An Algorithm for Computing Values of Options on the Maximum or Minimum of Several Assets

Phelim P. Boyle and Y. K. Tse*

Abstract

An approximate method is developed for computing the values of European options on the maximum or the minimum of several assets. The method is very fast and is accurate for parameter ranges that are often of the most interest. The approach casts the problem in terms of order statistics and can be used to handle situations where the initial asset prices, the asset variances, and the covariances are all unequal. Numerical values are given to illustrate the accuracy of the method.

I. Introduction

Options on the maximum and options on the minimum of several assets are of both theoretical and practical interest. Stulz (1982) developed closed form expressions for European options in the case of two underlying assets. Johnson (1987) extended these results to handle European options in the case of n assets. Boyle, Evnine, and Gibbs (1989) have used a multinomial lattice method to value American options when there are several underlying assets. When the number of assets exceeds two, however, the computations quickly become very burdensome.

Since a number of corporate securities contain imbedded options whose payoffs depend on two or more state variables, there is some interest in obtaining methods to price these options. In addition, the quality option, which is present in a number of important futures contracts, can be valued in terms of options on the minimum of the set of deliverable assets. Under the quality option, the short position can deliver any one of a set of acceptable assets, and the existence of this option reduces the futures price. Several recent papers have examined the impact of the quality option. Gay and Manaster (1984) analyzed its impact in the case of wheat futures contracts by assuming that there were just two deliverable types of

* First author, School of Accountancy, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada, and University of Illinois, Urbana-Champaign; second author, Department of Economics, National University of Singapore, Singapore 0511, and University of Illinois, Urbana-Champaign. This paper was written while both authors were visiting the University of Illinois, Urbana-Champaign, whose hospitality is gratefully acknowledged. The authors are grateful to Ruth Cornale for research assistance and also to two anonymous *JFQA* referees and to *JFQA* Managing Editor Paul Malatesta for constructive comments. Part of this research was funded by a grant from the Natural Sciences and Engineering Research Council of Canada.

wheat. Boyle (1989) extended this analysis to the case of n deliverable assets and obtained numerical results by imposing very strong symmetry conditions on the problem. He assumed that all the assets had the same initial price, the same variance, and that all the covariances were equal.

The quality option is also of considerable importance in the case of the Treasury bond futures contract. Cheng (1985), Hemler (1988), and Chowdry (1986), among others, have analyzed this situation. For tractability, these authors assumed a multivariate lognormal diffusion process. Carr (1988) analyzed the impact of the delivery option on bond futures prices using a more realistic interest rate model: he assumed mean-reverting diffusion processes for the spot interest rate and the long-term interest rate.

The aim of the current paper is to present an approximation method that computes the value of options on the maximum or the minimum of several assets. The method is applicable to the situation in which the asset prices follow a multivariate lognormal distribution. It can handle cases in which the asset prices are unequal, and it does not require that the variance-covariance matrix of the asset returns has any particular structure. The basic idea is to analyze the problem of valuing options on the maximum or the minimum in terms of order statistics. The algorithm uses an approximation method due to Clark (1961) for computing the first four moments of the maximum (or the minimum) of n jointly normal random variables. Lerman and Manski (1981) provide evidence of the accuracy of the Clark approach.

In the next section, we describe the Clark approach. It is a recursive procedure that only involves the computation of the univariate normal cumulative distribution function at each step. This method can be used to approximate the first four moments of the extreme order statistics of a set of multivariate normal random variables. We show how these moments can be used to obtain the expected values of the extreme order statistics in the case of a multivariate lognormal distribution. As shown in Section III, this can then be used to derive the value of the quality option, since the futures price in this case is related to a European call option on the minimum of n assets with a strike price of zero. This call option value can be computed in terms of the expectation of the lowest order statistic. Some numerical values are given and compared with those computed by other methods.

In Section IV, we describe how the Clark approach can be modified to deal with censored distributions. The objective here is to obtain a method to value options with a nonzero strike price. The details are given in the Appendix. We present numerical results to illustrate the accuracy of the approximation. For plausible parameter values, the algorithm gives values that are within 1 or 2 percent of the accurate value. We indicate the range of accuracy of the approximation. The final section contains some concluding comments.

II. The Clark Algorithm

In this section, we describe the Clark algorithm and indicate how it can be used to obtain the first four moments of the maximum of a set of normal variates.

