Ch. 5 Limits

Provisional Def: Fn f approaches limit l near a, if we can make $f(x)$ as close as we like to l by requiring that x be sufficiently close to, but unequal to, a.

example

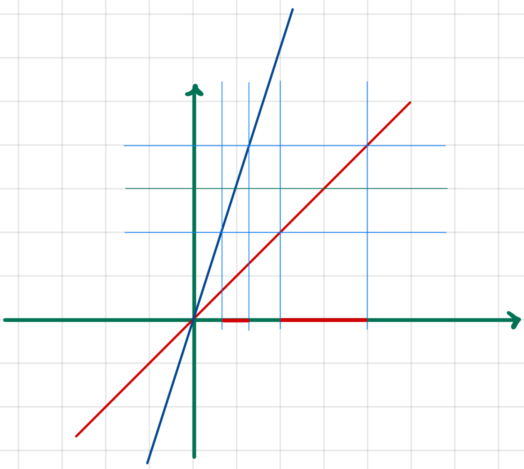
$f(x) = 3x$

$|3x - l| < \epsilon \Rightarrow -\epsilon < 3x - l < \epsilon$

$\Rightarrow l - \epsilon < 3x < l + \epsilon$

$\Rightarrow \frac{l-\epsilon}{3} < x < \frac{l+\epsilon}{3}$

equivalently, $-\frac{\epsilon}{3} < x - \frac{l}{3} < \frac{\epsilon}{3}$



$l = 15, \epsilon = 0.1 \Rightarrow 5 - \frac{1}{30} < x < 5 + \frac{1}{30}$

$\Rightarrow -\frac{1}{30} < x - 5 < \frac{1}{30}$

If $f(x) = 3 \cdot 10^6 x$ then

$l - \epsilon < 3 \cdot 10^6 x < l + \epsilon$

$-\frac{\epsilon}{3 \cdot 10^6} < x - \frac{l}{3 \cdot 10^6} < \frac{\epsilon}{3 \cdot 10^6}$

$f(x) = x^2$

$|x^2 - 9| < \epsilon$

$|x-3||x+3| < \epsilon$

Assume $|x-3| < 1 \Rightarrow -1 < x - 3 < 1 \Rightarrow 2 < x < 4$

$\Rightarrow 5 < x + 3 < 7 \Rightarrow |x+3| < 7$

$\Rightarrow |x^2 - 9| = |x-3||x+3| < 7|x-3| < \epsilon$

$\Rightarrow |x-3| < \frac{\epsilon}{7}$ provided $|x-3| < 1$

Let's do these calc. again using $|x-3| < 10$

$\Rightarrow -10 < x - 3 < 10 \Rightarrow -7 < x < 13 \Rightarrow -4 < x + 3 < 16$

$\Rightarrow |x+3| < 16$

$\Rightarrow |x^2 - 9| = |x-3||x+3| < 16|x-3| < \epsilon$

$\Rightarrow |x-3| < \frac{\epsilon}{16}$

more general argument to show $f(x) = x^2$ approaches a^2 near $x=a$ for any a.

Assume $|x-a|<1$

problem 1-12 $\Rightarrow |x|-|a| \le |x-a| < 1$

$\Rightarrow |x| < 1+|a|$

$\Rightarrow |x+a| \le |x|+|a| < 2|a|+1$

$*$ $x \le |x|$, $a \le |a|$ $\Rightarrow |x+a| \le ||x|+|a|| = |x|+|a|$

$\Rightarrow |x^2-a^2| = |x-a||x+a| < |x-a|(2|a|+1)$

$\Rightarrow |x^2-a^2|<\epsilon$ for $|x-a|(2|a|+1)<\epsilon$

$\Rightarrow |x-a| < \dfrac{\epsilon}{2|a|+1}$ provided that $|x-a|<1$

$\Rightarrow |x-a| < \min\left(1, \dfrac{\epsilon}{2|a|+1}\right)$

Consider $f(x) = \dfrac{1}{x}$, $x \neq 0$

$\left|\dfrac{1}{x} - \dfrac{1}{3}\right| = \left|\dfrac{3-x}{3x}\right| = \dfrac{1}{3}\cdot\dfrac{1}{|x|}\cdot|x-3|$

Assume $|x-3|<1 \Rightarrow 2<x<4 \Rightarrow \dfrac{1}{4} < \dfrac{1}{x} < \dfrac{1}{2}$

$\Rightarrow x>0 \Rightarrow \dfrac{1}{|x|} < \dfrac{1}{2}$

$\Rightarrow \left|\dfrac{1}{x}-\dfrac{1}{3}\right| < \dfrac{1}{6}|x-3| < \epsilon$

$\left|\dfrac{1}{x} - \dfrac{1}{3}\right|<\epsilon \Rightarrow |x-3|<6\epsilon$ provided $|x-3|<1$

$|x-3| < \min(1,6\epsilon)$

Consider $f(x)=\dfrac{1}{x}$ and the limit $\ell = -\dfrac{1}{3}$.

