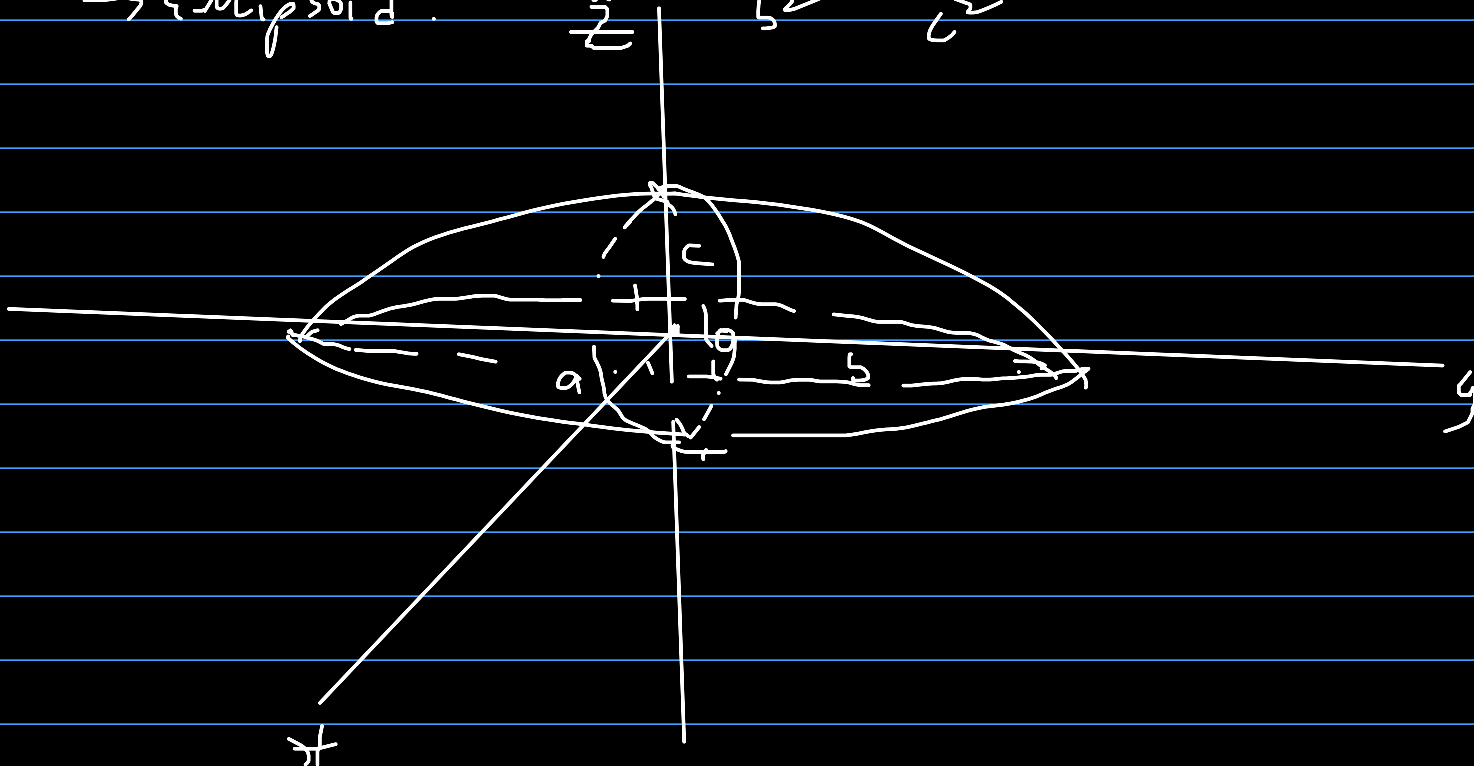
Seminar W10 – 973, 974

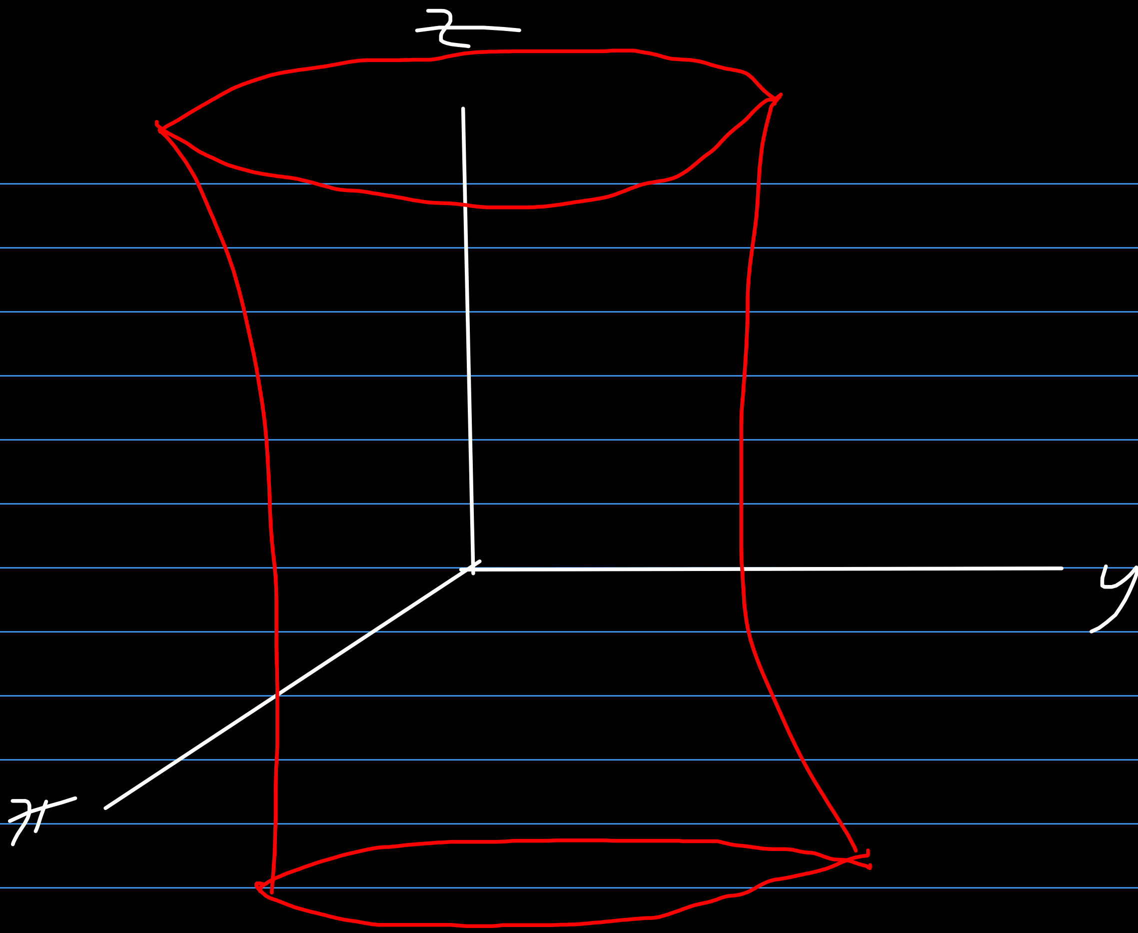
