Master equation for a kinetic model of trading market and its analytic solution

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We analyze an ideal gas like model of a trading market with quenched random saving factors for its agents and show that the steady state income (m) distribution $P(m)$ in the model has a power law tail with Pareto index ν exactly equal to unity, confirming the earlier numerical studies on this model. The analysis starts with the development of a master equation for the time development of $P(m)$. Precise solutions are then obtained in some special cases.

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I. INTRODUCTION

The distribution of wealth among individuals in an economy has been an important area of research in economics, for more than a hundred years. Pareto [1] first quantified the high-end of the income distribution in a society and found it to follow a power-law $P(m) \sim m^{-(1+\nu)}$, where P gives the normalized number of people with income m, and the exponent ν, called the Pareto index, was found to have a value between 1 and 3.

Considerable investigations with real data during the last ten years revealed that the tail of the income distribution indeed follows the above mentioned behavior and the value of the Pareto index ν is generally seen to vary between 1 and 2.5 [2, 3, 4]. It is also known that typically less than 10% of the population in any country possesses about 40% of the total wealth of that country and they follow the above law. The rest of the low income population, in fact the majority (90% or more), follow a different distribution which is debated to be either Gibbs [3, 5] or log-normal [4].

Much work has been done recently on models of markets, where economic (trading) activity is analogous to some scattering process [5, 6, 7, 8, 9, 10, 11]. We put our attention to models where introducing a saving factor for the agents, a wealth distribution similar to that in the real economy can be obtained [6, 7]. Savings do play an important role in determining the nature of the wealth distribution in an economy and this has already been observed in some recent investigations [12]. Two variants of the model have been of recent interest; namely, where the agents have the same fixed saving factor [6], and where the agents have a quenched random distribution of saving factors [7]. While the former has been understood to a certain extent (see e.g, [13, 14]), and argued to resemble a gamma distribution [14], attempts to analyze the latter model are still incomplete (see however, [15]). Further numerical studies [16] of time correlations in the model seem to indicate even more intriguing features of the model. In this paper, we intend to analyze the second market model with randomly distributed saving factor, using a master equation type approach similar to kinetic models of condensed matter.

II. THE MODEL

The market consists of N (fixed) agents, each having money $m_i(t)$ at time t ($i = 1, 2, \ldots, N$). The total money M ($= \sum_i^N m_i(t)$) in the market is also fixed. Each agent i has a saving factor λ_i ($0 \leq \lambda_i < 1$) such that in any trading (considered as a scattering) the agent saves a fraction λ_i of its money $m_i(t)$ at that time and offers the rest $(1 - \lambda_i)m_i(t)$ for random trading. We assume each trading to be a two-body (scattering) process. The evolution of money in such a trading can be written as:

$$m_i(t+1) = \lambda_i m_i(t) + \epsilon_{ij}\left[(1-\lambda_i)m_i(t) + (1-\lambda_j)m_j(t)\right], \quad (1)$$

$$m_j(t+1) = \lambda_j m_j(t) + (1-\epsilon_{ij})\left[(1-\lambda_i)m_i(t) + (1-\lambda_j)m_j(t)\right] \quad (2)$$

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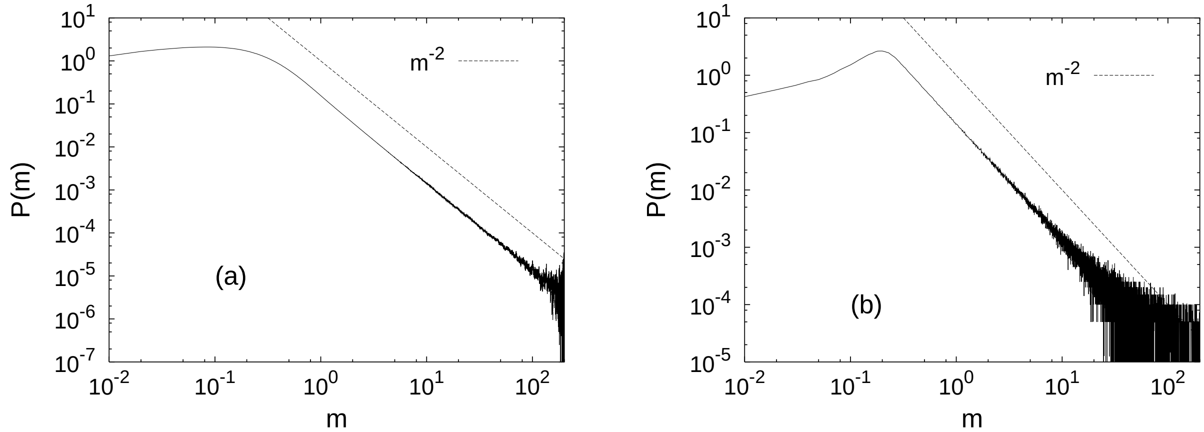

