Ch 12 Appendix - Parametric Representation of Curves

1. (a) point-slope form of tangent line at $(a, f(a))$: $y - f(a) = (x-a)f'(a)$

tangent line: all points of form $(x, f(a) + f'(a)(x-a))$

we can think of a vector-valued fn $\vec{c}(x) = (x, f(a) + f'(a)(x-a))$

and reparameterize it

$s(x) = x - a$

$\rightarrow x(s) = s + a$

$\rightarrow (\vec{c} \circ x)(s) = \vec{c}(x(s)) = (s+a, f(a) + sf'(a))$

(b) $\vec{c}(t) = (t, f(t))$

$\vec{c}\,'(t) = (1, f'(t))$

The tangent line of $\vec{c}$ at $(a, f(a))$ is defined as all points satisfying

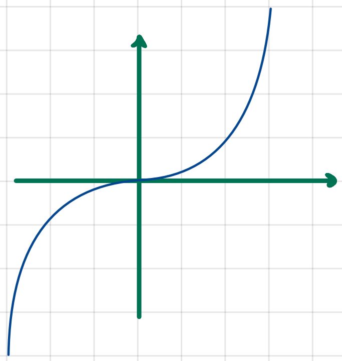
$\vec{c}(a) + s\vec{c}\,'(a) = (a, f(a)) + s(1, f'(a))$

$= (a+s, f(a) + sf'(a))$

But this is the same set of points that make up the tangent line of f at $(a, f(a))$.

2. $\vec{c}(t) = (f(t), t^2)$

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x \leq 0 \end{cases}$$



$$\vec{c}(t) = \begin{cases} (t^2, t^2) & t \geq 0 \\ (-t^2, t^2) & t \leq 0 \end{cases}$$

reparametrize: $x(t) = t^2 \to t(x) = \sqrt{x}$

$$\vec{d}(t) = c(\sqrt{t}) = \begin{cases} (t, t) & t \geq 0 \\ (-t, t) & t \leq 0 \end{cases}$$

The graph of c is the set of points (x, y) s.t. $x = y = t^2$ for $t \geq 0$ and $-x = y = t^2$ for $t \leq 0$

Therefore, $y(x) = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$

$\to$ $y(x) = |x|$

we know this fn is not diff at 0.

$$c'(t) = \begin{cases} (2t, 2t) & t \geq 0 \\ (-2t, 2t) & t \leq 0 \end{cases}$$

$c'(0) = (0, 0)$

3. $\left.\begin{array}{l} x = u(t) \\ y = v(t) \end{array}\right\}$ parametric repres. of a curve

u is one-one on some interval

(a) $f(x) = (v \circ u^{-1})(x) = v(u^{-1}(x))$

Intuitively, given an x-coord., we can obtain the time t (in the one-one interval) using $u^{-1}(x)$. With that time we can find $v(t)$.

$c(t) = (u(t), v(t))$

$t(x) = u^{-1}(x)$ is a reparametrization.

$\to d(x) = c(u^{-1}(x)) = (x, v(u^{-1}(x)))$

$\to y = f(x) = v(u^{-1}(x))$

(b) u diff. on this interval
$u'(t) \neq 0$ $\quad \to \quad$ at $x = u(t)$ we have $f'(x) = \dfrac{v'(t)}{u'(t)}$

Proof

$u'(t) \neq 0 \to (u^{-1})'(x) = \dfrac{1}{u'(u^{-1}(x))}$

$$f'(x) = v'(u^{-1}(x)) \cdot (u^{-1})'(x)$$

$$= v'(t) \cdot (u^{-1})'(u(t))$$

$$= v'(t) \cdot \frac{1}{u'(t)}$$

(c) $$f''(x) = \frac{v''(t)\,(u^{-1})'(x)\,u'(t) - v'(t)\,u''(t) \cdot (u^{-1})'(x)}{(u'(t))^2}$$

$$= \frac{v''(t) \cdot \cancel{u'(t)} \cdot \dfrac{1}{\cancel{u'(t)}} - u''(t)\,v'(t) \cdot \dfrac{1}{u'(t)}}{(u'(t))^2}$$

$$= \frac{v''(t)\,u'(t) - u''(t)\,v'(t)}{(u'(t))^3}$$

4. $x^{2/3}+y^{2/3}=1$ (1)

(i) $\cancel{\frac{2}{3}}x^{-\frac{1}{3}}+\cancel{\frac{2}{3}}y^{-\frac{1}{3}}y'=0$

$$y' = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}} \quad (2)$$

Note that if we solve for y explicitly,

$y^{2/3}=1-x^{2/3}$

$y=\pm(1-x^{2/3})^{3/2}=\pm\sqrt{(1-x^2)^3}$

we obtain two functions. Either one solves (1).

note $x^{2/3}+y^{2/3}=1$ is an equation. The solutions form a set of points in the plane. If we think of x as an independent variable then it can be that y is a function of x in a certain interval.

$c(t)=(\cos^3 t, \sin^3 t)$ is a parametric curve.

