Crowd-Anticrowd Theory of Multi-Agent Market Games

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February 1, 2008

Abstract

We present a dynamical theory of a multi-agent market game, the so-called Minority Game (MG), based on crowds and anticrowds. The time-averaged version of the dynamical equations provides a quantitatively accurate, yet intuitively simple, explanation for the variation of the standard deviation ('volatility') in MG-like games. We demonstrate this for the basic MG, and the MG with stochastic strategies. The time-dependent equations themselves reproduce the essential dynamics of the MG.

Agent-based games have great potential application in the study of fluctuations in financial markets. Challet and Zhang's Minority Game (MG) [1, 2] offers possibly the simplest example and has been the subject of much research [2]. The MG comprises an odd number of agents N choosing repeatedly between option 0 (e.g. buy) and option 1 (e.g. sell). The winners are those in the minority group, e.g. sellers win if there is an excess of buyers. The outcome at each timestep represents the winning decision, 0 or 1. A common bit-string of the m most recent outcomes is made available to the agents at each timestep [3]. The agents randomly pick s strategies at the beginning of the game, with repetitions allowed - each strategy is a bit-string of length 2^m which predicts the next outcome for each of the 2^m possible histories. Agents reward successful strategies with a (virtual) point. At each turn of the basic MG, the agent uses her most successful strategy, i.e. the one with the most virtual points. Here we develop a dynamical theory for MG-like games based on the formation of crowds and anticrowds.

The number of agents holding a particular combination of strategies can be written as a $D \times D \times \ldots$ (s terms) dimensional matrix Ω, where D is the total number of available strategies. For $s = 2$, this is simply a $D \times D$ matrix where the entry (i, j) represents the number of agents who picked strategy i and then j. The strategy labels are given by the decimal representation of the strategy plus unity, for example the strategy 0101 for $m = 2$ has strategy label 5+1=6. Ω is fixed at the beginning of the game ('quenched disorder') and can represent either the full strategy space or the reduced strategy space [1], depending on the choice of D. Σ is another time-independent matrix, containing all the strategies in the required space in their binary form: $\Sigma_{r,h+1}$ describes the prediction of strategy r given the history h (where h is the decimal corresponding to the m-bit binary history string).

We introduce a vector $\underline{n}(t)$: this contains the number of agents using each strategy at time t, in order of increasing strategy label. The vector $\underline{S}(t)$ contains the virtual score for each strategy at time t in order of increasing strategy label. The vector $\underline{R}(t)$ lists the strategy label in order of best-to-worst virtual points score at time t; if any strategies are tied in points then the strategy with the lower-value label is listed first. The vector $\underline{\rho}(t)$ shows the rank of the strategy listed in order of increasing strategy label at time t. Hence $\underline{R}(t)$ and $\underline{\rho}(t)$ can be found from $\underline{S}(t)$ using simple sort operations. The vector $\underline{n}(t)$ is the sum of two terms

$$\underline{n}(t) = \underline{n}^0(t) + \underline{n}^d(t) \quad . \tag{1}$$

Here $\underline{n}^0(t)$ gives the number of agents using each strategy; however where any strategies are tied in virtual score, $\underline{n}^0(t)$ assumes that the agent will use the strategy with the lower-value label by virtue of the definition of $\underline{R}(t)$. The term $\underline{n}^d(t)$ accounts for tied strategies, and hence provides a correction to $\underline{n}^0(t)$. $\underline{n}^0(t)$ is given by

$$\underline{n}^0(t)_r = \sum_{i=\rho(t)_r}^{2^{m+1}} [\widehat{F}(\Omega)]_{r,R(t)_i} \tag{2}$$

where $[\widehat{F}(\Omega)]_{\alpha,\beta} = \Omega_{\alpha,\beta} + \Omega_{\beta,\alpha} - \delta_{\alpha,\beta}\Omega_{\alpha,\beta}$. The vector $\underline{n}^d(t)$ is given by

$$\underline{n}^d(t)_r = \sum_{r' \neq r} \delta_{s(t)_{r'},s(t)_r} Sgn(r'-r) Bin_{r',r} \tag{3}$$

where: $Bin_{r',r} \sim B[(\widehat{F}(\Omega))_{r',r}, \frac{1}{2}]$ and $Bin_{r',r} = Bin_{r,r'}$. The standard notation Bin represents the binary distribution. Note the condition $Bin_{r',r} = Bin_{r,r'}$ which guarantees conservation of agents, as in the basic MG. The outcome parameter $\Upsilon(t)$ denotes which choice, 0 or 1, is the minority (and hence winning) decision at time t:

$$\Upsilon(t) = \mathcal{H}[-[\underline{n}(t)^T \Sigma']_{h(t)+1}] \tag{4}$$

where $\Sigma' = 2\Sigma - 1$. The history, i.e. bit-string of the m most recent outcomes, and the virtual scores of the strategies are updated as follows:

$$h(t+1) = 2[h(t) - 2^{m-1}\mathcal{H}[h(t) - 2^{m-1}]] + \Upsilon(t) \tag{5}$$

where $\mathcal{H}$ is the Heaviside function, and

$$\underline{S}(t+1) = \underline{S}(t) + \Sigma'_{h(t)+1}[2\Upsilon(t) - 1] \quad . \tag{6}$$

Equations (1-6) are a set of time-dependent equations which reproduce the essential dynamics of the basic MG, and can be easily extended to describe MG generalizations. Iterating these equations is equivalent to running a numerical simulation, but is far easier and can even be done analytically. A slight difference may arise as a result of the method chosen for tie-breaking between strategies with equal virtual points: a numerical program will typically break this tie using a separate coin-toss for each agent, whereas the dynamical equations group together those agents using the same pair of strategies and then assign a proportion of that group to a particular strategy using a coin-toss. This difference is typically unimportant.

As an example of the implementation of these equations, consider a time t_e during the following game: $m = 2$, $s = 2$ and $N = 101$ in the reduced strategy space, with a strategy configuration Ω and strategy score given as follows:

$$\Omega = \begin{pmatrix} 2 & 3 & 2 & 3 & 5 & 3 & 1 & 1 \\ 1 & 3 & 2 & 2 & 2 & 1 & 2 & 1 \\ 1 & 0 & 2 & 0 & 1 & 3 & 1 & 3 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 3 \\ 4 & 5 & 1 & 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 2 & 0 & 4 \\ 1 & 2 & 1 & 2 & 0 & 0 & 2 & 4 \\ 1 & 2 & 2 & 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

$$\underline{S}(t_e) = \begin{pmatrix} 3 \\ -1 \\ -3 \\ 1 \\ -1 \\ 3 \\ 1 \\ -3 \end{pmatrix}, \text{ with } \Sigma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} .$$

Using these values for Ω and $\underline{S}(t_e)$ we can obtain values for $\underline{n}(t)$ and ultimately $\underline{S}(t_e+1)$. Ω and $\underline{S}(t_e)$ imply that

$$\underline{n}^0(t_e) = \begin{pmatrix} 31 \\ 15 \\ 7 \\ 13 \\ 5 \\ 15 \\ 13 \\ 2 \end{pmatrix}, \text{ and } \underline{n}^d(t_e) = \begin{pmatrix} -3 \\ -2 \\ -5 \\ 0 \\ 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$

with probability $\frac{105}{65536}$, yielding $\underline{n}(t_e)$ when summed. (When two strategies are tied, agents holding these strategies each flip a coin to decide which strategy to use. The separate probabilities for all tied strategies, when multiplied together, yield the probability of the current $\underline{n}^d(t)$ being chosen.)

Suppose $h(t_e) = 2$, i.e. the last two minority groups were '1' then '0'. Hence $\Upsilon(t_e) = 0$, $h(t_e+1) = 0$ and consequently

$$\underline{S}(t_e+1) = \begin{pmatrix} 4 \\ -2 \\ -2 \\ 0 \\ 0 \\ 2 \\ 2 \\ -4 \end{pmatrix} .$$

An expression for the time-averaged quantity called the 'volatility' (standard deviation of the number of agents choosing one particular group) can be easily found using the above formalism:

$$\sigma_{MG} = \frac{\left[\sum_{t=t_1}^{t_2} \left[\varepsilon(t) - \overline{\varepsilon}\right]^2\right]^{\frac{1}{2}}}{t_2 - t_1} \qquad (7)$$

where $\varepsilon(t) = [\underline{n}(t)^T \Sigma]_{h(t)+1}$ and $\overline{\varepsilon}$ is the time-average of $\varepsilon(t)$ from time t_1 to t_2. Here t_1 and t_2 denote the time window over which the volatility is calculated. In the reduced strategy space [1] a similar quantity to this standard deviation can also be written down using our previously introduced (time-averaged) crowd-anticrowd framework [4]:

$$\sigma_{CA} = \frac{\sum_{t=t_1}^{t_2} \left[\frac{1}{4} \sum_{r=1}^{2^m} [\underline{n}(t)_r - \underline{n}(t)_{2^{m+1}+1-r}]^2\right]^{\frac{1}{2}}}{t_2 - t_1} . \qquad (8)$$

