Review of Derivatives Research, 7, 129–168, 2004

Assessing the Least Squares Monte-Carlo Approach to American Option Valuation

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Abstract. A detailed analysis of the Least Squares Monte-Carlo (LSM) approach to American option valuation suggested in Longstaff and Schwartz (2001) is performed. We compare the specification of the cross-sectional regressions with Laguerre polynomials used in Longstaff and Schwartz (2001) with alternative specifications and show that some of these have numerically better properties. Furthermore, each of these specifications leads to a trade-off between the time used to calculate a price and the precision of that price. Comparing the method-specific trade-offs reveals that a modified specification using ordinary monomials is preferred over the specification based on Laguerre polynomials. Next, we generalize the pricing problem by considering options on multiple assets and we show that the LSM method can be implemented easily for dimensions as high as ten or more. Furthermore, we show that the LSM method is computationally more efficient than existing numerical methods. In particular, when the number of assets is high, say five, Finite Difference methods are infeasible, and we show that our modified LSM method is superior to the Binomial Model.

Keywords: American options, Monte Carlo simulation, Least Squares Monte-Carlo, multiple underlying assets.

JEL classification: C15, G12, G13

It is well known that the price of an option generally depends on the strike price, the value of the underlying asset, the volatility of this asset, the amount of dividends paid on it, the interest rate and the time to maturity. Hull (1997) gives a thorough discussion of the effect of each of these factors. The most important solution to the pricing problem is, of course, the formula of Black and Scholes (1973) yielding the price of a European style option when the only stochastic factor is the asset price. For the American style option no analytical solution exists even in the simple framework of Black–Scholes. In reality this poses a big problem since all the equity options as well as the most traded of all the index options traded on the Chicago Board of Options Exchange (CBOE) are American. Instead, one has to resort to numerical solutions, the most famous of which is without doubt the Binomial Model suggested by Cox, Ross, and Rubinstein (1979).

A major problem with the Binomial approach from Cox, Ross, and Rubinstein (1979), as well as many other numerical methods, is the fact that the only factor treated as unknown is the price of the underlying asset. The other determining factors are treated as constants. This is an assumption which is clearly not valid for the interest rate and might not even be so for the amount of dividends paid. The problem is, however, that handling more than a couple of stochastic factors in the Binomial Model is computationally infeasible, since the number of nodes required grows exponentially in the number of factors. This is known as the curse of dimensionality, and it makes it difficult to allow for multiple stochastic factors such as interest rates, dividends, volatilities, or multiple underlying assets.

One alternative suggestion is to use simulation techniques. Simulation techniques have been used to price options for quite some time (see Boyle (1977)), and they have been shown to be applicable in situations with multiple stochastic factors (see Barraquand (1995)). However, in most cases the options have been European and not American, and it has been the general idea until very recently that it would be impossible from a computational point of view to use simulation methods to price American options (see Campbell, Lo, and MacKinlay (1996) and Hull (1997)). The reason is the positive probability of early exercise of an American option, which means that when pricing the option one also needs to calculate the optimal early exercise policy. This exercise strategy should be calculated recursively, but when simulation techniques are used at any time along any of the paths there is only one future path, and using these values would lead to biased results.

One of the first to propose solutions to the problem of pricing American options using simulation, and in particular of determining the optimal early exercise strategy, was Tilley (1993). In that paper, a simulation algorithm that mimics the standard lattice method is presented. Barraquand and Martineau (1995) develop a method which they call Stratified State Aggregation along the Payoff. The method is quite closely related to that of Tilley (1993) but easier to extend. The idea is to partition the state space of simulated paths into a number of cells, in such a way that the payoff from the option is approximately equal for all paths in the particular cell. From the simulated paths the probabilities of moving to different cells next period conditional on the current cell can be calculated. With these probabilities the expected value of keeping the option until next period can be evaluated and a strategy for optimal exercise determined.[1] Broadie and Glasserman (1997) also use simulation to price American options, but their approach is more closely related to the Binomial method.

An alternative formulation is in terms of optimal stopping times along the lines of Carriere (1996). Using a backwards induction theorem it is shown that pricing an American option is equivalent to calculating a number of conditional expectations. These are generally difficult to calculate, but it is shown how to combine simulation methods with somewhat advanced regression methods to get an approximation. A simulation based method to price American options with many similarities to the one of Carriere (1996), but which is supposedly simpler, has recently been proposed by Longstaff and Schwartz (2001) (henceforth LS). The idea is to estimate the conditional expectation of the payoff from keeping the option alive at each possible exercise point from a simple least squares cross-sectional regression using the simulated paths. The paper shows how to price different types of path dependent options using this Least Squares Monte Carlo (LSM) approach.

In this paper we provide a thorough numerical analysis of the LSM method. We consider the particular specification used in LS for the case of a simple Black–Scholes model and we compare the estimates obtained using different specifications of the cross-sectional regressions. Based on the numerical stability this leads us to suggest a numerically simpler specification for the cross-sectional regression than the one used by LS. The value of using our suggested specification is further emphasized when the trade-off between computational time and precision is examined. Next, we give details on how the LSM method can be extended to handle multiple stochastic factors and compare the results to those from existing numerical methods and we argue that the LSM method is much easier to extend. We further find that when the number of assets is high, say five, thus leaving Finite Difference methods infeasible in practice, our modified LSM method is superior to the Binomial Model in terms of the trade-off between computational time and precision.

The structure of the paper is as follows. Section 1 repeats how paths of stock prices can be simulated and we motivate why simulation methods do not suffer from the curse of dimensionality. The section also describes how to price European options and the general idea underlying the method proposed in LS is presented along with a small example. In Section 2 we examine different ways of implementing the LSM method. We compare the properties of the estimated price and the specification that balances time used and precision in the best way is found. Section 3 extends the LSM method to the case of options written on several assets. This extension illustrates the point that simulation techniques easily accommodate additional stochastic factors. Furthermore, the section shows that in a multivariate setting simulation methods may be computationally superior to other numerical methods. Section 4 concludes.

1. Option Pricing in a Black–Scholes Economy Using Simulation

As mentioned above, numerical methods like the Binomial Model (BM) and the Finite Difference Method (FDM) are not easily extended to more than a couple of stochastic factors. In the BM the major problem is that the number of nodes grows exponentially in the number of stochastic factors, L, since there will be 2^L branches emanating from each knot in the tree even for the simplest possible specification. Other numerical methods, including the FDM, simply cannot be extended to more than two or at the most tree stochastic factors. A possible solution to this curse of dimensionality is to use simulation. Here the state variables change according to some pre-specified distributions in each step and not by some factor of proportionality as in the BM, and the values next period are simply determined by a random draw from this distribution. A simulation consists of a large number of such simulated paths, $M \gg 1$, and as an estimator the average of the prices across the paths is used. Thus, the number of nodes remains constant through time and grows only linearly in the number of stochastic factors. Even with $M = 100{,}000$ the number of nodes is not necessarily computationally difficult to handle.

In the following we will show how to price options using simulation. We will assume that the only stochastic factors are the prices of the underlying assets, which we will think of as stocks. Furthermore, we will assume that the risk neutral dynamics of these stock prices can be specified as correlated Geometric Brownian Motions (GBM). We start by repeating the basics on how to simulate stock prices with these dynamics and we show how correlation can be introduced. We also describe the concept of a Brownian Bridge, a modification of the Wiener process which will be very useful in our application. Finally, we show how European options can be priced using simulation and we motivate the LSM method and provide a small example.

1.1. Simulating from a Geometric Brownian Motion

To simulate realizations from a GBM given by the Stochastic Differential Equation (SDE)

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \tag{1}$$

where W is a standard Wiener process and r and σ are assumed constant, we make use of the well known solution to (1). Given a starting level of $S(0)$ this is

$$S(t) = S(0)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right\}. \tag{2}$$

From the properties of the Wiener process, simulated values of $S(t)$ at a single point in time can then be obtained from the formula:

$$S(t) = S(0)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}Z(t)\right\}, \tag{3}$$

where $Z(t) \sim N(0,1)$. A sequence of values at discrete dates $0 < t_1 \leqslant t_2 \leqslant \cdots \leqslant t_N = T$ is obtained by setting

$$S(t_{i+1}) = S(t_i)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}Z(t_{i+1})\right\}, \tag{4}$$

where $Z(t_{i+1}) \sim IIN(0,1)$.

1.1.1. Simulating from Correlated Processes From (4) it is seen that the problem of simulating from a GBM is reduced to that of drawing from a standard normal distribution. Thus, if we need to simulate values from correlated processes the generalized version of (4) reduces the problem to that of drawing a random vector $\widetilde{Z}(t) = (\widetilde{Z}^1(t), \widetilde{Z}^2(t), \ldots, \widetilde{Z}^L(t))$ with a multivariate normal distribution, $N(0, \Sigma)$, where Σ is the covariance matrix with typical element $\Sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$. An easy way to do this is to draw L independent standard normally distributed random variables $Z(t) = (Z^1(t), Z^2(t), \ldots, Z^L(t))$ and choose a decomposition of Σ, C, such that $\Sigma = CC'$. It then follows that $CX \sim N(0, \Sigma)$. In most cases a suitable choice of decomposition would be a Cholesky decomposition, and since the matrix C is then lower triangular $\widetilde{Z}(t)$ can be calculated as

$$\begin{aligned}
\widetilde{Z}^1(t) &= C_{11}Z^1(t),\\
\widetilde{Z}^2(t) &= C_{21}Z^1(t) + C_{22}Z^2(t),\\
&\ \vdots\\
\widetilde{Z}^L(t) &= C_{L1}Z^1(t) + \cdots + C_{LL}Z^L(t).
\end{aligned} \tag{5}$$

Realizations of the multivariate stock prices are then easily simulated by using (4) for each stock with the values from (5).

1.1.2. Brownian Bridges There exists a modification of the Wiener process known as the Brownian Bridge process or a tied down Wiener process, which will be very useful in our application. A Brownian Bridge is a process which passes through an initial point x

and some given point y at a future time T. Following Kloeden, Platen, and Schurz (1994) we can construct this process from the formula

$$B(t) = x + W(t) - \frac{t}{T}\big(W(T) - y + x\big). \tag{6}$$

The process is Gaussian with mean $\mu(t) = x - (t/T)(x - y)$ and variance $\sigma^2(t,t) = t(T-t)/T$.

For our purpose we will assume that $x = 0$ and we will take the point, y, the process has to pass through at time T to be a draw from a $N(0, T)$ distribution. The Brownian bridge formula in (6) now allows us to start at time T, and calculate a sequence of stock prices at discrete dates $T = t_N > t_{N-1} \geqslant \cdots \geqslant t_1 \geqslant 0$ recursively backwards as

$$S(t_i) = S(0)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t_i + \sigma\left(\sqrt{\frac{t_i(t_{i+1}-t_i)}{t_{i+1}}}Z(t_i) + \frac{t_i}{t_{i+1}}Z(t_{i+1})\right)\right\}, \tag{7}$$

where $Z(t_i) \sim \mathit{IIN}(0,1)$. The Brownian bridge process generalizes easily to the correlated processes above, since all that is needed is to apply the "transformation" $\sqrt{t_i(t_{i+1}-t_i)/t_{i+1}}Z(t_i) + (t_i/t_{i+1})Z(t_{i+1})$ to the standard normally distributed random variables $Z(t) = (Z^1(t), Z^2(t), \ldots, Z^L(t))$ before they are run through (5).

For many types of options the pricing algorithm is recursive. In these situations the Brownian Bridge process enables us to minimize the use of memory in the simulations, since all that is needed to calculate period t_i stock prices are the values from period t_{i+1}.

1.2. Pricing European Options Using Simulation

Simulation methods can be used immediately to price European options and the method was actually introduced to finance even before the BM was suggested (see Boyle (1977)). To understand how simulation can be used, remember that under standard assumptions the price of a European option is the expectation under the risk neutral measure of the present value of its payoff. For a general specification of the payoff, $G(X, S(T), T)$, where X is the strike price, this can be written as

$$p \equiv p\big(S(0), T\big) = E\big[e^{-rT}G\big(X, S(T), T\big)\big]. \tag{8}$$

From (8) is clear that all one needs is the value of the stock price on the day the option expires, and an obvious estimate of the price can be calculated using the formula

$$\overline{p}_M = \frac{1}{M}\sum_{j=1}^{M} e^{-rT}G\big(X, S_j(T), T\big), \tag{9}$$

where M is the number of simulated paths and $S_j(T)$ is the value of the underlying stock at expiration of the option for path number j.

The method corresponds to a very crude form of numerical integration, and it is well known that the estimated price is unbiased and asymptotically normal. Obviously, this holds in multiple dimensions too and for quite general payoff functions. Thus, the method can provide us with a benchmark value for the European options in the case of multiple stochastic factors (see Section 3).

1.3. Pricing American Options Using Simulation

When it comes to pricing American options the major problem is the determination of the optimal early exercise strategy. To see why, consider pricing an American put option, the price of which we may write as

$$P \equiv P\big(S(0), T\big) = \max_{\tau \leqslant T} E\big[e^{-r\tau} G\big(X, S(\tau), \tau\big)\big], \tag{10}$$

where the maximization is over stopping times $\tau \leqslant T$ adapted to the filtration generated by the relevant stock price process $S(t)$. The problem with (10) is that at any possible exercise time, the holder of an American option should compare the payoff from immediate exercise to the expected payoff from keeping the option alive. The optimal decision is to exercise if the value of immediate exercise is positive and larger than the expected payoff from continuation.

In a simulation approach the problem is that we cannot simply use next period values of the underlying asset to determine the expected pathwise payoff from continuation and estimate the price by

$$\overline{P}_M = \frac{1}{M} \sum_{j=1}^{M} \max_{\tau} \big[e^{-r\tau} G\big(X, S_j(\tau), \tau\big)\big]. \tag{11}$$

This corresponds to assuming perfect foresight on behalf of the holder of the option and it would lead to biased price estimates (see Broadie and Glasserman (1997)). At first sight it could seem that the only way to circumvent this bias would be to simulate *several* paths from each possible exercise point, thus resulting in the curse of dimensionality that the lattice methods suffer from.

However, LS recently suggested to estimate the conditional expectation of the payoff from continuation using the cross-sectional information in the simulation. Thus, letting $g(x) = E[y|x]$, where y is the payoff from continuation and x represents the current state, an approximation to the conditional expectation of continuation can be used to determine the optimal exercise strategy. The motivation for their approach can be given in terms of Hilbert spaces, the space of square-integrable functions with the norm $\langle f(x), h(x)\rangle = \int f(x)h(x)dx$. A property of Hilbert spaces is that any function $g(x)$ belonging to this space can be represented as a countable linear combination of bases for this vector space. Thus, we can write

$$g(x) = \sum_{k=0}^{\infty} a_k \phi_k(x), \tag{12}$$

where $\{\phi_k(x)\}_{k=0}^{\infty}$ form a basis (see Royden (1988)).

