

## Problems, Prologue

i) ii)  $a \cdot x = a, a \neq 0 \Rightarrow x = 1$

$a^{-1} \cdot a \cdot x = a \cdot a^{-1}$ , from P7  $a^{-1}$  exists since  $a \neq 0$   
 $x = 1$

iii)  $x^2 - y^2 = (x-y)(x+y)$

use distributive law

$$x(x+y) + (-y)(x+y)$$

$$x \cdot x + xy + (-y)x + (-y)y$$

$$x^2 + xy + (-xy) + (-y^2)$$

$$x^2 - y^2$$

$$x^2 - y^2 = x^2 + xy - xy - y^2$$

$$= x(x+y) + (-y)(x+y)$$

$$= (x+y)(x-y)$$

iii)  $x^2 = y^2$  then  $x = y$  or  $x = -y$

$$|x|^2 = |y|^2$$

$$x^2 - y^2 = 1 = y \cdot y \cdot x^{-1} \cdot x^{-1}$$

$$(y \cdot x^{-1}) \cdot (y \cdot x^{-1}) = 1$$

$y \cdot x^{-1} = (y \cdot x^{-1})^{-1}$ , so the number  $y \cdot x^{-1}$  is equal to its inverse

$$\begin{cases} |x| = \sqrt{x^2} \\ |x|^2 = x^2 \end{cases}$$

$$x^2 - y^2 = 0$$

$$x^2 - xy + xy - y^2 = 0$$

$$x(x-y) + y(x-y)$$

$$(x-y)(x+y) = 0$$

$$x-y = 0 \text{ or } x+y = 0$$

$$x = y \text{ or } x = -y$$

if  $a = a^{-1}$  then  $a \cdot a = 1 \Rightarrow a^2 = 1$

$$|a|^2 = 1$$

$$|a| = 1 \text{ but } |a| = \begin{cases} a & a > 0 \text{ so } a = 1 \text{ or } a = -1 \\ -a & a \leq 0 \end{cases}$$

Therefore,  $y \cdot x^{-1} = \pm 1$

If  $x$  and  $y$  have same sign, then  $y \cdot x^{-1} = 1$  and  $y$  here is the inverse of  $x^{-1}, x$ .

if  $x$  and  $y$  have different signs,  $y \cdot x^{-1} = -1$

so,  $y = -x$  ie  $x = -y$

}, inverse has same sign as number  
 $a \cdot a^{-1} = 1$

if  $a > 0$ , then  $a^{-1} > 0$

if  $a < 0$ , then  $a^{-1} < 0$

which property justifies this?

iv)  $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

use distributive law

$$x(x^2 + xy + y^2) + (-y)(x^2 + xy + y^2)$$

$$x^3 - y^3$$

v)  $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \dots + xy^{n-2} + y^{n-1})$

$$x^4 - y^4 = (x-y)(x^3 + x^2y + xy^2 + y^3)$$

$$= (x-y) \cdot \sum_{i=0}^3 x^i y^{3-i} = x \sum_{i=0}^3 x^i y^{3-i} + (-y) \sum_{i=0}^3 x^i y^{3-i} = x^4 + \sum_{i=1}^3 x^i y^{3-i} - y^4 - y \sum_{i=1}^3 x^i y^{3-i}$$

$$= x^4 - y^4 + \sum_{i=1}^3 x^i y^{3-i} - \sum_{i=1}^3 x^i y^{3-i}$$

if  $j = i - 2$  then  $\sum_{i=0}^3 x^i y^{3-i}$

replace 4 with  $n$  and 3 with  $n-1$

$$v) x^n - j^n = (x-j) \sum_{i=0}^{n-1} x^{n-1-i} j^i$$

$$= x \sum_{i=0}^{n-1} x^{n-1-i} j^i - j \sum_{i=0}^{n-1} x^{n-1-i} j^i$$

$$= x^n + x \sum_{i=1}^{n-1} x^{n-1-i} j^i - j^n - j \sum_{i=0}^{n-2} x^{n-1-i} j^i$$

$$= x^n + j^n + \sum_{i=1}^{n-1} x^{n-1-i} j^i - \sum_{i=0}^{n-2} x^{n-1-i} j^i$$

$$\sum_{i=1}^{n-1} x^{n-1-i} j^i = x^{n-1} j + x^{n-2} j^2 + \dots + x j^{n-1}$$

$$\sum_{i=0}^{n-2} x^{n-1-i} j^i = x^n j + x^{n-2} j^2 + \dots + x j^{n-1}$$

$$vi) x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

$$x^3 + y^3 = x^3 + x^2y + xy^2 - yx^2 - xy^2 - y^3 + 2y^3$$

$$\begin{aligned} &= x^3 + x^2y - xy^2 + xy^2 - xy^2 + y^3 \\ &= x(x^2 - xy + y^2) + y(x^2 - xy + y^2) \\ &= (x^2 - xy + y^2)(x+y) \end{aligned}$$

back solution

in  $x^3 - y^3$ , if  $y$  is  $-y$  then

$$x^3 - (-y)^3 = x^3 + y^3$$

only counts if the exponent is odd

$$= (x+y)(x^2 - xy + y^2)$$

note: we used the additive inverse in place of the original  $y$  in  $x^3 - y^3$  and obtained the expression we wanted:  $x^3 + y^3$

general case

→ n terms, exponents from 0 to  $n-1$

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \dots + x^2y^{n-3} + xy^{n-2} + y^{n-1})$$

$$x^n + y^n = x^n + x^{n-1}y + x^{n-2}y^2 + \dots + x^3y^{n-3} + x^2y^{n-2} + xy^{n-1} \\ + y^n - yx^{n-1} - y^2x^{n-2} - \dots - x^2y^{n-2} - xy^{n-1}$$

2) let  $x=y$

$$x^2 = xy$$

$$x^2 - y^2 = xy - y^2$$

$$x^2 - xy + xy - y^2 = xy - y^2$$

$$(x-y)(x+y) = y(x-y)$$

if we divide by  $x-y$ , we multiply by  $(x-y)^{-1}$  then implicitly,  $x-y \neq 0 \Rightarrow x \neq y$   
so we can do it.

3) i)  $\frac{a}{b} = \frac{ac}{bc}$  if  $b,c \neq 0$

$$a \cdot b^{-1} = a \cdot b^{-1} \cdot 1 = a \cdot b^{-1} \cdot (c \cdot c^{-1})$$

ii)  $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$  if  $b,d \neq 0$

$$a \cdot b^{-1} + c \cdot d^{-1} = a \cdot b^{-1} \cdot d \cdot d^{-1} + c \cdot d^{-1} \cdot b \cdot b^{-1} \\ + b \cdot d^{-1} (cd + cb)$$

iii)  $(ab)^{-1} = a^{-1}b^{-1}$  if  $a,b \neq 0$

$$(ab)^{-1} \cdot ab = 1$$

$$(ab)^{-1} = a^{-1}b^{-1}$$

iv)  $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{db}$  if  $b,d \neq 0$

$$(a \cdot b^{-1}) \cdot (c \cdot d^{-1})$$

associative law to multiplication.  $(a \cdot b^{-1} \cdot c) \cdot d^{-1}$

$$= (a \cdot c) \cdot (d^{-1} \cdot b^{-1})$$

$$vi) \frac{a}{b} \mid \frac{c}{d} = \frac{ad}{bc} \quad b,c,d \neq 0$$

$$(a \cdot b^{-1}) \cdot (c \cdot d^{-1})^{-1}$$

$$(a \cdot b^{-1}) \cdot c^{-1} \cdot (d^{-1})^{-1} = a \cdot b^{-1} \cdot c^{-1} \cdot d = \frac{ad}{bc}$$

vi) if  $b,d \neq 0$  then  $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$

$$ab^{-1} = cd^{-1}$$

$$ab^{-1}b = cd^{-1}b = a$$

$$ad = cd^{-1}bd = cb \quad ad - bc = (cd) \cdot b^{-1} \cdot d^{-1} = (cb) \cdot b^{-1} \cdot d^{-1}$$

$$ad - bc$$

$$(cd) \cdot b^{-1} \cdot d^{-1} = (cb) \cdot b^{-1} \cdot d^{-1}$$

$$= a \cdot b^{-1} = c \cdot d^{-1}$$

$$\frac{a}{b} \cdot \frac{b}{a} = ab^{-1} = ba^{-1}$$

$$ab^{-1}b = a = bc^{-1}b$$

$$a \cdot a^{-1} = b \cdot b^{-1}$$

$$a^2 = b^2, \text{ we know from } i) \text{ iii) that } a = \pm b$$

$$4) \text{i)} 4-x < 3-2x$$

$$4-x - (3-2x) < 0 \quad a-b < 0$$

$$1+x < 0$$

$$\begin{aligned} x-(-1) &< 0 \Rightarrow -(x-(-1)) > 0 \Rightarrow -1-x \text{ in P} \Rightarrow -1 > x \\ a-b < 0 & \quad -(-c-b) > 0 \quad b-a \text{ in P} \quad b > a \end{aligned}$$

