Long range dependence in financial markets

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Summary. The notions of self-similarity, scaling, fractional processes and long range dependence have been repeatedly used to describe properties of financial time series: stock prices, foreign exchange rates, market indices and commodity prices. We discuss the relevance of these concepts in the context of financial modelling, their relation with the basic principles of financial theory and possible economic explanations for their presence in financial time series.

1 Introduction

The study of statistical properties of financial time series has revealed a wealth of interesting stylized facts which seem to be common to a wide variety of markets, instruments and periods [15, 21, 30, 58]:

- **Excess volatility:** many empirical studies point out to the fact that it is difficult to justify the observed level of variability in asset returns by variations in "fundamental" economic variables. In particular, the occurrence of large (negative or positive) returns is not always explainable by the arrival of new information on the market [18].
- **Heavy tails:** the (unconditional) distribution of returns displays a heavy tail with positive excess kurtosis.
- **Absence of autocorrelations in returns:** (linear) autocorrelations of asset returns are often insignificant, except for very small intraday time scales ($\simeq$ 20 minutes) where microstructure effects come into play.
- **Volatility clustering:** as noted by Mandelbrot [48], "large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes." A quantitative manifestation of this fact is that, while returns themselves are uncorrelated, absolute returns $|r_t|$ or their squares display a positive, significant and slowly decaying autocorrelation function: $\text{corr}(|r_t|, |r_{t+\tau}|) > 0$ for τ ranging from a few minutes to a several weeks.

- **Volume/volatility correlation:** trading volume is positively correlated with market volatility. Moreover, trading volume and volatility show the same type of "long memory" behavior [43].

The dependence properties of asset returns and the phenomenon of volatility clustering have especially intrigued many researchers and oriented in a major way the development of stochastic models in finance –GARCH models and stochastic volatility models are intended primarily to model this phenomenon. Also, it has inspired much debate as to whether there is long-range dependence in volatility.

Since the 1990s we have witnessed a surge of interest in this topic with the availability of new sources of financial data. A large number of empirical studies on asset prices have investigated long range dependence properties of asset returns. The concepts of self-similarity, scaling, fractional processes and long range dependence have been repeatedly used to describe properties of financial time series such as stock prices, foreign exchange rates, market indices and commodity prices.

While there is a vast literature on long range dependence in asset prices, most authors tackle the questions either from a purely theoretical perspective or from a purely empirical one, rarely both. We will attempt to discuss the relevance of these notions in the context of financial modelling both at a conceptual level, in relation with the basic principles of financial theory, and at an empirical level, by comparing them to properties of market data. Finally, we will briefly discuss some possible economic explanations for the presence of such properties in financial time series.

2 Dependence properties of financial time series

Denote by S_t the price of a financial asset — a stock, an exchange rate or a market index — and $X_t = \ln S_t$ its logarithm. Given a *time scale* Δ, the log return at scale Δ is defined as:

$$r_t = X_{t+\Delta} - X_t = \ln(\frac{S_{t+\Delta}}{S_t}). \tag{1}$$

Δ may vary between a minute (or even seconds) for tick data to several days. Observations are sampled at discrete times $t_n = n\Delta$. Time lags will be denoted by the Greek letter τ; typically, τ will be a multiple of Δ in estimations. For example, if Δ =1 day, $\mathrm{corr}[r_{t+\tau}, r_t]$ denotes the correlation between the daily return at period t and the daily return τ periods later.

2.1 Empirical behavior of autocorrelation functions

A typical display of daily log-returns is shown in figure 1: the volatility clustering feature is seen graphically from the presence of sustained periods of high

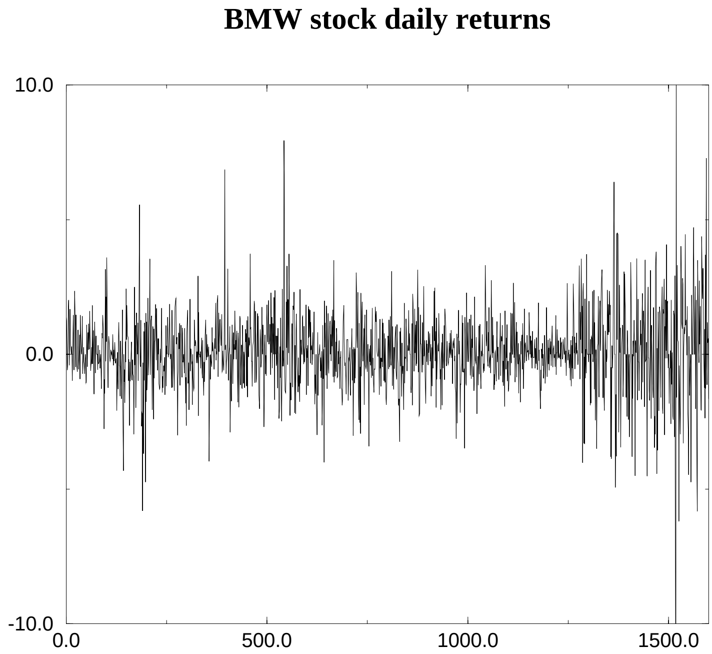

Fig. 1. Large changes cluster together: BMW daily log-returns. $\Delta = 1$ day.