In order to be able to use this in practice we need to approximate $g(x)$ using a finite linear combination which we denote $g_K(x)$, where K is the number of basis functions used beyond the intercept $\phi_0(\cdot) \equiv 1$. The simplest approximation concept is probably that of least squares regression where the coefficients $\{a_k\}_{k=0}^K$ are estimated using $M \geqslant K+1$ data points (y_j, x_j), $j = 1, \ldots, M$, by solving the minimization problem

$$\min_{\{a_k\}_{k=0}^K} \sum_{j=1}^{M} \big(a_0\phi_0(x_j) + a_1\phi_1(x_j) + \cdots + a_K\phi_K(x_j) - y_j\big)^2. \tag{13}$$

With the parameter estimates $\{\widehat{a}_k\}_{k=0}^K$ we estimate $g_K(x)$ with

$$\widehat{g}_K(x) = \sum_{k=0}^{K} \widehat{a}_k \phi_k(x). \tag{14}$$

Under very general conditions $\widehat{g}_K(x) \to g_K(x)$ as $M \to \infty$ and by definition $g_K(x) \to g(x)$ as $K \to \infty$. Thus, in principle the conditional expectation function $g(x)$ can be arbitrarily well approximated as both M and K tend to infinity.

1.3.1. The LSM Method in the Simple Black–Scholes Case Figure 1 shows six potential simulated paths and can be used to illustrate the general idea of the LSM method in case the option is an ordinary put option with strike price, X, equal to the initial stock level, $S(0)$, for which the payoff function is $G(X = S(0), S(\tau), \tau) = \max[X - S(\tau), 0]$. At time $t_2 = T$, that is at expiration, it is always optimal to exercise the option since it

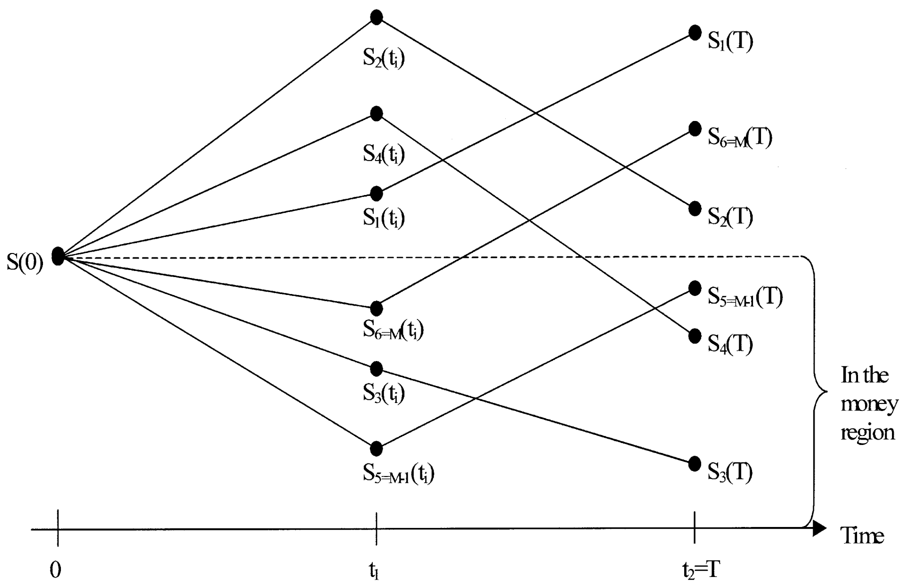


Figure 1. This figure shows six potential simulated stock price paths.

is worth nothing at any later time. Thus, the optimal exercise strategy along each path, $j = 1, \ldots, 6$, is $\tau_j = T$, and the value from following this strategy is known since this is just $\max[X - S_j(T), 0]$, $j = 1, \ldots, 6$. Because the European option corresponds to an American option with only one possible exercise date, the European price can be estimated using (9) with the relevant payoff function which simplifies to

$$p_{\text{LSM}} = \frac{1}{M} \sum_{j=1}^{M} e^{-rT} \max\bigl[X - S_j(T), 0\bigr]. \tag{15}$$

To calculate the optimal early exercise strategy of a more general American option, note that at time t_1 the option is in the money along paths number 3, 5 and 6, and it is necessary to determine whether early exercise is optimal or not along these paths. To this end we need to approximate the conditional expectation of continuing to hold the option until $t_2 = T$. Thus, in the regression in (13) we use the present value of holding the option, $e^{-r(t_2 - t_1)} \max[X - S_j(T), 0]$, as the y's and as explanatory variables we use transformations of $S_j(t_1)$, $j = 3, 5, 6$. Using the values from this regression in (14) yields an approximation to the conditional expectation. If the value of exercising is higher than the conditional expectation approximation along path number j we set $\tau_j = t_1$, otherwise $\tau_j = T$. If there is more than one exercise date prior to expiration this procedure is repeated recursively backwards through the simulation until time $t_0 = 0$, and with the approximation to the optimal early exercise strategy determined along each path, an estimate of the price of an American put option can be calculated as

$$P_{\text{LSM}} = \frac{1}{M} \sum_{j=1}^{M} \bigl[e^{-r\tau_j} \max\bigl(X - S_j(\tau_j), 0\bigr)\bigr]. \tag{16}$$

2. Assessing the LSM Method

In the paper by LS the method described above is used to price a number of different types of options. However, a closer examination of the performance of the algorithm even in the simplest Black–Scholes case and of the estimate in (16) was left for future research. At least three questions arise when it comes to actually implementing the method. First of all, it is of interest to examine the results of using different numbers of regressors and paths than those used in LS. As mentioned above both the number of paths and the number of regressors should tend to infinity in order to approximate the conditional expectation arbitrarily well, and we conjecture that the same will be a requirement for convergence of the price estimate. Secondly, the choice of Laguerre polynomials is somewhat arbitrary and alternative specifications of the cross-sectional regression model should be examined. Last but not least, from a practitioner's point of view computational time is limited, and when considering which regressors and which combination of M and K to use, the trade-off between precision and computational time should be considered.

This section deals with these three questions in sequence and as such one might claim that we are putting the LSM method to the ultimate test (see Longstaff and Schwartz (2001, Section 2.3)). However, in order to limit the computationally time we will consider a subset of the options in LS only. We limit attention to options with one year to expiration, and we set the number of possible exercise points to ten, although this number obviously could be increased to approximate the ordinary American option. The options all have a strike price of 40 and we consider three different initial stock prices spanning those in LS. To be precise we set $S(0) = 36, 40$, or 44. The volatility of the stock return is set to 40% and in all cases an interest rate of 6% is assumed.

2.1. Altering the Number of Regressors and Paths

In Longstaff and Schwartz (2001, Section 3) American style options are priced using 100,000 (50,000 plus 50,000 antithetic) simulated paths and the first 3 weighted Laguerre polynomials and a constant term in the regressions. However, other combinations of K and M could be considered and price estimates with different properties are likely to appear. To be precise, notice that the method potentially suffers from two types of biases. Firstly, there is an approximation bias as the conditional expectation function is estimated. This leads to a low bias which should vanish as the number of regressors tends to infinity. Secondly, there is a bias originating from using the same paths to estimate the conditional expectation function and to calculate the value of the option.[2] This leads to a high bias which potentially vanishes as the number of paths used is increased. However, the actual bias is generally not known and it will depend on both the number of simulated paths and the number of basis functions used as regressors.

In order to examine these aspects in more detail we formulate the cross-sectional regressions in the LSM method as

$$y(t_i) = \alpha + \sum_{k=1}^{K} \beta_k w\big(x(t_i)\big) L_{k-1}\big(x(t_i)\big) + u(t_i), \tag{17}$$

where $L_k(x(t_i))$ is the k'th Laguerre polynomial evaluated at $x(t_i)$ (these can be found in Panel A of Table 1), α and β_k are coefficients to be estimated. As in LS we use $w(x(t_i)) = \exp(-x(t_i)/2)$ and note that although $L_0 = 1$ corresponds to the constant term the first weighted Laguerre polynomial is $\exp(-x/2)$. Hence the indexing in (17). In our numerical exercise we consider increasing K from one to five and for each choice of K we increase the number of paths used in the simulations from 10,000 to 100,000 in increments of 10,000. Instead of using the exact formula for the Laguerre polynomials which can be found in Table 1, Panel A, we use the recursive formula for calculating the elements of $\{L_k\}_{k=0}^{K}$ which are found in Panel B of the same table, since this is computationally much easier. Also, in order to avoid numerical problems we normalize the stock prices by the strike price, and as a simple standard variance reduction technique we use *antithetic variates* as did LS. We use this procedure although the effect on the American price estimates is yet unknown. Finally, instead of reporting simply one price estimate for different combinations of K and M we report the mean and standard errors of 100 such estimates calculated with different seeds. The first price estimate is calculated using a seed

Table 1. Characteristics and Formulas for Orthogonal Polynomial Families

Panel A: General Characteristics

Polynomial Family	Weight	Interval	Definition
Laguerre	e^{-x}	$[0, \infty)$	$L_k(x) = \dfrac{e^x}{k!}\dfrac{d^k}{dx^k}(x^k e^{-x})$
General Chebyshev	$(1-(2x-1)^2)^{-0.5}$	$[0, 1]$	$T_k(x) = \cos(k\cos^{-1}(2x-1))$
Shifted Legendre	1	$(0, 1)$	$P_k(x) = \dfrac{(-1)^k}{2^k k!}\dfrac{d^k}{dx^k}((1-(2x-1)^2)^k)$

Panel B: Recursive Formulas (see Press et al. (1997) and Judd (1998))

First Member	Second Member	Third Member	General Formula
$L_0(x) = 1$	$L_1(x) = 1 - x$	$L_2(x) = \dfrac{x^2 - 4x + 2}{2}$	$L_{k+1}(x) = \dfrac{2k+1-x}{k+1}L_k(x) - \dfrac{k}{k+1}L_{k-1}(x)$
$T_0(x) = 1$	$T_1(x) = 2x - 1$	$T_2(x) = 8x^2 - 8x + 1$	$T_{k+1}(x) = (4x-2)T_k(x) - T_{k-1}(x)$
$P_0(x) = 1$	$P_1(x) = 2x - 1$	$P_2(x) = 6x^2 - 6x + 1$	$P_{k+1}(x) = \dfrac{2k+1}{k+1}xP_k(x) - \dfrac{k}{k+1}P_{k-1}(x)$

Notes: This table shows the general characteristics, in Panel A, and recursive formulas, in Panel B, for the different orthogonal polynomial families considered in the paper. In Panel A the column headed "Weight" gives the weighting function ensuring orthogonality, and the column headed "Interval" indicates the interval over which the respective family is orthogonal. Finally, the last column gives the explicit definition of the polynomials. In Panel B the first three members of the respective families are given together with the recursive formulas for the polynomial families from Press et al. (1997) and Judd (1998).

of 1,000,000, the second price with a seed equal to 2,000,000, and so on.[3] The benchmark with which we compare is a BM with 50,000 steps.

In Table 2 we report the results for the option with an initial stock level of 36. This is the option deepest in the money and it should be the option for which the performance of the LSM method should be most important. The first thing to note from the table is that irrespective of the number of paths used, convergence is by no means guaranteed when the number of regressors is increased. In particular, while the LSM method with a low number of regressors, say $K = 1$ or 2, underprices the options with more regressors, say $K = 3$ or 4, it produces upward biased prices. This corresponds to the approximation bias mentioned above which potentially vanishes as the number of regressors is increased. In light of this the suggestion in LS to increase the number of regressors until the estimated price no longer increases, using a given number of paths, is not useful as a rule of thumb for determining the number of basis functions needed to obtain an accurate approximation when the conditional expectation function is estimated. Furthermore, the last column of Table 2 shows that increasing the number of regressors can actually lead to a decrease in the estimated prices. This "non-monotonicity" further limits the practical use of Proposition 1 and the associated suggestions in LS. Indeed, for finite choices of K and N following this strategy and using $\widehat{g}_K(x)$ one might end with a biased estimate.[4]

Table 2. Price Estimates from the LSM Algorithm Using Various Numbers of Laguerre Polynomials in the Cross-Sectional Regression, K, and Various Numbers of Paths in the Simulation, M

	$K = 1$			$K = 2$			$K = 3$			$K = 4$			$K = 5$		
M	P_{LSM}	S.E.	Bias	P_{LSM}	S.E.	Bias	P_{LSM}	S.E.	Bias	P_{LSM}	S.E.	Bias	P_{LSM}	S.E.	Bias
10000	7.049	0.028	−0.022	7.067	0.030	−0.004	7.077	0.028	0.006	7.080	0.028	0.009	7.075	0.027	0.004
20000	7.050	0.020	−0.021	7.065	0.020	−0.006	7.073	0.020	0.002	7.074	0.020	0.003	7.069	0.026	−0.002
30000	7.044	0.016	−0.027	7.061	0.017	−0.010	7.070	0.016	−0.001	7.071	0.016	0.000	7.063	0.062	−0.008
40000	7.049	0.013	−0.022	7.066	0.012	−0.004	7.074	0.012	0.003	7.075	0.012	0.004	7.071	0.013	0.000
50000	7.047	0.013	−0.024	7.065	0.012	−0.006	7.073	0.011	0.002	7.074	0.011	0.003	7.070	0.014	−0.001
60000	7.045	0.011	−0.025	7.064	0.010	−0.007	7.072	0.010	0.001	7.073	0.010	0.002	7.070	0.011	−0.001
70000	7.046	0.010	−0.025	7.064	0.010	−0.007	7.072	0.009	0.001	7.073	0.009	0.002	7.069	0.014	−0.002
80000	7.045	0.010	−0.026	7.063	0.009	−0.008	7.071	0.008	0.000	7.072	0.009	0.001	7.068	0.010	−0.003
90000	7.047	0.010	−0.024	7.065	0.009	−0.006	7.073	0.009	0.002	7.074	0.009	0.003	7.065	0.046	−0.005
100000	7.045	0.009	−0.025	7.063	0.009	−0.008	7.071	0.008	0.000	7.072	0.008	0.001	7.069	0.009	−0.002

Regression in (19)
$$\ln RMSE = \underset{(0.514)}{0.5554} - \underset{(-5.16)}{0.4039} \ln M - \underset{(-3.65)}{0.3490} \ln K, \quad R^2 = 0.46$$

Notes: This table shows price estimates for an American put option with one year to expiration, ten possible exercise dates, and a strike price of 40. The stock level is 36 and the annualized volatility 40%. The interest rate is set to 6% annually. Price estimates are calculated for the different combinations of the number of regressors and the number of simulated paths. The reported values of P_{LSM} are means of 100 such price estimates calculated with different seeds in the random number generator. We also report the standard errors are of these 100 price estimates in the column headed S.E. The bias is the difference between P_{LSM} and the benchmark value from a Binomial Model with 50,000 steps, the value of which is 7.071.

