Seminar WG - 913

Conics

$$\ell : ax + by + c = 0 \rightarrow \text{a line}$$

$$\mathcal{C} : a_{11} x^2 + 2a_{12} xy + a_{22} y^2 + 2a_{10} x + 2a_{01} y + a_{00} = 0.$$

$\rightarrow$ conics

ellipse $\rightarrow$ the good

hyperbola $\rightarrow$ the bad

parabola $\rightarrow$ the ugly

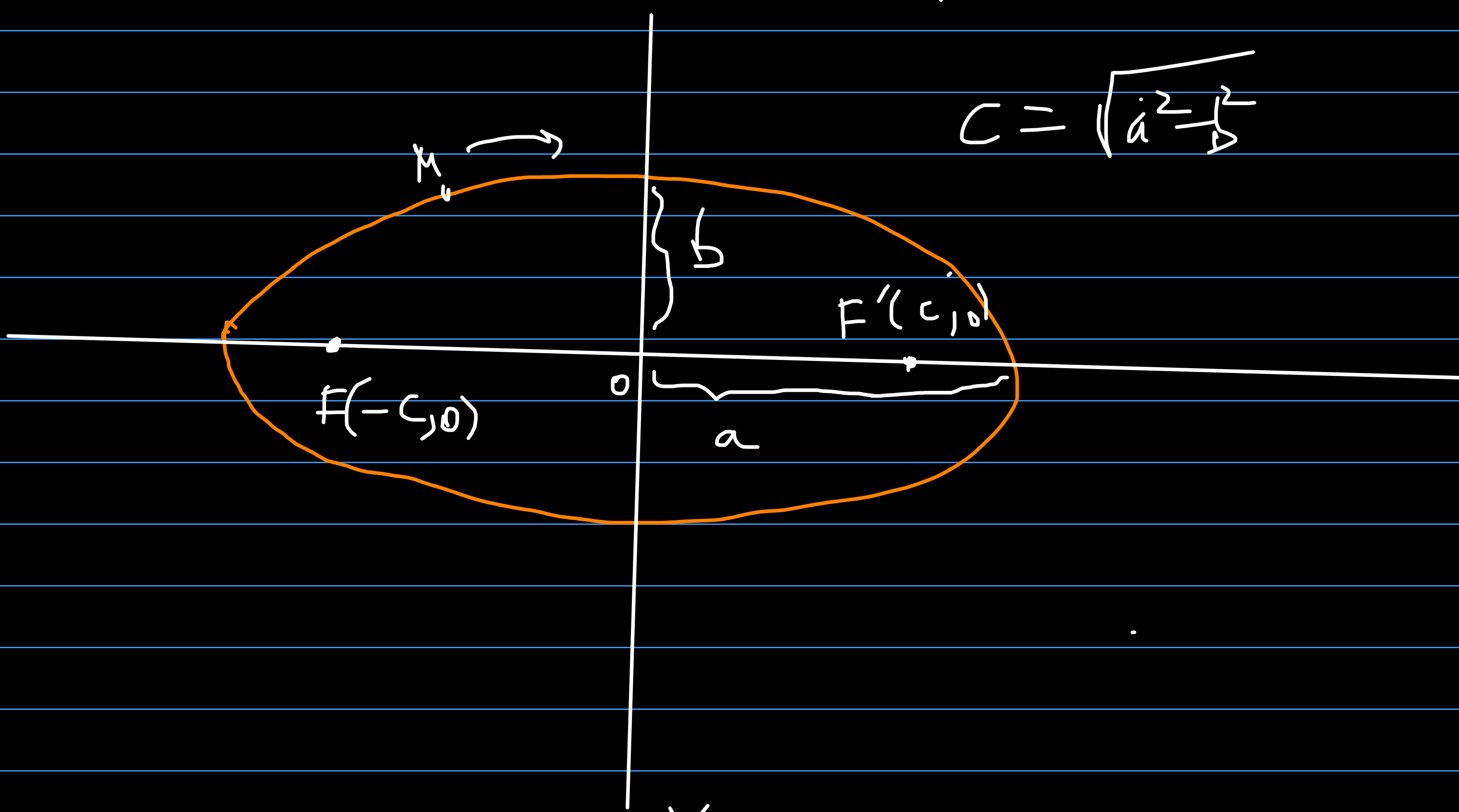
The ellipse: $\mathcal{E}: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

