AMS 507

Chapter 3
Conditional Probability & Independence

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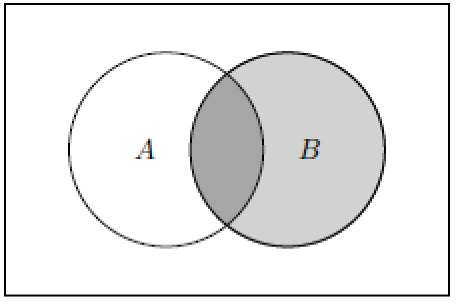
3.2 Conditional Probability

- Conditional probability of A given B:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \text{ if } P(B) > 0$$

Equivalently,

$$P(A \cap B) = P(B)P(A|B) \text{ if } P(B) > 0$$
$$= P(A)P(B|A) \text{ if } P(A) > 0$$



Example 3.2.1

BMI and age group

	Normal or Low BMI	Overweight	Obese	Total
Age < 30	0.09	0.06	0.05	0.20
Age ≥ 30	0.20	0.32	0.28	0.80
Total	0.29	0.38	0.33	1.00

(a) What is the probability that a person selected at random from the group will be obese?

(b) A person, selected at random from this group, is found to be obese. What is the probability that this person is younger than 30 years old?

Answer to Example 3.2.1

(a) Let A $= \{\text{obese}\}$. Then $P(A) = 0.33$.

(b) Let $B = \{\text{younger than 30 years old}\}$. Then

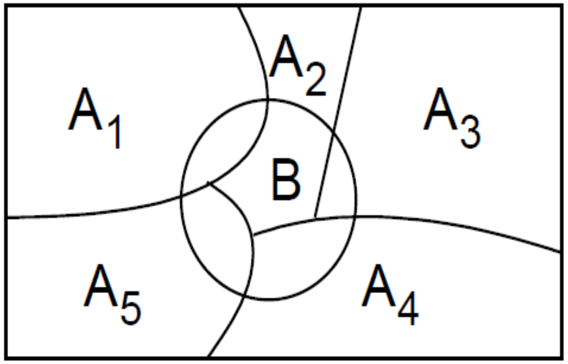
$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.05}{0.33} = 0.15$$

	Normal or Low BMI	Overweight	Obese	Total
Age < 30	0.09	0.06	0.05	0.20
Age ≥ 30	0.20	0.32	0.28	0.80
Total	0.29	0.38	0.33	1.00

3.3 Bayes' Formula

- If $A_1, A_2, \cdots, A_n$ is a partition of S. Then

$$P(B) = P(B \cap A_1) + \cdots + P(B \cap A_n)$$
$$= P(A_1)P(B|A_1) + \cdots + P(A_n)P(B|A_n)$$
$$= \sum_{i=1}^{n} P(A_i)P(B|A_i)$$



Bayes' Theorem:
Let $A_1, A_2, \dots, A_n$ be mutually exclusive & exhaustive. Then

$$P(A_r|B) = \frac{P(A_r \cap B)}{P(B)} = \frac{P(A_r)P(B|A_r)}{\sum_{i=1}^{n} P(A_i)P(B|A_i)}$$

Example 3.3.1

Probability of HIV infection in the US: 0.4%

Let $A_1 = \{\text{individual has HIV}\}$

$A_2 = \{\text{individual does not have HIV}\}$

$B = \{\text{positive HIV test result}\}$

$$P(A_1) = 0.004; P(A_2) = 0.996$$
$$P(B|A_1) = 0.997; P(B|A_2) = 0.015$$

If a test result is positive, what is the probability that the individual has the disease?

Answer to Example 3.3.1

$A_1 = \{\text{individual has HIV}\}$

$A_2 = \{\text{individual does not have HIV}\}$

$B = \{\text{positive HIV test result}\}$

$$P(A_1) = 0.004; P(A_2) = 0.996$$

$$P(B|A_1) = 0.997; P(B|A_2) = 0.015$$

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{P(A_1)P(B|A_1)}{\sum_{i=1}^{2} P(A_i)P(B|A_i)}$$

$$= \frac{(0.004)(0.997)}{(0.004)(0.997) + (0.996)(0.015)} = 0.21$$

Example 3.3.2

Box 1 contains 2 yellow and 4 green balls, whereas Box 2 contains 1 yellow and 1 green balls. A ball is randomly chosen from Box 1 and then transferred to Box 2, and a ball is then randomly selected from Box 2.

(a) What is the probability that the ball selected from Box 2 is yellow?

(b) What is the conditional probability that the transferred ball was yellow, given that a yellow ball is selected from Box 2?

Answer to Example 3.3.2

Box 1: 2 yellow, 4 green, Box 2: 1 yellow, 1 green

Let A be the event that the transferred ball is yellow and B the event that the ball from Box 2 is yellow.

