Ch13 - Integrals

Definition Let $a<b$. A partition of the interval $[a,b]$ is a finite collection of points in $[a,b]$, one of which is a, and one of which is b.

We shall denote the points in the partition $a=t_0<t_1<\dots<t_{n-1}<t_n=b$

Definition Suppose f bounded on $[a,b]$ and $P=\{t_0,\dots,t_n\}$ is partition of $[a,b]$.

so that m_i and M_i are defined

Let $m_i=\inf\{f(x): t_{i-1}\le x\le t_i\}$

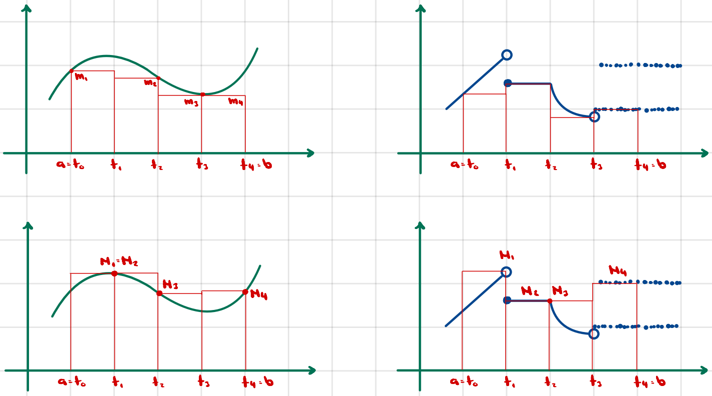
$M_i=\sup\{f(x): t_{i-1}\le x\le t_i\}$

f was not assumed continuous

Then

lower sum of f for $P = L(f,P)=\sum_{i=1}^{n} m_i(t_i-t_{i-1})$

upper sum of f for $P = U(f,P)=\sum_{i=1}^{n} M_i(t_i-t_{i-1})$



Note that $L(f,P)\le U(f,P)$ because for each i, $m_i(t_i-t_{i-1})\le M_i(t_i-t_{i-1})$

Lemma If Q and P are partitions and Q contains P then

$L(f,P)\le L(f,Q)$

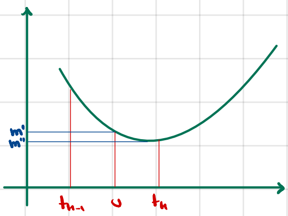
$U(f,P)\ge U(f,Q)$

Proof

Let's consider a special case first in which Q contains just one more point than P.

$P=\{t_0,\dots,t_n\}$

$Q=\{t_0,\dots,t_{k-1},u,t_k,\dots,t_n\}$



Let

$m'=\inf\{f(x): t_{k-1}\le x\le u\}$

$m''=\inf\{f(x): u\le x\le t_k\}$

Then

$L(f,P)=\sum_{i=1}^{n} m_i(t_i-t_{i-1})$

$L(f,Q)=\sum_{i=1}^{k-1} m_i(t_i-t_{i-1})+m'(u-t_{k-1})+m''(t_k-u)+\sum_{i=k+1}^{n} m_i(t_i-t_{i-1})$

The sums are the same except between t_{k-1} and t_k.

$m_{k-1}=\inf\{f(x): t_{k-1}\le x\le t_k\}$. But $\{f(x): t_{k-1}\le x\le t_k\}$ contains all the numbers in $\{f(x): t_{k-1}\le x\le u\}$ and extra ones. Therefore

$m_{k-1}\le m'$

Similarly, $m_{k-1}\le m''$

Therefore,

$$m_n(t_n - t_{n-1}) = m_n(t_n - u + u - t_{n-1}) = m_n(u - t_{n-1}) + m_n(t_n - u) \le m'(u - t_{n-1}) + m''(t_n - u)$$

This proves the special case

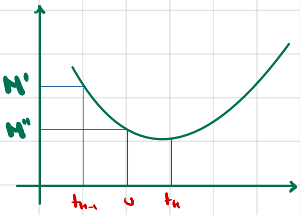
$$L(f,P) \le L(f,Q)$$

Let's prove, for this special case, that $U(f,P) \ge U(f,Q)$

Let

$M' = \sup\{f(x) : t_{n-1} \le x \le u\}$

$M'' = \sup\{f(x) : u \le x \le t_n\}$



$$U(f,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$$

$$U(f,Q) = \sum_{i=1}^{n-1} M_i(t_i - t_{i-1}) + m'(u - t_{n-1}) + m''(t_n - u) + \sum_{i=n+1}^{n} M_i(t_i - t_{i-1})$$

Again, the sums are the same except between t_{n-1} and t_n.