or low volatility. As noted above, the autocorrelation of returns is typically insignificant at lags between a few minutes and a month. An example is shown in figure 2 (left). This "spectral whiteness" of returns can be attributed to the activity of arbitrageurs who exploit linear correlations in returns via trend following strategies [49]. By contrast, the autocorrelation function of absolute returns remains positive over lags of several weeks and decays slowly to zero: figure 2 (right) shows this decay for SLM stock (NYSE). This observation is remarkably stable across asset classes and time periods and is regarded as a typical manifestation of volatility clustering [11, 16, 21, 30]. Similar behavior is observed for the autocorrelation of squared returns [11] and more generally for $|r_t|^\alpha$ [21, 22, 16] but it seems to be most significant for $\alpha = 1$ i.e. absolute returns [21].

GARCH models [11, 24] were among the first models to take into account the volatility clustering phenomenon. In a GARCH(1,1) model the (squared) volatility depends on last periods volatility:

$$r_t = \sigma_t \mathrm{var}\epsilon_t \quad \sigma_t^2 = a_0 + a\sigma_{t-1}^2 + b\mathrm{var}\epsilon_t^2 \qquad 0 < a + b < 1 \tag{2}$$

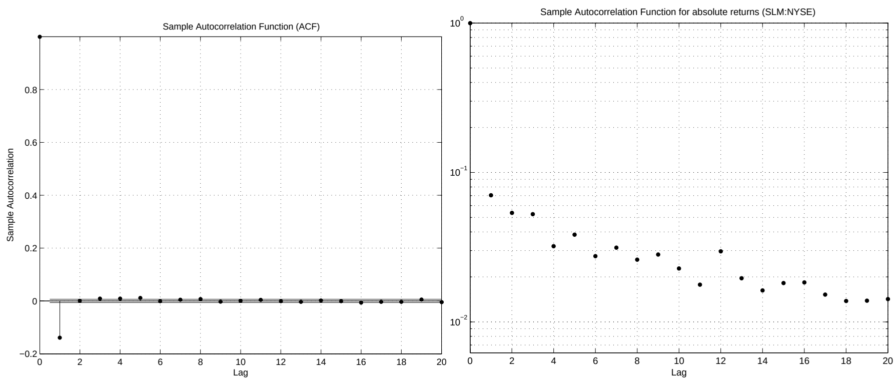

Fig. 2. SLM stock, NYSE, $\Delta = 5$ minutes. Left: autocorrelation function of log-returns. Right: autocorrelation of absolute log-returns.

leading to positive autocorrelation in the volatility process σ_t, with a rate of decay governed by $a+b$: the closer $a+b$ is to 1, the slower the decay of the autocorrelation of σ_t. The constraint $a+b<1$ allows for the existence of a stationary solution, while the upper limit $a+b=1$ corresponds to the case of an integrated process. Estimations of GARCH(1,1) on stock and index returns usually yield $a+b$ very close to 1 [11]. For this reason the volatility clustering phenomenon is sometimes called a "GARCH effect"; one should keep in mind however that volatility clustering is a "non-parametric" property and is not intrinsically linked to a GARCH specification.

While GARCH models give rise to exponential decay in autocorrelations of absolute or squared returns, the empirical autocorrelations are similar to a power law [16, 30]:

$$C_{|r|}(\tau) = \mathrm{corr}(|r_t|, |r_{t+\tau}|) \simeq \frac{c}{\tau^\beta}$$

with an exponent $\beta \leq 0.5$ [16, 5], which suggests the presence of "long-range" dependence in amplitudes of returns, discussed below.

2.2 Long range dependence

Let us recall briefly the commonly used definitions of long range dependence, based on the autocorrelation function of a process:

Definition 1 (Long range dependence). *A stationary process Y_t (with finite variance) is said to have* long range dependence *if its autocorrelation function $C(\tau) = \mathrm{corr}(Y_t, Y_{t+\tau})$ decays as a power of the lag τ:*

$$C(\tau) = \mathrm{corr}(Y_t, Y_{t+\tau}) \underset{\tau\to\infty}{\sim} \frac{L(\tau)}{\tau^{1-2d}} \qquad 0 < d < \frac{1}{2} \tag{3}$$

where L is slowly varying at infinity, i.e. verifies $\forall a > 0, \frac{L(at)}{L(t)} \to 1$ *as* $t \to \infty$*.*

By contrast, one speaks of "short range dependence" if the autocorrelation function decreases at a geometric rate:

$$\exists K > 0, c \in]0,1[, \quad |C(\tau)| \leq K c^{\tau} \tag{4}$$

Obviously, (3) and (4) are not the only possibilities for the behavior of the autocorrelation function at large lags: there are many other possible decays rates, intermediate between a power decay and a geometric decay. However, it is noteworthy that in all stochastic models used in the financial modeling literature, the behavior of returns and their absolute values fall within one of the two categories.

