# Sudoku Permutation Structure

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**Abstract**

To efficiently solve sudokus a lot of clever algorithms have been developed but none lead to a complete understanding of their structure. Why up to now the question of the fewest given that render a solution unique is still unsolved. In this paper I will establish an algebra to represent naturally sudokus based on permutation groups. It will as well allow to show that it needs at least 17 givens in a sudoku for having a unique solution.

## Contents

# 1 Sudoku Algebra

## 1.1 Representing a Sudoku

A sudoku has exactly each number from 1 to 9 in each line, column and 3x3 bloc. It is thus natural to start with square matrices of size three and construct the possible configuration in such a way to implement the three properties.

$$e_{ij} \in \mathbb{M}_3 \qquad (e_{ij})_{kl} = \delta_{kl}\delta_{jl} \qquad C = \sum_{i,j} e_{ij} \otimes e_{ji} \tag{1}$$

The central matrix C has by construction the desired properties which are maintained under permutation of the lines and columns inside a bloc. Thus every possible configuration for one number can be generated by applying a bloc permutation matrices $B^L$ on the lines and $B^C$ on the columns (see appendix 5.1). To obtain an entire sudoku we sum up over nine independent such configurations.

$$Sudoku = \sum_{n=1}^{9} n\, B_n^L C B_n^C \qquad B_n^L, B_n^C \in I_3 \otimes S_3 \tag{2}$$

Unfortunately this generates a lot of impossible sudokus because there is no guaranty that every location in a bloc is occupied once. In the next section we will work out the algebra of the bloc permutation matrices such that they fulfill this constraint as well. A bloc is denoted by the position $(l, c)$ of its bloc line and bloc column ($l, c \in \{0, 1, 2\}$).

$$C_{lc} = \sum_{n=1}^{9} n\, B_{nl}^L e_{cl} B_{nc}^C \qquad B_{nl}^L, B_{nc}^C \in S_3 \tag{3}$$

## 1.2 Possible Sets of Permutations

We start by writing down all the permutations of $S_3$ and pair them up with respect to their action.

$$S_3 = \left\{ \begin{array}{c} a = 123 \\ b = 132 \\ c = 312 \\ d = 213 \\ e = 231 \\ f = 321 \end{array} \right\} \qquad \begin{array}{cccc} \Rsh & 1 & 2 & 3 \\ 1 & a|b & c|d & e|f \\ 2 & e|d & a|f & c|b \\ 3 & c|f & e|b & a|d \end{array} \tag{4}$$

The table lists all the permutations which permute the number to the left into the location on top. Any set of nine permutations blocs $B_i = \{B_{ni} \,|\, \forall\, 1 \leq n \leq 9\}$ (for both lines and columns) needs to send exactly each element three times onto each location. This yields following overdetermined set of equations which have exactly four solutions.

$$\begin{array}{lll} N_a + N_b = 3 & N_e + N_d = 3 & N_c + N_f = 3 \\ N_c + N_d = 3 & N_a + N_f = 3 & N_e + N_b = 3 \\ N_e + N_f = 3 & N_c + N_b = 3 & N_a + N_d = 3 \end{array} \quad \begin{array}{c} N_a = N_c = N_e \\ N_b = N_d = N_f = 3 - N_a \end{array} \tag{5}$$

This allows to characterize each of the 3 sets $B_l^L$ and $B_c^C$by a number $N_i^L$, $N_i^C$ between 0 and 3.

## 1.3 Ordering the Sets into Similar or Complementary Subsets

Form the section before we know that the sets $B_l^L$ or $B_c^C$ can be split into 3 subsets of 3 permutations with following property.

- Sending the non null element $(e_{cl})_{cl}$ on the same line or column (Similar subsets).

- Sending the non null element $(e_{cl})_{cl}$ on three different lines or columns (Complementary subsets).

By pairing up similar sets from the lines with complementary sets of the columns, or vice a versa, we naturally obtain a full cover of the bloc.