$M_{n-1} = \sup\{f(x) : t_{n-1} \le x \le t_n\}$. But $\{f(x) : t_{n-1} \le x \le t_n\}$ contains all the numbers in $\{f(x) : t_{n-1} \le x \le u\}$ and extra ones. Therefore

$M_{n-1} \ge M'$, and similarly $M_{n-1} \ge M''$.

Therefore, $M_n(t_n - t_{n-1}) = M_n(t_n - u + u - t_{n-1}) = M_n(u - t_{n-1}) + M_n(t_n - u) \ge M'(u - t_{n-1}) + M''(t_n - u)$

Therefore $U(f,P) \ge U(f,Q)$

General Case

Given a partition P, if we add one point to the partition we obtain P_1.
Add another point, to P_1, and obtain P_2.
And so on until we have Q.

Then, at each step we can apply the results of the special case.

$$L(f,P) \le L(f,P_1) \le L(f,P_2) \le \dots \le L(f,Q)$$

$$U(f,P) \ge U(f,P_1) \ge U(f,P_2) \ge \dots \ge U(f,Q)$$

∎

Theorem 1 Let P_1 and P_2 be partitions of $[a,b]$, and let f be a fn which is bounded on $[a,b]$.
Then,
$$L(f,P_1) \le U(f,P_2)$$

Proof

Let partition P contain P_1 and P_2 (for example, by consisting of all the points in P_1 and P_2).
Then,
$$L(f,P_1) \le L(f,P) \le U(f,P) \le U(f,P_2)$$

Therefore, given any partition P', then for any other partition P we have $L(f,P) \le U(f,P')$.
Ie, $U(f,P')$ is an upper bound for all lower sums of f (on the same interval).

Therefore, $\forall P'$ $\sup\{L(f,P) : P \text{ partition of } [a,b]\} \le U(f,P')$

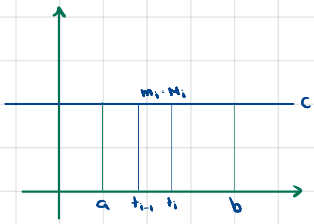
Symmetrically, $\sup\{L(f,P)\} \le \inf\{U(f,P)\}$

Therefore, for any partition P'

$$L(f,P') \le \sup\{L(f,P)\} \le \inf\{U(f,P)\} \le U(f,P')$$

Two Extreme Cases

$f(x) = c$, $\forall x \in [a,b]$. $P = \{t_0, \dots, t_n\}$ a partition.



Then, $m_i = M_i = c$ and $L(f,P) = U(f,P) = \sum_{i=1}^{n} c(t_i - t_{i-1}) = c(b-a)$

In this case, $\sup\{L(f,P)\} = \inf\{U(f,P)\}$.

All computations of lower and upper sums give the same value.

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1, & x \text{ rational} \end{cases}$$

in any interval there is always a rational and a irrational number

thus, for any P, in every interval $m_i = 0$, $M_i = 1$.

$$L(f,P) = \sum_{i=1}^{n} 0(t_i - t_{i-1}) = 0$$

$$U(f,P) = \sum_{i=1}^{n} 1(t_i - t_{i-1}) = b-a$$

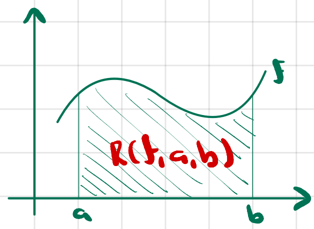
Now we have

$$L(f,P) = 0 = \sup\{L(f,P)\} < \inf\{U(f,P)\} = b-a = U(f,P)$$

Our goal is to compute "area" and our approach is to compute these sums. Intuitively, whatever "area" is it lies between the lower sums (actually their sup) and the upper sums (their inf). But the previous example seems problematic.
As the next definition shows, cases like these are "unmeasurable" for purposes of area computation.

