Mathematical Finance, Vol. 7, No. 2 (April 1997), 127–147

THE MARKET MODEL OF INTEREST RATE DYNAMICS[1]

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A class of term structure models with volatility of lognormal type is analyzed in the general HJM framework. The corresponding market forward rates do not explode, and are positive and mean reverting. Pricing of caps and floors is consistent with the Black formulas used in the market. Swaptions are priced with closed formulas that reduce (with an extra assumption) to exactly the Black swaption formulas when yield and volatility are flat. A two-factor version of the model is calibrated to the U.K. market price of caps and swaptions and to the historically estimated correlation between the forward rates.

KEY WORDS: term structure models, HJM framework, lognormality of rates, stochastic partial differential equations, caps, swaptions

1. INTRODUCTION

In most markets, caps and floors form the largest component of an average swap derivatives book. A cap/floor is a strip of caplets/floorlets each of which is a call/put option on a forward rate. Market practice is to price the option assuming that the underlying forward rate process is lognormally distributed with zero drift. Consequently, the option price is given by the Black futures formula, discounted from the settlement data.

In an arbitrage-free setting forward rates over consecutive time intervals are related to one another and cannot all be lognormal under one arbitrage-free measure. That probably is what led the academic community to a degree of skepticism toward the market practice of pricing caps, and sparked vigorous research with the aim of identifying an arbitrage-free term structure model.

The aim of this paper is to show that this market practice can be made consistent with an arbitrage-free term structure model. Consecutive quarterly or semiannual forward rates can all be lognormal while the model will remain arbitrage free. This is possible because each rate is lognormal under the forward (to the settlement date) arbitrage-free measure rather than under one (spot) arbitrage-free measure. Lognormality under the appropriate forward and not spot arbitrage-free measure is needed to justify the use of the Black futures formula with discount for caplet pricing. The market seems to interpret the concept of probability

[1]The authors would like to thank Citibank and the participants of the financial mathematics seminar at the University of New South Wales; mixing of the commercial and academic environments contributed greatly to this paper. A. Brace and M. Musiela would like to acknowledge the hospitality of the Isaac Newton Institute, where part of this work was carried out. D. Gątarek is grateful for financial support from the Australian Research Council.

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distribution in an intuitive rather than mathematical sense, and does not distinguish between the forward measures at different maturities.

We work with the term structure parametrization proposed by Musiela (1993) and later used by Musiela and Sondermann (1993), Brace and Musiela (1994a), Goldys, Musiela, and Sondermann (1994), and Musiela (1994). We denote by $r(t,x)$ the continuously compounded forward rate prevailing at time t over the time interval $[t+x, t+x+dx]$. There is an obvious relationship between the Heath, Jarrow, and Morton (1992) forward rates $f(t,T)$ and our $r(t,x)$, namely $r(t,x) = f(t, t+x)$. For all $T > 0$ the process

$$P(t,T) = \exp\left(-\int_0^{T-t} r(t,u)\,du\right) = \exp\left(-\int_t^T f(t,u)\,du\right),\ 0 \le t \le T,$$

describes price evolution of a zero coupon bond with maturity T. Time evolution of the discount function

$$x \longmapsto D(t,x) = P(t, t+x) = \exp\left(-\int_0^x r(t,u)\,du\right)$$

is described by the processes $\{D(t,x);\ t \ge 0\}$, $x \ge 0$. We make the usual mathematical assumptions. All processes are defined on the probability space $(\Omega, \{\mathcal{F}_t; t \ge 0\}, \mathbb{P})$, where the filtration $\{\mathcal{F}_t;\ t \ge 0\}$ is the $\mathbb{P}$-augmentation of the natural filtration generated by a d-dimensional Brownian motion $W = \{W(t);\ t \ge 0\}$. We assume that the process $\{r(t,x);\ t, x \ge 0\}$ satisfies

$$dr(t,x) = \frac{\partial}{\partial x}\left(\left(r(t,x) + \frac{1}{2}|\sigma(t,x)|^2\right)dt + \sigma(t,x)\cdot dW(t)\right), \tag{1.1}$$

where for all $x \ge 0$ the volatility process $\{\sigma(t,x);\ t \ge 0\}$ is $\mathcal{F}_t$-adapted with values in $\mathbb{R}^d$, while $|\ |$ and $\cdot$ stand for the usual norm and inner product in $\mathbb{R}^d$, respectively. We also assume that the function $x \longmapsto \sigma(t,x)$ is absolutely continuous and the derivative $\tau(t,x) = \frac{\partial}{\partial x}\sigma(t,x)$ is bounded on $\mathbb{R}_+^2 \times \Omega$. It follows easily that

$$dD(t,x) = D(t,x)\left((r(t,0) - r(t,x))\,dt - \sigma(t,x)\cdot dW(t)\right),$$

and hence $\sigma(t,x)$ can be interpreted as price volatility. Obviously we have $\sigma(t,0) = 0$.

The spot rate process $\{r(t,0);\ t \ge 0\}$ satisfies

$$dr(t,0) = \frac{\partial}{\partial x} r(t,x)\bigg|_{x=0} dt + \frac{\partial}{\partial x}\sigma(t,x)\bigg|_{x=0} \cdot dW(t)$$

and hence is not Markov, in general. The process

$$\beta(t) = \exp\left(\int_0^t r(s,0)\,ds\right),\ t \ge 0$$

represents the amount generated at time $t \geq 0$ by continuously reinvesting \$1 in the spot rate $r(s, 0)$, $0 \leq s \leq t$.

It is well-known that if for all $T > 0$ the process $\{P(t, T)/\beta(t); 0 \leq t \leq T\}$ is a martingale under $\mathbb{P}$ then there is no arbitrage possible between the zero coupon bonds $P(\cdot, T)$ of all maturities $T > 0$ and the savings account $\beta(\cdot)$. Note that, under (1.1), we can easily write that

$$\frac{P(t,T)}{\beta(t)} = P(0,T)\exp\left(-\int_0^t \sigma(s, T-s)\cdot dW(s) - \frac{1}{2}\int_0^t |\sigma(s, T-s)|^2\, ds\right), \tag{1.2}$$

where the right-hand side is a martingale. It also follows that

$$dP(t,T) = P(t,T)\left(r(t,0)dt - \sigma(t, T-t)\cdot dW(t)\right).$$

In Section 2 the existence of the model is established; cap and swaption formulas are derived in Section 3; and, finally, the calibration is described in Section 4.

2. THE MODEL

To specify the model, or equivalently, to define the volatility process $\sigma(t, x)$ in equation (1.1) we fix $\delta > 0$ (for example, $\delta = 0.25$) and assume that for each $x \geq 0$ the LIBOR rate process $\{L(t, x);\ t \geq 0\}$, defined by

$$1 + \delta L(t,x) = \exp\left(\int_x^{x+\delta} r(t,u)\, du\right), \tag{2.1}$$

has a lognormal volatility structure; i.e.,

$$dL(t,x) = \cdots dt + L(t,x)\gamma(t,x)\cdot dW(t), \tag{2.2}$$

where the deterministic function $\gamma\colon \mathbb{R}_+^2 \to \mathbb{R}^d$ is bounded and piecewise continuous. Using the Ito formula and (1.1) we get

$$\begin{aligned}
dL(t,x) &= \delta^{-1} d\exp\left(\int_x^{x+\delta} r(t,u)du\right)\\
&= \delta^{-1}\exp\left(\int_x^{x+\delta} r(t,u)\, du\right) d\left(\int_x^{x+\delta} r(t,u)\, du\right)\\
&\quad + \delta^{-1}\frac{1}{2}\exp\left(\int_x^{x+\delta} r(t,u)\, du\right)|\sigma(t, x+\delta) - \sigma(t,x)|^2\, dt\\
&= \delta^{-1}\exp\left(\int_x^{x+\delta} r(t,u)\, du\right)\\
&\quad \times \left(\left(r(t, x+\delta) - r(t,x) + \frac{1}{2}|\sigma(t, x+\delta)|^2 - \frac{1}{2}|\sigma(t,x)|^2\right) dt\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad + (\sigma(t, x+\delta) - \sigma(t,x)) \cdot dW(t)\Big) \\
&\quad + \delta^{-1}\frac{1}{2}\exp\left(\int_x^{x+\delta} r(t,u)\,du\right)|\sigma(t,x+\delta)-\sigma(t,x)|^2\,dt \\
&= \left(\frac{\partial}{\partial x}L(t,x) + \delta^{-1}(1+\delta L(t,x))\,\sigma(t,x+\delta)\right. \\
&\qquad \left.\cdot(\sigma(t,x+\delta)-\sigma(t,x))\right) dt \\
&\quad + \delta^{-1}(1+\delta L(t,x))(\sigma(t,x+\delta)-\sigma(t,x))\cdot dW(t).
\end{aligned}
$$

Consequently (2.2) holds for all $x \geq 0$ if and only if for all $x \geq 0$

$$\sigma(t, x+\delta) - \sigma(t,x) = \frac{\delta L(t,x)}{1+\delta L(t,x)}\gamma(t,x). \tag{2.3}$$

Under (2.3) the equation for $L(t,x)$ becomes

$$dL(t,x)=\left(\frac{\partial}{\partial x}L(t,x)+L(t,x)\gamma(t,x)\cdot\sigma(t,x+\delta)\right) dt+L(t,x)\gamma(t,x)\cdot dW(t). \tag{2.4}$$

Recurrence relationship (2.3) defines the HJM volatility process $\sigma(t,x)$ for all $x \geq \delta$ provided $\sigma(t,x)$ is defined on the interval $0 \leq x < \delta$. We set $\sigma(t,x) = 0$ for all $0 \leq x < \delta$ and hence, solving (2.3), we get for $x \geq \delta$

$$\sigma(t,x) = \sum_{k=1}^{[\delta^{-1}x]} \frac{\delta L(t,x-k\delta)}{1+\delta L(t,x-k\delta)}\gamma(t,x-k\delta). \tag{2.5}$$

