Homework #1

AMS 507, Fall 2025 **Due September 4**

1. Show the following inequality:

$$P(A \cap B) \geq P(A) + P(B) - 1$$

2. Let $\{A_n\}$ be a sequence of sets and P a probability function.

   (a) Show
   $$P(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i).$$

   (b) Show
   $$P(\cap_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - (n-1).$$

3. Give analytic proofs of the following equations:

   (a) For positive integers n and r,
   $$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \ 1 \leq r \leq n$$

   (b) For positive integers n and $n_1, n_2, \cdots, n_r$,
   $$\begin{aligned}\binom{n}{n_1, n_2, \cdots, n_r} &= \binom{n-1}{n_1 - 1, n_2, \cdots, n_r} + \binom{n-1}{n_1, n_2 - 1, \cdots, n_r} \\ &\quad + \cdots + \binom{n-1}{n_1, n_2, \cdots, n_r - 1}, \ n = n_1 + n_2 + \cdots + n_r\end{aligned}$$

4. A die is cast independently until the first 3 appears. If the casting stops on an even number of times, Michael wins; otherwise, Andrew wins.

   (a) Assuming the die is fair, what is the probability that Andrew wins?

   (b) Let p denote the probability of a 3. Show that the game favors Andrew, for all p, $0 < p < 1$.

5. In a box containing 36 strawberries, 2 of them are rotten. Brian randomly picked 5 of these strawberries.

   (a) What is the probability of having at least 1 rotten strawberry among the 5?

   (b) How many strawberries should be picked so that the probability of having exactly 2 rotten strawberries among them equals 2/35?

6. In a group of k people, where $2 \leq k \leq 365$, what is the probability that at least 2 in this group have the same birthday if

   (a) $k = 23$?

   (b) $k = 30$?

   (c) $k = 50$?

   Ignore February 29.

7. A sales person at a car dealer is showing cars to a prospective buyer. There are 9 models in the dealership. The customer wants to test-drive only 3 of them.

   (a) In how many ways could the 3 models be chosen if the order of test-driving is considered?

   (b) In how many ways could the 3 models be chosen if the order of test-driving is not important?

   (c) Suppose 6 of the models are new and the other 3 models are used. If the three cars to test-drive are randomly chosen, what is the probability that all 3 are new?

   (d) Is the answer to part (c) different depending on whether or not the order is considered?

8. If $\{E_n, n \geq 1\}$ is an increasing sequence of events, show that

$$\lim_{n \to \infty} P(E_n) = P\left(\lim_{n \to \infty} E_n\right).$$