Table 2 also shows that when the number of paths used is increased, for a reasonable number of regressors say $K = 3$ or 4, the effect of what could potentially be the high bias tends to vanish. However, the major benefit from increasing the number of paths used is to decrease the standard error of the estimates. This parallels the effect on the European price estimate for which $\sqrt{M}$ convergence can be shown. Obviously, a more appropriate measure of precision should take account of this and we suggest to use the root mean squared error (*RMSE*) given by

$$RMSE(P_{\text{LSM}}) = \sqrt{Var(P_{\text{LSM}}) + \left(E[P_{\text{LSM}}] - P\right)^2}, \tag{18}$$

rather than just the bias. This measure of precision is well known from basic statistics courses from which we, among other things, know that if the *RMSE* tends to zero the estimate is consistent. Panel A of Figure 2 plots the *RMSE* against the number of regressors and the number of paths (all variables have been log-transformed). The plot indicates that the *RMSE* is indeed lowered as both M and K are increased (perhaps with the exception of $K = 5$). The last row of the Table 2 reports the result of performing the following "panel" regression

$$\ln RMSE = \alpha + \beta_M \ln M \ + \beta_K \ln K. \tag{19}$$

Although the proposed relation only explains part of the variance in the estimates, the results provide further evidence that indeed both the number of paths and the number of regressors should be increased in order for the LSM method to converge to the simple Black–Scholes economy American option price.

For the at the money and out of the money options the results concerning the bias parallel those of Table 2 and we do not report the details (they are available upon request). However, as the determination of the early exercise strategy becomes less and less "important" the effect on the precision of the estimated price from increasing the number of regressors becomes smaller as Panels B and C of Figure 2 shows. This is further confirmed by the regression results from the regression in (19) shown below:

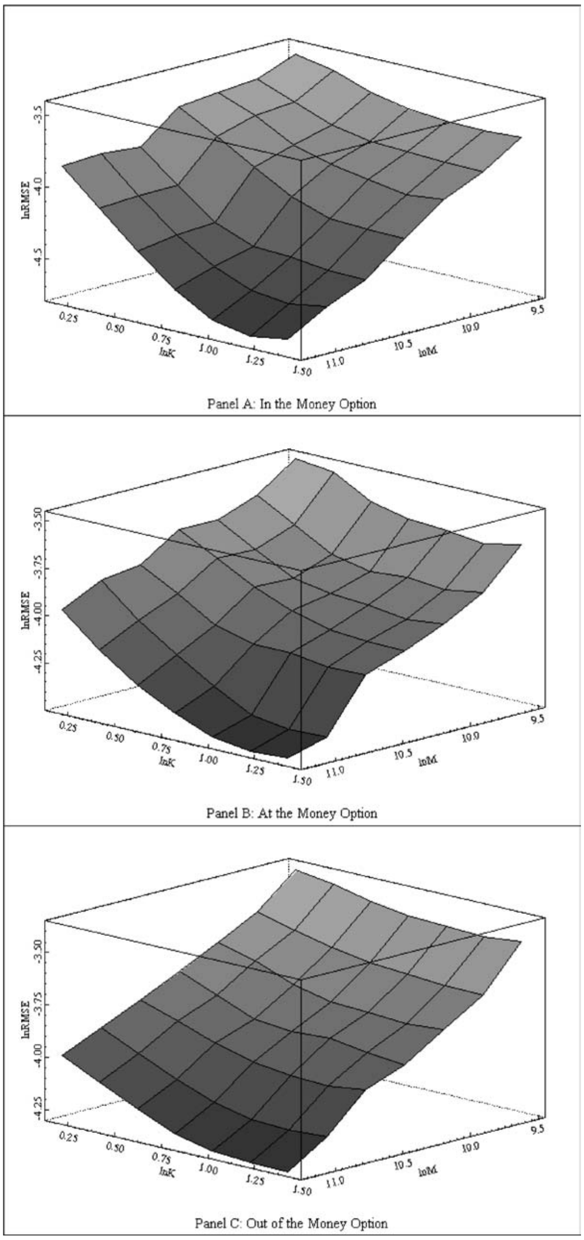
$$S = 40\text{:} \quad \ln RMSE = \underset{(1.91)}{1.1328} - \underset{(-8.41)}{0.4626} \ln M - \underset{(-3.77)}{0.2537} \ln K, \quad R^2 = 0.64,$$

$$S = 44\text{:} \quad \ln RMSE = \underset{(1.27)}{0.5157} - \underset{(-10.8)}{0.4065} \ln M - \underset{(-2.30)}{0.1060} \ln K, \quad R^2 = 0.72.$$

From this we note that as we move towards the out of the money region where the European feature dominates, the effect of increasing the number of regressors becomes less significant both statistically and in terms of the actual size, whereas the effect of increasing the number of paths becomes more and more significant at least from a statistical point of view.[5]

2.2. Using Alternative Polynomial Families

The different elements of the Laguerre family $\{L_k\}_{k=0}^{\infty}$ have the appealing property of being mutually orthogonal on the interval $[0; \infty)$ with respect to the weighting function



Panel A: In the Money Option

Panel B: At the Money Option

Panel C: Out of the Money Option

Figure 2. This figure shows plots of the *RMSE* from Equation (18) against the number of regressors, K, and the number of paths in the simulation, M, using the Laguerre polynomial family. All variables are in logarithms. Panel A reports results for an in the money option with a stock price of 36, Panel B reports results for an at the money option with a stock price of 40, and Panel C reports results for an out of the money option with a stock price of 44. See the notes to Table 2 for the other characteristics of the option.

$w(x) = \exp(-x)$, where two functions, ϕ_n and ϕ_m, are said to be mutually orthogonal on an interval $[a, b]$ with respect to a weighting function $w(x)$ if

$$\int_a^b \phi_n(x)\phi_m(x)w(x)dx = 0.$$

Heuristically this can be interpreted as a lack of correlation between the polynomial members, a property which is clearly appealing, although not required, in a regression context.[6] However, orthogonality is a property shared by other families of polynomials and it is not immediately clear whether using the Laguerre family to approximate $g(x)$ should be preferable compared to any of these. In fact, one criticism against the Laguerre polynomials is that the interval over which the conditional expectation is approximated is (0; 1) after normalization with the strike price and not $[0; \infty)$, and the regressors in (17) are in fact highly correlated over this interval as indicated by Panel A of Figure 3. Therefore, in this section we examine the results from using two different families of orthogonal polynomials, the general Chebyshev family, $\{T_k\}_{k=0}^{\infty}$, and the shifted Legendre family, $\{P_k\}_{k=0}^{\infty}$, both of which are orthogonal on the interval of interest, (0; 1). For completeness we will also consider using simple monomials.

The characteristics of the general Chebyshev and shifted Legendre polynomials can be found in Table 1, along with recursive formulas showing that these polynomials are too made up of monomials with different weighting schemes. Panels B and C of Figure 3 plot the first three members of each family on the interval of interest, (0; 1), over which they are orthogonal. Chebyshev polynomials are particularly useful when it comes to approximation of smooth nonperiodic functions, and it can be shown that the coefficients will eventually drop of for smooth functions (see Judd (1998, Theorem 6.4.2)). On the other hand, the major advantage with the Legendre polynomials is their relative simplicity, and since no computationally intensive weights have to be calculated the members are simply special combinations of monomials, $\{x^k\}_{k=0}^{\infty}$. We report the results from performing the same analysis as with the Laguerre polynomials in Tables 3 and 4 for the in the money option.

The first thing to note from Table 3 is that the number of regressors and the number of paths should both be increased to achieve arbitrarily close approximations with the Chebyshev polynomials, as it was the case with the Laguerre polynomials. However, unlike the situation with the Laguerre regressors when the Chebyshev regressors are used we do not experience the problems with $K = 5$. The table also shows that the major contribution from increasing the number of paths is once again to lower the standard errors, and Panel A of Figure 4 together with the regression results in the last row of Table 3 confirms that using the *RMSE* as a measure of precision both K and M should tend to infinity to obtain convergence of the estimated price as before. Table 4 shows that much the same conclusions hold when Legendre polynomials are used as regressors. However, for small numbers of regressors, say $K = 1$ or 2, the bias is smaller by an order of magnitude when compared to using the same number of Chebyshev regressors. Thus, the Legendre polynomials seem to be better behaved than either of the other orthogonal polynomial families. Note, that if we were to use ordinary monomials the result would be the same as in Table 4 and Figure 5.[7]

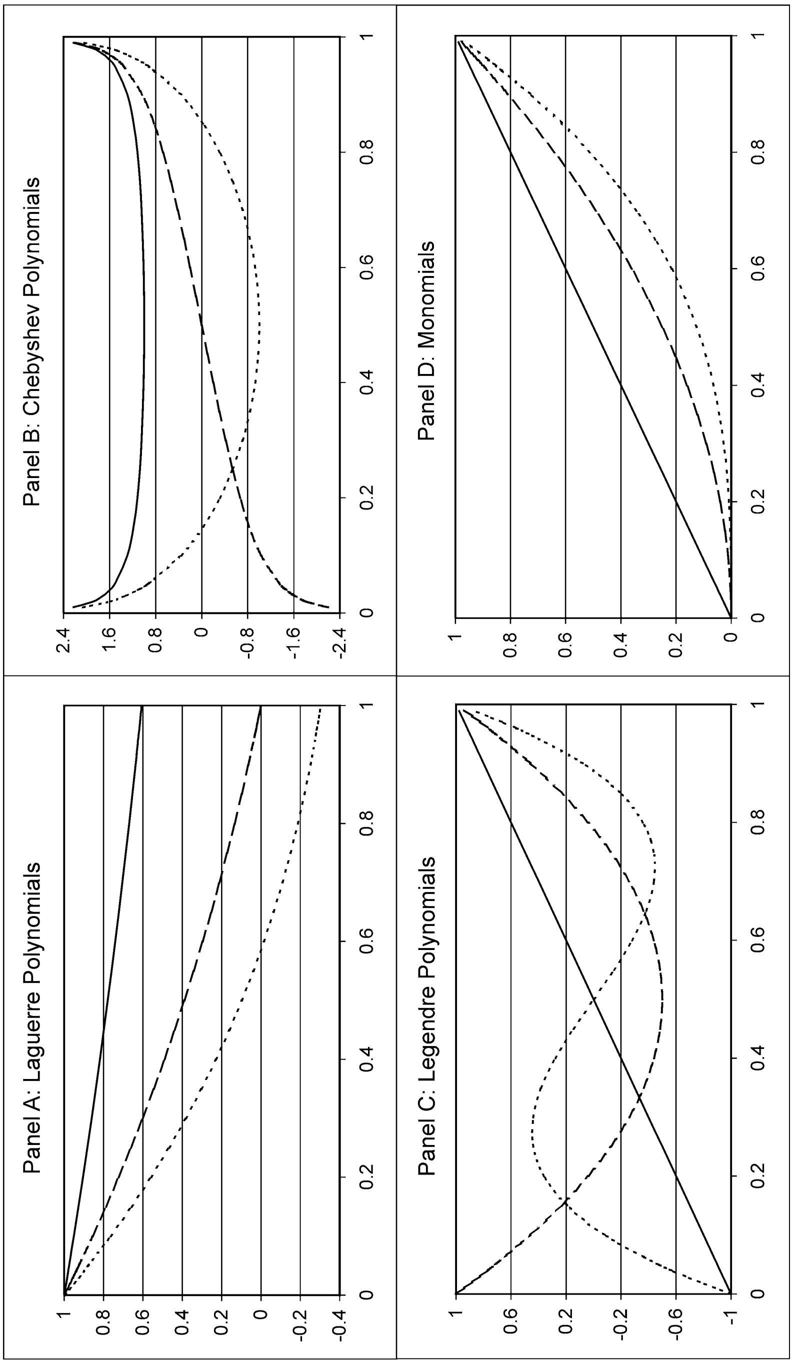


Figure 3. This figure shows the first three members of the polynomial families considered plotted on the interval (0; 1), the formulas of which can be found in Panel B of Table 1. The solid line corresponds to $k = 1$, the dashed line to $k = 2$, and the dotted line to $k = 3$.

Table 3. Price Estimates from the LSM Algorithm Using Various Numbers of Chebyshev Polynomials in the Cross-Sectional Regression, K, and Various Numbers of Paths in the Simulation, M

	$K = 1$			$K = 2$			$K = 3$			$K = 4$			$K = 5$		
M	P_{LSM}	S.E.	Bias	P_{LSM}	S.E.	Bias	P_{LSM}	S.E.	Bias	P_{LSM}	S.E.	Bias	P_{LSM}	S.E.	Bias
10000	6.730	0.041	−0.341	7.042	0.028	−0.029	7.062	0.029	−0.009	7.078	0.028	0.007	7.081	0.028	0.010
20000	6.730	0.029	−0.341	7.040	0.019	−0.031	7.060	0.021	−0.011	7.073	0.020	0.002	7.074	0.020	0.003
30000	6.734	0.024	−0.337	7.035	0.016	−0.036	7.055	0.017	−0.016	7.071	0.016	0.000	7.071	0.016	0.000
40000	6.735	0.020	−0.336	7.040	0.013	−0.031	7.061	0.012	−0.010	7.074	0.012	0.003	7.075	0.012	0.004
50000	6.737	0.019	−0.334	7.038	0.014	−0.033	7.059	0.013	−0.011	7.073	0.011	0.002	7.073	0.011	0.002
60000	6.735	0.016	−0.336	7.036	0.011	−0.035	7.059	0.011	−0.012	7.072	0.010	0.001	7.072	0.010	0.001
70000	6.734	0.016	−0.337	7.037	0.010	−0.034	7.059	0.010	−0.012	7.072	0.009	0.001	7.072	0.009	0.002
80000	6.731	0.014	−0.340	7.036	0.010	−0.035	7.057	0.009	−0.014	7.071	0.008	0.000	7.072	0.009	0.001
90000	6.736	0.015	−0.335	7.037	0.010	−0.034	7.059	0.009	−0.011	7.073	0.009	0.002	7.074	0.009	0.003
100000	6.736	0.011	−0.335	7.037	0.008	−0.034	7.058	0.009	−0.013	7.071	0.009	0.000	7.072	0.008	0.001

Regression in (19) $\ln RMSE = \underset{(2.06)}{1.7370} - \underset{(-3.78)}{0.2947} \ln M - \underset{(-22)}{2.0998} \ln K,\ R^2 = 0.91$

Notes: See Table 2.

