AMS 507

Chapter 4
Random Variables

Sections 1-5

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4.1 Random Variables

- <u>Def.</u> Random variable (RV): a function from S to the real space $\mathbb{R}$.

$$X\colon S \to \mathbb{R}$$

$$P(X = x) = P(\{s \in S : X(s) = x\})$$

Example 4.1.1

Let X be the number of heads obtained in three tosses of a fair coin.

Outcome	X
HHH	3
HHT	2
HTH	2
HTT	1
THH	2
THT	1
TTH	1
TTT	0

Value of X	Event
X = 0	{TTT}
X = 1	{HTT, THT, TTH}
X = 2	{HHT, HTH, THH}
X = 3	{HHH}

$$P(X = 0) = \frac{1}{8}$$

$$P(X = 1) = \frac{3}{8}$$

$$P(X = 2) = \frac{3}{8}$$

$$P(X = 3) = \frac{1}{8}$$

Types of Random Variables

- Discrete Random Variables have a countable number of possible values.
- Continuous Random Variables can take on any value in an interval and cannot be enumerated.

eg) X: # heads obtained in three tosses of a coin: discrete
X: amount of precipitation produced by a storm: continuous

4.2 Discrete Random Variables

- Probability mass function (pmf)

1. $p(x) = P(X = x)$
2. $p(x) \geq 0 \quad {}^{\forall}x$
3. For any $\sum_{\text{all } x} p(x) = 1$

Example 4.2.1

Which of the following is a pmf?

- $p(x) = \frac{x-1}{3}$ for $x = 0, 1, 2, 3$

$$f(0) = -\frac{1}{3} < 0 \text{ , so not a pmf.}$$

- $p(x) = \frac{x^2}{12}$ for $x = 0, 1, 2, 3$

$$p(x) \geq 0 \text{ for all } x\text{, but}$$

$$\sum_{x=0}^{3} p(x) = 0 + \frac{1}{12} + \frac{4}{12} + \frac{9}{12} = \frac{14}{12} = \frac{7}{6} > 1$$

So it is not a pmf.

Cumulative Distribution Function

<u>Def.</u> Cumulative distribution function (cdf):

$$F(x) = P(X \leq x) \quad {}^{\forall}x$$

Theorem 4.2.1 A function $F(x)$ is a cdf $\Leftrightarrow$

1. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$
2. $F(x)$ is nondecreasing
3. ${}^{\forall}x_0, \lim_{x \downarrow x_0} F(x) = F(x_0)$: right continuous

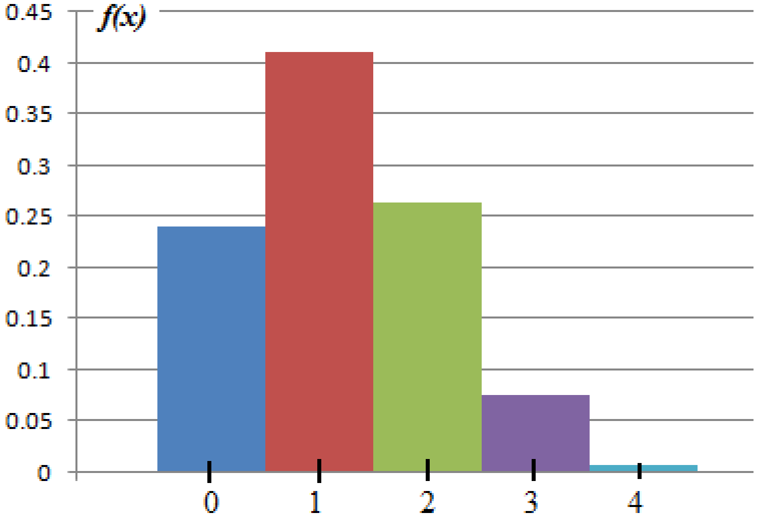
Example 4.2.2

Foreign made cars: 30%. Four cars are selected at random.

