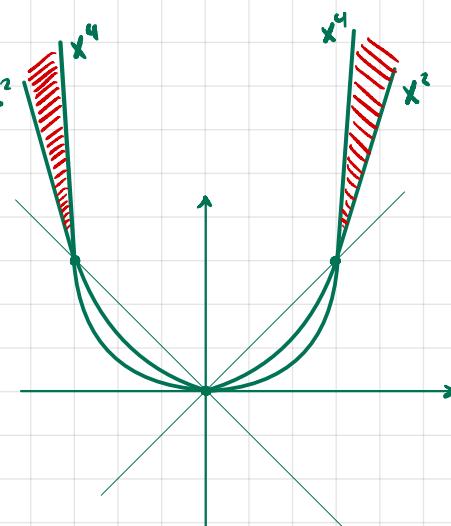
$$\begin{array}{c|ccccc} & + & + & + & - \\ \hline - & \textcircled{-} & \textcircled{+} & \textcircled{+} & \textcircled{-} \\ 0 & & & & \\ + & & & & \textcircled{-} \\ 1 & & & & \end{array}$$

$$\Rightarrow x < 0 \cup x > 1$$

$$x > -x^2 \Rightarrow x(1+x) > 0$$

$$\begin{array}{c|ccccc} & - & - & + & + \\ \hline - & \textcircled{-} & \textcircled{-} & \textcircled{+} & \textcircled{+} \\ 0 & & & & \\ + & & & & \textcircled{+} \\ 1 & & & & \end{array}$$

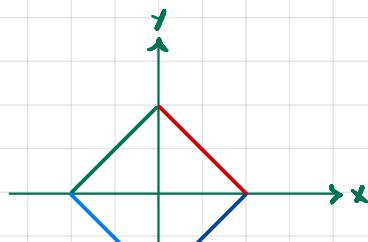
$$\Rightarrow x < -1 \cup x > 0$$



$$\Rightarrow x > 1 \cup x < -1$$

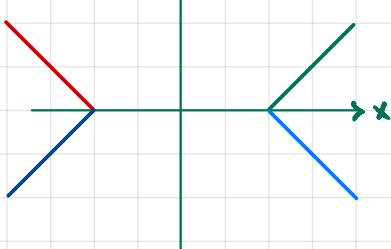
$$4. \text{ (i) } |x| + |y| = 1$$

$$\begin{aligned} x > 0, y > 0 &\Rightarrow x+y=1 \Rightarrow y=1-x \\ x > 0, y < 0 &\Rightarrow x-y=1 \Rightarrow y=x-1 \\ x < 0, y > 0 &\Rightarrow -x+y=1 \Rightarrow y=1+x \\ x < 0, y < 0 &\Rightarrow -x-y=1 \Rightarrow y=-x-1 \end{aligned}$$



$$\text{ii) } |x|-|y|=1$$

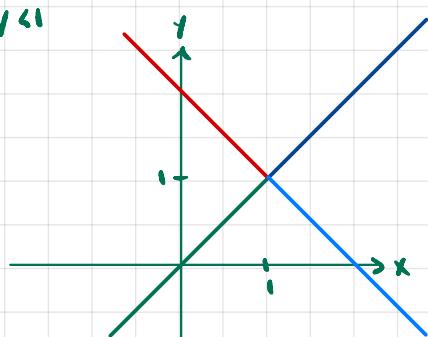
$$\begin{aligned} x > 0, y > 0 &\Rightarrow x-y=1 \Rightarrow y=x-1 \\ x > 0, y < 0 &\Rightarrow x+y=1 \Rightarrow y=1-x \\ x < 0, y > 0 &\Rightarrow -x-y=1 \Rightarrow y=-x-1 \\ x < 0, y < 0 &\Rightarrow -x+y=1 \Rightarrow y=1+x \end{aligned}$$



$$\text{iii) } |x-1|=|y-1|$$

$$|x-1| = \sqrt{(x-1)^2} = \sqrt{(y-1)^2} = |y-1|$$

$$\begin{aligned} x-1 < 0 &\Rightarrow |x-1| = 1-x \\ y-1 < 0 &\Rightarrow |y-1| = 1-y \Rightarrow x < 1, y < 1 \\ &\Rightarrow 1-x = 1-y \Rightarrow y = x \end{aligned}$$



$$\begin{aligned} x-1 < 0, y-1 > 0 &\Rightarrow x < 1, y > 1 \\ &\Rightarrow 1-x < 1 \Rightarrow x > 0 \end{aligned}$$

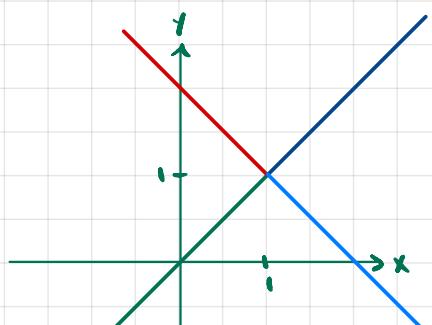
$$\begin{aligned} x-1 > 0, y-1 < 0 &\Rightarrow x > 1, y < 1 \\ &\Rightarrow x-1 > 1 \Rightarrow x > 2 \end{aligned}$$

$$x-1 > 0, y-1 > 0 \Rightarrow x > 1, y > 1 \\ \Rightarrow x-1 = y-1 \Rightarrow x = y$$

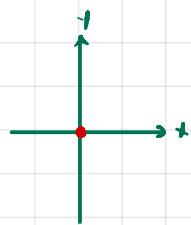
$$\text{iv) } |1-x|=|y-1|$$

$$\begin{aligned} 1-x \geq 0 &\Rightarrow x \leq 1 \\ y-1 \geq 0 &\Rightarrow y \geq 1 \end{aligned}$$

$$\begin{aligned} x \leq 1, y \geq 1 &\Rightarrow 1-x = y-1 \Rightarrow y = 2-x \\ x \leq 1, y \leq 1 &\Rightarrow 1-x = 1-y \Rightarrow x = y \\ x > 1, y \geq 1 &\Rightarrow x-1 = y-1 \Rightarrow x = y \\ x > 1, y < 1 &\Rightarrow x-1 = 1-y \Rightarrow y = 2-x \end{aligned}$$



$$\text{v) } x^2 + y^2 = 0 \Rightarrow x = y = 0$$

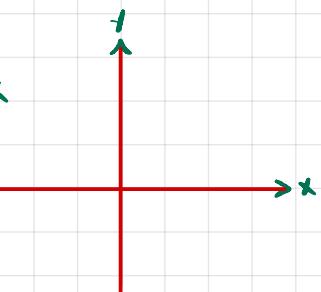


$$\text{vi) } xy=0 \Rightarrow x=0 \text{ or } y=0$$

note: $f_1, f_n \Rightarrow f_1 \cdot 0 = 0, f_n \cdot 0$,
where $0 \cdot f_1 = 0 \forall x$

proof

$$\begin{aligned} &f(x) \cdot 0(x) + f(x) \cdot 0(x) \\ &\text{distributive law for } f_m, p_q \\ &- f(x)(0(x) + 0(x)) \\ &- f(x) \cdot 0(x) = 0 \forall x \end{aligned}$$



How many roots does $ax^2 + bx + c$ have?

$$a > 0 \Rightarrow ax^2 + 2\sqrt{a}x + \frac{b}{2\sqrt{a}} + \left(\frac{b}{2\sqrt{a}}\right)^2 + c - \left(\frac{b}{2\sqrt{a}}\right)^2$$

$$\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 = \left(\frac{b}{2\sqrt{a}}\right)^2 - c = \frac{b^2 - 4ac}{4a}$$

$b^2 - 4ac < 0 \Rightarrow$ no solutions because left side $\geq 0 \forall x$ and right side < 0 .

$$b^2 - 4ac = 0 \Rightarrow \sqrt{a}x = -\frac{b}{2\sqrt{a}} \Rightarrow x = \frac{-b}{2a} \text{ one root}$$

$$b^2 - 4ac > 0 \Rightarrow \sqrt{a}x + \frac{b}{2\sqrt{a}} = \frac{2ax + b}{2\sqrt{a}} = \pm \sqrt{\frac{b^2 - 4ac}{4a}}$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ two roots}$$

$$\text{viii) } x^2 - 2x + y^2 - 4$$

Solution 1: long

$$\Rightarrow x^2 - 2x + y^2 - 4 = 0$$

$$\begin{aligned}\Delta &= 4 - 4 \cdot 1 \cdot (y^2 - 4) \\ &= 4 + 16 - 4y^2 \\ &= 20 - 4y^2\end{aligned}$$

$20 - 4y^2 < 0 \Rightarrow$ no solutions in x

$$4y^2 > 20 \Rightarrow y^2 > 5 \Rightarrow y > \sqrt{5} \cup y < -\sqrt{5}$$

$20 - 4y^2 = 0 \Rightarrow$ one solution in x

$$\Rightarrow y = \pm \sqrt{5} \Rightarrow x = \frac{2}{2} = 1$$

$$\begin{aligned}20 - 4y^2 \geq 0 \Rightarrow x &= \frac{2 \pm \sqrt{20 - 4y^2}}{2} = \frac{2 \pm 2\sqrt{5 - y^2}}{2} \\ &= 1 \pm \sqrt{5 - y^2}\end{aligned}$$

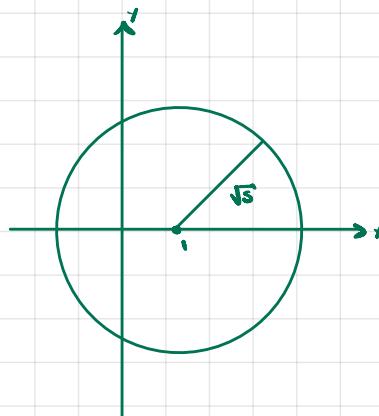
$$\begin{aligned}\Rightarrow x - 1 &= \pm \sqrt{5 - y^2} \\ \Rightarrow (x - 1)^2 &= 5 - y^2 \\ \Rightarrow (x - 1)^2 + y^2 &= (\sqrt{5})^2 \text{ (a circle)}\end{aligned}$$