If $x(t)=\cos^3 t$ and $y(t)=\sin^3 t$, then for any t,

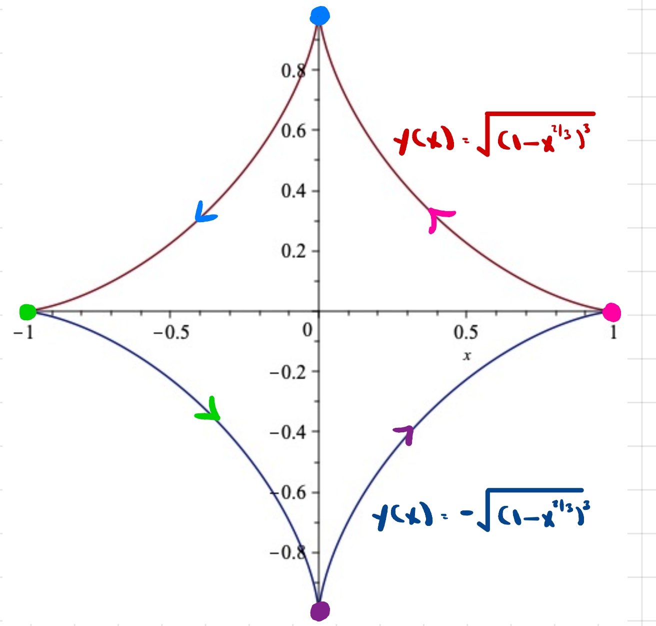
$$(\cos^3 t)^{2/3}+(\sin^3 t)^{2/3}=1.$$

That is, the set of points on the curve coincides w/ the set of points satisfying the initial equation we had.

The domain of both fns is

$(1-x^2)^3\geq 0 \to 1-x^2\geq 0 \to x^2\leq 1$

$\to |x|\leq 1 \to -1\leq x\leq 1$



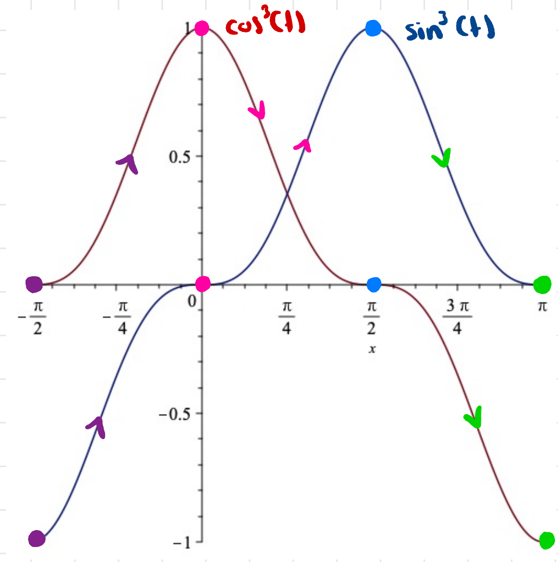
Note that $y(x)$ appears in (2), and so f' is also two functions, one for each of the two $y(x)$.

(ii) $x=\cos^3 t=u(t)$, $y=\sin^3 t=v(t)$ $\to$ $u'(t)=3\cos^2(t)(-\sin t)=-3\cos^2 t\sin t$, $v'(t)=3\sin^2 t\cos t$

consider the interval $(n\frac{\pi}{2},(n+1)\frac{\pi}{2}), n\in\mathbb{Z}$, for t

$u(t)$ is one-one on this interval, and $u'(t)\neq 0$

By 3b), if $y=f(x)$ then $y'(x)=f'(x)=\dfrac{\cancel{3}\sin^{\cancel{2}}t\,\cancel{\cos t}}{-\cancel{3}\cos^{\cancel{2}}t\,\cancel{\sin t}}=-\dfrac{\sin t}{\cos t}$



Interpretation

$(\cos^3 t,\sin^3 t)$ is one parametrization of the curve shown above.

$(t,\sqrt{(1-t^2)^3})$ and $(t,-\sqrt{(1-t^2)^3})$ are two parametrizations for two parts of the curve.

Note that since $x=\cos^3 t$, if we sub into (1) we get

$\cos^2 t+f(x)^{2/3}=1 \quad\to\quad f(x)^{2/3}=1-\cos^2 t=\sin^2 t$

$f(x)^{1/3}=\sin t$

$$\to f'(x)=-\frac{f(x)^{1/3}}{x^{1/3}}$$

5. $(x, y) = (u(t), v(t))$

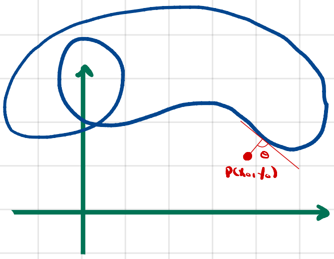
u, v diff.

$P = (x_0, y_0)$ point on plane → line from P to Q is perpendicular to tangent line of the curve at Q

$Q(u(\bar{t}), v(\bar{t}))$ point on curve closest to (x_0, y_0)

$u'(\bar{t})$ and $v'(\bar{t})$ not both zero.

Proof



$$D(t) = \sqrt{(u(t)-x_0)^2 + (v(t)-y_0)^2} \quad (1)$$

Note that D is defined for all t, it is bounded below, differentiable, and it has a min on any closed interval.