Clark (1961) derived exact expressions for the first four moments of the maximum of a pair of jointly normal variates as well as the correlation coefficient between the maximum of the pair and a third normal variate. Suppose that X_1 and X_2 have a bivariate normal distribution, where X_i has mean μ_i and standard deviation σ_i, $i = 1, 2$. The correlation coefficient between the two variables is ρ. Let Y denote the maximum of the pair (X_1, X_2). Denote the first four moments (about zero) of Y by $\mathrm{MOM}_j, j = 1, \ldots, 4$. Then

$$\mathrm{MOM}_1 = \mu_1 N(h) + \mu_2 [N(-h)] + \sigma\phi(h) ,$$

$$\mathrm{MOM}_2 = \left[\mu_1^2 + \sigma_1^2\right] N(h) + \left[\mu_2^2 + \sigma_2^2\right] [N(-h)] + \left[\mu_1 + \mu_2\right] \sigma\phi(h) ,$$

$$\begin{aligned}\mathrm{MOM}_3 = {} & \left[\mu_1^3 + 3\mu_1\sigma_1^2\right] N(h) + \left[\mu_2^3 + 3\mu_2\sigma_2^2\right] [N(-h)] \\ & + \Big\{\left[\mu_1^2 + \mu_1\mu_2 + \mu_2^2\right]\sigma \\ & + \Big[2\sigma_1^4 + \sigma_1^2\sigma_2^2 + 2\sigma_2^4 - 2\sigma_1^3\sigma_2\rho \\ & \quad - 2\sigma_1\sigma_2^3\rho - \sigma_1^2\sigma_2^2\rho^2\Big] [1/\sigma]\Big\}\phi(h) ,\end{aligned}$$

$$\begin{aligned}\mathrm{MOM}_4 = {} & \left[\mu_1^4 + 6\mu_1^2\sigma_1^2 + 3\sigma_1^4\right] N(h) + \left[\mu_2^4 + 6\mu_2^2\sigma_2^2 + 3\sigma_2^4\right] [N(-h)] \\ & + \Big\{\left[\mu_1^3 + \mu_1^2\mu_2 + \mu_1\mu_2^2 + \mu_2^3\right]\sigma - 3h\left[\sigma_1^4 - \sigma_2^4\right] \\ & + 4\mu_1\sigma_1^3\left[3\left[\sigma_1 - \rho\sigma_2\right][1/\sigma] - \left[\sigma_1 - \rho\sigma_2\right]^3\left[1/\sigma^3\right]\right] \\ & + 4\mu_2\sigma_2^3\left[3\left[\sigma_2 - \rho\sigma_1\right][1/\sigma] - \left[\sigma_2 - \rho\sigma_1\right]^3\left[1/\sigma^3\right]\right]\Big\}\phi(h) ,\end{aligned}$$

$$\begin{aligned}\text{where } \sigma^2 &= \sigma_1^2 - 2\sigma_1\sigma_2\rho + \sigma_2^2, \\ h &= [\mu_1 - \mu_2]/\sigma,\end{aligned}$$

$\phi(h)$ is the univariate standard normal density function,
and $N(h)$ is the univariate cumulative standard normal distribution function.

Suppose that the corrrelation coefficient between X_i and a third variable X_3 is ρ_{i3}, where $i = 1, 2$, then Clark obtained the following expression for the correlation coefficient between Y and X_3, ρ_{Y3},

$$\rho_{Y3} = \frac{\left[\sigma_1\rho_{13}N(h) + \sigma_2\rho_{23}[N(-h)]\right]}{\left[\mathrm{MOM}_2 - \mathrm{MOM}_1^2\right]^{0.5}} .$$

We remark that these expressions for the moments of Y and the correlation between Y and X_3 are exact. Furthermore, since Y is the maximum of two normal variates, the distribution of Y cannot be exactly normal.

Clark's exact results can be used to approximate the first four moments of the maximum of a set of n normal variates. The method proceeds recursively and the computations at each stage are very simple. Although the results are approximate, previous research (Clark (1961) and Lerman and Manski (1981)) attests to their accuracy over a range of assumptions.