$\left|\dfrac{1}{x} - \left(-\dfrac{1}{3}\right)\right| < \epsilon$

$\Rightarrow \left|\dfrac{1}{x} + \dfrac{1}{3}\right| = \left|\dfrac{3+x}{3x}\right| = \dfrac{1}{3}\cdot\dfrac{1}{|x|}\cdot|x+3| < \epsilon$

Assume $|x+3|<1 \Rightarrow -1<x+3<1 \Rightarrow -4<x<-2$

$\Rightarrow -\dfrac{1}{2} < \dfrac{1}{x} < -\dfrac{1}{4} \Rightarrow \dfrac{1}{|x|} < \dfrac{1}{2}$

$\Rightarrow \left|\dfrac{1}{x}+\dfrac{1}{3}\right| < \dfrac{1}{3}\cdot\dfrac{1}{2}\cdot|x+3| < \epsilon$

$\Rightarrow |x+3| < 6\epsilon$ provided $|x+3|<1$

in general

$\left|\dfrac{1}{x} - \dfrac{1}{a}\right| < \epsilon$

Say we were to assume $|x-a|<1$ but $|a|<1$.

$\Rightarrow$ $-1<x-a<1$, $-1<a<1$ $\Rightarrow a-1<x<a+1 \Rightarrow c<0<x<d$

But $\dfrac{1}{x}$ is not defined at $x=0$, so we wouldn't be able to say that $\left|\dfrac{1}{x} - \dfrac{1}{a}\right|<\epsilon$ whenever $|x-a|<\delta$ for $\delta>a$.

Instead we assume $|x-a| < \dfrac{|a|}{2}$

$\Rightarrow -\dfrac{|a|}{2} < x-a < \dfrac{|a|}{2}$

$a - \dfrac{|a|}{2} < x < a + \dfrac{|a|}{2}$

$a>0 \Rightarrow a=|a| \Rightarrow a - \dfrac{|a|}{2} = |a| - \dfrac{|a|}{2} = \dfrac{|a|}{2}$

Similarly, $a + \dfrac{|a|}{2} = \dfrac{3|a|}{2}$

$\Rightarrow \dfrac{|a|}{2} < x < \dfrac{3|a|}{2} \Rightarrow x\neq 0$

$a<0 \Rightarrow |a| = -a \Rightarrow a = -|a|$

$a - \dfrac{|a|}{2} = -|a| - \dfrac{|a|}{2} = -\dfrac{3|a|}{2}$

$a + \dfrac{|a|}{2} = -|a| + \dfrac{|a|}{2} = -\dfrac{|a|}{2}$

$\Rightarrow -\dfrac{3|a|}{2} < x < -\dfrac{|a|}{2} \Rightarrow x\neq 0$

$a - \dfrac{|a|}{2} < x < a + \dfrac{|a|}{2} \Rightarrow \dfrac{2}{2a+|a|} < \dfrac{1}{x} < \dfrac{2}{2a-|a|}$

$a>0 \Rightarrow a=|a| \Rightarrow \dfrac{2}{3|a|} < \dfrac{1}{x} < \dfrac{2}{|a|} \Rightarrow \dfrac{1}{|x|} < \dfrac{2}{|a|}$

$a<0 \Rightarrow a=-|a| \Rightarrow \dfrac{2}{-|a|} < \dfrac{1}{x} < -\dfrac{2}{3|a|} < 0$

$\Rightarrow \dfrac{1}{|x|} < \dfrac{2}{|a|}$

$\left|\dfrac{1}{x} - \dfrac{1}{a}\right| = \left|\dfrac{a-x}{ax}\right| = \dfrac{1}{|a|}\cdot\dfrac{1}{|x|}|x-a| < \dfrac{2}{|a|^2}|x-a| < \epsilon$

$|x-a| < \dfrac{\epsilon}{2}a^2$

e.g. $a=3 \Rightarrow \left|\dfrac{1}{x} - \dfrac{1}{3}\right|<\epsilon \iff |x-3| < \dfrac{9\epsilon}{2} = 4.5\epsilon$

provided $|x-3| < \dfrac{3}{2} = 1.5$

$\epsilon < \dfrac{1}{3} \Rightarrow 4.5\epsilon < 1.5 \Rightarrow$ smaller interval than initially assumed for $|x-3|$

example $f(x) = \sqrt{|x|}\sin\left(\frac{1}{x}\right)$

$\left|\sqrt{|x|}\sin\left(\frac{1}{x}\right) - 0\right| < \epsilon \Rightarrow \left|\sqrt{|x|}\sin\left(\frac{1}{x}\right)\right| < \epsilon$