Although there had been considerable development of statistical methods for processes with long-range dependence in the physical sciences, especially hydrology and agronomy, it was Granger [29] in 1966 who alerted the econometrics community to the ubiquity of time series with preponderance of spectral power near the origin, referring to this property as determining "the typical spectral shape of an economic variable".

2.3 Long range dependence and self-similarity

The long range dependence property (3) hinges upon the behavior of the autocorrelation function at *large* lags, a quantity which may be difficult to estimate empirically [9]. For this reason, models with long-range dependence are often formulated in terms of self-similar processes, which allow to extrapolate across time scales and deduce long time behavior from short time behavior, which is more readily observed. A stochastic process $(X_t)_{t\geq 0}$ is said to be self-similar if there exists $H > 0$ such that for any scaling factor $c > 0$, the processes $(X_{ct})_{t\geq 0}$ and $(c^H X_t)_{t\geq 0}$ have the same law:

$$(X_{ct})_{t\geq 0} \stackrel{d}{=} (c^H X_t)_{t\geq 0}. \tag{5}$$

H is called the self-similarity exponent of the process X. Note that a self-similar process cannot be stationary, so the above definition of long-range dependence cannot hold for a self-similar process, but eventually for its increments (if they are stationary).

In 1968 Mandelbrot and Van Ness [53] provided the connection between self-similar processes and long-range dependence in stationary time series via fractional Gaussian noise, and produced its spectral density $f(\lambda) \sim c_H |\lambda|^{1-2H} (\frac{1}{2} < H < 1)$ with an integrable pole at the origin, leading to the notion of "$1/f$-noise". Fractional Brownian motion is a typical example of self-similar process whose increments exhibit long range dependence: a fractional Brownian motion with self-similarity exponent $H \in]0,1[$ is a real centered Gaussian process with stationary increments $(B_t^H)_{t\geq 0}$ with covariance function:

$$\operatorname{cov}(B_t^H, B_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}). \tag{6}$$

For $H = 1/2$ we recover Brownian motion. For $H \neq 1/2$, the covariance of the increments decays very slowly, as a power of the lag; for $H > 1/2$ this leads to long-range dependence in the increments [53, 64].

But self-similarity does not imply long-range dependence in any way: α-stable Lévy processes provide examples of self-similar processes with *independent* increments. Nor is self-similarity implied by long range dependence: Cheridito [13] gives several examples of Gaussian processes with the same long range dependence features as fractional Brownian noise but with no self-similarity (thus very different "short range" properties and sample path behavior).

Comparing fractional Brownian motions and α-stable Lévy processes shows that self-similarity can have very different origins: it can arise from high variability, in situations where increments are independent and heavy-tailed (stable Lévy processes) or it can arise from *strong dependence* between increments even in absence of high variability, as illustrated by the example of fractional Brownian motion. These two mechanisms for self-similarity have been called the "Noah effect" and the "Joseph effect" by Mandelbrot [50]. By mixing these effects, one can construct self-similar processes where both long range dependence and heavy tails are present: fractional stable processes [64, 3] offer such examples.

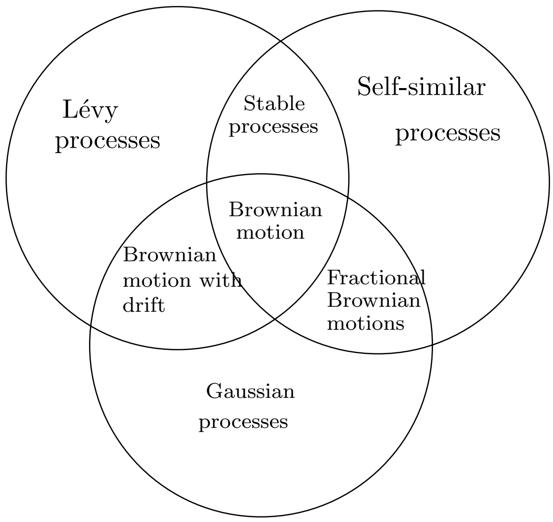

Fig. 3. Self-similar processes and their relation to Lévy processes and Gaussian processes.

2.4 Are stock prices self-similar?

As noted above, the example of fractional Brownian motion is thus misleading in this regard, since it conveys the idea that these two properties are associated. When testing for long range dependence in a model based on fractional Brownian motion, we thus test the joint hypothesis of self-similarity *and* long-range dependence and strict self-similarity is not observed to hold in asset returns [15, 16]. One should therefore distinguish general tests for self-similarity from tests of particular parametric models (such as α-stable Lévy processes or fractional Brownian motions).