Solution 2: complete square

$$x^2 - 2x + y^2 = 4$$

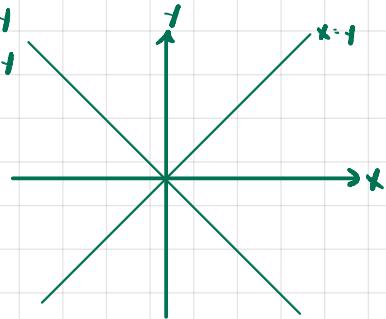
$$\Rightarrow x^2 - 2x + 1 + y^2 = 5$$

$$(x - 1)^2 + y^2 = 5, \text{ a circle}$$

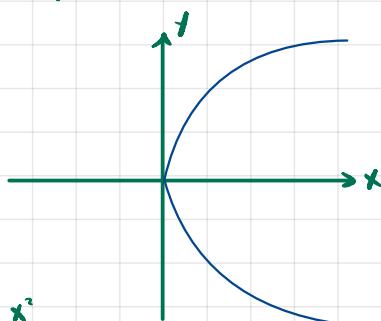


$$\text{viii) } x^2 = y^2$$

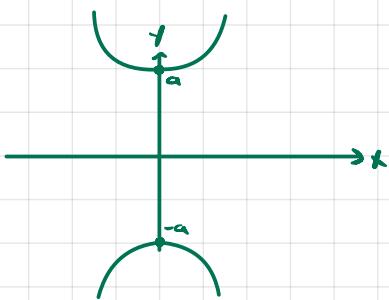
$$\pm x = \pm y \Rightarrow x = y \\ -x = y$$



$$\text{ix) } x \cdot y^2 = 0 \Rightarrow y = \pm \sqrt{x}$$



$$\text{ii) } \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$



$$x=0 \Rightarrow y^2 = a^2 \Rightarrow y = \pm a$$

$$y=0 \Rightarrow x^2 = -b^2 \Rightarrow \text{no soln} \Rightarrow y \neq 0$$

$$y^2 = a^2 + \frac{a^2}{b^2}x^2$$

$$y = \pm \sqrt{a^2 + \frac{a^2}{b^2}x^2}$$

$x=0 \Rightarrow$ the square root is minimized

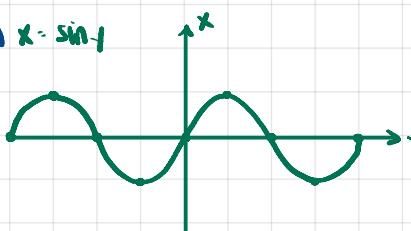
$$\Rightarrow y = \pm a$$

$$\text{iii) } x - |y|$$

$$y > 0 \Rightarrow x - y$$

$$y < 0 \Rightarrow x = -y$$

$$\text{iv) } x = \sin y$$



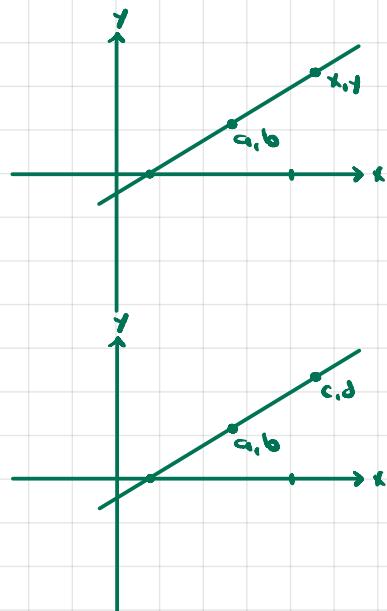
$$6. \text{ a) } f(a) = b$$

points $(x_1, f(x_1))$ that form slope m with a point (a, b)

$$\text{slope} = m = \frac{f - b}{x - a} = \frac{f(x) - b}{x - a}$$

$$\Rightarrow f(x) = b + m(x-a)$$

This is a linear function of x .



b) a+c

slope of line through $(a, b), (c, d)$

$$\text{slope} = m = \frac{d-b}{c-a}$$

points $(x_1, f(x_1))$ forming slope m with (a, b)

$$f(x) = b + \frac{d-b}{c-a}(x-a)$$

$$\text{c) } f(x) = mx + b$$

$$g(x) = m'x + b'$$

parallel $\Leftrightarrow m = m'$

$\Rightarrow A, B, C \in \mathbb{R}$

A, B not both zero

Show: $\{(x, y) : Ax + By + C = 0\}$ is a straight line.

Vertical line $\Rightarrow x = \text{constant} \forall y$

Case 1: $A, B, C \neq 0$

$$\Rightarrow By = -Ax - C \Rightarrow y = -\frac{A}{B}x - \frac{C}{B}, \text{ linear fn.}$$

Case 2: $A=0$

$$By = -C \Rightarrow y = -\frac{C}{B}, \text{ linear, horizontal fn}$$

Case 3: $B=0$

$$\Rightarrow x = -\frac{C}{A}, \text{ linear fn of } y, \text{ vertical line}$$

b) every straight line can be described as $\{(x, y) : Ax + By + C = 0\}$

A straight line means constant slope.

Consider two points on the straight line $(x_1, y_1), (x_2, y_2)$.

If $x_1 \neq x_2$ the slope is

$$\frac{y_2 - y_1}{x_2 - x_1} = m$$

Any other point (x, y) on the line forms

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} = m$$

$$m = 0 \Rightarrow y = y_1, \text{ ie } A = C = 0, B = 1$$

$$m \neq 0 \Rightarrow y - y_1 = mx - mx_1$$

$$mx - y - mx_1 + y_1 = 0$$

$$A = m$$

$$B = 1$$

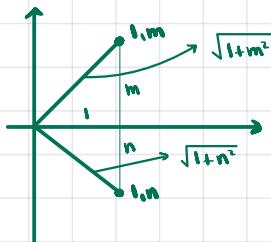
$$C = -mx_1 + y_1$$

8. a)

$$f(x) = mx + b$$

$$g(x) = nx + c$$

Prove: graphs are \perp if $mn = -1$



$$\text{if } \perp \text{ then } 1+n^2 + 1+m^2 = (m-n)^2 = m^2+n^2 - 2mn \\ \Rightarrow mn = -1$$

b) Two straight lines

$$\{(x_1, y_1) : Ax + By + C = 0\}$$

$$\{(x_2, y_2) : A'x + B'y + C' = 0\}$$

$$\text{Prove } \perp \Leftrightarrow AA' + BB' = 0$$

$$y = -\frac{C}{B} - \frac{A}{B}x$$

$$y = -\frac{C'}{B'} - \frac{A'}{B'}x$$

$$\frac{AA'}{BB'} = -1 \Rightarrow AA' + BB' = 0$$

9. In problem 1-14 we proved

$$\text{i)} x_1 f_1 + x_2 f_2 \leq \sqrt{x_1^2 + x_2^2} \sqrt{f_1^2 + f_2^2} \quad \text{if } x_1 = \lambda f_1 \text{ and } x_2 = \lambda f_2, \lambda \geq 0 \\ \text{and } \lambda f_1 = f_2 = 0$$

also if $\exists \lambda \in \mathbb{R}, (x_1, f_1) = \lambda (f_1, f_2)$ then $(\lambda f_1 - x_1)^2 + (\lambda f_2 - x_2)^2 \geq 0$

$$\text{ii)} 2x_1 \leq x^2 + f^2 \text{ ie } (x-f)^2 \geq 0$$

$$\Rightarrow x_1 f_1 + x_2 f_2 \leq \sqrt{x_1^2 + x_2^2} \sqrt{f_1^2 + f_2^2}$$

$$\text{Prove } \sqrt{(x_1 + f_1)^2 + (x_2 + f_2)^2} \leq \sqrt{x_1^2 + x_2^2} + \sqrt{f_1^2 + f_2^2}$$

Solution 1

$$x_1(x_1 + f_1) + x_2(x_2 + f_2) \leq \sqrt{(x_1 + f_1)^2 + (x_2 + f_2)^2} \cdot \sqrt{x_1^2 + x_2^2}$$

$$y_1(x_1 + f_1) + y_2(x_2 + f_2) \leq \sqrt{(x_1 + f_1)^2 + (x_2 + f_2)^2} \cdot \sqrt{y_1^2 + y_2^2}$$

Add inequalities

$$(x_1 + f_1)^2 + (x_2 + f_2)^2 \leq \sqrt{(x_1 + f_1)^2 + (x_2 + f_2)^2} \cdot (\sqrt{x_1^2 + x_2^2} + \sqrt{f_1^2 + f_2^2})$$

$$\frac{(x_1 + f_1)^2 + (x_2 + f_2)^2}{\sqrt{(x_1 + f_1)^2 + (x_2 + f_2)^2}} = \sqrt{(x_1 + f_1)^2 + (x_2 + f_2)^2} \leq \sqrt{x_1^2 + x_2^2} + \sqrt{f_1^2 + f_2^2}$$

Solution 2

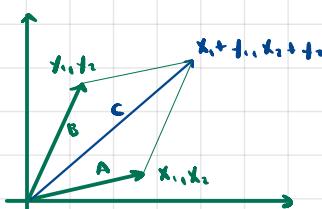
$$(x_1 + f_1)^2 + (x_2 + f_2)^2 = x_1^2 + x_2^2 + f_1^2 + f_2^2 + 2x_1 f_1 + 2x_2 f_2$$

$$= x_1^2 + x_2^2 + f_1^2 + f_2^2 + 2(x_1 f_1 + x_2 f_2)$$

$$\leq x_1^2 + x_2^2 + f_1^2 + f_2^2 + 2\sqrt{x_1^2 + x_2^2} \sqrt{f_1^2 + f_2^2}$$

$$= [\sqrt{x_1^2 + x_2^2} + \sqrt{f_1^2 + f_2^2}]^2$$

$$\Rightarrow \sqrt{(x_1 + f_1)^2 + (x_2 + f_2)^2} \leq \sqrt{x_1^2 + x_2^2} + \sqrt{f_1^2 + f_2^2}$$



$$C \leq A+B$$

$$\text{b) Prove } \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2} \leq \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$$

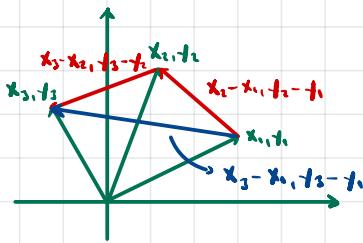
In a) let

$$x_{1a} = x_2 - x_1$$

$$x_{2a} = y_2 - y_1$$

$$y_{1a} = x_3 - x_2$$

$$y_{2a} = y_3 - y_2$$



Apply result from a)

