

Universal smooth min-entropy chain rules from entropic triangle inequalities

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Abstract

The von Neumann entropy of an n -partite system A_1^n given a system B can be written as the sum of the von Neumann entropies of the individual subsystems A_k given A_1^{k-1} and B . While it is known that such a chain rule does not hold for the smooth min-entropy, we prove a counterpart of this for a variant of the smooth min-entropy, which is equal to the conventional smooth min-entropy up to a constant. This enables us to lower bound the smooth min-entropy of an n -partite system in terms of, roughly speaking, equally strong entropies of the individual subsystems. We call this a *universal chain rule* for the smooth min-entropy, since it is applicable for all values of n . Our proof utilises the entropic triangle inequality based technique developed in [MD23] for analysing approximation chains. Additionally, we also prove an approximate version of the entropy accumulation theorem, which significantly relaxes the conditions required on the state to bound its smooth min-entropy. In particular, it does not require the state to be produced through a sequential process like previous entropy accumulation type bounds. In our upcoming paper [MD24], we use it to prove the security of parallel device independent quantum key distribution.

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1 Introduction

In the independent and identically distributed (i.i.d) regime, the rates of several interesting tasks like coding and randomness extraction are characterised by the von Neumann entropy [Sch95, Wil13]. The von Neumann entropy is the most commonly used and understood measure of uncertainty. It generally follows the behaviour one intuitively expects from a measure of uncertainty. Central to this paper is the intuitive expectation that the total uncertainty of two registers A_1 and A_2 given a register B should be the sum of the uncertainty of register A_1 given B and the uncertainty of register A_2 , conditioned on both A_1 and B . The von Neumann entropy, respects this intuition nicely and satisfies the following chain rule:

$$H(A_1 A_2 | B)_\rho = H(A_1 | B)_\rho + H(A_2 | A_1 B)_\rho. \quad (1)$$

More generally, we can decompose the von Neumann entropy of a large system $A_1^{n(1)}$ given B into a sum of the entropies of its parts as:

$$H(A_1^n | B)_\rho = \sum_{k=1}^n H(A_k | A_1^{k-1} B)_\rho. \quad (2)$$

⁽¹⁾The notation A_1^n denotes the set of registers A_1, A_2, \dots, A_n .

As we move beyond the i.i.d assumption and consider general processes, we move into the territory of one-shot information theory and need a multitude of different entropic measures to characterise and analyse various tasks. The smooth min-entropy [Ren06, RK05] is one of the most important information measures among these. It determines the amount of randomness that can be securely extracted from a (classical) register, when part of it is correlated with the adversary's state [TRSS10]. This is particularly useful in cryptography, where no assumptions (specifically i.i.d assumptions) can be placed on the adversary's actions. For a state ρ_{AB} , the min-entropy of system A conditioned on system B is defined as⁽²⁾

$$H_{\min}(A|B)_\rho := \sup \left\{ \lambda \in \mathbb{R} : \text{there exists a state } \sigma_B \text{ such that } \rho_{AB} \leq e^{-\lambda} \mathbb{1}_A \otimes \sigma_B \right\} \quad (3)$$

and the smooth min-entropy is defined as

$$H_{\min}^\epsilon(A|B)_\rho := \sup_{\tilde{\rho}} H_{\min}(A|B)_{\tilde{\rho}} \quad (4)$$

where the supremum is over all subnormalised states $\tilde{\rho}_{AB}$ which are ϵ -close to the state ρ in the purified distance. The smooth min-entropy behaves very differently from the von Neumann entropy. Informally speaking, it places a much larger weight on the worst case value of the conditioning register as compared to the von Neumann entropy, which places equal weight on all conditioning values. As a consequence, the smooth min-entropy exhibits behaviour that deviates from the intuitive expectations set by the von Neumann entropy. This is evident in the chain rule for smooth min-entropy [VDTR13], which is notably more complex than Eq. 1:

$$H_{\min}^{\epsilon_1 + 2\epsilon_2 + \delta}(A_1 A_2 | B)_\rho \geq H_{\min}^{\epsilon_1}(A_1 | B)_\rho + H_{\min}^{\epsilon_2}(A_2 | A_1 B)_\rho - k(\delta) \quad (5)$$

where $k(\delta) = O(\log \frac{1}{\delta})$. Moreover, it is easy to demonstrate that a smooth min-entropy counterpart for the relation in Eq. 2 cannot hold true. Mathematically, a relation of the following form cannot be valid for ϵ in a neighborhood of 0⁽³⁾:

$$H_{\min}^{g_1(\epsilon)}(A_1^n | B)_\rho \geq \sum_{k=1}^n H_{\min}^\epsilon(A_k | A_1^{k-1} B)_\rho - n g_2(\epsilon) - k(\epsilon), \quad (6)$$

where the functions g_1 , g_2 , and k are dependent solely on ϵ and $|A|$ (the dimension of the A_k registers, assumed to be constant in n) and are independent of n . Furthermore, $g_1(\epsilon)$ and $g_2(\epsilon)$ are required to be *small* functions of ϵ , meaning they are continuous and

⁽²⁾We use base e for exp and log, and *nat* units for entropies throughout this paper.

⁽³⁾Simply consider the classical distribution $p_{A_1^n B}$ where B is chosen uniformly at random from $\{0, 1\}$ and if $B = 0$ then A_1^n is set to the constant all zero string, otherwise each A_i is independently and randomly chosen bit. In this case, $H_{\min}^{g_1(\epsilon)}(A_1^n | B)_\rho = O(1)$ and for each k , $H_{\min}(A_k | A_1^{k-1} B)_\rho \geq \log 4/3$. For small ϵ , Eq. 6 would have a constant left-hand side and a linearly growing right-hand side.

approach 0 as ϵ tends to 0.

Eq. 6 is intimately related to the concept of *approximation chains* [MD23] and using them to bound entropies. For a state $\rho_{A_1^n B}$, a sequence of states $(\sigma_{A_1^k B}^{(k)})_{k=1}^n$ is called an ϵ -approximation chain of ρ if for every $1 \leq k \leq n$, we have $\rho_{A_1^k B} \approx_\epsilon \sigma_{A_1^k B}^{(k)}$. Often, direct bounds on the conditional entropies of the state ρ are unavailable; however, they can be obtained for the entropies of its approximation chain states. For instance, consider a state $\rho_{A_1^n B}$ where the A_k registers are produced sequentially. Ideally, each register A_k should be sampled independently from the rest. However, say imperfections introduce minor correlations between A_k and the earlier registers. In such a case, we might only be able to confirm that $\rho_{A_1^k B} \approx_\epsilon \rho_{A_k} \otimes \rho_{A_1^{k-1} B} =: \sigma_{A_1^k B}^{(k)}$. Despite these correlations, we expect the entropy of A_1^n given B to be large, as the A_k registers are almost independent of B . In such cases, a chain rule-like tool relating the entropy of ρ to the entropies of its approximation chain states would be helpful.

The impossibility of a bound of the form in Eq. 6 implies that $H_{\min}^\epsilon(A_1^n|B)_\rho$ cannot be lower bounded meaningfully in terms of the min-entropies of the approximation chain states of ρ , since these approximation chain states can simply be states satisfying $H_{\min}(A_k|A_1^{k-1}B)_{\sigma^{(k)}} = H_{\min}^\epsilon(A_k|A_1^{k-1}B)_\rho$ for every k . On the other hand, the von Neumann entropy of a state can easily be bounded in terms of its approximation chain by using the continuity of the conditional von Neumann entropy [AF04, Win16] to modify Eq. 2 and derive:

$$H(A_1^n|B)_\rho \geq \sum_{k=1}^n H(A_k|A_1^{k-1}B)_{\sigma^{(k)}} - nf(\epsilon) \quad (7)$$

where $f(\epsilon) = O\left(\epsilon \log \frac{|A|}{\epsilon}\right)$. The absence of a comparable bound for the smooth min-entropy severely limits us⁽⁴⁾.

There are multiple alternative definitions of the smooth min-entropy, which are equal to the one we defined above up to a constant [Ren06, TRSS10, ABJT20]. One of these, is the H_{\min}^\downarrow min-entropy and its smoothed variant $H_{\min}^{\downarrow, \epsilon}$, defined as:

$$H_{\min}^\downarrow(A|B)_\rho := \sup \left\{ \lambda \in \mathbb{R} : \rho_{AB} \leq e^{-\lambda} \mathbb{1}_A \otimes \rho_B \right\} \quad (8)$$

$$H_{\min}^{\downarrow, \epsilon}(A|B)_\rho := \sup_{\tilde{\rho}} H_{\min}^\downarrow(A|B)_{\tilde{\rho}} \quad (9)$$

⁽⁴⁾It is often useful to initially prove novel results using von Neumann entropies before translating them into the corresponding one-shot entropies. This approach separates the complexity of the problem into two distinct phases. The impossibility of Eq. 6, in particular, prevents us from porting arguments based on the von Neumann entropy into smooth min-entropy arguments.

where the second supremum is over all subnormalised states $\tilde{\rho}_{AB}$ which are ϵ -close to the state ρ in the purified distance. [TRSS10, Lemma 20] showed that this smooth min-entropy is equal to H_{\min}^ϵ up to a constant:

$$H_{\min}^{\epsilon/2}(A_1^n|B)_\rho - O\left(\log \frac{1}{\epsilon}\right) \leq H_{\min}^{\downarrow, \epsilon}(A_1^n|B)_\rho \leq H_{\min}^\epsilon(A_1^n|B)_\rho. \quad (10)$$

One can now ask whether $H_{\min}^{\downarrow, \epsilon}$ satisfies a chain rule like Eq. 6, that is, does

$$H_{\min}^{\downarrow, g_1(\epsilon)}(A_1^n|B)_\rho \geq \sum_{k=1}^n H_{\min}^{\downarrow, \epsilon}(A_k|A_1^{k-1}B)_\rho - ng_2(\epsilon) - k(\epsilon), \quad (11)$$

hold true for some g_1, g_2 and k as in Eq. 6? Remarkably, we prove that this is indeed the case. Establishing this in Theorem 3.4 is the first main result of this paper. We call this a *universal chain rule* for the smooth min-entropy to emphasise the fact that it is true and meaningful for a constant $\epsilon \in (0, 1)$ and an arbitrary $n \in \mathbb{N}$. This universal chain can be viewed as a smoothed generalisation of the chain rule for H_{\min}^\downarrow :

$$H_{\min}^\downarrow(A_1^n|B)_\rho \geq \sum_{k=1}^n H_{\min}^\downarrow(A_k|A_1^{k-1}B)_\rho, \quad (12)$$

which is well known and fairly simple to prove [Tom16, Proposition 5.5]. The universal chain rule in Eq. 11 is particularly interesting because it can be used to decompose the smooth min-entropy (both H_{\min}^ϵ and $H_{\min}^{\downarrow, \epsilon}$) into a sum of conditional entropies, which are equally strong.

We provide two proofs for this chain rule. We briefly describe the first proof technique here, since it is simpler and we use it to prove the approximate entropy accumulation theorem as well. This proof follows the entropic triangle inequality based approach developed in [MD23] for analysing approximation chains. The entropic triangle inequality allows us to bound the smooth min-entropy of the state ρ in Eq. 11 with the min-entropy of an auxiliary state σ as

$$H_{\min}^\delta(A_1^n|B)_\rho \geq H_{\min}(A_1^n|B)_\sigma - D_{\max}^\delta(\rho||\sigma). \quad (13)$$

Using the Generalised Golden Thompson inequality [SBT17], we identify a state $\sigma_{A_1^n B}$ satisfying

$$D_m(\rho_{A_1^n B}||\sigma_{A_1^n B}) \leq ng(\epsilon) \quad (14)$$

$$H_{\min}(A_1^n|B)_\sigma \geq \sum_{k=1}^n H_{\min}^{\downarrow, \epsilon}(A_k|A_1^{k-1}B)_\rho \quad (15)$$

where D_m is the measured relative entropy and $g(\epsilon)$ is a small function of ϵ . Selecting an appropriate state σ is the most non-trivial part of the proof. The bound in Eq. 14

can further be transformed into a smooth max-relative entropy bound using the substate theorem [JRS02, JN11]. Putting these bounds together into the triangle inequality then yields the universal chain rule.

The second major result in our paper is an *unstructured* approximate entropy accumulation theorem (Theorem 4.1). The entropy accumulation theorem (EAT) [DFR20], similar to a chain rule, decomposes the smooth min-entropy of a large system into a sum of the entropies of its parts. EAT can only be used for states $\rho_{A_1^n B_1^n E}$ satisfying the Markov chain $A_1^{k-1} \leftrightarrow B_1^{k-1} E \leftrightarrow B_k$ for every $k \in [n]$, which are produced by applying maps $\mathcal{M}_k : R_{k-1} \rightarrow A_k B_k R_k$ sequentially so that $\rho_{A_1^n B_1^n E} = \text{tr}_{R_k} \circ \mathcal{M}_n \circ \dots \circ \mathcal{M}_1(\rho_{R_0 E}^{(0)})$. In Theorem 4.1, we significantly relax the conditions on the structure of the state required to obtain an EAT like bound. We show that for any state as long as for every $k \in [n]$: $\rho_{A_1^k B_1^k E} \approx_\epsilon \mathcal{N}_k(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E R_k}^{(k)})$ for some state $\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E R_k}^{(k)}$ and a channel $\mathcal{N}_k : R_k \rightarrow A_k B_k$ which samples B_k independent of the previous registers⁽⁵⁾, we have the bound

$$H_{\min}^{\tilde{O}(\epsilon^{1/6})}(A_1^n | B_1^n E)_\rho \geq \sum_{k=1}^n \inf_{\omega_{R_k \tilde{R}_k}} H(A_k | B_k \tilde{R}_k)_{\mathcal{M}_k(\omega)} - n\tilde{O}(\epsilon^{1/12}) - \tilde{O}\left(\frac{1}{\epsilon^{5/12}}\right) \quad (16)$$

where \tilde{O} hides the logarithmic factors in ϵ and the infimum is over all states $\omega_{R_k \tilde{R}_k}$. This bound holds irrespective of the process which produces the state. In fact, the central motivation behind proving this theorem was for using it to prove the security of parallel device independent QKD. We do this in our upcoming companion work [MD24]. The proof approach for this theorem is very similar to that of Theorem 3.4.

2 Background

2.1 Notation

For a classical probability distribution p_{AB} , the conditional probability distribution $p_{A|B}$ is defined as $p_{A|B}(a|b) := \frac{p_{AB}(a,b)}{p_B(b)}$ when $p_B(b) > 0$. For the case when $p_B(b) = 0$, we define $p_{A|B}$ to be the uniform distribution for our purposes. The probability distribution $q_B p_{A|B}$ for a distribution q on random variable B is defined as $q_B p_{A|B}(b, a) := q_B(b) p_{A|B}(a|b)$.

The term normalised (subnormalised) quantum states is used for positive semidefinite operators with unit trace (trace less than 1). We denote the set of registers a quantum state describes (equivalently, its Hilbert space) using a subscript. Partial states of a

⁽⁵⁾It is worth noting two points. First, this condition is typically satisfied in applications of EAT. Second, if one strengthens the Markov chain condition for EAT, such that the Markov chain $A_1^{k-1} \leftrightarrow B_1^{k-1} E \leftrightarrow B_k$ holds for all inputs to the channels \mathcal{M}_k , then this seems to reduce to the independence condition used in this context.

quantum state are simply denoted by restricting the set of registers in the subscript. For example, if ρ_{AB} denotes a quantum state on registers A and B , then the partial states on registers A and B , will be denoted as ρ_A and ρ_B respectively.

The term “channel” is used for completely positive trace preserving (CPTP) linear maps between two spaces of Hermitian operators. A channel \mathcal{N} mapping registers A to B will be denoted by $\mathcal{N}_{A \rightarrow B}$.

We use the notation $[n]$ to denote the set $\{1, 2, \dots, n\}$. For n quantum registers (X_1, X_2, \dots, X_n) , the notation X_i^j for $i < j$ refers to the set of registers $(X_i, X_{i+1}, \dots, X_j)$. For a register A , $|A|$ represents the dimension of the underlying Hilbert space.

For two Hermitian operators X and Y , the operator inequality $X \geq Y$ is used to denote that $X - Y$ is a positive semidefinite operator and $X > Y$ denotes that $X - Y$ is a strictly positive operator. The notation $X \ll Y$ denotes that the support of operator X is contained in the support of Y . The identity operator on register A is denoted using $\mathbb{1}_A$.

2.2 Information theory

The trace norm is defined as $\|X\|_1 := \text{tr}((X^\dagger X)^{\frac{1}{2}})$. The fidelity between two positive operators P and Q is defined as $F(P, Q) = \|\sqrt{P}\sqrt{Q}\|_1^2$. The generalised fidelity between two subnormalised states ρ and σ is defined as

$$F_*(\rho, \sigma) := \left(\|\sqrt{\rho}\sqrt{\sigma}\|_1 + \sqrt{(1 - \text{tr } \rho)(1 - \text{tr } \sigma)} \right)^2. \quad (17)$$

The purified distance between two subnormalised states ρ and σ is defined as

$$P(\rho, \sigma) = \sqrt{1 - F_*(\rho, \sigma)}. \quad (18)$$

Throughout this paper, we use base e for both the functions \log and \exp . Therefore, all the entropies in this paper are in *nat* units. We follow the notation in Tomamichel’s book [Tom16] for Rényi entropies. The sandwiched α -Rényi relative entropy for $\alpha \in [\frac{1}{2}, 1) \cup (1, \infty]$ between the positive operator P and Q is defined as

$$\tilde{D}_\alpha(P||Q) = \begin{cases} \frac{1}{\alpha-1} \log \frac{\text{tr}(Q^{-\frac{\alpha'}{2}} P Q^{-\frac{\alpha'}{2}})^\alpha}{\text{tr}(P)} & \text{if } (\alpha < 1 \text{ and } P \not\leq Q) \text{ or } (P \ll Q) \\ \infty & \text{else.} \end{cases} \quad (19)$$

where $\alpha' = \frac{\alpha-1}{\alpha}$. In the limit $\alpha \rightarrow \infty$, the sandwiched divergence becomes equal to the max-relative entropy, D_{\max} , which is defined as

$$D_{\max}(P||Q) := \inf \{ \lambda \in \mathbb{R} : P \leq e^\lambda Q \}. \quad (20)$$

In the limit of $\alpha \rightarrow 1$, the sandwiched relative entropies equals the quantum relative entropy, $D(P||Q)$, which is defined as

$$D(P||Q) := \begin{cases} \frac{\text{tr}(P \log P - P \log Q)}{\text{tr}(P)} & \text{if } (P \ll Q) \\ \infty & \text{else.} \end{cases} \quad (21)$$

In this work, we will also need the measured relative entropy D_m . For two positive operators P and Q , it is defined as

$$D_m(P||Q) := \sup_{\mathcal{M}} D(\mathcal{M}(P)||\mathcal{M}(Q)) \quad (22)$$

where the supremum is taken over all measurement channels \mathcal{M} , that is, a channel such that $\mathcal{M}(\rho) = \sum_{x \in \mathcal{X}} \text{tr}(M_x \rho) |x\rangle \langle x|$ for some POVM elements $(M_x)_x$ and an alphabet \mathcal{X} .

We can use the sandwiched divergence to define the following conditional entropies for a subnormalised state ρ_{AB} :

$$\tilde{H}_\alpha^\uparrow(A|B)_\rho := \sup_{\sigma_B} -\tilde{D}_\alpha(\rho_{AB}||\mathbb{1}_A \otimes \sigma_B) \quad (23)$$

$$\tilde{H}_\alpha^\downarrow(A|B)_\rho := -\tilde{D}_\alpha(\rho_{AB}||\mathbb{1}_A \otimes \rho_B) \quad (24)$$

for $\alpha \in [\frac{1}{2}, 1) \cup (1, \infty]$. The supremum in the definition for $\tilde{H}_\alpha^\uparrow$ is over all quantum states σ_B on register B .

For $\alpha \rightarrow 1$, both these conditional entropies are equal to the von Neumann conditional entropy $H(A|B)$. $\tilde{H}_\infty^\uparrow(A|B)_\rho$ is usually called the min-entropy and is denoted as $H_{\min}(A|B)_\rho$. For a subnormalised state, it can also be defined as

$$H_{\min}(A|B)_\rho := \sup \left\{ \lambda \in \mathbb{R} : \text{there exists state } \sigma_B \text{ such that } \rho_{AB} \leq e^{-\lambda} \mathbb{1}_A \otimes \sigma_B \right\}. \quad (25)$$

Another version of the min-entropy, that will be relevant in this paper is given by $\tilde{H}_\infty^\downarrow(A|B)_\rho$. We denote it in this paper using $H_{\min}^\downarrow(A|B)_\rho$. It can alternatively be defined as

$$H_{\min}^\downarrow(A|B)_\rho := \sup \left\{ \lambda \in \mathbb{R} : \rho_{AB} \leq e^{-\lambda} \mathbb{1}_A \otimes \rho_B \right\} \quad (26)$$

$$= -\log \left\| \rho_B^{-\frac{1}{2}} \rho_{AB} \rho_B^{-\frac{1}{2}} \right\|_\infty \quad (27)$$

For a classical distribution p_{AB} , the min-entropies above simplify to

$$H_{\min}(A|B)_p = -\log \sum_b p(b) \max_a p(a|b) \quad (28)$$

$$H_{\min}^\downarrow(A|B)_p = -\log \max_{a,b} p(a|b). \quad (29)$$

The expressions for these entropies in the classical case clearly demonstrates the difference between H_{\min} and H_{\min}^\downarrow . Since H_{\min} is the negative log of the guessing probability [KRS09], it averages over the values of B . Whereas H_{\min}^\downarrow selects the worst possible value of B and then calculates the guessing probability for this value.

For the purpose of smoothing, we define the ϵ -ball around a subnormalised state ρ as the set

$$B_\epsilon(\rho) = \{\tilde{\rho} \geq 0 : P(\rho, \tilde{\rho}) \leq \epsilon \text{ and } \text{tr } \tilde{\rho} \leq 1\}. \quad (30)$$

The smooth max-relative entropy is defined as

$$D_{\max}^\epsilon(\rho||\sigma) := \min_{\tilde{\rho} \in B_\epsilon(\rho)} D_{\max}(\tilde{\rho}||\sigma) \quad (31)$$

We define the two version of smooth min-entropy of ρ_{AB} using the min-entropies defined above as

$$H_{\min}^\epsilon(A|B)_\rho := \max_{\tilde{\rho} \in B_\epsilon(\rho)} H_{\min}(A|B)_{\tilde{\rho}} \quad (32)$$

$$H_{\min}^{\downarrow, \epsilon}(A|B)_\rho := \max_{\tilde{\rho} \in B_\epsilon(\rho)} H_{\min}^\downarrow(A|B)_{\tilde{\rho}}. \quad (33)$$

Using [TRSS10, Lemma 20], these two are related as

$$H_{\min}^\epsilon(A|B)_\rho - \log\left(\frac{2}{\epsilon^2} + \frac{1}{1-\epsilon}\right) \leq H_{\min}^{\downarrow, 2\epsilon}(A|B)_\rho \leq H_{\min}^{2\epsilon}(A|B)_\rho. \quad (34)$$

That is, these entropies are essentially the same up to a constant term and a constant factor in the smoothing parameter.

2.3 Key lemmas and theorems

In this section, we state some important results, which we use throughout this paper. As stated in the introduction, the entropic triangle inequalities developed in [MD23] are central to analysing approximation chains and hence to this work. The following lemma states the two inequalities we use.

Lemma 2.1 (Entropic triangle inequality [MD23, Introduction and Lemma 3.5]). *For two states ρ_{AB} and η_{AB} and $\epsilon \in [0, 1)$, we have*

$$H_{\min}^\epsilon(A|B)_\rho \geq H_{\min}(A|B)_\eta - D_{\max}^\epsilon(\rho_{AB}||\eta_{AB}). \quad (35)$$

Further, for $\alpha \in (1, 2]$ and $\delta \in (0, 1)$ such that $\epsilon + \delta < 1$, we also have

$$H_{\min}^{\epsilon+\delta}(A|B)_\rho \geq \tilde{H}_\alpha^\uparrow(A|B)_\eta - \frac{\alpha}{\alpha-1} D_{\max}^\epsilon(\rho_{AB}||\eta_{AB}) - \frac{g_1(\delta, \epsilon)}{\alpha-1} \quad (36)$$

where $g_1(x, y) := -\log(1 - \sqrt{1-x^2}) - \log(1-y^2)$.

The quantum substate theorem stated below is one of the major results used to bound the smooth max-relative entropy in this paper. This theorem serves as a tool to convert bounds on the measured relative entropy to bounds on the smooth max-relative entropy.

Theorem 2.2 (Quantum substate theorem [JRS02, JN11]). *Let ρ and σ be two normalised states on the same Hilbert space. Then for any $\epsilon \in (0, 1)$, we have*

$$D_{\max}^{\sqrt{\epsilon}}(\rho||\sigma) \leq \frac{D_m(\rho||\sigma) + 1}{\epsilon} + \log \frac{1}{1 - \epsilon}. \quad (37)$$

Usually, the above theorem is stated with the relative entropy $D(\rho||\sigma)$ on the right-hand side instead of the measured relative entropy $D_m(\rho||\sigma)$. Since, in the proofs in this paper we are only able to derive bounds on the measured relative entropy, we need the stronger version stated above. It is, therefore, instructive to understand how it is possible to use $D_m(\rho||\sigma)$ in the bound above.

[JRS02, JN11] actually prove a bound on $D_{\max}^{\sqrt{\epsilon}}(\rho||\sigma)$ in terms of a divergence they call the *observational divergence* D_{obs} , which is defined as

$$D_{\text{obs}}(\rho||\sigma) := \sup \left\{ \text{tr}(P\rho) \log \frac{\text{tr}(P\rho)}{\text{tr}(P\sigma)} : \text{for } 0 \leq P \leq \mathbb{1} \right\}. \quad (38)$$

[JN11, Theorem 1] proves that

$$D_{\max}^{\sqrt{\epsilon}}(\rho||\sigma) \leq \frac{D_{\text{obs}}(\rho||\sigma)}{\epsilon} + \log \frac{1}{1 - \epsilon}. \quad (39)$$

Observe that for any binary measurement $(P, \mathbb{1} - P)$, we have

$$\begin{aligned} D_m(\rho||\sigma) &\geq \text{tr}(P\rho) \log \frac{\text{tr}(P\rho)}{\text{tr}(P\sigma)} + (1 - \text{tr}(P\rho)) \log \frac{1 - \text{tr}(P\rho)}{1 - \text{tr}(P\sigma)} \\ &\geq \text{tr}(P\rho) \log \frac{\text{tr}(P\rho)}{\text{tr}(P\sigma)} + (1 - \text{tr}(P\rho)) \log (1 - \text{tr}(P\sigma)) \\ &\geq \text{tr}(P\rho) \log \frac{\text{tr}(P\rho)}{\text{tr}(P\sigma)} - 1 \end{aligned}$$

which implies, that

$$D_{\text{obs}}(\rho||\sigma) \leq D_m(\rho||\sigma) + 1. \quad (40)$$

This allows us to use $D_m(\rho||\sigma)$ in the bound for the substate theorem.