Quadrics

→ Ellipsoid: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$



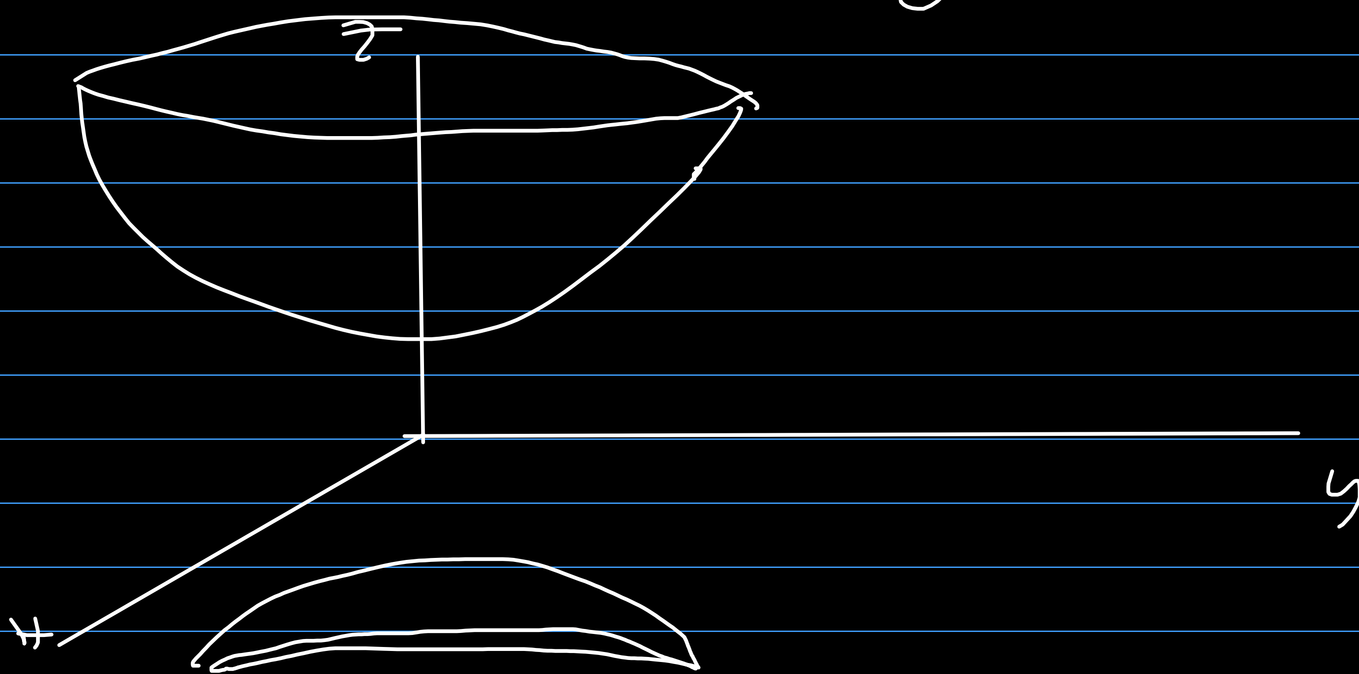
→ Hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

