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Abstract

Using an intuitive approach that also provides new intuition concerning the Black and Scholes equation, this paper extends the results of Johnson and Stulz to the pricing of options on the minimum or the maximum of *several* risky assets.

I. Introduction

By laborious calculation, Johnson [9] and Stulz [15] independently derived prices for options on the maximum and the minimum of two assets. This paper presents a simple, intuitive way, using the Cox and Ross [3] approach and a trick based on a device used by Margrabe [10], to write down the solution for the general case of an option on several assets. The result could be useful for pricing, among other things, currency option bonds, portfolio insurance, and the quality option in commodities contracts (see [6]).

In the next section, we first illustrate the procedure for the Black and Scholes equation, thereby obtaining some new intuition about this equation, and then develop the equations for calls on the maximum and the minimum. Section III is a summary.

II. The Pricing of a Call on the Maximum or the Minimum

First consider the Black and Scholes [1] equation. We can write the solution as

$$c = SN\left(d_1\right) - Xe^{-rT}N\left(d_2\right) ,$$

$$\text{where } d_2 = \frac{\log S/X + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} ,$$

$$d_1 = d_2 + \sigma\sqrt{T} ,$$

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$N(x)$ is the standard cumulative normal, and c, S, X, r, σ^2, and T are the call price, stock price, exercise price, risk-free rate, variance of the rate of return on the stock, and time to expiration, respectively. Now, using the Cox and Ross approach, $N(d_2)$ can be interpreted as the probability (in a risk-neutral world) that the call will be exercised, i.e., the probability that S^*, the stock price at expiration, will be greater than X given that the stock price is S today. In the Cox and Ross approach, we ordinarily come up with $N(d_1)$ by taking the expectation of S^* for $S^* > X$, changing variables, completing the square, etc., until the desired form is obtained. However, if we use a trick based on a device first introduced in Margrabe [10], the $N(d_1)$ term can be written down immediately. A call option can be thought of as an option to exchange cash for a common stock. Thus, it is like Margrabe's option to exchange one asset for another. Hence, it can be valued by a change in numeraire, just as in Margrabe's case. In our case, we use the stock price as numeraire. (Note that Garman and Hawkins [5] used this same approach with regard to currency options.) The call price measured in units of the stock price looks like a European put on a risky asset with current price $x \equiv Xe^{-rT}/S$, with unit exercise price, and a zero interest rate,

$$\frac{c}{S} = 1 \cdot N\left(-d_2'\right) - xN\left(-d_1'\right) ,$$

$$\text{where } d_2' = \frac{\log x - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$\text{and } d_1' = d_2' + \sigma\sqrt{T} .$$

In this world, the stock price measured in units of itself is just the risk-free asset, with a zero return. By Ito's lemma,

$$\frac{dx}{x} = \left(r - \mu + \sigma^2\right)dt - \sigma dz ,$$

where $t = -T$ is calendar time. Thus, the variance of the rate of return on x is just σ^2. In the Cox and Ross approach, we replace the drift term, $r - \mu + \sigma^2$, by the risk-free rate, which, in this case, is zero. Thus, $N(d_1) \equiv N(-d_2')$ is just the risk-neutral probability that this put will be exercised. Not only do we have this intuitive interpretation of $N(d_1)$, but, had we not known this factor, we could have written it down without going through any laborious calculations. While Smith [14] has noted that Boness [2] had a probability interpretation for $N(d_1)$, Boness evidently still had to do all the computations to find $N(d_1)$. These computations become extremely laborious when there are many possible exercise dates or many underlying assets. It should be noted that Merton ([11], [12]) also recognized the usefulness of the fact that an option price is generally linearly homogeneous in the stock price and the exercise price. We next apply our procedure to value calls on the maximum and the minimum.

We make the usual perfect market and European option assumptions. Consider n assets with current prices $S_1, S_2, \ldots, S_n$. We assume that each asset price follows geometric Brownian motion and that there are no dividends.