$$\text{ii)} 5-x^2 < 8 \quad a < b$$

$$8-(5-x^2) > 0 \quad b-a \text{ in P}, b-a > 0$$

$$3+x^2 > 0$$

$x^2 > -3$  but we know that  $x^2 \geq 0$  for all  $x$ , so  $x$  can't be a real number

$$\text{iii)} 6-x^2 < -2$$

$$x^2 > 8$$

$$\sqrt{x^2} > \sqrt{8}$$

$$|x| > \sqrt{8} \quad \text{if } x \geq 0, \text{ then } x > \sqrt{8}$$
$$x \leq 0, \text{ then } -x > \sqrt{8} \Rightarrow x < -\sqrt{8}$$

$$\text{iv)} (x-1)(x-3) > 0$$

$$a \cdot b > 0$$

$$\text{c1)} a > 0, b > 0$$

$$\begin{aligned} x-1 > 0 &\Rightarrow x > 1 & \text{c2)} a < 0, b < 0 \\ x-3 > 0 &\Rightarrow x > 3 & x < 1 &\Rightarrow x < 1 \\ &\Rightarrow x > 3 & x < 3 \end{aligned}$$

$$\text{v)} x^2-2x+2 > 0$$

$$(x-1)^2 > -1 \quad \text{valid for all } x \text{ since } a^2 \geq 0 \forall a$$

$$\text{vi)} x^2+x+1 > 2 \quad x > \frac{-1+\sqrt{5}}{2} \quad \text{or} \quad x < \frac{-1-\sqrt{5}}{2}$$

$$x^2+x-1 > 0$$

$$\Delta = 1 - 4 \cdot 1 \cdot (-1) = 5$$

$$x_0 = \frac{-1 \pm \sqrt{5}}{2}$$

$$\text{vii)} x^2-x+10 > 16$$

$$x^2-x-6 > 0$$

$$\Delta = 1 - 4 \cdot 1 \cdot (-6) = 25$$

$$x_0 = \frac{-1 \pm 5}{2} \Rightarrow 2 \Rightarrow -3$$

$$x > 2 \text{ or } x < -3$$

$$\text{viii) } x^2 + x + 1 > 0$$

$$\Delta = 1 - 4 \cdot 1 \cdot 1 = -3 \Rightarrow \text{no roots}$$

$$(x^2 + 2 \cdot \frac{1}{2} \cdot x + \frac{1}{4}) + \frac{3}{4}$$

$$(x + \frac{1}{2})^2 + \frac{3}{4} > 0 \quad \forall x$$

$$\text{ix) } (x-\pi)(x+5)(x-3) > 0$$

+++	$x > \pi, x > -5, x > 3 \Rightarrow x > \pi$
--+	$x < \pi, x < -5, x > 3 \Rightarrow \text{not possible}$
+--	$x > \pi, x < -5, x < 3 \Rightarrow \text{"}$
-+-	$x < \pi, x > -5, x < 3 \Rightarrow -5 < x < 3$



$$\text{x) } (x - \sqrt[3]{2})(x - \sqrt{2}) > 0$$

++	$x > \sqrt[3]{2}, x > \sqrt{2} \Rightarrow x > \sqrt{2}$
--	$x < \sqrt[3]{2}$

$$\text{xii) } 2^x < 8$$

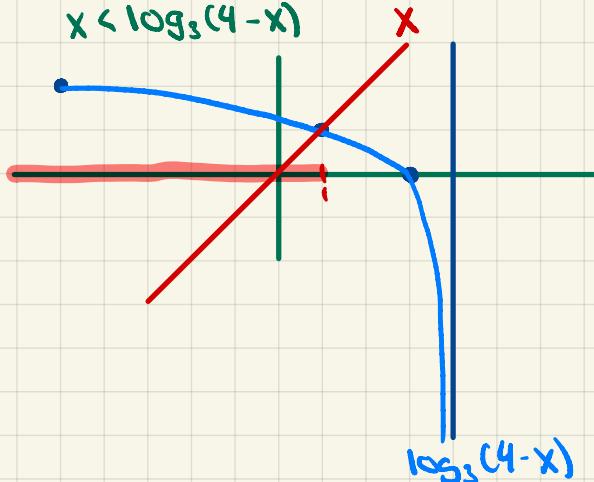
$$\log_2 2^x < \log_2 8$$

$$x < 3$$

$$\text{xiii) } x + 3^x < 4$$

$$3^x < 4 - x$$

$$x < \log_3(4-x)$$



$$\text{xiii) } \frac{1}{x} + \frac{1}{1-x} > 0$$

$$x \neq 0, 1-x \neq 0 \Rightarrow x \neq 1$$

Solution 1

$$\frac{1-x+x}{x(1-x)} = \frac{1}{x(1-x)} > 0$$

$$x(1-x) > 0$$

$\Rightarrow x > 0 \text{ and } x < 1$



Solution 2

$$\frac{1}{x} > -\frac{1}{1-x}$$



(1)  $x < 0 \Rightarrow \text{both sides negative}$

$$1 < -x | (1-x)$$

$\underbrace{< 0, \text{ could stop here}}$

$(1-x) \text{ is } > 0$

$$(1-x) < -x \Rightarrow 1 < 0$$

(2)  $x > 1$ , analogous to (1)  $\Rightarrow 1 < 0$

(3)  $0 < x < 1$

$$\frac{1}{x} > 0, -\frac{1}{1-x} < 0$$

$$1 > -\frac{x}{1-x}, 1-x > 0$$

$$1-x > -x \Rightarrow 0 < 1$$

Note: tricky in solution 2 is to figure out when the inequality is flipped when multiplying by an inverse

$$\text{xiv) } \frac{x-1}{x+1} > 0 \quad x \neq -1$$

$$++ \quad x-1 > 0 \Rightarrow x > 1 \Rightarrow x > 1$$

$$x+1 > 0 \Rightarrow x > -1$$

$$-- \quad x < 1 \Rightarrow x < -1$$

5)

i) if  $a < b$  and  $c < d$  then  $a+c < b+d$ 

$$b-a > 0$$

$$d-c > 0$$

$$(b-a) + (d-c) > 0$$

$$\Rightarrow (b+d) - (a+c) > 0 \Rightarrow b+d > a+c$$

ii) if  $a < b$  then  $-b < -a$ 

$$b-a > 0$$

from i) we can add

$$0+a < b$$

iii) if  $c < b$  and  $c > d$  then  $a-c < b-d$ 

$$b-a > 0$$

$$c-d > 0$$

$$(b-a) + (c-d) > 0$$

$$b-d + (- (a-c)) > 0$$

$$b-d > a-c$$

iv)  $a < b, c > 0 \Rightarrow ac < bc$ 

$$b-a > 0$$

$$c(b-a) > 0$$

$$bc - ac > 0 \Rightarrow bc > ac$$

v)  $a < b, c < 0 \Rightarrow ac > bc$ 

$$b-a > 0$$

$$c(b-a) < 0$$

$$bc - ac < 0$$

$$ac > bc$$

vi)  $a > 1 \Rightarrow a^2 > a$ 

$$a > 1 \Rightarrow a > 0, 1 < a$$

so by iv),  $1 \cdot a < a \cdot a$ vii)  $0 < a < 1 \Rightarrow a^2 < a$ 

$$0 < a < 1 \Rightarrow a \cdot a < 1 \cdot a$$

viii)  $0 \leq a < b, 0 \leq c < d \Rightarrow ac < bd$ if  $a=0$  or  $c=0$  then  $ac=0 < bd$ if  $c \neq 0$  and  $c \neq 0$ 

$$0 < ac < bc \text{ by iv)}$$

$$0 < bc < bd \text{ by iv)}$$

$$\Rightarrow ac < bd$$

ix)  $0 \leq a < b \Rightarrow a^2 < b^2$ special case of viii) where  $c=a, d=b$ x)  $a, b \geq 0, a^2 < b^2 \Rightarrow a < b$ assume  $a < b$  is false

then

$$i) a = b \Rightarrow a^2 = b^2 \times$$

$$ii) a > b \Rightarrow 0 \leq b \leq a \Rightarrow b^2 < a^2 \times$$

therefore,  $a < b$

6)

$$\text{a) } 0 \leq x < y \Rightarrow x^n < y^n \text{ for } n=1,2,3,\dots$$

From 5ix),  $x^2 < y^2$   
 so we have  $0 \leq x^2 < y^2$

From 5viii)  $0 \leq x < y, 0 \leq x^2 < y^2 \Rightarrow x \cdot x^2 < y \cdot y^2 \Rightarrow x^3 < y^3$