For a given run of the game $\sigma_{MG} \neq \sigma_{CA}$, however these quantities become quantitatively the same (within the limits of sample size) when averaged over initial configurations of strategies[4]. σ_{CA} mirrors the semi-analytic approach introduced to motivate the time-independent crowd-anticrowd theory of Ref. [4] (see Fig. 1 of Ref. [4]). Indeed, the dynamical equations can be linked more formally with our previous time-averaged approach[4]. Consider, for example, the situation where no two strategies are tied in virtual points and there are an equal number of agents having each possible pairing of strategies (low m limit and reduced strategy space), i.e. all elements in Ω are equal and non-zero. It is then easy to show that $\underline{n}^0(t)_r$ reduces to $\underline{n}^0{}_r = \frac{N}{(2^{m+1})^2}[1 + 2(2^{m+1} - \rho(t)_r)]$; this is precisely the vector of the quantity n_r introduced in Ref. [4] now written in order of increasing strategy label. If we allow for tied strategies, $\underline{n}^d(t)$ will be non-zero thus reducing the size of large crowds and increasing the size of the smaller crowds (and hence anticrowds), thereby leading to a smaller standard deviation.

We now turn to a comparison between the standard deviation or 'volatility' σ obtained from numerical simulations and our (time-averaged) crowd-anticrowd theory. We start with the basic MG. Figure 1 shows the spread of numerical values for different numerical runs (open circles), the full crowd-anticrowd theoretical calculation (large solid circles) and various limiting analytic curves (solid lines) for which closed-form expressions were given in Ref. [4]. Fuller details are provided in Ref. [4]. The time-averaged dynamics can be described using a quantity $P(r' = \bar{r})$ which represents the probability that any strategy r' is the anti-correlated partner of strategy r [4]. To produce the limiting analytic curves in Fig. 1, $P(r' = \bar{r})$ is taken to be either a delta-function or a flat distribution. The full theory takes the relevant form of $P(r' = \bar{r})$ from the game. The agreement is very good, confirming that our theory captures the essential physics.

In a variant of the basic MG, agents pick which strategy to use stochastically at each timestep. Focussing on $s = 2$, numerical simulations [5] found that the larger-than-random σ in the 'crowded' regime (i.e. small m) becomes smaller-than-random when the strategy-picking rule is made increasingly stochastic. Our crowd-anticrowd theory provides a quantitative explanation of this effect. Let θ be the probability that the agent uses the worst of her $s = 2$ strategies. Figure 2 shows a comparison between numerical simulation (open circles) and analytic expressions (monotonically-decreasing solid lines) obtained using our crowd-anticrowd theory (full details are given in Ref. [6]). These analytic expressions vary in their choice of $P(r' = \bar{r})$: the upper line σ_{delta} in

Fig. 2 assumes a delta function while the lower line σ_{flat} assumes a flat distribution. The theory agrees well in the range $\theta = 0 \rightarrow 0.35$ and provides a quantitative, yet physically intuitive, explanation for the previously unexplained transition in σ from larger-than-random to smaller-than-random as θ increases.

Above $\theta = 0.35$, the numerical data tend to flatten off while the analytic expressions predict a decrease in σ as $\theta \rightarrow 0.5$. This is because the analytic theory averages out the fluctuations in strategy-use at each time-step. In Ref. [6] we showed how to correct this shortcoming of the analytic theory. Consider $\theta = 0.5$; Fig. 2 inset (a) shows the measured numerical distribution in σ for $\theta = 0.5$, while inset (b) shows the result from the semi-analytic procedure introduced in Ref. [6]. The two distributions are in good agreement. Note that the non-zero average (4.7 for $N = 101, m = 2$ and $s = 2$) for each distribution lies *below* the random coin-toss limit $\sqrt{N}/2$. It is also possible to perform a fully analytic calculation of the average σ_θ in the $\theta \rightarrow 0.5$ limit [6]; this value (which is also 4.7 for $N = 101, m = 2$ and $s = 2$) is shown in Fig. 2.

In summary, we have demonstrated that the crowd-anticrowd approach can be applied to explain many aspects of MG games, yielding both time-averaged and time-dependent theories (see also Ref. [7]). Our efforts to develop such simplified market games in order to describe real-world financial markets are described elsewhere [8].