In addition to the quantum substate theorem, the following generalisation of the Golden-Thompson (GT) inequality is another major result enabling the proof approach used in this paper.

Theorem 2.3 (Generalised Golden-Thompson (GT) Inequality [SBT17]). *For a collection of Hermitian matrices $\{H_k\}_{k=1}^n$, we have*

$$\mathrm{tr} \exp \left(\sum_{k=1}^n H_k \right) \leq \int_{-\infty}^{\infty} dt \beta_0(t) \mathrm{tr} \left(e^{H_n} e^{\frac{1-it}{2} H_{n-1}} \dots e^{\frac{1-it}{2} H_2} e^{H_1} e^{\frac{1+it}{2} H_2} \dots e^{\frac{1+it}{2} H_{n-1}} \right) \quad (41)$$

where $\beta_0(t) := \frac{\pi}{2} (\cosh(\pi t) + 1)^{-1}$ is a probability density function.

Proof. Using [SBT17, Corollary 3.3], we have

$$\log \left\| \exp \left(\sum_{k=1}^n \frac{1}{2} H_k \right) \right\|_2 \leq \int_{-\infty}^{\infty} dt \beta_0(t) \log \left\| \prod_{k=1}^n \exp \left(\frac{1+it}{2} H_k \right) \right\|_2. \quad (42)$$

Expanding the norm gives

$$\frac{1}{2} \log \mathrm{tr} \left(\exp \left(\sum_{k=1}^n H_k \right) \right) \leq \int_{-\infty}^{\infty} dt \beta_0(t) \frac{1}{2} \log \mathrm{tr} \left(e^{H_n} e^{\frac{1-it}{2} H_{n-1}} \dots e^{\frac{1-it}{2} H_2} e^{H_1} e^{\frac{1+it}{2} H_2} \dots e^{\frac{1+it}{2} H_{n-1}} \right). \quad (43)$$

Using the concavity and monotonicity of log, we get the statement in the Theorem. \square

The GT inequality above is often used in conjunction with the following variational expressions for the relative entropies. This is also the case in this paper.

Lemma 2.4 ([Pet88, BFT17]). *For a normalised state ρ and a positive operator Q , the following variational forms hold true:*

$$D(\rho||Q) = \sup_{\omega>0} \{ \mathrm{tr}(\rho \log \omega) + 1 - \mathrm{tr} \exp(\log Q + \log \omega) \} \quad (44)$$

$$D_m(\rho||Q) = \sup_{\omega>0} \{ \mathrm{tr}(\rho \log \omega) + 1 - \mathrm{tr}(Q\omega) \}. \quad (45)$$

We will also use the following lemma, which is a quantum generalisation of the fact that if two probability distributions p_{AB} and q_{AB} are close to each other, then the probability distributions p_{AB} and $p_B q_{A|B}$ are also close to each other.

Lemma 2.5. *For a normalised state ρ_{AB} and a subnormalised state $\tilde{\rho}_{AB}$ such that $P(\rho_{AB}, \tilde{\rho}_{AB}) \leq \epsilon$, the state $\eta_{AB} := \rho_B^{1/2} \tilde{\rho}_B^{-1/2} \tilde{\rho}_{AB} \tilde{\rho}_B^{-1/2} \rho_B^{1/2}$ (Moore-Penrose pseudo-inverse) satisfies $P(\rho_{AB}, \eta_{AB}) \leq (\sqrt{2} + 1)\epsilon$. Note that if $\tilde{\rho}_B$ is full rank, then $\eta_B = \rho_B$.*

Proof. Note that since ρ_{AB} is normalised, we have $F(\tilde{\rho}_{AB}, \rho_{AB}) \geq 1 - \epsilon^2$. Let $|\tilde{\rho}\rangle_{ABR}$ be an arbitrary purification of $\tilde{\rho}_{AB}$. Observe that the pure state $|\eta\rangle := \rho_B^{1/2} \tilde{\rho}_B^{-1/2} |\tilde{\rho}\rangle_{ABR}$ is a purification of η_{AB} .

Let $|\tilde{\rho}\rangle_{ABR} = \sum_i \sqrt{\tilde{p}_i} |u_i\rangle_B \otimes |v_i\rangle_{AR}$ where all $p_i > 0$ be the Schmidt decomposition of $\tilde{\rho}_{ABR}$. This implies $\tilde{\rho}_B = \sum_i \tilde{p}_i |u_i\rangle \langle u_i|_B$. Then, using Uhlmann's theorem [Wat18, Theorem 3.22], we have

$$\begin{aligned}
F(\tilde{\rho}_{AB}, \eta_{AB}) &\geq |\langle \tilde{\rho} | \eta \rangle|^2 \\
&= \left| \langle \tilde{\rho} | \rho_B^{1/2} \tilde{\rho}_B^{-1/2} | \tilde{\rho} \rangle \right|^2 \\
&= \left| \text{tr} \left(\rho_B^{1/2} \tilde{\rho}_B^{-1/2} \tilde{\rho}_{ABR} \right) \right|^2 \\
&= \left| \text{tr} \left(\rho_B^{1/2} \tilde{\rho}_B^{1/2} \right) \right|^2 \\
&\geq F(\tilde{\rho}_B, \rho_B)^2 \\
&\geq (1 - \epsilon^2)^2 \\
&\geq 1 - 2\epsilon^2
\end{aligned}$$

where we have used the relation between the *pretty good* fidelity and fidelity [IRS17, Eq. 44] for the first inequality and the fact that $F(\tilde{\rho}_B, \rho_B) \geq F(\tilde{\rho}_{AB}, \rho_{AB})$. Further, we have

$$\begin{aligned}
P(\tilde{\rho}_{AB}, \eta_{AB}) &= \sqrt{1 - F_*(\tilde{\rho}_{AB}, \eta_{AB})} \\
&\leq \sqrt{1 - F(\tilde{\rho}_{AB}, \eta_{AB})} \\
&\leq \sqrt{2}\epsilon.
\end{aligned}$$

Using the triangle inequality, we get

$$\begin{aligned}
P(\rho_{AB}, \eta_{AB}) &\leq P(\rho_{AB}, \tilde{\rho}_{AB}) + P(\tilde{\rho}_{AB}, \eta_{AB}) \\
&\leq (\sqrt{2} + 1)\epsilon.
\end{aligned}$$

□

3 A universal chain rule for smooth min-entropy

In this section, we will prove the universal chain rule for $H_{\min}^{\downarrow, \epsilon}$. Specifically, we will show that for a state $\rho_{A_1^n B}$,

$$H_{\min}^{\downarrow, g_1(\epsilon)}(A_1^n | B)_\rho \geq \sum_{k=1}^n H_{\min}^{\downarrow, \epsilon}(A_k | A_1^{k-1} B)_\rho - n g_2(\epsilon) - k(\epsilon), \quad (46)$$

where g_1 and g_2 are small functions of ϵ (g_1 and g_2 are continuous and tend to 0 as $\epsilon \rightarrow 0$) and k is a general function of ϵ . It should be noted that these functions may depend on $|A|$, which is the size of the individual registers A_k . We begin by sketching the proof of such a chain rule in the classical case first. This will be beneficial for understanding the challenges that need to be solved in order to prove the statement in the quantum case. We will then generalise this proof to the quantum case. In Sec. 5, we provide an alternate proof for this chain rule.

3.1 Proof sketch for classical distributions

Consider the probability distribution $p_{A_1^n B}$. We will sketch a proof for a chain rule of the form in Eq. 46 for $\rho = p$. Let us first broadly describe the proof strategy. We will identify an auxiliary distribution $p_{A_1^n B}^{(\text{aux})}$, such that

$$D_{\max}^{g_1(\epsilon)}(p_{A_1^n B} \| p_{A_1^n B}^{(\text{aux})}) \leq n g_2(\epsilon) + k(\epsilon) \text{ and} \quad (47)$$

$$H_{\min}(A_1^n | B)_{p^{(\text{aux})}} \geq \sum_{k=1}^n H_{\min}^{\downarrow, \epsilon}(A_k | A_1^{k-1} B)_p \quad (48)$$

where g_1, g_2 and k are as in Eq. 46. Then, we can simply use the entropic triangle inequality to show that

$$H_{\min}^{g_1(\epsilon)}(A_1^n | B)_p \geq H_{\min}(A_1^n | B)_{p^{(\text{aux})}} - D_{\max}^{g_1(\epsilon)}(p_{A_1^n B} \| p_{A_1^n B}^{(\text{aux})}) \quad (49)$$

$$\geq \sum_{k=1}^n H_{\min}^{\downarrow, \epsilon}(A_k | A_1^{k-1} B)_p - n g_2(\epsilon) - k(\epsilon). \quad (50)$$

$p^{(\text{aux})}$ will be defined using the distributions which achieve $H_{\min}^{\downarrow, \epsilon}(A_k | A_1^{k-1} B)_p$. For $k \in [n]$, let $q_{A_1^k B}^{(k)}$ be the distribution, such that

$$\frac{1}{2} \left\| p_{A_1^k B} - q_{A_1^k B}^{(k)} \right\|_1 \leq \epsilon \quad (51)$$

$$H_{\min}^{\epsilon, \downarrow}(A_k | A_1^{k-1} B)_p = H_{\min}^{\downarrow}(A_k | A_1^{k-1} B)_{q^{(k)}} \quad (52)$$

It is easy to show that we also have

$$\frac{1}{2} \left\| p_{A_1^k B} - p_{A_1^{k-1} B} q_{A_k | A_1^{k-1} B}^{(k)} \right\|_1 \leq 2\epsilon. \quad (53)$$

We can define an auxiliary distribution as $q_{A_1^n B} := p_B \prod_{k=1}^n q_{A_k | A_1^{k-1} B}^{(k)}$. For this distribution, the conditional min-entropy satisfies (Eq. 29),

$$H_{\min}^{\downarrow}(A_k | A_1^{k-1} B)_q = H_{\min}^{\downarrow}(A_k | A_1^{k-1} B)_{q^{(k)}} = H_{\min}^{\epsilon, \downarrow}(A_k | A_1^{k-1} B)_p \quad (54)$$

for every k since $q_{A_k | A_1^{k-1} B} = q_{A_k | A_1^{k-1} B}^{(k)}$. Further, we can bound the min-entropy for this distribution as desired in Eq. 48 as

$$H_{\min}(A_1^n | B)_q \geq H_{\min}^{\downarrow}(A_1^n | B)_q \geq \sum_{k=1}^n H_{\min}^{\downarrow}(A_k | A_1^{k-1} B)_q \quad (55)$$

Next, we need to ensure that the smooth max-divergence between p and q is relatively small. Our strategy will be to bound the relative entropy between these two distributions

and use the substate theorem to convert that to a bound for the smooth max-divergence. We expect to be able to use the following chain rule for the relative entropy

$$D(p_{A_1^k B} \| q_{A_1^k B}) = D(p_{A_1^{k-1} B} \| q_{A_1^{k-1} B}) + D(p_{A_1^k B} \| p_{A_1^{k-1} B} q_{A_k | A_1^{k-1} B}) \quad (56)$$

to prove that the relative entropy distance between $p_{A_1^n B}$ and $q_{A_1^n B}$ is small. Using this repeatedly along with the fact that $q_{A_k | A_1^{k-1} B} = q_{A_k | A_1^{k-1} B}^{(k)}$, gives us

$$D(p_{A_1^n B} \| q_{A_1^n B}) = \sum_{k=1}^n D(p_{A_1^k B} \| p_{A_1^{k-1} B} q_{A_k | A_1^{k-1} B}^{(k)}). \quad (57)$$

If we could now show that each term inside the summation is some small function in ϵ , $g(\epsilon)$, we could show that the $D(p_{A_1^n B} \| q_{A_1^n B})$ is bounded by $ng(\epsilon)$. Eq. 53 shows that the two distributions in each of the terms are close to each other. However, the relative entropy between them need not be small or even bounded, since it could be that $p_{A_1^k B} \not\ll p_{A_1^{k-1} B} q_{A_k | A_1^{k-1} B}^{(k)}$. To circumvent this technicality, we need to *massage* the distributions $q_{A_k | A_1^{k-1} B}^{(k)}$ by a small amount. This is done by mixing these distributions with the uniform distribution u_{A_k} to produce the distributions

$$r_{A_k | A_1^{k-1} B}^{(k)} = (1 - \delta) q_{A_k | A_1^{k-1} B}^{(k)} + \delta u_{A_k}. \quad (58)$$

For these distributions, we can show that (Lemma B.2)

$$\frac{1}{2} \left\| p_{A_1^k B} - p_{A_1^{k-1} B} r_{A_k | A_1^{k-1} B}^{(k)} \right\|_1 \leq 2\epsilon + \delta \quad (59)$$

and

$$D(p_{A_1^k B} \| p_{A_1^{k-1} B} r_{A_k | A_1^{k-1} B}^{(k)}) \leq z(\epsilon, \delta) \quad (60)$$

where $z(\epsilon, \delta) = O\left((\epsilon + \delta) \log \frac{|A|}{\epsilon \delta}\right)$. Note that $z(\epsilon, \delta)$ can be made $O\left(\epsilon \log \frac{|A|}{\epsilon}\right)$ for $\delta = \epsilon$. Instead of using $q_{A_1^n B}$ as the auxiliary distribution, we can use $r_{A_1^n B} := p_B \prod_{k=1}^n r_{A_k | A_1^{k-1} B}^{(k)}$. We can use the same argument as above to bound the relative entropy between $p_{A_1^n B}$ and $r_{A_1^n B}$. This gives us

$$\begin{aligned} D(p_{A_1^n B} \| r_{A_1^n B}) &= \sum_{k=1}^n D(p_{A_1^k B} \| p_{A_1^{k-1} B} r_{A_k | A_1^{k-1} B}^{(k)}) \\ &\leq nz(\epsilon, \delta). \end{aligned} \quad (61)$$

Let $\mu := z(\epsilon, \delta)^{1/3}$, then using the substate theorem, we have

$$D_{\max}^{\mu}(p_{A_1^n B} \| r_{A_1^n B}) \leq n\mu + \frac{1}{\mu^2} + \log \frac{1}{1 - \mu^2}.$$

Using the entropic triangle inequality, we get

$$\begin{aligned}
H_{\min}^{\mu}(A_1^n|B)_p &\geq H_{\min}(A_1^n|B)_r - n\mu - \frac{1}{\mu^2} - \log \frac{1}{1-\mu^2} \\
&\geq \sum_{k=1}^n H_{\min}^{\downarrow}(A_k|A_1^{k-1}B)_r - n\mu - \frac{1}{\mu^2} - \log \frac{1}{1-\mu^2} \\
&\geq \sum_{k=1}^n H_{\min}^{\downarrow}(A_k|A_1^{k-1}B)_{q^{(k)}} - n\mu - \frac{1}{\mu^2} - \log \frac{1}{1-\mu^2} \\
&= \sum_{k=1}^n H_{\min}^{\downarrow,\epsilon}(A_k|A_1^{k-1}B)_p - n\mu - \frac{1}{\mu^2} - \log \frac{1}{1-\mu^2}
\end{aligned}$$

where in the third line, we use the quasi-concavity of H_{\min}^{\downarrow} [Tom16, Pg 73].

Thus, we have proven a chain rule for $H_{\min}^{\downarrow,\epsilon}$ of the desired form in the classical case. The primary challenge in generalizing this approach to the quantum setting lies in defining the auxiliary state. In the classical case, we could use the product of (suitably massaged) conditional distributions $q_{A_k|A_1^{k-1}B}^{(k)}$ to define the auxiliary distribution $r_{A_1^n B}$. However, the quantum generalisation of such a distribution is not unique. Moreover, such generalisations are all quite difficult to manipulate. A closely related problem is that we need to be able to prove a relative entropy bound similar to that in Eq. 61 for the auxiliary state. This also turns out to be quite challenging, especially because the quantum chain rules [SBT17, Theorem 4.1 and Proposition F.1] for the relative entropy do not yield something as simple and convenient as Eq. 56. We solve both of these problems indirectly. We will use a simple relative entropy bound between the state $\rho_{A_1^n B}$ and an unwieldy, yet simple exponential generalisation of the auxiliary probability distribution used above. Then, in order to convert this bound to a (measured) relative entropy bound between the state $\rho_{A_1^n B}$ and a simpler and more convenient auxiliary state, we employ the Generalised GT inequality (Lemma 3.1). Through this approach, we are able to delegate the pesky question of the correct generalisation of the auxiliary state to the GT inequality.

3.2 Proof for the quantum case

The following lemma uses the GT inequality as described above to create a generalisation of the auxiliary state used in the classical proof and also prove a measured relative entropy bound between the original state and this auxiliary state.

Lemma 3.1. *Suppose $\rho_{A_1^n B}$ is a normalised state and for $1 \leq k \leq n$ the normalised states $\bar{\rho}_{A_1^k B}^{(k)}$ are full rank and satisfy*

$$D\left(\rho_{A_1^k B} \parallel \bar{\rho}_{A_1^k B}^{(k)}\right) \leq \epsilon \quad (62)$$

for some $\epsilon > 0$. Let $\bar{\rho}_B^{(0)} := \rho_B$ for notational simplicity. Then, the subnormalised state⁽⁶⁾

$$\sigma_{A_1^n B} := \int_{-\infty}^{\infty} dt \beta_0(t) \prod_{k=0}^{n-1} \left[\left(\bar{\rho}_{A_1^k B}^{(k)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1^{k+1} B}^{(k+1)} \right)^{-\frac{1-it}{2}} \right] \cdot \bar{\rho}_{A_1^n B}^{(n)} \cdot \prod_{k=n-1}^0 \left[\left(\bar{\rho}_{A_1^{k+1} B}^{(k+1)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1^k B}^{(k)} \right)^{\frac{1+it}{2}} \right] \quad (63)$$

$$= \int_{-\infty}^{\infty} dt \beta_0(t) \rho_B^{\frac{1-it}{2}} \left(\bar{\rho}_B^{(1)} \right)^{-\frac{1-it}{2}} \left(\bar{\rho}_{A_1 B}^{(1)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1 B}^{(2)} \right)^{-\frac{1-it}{2}} \left(\bar{\rho}_{A_1^2 B}^{(2)} \right)^{\frac{1-it}{2}} \cdots \left(\bar{\rho}_{A_1^{n-1} B}^{(n)} \right)^{-\frac{1-it}{2}} \bar{\rho}_{A_1^n B}^{(n)} \cdot \left(\bar{\rho}_{A_1^{n-1} B}^{(n)} \right)^{-\frac{1+it}{2}} \cdots \left(\bar{\rho}_{A_1^2 B}^{(2)} \right)^{\frac{1+it}{2}} \left(\bar{\rho}_{A_1 B}^{(2)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1 B}^{(1)} \right)^{\frac{1+it}{2}} \left(\bar{\rho}_B^{(1)} \right)^{-\frac{1+it}{2}} \rho_B^{\frac{1+it}{2}}. \quad (64)$$

is such that

$$D_m(\rho_{A_1^n B} \| \sigma_{A_1^n B}) \leq n\epsilon. \quad (65)$$

Further, the partial states of σ are

$$\sigma_{A_1^k B} = \int_{-\infty}^{\infty} dt \beta_0(t) \prod_{j=0}^{k-1} \left[\left(\bar{\rho}_{A_1^j B}^{(j)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1^{j+1} B}^{(j+1)} \right)^{-\frac{1-it}{2}} \right] \cdot \bar{\rho}_{A_1^k B}^{(k)} \cdot \prod_{j=k-1}^0 \left[\left(\bar{\rho}_{A_1^{j+1} B}^{(j+1)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1^j B}^{(j)} \right)^{\frac{1+it}{2}} \right] \quad (66)$$

$$= \int_{-\infty}^{\infty} dt \beta_0(t) \rho_B^{\frac{1-it}{2}} \left(\bar{\rho}_B^{(1)} \right)^{-\frac{1-it}{2}} \left(\bar{\rho}_{A_1 B}^{(1)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1 B}^{(2)} \right)^{-\frac{1-it}{2}} \left(\bar{\rho}_{A_1^2 B}^{(2)} \right)^{\frac{1-it}{2}} \cdots \left(\bar{\rho}_{A_1^{k-1} B}^{(k)} \right)^{-\frac{1-it}{2}} \bar{\rho}_{A_1^k B}^{(k)} \cdot \left(\bar{\rho}_{A_1^{k-1} B}^{(k)} \right)^{-\frac{1+it}{2}} \cdots \left(\bar{\rho}_{A_1^2 B}^{(2)} \right)^{\frac{1+it}{2}} \left(\bar{\rho}_{A_1 B}^{(2)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1 B}^{(1)} \right)^{\frac{1+it}{2}} \left(\bar{\rho}_B^{(1)} \right)^{-\frac{1+it}{2}} \rho_B^{\frac{1+it}{2}}. \quad (67)$$

Hence, $\sigma_B = \rho_B$, and σ is normalised.

Proof. Our approach here is broadly based on the proof of [SBT17, Theorem 4.1]. Define the positive operator⁽⁷⁾

$$Q_{A_1^n B} := \exp \left(\sum_{k=1}^n \left(\log \bar{\rho}_{A_1^k B}^{(k)} - \log \bar{\rho}_{A_1^{k-1} B}^{(k)} \right) + \log \rho_B \right). \quad (68)$$

Note that if all the operators above commuted, $Q_{A_1^n B}$ (and also $\sigma_{A_1^n B}$) would be the probability distribution $\bar{\rho}_{A_n | A_1^{n-1} B}^{(n)} \bar{\rho}_{A_{n-1} | A_1^{n-2} B}^{(n-1)} \cdots \bar{\rho}_{A_1 | B}^{(1)} \rho_B$. As is often times the case with the

⁽⁶⁾In the following and throughout this paper, the product notation $\Pi_{j=0}^k A_k$ represents the operator product $A_0 A_1 \cdots A_k$ and the notation $\Pi_{j=k}^0 A_k$ represents the operator product $A_k A_{k-1} \cdots A_0$.

⁽⁷⁾By restricting ourselves to strictly positive operators $\bar{\rho}_{A_1^k B}^{(k)}$, we ensure that this expression is always well-defined.

quantum relative entropy, the best quantum generalisation of the above conditional distributions turns out to be an exponential⁽⁸⁾. For this operator, we have

$$\begin{aligned}
D(\rho_{A_1^n B} \| Q_{A_1^n B}) &= \text{tr}(\rho_{A_1^n B} (\log \rho_{A_1^n B} - \log Q_{A_1^n B})) \\
&= \text{tr} \rho_{A_1^n B} \left(\log \rho_{A_1^n B} - \sum_{k=1}^n \left(\log \bar{\rho}_{A_1^k B}^{(k)} - \log \bar{\rho}_{A_1^{k-1} B}^{(k)} \right) - \log \rho_B \right) \\
&= \sum_{k=1}^n \text{tr} \left(\rho_{A_1^n B} \left(\log \rho_{A_1^k B} - \log \bar{\rho}_{A_1^k B}^{(k)} \right) \right) - \sum_{k=1}^n \text{tr} \left(\rho_{A_1^n B} \left(\log \rho_{A_1^{k-1} B} - \log \bar{\rho}_{A_1^{k-1} B}^{(k)} \right) \right) \\
&= \sum_{k=1}^n \left(D(\rho_{A_1^k B} \| \bar{\rho}_{A_1^k B}^{(k)}) - D(\rho_{A_1^{k-1} B} \| \bar{\rho}_{A_1^{k-1} B}^{(k)}) \right) \\
&\leq \sum_{k=1}^n D(\rho_{A_1^k B} \| \bar{\rho}_{A_1^k B}^{(k)}) \\
&\leq n\epsilon
\end{aligned} \tag{69}$$

where in the second last line, we have used the fact that the relative entropy of two normalised states is positive and in the last line, we have used the bound in Eq. 62. Next, by using the variational expression for the relative entropy (Eq. 44), we see that

$$\begin{aligned}
n\epsilon &\geq D(\rho_{A_1^n B} \| Q_{A_1^n B}) \\
&= \sup_{\omega_{A_1^n B} > 0} \left\{ \text{tr}(\rho_{A_1^n B} \log \omega_{A_1^n B}) + 1 - \text{tr} \exp \left(\sum_{k=1}^n \left(\log \bar{\rho}_{A_1^k B}^{(k)} - \log \bar{\rho}_{A_1^{k-1} B}^{(k)} \right) + \log \rho_B + \log \omega_{A_1^n B} \right) \right\}.
\end{aligned} \tag{70}$$

The trace exponential above can be bounded using the Generalised GT inequality (Theorem 2.3) as

$$\begin{aligned}
&\text{tr} \exp \left(\sum_{k=1}^n \left(\log \bar{\rho}_{A_1^k B}^{(k)} - \log \bar{\rho}_{A_1^{k-1} B}^{(k)} \right) + \log \rho_B + \log \omega_{A_1^n B} \right) \\
&\leq \int_{-\infty}^{\infty} dt \beta_0(t) \text{tr} \left(\omega_{A_1^n B} \rho_B^{\frac{1-it}{2}} \left(\bar{\rho}_B^{(1)} \right)^{-\frac{1-it}{2}} \left(\bar{\rho}_{A_1 B}^{(1)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1 B}^{(2)} \right)^{-\frac{1-it}{2}} \left(\bar{\rho}_{A_1^2 B}^{(2)} \right)^{\frac{1-it}{2}} \dots \left(\bar{\rho}_{A_1^{n-1} B}^{(n)} \right)^{-\frac{1-it}{2}} \right. \\
&\quad \left. \bar{\rho}_{A_1^n B}^{(n)} \cdot \left(\bar{\rho}_{A_1^{n-1} B}^{(n)} \right)^{-\frac{1+it}{2}} \dots \left(\bar{\rho}_{A_1^2 B}^{(2)} \right)^{\frac{1+it}{2}} \left(\bar{\rho}_{A_1 B}^{(2)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1 B}^{(1)} \right)^{\frac{1+it}{2}} \left(\bar{\rho}_B^{(1)} \right)^{-\frac{1+it}{2}} \rho_B^{\frac{1+it}{2}} \right) \\
&= \int_{-\infty}^{\infty} dt \beta_0(t) \text{tr} \left(\omega_{A_1^n B} \prod_{k=0}^{n-1} \left[\left(\bar{\rho}_{A_1^k B}^{(k)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1^{k+1} B}^{(k+1)} \right)^{-\frac{1-it}{2}} \right] \cdot \bar{\rho}_{A_1^n B}^{(n)} \cdot \prod_{k=n-1}^0 \left[\left(\bar{\rho}_{A_1^{k+1} B}^{(k+1)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1^k B}^{(k)} \right)^{\frac{1+it}{2}} \right] \right) \\
&= \text{tr}(\omega_{A_1^n B} \sigma_{A_1^n B})
\end{aligned} \tag{71}$$

⁽⁸⁾Classically, we have that $D(P_{AB} \| Q_{AB}) - D(P_A \| Q_A) = D(P_{AB} \| P_A Q_{B|A})$. Quantumly, this equation is directly best generalised as $D(\rho_{AB} \| \sigma_{AB}) - D(\rho_A \| \sigma_A) = D(\rho_{AB} \| \exp(\log \sigma_{AB} - \log \sigma_A + \log \rho_A))$. Though, this is not very useful on its own.

for the state σ defined in the statement of the lemma. Plugging the bound above into Eq. 70, we get

$$\begin{aligned}
n\epsilon &\geq D(\rho_{A_1^n B} \| Q_{A_1^n B}) \\
&\geq \sup_{\omega_{A_1^n B} > 0} \left\{ \text{tr}(\rho_{A_1^n B} \log \omega_{A_1^n B}) + 1 - \text{tr}(\omega_{A_1^n B} \sigma_{A_1^n B}) \right\} \\
&= D_m(\rho_{A_1^n B} \| \sigma_{A_1^n B}).
\end{aligned} \tag{72}$$

□

We use the following theorem and its corollary to bound the relative entropy between two states which are close to each other and have a bounded max-relative entropy.