Keep applying 5viii) with  $0 \leq x < y, 0 \leq x^n < y^n, n=1,2,3,\dots$

$$\text{b) } x < y, n \text{ odd} \Rightarrow x^n < y^n$$

1)  $0 \leq x < y \Rightarrow x^n < y^n$  by 6a

2)  $x < 0, y > 0 \Rightarrow x = x^{n-1} \cdot x$

$x^{n-1} > 0, x < 0 \Rightarrow x^n < 0 \Rightarrow x^n < y^n > 0$

3)  $x < 0, y < 0, x < y$

$\Rightarrow 0 \leq -y < -x \Rightarrow (-y)^n < (-x)^n$ , for n odd

$(-y)^n = y^n$  for n odd, same for x, so  $-y^n < -x^n$

$\Rightarrow y^n < x^n$ , n odd

$$\text{c) } x^n = y^n, n \text{ odd} \Rightarrow x = y$$

$x < y, n \text{ odd} \Rightarrow x^n < y^n$  by 6b, contrad.

$y < x, n \text{ odd} \Rightarrow y^n < x^n$  by 6b, contrad.

$\Rightarrow x = y$  because one of  $x = y, x > y, x < y$  must hold

\* because Trichotomy law, the fact that  $x-y$  is a number, and the definitions of  $>$ ,  $<$ .

$$\text{d) } x^n = y^n, n \text{ even} \Rightarrow x = y \text{ or } x = -y$$

$$7. 0 < a < b \Rightarrow a < \sqrt{ab} < \frac{a+b}{2} < b$$

$$(a-b)^2 > 0 \Leftrightarrow a - 2\sqrt{ab} + b > 0 \Leftrightarrow \sqrt{ab} < \frac{a+b}{2}$$

$$0 < a < b \Rightarrow 0 < a < 2a < a+b \Rightarrow 0 < a < \frac{a+b}{2}$$

$$0 < a^2 < ab \Rightarrow 0 < a < \sqrt{ab}$$

$$0 < a < b \Rightarrow 0 < a < a+b < 2b \Rightarrow 0 < \frac{a}{2} < a < \frac{a+b}{2} < b$$

$$\Rightarrow 0 < a < \sqrt{ab} < \frac{a+b}{2} < b$$

8)

P(i) apply P'10 with  $b=0$

$\Rightarrow \forall a$ , one and only one of following holds:

$$a = 0$$

$$a > 0$$

$$a < 0$$

Alternatively,

suppose  $a > 0$  and  $-a > 0 \Rightarrow a + (-a) > 0$  contradicting def. of  $-a$ .

also if  $-a = 0$  then this also violates the def of  $-a$ , as  $a + (-a) = a$

therefore  $-a < 0$  from P'10

P(ii)  $\forall a, b, c \quad a < b, c < d$

using P'12,  $a+c < b+c$

$\Rightarrow a+c < b+c < b+d$

again P'12,  $b+c < b+d$

If  $a=0$  and  $c=0$  and  $b>0, d>0 \Rightarrow 0 < b+d$

P(iii) use P'13 with  $a=0$ :  $\forall b, c$ , if  $0 < b$  and  $0 < c$  then  $0 < bc$

$$9) \text{i)} \sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7} > 0$$

$$\text{ii)} 0 \leq |a| \leq |a| + |b| \quad \forall a, b$$

if  $b \geq 0$  then  $0 \leq |a+b| = |a| + |b|$

if  $b < 0$  then  $0 \leq |a+b| < |a| + |b|$

$$\Rightarrow \forall b \quad |a+b| \leq |a| + |b|$$

$$\Rightarrow \forall b \quad |a+b| - |a| - |b| \leq 0$$

so, eliminate outer absolute operator:

$$|(|a+b| - |a| - |b|)| = |a| + |b| - |a+b| \quad \text{ii)} |x-3| < 8$$

iii)  $a+b$  is a number, edit if d

now we are in the case of ii)

$$|(|d| + |c| - |d+c|)| \geq 0$$

$$= |d| + |c| - |d+c| = |a+b| + |c| - |a+b+c|$$

$$\text{iv)} |x^2 - 2xy + y^2| = |(x-y)^2| = (x-y)^2$$

$$\text{v)} |(|\sqrt{2} + \sqrt{3}| - |\sqrt{5} - \sqrt{7}|)|$$

$$\sqrt{2} + \sqrt{3} - (\sqrt{7} - \sqrt{5})$$

$$10) \text{ii)} |a+b| - |b|$$

$$a+b \geq 0 \Rightarrow a \geq -b$$

$$b \geq 0 \Rightarrow a+b - b = a$$

$$b < 0 \Rightarrow a+b + b = a + 2b$$

$$a+b < 0 \Rightarrow a < -b$$

$$b \geq 0 \Rightarrow -a - b - b = -a - 2b$$

$$b < 0 \Rightarrow -a - b + b = -a$$

$$\text{iii)} |(x(-1))|$$

$$x \geq 0 \Rightarrow |x(-1)|$$

$$x \geq 1 \Rightarrow x(-1) \text{ if } x \geq 1$$

$$0 \leq x < 1 \Rightarrow 1-x$$

$$x < 0 \Rightarrow |1-x(-1)|$$

$$-x-1 \geq 0 \Rightarrow x \leq -1 \Rightarrow -x-1$$

$$-x-1 < 0 \Rightarrow x > -1 \Rightarrow x+1$$

$$\text{iv)} |x(-1)x^2| = |x| - x^2$$

$$x \geq 0 \Rightarrow x - x^2$$

$$x < 0 \Rightarrow -x - x^2$$

$$10) \text{iv)} a - \underbrace{|(a-|a|)|}$$

$$a \geq 0 \Rightarrow 0$$

$$a < 0 \Rightarrow 2a < 0$$

$$\text{so, } a \geq 0 \Rightarrow a$$

$$a < 0 \Rightarrow a + 2a = 3a$$

$$\text{ii)} |x-3| = 8$$

$$x \geq 3, x = 11$$

$$x < 3, 3-x = 8 \Rightarrow x = -5$$

$$x \geq 3 \Rightarrow x-3 < 8 \Rightarrow x < 11 \Rightarrow [3, 11[$$

$$x < 3 \Rightarrow 3-x < 8 \Rightarrow x > -5 \Rightarrow ]-5, 3[ \\ \Rightarrow ]-5, 11[$$

$$\text{iii)} |x+4| < 2$$

$$x+4 \geq 0 \Rightarrow x \geq -4$$

$$x+4 < 2 \Rightarrow x < -2$$

$$[-4, -2[$$

$$x+4 < 0 \Rightarrow x < -4$$

$$-x-4 < 2 \Rightarrow x > -6$$

$$]-6, -4[$$

$$\Rightarrow ]-5, -2[$$

$$\text{iv)} |x-1| + |x-2| > 1$$



$$x \geq 2$$

$$x-1 + x-2 = 2x-3 > 1$$

$$x \leq 1 \Rightarrow 1-x - x+2 > 1$$

$$2x < 2$$

$$x < 1$$

$$x > 2$$

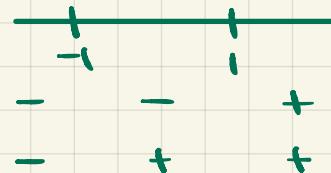
$$\text{so, } x < 1 \text{ or } x > 2$$

$$1 < x < 2$$

$$x-1 - x+2 > 1 \Rightarrow 1 > 1$$

actually  $x-1 - x+2 = 1$  always

$$\text{II) } \sqrt{|x-1|+|x+1|} < 2$$



$$x < -1 \Rightarrow 1-x-x-1 = -2x < 2 \Rightarrow x > -1 \text{ contradiction.}$$

