

Minimal Agent Based Model for Financial Markets I: Origin and Self-Organization of Stylized Facts

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Abstract

We introduce a minimal Agent Based Model for financial markets to understand the nature and Self-Organization of the Stylized Facts. The model is minimal in the sense that we try to identify the essential ingredients to reproduce the main most important deviations of price time series from a Random Walk behavior. We focus on four essential ingredients: fundamentalist agents which tend to stabilize the market; chartist agents which induce destabilization; analysis of price behavior for the two strategies; herding behavior which governs the possibility of changing strategy. Bubbles and crashes correspond to situations dominated by chartists, while fundamentalists provide a long time stability (on average). The Stylized Facts are shown to correspond to an intermittent behavior which occurs only for a finite value of the number of agents N . Therefore they correspond to finite size effect which, however, can occur at different time scales. We propose a new mechanism for the Self-Organization of this state which is linked to the existence of a threshold for the agents to be active or not active. The feedback between price fluctuations and number of active agents represent a crucial element for this state of Self-Organized-Intermittency. The model can be easily generalized to consider more realistic variants.

1 Introduction

In the past years there has been a large interest in the development of Agent Based Models (ABM) aimed at reproducing and understanding the Stylized Facts (SF) observed in the financial time series. The simplest model of price time series is the Random Walk (RW) introduced by Louis Bachelier in 1900 [1]. The availability of large amounts of data has revealed a variety of systematic deviations from the RW which are the SF and are relatively common to all markets. As we are going to see the properties of the SF resemble Critical Phenomena and Complex Systems with possible relations to various problems in

Statistical Physics. Many questions are however open and also how far one can go with these similarities represents a highly debated problem. The main SF, discussed in detailed in [2, 3, 4] are:

Absence of linear auto-correlations: It is not possible to predict the direction of the next price change from the previous one. Only at very short times (minutes or less) some correlations, positive or negative, can be present for technical reasons. This property corresponds to the condition of an efficient market (no simple arbitrage), which implies that it is not possible to make a profit without risk. If such a possibility would occur the market would immediately react and eliminate it.

Heavy Tails: The distribution of returns (price increments) has a shape which is markedly different from the Gaussian case (RW). The probability for large positive or negative fluctuations is much larger than for a Gaussian distribution. These fat tails have been fitted in various ways and in some range they can be approximated by a power law with an exponent ranging from 3 to 5. This behavior is observed for time scales ranging from minutes to weeks, while for very long times, a Gaussian distribution seems to be recovered.

Volatility Clustering: The variance of the returns shows a long range correlation in time. According to Mandelbrot [5, 6]: “large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes”. The correlation of the absolute values of the square of the returns is usually fitted by a power law (with exponents ranging from about zero to 1.0 and most probable value around 0.3) which can extend from minutes to weeks. These three main SF have been interpreted in term of various phenomenological stochastic models which permit in some cases an estimate of the risk beyond the Black and Scholes model which corresponds to the simple RW [6, 7, 8, 9, 10].

In our opinion it would be important to add the following additional elements which are also essential to build a conceptual framework necessary to understand market dynamics.

Nonstationarity and Time Scales: A characteristic of the price dynamics is that it is not clear if one can define a probability distribution which is stationary in time and, in any case, it is not easy to estimate the appropriate time scale [11]. This is a crucial point that should be carefully analyzed in any statistical treatment of the data.

Self-Organization: A point which is usually neglected is the fact that the stylized facts usually correspond to a particular situation of the market. If the market is pushed outside such a situation, it will evolve to restore it spontaneously. The question is why and how all markets seem to self-organize in this special state. The answer to this question represents a fundamental point in understanding the origin of the SF.

In this paper we discuss a minimal ABM [12] which includes the following elements (first considered by Lux and Marchesi in [13, 14]):

- **fundamentalists:** these agents have as reference a fundamental price p_f derived from standard economic analysis of the value of the stock. Their strategy is to trade on the fluctuations from this reference value and bet on the fact that the price will finally converge towards the reference value.

These traders are mostly institutional and their time scale is relatively long. Their effect is a tendency to stabilize the price around the reference value.

- chartists: they consider only the price time series and tend to follow trends in the positive or negative direction. In this respect they induce a destabilizing tendency in the market. These traders have usually a time horizon shorter than the fundamentalists and they are responsible for the large price fluctuations corresponding to bubbles or crashes.
- herding effect: this is the tendency to follow the strategy of the other traders. It should be noted that traders can change their strategy from fundamentalist to chartist and vice-versa depending on various elements.
- price behavior: each trader looks at the price from her perspective and derives a signal from the price behavior. This signal will be crucial in deciding her trading strategy.

These four elements are, in our opinion, the essential irreducible ones that an ABM should possess and we are going to describe in detail their combined effects. Of course real market are extremely more complex but the study of these elements represents a basis on which one might eventually add more realistic features. In this perspective we will try to construct a model which is the simplest possible containing these four elements. This permits to obtain a detailed understanding of the origin and nature of the SF and also to discuss their self-organization. The main result of this paper is that the SF correspond to finite size effect in the sense that they disappear in the limit of large N (total number of agents) or large time. Given that the agents' strategies can be defined for different time scales this finite size effects can act at different time and resemble power laws and critical exponents. However, in our perspective, these exponents are essentially a fit to the data without the important property of universality. This situation has important implication both for the microscopic understanding of the SF as well as for the analysis and interpretation of experimental data. The concept of self-organization arises very naturally from the introduction of a threshold in the agents' activity. Namely a price which is very stable (which corresponds to a very large N) demotivates agents to trade this stock and will naturally lead to a decrease of the number of agents. On the other hand a small number of agents leads to large fluctuations in the price which presents opportunities of arbitrage and this will attract more traders. So, in the end, the system will self-organizes around the number of traders which corresponds to a situation of intermittency, leading to the SF. This phenomenon does not correspond to self organized criticality precisely but rather to a Self Organized Intermittency (SOI) which occurs for a finite number of agents (on average).

In Sec. 2 we define the basic elements of the minimal ABM in the perspective of its maximum simplification.

In Sec. 3 we consider in detail the fluctuations induced by the herding dynamics.

In Sec. 4 we discuss the properties of the SF as arising from the model for a specific range of parameters.

In Sec. 5 we give an interpretation to the SF in terms of semi-analytical considerations.

In Sec. 6 we discuss the self-organization of the market dynamics towards the self-organized regime with the SF.

In Sec. 7 we present our conclusions.

In the next paper [15] we are going to focus on various statistical properties from both a numerical and analytical point of view. We will also discuss in some detail the differences between linear and multiplicative dynamics. In the present paper we only consider the linear dynamics for simplicity.

2 Definition of a minimal ABM with Fundamentalists and Chartists

In this section we are going to describe in detail the ABM introduced in [12]. The model is constructed to be as simple as possible but still able to reproduce the SF of financial time series. This simplicity is very important because it permits to derive a microscopic explanation of the SF in term of the model parameters. Elements which consider more realistic situations however can be added in a systematic way. This model is inspired by the well known Lux-Marchesi (LM) model [13, 14] but it is much simpler with respect both to the number of parameters and the rules for the dynamics.