Table 4. Price Estimates from the LSM Algorithm Using Various Numbers of Legendre Polynomials in the Cross-Sectional Regression, K, and Various Numbers of Paths in the Simulation, M

	$K=1$			$K=2$			$K=3$			$K=4$			$K=5$		
M	P_{LSM}	S.E.	Bias	P_{LSM}	S.E.	Bias	P_{LSM}	S.E.	Bias	P_{LSM}	S.E.	Bias	P_{LSM}	S.E.	Bias
10000	7.018	0.030	−0.053	7.069	0.029	−0.002	7.078	0.029	0.007	7.081	0.028	0.010	7.081	0.028	0.010
20000	7.018	0.019	−0.053	7.066	0.020	−0.005	7.073	0.020	0.002	7.074	0.020	0.003	7.074	0.019	0.003
30000	7.012	0.016	−0.059	7.063	0.017	−0.008	7.070	0.016	0.000	7.071	0.016	0.000	7.072	0.016	0.001
40000	7.018	0.014	−0.053	7.068	0.012	−0.003	7.074	0.012	0.003	7.075	0.012	0.004	7.075	0.012	0.004
50000	7.015	0.014	−0.056	7.067	0.011	−0.004	7.073	0.011	0.002	7.074	0.011	0.003	7.074	0.011	0.003
60000	7.014	0.011	−0.057	7.066	0.010	−0.005	7.072	0.010	0.001	7.073	0.010	0.002	7.072	0.010	0.001
70000	7.015	0.010	−0.056	7.066	0.010	−0.005	7.072	0.009	0.001	7.073	0.009	0.002	7.073	0.009	0.002
80000	7.013	0.010	−0.058	7.065	0.009	−0.006	7.071	0.008	0.000	7.072	0.009	0.001	7.072	0.009	0.001
90000	7.015	0.010	−0.056	7.067	0.009	−0.004	7.073	0.009	0.002	7.074	0.009	0.003	7.074	0.009	0.003
100000	7.015	0.009	−0.056	7.065	0.009	−0.006	7.071	0.009	0.000	7.072	0.008	0.001	7.072	0.008	0.001

Regression in (19) $\ln RMSE = \underset{(1.91)}{1.4577} - \underset{(-6.12)}{0.4317} \ln M - \underset{(-10.8)}{0.9307} \ln K, \; R^2 = 0.77$

Notes: See Table 2.

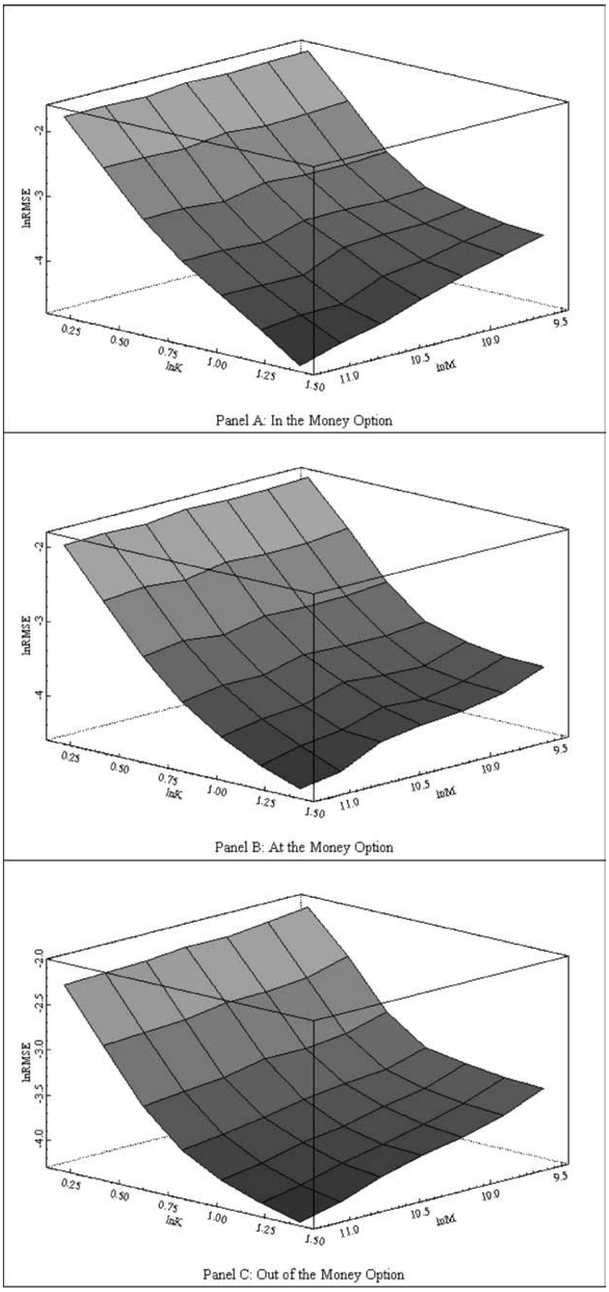


Figure 4. This figure shows plots of the *RMSE* from Equation (18) against the number of regressors, K, and the number of paths in the simulation, M, using the Chebyshev polynomial family. All variables are in logarithms. For the option characteristics see the notes to Figure 2.

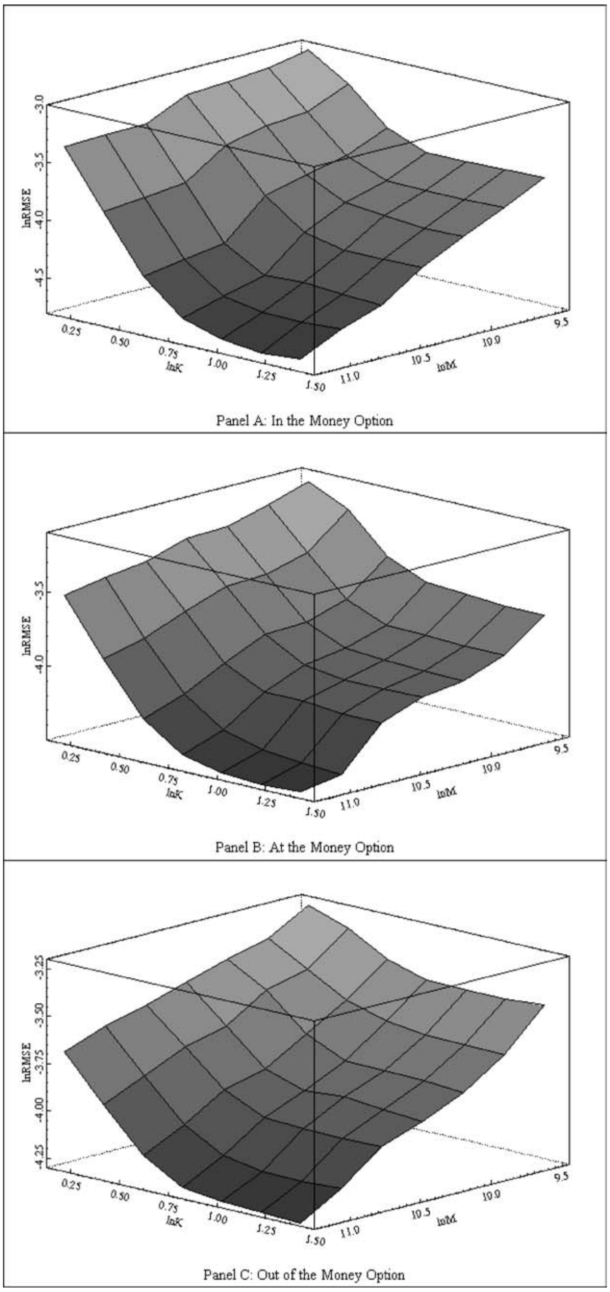


Figure 5. This figure shows plots of the *RMSE* from Equation (18) against the number of regressors, K, and the number of paths in the simulation, M, using the Legendre polynomial family. All variables are in logarithms. For the option characteristics see the notes to Figure 2.

When considering at the money and out of the money options we find the same pattern as with the Laguerre polynomials. Panels B and C of Figures 4 and 5 show the plots of the *RMSE* against K and M, and the results of the regressions in (19) for the at the money and out of the money regressions are shown below:

Results for the Chebyshev Polynomials

$$S = 40: \quad \ln RMSE = \underset{(1.71)}{1.4574} - \underset{(-3.69)}{0.2909} \ln M - \underset{(-19.4)}{1.8759} \ln K, \quad R^2 = 0.89$$

$$S = 44: \quad \ln RMSE = \underset{(1.65)}{1.3524} - \underset{(-4.16)}{0.3155} \ln M - \underset{(-16.1)}{1.4916} \ln K, \quad R^2 = 0.85$$

Results for the Legendre Polynomials

$$S = 40: \quad \ln RMSE = \underset{(1.76)}{1.4036} - \underset{(-7.44)}{0.4094} \ln M - \underset{(-10.4)}{0.7019} \ln K, \quad R^2 = 0.78$$

$$S = 44: \quad \ln RMSE = \underset{(2.23)}{0.8497} - \underset{(-11.5)}{0.4059} \ln M - \underset{(-9.86)}{0.4253} \ln K, \quad R^2 = 0.83$$

Again we see that as we move out of the money, the effect from increasing the number of regressors becomes less and less significant.

2.3. The Optimal Choice when the Trade-off between Computational Time and Precision Is Considered

The section above showed that when the number of paths and the number of regressors are increased the *RMSE* of the estimate is decreased. However, altering the cross-sectional specification could have a dramatic effect on the computational time. To see why, note that the regressions are by far the most time consuming part of the algorithm and adding regressors will therefore lead to a significant increase in the computational time used. Increasing the number of paths in the simulation will also increase the time needed to calculate the price estimates as more observations are used. Both of these effects will be particularly important for options deep in the money where a large proportion of the observations is used in the regressions. Furthermore, although the evaluation of each Laguerre polynomial can be facilitated using the available recursive formulas, the weights contain the exponential function, which is relatively computationally intensive to calculate. Changing the weighting scheme thus also effects the computational time used.

Since computational time is a relevant constraint in real life, we suggest that when examining which specification of the cross-sectional regressions is preferred, the precision of the estimates should be compared to the time used to calculate it and the specification which balances the two in an optimal way be chosen as the preferred one (on this see also Broadie and Detemple (1996)). As a measure of precision we again use the *RMSE* from (18), and we measure computational time as the number of prices it is possible to calculate per second on a Pentium 4, 1.5 GHz, with 256 MB of RAM using *Ox*. Figure 6 plots precision against computational time for the different polynomial families from above and for the different combinations of the number of regressors, K, and number of paths, M, for the in the money option. We also include results for the monomials.

From Panel A of Figure 6 we see that choosing $K = 1$ or larger than 4 for the Laguerre polynomials is never preferred. Instead, either $K = 2$ or 3 is the preferred specification depending on the preferences between time and precision. If speed is valued relatively high a specification with $K = 2$ is the preferred one even though Table 2 showed that this specification consistently underestimates the true price. For the Chebyshev specification in Panel B we see that the preferred specification should have $K = 4$ or 5. However, comparing the two panels we see that the preferred Chebyshev specifications have less favorable trade-offs between precision and speed. This is due to the larger number of regressors needed to get precise estimates.

Panel C of Table 6 shows the results for the Legendre specification, from which it is seen that the preferred specification has $K = 2$ or 3. When this panel is compared to Panel A we see that the preferred specification using Legendre polynomials dominates the preferred specification using Laguerre regressors. Generally, calculating a price estimate with Legendre polynomials takes around 90% of the time used to calculate a price estimate with the same number of paths and Laguerre regressors (the actual computational times are available upon request). As mentioned above the Legendre family consists of special combinations of ordinary monomials. For this reason the actual price estimates would be the same whether Legendre regressors or simple monomials are used. The trade-off for the monomials is shown in Panel D of Table 6. Comparing this panel to Panel C it is seen that the preferred specification should use the simple monomials. In real terms the reduction in actual computation time could be as large as 20%, and we note that even further reductions could be achieved if, e.g., Horners method is used to evaluate the conditional expectations (see Judd (1998)).

Figures 7 and 8 report the corresponding results for the at the money and out of the money option allowing us to compare the models across moneyness. The figures show that the ranking results are stable across moneyness.

2.4. The LSM Method in the Black–Scholes Model

In this section we have provided a detailed analysis of the LSM method applied to the simple American put option using the specification in Longstaff and Schwartz (2001, Section 3). The results are compared to those obtained using alternative specifications for the cross-sectional regressions with different polynomial families and we have shown that equivalent pricing results can be obtained. Our results complement the extensive numerical analysis referred to, but not reported on in LS. However, the analysis shows that choosing Laguerre regressors is not necessarily optimal from the perspective of numerical stability and instead our conclusion is that numerically more reliable results can be obtained by using other polynomial families, in particular the Legendre family. We attribute this to the fact that these polynomials, unlike the Laguerre polynomials, are orthogonal on the interval of interest for the conditional expectation approximation.

In this section we have further argued that the preferred specification in terms of the choice of polynomial family, number of regressors and number of paths should be the one

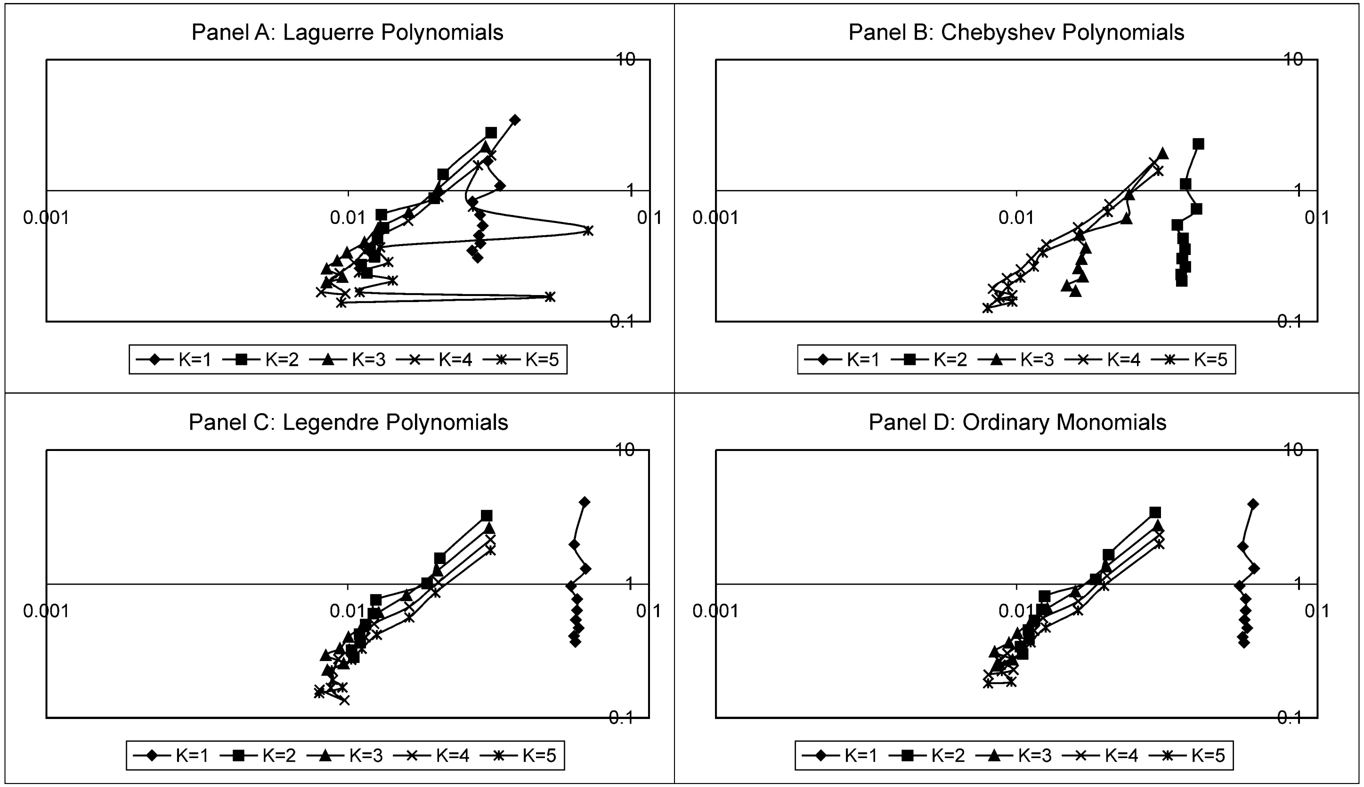


Figure 6. This figure reports the number of option prices that can be calculated in one second (y-axis) plotted against *RMSE* (x-axis) using different numbers of regressors for the different polynomial families. Preferred specifications are in the upper-left corner. Results are for the in the money option with a stock price of 36. See the notes to Table 2 for the other characteristics of the option.