$$-1 < x < 1 \Rightarrow 1-x+x+1 = 2 \text{ always } > 2 \quad \times$$

$$x \geq 1 \Rightarrow x-1+x+1 = 2x < 2 \Rightarrow x < 1 \quad \times$$

no solutions

$$\text{VII) } |x-1|+|x+1| < 1$$

again, three cases

$$\text{i) } x < -1, -2x < 1 \quad x > -\frac{1}{2} \quad \times$$

$$\text{ii) } -1 < x < 1, = 2 \text{ always } \quad \times$$

$$\text{iii) } x \geq 1, 2x < 1, x < \frac{1}{2} \quad \times$$

no solutions

$$\text{VIII) } |x-1| \cdot |x+1| = 0$$

$$|x-1| = 0 \Rightarrow x = 1$$

$$|x+1| = 0 \Rightarrow x = -1$$

$$\text{IX) } |x-1||x+2| = 3$$



$$-2 < x < 1$$

$$\begin{aligned} & (1-x)(x+2) \\ & = x+2-x^2-2x = -x^2+3x+2 = 3 \\ & x^2-3x-1=0 \\ & \Delta = 3-4 \cdot 1 \cdot 1 = -1 \\ & \text{no solutions} \end{aligned}$$

$$x < -2$$

$$\begin{aligned} & (1-x)(-x-2) \\ & = -x-2+x^2+2x = x^2+x-2 = 3 \\ & x^2+x-5=0 \\ & \Delta = 1-4 \cdot 1 \cdot (-8) = 21 \quad x = \frac{-1 \pm \sqrt{21}}{2} \end{aligned}$$

$$x = \frac{-1-\sqrt{21}}{2}$$

$$x > 1$$

$$\begin{aligned} & (x-1)(x+2) = x^2+2x-x-2 \\ & = x^2+x-2 = 3 \\ & x^2+x-5=0 \end{aligned}$$

$$x = \frac{-1+\sqrt{21}}{2}$$

12)

$$\text{i)} |xy| = |x||y|$$

$$xy \geq 0$$

$$\Rightarrow \text{if } x \geq 0, y \geq 0$$

$$xy = |x||y|$$

$$\text{ii)} x < 0, y < 0$$

$$xy = (-x)(-y) = |x||y|$$

$$xy < 0$$

$$\Rightarrow \text{if } x < 0, y > 0$$

$$-xy = (-x)y = |x||y|$$

$$\text{iii)} x > 0, y < 0$$

$$-xy = x(-y) = |x||y|$$

$$\text{iv)} \left| \frac{1}{x} \right| = \frac{1}{|x|} \text{ if } x \neq 0$$

$$x^{-1} > 0 \Rightarrow x \cdot x^{-1} = 1 \text{ and } x > 0$$

$$\left| \frac{1}{x} \right| = \frac{1}{|x|} = \frac{1}{|x|}$$

$$x^{-1} < 0 \Rightarrow x \cdot x^{-1} = 1 \Rightarrow x < 0$$

$$\left| \frac{1}{x} \right| = \frac{1}{|(-x)|} = \frac{1}{|x|}$$

$$\text{v)} |x|/|y| = |x|/|y| \text{ if } y \neq 0$$

$$x \geq 0, y > 0 \Rightarrow \frac{x}{y} = |x|/|y|$$

$$x < 0, y > 0 \Rightarrow -\frac{x}{y} = -(\frac{x}{|y|}) = |x|/|y|$$

$$x \geq 0, y < 0 \Rightarrow \frac{x}{|(-y)|} = -(\frac{x}{|y|}) = |x|/|y|$$

$$x < 0, y < 0 \Rightarrow -\frac{x}{-y} = \frac{x}{|y|} = |x|/|y|$$

$$\text{vi)} |x-y| \leq |x| + |y|$$

$$|x-y| = \begin{cases} x-y & \text{if } x \geq y \\ y-x & \text{if } x < y \end{cases}$$

$$|x-y| \leq |x| - y = |x| + (-y) \leq |x| + |y|$$

because  $x \leq |x|$

$$\text{vii)} |x|-|y| \leq |x-y|$$

$$\begin{aligned} |x| &= |y - (y-x)| \leq |y| + |y-x| \\ &\Rightarrow |x| - |y| \leq |y-x| \end{aligned}$$

$$\text{viii)} |(|x|-|y|)| \leq |x-y|$$

$$| |x|-|y| | = \begin{cases} |x|-|y| & \text{if } |x| \geq |y| \\ |y|-|x| & \text{if } |x| < |y| \end{cases}$$

$$\text{ix)} \rightarrow \forall x, y \quad |x|-|y| \leq |x-y| \text{ and } |y|-|x| \leq |y-x| = |x-y|$$

$$\rightarrow | |x|-|y| | \leq |x-y|$$

$$\text{x)} |x+y+z| \leq |x| + |y| + |z|$$

$$|x+y+z| \leq |x+y| + |z| \leq |x| + |y| + |z|$$

$$|x+y+z| = \begin{cases} x+y+z & \text{if } x \geq 0 \\ -(x+y+z) & \text{if } x < 0 \end{cases}$$

$$\text{if } x+y+z \geq 0$$

$$x \leq |x|, y \leq |y|, z \leq |z|$$

$$\Leftrightarrow x = |x| \text{ if } x \geq 0$$

(...)

$$\text{so, } x+y+z = |x| + |y| + |z| \text{ if } x, y, z \geq 0$$

$$\text{if } x+y+z < 0$$

$$|x+y+z| = -(x+y+z) < |x| + |y| + |z| \geq 0$$

Alternatively:

$$|x-y| - |x+y| \leq | |x| + |y| | = |x| + |y|$$

$$x \leq |x|, -y \leq |y|$$

$$13) \max(x, y) = \frac{x+y+|y-x|}{2}$$

$$\min(x, y) = \frac{x+y-|y-x|}{2}$$

MAX

$$1) y > x$$

$$\max(x, y) = \frac{x+y+|y-x|}{2} = y$$

$$y < x$$

$$\max(x, y) = \frac{x+y+|x-y|}{2} = x$$

analogous for min

$$\max(x, y, z) = \max(x, \max(y, z))$$

$$\max(x, \max(y, z)) = (x + m_{yz} + |m_{yz} - x|)/2$$

$$= \left( x + \frac{z+y+|y-z|}{2} + \left| \frac{z+y+|y-z|}{2} - x \right| \right) / 2$$

$$= \left( \frac{zx + z + y + |y-z|}{2} + \left| \frac{z+y-2x+|y-z|}{2} \right| \right) / 2$$

14)

$$a) |a| = 1-a$$

$$a \geq 0$$

$$|a| = a$$

$$1-a = -(a) = a$$

$$a < 0$$

$$|a| = -a$$

$$1-a = -a$$

$$b) -b \leq a \leq b \Leftrightarrow |a| \leq b$$

$$\Rightarrow -b \leq a \leq b$$

$$\begin{aligned} \text{if } c > 0 \text{ then } |a| = a \leq b \\ \text{if } c < 0 \text{ then } |a| = -a \leq b \Rightarrow |a| \leq b \end{aligned}$$

$$\Leftarrow |a| \leq b, \text{ first, } b \geq 0 \text{ since } |c| \geq 0$$

$$c > 0 \Rightarrow a \leq b, \text{ also } c > -b \text{ since } c > 0 \text{ and } b > 0 \Rightarrow -b < 0$$

$$c < 0 \Rightarrow a < b \text{ since } c < 0, b > 0; -a < b \text{ by assumption, } \Rightarrow a > -b$$

Therefore, if  $b = |a|: -|a| \leq a \leq |a| \Leftrightarrow |a| \leq b$

$$14) c) |a+b| \leq |a| + |b|$$

$$|a| \leq b \Rightarrow -b \leq a \leq b$$

$$\Rightarrow |a| \leq |a| \Rightarrow -|a| \leq a \leq |a|$$

so,

$$-|a| \leq a \leq |a|$$

$$-|b| \leq b \leq |b|$$

$$a+b \leq |a| + |b|$$

$$a+b \geq -|a|-|b|$$

$$\Rightarrow -(|a| + |b|) \leq a+b \leq |a| + |b|$$

$$\Rightarrow b \text{ by 14b) } |a+b| \leq |a| + |b|$$

15)  $x$  and  $y$  not both 0 then

$$x^2 + xy + y^2 > 0$$

$$x^4 + x^3y + x^2y^2 + xy^3 + y^4 > 0$$

→ it either is zero, trivial solution

→ it neither zero

i)  $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

$$x^3 - y^3 > 0 \Rightarrow x^3 > y^3$$

$$\Rightarrow x > y \Rightarrow x-y > 0$$

$$\Rightarrow x^2 + xy + y^2 > 0$$

$$x^3 - y^3 < 0 \Rightarrow x^3 < y^3 \Rightarrow x < y$$

$$\Rightarrow x-y < 0$$

$$\Rightarrow x^2 + xy + y^2 > 0$$

$$x^3 - y^3 = 0 \Rightarrow x = y$$

$$\Rightarrow x^2 + x \cdot x + x^2 = 3x^2 > 0$$

Summary: using the factorization, we know that

$x^3 - y^3$  has to be  $< 0, > 0$ , or  $= 0$  and this has implications for the factors. Which one factor is what we are investigating.