Therefore the process $\{L(t,x); t, x \geq 0\}$ must satisfy the following equation

$$
\begin{aligned}
dL(t,x) = &\left(\frac{\partial}{\partial x}L(t,x) + L(t,x)\gamma(t,x)\cdot\sigma(t,x) + \frac{\delta L^2(t,x)}{1+\delta L(t,x)}|\gamma(t,x)|^2\right) dt \\
&+ L(t,x)\gamma(t,x)\cdot dW(t).
\end{aligned} \tag{2.6}
$$

The preceding approach to the term structure modeling is quite different from the traditional one based on the instantaneous continuously compounded spot or forward rates, and therefore we believe its motivations and origins are worth mentioning. The change of focus from the instantaneous continuously compounded rates to the instantaneous effective annual rates was first proposed by Sandmann and Sondermann (1993) in response to the impossibility of pricing a Eurodollar futures contract with a lognormal model of the instantaneous continuously compounded spot rate. An HJM-type model based on the instantaneous effective annual rates was introduced by Goldys et al. (1994). A lognormal volatility structure was assumed on the effective annual rate $j(t,x)$ which is related to the instantaneous continuously compounded forward rate $r(t, x)$ via the formula

$$1 + j(t, x) = e^{r(t,x)}.$$

The case of nominal annual rates $q(t, x)$ corresponding to $r(t, x)$, i.e.,

$$(1 + \delta q(t, x))^{1/\delta} = e^{r(t,x)}$$

was studied by Musiela (1994). It turns out that the HJM volatility process $\sigma(t, x)$ takes the form

$$\sigma(t, x) = \int_0^x \delta^{-1}(1 - e^{-\delta r(t,u)})\gamma(t, u)\, du. \tag{2.7}$$

Obviously for $\delta = 1$ we obtain the Goldys et al. (1994) model and for $\delta = 0$ we get

$$\sigma(t, x) = \int_0^x r(t, u)\gamma(t, u)\, du,$$

and hence the HJM lognormal model, which is known to explode (for $\delta > 0$ no explosion occurs).

Unfortunately these models do not give closed form pricing formulas for options. In order to price a caplet, for example, one would have to use some numerically intensive algorithms. This would not be practical for model calibration, where an iterative procedure is needed to identify the volatility $\gamma(t, x)$ which returns the market prices for a large number of caps and swaptions.

A key piece in the term structure puzzle was found by Miltersen, Sandmann, and Sondermann (1994). First, attention was shifted from the instantaneous rates $q(t, x)$ to the nominal annual rates $f(t, x, \delta)$ defined by

$$(1 + f(t, x, \delta))^{\delta} = \exp(\int_x^{x+\delta} r(t, u)\, du). \tag{2.8}$$

More importantly, however, it was shown that for $\delta = 1$ the model prices a yearly caplet according to the market standard. Unfortunately the volatility $\sigma(t, x)$ was not completely identified, leaving open the question of the model specification for maturities different from $x = i\delta$ and the existence of solution to equation (1.1). These problems were only partially addressed in Miltersen, Sandmann, and Sondermann (1995), where a model based on the effective rates $f(t, T, \delta)$ defined by

$$1 + \delta f(t, T, \delta) = \exp\left(\int_{T-t}^{T+\delta-t} r(t, u)\, du\right) \tag{2.9}$$

was analyzed.

As explained earlier, we assume a lognormal volatility structure on the LIBOR rate $L(t,x)$, defined by (2.1), for all $x \geq 0$ and a fixed $\delta > 0$. This leads to the volatility $\sigma(t,x)$ given in (2.5) and (2.6) for $L(t,x)$. To prove existence and uniqueness of the solution to (2.6) we need the following result.

LEMMA 2.1. *For all $x \geq 0$ let $\{\xi(t,x);\, t \geq 0\}$ be an adapted bounded stochastic process with values in $\mathbb{R}^d$, $a(\cdot,x)\colon \mathbb{R}_+ \to \mathbb{R}^d$ be a deterministic bounded and piecewise continuous function, and let*

$$M(t,x) = \int_0^t a(s,x)\cdot dW(s).$$

For all $x \geq 0$ the equation

$$\begin{aligned}(2.10)\qquad dy(t,x) &= y(t,x)a(t,x)\\ &\quad\times\left(\left(\frac{\delta y(t,x)}{1+\delta y(t,x)}a(t,x)+\xi(t,x)\right)dt + dW(t)\right),\ y(0,x)>0,\end{aligned}$$

where $\delta > 0$ is a constant, has a unique strictly positive solution on $\mathbb{R}_+$. Moreover, if for some $k \in \{0,1,2,\ldots\}$, $y(0,\cdot) \in C^k(\mathbb{R}_+)$ and for all $t \geq 0$, $a(t,\cdot)$, $M(t,\cdot)$ and $\xi(t,\cdot) \in C^k(\mathbb{R}_+)$ then for all $t \geq 0$, $y(t,\cdot) \in C^k(\mathbb{R}_+)$.

Proof. Since the right-hand side in (2.10) is locally Lipschitz continuous (with respect to y) on $\mathbb{R} - \{-\delta^{-1}\}$ and Lipschitz continuous on $\mathbb{R}_+$, there exists a unique (possibly exploding) strictly positive solution to (2.10). By the Ito formula

$$\begin{aligned}(2.11)\qquad y(t,x) \;=\; y(0,x)\exp\Bigg(&\int_0^t a(s,x)\cdot dW(s) + \int_0^t a(s,x)\\ &\cdot\left(\frac{\delta y(s,x)}{1+\delta y(s,x)}a(s,x)+\xi(s,x)-\frac{1}{2}a(s,x)\right)ds\Bigg)\end{aligned}$$

for all $t < \tau = \inf\{t\colon y(t,x) = \infty \text{ or } y(t,x) = 0\}$. But if $y(t,x) = 0$ for some $t < \infty$ then $y(s,x) = 0$ for all $s \geq t$ and hence $\tau = \inf\{t\colon y(t,x) = \infty\}$. Moreover, because

$$\int_0^t |a(s,x)|^2\, ds < \infty$$

for all $t < \infty$ we deduce that $\tau = \infty$. Thus (2.11) is equivalent to the following Volterra-type integral equation for $\ell(t,x) = \log y(t,x)$

$$\begin{aligned}(2.12)\quad \ell(t,x) \;=\; &\ell(0,x) + \int_0^t a(s,x)\cdot dW(s)\\ &+ \int_0^t a(s,x)\cdot\left(\frac{\delta e^{\ell(s,x)}}{1+\delta e^{\ell(s,x)}}a(s,x)+\xi(s,x)-\frac{1}{2}a(s,x)\right)ds.\end{aligned}$$

Because the right-hand side in (2.12) is globally Lipschitz continuous with respect to ℓ, we deduce using the standard fix-point arguments that there exists a unique pathwise solution to equation (2.12). Moreover for any $t \geq 0$, $\ell(t, \cdot) \in C^k(\mathbb{R}_+)$ provided $\ell(0, \cdot)$, $a(t, \cdot)$, $\xi(t, \cdot) \in C^k(\mathbb{R}_+)$ for $t \geq 0$. □

THEOREM 2.1. *Let $\gamma : \mathbb{R}_+^2 \to \mathbb{R}^d$ be a deterministic bounded and piecewise continuous function, $\delta > 0$ be a constant and let*

$$M(t, x) = \int_0^t \gamma(s, x + t - s) \cdot dW(s).$$

Equation (2.6) admits a unique nonnegative solution $L(t, x)$ for any $t \geq 0$ and any nonnegative initial condition $L(0, \cdot) = L_0$. If $L_0 > 0$ then $L(t, \cdot) > 0$ for all $t > 0$. If $L_0 \in C^k(\mathbb{R}_+)$ and for all $t \geq 0$, $\gamma(t, \cdot) \in C^k(\mathbb{R}_+)$, $M(t, \cdot) \in C^k(\mathbb{R}_+)$, $(\partial^j / \partial x^j)\gamma(t, x)|_{x=0} = 0$, $j = 0, 1, \ldots, k$ then for all $t \geq 0$, $L(t, \cdot) \in C^k(\mathbb{R}_+)$.

Proof. By the solution to (2.6) we mean the so-called mild solution (cf. Da Prato and Zabczyk 1992); i.e., $L(t, x)$ is a solution if for all $t, x \geq 0$

$$\begin{aligned} L(t, x) \;=\; & L(0, x + t) + \int_0^t L(s, x + t - s)\gamma(s, x + t - s) \cdot \sigma(s, x + t - s)\, ds \\ & + \int_0^t \frac{\delta L^2(s, x + t - s)}{1 + \delta L(s, x + t - s)} |\gamma(s, x + t - s)|^2\, ds \\ & + \int_0^t L(s, x + t - s)\gamma(s, x + t - s) \cdot dW(s). \end{aligned}$$

This holds true for $0 \leq x < \delta$ because the process $L(t, x - t)$, $0 \leq t \leq x$, $x > 0$, is a solution to (2.10) with $a(t, x) = \gamma(t, (x - t) \vee 0)$ and $\xi(t, x) = 0$. For $\delta \leq x \leq 2\delta$ the process $L(t, x - t)$, $0 \leq t \leq x$ satisfies (2.10) with $a(t, x) = \gamma(t, (x - t) \vee 0)$ and

$$\xi(t, x) = \frac{\delta L(t, (x - \delta - t) \vee 0)}{1 + \delta L(t, (x - \delta - t) \vee 0)}.$$

By induction we prove that equation (2.6) admits a unique solution for any $x > 0$ and $0 \leq t \leq x$. Also by induction, using (2.5), we deduce that the corresponding $a(t, \cdot)$ and $\xi(t, \cdot)$ satisfy the assumptions of regularity in Lemma 2.1 and hence $L(t, \cdot)$ is smooth as well. □

