AMS 507

Chapter 2
Axioms of Probability

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2.2 Sample Space and Events

- Def. Sample Space (S): The set of all possible outcomes of the experiment
- Def. Event: A set of outcomes contained in S
- If A is an event, then $A \subset S$.

$$A \subset B \Leftrightarrow x \in A \Longrightarrow x \in B$$
$$A = B \Leftrightarrow A \subset B \,\&\, B \subset A$$

Example 2.2.1

(a) Tossing a coin twice: $S = \{HH, HT, TH, TT\}$

- The event of getting exactly one head: $A = \{HT, TH\}$
- The event that at least one of the flips results in a head: $B = \{HH, HT, TH\}$

(b) Tossing a die: $S = \{1, 2, 3, 4, 5, 6\}$

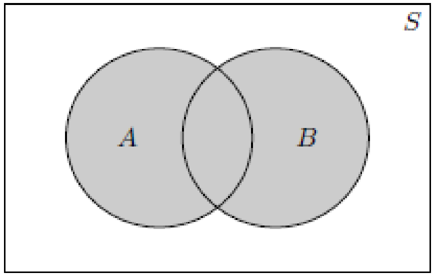
- The outcome is an odd number: $\{1, 3, 5\}$
- The outcome is less than 4: $\{1, 2, 3\}$
- The outcome is a 3: $\{3\}$

(c) Roll two dice, and record the sum. Then the sample space is

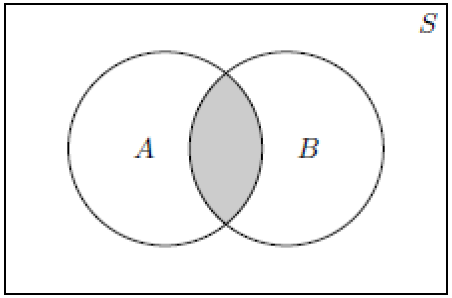
$$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

Set Operations

- Union of two events A & B ($A \cup B$; A or B): The event consisting of all outcomes that are either in A or B or both
  $A \cup B = \{x\text{: } x \in A \text{ or } x \in B\}$

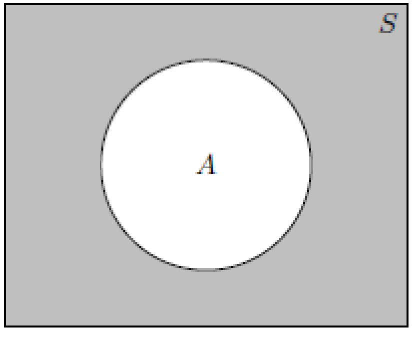


- Intersection ($A \cap B$; A and B): The event consisting of all outcomes that are in both A and B
  $A \cap B = \{x\text{: } x \in A \text{ and } x \in B\}$



- <u>Complement</u> (A^C): not A

$$A^C = \{x\colon\ x \notin A\ \}$$



Example 2.2.2

Tossing a coin twice

A: Tail at the second toss

B: At least one head

$A = \{HT, TT\}, \qquad B = \{HH, HT, TH\}$

$A \cup B = \{HH, HT, TH, TT\} = S$

$A \cap B = \{HT\}$

$A^C = \{\mathrm{HH}, \mathrm{TH}\}$

$B^C = \{\mathrm{TT}\}$

- For events $A_1, A_2, \cdots$ in S,

$$\bigcup_{i=1}^{\infty} A_i = \left\{x \in S\colon \quad {}^{\exists} i \text{ such that } x \in A_i\right\}$$

Type equation here.

$$\bigcap_{i=1}^{\infty} A_i = \left\{x \in S\colon\ x \in A_i \quad {}^{\forall} i\right\}$$

Properties of set operations

- Commutative

  (a) $A \cup B = B \cup A$

  (b) $A \cap B = B \cap A$

- Associative

  (a) $(A \cup B) \cup C = A \cup (B \cup C)$

  (b) $(A \cap B) \cap C = A \cap (B \cap C)$

- Distributive

  (a) $(\bigcup_n A_n) \cap B = \bigcup_n (A_n \cap B)$

  (b) $(\bigcap_n A_n) \cup B = \bigcap_n (A_n \cup B)$

Properties of set operations, Proof

- Commutative

  (a) $x \in A \cup B \Leftrightarrow x \in A \text{ or } x \in B$

  $\Leftrightarrow x \in A \text{ or } x \in B \Leftrightarrow x \in B \cup A$

  (b) $x \in A \cap B \Leftrightarrow x \in A \,\&\, x \in B$

  $\Leftrightarrow x \in A \,\&\, x \in B \Leftrightarrow x \in B \cap A$

- Associative

  (a) $x \in (A \cup B) \cup C \Leftrightarrow x \in A \cup B \text{ or } x \in C$

  $\Leftrightarrow (x \in A \text{ or } x \in B) \text{ or } x \in C$

  $\Leftrightarrow x \in (A \text{ or } B) \text{ or } x \in C$

  $\Leftrightarrow x \in (A \cup B) \text{ or } x \in C$

  $\Leftrightarrow x \in A \cap (B \cap C)$

  (b) Similar

Properties of set operations, Proof (continued)