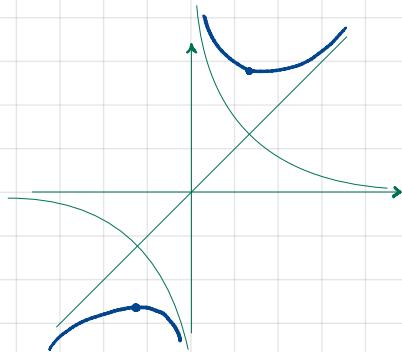
$$[(x_2 - x_1 + x_3 - x_2)^2 + (y_2 - y_1 + y_3 - y_2)^2]^{\frac{1}{2}} \leq [(x_2 - x_1)^2 + (y_2 - y_1)^2]^{\frac{1}{2}} + [(x_3 - x_2)^2 + (y_3 - y_2)^2]^{\frac{1}{2}}$$

$$\Rightarrow \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2} \leq \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$$

i. ii) $f(x) = x + \frac{1}{x}$

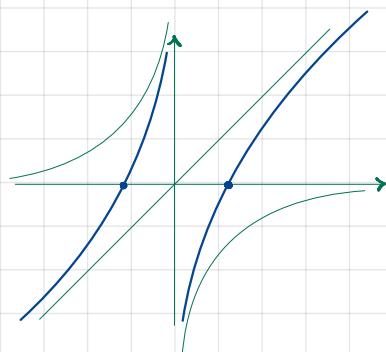
$$f(-x) = -x - \frac{1}{x} = -(x + \frac{1}{x}) = -f(x)$$

$$\Rightarrow f(x) = -f(-x), \text{ f is odd}$$

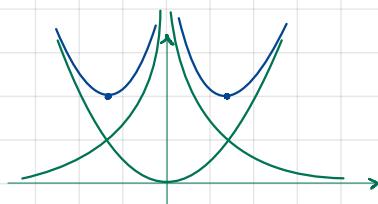


iii) $f(x) = x - \frac{1}{x}$

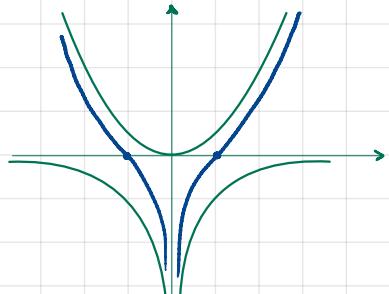
$$f(-x) = -x + \frac{1}{x} = -(x - \frac{1}{x}) = -f(x), \text{ odd fn}$$



iv) $f(x) = x^2 + \frac{1}{x^2}$

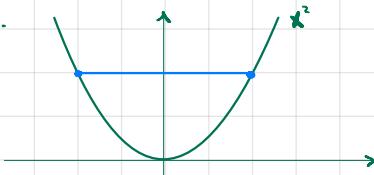


v) $f(x) = x^2 - \frac{1}{x^2}$



ii) f even

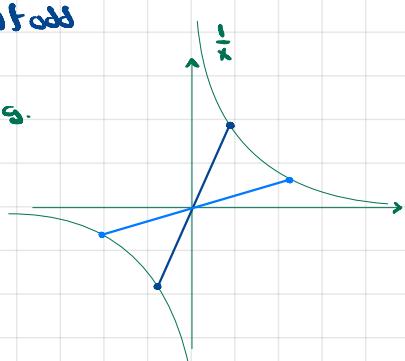
e.g.



symmetric about y axis

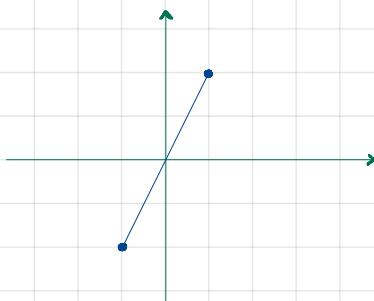
iii) f odd

e.g.



symmetric about the origin

iv) f nonnegative

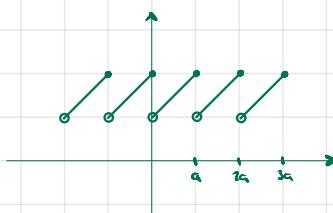


A fn can't be symmetric about the origin if it's nonneg.
no odd fns are nonnegative.

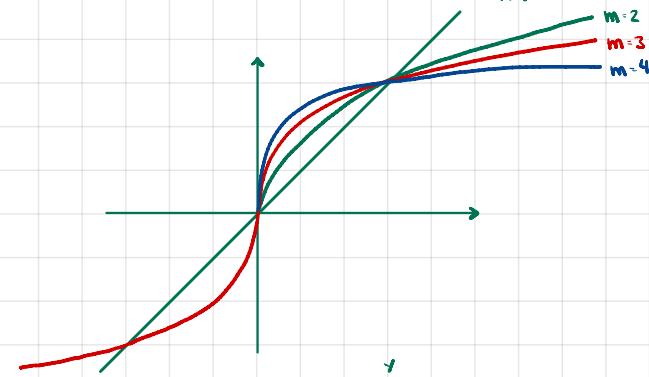
$$\text{odd} \Rightarrow f(x) = -f(-x)$$

$$f(x) > 0 \Rightarrow -f(-x) > 0 \Rightarrow f(-x) < 0$$

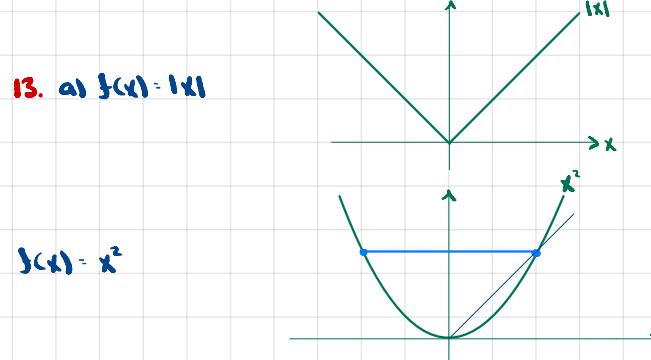
v) $f(x) = f(x+a) \forall x$, ie a periodic fn, w/ period a



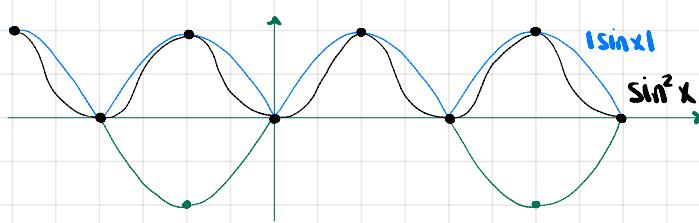
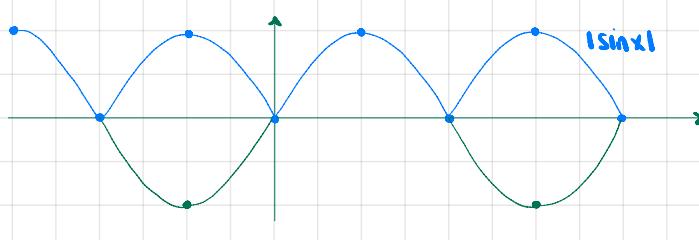
$$12. f(x) = \sqrt[m]{x} \quad m=1,3,5,7$$



$$13. a) f(x) = |x|$$

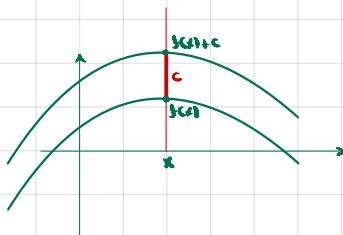


$$b) f(x) = |\sin x|$$

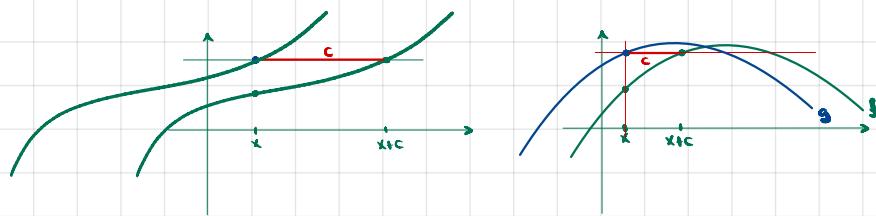


$$14. ii) g(x) = f(x) + c$$

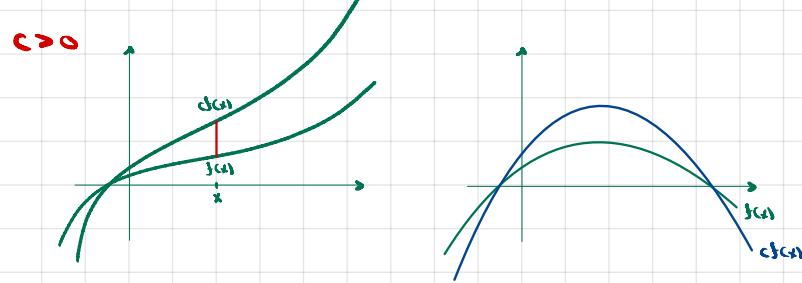
g shifted up or down at each x by $|c|$ units



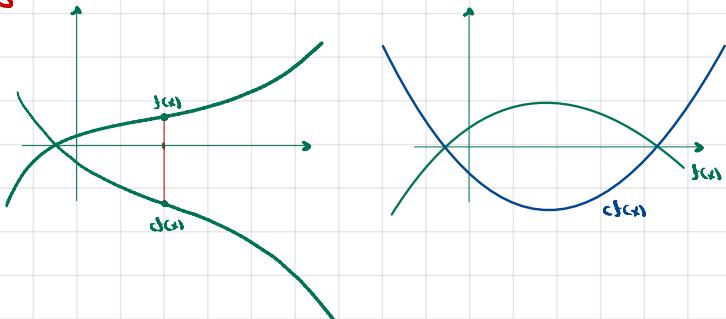
$$iii) g(x) = f(x+c)$$



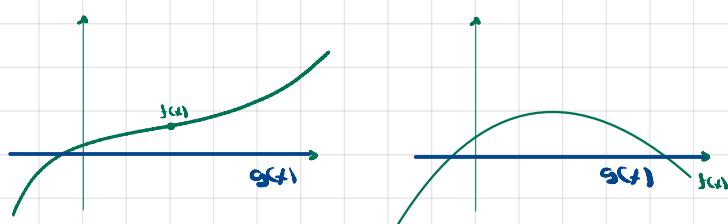
$$iv) g(x) = cf(x)$$



$$c < 0$$



$$c = 0$$



$$g(x) = f(x) + (c-1)f(x)$$

$c \neq 0 \Rightarrow$ The larger $|f(x)|$ the less the difference $|g(x) - f(x)|$

$$g(x) - f(x) = (c-1)f(x)$$

$$c > 1 \Rightarrow |g(x) - f(x)| = (c-1)|f(x)|$$

$$c < 1 \Rightarrow |g(x) - f(x)| = (1-c)|f(x)|$$

f and g coincide only when $f(x) - g(x) = 0$, the roots.