Theorem 3.2 ([Aud14, Theorem 4]). *For two normalised quantum states ρ and σ , and $\delta \in (0, 1]$, we have*

$$D(\rho \| (1 - \delta)\sigma + \delta\rho) \leq \frac{1}{2} \|\rho - \sigma\|_1 \log \frac{1}{\delta}. \tag{73}$$

Corollary 3.3. *Suppose two normalised quantum states ρ_{AB} and σ_{AB} are such that*

$$\frac{1}{2} \|\rho_{AB} - \sigma_{AB}\|_1 \leq \epsilon \tag{74}$$

for $\epsilon \in [0, 1]$ and

$$\delta\tau_A \otimes \rho_B \leq \sigma_{AB} \tag{75}$$

where $\delta \in (0, 1)$ and τ_A is the completely mixed state on register A . Then,

$$D(\rho_{AB} \| \sigma_{AB}) \leq \frac{\epsilon}{1 - \delta/|A|^2} \log \frac{|A|^2}{\delta}. \tag{76}$$

Proof. Note that

$$\frac{\delta}{|A|^2} \rho_{AB} \leq \delta\tau_A \otimes \rho_B \leq \sigma_{AB}.$$

So, we can write σ as

$$\sigma_{AB} = \left(1 - \frac{\delta}{|A|^2}\right) \sigma'_{AB} + \frac{\delta}{|A|^2} \rho_{AB} \tag{77}$$

for some normalised quantum state σ'_{AB} . For this state, we have

$$\begin{aligned} \frac{1}{2} \|\rho_{AB} - \sigma'_{AB}\|_1 &= \frac{1}{2} \left\| \rho_{AB} - \frac{1}{1 - \delta/|A|^2} (\sigma_{AB} - \frac{\delta}{|A|^2} \rho_{AB}) \right\|_1 \\ &= \frac{1}{2} \frac{1}{1 - \delta/|A|^2} \left\| \rho_{AB} - \frac{\delta}{|A|^2} \rho_{AB} - (\sigma_{AB} - \frac{\delta}{|A|^2} \rho_{AB}) \right\|_1 \\ &\leq \frac{\epsilon}{1 - \delta/|A|^2}. \end{aligned}$$

Using Theorem 3.2, we have that

$$\begin{aligned} D(\rho_{AB} \| \sigma_{AB}) &\leq \frac{1}{2} \|\rho_{AB} - \sigma'_{AB}\|_1 \log \frac{|A|^2}{\delta} \\ &\leq \frac{\epsilon}{1 - \delta/|A|^2} \log \frac{|A|^2}{\delta}. \end{aligned}$$

□

Now we are ready to state and prove the universal chain rule for the smooth min-entropy for quantum states.

Theorem 3.4. *For a normalised quantum state $\rho_{A_1^n B}$ such that for all $k \in [n]$ the dimension $|A_k| = |A|$, and $\epsilon \in (0, 1)$ such that $\mu = \left(\frac{2\epsilon}{1 - \epsilon/|A|^2} \log \frac{|A|^2}{\epsilon} \right)^{1/3} = O\left(\left(\epsilon \log \frac{|A|}{\epsilon}\right)^{1/3}\right)$ lies in $(0, 1)$, we have the chain rule*

$$H_{\min}^{\downarrow, 2\mu + \epsilon/2}(A_1^n | B)_\rho \geq \sum_{k=1}^n H_{\min}^{\downarrow, \epsilon/2}(A_k | A_1^{k-1} B)_\rho - n\mu - \frac{1}{\mu^2} - \log \frac{1}{1 - \mu^2} - \log \left(\frac{2}{\mu^2} + \frac{1}{1 - \mu} \right) \quad (78)$$

Proof. Case 1: Let's first consider states $\rho_{A_1^n B}$, which are full rank.

For every $k \in [n]$, define $\lambda_k := H_{\min}^{\downarrow, \epsilon}(A_k | A_1^{k-1} B)_\rho$ and let the state $\tilde{\rho}_{A_1^k B}^{(k)}$ be such that

$$P(\rho_{A_1^k B}, \tilde{\rho}_{A_1^k B}^{(k)}) \leq \epsilon \quad (79)$$

$$\tilde{\rho}_{A_1^k B}^{(k)} \leq e^{-\lambda_k} \mathbb{1}_{A_k} \otimes \tilde{\rho}_{A_1^{k-1} B}^{(k)}. \quad (80)$$

Without loss of generality, we will choose the states $\tilde{\rho}_{A_1^k B}^{(k)}$ to be normalised states, since dividing $\tilde{\rho}_{A_1^k B}^{(k)}$ by a normalising factor $N \leq 1$ only decreases its purified distance from ρ and leaves the operator inequality above invariant. Further, let $\delta \in (0, 1)$ be a small parameter. For every $k \in [n]$, we define the normalised states,

$$\bar{\rho}_{A_1^k B}^{(k)} := (1 - \delta) \tilde{\rho}_{A_1^k B}^{(k)} + \delta \tau_{A_k} \otimes \rho_{A_1^{k-1} B} \quad (81)$$

where τ_{A_k} is the completely mixed state on register A_k . Also, define $\bar{\rho}_B^{(0)} := \rho_B$ for notational convenience. Since, $\rho_{A_1^{k-1}B}$ is full rank, $\bar{\rho}_{A_1^k B}^{(k)}$ are also full rank for all k .

For each $k \in [n]$, these states satisfy

$$\begin{aligned} \frac{1}{2} \left\| \rho_{A_1^k B} - \bar{\rho}_{A_1^k B}^{(k)} \right\|_1 &\leq \frac{1-\delta}{2} \left\| \rho_{A_1^k B} - \tilde{\rho}_{A_1^k B}^{(k)} \right\|_1 + \frac{\delta}{2} \left\| \rho_{A_1^k B} - \tau_{A_k} \otimes \rho_{A_1^{k-1} B} \right\|_1 \\ &\leq \epsilon + \delta \end{aligned} \quad (82)$$

and

$$D(\rho_{A_1^k B} \| \bar{\rho}_{A_1^k B}^{(k)}) \leq \frac{\epsilon + \delta}{1 - \delta/|A|^2} \log \frac{|A|^2}{\delta}. \quad (83)$$

using Corollary 3.3. Let's define the right-hand side above as

$$z(\epsilon, \delta) := \frac{\epsilon + \delta}{1 - \delta/|A|^2} \log \frac{|A|^2}{\delta}. \quad (84)$$

Note that this can be made small ($O(\epsilon \log 1/\epsilon)$) by, for example, choosing $\delta = \epsilon$.

Using Lemma 3.1, we have that the state σ defined as

$$\sigma_{A_1^n B} := \int_{-\infty}^{\infty} dt \beta_0(t) \prod_{k=0}^{n-1} \left[\left(\bar{\rho}_{A_1^k B}^{(k)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1^{k+1} B}^{(k+1)} \right)^{-\frac{1-it}{2}} \right] \cdot \bar{\rho}_{A_1^n B}^{(n)} \cdot \prod_{k=n-1}^0 \left[\left(\bar{\rho}_{A_1^{k+1} B}^{(k+1)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1^k B}^{(k)} \right)^{\frac{1+it}{2}} \right] \quad (85)$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} dt \beta_0(t) \rho_B^{\frac{1-it}{2}} \left(\bar{\rho}_B^{(1)} \right)^{-\frac{1-it}{2}} \left(\bar{\rho}_{A_1 B}^{(1)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1 B}^{(2)} \right)^{-\frac{1-it}{2}} \left(\bar{\rho}_{A_1^2 B}^{(2)} \right)^{\frac{1-it}{2}} \cdots \left(\bar{\rho}_{A_1^{n-1} B}^{(n)} \right)^{-\frac{1-it}{2}} \\ &\quad \bar{\rho}_{A_1^n B}^{(n)} \cdot \left(\bar{\rho}_{A_1^{n-1} B}^{(n)} \right)^{-\frac{1+it}{2}} \cdots \left(\bar{\rho}_{A_1^2 B}^{(2)} \right)^{\frac{1+it}{2}} \left(\bar{\rho}_{A_1 B}^{(2)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1 B}^{(1)} \right)^{\frac{1+it}{2}} \left(\bar{\rho}_B^{(1)} \right)^{-\frac{1+it}{2}} \rho_B^{\frac{1+it}{2}} \end{aligned} \quad (86)$$

is normalised and satisfies

$$D_m(\rho_{A_1^n B} \| \sigma_{A_1^n B}) \leq n z(\epsilon, \delta). \quad (87)$$

Further,

$$\sigma_{A_1^k B} = \int_{-\infty}^{\infty} dt \beta_0(t) \prod_{j=0}^{k-1} \left[\left(\bar{\rho}_{A_1^j B}^{(j)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1^{j+1} B}^{(j+1)} \right)^{-\frac{1-it}{2}} \right] \cdot \bar{\rho}_{A_1^k B}^{(k)} \cdot \prod_{j=k-1}^0 \left[\left(\bar{\rho}_{A_1^{j+1} B}^{(j+1)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1^j B}^{(j)} \right)^{\frac{1+it}{2}} \right] \quad (88)$$

for $k \in [n]$. Using the substate theorem⁽⁹⁾, we have that for $\mu := z(\epsilon, \delta)^{1/3}$ (we omit the dependence of μ on ϵ and δ for clarity)

$$D_{\max}^{\mu}(\rho_{A_1^n B} \| \sigma_{A_1^n B}) \leq \frac{D_m(\rho_{A_1^n B} \| \sigma_{A_1^n B}) + 1}{\mu^2} + \log \frac{1}{1 - \mu^2} \quad (89)$$

$$\leq n\mu + \frac{1}{\mu^2} + \log \frac{1}{1 - \mu^2}. \quad (90)$$

A simple application of the entropic triangle inequality in Eq. 35 gives us that

$$\begin{aligned} H_{\min}^{\mu}(A_1^n | B)_{\rho} &\geq H_{\min}(A_1^n | B)_{\sigma} - D_{\max}^{\mu}(\rho_{A_1^n B} \| \sigma_{A_1^n B}) \\ &\geq H_{\min}(A_1^n | B)_{\sigma} - n\mu - \frac{1}{\mu^2} - \log \frac{1}{1 - \mu^2} \\ &\geq H_{\min}^{\downarrow}(A_1^n | B)_{\sigma} - n\mu - \frac{1}{\mu^2} - \log \frac{1}{1 - \mu^2} \\ &\geq \sum_{k=1}^n H_{\min}^{\downarrow}(A_k | A_1^{k-1} B)_{\sigma} - n\mu - \frac{1}{\mu^2} - \log \frac{1}{1 - \mu^2} \end{aligned} \quad (91)$$

where we have used the chain rule for H_{\min}^{\downarrow} (Eq. 12). We will now show that σ is such that $H_{\min}^{\downarrow}(A_k | A_1^{k-1} B)_{\sigma} \geq H_{\min}^{\downarrow, \epsilon}(A_k | A_1^{k-1} B)_{\rho}$.

Using the quasi-concavity of H_{\min}^{\downarrow} for $\bar{\rho}^{(k)}$ [Tom16, Pg 73], we have

$$\begin{aligned} H_{\min}^{\downarrow}(A_k | A_1^{k-1} B)_{\bar{\rho}^{(k)}} &\geq \min \left\{ H_{\min}^{\downarrow}(A_k | A_1^{k-1} B)_{\bar{\rho}^{(k)}}, H_{\min}^{\downarrow}(A_k | A_1^{k-1} B)_{\tau_{A_k} \otimes \rho_{A_1^{k-1} B}} \right\} \\ &\geq \min \left\{ H_{\min}^{\downarrow}(A_k | A_1^{k-1} B)_{\bar{\rho}^{(k)}}, \log |A| \right\} \\ &\geq H_{\min}^{\downarrow}(A_k | A_1^{k-1} B)_{\bar{\rho}^{(k)}}. \end{aligned}$$

Therefore, we have that

$$\bar{\rho}_{A_1^k B}^{(k)} \leq e^{-\lambda_k} \mathbb{1}_{A_k} \otimes \bar{\rho}_{A_1^{k-1} B}^{(k)}. \quad (92)$$

⁽⁹⁾It is quite remarkable that the substate theorem works with a D_m bound and the generalised GT inequality only yields a D_m bound. All our proofs exploit this fact. Whether it is an incredible coincidence or an indication of the tightness of these bounds: you decide.

This implies that

$$\begin{aligned}
\sigma_{A_1^k B} &= \int_{-\infty}^{\infty} dt \beta_0(t) \prod_{j=0}^{k-1} \left[\left(\bar{\rho}_{A_1^j B}^{(j)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1^j B}^{(j+1)} \right)^{-\frac{1-it}{2}} \right] \cdot \bar{\rho}_{A_1^k B}^{(k)} \cdot \prod_{j=k-1}^0 \left[\left(\bar{\rho}_{A_1^j B}^{(j+1)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1^j B}^{(j)} \right)^{\frac{1+it}{2}} \right] \\
&\leq e^{-\lambda_k} \int_{-\infty}^{\infty} dt \beta_0(t) \prod_{j=0}^{k-1} \left[\left(\bar{\rho}_{A_1^j B}^{(j)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1^j B}^{(j+1)} \right)^{-\frac{1-it}{2}} \right] \cdot \mathbb{1}_{A_k} \otimes \bar{\rho}_{A_1^{k-1} B}^{(k)} \cdot \prod_{j=k-1}^0 \left[\left(\bar{\rho}_{A_1^j B}^{(j+1)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1^j B}^{(j)} \right)^{\frac{1+it}{2}} \right] \\
&\leq e^{-\lambda_k} \mathbb{1}_{A_k} \otimes \int_{-\infty}^{\infty} dt \beta_0(t) \prod_{j=0}^{k-2} \left[\left(\bar{\rho}_{A_1^j B}^{(j)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1^j B}^{(j+1)} \right)^{-\frac{1-it}{2}} \right] \cdot \bar{\rho}_{A_1^{k-1} B}^{(k-1)} \cdot \prod_{j=k-2}^0 \left[\left(\bar{\rho}_{A_1^j B}^{(j+1)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1^j B}^{(j)} \right)^{\frac{1+it}{2}} \right] \\
&= e^{-\lambda_k} \mathbb{1}_{A_k} \otimes \sigma_{A_1^{k-1} B}.
\end{aligned}$$

Or, equivalently that $H_{\min}^{\downarrow}(A_k|A_1^{k-1}B)_{\sigma} \geq \lambda_k = H_{\min}^{\downarrow, \epsilon}(A_k|A_1^{k-1}B)_{\rho}$. Plugging this bound in Eq. 91, we get

$$H_{\min}^{\mu}(A_1^n|B)_{\rho} \geq \sum_{k=1}^n H_{\min}^{\downarrow, \epsilon}(A_k|A_1^{k-1}B)_{\rho} - n\mu - \frac{1}{\mu^2} - \log \frac{1}{1-\mu^2}. \quad (93)$$

Using Eq. 34, we can upper bound the left-hand side with $H_{\min}^{\downarrow, 2\mu}(A_1^n|B)$ to derive

$$H_{\min}^{\downarrow, 2\mu}(A_1^n|B)_{\rho} \geq \sum_{k=1}^n H_{\min}^{\downarrow, \epsilon}(A_k|A_1^{k-1}B)_{\rho} - n\mu - \frac{1}{\mu^2} - \log \frac{1}{1-\mu^2} - \log \left(\frac{2}{\mu^2} + \frac{1}{1-\mu} \right). \quad (94)$$

Case 2: The above proves the Theorem for the case when ρ was full rank. If ρ was not full rank, then for an arbitrary $\nu \in (0, 1)$, the state $\rho'_{A_1^n B} := (1-\nu)\rho_{A_1^n B} + \nu\tau_{A_1^n B}$, which has full support, satisfies

$$H_{\min}^{\downarrow, 2\mu}(A_1^n|B)_{\rho'} \geq \sum_{k=1}^n H_{\min}^{\downarrow, \epsilon}(A_k|A_1^{k-1}B)_{\rho'} - n\mu - \frac{1}{\mu^2} - \log \frac{1}{1-\mu^2} - \log \left(\frac{2}{\mu^2} + \frac{1}{1-\mu} \right) \quad (95)$$

which implies that

$$H_{\min}^{\downarrow, 2\mu + \sqrt{2\nu}}(A_1^n|B)_{\rho} \geq \sum_{k=1}^n H_{\min}^{\downarrow, \epsilon - \sqrt{2\nu}}(A_k|A_1^{k-1}B)_{\rho} - n\mu - \frac{1}{\mu^2} - \log \frac{1}{1-\mu^2} - \log \left(\frac{2}{\mu^2} + \frac{1}{1-\mu} \right) \quad (96)$$

for every $\nu \in (0, 1)$. Unfortunately, since H_{\min}^{\downarrow} is not continuous, we cannot use continuity to claim the above for ρ itself. We can, however, simply choose $\nu = \frac{\epsilon^2}{8}$, which gives us

$$H_{\min}^{\downarrow, 2\mu + \epsilon/2}(A_1^n|B)_{\rho} \geq \sum_{k=1}^n H_{\min}^{\downarrow, \epsilon/2}(A_k|A_1^{k-1}B)_{\rho'} - n\mu - \frac{1}{\mu^2} - \log \frac{1}{1-\mu^2} - \log \left(\frac{2}{\mu^2} + \frac{1}{1-\mu} \right) \quad (97)$$

where $\mu = z(\epsilon, \delta)^{1/3} = \left(\frac{\epsilon + \delta}{1 - \delta/|A|^2} \log \frac{|A|^2}{\delta} \right)^{1/3}$. To derive the bound in the Theorem, we make the concrete choice $\delta = \epsilon$. \square

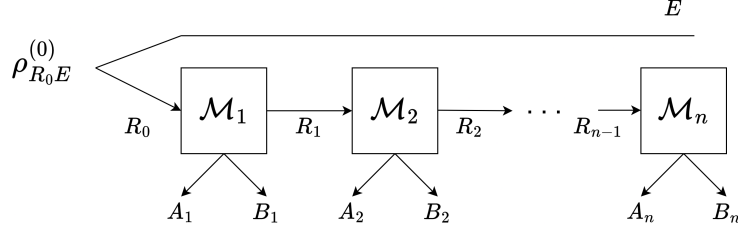


Figure 1: The setting for entropy accumulation. The channels \mathcal{M}_k are sequentially applied to the registers R_{k-1} and produce the secret information A_k and the side information B_k .

4 Unstructured approximate entropy accumulation

The entropy accumulation theorem (EAT) [DFR20] (also see [MFSR22]) is one of the most important tools available to bound the smooth min-entropy produced by a sequential process. It is useful for proving the security of quantum key distribution (QKD) [DFR20], device-independent QKD (DI-QKD) and randomness expansion [AFDF⁺18, AFRV19, LLR⁺21]. EAT can be used to bound the smooth min-entropy of a state produced by a process which has the structure shown in Fig. 1. In such a process, the output state $\rho_{A_1^n B_1^n E} = \mathcal{M}_n \circ \dots \circ \mathcal{M}_1(\rho_{R_0 E}^{(0)})$ is produced by sequentially applying the channels \mathcal{M}_k to the registers R_{k-1} for $k \in [n]$ to produce the registers A_k, B_k and R_k . In cryptographic applications of this theorem, the registers A_k represent partially secret information produced during the k th round of the protocol and the registers B_k denote the side information about these registers, which is leaked to the adversary. EAT provides a lower bound for the smooth min-entropy of the secret information conditioned on the adversary's state in these applications. Specifically, according to the theorem, as long as the output of the process in Fig. 1, $\rho_{A_1^n B_1^n E}$ satisfies the Markov chain

$$A_1^{k-1} \leftrightarrow B_1^{k-1} E \leftrightarrow B_k \quad (98)$$

for every $k \in [n]$, the smooth min-entropy satisfies

$$H_{\min}^{\delta}(A_1^n | B_1^n E)_{\rho} \geq \sum_{k=1}^n \inf_{\omega_{R_{k-1} \tilde{R}}} H(A_k | B_k \tilde{R})_{\mathcal{M}_k(\omega)} - c\sqrt{n} \quad (99)$$

where the infimum is over all input states to the channel \mathcal{M}_k and $c > 0$ is a constant that only depends on $|A|$ (the size of registers A_k) and δ . In this section, we will prove Theorem 4.1, which is an approximate version of EAT. We show that for any state $\rho_{A_1^n B_1^n E}$ whose partial states $\rho_{A_1^k B_1^k E}$ can be ϵ -approximated as the output of channels \mathcal{M}_k , which sample the side information B_k independent of the previous registers $A_1^{k-1} B_1^{k-1} E$ we can recover a statement similar to EAT⁽¹⁰⁾. Crucially, Theorem 4.1 does not require $\rho_{A_1^n B_1^n E}$

⁽¹⁰⁾This seems to be equivalent to requiring that all outputs of the channel \mathcal{M}_k satisfy $A_1^{k-1} \leftrightarrow B_1^{k-1} E \leftrightarrow B_k$.

to be produced by a structured process like in Fig. 1; the state may even be produced by a completely parallel process. In fact, our central motivation to develop this theorem was to prove the security of parallel device independent QKD. We do this in our upcoming work [MD24]. For this reason, we call this theorem the *unstructured* approximate EAT.

Previously, we also proved an approximate entropy accumulation theorem in [MD23, Theorem 5.1]. The key difference between these two theorems lies in their applicability: [MD23, Theorem 5.1] applies only to states ρ produced by a sequential process of the same structure as in EAT (Fig. 1), while Theorem 4.1 can be applied to states generated by a completely unstructured or parallel process. Furthermore, [MD23] considers approximation at the level of channels. Specifically, it considers the diamond norm approximation of the channels \mathcal{M}_k producing the output state ρ in Fig. 1 by *nicer* channels \mathcal{M}'_k , which satisfy certain Markov chain conditions. This is a strong approximation condition, since it requires the outputs of \mathcal{M}_k and \mathcal{M}'_k to be approximately equal for all input states. In contrast, in Theorem 4.1, we only need the trace distance between $\rho_{A_1^k B_1^k E}$ and the output of \mathcal{M}_k to be small for a single input state. We pay the price for these weaker conditions in terms of a much stronger condition on the side information produced by the approximation channels. Additionally, the smoothing parameter in Theorem 4.1 depends on the approximation parameter ϵ and cannot be made arbitrarily small. In comparison, the smoothing parameter for the smooth min-entropy in [MD23, Theorem 5.1] is independent of the approximation parameter and can be chosen arbitrarily small. Consequently, Theorem 4.1 may have limited utility in cryptographic scenarios where experimental noise or imperfections need to be accounted for, as cryptographic protocols typically require the smoothing parameter to be independent of the noise parameters. Under an analysis using Theorem 4.1, however, the noise parameters would determine the approximation parameter and hence the smoothing parameter. Nonetheless, we believe that Theorem 4.1 will be a valuable theoretical tool in scenarios where the approximation parameter can be made arbitrarily small, as demonstrated in the security analysis of parallel DI-QKD [MD24].

Theorem 4.1. *Let $\epsilon \in (0, 1)$ and for every $k \in [n]$ the registers A_k and B_k be such that $|A_k| = |A|$ and $|B_k| = |B|$. Suppose, the state $\rho_{A_1^n B_1^n E}$ is such that for every $k \in [n]$, there exists a channel $\mathcal{M}_k : R_k \rightarrow A_k B_k$ such that for all inputs X_{R_k} ,*

$$\text{tr}_{A_k} \circ \mathcal{M}_k (X_{R_k}) = \text{tr}(X) \theta_{B_k}^{(k)} \quad (100)$$

for some state $\theta_{B_k}^{(k)}$ (11), and a state $\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E R_k}^{(k,0)}$ for which

$$\frac{1}{2} \left\| \rho_{A_1^k B_1^k E} - \mathcal{M}_k(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E R_k}^{(k,0)}) \right\|_1 \leq \epsilon. \quad (101)$$

⁽¹¹⁾This condition can be relaxed to requiring channels \mathcal{M}_k such that for all input states $\sigma_{A_1^{k-1} B_1^{k-1} E R_k}$, the output state $\mathcal{M}_k(\sigma)$ satisfies the Markov chain condition $A_1^{k-1} \leftrightarrow B_1^{k-1} E \leftrightarrow B_k$. This condition seems to imply the independence condition used in the theorem.

Then, as long as $\mu := \left(\frac{4\sqrt{\epsilon} + \epsilon}{1 - \epsilon / (|A||B|)^2} \log \frac{|A|^2|B|^2}{\epsilon} \right)^{1/3} = O\left(\epsilon^{1/6} \left(\log \frac{|A|^2|B|^2}{\epsilon} \right)^{1/3}\right)$ lies in $(0, 1)$, we have the following lower bound for the smooth min-entropy of ρ :

$$H_{\min}^{\mu+\epsilon'}(A_1^n | B_1^n E)_\rho \geq \sum_{k=1}^n \inf_{\omega_{R_k \tilde{R}_k}} H(A_k | B_k \tilde{R}_k)_{\mathcal{M}_k(\omega)} - 3n\sqrt{\mu} \log(1 + 2|A|) - \frac{\log(1 + 2|A|)}{\sqrt{\mu}} \left(\frac{2}{\mu^2} + 2 \log \frac{1}{1 - \mu^2} + g_1(\epsilon', \mu) \right) \quad (102)$$

where the infimum is over all the input states $\omega_{R_k \tilde{R}_k}$ to the channel \mathcal{M}_k and $g_1(x, y) := -\log(1 - \sqrt{1 - x^2}) - \log(1 - y^2)$.

The proof of this theorem follows almost the same approach as that of Theorem 3.4. The major difference being that here we use the stronger triangle inequality (Eq. 36) to bound the smooth min-entropy with the α -Rényi conditional entropy of an auxiliary state. We then use the chain rule for these entropies along with the independence conditions on B_k to derive the lower bound above similar to [DFR20].

