Sudoku Permutation Structure

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Abstract

To efficiently solve sudokus a lot of clever algorithms have been developed but none lead to a complete understanding of their structure. Why up to now the question of the fewest given that render a solution unique is still unsolved. In this paper I will establish an algebra to represent naturally sudokus based on permutation groups. It will as well allow to show that it needs at least 17 givens in a sudoku for having a unique solution.

Contents

1 Representing a Sudoku

A sudoku has exactly each number from 1 to 9 in each line, column and 3x3 bloc. Every possible configuration for one number can be generated as follows by the use of bloc permutation matrices.

$$C = \left(\begin{array}{ccc@{\quad}ccc@{\quad}ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \tag{1}$$

$$B = \begin{pmatrix} b^1 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & b^3 \end{pmatrix} \; b^i \in S_3 \qquad Sudoku = \sum_{j=1}^{9} j\, B_L^j C B_C^j \tag{2}$$

The central matrix C has the desired properties which are maintained under permutation of the lines and columns inside a bloc. It follows that to generate all possible configuration for one number it is sufficient to permute lines and columns by bloc permutation matrices $B_L$and B_C.

To obtain an entire sudoku we sum up over nine independent such configurations. Unfortunately this generates a lot of impossible sudokus because there is no guaranty that every location in a bloc is occupied once. This will be implemented by additional constrains in the next sections.

2 Sudoku Algebra

This part is devoted to work out the algebra of the bloc permutation matrices sets covering each entry exactly once.

2.1 Possible Sets of Permutations

Now we will analyze the conditions necessary to cover each spot in a bloc exactly once. We start by writing down all the permutations of S_3 and pair them up with respect to their action.

$$S_3 = \left\{ \begin{array}{c} a = 123 \\ b = 132 \\ c = 312 \\ d = 213 \\ e = 231 \\ f = 321 \end{array} \right\} \qquad \begin{array}{cccc} \Rsh & 1 & 2 & 3 \\ 1 & a|b & c|d & e|f \\ 2 & e|d & a|f & c|b \\ 3 & c|f & e|b & a|d \end{array} \tag{3}$$

The table lists all the permutations which permute the number to the left into the location on top. Any set of nine permutations blocs $B^i = \left\{ b^{ij} \mid \forall 1 \leq j \leq 9 \right\}$ (for both lines and columns) needs to send exactly each element three times onto each location. This yields following overdetermined set of equations which have exactly four solutions.

$$\begin{array}{lll} N_a + N_b = 3 & N_e + N_d = 3 & N_c + N_f = 3 \\ N_c + N_d = 3 & N_a + N_f = 3 & N_e + N_b = 3 \\ N_e + N_f = 3 & N_c + N_b = 3 & N_a + N_d = 3 \end{array} \qquad \begin{array}{c} N_a = N_c = N_e \\ N_b = N_d = N_f = 3 - N_a \end{array} \tag{4}$$

This allows to characterize each of the 3 sets B_L^i and $B_C^i$by a number N_L^i, N_C^i between 0 or 3.

2.2 Ordering the Sets into Similar or Complementary Subsets

Under similar we understand a subset of three permutations acting identically on an element. The 4 possible sets established before are split into two types, those which by construction are similar for all 3 elements (type I) and those which are not (type II).

$$I \left\{ \begin{array}{llll} E^3 = & aaa & ccc & eee \\ E^0 = & bbb & ddd & fff \end{array} \right. \qquad II \left\{ \begin{array}{lllll} E^1 = & abb & cdd & eff & Similar\ on\ 1 \\ & cbb & edd & aff & Similar\ on\ 2 \\ & ebb & add & cff & Similar\ on\ 3 \\ & & & & \\ E^2 = & baa & fee & dcc & Similar\ on\ 1 \\ & faa & dee & bcc & Similar\ on\ 2 \\ & daa & bee & fcc & Similar\ on\ 3 \end{array} \right. \tag{5}$$

For the type II sets the 3 orderings are related in a cyclic way by the permutation $T = (174)$ as shown below. For type I sets we define T as being the identity.