By assumption $D(t)$ has a global min at $\bar{t}$. This is an interior point of $\mathbb{R}$, therefore $D'(\bar{t}) = 0$

$$D'(\bar{t}) = \frac{1}{2} \frac{\left[2(u(\bar{t})-x_0)u'(\bar{t}) + 2(v(\bar{t})-y_0)v'(\bar{t})\right]}{\sqrt{(u(\bar{t})-x_0)^2 + (v(\bar{t})-y_0)^2}} = 0$$

$$u'(\bar{t})(u(\bar{t})-x_0) + v'(\bar{t})(v(\bar{t})-y_0) = 0$$

$$(u'(\bar{t}), v'(\bar{t})) \cdot (u(\bar{t})-x_0, v(\bar{t})-y_0) = 0$$

$\rightarrow \cos\theta = 0 \rightarrow \theta = \pi/2$

The vector $(u'(\bar{t}), v'(\bar{t}))$ is tangent to curve at Q.

An alternative way to see this, w/ dot product is

Case 1: $u'(\bar{t}) \neq 0$

Then

$$\frac{v'(\bar{t})}{u'(\bar{t})} \cdot \frac{v(\bar{t})-y_0}{u(\bar{t})-x_0} = -1$$

From 4b, we know that $\frac{v'(t)}{u'(t)} = y'(x)$ is the slope of the tangent at $(u(\bar{t}), v(\bar{t}))$. $\frac{v(\bar{t})-y_0}{u(\bar{t})-x_0}$ is the slope of the line from P to Q.

From 4-8a, if two lines have slopes m and n, and $mn = -1$ then the lines are perpendicular.

6. (a)

f is a fn.

graph of f in polar coord.: all points w/ polar coord. $(f(\theta), \theta) = (r, \theta)$

we want the slope of the tangent line at $(f(\theta), \theta)$.

Note

$(f(\theta), \theta)$ are just coordinates, not a vector or a vector-valued fn.

$c(\theta) = (x(\theta), y(\theta)) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$ is a v.v. fn

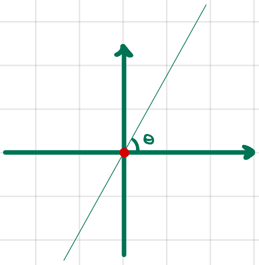
It is the param. representation of a curve which is the graph of f in polar coordinates.

As we proved in problem 3b, in intervals of θ where $x(\theta)$ is one-one and $x'(\theta) \neq 0$, we have

$$y'(x) = \frac{y'(\theta)}{x'(\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \quad (1)$$

(b) $f(\theta) = 0$

f diff at θ



$x = x(\theta) = f(\theta)\cos\theta = 0$

$$y'(0) = \frac{f'(\theta)\sin\theta + \cancelto{0}{f(\theta)}\cos\theta}{f'(\theta)\cos\theta - \cancelto{0}{f(\theta)}\sin\theta}$$

$$= \frac{\sin\theta}{\cos\theta}, \text{ since } f(\theta) = 0$$

Note that the above result relies on $x'(\theta) = f'(\theta)\cos\theta \neq 0$

If $\cos\theta = 0$ then $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$. The slope of the tangent isn't defined since it is vertical.

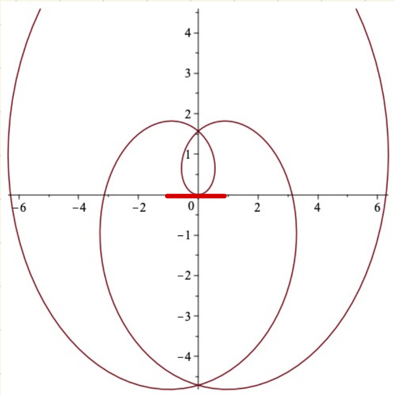
If $f'(\theta) = 0$ then $f(\theta) = \text{constant} = \alpha$, and if $\alpha \neq 0$ this is a circle.

$c(\theta) \neq (0,0)$ for all θ unless $f(\theta) = 0$ in which cases the curve is the single point $(0,0)$.

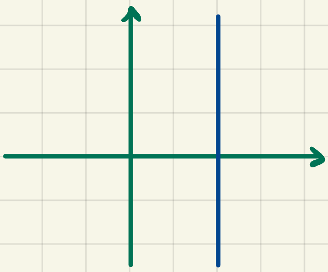
In either case there is no derivative at $(0,0)$.

We can observe this scenario in the context of some actual graphs of curves

Spiral of Archimedes $r = \theta$



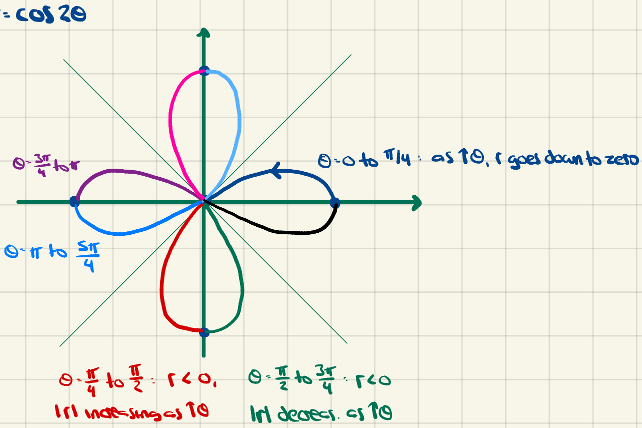
$r = \dfrac{a}{\cos\theta} = f(\theta)$ note that $c(\theta) = (x(\theta), y(\theta)) = (a, a\tan\theta)$



$x'(\theta) = 0$, so there is no defined $\frac{dy}{dx}$.