Proof. Case 1: First let's consider states $\rho_{A_1^n B_1^n E}$, which have full rank. Let $\nu \in (0, 1)$ be an arbitrarily chosen small parameter. For every $k \in [n]$, define the states

$$\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E R_k}^{(k,0)} := (1 - \nu) \tilde{\rho}_{A_1^{k-1} B_1^{k-1} E R_k}^{(k,0)} + \nu \tau_{A_1^{k-1} B_1^{k-1} E R_k} \quad (103)$$

$$\tilde{\rho}_{A_1^k B_1^k E}^{(k)} := \mathcal{M}_k \left(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E R_k}^{(k,0)} \right) \quad (104)$$

We make this modification to $\tilde{\rho}^{(k)}$ states so that $\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E}^{(k)}$ is full rank. For each k these states satisfy

$$\begin{aligned} & \frac{1}{2} \left\| \rho_{A_1^k B_1^k E} - \tilde{\rho}_{A_1^k B_1^k E}^{(k)} \right\|_1 \\ &= \frac{1}{2} \left\| \rho_{A_1^k B_1^k E} - (1 - \nu) \mathcal{M}_k \left(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E R_k}^{(k,0)} \right) - \nu \mathcal{M}_k \left(\tau_{A_1^{k-1} B_1^{k-1} E R_k} \right) \right\|_1 \\ &\leq \frac{1 - \nu}{2} \left\| \rho_{A_1^k B_1^k E} - \mathcal{M}_k \left(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E R_k}^{(k,0)} \right) \right\|_1 + \nu \\ &\leq \epsilon + \nu. \end{aligned} \quad (105)$$

Now, for each k , we define the states

$$\omega_{A_1^{k-1} B_1^{k-1} R_k E}^{(k,0)} := \rho_{A_1^{k-1} B_1^{k-1} E}^{1/2} \left(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E}^{(k,0)} \right)^{-1/2} \tilde{\rho}_{A_1^{k-1} B_1^{k-1} R_k E}^{(k,0)} \left(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E}^{(k,0)} \right)^{-1/2} \rho_{A_1^{k-1} B_1^{k-1} E}^{1/2} \quad (106)$$

$$\omega_{A_1^k B_1^k E}^{(k)} := \rho_{A_1^{k-1} B_1^{k-1} E}^{1/2} \left(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E}^{(k)} \right)^{-1/2} \tilde{\rho}_{A_1^k B_1^k E}^{(k)} \left(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E}^{(k)} \right)^{-1/2} \rho_{A_1^{k-1} B_1^{k-1} E}^{1/2} \quad (107)$$

$$= \mathcal{M}_k \left(\omega_{A_1^{k-1} B_1^{k-1} R_k E}^{(k,0)} \right). \quad (108)$$

Since, we defined $\tilde{\rho}_{A_1^{k-1}B_1^{k-1}E}^{(k)}$ to be full rank, we have that

$$\omega_{A_1^{k-1}B_1^{k-1}E}^{(k)} = \rho_{A_1^{k-1}B_1^{k-1}E}. \quad (109)$$

Using Lemma 2.5, we have that

$$\begin{aligned} \frac{1}{2} \left\| \rho_{A_1^k B_1^k E} - \omega_{A_1^k B_1^k E}^{(k)} \right\|_1 &\leq (\sqrt{2} + 1) P(\rho_{A_1^k B_1^k E}, \tilde{\rho}_{A_1^k B_1^k E}) \\ &\leq (\sqrt{2} + 1) \sqrt{2(\epsilon + \nu)} \\ &\leq 4\sqrt{\epsilon + \nu}. \end{aligned} \quad (110)$$

Let $\delta \in (0, 1)$ be a small parameter (to be set equal to ϵ later). Finally, for every $k \in [n]$, we define the states

$$\bar{\rho}_{A_1^k B_1^k E}^{(k)} := (1 - \delta) \omega_{A_1^k B_1^k E}^{(k)} + \delta \tau_{A_k B_k} \otimes \rho_{A_1^{k-1} B_1^{k-1} E} \quad (111)$$

$$= (1 - \delta) \omega_{A_1^k B_1^k E}^{(k)} + \delta \tau_{A_k B_k} \otimes \omega_{A_1^{k-1} B_1^{k-1} E}^{(k)} \quad (112)$$

where $\tau_{A_k B_k}$ is the completely mixed state on the registers A_k and B_k . Also, define $\bar{\rho}_E^{(0)} := \rho_E$. Let $\Delta_k : R_k \rightarrow A_k B_k$ be the map which traces out the register R_k and simply outputs $\tau_{A_k B_k}$ and let $\mathcal{M}_k^\delta := (1 - \delta) \mathcal{M}_k + \delta \Delta_k$. Then, we have that

$$\begin{aligned} \bar{\rho}_{A_1^k B_1^k E}^{(k)} &= (1 - \delta) \omega_{A_1^k B_1^k E}^{(k)} + \delta \tau_{A_k B_k} \otimes \omega_{A_1^{k-1} B_1^{k-1} E}^{(k)} \\ &= ((1 - \delta) \mathcal{M}_k + \delta \Delta_k) \left(\omega_{A_1^{k-1} B_1^{k-1} R_k E}^{(k,0)} \right) \\ &= \mathcal{M}_k^\delta \left(\omega_{A_1^{k-1} B_1^{k-1} R_k E}^{(k,0)} \right). \end{aligned} \quad (113)$$

We also have that

$$\frac{1}{2} \left\| \bar{\rho}_{A_1^k B_1^k E}^{(k)} - \rho_{A_1^k B_1^k E} \right\|_1 \leq 4\sqrt{\epsilon + \nu} + \delta. \quad (114)$$

Using Corollary 3.3, this gives us

$$D(\rho_{A_1^k B_1^k E} \| \bar{\rho}_{A_1^k B_1^k E}^{(k)}) \leq \frac{4\sqrt{\epsilon + \nu} + \delta}{1 - \delta/(|A||B|)^2} \log \frac{|A|^2 |B|^2}{\delta}. \quad (115)$$

We define the above bound as $z(\epsilon + \nu, \delta)$. Once again, using Lemma 3.1 (for $A_k \leftarrow A_k B_k$,

$B \leftarrow E$ and $\bar{\rho}_{A_1^k B}^{(k)} \leftarrow \bar{\rho}_{A_1^k B_1^k E}^{(k)}$ we have that auxiliary state

$$\sigma_{A_1^n B_1^n E} := \int_{-\infty}^{\infty} dt \beta_0(t) \prod_{k=0}^{n-1} \left[\left(\bar{\rho}_{A_1^k B_1^k E}^{(k)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1^k B_1^k E}^{(k+1)} \right)^{-\frac{1-it}{2}} \right] \cdot \bar{\rho}_{A_1^n B_1^n E}^{(n)} \cdot \prod_{k=n-1}^0 \left[\left(\bar{\rho}_{A_1^k B_1^k E}^{(k+1)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1^k B_1^k E}^{(k)} \right)^{\frac{1+it}{2}} \right] \quad (116)$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} dt \beta_0(t) \rho_E^{\frac{1-it}{2}} \left(\bar{\rho}_E^{(1)} \right)^{-\frac{1-it}{2}} \left(\bar{\rho}_{A_1 B_1 E}^{(1)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1 B_1 E}^{(2)} \right)^{-\frac{1-it}{2}} \left(\bar{\rho}_{A_1^2 B_1^2 E}^{(2)} \right)^{\frac{1-it}{2}} \cdots \left(\bar{\rho}_{A_1^{n-1} B_1^{n-1} E}^{(n)} \right)^{-\frac{1-it}{2}} \\ &\quad \bar{\rho}_{A_1^n B_1^n E}^{(n)} \cdot \left(\bar{\rho}_{A_1^{n-1} B_1^{n-1} E}^{(n)} \right)^{-\frac{1+it}{2}} \cdots \left(\bar{\rho}_{A_1^2 B_1^2 E}^{(2)} \right)^{\frac{1+it}{2}} \left(\bar{\rho}_{A_1 B_1 E}^{(2)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1 B_1 E}^{(1)} \right)^{\frac{1+it}{2}} \left(\bar{\rho}_E^{(1)} \right)^{-\frac{1+it}{2}} \rho_E^{\frac{1+it}{2}} \end{aligned} \quad (117)$$

is a normalised state satisfying

$$D_m(\rho_{A_1^n B_1^n E} \| \sigma_{A_1^n B_1^n E}) \leq nz(\epsilon + \nu, \delta) \quad (118)$$

and

$$\begin{aligned} &\sigma_{A_1^k B_1^k E} \\ &= \int_{-\infty}^{\infty} dt \beta_0(t) \prod_{j=0}^{k-1} \left[\left(\bar{\rho}_{A_1^j B_1^j E}^{(j)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1^j B_1^j E}^{(j+1)} \right)^{-\frac{1-it}{2}} \right] \cdot \bar{\rho}_{A_1^k B_1^k E}^{(k)} \cdot \prod_{j=k-1}^0 \left[\left(\bar{\rho}_{A_1^j B_1^j E}^{(j+1)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1^j B_1^j E}^{(j)} \right)^{\frac{1+it}{2}} \right] \end{aligned} \quad (119)$$

$$= \mathcal{M}_k^\delta \left(\int_{-\infty}^{\infty} dt \beta_0(t) \prod_{j=0}^{k-1} \left[\left(\bar{\rho}_{A_1^j B_1^j E}^{(j)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1^j B_1^j E}^{(j+1)} \right)^{-\frac{1-it}{2}} \right] \cdot \omega_{A_1^{k-1} B_1^{k-1} R_k E}^{(k,0)} \cdot \prod_{j=k-1}^0 \left[\left(\bar{\rho}_{A_1^j B_1^j E}^{(j+1)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1^j B_1^j E}^{(j)} \right)^{\frac{1+it}{2}} \right] \right) \quad (120)$$

Let $\sigma_{A_1^{k-1} B_1^{k-1} R_k E}^{(k,0)}$ be the input state for \mathcal{M}_k^δ above, so that

$$\sigma_{A_1^k B_1^k E} = \mathcal{M}_k^\delta \left(\sigma_{A_1^{k-1} B_1^{k-1} R_k E}^{(k,0)} \right) \quad (121)$$

Let's define $\mu := z(\epsilon + \nu, \delta)^{1/3}$. Using the substate theorem (Theorem 2.2), we get the following bound from the above relative entropy bound

$$D_{\max}^\mu(\rho_{A_1^n B_1^n E} \| \sigma_{A_1^n B_1^n E}) \leq n\mu + \frac{1}{\mu^2} + \log \frac{1}{1 - \mu^2}. \quad (122)$$

Let $\epsilon' \in (0, 1)$ be an arbitrary small parameter such that $\epsilon' + \mu < 1$ and let $\alpha \in (1, 2]$. We can now use the entropic triangle inequality in Eq. 36 to derive

$$H_{\min}^{\mu+\epsilon'}(A_1^n | B_1^n E)_\rho \geq \tilde{H}_\alpha^\dagger(A_1^n | B_1^n E)_\sigma - \frac{\alpha}{\alpha-1} n\mu - \frac{1}{\alpha-1} \left(\frac{\alpha}{\mu^2} + \alpha \log \frac{1}{1 - \mu^2} + g_1(\epsilon', \mu) \right) \quad (123)$$

Moreover, using Eq. 120, we can show that B_k is independent of $A_1^{k-1}B_1^{k-1}E$ in σ . For $\sigma_{A_1^{k-1}B_1^kE} = \text{tr}_{A_k} \circ \mathcal{M}_k^\delta(\sigma_{A_1^{k-1}B_1^{k-1}R_kE}^{(k,0)})$, we have

$$\sigma_{A_1^{k-1}B_1^kE} = (1 - \delta) \text{tr}_{A_k} \circ \mathcal{M}_k(\sigma_{A_1^{k-1}B_1^{k-1}R_kE}^{(k,0)}) + \delta \tau_{B_k} \otimes \sigma_{A_1^{k-1}B_1^{k-1}E}^{(k,0)} \quad (124)$$

$$= \left((1 - \delta) \theta_{B_k}^{(k)} + \delta \tau_{B_k} \right) \otimes \sigma_{A_1^{k-1}B_1^{k-1}E}. \quad (125)$$

In particular, we have that σ satisfies the Markov chain $A_1^{k-1} \leftrightarrow B_1^{k-1}E \leftrightarrow B_k$. So, we can use [DFR20, Corollary 3.5] to show that for every $k \in [n]$

$$\begin{aligned} \tilde{H}_\alpha^\downarrow(A_1^k|B_1^kE)_\sigma &\geq \tilde{H}_\alpha^\downarrow(A_1^{k-1}|B_1^{k-1}E)_\sigma + \inf_{\omega_{R_k\tilde{R}_k}} \tilde{H}_\alpha^\downarrow(A_k|B_k\tilde{R}_k)_{\mathcal{M}_k^\delta(\omega)} \\ &\geq \tilde{H}_\alpha^\downarrow(A_1^{k-1}|B_1^{k-1}E)_\sigma + \inf_{\omega_{R_k\tilde{R}_k}} \tilde{H}_\alpha^\downarrow(A_k|B_k\tilde{R}_k)_{\mathcal{M}_k(\omega)} \end{aligned} \quad (126)$$

where we use the quasi-concavity of $\tilde{H}_\alpha^\downarrow$ [Tom16, Pg 73] in the second line. Consecutively using this bound in Eq. 123 gives us

$$\begin{aligned} H_{\min}^{\mu+\epsilon'}(A_1^n|B_1^nE)_\rho &\geq \sum_{k=1}^n \inf_{\omega_{R_k\tilde{R}_k}} \tilde{H}_\alpha^\downarrow(A_k|B_k\tilde{R}_k)_{\mathcal{M}_k(\omega)} - \frac{\alpha}{\alpha-1} n\mu - \frac{1}{\alpha-1} \left(\frac{\alpha}{\mu^2} + \alpha \log \frac{1}{1-\mu^2} + g_1(\epsilon', \mu) \right) \\ &\geq \sum_{k=1}^n \inf_{\omega_{R_k\tilde{R}_k}} H(A_k|B_k\tilde{R}_k)_{\mathcal{M}_k(\omega)} - n(\alpha-1) \log^2(1+2|A|) - \frac{\alpha}{\alpha-1} n\mu \\ &\quad - \frac{1}{\alpha-1} \left(\frac{\alpha}{\mu^2} + \alpha \log \frac{1}{1-\mu^2} + g_1(\epsilon', \mu) \right) \end{aligned}$$

where in the second line we have used [DFR20, Lemma B.9] which is valid as long as $\alpha < 1 + 1/\log(1+2|A|)$. Lastly, we choose $\alpha = 1 + \frac{\sqrt{\mu}}{\log(1+2|A|)}$ and use $\alpha < 2$ as an upper bound to derive

$$\begin{aligned} H_{\min}^{\mu+\epsilon'}(A_1^n|B_1^nE)_\rho &\geq \sum_{k=1}^n \inf_{\omega_{R_k\tilde{R}_k}} H(A_k|B_k\tilde{R}_k)_{\mathcal{M}_k(\omega)} - 3n\sqrt{\mu} \log(1+2|A|) \\ &\quad - \frac{\log(1+2|A|)}{\sqrt{\mu}} \left(\frac{2}{\mu^2} + 2 \log \frac{1}{1-\mu^2} + g_1(\epsilon', \mu) \right) \end{aligned} \quad (127)$$

where $\mu = z(\epsilon + \nu, \delta)^{1/3}$. The above bound holds true for all $\nu > 0$. Therefore, it also holds for $\nu \rightarrow 0$. To derive the bound in the theorem statement, we fix $\delta = \epsilon$.

Case 2: Lastly, if $\rho_{A_1^n B_1^n E}$ were not full rank then we can always select a full rank state $\rho'_{A_1^n B_1^n E}$ ε -close to ρ , which would satisfy Eq. 101 and hence the above inequality with $\epsilon \rightarrow \epsilon + \varepsilon$. Now we can take $\varepsilon \rightarrow 0$ and use the continuity of H_{\min} to arrive at the above bound for such ρ . \square

In Appendix C, we incorporate testing into the approximate EAT proven above. Testing allows one to condition the state on a classical event and prove a smooth min-entropy lower bound for the conditional state. This enables one to prove much stronger and tighter bounds using EAT.

5 Alternative proof for the universal chain rule

The proofs so far in this paper have followed the approach of creating an auxiliary state using certain conditional states, proving that this state has a small relative entropy from the original state and reducing the original problem to a simpler one in terms of this auxiliary state. This is also the general approach we follow in [MD23]. In [MD23, Section 4] we provide a simple and intuitive alternate proof technique that lower bounds the smooth min-entropy for the classical approximately independent registers problem in an *element-wise* fashion. At the time of writing [MD23], we were unable to generalise this proof to the quantum setting. However, using the ideas developed in this paper, we can now generalise this proof. Here we use it to provide an alternative proof of the universal chain rule for the smooth min-entropy.

5.1 Classical proof

We begin by once again first sketching the proof in the classical case. Recall that we would like to prove that

$$H_{\min}^{g_1(\epsilon)}(A_1^n|B)_p \geq \sum_{k=1}^n H_{\min}^{\downarrow, \epsilon}(A_k|A_1^{k-1}B)_p - ng_2(\epsilon) - k(\epsilon). \quad (128)$$

Let $\lambda_k := H_{\min}^{\downarrow, \epsilon}(A_k|A_1^{k-1}B)_p$. Following Sec. 3.1, for $k \in [n]$, let $q_{A_1^k B}^{(k)}$ be the distribution, such that

$$\frac{1}{2} \left\| p_{A_1^k B} - q_{A_1^k B}^{(k)} \right\|_1 \leq \epsilon \quad (129)$$

$$H_{\min}^{\epsilon, \downarrow}(A_k|A_1^{k-1}B)_p = \lambda_k = H_{\min}^{\downarrow}(A_k|A_1^{k-1}B)_{q^{(k)}}. \quad (130)$$

Using Eq. 53, we also have that

$$\frac{1}{2} \left\| p_{A_1^k B} - p_{A_1^{k-1} B} q_{A_k|A_1^{k-1} B}^{(k)} \right\|_1 \leq 2\epsilon. \quad (131)$$

We will use the following lemma from [MD23].

Lemma 5.1 ([MD23, Lemma 4.1]). *Suppose p, q are probability distributions on \mathcal{X} such that $\frac{1}{2} \|p - q\|_1 \leq \epsilon$, then $S \subseteq \mathcal{X}$ defined as $S := \{x \in \mathcal{X} : p(x) \leq (1 + \epsilon^{1/2})q(x)\}$ is such that $q(S) \geq 1 - \epsilon^{1/2}$ and $p(S) \geq 1 - \epsilon^{1/2} - \epsilon$.*

Using the Lemma above, for every $k \in [n]$, we know that the set

$$\begin{aligned} B_k &:= \left\{ (a_1^n, b) : p(a_1^k | b) > (1 + \sqrt{2\epsilon}) p(a_1^{k-1} | b) q^{(k)}(a_k | a_1^{k-1} b) \right\} \\ &= \left\{ (a_1^n, b) : p(a_k | a_1^{k-1} b) > (1 + \sqrt{2\epsilon}) q^{(k)}(a_k | a_1^{k-1} b) \right\} \end{aligned}$$

satisfies $\Pr_\rho(B_k) \leq 3\sqrt{\epsilon}$. We can now define $L = \sum_{k=1}^n \chi_{B_k}$, which is a random variable that simply counts the number of bad sets B_k an element (a_1^n, b) belongs to. Using the Markov inequality, we have

$$\Pr_p \left[L > n\epsilon^{\frac{1}{4}} \right] \leq \frac{\mathbb{E}_p[L]}{n\epsilon^{\frac{1}{4}}} \leq 3\epsilon^{\frac{1}{4}}.$$

We can define the bad set $\mathcal{B} := \left\{ (a_1^n, b) : L(a_1^n, b) > n\epsilon^{\frac{1}{4}} \right\}$. Then for the subnormalised distribution $\tilde{p}_{A_1^n B}$ defined as

$$\tilde{p}_{A_1^n B}(a_1^n, b) = \begin{cases} p_{A_1^n B}(a_1^n, b) & (a_1^n, b) \notin \mathcal{B} \\ 0 & \text{else} \end{cases},$$

we have $P(\tilde{p}_{A_1^n B}, p_{A_1^n B}) \leq \sqrt{6}\epsilon^{1/8}$. Further, note that for every $(a_1^n, b) \notin \mathcal{B}$, we have

$$\begin{aligned} p(a_1^n | b) &= \prod_{k=1}^n p(a_k | a_1^{k-1}, b) \\ &= \prod_{k: (a_1^n, b) \notin B_k} p(a_k | a_1^{k-1}, b) \prod_{k: (a_1^n, b) \in B_k} p(a_k | a_1^{k-1}, b) \\ &\leq (1 + \sqrt{2\epsilon})^n \prod_{k: (a_1^n, b) \notin B_k} q^{(k)}(a_k | a_1^{k-1}, b) \prod_{k: (a_1^n, b) \in B_k} e^{\log |A| - \lambda_k} \\ &\leq (1 + \sqrt{2\epsilon})^n \prod_{k: (a_1^n, b) \notin B_k} e^{-\lambda_k} \prod_{k: (a_1^n, b) \in B_k} e^{\log |A| - \lambda_k} \\ &\leq (1 + \sqrt{2\epsilon})^n e^{n\epsilon^{1/4} \log |A|} \prod_{k=1}^n e^{-\lambda_k} \end{aligned}$$

where in the third line we have used the fact that if $(a_1^n, b) \notin B_k$, then $p(a_k | a_1^{k-1} b) \leq (1 + \sqrt{\epsilon}) q^{(k)}(a_k | a_1^{k-1} b)$ and that $\lambda_k \leq \log |A|$. In the last line we have used the fact that for $(a_1^n, b) \notin \mathcal{B}$, we have $|\{k \in [n] : (a_1^n, b) \in B_k\}| = L(a_1^n, b) \leq n\epsilon^{\frac{1}{4}}$. This proves the following lower bound for the smooth min-entropy of ρ

$$H_{\min}^{\downarrow, \sqrt{6}\epsilon^{1/8}}(A_1^n | B) \geq \sum_{k=1}^n H_{\min}^{\downarrow, \epsilon}(A_k | A_1^{k-1} B) - n\epsilon^{1/4} \log |A| - n \log(1 + \sqrt{\epsilon}). \quad (132)$$

There are many hurdles to generalising this proof to the quantum setting. Firstly, the correct generalisation of Lemma 5.1 is quite difficult. Secondly, the proof above bounds the

conditional probability $p(a_1^n|b)$ for every element $(a_1^n, b) \notin \mathcal{B}$ differently depending on which bad sets $(\mathcal{B}_k)_k$ it belongs in. It is not obvious how such an argument can be carried out for quantum states. For example, one cannot simply use the eigenvalues of $\rho_{A_1^n B}$ instead of $p(a_1^n|b)$ since the eigenvectors for each of the partial states $\rho_{A_1^k B}$ differ. Lastly, in the quantum case it does not seem that the Markov inequality can be used to identify the bad sets and the smoothed state as we did above.

The correct generalisation of Lemma 5.1 was proven by [fu23]. We state and reproduce prove it here in Lemma 5.5. In order to circumvent the second and third hurdles mentioned above, we modify the proof above.

Instead of figuring out the bad set \mathcal{B} and eliminating the elements belonging to this set to construct the smoothed distribution, we use the substate theorem and the entropic triangle inequality. Define the auxiliary subnormalised distribution

$$q(a_1^n b) := \delta^{L(a_1^n b)} p(a_1^n b) \quad (133)$$

for some small $\delta \in (0, \frac{1}{|A|})$. The $\delta^{L(a_1^n b)}$ factor in q simply guarantees that elements which have a large L (are *bad*) are significantly damped. This ensures that for all a_1^n, b , we have that

$$\begin{aligned} q(a_1^n b) &= \delta^{L(a_1^n b)} p(a_1^n b) \\ &= p(b) \prod_{k=1}^n \delta^{\chi_{B_k}(a_1^n b)} p(a_k | a_1^{k-1} b) \\ &\leq p(b) \prod_{k=1}^n (1 + \sqrt{2\epsilon}) e^{-\lambda_k}. \end{aligned} \quad (134)$$

The last inequality above is guaranteed both when $a_1^n, b \notin \mathcal{B}_k$ and when $a_1^n, b \in \mathcal{B}_k$ since we chose $\delta < \frac{1}{|A|}$. This gives us a lower bound the min-entropy of q . Moreover, we can easily show that the relative entropy between p and q is small:

$$\begin{aligned} D(p_{A_1^n B} \| q_{A_1^n B}) &= \mathbb{E}_p \left[\log \frac{p(A_1^n B)}{q(A_1^n B)} \right] \\ &= \mathbb{E}_p \left[L \log \frac{1}{\delta} \right] \\ &= 3n\sqrt{\epsilon} \log \frac{1}{\delta}. \end{aligned} \quad (135)$$

Now, we can bound the smooth min-entropy of p in terms of the min-entropy of q by using the substate theorem and the entropic triangle inequality. We will generalise this proof to the quantum case.

5.2 Lemmas

In this section, we will primarily use the following variant of the smooth min-entropy, which has previously appeared in [ABJT20]:

$$\bar{H}_{\min}^{\epsilon}(A|B)_{\rho} := \sup \left\{ \lambda \in \mathbb{R} : \text{for } \tilde{\rho}_{AB} \in B_{\epsilon}(\rho_{AB}) \text{ such that } \tilde{\rho}_{AB} \leq e^{-\lambda} \mathbb{1}_A \otimes \rho_B \right\}. \quad (136)$$

Note that $\bar{H}_{\min}^{\epsilon}(A|B)_{\rho} \geq -\log |A|$ because $\rho_{AB} \leq |A| \mathbb{1}_A \otimes \rho_B$. Also, $\bar{H}_{\min}^{\epsilon}(A|B)_{\rho} \leq \log \frac{|A|}{1-\epsilon^2}$. To see this note that for any $\tilde{\rho}_{AB}$ and λ , such that

$$\begin{aligned} \tilde{\rho}_{AB} &\leq e^{-\lambda} \mathbb{1}_A \otimes \rho_B \\ \Rightarrow \text{tr } \tilde{\rho}_{AB} &\leq e^{-\lambda} |A| \\ \Rightarrow \lambda &\leq \log \frac{|A|}{\text{tr } \tilde{\rho}_{AB}} \end{aligned} \quad (137)$$

where we have simply taken a trace in the second line. Finally, use the fact that $\text{tr } \tilde{\rho}_{AB} \geq 1 - \epsilon^2$ for all $\tilde{\rho}_{AB} \in B_{\epsilon}(\rho_{AB})$.

This smooth min-entropy can be lower bounded using the $H_{\min}^{\downarrow, \epsilon}$ min-entropy as the following lemma shows.

Lemma 5.2. *For a normalized quantum state ρ_{AB} and $\epsilon \geq 0$, we have*

$$\bar{H}_{\min}^{2\epsilon}(A|B)_{\rho} \geq H_{\min}^{\downarrow, \epsilon}(A|B)_{\rho}. \quad (138)$$

Proof. Let $\lambda = H_{\min}^{\downarrow, \epsilon}(A|B)_{\rho}$ and the state $\tilde{\rho}_{AB} \in B_{\epsilon}(\rho_{AB})$ be such that

$$\tilde{\rho}_{AB} \leq e^{-\lambda} \mathbb{1}_A \otimes \tilde{\rho}_B. \quad (139)$$

By Lemma A.1, we have that $\eta_{AB} := \rho_B^{1/2} U_B \tilde{\rho}_B^{-1/2} \tilde{\rho}_{AB} \tilde{\rho}_B^{-1/2} U_B^{\dagger} \rho_B^{1/2}$ satisfies $P(\rho_{AB}, \eta_{AB}) \leq 2\epsilon$. Clearly, this state also satisfies

$$\eta_{AB} \leq e^{-\lambda} \mathbb{1}_A \otimes \rho_B. \quad (140)$$

□

We also require the following operator inequalities relating an operator and its compressions.