Assume we have n jointly normal variates: $X_1, X_2, \ldots, X_n$ with known means, variances, and correlation coefficients. Let Y denote the maximum of these n variates. The following definitions are useful.

$$Y_1 = \max\left[X_1, X_2\right] ,$$

$$Y_2 = \max\left[X_1, X_2, X_3\right] = \max\left[Y_1, X_3\right] ,$$

$$\vdots$$

$$Y_i = \max\left[X_1, X_2, \ldots, X_{i+1}\right] = \max\left[Y_{i-1}, X_{i+1}\right] .$$

Hence,

$$Y = Y_{n-1} = \max\left[Y_{n-2}, X_n\right] .$$

By applying Clark's algorithm at each step, we can set up a recursive procedure to compute the first four moments of Y. We begin by computing the mean and variance of Y_1. In addition, we obtain the correlation coefficients of Y_1 with the remaining $(n-2)$ variates. We now *assume* the joint distribution of Y_1 and the variates $X_3, \ldots, X_n$ is multivariate normal. This assumption is obviously not correct but the virtue of Clark's method is that it nonetheless enables us to obtain quite accurate answers. We proceed in an iterative fashion until $Y_{n-1} = Y$. At this stage, we apply Clark's algorithm to obtain the first four moments of Y.

III. The Quality Option

The algorithm just described can be extended to compute the value of the futures price in the presence of a quality option. It can be shown that, in some circumstances, this futures price can be expressed in terms of a European call option, with a zero strike price, on the minimum of the assets in the deliverable set. Boyle (1989) uses this approach and we follow his notation and assumptions. The European call, with the strike price equal to zero, on the minimum of the n assets, can be expressed in terms of the expected value of the minimum of the n assets. Since the asset prices are assumed to follow a lognormal distribution, we need to modify the procedure described above since it is valid for a multivariate normal distribution. The modification exploits the functional relationship between a given (multivariate) lognormal distribution and its associated (multivariate) normal distribution. Clark's procedure is used to estimate the first four moments of the minimum[1] of the associated multivariate normal distribution. These moments are used to derive a Taylor series expression for the expected value of the minimum of the n assets.

The European call option on the minimum of n assets with zero strike price is denoted by

$$\mathrm{EUR}\left[t,\ \left(A_1(t), A_2(t), \ldots, A_n(t)\right),\ 0,\ t+T\right] ,$$

[1] Although we described Clark's algorithm in terms of the maximum of the n variates, it can readily be modified to handle the minimum of n variates. The details are provided later in this section.

where t denotes current time, $(t+T)$ denotes the expiration date of the option, and $A_i(t)$ denotes the current price of asset i. The value of this European call option may be computed using the risk-neutral approach pioneered by Cox and Ross (1976). Under this approach, the current option price is equal to its expected terminal value under the equivalent martingale measure. Hence, the European call can be written as the discounted expectation of the minimum of the n assets at the expiration date; i.e.,

$$e^{-RT}\hat{E}\Big[\min\Big(A_1(t+T),\ A_2(t+T), \ldots, A_n(t+T)\Big)\Big] ,$$

where R is the (assumed constant) riskless rate, and $\hat{E}$ denotes the expectation over the risk-adjusted distribution of terminal asset prices.

To simplify the notation, we let

$$A_i(t+T) = A_i \qquad i = 1, \ldots, n ,$$

$$\text{and } B_i = \log_e\big(A_i\big) \qquad i = 1, \ldots, n .$$

The B_i variates have, thus, a multivariate normal distribution. The required expectation becomes

$$\begin{aligned} &\hat{E}\Big[\min\Big(\exp\big[B_1\big],\ \exp\big[B_2\big], \ldots, \exp\big[B_n\big]\Big)\Big] \\ &= \hat{E}\Big[\exp\Big(\min\Big(B_1, B_2, \ldots, B_n\Big)\Big)\Big] \\ &= \hat{E}\Big[\exp\Big(-\max\Big(-B_1,\ -B_2, \ldots, -B_n\Big)\Big)\Big] \\ &= \hat{E}\left[\exp(-W)\right] , \end{aligned}$$

where $W = \max(-B_1, -B_2, \ldots, -B_n)$.