Definition A function f which is bounded on $[a,b]$ is integrable on $[a,b]$ if

$$\sup\{L(f,P) : P \text{ a partition of } [a,b]\} = \inf\{U(f,P) : P \text{ a partition of } [a,b]\} = \int_a^b f$$



This common number is called the integral of f on $[a,b]$.

$\int_a^b f$ is also called the area of $R(f,a,b)$ when $f(x) \geq 0$ for all x in $[a,b]$.

If f is integrable, then

$$L(f,P) \leq \int_a^b f \leq U(f,P) \quad \text{for all } P$$

$\int_a^b f$ is the unique number w/ this property

At this point,

we don't know how to determine, in general, which functions are integrable, or how to find $\int_a^b f$

we've determined that if $f(x) = c$ then f is integrable and $\int_a^b f = c(b-a)$, and if $f(x) = 0$ for $x \in \mathbb{Q}$ and 1 for $x \in \mathbb{R} - \mathbb{Q}$ then f is not integrable.

Theorem 2

f bounded on $[a,b]$ $\rightarrow$ $\left[f \text{ integrable on } [a,b] \leftrightarrow \forall \varepsilon > 0, \exists \text{ partition } P \text{ of } [a,b] \text{ such that } U(f,P) - L(f,P) < \varepsilon \right]$

Proof

this is just a restatement of the definition of integrability, convenient because no inf and sup.

Assume f bounded on $[a,b]$ (so that lower and upper sums are defined)

Assume $\forall \varepsilon > 0$ $\exists$ partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$

Let P' be any partition of $[a,b]$

Then $L(f,P) \leq \sup\{L(f,p)\} \leq \inf\{U(f,P)\} \leq U(f,P)$

$\sup\{L(f,p)\} - \inf\{U(f,P)\} \leq \sup\{L(f,p)\} - L(f,P) \leq U(f,P) - L(f,P) < \varepsilon$

$\rightarrow \sup\{L(f,p)\} = \inf\{U(f,P)\}$

$\rightarrow$ f integrable on $[a,b]$

Now assume f integrable on $[a,b]$.

Then $\sup\{L(f,p)\} = \inf\{U(f,P)\}$

We can choose P' such that $U(f,P'')$ is arbitrarily close to $\inf\{U(f,P)\}$, and $L(f,P')$ arbitrarily close to $\sup\{L(f,p)\}$. Therefore, $\forall \varepsilon > 0$ there are P' and P'' such that $U(f,P'') - L(f,P') < \varepsilon$.

Let P contain P' and P''.

According to the lemma,

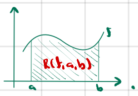
$$U(f,P) \le U(f,P'')$$
$$L(f,P) \ge L(f,P')$$

ie $L(f,P') \le L(f,P) \le U(f,P) \le U(f,P'')$

But then

$$U(f,P) - L(f,P) \le U(f,P'') - L(f,P') < \epsilon$$

Recap so far

For now, we're interested in areas of specific types of regions 

.

We define partition, lower sum, and upper sum.

Our assumption is that f is bounded on the interval of interest, so we can pick out m_i and M_i in each partition subinterval.

We prove some theorems about relationships between upper/lower sums for different partitions.

In particular, lower sums are smaller than or equal to upper sums, no matter the partition (Th. 1).

Because of this we discuss the relationship between the supremum of the lower sums and the infimum of the upper sums.

We define integrability of f based on this relationship: f is integrable if $\sup\{L(f,P)\} = \inf\{U(f,P)\}$.

The integral $\int_a^b f$ is defined as the unique number equal to both $\sup\{L(f,P)\}$ and $\inf\{U(f,P)\}$.

We then restate the condition $\sup\{L(f,P)\} = \inf\{U(f,P)\}$ differently. If this condition is true it means we can choose partitions P' and P'' such that the upper and lower sums differ by less than ϵ, for any $\epsilon > 0$.