X: number of foreign made cars. F: foreign made, D: domestic

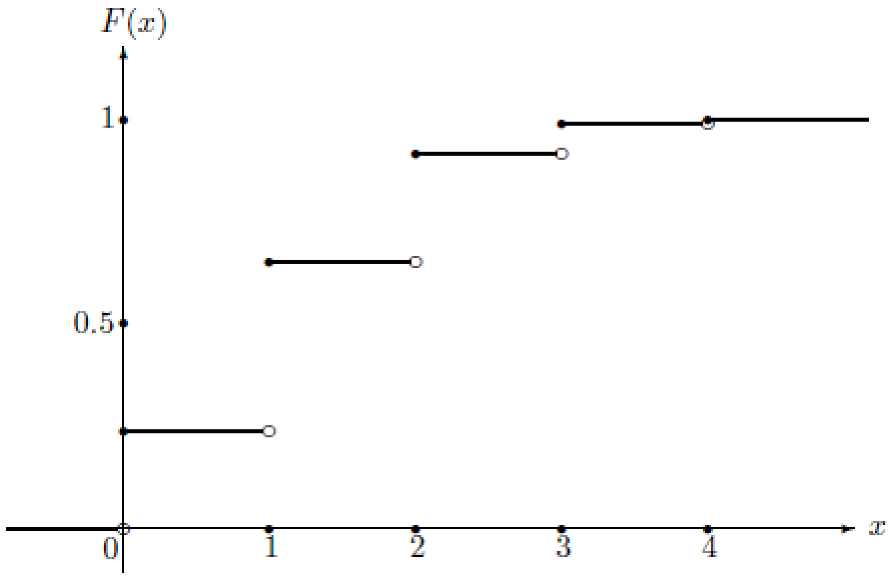
$X = 0$	$X = 1$	$X = 2$	$X = 3$	$X = 4$
DDDD	DDDF	DDFF	DFFF	FFFF
	DDFD	DFDF	FDFF	
	DFDD	DFFD	FFDF	
	FDDD	FDDF	FFFD	
		FDFD		
		FFDD		

$$P(X = 0) = P(DDDD) = 0.7^4 = 0.2401$$
$$P(X = 1) = 4 \cdot 0.7^3 \cdot 0.3 = 0.4116$$
$$P(X = 2) = 6 \cdot 0.7^2 \cdot 0.3^2 = 0.2646$$
$$P(X = 3) = 4 \cdot 0.7 \cdot 0.3^3 = 0.0756$$
$$P(X = 4) = 0.3^4 = 0.0081$$



Example 4.2.2 (continued)

x	$p(x)$	$F(x) = P(X \leq x)$
0	0.2401	0.2401
1	0.4116	0.6517
2	0.2646	0.9163
3	0.0756	0.9919
4	0.0081	1



$p(0)$ =0.2401, $p(1)$ =0.4116, $p(2)$ =0.2646, $p(3)$ =0.0756, $p(4)$ =0.0081

$F(0) = f(0) = 0.2401$

$F(1) = P(X \leq 1) = p(0) + p(1) = 0.2401 + 0.4116 = 0.6517$

$F(2) = P(X \leq 2) = p(0) + p(1) + p(2) = 0.9163$

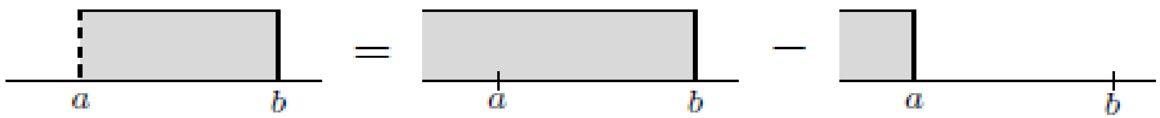
$F(3) = P(X \leq 3) = p(0) + p(1) + p(2) + p(3) = 0.9919$