$\left|\sin\left(\frac{1}{x}\right)\right| \leq 1 \ \forall x \neq 0$ (and we are considering only x near 0, not equal to 0)

$\Rightarrow \left|\sqrt{|x|}\sin\left(\frac{1}{x}\right)\right| \leq \sqrt{|x|} \quad \forall x \neq 0$

$\sqrt{|x|} < \epsilon \ \forall x \neq 0 \Rightarrow \left|\sqrt{|x|}\sin\left(\frac{1}{x}\right)\right| < \epsilon$

$|x| < \epsilon^2 \ \forall x \neq 0 \Rightarrow \left|\sqrt{|x|}\sin\left(\frac{1}{x}\right)\right| < \epsilon$

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example $f(x) = \sin\left(\frac{1}{x}\right)$

$|f(x) - 0| < \epsilon = \left|\sin\left(\frac{1}{x}\right)\right| < \epsilon$

Let $\epsilon = \frac{1}{2}$

Consider any interval containing $x = 0$.

$x^* = \dfrac{1}{\frac{\pi}{2} + 2\pi n}$ is in the interval for some n.

$f(x^*) = f\left(\frac{\pi}{2} + 2\pi n\right) = 1 > \frac{1}{2}$.

Ie, it is not true that for any ϵ we can have

$\left|\sin\left(\frac{1}{x}\right)\right| < \epsilon$ if x is within some interval containing zero. As we showed above, $\epsilon = \frac{1}{2}$ is an example where this isn't possible

$\Rightarrow$ 0 is not the limit of f as x approaches 0.

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we can show that there is no limit at $x=0$, ie f does not approach any value as x approaches zero.

$|f(x) - \ell| < \epsilon \Rightarrow$

Let $\epsilon = \frac{1}{2} \Rightarrow |\sin(1/x) - \ell| < \frac{1}{2}$

$\Rightarrow \ell - \frac{1}{2} < \sin(1/x) < \ell + \frac{1}{2}$

In any interval containing $x=0$ we can find an x such that $f(x) = 1$, and another with $f(x) = -1$.

No matter which ℓ we choose, $|\sin(1/x) - \ell|$ will be at least 1 for one of the points. $1 > \epsilon = \frac{1}{2}$. Ie, it is impossible to get $\sin(1/x)$ arbitrarily close to ℓ by having x arbitrarily close to zero.

$\Rightarrow$ There is no limit of $\sin(1/x)$ at $x=0$.

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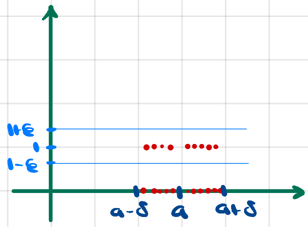
example: $f(x) = \begin{cases} 0, & x \in Q' \\ 1, & x \in Q \end{cases}$



no matter what a is we cannot get $f(x)$ as close to ℓ as we want simply by taking values from a small enough interval around a.

$\Rightarrow$ There is no limit at a.

Def: The fn f approaches the limit ℓ near a means: for every $\epsilon > 0$ there is some $\delta > 0$ such that $\forall x$ if $0 < |x - a| < \delta$ then $|f(x) - \ell| < \epsilon$.

* when f does not approach ℓ at a then the negation of the above definition reads:

there is some $\epsilon > 0$ such that for every $\delta > 0$ there is some x such that $0 < |x - a| < \delta$ but not $|f(x) - \ell| < \epsilon$.

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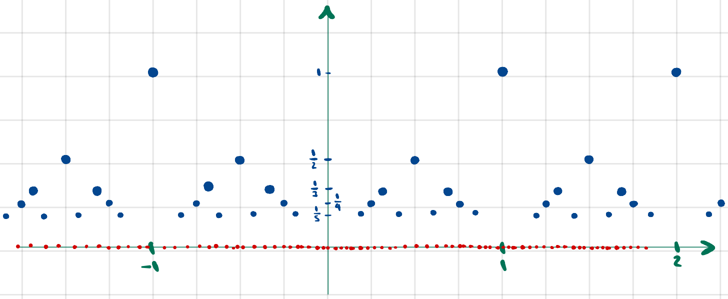
example

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q}' \\ 1/q & x = p/q \text{ in lowest terms } (p, q \in \mathbb{Z}, \text{ no common factor, } q > 0) \end{cases}$$



For any $0 < a < 1$, f approaches 0 at a

Let $\epsilon > 0$

Let $n \in \mathbb{N}$ be just large enough such that $\frac{1}{n} \leq \epsilon$
$m \in \mathbb{N}, m < n \Rightarrow \frac{1}{m} > \epsilon$

Consider the expression $|f(x) - 0| < \epsilon$

$x \in \mathbb{Q}' \Rightarrow |f(x) - 0| = 0 < \epsilon$ is true.