Lemma 5.3 (Asymmetric pinching [MD23, Lemma 5.2]). *For $t > 0$, a positive semidefinite operator $X \geq 0$ and orthogonal projections Π and $\Pi_{\perp} = \mathbb{1} - \Pi$, we have that*

$$X \leq (1+t)\Pi X \Pi + \left(1 + \frac{1}{t}\right)\Pi_{\perp} X \Pi_{\perp}. \quad (141)$$

Lemma 5.4. *For a positive operator $X \geq 0$ and orthogonal projectors Π and $\Pi_\perp = \mathbb{1} - \Pi$, we have*

$$\Pi_\perp X \Pi_\perp \leq 2X + 2\Pi X \Pi. \quad (142)$$

Proof. We will write the operator X as the block matrix

$$X = \begin{pmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{pmatrix} \quad (143)$$

where the blocks are partitioned according to the direct sum $\text{im}(\Pi) \oplus \text{im}(\Pi_\perp)$. The statement in the Lemma is now equivalent to proving that

$$0 \leq \begin{pmatrix} 4X_1 & 2X_2 \\ 2X_2^* & X_3 \end{pmatrix}. \quad (144)$$

Call the matrix above X' . Note that using the Schur's complement [Bha07, Exercise 1.3.5] $X \geq 0$ implies that $X_3 \geq 0$, $\text{im}(X_2^*) \subseteq \text{im}(X_3)$ and

$$X_1 \geq X_2 X_3^{-1} X_2^*. \quad (145)$$

Using Schur's complement characterization again for X' , we see that $X' \geq 0$ is equivalent to $X_3 \geq 0$, $\text{im}(X_2^*) \subseteq \text{im}(X_3)$ and

$$4X_1 \geq (2X_2)X_3^{-1}(2X_2)^*$$

which are all true because of the corresponding relations for X itself. \square

The following lemma correctly generalises Lemma 5.1. It was proven by user:fedja in response to a question by user:noel (pseudonym used by AM) on MathOverflow [fu23]. It has been reproduced here for the purpose of completeness.

Lemma 5.5 ([fu23]). *For $\epsilon \in [0, 1]$, and subnormalized quantum states ρ and σ on the finite dimensional Hilbert space \mathcal{X} such that $\frac{1}{2} \|\rho - \sigma\|_1 \leq \epsilon$, there exists an orthogonal projector Π such that*

$$\Pi \rho \Pi \leq (1 + g_1(\epsilon)) \sigma \quad (146)$$

and

$$\text{tr}((\mathbb{1} - \Pi)\rho) \leq g_2(\epsilon) \quad (147)$$

for the small functions

$$g_1(\epsilon) := \frac{8}{3}(1 + \epsilon^{1/3})\epsilon^{1/3} \log \frac{1}{\epsilon} + (1 + \epsilon^{1/3} + \epsilon^{2/3})\epsilon^{1/3} = O\left(\epsilon^{1/3} \log \frac{1}{\epsilon}\right) \quad (148)$$

$$g_2(\epsilon) := 4(1 + \epsilon^{1/3})\epsilon^{1/3} + 2\epsilon = O(\epsilon^{1/3}) \quad (149)$$

Proof. Let $\sigma = \sum_i q_i |x_i\rangle \langle x_i|$ be the eigenvalue decomposition of σ . Let $\delta_1 \in (0, 1)$ be a small parameter, which will be specified later. For every $k \geq 0$, define

$$\mathcal{X}_k := \text{span} \left\{ |x_i\rangle : q_i \in ((1 + \delta_1)^{-(k+1)}, (1 + \delta_1)^{-k}] \right\} \quad (150)$$

$$d_k := \dim(\mathcal{X}_k) \quad (151)$$

We have $\mathcal{X} = \bigoplus_{k=0}^{\infty} \mathcal{X}_k$ and $\sigma = \bigoplus_{k=0}^{\infty} \sigma|_{\mathcal{X}_k}$. Let P_k be the projector on the space \mathcal{X}_k for every k . Note that P_k commute with σ . If we restrict σ to the space \mathcal{X}_k , we have

$$\frac{1}{(1 + \delta_1)^{k+1}} \mathbb{1}_{\mathcal{X}_k} \leq \sigma|_{\mathcal{X}_k} \leq \frac{1}{(1 + \delta_1)^k} \mathbb{1}_{\mathcal{X}_k} \quad (152)$$

for every k . Further, for any projector Π_k in the space \mathcal{X}_k , we have

$$\begin{aligned} \|\sigma|_{\mathcal{X}_k}^{-\frac{1}{2}} \Pi_k \sigma|_{\mathcal{X}_k} \Pi_k \sigma|_{\mathcal{X}_k}^{-\frac{1}{2}}\|_{\infty} &\leq \|\sigma|_{\mathcal{X}_k}\|_{\infty} \|\sigma|_{\mathcal{X}_k}^{-1}\|_{\infty} \\ &\leq 1 + \delta_1 \end{aligned}$$

which implies that for any projector Π_k in \mathcal{X}_k

$$\Pi_k \sigma|_{\mathcal{X}_k} \Pi_k \leq (1 + \delta_1) \sigma|_{\mathcal{X}_k}. \quad (153)$$

We will choose the projector Π to satisfy the lemma to be of the form $\Pi = \bigoplus_{k=0}^{\infty} \Pi_k$, where each Π_k is a projector in the space \mathcal{X}_k . With such a projector, we have

$$\begin{aligned} \Pi \sigma \Pi &= \bigoplus_{k=0}^{\infty} \Pi_k \sigma|_{\mathcal{X}_k} \Pi_k \\ &\leq (1 + \delta_1) \bigoplus_{k=0}^{\infty} \sigma|_{\mathcal{X}_k} \\ &= (1 + \delta_1) \sigma. \end{aligned} \quad (154)$$

Define $\Delta := (\rho - \sigma)^+$ to be the positive part of $(\rho - \sigma)$ and observe that $\text{tr}(\Delta) \leq 2\epsilon$. Note that $\rho \leq \sigma + \Delta$. Further, let $\Delta_k := P_k \Delta P_k$ and $\mu_k := \text{tr}(\Delta P_k)$ for every k . Then, we have that

$$\begin{aligned} \sum_{k=0}^{\infty} \mu_k &= \sum_{k=0}^{\infty} \text{tr}(\Delta P_k) \\ &= \text{tr}(\Delta) \\ &\leq 2\epsilon. \end{aligned} \quad (155)$$

Let $\Delta_k = \sum_{i=1}^{d_k} \nu_{ki} |y_{ki}\rangle \langle y_{ki}|$ be the eigenvalue decomposition of Δ_k . Let $\delta_2 \in (0, 1)$ and $K \in \mathbb{N}$ be two more parameters to be chosen later. Define the subspace \mathcal{Y}_k of \mathcal{X}_k as

$$\mathcal{Y}_k := \text{span} \left\{ |y_{ki}\rangle : \nu_{ki} \leq \delta_2 (1 + \delta_1)^{-(k+1)} \right\}. \quad (156)$$

Note that the codimension of \mathcal{Y}_k in the space \mathcal{X}_k , is at most $\frac{\mu_k}{\delta_2}(1 + \delta_1)^{k+1}$. We further restrict the subspace \mathcal{Y}_k to the space

$$\mathcal{Z}_k := \mathcal{Y}_k \cap \bigcap_{j=0}^{k-K-1} \ker(P_j \Delta). \quad (157)$$

Since, rank of P_j is d_j , the codimension of \mathcal{Z}_k in \mathcal{Y}_k is at most $\sum_{j=0}^{k-K-1} d_j$. Define Π_k to be the projector on \mathcal{Z}_k . Note that since \mathcal{Z}_k is a subspace of \mathcal{X}_k , Π_k is a projector in the space \mathcal{X}_k . We define the projector

$$\Pi := \bigoplus_{k=0}^{\infty} \Pi_k, \quad (158)$$

which is of the form promised and show that this satisfies the conditions of the lemma.

We begin by showing that $\Pi \Delta \Pi$ can be bounded by a small multiple of $\Pi \sigma \Pi$, specifically

$$\Pi \Delta \Pi \leq (2K + 1) \delta_2 \Pi \sigma \Pi.$$

It is sufficient to show that for $|v\rangle = \bigoplus_{k=0}^{\infty} |v_k\rangle$ such that $|v_k\rangle \in \mathcal{Z}_k$ for every k , we have

$$\langle v | \Delta | v \rangle \leq (2K + 1) \delta_2 \langle v | \sigma | v \rangle. \quad (159)$$

The left-hand side above can be expanded as

$$\begin{aligned} \langle v | \Delta | v \rangle &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \langle v_i | \Delta | v_j \rangle \\ &= \sum_{i=0}^{\infty} \langle v_i | \Delta | v_i \rangle + 2 \sum_{i=0}^{\infty} \sum_{j=0}^{i-1} \operatorname{Re} \{ \langle v_i | \Delta | v_j \rangle \} \\ &\leq \sum_{i=0}^{\infty} \langle v_i | \Delta | v_i \rangle + 2 \sum_{i=0}^{\infty} \sum_{j=0}^{i-1} | \langle v_i | \Delta | v_j \rangle | \\ &= \sum_{i=0}^{\infty} \langle v_i | \Delta | v_i \rangle + 2 \sum_{i=0}^{\infty} \sum_{j=i-K}^{i-1} | \langle v_i | \Delta | v_j \rangle | + 2 \sum_{i=0}^{\infty} \sum_{j=0}^{i-K-1} | \langle v_i | \Delta | v_j \rangle |. \end{aligned} \quad (160)$$

The first summation above can be bounded as

$$\begin{aligned} \sum_{i=0}^{\infty} \langle v_i | \Delta | v_i \rangle &\leq \delta_2 \sum_{i=0}^{\infty} \frac{1}{(1 + \delta_1)^{i+1}} \langle v_i | v_i \rangle \\ &\leq \delta_2 \sum_{i=0}^{\infty} \langle v_i | \sigma | v_i \rangle \end{aligned} \quad (161)$$

where we have used the fact that $|v_i\rangle \in \mathcal{Y}_i$ and the maximum eigenvalue of $P_i \Delta P_i = \Delta_i$ in this subspace is at most $\delta_2 \frac{1}{(1 + \delta_1)^{i+1}}$, and the fact that the minimum eigenvalue of σ in

$\mathcal{Z}_i \subseteq \mathcal{X}_i$ is at least $\frac{1}{(1+\delta_1)^{i+1}}$.

We can bound the second term as

$$\begin{aligned}
\sum_{i=0}^{\infty} \sum_{j=i-K}^{i-1} |\langle v_i | \Delta | v_j \rangle| &\leq \sum_{i=0}^{\infty} \sum_{j=i-K}^{i-1} \left\| \Delta^{\frac{1}{2}} |v_j\rangle \right\| \left\| \Delta^{\frac{1}{2}} |v_i\rangle \right\| \\
&\leq \sum_{i=0}^{\infty} \frac{1}{2} \sum_{j=i-K}^{i-1} \left(\left\| \Delta^{\frac{1}{2}} |v_j\rangle \right\|^2 + \left\| \Delta^{\frac{1}{2}} |v_i\rangle \right\|^2 \right) \\
&= K \sum_{i=0}^{\infty} \left\| \Delta^{\frac{1}{2}} |v_i\rangle \right\|^2 \\
&= K \sum_{i=0}^{\infty} \langle v_i | \Delta | v_i \rangle
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality in the first line along with the fact that $\Delta \geq 0$ and the AM-GM inequality in the second line. This summation can now be bounded using Eq. 161.

The last term in Eq. 160 is a summation over inner products $|\langle v_i | \Delta | v_j \rangle| = |\langle v_j | P_j \Delta | v_i \rangle|$ for $j \leq i - K - 1$. By our definition for the subspace \mathcal{Z}_i , we have that $|v_i\rangle \in \ker(P_j \Delta)$ for these choices of the index j .

Putting these together, we have the bound

$$\begin{aligned}
\langle v | \Delta | v \rangle &\leq (2K + 1) \delta_2 \sum_{i=0}^{\infty} \langle v_i | \sigma | v_i \rangle \\
&= (2K + 1) \delta_2 \langle v | \sigma | v \rangle
\end{aligned}$$

for every $|v\rangle = \Pi |v\rangle$. Above we have used the fact that $|v_i\rangle \in \mathcal{X}_i$ and $\sigma = \bigoplus_{i=0}^{\infty} \sigma|_{\mathcal{X}_i}$. Therefore, we have

$$\Pi \Delta \Pi \leq (2K + 1) \delta_2 \Pi \sigma \Pi \quad (162)$$

and also

$$\begin{aligned}
\Pi \rho \Pi &\leq \Pi (\sigma + \Delta) \Pi \\
&\leq (1 + (2K + 1) \delta_2) \Pi \sigma \Pi.
\end{aligned} \quad (163)$$

Using Eq. 154 we can remove the sandwiching projectors from σ to get the bound

$$\Pi \rho \Pi \leq (1 + (2K + 1) \delta_2) (1 + \delta_1) \sigma. \quad (164)$$

Now, we will show that $\text{tr}(\Pi\sigma)$ is large. Recall that the codimension of \mathcal{Z}_k in \mathcal{X}_k is at most

$$\frac{\mu_k}{\delta_2}(1+\delta_1)^{k+1} + \sum_{j=0}^{k-K-1} d_j. \quad (165)$$

Hence,

$$\begin{aligned} \text{tr}((\mathbb{1} - \Pi)\sigma) &= \sum_{k=0}^{\infty} \text{tr}((P_k - \Pi_k)\sigma|_{\mathcal{X}_k}) \\ &\leq \sum_{k=0}^{\infty} \frac{1}{(1+\delta_1)^k} \text{tr}(P_k - \Pi_k) \\ &\leq \sum_{k=0}^{\infty} \frac{1}{(1+\delta_1)^k} \left(\frac{\mu_k}{\delta_2}(1+\delta_1)^{k+1} + \sum_{j=0}^{k-K-1} d_j \right) \\ &= \frac{1}{\delta_2}(1+\delta_1) \sum_{k=0}^{\infty} \mu_k + \sum_{j=0}^{\infty} d_j \sum_{k=j+K+1}^{\infty} (1+\delta_1)^{-k} \\ &= \frac{2\epsilon(1+\delta_1)}{\delta_2} + \frac{1}{\delta_1}(1+\delta_1)^{-(K-1)} \sum_{j=0}^{\infty} d_j \frac{1}{(1+\delta_1)^{j+1}} \\ &= (1+\delta_1) \left(\frac{2\epsilon}{\delta_2} + \frac{1}{\delta_1}(1+\delta_1)^{-K} \right) \end{aligned} \quad (166)$$

where in the second line we have used $\sigma|_{\mathcal{X}_k} \leq (1+\delta_1)^{-k} \mathbb{1}_{\mathcal{X}_k}$, in the third line we have used the bound on the codimension of \mathcal{Z}_k in Eq. 165, in the fifth line we have used the bound in Eq. 155 and in the last line we have used the fact that $\sum_{j=0}^{\infty} d_j(1+\delta_1)^{-(j+1)} \leq \text{tr} \sigma \leq 1$.

Now all that is left to do is to select the parameters δ_1, δ_2 and K . We choose

$$\delta_1 := \epsilon^{1/3} \quad (167)$$

$$\delta_2 := \epsilon^{2/3} \quad (168)$$

$$K := \left\lfloor \frac{2}{\epsilon^{1/3}} \log \frac{1}{\epsilon^{2/3}} \right\rfloor. \quad (169)$$

For these parameters, we get

$$\Pi\rho\Pi \leq \left(1 + \frac{8}{3}\epsilon^{1/3} \log \frac{1}{\epsilon} + \epsilon^{2/3}\right) (1 + \epsilon^{1/3})\sigma. \quad (170)$$

Note that $(1 + \delta_1)^{-1} \leq (1 - \delta_1/2) \leq e^{-\delta_1/2}$ since $\delta_1 \in (0, 1)$. This implies

$$\begin{aligned}
\text{tr}((\mathbb{1} - \Pi)\sigma) &\leq (1 + \epsilon^{1/3}) \left(2\epsilon^{1/3} + \frac{1}{\epsilon^{1/3}} e^{-K \frac{\epsilon^{1/3}}{2}} \right) \\
&\leq (1 + \epsilon^{1/3}) \left(2\epsilon^{1/3} + \frac{1}{\epsilon^{1/3}} e^{-\frac{\epsilon^{1/3}}{2} \left(\frac{2}{\epsilon^{1/3}} \log \frac{1}{\epsilon^{2/3}} - 1 \right)} \right) \\
&\leq (1 + \epsilon^{1/3}) \left(2\epsilon^{1/3} + \frac{1}{\epsilon^{1/3}} 2\epsilon^{2/3} \right) \\
&\leq 4(1 + \epsilon^{1/3})\epsilon^{1/3}.
\end{aligned}$$

Finally, using the fact that $\frac{1}{2} \|\rho - \sigma\|_1 \leq \epsilon$, we get

$$\begin{aligned}
\text{tr}((\mathbb{1} - \Pi)\rho) &\leq \text{tr}((\mathbb{1} - \Pi)\sigma) + 2\epsilon \\
&\leq 4(1 + \epsilon^{1/3})\epsilon^{1/3} + 2\epsilon.
\end{aligned} \tag{171}$$

□

5.3 Proof

In this section, we will generalise the classical proof presented earlier to the quantum case. We will prove a statement of the following form for all normalised quantum states $\rho_{A_1^n B}$:

$$H_{\min}^{g'_1(\epsilon)}(A_1^n | B)_\rho \geq \sum_{k=1}^n \bar{H}_{\min}^\epsilon(A_k | A_1^{k-1} B)_\rho - n g'_2(\epsilon) - k(\epsilon) \tag{172}$$

where g'_1 and g'_2 are small functions of ϵ and k is a general function of ϵ . This will be sufficient to prove the universal chain rule.

For every k , let $\lambda_k := \bar{H}_{\min}^\epsilon(A_k | A_1^{k-1} B)_\rho$ and $\tilde{\rho}_{A_1^k B}^{(k)}$ be the subnormalised state such that

$$P(\rho_{A_1^k B}, \tilde{\rho}_{A_1^k B}^{(k)}) \leq \epsilon \tag{173}$$

$$\tilde{\rho}_{A_1^k B}^{(k)} \leq e^{-\lambda_k} \mathbb{1}_{A_k} \otimes \rho_{A_1^{k-1} B}. \tag{174}$$

Using the Fuchs-van de Graaf inequality, for each k , we also have

$$\frac{1}{2} \left\| \rho_{A_1^k B} - \tilde{\rho}_{A_1^k B}^{(k)} \right\|_1 \leq \epsilon. \tag{175}$$

We will now define the projectors $P_{A_1^k B}^{(k,l)}$ for every $k \in [n]$ and the label $l \in \{g, b\}$ ($\{good, bad\}$). We define the *good* projector $P_{A_1^k B}^{(k,g)}$ to be the projector given by Lemma 5.5 when applied

to the states $\rho_{A_1^k B}$ and $\tilde{\rho}_{A_1^k B}^{(k)}$, so that it satisfies

$$P_{A_1^k B}^{(k,g)} \rho_{A_1^k B} P_{A_1^k B}^{(k,g)} \leq (1 + g_1(\epsilon)) \tilde{\rho}_{A_1^k B}^{(k)} \quad (176)$$

and

$$\text{tr}(P_{A_1^k B}^{(k,g)} \rho_{A_1^k B}) \geq 1 - g_2(\epsilon) \quad (177)$$

for g_1 and g_2 as defined in Lemma 5.5. Further, define its orthogonal complement as the *bad* projector,

$$P_{A_1^k B}^{(k,b)} := \mathbb{1}_{A_1^k B} - P_{A_1^k B}^{(k,g)}. \quad (178)$$

The label l in $P_{A_1^k B}^{(k,l)}$ will allow us to succinctly refer to these two projectors together. Define the subnormalised state

$$\eta_{A_1^n B} := P_{A_1 B}^{(1,g)} P_{A_1^2 B}^{(2,g)} \dots P_{A_1^n B}^{(n,g)} \rho_{A_1^n B} P_{A_1^n B}^{(n,g)} \dots P_{A_1^2 B}^{(2,g)} P_{A_1 B}^{(1,g)}. \quad (179)$$

For this state, observe that

$$\begin{aligned} & P_{A_1 B}^{(1,g)} P_{A_1^2 B}^{(2,g)} \dots P_{A_1^n B}^{(n,g)} \rho_{A_1^n B} P_{A_1^n B}^{(n,g)} \dots P_{A_1^2 B}^{(2,g)} P_{A_1 B}^{(1,g)} \\ & \leq (1 + g_1(\epsilon)) P_{A_1 B}^{(1,g)} P_{A_1^2 B}^{(2,g)} \dots P_{A_1^{n-1} B}^{(n-1,g)} \tilde{\rho}_{A_1^{n-1} B}^{(n)} P_{A_1^{n-1} B}^{(n-1,g)} \dots P_{A_1^2 B}^{(2,g)} P_{A_1 B}^{(1,g)} \\ & \leq e^{-\lambda_n} (1 + g_1(\epsilon)) P_{A_1 B}^{(1,g)} P_{A_1^2 B}^{(2,g)} \dots P_{A_1^{n-1} B}^{(n-1,g)} \mathbb{1}_{A_n} \otimes \rho_{A_1^{n-1} B} P_{A_1^{n-1} B}^{(n-1,g)} \dots P_{A_1^2 B}^{(2,g)} P_{A_1 B}^{(1,g)} \\ & = e^{-\lambda_n} (1 + g_1(\epsilon)) \mathbb{1}_{A_n} \otimes P_{A_1 B}^{(1,g)} P_{A_1^2 B}^{(2,g)} \dots P_{A_1^{n-1} B}^{(n-1,g)} \rho_{A_1^{n-1} B} P_{A_1^{n-1} B}^{(n-1,g)} \dots P_{A_1^2 B}^{(2,g)} P_{A_1 B}^{(1,g)} \\ & \leq \dots \\ & \leq e^{-\sum_{k=1}^n \lambda_k} (1 + g_1(\epsilon))^n \mathbb{1}_{A_1^n} \otimes \rho_B \end{aligned}$$

which implies that $H_{\min}(A_1^n | B)_\eta \geq \sum_{k=1}^n \bar{H}_{\min}^\epsilon(A_k | A_1^{k-1} B)_\rho - n \log(1 + g_1(\epsilon))$. Moreover, the distance between η and ρ is

$$\begin{aligned} P(\rho_{A_1^n B}, \eta_{A_1^n B}) &= P(\rho_{A_1^n B}, P_{A_1 B}^{(1,g)} P_{A_1^2 B}^{(2,g)} \dots P_{A_1^n B}^{(n,g)} \rho_{A_1^n B} P_{A_1^n B}^{(n,g)} \dots P_{A_1^2 B}^{(2,g)} P_{A_1 B}^{(1,g)}) \\ &\leq P(\rho_{A_1^n B}, P_{A_1 B}^{(1,g)} \rho_{A_1^n B} P_{A_1 B}^{(1,g)}) \\ &\quad + P(P_{A_1 B}^{(1,g)} \rho_{A_1^n B} P_{A_1 B}^{(1,g)}, P_{A_1 B}^{(1,g)} P_{A_1^2 B}^{(2,g)} \dots P_{A_1^n B}^{(n,g)} \rho_{A_1^n B} P_{A_1^n B}^{(n,g)} \dots P_{A_1^2 B}^{(2,g)} P_{A_1 B}^{(1,g)}) \\ &\leq P(\rho_{A_1^n B}, P_{A_1 B}^{(1,g)} \rho_{A_1^n B} P_{A_1 B}^{(1,g)}) + P(\rho_{A_1^n B}, P_{A_1^2 B}^{(2,g)} \dots P_{A_1^n B}^{(n,g)} \rho_{A_1^n B} P_{A_1^n B}^{(n,g)} \dots P_{A_1^2 B}^{(2,g)}) \\ &\leq \dots \\ &\leq \sum_{k=1}^n P(\rho_{A_1^n B}, P_{A_1^k B}^{(k,g)} \rho_{A_1^n B} P_{A_1^k B}^{(k,g)}) \\ &\leq n \sqrt{2g_2(\epsilon)} \end{aligned}$$

where we have used the gentle measurement lemma [Wat18, Proposition 3.14] for the last line. The distance between η and ρ grows like n times a small function and the min-entropy of η satisfies $H_{\min}(A_1^n|B)_\eta \geq \sum_{k=1}^n \bar{H}_{\min}^\epsilon(A_k|A_1^{k-1}B)_\rho - n \log(1 + g_1(\epsilon))$. If we were able to show that the relative entropy distance between these two states also grew like n times a small function, similar to the purified distance then we could use the substate theorem and the entropic triangle inequality to prove a bound of the required form. However, we can't directly prove that the relative entropy between these two states is small because in general the projectors sandwiching the state ρ in η could lead to a situation, where $\rho \not\ll \eta$, which would cause their relative entropy diverge. To remedy this, one could imagine adding a small amount of the complement projector to these projectors, so that we sandwich with $P_{A_1^k B}^{(k,g)} + \delta P_{A_1^k B}^{(k,b)}$ instead of simply $P_{A_1^k B}^{(k)}$. Under no further assumptions on ρ , the question of whether the divergence of ρ and η is finite or not in this case reduces to the question: given a state σ , a projector Π and its complement $\Pi_\perp := 1 - \Pi$, is

$$\sigma \ll (\Pi + \delta \Pi_\perp) \sigma (\Pi + \delta \Pi_\perp)? \quad (180)$$

The answer to this is easily seen to be negative when one consider the pure state $\sigma = |u\rangle\langle u|$. In this case, the above is equivalent to asking if $|u\rangle\langle u| \ll |v\rangle\langle v|$ for some vector $|v\rangle \neq |u\rangle$, which is not true.

The next simple remedy that comes to mind is to use a “pinching type” disturbance to ensure that the divergence is finite. For $\delta \in (0, 1)$, a projector Π and its complement Π_\perp , define the CP map $\mathcal{P}_\delta[\Pi]$ as

$$\mathcal{P}_\delta[\Pi](\sigma) := \Pi \sigma \Pi + \delta \Pi_\perp \sigma \Pi_\perp. \quad (181)$$

Then, by the pinching inequality, for every Π , we have

$$\begin{aligned} \sigma &\leq \frac{2}{\delta} (\delta \Pi \sigma \Pi + \delta \Pi_\perp \sigma \Pi_\perp) \\ &\leq \frac{2}{\delta} \mathcal{P}_\delta[\Pi](\sigma) \end{aligned}$$

which implies $D_{\max}(\sigma \| \mathcal{P}_\delta[\Pi](\sigma)) \leq O(\log 1/\delta)$ - a bound that should be sufficient for our purposes. Therefore, one can try proving the chain rule using the subnormalised state

$$\begin{aligned} &\mathcal{P}_\delta[P_{A_1 B}^{(1,g)}] \circ \mathcal{P}_\delta[P_{A_1^2 B}^{(2,g)}] \circ \dots \circ \mathcal{P}_\delta[P_{A_1^n B}^{(n,g)}](\rho_{A_1^n B}) \\ &= \sum_{l_1^n \in \{g,b\}^n} \delta^{\omega_b(l_1^n)} P_{A_1 B}^{(1,l_1)} P_{A_1^2 B}^{(2,l_2)} \dots P_{A_1^n B}^{(n,l_n)} \rho_{A_1^n B} P_{A_1^n B}^{(n,l_n)} \dots P_{A_1^2 B}^{(2,l_2)} P_{A_1 B}^{(1,l_1)} \end{aligned} \quad (182)$$

where $\omega_b(l_1^n) := |\{i : l_i = b\}|$ is the weight of b labels in the string l_1^n . In our proof below, we do not attempt to directly identify the correct modified state ourselves. Similar to the proofs before, we instead leave this question for the generalised GT inequality. We will use an exponential generalisation of the distribution q in Eq. 133. Variants of both of the above remedies make an appearance during the following proof.