In our model we consider a population of N interacting agents which are divided into two main categories: fundamentalists and chartists. The fundamentalist agents tend to stabilize the market driving the price towards a sort of reference value which we call the fundamental price (p_f). This kind of agents can be identified with long-term traders and institutional traders [16]. Chartists instead are short-term traders who look for detecting a local trend in the price fluctuation. This kind of agents tries to gain from detecting a trend and so they are responsible for the formation of market bubbles and crashes which destabilize the market. In the LM model the destabilizing effect made by chartists is implemented dividing the chartists agents into two subcategories: optimists which always buy and pessimists which always sell. In our model we can overcome this complication by simplifying the chartists' behavior. In fact, the description of chartists is in term of the recently introduced potential method [17, 18, 19, 20]. Chartist agents detect a trend by looking at the distance between the price and its smoothed profile which in our case is the moving average computed on the previous M time steps. Chartists try to follow the trend and bet that the price will moreover move away from the actual price. In this way they create a local bubble which destabilizes the market. The stochastic equation for the price which describes the chartists behavior can be written in terms of a RW with a force whose center is the distance between the price and the moving average:

$$p(t+1) = p(t) + \frac{b}{M-1} F(p(t) - p_M(t)) + \sigma \xi(t) \quad (1)$$

$$p_M(t) = \frac{1}{M} \sum_{i=t}^M p(i) \quad (2)$$

where b is a parameter which gives the strength of the force, p_M is the moving average performed on the previous M steps of the price, ξ is a white noise and σ is the amplitude of this noise. The factor $M-1$ in the denominator of the force term makes the potential independent of the particular choice of the moving

average [17]. This equation has been used in previous papers to analyze real data [17, 19, 20]. The conclusion is that it is indeed possible to observe this kind of forces also in real stock-prices. For our model we decided to adopt a simple linear expression for the force ($F = p(t) - p_M(t)$).

By integrating the force it is also possible to obtain the effective potential which describes the chartists' behavior. For the case of a simple linear force the potential is quadratic. In Fig. 1 we show the stochastic process described in Eq. (2) with a linear force and the corresponding potential. A random walk is also plotted for comparison. Depending on the sign of the parameter b one can obtain both attractive and repulsive potential. Here we are considering only the repulsive potential which describes the trend-follower behavior of chartists, because the stabilizing attractive case is already carried out by the fundamentalist agents. In fact, fundamentalists try to stabilize the market driving the price towards the fundamental price p_f which in our model is constant in time. The stochastic equation which describes the fundamentalists' behavior can be written in terms of a random walk with a further term which is responsible for the stabilizing action of fundamentalists:

$$p(t+1) = p(t) + \gamma(p_f - p(t)) + \sigma\xi(t) \quad (3)$$

where γ is the strength of the fundamentalists' action. For the moment the total number of agents, N , is kept fixed and there are N_f fundamentalists and N_c chartists but agents can decide to change their mind during the simulation by switching their strategy from fundamentalist to chartist and viceversa. The probability to change strategy is based on two terms. The first is an herding term: agents tend to imitate other people behavior proportionally to the relative number of agents in the arrival class. The second is a term, characterized by the parameter k , which leaves the agents the possibility to change strategy on the basis of considerations on the price behavior, independently on what the other agents are doing. The price signal which appears in the transition probabilities is proportional to $|p(t) - p_f|$ for the probability to become fundamentalist and $|p(t) - p_M(t)|$ for that to become chartists. The mathematical expression for the transition probabilities is:

$$P_{cf} \propto (K + \frac{N_f}{N}) \exp(\gamma|p(t) - p_f|) \quad (4)$$

$$P_{fc} \propto (K + \frac{N_c}{N}) \exp(\frac{b}{M-1}|p(t) - p_M(t)|) \quad (5)$$

For a realistic representation of the market the fundamentalists should dominate for very long times with intermittent appearance of bubbles or crashes due to the chartists. This is due to the fact that fundamentalists are usually big institutional traders which have on average a major weight in the market. In order to properly reproduce this effect we can introduce an asymmetry in the herding dynamics and, neglecting the price modulation, Eqs. (4) and (5) can be written in the following way:

$$P_{cf} \propto (K + \frac{N_f}{N})(1 + \delta) \quad (6)$$

$$P_{fc} \propto (K + \frac{N_c}{N})(1 - \delta) \quad (7)$$

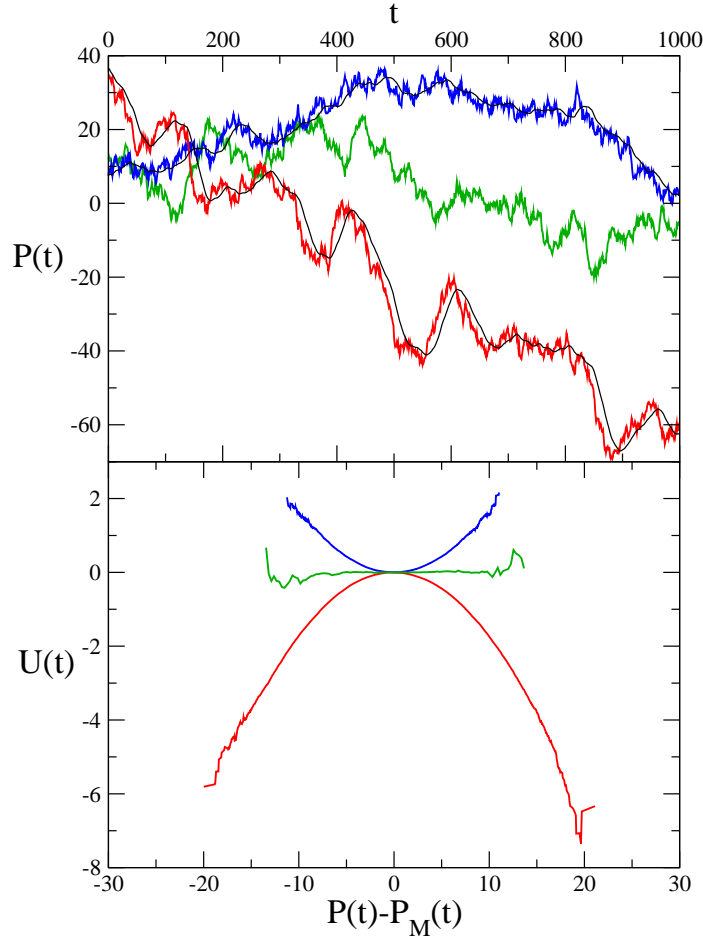


Figure 1: The upper plot shows the price fluctuations for the process described by Eqs.(2) and (2) with a linear force F for different values of the force strength parameter b . The values of the parameter b are $b = -1$ (blue line), $b = 0$ (green line) and $b = 1$ (red line) which correspond respectively to attractive, flat and repulsive potentials as shown in the lower plot. In our model we are considering only the repulsive potential which describes the behavior of the trend-follower chartists.

where the positive parameter δ define the asymmetry of the model. In the case $\delta = 0$ the model reduces to the symmetric ants model by Kirman [21]. In Sec. 3 we will analyze in detail this asymmetric model and the corresponding equilibrium distribution for the population of chartists and fundamentalists. For the price formation we are going to use a simple linearized form of the Walras' Law [22]:

$$p(t+1) - p(t) = \Delta p \propto ED(t) \quad (8)$$

where ED is the excess demand of the market at time t . This linear interpretation, valid for small price increments, ($\Delta p \ll 1$) is a technical simplification from the more realistic multiplicative dynamics and it is very useful for an analytical treatment. The more general multiplicative case will be discussed in detail in a forthcoming paper [15]. With respect to the dynamics of the ABM the multiplicative case will turn out to be less stable in terms of parameters while the linear case is mathematically less problematic.