The 4 possible sets established before are split into two types, those which by construction are similar for all 3 elements (type I) and those which are not (type II). Notice that by relabeling of $a \leftrightarrow b$, $c \leftrightarrow d$ and $e \leftrightarrow f$ we obtain exactly the same result, implying that the sets $E^0$and $E^3$are interchangeable as are $E^1$and $E^2$.

$$I \begin{cases} E^3 = & aaa & ccc & eee \\ E^0 = & bbb & ddd & fff \end{cases} \qquad II \begin{cases} E^1 = & abb & cdd & eff & Similar\ on\ 1 \\ & cbb & edd & aff & Similar\ on\ 2 \\ & ebb & add & cff & Similar\ on\ 3 \\ \\ E^2 = & baa & fee & dcc & Similar\ on\ 1 \\ & faa & dee & bcc & Similar\ on\ 2 \\ & daa & bee & fcc & Similar\ on\ 3 \end{cases} \tag{6}$$

For the type II sets the 3 orderings are related in a cyclic way by the permutation $T = (174)$ as shown below. For type I sets we define $T$ as being the identity.

$$\text{Example : } TE^1 = (174)(abb\, cdd\, eff) = (cbb\, edd\, aff) \tag{7}$$

Complementary subsets can be obtained by permuting a similar ordering (for both types) with $U = (258)(396)$ as show below. We notice that $T$ and $U$ are independent permutations.

$$\text{Example : } UE^3 = (258)(396)(aaa\, ccc\, eee) = (aec\, cae\, eca) \tag{8}$$

For each type the number of possible permutations keeping the similarity or complementarity property is listed bellow.

| | Type I | | Type II | |
|---|---|---|---|---|
| Similar | aaa ccc eee | $S_I^S = 3!$ | abb cdd eff | $S_{II}^S = 3^3\,3!$ |
| Complementary | ace ace ace | $S_I^C = (3!)^3$ | afd cbf edb | $S_{II}^C = 3^3(3!)^3$ |

The number of complementary orderings for the second type is a bit tricky. Starting from the similar ordering we notice that for the first set we have $3^3$ possibilities, for the second $2^3$and 1 for the last. Then we eliminate the permutation of identical elements and multiply with the number of permutations inside each set and obtain $(3!)^3(3!)^3/2^3 = 3^3(3!)^3$.

## 1.4 Constructing fully covered blocs

From now on we always assume that $B_l^L$and $B_c^C$ ($E_l^L, E_c^C \in \{E^0, E^1, E^2, E^3\}$) are ordered similar on 1 as defined in the section before. When we apply a permutation on a set we mean permuting the $n$ indices as follows.

$$C_{lc} = \sum_{n=1}^{9} n\, B^L_{A(n)l} e_{cl} B^C_{B(n)c} \equiv \sum_{n=1}^{9} n\, (A^{-1}B_l^L)_n e_{cl} (B^{-1}B_c^C)_n \quad A, B \in S_9 \tag{9}$$

This is very tedious to write out all the time, so because we are only interested on how the permutations from lines are paired up with those of the columns we will define a sum operation on two sets as $(a, b) + (c, d) = ((a, c), (b, d))$. Note that applying a permutation on one side of this sum operation is equivalent to apply the inverse of the permutation on the other side as shown below. We obtain the same pairs but in a different order.

$$(a,b,c) + (123)(d,e,f) = ((a,f),(b,d),(c,e)) \equiv (123)^{-1}(a,b,c) + (d,e,f) \tag{10}$$

For the bloc $(l,c)$ we must order the line set in a similar way on $l$ and the column set in a complementary way on c. Then we are free to permute the elements inside each subset and the subsets themself. After the pairing up operation we are free to permute the pairs as we want.

$$P_l\left(L_l T^c_{L_l} E^L_l + U C_c T^l_{C_c} E^C_c\right) \qquad L_l, C_c \in S_{3\times 3} = \sum_{i,j} e_{ij} \otimes S_3 \quad P_l \in S_9 \tag{11}$$

$$\text{Example : } TE^1 + UE^3 = ((ca)(be)(bc)(ea)(de)(dc)(aa)(fe)(fc)) \tag{12}$$

When written above$l$, $c$ are not indices on the matrices but exponents. Of course in a sudoku for each of the 6 sets (3 for the bloc lines and 3 for the bloc columns) only one permutation is allowed and we have now to figure out those consistent for all 9 blocs.