Since $B_1, B_2, \ldots, B_n$ are jointly normal, $-B_1, -B_2, \ldots, -B_n$ are also jointly normal. We can use Clark's procedure to compute the first four moments of W. We denote the mean of W by μ and higher order moments about the mean by μ_i, for $i = 2$, 3, and 4. The required expectation can be written as a Taylor series expansion in terms of these moments, i.e.,

$$\hat{E}(\exp[-W]) = \exp(-\mu)\left[1 + \frac{\mu_2}{2!} - \frac{\mu_3}{3!} + \frac{\mu_4}{4!}\right] . \tag{1}$$

It may be of interest to compare the numerical values of the size of the quality option with those published recently by Boyle (1989) using a different technique. Boyle (1989) also estimated the expected value of the lowest order statistic of the lognormal distribution of asset prices and computed the value of the quality option by discounting its expected terminal value in a risk-neutral world; however, the method used results from the statistical literature on order statistics, which assumed very strong symmetry. In particular, he had to assume that all the assets had the same initial value, that all asset returns had the same variance, and that the correlations between each pair of asset returns were equal. The method used in the present paper does not impose these restrictions. Table 1 compares the results obtained using Equation (1) of this paper with those published by Boyle (1989). For all computations, the assets have the same initial value of $40 and the same standard deviation of 25 percent per year. The time to maturity is 9 months and the risk-free rate of return is 10 percent per year. In addition, the correlation coefficient between each pair of assets is assumed to be equal to 0.95. Table 1 compares the two methods as the number of assets in the deliverable set increases. For 20 assets, the option value according to Equation (1) is 36.522, whereas Boyle (1989) computes it as 36.511, giving a relative error[2] of 0.03 percent.

For small numbers of deliverable assets, the agreement is exceptionally good and even for 50 assets the relative error is only 0.06 percent. One advantage of the procedure developed in this paper is that it can handle unequal variances, covariances, and initial asset prices. The procedure proposed by Boyle (1989) imposes strong symmetry in that the variances and correlations are assumed to be equal.

IV. Options on the Maximum and Minimum of Several Assets

The procedure developed in the previous section to compute the value of a European call, with a zero strike price, on the minimum of several assets also could be used to compute the price of a European call, with a zero strike price, on a maximum of n assets. In addition, the procedure can be extended to value European options on the maximum or the minimum of several assets when the strike price is nonzero. In this case, we need to compute the moments of the extreme order statistics of a censored distribution. The technical development of the procedure is given in the Appendix. To illustrate the method, we compare the results with the accurate values[3] obtained by integrating the multivariate normal density. The present method is much simpler from a computational viewpoint.

To examine the accuracy of the proposed approximation, we compared the approximate values with the accurate results for a range of values of the underlying parameters. Tables 2 through 7 give the results when there are three underlying assets. Because of the nature of the problem, there are a large number of degrees of freedom and we can only present a subset of the results obtained. In Tables 2 through 7, we provide accurate and approximate values for European call options on the maximum and European options on the minimum of three assets. The agreement is very good for the range of parameters considered. Table 2 considers a symmetric situation in which the initial asset prices, the asset return standard deviations, and the pairwise correlation coefficients are equal. The other tables introduce varying degrees of asymmetry in the initial conditions. In all cases, the approximate values are within pennies of the exact answers. The agreement between the approximate option values and the accurate option values is especially close for the base case presented in Table 2. Thus, for example, in the case of the at-the-money call option on the maximum of the three assets, the

[2] Relative error was computed as the ratio of the option value obtained by Equation (1) minus the option value obtained by Boyle (1989) to the option value obtained by Boyle (1989).

[3] Accurate values were obtained from Johnson (1987).

approximate value of 8.984 is very close to the accurate value of 8.986, giving a relative error[4] of 0.02 percent. The largest relative errors occur in Table 5, which is based on unequal standard deviations and correlation coefficients of 0.6, 0.4, and 0.6. Even in this case, the approximation produces good agreement with the accurate option value for calls on the maximum. The largest relative error of 0.383 percent occurs when the strike price is 45. In the case of call options on the minimum, the relative errors become larger as the strike price increases. For these parameter values, however, both option prices are small in absolute terms, so that the absolute difference is not so dramatic.

To test the robustness of the approximation, we carried out further numerical simulations. To conserve space, we report the general conclusions from these sensitivity tests rather than the detailed numerical values. In one set of tests, we used different interest rate assumptions; 5 percent p.a. and 15 percent p.a. This did not change the accuracy of the approximation very much. We also conducted the same comparison for different time-to-maturity assumptions. The conclusion here is that, for times to maturity of up to 5 years, the approximation is still very good. The maximum relative error is the order of half a percent. For longer terms to maturity, however, the approximation is less accurate. For example, for a twenty-year maturity, the relative error is around 8 percent. Similar results were obtained when we used four assets and compared the approximate values with the accurate ones. The overall conclusion is that the approximation is quite accurate as long as the time to maturity is five years or less.