→ locus of points M in the plane so that

$$MF+MF'=2a$$

where F and F' are two fixed points, called foci.

$$c=\sqrt{a^2-b^2}$$



$$T_{\mathcal{E}}(x_0,y_0):\ \frac{x x_0}{a^2}+\frac{y y_0}{b^2}=1$$

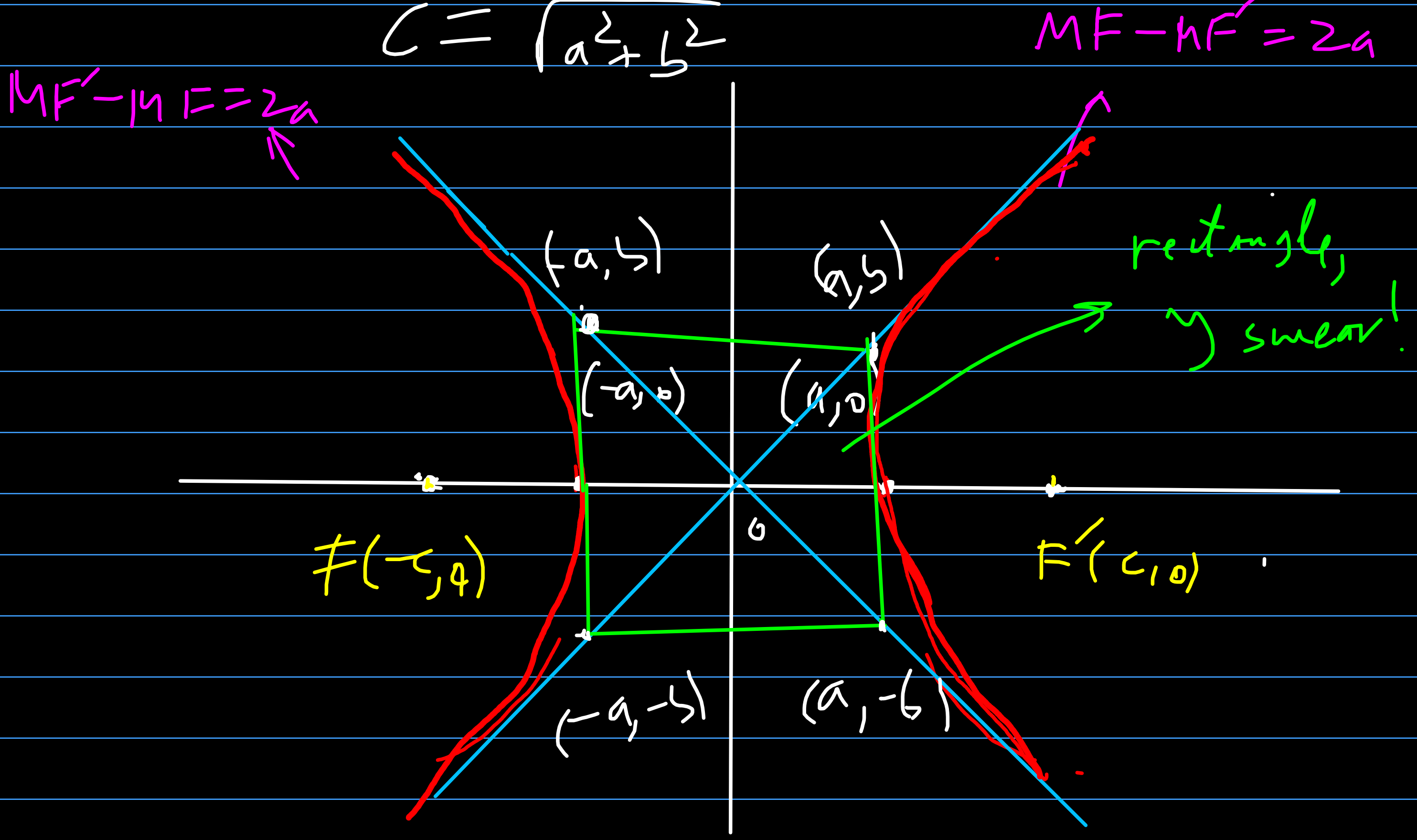
The hyperbola: $\mathcal{H}: \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

→ locus of points M in the plane so that

$$|MF - MF'| = 2a$$

where F and F' are fixed points called the foci.