A consequence of selfsimilarity is that for any $c, t > 0$, X_{ct} and $c^H X_t$ have the same distribution. Choosing $c = 1/t$ yields

$$\forall t > 0, \quad X_t \stackrel{d}{=} t^H X_1, \tag{7}$$

so the distribution of X_t, for any t, is completely determined by the distribution of X_1:

$$F_t(x) = \mathbb{P}(t^H X_1 \leq x) = F_1(\frac{x}{t^H}). \tag{8}$$

In particular if the tail of F_1 decays as a power of x, then the tail of F_t decays in the same way:

$$\mathbb{P}(X_1 \geq x) \underset{x\to\infty}{\sim} \frac{C}{x^\alpha} \Rightarrow \ [\forall t > 0,\ \mathbb{P}(X_1 \geq x) \underset{x\to\infty}{\sim} C\frac{t^{\alpha H}}{x^\alpha} = \frac{C(t)}{x^\alpha}]. \tag{9}$$

If F_t has a density ρ_t we obtain, by differentiating (8), the following relation for the densities:

$$\rho_t(x) = \frac{1}{t^H}\rho_1\left(\frac{x}{t^H}\right). \tag{10}$$

Substituting $x = 0$ in (10) yields the following scaling relation:

$$\forall t > 0,\ \rho_t(0) = \frac{\rho_1(0)}{t^H}. \tag{11}$$

Let us now consider the moments of X_t. From (7) it is obvious that $E[|X_t|^k] < \infty$ if and only if $E[|X_1|^k] < \infty$ in which case

$$E[X_t] = t^H E[X_1], \quad \mathrm{var}(X_t) = t^{2H}\mathrm{var}(X_1), \tag{12}$$
$$E[|X_t|^k] = t^{kH} E[|X_1|^k]. \tag{13}$$

Assume that the log-price $X_t = \ln S_t$ is a process with *stationary increments*. Since $X_{t+\Delta} - X_t$ has the same law as X_Δ, the density and moments of X_Δ can be estimated from a sample of increments.

The relation (11) has been used by several authors to test for self-similarity and estimate H from the behavior of the density of returns at zero: first one estimates $\rho_t(0)$ using the empirical histogram or a kernel estimator and then obtains an estimate of H as the regression coefficient of $\ln \rho_t(0)$ on $\ln t$:

$$\ln \hat{\rho}_t(0) = H \ln \frac{t}{\Delta} + \ln \hat{\rho}_{\Delta}(0) + \epsilon. \tag{14}$$

Applying this method to S&P 500 returns, Mantegna and Stanley [54] obtained $H \simeq 0.55$ and concluded towards evidence for an α-stable model with $\alpha = 1/H \simeq 1.75$. However, the scaling relation (11) holds for any self-similar process with exponent H and does not imply in any way that the process is a (stable) Lévy process. For example, (11) also holds for a fractional Brownian motion with exponent H — a Gaussian process with correlated increments having long range dependence! Scaling behavior of $\rho_t(0)$ is simply a necessary but not a sufficient condition for self-similarity: even if (11) is verified, one cannot conclude that the data generating process is self-similar and even less that it is an α-stable process.

Another method which has often been used in the empirical literature to test self-similarity is the "curve collapsing" method: one compares the aggregation properties of empirical densities with (10). Using asset prices sampled at interval Δ, one computes returns at various time horizons $n\Delta, n = 1 \ldots M$ and estimates the marginal density of these returns (via a histogram or a smooth kernel estimator). The scaling relation (10) then implies that the densities $\hat{\rho}_{n\Delta}(x)$ and $\frac{1}{n^H}\hat{\rho}_{\Delta}(\frac{x}{n^H})$ should coincide, a hypothesis which can be tested graphically and also more formally using a Kolmogorov–Smirnov test.

Although self-similarity is not limited to α-stable processes, rejecting self-similarity also leads to reject the α-stable Lévy process as a model for log-prices. If the log-price follows an α-stable Lévy process, daily, weekly and monthly returns should also be α-stable (with the same α). Empirical estimates [1, 10] show a value of α which increases with the time horizon. Finally, various estimates of tail indices for most stocks and exchange rates [1, 35, 45, 33, 44, 46] are often found to be larger than 2, which rules out infinite variance and stable distributions.

2.5 Dependence in stock returns

The volatility clustering feature indicates that asset returns are not independent across time; on the other hand the absence of linear autocorrelation shows that their dependence is nonlinear. Whether this dependence is "short range" or "long range" has been the object of many empirical studies.

The idea that stock returns could exhibit long range dependence was first suggested by Mandelbrot [49] and subsequently observed in many empirical studies using R/S analysis [52]. Such tests have been criticized by Lo [42] who pointed out that, after accounting for short range dependence, they might yield a different result and proposed a modified test statistic. Lo's statistic highly depends on the way "short range" dependence is accounted for and shows a bias towards rejecting long range dependence [66]. The final empirical conclusions are therefore less clear [67].

However, the absence of long range dependence in returns may be compatible with its presence in absolute returns or "volatility". As noted by Heyde [32], one should distinguish long range dependence in signs of increments, when $\mathrm{sign}(r_t)$ verifies (3), from long range dependence in amplitudes, when $|r_t|$ verifies (3). Asset returns do not seem to possess long range dependence in signs [32]. Many authors have thus suggested models, such as Fractionally Integrated GARCH models [6], in which returns have no autocorrelation but their amplitudes have long range dependence [5, 23].

It has been argued [39, 7] that the decay of $C_{|r|}(\tau)$ can also be reproduced by a superposition of several exponentials, indicating that the dependence is characterized by multiple time scales. In fact, an operational definition of long range dependence is that the time scale of dependence in a sample of length T is found to be of the order of T: dependence extends over the whole sample.[1] Interestingly, the largest time scale in [39] is found to be of the order of...the sample size, a prediction which would be compatible with long-range dependence!