V. Concluding Remarks

The approximate method described in this note is very convenient for evaluating options on the minimum of several assets when the strike price is zero. Such options can be used to compute the price of certain futures contracts when there is a quality option. The method was extended to evaluate European options on the maximum or the minimum of several assets when the strike price is nonzero. We provided numerical examples to illustrate the accuracy of the procedure. It is hoped this approach may be a useful supplement to the more accurate methods available that involve extensive computation when there are several assets.

Appendix: Evaluation of the European Call Options on $\max(A_1, A_2, \ldots, A_n)$ and $\min(A_1, A_2, \ldots, A_n)$

First consider the option on $\max(A_1, A_2, \ldots, A_n)$. Let K be the strike price, and P be the payoff of the call option at maturity. Then

$$\begin{aligned} P &= \max\left[\max\left(A_1, A_2, \ldots, A_n\right) - K, 0\right] \\ &= -K + \max\left[\max\left(A_1, A_2, \ldots, A_n\right), K\right] \\ &= -K + \max\left[\exp(V), K\right], \end{aligned}$$

[4] Relative error was computed as the ratio of the approximate value minus the accurate value to the accurate value.

TABLE 1

Option Values, with Zero Strike Price, for Different Numbers of Deliverable Assets; Comparison of Results Obtained Using Equation (1) with Those of Boyle (1989)

Number of Deliverable Assets	Option Value Equation (1)	Option Value Boyle (1989)	Relative Error
2	38.908	38.908	0.000%
3	38.371	38.374	−0.008%
4	38.029	38.033	−0.011%
5	37.782	37.786	−0.011%
10	37.102	37.100	0.005%
15	36.752	36.746	0.016%
20	36.522	36.511	0.030%
25	36.351	36.338	0.036%
30	36.217	36.201	0.044%
35	36.107	36.089	0.050%
40	36.013	35.994	0.053%
45	35.933	35.912	0.058%
50	35.862	35.840	0.061%

Note: All assets have the same initial price, $40. The standard deviation of returns on all the assets is 25 percent p.a., and the correlation coefficient between each pair of asset returns is 0.95. The time to option expiration is 9 months.

TABLE 2

Comparison of European Call Prices on the Maximum and the Minimum of Three Assets with Accurate Values

	Asset		
	One	Two	Three
Current Asset Prices	40	40	40
Volatilities	30%	30%	30%
Correlation Matrix	1.0	0.9	0.9
	0.9	1.0	0.9
	0.9	0.9	1.0

	Call on Maximum			Call on Minimum		
Strike Price	Option Value Approximate	Option Value Johnson (1987)	Relative Error	Option Value Approximate	Option Value Johnson (1987)	Relative Error
30	16.351	16.351	0.000%	10.396	10.405	−0.086%
35	12.383	12.384	−0.008%	7.086	7.094	−0.113%
40	8.984	8.986	−0.022%	4.581	4.588	−0.153%
45	6.267	6.270	−0.048%	2.835	2.840	−0.176%
50	4.226	4.229	−0.071%	1.694	1.698	−0.236%

Note: Interest rate 10 percent p.a. continuously compounded; time to expiration 1 year; equal asset prices; equal volatilities; equal correlations.

TABLE 3

Comparison of European Call Prices on the Maximum and the Minimum of Three Assets with Accurate Values

	Asset: One	Two	Three
Current Asset Prices	40	40	40
Volatilities	25%	30%	35%
Correlation Matrix	1.0	0.9	0.9
	0.9	1.0	0.9
	0.9	0.9	1.0

	Call on Maximum			Call on Minimum		
Strike Price	Option Value Approximate	Option Value Johnson (1987)	Relative Error	Option Value Approximate	Option Value Johnson (1987)	Relative Error
30	16.703	16.687	0.096%	10.172	10.178	−0.059%
35	12.682	12.661	0.166%	6.914	6.917	−0.043%
40	9.235	9.223	0.130%	4.441	4.427	0.316%
45	6.490	6.496	−0.092%	2.715	2.681	1.268%
50	4.438	4.462	−0.538%	1.593	1.545	3.107%

Note: Interest rate 10 percent p.a. continuously compounded; time to expiration 1 year; equal asset prices; unequal volatilities; equal correlations.