$F(4) = P(X \leq 4) = 1$

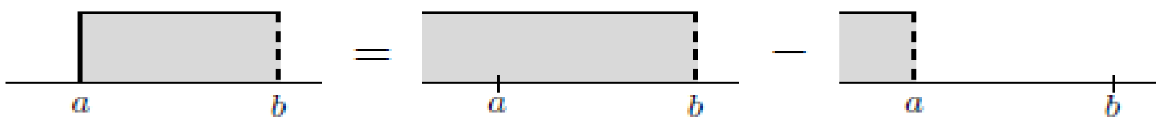
For $a < b$,

$$P(a \le X \le b) = P(X \le b) - P(X < a) = F(b) - F(a^-)$$

$$P(a < X \le b) = P(X \le b) - P(X \le a) = F(b) - F(a)$$



$$P(a \le X < b) = P(X < b) - P(X < a) = F(b^-) - F(a^-)$$



$$P(a < X < b) = P(X < b) - P(X \le a) = F(b^-) - F(a)$$

Example 4.2.4

x	$p(x)$	$F(x)$
0	0.1	0.1
1	0.2	0.3
2	0.3	0.6
3	0.2	0.8
4	0.2	1

$$P(1 \le X \le 3) = F(3) - F(0) = 0.8 - 0.1 = 0.7$$
$$P(1 < X \le 3) = F(3) - F(1) = 0.8 - 0.3 = 0.5$$
$$P(1 \le X < 3) = F(2) - F(0) = 0.6 - 0.1 = 0.5$$
$$P(1 < X < 3) = F(2) - F(1) = 0.6 - 0.3 = 0.3$$

Example 4.2.5

Tossing a coin until a head appears.

Let $p = P(H)$, and X: #tosses required to get a head.

$\Rightarrow P(X = x) = p(1-p)^{x-1}$ for $x = 1, 2, \cdots$

$$P(X \le x) = \sum_{k=1}^{x} P(X = k) = p \sum_{k=1}^{x} (1-p)^{k-1}$$

$$= p \frac{1-(1-p)^x}{1-(1-p)} = 1-(1-p)^x,$$

$$x = 1, 2, \cdots$$

(1) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

(2) nondecreasing (3) $\lim_{\epsilon \downarrow 0} F(x+\epsilon) = F(x)$

Thus, $F(x)$ is a cdf.

4.3 Expected Value

Def Mean (Expected value) of a discrete r.v. X:

$$E(X) = \mu = \sum_{\text{all } x} x p(x)$$

Example 4.3.1

What is the expected number of heads in three tosses of a fair coin?

Let X: # heads in three tosses of a fair coin.

$$E(X) = \sum_{x=0}^{3} xp(x)$$

$$= 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = 1.5$$

4.4 Expectation as a Function of RV's

$$E[h(X)] = \sum_{\text{all } x} h(x)p(x)$$

Example 4.4.1

In flipping 3 balanced coins find $E(X^3 - X)$

$$E(X^3 - X) = \sum_{x=0}^{3} (x^3 - x)p(x)$$

$$= (0-0) \cdot \frac{1}{8} + (1-1) \cdot \frac{3}{8} + (8-2) \cdot \frac{3}{8}$$

$$+ (27-3) \cdot \frac{1}{8} = \frac{42}{8} = 5.25$$

Example 4.4.2

$h(x) = 10 + 2x + x^2$. Find $E[h(X)]$.

x	$p(x)$	$h(x)$	$h(x)p(x)$
2	0.5	18	**9.0**
3	0.3	25	**7.5**
4	0.2	34	**6.8**
Total	**1**	**77**	**23.3**

$$E[h(X)] = \sum_{x=2}^{4} h(x)p(x)$$
$$= 18(0.5) + 25(0.3) + 34(0.2) = 23.3$$

Example 4.4.3

Let $Y = g(X) = aX + b$.

Then $E(Y) = E(aX + b)$.

<u>Proof</u>

$$E(Y) = E(aX + b) = \sum (ax + b)p(x)$$

$$= a\sum xp(x) + b\sum p(x)$$

$= aE(X) + b$

Transformation of Discrete RV

X is a rv with cdf $F_X(x) \Rightarrow$ any function $Y = g(X)$ is a rv.

For all set A, $\mathrm{P}(\mathrm{Y} \in \mathrm{A}) = P[g(X) \in \mathrm{A}]$.