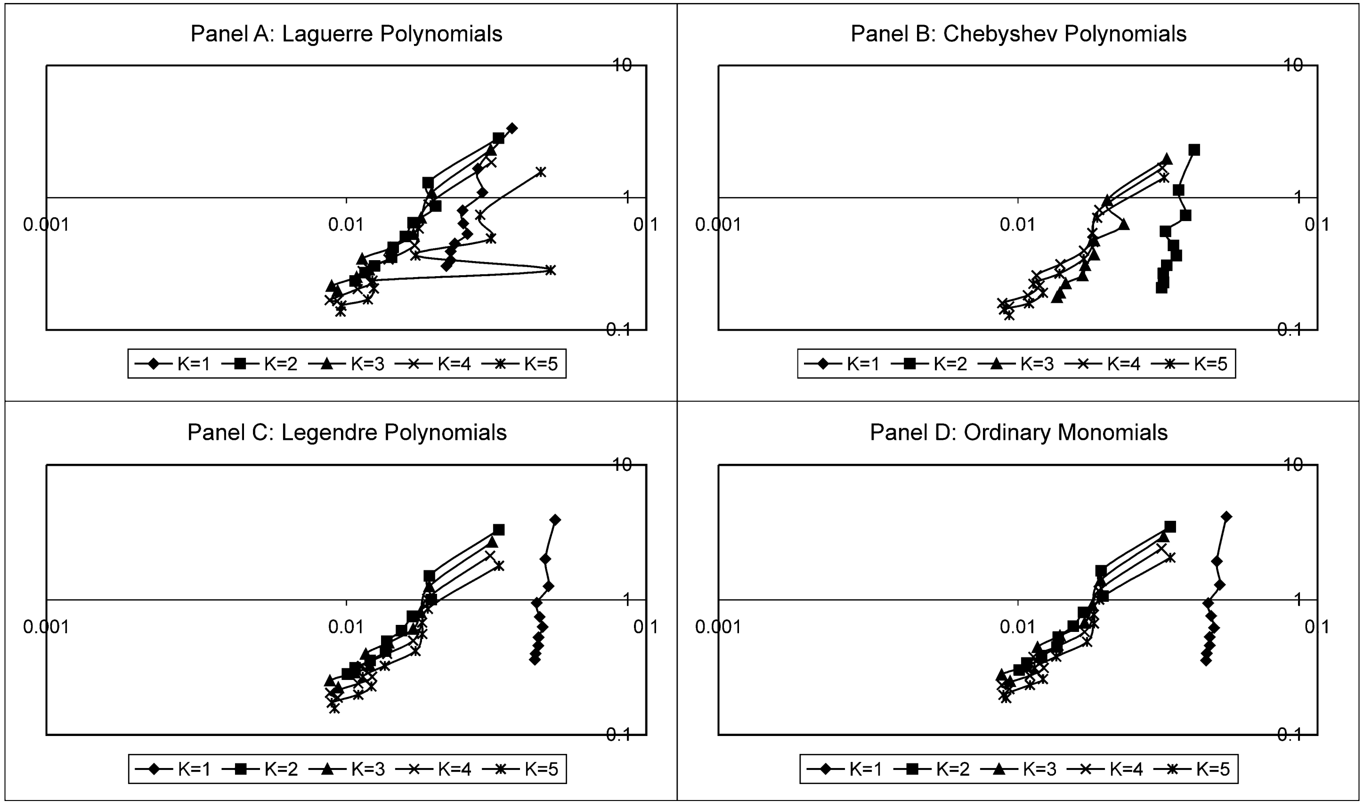


Figure 7. This figure reports the number of option prices that can be calculated in one second (y-axis) plotted against *RMSE* (x-axis) using different numbers of regressors for the different polynomial families. Preferred specifications are in the upper-left corner. Results are for the at the money option with a stock price of 40. See the notes to Table 2 for the other characteristics of the option.

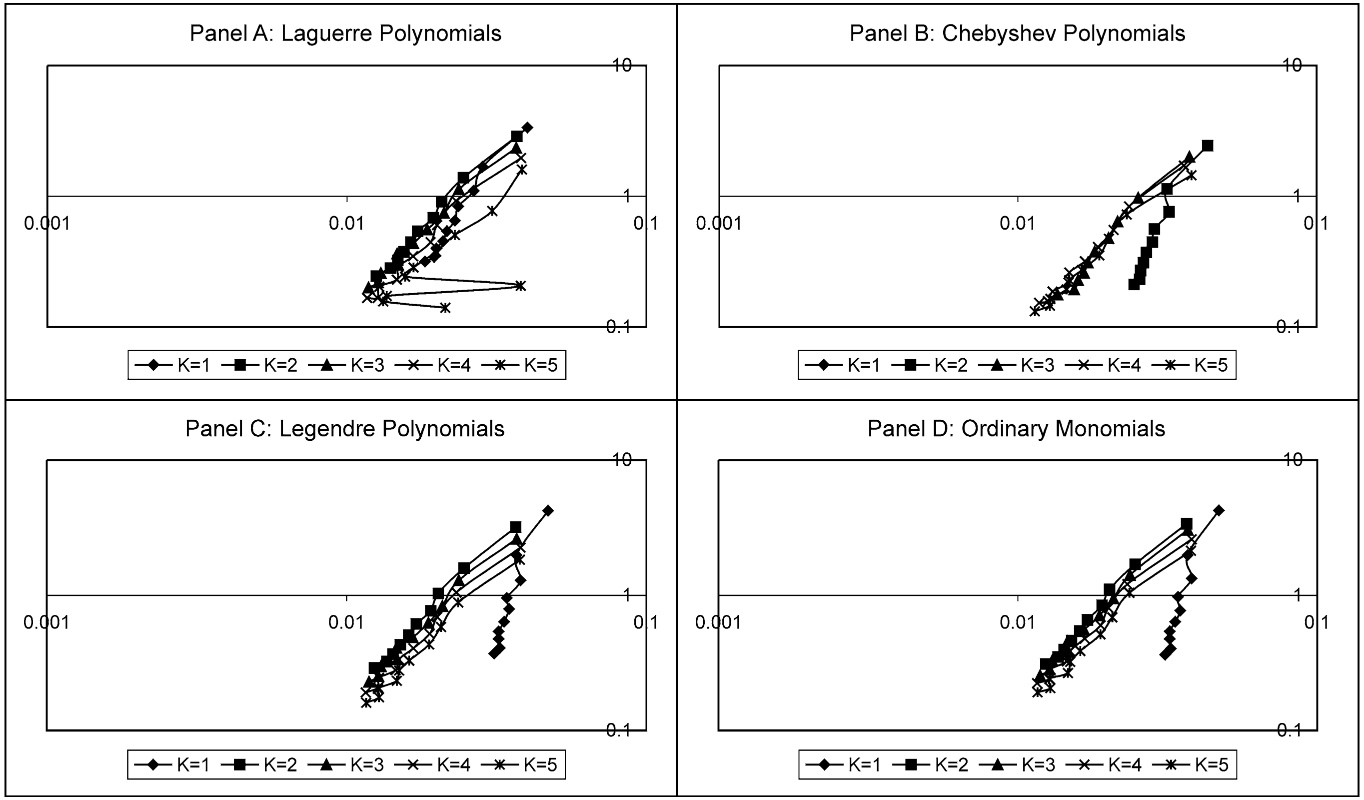


Figure 8. This figure reports the number of option prices that can be calculated in one second (y-axis) plotted against *RMSE* (x-axis) using different numbers of regressors for the different polynomial families. Preferred specifications are in the upper-left corner. Results are for the out of the money option with a stock price of 44. See the notes to Table 2 for the other characteristics of the option.

with the most favorable trade-off between precision and computational time. Considering this trade-off we conclude that using the Legendre family or even using simple monomials is preferred to the computationally more complicated polynomial families like Laguerre or Chebyshev. In light of the equally good pricing performance we recommend that these be used as regressors when using the LSM method to price American put options, and for this reason in what follows we will simply use the monomials as regressors.

3. Pricing Options on Multiple Assets

It is well known that the existing numerical methods are superior to simulation based methods in the simple Black–Scholes setting. Very precise estimates can be calculated with, e.g., the BM in short time, and our computational effort in the previous section may seem futile. However, in this section we will show that the interest into the use of simulation based methods can be justified exactly for the reasons mentioned in the introduction, that is by extending the number of stochastic factors. Our numerical analysis will show that this justification can be made both on the grounds of an easier implementation and on the grounds of a preferable trade-off between computational time and precision.

An easy example of multiple stochastic factors is the case in which there is not only one but several underlying assets. Options on multiple underlying assets typically have payoffs which are made a function of the maximum of the asset prices, the minimum of these, or the of average of them. In this section, we will examine different payoff functions along these lines. To be precise we will consider pricing American put options on several assets with payoffs given by any one of the following four specifications:

$$\text{Maximum options:} \quad \max\bigl(0, X - \max\bigl(S^1, \ldots, S^L\bigr)\bigr), \tag{20a}$$

$$\text{Minimum options:} \quad \max\bigl(0, X - \min\bigl(S^1, \ldots, S^L\bigr)\bigr), \tag{20b}$$

$$\text{Arithmetic Average options:} \quad \max\left(0, X - \frac{1}{L}\sum_{l=1}^{L} S^l\right), \tag{20c}$$

$$\text{Geometric Average options:} \quad \max\left(0, X - \left(\prod_{l=1}^{L} S^l\right)^{1/L}\right), \tag{20d}$$

where S^l, $l = 1, \ldots, L$ are the prices of the underlying assets, L being the dimension, and X is the strike price as before.

These types of payoff functions have been used widely in the literature and Boyle and Tse (1990) give examples on where these types of options are traded. Analytical results for European options on the maximum or minimum of several assets were given in Johnson (1987), and since the product of lognormals is lognormal, exact formulas for the European price of an option on the geometric average can be derived, too. However, as it was the case in the simple one dimensional Black–Scholes model considered previously, no analytical solutions exist for the American options and numerical methods must be used in all cases.

In this section, we start by discussing the existing methods and give a detailed description of how to generalize the LSM method. The rest of this section provides pricing results on the BM and the LSM methods for $L = 3$, and we finish by considering the performance of the two methods for the special case of payoff function (20d) for which credible benchmark values can be calculated in high dimensions (see Section 3.3).

3.1. Numerical Methods on Several Assets

Many of the papers extending other numerical procedures to allow for several underlying state variables have used the example with several underlying assets (see Boyle (1988) and Boyle, Evnine, and Gibbs (1989) for extensions of the BM). In the following we will use the jump sizes and the jump probabilities from the Binomial Model of Boyle, Evnine, and Gibbs (1989) as a possible benchmark model. In principle, the resulting tree is a straightforward generalization of the one dimensional tree, because the specification of the jump sizes assures that the "individual" stock trees are recombining. However, there will be 2^L branches emanating from each knot in the tree, and thus both the memory requirement and the computational time are exponential in the number of assets. Furthermore, as the number of stochastic factors increases all the jump sizes and probabilities change and will have to be recalculated. Thus, in terms of computational work there is no easy way to generalize the BM. In three dimensions the method was used in Boyle, Evnine, and Gibbs (1989, Table 2) to calculate European prices of options with the payoff functions in (20a)–(20d). With our implementation of their method we get exactly the same European prices but in the following we will extend the analysis to American options.[8]

We refrain from implementing the Finite Difference method since this involves much tedious work, and is extremely difficult if not impossible to implement in more than three dimensions. On the other hand, in a simulation context multiple underlying assets were used in Barraquand and Martineau (1995), and it was mentioned briefly in LS. We now discuss in detail how to implement the LSM method in a multivariate setting. In particular we discuss which regressors to use.

3.1.1. The LSM Method on Several Assets The LSM method is easily extended to several underlying assets since only two issues have to be considered. First of all we have to specify how to simulate from correlated stochastic processes. This question was dealt with in Section 1.1.1, and from a computational point of view it is easy to generalize the procedure to arbitrary dimensions. Furthermore, as it was mentioned in that section the Brownian Bridge can be used as a computational efficient method for simulation, and even if the dimension is large, say $L = 10$, using as large a number of paths as $M = 100{,}000$ can be handled on an ordinary personal computer as it only requires storage of 1,000,000 stock values.

The second problem is to decide which regressors to use in the multidimensional cross-sectional regression. Obviously, this could be done on an ad hoc basis as in LS, but in a more flexible application this is not interesting. Instead we choose to work with the *complete set of polynomials* (see Judd (1998)). The complete set of polynomials of total degree K in L dimensions are given as

$$\mathcal{P}_K^L = \left\{ x_1^{i_1} \cdots x_L^{i_L} \,\middle|\, \sum_{l=1}^{L} i_l \leqslant K, 0 \leqslant i_1, \ldots, i_L \right\}. \tag{21}$$

Note that these are the terms that would appear in the Kth order Taylor expansion, and the set contains all products and cross products of order less than or equal to K. At first it might seem that the number of elements in $\mathcal{P}_K^L$ will grow rapidly. However, we know from Judd (1998) that the number of elements in $\mathcal{P}_K^L$ is polynomial only in the number of assets L given the degree K, and for $K = 2$ the number is $1 + L + L(L+1)/2$ while for $K = 3$ there will be a total of $1 + L + L(L+1)/2 + L^2 + L(L-1)(L-2)/6$ regressors.

Generating Multi-Indices. The exponents in (21) are sometimes referred to as a multi-index, and at first it may seem complicated to implement this in arbitrary dimensions. However, this is not so and the set can be generated from the following pseudo-code directly applicable to any matrix language.