$|f(1/n)| = 1/n \leq \epsilon$ by definition.

$m < n \Rightarrow f(1/m) > \epsilon \Rightarrow |f(1/m)| > \epsilon$

There are a limited number of such rational numbers.
$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \dots, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}$

One of these rational numbers is closest to a, ie $|p/q - a|$ is smallest for one p/q among these numbers.

Choose $\delta = |p/q - a|$ for p/q the number selected above.

$0 < |x - a| < \delta \Rightarrow x$ not one of $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \dots, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}$

$\Rightarrow |f(x)| < \epsilon \ \forall x \in (a - \delta, a + \delta)$

$\Rightarrow f$ approaches 0 as x approaches a

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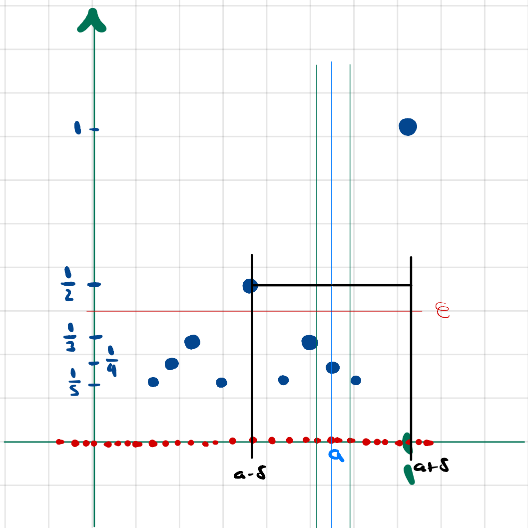
e.g. consider $a = 0.75$, $\epsilon = 0.4$

$n = 3 \Rightarrow \frac{1}{n} < 0.4$

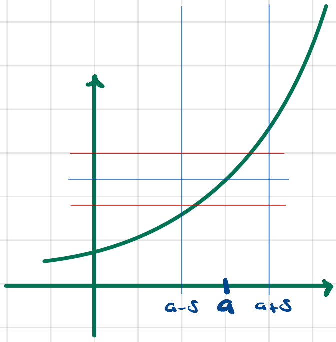
$\frac{1}{2}$ only possible x such that $f(x) > \epsilon$



Theorem 1: A fn cannot approach two different limits near a

In other words, if f approaches ℓ near a, and f approaches m near a then $\ell = m$.

Proof:



f approaches ℓ near a

$\forall \epsilon > 0 \ \exists \delta_1 > 0, \ |x-a| < \delta_1 \Rightarrow |f(x) - \ell| < \epsilon$ (1)

f approaches m near a

$\forall \epsilon > 0, \exists \delta_2 > 0, \ |x-a| < \delta_2 \rightarrow |f(x) - m| < \epsilon$ (2)

$\Rightarrow \forall \epsilon > 0, \ |x-a| < \min(\delta_1, \delta_2) \Rightarrow |f(x) - m| < \epsilon$ and $|f(x) - \ell| < \epsilon$

Call $\min(\delta_1, \delta_2) = \delta$

Assume $m \neq \ell$

$|x-a| < \delta \Rightarrow m - \epsilon < f(x) < m + \epsilon$

$|x-a| < \delta \Rightarrow \ell - \epsilon < f(x) < \ell + \epsilon$

choose $\epsilon = \frac{|m-\ell|}{2}$

$|x-a| < \delta \Rightarrow \quad |f(x) - m| < \frac{|m-\ell|}{2}$

$\quad |f(x) - \ell| < \frac{|m-\ell|}{2}$

$$\begin{aligned} |m-\ell| &= |m - f(x) + f(x) - \ell| \\ &= |(m - f(x)) + (f(x) - \ell)| \\ &\leq |m - f(x)| + |f(x) - \ell| \\ &< \frac{|m-\ell|}{2} + \frac{|m-\ell|}{2} = |m-\ell| \end{aligned}$$