In our model the excess demand ED is simply proportional to the price signal of chartists and fundamentalists:

$$ED = ED_f + ED_c = \frac{N_f}{N} \gamma (p_f - p(t)) + \frac{N_c}{N} \frac{b}{M-1} (p(t) - p_M(t)) \quad (9)$$

then adding a noise term we have the complete equation for the price formation:

$$p(t+1) - p(t) = ED + \sigma \xi(t) \quad (10)$$

In this case we have that the volume of the agents' actions is proportional to the signal they perceive. This situation is much simpler than the corresponding price formation for the Lux-Marchesi model where prices can only vary of a fixed amount (tick) and the connection with the excess demand is implemented in a probabilistic way.

3 Herding Dynamics

3.1 Symmetric case

In his seminal paper Kirman [21] proposed a simple dynamical model to explain a peculiar behavior observed in ant colonies. Having two identical food sources, ants prefer alternatively only one of these, and (almost) periodically switch from one source to the other. In Kirman's model this effect is caused by an *herding dynamics*, in which the evolution is stochastic and based on the meetings of the ants. In particular an ant can recruit a companion, and bring it to its preferred source, with a certain probability.

This model can be used to take into account the herding dynamics of a population of N agents (with N fixed), and it is formalized as follows.

Let us suppose the existence of two microscopic states: an agent at time t can be either a *fundamentalist* or a *chartist*. Defining $N_c = N_c(t)$ as the number of chartists at time t (and analogously $N_f = N_f(t) = N - N_c(t)$ as the the number of fundamentalists), the quantity

$$x = \frac{N_c}{N} \quad (11)$$

varying between 0 and +1 describes the macroscopic state of the system.

A possibility to change opinion is given to each agent at any given time step if she meets an agent with opposite views who succeeds in recruiting her. Under the hypothesis that the transition probabilities are small enough, in a single time step there will be no more than one change of opinion. In this case we can write the transition rates as

$$P_{cf} = \beta \frac{N_f}{N} \quad P_{fc} = \beta \frac{N_c}{N} \quad (12)$$

where P_{cf} is the probability than *one* agent passes from the chartist group to the fundamentalist one, and β regulates the speed of the process.

Roughly, this is the approach adopted by Lux and Marchesi to rule the transitions between agents in their model [14, 13], apart an exponential term depending on agents' utilities that plays a minor role in the dynamics. If we adopt the rates of Eq.(12) the dynamics will admit two absorbing states ($N_c = 0$ and $N_c = N$). Lux and Marchesi assign a lower limit of 4 for each class of agents, while Alfarano et al. [23], following Kirman [21], suggested a Poissonian term (that is, independent from the number of agents) to avoid these fixed points. This new term can be seen as a spontaneous tendency to change one's mind, independently on the others. This corresponds essentially on the term K we have discussed in relation of Eqs. (4-7). The simplest model is therefore given by the following transition probabilities. We will adopt this last solution and write the transition probabilities as

$$P_{cf} = \beta(K + \frac{N_f}{N}) \quad P_{fc} = \beta(K + \frac{N_c}{N}). \quad (13)$$

Simulating this process one can observe two different behaviors depending on the choice of the parameters, and in particular on $\epsilon \equiv KN$. It can be shown [21] that if $\epsilon < 1$ the system spends most of the time in the two *metastable states* $x \approx 0$ and $x \approx 1$, sometimes switching from one to the other. In the case

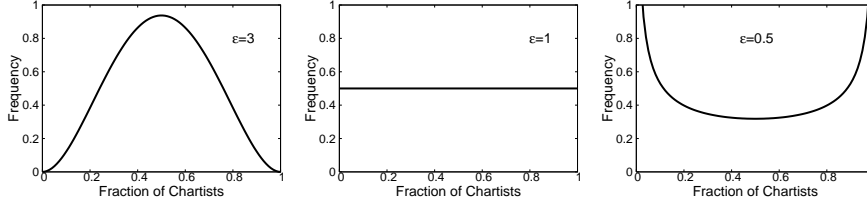


Figure 2: The functional form of the distribution Eq. (14) changes with $\epsilon = KN$. We show the cases $\epsilon = 3.0; 1.0$ and 0.5 . Only for $\epsilon < 1$ the system shows an alternation between two metastable states.

$\epsilon \geq 1$, that is, if the self-conversion term is large enough, the systems fluctuates around $x = 0.5$.

Kirman explicitly derived the functional form of the equilibrium function for x ,

$$P_s(x) \sim x^{\epsilon-1}(1-x)^{\epsilon-1}. \quad (14)$$

In Fig. 2 the three cases $\epsilon > 1$, $\epsilon = 1$ and $\epsilon < 1$ are shown. In the first case the distribution has a single peak for $x = 0.5$. By decreasing ϵ the distribution gets smoother and smoother, until it becomes uniform for $\epsilon = 1$. If one further decreases ϵ , the distribution develops two peaks and becomes bimodal.

3.2 Asymmetric case

It is possible to generalize the symmetric herding model described above to the asymmetric case, in order to take account of the fact that institutional traders, that have more impact on the market than individuals, usually adopt long-term strategies [16], as fundamentalists do. Here we propose a further generalization of the model proposed of Alfarano et al. [24], which is more convenient for our purposes.

Our asymmetric dynamics is given by:

$$\begin{aligned} p_{f \rightarrow c} &= \beta(1 - \delta) \left[K_1 + \frac{N_c}{N} \right] \\ p_{c \rightarrow f} &= \beta(1 + \delta) \left[K_2 + \frac{N_f}{N} \right] \end{aligned} \quad (15)$$

where $\delta > 0$ regulates the asymmetry between the two metastable states (the model described in [24] can be recovered in the case $\delta = 0$).

We are going to see that the asymmetry introduced by the term δ is more realistic than the one due to K_1 and K_2 with respect to the objective of having the market dominated (on average) by fundamentalists at very large times.

It is possible to find the Fokker-Planck equation associated to the process given by Eq.(15). Following Alfarano et al. [24], we consider the Master Equation in the case in which, on average, we have no more than a change of opinion in a single time step (this approximation is valid if $\beta \ll 1$):

$$\frac{\Delta P_{N_c}(t)}{\Delta t} = P_{N_c+1} \pi(N_c + 1 \rightarrow N_c) - P_{N_c} \pi(N_c \rightarrow N_c - 1)$$

$$-P_{N_c}\pi(N_c \rightarrow N_c + 1) + P_{N_c-1}\pi(N_c - 1 \rightarrow N_c). \quad (16)$$

where $P_{N_c}(t)$ is the probability to have N_c chartists at time t and π is the rate of the transition in the brackets. In the continuous limit we can substitute the transition rates in Eq.(16), finding after some passages the Fokker-Planck equation

$$\frac{\partial p(x, t)}{\partial t} = -\frac{\partial}{\partial x}A(x)p(x, t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}D(x)p(x, t) \quad (17)$$

with drift and diffusion function given respectively by

$$A(x) = \beta \left[-2\delta x(1-x) + (1-\delta)K_1(1-x) - (1+\delta)K_2x - \delta\frac{1}{N^2} \right]$$

and

$$D(x) = \beta \left[\frac{2}{N}x(1-x) + \frac{(1-\delta)}{N}K_1(1-x) + \frac{(1+\delta)}{N}K_2x - 2\frac{\delta}{N^2}(x - \frac{1}{2}) \right].$$

Given these functions, the asymptotic stable distribution for the fraction x of chartists is given by the textbook formula

$$p_s(x, N) = \frac{C}{D(x)} \exp \left[\int^x \frac{2A(y)}{D(y)} dy \right] \quad (18)$$

where C is a normalization constant (this expression can be obtained setting the temporal derivative of $p_s(x, N)$ equal to zero and integrating twice the Fokker-Planck equation, see for example [25]).