In other words, since we can get $L(f,P)$ as close to $\int_a^b f$ as we want, and the same for $U(f,P)$, then we can make $L(f,P)$ and $U(f,P)$ as close as we want.

At this point we can take specific functions and try to figure out if f is integrable, and if so, what the unique $\int_a^b f$ is.

We show

$$\int_a^b f = c(b-a) \quad \text{if } f(x) = c \text{ for all } x$$

$$\int_a^b f = \frac{b^2}{2} - \frac{a^2}{2} \quad \text{if } f(x) = x \text{ for all } x$$

$$\int_a^b f = \frac{b^3}{3} - \frac{a^3}{3} \quad \text{if } f(x) = x^2 \text{ for all } x$$

The notation $\int_a^b f$ is inconvenient. The integrable function's formula doesn't appear directly.

$\int_a^b f(x)\,dx$ means the same as $\int_a^b f$

dx has no meaning in isolation. w/ limits, $x \to$ also had no meaning in isolation.

Observations

The integrals of most fns are impossible to determine exactly. It is nonetheless important to at least know if a fn is integrable.

It is possible to precisely specify which functions are integrable. The theorems are too difficult for the present.

We can specify a criterion which lets us know that a certain group of functions is integrable: continuous fns.

Theorem 3 f cont. on $[a,b] \to f$ integrable on $[a,b]$

Proof

f cont. on $[a,b] \to f$ bounded on $[a,b]$

Recall from ch. 8 appendix, "Unif. Continuity"

Theorem 1 f cont. on $[a,b] \to f$ u.c. on $[a,b]$

Therefore, f is u.c. on $[a,b]$, so $\exists \delta > 0$ s.t. $\forall x \forall y \in [a,b]$

$$|x-y| < \delta \to |f(x)-f(y)| < \frac{\epsilon}{2(b-a)}$$

Choose a partition $P = \{t_0, \ldots, t_n\}$ of $[a,b]$ such that $|t_i - t_{i-1}| < \delta$.

Then for each i, $\forall x \forall y \in [t_{i-1}, t_i] \to |f(x)-f(y)| < \frac{\epsilon}{2(b-a)}$

Since f is cont, it takes on a max and min value in each $[t_{i-1}, t_i]$.

Therefore, $\exists x_i,\ x_i \in [t_{i-1}, t_i] \wedge f(x_i) = \inf\{f(x) : x \in [t_{i-1}, t_i]\} = m_i$

$\exists y_i,\ y_i \in [t_{i-1}, t_i] \wedge f(y_i) = \sup\{f(x) : x \in [t_{i-1}, t_i]\} = M_i$

Therefore $|M_i - m_i| < \frac{\epsilon}{2(b-a)} < \frac{\epsilon}{b-a}$

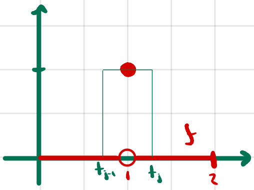
$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) < \frac{\epsilon}{b-a} \sum_{i=1}^{n} (t_i - t_{i-1}) = \epsilon$$

Hence f is integrable on $[a,b]$.

■

Example

$$f(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$$



For $x \in [0,2]$

$P = \{t_0, \dots, t_n\}$ partition of $[0,2]$ such that $t_{j-1} < 1 < t_j$

Lets show that f is integrable.

$$L(f,P) = \sum_{i=1}^{j-1} \cancelto{0}{m_i}(t_i - t_{i-1}) + \cancelto{0}{m_j}(t_j - t_{j-1}) + \sum_{i=j+1}^{n} \cancelto{0}{m_i}(t_i - t_{i-1}) = 0$$

$$U(f,P) = \sum_{i=1}^{j-1} \cancelto{0}{M_i}(t_i - t_{i-1}) + (t_j - t_{j-1}) + \sum_{i=j+1}^{n} \cancelto{0}{M_i}(t_i - t_{i-1}) = t_j - t_{j-1}$$

$$U(f,P) - L(f,P) = t_j - t_{j-1}$$

If P is chosen such that $t_i - t_{i-1} < \epsilon$ for all i then $U(f,P) - L(f,P) < \epsilon$.

