LECTURE 10 - DYNAMICAL SYSTEMS

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Planar dynamical systems (cont)

1) We consider the idealized eq. of a pendulum

$\ddot{\theta} + \omega^2 \sin\theta = 0$, where $\omega > 0$ parameter

↳ this is a second order nonlinear eq.

$\sin\theta \simeq \theta$ good approx. if θ is small

time t $\quad \nu\dot{\theta}$ friction with the air

$\nu = 0$ no friction $\Leftrightarrow$ idealized

Study the stability of the equilibrium points.

First we write this second order eq. as a planar system in θ and $\dot{\theta}$.

$x = \theta, y = \dot{\theta} \qquad \ddot{\theta} + \sin\theta = 0$

$\begin{cases} \dot{x} = y \\ \dot{y} = -\sin x \end{cases}$ Find the eq. points: $\begin{cases} y = 0 \\ -\sin x = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x = k\pi, k \in \mathbb{Z} \end{cases}$

We obtained $(k\pi, 0), \forall k \in \mathbb{Z}$

Physically, there are 2 eq. positions: $(0,0), (\pi, 0)$.

We use the linearization method to study their stability.

$f(x,y) = (y, -\sin x) \qquad Jf(x,y) = \begin{pmatrix} 0 & 1 \\ -\cos x & 0 \end{pmatrix}$

$\eta_1^* = (\pi, 0) \Rightarrow A_1 = Jf(\pi,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \det(A - \lambda I_2) = 0 \sim \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - 1 = 0 \Leftrightarrow \lambda = \pm 1 \Rightarrow$

$\Rightarrow$ eigenvalues of A_1: $\lambda_1 = -1, \lambda_2 = 1$

$\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq 0, \lambda_2 \neq 0 \Rightarrow \eta_1^*$ is hyperbolic $\Rightarrow$ the linearization method works $\Big\} \Rightarrow$

$\lambda_1 < 0 < \lambda_2 \Rightarrow$ the lin. system $\dot{x} = A_1 x$ is a saddle

L.M.

$\Rightarrow \eta_1^* = (\pi, 0)$ is unstable

$\eta_2^* = (0,0) \Rightarrow A_2 = Jf(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$ complex conjugated, $Re(\lambda_1) = Re(\lambda_2) = 0 \Big\} \Rightarrow$

$\Rightarrow \eta_2^*$ is not hyperbolic, thus LM fails

Th: Assume that the eigenvalues of $Jf(\eta^*)$ are $\lambda_{1,2} = \pm i\beta, \beta \in \mathbb{R}^*$. If the planar system $\dot{x} = f(x)$ has a first integral H well defined in a neighborhood of its eq. point η^* then: η^* is stable.

$\begin{cases} \dot{x} = y \\ \dot{y} = -\sin x \end{cases} \qquad \frac{\partial H}{\partial x} \cdot (y) + \frac{\partial H}{\partial y} \cdot (-\sin x) = 0, \forall (x,y) \in V$
$\qquad \forall \in \mathcal{V}(0,0)$

$H = ?$

$\frac{dy}{dx} = \frac{-\sin x}{y}$ separable eq. $\int y\,dy = -\int \sin x\,dx$, $\frac{y^2}{2} = \cos x + c, c \in \mathbb{R}$

$H(x,y) = \frac{y^2}{2} - \cos x, \forall (x,y) \in \mathbb{R}^2 \qquad \frac{\partial H}{\partial x} = \sin x , \frac{\partial H}{\partial y} = y$

Check: $\frac{\partial H}{\partial x} y + \frac{\partial H}{\partial y} \cdot (-\sin x) = 0, \forall (x,y) \in \mathbb{R}^2$

Then: $H: \mathbb{R}^2 \to \mathbb{R}$ - global first int, it is well def. in a neighborhood of $(0,0) \Rightarrow$

$\Rightarrow \eta_2^* = (0,0)$ is stable

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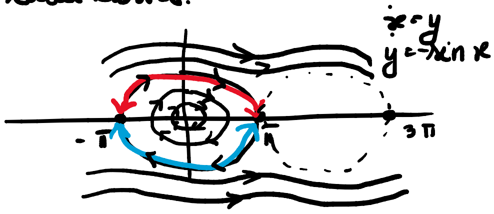
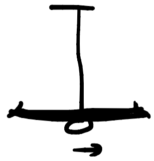
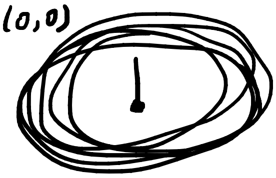
In Lab 5 we rep. the level curves of H in a small region containing the origin, and we have that they are closed curves.