The explicit expression of the equilibrium distribution can be derived by appropriate analytical computer codes and it is rather complex. Nevertheless a simple approximation will allow us to derive the distribution $\tilde{p}_s(x, N)$ which can be easily calculated and it illustrates clearly the role played by the asymmetry parameter δ .

The idea is to disregard the terms of order $1/N^2$ in the drift and in the diffusion functions. This approximation is valid for $N \gg 1$ and, since we will deal with at least ~ 50 agents, we can expect that it is rather appropriate to our case.

Moreover, we are interested only in the case $K_1N = K_2N < 1$ in order to avoid the unimodality of the asymptotic distribution. This can be obtained setting $K_1 = K_2 = r/N$, with $r < 1$. The reasons for adopting a parametrization in which K is inversely proportional to N are:

- (i) qualitatively one may expect that the probability to neglect the herding behavior decreases when N increases.
- (ii) in the following we will consider the properties of the model as a function of N and we believe it is realistic that the system always stays in the bimodal case. This requires that $\epsilon = KN$ should not exceed the value one even for large values of N .

With this choice and using the approximation described above, the drift and the diffusion functions become respectively

$$\tilde{A}(x) = \beta \left[-2\delta x(1-x) + \frac{r}{N} \left(1 - \delta - 2x \right) \right] \quad (19)$$

and

$$\tilde{D}(x) = \beta \left[\frac{2}{N} x(1-x) \right] \quad (20)$$

Now we can easily solve the integral in Eq.(18) and derive

$$\tilde{p}_s(x, N) \propto x^{r(1-\delta)-1} (1-x)^{r(1+\delta)-1} \exp(-2\delta Nx). \quad (21)$$

The form of Eq. (21) clearly shows that, by increasing N , the values of x near to 1 are exponentially suppressed. In other words, if N is small the equilibrium distribution will be bimodal, because $\delta \ll 1$ and $r < 1$ (compare the Eqs. (21) and (14)), while by increasing N the asymmetry, regulated by the parameter δ , becomes more and more important.

We have also simulated the process described by the Eq. (15) and we have compared the distributions obtained by integrating the Eq. (18) with the complete drift and diffusion functions. As it is clear from Fig. 3, the agreement between theory and simulation is very good.

3.3 Finite size effects

Alfarano et al. [23] computed the time that on average must be waited to see the switch from one metastable state to the other in the symmetric case:

$$T_0 = \frac{N}{\beta} \frac{\pi}{(1-2\epsilon)} \frac{\cos(\pi\epsilon)}{\sin(\pi\epsilon)}.$$

The presence of this characteristic temporal scale implies that this simple model would give exponential correlation functions for volatility clustering. However different time scales in agents' strategies can lead to a superposition of different characteristic times and therefore to long-range relaxation.

Examining the functional form of T_0 one can see that for $\epsilon \rightarrow 0$ the mean first passage time diverges, because of the emergence of the two absorbing states. Moreover, keeping ϵ constant and increasing N the time spent in one of the metastable states gets longer.

We have simulated the process given by Eq.(12) using different values of N keeping ϵ fixed and equal to 0.5. For the velocity parameter β we choose the value 0.02. As expected, the system oscillates between the two metastable states. As shown in Fig. 4, the only difference is the frequency of the passages from one state to the other. For small values of N ($N = 50$) the rate to change strategy is very high and this leads to an unrealistic situation in which fluctuations are too fast. On the other hand, for large values of N ($N = 5000$) the system get essentially locked in one of the two states and the fluctuations become frozen. Only for an intermediate values of N ($N = 500$) the system shows an intermittent behavior which resembles experimental observations and will lead to the SF. This finite size effect was first noted in ?? in relation to the LM model. In conclusion, this simple model explains why, in the limit $N \rightarrow \infty$, models like the Lux-Marchesi one loose their property to generate the SF.

This result is based on very general properties and it is easy to expect that it will be a general one for the entire class of model we are considering. This implies that in this kind of models, given a fixed temporal scale, the system will be able to generate the "right" amount of fluctuations only within a specific range of

finite values of N . This is a very important observation from both a conceptual and a practical point of view. In fact in this class of models the quasi-critical state linked to the SF, corresponds to a finite size effect (with respect to N and to t) in the sense of Statistical Physics. Therefore we are not in presence of a real critical behavior characterized by universal power law exponents. The fact that in some cases one can fit the experimental data with power laws can be easily understood by considering that different agents might operate at different time scales. This would lead to a superpositions of finite size effects corresponding to different time scale switch may appear as a sort of effective power law exponent. The possibility that a suitable coupling between different time scale exist leading to genuine critical behavior is of course open. However, this is not the case for the class of model we are considering. In this perspective the variability of the effective exponents and their breakdown observed in various data [2] can be a genuine effect not simply due to limitation or problems with a database. This of course change the perspective of the data analysis and of their comparison with the models. We expect that the behavior of different markets is reasonably similar because the key elements are essentially the same but without a strict universality.

In the asymmetric case one must consider two different temporal scales, say T_1 and T_2 , the first referring to the formation of the chartists' bubble and the second relative to its duration. We have investigated with numerical simulations their dependence on N and δ , finding, as expected, a divergence of T_1 in the limit $N \rightarrow \infty$. Therefore the introduction of an asymmetry in the agents dynamics does not change the finite size effect associated to the quasi-critical behavior and the SF.

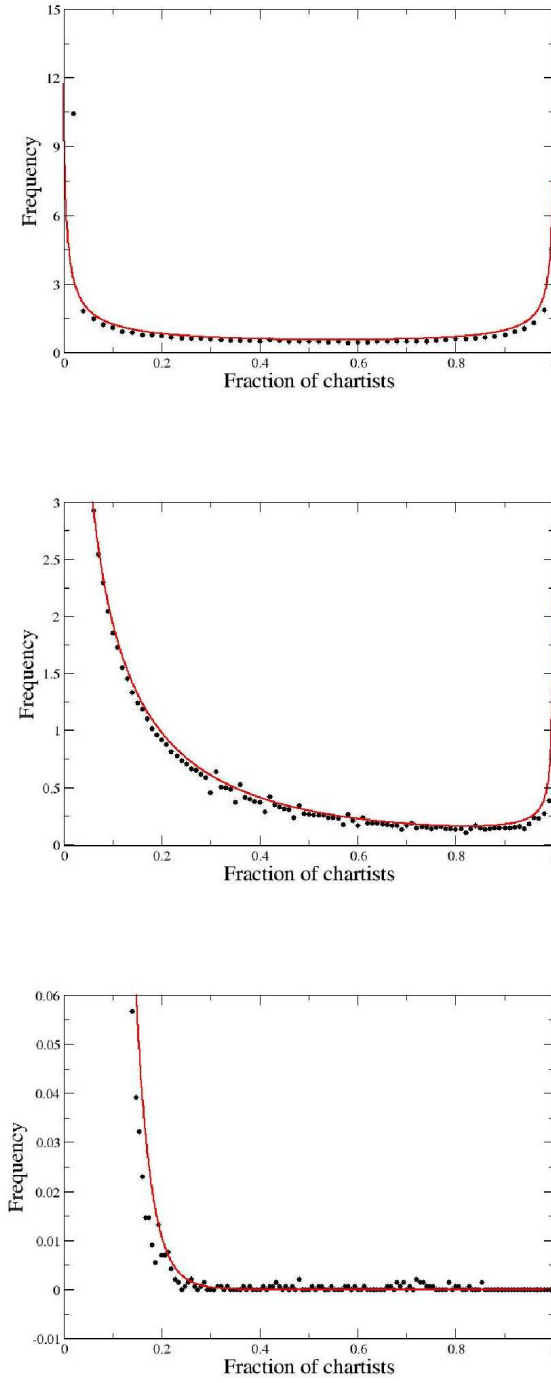


Figure 3: Comparison between theory (red line) and simulation (black circles) for the asymmetric model described by the transition rates given by Eq. (15). From top to bottom the three cases $N = 50$, $N = 500$ and $N = 5000$ are shown, the other relevant parameter δ being fixed and equal to 0.003. The variation of the vertical scale underlines the progressive compression of the distribution towards the state in which fundamentalists are dominant.