We thank A. Short for useful discussions.

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FIG. 1. Theoretical crowd-anticrowd calculation (solid circles) and numerical simulations (open circles) for the standard deviation σ in basic MG with $s = 2$ and $N = 101$. 16 numerical runs are shown for each m. Solid lines correspond to analytic expressions representing special cases witihin the time-averaged crowd-anticrowd theory of Ref. [4].

FIG. 2. Theoretical crowd-anticrowd calculation and numerical simulations (circles) for σ vs. the probability parameter θ in the stochastic MG. Here $N = 101, m = 2$ and $s = 2$. Monotonically decreasing solid lines correspond to analytic expressions σ_{delta} and σ_{flat} (see text). Dashed line shows random coin-toss value. Solid arrow indicates theoretical value $\sigma_{\theta \to 0.5} = 4.7$ for $\theta \to 0.5$. Inset shows distribution of σ values at $\theta = 0.5$ for several thousand randomly-chosen initial strategy configurations: (a) numerical simulation, (b) semi-analytic theory of Ref. [6].

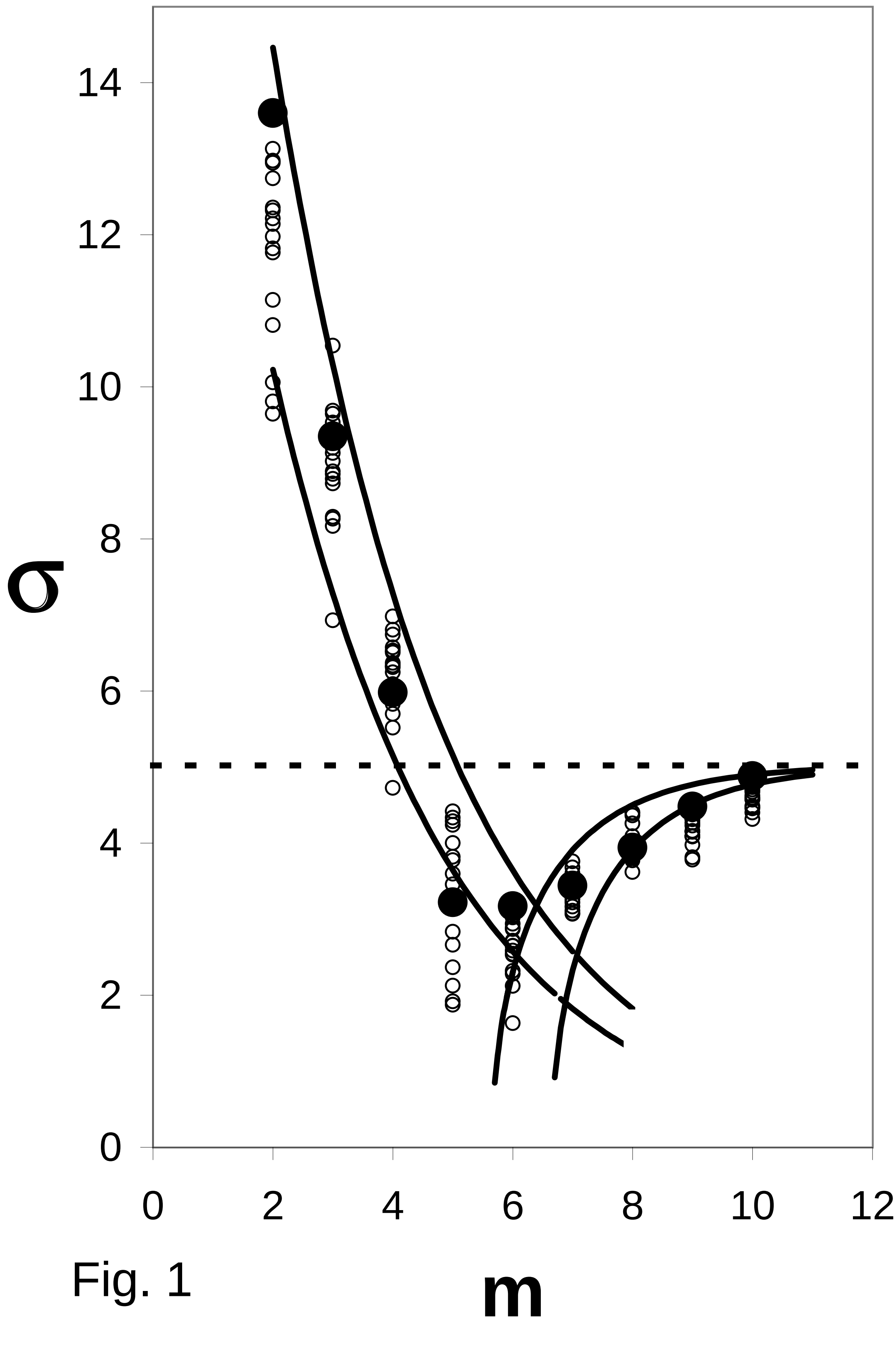

Fig. 1

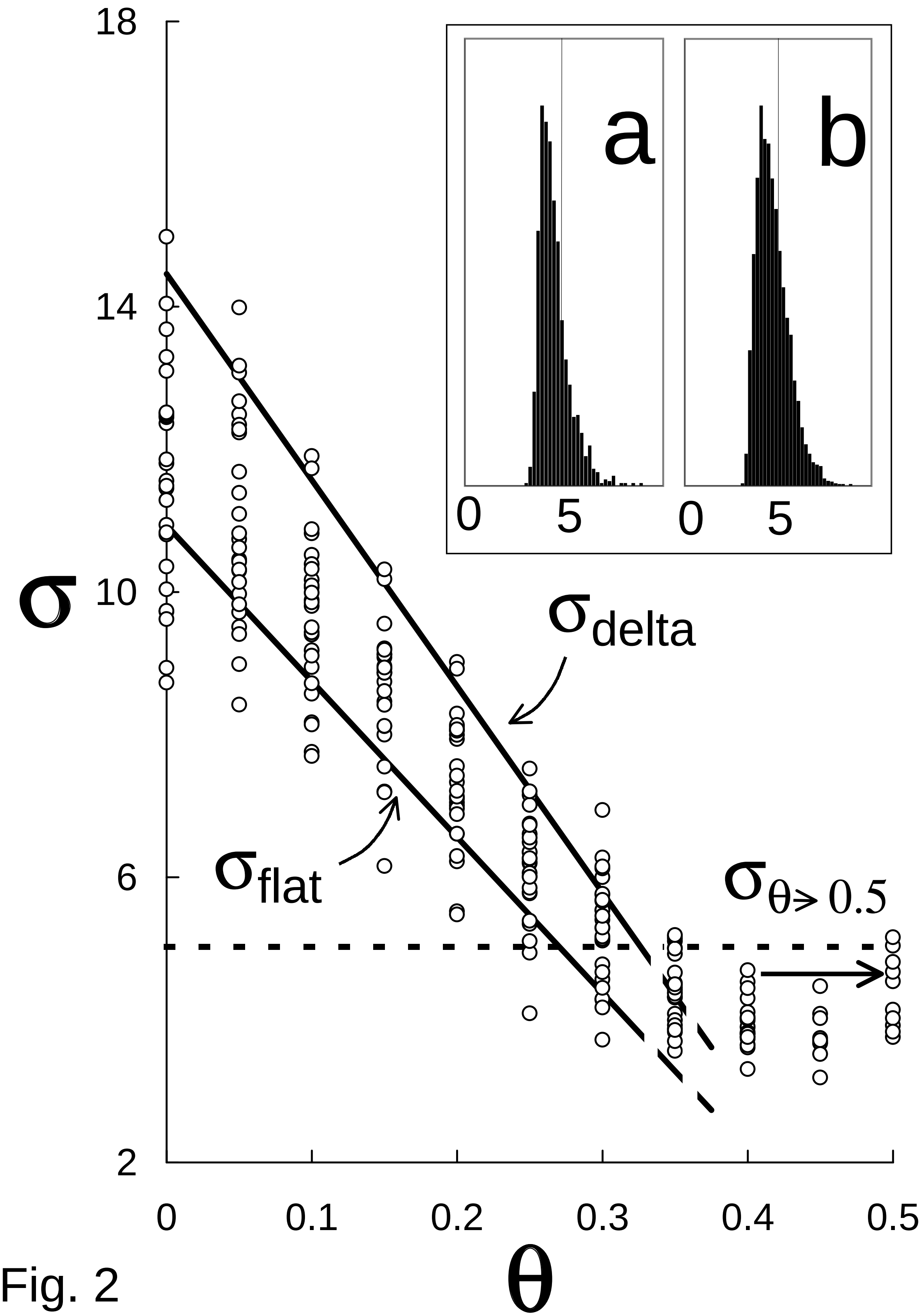

Fig. 2