$$c = \sqrt{a^2+b^2}$$



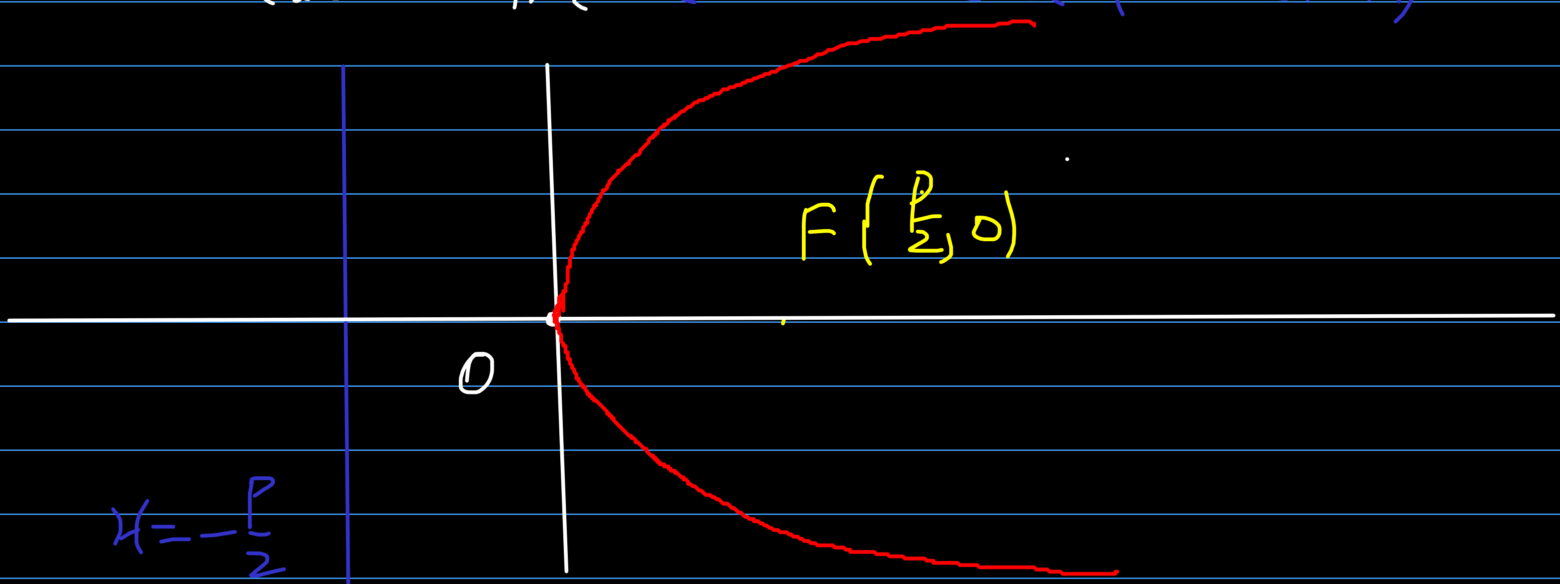
the asymptotes to this curve are:

$$y = \pm \frac{b}{a} x$$

$$T_{\mathcal{H}}(x_0 y_0) : \frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1$$

<u>The parabola</u>: $\mathcal{P} : y^2 = 2px$

→ the locus of points in the plane that are equidistant to a fixed point, called the focus and a fixed line called the director line (directrix)



$$T_{\mathcal{P}}(x_0, y_0) : \quad y y_0 = p(x + x_0)$$

9.2. Find the equations of the tangent lines to the ellipse $\mathcal{E} : x^2 + 4y^2 - 20 = 0$ which are orthogonal to the line

$$d : 2x - 2y - 13 = 0$$

Let (x_0, y_0) be a point on $\mathcal{E}$

$$T_{\mathcal{E}}(x_0, y_0) : \frac{x x_0}{20} + \frac{y y_0}{5} = 1 \iff$$

$$\iff x x_0 + 4 y y_0 - 20 = 0$$

$$\Rightarrow m_{T_{\mathcal{E}}(x_0, y_0)} = -\frac{x_0}{4 y_0}, \text{ if } y_0 \neq 0$$

$$m_d = 1$$

$d \perp T_{\mathcal{E}}(x_0, y_0) \Leftrightarrow m_d \cdot m_{T_{\mathcal{E}}(x_0,y_0)} = -1 \Leftrightarrow$