- Distributive

(a) $x \in (\bigcup_n A_n) \cap B \Leftrightarrow x \in \bigcup_n A_n \;\&\; x \in B$

$\Leftrightarrow (x \in A_n \text{ for some } n) \;\&\; x \in B$

$\Leftrightarrow (x \in A_n \;\&\; x \in B) \text{ for some } n$

$\Leftrightarrow x \in A_n \cap B \text{ for some } n$

$\Leftrightarrow x \in \bigcup_n (A_n \cap B)$

(b) Similar

Example 2.2.3

- Let $S = (0, 1]$ and $A_i = \left[\frac{1}{i}, 1\right], i = 1, 2, \cdots$. Then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \left[\frac{1}{i}, 1\right] = \lim_{n\to\infty} \bigcup_{i=1}^{n} \left[\frac{1}{i}, 1\right] = \lim_{n\to\infty} \left[\frac{1}{n}, 1\right] = (0, 1]$$

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \left[\frac{1}{i}, 1\right] = \lim_{n\to\infty} \bigcap_{i=1}^{n} \left[\frac{1}{i}, 1\right] = \lim_{n\to\infty} \{1\} = \{1\}$$

De Morgan's Law

$(A \cup B)^C = A^C \cap B^C,\ (A \cap B)^C = A^C \cup B^C$

(a) $\left(\bigcup_{i=1}^{n} A_i\right)^C = \bigcap_{i=1}^{n} A_i^C$

(b) $\left(\bigcap_{i=1}^{n} A_i\right)^C = \bigcup_{i=1}^{n} A_i^C$

<u>Proof</u>

(a) $x \in \left(\bigcup_{i=1}^{n} A_i\right)^C \Leftrightarrow x \notin \bigcup_n A_n$

$$\Leftrightarrow x \notin A_i \quad {}^{\forall} i$$

$$\Leftrightarrow x \in A_i^C \quad {}^{\forall} i$$

$$\Leftrightarrow x \in \bigcap_{i=1}^{n} A_i^C$$

(b) Similar

Example 2.2.2 (revisited)

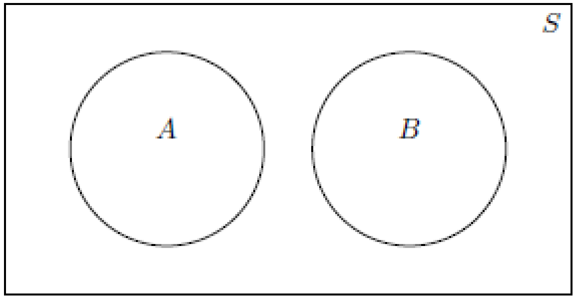
$$A = \{HT, TT\}, \qquad B = \{HH, HT, TH\}$$

$$(A \cup B)^C = \emptyset = A^C \cap B^C$$

where Ø is the empty set

$$(A \cap B)^C = \{\mathrm{HH}, \mathrm{TH}, \mathrm{TT}\} = A^C \cup B^C$$

- A and B are mutually exclusive (disjoint) if A and B have no outcomes in common.



Example 2.2.2 (continued)

A: Tail at the second toss

D: Two heads

$A = \{HT, TT\}, \qquad D = \{HH\}$

A and *D* are disjoint.

Example 2.2.3

- Let $S = (0, 1]$ and $A_i = \left[\frac{1}{i}, 1\right], i = 1, 2, \cdots$. Then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \left[\frac{1}{i}, 1\right] = \lim_{n\to\infty} \bigcup_{i=1}^{n} \left[\frac{1}{i}, 1\right] = \lim_{n\to\infty} \left[\frac{1}{n}, 1\right] = (0, 1]$$

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \left[\frac{1}{i}, 1\right] = \lim_{n\to\infty} \bigcap_{i=1}^{n} \left[\frac{1}{i}, 1\right] = \lim_{n\to\infty} \{1\} = \{1\}$$

Example 2.2.4

- Let $A_i = [\frac{1}{i+1}, \frac{1}{i}), i = 1, 2, \cdots.$ Then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} [\frac{1}{i+1}, \frac{1}{i}) = (0, 1)$$

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} [\frac{1}{i+1}, \frac{1}{i}) = \emptyset$$

- <u>Def.</u> If $A_1, A_2, \cdots$ are disjoint and $\bigcup_{i=1}^{\infty} A_i = S$, then $A_1, A_2, \cdots$ form a partition of S.
- From Example 2.2.4, $A_1, A_2, \cdots$, where

$$A_i = [\frac{1}{i+1}, \frac{1}{i})$$

Form a partition of $(0, 1)$.