$y'(\theta) = \dfrac{a}{\cos^2\theta} \neq 0$, so $\dfrac{dx}{dy} = \dfrac{0}{a/\cos^2\theta} = 0$

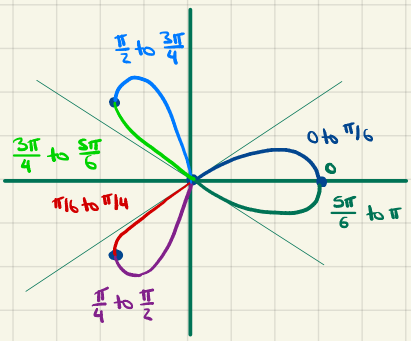
$r = \cos 2\theta$



$c(\theta) = (\cos(2\theta)\cdot\cos\theta, \cos(2\theta)\sin\theta)$

$f(\theta) = 0$? $\cos(2\theta) = 0 \to 2\theta = \frac{\pi}{2} + k\pi \to \theta = \frac{\pi}{4} + k\frac{\pi}{2} \to \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$r = \cos 3\theta = f(\theta)$



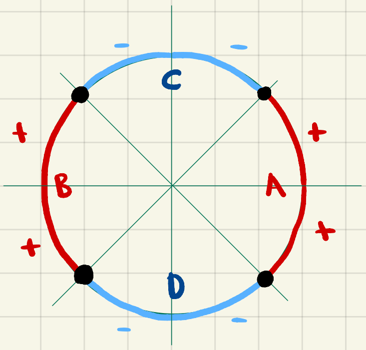
$f(\theta) = 0 \to \cos(3\theta) = 0 \to 3\theta = \frac{\pi}{2} + k\pi \to \theta = \frac{\pi}{6} + k\frac{\pi}{3}$

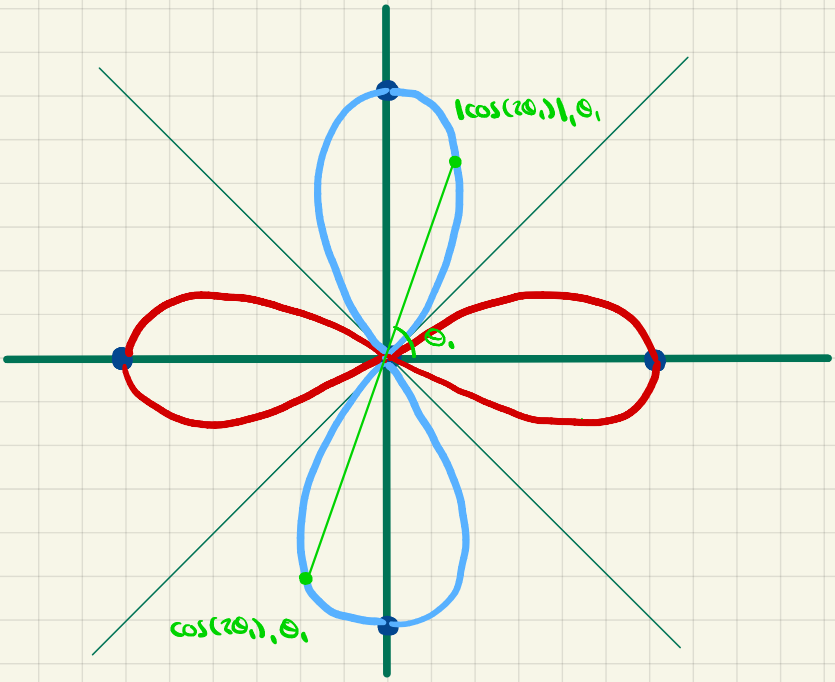
$\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

Note, however, that $\cos(\pi/2) \cdot \cos(3\pi/2) = 0$, where the tangent is vertical.

Also,

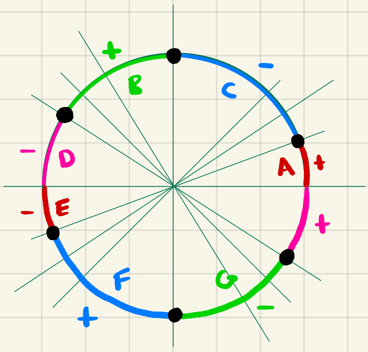
$r = |\cos 2\theta|$

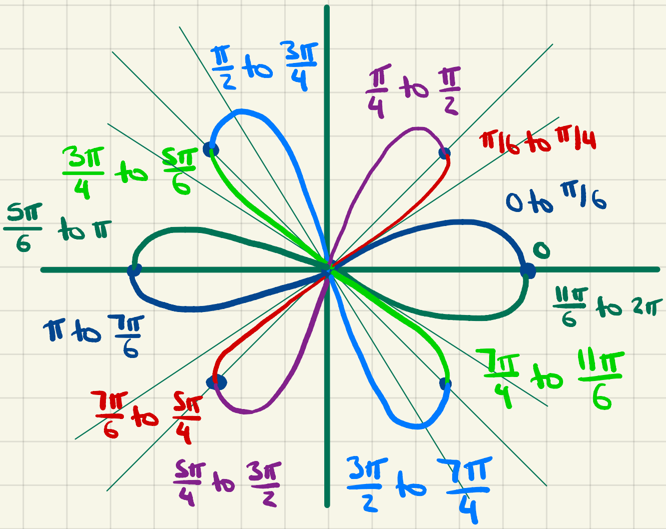




$|\cos 2\theta| = 0$ at same θ as the case of $\cos 2\theta$.