COROLLARY 2.1. *If for some $k \in \mathbb{N}$ and all $t \geq 0$, $\gamma(t, \cdot) \in C^k(\mathbb{R}_+)$ and $(\partial^j / \partial x^j)\gamma(t, x)|_{x=0} = 0$, $j = 1, \ldots, k$ then equation (1.1) has a unique solution $r(t, \cdot) \in C^{k-1}(\mathbb{R}_+)$ for any positive initial condition $r(0, \cdot) \in C^{k-1}(\mathbb{R}_+)$.*

Proof. Consider (1.1) as an equation with fixed volatility processes $\sigma(t,x)$ given by (2.5) and (2.6). □

REMARK 2.1. Volatility $\sigma(t,x)$ given in (2.5) is not differentiable with respect to x for some functions γ (for example, piecewise constant with respect to x). In such a case the term structure dynamics cannot be analyzed in the HJM framework (1.1). However this difficulty is rather technical. Property (1.2) is sufficient to eliminate arbitrage. By putting $T = t$ in (1.2) we may also use it to define the numeraire (savings account) in terms of the price volatility σ. It is also easy to see that for all $t \geq 0$

$$\begin{aligned}
P(t,t+\delta) &= \beta(t)P(0,t+\delta)\exp\biggl(-\int_0^t \sigma(s,t+\delta-s)\cdot dW(s) \\
&\qquad\qquad - \frac{1}{2}\int_0^t |\sigma(s,t+\delta-s|^2\, ds\biggr) \\
&= \beta(t)P(0,t+\delta)\exp\biggl(-\int_0^{t+\delta} \sigma(s,t+\delta-s)\cdot dW(s) \\
&\qquad\qquad - \frac{1}{2}\int_0^{t+\delta} |\sigma(s,t+\delta-s)|^2\, ds\biggr) \\
&= \beta(t)/\beta(t+\delta)
\end{aligned}$$

because $\sigma(t,x) = 0$ for $0 \leq x < \delta$. Solving the recurrence relationship

$$\beta(t+\delta) = \beta(t)P(t,t+\delta)^{-1} \tag{2.13}$$

we get

$$\beta(t) = \prod_{k=0}^{[\delta^{-1}t]} P((t-(k+1)\delta)^+, t-k\delta)^{-1}. \tag{2.14}$$

The discounted by $\{\beta(t), t \geq 0\}$ zero coupon bond prices $\{P(t,T); 0 \leq t \leq T\}$ satisfy (1.2) and hence there is no arbitrage.

REMARK 2.2. Regularity of γ has an important influence on the short rate $r(t,0)$ dynamics. If the process $\{r(t,0); t \geq 0\}$ is a semimartingale, then it satisfies

$$dr(t,0) = \frac{\partial}{\partial x} r(t,x)|_{x=0}dt. \tag{2.15}$$

Consequently, the short rate is a process of finite variation and therefore it cannot be strong Markov, except for the deterministic case (cf. Çinlar and Jacod 1981, Remark 3.41). The LIBOR process $\{L(t,0); t \geq 0\}$ satisfies (2.15) as well.

REMARK 2.3. It follows from (2.11) and Theorem 2.1 that the process $\{L(t,x); t, x \geq 0\}$ satisfies

$$\begin{aligned} L(t,x) \;=\; L(0,x+t)\exp\Bigg(\int_0^t \gamma(s,x+t-s)\cdot dW(s) + \int_0^t \gamma(s,x+t-s) \\ \cdot\left(\frac{\delta L(s,x+t-s)}{1+\delta L(s,x+t-s)}\gamma(s,x+t-s)\right. \\ \left. + \sigma(s,x+t-s) - \frac{1}{2}\gamma(s,x+t-s)\right) ds\Bigg) \end{aligned}$$

and

$$\begin{aligned} |\gamma(s,x+t-s)\cdot\sigma(s,x+t-s)| \;\leq\; \sum_{k=1}^{[\delta^{-1}(x+t-s)]} |\gamma(s,x+t-s)| \\ \times |\gamma(s,x+t-s-k\delta)|. \end{aligned}$$

Therefore

$$L_1(t,x) \leq L(t,x) \leq L_2(t,x),$$

where

$$\begin{aligned} L_1(t,x) \;&=\; L(0,x+t)\exp\Bigg(\int_0^t \gamma(s,x+t-s)\cdot dW(s) \\ &\qquad - \int_0^t \left(\alpha(s,x+t-s) + \frac{1}{2}|\gamma(s,x+t-s)|^2\right) ds\Bigg), \\ L_2(t,x) \;&=\; L(0,x+t)\exp\Bigg(\int_0^t \gamma(s,x+t-s)\cdot dW(s) \\ &\qquad + \int_0^t \left(\alpha(s,x+t-s) + \frac{1}{2}|\gamma(s,x+t-s)|^2\right) ds\Bigg) \end{aligned}$$

while

$$\alpha(t,x) = \sum_{k=1}^{[\delta^{-1}x]} |\gamma(t,x)||\gamma(t,x-k\delta)|.$$

Consequently the LIBOR rate is bounded from below and above by lognormal processes. The estimate from above can be used to show that the Eurodollar futures price is well defined. The most common Eurodollar futures contract relates to the LIBOR rate. The futures payoff at time T is equal to $\delta L(T,0)$ and hence the Eurodollar futures price at time $t \leq T$ is $E(\delta L(T,0) \mid \mathcal{F}_t)$. Because $L(T,0) \leq L_2(T,0)$ and

$$EL_2(T,0) = L(0,T)\exp\left(\int_0^T (\alpha(s,T-s) + |\gamma(s,T-s)|^2)\, ds\right) < \infty,$$

we conclude that the expectation is finite.

REMARK 2.4. For $n = 1, 2, \ldots$ and $t \geq 0$ define

$$y_n(t) = L(t, (n\delta - t \vee 0), \gamma_n(t) = \gamma(t, (n\delta = t) \vee 0)$$

and assume that $\gamma(t, 0) = 0$. It follows easily that the processes $\{y_n(t); t \geq 0\}, n = 1, 2, \ldots$ satisfy the following closed system of stochastic equations:

$$dy_n(t) = y_n(t)\gamma_n(t) \cdot \left(\sum_{j=[\delta^{-1}t]+1}^{n} \frac{\delta y_j(t)}{1 + \delta y_j(t)} \gamma_j(t) dt + dW(t) \right).$$

We conclude this section with a study indicating that our model will typically generate mean reverting behavior. Interest rates tend to drop when they are too high and tend to rise when they are too low. This property, well supported by empirical evidence, is known as mean reversion. We assume that $|\gamma(t, x)| \leq \beta(x)$, where

$$\text{(A1)} \qquad \sup_{0 \leq x \leq \delta} \sum_{k=0}^{\infty} \beta(x + k\delta) < \infty,$$

$$\text{(A2)} \qquad \int_0^\infty (x + 1)\beta^2(x) dx < \infty.$$

PROPOSITION 2.1. *Assume (A1)–(A2). Then for any $p \geq 1$ and any deterministic initial condition $L(0, \cdot) \in C_b(\mathbb{R}_+)$*

$$\sup_{t \geq 0} \sup_{x \geq 0} EL^p(t, x) < \infty.$$

Proof. Let α and L_2 be as in Remark 2.3. By (A1) and (A2),

$$\sup_{t \geq 0} \sup_{x \geq 0} \int_0^t \left(\alpha(t, x + t - s) + \frac{1}{2} |\gamma(s, x + t - s)|^2 \right) ds < \infty,$$

and

$$E \left(\int_0^t \gamma(s, x + t - s) \cdot dW(s) \right)^2 \leq \int_0^\infty \beta^2(x + s)\, ds \leq \int_0^\infty \beta^2(x)\, dx < \infty.$$

Since $\log L_2$ is Gaussian,

$$\sup_{t \geq 0} \sup_{x \geq 0} EL_2^p(t, x) < \infty$$

for any $p \geq 1$. Since $L \leq L_2$

$$\sup_{t\geq 0}\sup_{x\geq 0} EL^p(t,x) < \infty. \qquad \square$$

Additionally, assume

(A3) $$\gamma(t,x) = \gamma(x),$$

(A4) $$\int_0^\infty x|\gamma'(x)|^2\,dx < \infty,$$

(A5) $$\sup_{0\leq x\leq\delta}\sum_{k=0}^{\infty}|\gamma'(x+k\delta)| < \infty,$$

(A6) $$\int_0^\infty |\gamma(x)|\,dx = C < \frac{1}{K}.$$

Assumption (A3) implies that L is a time-homogeneous Markov process. Hence we can study the notion of invariant measures. The proof of existence of an invariant measure will follow the standard Krylov–Bogoliubov scheme: Feller property and tightness of family of distributions $\mathcal{L}(L(t))_{t\geq 0}$ implies existence of an invariant measure. For details we refer to DaPrato and Zabczyk (1992).

Let $C_0(\mathbb{R}) = \{u \in C(\mathbb{R})\colon u(x) \to 0 \text{ as } x \to \infty\}$ and let $C^\alpha(\mathbb{R}) = \{u \in C(\mathbb{R})\colon |u(x) - u(z)| \leq C|x-z|^\alpha\}$ for any $0 < \alpha \leq 1$. The Hölder norm in $C^\alpha(\mathbb{R})$ will be denoted by $\|\cdot\|_\alpha$. The following result will be useful.