TABLE 4

Comparison of European Call Prices on the Maximum and the Minimum of Three Assets with Accurate Values

	Asset: One	Two	Three
Current Asset Prices	40	40	40
Volatilities	30%	30%	30%
Correlation Matrix	1.0	0.6	0.4
	0.6	1.0	0.6
	0.4	0.6	1.0

	Call on Maximum			Call on Minimum		
Strike Price	Option Value Approximate	Option Value Johnson (1987)	Relative Error	Option Value Approximate	Option Value Johnson (1987)	Relative Error
30	20.046	20.018	0.140%	7.184	7.214	−0.416%
35	15.758	15.730	0.178%	4.323	4.345	−0.506%
40	11.855	11.832	0.194%	2.408	2.419	−0.455%
45	8.536	8.520	0.188%	1.259	1.262	−0.238%
50	5.901	5.895	0.102%	0.626	0.626	0.000%

Note: Interest rate 10 percent p.a. continuously compounded; time to expiration 1 year; equal asset prices; equal volatilities; unequal correlations.

TABLE 5

Comparison of European Call Prices on the Maximum and the Minimum of Three Assets with Accurate Values

	Asset		
	One	Two	Three
Current Asset Prices	40	40	40
Volatilities	25%	30%	35%
Correlation Matrix	1.0	0.6	0.4
	0.6	1.0	0.6
	0.4	0.6	1.0

	Call on Maximum			Call on Minimum		
Strike Price	Option Value Approximate	Option Value Johnson (1987)	Relative Error	Option Value Approximate	Option Value Johnson (1987)	Relative Error
30	20.185	20.153	0.159%	7.172	7.172	0.000%
35	15.885	15.847	0.240%	4.326	4.311	0.348%
40	11.969	11.929	0.335%	2.409	2.379	1.261%
45	8.644	8.611	0.383%	1.258	1.220	3.115%
50	6.013	5.995	0.300%	0.625	0.589	6.112%

Note: Interest rate 10 percent p.a. continuously compounded; time to expiration 1 year; equal asset prices; unequal volatilities; unequal correlations.

TABLE 6

Comparison of European Call Prices on the Maximum and the Minimum of Three Assets with Accurate Values

	Asset		
	One	Two	Three
Current Asset Prices	40	45	50
Volatilities	30%	30%	30%
Correlation Matrix	1.0	0.9	0.9
	0.9	1.0	0.9
	0.9	0.9	1.0

	Call on Maximum			Call on Minimum		
Strike Price	Option Value Approximate	Option Value Johnson (1987)	Relative Error	Option Value Approximate	Option Value Johnson (1987)	Relative Error
30	23.748	23.765	−0.072%	12.677	12.686	−0.071%
35	19.431	19.448	−0.087%	9.076	9.084	−0.088%
40	15.419	15 436	−0.110%	6.194	6.200	−0.097%
45	11.866	11.882	−0.135%	4.057	4.061	−0.098%
50	8.872	8.888	−0.180%	2.569	2.570	−0.039%

Note: Interest rate 10 percent p.a. continuously compounded; time to expiration 1 year; unequal asset prices; equal volatilities; equal correlations.

TABLE 7

Comparison of European Call Prices on the Maximum and the Minimum of Three Assets with Accurate Values

	Asset		
	One	Two	Three
Current Asset Prices	40	45	50
Volatilities	30%	30%	30%
Correlation Matrix	1.0	0.6	0.4
	0.6	1.0	0.6
	0.4	0.6	1.0

	Call on Maximum			Call on Minimum		
Strike Price	Option Value Approximate	Option Value Johnson (1987)	Relative Error	Option Value Approximate	Option Value Johnson (1987)	Relative Error
30	26.954	26.955	−0.004%	9.903	9.973	−0.702%
35	22.511	22.510	0.004%	6.538	6.600	−0.939%
40	18.248	18.245	0.016%	4.031	4.078	−1.153%
45	14.325	14.321	0.028%	2.342	2.373	−1.306%
50	10.891	10.889	0.018%	1.296	1.314	−1.370%

Note: Interest rate 10 percent p.a. continuously compounded; time to expiration 1 year; unequal asset prices; equal volatilities; unequal correlations.

where

$$V = \max\left(B_1, B_2, \ldots, B_n\right), \tag{A.1}$$
$$\text{and} \quad B_i = \log_e\left(A_i\right).$$

Hence, $\hat{E}(P) = -K+\hat{E}[\max(\exp(V),K)]$, and the value of the call option is $e^{-RT}\hat{E}(P)$. Thus, we need to evaluate $\hat{E}[\text{Max}(\exp(V),K)]$.