Consider a call on the maximum of the n assets with exercise price X and time to maturity T. Then, by the Cox and Ross [3] approach, we know one term in the expression for the price, $c_{\max}$, of this call must be the negative of the discounted exercise price multiplied by the probability (in a risk-neutral world) that at least one of the asset prices will be greater than X. Hence, one term is

$$-Xe^{-rT}\left[1 - \text{Prob}\left(S_1^*, S_2^*, \ldots, S_n^* < X\right)\right],$$

where asterisks denote values at maturity. This probability expression is simply

$$1 - N_n\left(-d_2\left(S_1, X, \sigma_1^2\right), -d_2\left(S_2, X, \sigma_2^2\right), \ldots, -d_2\left(S_n, X, \sigma_n^2\right), \rho_{12}, \rho_{13}, \ldots\right),$$

$$\text{where, in general, } d_2\left(s, x, \sigma^2\right) = \frac{\log \frac{s}{x} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}},$$

N_i is the i-variate standard cumulative normal, σ_i^2 is the variance of the rate of return on the ith asset, and ρ_{ij} is the correlation coefficient for the returns on the ith and jth assets.

We still need to identify the positive terms in the expression for $c_{\max}$. We identify each of these terms, one by one, using a change of numeraire. Using the ith asset as numeraire means that we measure $c_{\max}$ in units of S_i, the ith asset price, i.e., divide by S_i. Then the call on the maximum in units of S_i is transformed into a complex security that consists of (1) a European put with unit exercise price, where the put is exercised by surrendering a risky asset, the current price of which is $x_i \equiv Xe^{-rT}/S_i$, provided

$$S_i^* = \max_{j=1,n} S_j^*,$$

(2) an option to exchange the risky asset for S_2, provided

$$S_2^* = \max_{j=1,n} S_j^*, \ldots,$$

and (n) an option to exchange the risky asset for S_n, provided

$$S_n^* = \max_{j=1,n} S_j^*.$$

In this transformed world, the interest rate is zero. The point of making this transformation is that we can immediately identify the positive term in the put as the risk-neutral joint probability that $x_i^* < 1$, $S_1^*/S_i^* < 1, \ldots, S_n^*/S_i^* < 1$. Thus, this term, which, when multiplied by S_i, is the ith term in the expression for $c_{\max}$, can be written as

$$N_n\left(-d_2'\left(x_i, 1, \sigma_i^2\right), -d_2'\left(S_1, S_i, \sigma_{1i}^2\right), \ldots, -d_2'\left(S_n, S_i, \sigma_{ni}^2\right), \rho_{i1i}, \rho_{i2i}, \ldots\right),$$

or, equivalently, as

$$N_n\left(d_1\left(S_i, X, \sigma_i^2\right), d_1'\left(S_i, S_1, \sigma_{1i}^2\right), \ldots, d_n'\left(S_i, S_n, \sigma_{ni}^2\right), \rho_{i1i}, \rho_{i2i}, \ldots\right),$$

where primes indicate that the interest rate is set to zero, where, in general,

$$\sigma_{ij}^2 = \sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2 ,$$

and where the correlation coefficients are to be determined. Thus, we can identify each of the positive terms in the expression for $c_{\max}$ as the ith asset price multiplied by a probability term defined in terms of various d_1s. We therefore have

$$\begin{aligned}(1)\qquad c_{\max} = {} & S_1 N_n\Big(d_1\left(S_1, X, \sigma_1^2\right), d_1'\left(S_1, S_2, \sigma_{12}^2\right), \ldots , \\ & \quad d_1'\left(S_1, S_n, \sigma_{1n}^2\right), \rho_{112}, \rho_{113}, \ldots\Big) \\ & + S_2 N_n\Big(d_1\left(S_2, X, \sigma_1^2\right), d_1'\left(S_2, S_1, \sigma_{12}^2\right), \ldots , \\ & \quad d_1'\left(S_2, S_n, \sigma_{2n}^2\right), \rho_{212}, \rho_{223}, \ldots\Big) \\ & + \ldots \\ & + S_n N_n\Big(d_1\left(S_n, X, \sigma_n^2\right), d_1'\left(S_n, S_1, \sigma_{1n}^2\right), \ldots , \\ & \quad d_1'\left(S_n, S_{n-1}, \sigma_{n-1n}^2\right), \rho_{n1n}, \rho_{n2n}, \ldots\Big) \\ & - Xe^{-rT}\Big(1 - N_n\Big(-d_2\left(S_1, X, \sigma_1^2\right), -d_2\left(S_2, X, \sigma_2^2\right), \ldots , \\ & \quad -d_2\left(S_n, X, \sigma_n^2\right), \rho_{12}, \rho_{13}, \ldots\Big)\Big) ,\end{aligned}$$