$\Leftrightarrow -\frac{x_0}{4y_0} = -1 \Leftrightarrow x_0 = 4y_0$

Because $x_0^2 + 4y_0^2 - 20 = 0$

we have:

$$16y_0^2 + 4y_0^2 - 20 = 0$$

$$\Rightarrow y_0^2 = 1 \Rightarrow y_0 = \pm 1$$

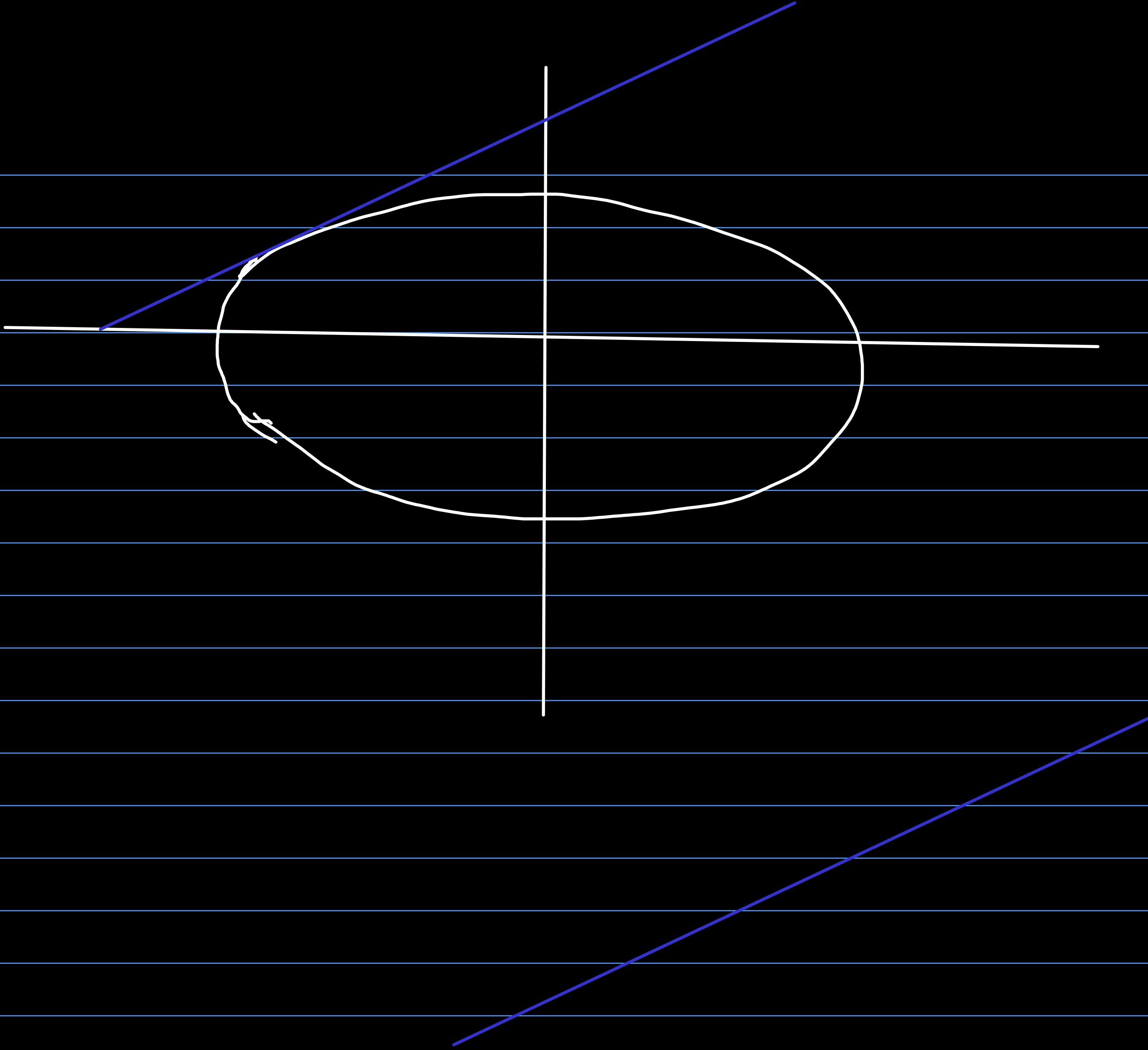
$$\Rightarrow x_0 = \pm 4$$

the two tangents are

$T_{\mathcal{E}}(4,1): \frac{x}{5} + \frac{y}{5} = 1 \Leftrightarrow x + y - 5 = 0$

$T_{\mathcal{E}}(-4,-1): -4x - 4y - 20 = 0 \Leftrightarrow$

$\Leftrightarrow x + y + 5 = 0$



9.9. Find the equation of the tangent line to the parabola

$$P: y^2 - 36x = 0$$

passing through $P(2,9)$.