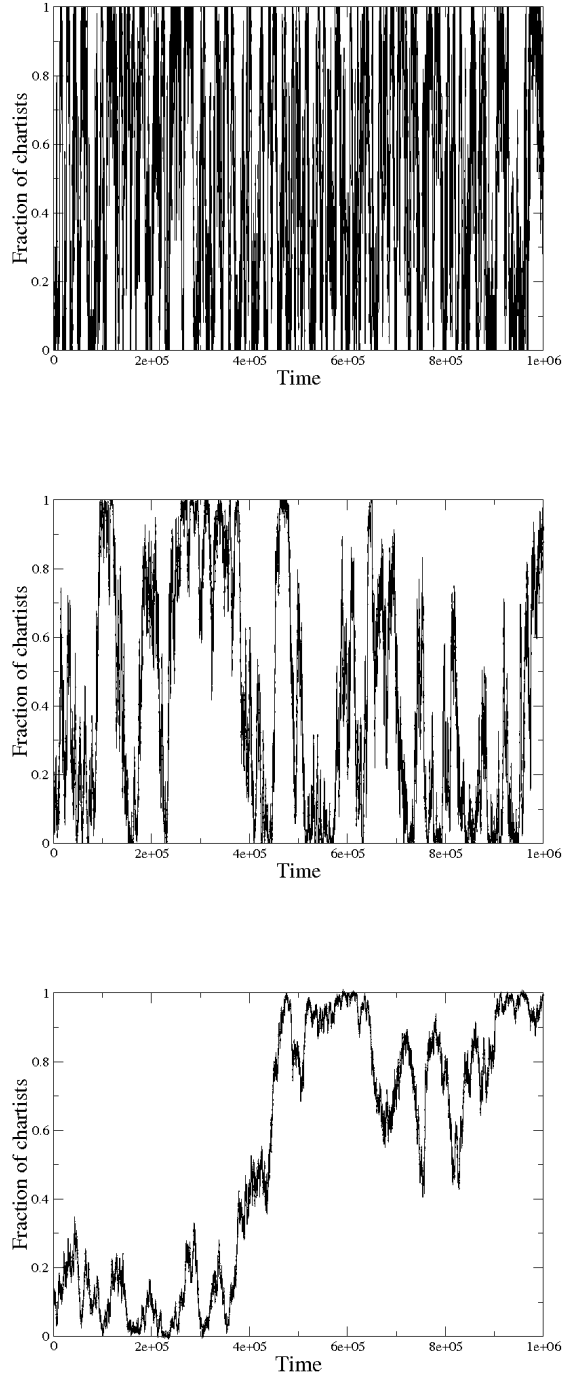


Figure 4: Fraction of chartists x for the dynamics given by Eq.(12). We have considered different values of N but the same temporal scale for the horizontal axis, $T = 10^6$ time steps. From top to bottom we have $N = 50$, 500 and 5000 respectively. One can see that the cases $N = 50$ and $N = 5000$ are unrealistic, corresponding to too many or too few fluctuations. Only the case $N = 500$ shows the right intermittency which leads to Stylized Facts.

4 Stylized Facts from the Model

In this section we discuss the results of some simulations of the minimal model described in Sec. 2. In particular we will see that this model is able to reproduce the main SF of financial markets listed in the introduction. In order to clarify all the elements of the model we are going to discuss increasingly complex cases.

(a) Single agent

Considering that in the simplest model all agents are statistically identical (no real heterogeneity) we can fix our parameters in such a way that even a single stochastic agent can lead to an interesting dynamics. This single agent can be chartist or fundamentalist and the herding term is not active in this case. For simplicity we have also neglected the exponential term related to the price behavior in the transition probabilities. When the agent switches her behavior from chartist to fundamentalist, the market dynamics is in turn given by Eq. (2) or (3). If the agent is chartists she follows the market trend and creates bubbles, in this case the price fluctuations are larger with respect to a simple random walk. Otherwise if she is fundamentalist the price is driven towards the fundamental price and the fluctuations are smaller than the random walk ones. In this case the price is not diffusive and it remains almost constant, oscillating around the fundamental price p_f . In Fig. 5 we show the results of the simulation for the one-agent model. We can observe that the price dynamics displays local bubbles corresponding to periods in which the agent is a chartist. The price is instead almost constant when the agent is fundamentalist. Also we can observe periods of high or low volatility depending on the strategy of the agent. This dynamics clearly leads to fat tails in the distribution of price increments and also to a certain volatility clustering that certainly in this case is not due to the herding dynamics. This simple example shows that, once we have a full control of the model parameters, we can trace and reproduce some SF even with an extremely minimal model.

(b) Many statistically equivalent agents ($N = 100$)

We now consider the more realistic dynamics with a larger number of agents. In principle we can tune the parameters to obtain the SF for any preassigned value of N . For example in Fig. 6 (upper) we show the dynamics of the case $N = 100$ still without the exponential price term. Also in this case periods of high or low volatility correspond to regions in which chartist or fundamentalist agents dominate. We have analyzed the SF for this model and both fat tails and volatility clustering are shown in Fig. 6 (lower). The square price-fluctuations shows a positive autocorrelation which (unlike real data) decays exponentially. As we have discussed this depends on the fact that this model has a single characteristic time scale.

(c) Many heterogeneous agents

The limitation of a single time scale can be easily removed by introducing a real heterogeneity in the time scales of the agents' strategies. In particular we have introduced a distribution of values for the parameter M which is the number of steps agents consider for the estimation of the moving average of the price. We adopt a distribution of values also for the parameter b which gives the strength of the action of chartists. In this case we also introduce the exponential price