LEMMA 2.2. *A family of functions $\Gamma \subset C_0(\mathbb{R}_+)$ is relatively compact in $C_0(\mathbb{R}_+)$ if and only if the following conditions are satisfied:*

- *(i) The family Γ is equicontinuous on any bounded set.*
- *(ii) There exists a function $R\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $R(u) \to 0$ as $u \to \infty$ and $|f(u)| \leq R(u)$ for any $f \in \Gamma$ and $u \geq 0$.*

THEOREM 2.2. *Assume (A1)–(A5). Let L be the solution of (2.6) and let*

$$\sup_{0\leq x<\infty} |\log L(0,x)| < \infty.$$

Then

$$\sup_{t\geq 0} E\|\log L(t)\| < \infty.$$

If, moreover, (A6) is satisfied than there exists an invariant measure for the process L, concentrated on the closed set

$$U = \{u \in C(\mathbb{R}) : u > 0 \text{ and } u(x) \to 1 \text{ as } x \to \infty\}.$$

Proof. Consider the process $l(t,x) = \log L(t,x)$ which can be represented as

$$\begin{aligned}(2.16) \qquad l(x,t) \;=\; & l_0(x+t) + \int_0^t F(l(t-s))(x+t-s)\cdot\gamma(x+t-s)\,ds \\ & - \frac{1}{2}\int_0^t |\gamma(x+t-s)|^2\,ds + M(t,x)\end{aligned}$$

for any $t \geq 0$, where M is defined by

$$M(t,x) = \int_0^t \gamma(x+t-s)\cdot dW(s)$$

and

$$F(l)(x) = \sum_{k=0}^{[\delta^{-1}x]} \frac{\delta e^{l(x-k\delta)}}{1+\delta e^{l(x-k\delta)}}\gamma(x-k\delta)$$

for any $l \in C(\mathbb{R})$. By (A1), $\gamma\cdot F\colon\ C_0(\mathbb{R}_+) \to C_0(\mathbb{R}_+)$ is a Lipschitz continuous function. By the standard fix-point method $l(t)$ depends continuously on the initial condition in the space $C_0(\mathbb{R}_+)$. Therefore the process l is a Feller process. Notice that

$$M'_x(t,x) = \int_0^t \gamma'(x+t-s)\cdot dW(s).$$

By the Ito formula

$$(2.17) \quad E\int_0^\infty M^2(t,x)\,dx \leq \int_0^\infty\int_0^\infty |\gamma(x+s)|^2\,dxds = \sqrt{2}\int_0^\infty x|\gamma(x)|^2\,dx$$

and

$$(2.18) \quad E\int_0^\infty M'_x(t,x)^2\,dx \leq \int_0^\infty\int_0^\infty |\gamma'(x+s)|^2\,dxds = \sqrt{2}\int_0^\infty x|\gamma'(x)|^2\,dx.$$

By (2.16), for $t > 0$,

$$E\|l(t)\| \leq \|l(0)\| + K\int_0^\infty |\gamma(x)|\,dx + \frac{1}{2}\int_0^\infty |\gamma(x)|^2\,dx + E\sup_{x\geq 0}|M(t,x)|.$$

By (2.17), (2.18), and the Sobolev imbedding

$$\begin{aligned} E\sup_{x\geq u}|M(t,x)|^2 \;\leq\; & C_1\left(\int_0^t\int_u^\infty |\gamma(x+s)|^2\,dxds + \int_0^t\int_u^\infty |\gamma'(x+s)|^2\,dxds\right) \\ \leq\; & R(u) < \infty\end{aligned}$$

and for an $\alpha < \frac{1}{2}$

$$\begin{aligned} E\|M(t)\|_\alpha^2 &\leq C_1 \left(\int_0^t \int_0^\infty |\gamma(x+s)|^2 \, dx ds + \int_0^t \int_0^\infty |\gamma'(x+s)|^2 \, dx ds \right) \\ &\leq R < \infty, \end{aligned}$$

where R and $R(u)$ are independent of t and $R(u) \to 0$ as $u \to \infty$. Thus

$$\sup_{t \geq 0} E\|l(t)\| < \infty.$$

Let (A6) be also satisfied. Assume from now that $l_0 = 0$. In order to prove existence of an invariant measure for the process l we will prove that the family of laws $\mathcal{L}(l(t))_{t\geq 0}$ is tight. Again by (2.16)

$$\begin{aligned} \sup_{t\geq 0} E \sup_{x \geq u} |l(t,x)| &\leq K \int_u^\infty |\gamma(x)| \, dx + \frac{1}{2} \int_u^\infty |\gamma(x)|^2 \, dx \\ &\quad + E \sup_{x\geq u} |M(t,x)| \to 0 \end{aligned} \tag{2.19}$$

as $u \to \infty$. Moreover for any $\psi \in C(\mathbb{R})$

$$\frac{|F(\psi)(x) - F(\psi)(u)|}{|x-u|^\alpha} \leq C_1 + CK \sup_{x,u\geq 0} \frac{|\psi(x) - \psi(u)|}{|x-u|^\alpha} \tag{2.20}$$

for a certain constant C_1. By (A2), (A4), and (A6)

$$\sup_{t\geq 0} E\|l(t)\|_\alpha \leq C_2 + CK \sup_{t \geq 0} E\|l(t)\|_\alpha.$$

Since $CK < 1$

$$\sup_{t\geq 0} E\|l(t)\|_\alpha \leq \frac{C_2}{1 - CK}. \tag{2.21}$$

By (2.19), (2.21), and Lemma 2.2, the family $\mathcal{L}(l(t))_{t\geq 0}$ is tight on $C_0(\mathbb{R})$. Since l is a Feller process, by the standard Krylov–Bogoliubov technique there exists an invariant measure for the process l, concentrated on $C_0(\mathbb{R})$. Existence of invariant measures for l on $C_0(\mathbb{R})$ is equivalent to existence of invariant measures for L on U. □

3. DERIVATIVES PRICING

In this section we derive formulas for caps and swaptions at different compounding frequencies (for example, quarterly and semiannually).

Consider a payer forward swap on principal 1 settled quarterly in arrears at times $T_j = T_0 + j\delta$, $j = 1, \ldots, n$. The LIBOR rate received at time T_j is set at time T_{j-1} at the level (cf. (2.1))

$$L(T_{j-1}, 0) = \delta^{-1}(P(T_{j-1}, T_j)^{-1} - 1).$$

The swap cash flows at times T_j, $j = 1, \ldots, n$ are $L(T_{j-1}, 0)\delta$ and $-\kappa\delta$ and hence the time $t\,(t \leq T_0)$ value of the swap is (cf. Brace and Musiela 1994)

$$E\left(\sum_{j=1}^{n} \frac{\beta(t)}{\beta(T_j)}(L(T_{j-1}, 0) - \kappa)\delta \mid \mathcal{F}_t\right) = P(t, T_0) - \sum_{j=1}^{n} C_j P(t, T_j), \tag{3.1}$$

where $C_j = \kappa\delta$ for $j = 1, \ldots, n-1$ and $C_n = 1 + \kappa\delta$.

The forward swap rate $\omega_{T_0}(t, n)$ at time t for maturity T_0 is that value of the fixed rate κ which makes the value of the forward swap zero; i.e.,

$$\omega_{T_0}(t, n) = \left(\delta \sum_{j=1}^{n} P(t, T_j)\right)^{-1} (P(t, T_0) - P(t, T_n)). \tag{3.2}$$

In a forward cap (res. floor) on principal 1 settled in arrears at times T_j, $j = 1, \ldots, n$ the cash flows at times T_j are $(L(T_{j-1}, 0) - \kappa)^+\delta$ (res. $(\kappa - L(T_{j-1}, 0))^+\delta$). The cap price at time $t \leq T_0$ is

$$\begin{aligned} \mathrm{Cap}(t) &= \sum_{j=1}^{n} E\left(\frac{\beta(t)}{\beta(T_j)}(L(T_{j-1}, 0) - \kappa)^+\delta \mid \mathcal{F}_t\right) \\ &= \sum_{j=1}^{n} P(t, T_j) E_{T_j}\left((L(T_{j-1}, 0) - \kappa)^+\delta \mid \mathcal{F}_t\right), \end{aligned}$$

where E_T stands for the expectation under the forward measure $\mathbb{P}_T$ defined by (cf. Musiela (1995))

$$\begin{aligned} \mathbb{P}_T &= \exp\left(-\int_0^T \sigma(t, T-t) \cdot dW(t) - \frac{1}{2}\int_0^T |\sigma(t, T-t)|^2\, dt\right)\mathbb{P} \\ &= (P(0, T)\beta(t))^{-1}\, \mathbb{P}. \end{aligned} \tag{3.3}$$

The process

$$K(t, T) = L(t, T-1) \quad 0 \leq t \leq T \tag{3.4}$$

satisfies (cf. (2.3) and Theorem 2.1)

$$\begin{aligned} dK(t,T) &= K(t,T)\gamma(t,T-t) \\ &\quad \cdot \left(\left(\frac{\delta K(t,T)}{1+\delta K(t,T)} \gamma(t,T-t) + \sigma(t,T-t) \right) dt + dW(t) \right) \\ &= K(t,T)\gamma(t,T-t) \cdot (\sigma(t,T+\delta-t)dt + dW(t)) \,. \end{aligned}$$

Moreover, the process

$$W_T(t) = W(t) + \int_0^t \sigma(s,T-s)\,ds \tag{3.5}$$

is a Brownian motion under $\mathbb{P}_T$. Consequently

$$dK(t,T) = K(t,T)\gamma(t,T-t) \cdot dW_{T+\delta}(t) \tag{3.6}$$

and hence $K(t,T)$ is lognormally distributed under $\mathbb{P}_{T+\delta}$. It follows that

$$\begin{aligned} E_{T+\delta}\left((L(T,0)-\kappa)^+ \mid \mathcal{F}_t \right) &= E_{T+\delta}\left((K(T,T)-\kappa)^+ \mid \mathcal{F}_t \right) \\ &= K(t,T)N(h(t,T)) - \kappa N(h(t,T)-\zeta(t,T)), \end{aligned}$$

where

$$h(t,T) = \left(\log \frac{K(t,T)}{\kappa} + \frac{1}{2}\zeta^2(t,T) \right) \Big/ \zeta(t,T),$$
$$\zeta^2(t,T) = \int_t^T |\gamma(s,T-s)|^2\,ds$$

and hence we have the following result.

