

# Theorems from Vector Calculus

In the following  $\phi$ ,  $\psi$ , and  $\mathbf{A}$  are well-behaved scalar or vector functions,  $V$  is a three-dimensional volume with volume element  $d^3x$ ,  $S$  is a closed two-dimensional surface bounding  $V$ , with area element  $da$  and unit outward normal  $\mathbf{n}$  at  $da$ .

$$\int_V \nabla \cdot \mathbf{A} d^3x = \int_S \mathbf{A} \cdot \mathbf{n} da \quad (\text{Divergence theorem})$$

$$\int_V \nabla \psi d^3x = \int_S \psi \mathbf{n} da$$

$$\int_V \nabla \times \mathbf{A} d^3x = \int_S \mathbf{n} \times \mathbf{A} da$$

$$\int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) d^3x = \int_S \phi \mathbf{n} \cdot \nabla \psi da \quad (\text{Green's first identity})$$

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) d^3x = \int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \mathbf{n} da \quad (\text{Green's theorem})$$

In the following  $S$  is an open surface and  $C$  is the contour bounding it, with line element  $d\mathbf{l}$ . The normal  $\mathbf{n}$  to  $S$  is defined by the right-hand side rule in relation to the sense of the line integral around  $C$ .

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} da = \oint_C \mathbf{A} \cdot d\mathbf{l} \quad (\text{Stokes's theorem})$$

$$\int_S \mathbf{n} \times \nabla \psi da = \oint_C \psi d\mathbf{l}$$

# Vector Formulas

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\nabla \times \nabla \psi = 0$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

$$\nabla \cdot (\psi \mathbf{a}) = \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a}$$

$$\nabla \times (\psi \mathbf{a}) = \nabla \psi \times \mathbf{a} + \psi \nabla \times \mathbf{a}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

If  $\mathbf{x}$  is the coordinate of a point with respect to some origin, with magnitude  $r=|\mathbf{x}|$ , and  $\mathbf{n}=\mathbf{x}/r$  is a unit radial vector, then

$$\nabla \cdot \mathbf{x} = 3 \quad \nabla \times \mathbf{x} = 0$$

$$\nabla \cdot \mathbf{n} = \frac{2}{r} \quad \nabla \times \mathbf{n} = 0$$

$$(\mathbf{a} \cdot \nabla)\mathbf{n} = \frac{1}{r} [\mathbf{a} - \mathbf{n}(\mathbf{a} \cdot \mathbf{n})] \equiv \frac{\mathbf{a}_\perp}{r}$$

# Explicit Forms of Vector Operations

Let  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and  $A_1, A_2, A_3$  be the corresponding components of  $\mathbf{A}$ . Then

**Cartesian**  
( $x_1, x_2, x_3 = x, y, z$ )

$$\begin{aligned}\nabla\psi &= \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3} \\ \nabla \cdot \mathbf{A} &= \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} \\ \nabla \times \mathbf{A} &= \mathbf{e}_1 \left( \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right) + \mathbf{e}_2 \left( \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right) + \mathbf{e}_3 \left( \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right) \\ \nabla^2\psi &= \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}\end{aligned}$$


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**Cylindrical**  
( $\rho, \phi, z$ )

$$\begin{aligned}\nabla\psi &= \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho} \frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z} \\ \nabla \cdot \mathbf{A} &= \frac{1}{\rho} \frac{\partial}{\partial\rho} (\rho A_1) + \frac{1}{\rho} \frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z} \\ \nabla \times \mathbf{A} &= \mathbf{e}_1 \left( \frac{1}{\rho} \frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z} \right) + \mathbf{e}_2 \left( \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho} \right) + \mathbf{e}_3 \left( \frac{\partial}{\partial\rho} (\rho A_2) - \frac{\partial A_1}{\partial\phi} \right) \\ \nabla^2\psi &= \frac{1}{\rho} \frac{\partial}{\partial\rho} \left( \rho \frac{\partial\psi}{\partial\rho} \right) + \frac{1}{\rho^2} \frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}\end{aligned}$$


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**Spherical**  
( $r, \theta, \phi$ )

$$\begin{aligned}\nabla\psi &= \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r} \frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r \sin\theta} \frac{\partial\psi}{\partial\phi} \\ \nabla \cdot \mathbf{A} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_1) + \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta A_2) + \frac{1}{r \sin\theta} \frac{\partial A_3}{\partial\phi} \\ \nabla \times \mathbf{A} &= \mathbf{e}_1 \frac{1}{r \sin\theta} \left[ \frac{\partial}{\partial\theta} (\sin\theta A_3) - \frac{\partial A_2}{\partial\phi} \right] \\ &\quad + \mathbf{e}_2 \left[ \frac{1}{r \sin\theta} \frac{\partial A_1}{\partial\phi} - \frac{1}{r} \frac{\partial}{\partial r} (r A_3) \right] + \mathbf{e}_3 \frac{1}{r} \left[ \frac{\partial}{\partial r} (r A_2) - \frac{\partial A_1}{\partial\theta} \right] \\ \nabla^2\psi &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial\psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial\psi}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2} \\ &\quad \left[ \text{Note that } \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial\psi}{\partial r} \right) \equiv \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\psi) \right]\end{aligned}$$