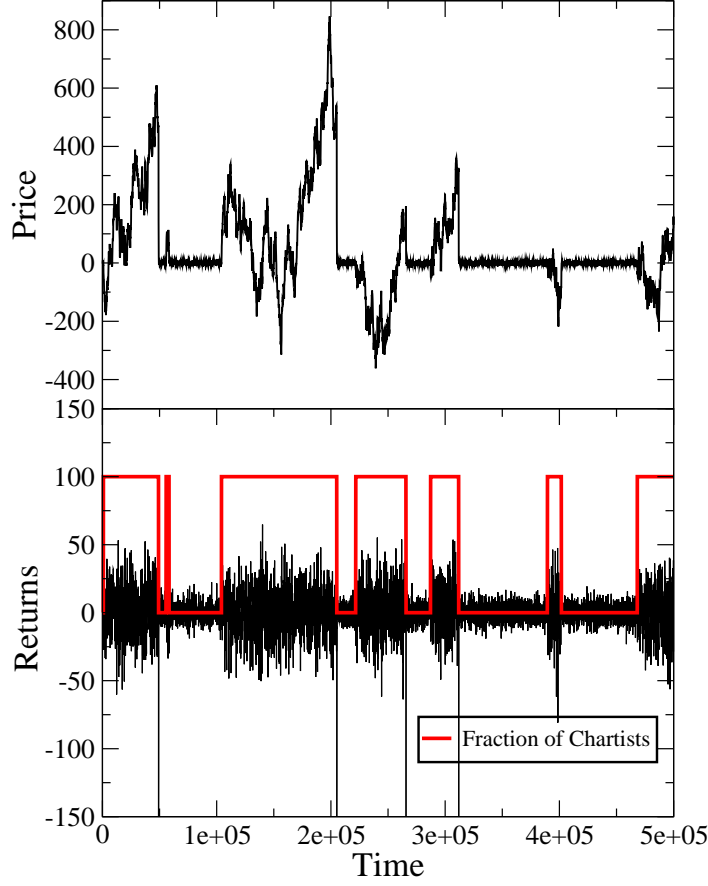


Figure 5: Even a one agent model subjected to the stochastic dynamics can lead to non trivial results in terms of fat tails and volatility clustering. Here we can see the corresponding price and returns fluctuations. In this case we have also simplified the probability to change strategy by neglecting the exponential term corresponding to the price behavior. In the figure below the large returns are clearly related to the periods in which the agent is a chartist. The price behavior may look unrealistic because we consider a case with a fixed value of p_f . If one would also introduce a random walk behavior for p_f the price behavior would look much more realistic.

term in the rate equations in order to have the most complete version of our model. In Fig. 7 we report the SF corresponding to a uniform M distribution between 10 and 50 time steps and a uniform b distribution between 0 and 2. The comparison of these results with those of the simplified model discussed before permits to draw the following conclusions:

- The exponential term related to the price behavior in the transition probabilities enhances the bubbles and crashes corresponding to the chartists action and it amplifies the fat tail phenomenon as shown by comparing Fig. 7 (lower) to Fig. 6 (lower). This is a qualitative argument because the two models are actually different and a precise comparison keeping the same effective parameters is not possible.
- The real heterogeneity of the agents time scales produces in this case a distribution of transition rates which leads to a quasi-power law behavior for the volatility correlations as shown in the inset of Fig. 7 (upper). On the other hand this different time scales have the effect of decorrelating the agents dynamics and the overall amplitude of the volatility is reduced.
- The short time anti-correlation for the returns (Fig. 7 upper), can be related to the long time predominance of fundamentalist and will be discussed in detail in paper II [15].

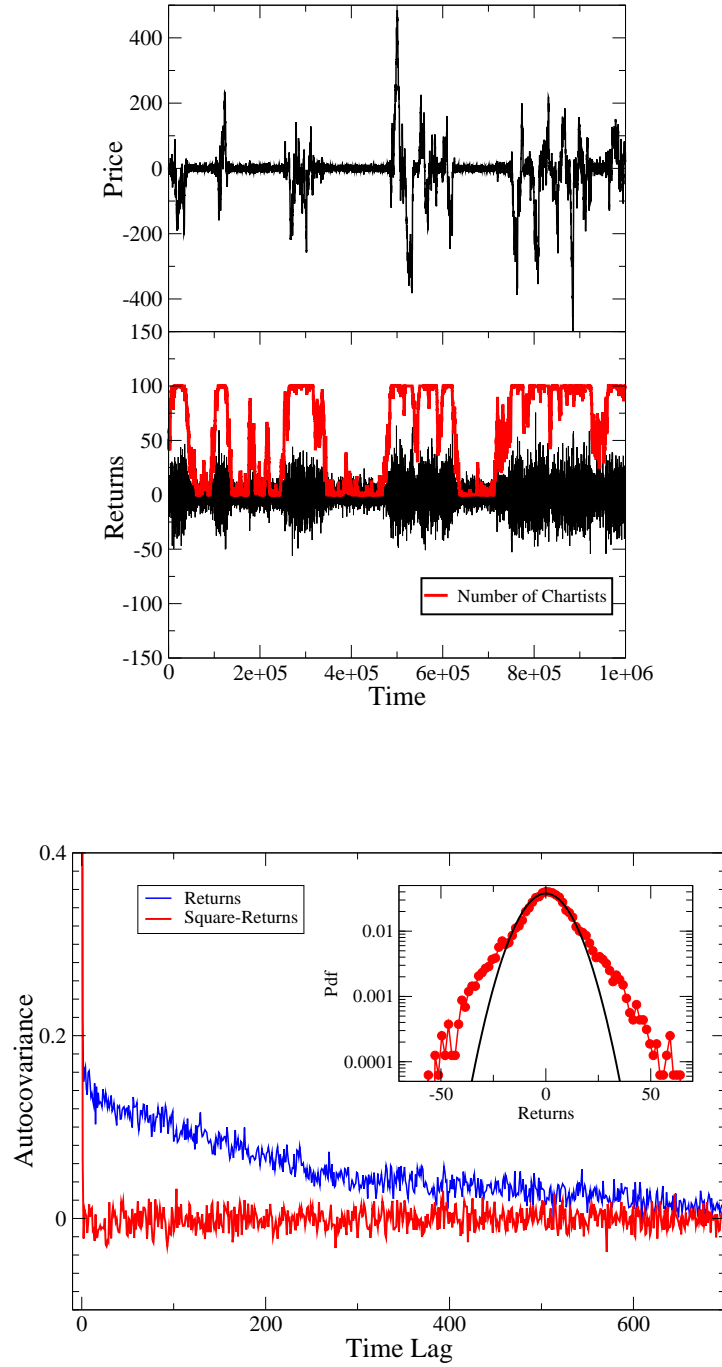


Figure 6: We can adjust the parameters of the model in such a way that the SF appear for any preassigned number of agents. The present case refers to $N = 100$. Also in this case we have omitted the exponential term of the price behavior in the probability to change strategy.