2.3 Axioms of Probability

- Probability Axioms
  1. $P(A) \geq 0$
  2. $P(S) = 1$
  3. For any mutually exclusive (pairwise disjoint) events $A_1, A_2, \cdots,$

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

Example 2.3.1

Tossing a fair coin: $S = \{H, T\}$

$$P(H) = P(T) = \frac{1}{2}$$

$$P(S) = P(H) + P(T)$$

Example 2.3.2

Tossing two balanced coins:

$$S = \{HH, HT, TH, TT\}$$

$$P(S) = P(HH) + P(HT) + P(TH) + P(TT)$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

Let A be the event of obtaining at least one tail. Then

$$A = \{HT, TH, TT\}$$

$$P(A) = P(HT) + P(TH) + P(TT) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

Example 2.3.3

Tossing a fair coin until a tail is obtained

$$S = \{T, HT, HHT, \cdots\}$$

$$P(S) = P(T) + P(HT) + P(HHT) + \cdots$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

2.4 Some Simple Propositions

Theorem 2.4.1 If P is a probability function and $A \subset S$, then

1. $P(\emptyset) = 0$
2. $0 \leq P(A) \leq 1$
3. $P(A^C) = 1 - P(A)$

Proof

Proof of Theorem 2.4.1

1. $1 = P(\emptyset \cup \mathrm{S}) = P(\emptyset) + P(S) = P(\emptyset) + 1$

   $\therefore P(\emptyset) = 0$

2. $1 = P(S) = P(A \cup A^C) = P(A) + P(A^C)$

$$P(A) \geq 0 \text{ and } P(A^C) \geq 0$$

$$\therefore 0 \leq P(A) \leq 1$$

3. From part 2, $P(A^C) = 1 - P(A)$

<u>Theorem 2.4.2</u> If P is a probability function and $A \subset S$ & $B \subset S$, then

1. $P(B \cap A^C) = P(B) - P(A \cap B)$
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
3. $A \subset B \Longrightarrow P(A) \leq P(B)$

<u>Proof</u>

Proof of Theorem 2.4.2

1.

$$B = (B \cap A) \cup (B \cap A^C)$$
$$\therefore P(B) = P[(B \cap A) \cup (B \cap A^C)]$$
$$= P(B \cap A) + P(B \cap A^C)$$
$$\therefore\ P(B \cap A^C) = P(B) - P(A \cap B)$$

2.

$$P(A \cup B) = P(A) + P(B \cap A^C) = P(A) + P(B) - P(A \cap B)$$

3.

$$A \subset B$$
$$\Longrightarrow 0 \leq P(B \cap A^C) = P(B) - P(A \cap B) = P(B) - P(A)$$
$$\therefore\ P(A) \leq P(B)$$

Example 2.4.1

Let 80% of freshmen in a college take statistics, 50% take physics and 40% take both statistics and physics.

(a) What is the probability of taking at least one of these courses?

Let $A = \{\text{taking statistics}\}, \quad B = \{\text{taking physics}\}$. Then

$$P(A) = 0.8, P(B) = 0.5, P(A \cap B) = 0.4$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8 + 0.5 - 0.4$$
$$= 0.9$$

(b) What is the probability of taking only one of these courses?

$$P(A \cup B) - P(A \cap B) = 0.9 - 0.4 = 0.5$$

- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$
  $-P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$
- $P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i_1 < i_2} P(A_{i_1} \cap A_{i_2})$
  $+ \cdots + (-1)^{k+1} \sum_{i_1 < i_2 < \cdots < i_k} P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k})$
  $+ \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n)$

Example 2.4.2

Suppose we know that $P(A) = 0.2$ and $P(B) = 0.6$.

(a) What is the largest *possible* value of $P(A \cap B)$?

0.2 when $A \subset B$

(b) What is the largest *possible* value of $P(A \cup B)$?

$$P(A) + P(B) = 0.2 + 0.6 = 0.8$$

(c) What is the smallest *possible* value of $P(A \cap B)$?

0 when A and B are disjoint

(d) What is the smallest *possible* value of $P(A \cup B)$?

0.6 when $A \subset B$

Example 2.4.3

Suppose we know that $P(A) = 0.7$ and $P(B) = 0.6$.