$$TE^1 = (174)\{abb\,cdd\,eff\} = \{cbb\,edd\,aff\} \tag{6}$$

By complementary is meant a subset sending an element on each of the three possible locations. They can be obtained by permuting a similar ordering (for both types) with $U = (258)(396)$ as show below. We notice that T and U are independent permutations.

$$UE^3 = (258)(396)\{aaa\,ccc\,eee\} = \{aec\,cae\,eca\} \tag{7}$$

2.3 Constructing fully covered blocs

A bloc in the C matrix is described by the position (l, c) of its non null entry $(l, c \in \{0, 1, 2\})$. Lets determine the possible permutations of $E_l^L$and E_c^C, the sets of permutations acting on the lines respectively columns $(E_l^L, E_c^C \in \{E^0, E^1, E^2, E^3\})$, to obtain a full covering of the bloc. The pairing up of E_l^L ordered in a similar way on l and E_c^C ordered in a complementary way on c gives by construction the desired result. We need to introduce two additional permutations to shuffle the elements inside the subsets and one permutation to shuffle the obtained pairs.

$$P_l \left(L_l T^c_{L_l} E_l^L + U C_c T^l_{C_c} E_c^C \right) \qquad L_l, C_c \in S_{3x3} \quad P_l \in S_9 \tag{8}$$

Were the $+$ operation has to be understand as a pairing up operation as for example $\{a, b\} + \{c, d\} = \{(a, c), (b, d)\}$and l, c are not indices on the matrices but exponents when written above. Note that applying a permutation on one side of this sum operation is equivalent to apply the inverse of the permutation on the other side as shown below.

$$\{a, b, c\} + (123)\{d, e, f\} = \{(a, f), (b, d), (c, e)\} \equiv (123)^{-1}\{a, b, c\} + \{d, e, f\} \tag{9}$$

Of course in a sudoku for each of the 6 sets (3 for the bloc lines and 3 for the bloc colums) only one permutation is allowed and we have now to figure out those consistent for all 9 blocs.

2.4 Consistency Equations

The set of equations obtained for two bloc lines and three bloc colums is sufficient to determine the entire sudoku because the result can be applied to any pair of bloc lines by permutation of those. Moving all the permutations in the equation 8 on the colums using 9 and asking that those are the same for the two lines we obtain following system.

$$
\begin{aligned}
L_0 U C_0 &= P_1 L_1 U C_0 T_{C_0} \\
T_{L_0} L_0 U C_1 &= P_1 T_{L_1} L_1 U C_1 T_{C_1} \\
T_{L_0}^2 L_0 U C_2 &= P_1 T_{L_1}^2 L_1 U C_2 T_{C_2}
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
P_1 &= L_1 U C_0 T_{C_0} C_0^{-1} U^{-1} L_0^{-1} \\
(C_0 T_{C_0} C_0^{-1})(U L_0 T_{L_0} L_0^{-1} U^{-1}) &= (U L_1 T_{L_1} L_1^{-1} U^{-1})(C_1 T_{C_1} C_1^{-1}) \\
(C_2 T_{C_2} C_2^{-1})(U L_0 T_{L_0} L_0^{-1} U^{-1}) &= (U L_1 T_{L_1} L_1^{-1} U^{-1})(C_0 T_{C_0} C_0^{-1})
\end{aligned}
\tag{11}
$$

Those equations are exclusively products of 3-cycles in S_9. As long as not all terms are non-vanishing those equations can only be solved if each cycle acts on the same 3 elements.

If all sets are of type I the equations are trivially satisfied because all the T matrices are identity. Otherwise it needs at least one type II set on each side of the equality.

2.5 Sudoku Type Classification

For each bloc line and colums we have two possible types, but by interchange of bloc lines or colums and possible rotation of the entire sudoku only the following 10 sudokus have different consistency equations.