$r = |\cos 3\theta|$





$|\cos(3\theta)| = 0$

$\cos 3\theta = 0 \to 3\theta = \frac{\pi}{2} + n\pi \to \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

$-\cos 3\theta = 0 \to 3\theta = \frac{\pi}{2} + n\pi \to \theta = \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

$A = [0, \frac{\pi}{6}]$

$B = [\frac{\pi}{2}, \frac{5\pi}{6}]$

$C = [\frac{\pi}{6}, \frac{\pi}{2}]$

$D = [\frac{5\pi}{6}, \pi]$

Same as the case of $r = \cos 3\theta$

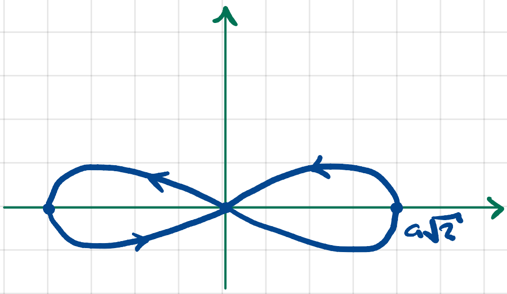
$$r = \begin{cases} \cos 3\theta & \theta \in A \cup B \\ -\cos 3\theta & \theta \in C \cup D \end{cases}$$

$r^2 = 2a^2\cos(2\theta)$

$2a^2\cos(2\theta) = 0$

$\to \cos 2\theta = 0$

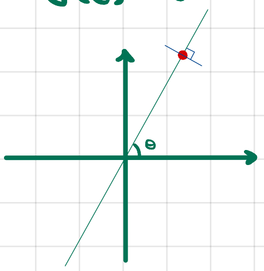
$\to \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$



(c) $(f(\theta), \theta)$ furthest point of graph of f from origin

As we showed in problem 5, given a point P (in our case, $(0,0)$), if Q is on the graph of a curve and is either the closest or farthest point from P, then $(u'(\theta), v'(\theta)) \cdot (u(\theta)-0, v(\theta)-0) = 0$

or $\frac{v'(\theta)}{u'(\theta)} \cdot \frac{v(\theta)}{u(\theta)} = -1 \to \frac{v'(\theta)}{u'(\theta)} \cdot \frac{\sin\theta}{\cos\theta} = -1 \to f'(x) = -\frac{\cos\theta}{\sin\theta}$



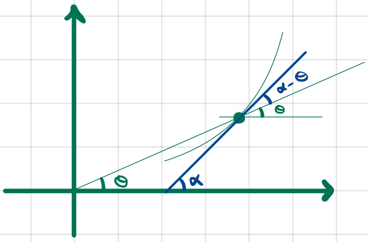
The tangent at $(u(\theta), v(\theta))$ is perpendicular to the line from origin to $(u(\theta), v(\theta))$, and has slope $\cos\theta/\sin\theta = \text{cotg}(\theta) = 1/\tan\theta$.

Note also that we can obtain the same information by computing the derivative directly

$$y'(x) = \frac{v'(\theta)}{u'(\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = -\frac{\cos\theta}{\sin\theta}$$

(d) $\tan(\alpha-\theta) = \frac{f(\theta)}{f'(\theta)}$

Proof



A parametrization of the curve: $(x(\theta), y(\theta)) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$

$$\tan\alpha = y'(x) = \frac{x'(\theta)}{y'(\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

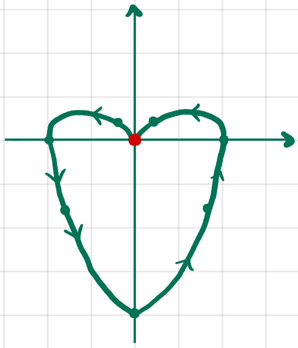
$\tan\theta = \frac{\sin\theta}{\cos\theta}$

From trigonometry, $\tan(\alpha-\theta) = \frac{\tan\alpha - \tan\theta}{1+\tan\alpha\tan\theta}$

Subbing in, we obtain

$$\tan(\alpha-\theta) = \frac{f(\theta)}{f'(\theta)}$$

7. (a) $r = f(\theta) = 1 - \sin\theta$, also described by $(x^2+y^2+y)^2 = x^2+y^2$



(i) Assume $y = f(x)$

$$\cancel{2}(x^2+y^2+y)(2x+2yy'+y') = \cancel{2}x + \cancel{2}yy'$$

$$2x(x^2+y^2+y) + y'(x^2+y^2+y)(2y+1) = x + yy'$$

$$y'(x^2+y^2+y)(2y+1) - y'y = x - 2x(x^2+y^2+y) = x(1-2(x^2+y^2+y))$$

$$y'[(x^2+y^2+y)(2y+1) - y] = x(1-2(x^2+y^2+y))$$

$$y' = \frac{x(1-2(x^2+y^2+y))}{(x^2+y^2+y)(2y+1) - y}$$

(ii) $$\tan(\alpha - \theta) = \frac{\tan\alpha - \tan\theta}{1+\tan\alpha\tan\theta} = \frac{f(\theta)}{f'(\theta)} = \frac{1-\sin\theta}{-\cos\theta}$$

