Stony Brook University

DATA ANALYSIS

AMS 572

Basic Concepts of Inference

Applied Math and Stats, Stony Brook University

Silvia Sharna

FAR BEYOND

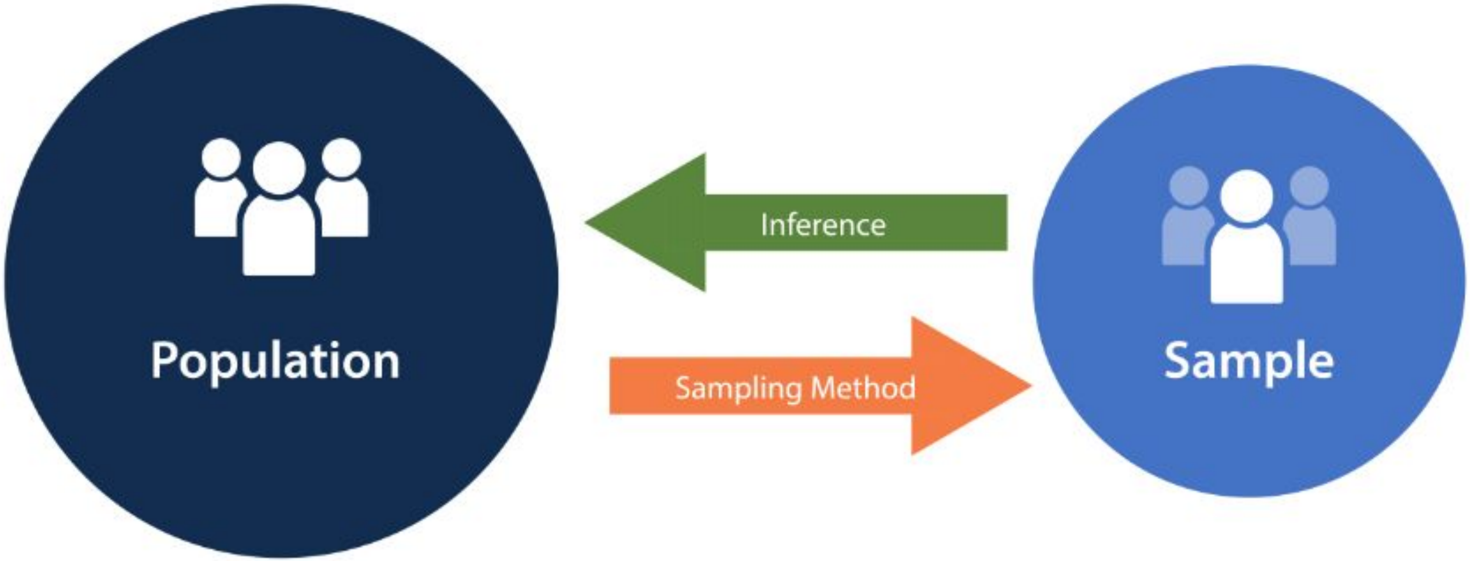
Sampling

Representative Sample !

- A function of statistics is the provision of techniques for making inductive inferences, i.e, generalizations beyond the actual data in hand
- The goal of inductive inference is to find out something about a target population by examining a sample of it. (Read Chapter 3 of sampling designs).
- Inductive inference is accomplished by constructing a model that describes the origin of the data and a model for data collection (sampling)

Stony Brook University

Representative Sample !



FAR BEYOND

Stony Brook University

Definition: The random variables $X_1, \ldots, X_n$ are called a *random sample* of size n from the population f (x) if $X_1, \ldots, X_n$ are independent and identically distributed (iid) with marginal pdf or pmf f (x)

Note: The definition implies that the joint pdf or pmf of $X_1, \ldots, X_n$ is

$$\prod_{i=1}^{n} f(x_i)$$

FAR BEYOND

Statistics and sample moments

- A *statistic* is a function of observable random variables. The probability distribution of a statistic is called its sampling distribution
- Sample mean:

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

- Sample variance:

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2, n > 1$$

- *Estimand*: parameter of interest we are trying to estimate; a constant; Eg μ
- *Estimator*: the statistic used to estimate the estimand; a random variable; Eg $\bar{X}$
- *Estimate*: a realization of an estimator from an observed data set; Eg $\bar{x} = 36.3$

Point estimation

- Method of Maximum Likelihood Estimator
- Method of Moments Estimator

Point estimation

- Method of Maximum Likelihood

Definition: Let $X_1, \ldots, X_n$ be a sample with joint p.d.f or p.m.f $f(\mathbf{x}|\theta)$. Given that $\mathbf{X} = \mathbf{x}$ is observed, the function of θ defined by

$$L(\theta|\mathbf{x}) = f(\mathbf{x}|\theta)$$

is called the likelihood function.