(a)

$$P(B) = P(A \cap B) + P(A^C \cap B)$$
$$= P(B|A)P(A) + P(B|A^C)P(A^C) = \frac{2}{3} \cdot \frac{2}{6} + \frac{1}{3} \cdot \frac{4}{6} = \frac{4}{9}$$

(b)

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{(2/3)(2/6)}{4/9} = \frac{1}{2}$$

3.4 Independent Events

- A and B are independent if
$$P(A|B) = P(A).$$

Example 3.4.1

Tossing a fair die

Let $A = \{2, 4, 6\},\ B = \{1, 2, 3\},\ C = \{1, 2, 3, 4\}$.

Then $P(A) = \frac{1}{2}, P(A|B) = \frac{1}{3}, P(A|C) = \frac{1}{2}$.

A and B are dependent & A and C are independent.

- A and B are independent

  $\Leftrightarrow P(A \cap B) = P(A)P(B)$

  Proof) $P(A \cap B) = P(A|B)P(B)$

  and by independence $P(A|B) = P(A)$

  Therefore, $P(A \cap B) = P(A)P(B)$

Theorem 3.4.1

If A and B are independent, then the following pairs are independent.

(a) A and B^C

(b) A^C and B

(c) A^C and B^C

Proof

(a) Since A and B are independent,

$$P(A \cap B^C) = P(B^C|A)P(A) = [1 - P(B|A)]P(A)$$
$$= [1 - P(B)]P(A) = P(B^C)P(A)$$

(b) and (c) Similar

Example 3.4.1 (continued)

$A = \{2, 4, 6\}, \qquad B = \{1, 2, 3\}, \qquad C = \{1, 2, 3, 4\}$

$A \cap B = \{2\}$. A and B are dependent because

$$P(A \cap B) = \frac{1}{6} \neq P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$A \cap C = \{2, 4\}$. A and C are independent because

$$P(A \cap C) = \frac{1}{3} = P(A)P(C) = \frac{1}{2} \cdot \frac{2}{3}$$

- $A_1, A_2, \cdots, A_n$ are independent if, for any k, and every subset of indices $i_1, i_2, \cdots, i_k$,

$$P\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\right)$$
$$= P\left(A_{i_1}\right) P\left(A_{i_2}\right) \cdots P\left(A_{i_k}\right).$$

Example 3.4.2

Tossing two dice:

$$S = \{(1,1),(1,2),\cdots,(1,6),(2,1),\cdots,(2,6),\cdots,(6,1),\cdots,(6,6)\}$$

Let

$A = \{\text{doubles appear}\}$, $B = \{\text{the sum is between 7 and 10}\}$,

$$C = \{\text{The sum is 2 or 7 or 8}\}$$

Are A, B and C independent?

Answer to Example 3.4.2

$S = \{(1,1),(1,2),\cdots,(1,6),(2,1),\cdots,(2,6),\cdots,(6,1),\cdots,(6,6)\}$

$A = \{\text{doubles appear}\}$, $B = \{\text{the sum is between 7 and 10}\}$,

$$C = \{\text{The sum is 2 or 7 or 8}\}$$

$$P(A) = \frac{1}{6}, \qquad P(B) = \frac{1}{2}, \qquad P(C) = \frac{1}{3}$$

$$P(A \cap B \cap C) = P\{(4,4)\} = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{2} \cdot \frac{1}{2} = P(A)P(B)P(C)$$

However,

$$P(A \cap B) = P\{(4,4),(5,5)\} = \frac{1}{18} \neq P(A)P(B) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$$

Therefore, A, B and C are not independent.

Example 3.4.3

Let the sample space S consists of all possible permutations of the colors red (r), blue (b), and green (g) along with the three triples of each color. Thus,

$A = \{\text{rrr}, \text{bbb}, \text{ggg}, \text{rbg}, \text{rgb}, \text{brg}, \text{bgr}, \text{grb}, \text{gbr}\}.$

Define $R_i = \{\text{the } i\text{th place in the triple is occupied by } r\}.$

Are R_1, R_2 and R_3 independent?

Answer to Example 3.4.3

$$P(R_i) = \frac{1}{3}, \qquad i = 1, 2, 3$$

and

$$P(R_i \cap R_j) = \frac{1}{9} \quad {}^{\forall} i \neq j$$

However,

$$P(R_1 \cap R_2 \cap R_3) = P(\mathrm{rrr}) = \frac{1}{9}$$

$$\neq P(R_1)P(R_2)P(R_3) = \frac{1}{27}$$

Hence, R_1, R_2 and R_3 are not independent.