$\Rightarrow |m-\ell| < |m-\ell|$, a contradiction

$\Rightarrow$ The statement: $\forall \epsilon > 0, \exists \delta = \min(\delta_1, \delta_2), \ |x-a| < \delta \Rightarrow |f(x) - m| < \epsilon, \ |f(x) - \ell| < \epsilon, \ \forall \epsilon$

is false. We found a ϵ for which there is no δ such that $|x-a| < \delta \Rightarrow |f(x) - m| < \epsilon, \ |f(x) - \ell| < \epsilon$

$\rightarrow$ Because limit of f at a is unique, we can denote it as $\lim_{x \to a} f(x)$

$\lim_{x \to a} f(x) = \ell$ "the limit of f(x) as x approaches a is ℓ"
"f approaches ℓ near a"

$\rightarrow$ It is possible that near a f doesn't approach a limit.

ie $\lim_{x \to a} f(x) = \ell$ is false for every ℓ

This means $\lim_{x \to a} f(x)$ does not exist

$\lim_{x\to a} x+f^3 = \ell$ s.t. $|x-a|<\delta \Rightarrow |x+f^3-\ell|<\epsilon$

If we have a guess for what the limit is, we can verify if it is:

$|x+f^3-a-f^3| = |x-a| < \delta$ by assumption

Given a $\epsilon>0$, choose $\delta = \frac{\epsilon}{2} \Rightarrow |x+f^3-(a-f^3)| < \epsilon$

$\Rightarrow \lim_{x\to a}(x+f^3) = a+f^3$

If we don't know what ℓ is

Consider 1. $\lim_{x\to a} f(x)$

2. $\lim_{h\to 0} f(a+h)$

(1) is the number ℓ s.t. $\forall\epsilon, \exists\delta$ s.t. $|x-a|<\delta \Rightarrow |f(x)-\ell|<\epsilon$

(2) is the number ℓ s.t. $\forall\epsilon, \exists\delta$ s.t. $|h|<\delta \Rightarrow |f(a+h)-\ell|<\epsilon$

Let $h = x-a$. Then $x = h+a$

(1) is now ℓ s.t. $|h|<\delta \Rightarrow |f(a+h)-\ell|<\epsilon$

Theorem 2

$\lim_{x\to a} f(x) = \ell$, $\lim_{x\to a} g(x) = m$ $\Rightarrow$

$\lim_{x\to a}(f+g)(x) = \ell+m$ (1)

$\lim_{x\to a}(f\cdot g)(x) = \ell\cdot m$ (2)

$m\neq 0 \Rightarrow \lim_{x\to a}\frac{1}{g} = \frac{1}{m}$ (3)

To prove these, we need following results from Ch.1 problems:

Lemma

1. $|x-x_0|<\frac{\epsilon}{2}, |y-y_0|<\frac{\epsilon}{2} \Rightarrow |(x+y)-(x_0+y_0)|<\epsilon$

2. $|x-x_0|<\min\left(1, \frac{\epsilon}{2(|y_0|+1)}\right)$, $|y-y_0|<\frac{\epsilon}{2(|x_0|+1)}$ $\Rightarrow |xy-x_0y_0|<\epsilon$

3. $y_0\neq 0, |y-y_0|<\min\left(\frac{|y_0|}{2}, \frac{\epsilon|y_0|^2}{2}\right)$

$\Rightarrow y\neq 0, \left|\frac{1}{y}-\frac{1}{y_0}\right|<\epsilon$

Proof (1)

$\forall\epsilon, \exists\delta_1$ s.t. $|x-a|<\delta_1 \Rightarrow |f(x)-\ell|<\epsilon$

$\forall\epsilon, \exists\delta_2$ s.t. $|x-a|<\delta_2 \Rightarrow |g(x)-m|<\epsilon$

Let $\delta = \min(\delta_1,\delta_2)$

$\forall\epsilon, |x-a|<\delta \Rightarrow |f(x)-\ell|<\frac{\epsilon}{2}, |g(x)-m|<\frac{\epsilon}{2}$

By part (1) of the lemma,

$|f(x)+g(x)-(m+\ell)|<\epsilon$

$\Rightarrow \lim_{x\to a}(f+g)(x) = m+\ell$

Proof of (2)

By assumption
$\forall\epsilon>0, \exists\delta_1,\delta_2$ s.t.