Lemma 5.6. For a full rank state $\rho_{A_1^n B}$ and $\epsilon \in (0, 1)$, we have the chain rule

$$\begin{aligned} H_{\min}^{\mu}(A_1^n|B)_{\rho} &\geq \sum_{k=1}^n \bar{H}_{\min}^{\epsilon}(A_k|A_1^{k-1}B)_{\rho} - n \log \left(1 + 4|A|^2 \frac{\epsilon}{1-\epsilon^2} \right) - n \log(1 + g_1(\epsilon)) \\ &\quad - n \log(1 + \epsilon) - n(\mu + \mu^3) - \frac{1}{\mu^2} - \log \frac{1}{1-\mu^2} \end{aligned} \quad (183)$$

where

$$\mu := \left(8(1 + \epsilon^{1/3}) \log \frac{1}{\epsilon} \right)^{1/3} \epsilon^{1/9} = O \left(\epsilon^{1/9} \left(\log \frac{1}{\epsilon} \right)^{1/3} \right) \quad (184)$$

$$g_1(\epsilon) := \frac{8}{3}(1 + \epsilon^{1/3}) \epsilon^{1/3} \log \frac{1}{\epsilon} + (1 + \epsilon^{1/3} + \epsilon^{2/3}) \epsilon^{1/3} = O \left(\epsilon^{1/3} \log \frac{1}{\epsilon} \right) \quad (185)$$

as long as $\mu \in (0, 1)$.

Proof. We retain the definitions of $\tilde{\rho}_{A_1^k B}$, λ_k , and $P_{A_1^k B}^{(k,l)}$ from the discussion above. Following the classical proof in Sec. 5.1, define $L_{A_1^n B} := \sum_{k=1}^n P_{A_1^k B}^{(k,b)}$ as the sum of the “bad” projectors. Then, we have that

$$\begin{aligned} \text{tr}(L_{A_1^n B} \rho_{A_1^n B}) &= \sum_{k=1}^n \text{tr}(P_{A_1^k B}^{(k,b)} \rho_{A_1^k B}) \\ &\leq n g_2(\epsilon). \end{aligned} \quad (186)$$

Let $\delta \in (0, 1)$ be a parameter to be chosen later. The above then implies that

$$\begin{aligned} &n g_2(\epsilon) \log \frac{1}{\delta} \\ &\geq D(\rho_{A_1^n B} \| \exp(\log \rho_{A_1^n B} + \log(\delta) L_{A_1^n B})) \\ &= \sup_{\omega_{A_1^n B} > 0} \left\{ \text{tr}(\rho_{A_1^n B} \log \omega_{A_1^n B}) + 1 - \text{tr} \exp \left(\log \omega_{A_1^n B} + \log(\delta) \sum_{k=1}^n P_{A_1^k B}^{(k,b)} + \log \rho_{A_1^n B} \right) \right\} \end{aligned} \quad (187)$$

For the trace of exponential term in the last expression above, we can use the generalised Golden-Thompson inequality (Theorem 2.3) as

$$\begin{aligned} &\text{tr} \exp \left(\log \omega_{A_1^n B} + \log(\delta) \sum_{k=1}^n P_{A_1^k B}^{(k,b)} + \log \rho_{A_1^n B} \right) \\ &\leq \int_{-\infty}^{\infty} dt \beta_0(t) \text{tr} \left(\omega_{A_1^n B} e^{\frac{1-it}{2} \log(\delta) P_{A_1 B}^{(1,b)}} \dots e^{\frac{1-it}{2} \log(\delta) P_{A_1^n B}^{(n,b)}} \rho_{A_1^n B} e^{\frac{1+it}{2} \log(\delta) P_{A_1^n B}^{(n,b)}} \dots e^{\frac{1+it}{2} \log(\delta) P_{A_1 B}^{(1,b)}} \right) \\ &= \int_{-\infty}^{\infty} dt \beta_0(t) \text{tr} \left(\omega_{A_1^n B} (\delta^{\frac{1-it}{2}} P_{A_1 B}^{(1,b)} + P_{A_1 B}^{(1,g)}) \dots (\delta^{\frac{1-it}{2}} P_{A_1^n B}^{(n,b)} + P_{A_1^n B}^{(n,g)}) \rho_{A_1^n B} \right. \\ &\quad \left. (\delta^{\frac{1+it}{2}} P_{A_1^n B}^{(n,b)} + P_{A_1^n B}^{(n,g)}) \dots (\delta^{\frac{1+it}{2}} P_{A_1 B}^{(1,b)} + P_{A_1 B}^{(1,g)}) \right) \end{aligned} \quad (188)$$

Define the subnormalised state

$$\eta_{A_1^n B} := \int_{-\infty}^{\infty} dt \beta_0(t) (\delta^{\frac{1-it}{2}} P_{A_1 B}^{(1,b)} + P_{A_1 B}^{(1,g)}) \cdots (\delta^{\frac{1-it}{2}} P_{A_1^n B}^{(n,b)} + P_{A_1^n B}^{(n,g)}) \rho_{A_1^n B} \\ (\delta^{\frac{1+it}{2}} P_{A_1^n B}^{(n,b)} + P_{A_1^n B}^{(n,g)}) \cdots (\delta^{\frac{1+it}{2}} P_{A_1 B}^{(1,b)} + P_{A_1 B}^{(1,g)}) \quad (189)$$

Observe that this is just a clever (and correct) way of implementing the first remedy we discussed in the motivation. From Eq. 188, we have

$$\text{tr} \exp \left(\log \rho_{A_1^n B} + \log(\delta) \sum_{k=1}^n P_{A_1^k B}^{(k,b)} + \log \omega_{A_1^n B} \right) \leq \text{tr}(\omega_{A_1^n B} \eta_{A_1^n B}) \quad (190)$$

Plugging this in Eq. 187, we get

$$ng_2(\epsilon) \log \frac{1}{\delta} \geq \sup_{\omega_{A_1^n B} > 0} \{ \text{tr}(\rho_{A_1^n B} \log \omega_{A_1^n B}) + 1 - \text{tr}(\omega_{A_1^n B} \eta_{A_1^n B}) \} \\ = D_m(\rho_{A_1^n B} \| \eta_{A_1^n B}). \quad (191)$$

Let $Z := \text{tr}(\eta_{A_1^n B})$. Then, using the data processing inequality on the above $D_m(\rho_{A_1^n B} \| \eta_{A_1^n B})$ bound, we see that

$$\log \frac{1}{Z} \leq ng_2(\epsilon) \log \frac{1}{\delta}. \quad (192)$$

We can also upper bound Z as

$$\text{tr}(\eta_{A_1^n B}) = \int_{-\infty}^{\infty} dt \beta_0(t) \text{tr} \left((\delta^{\frac{1-it}{2}} P_{A_1 B}^{(1,b)} + P_{A_1 B}^{(1,g)}) \cdots (\delta^{\frac{1-it}{2}} P_{A_1^n B}^{(n,b)} + P_{A_1^n B}^{(n,g)}) \rho_{A_1^n B} \right. \\ \left. (\delta^{\frac{1+it}{2}} P_{A_1^n B}^{(n,b)} + P_{A_1^n B}^{(n,g)}) \cdots (\delta^{\frac{1+it}{2}} P_{A_1 B}^{(1,b)} + P_{A_1 B}^{(1,g)}) \right) \\ \leq \int_{-\infty}^{\infty} dt \beta_0(t) \left\| \delta P_{A_1 B}^{(1,b)} + P_{A_1 B}^{(1,g)} \right\|_{\infty} \text{tr} \left((\delta^{\frac{1-it}{2}} P_{A_1^2 B}^{(2,b)} + P_{A_1^2 B}^{(2,g)}) \cdots (\delta^{\frac{1-it}{2}} P_{A_1^n B}^{(n,b)} + P_{A_1^n B}^{(n,g)}) \rho_{A_1^n B} \right. \\ \left. (\delta^{\frac{1+it}{2}} P_{A_1^n B}^{(n,b)} + P_{A_1^n B}^{(n,g)}) \cdots (\delta^{\frac{1+it}{2}} P_{A_1^2 B}^{(2,b)} + P_{A_1^2 B}^{(2,g)}) \right) \\ \leq \int_{-\infty}^{\infty} dt \beta_0(t) \text{tr} \left((\delta^{\frac{1-it}{2}} P_{A_1^2 B}^{(2,b)} + P_{A_1^2 B}^{(2,g)}) \cdots (\delta^{\frac{1-it}{2}} P_{A_1^n B}^{(n,b)} + P_{A_1^n B}^{(n,g)}) \rho_{A_1^n B} \right. \\ \left. (\delta^{\frac{1+it}{2}} P_{A_1^n B}^{(n,b)} + P_{A_1^n B}^{(n,g)}) \cdots (\delta^{\frac{1+it}{2}} P_{A_1^2 B}^{(2,b)} + P_{A_1^2 B}^{(2,g)}) \right) \\ \leq \cdots \\ \leq \text{tr} \rho_{A_1^n B} \\ = 1. \quad (193)$$

We can define $\mu(\epsilon, \delta) := (g_2(\epsilon) \log \frac{1}{\delta})^{1/3}$ (we will use the shorthand μ going forth). Using the substate theorem, we have

$$\begin{aligned}
D_{\max}^{\mu} \left(\rho_{A_1^n B} \parallel \frac{\eta_{A_1^n B}}{Z} \right) &\leq \frac{D_m \left(\rho_{A_1^n B} \parallel \frac{\eta_{A_1^n B}}{Z} \right) + 1}{\mu^2} + \log \frac{1}{1 - \mu^2} \\
&\leq \frac{D_m \left(\rho_{A_1^n B} \parallel \eta_{A_1^n B} \right) - \log \frac{1}{Z} + 1}{\mu^2} + \log \frac{1}{1 - \mu^2} \\
&\leq \frac{n\mu^3 + 1}{\mu^2} + \log \frac{1}{1 - \mu^2} \\
&\leq n\mu + \frac{1}{\mu^2} + \log \frac{1}{1 - \mu^2}
\end{aligned} \tag{194}$$

where we have used $Z \leq 1$ in the third line.

We can get rid of Z normalisation factor by using Eq. 192

$$\begin{aligned}
D_{\max}^{\mu} \left(\rho_{A_1^n B} \parallel \eta_{A_1^n B} \right) &= D_{\max}^{\mu} \left(\rho_{A_1^n B} \parallel \frac{\eta_{A_1^n B}}{Z} \right) + \log \frac{1}{Z} \\
&\leq n(\mu + \mu^3) + \frac{1}{\mu^2} + \log \frac{1}{1 - \mu^2}
\end{aligned} \tag{195}$$

Though η has a nice form, it is still difficult to use the properties of the projectors $P_{A_1^k B}^{(k, g)}$ (Eq. 176) with it. We will now show how we can dominate η using a state of the form in Eq. 182.

Let X be an arbitrary positive operator, and Π and $\Pi_{\perp} = \mathbb{1} - \Pi$ be orthogonal projectors. Asymmetric pinching (Lemma 5.3) with the projectors Π and Π_{\perp} , and parameter $\sqrt{\delta}$ shows that for all $t \in \mathbb{R}$

$$\begin{aligned}
(\Pi + \delta^{\frac{1-it}{2}} \Pi_{\perp}) X (\Pi + \delta^{\frac{1+it}{2}} \Pi_{\perp}) &\leq (1 + \sqrt{\delta}) \Pi X \Pi + \left(1 + \frac{1}{\sqrt{\delta}} \right) \delta \Pi_{\perp} X \Pi_{\perp} \\
&= (1 + \sqrt{\delta}) P_{\sqrt{\delta}}[\Pi](X)
\end{aligned} \tag{196}$$

where $P_{\sqrt{\delta}}[\Pi](X) = \Pi X \Pi + \sqrt{\delta} \Pi_{\perp} X \Pi_{\perp}$ as defined in Eq. 181.

Using Eq. 196 repeatedly, we have that for every $t \in \mathbb{R}$

$$\begin{aligned}
& (\delta^{\frac{1-it}{2}} P_{A_1 B}^{(1,b)} + P_{A_1 B}^{(1,g)}) \dots (\delta^{\frac{1-it}{2}} P_{A_1^n B}^{(n,b)} + P_{A_1^n B}^{(n,g)}) \rho_{A_1^n B} (\delta^{\frac{1+it}{2}} P_{A_1^n B}^{(n,b)} + P_{A_1^n B}^{(n,g)}) \dots (\delta^{\frac{1+it}{2}} P_{A_1 B}^{(1,b)} + P_{A_1 B}^{(1,g)}) \\
& \leq (1 + \sqrt{\delta}) (\delta^{\frac{1-it}{2}} P_{A_1 B}^{(1,b)} + P_{A_1 B}^{(1,g)}) \dots (\delta^{\frac{1-it}{2}} P_{A_1^{n-1} B}^{(n-1,b)} + P_{A_1^{n-1} B}^{(n-1,g)}) P_{\sqrt{\delta}}[P_{A_1^n B}^{(n,g)}](\rho_{A_1^n B}) \\
& \quad (\delta^{\frac{1+it}{2}} P_{A_1^{n-1} B}^{(n-1,b)} + P_{A_1^{n-1} B}^{(n-1,g)}) \dots (\delta^{\frac{1+it}{2}} P_{A_1 B}^{(1,b)} + P_{A_1 B}^{(1,g)}) \\
& \leq \dots \\
& \leq (1 + \sqrt{\delta})^n P_{\sqrt{\delta}}[P_{A_1 B}^{(1,g)}] \circ \dots \circ P_{\sqrt{\delta}}[P_{A_1^n B}^{(n,g)}](\rho_{A_1^n B})
\end{aligned} \tag{197}$$

which implies that

$$\eta_{A_1^n B} \leq (1 + \sqrt{\delta})^n P_{\sqrt{\delta}}[P_{A_1 B}^{(1,g)}] \circ \dots \circ P_{\sqrt{\delta}}[P_{A_1^n B}^{(n,g)}](\rho_{A_1^n B}). \tag{198}$$

Let's define $\sigma_{A_1^n B} := P_{\sqrt{\delta}}[P_{A_1 B}^{(1,g)}] \circ \dots \circ P_{\sqrt{\delta}}[P_{A_1^n B}^{(n,g)}](\rho_{A_1^n B})$.

This bound essentially says that the D_{\max} between these two states is small:

$$D_{\max}(\eta_{A_1^n B} \| \sigma_{A_1^n B}) \leq n \log(1 + \sqrt{\delta}) \tag{199}$$

Combining this with the smooth D_{\max} bound between ρ and η (Eq. 195) shows that the smooth D_{\max} between ρ and σ is also small confirming our intuition from the motivation.

The only thing left to do now is to show that min-entropy of the state σ is large, i.e., $\geq \sum_{k=1}^n \lambda_k$ (recall $\lambda_k := \bar{H}_{\min}^\epsilon(A_k | A_1^{k-1} B)_\rho$). Note that

$$\sigma_{A_1^n B} = \sum_{l_1^n \in \{g,b\}^n} \delta^{\frac{1}{2}\omega_b(l_1^n)} P_{A_1 B}^{(1,l_1)} P_{A_1^2 B}^{(2,l_2)} \dots P_{A_1^n B}^{(n,l_n)} \rho_{A_1^n B} P_{A_1^n B}^{(n,l_n)} \dots P_{A_1^2 B}^{(2,l_2)} P_{A_1 B}^{(1,l_1)} \tag{200}$$

where $\omega_b(l_1^n) := |\{i : l_i = b\}|$ the weight of b labels in the string l_1^n . We will now bound the min-entropy of each of the terms in this summation. Roughly speaking, whenever the label $l_k = g$ (a good projector is applied), λ_k amount of min-entropy will be accumulated but when the projector is bad $O(\log|A|)$ amount of min-entropy will be lost. The reason, we are still able to accumulate a large amount of min-entropy for the state $\sigma_{A_1^n B}$ is because these terms are also weighted with the factor $\delta^{\frac{1}{2}\omega_b(l_1^n)}$. If the number of bad projectors in a term is large, then this factor ensures that the contribution of this term is small.

Claim 5.7. *For every $k \in [n]$ and $l_k \in \{g, b\}$, we have*

$$P_{A_1^k B}^{(k,l_k)} \rho_{A_1^k B} P_{A_1^k B}^{(k,l_k)} \leq \begin{cases} (1 + g_1(\epsilon)) e^{-\lambda_k} \mathbb{1}_{A_k} \otimes \rho_{A_1^{k-1} B} & \text{if } l_k = g \\ 4|A|(1 + g_1(\epsilon)) \mathbb{1}_{A_k} \otimes \rho_{A_1^{k-1} B} & \text{if } l_k = b \end{cases} \tag{201}$$

which can succinctly be written as

$$P_{A_1^k B}^{(k, l_k)} \rho_{A_1^k B} P_{A_1^k B}^{(k, l_k)} \leq (1 + g_1(\epsilon)) e^{-\lambda_k \delta(l_k, g)} (4|A|)^{\delta(l_k, b)} \mathbb{1}_{A_k} \otimes \rho_{A_1^{k-1} B} \quad (202)$$

where $\delta(x, y)$ is the Kronecker delta function ($\delta(x, y) = 1$ if $x = y$ else it is 0).

Proof. Let's first consider the case when $l_k = g$. In this case, we have

$$\begin{aligned} P_{A_1^k B}^{(k, g)} \rho_{A_1^k B} P_{A_1^k B}^{(k, g)} &\leq (1 + g_1(\epsilon)) \tilde{\rho}_{A_1^k B}^{(k)} \\ &\leq (1 + g_1(\epsilon)) e^{-\lambda_k} \mathbb{1}_{A_k} \otimes \rho_{A_1^{k-1} B} \end{aligned} \quad (203)$$

where the first line follows from the definition of the good projectors (Eq. 176) and the second line follows from Eq. 174.

When $l_k = b$, we have

$$\begin{aligned} P_{A_1^k B}^{(k, b)} \rho_{A_1^k B} P_{A_1^k B}^{(k, b)} &\leq 2\rho_{A_1^k B} + 2P_{A_1^k B}^{(k, g)} \rho_{A_1^k B} P_{A_1^k B}^{(k, g)} \\ &\leq 2\rho_{A_1^k B} + 2(1 + g_1(\epsilon)) \tilde{\rho}_{A_1^k B}^{(k)} \\ &\leq 2(1 + g_1(\epsilon)) \left(|A| \mathbb{1}_{A_k} \otimes \rho_{A_1^{k-1} B} + e^{-\lambda_k} \mathbb{1}_{A_k} \otimes \rho_{A_1^{k-1} B} \right) \\ &\leq 4|A|(1 + g_1(\epsilon)) \mathbb{1}_{A_k} \otimes \rho_{A_1^{k-1} B} \end{aligned} \quad (204)$$

where in the first line we have used Lemma 5.4, in the second line we have used Eq. 176, in the third line we have used $\rho_{A_1^k B} \leq |A| \mathbb{1}_{A_k} \otimes \rho_{A_1^{k-1} B}$ and Eq. 174, and in the last line we use $\lambda_k \geq -\log |A|$. \square

For every k , let $c_k(l_k) := e^{-\lambda_k \delta(l_k, g)} (4|A|)^{\delta(l_k, b)}$ so that

$$P_{A_1^k B}^{(k, l_k)} \rho_{A_1^k B} P_{A_1^k B}^{(k, l_k)} \leq (1 + g_1(\epsilon)) c_k(l_k) \mathbb{1}_{A_k} \otimes \rho_{A_1^{k-1} B}. \quad (205)$$

Now, observe that for each term in the summation in Eq. 200

$$\begin{aligned} &P_{A_1 B}^{(1, l_1)} P_{A_1^2 B}^{(2, l_2)} \dots P_{A_1^n B}^{(n, l_n)} \rho_{A_1^n B} P_{A_1^n B}^{(n, l_n)} \dots P_{A_1^2 B}^{(2, l_2)} P_{A_1 B}^{(1, l_1)} \\ &\leq (1 + g_1(\epsilon)) c_n(l_n) P_{A_1 B}^{(1, l_1)} P_{A_1^2 B}^{(2, l_2)} \dots P_{A_1^{n-1} B}^{(n-1, l_{n-1})} \mathbb{1}_{A_n} \otimes \rho_{A_1^{n-1} B} P_{A_1^{n-1} B}^{(n-1, l_{n-1})} \dots P_{A_1^2 B}^{(2, l_2)} P_{A_1 B}^{(1, l_1)} \\ &= (1 + g_1(\epsilon)) c_n(l_n) \mathbb{1}_{A_n} \otimes P_{A_1 B}^{(1, l_1)} P_{A_1^2 B}^{(2, l_2)} \dots P_{A_1^{n-1} B}^{(n-1, l_{n-1})} \rho_{A_1^{n-1} B} P_{A_1^{n-1} B}^{(n-1, l_{n-1})} \dots P_{A_1^2 B}^{(2, l_2)} P_{A_1 B}^{(1, l_1)} \\ &\leq \dots \\ &\leq (1 + g_1(\epsilon))^n \left(\prod_{k=1}^n c_k(l_k) \right) \mathbb{1}_{A_1^n} \otimes \rho_B. \end{aligned} \quad (206)$$

Plugging this bound into the expression for σ in Eq. 200, we get

$$\begin{aligned}\sigma_{A_1^n B} &= \sum_{l_1^n \in \{g, b\}^n} \delta^{\frac{1}{2}\omega_b(l_1^n)} P_{A_1 B}^{(1, l_1)} P_{A_1^2 B}^{(2, l_2)} \dots P_{A_1^n B}^{(n, l_n)} \rho_{A_1^n B} P_{A_1^n B}^{(n, l_n)} \dots P_{A_1^2 B}^{(2, l_2)} P_{A_1 B}^{(1, l_1)} \\ &\leq (1 + g_1(\epsilon))^n \left(\sum_{l_1^n \in \{g, b\}^n} \delta^{\frac{1}{2}\omega_b(l_1^n)} \prod_{k=1}^n c_k(l_k) \right) \mathbb{1}_{A_1^n} \otimes \rho_B.\end{aligned}\quad (207)$$

Let us now bound the expression

$$\begin{aligned}\sum_{l_1^n \in \{g, b\}^n} \delta^{\frac{1}{2}\omega_b(l_1^n)} \prod_{k=1}^n c_k(l_k) &= \sum_{l_1^n \in \{g, b\}^n} \prod_{k=1}^n \sqrt{\delta}^{\delta(l_k, b)} \prod_{k=1}^n e^{-\lambda_k \delta(l_k, g)} (4|A|)^{\delta(l_k, b)} \\ &= \sum_{l_1^n \in \{g, b\}^n} \prod_{k=1}^n e^{-\lambda_k \delta(l_k, g)} (4|A|\sqrt{\delta})^{\delta(l_k, b)} \\ &= \prod_{k=1}^n \left(e^{-\lambda_k} + 4|A|\sqrt{\delta} \right) \\ &= \prod_{k=1}^n e^{-\lambda_k} \left(1 + 4|A|e^{\lambda_k} \sqrt{\delta} \right) \\ &\leq \prod_{k=1}^n e^{-\lambda_k} \left(1 + 4|A|^2 \frac{\sqrt{\delta}}{1 - \epsilon^2} \right) \\ &= \left(1 + 4|A|^2 \frac{\sqrt{\delta}}{1 - \epsilon^2} \right)^n e^{-\sum_{k=1}^n \lambda_k}\end{aligned}\quad (208)$$

where we have used $\lambda_k \leq \log \frac{|A|}{1 - \epsilon^2}$ in the second last line. Combining Eq. 207 and Eq. 208, we get

$$D_{\max}(\sigma_{A_1^n B} \| \mathbb{1}_{A_1^n} \otimes \rho_B) \leq - \sum_{k=1}^n \lambda_k + n \log \left(1 + 4|A|^2 \frac{\sqrt{\delta}}{1 - \epsilon^2} \right) + n \log(1 + g_1(\epsilon)). \quad (209)$$

Further, combining this with Eq. 199 gives

$$D_{\max}(\eta_{A_1^n B} \| \mathbb{1}_{A_1^n} \otimes \rho_B) \leq - \sum_{k=1}^n \lambda_k + n \log \left(1 + 4|A|^2 \frac{\sqrt{\delta}}{1 - \epsilon^2} \right) + n \log(1 + g_1(\epsilon)) + n \log(1 + \sqrt{\delta}) \quad (210)$$

which in term implies that

$$\begin{aligned}H_{\min}(A_1^n | B)_\eta &\geq \sum_{k=1}^n \lambda_k - n \log \left(1 + 4|A|^2 \frac{\sqrt{\delta}}{1 - \epsilon^2} \right) - n \log(1 + g_1(\epsilon)) - n \log(1 + \sqrt{\delta}) \\ &\geq \sum_{k=1}^n \bar{H}_{\min}^\epsilon(A_k | A_1^{k-1} B)_\rho - n \log \left(1 + 4|A|^2 \frac{\sqrt{\delta}}{1 - \epsilon^2} \right) - n \log(1 + g_1(\epsilon)) - n \log(1 + \sqrt{\delta}).\end{aligned}$$

Finally, we can use the entropic triangle inequality in Eq. 35 along with Eq. 195 to get

$$\begin{aligned} H_{\min}^{\mu}(A_1^n|B)_{\rho} &\geq \sum_{k=1}^n \bar{H}_{\min}^{\epsilon}(A_k|A_1^{k-1}B)_{\rho} - n \log \left(1 + 4|A|^2 \frac{\sqrt{\delta}}{1-\epsilon^2} \right) - n \log(1 + g_1(\epsilon)) \\ &\quad - n \log(1 + \sqrt{\delta}) - n(\mu + \mu^3) - \frac{1}{\mu^2} - \log \frac{1}{1-\mu^2} \end{aligned} \quad (211)$$

where $\mu = (g_2(\epsilon) \log \frac{1}{\delta})^{1/3}$. We choose the parameter $\delta = \epsilon^2$, so that we have

$$\begin{aligned} H_{\min}^{\mu}(A_1^n|B)_{\rho} &\geq \sum_{k=1}^n \bar{H}_{\min}^{\epsilon}(A_k|A_1^{k-1}B)_{\rho} - n \log \left(1 + 4|A|^2 \frac{\epsilon}{1-\epsilon^2} \right) - n \log(1 + g_1(\epsilon)) \\ &\quad - n \log(1 + \epsilon) - n(\mu + \mu^3) - \frac{1}{\mu^2} - \log \frac{1}{1-\mu^2} \end{aligned} \quad (212)$$

for $\mu = (8(1 + \epsilon^{1/3}) \log \frac{1}{\epsilon})^{1/3} \epsilon^{1/9} = O(\epsilon^{1/9} (\log \frac{1}{\epsilon})^{1/3})$. \square

Theorem 5.8. *For a state $\rho_{A_1^n B}$ and $\epsilon \in (0, 1)$, we have the chain rule*

$$\begin{aligned} H_{\min}^{\downarrow, 2\mu+\epsilon/4}(A_1^n|B)_{\rho} &\geq \sum_{k=1}^n H_{\min}^{\downarrow, \epsilon/4}(A_k|A_1^{k-1}B)_{\rho} - n \log \left(1 + 4|A|^2 \frac{\epsilon}{1-\epsilon^2} \right) - n \log(1 + g_1(\epsilon)) \\ &\quad - n \log(1 + \epsilon) - n(\mu + \mu^3) - \frac{1}{\mu^2} - \log \frac{1}{1-\mu^2} - \log \left(\frac{2}{\mu^2} + \frac{1}{1-\mu} \right) \end{aligned} \quad (213)$$

where

$$\mu := \left(8(1 + \epsilon^{1/3}) \log \frac{1}{\epsilon} \right)^{1/3} \epsilon^{1/9} = O \left(\epsilon^{1/9} \left(\log \frac{1}{\epsilon} \right)^{1/3} \right) \quad (214)$$

$$g_1(\epsilon) := \frac{8}{3}(1 + \epsilon^{1/3})\epsilon^{1/3} \log \frac{1}{\epsilon} + (1 + \epsilon^{1/3} + \epsilon^{2/3})\epsilon^{1/3} = O \left(\epsilon^{1/3} \log \frac{1}{\epsilon} \right) \quad (215)$$

as long as $\mu \in (0, 1)$.