## 1.5 Consistency Equations

The set of equations obtained for two bloc lines and three bloc columns is sufficient to determine the entire sudoku because the result can be applied to any pair of bloc lines by permutation of those. Moving all the permutations in the equation 11 on the columns using 10 and asking that those are the same for the two lines we obtain following system.

$$\begin{array}{c} L_0^{-1} U C_0 = P_1 L_1^{-1} U C_0 T_{C_0} \\ T^2_{L_0} L_0^{-1} U C_1 = P_1 T^2_{L_1} L_1^{-1} U C_1 T_{C_1} \\ T_{L_0} L_0^{-1} U C_2 = P_1 T_{L_1} L_1^{-1} U C_2 T_{C_2} \end{array} \tag{13}$$

$$\begin{array}{c} P_1 = L_0^{-1} U C_0 T^2_{C_0} C_0^{-1} U^{-1} L_1 \\ (U^{-1} L_1 T_{L_1} L_1^{-1} U)(C_0 T_{C_0} C_0^{-1}) = (C_1 T_{C_1} C_1^{-1})(U^{-1} L_0 T_{L_0} L_0^{-1} U) \\ (U^{-1} L_1 T_{L_1} L_1^{-1} U)(C_2 T_{C_2} C_2^{-1}) = (C_0 T_{C_0} C_0^{-1})(U^{-1} L_0 T_{L_0} L_0^{-1} U) \end{array} \tag{14}$$

Those equations are exclusively products of 3-cycles in $S_9$. As long as not all terms are non-vanishing those equations can only be solved if each cycle acts on the same 3 elements.

If all sets are of type I the equations are trivially satisfied because all the $T$ matrices are identity. Otherwise it needs at least one type II set on each side of the equality.

## 1.6 Sudoku Type Classification

For each bloc line and columns we have two possible types, but by interchange of bloc lines or columns and possible rotation of the entire sudoku only the following 10 sudokus have different consistency equations.

$$\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
\begin{array}{cccc} & I & I & I \\ I & & & \\ I & & V & \\ I & & & \end{array} &
\begin{array}{cccc} & I & I & II \\ I & & & \\ I & & X & \\ I & & & \end{array} &
\begin{array}{cccc} & I & II & II \\ I & & & \\ I & & X & \\ I & & & \end{array} &
\begin{array}{cccc} & II & II & II \\ I & & & \\ I & & V & \\ I & & & \end{array} &
\begin{array}{cccc} & I & I & II \\ I & & & \\ I & & X & \\ II & & & \end{array} \\
6 & 7 & 8 & 9 & 10 \\
\begin{array}{cccc} & I & II & II \\ I & & & \\ I & & X & \\ II & & & \end{array} &
\begin{array}{cccc} & II & II & II \\ I & & & \\ I & & X & \\ II & & & \end{array} &
\begin{array}{cccc} & I & II & II \\ I & & & \\ II & & V & \\ II & & & \end{array} &
\begin{array}{cccc} & II & II & II \\ I & & & \\ II & & X & \\ II & & & \end{array} &
\begin{array}{cccc} & II & II & II \\ II & & & \\ II & & V & \\ II & & & \end{array}
\end{array} \tag{15}$$

- (1) Trivially satisfies all the equations.
- (4) Can be satisfied with $C_0 = C_1 = C_2$.
- (8) + (10) Can be satisfied in several ways.
- (2) + (3) + (5) + (6) Leads to $T = I$ which is impossible, thus those types are not allowed.
- (7) + (9) Considering line one and three leads to having $(UL_1T_{L_1}L_1^{-1}U^{-1})(C_0T_{C_0}C_0^{-1})$ and $(UL_1T_{L_1}^{-1}L_1^{-1}U^{-1})(C_0T_{C_0}C_0^{-1})$ of which one must be the identity because those are cycles on three elements ( $(123)^2 = (132) \quad (123)(132) = I$ ). Thus we again get $T = I$ which is impossible.

Thus only four possible combinations remain, which greatly restricts the number of possible sudokus.