We know that closed orbits correspond to periodic solutions. So, if the init. position is close to $(0,0)$



Polar coordinates

Consider the following planar systems

(i) $\begin{cases}\dot{x} = -y\\ \dot{y} = x\end{cases}$ (ii) $\begin{cases}\dot{x} = -y - x(x^2+y^2)\\ \dot{y} = x - y(x^2+y^2)\end{cases}$ (iii) $\begin{cases}\dot{x} = -y + xy\\ \dot{y} = x - x^2\end{cases}$ (iv) $\begin{cases}\dot{x} = -y + x(x^2+y^2)\\ \dot{y} = x + y(x^2+y^2)\end{cases}$

a) Specify the type of (i).

b) Show that $(0,0)$ - non-hyperbolic eq. point of (ii), (iii), (iv).

c) Passing to polar coord, rep. the phase portrait of each system.

d) Reading the phase portrait, specify the stability of $(0,0)$.

a) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$, $\lambda^2 = -1 \Rightarrow \lambda_{1,2} = \pm i$; This lin. system is a center

b) $f(x,y) = (-y - x^3 - xy^2, x - yx^2 - y^3)$

$f(0,0) = (0,0) \Rightarrow (0,0)$ is an eq. point

$Jf(x,y) = \begin{pmatrix} -3x^2 - y^2 & -1-2xy \\ 1-2xy & -x^2-3y^2 \end{pmatrix} \Rightarrow Jf(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = A$

$\lambda_{1,2} = \pm i \Rightarrow (0,0)$ is not hyperbolic for (ii)

The same for (iii) and (iv). $Jf(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

c) Polar coord.: $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$ $\begin{cases} \rho^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases}$

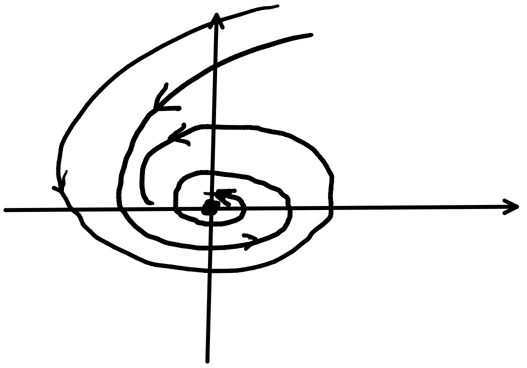
$\begin{cases} \rho\dot{\rho} = x\dot{x} + y\dot{y} \\ \frac{\dot{\theta}}{\cos^2\theta} = \frac{\dot{y}x - y\dot{x}}{x^2} \end{cases}$ $\Rightarrow$ (ii) $\begin{cases} \rho\dot{\rho} = \cancel{-xy} - x^2(x^2+y^2) \cancel{+ yx} - y^2(x^2+y^2) \\ \frac{\dot{\theta}}{\cos^2\theta} = \frac{x^2 \cancel{- xy(x^2+y^2)} + y^2 \cancel{+ xy(x^2+y^2)}}{x^2} \end{cases}$ $\Leftrightarrow$ $\begin{cases} \rho\dot{\rho} = -(x^2+y^2)^2 = -\rho^4 \\ \frac{\dot{\theta}}{\cos^2\theta} = \frac{\rho^2}{\rho^2\cos^2\theta} \end{cases}$ $\Leftrightarrow$

$\Leftrightarrow \begin{cases} \dot{\rho} = -\rho^3 \\ \dot{\theta} = 1 \end{cases}$

$0 \longleftarrow$ $-$ ρ decreases $\rho \searrow \Rightarrow$ approach the origin