2)  $x^5 - y^5 = (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$

$$x^5 - y^5 > 0 \Rightarrow x > y \Rightarrow x-y > 0$$

$$\Rightarrow x^4 + \dots + y^4 > 0$$

(analogous to part 1)

16) a)  $(x+y)^2 = x^2 + y^2$  only when  $x=0$  or  $y=0$

$$(x+y)^3 = x^3 + y^3$$
 only when  $x=0$  or  $y=0$  or  $x=-y$

ii)  $(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$   
 $= x^2 + y^2 \Rightarrow 2xy = 0 \Rightarrow x=0$  or  $y=0$

iii)  $(x+y)(x^2 + 2xy + y^2) = x^3 + 2x^2y + xy^2 + yx^2 + 2xy^2 + y^3$   
 $= x^3 + y^3 \Rightarrow 2x^2y + xy^2 + yx^2 + 2xy^2 = 0$   
 $\Rightarrow xy(2x+y+x+2y) = 0$   
 $\Rightarrow xy(3x+3y) = 0$   
 $\Rightarrow x=0$  or  $y=0$  or  $x=-y$

b)  $x^2 + 2xy + y^2 = (x+y)^2 \geq 0$

Show  $4x^2 + 6xy + 4y^2 > 0$  unless  $x=0, y=0$

$$4(x^2 + 2xy + y^2) - 2xy$$
$$4(x+y)^2 - 2xy$$

$$2xy = (x+y)^2 - x^2 - y^2$$

$$4(x+y)^2 - (x+y)^2 + x^2 + y^2$$
$$= 3(x+y)^2 + x^2 + y^2, \text{ which is } \geq 0 \forall x, y \neq 0$$

c)  $(x+y)^4 = x^4 + y^4$

$$(x+y)^2(x+y)^2$$
$$(x^2 + 2xy + y^2)(x^2 + 2xy + y^2)$$

$$x^4 + 2x^3y + x^2y^2 +$$
$$2x^2y + 4x^2y^2 + 2xy^3$$
$$+ y^2x^2 + 2xy^3 + y^4$$
$$= x^4 + y^4 + 4x^3y + 6x^2y^2 + 4xy^3$$

$$4x^3y + 6x^2y^2 + 4xy^3 = 0$$

$$xy(4x^2 + 6xy + 4y^2) = 0$$

$$x=0$$
 or  $y=0$  or  $x=0, y=0$

16) d)  $(x+y)^5 = x^5 + y^5$  when?

$$(x+y)^5 = (x+y)^4(x+y)$$

$$(x^4 + y^4 + 4x^3y + 6x^2y^2 + 4xy^3)(x+y)$$

$$x^5 + y^5 + 4x^4y + 6x^3y^2 + 4x^2y^3 + 4xy^4 + y^5$$

$$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$= x^5 + y^5 + 5xy(y^3 + 2x^2y + 2x^3 + x^2y^2)$$

$$5xy(x^3 + 2x^2y + 2xy^2 + y^3) = 0$$

$$x=0 \text{ or } y=0 \text{ or } x^3 + 2x^2y + 2xy^2 + y^3 = 0$$

$$x^3 + 3x^2y + 3xy^2 + y^3 - x^2y - xy^2 = 0$$

$$(x+y)^3 = x^2y + xy^2 = xy(x+y)$$

which is true if  $x+y=0 \Rightarrow x=-y$

or if  $xy \neq 0$ ,  $(x+y)^3 = xy$

$\Rightarrow x^2 + xy + y^2 = 0$  but from (1) this expression is  $> 0$   
unless  $x=y=0$

so,  $x=-y=0$  or  $x=-y$  or  $x=0$  or  $y=0$

$\Rightarrow x=0$  or  $y=0$  or  $x=-y$

We can now guess as to when  $(x+y)^n = x^n + y^n$   
so far we have

$$(x+y)^2 = x^2 + y^2 \text{ if } x=0 \text{ or } y=0$$

$$(x+y)^3 = x^3 + y^3 \text{ if } x=0 \text{ or } y=0 \text{ or } x=-y$$

$$(x+y)^4 = x^4 + y^4 \text{ if } x=0 \text{ or } y=0$$

$$(x+y)^5 = x^5 + y^5 \text{ if } x=0 \text{ or } y=0 \text{ or } x=-y$$

so guess,  $(x+y)^n = x^n + y^n$  when  
it's even,  $x$  &  $y$  is zero

it's odd,  $x$  &  $y$  is zero or  $x=-y$

17) a) smallest possible  $2x^2 - 3x + 4$

$$2(x^2 - 1.5x + 2)$$

$$2(x^2 - 2 \cdot 1 \cdot 0.75x + 0.75^2) + 46/16$$

$$2(x^2 - 2 \cdot 1 \cdot 0.75x + 0.75^2) + 46/16$$

$$2(x - 3/4)^2 + 23/16$$

$x = 3/4$  gives smallest value of  $23/16$

b)  $x^2 - 3x + 2y^2 + 4y + 2$

$$(x^2 - 2 \cdot 1 \cdot 1.5x + 1.5^2) - 1.5^2 + 2(y^2 + 2 \cdot 1 \cdot y + 1)$$

$$(x - 1.5)^2 + 2(y + 1)^2 - 1.5^2$$

$x = 1.5, y = -1$ , smallest value is  $-1.5^2$

c)  $x^2 + 4xy + 5y^2 - 4x - 6y + 7$

$$x^2 + 4x(y - 1) + 5y^2 - 6y + 7$$

$$[x^2 + 4x(y - 1) + 4(y - 1)^2]$$

$$+ 5y^2 - 6y + 7 - 4(y - 1)^2$$

$$(x + 2(y - 1))^2 + \underbrace{5y^2 - 6y + 7 - 4(y - 1)^2}_{y^2 + 2y + 3} - 4y^2 + 8y - 4$$

$$y^2 + 2y + 3 \Rightarrow y^2 + 2y + 1 + 2 \Rightarrow (y + 1)^2 + 2$$

$$(x + 2(y - 1))^2 + (y + 1)^2 + 2$$

smallest value is 2 when  $y = -1$   
and  $(x + (-1))^2 \geq 0 \Rightarrow x = 0$

18) a)  $b^2 - 4ac \geq 0$

$$x^2 + bx + c = 0$$

$$\text{i)} \left( \frac{-b + \sqrt{b^2 - 4c}}{2} \right)^2 + b \left( \frac{-b + \sqrt{b^2 - 4c}}{2} \right) + c$$

$$= (b^2 - 4c - 2b\sqrt{b^2 - 4c} + b^2)/4 + (-b^2 + b\sqrt{b^2 - 4c} + 2c)/2$$
  
~~= [b<sup>2</sup> - 4c - 2b<sup>2</sup> $\cancel{\sqrt{b^2 - 4c}} + b^2 + 2b<sup>2</sup> $\cancel{\sqrt{b^2 - 4c}} + 4c]/4$$~~   
 $= 0$

ii) For  $(-b - \sqrt{b^2 - 4c})/2$  analogous.

18) b)  $b^2 - 4c < 0$

$$x^2 + bx + c > 0$$

$$x^2 + bx + 0.25b^2 + c - 0.25b^2 \\ \underbrace{(x + 0.5b)^2}_{> 0} - \underbrace{(b^2 - 4c)/4}_{> 0}$$

c)  $x^2 + xy + y^2 > 0$  if  $x \neq 0, y \neq 0$

$\forall y \neq 0, y^2 - 4y^2 = -3y^2 < 0$   
so, from c),  $x^2 + xy + y^2 > 0 \quad \forall x$   
analogous to show  $\forall x \neq 0, x^2 + xy + y^2 > 0 \quad \forall y$

d)  $x^2 + \alpha xy + y^2 > 0$  whenever  $x \neq 0$  or  $y \neq 0$

if  $y = 0$ , then  $\forall x \neq 0$  satisfies the inequality.  
 $\forall y \neq 0, \alpha^2 y^2 - 4y^2 = y^2(\alpha^2 - 4) = b^2 - 4c$

if  $y^2(\alpha^2 - 4) < 0$  then  $x^2 + \alpha xy + y^2 > 0$   
 $\Rightarrow \alpha^2 - 4 < 0 \Rightarrow \alpha^2 < 4 \Rightarrow \alpha < 2, \alpha > -2$