Many of these studies test for long range dependence in returns, volatility,.. by examining sample autocorrelations, Hurst exponents etc. but if time series of asset returns indeed possess the two features of heavy tails *and* long range dependence, then many of the standard estimation procedures for these quantities may fail to work [9, 61]. For example, sample autocorrelation functions may fail to be consistent estimators of the true autocorrelation of returns in the price generating process: Resnick and van der Berg [62] give examples of such processes where sample autocorrelations converge to *random* values as sample size grows! Also, in cases where the sample ACF is consistent, its estimation error can have a heavy-tailed asymptotic distribution, leading to large errors. The situation is even worse for autocorrelations of squared returns [62]. Thus, one must be cautious in identifying behavior of *sample* autocorrelation with the autocorrelations of the return process.

Slow decay of sample autocorrelation functions may possibly arise from other mechanism than long-range dependence. For example, Mikosch & Starica [56] note that nonstationarity of the returns may also generate spurious effects which can be mistaken for long-range dependence in the volatility. However, we will not go to the extreme of suggesting, as in [56], that the slow decay of sample autocorrelations of absolute returns is a pure artefact due to non-stationarity. "Non-stationarity" does not suggest a modeling approach and it seems highly unlikely that unstructured non-stationarity would lead to such a robust, stylized behavior for the sample autocorrelations of absolute returns, stable across asset classes and time periods. The robustness of these empirical facts call for an explanation, which "non-stationarity" does not provide. Of course, these mechanisms are not mutually exclusive: a recent

[1] On this point, see also [51].

study by Granger and Hyng [27] illustrates the interplay of these two effects by combining an underlying long memory process with occasional structural breaks.

3 Fractional processes and arbitrage constraints

A fallacy often encountered in the literature is that "long range dependence in returns is incompatible with absence of arbitrage", therefore ruled out by financial theory. This idea is so widespread that it is worthwhile discussing it here.

The problem stems from the fact that fractional Brownian motions and several related fractional processes do not belong to the class of *semimartingales*. We will review this notion briefly and discuss its implications for fractional models in finance.

3.1 Stochastic integrals and trading gains

Let us consider a financial asset whose price is modeled by a stochastic process S_t defined on a probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$. If an investor trades at times $T_0 = 0 < T_1 < \cdots < T_n < T_{n+1} = T$, detaining a quantity ϕ_i of the asset during the period $]T_i, T_{i+1}]$ then the capital gain resulting from fluctuations in the market price is given by

$$\sum_{i=0}^{n} \phi_i (S_{T_{i+1}} - S_{T_i}). \tag{15}$$

This nonanticipative quantity, which represents the capital gain of the investor following the strategy ϕ, is called the stochastic integral of the process

$$\phi = \sum_{i=0}^{n} \phi_i 1_{]T_i, T_{i+1}]} \tag{16}$$

with respect to S and denoted by $\int_0^T \phi_t dS_t$. Here the trading times T_i can be nonanticipative random times –buys or sells can be triggered by recent price behavior– and ϕ_i are nonanticipative bounded random variables. ϕ is then called a *simple predictable process*: such processes are the mathematical representations of realistic trading strategies, which consist in buying and selling a finite number of times in $[0, T]$. Denote the set of simple predictable processes by $\mathbb{S}([0, T])$.

In the setting of Ito integration theory, stochastic integration is developed with respect to a class of stochastic processes known as *semimartingales*: these processes can be defined via their decomposition as a bounded variation process (signal) plus a local martingale (noise) [20] or, alternatively, as processes S for which the stochastic integral defined by (15) is continuous[2]

[2] The fact that these two definitions of semimartingales coincide is a deep result, due to Dellacherie-Mokobodski-Meyer, see [60].

[55, 60] in the following sense: for any sequence of simple predictable processes $(\phi^n) \in \mathbb{S}([0,T])$ if

$$\sup_{(t,\omega)\in[0,T]\times\Omega} |\phi^n_t(\omega) - \phi_t(\omega)| \underset{n\to\infty}{\to} 0 \text{ then } \quad \int_0^T \phi^n dS \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} \int_0^T \phi dS. \quad (17)$$

This stability property of stochastic integrals then allows to extend the set of integrands to processes ϕ which can be expressed as limits of *nonanticipative Riemann sums* –as in (15)– of simple predictable processes. This allows to consider more general strategies in a set $\mathcal{A}$ larger than $\mathbb{S}([0,T])$ and allow for "continuous trading". An important point is that the Ito integral is both nonanticipative and can be interpreted, as in (15), in terms of the gains from trading. This important remark, first pointed out in [31], isolates Ito integration theory as the appropriate one for use in financial modelling.