FIG. 1: Steady state money distribution $P(m)$ against m in a numerical simulation of a market with $N = 200$, following equations (1) and (2) with (a) ϵ_{ij} randomly distributed in the interval 0 to 1 and (b) $\epsilon_{ij} = 1/2$. The dotted lines correspond to $m^{-(1+\nu)}$; $\nu = 1$.

where each $m_i \geq 0$ and ϵ_{ij} is a random fraction ($0 \leq \epsilon \leq 1$). Typical numerical results for the steady state money distribution in such a model is shown in Fig. 1(a) for uniform distribution of λ_i ($0 \leq \lambda_i < 1$) among the agents.

III. DYNAMICS OF MONEY EXCHANGE

We will now investigate the steady state distribution of money resulting from the above two equations representing the trading and money dynamics. We will now solve the dynamics of money distribution in two limits. In one case, we study the evolution of the mutual money difference among the agents and look for a self-consistent equation for its steady state distribution. In the other case, we develop a master equation for the money distribution function.

A. Distribution of money difference

Clearly in the process as considered above, the total money $(m_i + m_j)$ of the pair of agents i and j remains constant, while the difference Δm_{ij} evolves as

$$(\Delta m_{ij})_{t+1} \equiv (m_i - m_j)_{t+1} = \left(\frac{\lambda_i + \lambda_j}{2}\right)(\Delta m_{ij})_t + \left(\frac{\lambda_i - \lambda_j}{2}\right)(m_i + m_j)_t + (2\epsilon_{ij} - 1)[(1-\lambda_i)m_i(t) + (1-\lambda_j)m_j(t)]. \quad (3)$$

Numerically, as shown in Fig. 1, we observe that the steady state money distribution in the market becomes a power law, following such tradings when the saving factor λ_i of the agents remain constant over time but varies from agent to agent widely. As shown in the numerical simulation results for $P(m)$ in Fig. 1(b), the law, as well as the exponent, remains unchanged even when $\epsilon_{ij} = 1/2$ for every trading. This can be justified by the earlier numerical observation [6, 7] for fixed λ market ($\lambda_i = \lambda$ for all i) that in the steady state, criticality occurs as $\lambda \to 1$ where of course the dynamics becomes extremely slow. In other words, after the steady state is realized, the third term in (3) becomes unimportant for the critical behavior. We therefore concentrate on this case, where the above evolution equation for Δm_{ij} can be written in a more simplified form as

$$(\Delta m_{ij})_{t+1} = \alpha_{ij}(\Delta m_{ij})_t + \beta_{ij}(m_i + m_j)_t, \quad (4)$$

where $\alpha_{ij} = \frac{1}{2}(\lambda_i + \lambda_j)$ and $\beta_{ij} = \frac{1}{2}(\lambda_i - \lambda_j)$. As such, $0 \leq \alpha < 1$ and $-\frac{1}{2} < \beta < \frac{1}{2}$.

The steady state probability distribution D for the modulus $\Delta = |\Delta m|$ of the mutual money difference between any two agents in the market can be obtained from (4) in the following way provided Δ is very much larger than the average money per agent $= M/N$. This is because, using eqn. (4), large Δ can appear at $t+1$, say, from 'scattering' from any situation at t for which the right hand side of eqn. (4) is large. The possibilities are (at t) m_i large (rare) and m_j not large, where the right hand side of eqn. (4) becomes $\sim (\alpha_{ij} + \beta_{ij})(\Delta_{ij})_t$; or m_j large (rare) and m_i not large (making the right hand side of eqn. (4) becomes $\sim (\alpha_{ij} - \beta_{ij})(\Delta_{ij})_t$); or when m_i and m_j are both large, which is a much rarer situation than the first two and hence is negligible. Then if, say, m_i is large and m_j is not, the right hand side of (4) becomes $\sim (\alpha_{ij} + \beta_{ij})(\Delta_{ij})_t$ and so on. Consequently for large Δ the distribution D satisfies