$\Rightarrow \forall y \neq 0$ , if  $\alpha \in ]-2, 2[$  then ineq. is  $> 0 \quad \forall x$

analogously we reach,

$\forall x \neq 0$  if  $\alpha \in ]-2, 2[$ , ineq. is  $> 0$  for  $y$

e)  $x^2 + bx + c, cx^2 + bx + c, c > 0$

$$\text{i)} x^2 + bx + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2$$

$$\underbrace{(x + \frac{b}{2})^2}_{\geq 0} + (c - \left(\frac{b}{2}\right)^2)$$

if  $x = -b/2$ , the value of the expr. is  $c - \frac{b^2}{4}$

$$\text{ii)} \left(cx^2 + 2 \cdot \sqrt{c} \cdot x \cdot \frac{b}{2\sqrt{c}} + \left(\frac{b}{2\sqrt{c}}\right)^2\right) + c - \left(\frac{b}{2\sqrt{c}}\right)^2$$

$$\left(\sqrt{c}x + \frac{b}{2\sqrt{c}}\right)^2 + c - \left(\frac{b}{2\sqrt{c}}\right)^2$$

so,  $x = -\frac{b}{2c} \Rightarrow \min \text{ value of } \frac{-b^2 + 4ac}{4a}$

## Recap of Derivations related to quadratic expression

given  $ax^2 + bx + c$

we'd like to find analytical expressions for value of  $x$  when the expression is zero (roots), and when the expression has min value

First, we factorize

$$ax^2 + 2\sqrt{a}x \cdot \frac{b}{2\sqrt{a}} + \left(\frac{b}{2\sqrt{a}}\right)^2 + c - \left(\frac{b}{2\sqrt{a}}\right)^2$$

$$\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 + c - \left(\frac{b}{2\sqrt{a}}\right)^2$$

set the expression to zero to find roots

$$\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 = \frac{b^2}{4a} - c = \frac{b^2 - 4ac}{4a}$$

$$\frac{2ax + b}{2\sqrt{a}} = \pm \frac{\sqrt{b^2 - 4ac}}{2\sqrt{a}}, b^2 - 4ac \geq 0$$

$$2ax + b = \pm \sqrt{b^2 - 4ac}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

note the assumptions required to derive this:  $a > 0, b^2 - 4ac \geq 0$

we know more about  $b^2 - 4ac$  by looking at:

$$\underbrace{\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2}_{\geq 0} + c - \underbrace{\left(\frac{b}{2\sqrt{a}}\right)^2}_{\frac{4ac - b^2}{4a} \geq 0} = 4ac - b^2 \geq 0$$

so, if  $b^2 - 4ac < 0$  (with  $a > 0$ ),  
 $ax^2 + bx + c > 0$

if  $b^2 - 4ac \geq 0$ , we have at least one root.

finally, notice that the minimum value for  $\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2$  is zero. This happens when  $\sqrt{a}x = -b/2\sqrt{a} \Rightarrow x = -\frac{b}{2a}$ . The expression has its minimum value at that value of  $x$ , that minimum is  $c - \left(\frac{b}{2\sqrt{a}}\right)^2 = \frac{-b^2 + 4ac}{4a}$

\* Note alternative Factorization

$$cx^2 + bx + c =$$

$$= a \left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

i.e. it has min  $x$  as  $C - \frac{b^2}{4}$

$$= \frac{c}{a} - \frac{b^2}{a^2 \cdot 4} \Rightarrow$$

min for whole expr. is

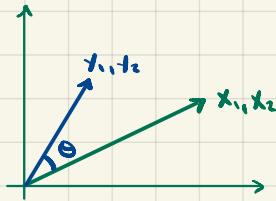
$$a \left(\frac{c}{a} - \frac{b^2}{4a^2}\right) = c - \frac{b^2}{4a}, \text{ same}$$

as definitive defn. of this at bottom of page.

$$19. \text{ Schwarz Inequality } x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}$$

$$\begin{aligned} \text{a) } x_1 &= \lambda y_1 \\ x_2 &= \lambda y_2 \\ \lambda &\geq 0 \end{aligned} \Rightarrow x_1y_1 + x_2y_2 = \sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}$$

vector interpretation



$$\begin{aligned} \langle x_1, y_1 \rangle &= \langle x_2, y_2 \rangle = |\vec{x}| |\vec{y}| \cos \theta \\ \cos \theta < 1 &\Rightarrow x_1y_1 + x_2y_2 \leq |\vec{x}| |\vec{y}| \end{aligned}$$

$$\text{ii) } x_1y_1 + x_2y_2 = \lambda y_1^2 + \lambda y_2^2 = \lambda(y_1^2 + y_2^2) = \lambda \sqrt{y_1^2 + y_2^2} \sqrt{y_1^2 + y_2^2} = \sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}$$

$$x_1 = \lambda y_1 \Rightarrow y_1 = \frac{x_1}{\lambda}$$

$$\sqrt{x_1^2 + x_2^2} = \lambda \sqrt{y_1^2 + y_2^2}$$

$$\text{iii) } y_1 = y_2 = 0 \Rightarrow x_1y_1 + x_2y_2 = 0 = 0 \sqrt{x_1^2 + x_2^2}$$

$$\text{iii) } \# \lambda \text{ s.t. } x_1 = \lambda y_1, x_2 = \lambda y_2$$

vector interpretation

i.e. for  $\vec{x} = \langle x_1, y_1 \rangle, \vec{y} = \langle x_2, y_2 \rangle, c_1 \vec{x} + c_2 \vec{y} = 0 = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ . For there to be non-trivial sol'n, determinant must be 0.  $\Rightarrow x_1y_2 - x_2y_1 = 0 \Rightarrow \frac{x_1}{y_1} = \frac{x_2}{y_2} = \lambda$ .

$\Rightarrow$  Since we assume there is no such  $\lambda$ , the only soln is  $(0, 0)$ .  $\Rightarrow \vec{x}, \vec{y}$  linearly indep.

For some  $\lambda$ ,  $(\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2 = \lambda^2(y_1^2 + y_2^2) - 2\lambda(x_1y_1 + x_2y_2) + (x_1^2 + x_2^2) \geq 0$ , a second degree poly. in  $\lambda$ , given  $x_1, y_1, x_2, y_2$ . The inequality represents all the  $x_1, x_2, y_1, y_2$  (i.e. vectors  $\vec{x}$  and  $\vec{y}$ ) we can choose such that they're lin. indep. Note if we instead choose  $x_1 = \lambda y_1$  and  $x_2 = \lambda y_2$ , then  $\vec{x}$  and  $\vec{y}$  are lin. dep. and the eq. has roots, i.e. the inequality equality.

$$\Delta = b^2 - 4ac = 4(x_1y_1 + x_2y_2)^2 - 4(y_1^2 + y_2^2)(x_1^2 + x_2^2) < 0$$

$$\Rightarrow (x_1y_1 + x_2y_2)^2 < (y_1^2 + y_2^2)(x_1^2 + x_2^2)$$

$$\Rightarrow x_1y_1 + x_2y_2 < \sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}$$

$$\text{Note } |\lambda \langle y_1, y_2 \rangle - \langle x_1, x_2 \rangle|^2 = \langle \lambda y_1 - x_1, \lambda y_2 - x_2 \rangle \cdot \langle \lambda y_1 - x_1, \lambda y_2 - x_2 \rangle$$

$$= (\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2 > 0 \Rightarrow \text{the norm of a linear combination of } \vec{x} \text{ and } \vec{y} \text{ is } > 0, \text{ no matter what } \lambda \neq 0 \text{ is. This is a way of saying they're lin. indep.}$$

b) Use  $2xy \leq x^2 + y^2$  with  $x = \frac{x_i}{\sqrt{x_i^2 + x_2^2}}$ ,  $y = \frac{y_i}{\sqrt{y_i^2 + y_2^2}}$  with  $i=1$  then  $i=2$ .

$$\frac{2x_1y_1}{\sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)}} \leq \frac{x_1^2}{x_1^2 + x_2^2} + \frac{y_1^2}{y_1^2 + y_2^2}$$

$$\frac{2x_2y_2}{\sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)}} \leq \frac{x_2^2}{x_1^2 + x_2^2} + \frac{y_2^2}{y_1^2 + y_2^2}$$

Add

$$\frac{2(x_1y_1 + x_2y_2)}{\sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)}} \leq 2 \Rightarrow x_1y_1 + x_2y_2 \leq \sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)}$$

Where does the initial inequality come from?  $2xy \leq x^2 + y^2$

$$(x-y)^2 = x^2 - 2xy + y^2 \geq 0 \Rightarrow 2xy \leq x^2 + y^2$$

$$2 \frac{\langle x_1, x_2 \rangle}{\sqrt{x_1^2 + x_2^2}} \cdot \frac{\langle y_1, y_2 \rangle}{\sqrt{y_1^2 + y_2^2}} \leq \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2} + \frac{y_1^2 + y_2^2}{y_1^2 + y_2^2}$$

$$\Rightarrow \frac{2(x_1y_1 + x_2y_2)}{\sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)}} \leq 2$$

c) Prove  $(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2$

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = x_1^2y_1^2 + x_1^2y_2^2 + x_2^2y_1^2 + x_2^2y_2^2$$

$$= (x_1^2y_1^2 + 2x_1x_2y_1y_2 + x_2^2y_2^2) + (x_1^2y_2^2 - 2x_1x_2y_1y_2 + x_2^2y_1^2)$$

$$= (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2 \geq (x_1y_1 + x_2y_2)^2$$

Take square root  $\Rightarrow x_1y_1 + x_2y_2 \leq \sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)}$

Note that we've proved this fact regarding Schwarz Inequality:

$\rightarrow \vec{x} \cdot \lambda \vec{y} \Rightarrow$  equality holds in Schwarz Ineq.