PROPOSITION 3.1. *The cap price at time $t \le T_0$ is*

$$\begin{aligned} \mathrm{Cap}(t) &= \sum_{j=1}^{n} \delta P(t,T_j) \\ &\quad \times \left(K(t,T_{j-1})N(h(t,T_{j-1})) - \kappa N(h(t,T_{j-1}) - \zeta(t,T_{j-1})) \right). \end{aligned}$$

REMARK 3.1. The preceding $\mathrm{Cap}(t)$ formula corresponds to the market Black futures formula with discount from the settlement date. It was originally derived using a different approach and model set-up by Miltersen et al. (1994).

A payer swaption at strike κ maturing at time T_0 gives the right to receive at time T_0 the cash flows of the corresponding forward payer swap settled in arrears or, alternatively, discounted from the settlement dates $T_j = T_0 + j\delta,\ j = 1, \dots, n$ to T_0 value of the cash flows defined by $(\omega_{T_0}(T_0, n) - \kappa)^+\delta$, where $\omega_{T_0}(T_0, n)$ is given in (3.2). Hence the time $t \leq T_0$ price of the option is

$$
\begin{aligned}
(3.7)\qquad & E\left(\frac{\beta(t)}{\beta(T_0)} E\left(\sum_{j=1}^{n} \frac{\beta(T_0)}{\beta(T_j)} (\omega_{T_0}(T_0, n) - \kappa)^+\delta \mid \mathcal{F}_{T_0}\right) \mid \mathcal{F}_t\right) \\
& = E\left(\frac{\beta(t)}{\beta(T_0)} \left(1 - \sum_{j=1}^{n} C_j P(T_0, T_j)\right)^+ \mid \mathcal{F}_t\right) \\
& = E\left(\frac{\beta(t)}{\beta(T_0)} \left(E\left(\sum_{j=1}^{n} \frac{\beta(T_0)}{\beta(T_j)} (L(T_{j-1}, 0) - \kappa)\delta \mid \mathcal{F}_{T_0}\right)\right)^+ \mid \mathcal{F}_t\right),
\end{aligned}
$$

where $C_j = \kappa\delta$ for $j = 1, \dots, n-1$ and $C_n = 1 + \kappa\delta$ (cf. Brace and Musiela 1994b). Let

$$
(3.8)\qquad A = \{\omega_{T_0}(T_0, n) \geq \kappa\} = \left\{\sum_{j=1}^{n} C_j P(T_0, T_j) \leq 1\right\}
$$

be the event that the swaption ends up in the money. The second expression in (3.7) can be written as follows

$$
\begin{aligned}
(3.9)\qquad & P(t, T_0)\mathbb{P}_{T_0}(A \mid \mathcal{F}_t) - \sum_{j=1}^{n} C_j E\left(\frac{\beta(t)}{\beta(T_0)} E\left(\frac{\beta(T_0)}{\beta(T_j)} \mid \mathcal{F}_{T_0}\right) I_A \mid \mathcal{F}_t\right) \\
& = P(t, T_0)\mathbb{P}_{T_0}(A \mid \mathcal{F}_t) - \sum_{j=1}^{n} C_j P(t, T_j)\mathbb{P}_{T_j}(A \mid \mathcal{F}_t).
\end{aligned}
$$

Also for all $j = 1, \dots, n$

$$
\begin{aligned}
(3.10)\qquad P(t, T_{j-1})\mathbb{P}_{T_{j-1}}(A \mid \mathcal{F}_t) &= E\left(\frac{\beta(t)}{\beta(T_{j-1})} I_A \mid \mathcal{F}_t\right) \\
&= E\left(\frac{\beta(t)}{\beta(T_j)} \frac{1}{P(T_{j-1}, T_j)} I_A \mid \mathcal{F}_t\right) \\
&= E\left(\frac{\beta(t)}{\beta(T_j)} (1 + \delta K(T_{j-1}, T_{j-1})) I_A \mid \mathcal{F}_t\right) \\
&= P(t, T_j)\mathbb{P}_{T_j}(A \mid \mathcal{F}_t) \\
&\quad + \delta P(t, T_j) E_{T_j}(K(T_{j-1}, T_{j-1}) I_A \mid \mathcal{F}_t) \\
&= P(t, T_j)\mathbb{P}_{T_j}(A \mid \mathcal{F}_t) \\
&\quad + \delta P(t, T_j) E_{T_j}(K(T_0, T_{j-1}) I_A \mid \mathcal{F}_t),
\end{aligned}
$$

where the last equality holds because the process $\{K(t, T_{j-1}); 0 \le t \le T_{j-1}\}$ is a martingale under the measure $\mathbb{P}_{T_j}$ and the event A is $\mathcal{F}_{T_0}$ measurable (see (3.4)–(3.6) and (3.8)). Consequently we have the following result.

THEOREM 3.1. *The payer swaption price at time $t \le T_0$ is*

$$Ps(t) = \delta \sum_{j=1}^{n} P(t, T_j) E_{T_j} \left((K(T_0, T_{j-1}) - \kappa) I_A \mid \mathcal{F}_t\right).$$

To simplify further the preceding swaption formula we need to analyze first the relationships between the forward measures $\mathbb{P}_{T_j}$, defined in (3.3), as well as the corresponding forward Brownian motions W_{T_j}, given in (3.5), for $j = 1, 2, \dots, n$. We have

$$\begin{aligned} dW_{T_j}(t) &= dW(t) + \sigma(t, T_j - t)dt \\ &= dW_{T_{j-1}}(t) + (\sigma(t, T_j - t) - \sigma(t, T_{j-1} - t))\, dt \\ &= dW_{T_{j-1}}(t) + \frac{\delta K(t, T_{j-1})}{1 + \delta K(t, T_{j-1})} \gamma(t, T_{j-1} - t)\, dt. \end{aligned} \tag{3.11}$$

Also, because the process $\{K(t, T_{j-1});\ 0 \le t \le T_{j-1}\}$ satisfies

$$dK(t, T_{j-1}) = K(t, T_{j-1})\gamma(t, T_{j-1} - t) \cdot dW_{T_j}(t),$$

we have

$$\begin{aligned} d\frac{\delta K(t, T_{j-1})}{1 + \delta K(t, T_{j-1})} &= \frac{\delta K(t, T_{j-1})}{(1 + \delta K(t, T_{j-1}))^2} \gamma(t, T_{j-1} - t) \cdot dW_{T_j}(t) \\ &\quad - \frac{\delta^2 K^2(t, T_{j-1})}{(1 + \delta K(t, T_{j-1}))^3} |\gamma(t, T_{j-1} - t)|^2\, dt \\ &= \frac{\delta K(t, T_{j-1})}{(1 + \delta K(t, T_{j-1}))^2} \gamma(t, T_{j-1} - t) \cdot dW_{T_{j-1}}(t), \end{aligned} \tag{3.12}$$

and hence the process $\{(1+\delta K(t, T_{j-1}))^{-1}\delta K(t, T_{j-1}); 0 \le t \le T_{j-1}\}$ is a supermartingale under the measure $\mathbb{P}_{T_j}$ and a martingale under the measure $\mathbb{P}_{T_{j-1}}$.

Let for $t \le T_0$

$$F_{T_0}(t, T_k) = \frac{P(t, T_k)}{P(t, T_0)} \tag{3.13}$$

denote the forward price at time t for settlement at time T_0 on a T_k maturity zero coupon bond. Because we have

$$F_{T_0}(t, T_k) = \prod_{i=1}^{k} F_{T_{i-1}}(t, T_i) = \left(\prod_{i=1}^{k} (1 + \delta K(t, T_{i-1}))\right)^{-1}$$

the event A, defined in (3.8), can be written as follows

$$
\begin{aligned}
(3.14)\quad A &= \left\{ \sum_{k=1}^{n} C_k \left(\prod_{i=1}^{k} (1 + \delta K(T_0, T_{i-1})) \right)^{-1} \le 1 \right\} \\
&= \left\{ \sum_{k=1}^{n} C_k \left(\prod_{i=1}^{k} (1 + \delta K(t, T_{i-1}) \exp \left(\int_t^{T_0} \gamma(s, T_{i-1} - s) \cdot dW_{T_i}(s) \right.\right.\right. \\
&\qquad \left.\left.\left. - \frac{1}{2} \int_t^{T_0} |\gamma(s, T_{i-1} - s)|^2 \, ds \right) \right)^{-1} \le 1 \right\}.
\end{aligned}
$$

Moreover, we deduce from (3.11) that for $t \le T_0$ and $i, j = 1, \ldots, n$

$$
\begin{aligned}
(3.15)\quad dW_{T_i}(t) &= dW_{T_j}(t) + \sum_{\ell=0}^{i-1} \frac{\delta K(t, T_\ell)}{1 + \delta K(t, T_\ell)} \gamma(t, T_\ell - t) \, dt \\
&\quad - \sum_{\ell=0}^{j-1} \frac{\delta K(t, T_\ell)}{1 + \delta K(t, T_\ell)} \gamma(t, T_\ell - t) \, dt.
\end{aligned}
$$

Consequently we can write

$$
\begin{aligned}
(3.16)\quad X_i &= \int_t^{T_0} \gamma(s, T_{i-1} - s) \cdot dW_{T_i}(s) = \int_t^{T_0} \gamma(s, T_{i-1} - s) \cdot dW_{T_j}(s) \\
&\quad + \sum_{\ell=1}^{i} \int_t^{T_0} \frac{\delta K(s, T_{\ell-1})}{1 + \delta K(s, T_{\ell-1})} \gamma(s, T_{\ell-1} - s) \cdot \gamma(s, T_{i-1} - s) \, ds \\
&\quad - \sum_{\ell=1}^{j} \int_t^{T_0} \frac{\delta K(s, T_{\ell-1})}{1 + \delta K(s, T_{\ell-1})} \gamma(s, T_{\ell-1} - s) \cdot \gamma(s, T_{i-1} - s) \, ds.
\end{aligned}
$$

We will approximate the conditional on $\mathcal{F}_t$ distribution of $X_1, \ldots, X_n$ under the measure $\mathbb{P}_{T_j}$ (for each $j = 1, \ldots, n$) by the distribution of the random vector $X_1^j, \ldots, X_n^j$, where

$$
\begin{aligned}
(3.17)\quad X_i^j &= \int_t^{T_0} \gamma(s, T_{i-1} - s) \cdot dW_{T_j}(s) + \sum_{\ell=1}^{i} \frac{\delta K(t, T_{\ell-1})}{1 + \delta K(t, T_{\ell-1})} \Delta_{\ell i} \\
&\quad - \sum_{\ell=1}^{j} \frac{\delta K(t, T_{\ell-1})}{1 + \delta K(t, T_{\ell-1})} \Delta_{\ell i}
\end{aligned}
$$

and

$$
(3.18)\quad \Delta_{\ell i} = \int_t^{T_0} \gamma(s, T_{\ell-1} - s) \cdot \gamma(s, T_{i-1} - s) \, ds.
$$

In view of (3.12) this approximation corresponds to Wiener chaos order 0 approximation of the process

$$\frac{\delta K(s, T_\ell)}{1 + \delta K(s, T_\ell)} \quad s \le T_\ell \tag{3.19}$$

under the measure $\mathbb{P}_{T_\ell}$. A more accurate approximation involving Wiener chaoses of order 0 and 1 may be used as well. We found, however, that the contribution of order 1 Wiener chaos is not very significant; so we can simply replace the process (3.19) by its value at t in formula (3.15) or, because of (3.12), by the conditional expectation under $\mathbb{P}_{T_{\ell-1}}$ given $\mathcal{F}_t$.