# 2 Constraints due to Givens in a Sudoku

All the permutations involved are in $S_3$, which have two degrees of freedom. A given fixes one of the three to permuting elements, so there are two possibilities left. For example if we know that for $P \in S_3$ we have $P(1) = 2$ then the possibilities are $P = (123)$ and $P = (12)(3)$.

A given reduces to two the possibilities in the corresponding bloc line and bloc column. If we have twice the same number in a bloc line or column then we have fixed both degrees of freedom and entirely determined one of the matrices in the set.

## 2.1 Inequalities from Givens

Lets consider one set, we denote by $N_{ab}$ the number of constraints of the type $a \mid b$ and so on for all the 9 possible constraints (see 4). The variables $0 \leq n_{ab} \leq N_{ab}$ represent the possible choices with the constraint $N_a \geq N_{ab} - n_{ab}$ and $N_b = 3 - N_a \geq n_{ab}$. Combining this for all the possible constraints yields the following inequalities.

$$N_a \geq N_{ab} + N_{af} + N_{ad} - n_{ab} - n_{ad} - n_{af} \qquad N_a \leq 3 - n_{ab} - n_{bc} - n_{be} \tag{16}$$

$$N_a \geq N_{cd} + N_{bc} + N_{cf} - n_{bc} - n_{cd} - n_{cf} \qquad N_a \leq 3 - n_{ad} - n_{cd} - n_{de} \tag{17}$$

$$N_a \geq N_{ef} + N_{de} + N_{be} - n_{be} - n_{de} - n_{ef} \qquad N_a \leq 3 - n_{af} - n_{cf} - n_{ef} \tag{18}$$

We can see that the inequalities on the left are totally independent from each other, as are those on the right. A hint in a bloc line or column can be in three different blocs, the N's having same color represent hints in a same bloc. Out of the six we only need to keep the one giving the lowest upper bound and the one giving the highest lower bound. The three $n$ variables in a inequality on the left appear each once in a different inequality on the right.

## 2.2 Determination with Hints on Different Numbers

The inequalities a totaly symmetric with respect to the interchange $N \leftrightarrow n$, so we only need to determine how to fix $N_a = 3 \equiv 0$ and $N_a = 2 \equiv 1$ because they equivalent.

- Case $N_a = 3$ : we need a configuration giving $N_a \geq 3$ and forbiding to set a $n$ to a non zero value. The only way is to have two lower bounds acting on the same upper bound as follows.

  $$N_a \geq 3 - n_{ab} \qquad N_a \leq 3 - n_{ab} - n_{bc} \qquad N_a \geq 3 - n_{bc} \qquad N_{ab} = N_{bc} = 3 \tag{19}$$

  Those six hints necesseraly need to be in two different blocs. The six hints enterly fix the permutations the type I set.

- Case $N_a = 2:$ we need a configuration enforcing $N_a \geq 2$ and $N_a \leq 2$. The only way is to enforce having at least one none zero n, implying that we will have two none zero n's which have to be in two different upper bounds.

$$N_a \geq 1 + 3 - n_{ab} - n_{ad} \qquad N_a \leq 3 - n_{ab} \qquad N_{ab} = 1 \quad N_{ad} = 3 \tag{20}$$

$$N_a \geq 2 - n_{cd} \qquad N_a \leq 3 - n_{ad} - n_{cd} \qquad N_{cd} = 2 \tag{21}$$

  If we had only four or five hints we could continue increasing the n's and have multiples solutions. The only possiblity is with 6 hints as shown above, we would need to increase two n's acting in the same upper bound and then giving a contradiction. The six hints do not entirely fix the permutation.

### 2.3 Determination with Hints on the Same Number

If we already knew one of the nine matrice due to two givens on the same number in a same bloc row or column we would get inequalities of the following type.

$$N_a \geq 1 + N_{ab} + N_{af} + N_{ad} - n_{ab} - n_{ad} - n_{af} \qquad N_a \leq 3 - n_{ab} - n_{bc} - n_{be} \tag{22}$$

$$N_a \geq 1 + N_{cd} + N_{bc} + N_{cf} - n_{bc} - n_{cd} - n_{cf} \qquad N_a \leq 3 - n_{ad} - n_{cd} - n_{de} \tag{23}$$

$$N_a \geq 1 + N_{ef} + N_{de} + N_{be} - n_{be} - n_{de} - n_{ef} \qquad N_a \leq 3 - n_{af} - n_{cf} - n_{ef} \tag{24}$$

By the same reasoning as before we would obtain that it needs 4 additional givens to determine the set. If we had twice two givens on the same number, thus fixing two matrices, we would have the inequalities with a 2 instead of a 1. This then yields again that we would need 2 more givens to obtain a unique solution. Conclusively we always need 6 givens in a bloc line or bloc column to determine the corresponding set.