Point estimation

- Method of Maximum Likelihood

Definition: For a given observed sample $\mathbf{x}$, let $\hat{\theta}(\mathbf{x})$ be a value in the parameter space at which $L(\theta|\mathbf{x})$ attains its maximum. $\hat{\theta}(\mathbf{x})$ is called the maximum likelihood estimator (MLE) of θ.

Example: Suppose $X_i \sim N(\mu, \sigma^2)$ for $i = 1, \ldots, n$. Derive the MLE for μ and σ^2.

Point estimation

- Method of Moments

Method of moment method is one of the oldest method, and is simple to use. This method almost always yields some sort of estimate (MME).

Point estimation

- Method of Moments

Work by equating the sample moments to the population moments:

$$E(X) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$E(X^2) = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$\vdots$$

$$E(X^k) = \frac{1}{n}\sum_{i=1}^{n} X_i^k$$

Point estimation

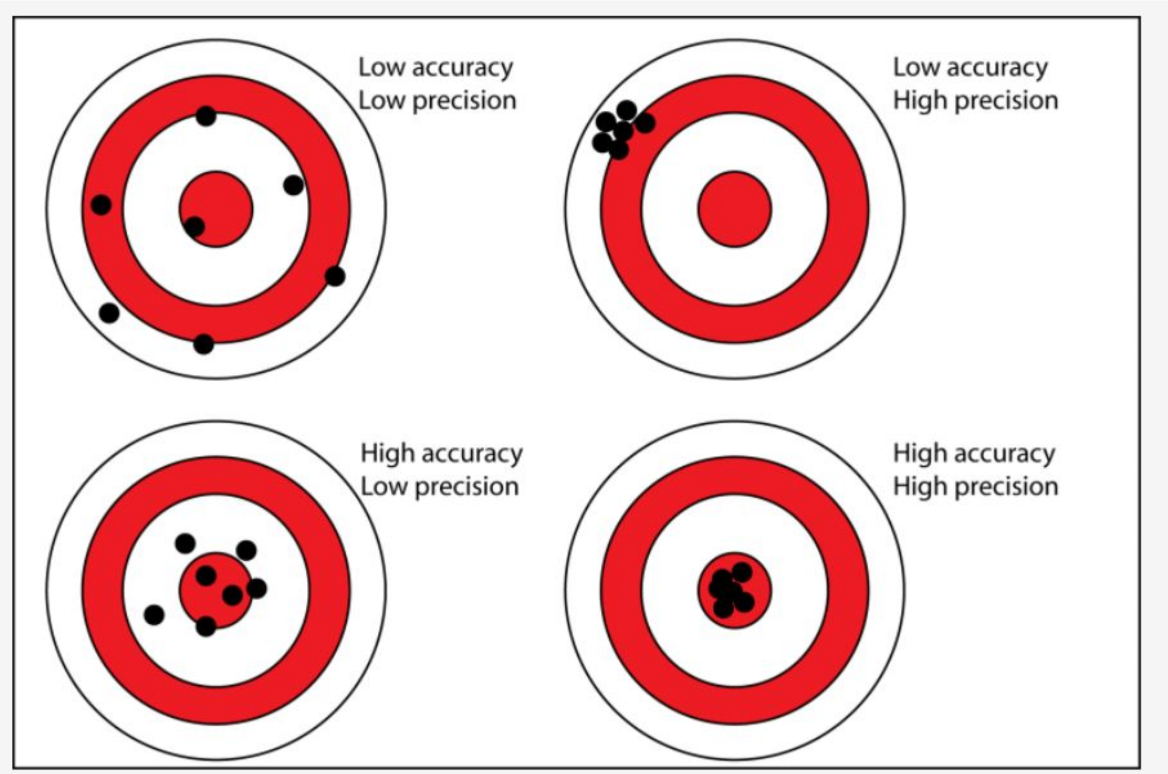
- Method of Moments

Example: Suppose $X_i \sim N(\mu, \sigma^2)$ for $i = 1, \ldots, n$. Derive the MME for μ and σ^2.

Stony Brook University

Methods of Evaluating Estimators

FAR BEYOND


Low accuracy
Low precision
Low accuracy
High precision
High accuracy
Low precision
High accuracy
High precision

Methods of Evaluating Estimators

Definition: The bias of an estimator $\hat{\theta}$ of θ is

$$\text{bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$$

(measures accuracy)

Example: Are the MLEs of μ and σ^2 for normal distribution unbiased? If not, find an unbiased estimator for these parameters.