Fractional Brownian motion with $H \neq 1/2$ is *not* a semimartingale. This result can be shown in several ways, see [63]. The implication is that, in a model where the stock price is described by a function of a FBM, one cannot extend the construction of the gains process (15) to strategies beyond $\mathbb{S}([0,T])$ in a continuous way. Alternative constructions of the stochastic integral do exist: various extensions have been proposed which allow to construct stochastic integrals with respect to fractional Brownian motion and several authors have attempted to construct financial models using them. But it should be noted from the onset that such approaches are doomed to produce results whose financial interpretation is dubious: the *only* integral which can be interpreted in terms of the capital gain of a trading strategy is (15), and any other one will either anticipate on the future and/or not coincide with the gains of simple strategies.

However, it should be kept in mind that the set $\mathbb{S}([0,T])$ already contains any reasonable trading strategy. And, for computing the gain $\int_0^T \phi.dS$ of such strategies there is no need for S to be a semimartingale. Thus fractional processes with any sample path structure can be used as long as we limit ourselves to constructive problems (as opposed to existence theorems such as martingale representations) based on simple strategies.

In fact, $\mathbb{S}([0,T])$ already contains too many unrealistic strategies, which require to trade very frequently since the number of trades n can be arbitrarily large. One can define restricted sets where such infinitely frequent trading is excluded (see below).

3.2 Martingales, semi-martingales and arbitrage

An arbitrage strategy is defined as a strategy $\phi \in \mathcal{A}$ which realizes a possibly non-zero gain by starting from a zero initial capital:

$$\phi_0 = 0 \qquad \mathbb{P}(\int_0^T \phi_t dS_t) > 0 \quad (18)$$

Note that the definition of arbitrage depends on the set of possible strategies $\mathcal{A}$ and on the definition of the stochastic integral.

Arbitrage pricing theory is based on a fundamental result of Harrison and Pliska [31], who show a model of price evolution is arbitrage-free if and only if the (discounted) price S_t of any asset can be represented as the conditional expectation of its final value S_T with respect to some probability measure $\mathbb{Q} \sim \mathbb{P}$:

$$\exists \mathbb{Q} \sim \mathbb{P}, \qquad S_t = E^{\mathbb{Q}}[S_T|\mathcal{F}_t] \tag{19}$$

In particular, S_t is a martingale under $\mathbb{Q}$. A precise statement of this results involves the specification of the set of admissible strategies $\mathcal{A}$, which is taken to be much larger than $\mathbb{S}([0,T])$ [19], usually containing all predictable processes ϕ such that $\int_0^t \phi dS$ is bounded.

Under the (real-world) model $\mathbb{P}$, S_t is not a martingale necessarily, but it is a still a semi-martingale: this property is preserved under equivalent changes of measure [60]. Since fractional Brownian motion is not a semimartingale, a model in which the (log)-price are described by a fractional Brownian motion is not arbitrage-free, in the sense that there exists a strategy $\phi \in \mathcal{A}$ verifying (18).

But this result and the fact that fractional Brownian motions fail to be semimartingales crucially depend on the *local* behavior of its sample paths, not on its long range dependence property. Cheridito [12] and Rogers [63] give several examples of Gaussian processes with the same long range dependence features as fractional Brownian motion, but which are semimartingales and lead to arbitrage-free models. A starting point for such constructions is the moving average represenation for fractional Brownian motion:

$$B_t^H = k \int_{-\infty}^{\infty} \{((t-s)^+)^{H-1/2} - (-s)^{H-1/2}\} dW_s,$$

where W_t is a Brownian motion, $H \in (0,1)$ is the self-similarity parameter and k is a suitable normalizing constant. Rogers [63] proposes a model which has the same long range dependence properties as Brownian motion but *is* a semimartingale, in the following way:

$$X_t = \int_{-\infty}^{t} \phi(t-s) dW_s + \int_{-\infty}^{0} \phi(-s) dW_s.$$

where $\phi \in C^2(\mathbf{R}), \phi(0) = 1,\ \phi'(0) = 1$ and $\lim_{t\to\infty} \phi''(t) t^{5/2-H} \in (0,\infty)$. An example of a kernel verifying this property is

$$\phi(t) = (\epsilon + t^2)^{(2H-1)/4}$$

Also, a closer look shows that *even* fractional Brownian motion and fractional processes are not ruled out by arbitrage considerations. The fact that

FBM is not a semimartingale implied the existence of $\phi \in \mathcal{A}$ verifying (18): Rogers [63] offers examples of such strategies. However, these strategies can only be performed if it is possible to buy and sell within arbitrarily small time intervals. Arbitrage can be ruled out from fractional Brownian models by introducing a minimal amount of time h ¿ 0 that must lie between two consecutive transactions i.e. considering strategies in

$$\mathbb{S}^h([0,T]) := \{\sum_{i=0}^{n} \phi_i 1_{]T_i,T_{i+1}]} \in \mathbb{S}([0,T]), \quad \inf_i (T_{i+1} - T_i) > h\} \tag{20}$$

As shown by Cheridito [12], no arbitrage can be constructed using strategies in $\bigcap_{h>0} \mathbb{S}^h([0,T])$, i.e. no matter how frequently one trades.

Thus, the semi-martingale property is more a question of theoretical convenience, allowing not to worry constantly about the class of admissible strategies in theoretical developments, rather than a constraint on the models to be used. Finally, we have noted that long-range dependence has no relation with the semi-martingale property –which is a property of the fine structure of sample paths– and only when it is coupled to self-similarity (in the case of fractional processes) does it interfere with the semimartingale property.