$$\begin{aligned} D(\Delta) &= \int d\Delta' \; D(\Delta') \, \langle \delta(\Delta - (\alpha+\beta)\Delta') + \delta(\Delta - (\alpha-\beta)\Delta') \rangle \\ &= 2 \langle \left(\frac{1}{\lambda}\right) \; D\left(\frac{\Delta}{\lambda}\right) \rangle, \end{aligned} \tag{5}$$

where we have used the symmetry of the β distribution and the relation $\alpha_{ij} + \beta_{ij} = \lambda_i$, and have suppressed labels i, j. Here $\langle \ldots \rangle$ denote average over λ distribution in the market. Taking now a uniform random distribution of the saving factor λ, $\rho(\lambda) = 1$ for $0 \le \lambda < 1$, and assuming $D(\Delta) \sim \Delta^{-(1+\gamma)}$ for large Δ, we get

$$1 = 2 \int d\lambda \; \lambda^{\gamma} = 2(1+\gamma)^{-1}, \tag{6}$$

giving $\gamma = 1$. No other value fits the above equation. This also indicates that the money distribution $P(m)$ in the market also follows a similar power law variation, $P(m) \sim m^{-(1+\nu)}$ and $\nu = \gamma$. We will now show in a more rigorous way that indeed the only stable solution corresponds to $\nu = 1$, as observed numerically [7, 8, 9].

B. Master equation and its analysis

We now proceed to develop a Boltzmann-like master equation for the time development of $P(m,t)$, the probability distribution of money in the market. We again consider the case $\epsilon_{ij} = \frac{1}{2}$ in (1) and (2) and rewrite them as

$$\begin{pmatrix} m_i \\ m_j \end{pmatrix}_{t+1} = \mathcal{A} \begin{pmatrix} m_i \\ m_j \end{pmatrix}_t \tag{7}$$

where

$$\mathcal{A} = \begin{pmatrix} \mu_i^+ & \mu_j^- \\ \mu_i^- & \mu_j^+ \end{pmatrix}; \quad \mu^{\pm} = \frac{1}{2}(1 \pm \lambda). \tag{8}$$

Collecting the contributions from terms scattering in and subtracting those scattering out, we can write the master equation for $P(m,t)$ as (cf. [11])

$$\begin{aligned} P(m, t+\Delta t) - P(m,t) &= \langle \int dm_i \int dm_j \; P(m_i,t) P(m_j,t) \\ &\quad \times \{ [\delta(\{\mathcal{A}\,\mathbf{m}\}_i - m) + \delta(\{\mathcal{A}\,\mathbf{m}\}_j - m)] - [\delta(m_i - m) + \delta(m_j - m)] \} \rangle \\ &= \langle \int dm_i \int dm_j \; P(m_i,t) P(m_j,t) \\ &\quad \times [\delta(\mu_i^+ m_i + \mu_j^- m_j - m) + \delta(\mu_i^- m_i + \mu_j^+ m_j - m) - \delta(m_i - m) + \delta(m_j - m)] \rangle. \end{aligned} \tag{9}$$

The above equation can be rewritten as

$$\frac{\partial P(m,t)}{\partial t} + P(m,t) = \langle \int dm_i \int dm_j \; P(m_i,t) P(m_j,t) \; \delta(\mu_i^+ m_i + \mu_j^- m_j - m) \rangle, \tag{10}$$

which in the steady state gives

$$P(m) = \langle \int dm_i \int dm_j \; P(m_i) P(m_j) \; \delta(\mu_i^+ m_i + \mu_j^- m_j - m) \rangle. \tag{11}$$