Let $l_m : y - 9 = m(x-2)$ be a general line that contains $P(2, 9)$.

$$l_m \cap \mathcal{P} : \begin{cases} y^2 - 36x = 0 \\ y - 9 = m(x-2) \end{cases} \iff$$

$$\iff \begin{cases} y = mx - 2m + 9 \\ y^2 - 36x = 0 \end{cases} \iff$$

$$\iff \begin{cases} y = mx - 2m + 9 \\ (mx - 2m + 9)^2 - 36x = 0 \end{cases} \iff$$

$$\iff \begin{cases} y = mx - 2m + 9 \\ m^2x^2 + x(-4m^2 + 18m - 36) + \\ + 4m^2 + 81 - 36m = 0 \end{cases}$$

l_m tangent to $\mathcal{P} \iff |l_m \cap \mathcal{P}| = 1 \iff$

$\Leftrightarrow$ the equation

$$E: m^2x^2 + x(-4m^2+18m-36) + $$
$$+ 4m^2 + 81 - 36m = 0$$

has a unique solution $\Leftrightarrow$

$$\Leftrightarrow \quad \Delta_E = 0$$

$$\Delta_E = (-4m^2+18m-36)^2 -$$
$$- 4m^2(4m^2-36m+81) =$$

$$= 4(-2m^2+9m-18)^2 -$$
$$- 4 \cdot m^2(-4m^2-36m+81) =$$
$$= 288m^2 - 1296m + 1296.$$

$$\Rightarrow m_1 = 3 \qquad m_2 = \frac{3}{2}$$

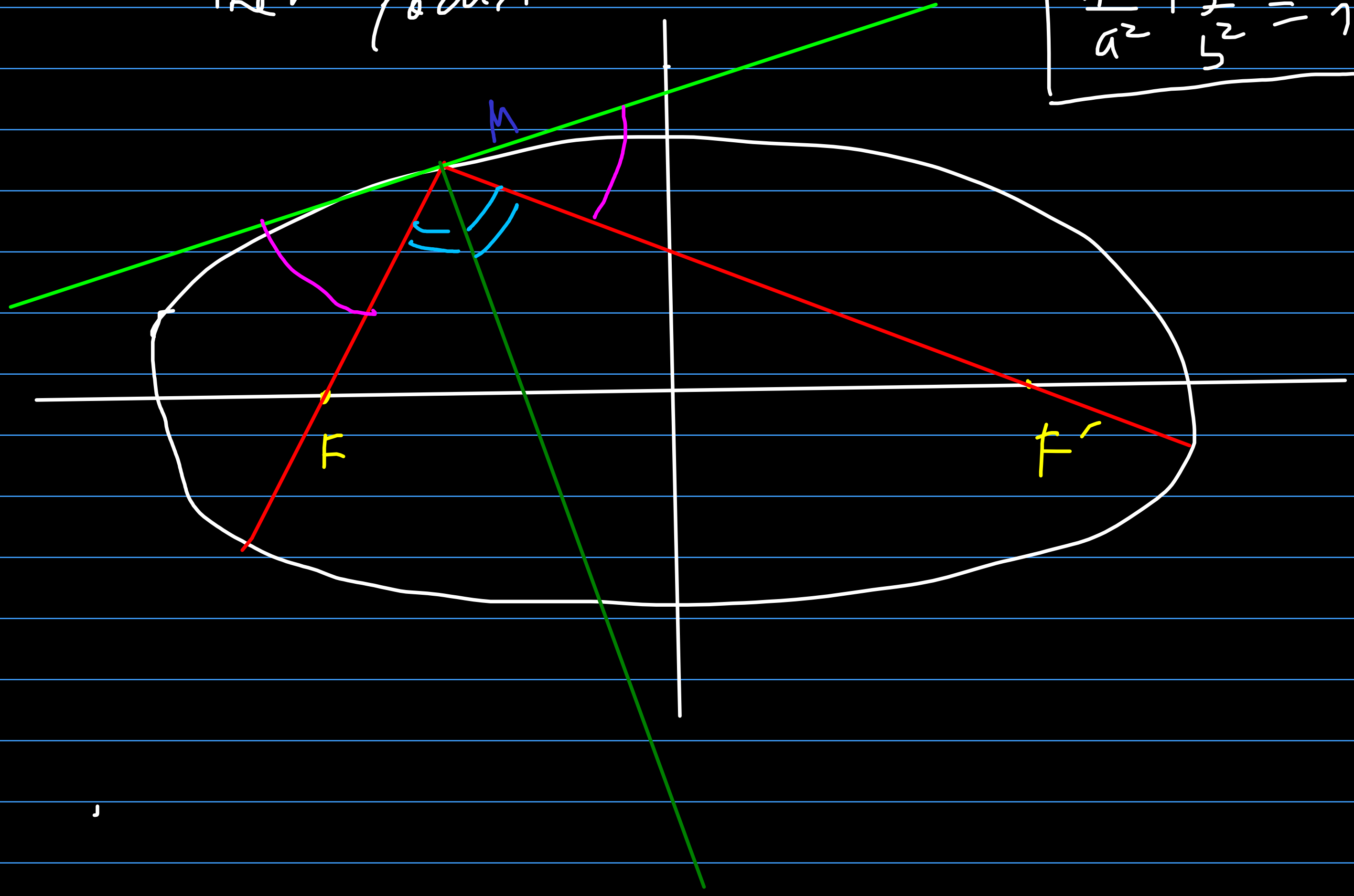
$\Rightarrow$ the tangents that we want are