$|x-a|<\delta_1 \Rightarrow |f(x)-\ell|<\min\left(1, \frac{\epsilon}{2(|m|+1)}\right)$

$|x-a|<\delta_2 \Rightarrow |g(x)-m|<\frac{\epsilon}{2(|\ell|+1)}$

Let $\delta = \min(\delta_1,\delta_2)$

$\Rightarrow |x-a|<\delta \Rightarrow |f(x)-\ell|<\min\left(1, \frac{\epsilon}{2(|m|+1)}\right)$

$|g(x)-m|<\frac{\epsilon}{2(|\ell|+1)}$

$\Rightarrow$ by lemma part (2)

$|f(x)g(x)-m\ell|<\epsilon$

$\Rightarrow \lim_{x\to a}(f\cdot g)(x) = m\ell$

Proof of (3)

$\forall\epsilon>0$

Let $\epsilon_0 = \min\left(\frac{|m|}{2}, \frac{\epsilon|m|^2}{2}\right), m\neq 0$

$\Rightarrow \exists\delta>0$ s.t. $|x-a|<\delta \Rightarrow |g(x)-m|<\epsilon_0$

$\Rightarrow g(x)\neq 0, \left|\frac{1}{g(x)}-\frac{1}{m}\right|<\epsilon$

$\Rightarrow \lim_{x\to a}\frac{1}{g(x)} = \frac{1}{m}, g(x)\neq 0$

→ Prove $\lim_{x\to a} \frac{x^3+7x^5}{x^2+1} = \frac{a^3+7a^5}{a^2+1}$

by using Theorem 2.

$\lim_{x\to a} 7 = 7$

proof: $\forall \epsilon, \forall \delta \Rightarrow |x-a|<\delta \Rightarrow |f(x)-7| = 0 < \epsilon$

$\lim_{x\to a} 1 = 1$

$\lim_{x\to a} x = a$

proof: $\forall \epsilon>0, \forall \delta<\epsilon \Rightarrow |x-a|<\delta \Rightarrow |f(x)-a| = |x-a| < \delta$

$\lim_{x\to a} x^2 = \lim_{x\to a} x\cdot x = \lim_{x\to a} x \cdot \lim_{x\to a} x = a\cdot a = a^2$

Assume $\lim_{x\to a} x^n = a^n$

$\lim_{x\to a} x^{n+1} = \lim_{x\to a} x\cdot x^n = \lim_{x\to a} x \cdot \lim_{x\to a} x^n = a\cdot a^n = a^{n+1}$

By induction, $\lim_{x\to a} x^n = a^n$

$\lim_{x\to a} \frac{x^3+7x^5}{x^2+1} = \lim_{x\to a} \left(\frac{1}{x^2+1}\right)\cdot (x^3+7x^5)$

$= \lim_{x\to a} \frac{1}{x^2+1} \cdot \lim_{x\to a} (x^3+7x^5) = \frac{1}{a^2+1} a^3+7a^5$

$\lim_{x\to a} \frac{1}{x^2+1} = \frac{1}{a^2+1}$ by Th.2(3)

$\lim_{x\to a} (x^2+1) \underset{\text{Th.2 (1)}}{=} a^2+1 \neq 0 \ \forall a$

$\lim_{x\to a} (x^3+7x^5) = a^3+7a^5$ Th.2(1)

Now let's actually find a δ such that $\forall \epsilon>0$

$|x-a|<\delta \Rightarrow \left|\frac{x^3+7x^5}{x^2+1} - \frac{a^3+7a^5}{a^2+1}\right| < \epsilon$

→ Find $\lim_{x\to a} (x^2+1)$

$|x^2+1-(a^2+1)| = |x^2-a^2| = |x-a||x+a|$

Assume $|x-a|<1$

$|x|-|a| \leq |x-a| < 1 \Rightarrow |x| \leq 1+|a|$

$|x+a| \leq |x|+|a| \leq 1+2|a|$

$|x^2-a^2| = |x+a||x-a| \leq |x-a|(1+2|a|)$

if $|x-a|(1+2|a|)<\epsilon$ then $|x^2-a^2|<\epsilon, \forall\epsilon>0$

$\Rightarrow |x-a| < \frac{\epsilon}{1+2|a|}$

$\forall\epsilon>0$ if we choose $0<\delta<\frac{\epsilon}{1+2|a|}$ then $|x^2-a^2|<\epsilon$

$\Rightarrow |(x^2+c)-(a^2+c)|<\epsilon$

$\Rightarrow \lim_{x\to a} (x^2+c) = a^2+c$

Find $\lim_{x\to a} x^3+7x^5$

$\lim_{x\to a} x^n = a^n$, $\lim_{x\to a} 7 = 7$, as proved previously.

