AMS 507

Chapter 1
Combinatorial Analysis

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1.2 Basic Principle of Counting

- Product Rule

Suppose a set consists of ordered collections of k elements.
There are n_1 possible choices for the 1st element.
There are n_2 possible choices for the 2nd element.
……
There are n_k possible choices for the k^{th} element.

$\Rightarrow$There are $n_1 n_2 \cdots n_k$ possible k combinations.

Example 1.2.1

If a test consists of 12 true-false questions, in how many different ways can a student mark the test paper with one answer to each question?

Answer: $2 \cdot 2 \cdots 2 = 2^{12} = 4{,}096$

Example 1.2.2

A license plate has exactly 6 characters, each of which can be a digit or a letter excluding letters O and I. (Letters and digits can be repeated.)

(a) How many different license plates are possible?

Let $A_i = \{0, 1, 2, \cdots\ 9, A, B, \cdots H, J, \cdots N, P,\ \cdots, Z\}$,

$i = 1, \cdots, 6$

$n_1 \cdots n_6 = 34 \times 34 \times 34 \times 34 \times 34 \times 34 = 34^6 = 1{,}544{,}804{,}416$

(b) What if the first 3 characters must be digits, and the last 3 must be letters excluding O and I?

$$\text{Number of possible license plates} \\ = 10 \times 10 \times 10 \times 24 \times 24 \times 24 = 13{,}824{,}000$$

1.3 Permutations (order is taken into account)

- Def. Permutation: An ordered sequence of k objects from a set of n distinct objects
- Number of different ways to order n distinct objects: $n!$
- Number of different ways to order k objects from a set of n distinct objects:

$$_{n}P_{k} = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Example 1.3.1

- How many different batting orders are possible for a baseball team consisting of 9 players?

$9! = 362{,}880$ possible batting orders

Example 1.3.2

A committee consists of 10 members. Find the number of possible choices of a chair, vice chair, and secretary.

$$_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10 \cdot 9 \cdot 8 = 720$$

1.4 Combinations (order is not considered)

- Def. Combination: Any unordered subset of k objects from a set of n distinct objects

$$ {}_nC_k = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!} $$

Example 1.4.1

A committee consists of 10 members. Find the number of possible choices of 3 representatives.

$$\binom{10}{3} = \frac{10!}{3!(10-3)!} = \frac{10 \cdot 9 \cdot 8}{3!} = \frac{720}{6} = 120$$

Example 1.4.2

Pick 3 cards in succession from a full deck of 52.

(a) If the order or arrangement of the selection is important, how many possible outcomes?

$$_{52}P_3 = \frac{52!}{(52-3)!} = \frac{52!}{49!} = 52 \cdot 51 \cdot 50 = 132{,}600$$

(b) If the order or arrangement of the selection is not important, how many possible outcomes?

$$\binom{52}{3} = \frac{_{52}P_3}{3!} = \frac{52!}{3!\,(52-3)!} = \frac{52 \cdot 51 \cdot 50}{3 \cdot 2} = 22{,}100$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \qquad k = 1, \cdots, n$$

Proof

Binomial Theorem

<u>Proof</u> Mathematical Induction

(i) When $n = 1$, $x + y = \binom{1}{0}x^0y^1 + \binom{1}{1}x^1y^0 = y + x$

(ii) Assume the equation for $n - 1$.

(iii) $(x + y)^n = (x + y)(x + y)^{n-1}$

$$= (x + y)\sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k}$$

$$= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{k+1}y^{n-1-k} + \sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-k}$$

$$= \sum_{i=1}^{n}\binom{n-1}{i-1}x^i y^{n-i} + \sum_{i=0}^{n-1}\binom{n-1}{i}x^i y^{n-i}$$

$$= x^n + \sum_{i=1}^{n-1}\left[\binom{n-1}{i-1} + \binom{n-1}{i}\right]x^i y^{n-i} + y^n$$

$$= x^n + \sum_{i=1}^{n-1}\binom{n}{i}x^i y^{n-i} + y^n$$

$$= \sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i}$$

Example 1.4.4

Suppose that a set S has n elements. Find the number of subsets that can be formed from the elements of S.

Since the number of subsets with i elements is $\binom{n}{i}$, $i = 0, 1, \cdots, n,$ the number of subsets that can be formed from the elements is

$$\sum_{i=1}^{n} \binom{n}{i} = \sum_{i=1}^{n} \binom{n}{i} 1^i 1^{n-i} = (1+1)^n = 2^n$$

1.5 Multinomial Coefficients

$$\binom{n}{n_1, n_2, \cdots, n_k} = \frac{n!}{n_1!\; n_2! \cdots n_k!}, \text{ where } \sum_{i=1}^{k} n_i = n$$

- Multinomial Theorem

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{(n_1, n_2, \cdots, n_k) \\ n_1 + \cdots + n_k = n}} \binom{n}{n_1, n_2, \cdots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

Example 1.5.1

How many different spellings of “statistics” are there?

The word consists of 3 s’s and t’s each, 2 I’s, and 1 a and c. Therefore, there are

$$\frac{10!}{3!\,3!\,1!\,2!\,1!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 2} = 50{,}400$$

ways.

Example 1.5.2

Verify the following identities for $n \geq 2$.

(a)

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0$$

(b)

$$\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}$$

(c)

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} k = 0$$

Answer to Example 1.5.2

For $n \geq 2$,

(a)

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k = \sum_{k=0}^{n}\binom{n}{k}(-1)^k\, 1^{n-k}$$
$$= (-1+1)^n = 0$$

(b)

$$k\binom{n}{k} = k\frac{n!}{k!\,(n-k)!} = n\frac{(n-1)!}{(k-1)!\,(n-k)!} = n\binom{n-1}{k-1}$$

$$\therefore \sum_{k=1}^{n} k\binom{n}{k} = n\sum_{k=1}^{n}\binom{n-1}{k-1} = n\sum_{j=0}^{n-1}\binom{n-1}{j}$$
$$= n\sum_{j=0}^{n-1}\binom{n-1}{j}1^j 1^{n-1-j} = n2^{n-1}$$

Answer to Example 1.5.2 (continued)

For $n \geq 2$,

(c) From parts (b) and (a),

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1} k = n \sum_{k=1}^{n} \binom{n-1}{k-1} (-1)^{k+1}$$

$$= n \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^{j} = 0$$

Example 1.5.3

Show

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

<u>Proof</u>

(1) In tossing a fair coin n times, there are $\binom{n}{k}$ ways to obtain k heads. (2) In tossing a fair coin n times, there are $\binom{n}{k}$ ways to obtain k tails.

If we combine (1) and (2), there are $\sum_{k=0}^{n} \binom{n}{k}^2$ ways for all k.

(3) If we add the two trials in (1) and (2), we obtain n heads (or n tails) in $2n$ tosses. This is $\binom{2n}{n}$.