Hence f is integrable on $[a,b]$.

Theorem 4 Let $a<c<b$. Then,

f integrable on $[a,b]$ $\leftrightarrow$ f integrable on $[a,c]$ and $[c,b]$

f integrable on $[a,b]$ $\rightarrow$ $\int_a^b f = \int_a^c f + \int_c^b f$

Proof

Assume f integrable on $[a,b]$.

let $\epsilon>0$.

let $P' = \{t_0,\dots,t_n\}$ s.t. $U(f,P')-L(f,P')<\epsilon$.

let P contain P', plus the point c if it's not already in P'. Thus $P = \{t_0,\dots,t_m,c,t_{m+1},\dots,t_n\}$

Lemma $\rightarrow$ $L(f,P') \leq L(f,P)$
$U(f,P') \geq U(f,P)$

ie $L(f,P') \leq L(f,P) \leq U(f,P) \leq U(f,P')$. Therefore $U(f,P)-L(f,P) \leq U(f,P')-L(f,P')<\epsilon$.

---

let $Q_1 = \{t_0,\dots,t_m,c\}$, ie $Q_1 \subset P$ and $Q_2 = \{t_{m+1},\dots,t_n\}$, ie $Q_2 \subset P$.

$U(f,P)-L(f,P)$

$$= \sum_{i=0}^{m} M_i(t_i-t_{i-1}) + M_c(c-t_m) + \sum_{i=m+1}^{n} M_i(t_i-t_{i-1}) - \sum_{i=0}^{m} m_i(t_i-t_{i-1}) - m_c(c-t_m) - \sum_{i=m+1}^{n} m_i(t_i-t_{i-1}) < \epsilon$$

$$= \left(\sum_{i=0}^{m} M_i(t_i-t_{i-1}) + M_c(c-t_m) - \sum_{i=0}^{m} m_i(t_i-t_{i-1}) - m_c(c-t_m)\right) + \left(\sum_{i=m+1}^{n} M_i(t_i-t_{i-1}) - \sum_{i=m+1}^{n} m_i(t_i-t_{i-1})\right) < \epsilon$$

$$= U(f,Q_1) - L(f,Q_1) + U(f,Q_2) - L(f,Q_2) < \epsilon$$

Since $[U(f,Q_1)-L(f,Q_1)]$ and $[U(f,Q_2)-L(f,Q_2)]$ are nonnegative, each is $<\epsilon$.

Thus f is integrable on $[a,c]$ and $[c,b]$.

Thus, $L(f,Q_1) \leq \int_a^c f \leq U(f,Q_2)$ and $L(f,Q_2) \leq \int_a^b f \leq U(f,Q_2)$

Hence
$$L(f,P) = L(f,Q_1)+L(f,Q_2) \leq \int_a^c f + \int_c^b f \leq U(f,Q_1)+U(f,Q_2) = U(f,P)$$

And this is true for any P. But then $\int_a^b f = \int_a^c f + \int_c^b f$

■

Now assume f integrable on $[a,c]$ and $[c,b]$

Let Q_1 be a partition of $[a,c]$ and Q_2 a partition of $[c,b]$ such that

$$U(f,Q_1) - L(f,Q_1) < \frac{\epsilon}{2}$$

$$U(f,Q_2) - L(f,Q_2) < \frac{\epsilon}{2}$$

Let P be the partition of $[a,b]$ containing Q_1 and Q_2.

Then

$$[U(f,Q_1) + U(f,Q_2)] - [L(f,Q_1) + L(f,Q_2)] < \epsilon$$

$$U(f,P) - L(f,P) < \epsilon$$

$\rightarrow$ f integrable on $[a,b]$.

■

We defined $\int_a^b f$ for $a<b$ as the number

$\sup\{L(f,P) : P \text{ a partition of } [a,b]\} = \inf\{U(f,P) : P \text{ a partition of } [a,b]\} = \int_a^b f$

We now define $\int_b^a f = -\int_b^a f$ and $\int_a^a f = 0$

Theorem f integrable on interval containing a,b,c $\rightarrow$ $\int_a^c f + \int_c^b f = \int_a^b f$

Proof

Case 1: $a<c<b$. This was proved in Theorem 4.