(2) where

$$d_1'\left(S_i, S_j, \sigma_{ij}^2\right) = \frac{\log \dfrac{S_i}{S_j} + \dfrac{1}{2}\sigma_{ij}^2 T}{\sigma_{ij}\sqrt{T}} ,$$

and where the triple indexed correlation coefficients are found as follows. We have

$$\begin{aligned}(3)\qquad \mathrm{Cov}\left(\log S_1^*, \log \frac{S_i^*}{S_j^*}\right) &= \mathrm{Var}\left(\log S_i^*\right) - \mathrm{Cov}\left(\log S_i^*, \log S_j^*\right) \\ &= \sigma_i^2 - \rho_{ij}\sigma_i\sigma_j .\end{aligned}$$

But the left-hand side can also be defined as $\sigma_i\sigma_{ij}\rho_{iij}$. Thus,

$$(4)\qquad \rho_{iij} = \frac{\sigma_i - \rho_{ij}\sigma_j}{\sigma_{ij}} .$$

Similarly,
$$\mathrm{Cov}\left(\log \frac{S_i^*}{S_k^*}, \log \frac{S_i^*}{S_j^*}\right) = \sigma_i^2 - \rho_{ij}\sigma_i\sigma_j - \rho_{ik}\sigma_i\sigma_k + \rho_{jk}\sigma_j\sigma_k ,$$

(5) so that
$$\rho_{ijk} = \frac{\sigma_i^2 - \rho_{ij}\sigma_i\sigma_j - \rho_{ik}\sigma_i\sigma_k + \rho_{jk}\sigma_j\sigma_k}{\sigma_{ij}\sigma_{ik}} .$$

Tilley and Latainer [16] state an equation resembling (1). However, they leave the factors multiplying $S_1, \dots, S_n$, and Xe^{-rT} undefined.

For $n = 2$, the expression simplifies to

$$
\begin{aligned}
c_{\max} = {} & S_1 N_2\left(d_1\left(S_1, X, \sigma_1^2\right), d_1'\left(S_1, S_2, \sigma_{12}^2\right), \rho_{112}\right) \\
(6) \qquad & + S_2 N_2\left(d_1\left(S_2, X, \sigma_2^2\right), d_1'\left(S_2, S_1, \sigma_{12}^2\right), \rho_{212}\right) \\
& - Xe^{-rT}\left(1 - N_2\left(-d_2\left(S_1, X, \sigma_1^2\right), -d_2\left(S_2, X, \sigma_2^2\right), \rho_{12}\right)\right),
\end{aligned}
$$

which is consistent with the equations in Johnson [9] and Stulz [15], correcting for the typographical error in equation (11) of Stulz ($\sigma^2\sqrt{\tau}$ should be $\sigma^2\tau$).