$$\text{II) } g(x) = f(cx)$$



$$c = 0 \Rightarrow g(x) = f(0)$$

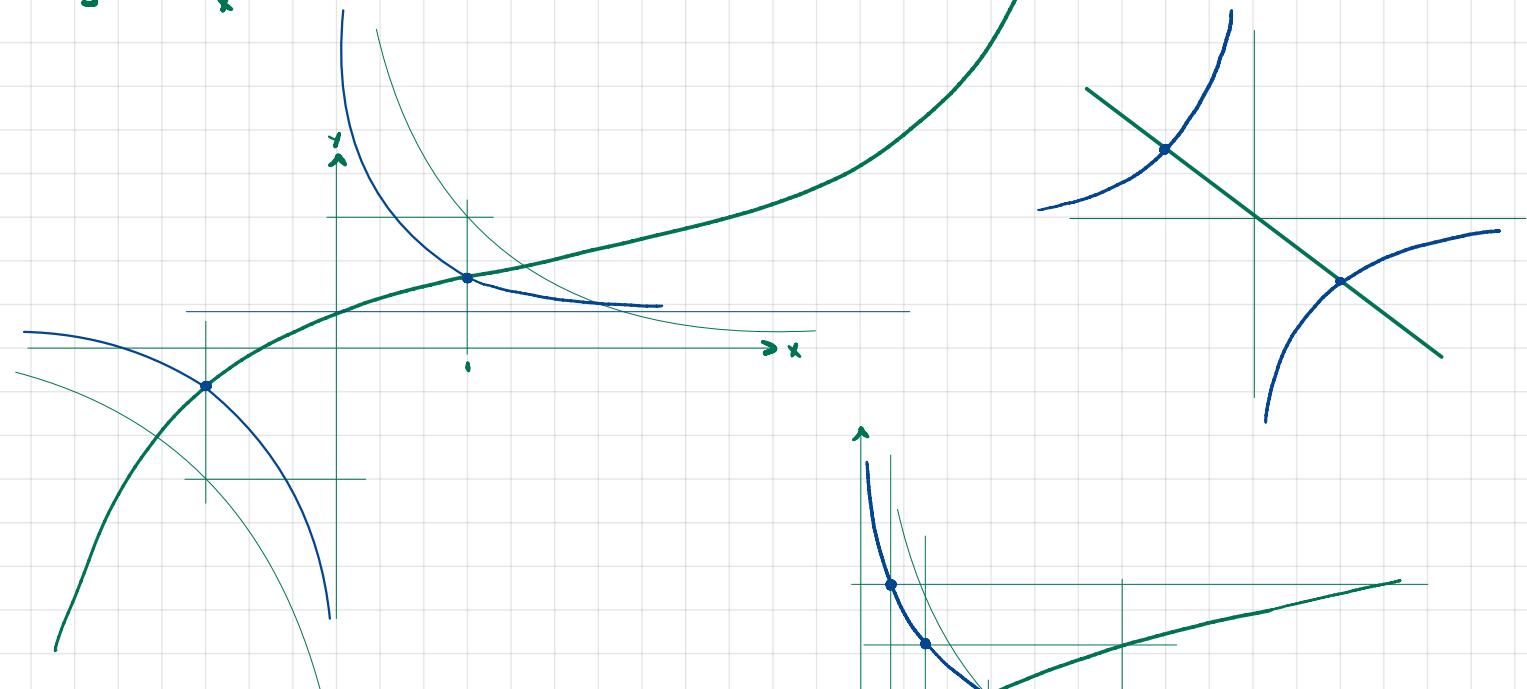
$$c = 1 \Rightarrow g(x) = f(x)$$

$0 < c < 1 \Rightarrow g$ "stretches" \downarrow along the x-axis direction

$c > 1 \Rightarrow g$ "compresses" \uparrow along the x-axis

For $c < 0$ we have three analogous cases, but w.r.t. to the $f_{\text{in}} - f(x)$.

$$\text{III) } g(x) = f\left(\frac{1}{x}\right)$$



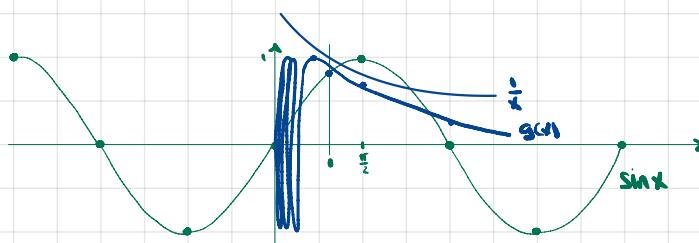
$$g(x) = f\left(\frac{1}{x}\right) = \frac{1}{x} = x \circ x^2 - 1 \Rightarrow x = \pm 1$$

$$g(0^+) = f\left(\frac{1}{0^+}\right) = f(\infty)$$

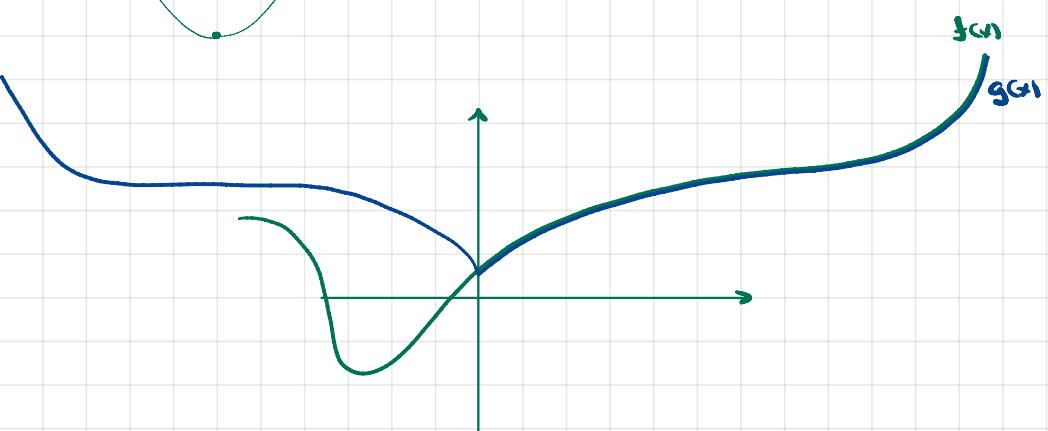
$$g(0^-) = f\left(\frac{1}{0^-}\right) = f(-\infty)$$

$$g(\infty) = f\left(\frac{1}{\infty}\right) = f(0)$$

$$g(-\infty) = f\left(\frac{1}{-\infty}\right) = f(0^-)$$

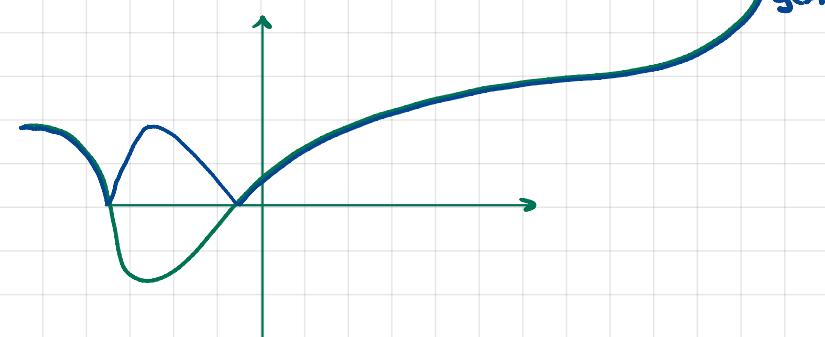


vii) $g(x) = f(|x|)$



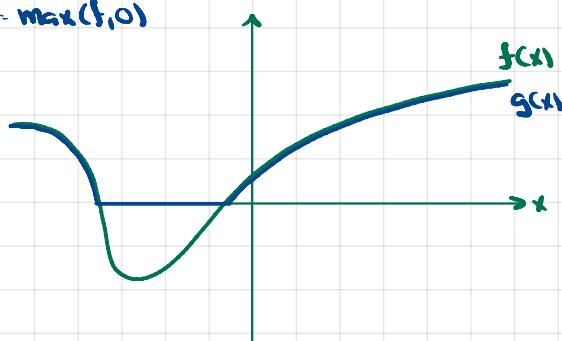
g reflects the portion of f for $x > 0$ across the x -axis

viii) $g(x) = |f(x)|$



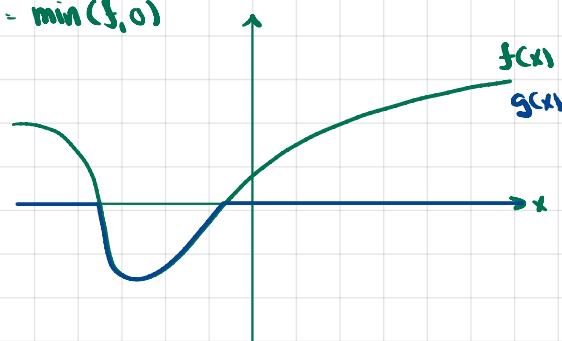
g matches f if $f(x) > 0$. It reflects the negative portions of f across the x -axis.

viii) $g(x) = \max(f, 0)$



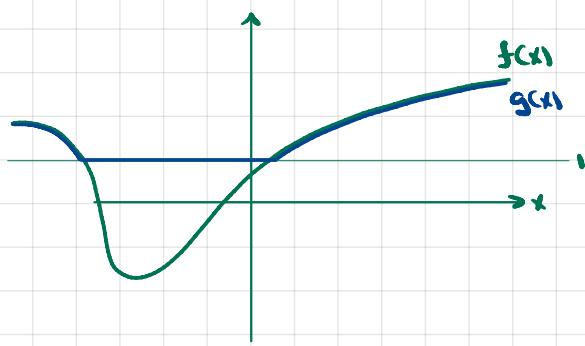
g keeps positive parts of f , where f is ≤ 0 , g is 0.

ix) $g(x) = \min(f, 0)$



g keeps negative parts of f and is zero elsewhere.

$$x) g(x) = \max(f(x), 0)$$



g keeps portions of f s.t. $f(x) > 0$ and is 1 elsewhere else.

$$15. f(x) = ax^2 + bx + c = a\left(x + \frac{b}{a}x + \frac{c}{a}\right) = ag(x)$$

$$\begin{aligned} g(x) &= x^2 + 2x \cdot \frac{b}{2a} + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2 \\ &= \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} \\ &= \underbrace{\left(x + \frac{b}{2a}\right)^2}_{\geq 0} + \underbrace{\frac{4ac-b^2}{4a^2}}_{\text{constant}} \end{aligned}$$

$$\Rightarrow g(x) \text{ min when } x = -\frac{b}{2a} \Rightarrow g(x_{\min}) = \frac{4ac-b^2}{4a^2}$$

roots

$$g(x) = 0 \Rightarrow \frac{b^2-4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

$$\text{case 1: } b^2-4ac = 0 \Rightarrow \text{one root at } x_r = -\frac{b}{2a} = x_{\min}$$

$$g(x) \geq 0 \forall x$$

$$\text{case 2: } b^2-4ac > 0 \Rightarrow \text{two roots}$$

$$x_r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x > x_{r+}, x < x_{r-} \Rightarrow g(x) > 0$$