(a) What is the largest *possible* value of $P(A \cap B)$?

$$0.6 \text{ when } A \subset B$$

(b) What is the largest *possible* value of $P(A \cup B)$?

$$1 \text{ because } P(A) + P(B) = 1$$

(c) What is the smallest *possible* value of $P(A \cap B)$?

$$1 = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.7 + 0.6 - P(A \cap B) \Longrightarrow P(A \cap B) = 0.3$$

(d) What is the smallest *possible* value of $P(A \cup B)$?

$$0.7 \text{ when } A \subset B$$

2.5 Equally Likely Outcomes

- If there are n equally likely outcomes, and s are labeled success, then the probability of a successful outcome is given as s/n.
- Suppose that each outcome of S={a_1, a_2, …, a_n} is “equally likely” (meaning that $P(\{a_i\}) = P(\{a_j\})$ for each $i \neq j$)
- Then, from the axioms of probability, we get that

$$P(E) = \frac{|E|}{|S|}$$

Show why.

Example 2.5.1

Tossing a fair die: $S = \{1, 2, 3, 4, 5, 6\}$

Probability of getting an odd number A$= \{1,\ 3,\ 5\}$ is

$$P(A) = \frac{\#\text{elelments in } A}{\#\text{elements in } S} = \ \frac{3}{6} = \frac{1}{2}.$$

Example 2.5.2

Probability of drawing a king from a deck of 52 cards:

Let $A = \{\text{king}\}$. Then

$$P(A) = \frac{\#\text{elelments in } A}{\#\text{elements in } S} = \frac{4}{52} = \frac{1}{13}$$

Example 2.5.3

Tossing a fair die twice

Let $A = \{\text{sum of the numbers is } 6\}$

$= \{(1,5),(2,4),(3,3),(4,2),(5,1)\}.$

The probability of A is

$$P(A) = \frac{\#\text{elelments in } A}{\#\text{elements in } S} = \frac{5}{36}$$

Example 2.5.4

A committee has 8 male members and 12 female members. Choose 5 representatives in this committee at random. Let $D_i =$ {exactly i of the 5 representatives are females}, $i = 0, 1, \cdots, 5$.

(a) Probability that exactly 3 of the 5 representatives are females?

(b) Probability that at least 3 of the 5 representatives are females?

Answer to Example 2.5.4

A committee has 8 male members and 12 female members. Choose 5 representatives in this committee at random. Let D_i ={exactly i of the 5 representatives are females}, $i = 0, 1, \cdots, 5$.

(a)Probability that exactly 3 of the 5 representatives are females?

$$P(D_3) = \frac{\binom{12}{3}\binom{8}{2}}{\binom{20}{5}} = \frac{\frac{12!}{3!\,9!} \cdot \frac{8!}{2!\,6!}}{\frac{20!}{5!\,15!}} = 0.3973$$

Answer to Example 2.5.4 (continued)

A committee has 8 male members and 12 female members. Choose 5 representatives in this committee at random. Let D_i ={exactly i of the 5 representatives are females}, $i = 0, 1, \cdots, 5$.

(b) Probability that at least 3 of the 5 representatives are females?

$$P(D_3 \cup D_4 \cup D_5) = \frac{\binom{12}{3}\binom{8}{2}}{\binom{20}{5}} + \frac{\binom{12}{4}\binom{8}{1}}{\binom{20}{5}} + \frac{\binom{12}{5}\binom{8}{0}}{\binom{20}{5}}$$

$$= 0.7038$$

Example 2.5.5 Birthday Problem

If n students are present in a classroom, what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than ½? Assume no February 29.

Answers to the Birthday Problem

Let A be the event that no two people have the same birthday in class. Then

$$P(A) = \frac{{}_{365}P_k}{365^k} = \frac{365!}{(365-k)!\,(365)^k}$$

$$= \exp\left[\sum_{i=1}^{365} \log i - \sum_{i=1}^{365-k} \log i - k \log 365\right]$$

$$P(A^C) = 1 - P(A) = 1 - \frac{{}_{365}P_k}{365^k}$$

(a) $1 - \frac{{}_{365}P_{23}}{365^{23}} = 0.507$

(b) $1 - \frac{{}_{365}P_{30}}{365^{30}} = 0.706$

(c) $1 - \frac{{}_{365}P_{50}}{365^{50}} = 0.970$

Birthday Problem (continued)

```
logfact<-function(n) {
  x=0
  for (i in 2:n) {
    x=x+log(i)
    }
  return(x)
}
#======================
bdaypr<-function(k) {
  a=logfact(365)-logfact(365-k)-k*log(365)
  1-exp(a)
}
```

bdaypr(23) → 0.5072972

Bdaypr(50) → 0.9703736

Bdaypr(75) → 0.9997199