### 2.4 Determination Knowing the Type

If we know that a set is of type type I it needs 4 more hints to determine it.

$$N_a \geq 3 - n_{ab} \qquad N_a \leq 3 - n_{ab} - n_{bc} \qquad N_a \geq 1 - n_{bc} \qquad N_{ab} = 3 \quad N_{bc} = 1 \tag{25}$$

If its of type II we only need two more hints to determine it.

$$N_a \geq 1 + 1 - n_{ab} \qquad N_a \leq 2 - n_{ab} - n_{bc} \qquad N_a \geq 1 + 1 - n_{bc} \qquad N_{ab} = 1 \quad N_{bc} = 1 \tag{26}$$

### 2.5 Column-Line Constraints from Givens

## 3 The Four Sudoku Types

We are now going to analyze each of the four possible types of sudokus in order to enumerate them and determine the fewest givens for a unique solution.

$$\begin{array}{cccc} & I & I & I \\ I & & & \\ I & & (1) & \\ I & & & \end{array} \qquad \begin{array}{cccc} & II & II & II \\ I & & & \\ I & & (2) & \\ I & & & \end{array} \qquad \begin{array}{cccc} & I & II & II \\ I & & & \\ II & & (3) & \\ II & & & \end{array} \qquad \begin{array}{cccc} & II & II & II \\ II & & & \\ II & & (4) & \\ II & & & \end{array} \tag{27}$$

### 3.1 Type 1 Sudokus

The consistency equations are trivial to satisfy, the possible permutations for lines are independent of each other as are those between columns.

| Lines | Possibilities |
|---|---|
| aaa ccc eee | $2 \cdot 9!/(3!)^3$ |
| aaa ccc eee | $2 \cdot 3!$ |
| aaa ccc eee | $2 \cdot 3!$ |

| Columns | ace ace ace | ace ace ace | ace ace ace |
|---|---|---|---|
| Possibilities | $2 \cdot (3!)^3$ | $2 \cdot (3!)^3$ | $2 \cdot (3!)^3$ |

$$N_1 = 2^6 \frac{9!}{(3!)^3} (3!)^{3\times 3} (3!)^2 = 39007939461120 \simeq 3.9 \cdot 10^{13} \tag{28}$$

For every column we have 6 independent degrees of freedom, leading to the fact that we would at least need 18 givens to have a unique solutions.

### 3.2 Type 2 Sudokus

If we take the identity for the first line the we are free to permute the first column as used but the second and third column have only 4 degrees of freedom. We are free to cycle permute the T cycle which gives three possibilities, and swap the three left pairs.

$$P_1 = U C_0 T^2_{C_0} C_0^{-1} U^{-1} L_1 \quad (C_0 T_{C_0} C_0^{-1}) = (C_1 T C_1^{-1}) \quad (C_2 T C_2^{-1}) = (C_0 T_{C_0} C_0^{-1}) \tag{29}$$

| Lines | Possibilities |
|---|---|
| aaa ccc eee | $2 \cdot 9!/(3!)^3$ |
| eaa acc cee | $2 \cdot 3!$ |
| caa ecc aee | $2 \cdot 3!$ |

| Columns | afd cbf edb | afd cbf edb | afd cbf edb |
|---|---|---|---|
| Possibilities | $2^{10} \cdot 3^6 \cdot 17$ | | |

$$N_2 = 2^3 \frac{9!}{(3!)^3} \cdot 2^{10} \cdot 3^6 \cdot 17 \cdot (3!)^2 \cdot 2 = 12280277237760 \simeq 1.2 \cdot 10^{13} \tag{30}$$