Obviously the conditional on $\mathcal{F}_t$ distribution of $X_1^j, \ldots, X_n^j$ under the measure $\mathbb{P}_{T_j}$ is $N(\mu^j, \Delta)$, where $\Delta_{\ell i}$ is given in (3.18) and

$$\mu_i^j = \sum_{\ell=1}^{i} \frac{\delta K(t, T_{\ell-1})}{1 + \delta K(t, T_{\ell-1})} \Delta_{\ell i} - \sum_{\ell=1}^{j} \frac{\delta K(t, T_{\ell-1})}{1 + \delta K(t, T_{\ell-1})} \Delta_{\ell i}. \tag{3.20}$$

In practice the first eigenvalue of the matrix Δ is approximately 50 times larger than the second, and therefore we can assume that Δ is of rank 1, or equivalently that

$$\Delta_{\ell i} = \Gamma_\ell \Gamma_i \tag{3.21}$$

for some positive constants $\Gamma_1, \ldots \Gamma_n$. Set $d_0 = 0$ and for $i \ge 1$

$$d_i = \sum_{\ell=1}^{i} \frac{\delta K(t, T_{\ell-1})}{1 + \delta K(t, T_{\ell-1})} \Gamma_\ell, \tag{3.22}$$

then it follows from (3.20) and (3.21) that

$$\mu_i^j = \Gamma_i (d_i - d_j). \tag{3.23}$$

For all $j = 1, \ldots, n$ the function

$$f_j(x) = 1 - \sum_{k=1}^{n} C_k \left(\prod_{i=1}^{k} \left(1 + \delta K(t, T_{i-1}) \exp\left(\Gamma_i (x + d_i - d_j) - \frac{1}{2}\Gamma_i^2 \right) \right) \right)^{-1}$$

satisfies $f_j'(x) > 0$, $f_j(-\infty) = -n\delta\kappa$, $f(\infty) = 1$. Hence there is a unique point s_j such that $f_j(s_j) = 0$. Moreover, if s_0 is the solution with $j = 0$, clearly $s_j = s_0 + d_j$. Also $f_j(x) \ge 0$ for $x \ge s_j$ and therefore, using (3.14), (3.17), and (3.23) we deduce that

$$\mathbb{P}_{T_j}(A \mid \mathcal{F}_t) = \mathbb{P}_{T_j}(X_j^j \ge \Gamma_i s_j \mid \mathcal{F}_t) = N(-s_0 - d_j). \tag{3.24}$$

Moreover, standard arguments yield

$$
\begin{aligned}
(3.25)\quad E_{T_j}(K(T_0, T_{j-1})I_A \mid \mathcal{F}_t) \;=\; & E_{T_j}\Bigg(K(t, T_{j-1}) \exp\bigg(\int_t^{T_0} \gamma(s, T_{j-1}-s)\cdot dW_{T_j}(s) \\
& \qquad - \frac{1}{2}\int_t^{T_0} |\gamma(s, T_{j-1}-s)|^2\, ds\bigg) I_A \mid \mathcal{F}_t\Bigg) \\
=\; & K(t, T_{j-1})N(-s_0 - d_j + \Gamma_j)
\end{aligned}
$$

and finally we obtain the following result.

THEOREM 3.2. *The price at time $t \le T_0$ of the payer swaption can be approximated by*

$$
Psa(t) = \delta \sum_{j=1}^{n} P(t, T_j)(K(t, T_{j-1})N(-s_0 - d_j + \Gamma_j) - \kappa N(-s_0 - d_j)),
$$

where s_0 is given by

$$
\sum_{k=1}^{n} C_k \left(\prod_{i=1}^{k}\left(1 + \delta K(t, T_{i-1}) \exp\left(\Gamma_i(s_0 + d_i) - \frac{1}{2}\Gamma_i^2\right)\right)\right)^{-1} = 1,
$$

$C_k = \kappa\delta$, $k = 1, \ldots, n-1$, $C_n = 1 + \kappa\delta$, while Γ_i and d_i are defined in (3.21) and (3.22) respectively.

Proof. Follows from Theorem 3.1 and formulas (3.24) and (3.25). □

In the US, UK, and Japanese markets caps correspond to rates compounded quarterly, while swaptions are semiannual. In the German market caps are quarterly and swaptions annual. We deal with this problem by assuming lognormal volatility structure on the quarterly rates. The forward swap rate at time $t \le T_0$ is

$$
(3.26)\qquad \omega_{T_0}^{(k)}(t, n) = \left(k\delta \sum_{j=1}^{n} P(t, T_{kj})\right)^{-1} (P(t, T_0) - P(t, T_{kn}))
$$

and hence the time $t \le T_0$ price of a payer swaption at strike κ maturing at time T_0 is

$$
\begin{aligned}
Ps^{(k)}(t) \;&=\; E\left(\sum_{j=1}^{n} \frac{\beta(t)}{\beta(T_{kj})}(\omega_{T_0}^{(k)}(T_0, n) - \kappa)^+ k\delta \mid \mathcal{F}_t\right) \\
&=\; E\left(\frac{\beta(t)}{\beta(T_0)}\left(1 - \sum_{j=1}^{n} C_j^{(k)} P(T_0, T_{kj})\right)^+ \mid \mathcal{F}_t\right) \\
&=\; P(t, T_0)\mathbb{P}_{T_0}(A \mid \mathcal{F}_t) - \sum_{j=1}^{n} C_j^{(k)} P(t, T_{kj})\mathbb{P}_{T_{kj}}(A \mid \mathcal{F}_t),
\end{aligned}
$$

where $C_j^{(k)} = k\kappa\delta$ for $j = 1, \ldots, n-1$ and $C_n^{(k)} = 1 + k\kappa\delta$, while

$$A = \{\omega_{T_0}^{(k)}(T_0, n) \geq \kappa\} = \left\{ \sum_{j=1}^{n} C_j^{(k)} P(T_0, T_{kj}) \leq 1 \right\}.$$

From (3.10) it follows that for all j

$$\begin{aligned} P(t, T_{k(j-1)})\mathbb{P}_{T_{k(j-1)}}(A \mid \mathcal{F}_t) \ = \ & P(t, T_{kj})\mathbb{P}_{T_{kj}}(A \mid \mathcal{F}_t) \\ & + \delta \sum_{i=1}^{k} P(t, T_{k(j-1)+i}) E_{T_{k(j-1)+i}} \\ & \times (K(T_0, T_{k(j-1)+i-1}) I_A \mid \mathcal{F}_t), \end{aligned}$$

and hence

$$\begin{aligned} Ps^{(k)}(t) \ = \ \delta \sum_{j=1}^{n} \Bigg(& \sum_{i=k(j-1)+1}^{kj} P(t, T_i) E_{T_i} \left(K(T_0, T_{i-1}) I_A \mid \mathcal{F}_t\right) \\ & - k\kappa P(t, T_{kj}) \mathbb{P}_{T_{kj}}(A \mid \mathcal{F}_t) \Bigg). \end{aligned} \tag{3.27}$$

Repeating arguments used in the proof of Theorem 3.1 we deduce the following swaption approximation formula.

THEOREM 3.3. *Let k and δ be such that $(k\delta)^{-1}$ is the compounding frequency per year of the swap rate $\omega_{T_0}^{(k)}(t, n)$, given in (3.26). The time $t \leq T_0$ price of a payer swaption can be approximated by*

$$\begin{aligned} Psa^{(k)}(t) \ = \ \delta \sum_{j=1}^{n} \Bigg(& \sum_{i=k(j-1)+1}^{kj} P(t, T_i) K(t, T_{i-1}) N\left(-s_0^{(k)} - d_i + \Gamma_i\right) \\ & - k\kappa P(t, T_{kj}) N\left(-s_0^{(k)} - d_{kj}\right) \Bigg), \end{aligned} \tag{3.28}$$

where $s_0^{(k)}$ is given by

$$\sum_{j=1}^{n} C_j^{(k)} \left(\prod_{i=1}^{kj} \left(1 + \delta K(t, T_{i-1}) \exp\left(\Gamma_i \left(s_0^{(k)} + d_i\right) - \frac{1}{2}\Gamma_i^2\right)\right)\right)^{-1} = 1,$$

$C_j^{(k)} = k\kappa\delta$ for $j = 1, \ldots, n-1$, $C_n^{(k)} = 1 + k\kappa\delta$, and Γ_i and d_i are defined in (3.21) and (3.22) respectively.