$$l_3: y - 9 = 3(x-3)$$

$$l_{\frac{3}{2}} : y - 9 = \frac{3}{2}(x - 2)$$

9.12. Show that a ray of light through a focus of an ellipse reflects to a ray that passes through the other focus.

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$



We have to prove that for every point $M(x_0, y_0)$

on the ellipse, the normal $N_{\mathcal{E}}(x_0, y_0)$ is the bisector of the angle $\widehat{FMF'}$

We will show that $\forall\, T \in N_{\mathcal{E}}(x_0, y_0)$:

$$\mathrm{dist}(T, MF) = \mathrm{dist}(T, MF')$$

$$N_{\mathcal{E}}(x_0, y_0): \frac{x - x_0}{f'_x(x_0, y_0)} = \frac{y - y_0}{f'_y(x_0, y_0)}$$

$$\Leftrightarrow \frac{x - x_0}{\frac{2x_0}{a^2}} = \frac{y - y_0}{\frac{2y_0}{b^2}} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = x_0 + \frac{2x_0}{a^2} \cdot \lambda \\ y = y_0 + \frac{2y_0}{b^2} \cdot \lambda \end{cases}, \lambda \in \mathbb{R}$$

$M(x_0, y_0)$, $F(-c, 0)$, $F'(c, 0)$

$$MF: \quad \frac{x+c}{x_0+c} = \frac{y}{y_0} \Leftrightarrow$$

$$\Leftrightarrow y_0 x - (x_0+c)\cdot y + c y_0 = 0$$

$$MF': \quad \frac{x-c}{x_0-c} = \frac{y}{y_0} \Leftrightarrow$$

$$\Leftrightarrow y_0 x - (x_0-c)\cdot y - c y_0 = 0$$

$$\operatorname{dist}(T, MF) = \frac{\left| y_0\left(x_0 + \frac{2x_0}{a^2}\right) - (x_0+c)\cdot\left(y_0 + \frac{2y_0}{b^2}\right) + c y_0 \right|}{\sqrt{y_0^2 + (x_0+c)^2}}$$

$$\operatorname{dist}(T, MF') = \frac{\left| y_0\left(x_0 + \frac{2x_0}{a^2}\right) - (x_0-c)\cdot\left(y_0 + \frac{2y_0}{b^2}\right) - c y_0 \right|}{\sqrt{y_0^2 + (x_0-c)^2}}$$

We show that

$$\frac{\left|y_0\left(x_0+\frac{2x_0}{a^2}\right)-(x_0+c)\cdot\left(y_0+\frac{2y_0}{b^2}\right)+cy_0\right|}{\sqrt{y_0^2+(x_0+c)^2}}=\frac{\left|y_0\left(x_0+\frac{2x_0}{a^2}\right)-(x_0-c)\cdot\left(y_0+\frac{2y_0}{b^2}\right)-cy_0\right|}{\sqrt{y_0^2+(x_0-c)^2}}$$

(also keeping in mind that

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1 \quad \text{and} \quad c=\sqrt{a^2-b^2}$$
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