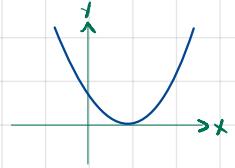
$$x_{r-} < x < x_{r+} \Rightarrow g(x) < 0$$

$$\text{case 3: } b^2-4ac < 0 \Rightarrow \text{no roots}$$

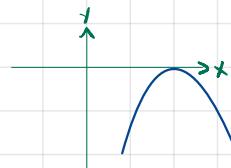
$$g(x) > 0 \forall x$$

Now consider $f(x)$

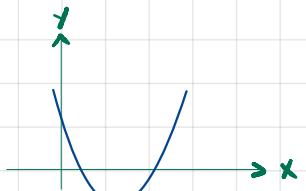
$$\text{case 1: } a > 0, b^2-4ac = 0$$



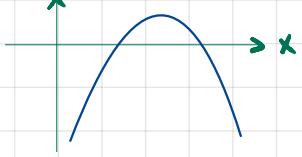
$$\text{case 2: } a < 0, b^2-4ac = 0$$



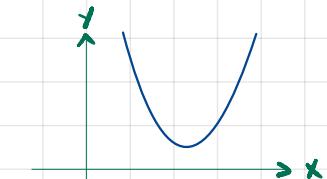
$$\text{case 3: } a > 0, b^2-4ac > 0$$



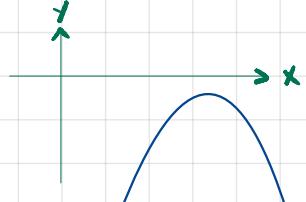
$$\text{case 4: } a < 0, b^2-4ac > 0$$



$$\text{case 5: } a > 0, b^2-4ac < 0$$



$$\text{case 6: } a < 0, b^2-4ac < 0$$



16. Assume A and C not both zero.

Show $\{(x,y) : Ax^2 + Bxy + Cy^2 + Dx + E = 0\}$

is either

parabola

ellipse

hyperbola

or degenerate cases

two lines

one line

a point

\emptyset

* There are 3⁵ possible cases

case 1: $C=0$ $Ax^2 + Bxy + Dy + E = 0$ 3⁴ cases

$$\Rightarrow y = -\frac{A}{D}x^2 - \frac{B}{D}x - \frac{E}{D}$$

= $ax^2 + bx + c$ parabola

$$A=0 \quad y = -\frac{B}{D}x - \frac{E}{D} = bx + c \text{ line}$$

$$A=0, B=0 \quad y = -\frac{E}{D} \text{ c line}$$

$$D=0 \Rightarrow Ax^2 + Bxy + E = 0$$

use & back in the cases of problem 15.

0 roots $\Rightarrow \emptyset$

1 " \Rightarrow line

2 " \Rightarrow two lines

case 2: $A=0$ $Bxy + Cy^2 + Dy + E = 0$ 3⁴ - 3³ cases

symmetric to case 1.

$$3^5 - (3^4 + 3^4 - 3^3) = 3^4 + 3^3 = 81 + 27 = 108$$

cases left

case 3: $A > 0, C > 0, B=D=0$ 3 cases

$$\frac{x^2}{C} + \frac{B}{AC}x + \frac{y^2}{A} + \frac{D}{AC}y + \frac{E}{AC} = 0$$

$$\frac{x^2}{C} + \frac{y^2}{A} = -\frac{E}{AC}$$

$$\Rightarrow E > 0 \Rightarrow \emptyset$$

$$\Rightarrow E = 0 \Rightarrow (x,y) = (0,0)$$

$$\Rightarrow E < 0 \Rightarrow \frac{x^2}{-EIA} + \frac{y^2}{-EIC} = 1$$

$$\frac{x^2}{a} + \frac{y^2}{b} = 1 \text{ ellipse}$$

* circle if $A=C$

case 4: $A > 0, C < 0, B=D=0$ 3 cases

$$\frac{x^2}{C} + \frac{y^2}{A} = -\frac{E}{AC}$$

$$E=0 \Rightarrow (x,y) = (0,0)$$

$$E > 0 \Rightarrow \frac{y^2}{-EIC} + \frac{x^2}{-EIA} = 1$$

$$\frac{y^2}{a} - \frac{x^2}{b} = 1 \text{ hyperbola}$$

$$E < 0 \Rightarrow \emptyset$$

case 5: $A < 0, C > 0, B=D=0$ 3 cases

B) symmetric w/ case 4 are have either single point (0,0), empty set, or hyperbolas.

case 6: $A < 0, C < 0, B=D=0$ 3 cases

$$\frac{x^2}{C} + \frac{y^2}{A} = -\frac{E}{AC}$$

$$E=0 \Rightarrow (0,0)$$

$$E < 0 \Rightarrow \emptyset$$

$$E > 0 \Rightarrow \frac{y^2}{-EIC} + \frac{x^2}{-EIA} = 1$$

$$\Rightarrow \frac{x^2}{a} + \frac{y^2}{b} = 1 \text{ ellipse}$$

What's left are cases of B and D not both zero.

8 possibilities \times 4 cases involving A, B, B=D=0, E + 3 cases

Jct E = 96 cases

CASE 1: A > 0, C > 0, B > 0, D = 0, E

$$\frac{x^2}{C} + \frac{B}{AC}x + \frac{y^2}{A} + \frac{E}{AC} = 0$$

$$\frac{x^2}{C} + 2 \cdot \frac{x}{\sqrt{C}} \cdot \frac{B}{2A\sqrt{C}} + \left(\frac{B}{2A\sqrt{C}}\right)^2 - \left(\frac{B}{2A\sqrt{C}}\right)^2 + \frac{y^2}{A} + \frac{E}{AC}$$

$$\left(\frac{x}{\sqrt{C}} + \frac{B}{2A\sqrt{C}}\right)^2$$

$$Ax^2 + By^2 + C + D_1 + E = 0$$

A > 0, C > 0

$$A(x^2 + \frac{B}{A}x) + C(y^2 + \frac{D}{C}y) + E = 0$$

$$A(x^2 + 2x \frac{B}{2A} + (\frac{B}{2A})^2) + C(y^2 + 2y \frac{D}{2C} + (\frac{D}{2C})^2) + E - A(\frac{B}{2A})^2 - C(\frac{D}{2C})^2 = 0$$

$$A(x + \frac{B}{2A})^2 + C(y + \frac{D}{2C})^2 - A(\frac{B}{2A})^2 - C(\frac{D}{2C})^2 = E$$

$$A(x + \frac{B}{2A})^2 + C(y + \frac{D}{2C})^2 = F$$

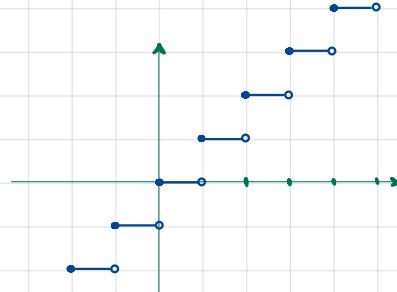
F = 0 \Rightarrow single point

F < 0 \Rightarrow \emptyset

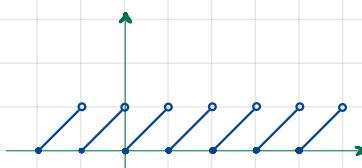
F > 0 \Rightarrow

$$\text{II. } [x] = \text{largest } z \leq x$$

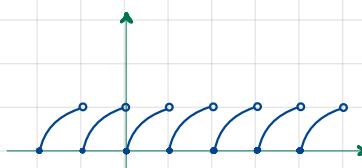
$$\text{i) } f(x) = [x]$$



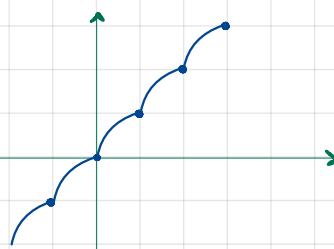
$$\text{ii) } f(x) = x - [x]$$



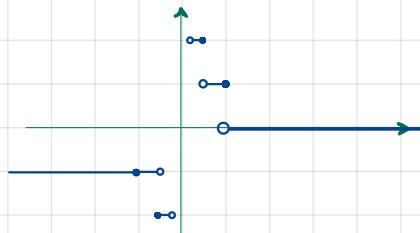
$$\text{iii) } f(x) = \sqrt{x - [x]}$$



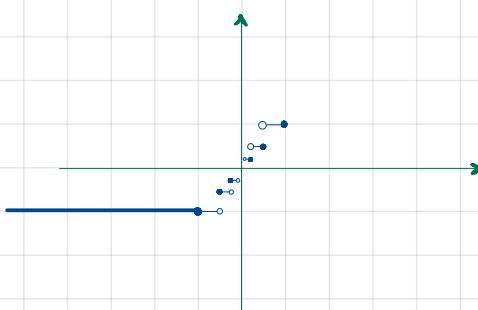
$$\text{iv) } f(x) = [x] + \sqrt{x - [x]}$$