REMARK 3.2. If one chooses $\delta = 0.25$, for example, in a market with quarterly and semiannual caps and swaptions, then formula (3.28) can be used to price the semiannual caps and swaptions and hence it can also be used to jointly calibrate to both quarterly and semiannual volatility inputs.

To analyze differences between the exact swaption value, computed by simulation, and an approximate value computed using formula (3.28) with $k = 1$ and $t = 0$, a one-factor model was fitted to U.S. cap and swaption data on 12 July 1994 generating a typical volatility structure. Simulation prices were generated under the $\mathbb{P}_{T_n}$ forward measure using the exact formula

$$P(0, T_n)E_{T_n}\left(\sum_{j=0}^{n-1} C_j \prod_{i=j+1}^{n} (1 + \delta K(T_0, T_{i-1})) + C_n\right)^+ \tag{3.29}$$

with $C_0 = 1$, $C_j = -\kappa\delta$, $j = 1, \ldots, n-1$, $C_n = -(1 + \kappa\delta)$ and

$$K(t, T_{i-1}) = K(0, T_{i-1}) \exp\left(\int_0^t \gamma(s, T_{i-1} - s) \cdot dW_{T_i}(s) - \frac{1}{2}\int_0^t |\gamma(s, T_{i-1} - s)|^2\, ds\right),$$

$$W_{T_{i-1}}(t) = W_{T_i}(t) - \int_0^t \frac{\delta K(s, T_{i-1})}{1 + \delta K(s, T_{i-1})}\gamma(s, T_{i-1} - s)\, ds.$$

The preceding equations permit the recursive calculation of the Brownian motions $W_{T_0}(t), \ldots, W_{T_{n-1}}(t)$ for $0 \leq t \leq T_0$. For each simulation of $W_{T_n}(t)$ on $[0, T_0]$ that gives values of $K(T_0, T_{i-1})$, $i = 1, \ldots, n$, substitution in (3.29) gives the corresponding value of the swaption. The simulation procedure, which involves Riemann and stochastic integration steps, was checked by back calculating the cap prices used in parametrization. The simulation prices coincided with the closed form prices calculated using the Cap(0) formula of Proposition 3.1. Table 3.1 gives the swaption prices for a range of strikes, option maturities, and swap lengths. Two standard deviation errors of simulated prices are in brackets. Bid and ask spreads, estimated by professional dealers at Citibank London, are in the last column.

We also compared formula (3.28) with the market formula for pricing swaptions, based on assuming the underlying swap rate is lognormal, and given by

$$\delta \sum_{j=1}^{n} P(0, T_j)(\omega_{T_0}(0, n)N(h) - \kappa N(h - \gamma\sqrt{T_0})), \tag{3.30}$$

where

$$h = \left(\log \frac{\omega_{T_0}(0, n)}{\kappa} + \frac{1}{2}\gamma^2 T_0\right) \Big/ \gamma\sqrt{T}_0.$$

TABLE 3.1
Accuracy of the Formula $Psa^{(1)}(0)$

Option maturity × Swap length	Strike	Simulation price	$Psa^{(1)}(0)$	Spreads
	6.00%	159.71(0.25)	160.17	4
0.25 × 2	7.00%	39.62(0.25)	40.39	2.5
	8.00%	4.25(0.25)	4.61	2
	6.25%	237.79(0.25)	238.09	5
0.25 × 3	7.25%	59.33(0.25)	60.54	3.5
	8.25%	6.11(0.25)	6.75	2
	6.60%	361.24(0.25)	362.72	10
0.25 × 5	7.60%	79.49(0.25)	81.16	6
	8.60%	5.84(0.25)	6.39	3
	6.70%	386.34(0.25)	389.79	11
0.5 × 5	7.70%	127.34(0.25)	131.43	8
	8.70%	25.99(0.25)	28.12	4
	6.60%	187.23(0.25)	188.56	7
1 × 2	7.60%	92.93(0.25)	94.76	5
	8.60%	40.29(0.25)	41.99	3
	6.75%	230.00(0.25)	231.80	10
2 × 2	7.75%	140.17(0.25)	142.60	7
	8.75%	80.54(0.25)	82.98	6
	7.50%	359.41(0.26)	363.60	20
2 × 5	8.50%	189.24(0.25)	194.47	16
	9.50%	91.64(0.25)	95.79	10
	7.00%	227.75(0.25)	230.28	11
3 × 2	8.00%	148.15(0.25)	151.14	8
	9.00%	92.68(0.25)	95.67	6
	7.00%	323.71(0.25)	327.13	16
3 × 3	8.00%	204.93(0.25)	208.80	12
	9.00%	123.43(0.25)	127.39	9
	7.00%	502.65(0.40)	506.34	27
5 × 5	8.00%	331.56(0.37)	336.90	22
	9.00%	209.39(0.34)	215.33	22
	10.00%	127.85(0.31)	133.44	18

The difference between calculated and simulated prices is well within spreads. All prices are in basis points (1 bp = \$100 per \$1M face value).

Note that because

$$E \sum_{j=1}^{n} \frac{1}{\beta(T_j)} (\omega_{T_0}(T_0, n) - \kappa)^+ \delta$$

$$= \delta \sum_{j=1}^{n} P(0, T_j) E_{T_j} (\omega_{T_0}(T_0, n) - \kappa)^+$$

TABLE 3.2
Black vs Calculated Price

Option maturity × Swap length	Strike	Black price	$Psa^{(1)}(0)$
	8%	183.88	183.88
0.25 × 1	10%	36.59	36.59
	12%	1.35	1.35
	8%	344.05	344.05
1 × 2	10%	129.36	129.35
	12%	34.87	34.87
	8%	748.02	747.97
1 × 5	10%	281.24	281.14
	12%	75.82	75.73
	8%	1204.52	1204.19
1 × 10	10%	452.88	452.20
	12%	122.08	121.60
	8%	473.29	473.21
3 × 3	10%	262.20	262.09
	12%	136.27	136.17

When yield and volatility are flat (10% and 20% respectively) the Black swaption formula and $Psa^{(1)}(0)$ are almost identical. All prices are in basis points (1 bp = \$100 per \$1M face value).

the market seems to identify the forward measures $\mathbb{P}_{T_j}$, $j = 1, \dots, n$ with the forward measure $\mathbb{P}_{T_0}$ and assumes lognormality of the swap rate process $\omega_{T_0}(t, n)$, $0 \le t \le T_0$ under the measure $\mathbb{P}_{T_0}$. In fact, formula (3.28) reduces to (3.30) if $d_i = 0$, $\Gamma_i = \Delta_{ii}^{1/2} = \gamma\sqrt{T}_0$, and $K(0, T_i) = K$. We assumed constant 10% yield (compounded quarterly) and 20% volatility in formulas (3.28) and (3.30). Table 3.2 gives the swaption prices.

4. MODEL CALIBRATION

To calibrate the model we used data from the U.K. market for Friday, 3 Feb 95. Market cash, futures, and swap rates are given in Table 4.1, together with the corresponding zero coupon discount function (ZCDF). Cap and swaption volatilities, given in Table 4.3 (or 4.4), together with the historically estimated correlation between the forward rates, given in Table 4.2, were used to compute the model volatilities. We assumed a two-factor model with a piecewise constant volatility structure $\gamma(t, x) = f(t)\gamma(x)$, where $\gamma(x) = (\gamma_1(x), \gamma_2(x))$ and $f : \mathbb{R}_+ \to \mathbb{R}$. If $f \equiv 1$ the volatility is time homogeneous so f represents the term structure of volatility. Because in the U.K. market caps are quarterly while swaptions are semiannual, we used the cap formula from Proposition 3.1 with $\delta = 0.25$ and the swaption formula from Theorem 3.3 with $k = 2$. Computed volatility functions for 3 Feb 95 are given in Table 4.3. As a comparison a one-factor normal HJM model was fitted to the same set of data. Normal volatilities for 3 Feb 95 are given in Table 4.4 (formulas 3.2 and 6.1 from

TABLE 4.1
GBP Yield Curve for 3 Feb 1995

Market rates

Tenor	Rate	Tenor	Rate
Cash 1 Month	6.68750%	Future 18 Dec 96	90.94
Cash 2 Month	6.75000%	Future 19 Mar 97	90.90
Cash 3 Month	6.78125%	Future 18 Jun 97	90.88
Cash 6 Month	7.12500%	Future 17 Sep 97	90.85
Cash 9 Month	7.50000%	Future 17 Dec 97	90.85
Future 15 Mar 95	92.94	Swap 2 year	8.265%
Future 21 Jun 95	92.26	Swap 3 year	8.550%
Future 20 Sep 95	91.83	Swap 4 year	8.655%
Future 20 Dec 95	91.52	Swap 5 year	8.770%
Future 20 Mar 96	91.31	Swap 7 year	8.910%
Future 19 Jun 96	91.15	Swap 10 year	8.920%
Future 18 Sep 96	91.04		

Zero Coupon Discount Function (ZCDF)

Tenor: x	ZCDF (x)	Tenor: x	ZCDF (x)	Tenor: x	ZCDF (x)
0.00000000	1.00000000	2.37260274	0.82165363	7.00821918	0.53980408
0.07671233	0.99489605	2.62191781	0.80338660	7.50684932	0.51675532
0.10958904	0.99268989	2.87123288	0.78546824	8.00547945	0.49467915
0.37808219	0.97422289	3.12054795	0.76794952	8.50410959	0.47353468
0.62739726	0.95577923	3.49863014	0.74363879	9.00547945	0.45317677
0.87671233	0.93669956	4.00273973	0.71105063	9.50410959	0.43378439
1.12602740	0.91730596	4.49863014	0.67976222	10.00821918	0.41501669
1.37534247	0.89785353	5.00273973	0.64895804	10.50821918	0.39720417
1.62465753	0.87847062	5.50136986	0.62032221	11.00821918	0.38015617
1.87397260	0.85927558	6.01095890	0.59213852	11.50821918	0.36383986
2.12328767	0.84029504	6.50136986	0.56589374		

The zero coupon discount function is calculated from the market rates at various tenors. Intermediate rates can be found by splining.