For any given θ, we know $\tan\theta = \sin\theta/\cos\theta$, and we know $\tan(\alpha-\theta)$. We should be able to solve for $\tan\alpha$.

$$-\cos\theta(\tan\alpha - \tan\theta) = 1 - \sin\theta + \tan\alpha\tan\theta - \tan\alpha\tan\theta\sin\theta$$

$$\tan\alpha(-\cos\theta - \tan\theta + \tan\theta\sin\theta) = 1 - \sin\theta - \cos\theta\tan\theta = 1 - 2\sin\theta$$

$$\tan\alpha\left(\frac{-\cos^2\theta - \sin\theta + \sin^2\theta}{\cos\theta}\right) = 1 - 2\sin\theta$$

$$\tan\alpha = \frac{\cos\theta(1-2\sin\theta)}{\sin^2\theta - \cos^2\theta - \sin\theta}$$

Interpletation

In 6d, we obtained a formula for $\tan(\alpha-\theta)$ based on knowledge of $\tan\theta$ and $\tan\alpha$. Tanθ is always known if we know the point we are at: $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$.

$\tan\alpha$, on the other hand, is the slope of the tangent, and this we found in 3b to be $y'(x) = \frac{y'(\theta)}{x'(\theta)}$.

$$(x(\theta), y(\theta)) = ((1-\sin\theta)\cos\theta, (1-\sin\theta)\sin\theta)$$

$$x'(\theta) = -\cos\theta\cos\theta + (1-\sin\theta)(-\sin\theta)$$
$$= -\cos^2\theta - \sin\theta + \sin^2\theta$$

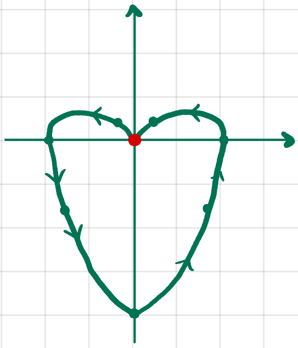
$$y'(\theta) = -\cos\theta\sin\theta + (1-\sin\theta)\cos\theta$$
$$= -\cos\theta\sin\theta + \cos\theta - \sin\theta\cos\theta$$
$$= \cos\theta - 2\sin\theta\cos\theta$$

$$y'(x) = \frac{y'(\theta)}{x'(\theta)} = \frac{\cos\theta - 2\sin\theta\cos\theta}{-\cos^2\theta - \sin\theta + \sin^2\theta}$$

So this direct calculation is yet another way to obtain $y'(x)$.

(b) tangent lines at the origin

$r = 1 - \sin\theta = 0 \to \sin\theta = 1 \to \theta = \frac{\pi}{2}$



$x'(\theta) = \sin^2\theta - \cos^2\theta - \sin\theta$
$y'(\theta) = \cos\theta(1 - 2\sin\theta)$

$x'(\pi/2) = 1 - 0 - 1 = 0 \to$ we can't use the formula $y'(x) = \frac{y'(\theta)}{x'(\theta)}$.

$y'(\pi/2) = 0(1-0) = 0 \to$ can't use $x'(y) = \frac{x'(\theta)}{y'(\theta)}$

How can we show that the tangent is vertical?

Let's compute $\lim_{x\to0^+} y'(x)$ and $\lim_{x\to0^-} y'(x)$.

$$y'(x) = \frac{\cos\theta - 2\sin\theta\cos\theta}{\sin^2\theta - \cos^2\theta - \sin\theta}$$

We want to compute $\lim_{\substack{\theta\to\frac{\pi}{2}^+ \\ x\to0^+}} \frac{\cos\theta - 2\sin\theta\cos\theta}{\sin^2\theta - \cos^2\theta - \sin\theta}$. But first let's just show that $x\to0$ when $\theta\to\pi/2$.

$\lim_{\theta\to\theta_0} \cos\theta = \cos\theta_0 \iff \forall\varepsilon>0\ \exists\delta>0\ \forall\theta\ |\theta-\theta_0|<\delta \to |\cos(\theta) - \cos(\theta_0)| < \varepsilon \to \cos\theta_0 - \varepsilon < \cos\theta < \cos\theta_0 + \varepsilon$

Let $\theta - \theta_0 = h$. Then $\forall\varepsilon>0\ \exists\delta>0\ \forall h\ |h|<\delta \to |\cos(\theta_0+h) - \cos\theta_0| < \varepsilon$

$\lim_{h\to0} \cos(\theta_0+h) = \cos(\theta_0)$

Similarly, $\forall\varepsilon>0\ \exists\delta\ \forall\theta\ |\theta-\theta_0|<\delta \to |\sin(\theta) - \sin(\theta_0)| < \varepsilon \to \sin(\theta_0) - \varepsilon < \sin(\theta) < \sin(\theta_0) + \varepsilon$
$\to -\sin\theta_0 - \varepsilon < -\sin\theta < -\sin\theta_0 + \varepsilon$
$\to 1 - \sin\theta_0 - \varepsilon < 1 - \sin\theta < 1 - \sin\theta_0 + \varepsilon$

