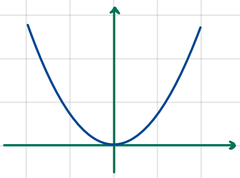
Ch 8 Appendix - Uniform continuity

Definition function f is uniformly continuous on an interval A if for every $\epsilon > 0$ there is some $\delta > 0$ such that for all x and y in A,

$$|x-y| < \delta \rightarrow |f(x)-f(y)| < \epsilon$$

Examples



$f(x) = x^2$, cont. but not u.c. on $\mathbb{R}$

Proof

consider $a \in \mathbb{R}$.

Let's say you choose an $\epsilon > 0$ and you have $\delta > 0$ s.t.

$|x-a| < \delta \rightarrow |f(x)-f(a)| < \epsilon$ (1)

consider $\frac{\delta}{2}$. (1) is still true if we sub $\frac{\delta}{2}$ for δ.

Assume (1) is true for all a.

Then, since $\left|a + \frac{\delta}{2} - a\right| = \left|\frac{\delta}{2}\right| < \delta$,

$|f(a+\delta/2) - f(a)| < \epsilon$

$|(a+\delta/2)^2 - a^2| < \epsilon$

$\left|a\delta + \frac{\delta^2}{4}\right| < \epsilon$

But is this really true for all a?

$\left|a\delta + \frac{\delta^2}{4}\right| > |a\delta|$

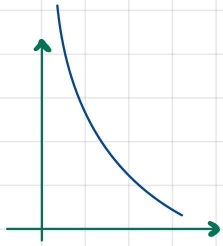
If $a > \frac{\epsilon}{\delta}$ then $|a\delta| > \epsilon$

Hence $|f(a+\delta/2) - f(a)| > \epsilon$

$\perp$

Therefore (1) is not true for all a.

$f(x) = \frac{1}{x}$, $x \in (0,1)$

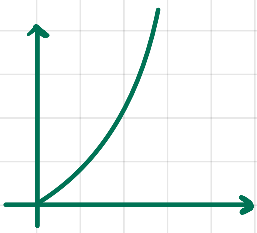


not u.c. on $(0,1)$, but c. on $(0,1)$

$f(x) = x$, u.c. on $\mathbb{R}$.

$f(x) = x^2$, $x \in [0,2]$

u.c. on $[0,2]$



$|x^2 - a^2| = |(x-a)(x+a)|$

$|x-a| < 1 \rightarrow a-1 < x < a+1$

$\rightarrow a < x+1 < a+2$

$\rightarrow |x+1| < a+2$

$|(x-a)(x+a)| \leq |x-a||x+a| < (a+2)|x-a|$ if $|x-a| < 1$.

$|x^2 - a^2| < (a+2)|x-a| < \epsilon$

$|x-a| < \frac{\epsilon}{a+2}$

Therefore, if $|x-a| < \min(1, \epsilon/(a+2))$ then

$|x^2 - a^2| < \epsilon$.

$\forall a \in [0,2]$, $|x-a| < \min(1, \epsilon/4) \rightarrow |x^2 - a^2| < \epsilon$.

Ie, $f(x) = x^2$ is unif. cont. on a closed interval such as $[0,2]$.

Lemma Let $a<b<c$ and let f be continuous on $[a,c]$. Let $\epsilon>0$ and suppose the following statements are true

i) x, y both in $[a,b]$, $|x-y|<\delta_1 \rightarrow |f(x)-f(y)|<\epsilon$
ii) x, y both in $[b,c]$, $|x-y|<\delta_2 \rightarrow |f(x)-f(y)|<\epsilon$

Then, $\exists \delta>0$ s.t.

x, y both in $[a,c]$, $|x-y|<\delta \rightarrow |f(x)-f(y)|<\epsilon$

Proof

f is cont. at b so $\exists \delta_3>0$ s.t.

$$\forall x,\ |x-b|<\delta_3 \rightarrow |f(x)-f(b)|<\frac{\epsilon}{2}$$

Take two such values:

$$|x_1-b|<\delta_3 \text{ and } |x_2-b|<\delta_3 \quad \rightarrow \quad |f(x_1)-b|-|f(x_2)-b| \leq |f(x_1)-f(x_2)| \leq |f(x_1)-b|+|f(x_2)-b|<\epsilon$$

$$\rightarrow |f(x_1)-f(x_2)|<\epsilon$$

Let $\delta=\min(\delta_1, \delta_2, \delta_3)$

Now for any x and y in $[a,c]$ s.t. $|x-y|<\delta$ there are three cases

1) x and y in $[a,b]$. By i), $|f(x)-f(y)|<\epsilon$
2) x and y in $[b,c]$. By ii), $|f(x)-f(y)|<\epsilon$
3) x in $[a,b]$, y in $[b,c]$, or vice-versa.
   Then since $|x-b|<\delta$ and $|y-b|<\delta$, $|f(x)-f(y)|<\epsilon$

Theorem 1 f cont. on $[a,b] \rightarrow f$ u.c. on $[a,b]$

Proof

Let's introduce a new term: ϵ-good. f is ϵ-good on $[a,b]$ if there is some $\delta>0$ such that for all y and z in $[a,b]$,
$|y-z|<\delta \rightarrow |f(y)-f(z)|<\epsilon$
We want to prove that f is ϵ-good on $[a,b]$ for all $\epsilon>0$.

Let $\epsilon>0$.
Let $A=\{x : a \leq x \leq b \text{ and } f \text{ is } \epsilon\text{-good on } [a,x]\}$
Then $A \neq \emptyset$, since $a \in A$.
A bounded above by b.
$\rightarrow$ A has l.u.b. α
Let's prove $\alpha=b$.

Assume $\alpha<b$.
f cont at α so $\exists \delta_0>0$ s.t. $|y-\alpha|<\delta \rightarrow |f(y)-f(\alpha)|<\frac{\epsilon}{2}$
Therefore if $|z-\alpha|<\delta_0$ then $|f(y)-f(z)|<\epsilon$
Therefore f ϵ-good on $[\alpha-\delta_0, \alpha+\delta_0]$
But f also ϵ good on $[a, \alpha-\delta_0]$

By the lemma above f ϵ-good on $[a, \alpha+\delta_0]$.
So $\alpha+\delta_0$ in A. $\perp$.
Therefore $\alpha \geq b$. But $\alpha>b$ leads to $\perp$ because b is u.b.
Finally, since f cont. at b, $\exists \delta_0>0$, $b-\delta_0<y<b \rightarrow |f(y)-f(b)|<\frac{\epsilon}{2}$
So f ϵ-good on $[b-\epsilon_0, b]$. Since f ϵ-good on $[a, b-\epsilon_0]$, by the lemma, f ϵ-good on $[a,b]$.