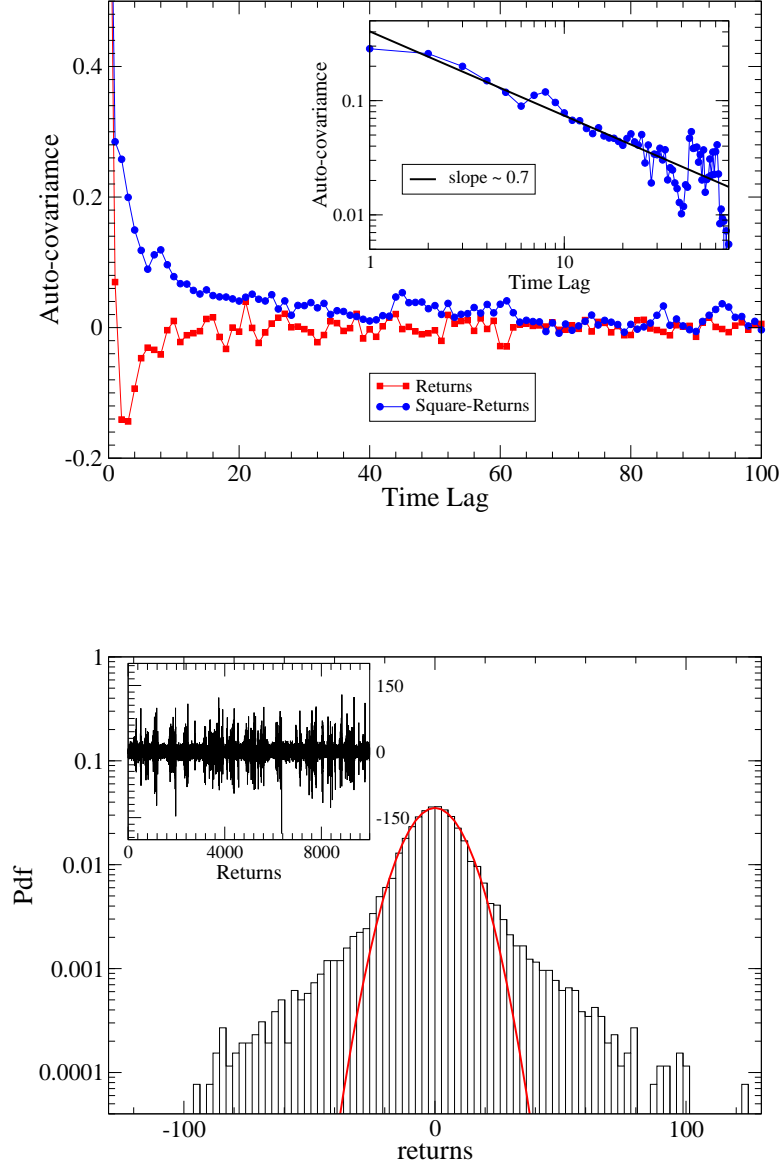


Figure 7: Here we show the results for the most complete case we consider and in this case we adopt $N = 500$. This corresponds to including the exponential price term and introducing a real heterogeneity in the agents respect to their time horizons and impact on the market (see text discussion for details). We can see that the heterogeneity suppresses somewhat the volatility clustering because a lack of coherence between the agents behavior but the shape of the volatility time correlation (see insert) in this case resembles power law. The exponential price term increases the size of the fat tails as shown in the lower figure.

5 Microscopic origin of the Stylized Facts

As pointed out before, the intermittent behavior that characterizes the oscillations of agents' strategies is crucial to generate the Stylized Facts. One must choose a certain temporal window *and* a correct value of N to obtain the "right" amount of fluctuations (the other parameters being fixed). In fact, while a low value of N produces too many fluctuations, a high value of N will prevent the formation of the chartists bubble at all. Therefore the Stylized Facts in our model can be regarded as finite size effects, that is, they disappear in the thermodynamic limit. Given the general nature of these results we may conjecture that it should apply to a broad class of agents model and it is not a special property of our specific model.

This has important consequences for both data analysis and for the structure of the models proposed to investigate the origin of Stylized Facts. The fact that the exponents of the power laws characterizing financial markets don't seem to be universal [2] can be easily explained in this framework. The apparent power laws observed in many data find a natural explanation in the presence of a distribution of agents' strategies in terms of their time horizons and strength. Concerning universality, even if this is not strictly present in this model, it is reasonable to expect a certain similarity in all markets. This is due to the fact that the key elements to generate the SF are of very general nature as we are going to explain in the following discussion.

The simplicity of the model permits to interpret the origin of Stylized Facts directly from the agents' strategies in the market. To this purpose it will be useful to define the concept of *effective action*. In our model agents not only decide between selling and buying, they operate in the market proportionally to the *signal* $p - p_f$ or $p - p_M$, selling or buying quantities which are proportional to this signal. Besides this, while in Statistical Mechanics the study of the dynamics of a model is usually done with a fixed value of N (and eventually infinite), nothing in financial markets permits us to justify this assumption. An agent can either enter or exit from the market on the basis of various considerations, and in addition she can vary the volume of the exchanges in the market. In summary, we must consider the effective action present in the market as the sum of these two effects, the fact that an agent can operate with different volumes and the fact that agents can enter or exit from the market.

Let us now suppose that for some reason at time t there is a price fluctuation (in any direction) Δp . Following our line of reasoning, this will produce an increase of the effective action, because both fundamentalists and chartists will see a signal in the market, and this action will produce more price fluctuations, and so on. So action leads to more action. On the contrary, if the market doesn't show opportunities to be profitable, agents will be discouraged from operating and so periods of low fluctuations will be followed by periods of low fluctuations. This mechanism resembles the GARCH process, in which the volatility at time $t + 1$ depends on the volatility at time t and also on the return at time t [26]:

$$\sigma(t + 1) = f(\sigma(t); \Delta p(t)) \quad (22)$$

We recall that the GARCH process was originally proposed as a phenomenological scheme to reproduce the phenomena of volatility clustering and fat tails. In our case we propose a microscopic interpretation of something similar to the GARCH process which, however, is now related to the specific agents dynamics.

On the other hand our model doesn't show any appreciable linear correlation between price increments, even if the arbitrage condition is not explicitly implemented in the model. This can be understood in the following way. Consider a price increment (with sign) Δp at time t . The next price increment will depend not only from the previous fluctuation, as it is for volatility, but also on all the other variables of the system, like the number of chartists, the specific values of p_M and p_f and so on. Schematically we can write:

$$\Delta p(t+1) = f(\sigma; N_c; N_f; p_M; p_f). \quad (23)$$

All these additional variables are in general not correlated in a direct way with the price fluctuations, so they lead to a decorrelation of the price increments, as is, instead, in the case of volatility.

This line of reasoning explains qualitatively the presence of volatility clustering and the absence of linear correlations in financial markets. We believe that these considerations are of general nature and therefore they should be valid for all markets and models. Within our class of models, however, this general behavior does not reach the status of *universal behavior* in the sense of Statistical Physics.

6 Self Organized Intermittency

In the previous section we have seen that our model is able to generate the SF of financial markets and it is possible to control their origin and nature in great detail. However, in our model, as in most of the models in the literature, these SF occur only in a very specific and limited region of the models' parameters. This is a problem which is seldom discussed in the literature but in our opinion poses a very basic question: why the market dynamics evolves spontaneously, or self-organizes, in the specific region of parameters which corresponds to the SF? In this section we are going to discuss in detail a possible answer to this fundamental question.

In usual Critical Phenomena of Statistical Physics there is a basic difference between the parameters of the model (usually called coupling constants) and the number N of elements considered. Models with different coupling constants, but belonging to the same universality class, evolve towards the same critical properties in the asymptotic limit of very large N and very large time. This situation is called universality and it is often present in equilibrium critical phenomena in which the critical region requires an external fine tuning of various parameters. This is a typical situation of competition between order and disorder which occurs at the critical point.

Self-organization instead occurs in a vast class of models characterized by a non linear dissipative dynamics far from equilibrium. Popular examples of these self organized models are the fractal growth models like of Diffusion Limited Aggregation (DLA), the Dielectric Breakdown Model (DBM) and the Sandpile Model [27, 28, 29, 30]. In these models the nonlinear dynamics drives spontaneously system towards a critical state. This can occur from a variety of initial configurations and parameters and define a state that is always the same, the critical one. For these reasons this phenomenon has been named Self-Organized Criticality (SOC) [31]. It is important to note that this self-organization is also an asymptotic phenomenon, in the sense that it occurs in the limit of large N and large time.

For financial markets the possible analogy with SOC phenomena is very tempting. However we are going to see that there are basic fundamental differences with the above concepts which require the developments of a different theoretical framework. In this section we are going to introduce the first step along this line.

A very important observation is the fact that the SF appear only for a specific value of the number of agents N . This result may appear as problematic in view of the above discussion about self-organization. A finite value of N cannot lead to universality and so the presence of the SF in virtually all markets (with very different number of agents) appears rather mysterious. In this perspective a situation in which the SF are generated only in the limit of large N would have appear more natural. Also our model shows clearly that the intermittent behavior corresponding to the SF must necessary occur for a finite value of N . It is easy to realise that this conclusion has a general nature with respect to the four essential ingredients of our class of models. An additional problem, in trying to explain the self-organization of the SF, comes from the fact, that in addition to N , the model contains several other parameters. So, even if one would be able to produce the SF in the large N limit, the self organization with respect to the other parameters would remain unclear.