Proof. Case 1: If $\rho_{A_1^n B}$ is full rank, then we can use the lemma above along with Eq. 34 and Lemma 5.2 to show that

$$\begin{aligned} H_{\min}^{\downarrow, 2\mu}(A_1^n|B)_{\rho} &\geq \sum_{k=1}^n H_{\min}^{\downarrow, \epsilon/2}(A_k|A_1^{k-1}B)_{\rho} - n \log \left(1 + 4|A|^2 \frac{\epsilon}{1-\epsilon^2} \right) - n \log(1 + g_1(\epsilon)) \\ &\quad - n \log(1 + \epsilon) - n(\mu + \mu^3) - \frac{1}{\mu^2} - \log \frac{1}{1-\mu^2} - \log \left(\frac{2}{\mu^2} + \frac{1}{1-\mu} \right). \end{aligned} \quad (216)$$

Case 2: Now, we can follow the same argument as the one in Case 2 of the proof of Theorem 3.4 to derive the bound in the theorem statement for all states ρ . \square

6 Conclusion

We develop a powerful proof technique combining the entropic triangle inequality, the generalised GT inequality and the substate theorem for proving entropic bounds for approximation chains. We use this technique to prove novel chain rules- the universal smooth min-entropy chain rule and the unstructured approximate entropy accumulation theorem. Importantly, both of these chain rules can be used meaningfully for arbitrarily large number of systems.

As far as applications are concerned, we use the unstructured approximate EAT to prove the security of parallel DI-QKD in our upcoming work [MD24]. The universal chain rule should aid in transforming von Neumann entropy based arguments to one-shot arguments. Furthermore, it provides a straightforward solution to problems such as the approximately independent registers problem discussed in [MD23, Section 4]. It is also our conviction that the proof technique introduced in this paper will be useful for tackling other problems. For instance, it should be possible to use it to derive similar chain rules for one-shot variants (e.g., I_{\max}) of the (multipartite) mutual information.

Finally, we did not attempt to optimise our bounds here, but it should be interesting to study the absolute limits of the entropic error terms and the smoothing errors in our chain rules.

APPENDICES

A Additional lemmas

The following lemma provides a tighter bound for the distance as compared to Lemma 2.5. It is proven in [DBWR14, Lemma B.3]. We could replace the use of Lemma 2.5 in Theorem 4.1 and C.2 with the following. This would slightly improve some constants. We use Lemma 2.5 instead for notational clarity, since we do not need to keep track of the additional unitary introduced in the following lemma.

Lemma A.1. *For a normalised state ρ_{AB} and a subnormalised state $\tilde{\rho}_{AB}$ such that $P(\rho_{AB}, \tilde{\rho}_{AB}) \leq \epsilon$, there exists a unitary U_B on the register B such that the state*

$$\eta_{AB} := \rho_B^{1/2} U_B \tilde{\rho}_B^{-1/2} \tilde{\rho}_{AB} \tilde{\rho}_B^{-1/2} U_B^\dagger \rho_B^{1/2} \quad (217)$$

($\tilde{\rho}_B^{-1/2}$ is the Moore-Penrose pseudo-inverse above) satisfies $P(\rho_{AB}, \eta_{AB}) \leq 2\epsilon$. Note that if $\tilde{\rho}_B$ is full rank, then $\eta_B = \rho_B$.

Proof. Note that since ρ_{AB} is normalised, we have $F(\tilde{\rho}_{AB}, \rho_{AB}) \geq 1 - \epsilon^2$. Let $|\tilde{\rho}\rangle_{ABR}$ be an arbitrary purification of $\tilde{\rho}_{AB}$. Let U_B be any unitary for now and let η_{AB} be defined as in

Eq. 217 above. We will choose U_B so that $F(\tilde{\rho}_{AB}, \eta_{AB})$ is large.

Observe that the pure state $|\eta\rangle_{ABR} := \rho_B^{1/2} U_B \tilde{\rho}_B^{-1/2} |\tilde{\rho}\rangle_{ABR}$ is a purification of η_{AB} . Using Uhlmann's theorem [Wat18, Theorem 3.22], we have

$$\begin{aligned} F(\tilde{\rho}_{AB}, \eta_{AB}) &\geq |\langle \tilde{\rho} | \eta \rangle|^2 \\ &= \left| \langle \tilde{\rho} | \rho_B^{1/2} U_B \tilde{\rho}_B^{-1/2} | \tilde{\rho} \rangle \right|^2 \\ &= \left| \text{tr}(\rho_B^{1/2} U_B \tilde{\rho}_B^{-1/2} \tilde{\rho}_{ABR}) \right|^2 \\ &= \left| \text{tr}(U_B \tilde{\rho}_B^{1/2} \rho_B^{1/2}) \right|^2. \end{aligned}$$

Say the polar decomposition of $\tilde{\rho}_B^{1/2} \rho_B^{1/2} = V_B \left| \tilde{\rho}_B^{1/2} \rho_B^{1/2} \right|$. We can now select U_B to be V_B^\dagger , so that

$$\begin{aligned} F(\tilde{\rho}_{AB}, \eta_{AB}) &\geq \left(\text{tr} \left| \tilde{\rho}_B^{1/2} \rho_B^{1/2} \right| \right)^2 \\ &= F(\tilde{\rho}_B, \rho_B) \\ &\geq 1 - \epsilon^2 \end{aligned} \tag{218}$$

where we have used $F(\tilde{\rho}_B, \rho_B) \geq F(\tilde{\rho}_{AB}, \rho_{AB})$. Further, we have

$$\begin{aligned} P(\tilde{\rho}_{AB}, \eta_{AB}) &= \sqrt{1 - F_*(\tilde{\rho}_{AB}, \eta_{AB})} \\ &\leq \sqrt{1 - F(\tilde{\rho}_{AB}, \eta_{AB})} \\ &\leq \epsilon. \end{aligned}$$

Using the triangle inequality, for this choice of U_B , we get

$$\begin{aligned} P(\rho_{AB}, \eta_{AB}) &\leq P(\rho_{AB}, \tilde{\rho}_{AB}) + P(\tilde{\rho}_{AB}, \eta_{AB}) \\ &\leq 2\epsilon. \end{aligned}$$

□

B Classical approximate chain rule for the relative entropy

Lemma B.1. *Let p and p' be probability distributions over \mathcal{X} . Then, for a function $f : \mathcal{X} \rightarrow \mathbb{R}$, we have*

$$|\mathbb{E}_{X \sim p}[f(X)] - \mathbb{E}_{X \sim p'}[f(X)]| \leq \max_{x \in \mathcal{X}} |f(x)| \|p - p'\|_1. \tag{219}$$

Proof.

$$\begin{aligned}
|\mathbb{E}_{X \sim p}[f(X)] - \mathbb{E}_{X \sim p'}[f(X)]| &= \left| \sum_{x \in \mathcal{X}} p(x)f(x) - \sum_{x \in \mathcal{X}} p'(x)f(x) \right| \\
&= \left| \sum_{x \in \mathcal{X}} (p(x) - p'(x))f(x) \right| \\
&\leq \sum_{x \in \mathcal{X}} |p(x) - p'(x)| |f(x)| \\
&\leq \max_{x \in \mathcal{X}} |f(x)| \|p - p'\|_1
\end{aligned}$$

□

We will call the following Lemma an approximate chain rule for the relative entropy. It allows us to switch the probability distribution p_{AB} in the term $D(p_{AB} \| p_B q_{A|B})$ of the chain rule for $D(p_{AB} \| q_{AB})$ to $D(p'_{AB} \| p'_B q_{A|B})$, where $p \approx_\epsilon p'$ while incurring a penalty which depends only on the size of the register A .

Lemma B.2. *Suppose that $\delta > 0$ and q_{AB} is a probability distribution over $\mathcal{A} \times \mathcal{B}$ such that for all $a, b \in \mathcal{A} \times \mathcal{B}$, we have $q(a|b) \geq \delta$. Then, for $0 < \epsilon < \frac{1}{2}$ and a p'_{AB} such that $\frac{1}{2} \|p_{AB} - p'_{AB}\|_1 \leq \epsilon$, we have that*

$$D(p_{AB} \| q_{AB}) \leq D(p_B \| q_B) + D(p'_{AB} \| p'_B q_{A|B}) + z(\epsilon, \delta) \quad (220)$$

where $z(\epsilon, \delta) := h(2\epsilon) + 6\epsilon \log \frac{1}{\delta} + 4\epsilon \log |\mathcal{A}|$. Equivalently, we have

$$D(p_{AB} \| q_{AB}) \leq D(p_B \| q_B) + \inf_{\|p'_{AB} - p_{AB}\|_1 \leq 2\epsilon} D(p'_{AB} \| p'_B q_{A|B}) + z(\epsilon, \delta) \quad (221)$$

Proof. If $\text{supp}(p_B) \not\subseteq \text{supp}(q_B)$, then the right-hand side is infinite and the identity is trivially true. We suppose $\text{supp}(p_B) \subseteq \text{supp}(q_B)$ here on.

We will show that for every p'_{AB} , which is ϵ -close to p_{AB} the right-hand side in Eq. 220 is greater than the left-hand side. Classically, we have the chain rule

$$D(p_{AB} \| q_{AB}) = D(p_B \| q_B) + \mathbb{E}_{B \sim p_B} [D(p_{A|B} \| q_{A|B})]. \quad (222)$$

Note that both the above terms are finite ($q(a|b) \geq \delta$ is given). We will bound $\max_b D(p_{A|b} \| q_{A|b})$ and then use Lemma B.1 to create a bound for the expectation in terms of p' . For a given

$b \in \mathcal{B}$, we have

$$\begin{aligned}
|D(p_{A|b}||q_{A|b})| &= \left| \sum_a p(a|b) \log \frac{p(a|b)}{q(a|b)} \right| \\
&\leq \left| \sum_a p(a|b) \log p(a|b) \right| + \left| \sum_a p(a|b) \log \frac{1}{q(a|b)} \right| \\
&= H(A|B=b)_p + \sum_a p(a|b) \log \frac{1}{q(a|b)} \\
&\leq \log(|\mathcal{A}|) + \log \frac{1}{\delta}.
\end{aligned}$$

Now, using Lemma B.1,

$$\mathbb{E}_{B \sim p_B}[D(p_{A|B}||q_{A|B})] \leq \mathbb{E}_{B \sim p'_B}[D(p_{A|B}||q_{A|B})] + 2\epsilon \left(\log(|\mathcal{A}|) + \log \frac{1}{\delta} \right). \quad (223)$$

Finally, we need to change the $p_{A|B}$ in $D(p_{A|B}||q_{A|B})$ to $p'_{A|B}$.

$$\begin{aligned}
D(p_{A|b}||q_{A|b}) &= -H(A|B=b)_p + \mathbb{E}_{A \sim p_{A|b}} \left[\log \frac{1}{q(A|b)} \right] \\
&\leq -H(A|B=b)_{p'} + h \left(\frac{1}{2} \|p_{A|b} - p'_{A|b}\|_1 \right) + \frac{1}{2} \|p_{A|b} - p'_{A|b}\|_1 \log(|\mathcal{A}|) \\
&\quad + \mathbb{E}_{A \sim p'_{A|b}} \left[\log \frac{1}{q(A|b)} \right] + \|p_{A|b} - p'_{A|b}\|_1 \log \frac{1}{\delta} \\
&= D(p'_{A|b}||q_{A|b}) + h \left(\frac{1}{2} \|p_{A|b} - p'_{A|b}\|_1 \right) + \frac{1}{2} \|p_{A|b} - p'_{A|b}\|_1 \left(2 \log \frac{1}{\delta} + \log(|\mathcal{A}|) \right)
\end{aligned}$$

where we used the Fannes-Audenaert continuity bound [Wil13, Theorem 11.10.2] and Lemma B.1 in the second line. Taking the expectation over p'_B , we get

$$\begin{aligned}
&\mathbb{E}_{B \sim p'_B}[D(p_{A|B}||q_{A|B})] \\
&\leq \mathbb{E}_{B \sim p'_B} \left[D(p'_{A|B}||q_{A|B}) + h \left(\frac{1}{2} \|p_{A|B} - p'_{A|B}\|_1 \right) + \frac{1}{2} \|p_{A|B} - p'_{A|B}\|_1 \left(2 \log \frac{1}{\delta} + \log(|\mathcal{A}|) \right) \right] \\
&\leq \mathbb{E}_{B \sim p'_B} \left[D(p'_{A|B}||q_{A|B}) \right] + h \left(\frac{1}{2} \|p'_B p_{A|B} - p'_{AB}\|_1 \right) + \frac{1}{2} \|p'_B p_{A|B} - p'_{AB}\|_1 \left(2 \log \frac{1}{\delta} + \log |\mathcal{A}| \right) \\
&\leq \mathbb{E}_{B \sim p'_B} \left[D(p'_{A|B}||q_{A|B}) \right] + h(2\epsilon) + 2\epsilon \left(2 \log \frac{1}{\delta} + \log |\mathcal{A}| \right) \\
&= D(p'_{AB}||p'_B q_{A|B}) + h(2\epsilon) + 2\epsilon \left(2 \log \frac{1}{\delta} + \log |\mathcal{A}| \right)
\end{aligned}$$

where we have used the fact that if $\|p_{AB} - p'_{AB}\|_1 \leq \epsilon$, then $\|p'_B p_{A|B} - p'_{AB}\|_1 \leq 2\epsilon$. This can be derived using the triangle inequality. Putting this in Eq. 223, we get

$$\mathbb{E}_{B \sim p_B}[D(p_{A|B}||q_{A|B})] \leq D(p'_{AB}||p'_B q_{A|B}) + h(2\epsilon) + 6\epsilon \log \frac{1}{\delta} + 4\epsilon \log |\mathcal{A}|.$$

Therefore, using Eq. 222, we get

$$D(p_{AB}||q_{AB}) \leq D(p_B||q_B) + D(p'_{AB}||p'_B q_{A|B}) + h(2\epsilon) + 6\epsilon \log \frac{1}{\delta} + 4\epsilon \log |\mathcal{A}|.$$

□

C Testing for approximate entropy accumulation

To incorporate testing to Theorem 4.1, we follow [MFSR22], which is itself based on [DF19].

First, we will define the testing channels \mathcal{T}_k . These channels measure the outputs A_k and B_k of the state ρ and output a result X_k based on these measurements. Concretely, for every $k \in [n]$ the channel $\mathcal{T}_k : A_k B_k \rightarrow A_k B_k X_k$ is of the form

$$\mathcal{T}_k(\omega_{A_k B_k}) = \sum_{a,b} \Pi_{A_k}^{(a)} \otimes \Pi_{B_k}^{(b)} \omega_{A_k B_k} \Pi_{A_k}^{(a)} \otimes \Pi_{B_k}^{(b)} \otimes |x(a,b)\rangle \langle x(a,b)|_{X_k} \quad (224)$$

where $\{\Pi_{A_k}^{(a)}\}_a$ and $\{\Pi_{B_k}^{(b)}\}_b$ are orthogonal projectors and $x(\cdot)$ is some deterministic function which uses the measurements a and b to create the output register X_k .

Using these orthogonal projectors, we further define the orthogonal projectors on the registers $A_1^k B_1^k$

$$\Pi_{A_1^k B_1^k}^{(a_1^k, b_1^k)} := \bigotimes_{i=1}^k \left(\Pi_{A_i}^{(a_i)} \otimes \Pi_{B_i}^{(b_i)} \right) \quad (225)$$

for every a_1^k, b_1^k . Together these form a measurement on the registers $A_1^k B_1^k$.

Next we define the min-tradeoff functions. Let \mathbb{P} be the set of probability distributions over the alphabet of X registers. Let R be any register isomorphic to R_k . For a probability $q \in \mathbb{P}$ and a channel $\mathcal{N}_k : R_k \rightarrow A_k B_k$, we define the set

$$\Sigma_k(q|\mathcal{N}_k) := \{ \nu_{A_k B_k X_k R} = \mathcal{T}_k \circ \mathcal{N}_k(\omega_{R_k R}) : \text{for a state } \omega_{R_{k-1} R} \text{ such that } \nu_{X_k} = q \}. \quad (226)$$

Definition C.1. A function $f : \mathbb{P} \rightarrow \mathbb{R}$ is called a min-tradeoff function for the channels $\{\mathcal{N}_k\}_{k=1}^n$ if for every k , it satisfies

$$f(q) \leq \inf_{\nu \in \Sigma_k(q|\mathcal{N}_k)} H(A_k | B_k R)_\nu. \quad (227)$$

We will also need the definitions of the following simple properties of the min-tradeoff functions for our entropy accumulation theorem:

$$\max(f) := \max_{q \in \mathbb{P}} f(q) \quad (228)$$

$$\min(f) := \min_{q \in \mathbb{P}} f(q). \quad (229)$$

We now state the unstructured approximate EAT with testing.

Theorem C.2. Let $\epsilon \in (0, 1)$ and for every $k \in [n]$, the registers A_k and B_k be such that $|A_k| = |A|$ and $|B_k| = |B|$. Suppose, the state $\rho_{A_1^n B_1^n X_1^n E}$ is such that

1. The registers X_1^n can be recreated by applying the testing maps to the registers A_1^n and B_1^n , that is,

$$\rho_{A_1^n B_1^n X_1^n E} = \mathcal{T}_n \circ \cdots \circ \mathcal{T}_1(\rho_{A_1^n B_1^n E}) \quad (230)$$

2. For every $k \in [n]$, there exists a channel $\mathcal{M}_k : R_k \rightarrow A_k B_k$ and a state $\theta_{B_k}^{(k)}$ such that

$$\text{tr}_{X_k} \circ \mathcal{T}_k \circ \mathcal{M}_k = \mathcal{M}_k \quad (231)$$

$$\text{tr}_{A_k} \circ \mathcal{M}_k(X_{R_k}) = \text{tr}(X) \theta_{B_k}^{(k)} \quad \text{for all operators } X_{R_k} \quad (232)$$

and a state $\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E R_k}^{(k,0)}$ for which

$$\frac{1}{2} \left\| \rho_{A_1^k B_1^k E} - \mathcal{M}_k(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E R_k}^{(k,0)}) \right\|_1 \leq \epsilon. \quad (233)$$

Then, for an event Ω defined using X_1^n , an affine min-tradeoff function f for $\{\mathcal{M}_k\}_{k=1}^n$ such that for every $x_1^n \in \Omega$, $f(\text{freq}(x_1^n)) \geq h$, we have

$$\begin{aligned} H_{\min}^{\mu' + \epsilon'}(A_1^n | B_1^n E)_{\rho_{\Omega}} &\geq n(h - V(3\sqrt{\mu} + 4\epsilon) - g_2(2\epsilon)) \\ &\quad - \frac{V}{\sqrt{\mu}} \left(2 \log \frac{1}{P_{\rho}(\Omega) - \mu} + \frac{2}{\mu^2} + 2 \log \frac{1}{1 - \mu^2} + g_1(\epsilon', \mu') \right) \end{aligned} \quad (234)$$

where

$$\mu := \left(\frac{8\sqrt{\epsilon} + 2\epsilon}{1 - \epsilon^2/(|A||B|)^2} \log \frac{|A||B|}{\epsilon} \right)^{1/3} \quad (235)$$

$$\mu' := 2 \sqrt{\frac{\mu}{P_{\rho}(\Omega)}} \quad (236)$$

$$V := \log(1 + 2|A|) + \lceil \max(f) - \min(f) \rceil \quad (237)$$

$$g_1(x, y) := -\log(1 - \sqrt{1 - x^2}) - \log(1 - y^2) \quad (238)$$

$$g_2(x) := x \log \frac{1}{x} + (1 + x) \log(1 + x) \quad (239)$$

and $\epsilon' \in (0, 1)$ such that $\mu' + \epsilon' < 1$.

Proof. We first define the sets

$$\mathcal{A}_k := \left\{ X_{A_1^k B_1^k E} : X_{A_1^k B_1^k E} = \sum_{a_1^k, b_1^k} \Pi_{A_1^k B_1^k}^{(a_1^k b_1^k)} X_{A_1^k B_1^k E} \Pi_{A_1^k B_1^k}^{(a_1^k b_1^k)} \right\} \quad (240)$$

for every $0 \leq k \leq n$. It should be noted that each of these sets is a *unital algebra*, i.e., a vector space over the field \mathbb{C} closed under the matrix product containing identity $\mathbb{1}_{A_1^k B_1^k E}$. It is also easy to see that

$$\mathcal{A}_k \otimes \mathbb{1}_{A_{k+1} B_{k+1}} := \left\{ X_{A_1^k B_1^k E} \otimes \mathbb{1}_{A_{k+1} B_{k+1}} : X_{A_1^k B_1^k E} \in \mathcal{A}_k \right\} \subseteq \mathcal{A}_{k+1} \quad (241)$$

$$\text{tr}_{A_k B_k}(\mathcal{A}_k) := \left\{ X_{A_1^{k-1} B_1^{k-1} E} : X_{A_1^k B_1^k E} \in \mathcal{A}_k \right\} \subseteq \mathcal{A}_{k-1}. \quad (242)$$

Finally, it should be noted that for all operators $X_{A_1^k B_1^k E} \in \mathcal{A}_k$, we have

$$\text{tr}_{X_1^k} \circ \mathcal{T}_k \circ \cdots \circ \mathcal{T}_1(X) = X. \quad (243)$$

The above equation in particular implies that one can measure a state in \mathcal{A}_k using the channel $\mathcal{T}_k \circ \cdots \circ \mathcal{T}_1$ without disturbing the state. All the auxiliary states defined during this proof will be contained inside an appropriate set \mathcal{A}_k . Noting that these sets are, in fact, an algebra will make it easier to see this. For example, observe that each partial state $\rho_{A_1^k B_1^k E} \in \mathcal{A}_k$.

Case 1: To begin, we restrict our attention to states $\rho_{A_1^n B_1^n E}$, which have full rank. Let $\nu \in (0, 1)$ be an arbitrarily chosen small parameter. For every $k \in [n]$, define the states

$$\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E R_k}^{(k,0)} := (1 - \nu) \tilde{\rho}_{A_1^{k-1} B_1^{k-1} E R_k}^{(k,0)} + \nu \tau_{A_1^{k-1} B_1^{k-1} E R_k} \quad (244)$$

$$\tilde{\rho}_{A_1^{k-1} B_1^{k-1} X_1^{k-1} E R_k}^{(k,1)} := \mathcal{T}_{k-1} \circ \cdots \circ \mathcal{T}_1(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E R_k}^{(k,0)}) \quad (245)$$

$$\tilde{\rho}_{A_1^k B_1^k X_1^k E}^{(k)} := \mathcal{T}_k \circ \mathcal{M}_k \left(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} X_1^{k-1} E R_k}^{(k,1)} \right). \quad (246)$$

Since the maps $\text{tr}_{X_i} \circ \mathcal{T}_i$ are unital, for every k , we have

$$\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E}^{(k)} = \tilde{\rho}_{A_1^{k-1} B_1^{k-1} E}^{(k,1)} \quad (247)$$

$$\geq \nu \tau_{A_1^{k-1} B_1^{k-1} E}. \quad (248)$$

Thus, the states $\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E}^{(k)}$ are also full rank. Moreover, we have that

$$\tilde{\rho}_{A_1^k B_1^k E}^{(k)} \in \mathcal{A}_k. \quad (249)$$

For each k these states satisfy

$$\begin{aligned} \frac{1}{2} \left\| \rho_{A_1^k B_1^k X_1^k E} - \tilde{\rho}_{A_1^k B_1^k X_1^k E}^{(k)} \right\| &= \frac{1}{2} \left\| \rho_{A_1^k B_1^k X_1^k E} - \mathcal{T}_k \circ \mathcal{M}_k \left(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} X_1^{k-1} E R_k}^{(k,1)} \right) \right\|_1 \\ &= \frac{1 - \nu}{2} \left\| \mathcal{T}_k \circ \cdots \circ \mathcal{T}_1(\rho_{A_1^k B_1^k E}) - \mathcal{T}_k \circ \cdots \circ \mathcal{T}_1 \circ \mathcal{M}_k \left(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E R_k}^{(k,0)} \right) \right\|_1 + \nu \\ &\leq \frac{1 - \nu}{2} \left\| \rho_{A_1^k B_1^k E} - \mathcal{M}_k \left(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E R_k}^{(k,0)} \right) \right\|_1 + \nu \\ &\leq \epsilon + \nu. \end{aligned} \quad (250)$$

Now, for each $k \in [n]$, we define the normalised states

$$\omega_{A_1^{k-1}B_1^{k-1}R_kE}^{(k,1)} := \rho_{A_1^{k-1}B_1^{k-1}E}^{1/2} \left(\tilde{\rho}_{A_1^{k-1}B_1^{k-1}E}^{(k)} \right)^{-1/2} \tilde{\rho}_{A_1^{k-1}B_1^{k-1}R_kE}^{(k,1)} \left(\tilde{\rho}_{A_1^{k-1}B_1^{k-1}E}^{(k)} \right)^{-1/2} \rho_{A_1^{k-1}B_1^{k-1}E}^{1/2} \quad (251)$$

and

$$\omega_{A_1^k B_1^k E}^{(k)} := \mathcal{M}_k \left(\omega_{A_1^{k-1} B_1^{k-1} R_k E}^{(k,1)} \right) \quad (252)$$

$$= \rho_{A_1^{k-1} B_1^{k-1} E}^{1/2} \left(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E}^{(k)} \right)^{-1/2} \text{tr}_{X_k} \circ \mathcal{T}_k \circ \mathcal{M}_k \left(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} R_k E}^{(k,1)} \right) \left(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E}^{(k)} \right)^{-1/2} \rho_{A_1^{k-1} B_1^{k-1} E}^{1/2} \quad (253)$$

$$= \rho_{A_1^{k-1} B_1^{k-1} E}^{1/2} \left(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E}^{(k)} \right)^{-1/2} \tilde{\rho}_{A_1^k B_1^k E}^{(k)} \left(\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E}^{(k)} \right)^{-1/2} \rho_{A_1^{k-1} B_1^{k-1} E}^{1/2}. \quad (254)$$

We used $\text{tr}_{X_k} \circ \mathcal{T}_k \circ \mathcal{M}_k = \mathcal{M}_k$ above. Observe that $\omega_{A_1^k B_1^k E}^{(k)} \in \mathcal{A}_k$ (using Eq. 249, 241 and 242). Since, we defined $\tilde{\rho}_{A_1^{k-1} B_1^{k-1} E}^{(k)}$ to be full rank, we have that

$$\omega_{A_1^{k-1} B_1^{k-1} E}^{(k)} = \rho_{A_1^{k-1} B_1^{k-1} E}. \quad (255)$$

Using Lemma 2.5, we have that

$$\begin{aligned} \frac{1}{2} \left\| \rho_{A_1^k B_1^k E} - \omega_{A_1^k B_1^k E}^{(k)} \right\|_1 &\leq (\sqrt{2} + 1) P(\rho_{A_1^k B_1^k E}, \tilde{\rho}_{A_1^k B_1^k E}) \\ &\leq (\sqrt{2} + 1) \sqrt{2(\epsilon + \nu)} \\ &\leq 4\sqrt{\epsilon + \nu}. \end{aligned} \quad (256)$$

Let $\delta \in (0, 1)$ be a small parameter (to be set equal to ϵ later). Define the states

$$\rho_{A_k B_k}^{(k,\delta)} := (1 - \delta) \rho_{A_k B_k} + \delta \tau_{A_k B_k}. \quad (257)$$

Finally, for every $k \in [n]$, we define the states

$$\bar{\rho}_{A_1^k B_1^k E}^{(k)} := (1 - \delta) \omega_{A_1^k B_1^k E}^{(k)} + \delta \rho_{A_k B_k}^{(k,\delta)} \otimes \rho_{A_1^{k-1} B_1^{k-1} E} \quad (258)$$

$$= (1 - \delta) \omega_{A_1^k B_1^k E}^{(k)} + \delta \rho_{A_k B_k}^{(k,\delta)} \otimes \omega_{A_1^{k-1} B_1^{k-1} E}^{(k)} \quad (259)$$

Also, define $\bar{\rho}_E^{(0)} := \rho_E$. For each $0 \leq k \leq n$, the state $\bar{\rho}_{A_1^k B_1^k E}^{(k)} \in \mathcal{A}_k$ using Eq. 241. We have taken a slight diversion from the proof of Theorem 4.1 in defining the above state. This has been done to ensure we are able to bound the entropy in Eq. 294.