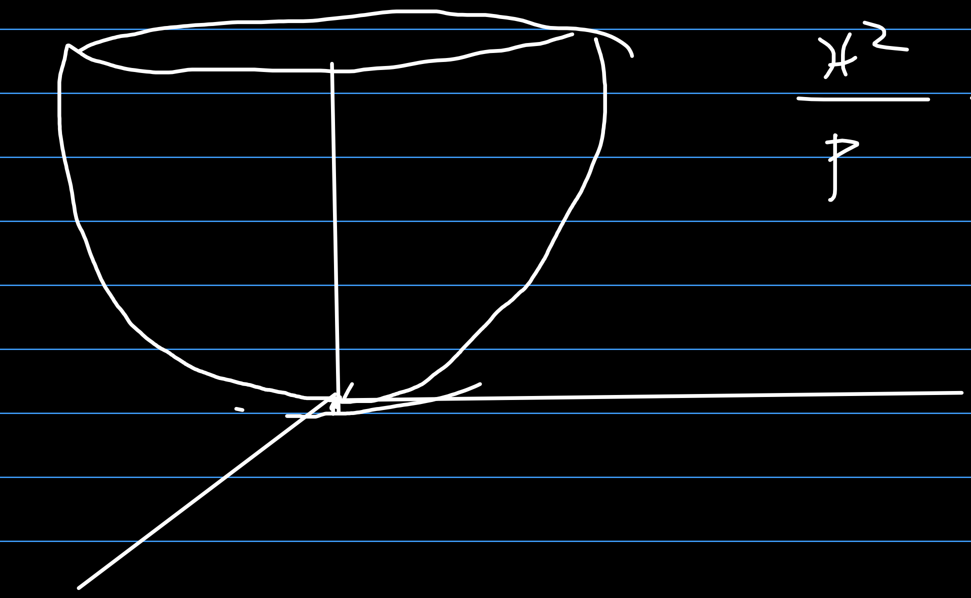


→ Hyperboloid of two sheets

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

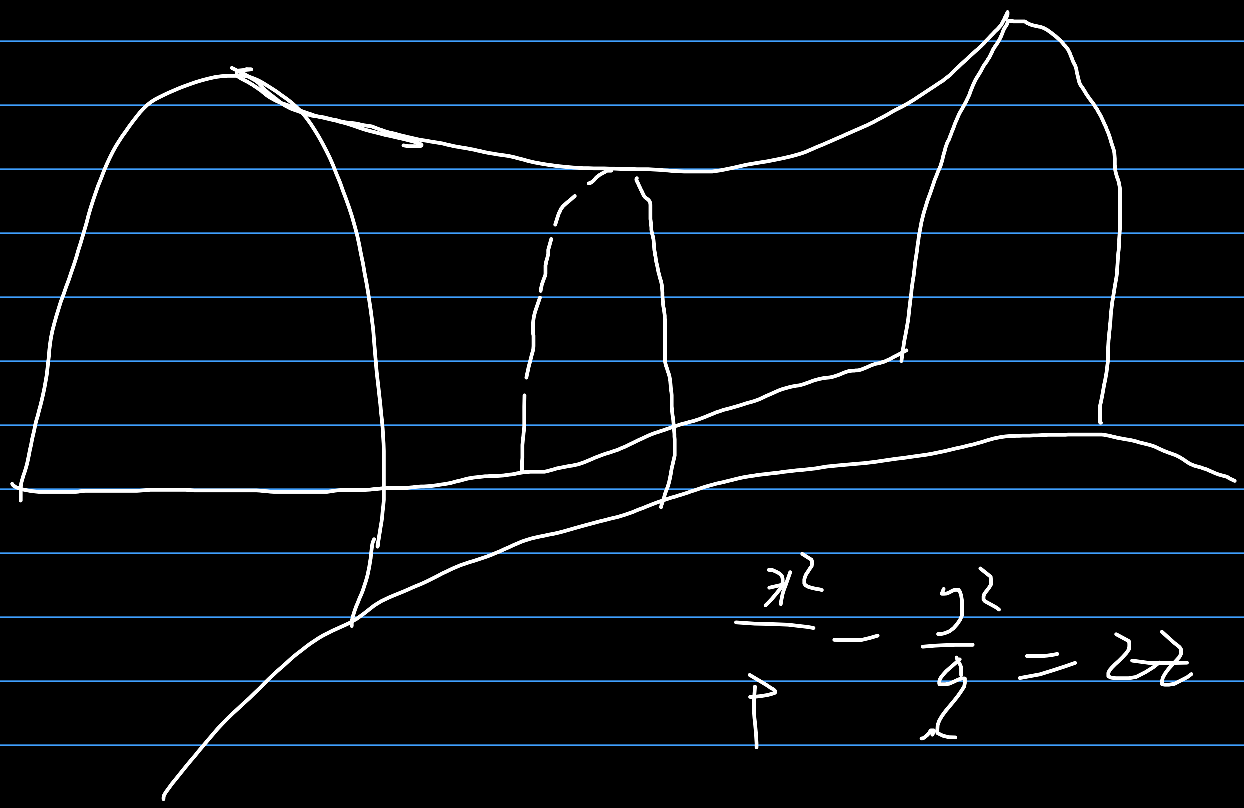


· Elliptic paraboloid



$$\frac{x^2}{p} + \frac{y^2}{q} = 2z$$

· Hyperbolic paraboloid



$$\frac{x^2}{p} - \frac{y^2}{q} = 2z$$

→ cones: elliptic, hyperbolic, parabolic

→ cylinder.