Similarly, for the option on the minimum we have

$$
\begin{aligned}
c_{\min} = {} & S_1 N_n\left(d_1\left(S_1, X, \sigma_1^2\right), -d_1'\left(S_1, S_2, \sigma_{12}^2\right), \dots, \right. \\
& \left. -d_1'\left(S_1, S_n, \sigma_{1n}^2\right), -\rho_{112}, -\rho_{113}, \dots, \rho_{123}, \dots\right) \\
(7) \qquad & + S_2 N_n\left(d_1\left(S_2, X, \sigma_2^2\right), -d_1'\left(S_2, S_1, \sigma_{12}^2\right), \dots, \right. \\
& \left. -d_1'\left(S_2, S_n, \sigma_{2n}^2\right), -\rho_{212}, -\rho_{223}, \dots, \rho_{213}, \dots\right) \\
& + \dots \\
& + S_n N_n\left(d_1\left(S_n, X, \sigma_n^2\right), -d_1'\left(S_n, S_1, \sigma_{1n}^2\right), \dots, \right. \\
& \left. -d_1'\left(S_n, S_{n-1}, \sigma_{n-1n}^2\right), -\rho_{n1n}, \rho_{n2n}, \dots, \rho_{n12}, \dots\right) \\
& - Xe^{-rT} N_n\left(d_2\left(S_1, X, \sigma_1^2\right), d_2\left(S_2, X, \sigma_2^2\right), \dots, \right. \\
& \left. d_2\left(S_n, X, \sigma_n^2\right), \rho_{12}, \rho_{13}, \dots\right),
\end{aligned}
$$

which for $n = 2$ reduces to

$$
\begin{aligned}
c_{\min} = {} & S_1 N_2\left(d_1\left(S_1, X, \sigma_1^2\right), -d_1'\left(S_1, S_2, \sigma_{12}^2\right), -\rho_{112}\right) \\
(8) \qquad & + S_2 N_2\left(d_1\left(S_2, X, \sigma_2^2\right), -d_1'\left(S_2, S_1, \sigma_{12}^2\right), -\rho_{212}\right) \\
& - Xe^{-rT} N_2\left(d_2\left(S_1, X, \sigma_1^2\right), d_2\left(S_2, X, \sigma_2^2\right), \rho_{12}\right),
\end{aligned}
$$

which agrees with the corrected version of equation (11) of Stulz.

Note that there is linear homogeneity in that

$$
(9) \qquad c_{\max} = S_1 \frac{\partial c_{\max}}{\partial S_1} + \dots S_n \frac{\partial c_{\max}}{\partial S_n} + X \frac{\partial c_{\max}}{\partial X}
$$

and similarly for $c_{\min}$. Also note that while $c_{\min}$ decreases as more assets are added, $c_{\max}$ increases and can have a very large value. Consider, for example, the case $S_1 = S_2 = \dots = S_n = X$ and T very large. Then the d_1 terms for $c_{\max}$ are very large while Xe^{-rT} is very small. Thus, $c_{\max}$ approaches $S_1 + S_2 + \dots + S_n$. If one had a thousand-year call on the maximum of all the stocks on the New York Stock Exchange, this would mean obtaining the best performing stock for an exercise price that is negligible in present value, and, provided no pair of stocks is perfectly positively correlated, the value of owning all the others together would very likely be small compared to the value of that best performing stock. In fact, for $n = 2$, $S_1 = S_2 = X = 40$, $r = 0.1$, $\sigma_1 = \sigma_2 = 0.3$, $\rho = 0.5$, and $T = 1$, 10, and 100 years, we obtain $c_{\max} = 9.96$, 40.54, and 74.65, respectively. Finally, note that the $n = 2$ identity, proved in Stulz [15] and Johnson [9],

$$c_{\max} + c_{\min} = c\left(S_1, X\right) + c\left(S_2, X\right), \tag{10}$$

where $c(S_i,X)$ is an ordinary call on S_i with exercise price X, can be extended to the general case by introducing calls on the second best, third best, etc.

Puts can be handled in the same way. See Geske [7], Geske and Johnson [8], and Schervish [13] for numerical methods for evaluating the multivariate normals. See Dothan and Williams [4] for a different application of complex options.

III. Summary

This paper develops equations for the prices of calls on the maximum and the minimum of several risky assets. The equations reduce to the results of Stulz [15] and Johnson [9] when there are only two assets. The technique used in this paper not only provides more intuition about the Black and Scholes equation, but can also be used to derive other results, such as those in Geske and Johnson [8], in a simple way.

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