Writing $m_i \mu_i^+ = xm$, we can decompose the range $[0,1]$ of x into three regions: $[0,\kappa]$, $[\kappa, 1-\kappa']$ and $[1-\kappa', 1]$. Collecting the relevant terms in the three regions, we can rewrite the equation for $P(m)$ above as

$$\begin{aligned} P(m) &= \langle \frac{m}{\mu^+ \mu^-} \int_0^1 dx P\left(\frac{xm}{\mu^+}\right) P\left(\frac{m(1-x)}{\mu^-}\right) \rangle \\ &= \langle \frac{m}{\mu^+ \mu^-} \left\{ P\left(\frac{m}{\mu^-}\right) \frac{\mu^+}{m} \int_0^{\frac{\kappa m}{\mu^+}} dy P(y) + P\left(\frac{m}{\mu^+}\right) \frac{\mu^-}{m} \int_0^{\frac{\kappa' m}{\mu^-}} dy P(y) + \int_{\kappa}^{1-\kappa'} dx P\left(\frac{xm}{\mu^+}\right) P\left(\frac{m(1-x)}{\mu^-}\right) \right\} \rangle \end{aligned} \tag{12}$$

where the result applies for κ and κ' sufficiently small. If we take $m \gg 1/\kappa$, $m \gg 1/\kappa'$ and $\kappa, \kappa' \to 0$ ($m \to \infty$), then

$$P(m) = \langle \frac{m}{\mu^+\mu^-} \left\{ P\left(\frac{m}{\mu^-}\right)\frac{\mu^+}{m} + P\left(\frac{m}{\mu^+}\right)\frac{\mu^-}{m} + \int_\kappa^{1-\kappa'} dx P\left(\frac{xm}{\mu^+}\right) P\left(\frac{m(1-x)}{\mu^-}\right) \right\} \rangle. \tag{13}$$

Assuming now as before, $P(m) = A/m^{1+\nu}$ for $m \to \infty$, we get

$$1 = \langle (\mu^+)^\nu + (\mu^-)^\nu \rangle \equiv \int\int d\mu^+ d\mu^- p(\mu^+) q(\mu^-) \left[(\mu^+)^\nu + (\mu^-)^\nu \right], \tag{14}$$

as the ratio of the third term in (13) to the other terms vanishes like $(m\kappa)^{-\nu}$, $(m\kappa')^{-\nu}$ in this limit and $p(\mu^+)$ and $q(\mu^-)$ are the distributions of the variables μ^+ and μ^-, which vary uniformly in the ranges $[\frac{1}{2}, 1]$ and $[0, \frac{1}{2}]$ respectively (cf. eqn (8)). The i, j indices, for μ^+ and μ^- are again suppressed here in (14) and we utilise the fact that μ_i^+ and μ_j^- are independent for $i \neq j$. An alternative way of deriving Eqn. (14) from Eqn. (11) is to consider the dominant terms ($\propto x^{-r}$ for $r > 0$, or $\propto \ln(1/x)$ for $r = 0$) in the $x \to 0$ limit of the integral $\int_0^\infty m^{(\nu+r)} P(m) \exp(-mx) dm$ (see Appendix A). We therefore get from Eqn. (14), after integrations, $1 = 2/(\nu + 1)$, giving $\nu = 1$.

IV. SUMMARY AND DISCUSSIONS

In our models [6, 7, 8, 9], we consider the ideal-gas-like trading markets where each agent is identified with a gas molecule and each trading as an elastic or money-conserving (two-body) collision. Unlike in a gas, we introduce a saving factor λ for each agents. Our model, without savings ($\lambda = 0$), obviously yield a Gibbs law for the steady-state money distribution. Our numerical results for various widely distributed (quenched) saving factor λ showed [7, 8, 9] that the steady state income distribution $P(m)$ in the market has a power-law tail $P(m) \sim m^{-(1+\nu)}$ for large income limit, where $\nu \simeq 1.0$. This observation has been confirmed in several later numerical studies as well [15, 16]. Since $Q(m) = \int_m^\infty P(m) dm$ can be identified with the inverse rank, our observation in the model with $\nu = 1$ suggests that the rank of any agent goes inversely with his/her income/wealth, fitting very well with the Zipf's original observation [17]. It has been noted from these numerical simulation studies that the large income group people usually have larger saving factors [7]. This, in fact, compares well with observations in real markets [12, 18]. The time correlations induced by the random saving factor also has an interesting power-law behavior [16]. A master equation for $P(m, t)$, as in (9), for the original case (eqns. (1) and (2)) was first formulated for fixed λ (λ_i same for all i), in [13] and solved numerically. Later, a generalized master equation for the same, where λ is distributed, was formulated and solved in [15].