Example 3.4.4

Tossing a coin three times

$$S = \{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{THH}, \mathrm{TTH}, \mathrm{THT}, \mathrm{HTT}, \mathrm{TTT}\}$$

Let H_i denote the event that the ith toss is a head. Are H_1, H_2 and H_3 independent?

Answer to Example 3.4.4

Let $H_i, i = 1, 2, 3$ denote the event that the ith toss is a head. Then

$$H_1 = \{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{HTT}\}$$
$$H_2 = \{\mathrm{HHH}, \mathrm{HHT}, \mathrm{THH}, \mathrm{THT}\}$$
$$H_3 = \{\mathrm{HHH}, \mathrm{THH}, \mathrm{HTH}, \mathrm{HTH}\}$$

$$P(H_i \cap H_j) = P(H_i)P(H_j) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \quad {}^{\forall} i \neq j$$

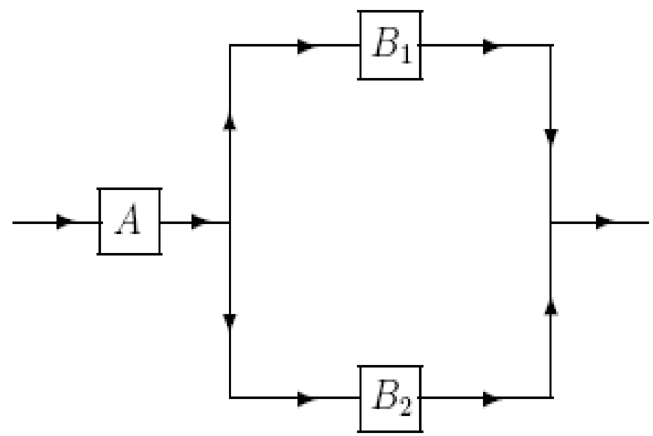
and

$$P(H_1 \cap H_2 \cap H_3) = P(H_1)P(H_2)P(H_3) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

Therefore, H_1, H_2 and H_3 are independent.

Example 3.4.5

A, B_1 and B_2 operate independently. The following system works if A works and either B_1 or B_2 works. What is the probability that the following system works?

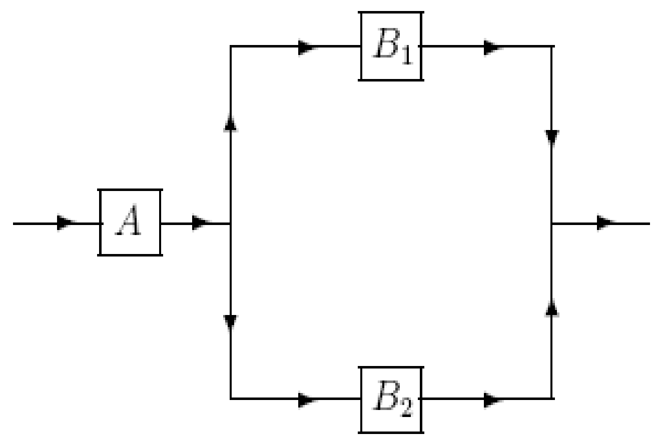


Let $P(A) = 0.9$, $P(B_1) = 0.8$ and $P(B_2) = 0.7$

Answer to Example 3.4.5

Let $P(A) = 0.9$, $P(B_1) = 0.8$ and $P(B_2) = 0.7$

$$\begin{aligned} P(\text{system working}) &= P[A \cap (B_1 \cup B_2)] \\ &= P(A)P(B_1 \cup B_2) \quad \text{(by independence)} \\ &= 0.9[P(B_1) + P(B_2) - P(B_1 \cap B_2)] \\ &= 0.9[P(B_1) + P(B_2) - P(B_1)P(B_2)] \\ &= 0.9[0.8 + 0.7 - (0.8)(0.7)] \\ &= 0.9(0.94) = 0.846 \end{aligned}$$



3.5 $P(\cdot\,|B)$ is a Probability

Theorem 3.5.1 $P(\cdot\,|B)$ is a Probability for any event $B \subset S$ with $P(B) > 0$.