Now, $x(\theta) = (1-\sin\theta)\cos\theta$

For $\theta_0 = \frac{\pi}{2}$ we have

$\forall\varepsilon>0\ \exists\delta>0\ \forall\theta\ |\theta - \pi/2| < \delta \to -\varepsilon < \cos\theta < \varepsilon \ \wedge\ -\varepsilon < 1 - \sin\theta < \varepsilon$

$\to -\varepsilon^2 < -\varepsilon\cos\theta < \cos\theta(1-\sin\theta) < \varepsilon\cos\theta < \varepsilon^2$

Therefore

$\forall\varepsilon>0\ \exists\delta>0\ \forall\theta\ |\theta - \pi/2| < \delta \to -\sqrt{\varepsilon} < \cos\theta < \sqrt{\varepsilon} \ \wedge\ -\sqrt{\varepsilon} < 1 - \sin\theta < \sqrt{\varepsilon}$

$\to -\sqrt{\varepsilon}\cos\theta < (1-\sin\theta)\cos\theta < \sqrt{\varepsilon}\cos\theta$

$\to -\varepsilon < \cos\theta(1-\sin\theta) < \varepsilon$

$\to |x| < \varepsilon$

That is, $\theta \to \frac{\pi}{2} \to x \to 0$

Back to our limit

$$\lim_{\substack{\theta \to \frac{\pi}{2}^+ \\ x \to 0^+}} \frac{\cos\theta - 2\sin\theta\cos\theta}{\sin^2\theta - \cos^2\theta - \sin\theta} = \lim_{\substack{\theta \to \frac{\pi}{2}^+ \\ x \to 0^+}} \frac{h(\theta)}{g(\theta)}$$

Note that we can't apply L'Hôspital rule

$$h'(\theta) = -\sin\theta - 2(\cos^2\theta - \sin^2\theta) \rightarrow h'(\pi/2) = -1 - 2(0-1) = -1+2 = 1$$

$$g'(\theta) = 2\sin\theta\cos\theta - 2\cos\theta(-\sin\theta) - \cos\theta \rightarrow g'(\pi/2) = 2\cdot 1\cdot 0 - 2\cdot 0\cdot(-1) - 0 = 0$$

$$\lim_{\substack{\theta \to \frac{\pi}{2}^+ \\ x \to 0^+}} h(\theta) = 0$$

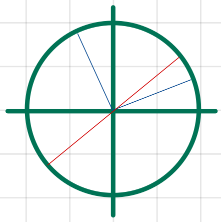
$$\lim_{\substack{\theta \to \frac{\pi}{2}^+ \\ x \to 0^+}} g(\theta) = 0$$

$$\lim_{\substack{x \to 0^+ \\ \theta \to \pi/2^+}} \frac{h'(\theta)}{g'(\theta)}$$ doesn't exist can't apply L'Hôpital's Rule

Let's try a change of variable

Let $\theta = \phi + \pi/2$.

$$\lim_{\phi \to 0^+} \frac{\cos(\phi+\pi/2) - 2\sin(\phi+\pi/2)\cos(\phi+\pi/2)}{\sin^2(\phi+\pi/2) - \cos^2(\phi+\pi/2) - \sin(\phi+\pi/2)} = \lim_{\phi \to 0^+} \frac{\cos(\phi+\pi/2) - \sin(2(\phi+\pi/2))}{\sin^2(\phi+\pi/2) - \cos^2(\phi+\pi/2) - \sin(\phi+\pi/2)}$$



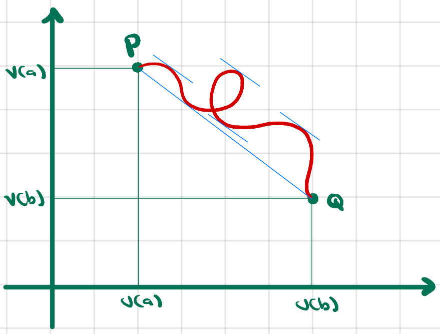
$\cos(\phi+\pi/2) = -\sin(\phi)$

$\sin(2\phi+\pi) = -\sin(2\phi)$ $\quad$ $\cos(2\phi+\pi) = -\cos(2\phi)$

$\sin(\phi+\pi/2) = \cos(\phi)$ $\quad$ $\cos(\phi+\pi/2) = -\sin\phi$

$$\lim_{\phi \to 0^+} \frac{-\sin(\phi) + \sin(2\phi)}{\cos^2(\phi) - \sin^2(\phi) - \cos(\phi)}$$

9. u, v continuous on $[a,b]$ diff on (a,b)



$P = (u(a), v(a))$
$Q = (u(b), v(b))$

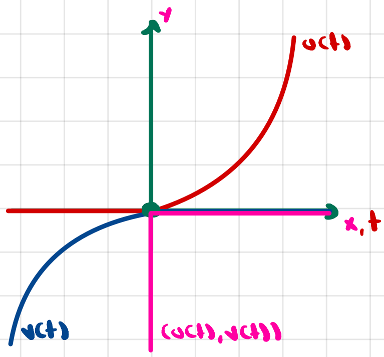
$c(t) = (u(t), v(t))$ is a parametric repres. of a curve from P to Q.