$\forall\epsilon>0$

$\forall \delta_1>0$, $|x-a|<\delta_1 \Rightarrow |7-7| < \min\left(1, \frac{\epsilon}{2(|a^5|+1)}\right)$

$|x-a|<\delta_2 \Rightarrow |x^5-a^5| < \frac{\epsilon}{2(7+1)}$

$|x-a|<\delta_2 \Rightarrow |7-7| < \min\left(1, \frac{\epsilon}{2(|a^5|+1)}\right)$, $|x^5-a^5| < \frac{\epsilon}{2(7+1)}$

lemma (2) $\Rightarrow |7x^5-7a^5|<\epsilon$

$\Rightarrow \lim_{x\to a} 7x^5 = 7a^5$

$\forall\epsilon>0 \ \exists\delta_1 : |x-a|<\delta_1 \Rightarrow |x^3-a^3|<\epsilon$

$\forall\epsilon>0, \exists\delta_2 : |x-a|<\delta_2 \Rightarrow |7x^5-7a^5|<\epsilon$

$\delta = \min(\delta_1,\delta_2) \Rightarrow \forall\epsilon>0, |x-a|<\delta \Rightarrow$ $|x^3-a^3|<\epsilon/2$
$|7x^5-7a^5|<\epsilon/2$

Lemma (1) $\Rightarrow |x^3+7x^5-a^3-7a^5|<\epsilon$

$\Rightarrow \lim_{x\to a} (x^3+7x^5) = a^3+7x^5$

$$\lim_{x\to a} \frac{1}{x^2+1}$$

$g(x) = x^2+1 \Rightarrow \lim_{x\to a} g(x) = a^2+1$

$\forall \epsilon>0 \ \exists \delta>0$ s.t. $|x-a|<\delta \Rightarrow |g(x)-(a^2+1)|<\epsilon$

* This δ is $0<\delta<\frac{\epsilon}{1+2|a|}$

Let $\epsilon = \min\left(\frac{|a^2+1|}{2}, \frac{\epsilon_0 |a^2+1|^2}{2}\right)$, $\forall \epsilon_0>0$

lemma (3) $\Rightarrow$ $g(a)\neq 0$, $\lim_{x\to a} \frac{1}{x^2+1} = \frac{1}{a^2+1}$

ie, $\forall \epsilon>0$, $0<\delta<\frac{1}{1+2|a|}\min\left(\frac{|a^2+1|}{2}, \frac{\epsilon_0|a^2+1|^2}{2}\right) \Rightarrow \left|\frac{1}{x^2+1}-\frac{1}{a^2+1}\right|<\epsilon_0$

At this point we have

$$\lim_{x\to a} \frac{x^3+7x^5}{x^2+1} = \lim_{x\to a}\left(\frac{1}{x^2+1}\right)(x^3+7x^5) = \lim_{x\to a} f(x)\cdot g(x)$$

with $\lim_{x\to a} f(x) = \frac{1}{a^2+1}$ $\quad \forall\epsilon>0$, $0<\delta<\frac{1}{1+2|a|}\min\left(\frac{|a^2+1|}{2}, \frac{\epsilon_0|a^2+1|^2}{2}\right) \Rightarrow \left|\frac{1}{x^2+1}-\frac{1}{a^2+1}\right|<\epsilon_0$

$\lim_{x\to a} g(x) = a^3+7a^5$ $\quad \forall\epsilon>0$, $\delta = \min(\delta_1,\delta_2) \Rightarrow |x-a|<\delta \Rightarrow |x^3+7x^5-a^3-7a^5|<\epsilon$

$\forall\epsilon$, $\delta = \min\left[\frac{1}{1+2|a|}, \delta_1, \delta_2\right] \Rightarrow$

$|x-a|<\delta \Rightarrow \left|f(x)-\frac{1}{a^2+1}\right| < \min\left(1, \frac{\epsilon_0}{2(|a^3-7a^5|+1)}\right)$

$$|g(x)-a^3-7a^5| < \frac{\epsilon_0}{2\left(\left|\frac{1}{a^2+1}\right|+1\right)}$$

Lemma (2) $\Rightarrow$ $\left|\frac{x^3+7x^5}{x^2+1} - \frac{a^3+7a^5}{a^2+1}\right| < \epsilon$

$$\Rightarrow \lim_{x\to a}\frac{x^3+7x^5}{x^2+1} = \frac{a^3+7a^5}{a^2+1}$$

In summary, to prove the above statement, we proved various limits separately. The final δ is the minimum of the δ's used in the limit expressions of x^2+1, x^3+7x^5. Note that for the latter expr., f. ex, the δ is also the min of δ's used in the limit expressions of x^3 and $7x^5$. We thus need to keep x close enough to a such that each limit required in the calculation of the limit of the entire expr. is true.

Note that the limit of, say x^3 involves some δ which we didn't specify explicitly. We know it exists because we proved that $\lim_{x\to a} x^n = a^n$ using induction and theorem 1.