TABLE 4.2
Forward Rate Correlations for GBP

	0	0.25	0.5	1	1.5	2	2.5	3	4	5	7	9
0	1.0000	0.6853	0.5320	0.3125	0.3156	0.2781	0.1835	0.0617	0.1974	0.1021	0.1029	0.0598
0.25	0.6853	1.0000	0.8415	0.6246	0.6231	0.5330	0.4278	0.3274	0.4463	0.2459	0.3326	0.2625
0.5	0.5320	0.8415	1.0000	0.7903	0.7844	0.7320	0.6346	0.4521	0.5812	0.3439	0.4533	0.3661
1	0.3125	0.6246	0.7903	1.0000	0.9967	0.8108	0.7239	0.5429	0.6121	0.4426	0.5189	0.4251
1.5	0.3156	0.6231	0.7844	0.9967	1.0000	0.8149	0.7286	0.5384	0.6169	0.4464	0.5233	0.4299
2	0.2781	0.5330	0.7320	0.8108	0.8149	1.0000	0.9756	0.5676	0.6860	0.4969	0.5734	0.4771
2.5	0.1835	0.4278	0.6346	0.7239	0.7286	0.9756	1.0000	0.5457	0.6583	0.4921	0.5510	0.4581
3	0.0617	0.3274	0.4521	0.5429	0.5384	0.5676	0.5457	1.0000	0.5942	0.6078	0.6751	0.6017
4	0.1974	0.4463	0.5812	0.6121	0.6169	0.6860	0.6583	0.5942	1.0000	0.4845	0.6452	0.5673
5	0.1021	0.2439	0.3439	0.4426	0.4464	0.4969	0.4921	0.6078	0.4845	1.0000	0.6015	0.5200
7	0.1029	0.3326	0.4533	0.5189	0.5233	0.5734	0.5510	0.6751	0.6452	0.6015	1.0000	0.9889
9	0.0598	0.2625	0.3661	0.4251	0.4299	0.4771	0.4581	0.6017	0.5673	0.5200	0.9889	1.0000

Forward rates were assumed constant on the intervals between the given terms. One year of data (1994) was used to calculate this table.

TABLE 4.3
Lognormal HJM Fit for 3 Feb 1995

Currency: GBP		A-T-M	Black	Market	Average error (%): 0.64	
Contract	Length	strike (%)	volatility (%)	price (bp)	Error (bp)	Error (%)
Cap	1	7.88	15.50	27	−0.0	−0.0
Cap	2	8.39	17.75	100	2.5	2.5
Cap	3	8.64	18.00	185	0.8	0.4
Cap	4	8.69	17.75	267	0.3	0.1
Cap	5	8.79	17.75	360	−7.4	−2.1
Cap	7	8.90	16.50	511	2.5	0.5
Cap	10	8.89	15.50	703	−0.0	−0.0
	Option maturity × Swap length					
Swaption	0.25 × 2	8.57	16.75	50	−0.6	−1.2
Swaption	0.25 × 3	8.75	16.50	73	−0.1	−0.1
Swaption	1 × 4	9.10	15.50	172	−0.4	−0.2
Swaption	0.25 × 5	8.90	15.00	103	0.1	0.1
Swaption	0.25 × 7	9.00	13.75	123	1.6	1.3
Swaption	0.25 × 10	8.99	13.25	151	−0.1	−0.1
Swaption	1 × 9	9.12	13.25	271	−1.7	−0.6
Swaption	2 × 8	9.16	12.75	312	1.2	0.4

Contracts to be fitted are on the left with their at-the-money strikes and market quoted Black volatilities. Prices and the fit, obtained with the volatility functions below, are on the right. Average error in fitting is 0.64%, and the largest single error is 2.5%. Note 1 bp = \$100 per \$M face value.

Tenor: x, t	$\gamma_1(x)$	$\gamma_2(x)$	$f(t)$
0.25	0.09481393	0.12146092	1.00000000
0.50	0.08498925	0.05117321	1.00000000
1.00	0.22939966	0.09100802	0.99168448
1.50	0.19166872	0.02876211	1.00388389
2.00	0.08232925	0.01172934	1.00388389
2.50	0.18548202	0.00047705	1.07602593
3.00	0.13817885	−0.01160086	1.07602593
4.00	0.08562258	−0.04673283	1.04727642
5.00	0.14547123	−0.04181446	1.02727799
7.00	0.08869328	−0.05459175	0.96660430
9.00	0.04121240	−0.03631021	0.93012459
11.00	0.15206796	−0.16626765	0.81425256

Piecewise constant on each internal.

Brace and Musiela (1994a) were used in the process of model calibration). Lognormal and normal HJM model fits, expressed in terms of the market cap and swaption prices, are given in Table 4.3 and 4.4, respectively.

Discount functions and volatilities for other days of the week 30 Jan to 3 Feb 1995 are available in spreadsheet format on request. The inhomogeneous component $f(t)$ varies over the first 5 years from 0.934 at 0.5 year on 2 Feb 95 to 1.133 at 2 years on 1 Feb 95.

TABLE 4.4
Normal HJM Fit for 3 Feb 1995

Currency: GBP		A-T-M	Black	Market	Average error (%): 0.55	
Contract	Length	strike (%)	volatility (%)	price (bp)	Error (bp)	Error (%)
Cap	1	7.88	15.50	27	0.00	0.00
Cap	2	8.39	17.75	100	2.4	2.4
Cap	3	8.64	18.00	185	−0.8	−0.5
Cap	4	8.69	17.75	267	−0.2	−0.1
Cap	5	8.79	17.75	360	−8.9	−2.5
Cap	7	8.90	16.50	511	−5.6	−1.1
Cap	10	8.89	15.50	703	1.7	0.2
	Option maturity × Swap length					
Swaption	0.25 × 2	8.57	16.75	50	−0.0	−0.1
Swaption	0.25 × 3	8.75	16.50	73	0.3	0.5
Swaption	1 × 4	9.10	15.50	172	0.0	0.0
Swaption	0.25 × 5	8.90	15.00	103	−0.0	−0.0
Swaption	0.25 × 7	9.00	13.75	123	−0.2	−0.1
Swaption	0.25 × 10	8.99	13.25	151	0.1	0.1
Swaption	1 × 9	9.12	13.25	271	−1.3	−0.5
Swaption	2 × 8	9.16	12.75	312	−0.2	−0.1

Contracts to be fitted are on the left with their at-the-money strikes and market quoted Black volatilities. Prices and the fit, obtained with the volatility function below, are on the right. Average error in fitting is 0.55%, and the largest single error is −2.5%. Note 1 bp = $100 per $M face value.

Normal HJM volatility

Tenor: x	$\sigma(x)/x$
0.25	0.01236511
0.50	0.01212989
1.00	0.01207662
1.50	0.01692911
2.00	0.01359211
3.00	0.01385645
4.00	0.01384691
5.00	0.01270641
7.00	0.01154330
11.00	0.01093066

Piecewise constant on each interval.

For maturities beyond 5 years the inhomogeneous component drops to 0.718 at 9 and 11 years on 31 Jan 95. The quality of fit can be defined as follows

	Fit Error (%)		
	Tolerable	Satisfactory	Good
Average error	$< 2\%$	$< 2\%$	$< 1\%$
Individual error	$< 8\%$	$< 5\%$	$< 3\%$

TABLE 4.5
Implied Black Volatility of Caps and Swaptions for 3 Feb 1995

Lognormal

Cap/Swap length	Cap	Swaption maturity 0.25	0.5	1.0	2.0	3.0	4.0	5.0
1	15.50	19.12	19.86	20.42	18.05	17.34	16.31	15.79
2	18.29	16.54	16.99	17.77	16.29	15.75	15.27	14.69
3	18.09	16.48	16.44	16.42	15.24	15.05	14.56	14.02
4	17.77	15.00	15.18	15.47	14.75	14.49	14.03	13.38
5	17.35	15.02	15.08	15.12	14.30	14.03	13.46	13.04
6	16.99	14.38	14.45	14.48	13.72	13.37	13.07	
7	16.59	13.93	13.94	13.87	13.06	13.03		
8	16.21	13.18	13.20	13.14	12.80			
9	15.81	12.60	12.82	13.17				
10	15.50	13.24	13.48					

Normal

Cap/Swap length	Cap	Swaption maturity 0.25	0.5	1.0	2.0	3.0	4.0	5.0
1	15.50	17.72	19.38	19.35	16.39	16.75	15.27	14.37
2	18.25	16.74	16.51	16.45	15.70	15.24	14.09	13.22
3	17.91	16.58	16.49	16.46	15.21	14.46	13.39	12.93
4	17.74	16.25	15.95	15.50	14.27	13.54	12.88	12.51
5	17.27	14.99	14.78	14.48	13.43	13.01	12.49	12.20
6	16.79	14.24	14.06	13.79	13.05	12.72	12.29	
7	16.30	13.73	13.62	13.49	12.85	12.57		
8	16.01	13.54	13.44	13.32	12.74			
9	15.76	13.38	13.30	13.19				
10	15.54	13.26	13.18					

With the determined parametrizations, the Black volatilities for at-the-money contracts change smoothly from maturity to maturity and between different underlying swap lengths. That property is important because many dealers presently value swaptions by building similar matrices (by various means) and then using the Black formulas.

The normal HJM model can be almost always fitted to the U.K. and U.S. caps and swaptions data with a one-factor homogeneous volatility; fitting the correlation with a second factor improves the overall fit. The lognormal HJM model frequently cannot fit a term structure of volatility in the lognormal case and may also indicate that the price volatility of the normal HJM is more stable over time than the yield volatility of the lognormal HJM. The implied Black volatilities of caps and swaptions, Table 4.5, for both models are quite similar with the lognormal volatilities being 1% to 1.5% greater than the normal at longer swaption maturities. That probably reflects the different impact of correlation on the two models.

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