$\rightarrow$  if one vector is 0, holds trivially.

1) if  $\vec{x}$  and  $\vec{y}$  l.i. then strict inequality.

2) proof starting at  $(x-y)^2 \geq 0$ ,  $x$  and  $y$  unit vectors

$$3) " " "(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2$$

Let's deduce the equality case from each proof

$$1) (\lambda x_1 - x_1)^2 + (\lambda y_2 - x_2)^2 = 0$$

$$\Rightarrow x_1y_1 + x_2y_2 = \sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}$$

ie there is a  $\lambda$  such that  $\vec{x} = \langle x_1, x_2 \rangle$  and  $\vec{y} = \langle y_1, y_2 \rangle$  have a lin. comb. = 0, with  $\lambda \neq 0$ .

What sort of vectors  $\vec{x}$  and  $\vec{y}$  are these? The ones for which Schwarz equality holds.

2)  $(\vec{x} - \vec{y})^2 = 0 \Rightarrow \vec{x} = \vec{y}$ , and Schwarz Equality

$\vec{x} = \vec{y}$  means we have the same unit vector

$$\text{But } \frac{x_1}{\sqrt{x_1^2 + x_2^2}} = \frac{y_1}{\sqrt{y_1^2 + y_2^2}} \Rightarrow x_1 = \lambda y_1$$

$$\lambda = \sqrt{\frac{x_1^2 + x_2^2}{y_1^2 + y_2^2}}$$

$$3) (x_1^2 + x_2^2)(y_1^2 + y_2^2) =$$

$$(x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2 =$$

$$= (x_1y_1 + x_2y_2)^2$$

$$\Rightarrow (x_1y_2 - x_2y_1)^2 = 0$$

$$\Rightarrow \frac{x_1}{y_1} = \frac{x_2}{y_2} = \lambda$$

$$b) 2xy \leq x^2 + y^2$$

d) equality of Schurz Ineq.

$$x = x_1 / \sqrt{x_1^2 + x_2^2} \quad y = y_1 / \sqrt{y_1^2 + y_2^2}$$

ii) we know that  $x^2 + 2xy + y^2 \geq 0$  because  $(x-y)^2 \geq 0$

$$i=1 \Rightarrow 2 \frac{x_1 y_1}{\sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}} \leq \frac{x_1^2}{x_1^2 + x_2^2} + \frac{y_1^2}{y_1^2 + y_2^2}$$

$$i=2 \Rightarrow 2 \frac{x_2 y_2}{\sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}} \leq \frac{x_2^2}{x_1^2 + x_2^2} + \frac{y_2^2}{y_1^2 + y_2^2}$$

Adding the two inequalities:

$$\frac{2x_1 y_1 + 2x_2 y_2}{\sqrt{x_1^2 + x_2^2} \cdot \sqrt{y_1^2 + y_2^2}} \leq \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2} + \frac{y_1^2 + y_2^2}{y_1^2 + y_2^2} = 2$$

$$\Rightarrow \frac{2(x_1 y_1 + x_2 y_2)}{\sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}} \leq 2$$

$$\Rightarrow x_1 y_1 + x_2 y_2 \leq \sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}$$

$$c) (x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1 y_1 + x_2 y_2)^2 + (x_1 y_2 - x_2 y_1)^2$$

$$x_1^2 y_1^2 + x_1^2 y_2^2 + x_2^2 y_1^2 + x_2^2 y_2^2$$

$$(x_1 y_1)^2 + 2x_1 y_1 x_2 y_2 + (x_2 y_2)^2 + x_1^2 y_2^2 + x_2^2 y_1^2 - 2x_1 y_1 x_2 y_2$$

$$(x_1 y_1 + x_2 y_2)^2 + (x_1 y_2 - x_2 y_1)^2 - 2x_1 y_1 x_2 y_2 + (x_2 y_1)^2$$

$$= (x_1 y_1 + x_2 y_2)^2 + (x_1 y_2 - x_2 y_1)^2 \geq (x_1 y_1 + x_2 y_2)^2$$

therefore, taking  $\sqrt{\cdot}$ :

$$\sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)} \geq x_1 y_1 + x_2 y_2$$

i) in c), we assumed  $c \lambda$  s.t.  $x_1 = \lambda y_1, x_2 = \lambda y_2$  and get equality i.  
also if  $y_1 = y_2 = 0$ , we get  $0 = 0 \forall x_1, x_2$

ii) we started from a known relationship between numbers:  $x^2 - 2xy + y^2 \geq 0$   
we assumed analytic values for  $x_1, y_1, y_2$  and get the inequality.

we could start with  $x^2 - 2xy + y^2 = 0$  and get the eq. in the Schurz Ineq. if we have  $(x-y)^2 = 0$   
 $\Rightarrow x = y$ . In this case  $\frac{x_1}{\sqrt{x_1^2 + x_2^2}} = \frac{y_1}{\sqrt{y_1^2 + y_2^2}}$

But this means our assumed values imply that  $x_i = \lambda y_i$  where

$$\lambda = \left[ \frac{(x_1^2 + x_2^2)}{(y_1^2 + y_2^2)} \right]^{1/2}$$

Finally,  $y_1 = y_2 = 0$  means that

$x_1 = y_1 = 0$  so we get the equality derived from the initial expr. of cell.

iii) In c) we started with  $(x_1^2 + x_2^2)(y_1^2 + y_2^2) = \dots$

$$= (x_1 y_1 + x_2 y_2)^2 + (x_1 y_2 - x_2 y_1)^2$$

If  $x_1 y_2 - x_2 y_1 = 0$  we obtain Schurz Equality.

This happens if  $y_1 = -y_2 = 0$   $\forall x_1, x_2$  or if:

$$1) y_1 \neq 0 \Rightarrow x_2 = \left( \frac{x_1}{y_1} \right) y_2$$

$$x_1 = \frac{x_2 y_1}{y_2} = \left( \frac{x_1}{y_1} \right) y_1, \lambda = x_1/y_1$$

$$2) y_2 \neq 0 \Rightarrow x_1 = \left( \frac{x_2}{y_2} \right) y_1$$

$$x_2 = \frac{x_1}{y_1} y_2 = \frac{x_2}{y_2} y_2, \lambda = x_2/y_2$$

## 20. Prove

$$|x - x_0| < \frac{\epsilon}{2}$$

$$|f - f_0| < \frac{\epsilon}{2}$$

$$|(x+y) - (x_0 + y_0)| < \epsilon$$

$$\Rightarrow |(x-f) - (x_0 - f_0)| < \epsilon$$

$$x \leq |x|, |y| \leq |y| \Rightarrow |x+y| \leq ||x|+|y|| = |x|+|y|$$

$$|(x+y) - (x_0 + y_0)| = |(x-x_0) + (f-f_0)|$$

$$\leq |x-x_0| + |f-f_0|$$

$$= \epsilon$$

$$|x-f| \cdot |x+f| \leq ||x|+|f|| = |x|+|f|$$

$$|(x-f) - (x_0 - f_0)| = |(x-x_0) - (f-f_0)|$$

$$\leq |x-x_0| + |f-f_0|$$

$$= \epsilon$$

## 21. Prove

$$|x-x_0| < \min\left(\frac{\epsilon}{2(|f_0|+1)}, 1\right)$$

$$\Rightarrow |x-f| < \epsilon$$

$$|f-f_0| < \frac{\epsilon}{2(|x_0|+1)}$$

$$|x| - |x_0| \leq |x-x_0| < 1$$

$$\Rightarrow |x| < 1 + |x_0| \Rightarrow |f-f_0| < \frac{\epsilon}{2|x|}$$

$$|x-x_0| < \frac{\epsilon}{2(|f_0|+1)}$$

$$(x-x_0)(f-f_0)$$

$$= x-f - x_0f + x_0f_0$$

$$= (x-f - x_0f_0) + 2x_0f_0 - x_0f - x_0f_0$$

$$\Rightarrow x-f - x_0f_0$$

$$= (x-x_0)(f-f_0) + x_0f + f-f_0 - 2x_0f_0$$

$$= x(f-f_0) - x_0(f-f_0) + \cancel{x_0f} + f-f_0 - 2x_0f_0$$

$$= x(f-f_0) + f_0(x-x_0)$$

$$|xf - x_0f_0| = |x(f-f_0) + f_0(x-x_0)|$$

$$\leq |x| |f-f_0| + |f_0| |x-x_0|$$

$$\leq |x| \frac{\epsilon}{2|x|} + \frac{|f_0|}{|x|} \frac{\epsilon}{2}$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$20) |x - x_0| < \frac{\epsilon}{2}, |y - y_0| < \frac{\epsilon}{2} \Rightarrow |(x+y) - (x_0 + y_0)| < \epsilon$$