$\mathcal{Y}: \quad f(x,y,z) = 0$

$$\Rightarrow T_{\mathcal{Y}}(x_0,y_0,z_0): f'_x(x_0,y_0,z_0)\cdot(x-x_0) + f'_y(x_0,y_0,z_0)\cdot(y-y_0) + f'_z(x_0,y_0,z_0)\cdot(z-z_0) = 0$$

$$N_{\mathcal{Y}}(x_0,y_0,z_0): \frac{x-x_0}{f'_x} = \frac{y-y_0}{f'_y} = \frac{z-z_0}{f'_z}$$

10.1. Find the intersection points between the ellipsoid $\mathcal{E}: \dfrac{x^2}{16} + \dfrac{y^2}{12} + \dfrac{z^2}{4} = 1$ with the line

$$\frac{x-4}{2} = \frac{y+6}{-3} = \frac{z+2}{-2}$$

and write the equations of the tangent planes and the normal lines to the ellipsoid at the intersection points.

$$\ell \cap \mathcal{E}: \begin{cases} x = 4+2t \\ y = -6-3t \\ z = -2-2t \\ \dfrac{x^2}{16} + \dfrac{y^2}{12} + \dfrac{z^2}{4} = 1 \end{cases} \Longleftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 4+2t \\ y = -6-3t \\ z = -2-2t \\ \dfrac{(4+2t)^2}{16} + \dfrac{(-6-3t)^2}{12} + \dfrac{(-2-2t)^2}{4} = 1 \end{cases}$$

$$\frac{(4+2t)^2}{16} + \frac{(-6-3t)^2}{12} + \frac{(-2-2t)^2}{4} = 1$$

$$\frac{16+16t+4t^2}{16} + \frac{36+36t+9t^2}{12} + \frac{4+8t+4t^2}{4} = 1$$

$$\Rightarrow \frac{4+4t+t^2}{4} + \frac{12+12t+3t^2}{4} + \frac{4+8t+4t^2}{4} = 1 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad 8t^2 + 24t + 16 = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad t^2 + 3t + 2 = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad t \in \left\{ \frac{-3 + \sqrt{9-8}}{2}, \frac{-3 - \sqrt{9-8}}{2} \right\} \Leftrightarrow$$