$$\text{vi) } f(x) = \left[\frac{1}{x} \right]$$

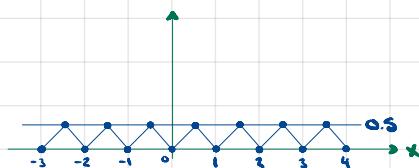


$$\text{vii) } f(x) = \left[\frac{1}{|x|} \right]$$

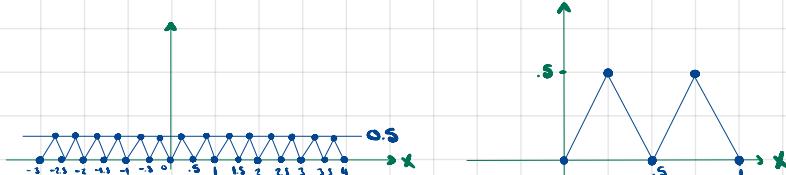


18. $\{x\}$ distance from x to nearest integer

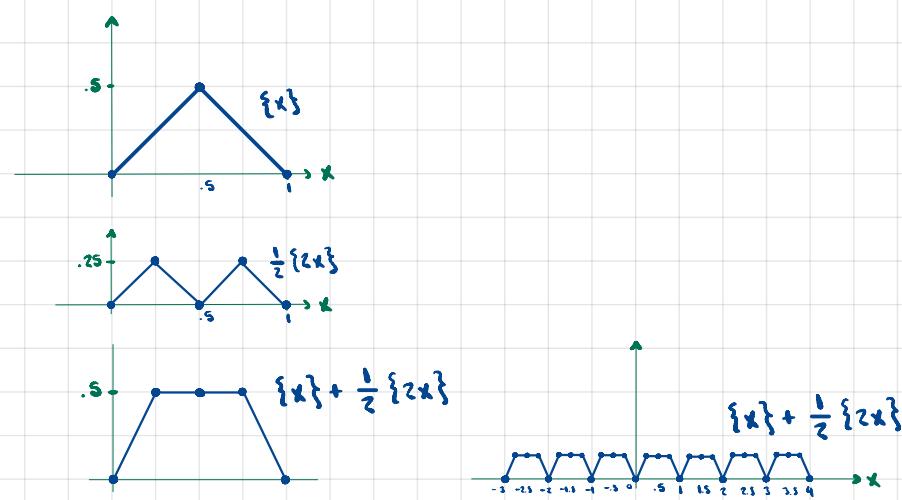
ii) $f(x) = \{x\}$



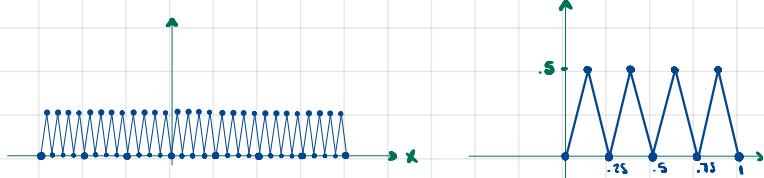
iii) $f(x) = \{2x\}$



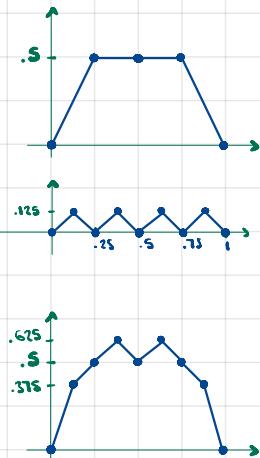
iv) $f(x) = \{x\} + \frac{1}{2}\{2x\}$



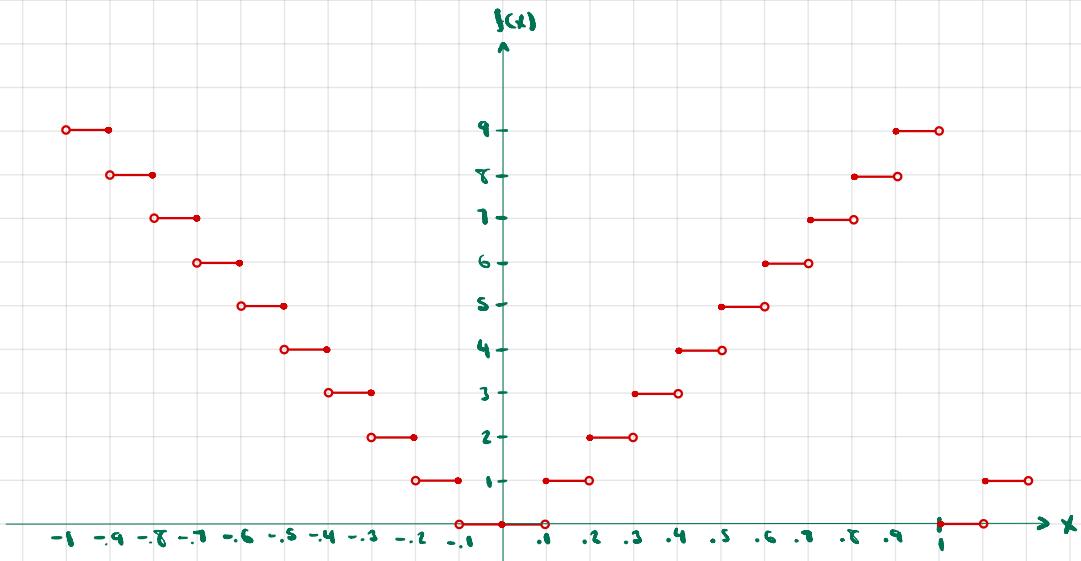
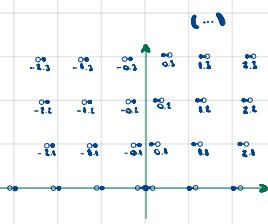
v) $f(x) = \{4x\}$



vi) $f(x) = \{x\} + \frac{1}{2}\{2x\} + \frac{1}{4}\{4x\}$



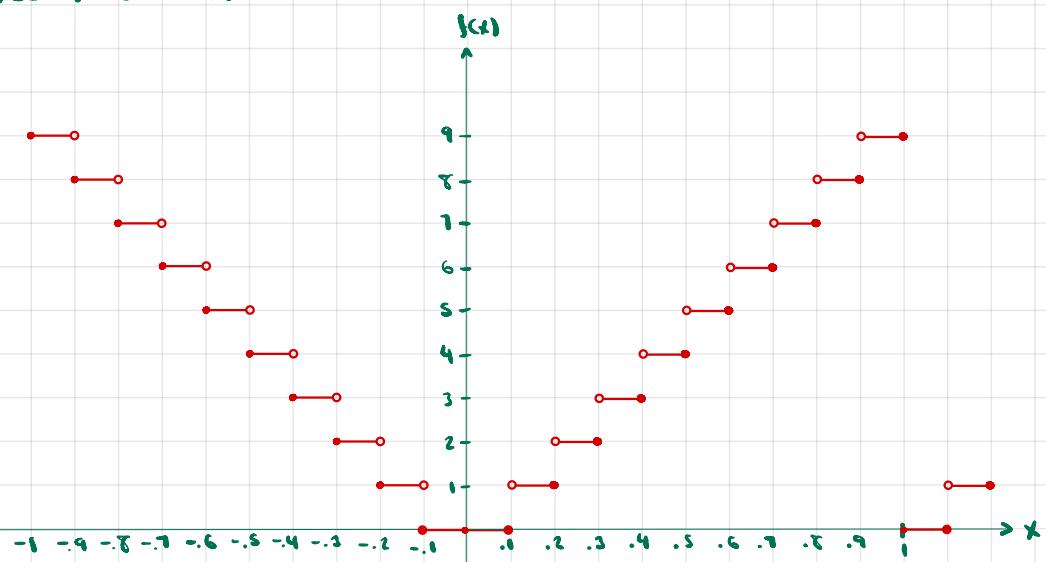
19.

ii) $f(x) = 1^{\text{st}} \text{ number of decimal expansion of } x$ 

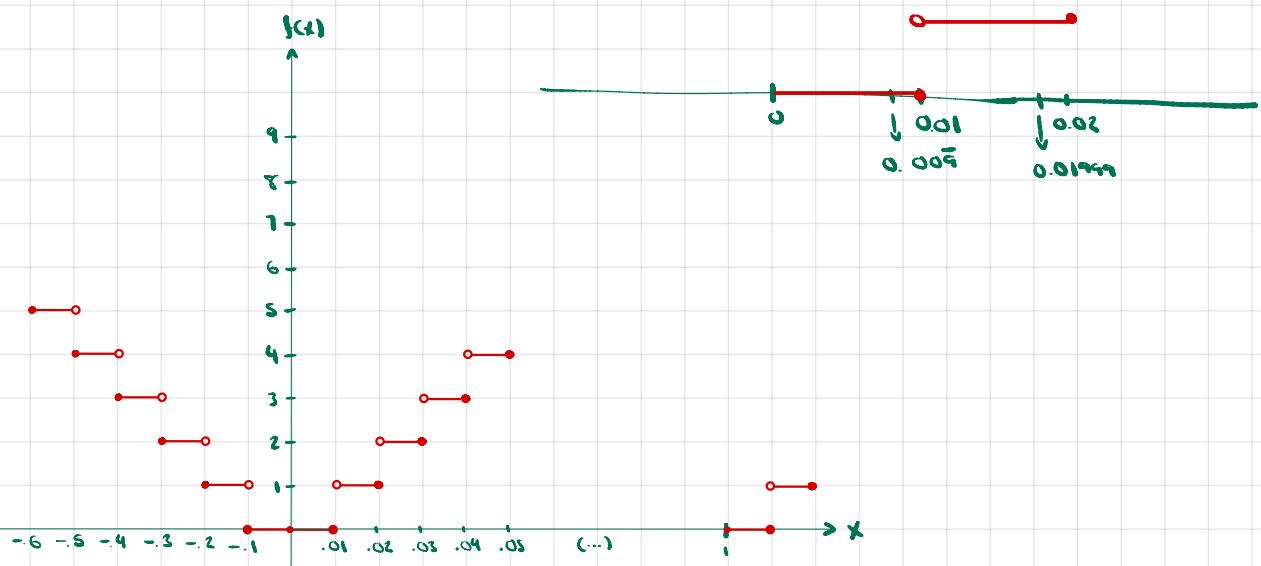
However, we are considering that, e.g., $0.\bar{0}9 = 1$.

$$\Rightarrow f(0.1) = f(0.0\bar{9}) = 0$$

$$f(0.2) = f(0.1\bar{9}) = 1$$



ii) $f(x)$ = 2nd number in decimal expansion of x



iii) $f(x)$ = numbers of 7's in decimal expansion of x if this number is finite,
0 otherwise

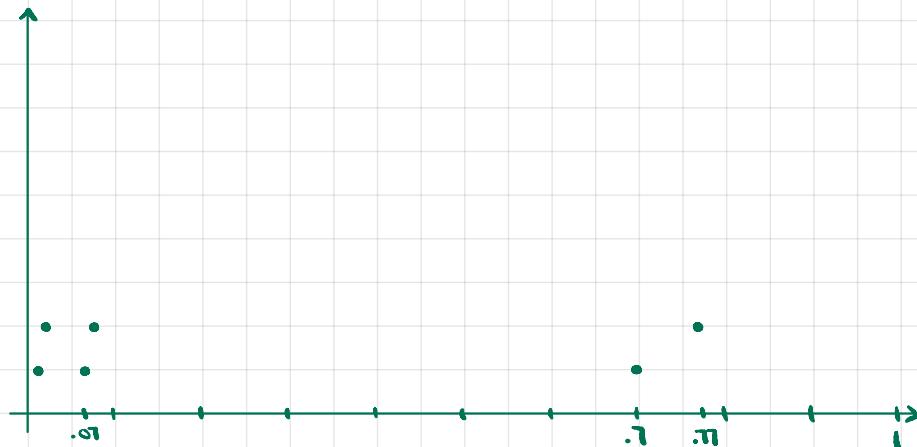
Infinite 7's

Rational numbers w/ infinite 7's

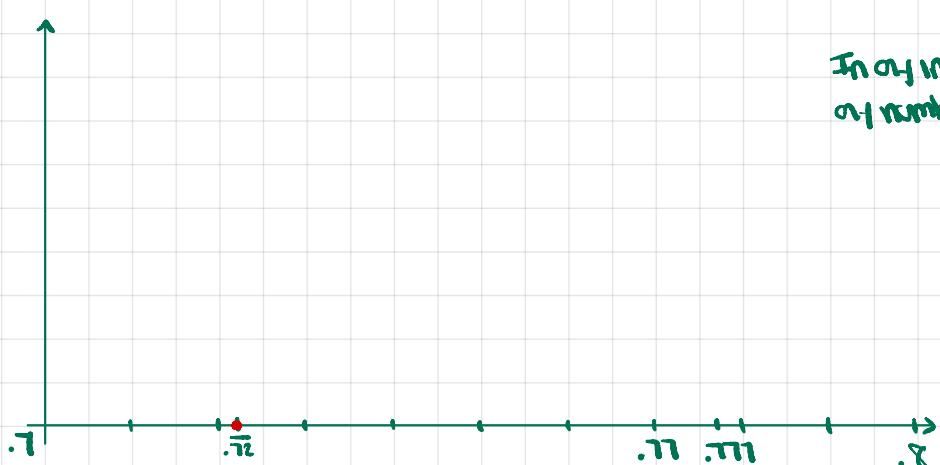
Fraction w/ repeating decimal repes. e.g. $0.\overline{7}$, $0.\overline{74}$, $0.\overline{247}$

Finite 7's

0.7	0.77
0.07	0.077
0.007	0.0077



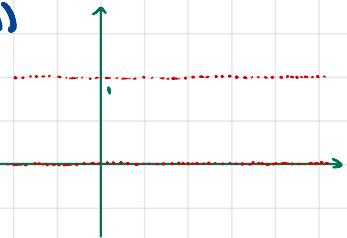
In any interval there are infinite numbers w/
any number of 7's, from zero to infinity.



$$.72 \Rightarrow 10^2 \cdot .72 = 72 + .72 \Rightarrow 72 = .72(10^2 - 1)$$

$$\Rightarrow .72 = \frac{72}{99}$$

i)



v) $f(x)$ = number obtained by replacing all digits in decimal expansion of x which come after first 7, i.e. only by 0.