From our studies we are going to propose a conceptually different mechanism of the self-organization phenomenon. Consider a certain market with its characteristic parameters. These values will govern the rate equation of the agents dynamics as described by Eqs. 4 and 5 and in Fig. 4. For any set of these parameters there will be a characteristic value of N which separates the regions with large fluctuations with the ones with small fluctuations, as shown for example in Fig. 4. The SF appears precisely around this finite value of N , which corresponds to the right amount of intermittency. So our basic question about the SO is now transformed in a question related to the number of agents. Why the system should have precisely this number of agents acting on the market? This question can now be answered in the following way. We have seen that if N is very large the system gets locked in the fundamentalist state and this leads to a very stable dynamics of the price. Such a situation will produce very small signals for the agent strategies. If we assume that an agent operates in a certain market only if her signal is larger than a minimum threshold, it is easy to realise that in the stable price situation, corresponding to large N , the number of active agents will decrease. On the other hand in the case of very few agents, the price dynamics shows very high fluctuations [12] and this produces large signals for the agents and will attract more agents in the market. Mathematically this concept can be implemented by introducing a threshold to the price fluctuations which corresponds to the decision of agents to be active or not active in the particular market. We have introduced this threshold in our model and a typical result is shown in Fig. 8. We can see that the variable number of agents evolves spontaneously towards the characteristic value of N corresponding to the intermittent state and the SF. Since this state is not precisely critical in the sense of Statistical Physics we propose to call this phenomenon *Self Organized intermittency* (SOI). This new vision has important implications in various directions. On one hand it seems to be a reasonably natural and robust explanation of the self-organization towards the quasi-critical state, on the other hand the quasi-critical state leading to the SF is unavoidably linked to finite size effect. A first consequence is that in the limit of large N , or in the limit of large time, the SF disappear. A fact which seems to be indeed reproduced by price time series [2]. Another consequence is that genuine critical exponents and universality can not be expected in this framework. The fact that several datasets can be fitted by power laws is however not surprising. Since different agents may have different time horizons in their trading strategies, one has a superposition of finite size effects which may appear as a power law in some range. This perspective implies that the discrepancy in the effective exponents for different dataset and for different time scales [2] may be a genuine intrinsic property instead of spurious effect due to the incompleteness of the data. A key element of the present scheme is the nonstationarity of the system with respect to the value of N which represent a major departure from usual critical phenomena. On the other hand this non stationarity appears to be a very important element in real financial markets.

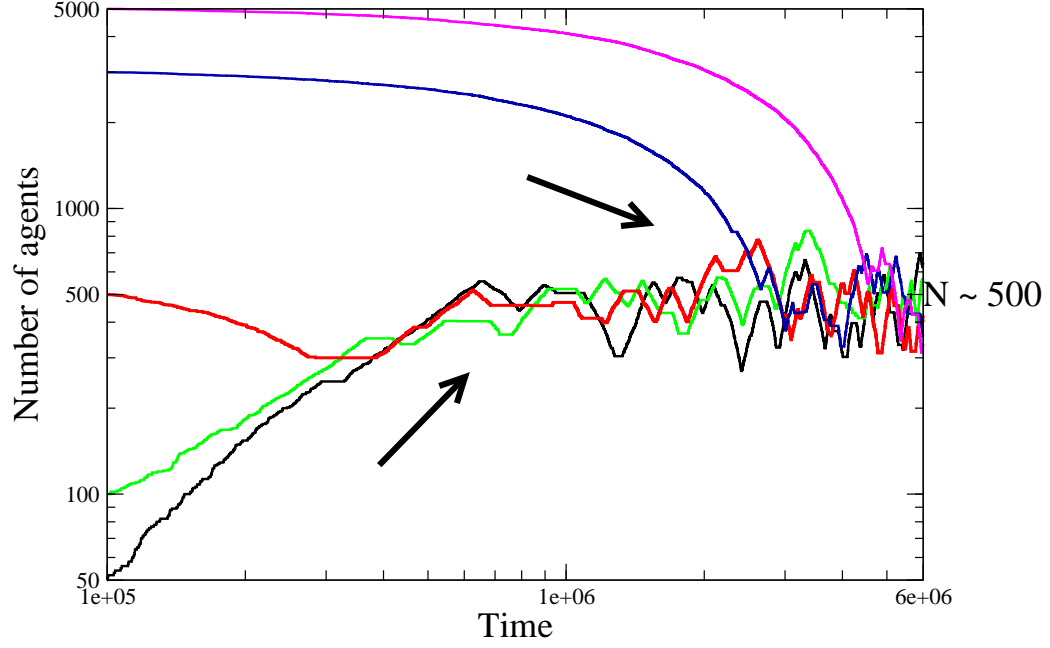


Figure 8: Self-Organized Internittency. We can observed the self-organized nature of the dynamics toward to the quasi-critical state corresponding to existence of Stylised Facts. We show five cases, characterized by different initial conditions, $N(t = 0) = 5000, 3000, 500, 100$ and 50 . If the number of the agents in the market is too large ($N = 5000, 3000$) the price will be stable and essentially driven by fundamentalists. In this case the market is not profitable and agents will not operate. So N will decrease. On the other hand a too small number of agents ($N = 50, 100$) will lead to large fluctuations and will attract more agents, increasing the value of N . In all cases the system converges to an intermediate situation ($N \approx 500$ in this case) which corresponds to the intermittent behavior which leads to Stylized Facts. Note that this regime is not real critical and universal in the sense of Statistical Physics.

7 Conclusions

Our starting point has been a detailed analysis of the LM model first in terms of the role played by the total number of players N and also on the study of the stability of the SF with respect to the variation of the other parameters. A particular puzzling point was the fact that the SF are present only for a finite value of N ($N = 500$) in the LM model) and not in the asymptotic limit $N \rightarrow \infty$. Starting from this consideration our aim has been to introduce a minimal ABM based on F and C which would permit to clarify these points and eventually also to discuss the self-organization. A basic technical simplification is the description of the chartists in terms of the newly introduced effective potential model. This and other similar simplifications permit us to analyze in detail the mathematical properties of the model. In order to achieve the deepest understanding also analytically of the dynamics we have focused for the moment on the linear returns. In paper II [15] we are going to consider also the extension to logarithmic returns but in the limit of small fluctuations the two approach essentially coincide. The main results of our model are:

- Detailed understanding of the origin of the SF with respect to the microscopic dynamics of the agents.
- Demonstration that in this class of model the SF correspond to finite size effects and not to universal critical exponents. This finite size effect, however, can be active at different time scales.
- Bubbles of chartists can be triggered spontaneously by a multiplicative cascade which can originate from tiny random fluctuations. This situation resembles in part the avalanches of the Sandpile Model in Statistical Physics [31].
- We have shown the importance of the non stationarity in the dynamics of the number of active agents N and we introduced a characteristic threshold to decide when an agent can enter or exit from the market.
- This threshold and the relative non stationarity are proposed to represent the key element in the self-organization mechanism. This self organization, however, leads to an intermittency related to finite size effects. For this reasons we define it as Self-Organized-Intermittency (SOI).

Starting from the minimal model introduced in this paper and considering that one can obtain a detailed microscopic understanding of its dynamics, it is easy to identify a number of realistic variances which can be introduced as generalizations of the model. In future works we will consider this variantes. However, the present model was aimed at a different target. The idea was to define the minimal set of elements which could lead to the SF and to the SO phenomenon.

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