Case 2: $a<b<c$

$$\int_a^c f + \int_c^b f = \int_a^c f - \int_b^c f = \int_a^b f + \int_b^c f - \int_b^c f = \int_a^b f$$

Case 3: $b<a<c$

$$\int_a^c f + \int_c^b f = \int_a^c f - \int_b^c f = \int_a^c f - \int_b^a f - \int_a^c f = -\int_b^a f = \int_a^b f$$

The remaining cases are similar ($b<c<a$, $c<a<b$, $c<b<a$)

■

Theorem 5 f,g integrable on $[a,b]$ $\rightarrow$ $f+g$ integrable on $[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

Proof

Since f and g are integrable, there are partitions P' and P'' such that

$$U(f,P') - L(f,P') < \epsilon/2$$

$$U(g,P'') - L(g,P'') < \epsilon/2$$

Let Q be the partition containing P' and P''. Then from the lemma

$$U(f,Q) \le U(f,P') \qquad U(g,Q) \le U(g,P'')$$
$$L(f,Q) \ge L(f,P') \qquad L(g,Q) \ge L(g,P'')$$

Hence

$$U(f,Q) + U(g,Q) \le U(f,P') + U(g,P'')$$
$$L(f,Q) + L(g,Q) \ge L(f,P') + L(g,P'')$$

$$\rightarrow U(f,Q) + U(g,Q) - \big(L(f,Q) + L(g,Q)\big) \le U(f,P') + U(g,P'') - \big(L(f,P') + L(g,P'')\big) < \epsilon$$

$\rightarrow U(f+g,Q) - L(f+g,Q) < \epsilon$ $\rightarrow$ $f+g$ integrable on $[a,b]$.

Let $P = \{t_0, \dots, t_n\}$ be any partition of $[a,b]$.

Let $m_i = \inf\{(f+g)(x) : t_{i-1} \le x \le t_i\}$ $\qquad M_i = \sup\{(f+g)(x) : t_{i-1} \le x \le t_i\}$
$m_i' = \inf\{f(x) : t_{i-1} \le x \le t_i\}$ $\qquad M_i' = \sup\{f(x) : t_{i-1} \le x \le t_i\}$
$m_i'' = \inf\{g(x) : t_{i-1} \le x \le t_i\}$ $\qquad M_i'' = \sup\{g(x) : t_{i-1} \le x \le t_i\}$

It can be shown that $m_i \ge m_i' + m_i''$ and $M_i \le M_i' + M_i''$

Therefore $L(f,P) + L(g,P) \le L(f+g,P)$ and $U(f+g,P) \le U(f,P) + U(g,P)$

Thus, $L(f,P) + L(g,P) \le L(f+g,P) \le U(f+g,P) \le U(f,P) + U(g,P)$

Since $L(f,P) \le \int_a^b f \le U(f,P)$ and $L(g,P) \le \int_a^b g \le U(g,P)$ then

$L(f+g,P) = L(f,P) + L(g,P) \le \int_a^b f + \int_a^b g \le U(f,P) + U(g,P) = U(f+g,P)$ (1)

But since $f+g$ is integrable then for any P, $L(f+g,P) \le \int_a^b (f+g) \le U(f+g,P)$ (2)

(1) and (2) are true for all P.

Since we know that $L(f+g,P) \le \sup\{L(f+g,P)\} = \inf\{U(f+g,P)\} = \int_a^b (f+g) \le U(f+g,P)$

then $\int_a^b (f+g) = \int_a^b f + \int_a^b g$. (we can argue this final step by contradiction for example).