1. Let K be the maximum polynomial order and let L be the number of assets. Define a matrix Λ of dimension $(L, (K+1)^L)$ and declare a counter i.
2. For $i = 1$ to L by 1 let the ith row of Λ be defined as

$$vecr\left(1_{(K+1)^{L-i},1} \cdot floor\left(range\left(0, \frac{K}{(K+1)^{i-1}}, \frac{1}{(K+1)^{i-1}}\right)\right)\right),$$

   where $vecr(A)$ returns the rows of A after each other, $1_{a,b}$ returns a matrix of dimension (a, b) filled with ones, $range(c, d, e)$ defines a row vector with elements from c to d incrementing by e, and $floor(B)$ returns the floor of each element in B.
3. Delete all columns of Λ for which the sum of the exponents is larger than K.

Note that for $L = 1$ and, say $K = 2$, we have

$$\Lambda = vecr\big(1_{1,1} \cdot floor\big(range(0, 2, 1)\big)\big) = vecr\big(1 \cdot floor(0\ 1\ 2)\big),$$

or, equivalently,

$$\Lambda = (0\ 1\ 2).$$

Likewise for $L = 2$ we would get

$$\Lambda = \begin{pmatrix} 0\ 1\ 2\ 0\ 1\ 0 \\ 0\ 0\ 0\ 1\ 1\ 2 \end{pmatrix}.$$

3.1.2. Comparing with Other Numerical Methods As a first test of our methods we compare the price estimates to those from Barraquand and Martineau (1995, Table 3) where American style put options on the maximum of three underlying assets are priced. The underlying assets all have an initial price of 40, and the volatilities are 20%, 30%, and

50%, respectively. The interest rate is set to 5% and exercise is possible at ten equidistant time steps. The results from using our method as well as the difference between our results and those from Barraquand and Martineau (1995) are shown in Table 5 with the option characteristics in the first three columns, with ρ being the correlation, T the number of months to expiration, and X the strike price as before. We do not report results on the case of $\rho = 1$ since this, obviously, corresponds to the one dimensional situation. Columns four and five report averages and standard errors from 100 prices calculated from the LSM method using $K = 3$ and $M = 100{,}000$, and columns six and seven report the European price estimates using the same simulated paths. In columns eight and nine we report the price estimates from a BM with 200 steps during the life of the option.

The differences between our simulated prices and those of Barraquand and Martineau (1995) are reported in columns ten and eleven from which we see that they are within one cent of each other. This is also the case when our results on the BM are compared to the results from Barraquand and Martineau (1995) on their variant of a lattice method, the differences of which are reported in the last two columns. Overall our prices are indistinguishable from the ones in Barraquand and Martineau (1995) with the numerical accuracy given in that paper.

3.2. *Pricing Results for $L = 3$*

In the following we will consider pricing options with a strike price $X = 40$, a time to maturity $T = 12$ months, and an interest rate of $r = 6\%$. We will further assume that the volatilities of the underlying assets are $\sigma_l = 40\%$, for $l = 1, 2, 3$. We will assume that the initial stock price is the same for all three assets but we allow it to be either 36, 40, or 44. The correlation between the assets is allowed to take on four different values, $\rho = 0$, 0.25, 0.5, or 0.75.

3.2.1. The Binomial Model and European Prices In one dimension we compared the BM results using 50,000 steps to the exact solutions from the Black–Scholes model and found a very precise estimate from the BM model. Thus, we did not hesitate to use the American price from a BM model with the same number of steps believing it to be a very close approximation to the true American price. In three dimensions we would potentially like to use the same strategy. However, using such a large number of steps is impossible because of the exponential growth in the amount of computational work. For $L = 3$ the memory requirement with 400 steps exceeds 1GB RAM and the computational time is three to four hours on the equivalent of a Pentium 4 1.5 GHz. Thus, this specification is clearly not feasible, and it becomes of interest to know how well the model can approximate the true value for a feasible number of steps.

In Tables 6 and 7 we report results from implementing the BM using 100, 200, and 400 steps. We compare the European results to those from a Numerical Integration (NI) benchmark using the average of 100 European prices calculated by simulation with $M = 100{,}000$

Table 5. Put Option Prices on the Maximum of Three Assets

Parameters			Results from Our Implementation						Comparison with SSAP Results			
ρ	T	X	P_{LSM}	S.E. (P_{LSM})	p_{LSM}	S.E. (p_{LSM})	P_{BM}	p_{BM}	$P_{\mathrm{SSAP}} - P_{\mathrm{LSM}}$	$p_{\mathrm{SSAP}} - p_{\mathrm{LSM}}$	$P_{\mathrm{PDE}} - P_{\mathrm{BM}}$	$p_{\mathrm{PDE}} - p_{\mathrm{BM}}$
0	1	35	0.000	0.000	0.000	0.000	0.000	0.000	0.00	0.00	0.00	0.00
0	1	40	0.226	0.001	0.132	0.001	0.227	0.130	0.00	0.00	0.00	0.00
0	1	45	5.000	0.000	2.260	0.006	5.000	2.260	0.00	0.01	0.00	0.00
0.5	1	35	0.002	0.000	0.002	0.000	0.002	0.002	0.00	0.00	0.00	0.00
0.5	1	40	0.487	0.002	0.387	0.003	0.488	0.386	0.00	0.00	−0.01	−0.01
0.5	1	45	5.000	0.000	3.007	0.004	5.000	3.007	0.00	0.00	0.00	−0.01
0	4	35	0.013	0.000	0.009	0.000	0.013	0.009	0.00	0.00	0.00	0.00
0	4	40	0.438	0.003	0.251	0.003	0.439	0.248	0.01	0.00	0.00	0.00
0	4	45	5.000	0.000	1.558	0.007	5.000	1.556	0.00	0.00	0.00	−0.01
0.5	4	35	0.089	0.001	0.078	0.001	0.089	0.078	0.00	0.00	0.00	−0.01
0.5	4	40	0.931	0.004	0.726	0.005	0.934	0.723	0.01	−0.01	0.00	0.00
0.5	4	45	5.000	0.000	2.660	0.007	5.000	2.659	0.00	0.00	0.00	−0.01
0	7	35	0.042	0.001	0.029	0.001	0.042	0.029	0.00	0.00	0.00	0.00
0	7	40	0.567	0.003	0.319	0.003	0.567	0.315	0.01	0.00	0.00	−0.01
0	7	45	5.000	0.000	1.416	0.007	5.000	1.418	0.00	0.00	0.00	−0.01
0.5	7	35	0.209	0.003	0.177	0.003	0.209	0.177	−0.01	−0.01	−0.01	−0.01
0.5	7	40	1.195	0.005	0.916	0.006	1.199	0.912	0.02	−0.01	−0.01	0.00
0.5	7	45	5.000	0.000	2.649	0.009	5.000	2.651	0.00	0.00	0.00	−0.02
Average differences:									0.00	0.00	0.00	0.00

Notes: This table reports prices from the LSM algorithm for put options on the maximum of three assets using the specifications from Barraquand and Martineau (1995, Table 3). Thus, the underlying stocks all have a price of 40 and the volatilities are 20%, 30%, and 50% respective. In the table ρ denotes the correlation between the stocks, T denotes the number of months to expiration, and X denotes the strike price. For all the options the interest rate is fixed at 5% and ten exercise points are considered during the life of the option. The estimates from the LSM algorithm are averages of 100 calculated price estimates using different seeds in the random number generator, and we also report the standard errors of these 100 estimates. For the American price estimates, P_{LSM}, we use monomials of maximum order $K = 3$. For these as well as for the European estimates, p_{LSM}, $M = 100{,}000$ paths are used in the simulations. For comparison we report the price estimates from a BM with 200 steps during the life of the option. The results from the LSM algorithm and the BM are compared to the estimates from Barraquand and Martineau (1995, Table 3) with a subscript "SSAP" denoting their simulated values and subscript "PDE" denoting values from their variant of a lattice method.

Table 6. Put Option Prices in 3 Dimensions with Non-Smooth Payoff Functions

Panel A: Maximum Options									
S	ρ	p_{LSM}	$p_{\mathrm{BM}_{100}}$	$p_{\mathrm{BM}_{200}}$	$p_{\mathrm{BM}_{400}}$	P_{LSM}	$P_{\mathrm{BM}_{100}}$	$P_{\mathrm{BM}_{200}}$	$P_{\mathrm{BM}_{400}}$
36	0	1.300 (0.009)	1.318 [20.222]	1.302 [2.223]	1.306 [5.966]	4 (·)	4 {0}	4 {0}	4 {0}
40	0	0.652 (0.006)	0.644 [−12.518]	0.648 [−5.610]	0.651 [−2.148]	1.113 (0.006)	1.160 {0.047}	1.133 {0.020}	1.125 {0.011}
44	0	0.314 (0.004)	0.318 [8.350]	0.319 [10.620]	0.317 [6.040]	0.492 (0.004)	0.514 {0.022}	0.504 {0.013}	0.498 {0.006}
36	0.25	2.006 (0.011)	2.026 [18.772]	2.010 [3.069]	2.012 [5.350]	4 (·)	4 {0}	4 {0}	4 {0}
40	0.25	1.164 (0.009)	1.157 [−7.626]	1.161 [−3.265]	1.163 [−1.078]	1.695 (0.008)	1.740 {0.045}	1.722 {0.027}	1.710 {0.015}
44	0.25	0.662 (0.007)	0.668 [9.822]	0.668 [10.089]	0.666 [6.026]	0.910 (0.006)	0.938 {0.028}	0.928 {0.018}	0.921 {0.011}
36	0.5	2.849 (0.012)	2.873 [19.462]	2.854 [4.292]	2.856 [5.265]	4 (·)	4 {0}	4 {0}	4 {0}
40	0.5	1.818 (0.011)	1.814 [−3.819]	1.816 [−1.658]	1.817 [−0.567]	2.351 (0.010)	2.409 {0.059}	2.383 {0.033}	2.372 {0.021}
44	0.5	1.145 (0.009)	1.155 [11.110]	1.154 [9.618]	1.150 [5.448]	1.427 (0.008)	1.467 {0.040}	1.452 {0.025}	1.442 {0.016}
36	0.75	3.957 (0.013)	3.990 [24.499]	3.967 [7.649]	3.966 [6.710]	4.790 (0.010)	4.919 {0.130}	4.859 {0.069}	4.835 {0.045}
40	0.75	2.718 (0.013)	2.721 [2.678]	2.720 [1.556]	2.719 [1.030]	3.182 (0.012)	3.252 {0.070}	3.225 {0.043}	3.211 {0.029}
44	0.75	1.850 (0.011)	1.867 [14.990]	1.862 [10.841]	1.857 [5.957]	2.117 (0.010)	2.168 {0.052}	2.152 {0.035}	2.140 {0.023}

to gauge how well the BM performs. The results show that we have to be extremely careful when using the BM. In particular, for the non-smooth payoffs from (20a) and (20b) in Table 6 using 200 or even 400 steps in a BM is not enough to produce a European price estimate within the 99% Confidence Interval ($CI_{0.99}$) for the NI benchmarks. For options with payoff function (20c) or (20d), Table 7 shows that as long as more than 200 steps are used the prices are actually within the $CI_{0.99}$ unless the options are deep in the money and have a payoff function like (20d). Using 400 steps in the BM results in European price estimates insignificantly different from the NI values. Thus, it seems that with a feasible number of steps the BM can generate sensible prices only with smooth payoff functions. These results are in agreement with what was found in Boyle, Evnine, and Gibbs (1989), and as the computational complexity grows with N^3 it poses a genuine problem.[9]

3.2.2. The LSM Method and American Prices The average of 100 American price estimates setting $K = 3$ and using the same $M = 100{,}000$ paths are shown in column 7 of Tables 6 and 7 with the corresponding standard errors in parenthesis below. In columns eight through ten we report the American prices from the BM and below these, in curly brackets, the bias, $P_{\mathrm{BM}} - P_{\mathrm{LSM}}$. We do not report any test statistics because not much is known about the asymptotics of the American prices from the LSM method.

The results from Table 6 indicate that there are large discrepancies between the price estimates for the non-smooth option payoff functions in (20a) and (20b), and although the bias seems to vanish as the number of steps in the BM increases at least for the max-

Table 6. (Continued)

Panel B: Minimum Options									
S	ρ	p_{LSM}	$p_{\mathrm{BM}_{100}}$	$p_{\mathrm{BM}_{200}}$	$p_{\mathrm{BM}_{400}}$	P_{LSM}	$P_{\mathrm{BM}_{100}}$	$P_{\mathrm{BM}_{200}}$	$P_{\mathrm{BM}_{400}}$
36	0	13.171 (0.019)	13.165 [−2.779]	13.166 [−2.346]	13.170 [−0.446]	13.306 (0.017)	13.397 {0.091}	13.396 {0.090}	13.398 {0.092}
40	0	10.779 (0.020)	10.761 [−9.006]	10.770 [−4.323]	10.775 [−1.980]	10.860 (0.019)	10.961 {0.101}	10.968 {0.108}	10.971 {0.111}
44	0	8.645 (0.021)	8.642 [−1.324]	8.647 [1.036]	8.647 [1.133]	8.676 (0.020)	8.807 {0.130}	8.810 {0.134}	8.809 {0.133}
36	0.25	12.151 (0.017)	12.145 [−3.969]	12.146 [−3.416]	12.151 [−0.315]	12.291 (0.016)	12.371 {0.079}	12.370 {0.078}	12.373 {0.082}
40	0.25	9.862 (0.017)	9.840 [−12.775]	9.852 [−5.948]	9.858 [−2.535]	9.950 (0.017)	10.027 {0.078}	10.036 {0.087}	10.041 {0.091}
44	0.25	7.871 (0.018)	7.867 [−2.737]	7.873 [0.939]	7.874 [1.288]	7.914 (0.018)	8.014 {0.100}	8.019 {0.105}	8.019 {0.105}
36	0.5	11.035 (0.014)	11.025 [−7.660]	11.027 [−6.078]	11.034 [−0.964]	11.191 (0.014)	11.245 {0.053}	11.245 {0.053}	11.250 {0.059}
40	0.5	8.866 (0.014)	8.838 [−19.375]	8.853 [−9.190]	8.860 [−4.107]	8.970 (0.015)	9.012 {0.042}	9.024 {0.054}	9.030 {0.060}
44	0.5	7.024 (0.015)	7.016 [−5.882]	7.024 [−0.099]	7.026 [0.866]	7.088 (0.014)	7.146 {0.058}	7.154 {0.066}	7.155 {0.067}
36	0.75	9.684 (0.010)	9.664 [−19.864]	9.671 [−13.664]	9.681 [−3.731]	9.863 (0.011)	9.880 {0.017}	9.885 {0.022}	9.894 {0.031}
40	0.75	7.667 (0.012)	7.628 [−31.941]	7.648 [−15.311]	7.658 [−7.049]	7.790 (0.011)	7.789 {−0.001}	7.808 {0.018}	7.816 {0.026}
44	0.75	6.000 (0.014)	5.983 [−12.262]	5.996 [−2.795]	5.999 [−0.351]	6.083 (0.012)	6.098 {0.015}	6.111 {0.028}	6.114 {0.030}

Notes: This table reports prices from the LSM method and the BM with different numbers of steps in 3 dimensions for options with payoff function (20a) in Panel A and (20b) in Panel B. The options all have a strike price of 40 and matures in one year. The interest rate is fixed at 6% and exercise is considered ten times. In the table S denotes the level of the underlying stocks and ρ denotes the correlation. For all stocks a volatility of 40% is assumed. In the simulations $M = 100{,}000$ paths were used and for the American options monomials of maximum order $K = 3$ were used. The reported prices are averages of 100 price estimates calculated with different seeds in the random number generator. We report the standard errors of these estimates in parentheses. For comparison we report the corresponding values from a BM with various steps. In squared brackets below the European values, denoted p, we report the t-values from testing the hypotheses of equality of these estimates and those from the LSM method for a given number of steps in the BM. In curly brackets below the American values, denoted P, we report the differences between the LSM and Binomial prices.

imum options, the differences are still of economic importance and may be as large as ten cents for the in the money minimum options. However, these were exactly the payoff functions for which the BM performed the worst, and we have no reason to believe that this should not also be the case when it comes to the American price estimates. Furthermore, these payoff functions are probably also the ones most difficult to approximate with the LSM method because of their non-smoothness. Thus, it is difficult to conclude anything about the properties of the price estimators for this type of payoff functions.