Let $\Delta_k : R_k \rightarrow A_k B_k$ be the map which traces out the register R_k and outputs $\rho_{A_k B_k}^{(k, \delta)}$. Further, let

$$\mathcal{M}_k^\delta := (1 - \delta) \mathcal{M}_k + \delta \Delta_k. \quad (260)$$

Note that similar to \mathcal{M}_k , \mathcal{M}_k^δ also satisfies

$$\mathcal{M}_k^\delta = \text{tr}_{X_k} \circ \mathcal{T}_k \circ \mathcal{M}_k^\delta. \quad (261)$$

Then, we have that

$$\begin{aligned} \bar{\rho}_{A_1^k B_1^k E}^{(k)} &= (1 - \delta) \omega_{A_1^k B_1^k E}^{(k)} + \delta \rho_{A_k B_k}^{(k, \delta)} \otimes \omega_{A_1^{k-1} B_1^{k-1} E}^{(k)} \\ &= ((1 - \delta) \mathcal{M}_k + \delta \Delta_k) \left(\omega_{A_1^{k-1} B_1^{k-1} R_k E}^{(k, 1)} \right) \\ &= \mathcal{M}_k^\delta \left(\omega_{A_1^{k-1} B_1^{k-1} R_k E}^{(k, 1)} \right). \end{aligned} \quad (262)$$

We also have that

$$\frac{1}{2} \left\| \bar{\rho}_{A_1^k B_1^k E}^{(k)} - \rho_{A_1^k B_1^k E} \right\|_1 \leq 4\sqrt{\epsilon + \nu} + \delta \quad (263)$$

and by the definition of $\bar{\rho}_{A_1^k B_1^k E}^{(k)}$

$$\delta^2 \tau_{A_k B_k} \otimes \rho_{A_1^{k-1} B_1^{k-1} E} \leq \bar{\rho}_{A_1^k B_1^k E}^{(k)} \quad (264)$$

Using Corollary 3.3, gives us that

$$D(\rho_{A_1^k B_1^k E} \| \bar{\rho}_{A_1^k B_1^k E}^{(k)}) \leq \frac{8\sqrt{\epsilon + \nu} + 2\delta}{1 - \delta^2 / (|A||B|)^2} \log \frac{|A||B|}{\delta}. \quad (265)$$

We define the above bound as $z(\epsilon + \nu, \delta)$. Using Lemma 3.1, for the normalised auxiliary state

$$\sigma_{A_1^n B_1^n E} := \int_{-\infty}^{\infty} dt \beta_0(t) \prod_{k=0}^{n-1} \left[\left(\bar{\rho}_{A_1^k B_1^k E}^{(k)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1^{k+1} B_1^{k+1} E}^{(k+1)} \right)^{-\frac{1-it}{2}} \right] \cdot \bar{\rho}_{A_1^n B_1^n E}^{(n)} \cdot \prod_{k=n-1}^0 \left[\left(\bar{\rho}_{A_1^k B_1^k E}^{(k+1)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1^k B_1^k E}^{(k)} \right)^{\frac{1+it}{2}} \right] \quad (266)$$

we have that

$$D_m(\rho_{A_1^n B_1^n E} \| \sigma_{A_1^n B_1^n E}) \leq n z(\epsilon + \nu, \delta) \quad (267)$$

and

$$\sigma_{A_1^k B_1^k E}$$

$$= \int_{-\infty}^{\infty} dt \beta_0(t) \prod_{j=0}^{k-1} \left[\left(\bar{\rho}_{A_1^j B_1^j E}^{(j)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1^{j+1} B_1^{j+1} E}^{(j+1)} \right)^{-\frac{1-it}{2}} \right] \cdot \bar{\rho}_{A_1^k B_1^k E}^{(k)} \cdot \prod_{j=k-1}^0 \left[\left(\bar{\rho}_{A_1^j B_1^j E}^{(j+1)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1^j B_1^j E}^{(j)} \right)^{\frac{1+it}{2}} \right] \quad (268)$$

$$= \mathcal{M}_k^{\delta} \left(\int_{-\infty}^{\infty} dt \beta_0(t) \prod_{j=0}^{k-1} \left[\left(\bar{\rho}_{A_1^j B_1^j E}^{(j)} \right)^{\frac{1-it}{2}} \left(\bar{\rho}_{A_1^{j+1} B_1^{j+1} E}^{(j+1)} \right)^{-\frac{1-it}{2}} \right] \cdot \omega_{A_1^{k-1} B_1^{k-1} R_k E}^{(k,1)} \cdot \prod_{j=k-1}^0 \left[\left(\bar{\rho}_{A_1^j B_1^j E}^{(j+1)} \right)^{-\frac{1+it}{2}} \left(\bar{\rho}_{A_1^j B_1^j E}^{(j)} \right)^{\frac{1+it}{2}} \right] \right) \quad (269)$$

for all $k \in [n]$. Let $\sigma_{A_1^{k-1} B_1^{k-1} R_k E}^{(k,0)}$ be the input state for \mathcal{M}_k^{δ} above, so that

$$\sigma_{A_1^k B_1^k E} = \mathcal{M}_k^{\delta} \left(\sigma_{A_1^{k-1} B_1^{k-1} R_k E}^{(k,0)} \right) \quad (270)$$

It is also easy to see using the properties of the algebras \mathcal{A}_k that $\sigma_{A_1^k B_1^k E} \in \mathcal{A}_k$ for each k . Thus, we can extend the state σ as follows using measurements $(\mathcal{T}_i)_i$

$$\sigma_{A_1^n B_1^n X_1^n E} = \mathcal{T}_n \circ \cdots \circ \mathcal{T}_1 \left(\sigma_{A_1^n B_1^n E} \right).$$

Note that the partial state

$$\begin{aligned} \sigma_{A_1^k B_1^k X_1^k E} &= \text{tr}_{A_{k+1}^n B_{k+1}^n X_{k+1}^n} \left(\mathcal{T}_n \circ \cdots \circ \mathcal{T}_1 \left(\sigma_{A_1^n B_1^n E} \right) \right) \\ &= \mathcal{T}_k \circ \cdots \circ \mathcal{T}_1 \left(\text{tr}_{A_{k+1}^n B_{k+1}^n X_{k+1}^n} \left(\mathcal{T}_n \circ \cdots \circ \mathcal{T}_{k+1} \left(\sigma_{A_1^n B_1^n E} \right) \right) \right) \\ &= \mathcal{T}_k \circ \cdots \circ \mathcal{T}_1 \left(\sigma_{A_1^k B_1^k E} \right) \end{aligned} \quad (271)$$

$$= \mathcal{T}_k \circ \mathcal{M}_k^{\delta} \circ \mathcal{T}_{k-1} \circ \cdots \circ \mathcal{T}_1 \left(\sigma_{A_1^{k-1} B_1^{k-1} R_k E}^{(k,0)} \right) \quad (272)$$

Let's define $\mu := z(\epsilon + \nu, \delta)^{1/3}$. Using the substate theorem (Theorem 2.2), we get the following bound from the above relative entropy bound

$$D_{\max}^{\mu}(\rho_{A_1^n B_1^n E} \| \sigma_{A_1^n B_1^n E}) \leq n\mu + \frac{1}{\mu^2} + \log \frac{1}{1 - \mu^2} \quad (273)$$

which using data processing also implies that

$$D_{\max}^{\mu}(\rho_{A_1^n B_1^n X_1^n E} \| \sigma_{A_1^n B_1^n X_1^n E}) \leq n\mu + \frac{1}{\mu^2} + \log \frac{1}{1 - \mu^2} \quad (274)$$

The bound above implies that there exists a state $\rho'_{A_1^n B_1^n X_1^n E}$, which is also classical on X_1^n such that

$$P \left(\rho_{A_1^n B_1^n X_1^n E}, \rho'_{A_1^n B_1^n X_1^n E} \right) \leq \mu \quad (275)$$

and

$$\rho'_{A_1^n B_1^n X_1^n E} \leq \frac{e^{n\mu + \frac{1}{\mu^2}}}{1 - \mu^2} \sigma_{A_1^n B_1^n X_1^n E}. \quad (276)$$

The registers X_1^n for ρ' can be chosen to be classical, since the channel measuring X_1^n only decreases the distance between ρ' and ρ , and the new state produced would also satisfy Eq. 276. As the registers X_1^n are classical for both σ and ρ' , we can condition these states on the event Ω . We will call the probability of the event Ω for the state σ and ρ' $P_\sigma(\Omega)$ and $P_{\rho'}(\Omega)$ respectively. Using [MD23, Lemma G.1] and the Fuchs-van de Graaf inequality, we have

$$P\left(\rho_{A_1^n B_1^n X_1^n E|\Omega}, \rho'_{A_1^n B_1^n X_1^n E|\Omega}\right) \leq 2\sqrt{\frac{\mu}{P_\rho(\Omega)}}. \quad (277)$$

Conditioning Eq. 276 on Ω , we get

$$P_{\rho'}(\Omega) \rho'_{A_1^n B_1^n X_1^n E|\Omega} \leq \frac{e^{n\mu + \frac{1}{\mu^2}}}{1 - \mu^2} P_\sigma(\Omega) \sigma_{A_1^n B_1^n X_1^n E|\Omega}. \quad (278)$$

Together, the above two equations imply that

$$D_{\max}^{\mu'}(\rho_{A_1^n B_1^n X_1^n E|\Omega} \| \sigma_{A_1^n B_1^n X_1^n E|\Omega}) \leq n\mu + \frac{1}{\mu^2} + \log \frac{P_\sigma(\Omega)}{P_{\rho'}(\Omega)} + \log \frac{1}{1 - \mu^2} \quad (279)$$

for $\mu' := 2\sqrt{\frac{\mu}{P_\rho(\Omega)}}$.

For $\epsilon' > 0$ such that $\mu' + \epsilon' < 1$ and $\alpha \in (1, 2]$, we can plug the above in the bound provided by Lemma 2.1 to get

$$\begin{aligned} H_{\min}^{\mu' + \epsilon'}(A_1^n | B_1^n E)_{\rho|\Omega} &\geq \tilde{H}_\alpha^\dagger(A_1^n | B_1^n E)_{\sigma|\Omega} - \frac{\alpha}{\alpha - 1} n\mu \\ &\quad - \frac{1}{\alpha - 1} \left(\alpha \log \frac{P_\sigma(\Omega)}{P_{\rho'}(\Omega)} + \frac{\alpha}{\mu^2} + \alpha \log \frac{1}{1 - \mu^2} + g_1(\epsilon', \mu') \right). \end{aligned} \quad (280)$$

Now, note that using Eq. 224 and 271, and [DFR20, Lemma B.7] we have

$$\tilde{H}_\alpha^\dagger(A_1^n | B_1^n E)_{\sigma|\Omega} = \tilde{H}_\alpha^\dagger(A_1^n X_1^n | B_1^n E)_{\sigma|\Omega}. \quad (281)$$

For every k , we introduce a register D_k of dimension $|D| := \lceil e^{\max(f) - \min(f)} \rceil$ and the channels $\mathcal{D}_k : X_k \rightarrow X_k D_k$ defined as

$$\mathcal{D}_k(\omega) := \sum_x \langle x | \omega | x \rangle |x\rangle \langle x| \otimes \nu_x \quad (282)$$

where for every x , the state ν_x is a mixture between a uniform distribution on $\{1, 2, \dots, \lfloor e^{\max(f)-f(\delta_x)} \rfloor\}$ and a uniform distribution on $\{1, 2, \dots, \lceil e^{\max(f)-f(\delta_x)} \rceil\}$, so that

$$H(D_k)_{\nu_x} = \max(f) - f(\delta_x) \quad (283)$$

where δ_x is the distribution with unit weight at element x . Define the state

$$\bar{\sigma}_{A_1^n B_1^n X_1^n D_1^n E} := \mathcal{D}_n \circ \dots \circ \mathcal{D}_1(\sigma_{A_1^n B_1^n X_1^n E}) \quad (284)$$

Now [MFSR22, Lemma 4.5] implies that this satisfies

$$\tilde{H}_\alpha^\uparrow(A_1^n X_1^n | B_1^n E)_{\bar{\sigma}_{|\Omega}} \geq \tilde{H}_\alpha^\uparrow(A_1^n X_1^n D_1^n | B_1^n E)_{\bar{\sigma}_{|\Omega}} - \max_{x_1^n \in \Omega} H_\alpha(D_1^n)_{\bar{\sigma}_{|x_1^n}} \quad (285)$$

For $x_1^n \in \Omega$, we have

$$\begin{aligned} H_\alpha(D_1^n)_{\bar{\sigma}_{|x_1^n}} &\leq H(D_1^n)_{\bar{\sigma}_{|x_1^n}} \\ &= \sum_{k=1}^n H(D_k)_{\nu_{x_k}} \\ &= \sum_{k=1}^n \max(f) - f(\delta_{x_k}) \\ &= n \max(f) - n f(\text{freq}(x_1^n)) \\ &\leq n \max(f) - n h. \end{aligned} \quad (286)$$

We can get rid of the conditioning on the right-hand side of Eq. 285 by using [DFR20, Lemma B.5]. This gives us

$$\tilde{H}_\alpha^\uparrow(A_1^n X_1^n D_1^n | B_1^n E)_{\bar{\sigma}_{|\Omega}} \geq \tilde{H}_\alpha^\uparrow(A_1^n X_1^n D_1^n | B_1^n E)_{\bar{\sigma}} - \frac{\alpha}{\alpha - 1} \log \frac{1}{P_\sigma(\Omega)} \quad (287)$$

Moreover, using Eq. 272, we can show that B_k is independent of $A_1^{k-1} X_1^{k-1} D_1^{k-1} B_1^{k-1} E$ in $\bar{\sigma}$. Firstly, for $\sigma_{A_1^{k-1} B_1^k E} = \text{tr}_{A_k} \circ \mathcal{M}_k^\delta(\sigma_{A_1^{k-1} B_1^{k-1} R_k E}^{(k,0)})$, we have

$$\sigma_{A_1^{k-1} B_1^k E} = (1 - \delta) \text{tr}_{A_k} \circ \mathcal{M}_k(\sigma_{A_1^{k-1} B_1^{k-1} R_k E}^{(k,0)}) + \delta \rho_{B_k}^{(k,\delta)} \otimes \sigma_{A_1^{k-1} B_1^{k-1} E} \quad (288)$$

$$= \left((1 - \delta) \theta_{B_k}^{(k)} + \delta \rho_{B_k}^{(k,\delta)} \right) \otimes \sigma_{A_1^{k-1} B_1^{k-1} E}. \quad (289)$$

Since, $\bar{\sigma}_{A_1^{k-1} X_1^{k-1} D_1^{k-1} B_1^k E} = \mathcal{D}_{k-1} \circ \mathcal{T}_{k-1} \dots \circ \mathcal{D}_1 \circ \mathcal{T}_1(\sigma_{A_1^{k-1} B_1^k E})$, we can easily see that B_k is independent of the other registers. In particular, $\bar{\sigma}$ satisfies the Markov chain $A_1^{k-1} X_1^{k-1} D_1^{k-1} \leftrightarrow B_1^{k-1} E \leftrightarrow B_k$. Let's define the channels

$$\bar{\mathcal{M}}_k := \mathcal{D}_k \circ \mathcal{T}_k \circ \mathcal{M}_k \quad (290)$$

$$\bar{\mathcal{M}}_k^\delta := \mathcal{D}_k \circ \mathcal{T}_k \circ \mathcal{M}_k^\delta. \quad (291)$$

Now we can use [DFR20, Corollary 3.5] to show that for every $k \in [n]$

$$\begin{aligned}
\tilde{H}_\alpha^\downarrow(A_1^k X_1^k D_1^k | B_1^k E)_{\bar{\sigma}} &\geq \tilde{H}_\alpha^\downarrow(A_1^{k-1} X_1^{k-1} D_1^{k-1} | B_1^{k-1} E)_{\bar{\sigma}} + \inf_{\omega_{R_k \tilde{R}_k}} \tilde{H}_\alpha^\downarrow(A_k X_k D_k | B_k \tilde{R}_k)_{\bar{\mathcal{M}}_k^\delta(\omega)} \\
&\geq \tilde{H}_\alpha^\downarrow(A_1^{k-1} X_1^{k-1} D_1^{k-1} | B_1^{k-1} E)_{\bar{\sigma}} \\
&\quad + \min \left\{ \inf_{\omega_{R_k \tilde{R}_k}} \tilde{H}_\alpha^\downarrow(A_k D_k | B_k \tilde{R}_k)_{\bar{\mathcal{M}}_k(\omega)}, \tilde{H}_\alpha^\downarrow(A_k D_k | B_k)_{\mathcal{D}_k \circ \mathcal{T}_k(\rho_{A_k B_k}^{(k,\delta)})} \right\}.
\end{aligned} \tag{292}$$

where we have used the quasi-concavity of Rényi conditional entropies [Tom16, Pg 73] and the fact that X_k are classical in the second line.

We now lower bound the two terms in the minimum above. Using [DFR20, Lemma B.9], for a state $\nu = \bar{\mathcal{M}}_k(\omega_{R_k \tilde{R}_k})$ and $1 < \alpha < \frac{1}{\log(1+2|A||D|)}$ we have that

$$\begin{aligned}
\tilde{H}_\alpha^\downarrow(A_k D_k | B_k \tilde{R}_k)_\nu &\geq H(A_k D_k | B_k \tilde{R}_k)_\nu - (\alpha - 1) \log^2(1 + 2|A||D|) \\
&= H(A_k | B_k \tilde{R}_k)_\nu + H(D_k | A_k B_k \tilde{R}_k)_\nu - (\alpha - 1) \log^2(1 + 2|A||D|) \\
&= H(A_k | B_k \tilde{R}_k)_\nu + H(D_k | X_k)_\nu - (\alpha - 1) \log^2(1 + 2|A||D|) \\
&= H(A_k | B_k \tilde{R}_k)_\nu + \sum_x \nu(x) (\max(f) - f(\delta_x)) - (\alpha - 1) \log^2(1 + 2|A||D|) \\
&= H(A_k | B_k \tilde{R}_k)_\nu + \max(f) - f(\nu_X) - (\alpha - 1) \log^2(1 + 2|A||D|) \\
&\geq \max(f) - (\alpha - 1) \log^2(1 + 2|A||D|)
\end{aligned} \tag{293}$$

For the second term, and $1 < \alpha < \frac{1}{\log(1+2|A||D|)}$, we have

$$\begin{aligned}
\tilde{H}_\alpha^\downarrow(A_k D_k | B_k)_{\mathcal{D}_k \circ \mathcal{T}_k(\rho_{A_k B_k}^{(k,\delta)})} &\geq H(A_k D_k | B_k)_{\mathcal{D}_k \circ \mathcal{T}_k(\rho_{A_k B_k}^{(k,\delta)})} - (\alpha - 1) \log^2(1 + 2|A||D|) \\
&\geq H(A_k D_k | B_k)_{\bar{\mathcal{M}}_k(\tilde{\rho}_{R_k}^{(k,0)})} - (\alpha - 1) \log^2(1 + 2|A||D|) \\
&\quad - 2(\epsilon + \delta) \log |A||D| - g(\epsilon + \delta) \\
&\geq \max(f) - (\alpha - 1) \log^2(1 + 2|A||D|) - 2(\epsilon + \delta) \log |A||D| - g_2(\epsilon + \delta)
\end{aligned} \tag{294}$$

where we have used [DFR20, Lemma B.9] in the first line, the AFW continuity bound [Wil13, Theorem 11.10.3] in the second line to convert the entropy on $\mathcal{D}_k \circ \mathcal{T}_k(\rho_{A_k B_k}^{(k,\delta)})$ to an entropy on $\bar{\mathcal{M}}_k(\tilde{\rho}_{R_k}^{(k,0)})$ and finally we use bound in Eq. 293 for states of this form.

Plugging these in Eq. 292, gives us that for every $k \in [n]$

$$\begin{aligned}
\tilde{H}_\alpha^\downarrow(A_1^k X_1^k D_1^k | B_1^k E)_{\bar{\sigma}} &\geq \tilde{H}_\alpha^\downarrow(A_1^{k-1} X_1^{k-1} D_1^{k-1} | B_1^{k-1} E)_{\bar{\sigma}} \\
&\quad + \max(f) - (\alpha - 1) \log^2(1 + 2|A||D|) \\
&\quad - 2(\epsilon + \delta) \log |A||D| - g_2(\epsilon + \delta)
\end{aligned} \tag{295}$$

Consecutively using this bound gives us

$$\begin{aligned}\tilde{H}_\alpha^\uparrow(A_1^n X_1^n D_1^n | B_1^n E)_{\bar{\sigma}} &\geq \tilde{H}_\alpha^\downarrow(A_1^n X_1^n D_1^n | B_1^n E)_{\bar{\sigma}} \\ &\geq n \max(f) - n(\alpha - 1) \log^2(1 + 2|A||D|) \\ &\quad - 2n(\epsilon + \delta) \log |A||D| - n g_2(\epsilon + \delta)\end{aligned}\quad (296)$$

Combining Eq. 281, 285, 286, 287 and 296, we get

$$\begin{aligned}\tilde{H}_\alpha^\uparrow(A_1^n | B_1^n E)_{\sigma|\Omega} &\geq nh - n(\alpha - 1) \log^2(1 + 2|A||D|) - 2n(\epsilon + \delta) \log |A||D| - n g_2(\epsilon + \delta) \\ &\quad - \frac{\alpha}{\alpha - 1} \log \frac{1}{P_\sigma(\Omega)}.\end{aligned}\quad (297)$$

Plugging this into Eq. 280, we get

$$\begin{aligned}H_{\min}^{\mu' + \epsilon'}(A_1^n | B_1^n E)_{\rho|\Omega} &\geq nh - n(\alpha - 1) \log^2(1 + 2|A||D|) - \frac{\alpha}{\alpha - 1} n \mu \\ &\quad - 2n(\epsilon + \delta) \log |A||D| - n g_2(\epsilon + \delta) \\ &\quad - \frac{1}{\alpha - 1} \left(\alpha \log \frac{1}{P_{\rho'}(\Omega)} + \frac{\alpha}{\mu^2} + \alpha \log \frac{1}{1 - \mu^2} + g_1(\epsilon', \mu') \right).\end{aligned}\quad (298)$$

Finally, we choose $\alpha = 1 + \frac{\sqrt{\mu}}{\log(1 + 2|A||D|)}$ and use the $\alpha < 2$ as an upper bound to derive

$$\begin{aligned}H_{\min}^{\mu' + \epsilon'}(A_1^n | B_1^n E)_{\rho|\Omega} &\geq n(h - 3\sqrt{\mu} \log(1 + 2|A||D|) - 2(\epsilon + \delta) \log |A||D| - g_2(\epsilon + \delta)) \\ &\quad - \frac{\log(1 + 2|A||D|)}{\sqrt{\mu}} \left(2 \log \frac{1}{P_\rho(\Omega) - \mu} + \frac{2}{\mu^2} + 2 \log \frac{1}{1 - \mu^2} + g_1(\epsilon', \mu') \right).\end{aligned}\quad (299)$$

Note that $\log(1 + 2|A||D|) \leq \log(1 + 2|A|) + \log|D| \leq \log(1 + 2|A|) + \lceil \max(f) - \min(f) \rceil = V$. Recall that $\mu = z(\epsilon + \nu, \delta)^{1/3}$. Since, ν can be chosen arbitrarily greater than 0, if we let $\nu \rightarrow 0$, the bound in Eq. 299 is still valid. Finally, choose $\delta = \epsilon$ to derive the bound in the theorem.

Case 2: Finally, for a state $\rho_{A_1^n B_1^n E}$ which satisfies the assumptions of the Theorem but is not full rank, we can consider the full rank states $\rho_{A_1^n B_1^n E}^{(\epsilon)} := (1 - \epsilon) \rho_{A_1^n B_1^n E} + \epsilon \tau_{A_1^n B_1^n E}$ for an arbitrarily small $\epsilon > 0$. $\rho_{A_1^n B_1^n E}^{(\epsilon)}$ will be full rank and satisfy the assumptions of the Theorem and the bound in Eq. 299 for $\epsilon \rightarrow \epsilon + \epsilon$. One can then prove the lower bound above for the state $\rho_{A_1^n B_1^n E}$ by taking the limit $\epsilon \rightarrow 0$. \square

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