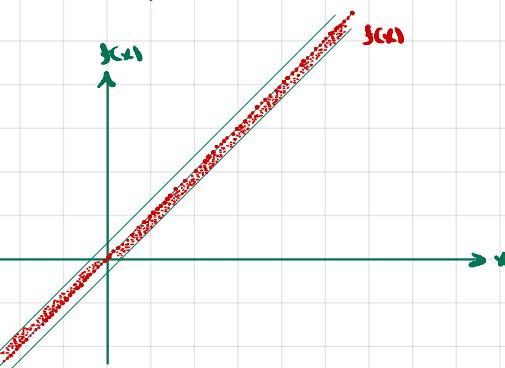
$$f(0.77) = 0.7$$

$$f(0.07) = 0.07$$

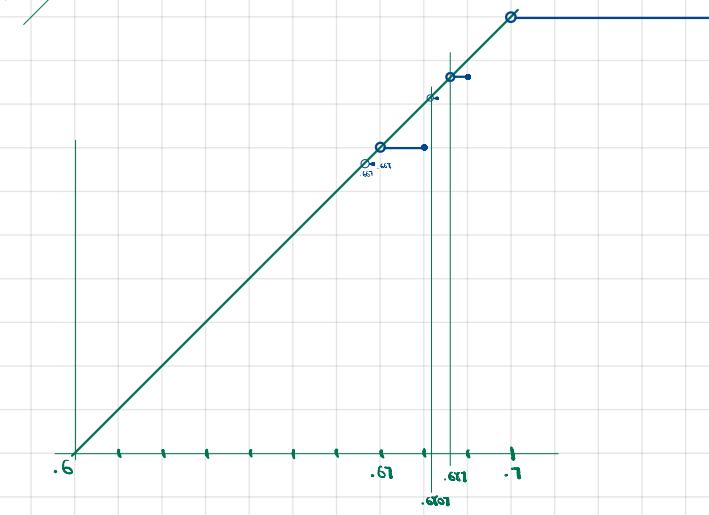
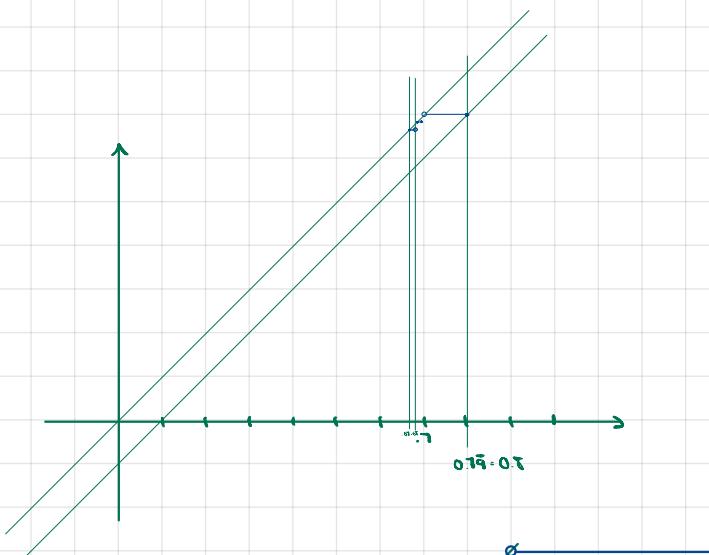
$$f(0.\overline{72}) = 0.7$$

$$f(0.7\overline{9}) = 0.7$$

Note: Initially I read the problem incorrectly. I read that only one digit could be set to zero: the one after the first 7.
i.e. remove at most 0.07 from numbers $a_1 \dots a_n$ in decimal expansion



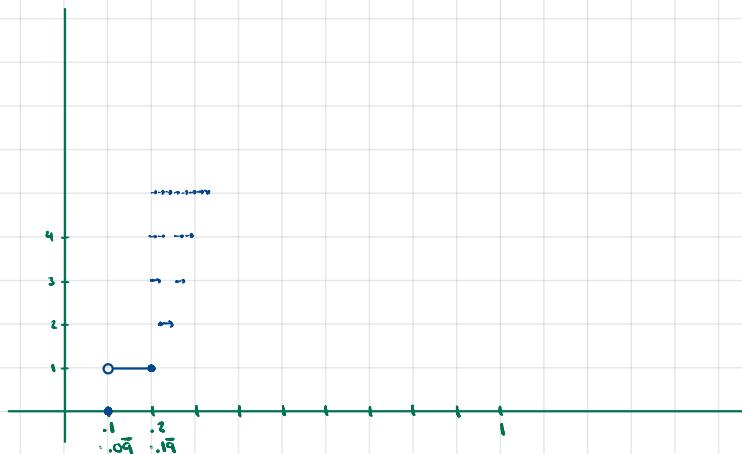
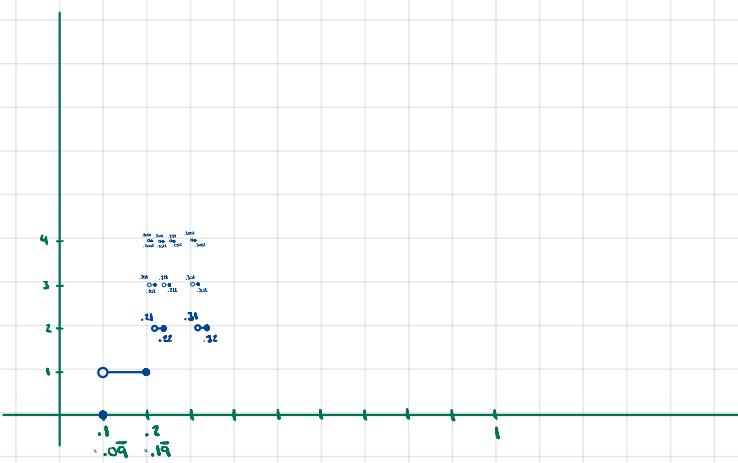
$0.0\overline{9}$ is max subtraction.



$f(x)$ is composed of infinite intervals like the ones above, with points on the line $y = x$ for numbers x with no 7s in them.

vii) $f(x) =$

- o if 1 never appears in decimal expansion of x
- n if 1 first appears in n^{th} place



20.

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q}' \\ \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \text{ in lowest terms} \end{cases}$$

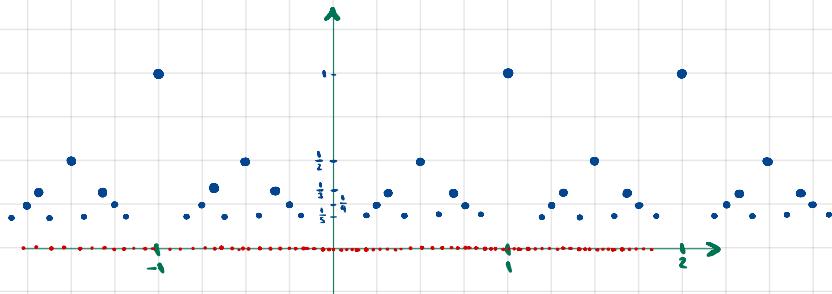
↑ no common factor, $q > 0$

$$q=1 \Rightarrow x = p \in \mathbb{Z}, f(x)=1, x = -1, 1, 2$$

$$q=2 \Rightarrow x = \frac{p}{2}, p \in \mathbb{Z}, f(x) = \frac{1}{2}, x = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$$

$$q=3 \Rightarrow x = \frac{p}{3}, p \in \mathbb{Z}, f(x) = \frac{1}{3}, x = -\frac{5}{3}, -\frac{4}{3}, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}$$

$$q=4 \Rightarrow x = \frac{p}{4}, p \in \mathbb{Z}, f(x) = \frac{1}{4}, x = -\frac{5}{4}, -\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}$$

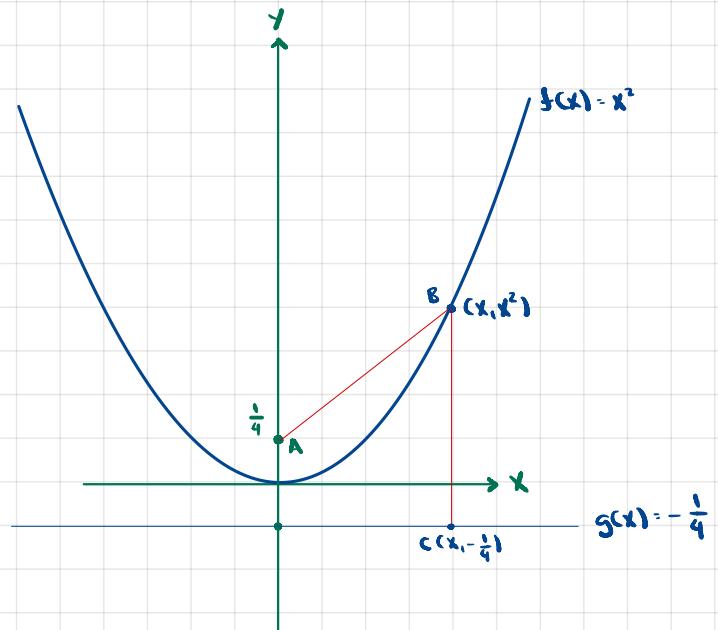


$$21. a) f(x) = x^2$$

$$\begin{aligned} AB &= \left(x^2 + (x^2 - \frac{1}{4})^2 \right)^{1/2} = \left(x^2 + x^4 - \frac{x^2}{2} + \frac{1}{16} \right)^{1/2} \\ &= \left(x^4 + \frac{x^2}{2} + \frac{1}{16} \right)^{1/2} \end{aligned}$$

$$BC = \left((x^2 + \frac{1}{4})^2 \right)^{1/2} = \left(x^4 + \frac{x^2}{2} + \frac{1}{16} \right)^{1/2}$$

$\Rightarrow AB = BC$, ie any point (x, x^2) is equidistant from $(0, 1/4)$ and the line $g(x) = -\frac{1}{4}$.



b) consider a point $A = (x, y)$, equidistant from P and g .