But the main conclusion of this discussion is that the question of the adequacy of stochastic processes with long range dependence, and in particular models based on fractional Brownian motion, for modeling asset prices is mainly an *empirical* one: theoretical restrictions imposed by arbitrage are quite weak and cannot be used as arguments to exclude a family of stochastic processes as possible models. On the empirical side, however, there is a lot of evidence pointing to positive dependence over large time horizons in *absolute* returns [5, 15, 16, 21, 43, 57] but not in the returns themselves, showing that it is more interesting to use fractional processes as models of *volatility* rather than for modeling prices directly [14, 6, 57].

4 Economic mechanisms for long range dependence

While fractional processes may mimick volatility clustering in financial time series, they do not provide any economic explanation for it. The fact that these observations are common to a wide variety of markets and time periods [15] suggest that common mechanisms may be at work in these markets. Many attempts have been made to trace back the phenomenon of long range dependence in volatility to economic mechanisms present in the markets generating this volatility.

Independently of the econometric debate on the “true nature” of the return generating process, one can take into account such empirical observations without pinpointing a specific stochastic model by testing for similar behavior of sample autocorrelations in such economic models and using sample autocorrelations for indirect inference [26] of the parameters of such models.

4.1 Heterogeneity in time horizons of economic agents

Heterogeneity in agent's time scale has been considered as a possible origin for various stylized facts [30]. Long term investors naturally focus on long-term behavior of prices, whereas traders aim to exploit short-term fluctuations. Granger [28] suggested that long memory in economic time series can be due to the aggregation of a cross section of time series with different persistence levels. This argument was proposed by Andersen & Bollerslev [2] as a possible explanation for volatility clustering in terms of aggregation of different information flows.

The effects of the diversity in time horizons on price dynamics have also been studied by Lebaron [38] in an artificial stock market, showing that the presence of heterogeneity in horizons may lead to an increase in return variability, as well as volatility-volume relationships similar to those of actual markets.

4.2 Evolutionary models

Several studies have considered modeling financial markets by analogy with ecological systems where various trading strategies co-exist and evolve via a "natural selection" mechanism, according to their relative profitability [4, 38]. The idea of these models, the prototype of which is the Santa Fe artificial stock market [4, 40], is that a financial market can be viewed as a population of agents, identified by their (set of) decision rules. A decision rule is defined as a mapping from an agents information set (price history, trading volume, other economic indicators) to the set of actions (buy, sell, no trade). The evolution of agents decision rule is often modeled using a genetic algorithm. The specification and simulation of such evolutionary models can be quite involved and specialized simulation platforms have been developed to allow the user to specify variants of agents strategies and evolution rules. Other evolutionary models represent the evolution by a deterministic dynamical system which, through the complex price dynamics it generate, are able to mimick some "statistical" properties of the returns process, including volatility clustering [34]. However due to the complexity of the models they are not amenable to a direct comparison with financial data.

4.3 Switching between trading strategies

Another mechanism leading to long range dependence is switching of agents trading behavior between two or more strategies. The economic literature contains examples where switching of economic agents between two behavioral patterns leads to large aggregate fluctuations [36]: in the context of financial markets, these behavioral patterns can be seen as trading rules and the resulting aggregate fluctuations as large movements in the market price i.e. heavy tails in returns. Recently, models based on this idea have also been shown to generate volatility clustering [37, 47].

Lux and Marchesi [47] study an agent-based model in which heavy tails of asset returns and volatility clustering arise from behavioral switching of market participants between fundamentalist and chartist behavior. Fundamentalists expect that the price follows the fundamental value in the long run. Noise traders try to identify price trends, which results in a tendency to herding. Agents are allowed to switch between these two behaviors according to the performance of the various strategies. Noise traders evaluate their performance according to realized gains, whereas for the fundamentalists, performance is measured according to the difference between the price and the fundamental value, which represents the anticipated gain of a "convergence trade". This decision-making process is driven by an exogenous fundamental value, which follows a Gaussian random walk. Price changes are brought about by a market maker reacting to imbalances between demand and supply. Most of the time, a stable and efficient market results. However, its usual tranquil performance is interspersed by sudden transient phases of destabilization. An outbreak of volatility occurs if the fraction of agents using chartist techniques surpasses a certain threshold value, but such phases are quickly brought to an end by stabilizing tendencies. This behavioral switching is believed be the cause of volatility clustering and heavy tails in the Lux-Marchesi model [47].

Kirman and Teyssière [37] have proposed a variant of [36] in which the proportion $\alpha(t)$ of fundamentalists in the market follows a Markov chain, of the type used in epidemiological models, describing herding of opinions. Simulation of this model exhibit autocorrelation patterns in absolute returns with a behavior similar to those observed in returns.