On the other hand, for the smooth payoff functions in (20c) and (20d) in Table 7 the BM and the LSM prices are very close to each other. These were the payoff functions for which the BM could provide a sensible benchmark, and we conclude that if the payoff functions are smooth, equivalent price estimates can be obtained with the LSM and BM models.

Table 7. Put Option Prices in 3 Dimensions with Smooth Payoff Functions

Panel A: Arithmetic Average Options

S	ρ	$p_{\mathbf{LSM}}$	$p_{\mathbf{BM}_{100}}$	$p_{\mathbf{BM}_{200}}$	$p_{\mathbf{BM}_{400}}$	$P_{\mathbf{LSM}}$	$P_{\mathbf{BM}_{100}}$	$P_{\mathbf{BM}_{200}}$	$P_{\mathbf{BM}_{400}}$
36	0	4.370	4.370	4.371	4.371	4.867	4.867	4.866	4.866
		(0.009)	[0.550]	[0.856]	[0.949]	(0.007)	{0.000}	{−0.001}	{−0.002}
40	0	2.606	2.608	2.608	2.607	2.826	2.827	2.826	2.826
		(0.009)	[2.182]	[1.663]	[1.506]	(0.007)	{0.001}	{0.000}	{−0.001}
44	0	1.469	1.471	1.471	1.470	1.565	1.566	1.565	1.565
		(0.008)	[2.793]	[2.275]	[1.904]	(0.007)	{0.001}	{0.000}	{0.000}
36	0.25	5.069	5.071	5.071	5.071	5.504	5.505	5.503	5.503
		(0.009)	[2.299]	[1.859]	[1.589]	(0.008)	{0.001}	{0.000}	{−0.001}
40	0.25	3.339	3.342	3.341	3.341	3.564	3.566	3.564	3.564
		(0.011)	[3.174]	[2.206]	[1.793]	(0.008)	{0.002}	{0.000}	{0.000}
44	0.25	2.132	2.135	2.134	2.133	2.247	2.250	2.249	2.248
		(0.010)	[3.190]	[2.331]	[1.812]	(0.009)	{0.003}	{0.001}	{0.001}
36	0.5	5.673	5.676	5.675	5.675	6.074	6.076	6.074	6.073
		(0.010)	[3.431]	[2.428]	[1.887]	(0.008)	{0.002}	{0.000}	{−0.001}
40	0.5	3.971	3.976	3.974	3.973	4.200	4.202	4.200	4.199
		(0.012)	[3.751]	[2.473]	[1.874]	(0.009)	{0.003}	{0.000}	{−0.001}
44	0.5	2.727	2.731	2.730	2.729	2.856	2.860	2.858	2.857
		(0.012)	[3.682]	[2.556]	[1.915]	(0.010)	{0.004}	{0.002}	{0.001}
36	0.75	6.215	6.219	6.217	6.216	6.594	6.596	6.594	6.592
		(0.010)	[3.637]	[2.291]	[1.549]	(0.009)	{0.003}	{0.000}	{−0.002}
40	0.75	4.539	4.543	4.542	4.541	4.769	4.772	4.769	4.768
		(0.013)	[3.724]	[2.348]	[1.653]	(0.009)	{0.004}	{0.001}	{−0.001}
44	0.75	3.273	3.278	3.276	3.276	3.414	3.417	3.415	3.414
		(0.014)	[3.630]	[2.332]	[1.635]	(0.012)	{0.003}	{0.001}	{0.000}

Panel B: Geometric Average Options

S	ρ	$p_{\mathbf{LSM}}$	$p_{\mathbf{BM}_{100}}$	$p_{\mathbf{BM}_{200}}$	$p_{\mathbf{BM}_{400}}$	$P_{\mathbf{LSM}}$	$P_{\mathbf{BM}_{100}}$	$P_{\mathbf{BM}_{200}}$	$P_{\mathbf{BM}_{400}}$
36	0	5.365	5.364	5.367	5.365	5.499	5.500	5.502	5.501
		(0.006)	[−1.319]	[3.467]	[1.296]	(0.007)	0.001	0.003	0.002
40	0	3.348	3.347	3.349	3.349	3.405	3.407	3.408	3.409
		(0.009)	[−1.602]	[0.078]	[0.918]	(0.007)	0.002	0.003	0.003
44	0	1.973	1.977	1.974	1.975	1.997	2.003	1.999	2.000
		(0.009)	[4.400]	[0.615]	[1.597]	(0.008)	0.005	0.002	0.003
36	0.25	5.785	5.786	5.788	5.787	5.989	5.988	5.989	5.987
		(0.007)	[0.989]	[3.579]	[1.703]	(0.007)	−0.001	0.000	−0.002
40	0.25	3.908	3.908	3.909	3.909	4.012	4.013	4.013	4.013
		(0.010)	[0.685]	[1.187]	[1.440]	(0.008)	0.001	0.001	0.001
44	0.25	2.557	2.562	2.559	2.559	2.611	2.616	2.612	2.613
		(0.010)	[4.753]	[1.502]	[1.884]	(0.010)	{0.004}	{0.001}	{0.001}
36	0.5	6.138	6.139	6.141	6.139	6.399	6.399	6.399	6.397
		(0.009)	[2.104]	[3.445]	[1.743]	(0.007)	{0.000}	{0.000}	{−0.002}
40	0.5	4.355	4.357	4.357	4.357	4.504	4.505	4.504	4.503
		(0.012)	[1.736]	[1.556]	[1.468]	(0.008)	{0.001}	{−0.001}	{−0.001}
44	0.5	3.030	3.036	3.032	3.032	3.115	3.120	3.116	3.116
		(0.012)	[4.887]	[1.875]	[1.896]	(0.011)	{0.005}	{0.001}	{0.001}
36	0.75	6.442	6.445	6.445	6.443	6.757	6.757	6.756	6.754
		(0.010)	[2.414]	[2.989]	[1.412]	(0.008)	{0.001}	{0.000}	{−0.002}
40	0.75	4.732	4.734	4.734	4.733	4.923	4.924	4.922	4.922
		(0.013)	[2.174]	[1.584]	[1.290]	(0.009)	{0.001}	{−0.001}	{−0.002}
44	0.75	3.432	3.438	3.434	3.434	3.549	3.553	3.549	3.548
		(0.014)	[4.588]	[1.776]	[1.572]	(0.012)	{0.005}	{0.000}	{0.000}

Notes: This table reports prices from the LSM method and the Binomial Model with different numbers of steps in 3 dimensions for options with payoff function (20c) in Panel A and (20d) in Panel B. See also the notes to Table 6.

3.3. *Pricing Results for General L*

The limitations of the BM as a benchmark when the dimension of the pricing problem increases are clear from the analysis above. However, as mentioned the case of an option on the geometric average in (20d) is somewhat special. This follows from the fact that the product of lognormals is lognormal, and therefore the pricing problem can be reduced to one dimension. To be specific, consider L individual GBM processes and denote by S_G the geometric average of these. Then it follows that the dynamics of S_G are given by

$$S_G(t) = S_G(0)\exp\left\{\left(r - \frac{1}{2L}\sum_{i=1}^{L}\sigma_i^2\right)t + \sqrt{\sum_{i=1}^{L}\sum_{j=1}^{L}\rho_{ij}\sigma_i\sigma_j}\sqrt{t}Z\right\}, \tag{22}$$

where σ_i is the volatility of stock i, ρ_{ij} is the correlation between stock i and j, and $Z \sim N(0, 1)$. Thus, we can calculate benchmark values for any dimension using, e.g., the BM in one dimension with, say 50,000 steps. In this section we will compare the multidimensional BM and the LSM method to this benchmark using a subset of the options above with $\rho = 0.5$. Since the volatilities are identical in this case as are the correlations, (22) simplifies to

$$S_G(t) = S_G(0)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{\frac{1 + (L-1)\rho}{L}}\sqrt{t}Z\right\}. \tag{23}$$

We will start by examining the pricing properties in higher dimensions, and we finish this section by comparing the trade-off between computational time and precision for the multivariate problems.

3.3.1. Pricing Properties In Table 8 we provide results from the LSM method using $K = 2$ for $L = 2, \ldots, 10$. As before we report averages and standard errors from 100 calculated prices using different seeds, and we compare the price estimates to the one-dimensional BM with 50,000 steps. The first part of each panel confirms that the BM is a valid benchmark disregarding the dimension of the pricing problem. In all cases the pricing errors are less than two-tenths of a cent and insignificantly different from the NI values found using simulation.

For the American price estimates we see that the bias of the LSM method is less than one cent in all the cases. However, the bias indicates that relative to the benchmark the LSM estimates increase. With a fixed number of paths and an increasing number of regressors this is to be expected as the high bias eventually becomes more and more pronounced.

3.3.2. The Optimal Choice when the Trade-off between Precision and Computational Time Is Considered for General L We have managed to implement the BM in up to $L = 6$ dimensions. However, this could be done using only 10 steps. Figures 9–11 plot the *RMSE* from (18) against computational time measured as the number of prices one can calculate per second for the in the money, the at the money, and the out of the money options, respectively, following the procedure outlined in Section 2.3.

Starting with the in the money option, Figure 9 shows that the BM has a better trade-off between precision and computational time for low dimensional problems. Indeed, in one dimension the estimate from the BM has less than 1 percent of the *RMSE* of the LSM estimate for a given amount of computational time. While the BM is also the preferred model

Table 8. Geometric Average Put Option Prices

Panel A: In the Money Option with a Stock Price Equal to 36

Dimension	p_{BM}	T-stat	p_{LSM}	S.E. (p_{LSM})	P_{BM}	Bias	P_{LSM}	S.E. (P_{LSM})
2	6.296	−2.538	6.298	0.009	6.580	−0.008	6.572	0.008
3	6.139	−1.368	6.138	0.009	6.396	−0.008	6.388	0.008
4	6.056	−0.578	6.055	0.009	6.300	−0.007	6.292	0.008
5	6.005	0.490	6.005	0.008	6.240	−0.008	6.232	0.007
6	5.970	−0.114	5.970	0.008	6.199	−0.006	6.194	0.008
7	5.945	−1.712	5.943	0.008	6.170	−0.005	6.165	0.007
8	5.925	−1.836	5.924	0.008	6.148	−0.002	6.146	0.006
9	5.910	−1.090	5.910	0.008	6.130	0.000	6.130	0.008
10	5.898	1.015	5.899	0.008	6.116	0.001	6.118	0.008

Panel B: At the Money Option with a Stock Price Equal to 40

Dimension	p_{BM}	T-stat	p_{LSM}	S.E. (p_{LSM})	P_{BM}	Bias	P_{LSM}	S.E. (P_{LSM})
2	4.552	−2.493	4.549	0.011	4.719	−0.008	4.711	0.008
3	4.357	−1.360	4.355	0.012	4.503	−0.007	4.496	0.009
4	4.253	−0.411	4.252	0.012	4.388	−0.006	4.382	0.009
5	4.188	0.773	4.189	0.010	4.317	−0.004	4.313	0.008
6	4.144	−0.137	4.144	0.011	4.269	−0.003	4.266	0.008
7	4.112	−1.714	4.110	0.011	4.234	−0.003	4.230	0.008
8	4.087	−1.193	4.086	0.010	4.207	−0.001	4.206	0.007
9	4.068	−0.766	4.067	0.010	4.186	0.001	4.188	0.008
10	4.053	0.865	4.054	0.011	4.169	0.002	4.172	0.009

Panel C: Out of the Money Option with a Stock Price Equal to 44

Dimension	p_{BM}	T-stat	p_{LSM}	S.E. (p_{LSM})	P_{BM}	Bias	P_{LSM}	S.E. (P_{LSM})
2	3.240	−2.575	3.237	0.012	3.339	−0.007	3.332	0.009
3	3.032	−1.441	3.030	0.012	3.115	−0.007	3.108	0.011
4	2.921	−0.502	2.921	0.012	2.997	−0.005	2.993	0.011
5	2.853	0.672	2.854	0.010	2.924	−0.002	2.922	0.009
6	2.806	−0.126	2.806	0.011	2.874	−0.002	2.873	0.010
7	2.772	−1.716	2.770	0.011	2.838	−0.003	2.835	0.010
8	2.747	−1.520	2.745	0.011	2.811	0.000	2.811	0.010
9	2.726	−0.924	2.725	0.011	2.789	0.003	2.792	0.010
10	2.710	0.944	2.711	0.011	2.772	0.004	2.776	0.010

Notes: This table reports prices and standard errors from the LSM method in multiple dimensions for the geometric average options with the payoff function from (20d). The underlying stocks all have a volatility of 40% and the correlation is 0.5. The options expire in one year, the strike price is 40, and we consider ten possible exercise dates. The interest rate is fixed at 6%. We use $M = 100{,}000$ paths and set $K = 2$ for the American prices and report the average of 100 estimates with different seeds together with the standard errors of these. The benchmark values are from the one-dimensional BM with 50,000 steps. In the column headed "T-stat" are t-values from testing the hypothesis of equality between the European prices, denoted p, from the BM benchmark and the LSM method. In the column headed "Bias" the differences between the LSM method and the BM American prices, denoted P, are reported.