Important Detail

$\to$ For $\lim_{x \to a} f(x)$ to be defined, $f(a)$ does not need to be defined, nor does it need to be defined at all $x \neq a$. However, there must be some $\delta > 0$ such that $f(x)$ is defined $\forall x : 0 < |x-a| < \delta$

Limits From Above/Below

$\lim_{x \to a^+} f(x) = \ell$ or $\lim_{x \to \downarrow a} f(x) = \ell$

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < x - a < \delta \Rightarrow |f(x) - \ell| < \varepsilon$

$\lim_{x \to \infty} f(x) = \ell$

$\forall \varepsilon \ \exists N \in \mathbb{R}$ s.t. $\forall x \ \ x > N \Rightarrow |f(x) - \ell| < \varepsilon$

$\lim_{x\to a}\sqrt{x}$

$f(x)=\sqrt{x}$

$\forall\epsilon>0$, we are looking for $\delta>0$: $|x-a|<\delta \Rightarrow |\sqrt{x}-\sqrt{a}|<\epsilon$

the domain of f is $[0,+\infty)$, so $x\geq 0$.

consider $0<\delta\leq 1$

$|x-a|<\delta<a \Rightarrow -a<-\delta<x-a<\delta<a$

$\Rightarrow 0<a-\delta<x<a+\delta<2a$

$$|\sqrt{x}-\sqrt{a}| = \frac{|\sqrt{x}-\sqrt{a}||\sqrt{x}+\sqrt{a}|}{|\sqrt{x}+\sqrt{a}|} = \frac{|x-a|}{|\sqrt{x}+\sqrt{a}|}$$

$a\geq 1 \Rightarrow |\sqrt{x}-\sqrt{a}|<|x-a|$

$\Rightarrow a\geq 1, \forall\epsilon>0 \Rightarrow |x-a|<\epsilon \Rightarrow |\sqrt{x}-\sqrt{a}|<\epsilon$

$\Rightarrow \lim_{x\to a} f(x)=\sqrt{a}$, $a\geq 1$

let's consider $0<a<1$.

$|x-a|<\delta<a<1 \Rightarrow -1<-a<-\delta<x-a<\delta<a<1$

$\Rightarrow a-1<0<a-\delta<x<a+\delta<2a<a+1$

Let $\delta<\min(a,1-a) \Rightarrow$

$|x-a|<\delta \Rightarrow a-\delta<x<a+\delta$

$a<\frac{1}{2} \Rightarrow \min(a,1-a)=a \Rightarrow 0<x<2a<1$

$a>\frac{1}{2} \Rightarrow \min(a,1-a)=1-a \Rightarrow 2a-1<x<1$

$\Rightarrow 0<x<1$

$|x-a|=|(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a})|=|\sqrt{x}+\sqrt{a}||\sqrt{x}-\sqrt{a}|<\delta$

$$\Rightarrow |\sqrt{x}-\sqrt{a}|<\frac{\delta}{\sqrt{x}+\sqrt{a}}$$

$0<a-\delta<x<a+\delta<1$

$0<x<1 \Rightarrow x<\sqrt{x}$

$0<a<1 \Rightarrow a<\sqrt{a}$ $\quad\Rightarrow |\sqrt{x}-\sqrt{a}|<\frac{\delta}{x+a}<\frac{\delta}{2a-\delta}$

$a-\delta<x<a+\delta$

$\Rightarrow 2a-\delta<x+a<2a+\delta$

$\forall\epsilon>0$, choose δ: $\frac{\delta}{2a-\delta}\leq\epsilon \Rightarrow \delta\leq 2a\epsilon-\delta\epsilon$

$\Rightarrow \delta(1+\epsilon)\leq 2a\epsilon \Rightarrow \delta\leq\frac{2a\epsilon}{1+\epsilon}$

$\delta<\min\left(\frac{2a\epsilon}{1+\epsilon}, \min(a,1-a)\right)$, $|x-a|<\delta$

$\Rightarrow |\sqrt{x}-\sqrt{a}|<\epsilon$

$\Rightarrow \lim_{x\to a}\sqrt{x}=\sqrt{a}$, $0<a<1$

$$< \frac{|x-a|}{\sqrt{a}} < \frac{\delta}{\sqrt{a}}<\epsilon \Rightarrow \delta<\sqrt{a}\,\epsilon$$

Therefore

$\forall\epsilon>0$, choose $\delta=\sqrt{a}\,\epsilon$, $|x-a|<\delta \Rightarrow |\sqrt{x}-\sqrt{a}|<\epsilon$

much simpler sol'n!