At some point on the curve, the tangent is parallel to line segment from P to Q.

Proof

Line segment: $g(t) = \dfrac{v(b)-v(a)}{u(b)-u(a)}(u(t)-u(a)) + v(a)$, $t \in [a,b]$, assuming $u(b) \neq u(a)$

Let $d(t) = v(t) - g(t)$.



Then $d(a) = v(a) - v(a) = 0$ and $d(b) = v(b) - v(b) = 0$.

Since d is diff, $\exists c, c \in (a,b) \wedge d'(c) = 0$.

$d'(t) = v'(t) - g'(t) = v'(t) - u'(t)\dfrac{v(b)-v(a)}{u(b)-u(a)} = 0$

If $u'(t) \neq 0$ then $\dfrac{v'(t)}{u'(t)} = \dfrac{v(b)-v(a)}{u(b)-u(a)}$

But $y'(x) = \dfrac{v'(t)}{u'(t)}$. Therefore $\exists c, c \in (a,b) \wedge y'(x) = \dfrac{v'(c)}{u'(c)} = \dfrac{v(b)-v(a)}{u(b)-u(a)}$

Alternative solution

Recall Cauchy Mean Value Theorem

f, g cont. on $[a,b]$ and diff on (a,b) $\rightarrow \exists x, x \in (a,b) \wedge [f(b)-f(a)]g'(x) = [g(b)-g(a)]f'(x)$

if $g(b) \neq g(a)$ and $g'(x) \neq 0$ we can rewrite this as $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(x)}{g'(x)}$

---

u and v are, by assumption, continuous on $[a,b]$ diff on (a,b)

Therefore, $\exists c, c \in (a,b) \wedge [u(b)-u(a)]v'(c) = [v(b)-v(a)]u'(c)$

P must be different from Q. Therefore either $u(b) \neq u(a)$ or $v(b) \neq v(a)$. Assume $u(b) \neq u(a)$.

If $u'(c) = 0$ then $v'(c) = 0$.

10. Definition of limit for vector-valued fn

$\lim_{t\to a} c(t) = \ell$ means position approaches ℓ as parameter approaches a

$\forall \epsilon > 0\ \exists \delta > 0\ \forall t\ 0 < |t-a| < \delta \to \|c(t) - \ell\| < \epsilon$ (1)

the size of the vector from ℓ to $c(t)$ can be made arbitrarily small

where $\ell = (\ell_1, \ell_2)$ is a vector.

---

norm $\|\vec{v}\| \equiv \sqrt{\vec{v}\cdot\vec{v}}$

$c(t) = (u(t), v(t))$

$\|c(t) - \ell\| = \|(u(t)-\ell_1, v(t)-\ell_2)\| = \sqrt{(u(t)-\ell_1)^2 + (v(t)-\ell_2)^2}$

$\|c(t)-\ell\|^2 = (u(t)-\ell_1)^2 + (v(t)-\ell_2)^2 = |u(t)-\ell_1|^2 + |v(t)-\ell_2|^2$

(a) $|u(t)-\ell_1| \leq \|c(t)-\ell\|$

$|v(t)-\ell_2| \leq \|c(t)-\ell\|$

Proof

$\|c(t)-\ell\|^2 = |u(t)-\ell_1|^2 + |v(t)-\ell_2|^2 \geq |u(t)-\ell_1|^2$

$\leftrightarrow |u(t)-\ell_1| \leq \|c(t)-\ell\| < \epsilon$

Analogously, $|v(t) - \ell_2| \leq \|c(t)-\ell\| < \epsilon$

Therefore

$\forall \epsilon > 0\ \exists \delta > 0\ \forall t\ 0 < |t-a| < \delta \to |u(t)-\ell_1| < \epsilon \wedge |v(t)-\ell_2| < \epsilon$

$\to \lim_{t\to a} u(t) = \ell_1$

$\lim_{t\to a} v(t) = \ell_2$

(b) Let $\lim_{t\to a} c(t) = (\lim_{t\to a} u(t), \lim_{t\to a} v(t)) = (\ell_1, \ell_2)$

$\lim_{t\to a} u(t) = \ell_1 \to \forall \epsilon > 0\ \exists \delta > 0\ \forall t\ 0 < |t-a| < \delta \to |u(t)-\ell_1| < \epsilon \to |u(t)-\ell_1|^2 < \epsilon^2$

$\lim_{t\to a} v(t) = \ell_2 \to \forall \epsilon > 0\ \exists \delta > 0\ \forall t\ 0 < |t-a| < \delta \to |v(t)-\ell_2| < \epsilon \to |v(t)-\ell_2|^2 < \epsilon^2$

Therefore $|u(t)-\ell_1|^2 + |v(t)-\ell_2|^2 = \|c(t)-\ell\|^2 < 2\epsilon^2 \to \|c(t)-\ell\| < \sqrt{2}\,\epsilon$

$\forall \epsilon > 0$, let $\epsilon_1 = \frac{\epsilon}{\sqrt{2}}$. Then $\exists \delta > 0\ \forall t\ 0 < |t-a| < \delta \to \|c(t)-\ell\| < \epsilon$

$\to \lim_{t\to a} c(t) = \ell$ as per definition (1) above.