Ghoulmie, Cont and Nadal [17] propose a model where agents compare a common information (signal) to an individual threshold, whose value is heterogeneous across agents. These thresholds are dynamically updated based on recent price volatility. It is shown in [17] that, without any chartist/fundamentalist competition nor any direct interaction between agents, this model is capable of generating volatility clustering while maintaining absence of linear correlations in returns. This model points to a link between investor inertia and volatility clustering and provide an economic explanation for the switching mechanism proposed in the econometrics literature as an origin of volatility clustering.

4.4 Investor inertia

As argued by Liu [41], though the presence of a Markovian regime switching mechanism in volatility can lead to volatility clustering, is not sufficient to generate long-range dependence in absolute returns. More important than the switching is the fact the time spent in each regime –the duration of regimes– should have a heavy-tailed distribution [59, 65]. By contrast with Markov switching, which leads to short range correlations, this mechanism has been called "renewal switching".

Bayraktar et al. [8] study a model where an order flow with random, heavy-tailed, durations between trades leads to long range dependence in returns. When the durations τ_n of the inactivity periods have a distribution of the form $\mathbb{P}(\tau_n \geq t) = t^{-\alpha}L(t)$, conditions are given under which, in the limit of a large number of agents randomly submitting orders, the price process in this models converges to a a process with Hurst exponent $H = (3-\alpha)/2 > 1/2$. In this model the randomness (and the heavy tailed nature) of the durations between trades are both exogenous ingredients, chosen in a way that generates long range dependence in the returns. However, as noted above, empirical observations point to clustering and persistence in *volatility* rather than in returns so such a result does not seem to be consistent with the stylized facts.

By contrast, as noted above, regime switching in *volatility* with heavy-tailed durations could lead to volatility clustering. Although in the agent-based models discussed above, it may not be easy to speak of well-defined "regimes" of activity, but Giardina and Bouchaud [25] argue that this is indeed the mechanism which generates volatility clustering in the Lux-Marchesi [47] and other models discussed above. In these models, agents switch between strategies based on their relative performance; Giardina and Bouchaud argue that this (cumulative) relative performance index actually behaves in time like a random walk, so the switching times can be interpreted as times when the random walk crosses zero: the interval between successive zero-crossings is then known to be heavy-tailed, with a tail exponent 3/2.

Ghoulmie, Cont and Nadal [17] show that investor inertia can also result from a threshold behavior of agents: an agent will not trade in the market unless the discrepancy between his anticipation of the value of the financial asset and the current market price reaches a certain threshold, which may be heterogeneous across agents. Figure 4 displays the evolution of the portfolio $\pi_i(t)$ of a typical agent in this model: short periods of activity (trading) are separated by long periods of inertia, where the portfolio remains constant. Such "renewal switching" between periods of high and low activity, with long durations of periods, can lead to long range dependence in volatilities [65].

5 Conclusion

Volatility clustering, manifested through slowly decaying autocorrelations for absolute returns, is a characteristic property of most time series of financial asset returns. Whether this "slow" decay corresponds to long range dependence is a difficult question subject to an ongoing statistical debate. But is definitely an *empirical* question: first principles of financial theory –such as absence of arbitrage– cannot be invoked to give any response to it.

As noted by many econometricians [67, 56], statistical analysis alone is not likely to provide a definite answer for the presence or absence of long-

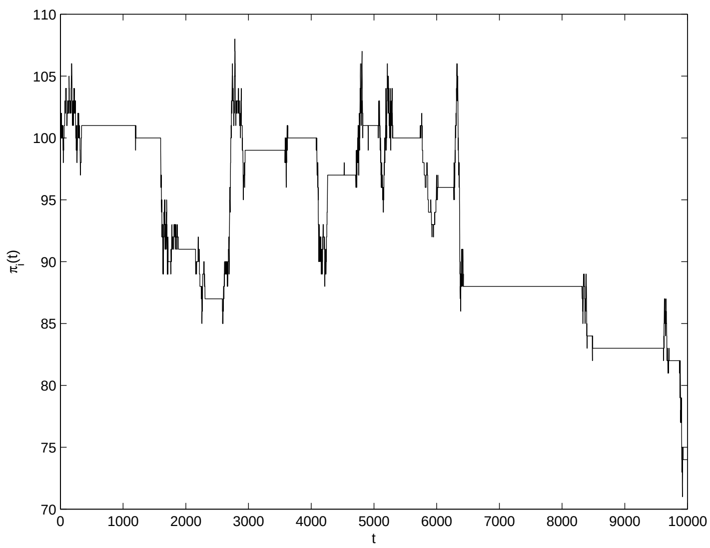

Fig. 4. Investor inertia: the evolution of the portfolio of a typical agent shows long periods of inactivity punctuated by bursts of activity.

range dependence phenomenon in stock returns or volatility, unless economic mechanisms are proposed to understand the origin of such phenomena.

Agent-based models, which seek to explain volatility clustering in terms of behavior of market participants, have been proposed in order to explain long range dependence in volatility. A common feature of these models seems to be the "switching" of the market between periods of high and low activity, with long durations of periods. As we have noted, such "renewal switching" can lead to long range dependence in volatilities if the market switches between regimes of high and low volatility. The link between such economic models and the realm of stochastic models in finance is intriguing and remains an active topic of research at the time of writing.

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