### 3.3 Type 3 Sudokus

$$P_1 = I \quad (U^{-1} L_1 T L_1^{-1} U) = (C_1 T C_1^{-1}) \quad (U^{-1} L_1 T L_1^{-1} U) = (C_2 T^2 C_2^{-1}) \tag{31}$$

| Lines | Possibilities |
|---|---|
| aaa ccc eee | $2 \cdot 9!/(3!)^3$ |
| abb cdd eff | $2 \cdot 3^3 \cdot 3!$ |
| abb cdd eff | $2 \cdot 3$ |

| Columns | ace ace ace | afd cbf edb | afd edb cbf |
|---|---|---|---|
| Possibilities | $2 \cdot (3!)^3$ | $2 \cdot 3 \cdot 2^3$ | $2 \cdot 3 \cdot 2^3$ |

$$N_3 = 2^6 \cdot 9! \cdot 2^6 \cdot 3^6 \cdot 3! \cdot (3!)^2 = 234047636766720 \simeq 2.3 \cdot 10^{14} \tag{32}$$

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### 3.4 Type 4 Sudokus

$$P_1 = T^2 \quad T(C_0 T C_0^{-1}) = (C_1 T C_1^{-1}) T \quad T(C_2 T C_2^{-1}) = (C_0 T C_0^{-1}) T \tag{33}$$

| Lines | Possibilities |
|---|---|
| abb cdd eff | $2 \cdot 9!/2^3$ |
| ebb add cff | $2 \cdot 3!$ |
| cbb edd aff | $2 \cdot 3!$ |

| Columns | afd cbf edb | efd abf cdb | cfd ebf adb |
|---|---|---|---|
| Possibilities | $2 \cdot 3^3 (3!)^3$ | $2 \cdot 3 \cdot 2^3$ | $2 \cdot 3 \cdot 2^3$ |

$$N_4 = 2^3 \cdot 9! \cdot 2^6 \cdot 3^5 \cdot (3!)^5 = 351071455150080 \simeq 3.5 \cdot 10^{14} \tag{34}$$

??

### 3.5 Total Number of Sudokus

The total number of Sudokus is much lower then found in [1] by Bertram Felgenhauer and Fraze Javis. They are probably overcounting sudokus because they multiply by the size of the equivalences classes of each used symmetry. When actually they should only multiply by the size of the equivalence classes of the combined symmetries.

# 4 Conclusion

In the more general case of a $N \times N$ Sudoku the generalisation of the system of equations 5 would contain $N^2$equations for $N!$ variables. Consequently the number of possible sets would increase factorialy and the generalized ordering of 6 would become difficult. Probably a clever notation could make it possible to establish rules in the general case.

# 5 Appendix

## 5.1 Basis, Central Matrix and Bloc Permutations

$$e_{00} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad e_{01} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \ldots \quad e_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{35}$$

$$C = \sum_{i,j} e_{ij} \otimes e_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & & & & & & & & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ & & & & & & & & \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{36}$$

$$B \in I_3 \otimes S_3 \qquad B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix} b_i \in S_3 \qquad B_i = b_i \tag{37}$$

For future use we calculated as well the total number of two and three sets for which a, c and e are at same position.

| abb cdd eff | Number of orderings | Number of orderings keeping a, c and e fixed |
|---|---|---|
| ace bdf bdf | $3 \cdot (3!)^3$ | $\rightarrow \times (3!)^2$ |
| acf bde bdf | $(3!)^4$ | $\rightarrow \times 2 \cdot 3!$ |
| ade bcf bdf | $(3!)^4$ | $\rightarrow \times 2 \cdot 3!$ |
| adf bce bdf | $(3!)^4$ | $\rightarrow \times 2 \cdot 3!$ |
| adf bcf bde | $(3!)^4$ | $\rightarrow \times 2^3$ |
| | $= 3^3(3!)^3$ | $1\ Set = 2^5 \cdot 3^4 \cdot 31 \qquad 2\ Sets = 2^{10} \cdot 3^4 \cdot 17$ |

# References


[1] Bertram Felgenhauer, Fraze Jarvis : Enumerating possible Sudoku grids