Methods of Evaluating Estimators

Definition: The mean squared error (MSE) of an estimator $\hat{\theta}$ of θ is

$$\mathrm{MSE}(\hat{\theta}) = E(\hat{\theta} - \theta)^2$$

(measures precision and accuracy)

In addition, $\mathrm{MSE}(\hat{\theta}) = \mathrm{Var}(\hat{\theta}) + \mathrm{bias}(\hat{\theta})^2$

Methods of Evaluating Estimators

Example 6.4 of Tamhane and Dunlop:
Compare the MSE of these estimators for σ^2,
$S_1^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$ and $S_2^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/n$.

It can be shown that

$$\mathrm{MSE}(S_1^2) = \frac{2\sigma^4}{n-1}$$

and

$$\mathrm{MSE}(S_2^2) = \frac{2n-1}{n^2}\sigma^4$$

Thus, $\mathrm{MSE}(S_1^2) > \mathrm{MSE}(S_2^2)$

- On average S_2^2 will be closer to σ^2 if MSE is used as criterion
- On average S_2^2 underestimates σ^2

Confidence Interval

Definition: Let $X_1, \ldots, X_n$ be random variables with joint p.d.f or p.m.f $f(\mathbf{x}|\theta)$. The random interval $(T_1(\mathbf{X}), T_2(\mathbf{X}))$ is called a $100(1-\alpha)\%$ confidence interval for θ if

$$P(T_1(\mathbf{X}) < \theta < T_2(\mathbf{X})) = 1 - \alpha$$

where $0 < \alpha < 1$.

- $T_1(\mathbf{X})$ and $T_2(\mathbf{X})$ are called the lower and upper confidence limits, respectively.
- $1 - \alpha$ is called the confidence coefficient.

Confidence Interval Interpretation

- If we draw 100 different random samples, on average $100(1-\alpha)\%$ of them will contain θ
- How can one decrease the width of confidence interval?

We will study the pivotal quantity method for deriving confidence interval in Chapter 7 lecture notes.

Hypothesis Testing

Stony Brook University

Hypothesis testing

1. Set up a hypothesis
2. Collect data
3. Infer from the data whether hypothesis is plausible

- Examples:
  Will folic acid supplementation reduce the risk of stroke?
  Is the return the same in project 1 and project 2?

Hypothesis testing

- Null hypothesis H_0
- Example of null hypothesis:
  The incidence of stroke will be the same in those taking folic acid supplements and those not taking folic acid supplements
  Investing in project 1 will yield the same return as investing in project 2

Null and Alternative

- In a test of a hypothesis, we are testing whether some population parameter has a particular value
- For example,

$$H_0 : \theta = \theta_0$$

where θ_0 is a known constant
- The alternative hypothesis is complement of null hypothesis

$$H_a : \theta \neq \theta_0$$

Stony Brook University

Test statistic

- Once the data are collected, we will compute a *test statistic* related to θ, say $S(\hat{\theta})$
- $S(\hat{\theta})$ is a random variable, since it is computed from a sample
- $S(\hat{\theta})$ will have a particular probability distribution under the assumption H_0, say $F_0[S(\hat{\theta})]$

FAR BEYOND

Test statistic

- Under F_0, we compute the probability that we would observe $S(\hat{\theta})$ or a value more extreme than $S(\hat{\theta})$ if the null H_0 was true
- If this probability is large, the data are consistent with H_0
- If this probability is small, H_0 is probably not true

Interpretation

- Usually if the probability is small, we conclude H_0 is not true; i.e., we "reject" H_0
- If the probability is large, we have not proven H_0. We say that "we failed to reject H_0"
- We can never prove H_0 is true!

Significance Level

- How do we decide if the probability is too small?
- Prior to seeing the data, we select a value α such that: if the computed probability is less than or equal to α, we reject H_0
- α is known as *significance level*

Stony Brook University

Critical Region and Value

- We have a statistic $S(\hat{\theta})$ with distribution F_0 under the null hypothesis
- We specify α and under F_0 determine a *critical region* or *rejection region* C_α such that

$$\Pr[S(\hat{\theta}) \in C_\alpha | H_0] = \alpha$$

- Values at the boundaries of C_α are called *critical values*

FAR BEYOND

Critical Region and Value

- From the data we compute $S(\hat{\theta})$
- If $S(\hat{\theta}) \in C_\alpha$, we reject H_0
- If $S(\hat{\theta}) \notin C_\alpha$, the data are consistent with H_0 and we do not reject H_0

Type I error, Type II error and Power

		Truth	
		H_0	H_a
Decision	Do not reject		Type II error
	Reject	Type I error	Power

Tests of Hypotheses: Seven Steps

1. Design study
2. Establish null hypothesis
3. Determine test statistic to be employed
4. Choose significance level α and establish C_α
5. Carry out study and collect data
6. Compute statistic from data
7. If statistic is in C_α, reject H_0

We will study step by step of setting up a hypothesis test in Chapter 7 slides.