$$|(x-y) - (x_0 - y_0)| < \epsilon$$

$$|(x+y) - (x_0 + y_0)|$$

$$= |(x-x_0) + (y-y_0)| \leq |x-x_0| + |y-y_0| < \epsilon$$

$$|(x-y) - (x_0 - y_0)|$$

$$\cdot |(x-x_0) - (y-y_0)| = |(y-x_0) + (y_0 - y)| \leq |x-x_0| + |y_0 - y| = |x-x_0| + |y-y_0| < \epsilon$$

$$|a| = |-a|$$



$$21) |x - x_0| < \min\left(\frac{\epsilon}{2(|y_0|+1)}, 1\right) \Rightarrow |xy - x_0 y_0| < \epsilon$$

$$|y - y_0| < \frac{\epsilon}{2(|x_0|+1)}$$

$$|x| - |x_0| \leq |x - x_0| < 1 \Rightarrow |x| \leq 1 + |x_0|$$

$$|y - y_0| < \frac{\epsilon}{2|x|} \Rightarrow |y - y_0||x| < \frac{\epsilon}{2} \quad (1)$$

$$|x - x_0| < \frac{\epsilon}{2(|y_0|+1)} \Rightarrow |x - x_0|(|y_0|+1) < \frac{\epsilon}{2} \quad (2)$$

sum (1) and (2)

$$|y - y_0||x| + |x - x_0|(|y_0|+1) < \epsilon$$

$$|xy - x_0 y_0 + x_0 y_0 - x_0 y_0| + |x - x_0| < \epsilon$$

$$|xy - x_0 y_0| < \epsilon$$

Alternative solution

$$|x| - |x_0| \leq |x - x_0| < 1 \Rightarrow |x| < 1 + |x_0|$$

$$|xy - x_0 y_0| = |x(y - y_0) + y_0(x - x_0)| \leq |x||y - y_0| + |y_0||x - x_0|$$

$$< |x| \cdot \frac{\epsilon}{2|x|} + |y_0| \cdot \frac{\epsilon}{2(|y_0|+1)} \Rightarrow |xy - x_0 y_0| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\underbrace{\frac{|y_0|}{|y_0|+1} \cdot \frac{\epsilon}{2}}_{< \frac{\epsilon}{2}} < \frac{\epsilon}{2}$$

$$22) \quad y_0 \neq 0 \quad |y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\epsilon |y_0|^2}{2}\right) \Rightarrow y \neq 0 \text{ and } \left|\frac{1}{y} - \frac{1}{y_0}\right| < \epsilon$$

$$|y_0| = |y + (y_0 - y)| \leq |y| + |y_0 - y|$$

$$|y - y_0| < \frac{|y_0|}{2} \Rightarrow |y_0| < |y| + \frac{|y-y_0|}{2} \Rightarrow |y_0|/2 < |y|$$

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| = \left|\frac{y_0 - y}{y y_0}\right| = \left|\frac{y - y_0}{y y_0}\right| = \frac{|y - y_0|}{|y||y_0|} < \frac{|y - y_0|}{|y_0|^2/2}$$

$$< \frac{\epsilon |y_0|^2}{2} / |y_0|^2/2 = \epsilon$$

$$23) \quad y_0 \neq 0 \quad \begin{matrix} y \neq 0 \\ |y - y_0| < ? \end{matrix}$$

$$|x - x_0| < ? \Rightarrow \left|\frac{x}{y} - \frac{x_0}{y_0}\right| < \epsilon$$

$$|x - x_0| < ?$$

From 21 we have:

$$\left|x \cdot \frac{1}{y} - x_0 \cdot \frac{1}{y_0}\right| < \epsilon \quad \text{if } |x - x_0| < \min\left(\frac{\epsilon}{2(|\frac{1}{y_0}|+1)}, 1\right)$$

$$|y|, |y + y_0| < \frac{\epsilon}{2(|y_0|+1)}$$

From 22 we have:

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| < \frac{\epsilon}{2(|y_0|+1)} \quad \text{if } |y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\epsilon}{2(|y_0|+1)} \cdot \frac{|y_0|^2}{2}\right) \\ = \min\left(\frac{|y_0|}{2}, \frac{\epsilon |y_0|^2}{4(|y_0|+1)}\right)$$

## Recap Problems 21-23

$$\rightarrow xy - x_0 y_0 = x(y - y_0) + y_0(x - x_0)$$

$$\rightarrow y_0 = y_0 + (y - y_0)$$

2S)

number: 0 or 1

operations

$$+ \begin{matrix} & b \\ & 0 \\ & 1 \\ a & 0 \\ 1 & 1 \\ \end{matrix}$$

$$\cdot \begin{matrix} & b \\ & 0 \\ & 1 \\ a & 0 \\ 1 & 0 \\ \end{matrix}$$

P1)

$$(a+b)+c = a+(b+c)$$

$$\begin{array}{c} a+b \\ + \begin{matrix} & 0 & 1 \\ t_{00} & 0 & 0 \\ t_{01} & 1 & 0 \\ t_{10} & 1 & 1 \\ t_{11} & 0 & 1 \\ \end{matrix} \\ \hline \begin{matrix} & 0 & 1 \\ & 0 & 1 \\ & 1 & 0 \\ & 1 & 1 \\ \end{matrix} \end{array} \quad \begin{array}{c} a \\ + \begin{matrix} & 0 & 1 \\ t_{00} & 0 & 0 \\ t_{01} & 1 & 0 \\ t_{10} & 1 & 1 \\ t_{11} & 0 & 1 \\ \end{matrix} \\ \hline \begin{matrix} & 0 & 1 \\ & 0 & 1 \\ & 1 & 0 \\ & 1 & 1 \\ \end{matrix} \end{array}$$

P2)

$$a+0 = 0+a = a$$

$$\begin{matrix} a \\ + \begin{matrix} & 0 & 1 \\ 0 & 0 & 1 \\ \end{matrix} \\ \hline \begin{matrix} & 0 & 1 \\ & 0 & 1 \\ & 1 & 0 \\ \end{matrix} \end{array}$$

P3)  $a+(-a) = (-a)+a = 0$

$$a=0, -a=0$$

$$a=1, -a=1$$

P4)  $0+0 = 0$

$$1+0 = 0+1 = 1$$

$$1+1 = 0$$

P5)

$$\begin{matrix} ab \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \end{matrix} \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{matrix} \begin{matrix} c \\ 0 \\ 0 \\ 0 \\ -1 \end{matrix}$$

$(ab) \cdot c$ ,  $c(ba)$  is the same: replace  $ab$  with  $ba$ ,  $c$  with  $a$

P6)  $a \cdot 1 = 1 \cdot a = a$

$$\begin{matrix} a \\ 1 \end{matrix} \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$$

P7)  $a \cdot c^{-1} = c^{-1} \cdot a = 1, \forall c \neq 0$

$$a \cdot 1 \quad c^{-1} = 1$$

P8)  $a \cdot b = b \cdot a \quad \forall a, b$

$$0 \cdot 0 = 0$$

$$0 \cdot 1 = 1 \cdot 0 = 0$$

$$1 \cdot 1 = 1$$

P9)  $a(b+c) = ab+ac$

$$\begin{matrix} a \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 0 \\ -1 \end{matrix} \begin{matrix} b \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ -1 \\ -1 \end{matrix} \begin{matrix} c \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ -1 \end{matrix} \begin{matrix} c(b+c) \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \end{matrix} \begin{matrix} ab+ac \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \end{matrix}$$