$$\Leftrightarrow t \in \{ -1, -2 \}$$

$$\Rightarrow \mathcal{E} \cap \ell = \{ (2, -3, 0), (0, 0, 2) \}$$

$$f(x, y, z) = \frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} - 1$$

$$f'_x(x_0, y_0, z_0) = \frac{x_0}{8}$$

$$f'_y(x_0, y_0, z_0) = \frac{y_0}{6}$$

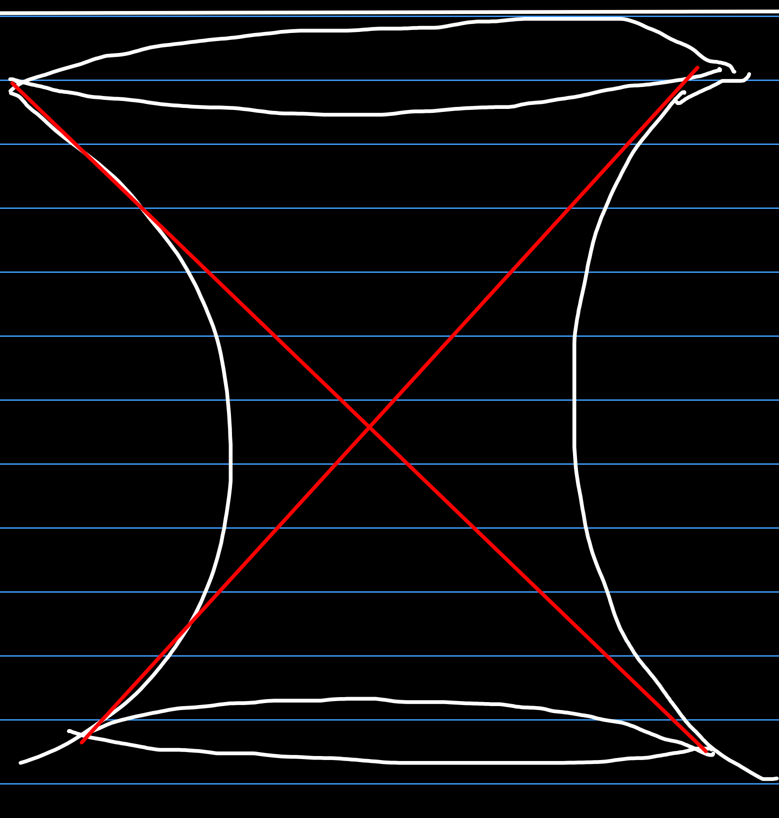
$$f'_z(x_0, y_0, z_0) = \frac{z_0}{2}$$

$T_\xi(2,-3,0): \frac{1}{4}\cdot(x-2)+\left(-\frac{1}{2}\right)\cdot(y+3)=0$

$T_\xi(0,0,2): \quad z-2=0$

$$N_\xi(2,-3,0): \begin{cases} \dfrac{x-2}{\frac{1}{4}} = \dfrac{y+3}{-\frac{1}{2}} \\ z=0 \end{cases}$$

$$N_\xi(0,0,2): \begin{cases} x=0 \\ y=0 \end{cases}$$



Finding rectilinear generatrices..

→ hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2}$$

$$\left(\frac{x}{a}-\frac{z}{c}\right)\cdot\left(\frac{x}{a}+\frac{z}{c}\right)=\left(1-\frac{y}{b}\right)\cdot\left(1+\frac{y}{b}\right)$$

$$d_\lambda : \begin{cases} \frac{x}{a}-\frac{z}{c}=\lambda\cdot\left(1-\frac{y}{b}\right) \\ \lambda\left(\frac{x}{a}+\frac{z}{c}\right)=1+\frac{y}{b} \end{cases}$$

$$d'_\mu : \begin{cases} \frac{x}{a}-\frac{z}{c}=\mu\cdot\left(1+\frac{y}{b}\right) \\ \mu\cdot\left(\frac{x}{a}+\frac{z}{c}\right)=1-\frac{y}{b} \end{cases}$$

— hyperbolic paraboloid

$$\frac{x^2}{p} - \frac{y^2}{q} = 2z \quad , p, q > 0$$

$$\left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}\right)\left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right) = 2z$$

$$d_\lambda : \frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}} = 2\lambda$$

$$\lambda \cdot \left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right) = z$$

$$d'_\mu : \frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}} = 2\mu$$

$$\mu \cdot \left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}\right) = z$$

10.2. Find the rectilinear generatrices of the quadric $4x^2 - 9y^2 = 36z$, which pass through the point $P(3\sqrt{2}, 2, 1)$

$$4x^2 - 9y^2 = 36z$$

$$(2x-3y)(2x+3y) = 36z$$

$$d_\lambda : \begin{cases} \pi_{1\lambda} : 2x - 3y = 2\lambda \\ \pi_{2\lambda} : \lambda \cdot (2x+3y) = 18z \end{cases}$$