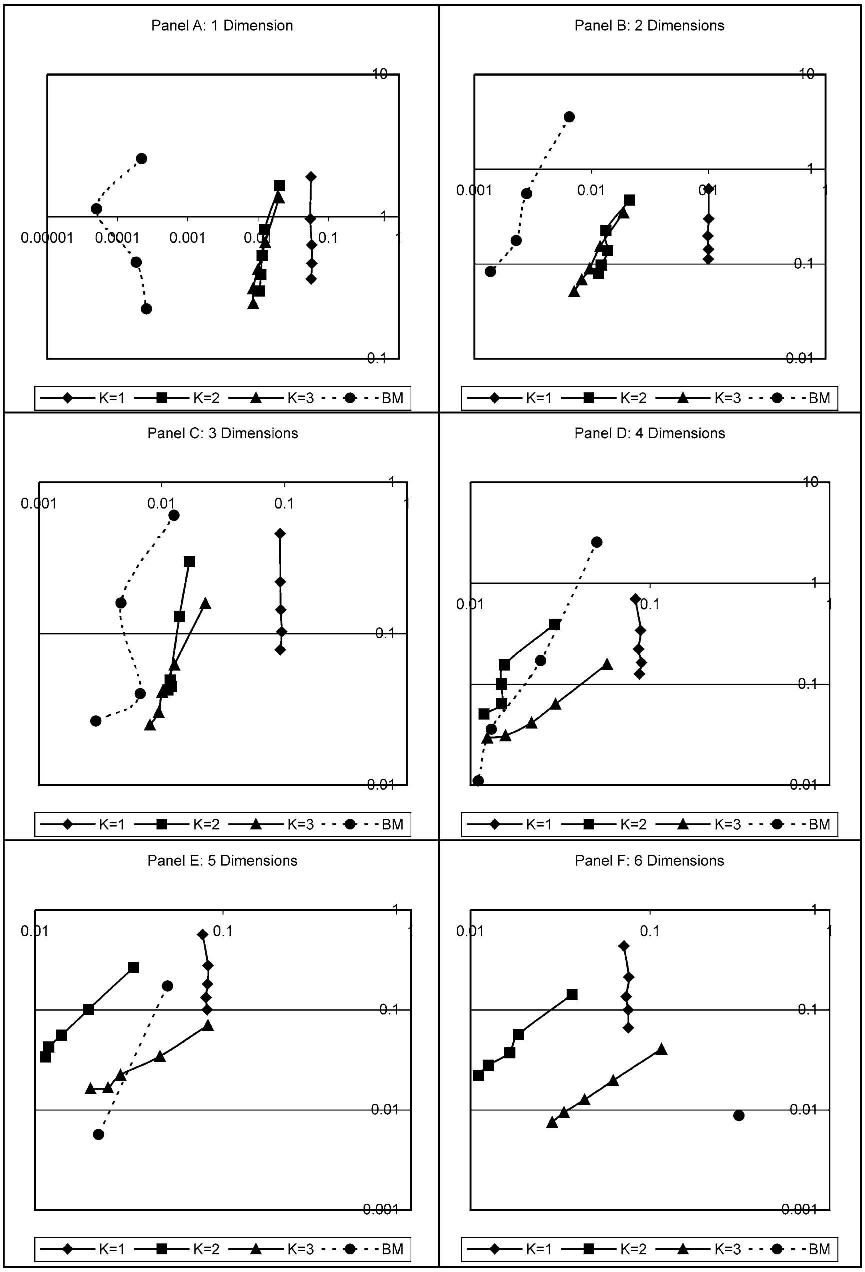


Figure 9. This figure reports the number of option prices that can be calculated in one second (y-axis) plotted against *RMSE* (x-axis) using different numbers of regressors for different dimensions. Preferred specifications are in the upper-left corner. Results are for the in the money option with a stock price of 36. See the notes to Table 8 for the other characteristics of the option.

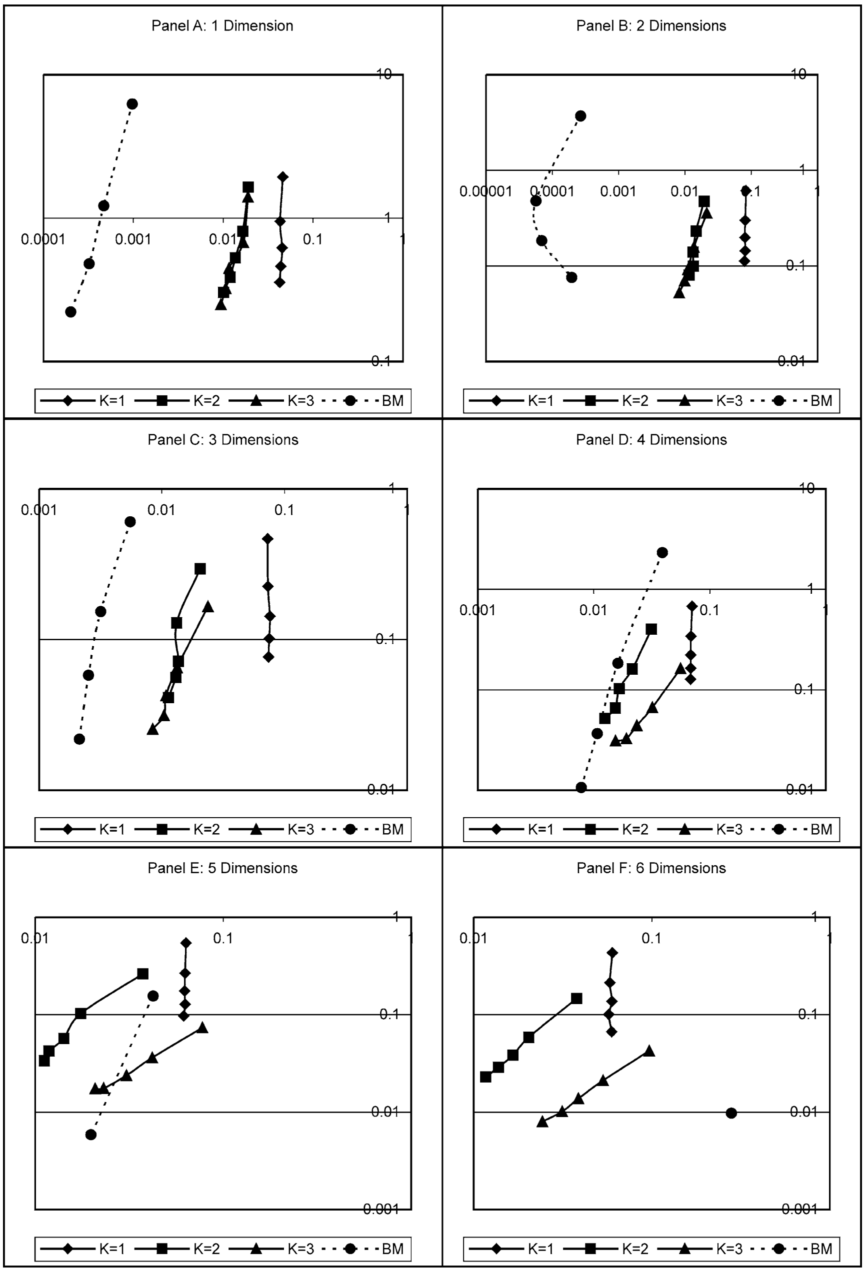


Figure 10. This figure reports the number of option prices that can be calculated in one second (y-axis) plotted against *RMSE* (x-axis) using different numbers of regressors for different dimensions. Preferred specifications are in the upper-left corner. Results are for the at the money option with a stock price of 40. See the notes to Table 8 for the other characteristics of the option.

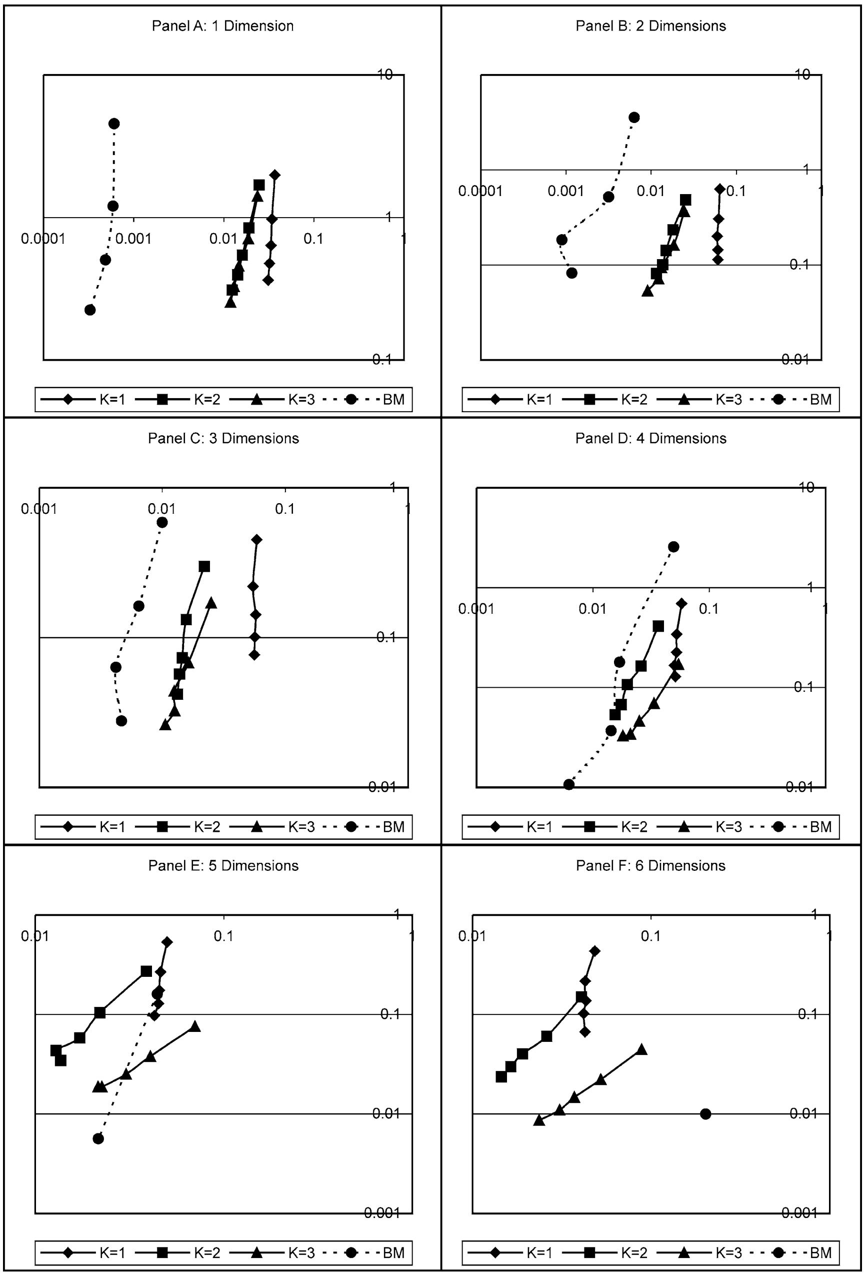


Figure 11. This figure reports the number of option prices that can be calculated in one second (y-axis) plotted against *RMSE* (x-axis) using different numbers of regressors for different dimensions. Preferred specifications are in the upper-left corner. Results are for the out of the money option with a stock price of 44. See the notes to Table 8 for the other characteristics of the option.

for $L = 2$ and 3, from the figure it is clear that as the dimension of the pricing problem is increased the LSM method becomes more and more competitive when compared to the BM. For $L = 4$ the LSM method potentially has a better trade-off, but for $L = 5$ or 6 the figure shows that the estimate from the LSM method using either $K = 1$ or 2 is the preferred model. In the highest dimension the *RMSE* of the LSM method could be as little as one tenth of the BM.

Figures 10 and 11 show that the results above hold across moneyness, and together with the results from Table 8 we can conclude that not only does the LSM method provide credible results in high dimensions, but the LSM method with $K = 2$ also has a preferable trade-off between computational time and precision to that of the BM for pricing problems with dimensions in excess of three or four.

4. Conclusion

In this paper a detailed analysis of the Least Squares Monte-Carlo (LSM) method suggested in Longstaff and Schwartz (2001) is performed. In the simple Black–Scholes type model we show that the specification used in LS is numerically inferior to using simple monomials. The specification of the cross-sectional regressions used in the original paper is compared with alternative specifications, each of which leads to a trade-off between the time used to calculate a price and the precision of that price. Comparing the method-specific trade-off reveals that the preferred specification uses $K = 2$ or 3 simple ordinary polynomials.

Next, we analyze how the LSM method fares when the number of stochastic factors is increased. As an easy exercise we consider pricing options on multiple assets. We argue that the LSM method is much easier to generalize to multiple dimensions than other of the known numerical procedures. We show that the LSM method can be used to price options on multiple assets as long as the payoff functions are smooth. Furthermore, we show that as the number of stochastic factors is increased the LSM method has a better trade-off between computational time and precision. Thus, for high dimensional problems the LSM method should be preferred to the Binomial Model.

Notes

1. Boyle, Broadie, and Glasserman (1997) have criticized the methods suggested by Tilley (1993) and by Barraquand and Martineau (1995). They conclude that the method suggested by Tilley gives rise to prices which are biased upwards, since the same simulated values are used to bundle the simulated paths and to estimate the optimal early exercise policy. They argue that the same should apply to the method suggested by Barraquand and Martineau, and they give a small example showing that convergence to the correct price is not guaranteed. Furthermore, Broadie and Glasserman (1997) argue that among a large class of estimators like the ones in Tilley (1993) and Barraquand and Martineau (1995) there is no unbiased estimator of the American put price.
2. This bias corresponds to what was noted by Boyle, Broadie, and Glasserman (1997) (see Note 1).
3. All computations were performed using the matrix language *Ox* (see Doornik (2001)) and using the random number generator *rann*$(\cdot)$.
4. Although Proposition 1 in Longstaff and Schwartz (2001) is formulated in terms of $\widehat{F}_M$ which corresponds to $\widehat{g}_K$ in our notation, the proof actually requires that F_M, which corresponds to our g_K function, is known.

This is obviously not a reasonable assumption in practice and if it were true there would be no need for any least-squares approximation.

5. In the limit we conjecture that β_M approaches 0.5, which is exactly the $\sqrt{M}$ consistency for the European estimate.
6. As pointed out by a referee, orthogonality is not necessary in the cross-sectional regressions since it is only the fitted values of the regression that matters (see also the discussion in Longstaff and Schwartz (2001, Section 8.3)).
7. We have implemented the LSM method using two additional families of orthogonal polynomials, the family of 2nd order general Chebyshev polynomials and the family of Hermite polynomials, in (17). The results for both these families were in between those of the Laguerre family and either the general Chebyshev or the Legendre family in terms of bias and *RMSE*. For this reason we do not report any details although they are available upon request.
8. We have also compared the method with that of Boyle (1988) and, although the method in that paper uses a five-point jump process and the specifications of the jump sizes and probabilities are therefore slightly different from the ones used here, we can replicate the European and American prices from that paper.
9. A solution could be to use extrapolation methods like Richardson Extrapolation (see Press, Teukolsky, Vetterling, and Flannery (1997)). However, Amin (1991) claims that these methods cannot be implemented to determine American option prices in an easy fashion. One problem is the oscillatory nature of the convergence which is found for options with payoff function (20a) in particular (see Carr (1998)). Thus, we refrain from using this procedure.

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