■

Theorem 6 f integrable on $[a,b]$ $\rightarrow$ for any number c, cf is integrable on $[a,b]$ and $\int_a^b cf = c\cdot\int_a^b f$

Proof

If $c>0$ then $cf = \sum_{i=1}^{c} f$, which is integrable by Th. 5.

$$\int_a^b cf = \int_a^b \left(f + \sum_{i=1}^{c-1} f\right) = \int_a^b f + \int_a^b \left(f + \sum_{i=1}^{c-2} f\right) = c\cdot\int_a^b f$$

If $c=0$ then $cf = 0$ and $\int_a^b 0 = 0$

If $c<0$ then $cf = |c|(-f) = \sum_{i=1}^{|c|}(-f)$

Note that for any partition P

$U(f,P) = \sum M_i \Delta t$

$\inf_t$
$\sup_t$

$L(f,P) = \sum m_i \Delta t$

$U(-f,P) = \sum(-m_i \Delta t) = -\sum m_i \Delta t$

$-\sup_t = \inf$
$-\inf_t = \sup$

$L(-f,P) = \sum(-M_i \Delta t) = -\sum M_i \Delta t$

f integrable $\rightarrow \exists P,\ U(f,P) - L(f,P) < \epsilon$ for any $\epsilon > 0$

$U(-f,P) - L(-f,P) = -\sum m_i \Delta t - \left(-\sum M_i \Delta t\right)$

$= \sum M_i \Delta t - \sum m_i \Delta t$

$= U(f,P) - L(f,P) < \epsilon$

$\rightarrow -f$ integrable

By Th. 5, $\sum_{i=1}^{|c|}(-f)$ integrable and $\int_a^b |c|(-f) = |c|\int_a^b (-f) = -|c|\int_a^b f$

Theorem 7 Suppose f integrable on $[a,b]$ and that $m \le f(x) \le M$ for all x in $[a,b]$.

Then, $m(b-a) \le \int_a^b f \le M(b-a)$

Proof

For any partition P, $L(f,P) = \sum m_i \Delta t \ge \sum m \Delta t = m(b-a)$

$U(f,P) = \sum M_i \Delta t \le \sum M \Delta t = M(b-a)$

Hence $m(b-a) \le L(f,P) \le \sup\{L(f,P)\} = \int_a^b f = \inf\{U(f,P)\} \le U(f,P) \le M(b-a)$

$m(b-a) \le \int_a^b f \le M(b-a)$

■

Theorem 8 f integrable on $[a,b]$ and F defined $F(x) = \int_a^x f = \int_a^x f(t)\,dt \rightarrow F$ continuous on $[a,b]$

Proof

Suppose $c \in [a,b]$.

Since f integr., f is bounded on $[a,b]$. Let M s.t. $\forall x, x \in [a,b] \rightarrow |f(x)| \le M$.

Let $h>0$. Then $F(c+h) - F(c) = \int_a^{c+h} f - \int_a^c f = \int_c^{c+h} f$

Since $-M \le f(x) \le M$ for all x, then from Th.7 we know $-Mh \le \int_c^{c+h} f \le Mh$

$-M|h| = -Mh \le F(c+h) - F(c) \le Mh = M|h|$ (1)

Now let $h<0$.

Note that $\int_c^{c+h} f = -\int_{c+h}^c f$

Apply Th.7 to $[c+h, c]$:

$\forall x, x \in [c+h, c] \rightarrow -M \le f(x) \le M$

$-M(-h) \le \int_{c+h}^c f \le M(-h) \rightarrow Mh \le \int_{c+h}^c f \le -Mh \rightarrow Mh \le F(c) - F(c+h) \le -Mh$

multiply by $-1 \rightarrow -M|h| = Mh \le F(c+h) - F(c) \le -Mh = M|h|$ (2)

Combine (1) and (2): $|F(c+h) - F(c)| \le M|h|$

For any $\varepsilon > 0$, we have

$|h| < \frac{\varepsilon}{M} \rightarrow |F(c+h) - F(c)| < \varepsilon$

$\rightarrow \lim_{h \to 0} F(c+h) = F(c)$

ie F cont. at c.

Note

We can change the value of a fn at finitely many points and the value of integrals of the fn don't change.

We've shown previously that for

$$g(x) = \begin{cases} c & x = x_0 \\ 0 & x \neq x_0 \end{cases}$$

$\int_a^b g = 0.$

For any integrable f, $f+g$ is integrable and $\int_a^b (f+g) = \int_a^b f + \int_a^b g = \int_a^b f$