We have formulated here a Boltzmann-type master equation for the distributed saving factor case in (1) and (2). Based on the observation that even in the case with $\epsilon = 1/2$ (with λ distributed in the range $0 \leq \lambda_i < 1$, $\lambda_i \neq \lambda_j$), in (1) and (2), the steady state money distribution has the same power-law behavior as in the general case and shows the same Pareto index, we solve the master equation for this special case. We show that the analytic results clearly support the power-law for $P(m)$ with the exponent value $\nu = 1$. Although our analysis of the solution of the master equation is for a special case and it cannot be readily extended to explore the wide universality of the Pareto exponent as observed in the numerical simulations of the various versions of our model [7, 15], let alone the quasi-universality for other ν values as observed in the real markets [2, 3, 4], the demonstration here that the master equation admits of a Pareto-like power law solution (for large m) with $\nu = 1$, should be significant.

Apart from the intriguing observation that Gibbs (1901) and Pareto (1897) distributions fall in the same category of models and can appear naturally in the century-old and well-established kinetic theory of gas, our study indicates the appearance of self-organized criticality in the simplest (gas-like) models so far, when the stability effect of savings is incorporated. This remarkable effect can be analyzed in terms of master equations developed here and can also be studied analytically in the special limits considered.

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APPENDIX A: ALTERNATIVE SOLUTION OF THE STEADY STATE MASTER EQUATION (11)

Let $S_r(x) = \int_0^\infty dm P(m) m^{\nu+r} \exp(-mx)$; $r \geq 0, x > 0$. If $P(m) = A/m^{1+\nu}$, then

$$\begin{aligned} S_r(x) &= A \int_0^\infty dm\; m^{r-1} \exp(-mx) \\ &\sim A\, \frac{x^{-r}}{r} \qquad \text{if } r > 0 \\ &\sim A \ln\left(\frac{1}{x}\right) \quad \text{if } r = 0. \end{aligned} \tag{A1}$$

From eqn. (11), we can write

$$\begin{aligned} S_r(x) &= \langle \int_0^\infty dm_i \int_0^\infty dm_j\; P(m_i)P(m_j)(m_i\mu_i^+ + m_j\mu_j^-)^{\nu+r} \exp[-(m_i\mu_i^+ + m_j\mu_j^-)x]\rangle \\ &\simeq \int_0^\infty dm_i\; Am_i^{r-1} \langle \exp(-m_i\mu_i^+ x)\left(\mu_i^+\right)^{\nu+r}\rangle \left[\int_0^\infty dm_j\; P(m_j)\langle\exp(-m_j\mu_j^- x)\rangle\right] \\ &\quad + \int_0^\infty dm_j\; Am_j^{r-1} \langle \exp(-m_j\mu_j^- x)\left(\mu_j^-\right)^{\nu+r}\rangle \left[\int_0^\infty dm_i\; P(m_i)\langle\exp(-m_i\mu_i^+ x)\rangle\right] \end{aligned} \tag{A2}$$

or,

$$\begin{aligned} S_r(x) &= \int_{\frac{1}{2}}^1 d\mu_i^+\; p(\mu_i^+) \left(\int_0^\infty dm_i\; Am_i^{r-1} \exp(-m_i\mu_i^+ x)\right) \left(\mu_i^+\right)^{\nu+r} \\ &\quad + \int_0^{\frac{1}{2}} d\mu_j^-\; q(\mu_j^-) \left(\int_0^\infty dm_j\; Am_j^{r-1} \exp(-m_j\mu_j^- x)\right) \left(\mu_j^-\right)^{\nu+r}, \end{aligned} \tag{A3}$$

since for small x, the terms in the square brackets in (A2) approach unity. We can therefore rewrite (A3) as

$$S_r(x) = 2\left[\int_{\frac{1}{2}}^1 d\mu^+ (\mu^+)^{\nu+r} S_r(x\mu^+) + \int_0^{\frac{1}{2}} d\mu^- (\mu^-)^{\nu+r} S_r(x\mu^-)\right]. \tag{A4}$$

Using now the forms of $S_r(x)$ as in (A1), and collecting terms of order x^{-r} (for $r > 0$) or of order $\ln(1/x)$ (for $r = 0$) from both sides of (A4), we get (14).

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