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
\begin{array}{cccc} & I & I & I \\ I & & & \\ I & & V & \\ I & & & \end{array} &
\begin{array}{cccc} & I & I & II \\ I & & & \\ I & & X & \\ I & & & \end{array} &
\begin{array}{cccc} & I & II & II \\ I & & & \\ I & & X & \\ I & & & \end{array} &
\begin{array}{cccc} & II & II & II \\ I & & & \\ I & & V & \\ I & & & \end{array} &
\begin{array}{cccc} & I & I & II \\ I & & & \\ I & & X & \\ II & & & \end{array} \\
6 & 7 & 8 & 9 & 10 \\
\begin{array}{cccc} & I & II & II \\ I & & & \\ I & & X & \\ II & & & \end{array} &
\begin{array}{cccc} & II & II & II \\ I & & & \\ I & & X & \\ II & & & \end{array} &
\begin{array}{cccc} & I & II & II \\ I & & & \\ II & & V & \\ II & & & \end{array} &
\begin{array}{cccc} & II & II & II \\ I & & & \\ II & & X & \\ II & & & \end{array} &
\begin{array}{cccc} & II & II & II \\ II & & & \\ II & & V & \\ II & & & \end{array}
\end{array}
\tag{12}
$$

- (1) Trivially satisfies all the equations.
- (4) Can be satisfied with $C_0 = C_1 = C_2$.
- (10) Can be satisfied in several ways.
- (2) + (3) + (5) + (6) Leads to $T = I$ which is impossible, thus those types are not allowed.
- (7) + (9) Considering line one and three leads to having $(U L_1 T_{L_1} L_1^{-1} U^{-1})(C_0 T_{C_0} C_0^{-1})$ and $(U L_1 T_{L_1}^{-1} L_1^{-1} U^{-1})(C_0 T_{C_0} C_0^{-1})$ of which one must be the identity because those are cycles on three elements ($(123)^2 = (132) \quad (123)(132) = I$). Thus we again get $T = I$ which is impossible.

Thus only four possible combinations remain, which greatly restricts the number of possible sudokus.

3 Number of Degrees of Freedom

3.1 Fixing the Sudoku Type

Test

$$
\begin{array}{cccc}
(1) & (2) & (3) & (4) \\
\begin{array}{cccc} & I & I & I \\ I & 2 & 2 & 2 \\ I & 2 & 2 & 1 \\ I & 2 & 2 & 0 \end{array} &
\begin{array}{cccc} & II & II & II \\ I & 2 & 2 & 2 \\ I & 2 & 2 & 2 \\ I & 2 & 2 & 0 \end{array} &
\begin{array}{cccc} & I & II & II \\ I & 2 & 2 & 2 \\ II & 2 & 2 & 2 \\ II & 2 & 2 & 0 \end{array} &
\begin{array}{cccc} & II & II & II \\ II & 2 & 2 & 2 \\ II & 2 & 2 & 2 \\ II & 2 & 2 & 2 \end{array}
\end{array}
\tag{13}
$$

- To ensure that we have a type (1) sudoku we need to fix entirely one set in the lines and two in the colums to be of type I. For the three left sets we only have to decide between the two possibilities for type I sets. This can be done using a total of $G = 6 + 4 + 4 + 1 = 15$ givens in the sudoku.
- To ensure that we have a type (2) sudoku we need to fix entirely two sets in the lines to be of type I and two sets in the colums to be of type II. For the two left sets we only have to decide between the two possibilities inside type I or II sets. This can be done using a total of $G = 6 + 6 + 2 + 2 = 16$ givens in the sudoku.
- To ensure that we have a type (3) sudoku we need to fix entirely one set in the lines and one in the colums to be of type I and one set in the colums to be of type II. For the two left sets we only have to decide between the two possibilities inside type II. This can be done using a total of $G = 6 + 6 + 2 + 2 = 16$ givens in the sudoku.
- To ensure that we have a type (4) sudoku we need to fix entirely one set in the lines and three sets in the colums to be of type II. For the two left sets we only have to decide between the two possibilities inside type II sets. This can be done using a total of $G = 6 + 6 + 6 = 18$ givens in the sudoku.

3.2 Fixing Permutation Degrees of Freedom

4 Outlook on $N \times N$ Sudokus

In the more general case of a $N \times N$ Sudoku the generalisation of the system of equations 4 would contain N^2equations for $N!$ variables. Consequently the number of possible sets would increase factorialy and the generalised ordering of 5 would become difficult.

Probably a clever notation could make it possible to establish rules in the general case.

References

[1] Bertram Felgenhauer, Fraze Jarvis : Enumerating possible Sudoku grids