Proof

1. Clearly $P(\cdot\,|B) \geq 0$.
2. $$P(S|B) = \frac{P(S\cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$
3. For any mutually exclusive events $A_1, A_2, \cdots,$

$$P\left(\bigcup_{i=1}^{\infty} A_i\,|B\right) = \frac{P[(\cup_{i=1}^{\infty} A_i) \cap B]}{P(B)} = \frac{P[\cup_{i=1}^{\infty}(A_i \cap B)]}{P(B)}$$

$$= \frac{\sum_{i=1}^{\infty} P(A_i \cap B)}{P(B)} = \sum_{i=1}^{\infty} \frac{P(A_i \cap B)}{P(B)} = \sum_{i=1}^{\infty} P(A_i|B)$$

Corollary

$$P(A^C|B) = 1 - P(A|B)$$

Example 3.5.1 (from Ross)

A female chimp has given birth. It is not certain, however, which of two male chimps is the father. Before any genetic analysis has been performed, it is believed that the probability that male number 1 (M1) is the father is p and the probability that male number 2 (M2) is the father is $1 - p$. DNA obtained from the mother, M1, and M2 indicates that on one specific location of the genome, the mother has the gene pair (A, A), M1 has the gene pair (a, a), and M2 has the gene pair (A, a). If a DNA test shows that the baby chimp has the gene pair (A, a), what is the probability that M1 is the father?

Answer to Example 3.5.1

The mother's gene pair: (A, A), M1's gene pair: (a, a), M2's gene pair (A, a). If the baby chimp's gene pair is (A, a), what is the probability that M1 is the father?

Let E denote the event that the gene pair is (A, a). Then

$$P(M1|E) = \frac{P(E|M1)P(M1)}{P(E|M1)P(M1) + P(E|M2)P(M2)}$$

$$= \frac{p \cdot 1}{p \cdot 1 + (1-p) \cdot \frac{1}{2}} = \frac{2p}{2p + 1 - p} = \frac{2p}{p+1}$$

Example 3.5.2 (from Ross)

Independent trials, each resulting in a success with probability p or a failure with probability $q = 1 - p$, are performed. Compute the probability that a run of n consecutive successes occurs before a run of m consecutive failures.

Example 3.5.3 Gambler's Ruin Problem

Two gamblers A and B bet on the outcomes of successive flips of a coin. If it comes up a head, A collects \$1 from B, and if it comes up a tail, B collects \$1 from A. Suppose $P(H) = p$. What is the probability that A winds up with all the money if she starts with \$$i$ and B starts with \$$(N - i)$?

Answer to Example 3.5.2

Let E denote the event that A winds up all the money, and H the event that the first flip lands head, and $q = 1 - p$. Then

$$P_i = P(E) = P(E|H)P(H) + P(E|H^C)P(H^C)$$
$$= pP(E|H) + qP(E|H^C) = pP_{i+1} + qP_{i-1}$$

$$(p+q)P_i = pP_{i+1} + qP_{i-1} \text{ or } p(P_{i+1} - P_i) = q(P_i - P_{i-1})$$

$$P_2 - P_1 = \frac{q}{p}(P_1 - P_0) = \frac{q}{p}P_1$$
$$P_3 - P_2 = \frac{q}{p}(P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1$$
$$\vdots$$
$$P_i - P_{i-1} = \frac{q}{p}(P_{i-1} - P_{i-2}) = \left(\frac{q}{p}\right)^{i-1} P_1$$
$$\vdots$$
$$P_N - P_{N-1} = \frac{q}{p}(P_{N-1} - P_{N-2}) = \left(\frac{q}{p}\right)^{N-1} P_1$$

Answer to Example 3.5.2 (continued)

$$P_2 - P_1 = \frac{q}{p}(P_1 - P_0) = \frac{q}{p}P_1 \qquad (1)$$

$$P_3 - P_2 = \frac{q}{p}(P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1 \qquad (2)$$

$$\vdots$$

$$P_i - P_{i-1} = \frac{q}{p}(P_{i-1} - P_{i-2}) = \left(\frac{q}{p}\right)^{i-1} P_1 \qquad (3)$$

Sum of LHS from (1) through (3):

$$P_i - P_1 = P_1\left[\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right]$$

$$P_i = P_1\left[1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right] = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)}P_1 & \text{if } p \neq 1/2 \\ iP_1 & \text{if } p = 1/2 \end{cases}$$

Answer to Example 3.5.2 (continued)

$$P_N = \begin{cases} \dfrac{1-(q/p)^N}{1-(q/p)} P_1 & \text{if } p \neq 1/2 \\ N P_1 & \text{if } p = 1/2 \end{cases}$$

Since $P_N = 1$,

$$P_1 = \begin{cases} \dfrac{1-(q/p)}{1-(q/p)^N} & \text{if } p \neq 1/2 \\ 1/N & \text{if } p = 1/2 \end{cases}$$

Therefore,

$$P_i = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)} P_1 & \text{if } q/p \neq 1 \\ i P_1 & \text{if } q/p = 1 \end{cases} = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)} \cdot \dfrac{1-(q/p)}{1-(q/p)^N} & \text{if } p \neq 1/2 \\ i \cdot \dfrac{1}{N} & \text{if } p = 1/2 \end{cases}$$

$$= \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq 1/2 \\ i/N & \text{if } p = 1/2 \end{cases}$$