We use Clark's algorithm to evaluate the mean and variance of V, and denote these by μ_V and σ_V^2, respectively. With the standardization transformation $Z = (V-\mu_V)/\sigma_V$, we have

$$\begin{aligned}\hat{E}\Big[\max\left(\exp[V], K\right)\Big] &= \hat{E}\Big[\max\left(\exp\left[\mu_V + \sigma_V Z\right], K\right)\Big] \\ &= \hat{E}\Big[\exp\left(\mu_V+\sigma_V Z^*\right)\Big],\end{aligned} \tag{A.2}$$

where Z^* is a censored random variable defined by

$$Z^* = \begin{cases} Z & \text{if } Z \geqslant K^* \\ K^* & \text{if } Z < K^*, \end{cases}$$

with $K^* = (\log_e[K]-\mu_V)/\sigma_V$. If we denote $\mu^* = \hat{E}(Z^*)$ and $\mu_i^* = \hat{E}(Z^*-\mu^*)^i$ for $i = 2, 3$, and 4, by a Taylor series expansion (A.2) can be written as

(A.3)
$$\hat{E}\left(\exp\left[\mu_V+\sigma_V Z^*\right]\right) = \exp\left[\mu_V\right]\left\{\exp\left[\sigma_V\mu^*\right]\left(1+\frac{\mu_2^*\sigma_V^2}{2!}+\frac{\mu_3^*\sigma_V^3}{3!}+\frac{\mu_4^*\sigma_V^4}{4!}\right)\right\}.$$

To compute the moments of Z^*, we approximate the density function of Z by a Gram-Charlier approximation (see Kendall and Stuart (1969)). We denote the density and distribution function of a standard normal variate by $\phi(\cdot)$ and $N(\cdot)$, respectively. The third and fourth central moments of Z, denoted by ν_3 and ν_4, respectively, can be computed from the moments of V. The Gram-Charlier expansion approximates the density function of $Z, f(\cdot)$, by the equation

$$\text{(A.4)} \qquad f(z) = \phi(z)\left[1 + \frac{\nu_3\left(z^3 - 3z\right)}{3!} + \frac{\left(\nu_4 - 3\right)\left(z^4 - 6z^2 + 3\right)}{4!}\right].$$

Then the moments μ^* and μ_i^* for $i = 2$, 3, and 4, can be evaluated from the integrals

$$\text{(A.5)} \qquad \hat{E}\left(Z^{*i}\right) = K^{*i}\int_{-\infty}^{K^*} f(z)\,dz + \int_{K^*}^{\infty} z^i f(z)\,dz \quad i = 1,\ldots,4.$$

Straightforward integration shows that (A.5) can be calculated using the formulae

(A.6)
$$\hat{E}\left(Z^{*i}\right) = H_{i0} + \frac{\nu_3\left(H_{i3} - 3H_{i1}\right)}{3!} + \frac{\left(\nu_4 - 3\right)\left(H_{i4} - 6H_{i2} + 3H_{i0}\right)}{4!} \quad i = 1,\ldots,4,$$

where $H_{ij} = K^{*i}J_j + I_{i+j}$ for $i = 1,\ldots,4$ and $j = 0,\ldots,4$, with I_i and J_i given by

$$\begin{aligned}
I_0 &= 1 - N(K^*)\\
I_1 &= \phi(K^*)\\
I_{i+1} &= iI_{i-1} + K^{*i}\phi(K^*) \qquad i = 1,\ldots,7,\\
J_0 &= N(K^*)\\
J_1 &= -I_1\\
J_2 &= 1 - I_2\\
J_3 &= -I_3\\
J_4 &= 3 - I_4.
\end{aligned}$$

Finally, to evaluate the option on $\min(A_1, A_2, \ldots, A_n)$, we only need to redefine V as $-\mathrm{Max}(-B_1, -B_2, \ldots, -B_n)$.

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