$$\vec{n}_{\pi_{1\lambda}} (2, -3, 0)$$

$$\vec{n}_{\pi_{2\lambda}} (2\lambda, 3\lambda, -18)$$

$$\vec{d}_\lambda = \vec{n}_{\pi_{1\lambda}} \times \vec{n}_{\pi_{2\lambda}} = \begin{vmatrix} i & j & k \\ 2 & -3 & 0 \\ 2\lambda & 3\lambda & -18 \end{vmatrix} =$$

$$= 54\vec{i} + 36\vec{j} + 72\lambda\vec{k}$$

$$\Rightarrow \vec{d}_\lambda(9, 6, 2\lambda)$$

$$P \in d_\lambda \Rightarrow \begin{cases} 2 \cdot 3\sqrt{2} - 6 = 2\lambda \\ \lambda \cdot (6\sqrt{2} + 6) = 18 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \lambda = 3\sqrt{2} - 3 \\ \lambda = \dfrac{18}{6\sqrt{2}+6} = \dfrac{3}{\sqrt{2}+1} \end{cases}$$

$$\Rightarrow \lambda = 3\sqrt{2} - 3$$

$\Rightarrow$ $d_{3\sqrt{2}-3}$ is a line that contains P

$$d'_\mu : \begin{cases} \lambda \cdot (2x - 3y) = 18z \\ 2x + 3y = 2\lambda \end{cases}$$

We do the same thing as above

10.3. Find the rectilinear generatrices of the hyperboloid of one sheet

$$(\mathcal{H}_1): \frac{x^2}{36}+\frac{y^2}{9}-\frac{z^2}{4}=1$$

which are parallel to the plane:

$$\pi: x+y+z=0$$

$$\frac{x^2}{36}-\frac{z^2}{4}=1-\frac{y^2}{9}$$

$$\left(\frac{x}{6}-\frac{z}{2}\right)\left(\frac{x}{6}+\frac{z}{2}\right)=\left(1-\frac{y}{3}\right)\cdot\left(1+\frac{y}{3}\right)$$

$$d_\lambda: \begin{cases} \frac{x}{6}-\frac{z}{2}=\lambda\cdot\left(1-\frac{y}{3}\right) \\ \lambda\cdot\left(\frac{x}{6}+\frac{z}{2}\right)=\left(1+\frac{y}{3}\right) \end{cases}$$

$$\vec{d_\lambda}=\left(\frac{1}{6}, \frac{\lambda}{3}, -\frac{1}{2}\right)\times\left(\frac{\lambda}{6}, -\frac{1}{3}, \frac{\lambda}{2}\right)=$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{6} & \frac{\lambda}{3} & -\frac{1}{2} \\ \frac{\lambda}{6} & -\frac{1}{3} & \frac{\lambda}{2} \end{vmatrix} =$$

$$= \vec{i}\left(\frac{\lambda^2}{6} - \frac{1}{6}\right) + \vec{j}\cdot\left(-\frac{\lambda}{6}\right) + \vec{k}\cdot\left(\frac{-1-\lambda^2}{18}\right)$$

$$d_\lambda \parallel \pi \Leftrightarrow d_\lambda \perp \vec{n_\pi} \Leftrightarrow$$

$$\Leftrightarrow \vec{d_\lambda}\cdot\vec{n_\pi} = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{\lambda^2}{6} - \frac{1}{6} - \frac{\lambda}{6} - \frac{1+\lambda^2}{18} = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{\lambda^2}{9} - \frac{\lambda}{6} - \frac{2}{9} = 0 \Leftrightarrow 6\lambda^2 - 9\lambda - 12 = 0$$

$$\Leftrightarrow 2\lambda^2 - 3\lambda - 4 = 0 \Leftrightarrow \lambda_{1,2} = \frac{3 \pm \sqrt{9+32}}{4}$$

$$d'_\mu : \frac{x}{6} - \frac{z}{2} = \mu \cdot \left(1 + \frac{y}{3}\right)$$

$$\mu\left(\frac{x}{6} + \frac{z}{2}\right) = 1 - \frac{y}{3}$$

We do the same thing here with this family.