2 $\|.\|_1$ small implies $D_{\max}$ small for some large projection

2.1 Classical case

Theorem 2.1. *Suppose p, q are probability distributions on $\mathcal{X}$ such that $\frac{1}{2}\|p-q\|_1 \leq \epsilon$, then $S \subseteq \mathcal{X}$ defined as $S := \{x \in \mathcal{X} : p(x) \leq (1+\epsilon^{1/2})q(x)\}$ is such that $q(S) \geq 1-\epsilon^{1/2}$ and $p(S) \geq 1-\epsilon^{1/2}-\epsilon$.*

Proof. For $S^c := \mathcal{X} \setminus S$, where S is the set defined above we have that

$$\begin{aligned}
\epsilon \geq \frac{1}{2}\|p-q\|_1 &= \max_{H \subseteq \mathcal{X}} |p(H)-q(H)| \\
&\geq q(S^c)\left|\frac{p(S^c)}{q(S^c)}-1\right| \\
&\geq q(S^c)\left(\frac{p(S^c)}{q(S^c)}-1\right) \\
&= q(S^c)\left(\frac{\sum_{x \in S^c} p(x)}{\sum_{x \in S^c} q(x)}-1\right) \\
&\geq q(S^c)\left(\frac{\sum_{x \in S^c}(1+\epsilon^{\frac{1}{2}})q(x)}{\sum_{x \in S^c} q(x)}-1\right) \\
&\geq q(S^c)\epsilon^{\frac{1}{2}}
\end{aligned}$$

which implies that $q(S^c) \leq \epsilon^{\frac{1}{2}}$. Now, the statement of the theorem follows. □

We can use this above fact to define B_k in the classical approx AEP proof and carry out the proof. We can also use coupling to carry out that proof.

Theorem 2.2. *Suppose ρ, σ are quantum states such that $\|\rho-\sigma\|_1 < \epsilon$, then there exists a projector P such that $\operatorname{tr}(P\rho) \geq 1-\epsilon^{1/2}-\epsilon$ and $\operatorname{tr}(P\sigma) \geq 1-\epsilon^{1/2}$ and $P\rho P \leq (1+\epsilon^{1/2})P\sigma P$.*

Proof. Let $\{v_i\}_{i \in \mathcal{I}}$ be the eigenvectors for $(1+\epsilon^{1/2})\sigma-\rho$. Define the probability distributions

$$\begin{aligned}
p(i) &:= \langle v_i, \rho v_i \rangle \\
q(i) &:= \langle v_i, \sigma v_i \rangle .
\end{aligned}$$

Then, using the classical theorem we can find a subset $S \subseteq \mathcal{I}$ of the such that $p(S) \geq 1-\epsilon^{1/2}-\epsilon$ and $q(S) \geq 1-\epsilon^{1/2}$ and for every $i \in S$, $p(i) \leq (1+\epsilon^{1/2})q(i)$. Note that $(1+\epsilon^{1/2})q(i)-p(i)$ is the eigenvalue of $(1+\epsilon^{1/2})\sigma-\rho$. This implies that for $\Pi_S := \sum_{i \in S} v_i v_i^*$

$$\Pi_S\left((1+\epsilon^{1/2})\sigma-\rho\right)\Pi_S \geq 0.$$

Moreover, $\operatorname{tr}(\Pi_S\rho) = p(S)$, $\operatorname{tr}(\Pi_S\sigma) = q(S)$. □