$$AP = \sqrt{(x-\alpha)^2 + (y-\beta)^2}$$

$$Ag = \sqrt{(y-\beta)^2}$$

$$AP = Ag \Rightarrow (x-\alpha)^2 + (y-\beta)^2 = (y-\beta)^2$$

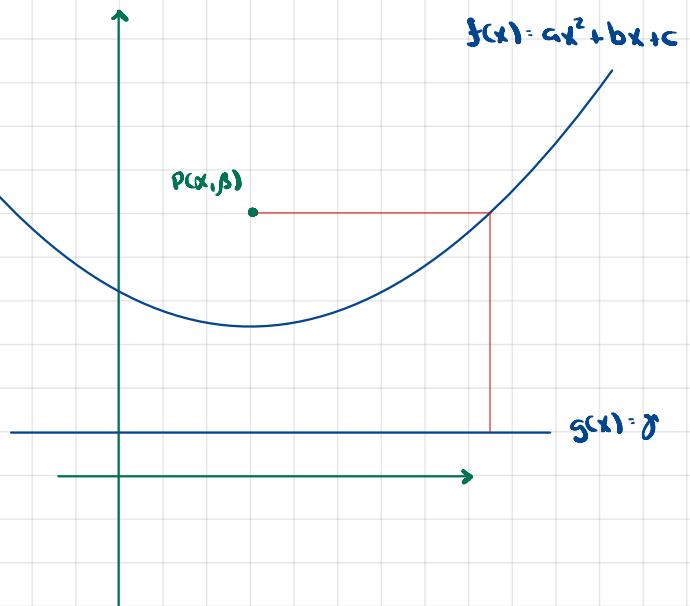
$$x^2 - 2x\alpha + \alpha^2 + y^2 - 2y\beta + \beta^2 = y^2 - 2y\beta + \beta^2$$

$$x^2 - 2x\alpha + \alpha^2 + \beta^2 - \beta^2 = 2y\beta - 2y\beta = 2y(\beta - \delta)$$

by assumption, $\beta \neq \delta$

$$\Rightarrow y = \frac{1}{2(\beta-\alpha)} x^2 - \frac{2\alpha}{2(\beta-\alpha)} x + \frac{\alpha^2 + \beta^2 - \beta^2}{2(\beta-\alpha)}$$

$$\Rightarrow y = f(x) = ax^2 + bx + c$$



* if $\beta = \delta$ we could be looking for points (x, y) such that $A = B$ in the figure:

$$\text{ie } \sqrt{(x-\alpha)^2 + (y-\beta)^2} = \sqrt{(y-\beta)^2}$$

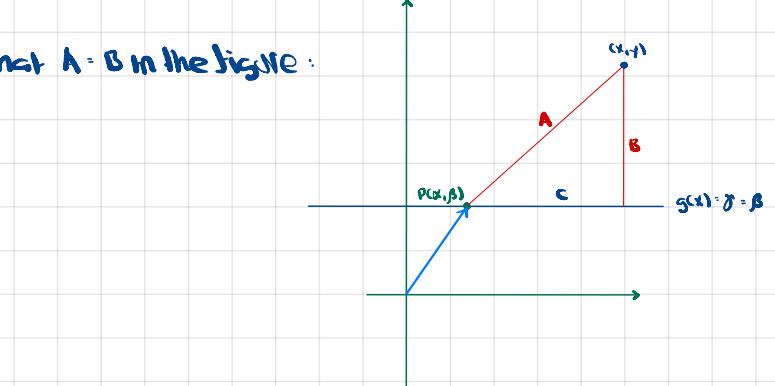
$\Rightarrow x = \alpha$, ie points on the x -axis.

* note that by the Triangle Inequality

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} \leq \sqrt{(y-\beta)^2} + \sqrt{(x-\alpha)^2}$$

$$A \leq B + C$$

equality only if $\vec{B} = \lambda \vec{C}$ or one of \vec{B}, \vec{C} is $\vec{0}$.



$$\begin{aligned} \vec{A} &= (x-\alpha, y-\beta) \\ \vec{B} &= (0, y-\beta) \\ \vec{C} &= (x-\alpha, 0) \end{aligned}$$

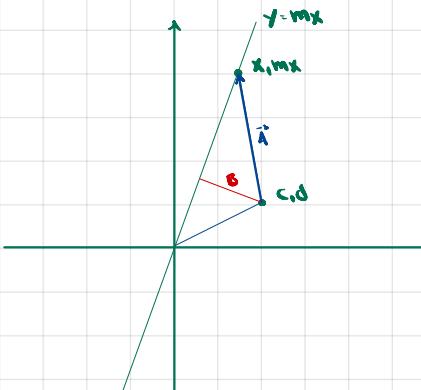
22. a) shows squared distance from (c, d) to (x, mx) is

$$x^2(m^2+1) + x(-2md-2c) + d^2 + c^2$$

$$\begin{aligned}\vec{A} &= (x, mx) - (c, d) \\ &= (x-c, mx-d)\end{aligned}$$

$$\|\vec{A}\| = \sqrt{(x-c)^2 + (mx-d)^2}$$

$$\begin{aligned}f(x) &= \|\vec{A}\|^2 = x^2 - 2xc + c^2 + m^2x^2 - 2mdx + d^2 \\ &= x^2(m^2+1) + x(-2md-2c) + d^2 + c^2 \\ &\text{2nd degree polynomial in } x, \text{ representing square of distance} \\ &\text{from point } (c, d) \text{ to arbitrary point on } f(x) = mx.\end{aligned}$$



$m^2+1 > 0 \Rightarrow$ parabola goes to $\pm\infty$ as $x \rightarrow \pm\infty$.

$$x_{\min} = -\frac{(-2md-2c)}{2(m^2+1)} = \frac{md+c}{m^2+1}$$

$$\begin{aligned}f(x_{\min}) &= \left(\frac{md+c}{m^2+1}\right)^2(m^2+1) - \frac{md+c}{m^2+1} \cdot 2(md+c) + d^2 + c^2 = \frac{(md+c)^2}{m^2+1} - 2 \frac{(md+c)^2}{m^2+1} + d^2 + c^2 \\ &= -\frac{(md+c)^2}{m^2+1} + d^2 + c^2 = \frac{-(m^2d^2 + 2mdc + c^2) + m^2d^2 + m^2c^2 + d^2 + c^2}{m^2+1} \\ &= \frac{m^2c^2 - 2mdc + d^2}{m^2+1} = \frac{(mc-d)^2}{m^2+1}\end{aligned}$$

$$\Rightarrow \text{min distance} = \frac{|mc-d|}{\sqrt{m^2+1}}$$

Note

$$\Delta = (-2md-2c)^2 - 4(m^2+1)(d^2+c^2)$$

$$= \cancel{4m^2d^2 + 8mdc + 4c^2} - \cancel{4m^2d^2} - \cancel{4m^2c^2} - \cancel{4d^2} - \cancel{4c^2}$$

$$\begin{aligned}&= 8mdc - 4m^2c^2 - 4d^2 \\ &= -4(m^2c^2 - 2mdc + d^2) \\ &= -4(mc-d)^2 \leq 0\end{aligned}$$

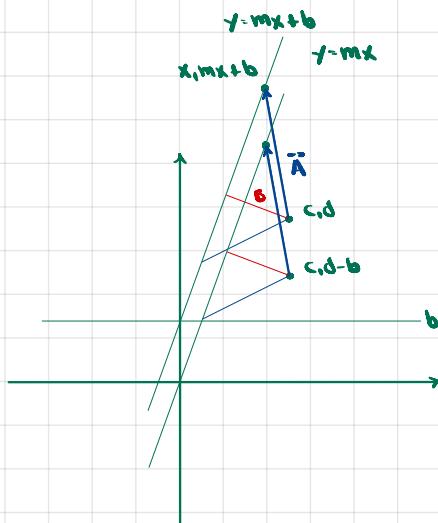
\Rightarrow either zero roots or one root

Assume $m > 0$

either $c = d = 0$ or
 $mc - d = 0 \Rightarrow mc = d \Rightarrow c$ and d have same sign then there is one root. This happens when $m = \frac{d}{c}$
 $\Rightarrow f(x) = \frac{d}{c}x$, $f(c) = d$, (c, d) on f , so there is one point where the distance is zero.

$\Rightarrow (c, d)$ in quadrants II or IV then no roots no matter what.

b) distance from (c,d) to $y = mx + b$



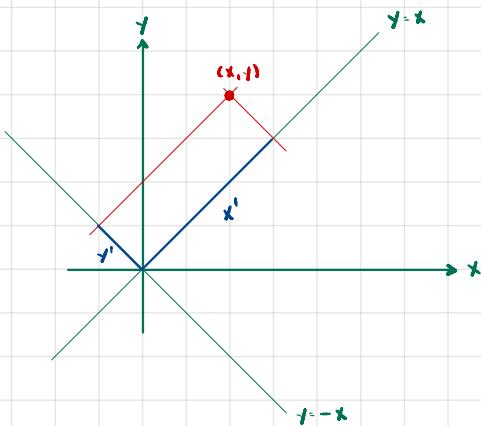
$$\begin{aligned}\vec{A} &= (x, mx+b) - (c, d) = (x-c, mx+b-d) = (x-c, mx-(d-b)) \\ &= (x, mx) - (c, d-b)\end{aligned}$$

$$\|\vec{A}\|^2 = x^2(m^2+1) + x(-2m(d-b) - 2c) + (d-b)^2 + c^2$$

$$\min \|\vec{A}\| = \frac{|mc - (d-b)|}{\sqrt{m^2+1}}$$

$$= \frac{|mc + b - d|}{\sqrt{m^2+1}}$$

23.



a)

distance from (x, y) to $y = x = x$

$$y' = \frac{|x - y|}{\sqrt{1^2 + 1^2}} = \frac{|x - y|}{\sqrt{2}} = \frac{|y - x|}{\sqrt{2}}$$

because $y > x$ in graph above

distance from (x, y) to $y = -x = -x$

$$x'' = \frac{|x - -y|}{\sqrt{1^2 + 1^2}} = \frac{|x + y|}{\sqrt{2}}$$

b) show $\left\{(x, y) : \left(\frac{x'}{\sqrt{2}}\right)^2 - \left(\frac{y'}{\sqrt{2}}\right)^2 = 1\right\} = \{(x, y) : x^2 + y^2 = 1\}$

$$\left(\frac{x'}{\sqrt{2}}\right)^2 = \frac{(x+y)^2}{4}$$

$$\left(\frac{y'}{\sqrt{2}}\right)^2 = \frac{(y-x)^2}{4}$$

$$\left(\frac{x'}{\sqrt{2}}\right)^2 - \left(\frac{y'}{\sqrt{2}}\right)^2 = \frac{(x+y)^2}{4} - \frac{(y-x)^2}{4} = \frac{y^2 + 2xy + x^2 - (y^2 - 2xy + x^2)}{4} = xy$$

$$\therefore \left(\frac{x'}{\sqrt{2}}\right)^2 - \left(\frac{y'}{\sqrt{2}}\right)^2 = xy$$