Let $\boldsymbol{\mathcal{X}}$ be the sample space of X and

$\boldsymbol{\mathcal{Y}}$ the sample space of Y. If g is 1-1,

$g^{-1}(A) = \{x \in \boldsymbol{\mathcal{X}}: g(x) \in A\}$

$g^{-1}(\{y\}) = \{x \in \boldsymbol{\mathcal{X}}: g(x) = y\}$

$P(Y \in A) = P[g(X) \in A] = P[X \in g^{-1}(A)]$

X is discrete $\Rightarrow \boldsymbol{\mathcal{X}}$ is countable $\Rightarrow$ $\boldsymbol{\mathcal{Y}} = \{y: y = g(x): x \in \boldsymbol{\mathcal{X}}\}$ is countable. $\therefore$ Y is discrete.

$p_Y(y) = P(Y = y) = \sum_{x \in g^{-1}(\boldsymbol{y})} P(X = x) = \sum_{x \in g^{-1}(\boldsymbol{y})} p_X(x)$ for $y \in \boldsymbol{\mathcal{Y}}$

Example 4.4.4

Let X be a discrete rv with pmf

$$p_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x},$$

$x = 0, 1, \cdots, n.$

Let $Y = n - X$. Then

$$p_Y(y) = P(Y = y) = P(X = n - y) = p_X(n - y)$$

$$= \binom{n}{n-y} p^{n-y} (1-p)^y = \binom{n}{y} (1-p)^y p^{n-y},$$

$y = 0, 1, \cdots, n.$

4.5 Variance

- <u>Variance</u> of a discrete r.v. X:

$$Var(X) = \sigma^2 = E[(X-\mu)^2] = \sum_{\text{all } x} (x-\mu)^2 f(x)$$

$$\text{sd}(X) = \sigma = \sqrt{\text{Var}(X)}$$

Alternatively $\quad \text{Var}(X) = E(X^2) - \mu^2$

$$= \sum_{\text{all } x} x^2 f(x) - \left[\sum_{\text{all } x} x f(x)\right]^2$$

$$\mathrm{Var}(X) = E[(X-\mu)^2] = E(X^2) - \mu^2$$

Proof:

$$\mathrm{Var}(X) = E(X-\mu)^2 = \sum_{\text{all } x} (x-\mu)^2 p(x)$$

$$= \sum_{\text{all } x} (x^2 - 2\mu x + \mu^2) p(x)$$

$$= \sum_{\text{all } x} x^2 p(x) - 2\mu \sum_{\text{all } x} x p(x) + \mu^2 \sum_{\text{all } x} p(x)$$

$$= \sum_{\text{all } x} x^2 p(x) - 2\mu^2 + \mu^2 = \sum_{\text{all } x} x^2 p(x) - \mu^2$$

$$= E(X^2) - \mu^2$$

Example 4.5.1

x	1	2	5	9
$p(x)$	0.3	0.4	0.2	0.1

$$E(X) = \mu = \sum_{\text{all } x} xp(x) = 1 \cdot 0.3 + 2 \cdot 0.4 + 5 \cdot 0.2 + 9 \cdot 0.1$$
$$= 3$$

$$\text{Var}(X) = E[(X-\mu)^2] = \sum_{\text{all } x} (x-\mu)^2 p(x)$$
$$= (1-3)^2(0.3) + (2-3)^2(0.4) + (5-3)^2(0.2)$$
$$+ (9-3)^2(0.1) = 6$$

Alternatively,

$$\text{Var}(X) = E(X^2) - \mu^2 = \sum_{\text{all } x} x^2 p(x) - \mu^2$$
$$= 1^2(0.3) + 2^2(0.4) + 5^2(0.2) + 9^2(0.1) - 3^2 = 15 - 9$$
$$= 6$$

Let $Y = g(X) = aX + b$.

Then $\text{Var}(Y) = a^2\text{Var}(X)$.

<u>Proof</u>

$$\text{Var}(Y) = E[(Y - \mu_Y)^2]$$
$$= E[\{(aX + b) - (a\mu_X + b)\}^2]$$
$$= E[a^2(X - \mu_X)^2]$$
$$= \sum_{\text{all } x} a^2(x - \mu_X)^2 p(x)$$
$$= a^2 \sum_{\text{all } x} (x - \mu_X)^2 p(x)$$
$$= a^2\text{Var}(X)$$