$\dot{\theta} > 0$, θ increases $\Rightarrow$ rotation around the origin $(0,0)$ in the trig sense

d) The eq. point $(0,0)$ is a global attractor



(iii) $\begin{cases} \dot{x} = -y + xy \\ \dot{y} = x - x^2 \end{cases}$ $\quad \begin{cases} \rho\dot{\rho} = x(-y+xy) + y(x-x^2) \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{x(x-x^2) - y(-y+xy)}{x^2} \end{cases}$ $\Leftrightarrow \begin{cases} \rho\dot{\rho} = -xy + x^2y + yx - x^2y = 0 \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{x^2 - x^3 + y^2 - xy^2}{x^2} = \dfrac{x^2+y^2 - x(x^2+y^2)}{x^2} \end{cases}$

$= \dfrac{(x^2+y^2)(1-x)}{x^2} \Rightarrow$

$\Rightarrow \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{\rho^2(1-\rho\cos\theta)}{\rho^2\cos^2\theta}$

$\Leftrightarrow \begin{cases} \dot\rho = 0 \\ \dot\theta = 1 - \rho\cos\theta \end{cases}$, $\rho = c$, ρ - the expr. of a first int. in polar coord; $\rho < 1 \Rightarrow \dot\theta > 0$

Try to find the eq. points and a first integral

$\begin{cases} \dot{x} = -y(1-x) \\ \dot{y} = x(1-x) \end{cases}$ $\quad \begin{cases} -y(1-x) = 0 \\ x(1-x) = 0 \end{cases}$ $\quad (0,0)$, $x = 1$ - line of eq. points

$\dfrac{dy}{dx} = \dfrac{x(1-x)}{-y(1-x)}$, $-y\,dy = x\,dx$ $\quad -\dfrac{y^2}{2} = \dfrac{x^2}{2} + k$, $k \in \mathbb{R} \Rightarrow x^2 + y^2 = -2k$, $k \in \mathbb{R}$

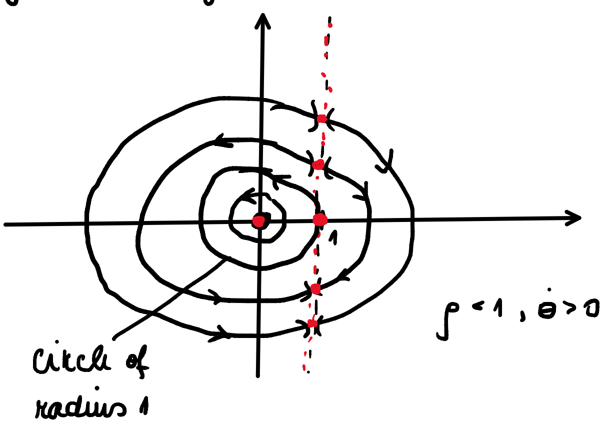
$H(x,y) = x^2 + y^2$, $(x,y) \in \mathbb{R}^2$

$\dfrac{\partial H}{\partial x}\cdot(-y+xy) + \dfrac{\partial H}{\partial y}\cdot(x - x^2) = 0 \Leftrightarrow 2x(-y+xy) + 2y(x-x^2) = 0$

$\Leftrightarrow \cancel{-2xy} + \cancel{2x^2y} + \cancel{2xy} - \cancel{2x^2y} = 0$, true $\Rightarrow H$ is a first integral, $\forall (x,y) \in \mathbb{R}^2$

$x^2 + y^2 = c$, $c \in \mathbb{R} \Rightarrow \rho = c$, $c \in \mathbb{R} \mapsto$ circles centered in $(0,0)$

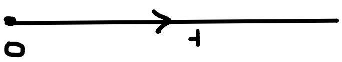
first we rep. the eq. points



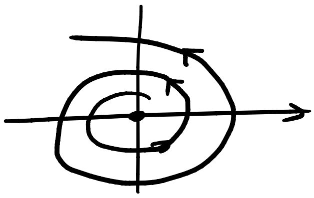
$\rho < 1$, $\dot\theta > 0$

From the $\exists!$ theorem we can deduce that for two orbits γ_1 and γ_2 of a planar system either $\gamma_1 = \gamma_2$ or $\gamma_1 \cap \gamma_2 = \emptyset$.
In this case $(0,0)$ is a stable eq. point.

(iv) $\begin{cases} \dot\rho = \rho^3 \\ \dot\theta = 1 \end{cases}$



$\rho > 0$, $\dot\theta > 0$



$(0,0)$ is a global repeller

Conclusion: We had 3 nonlinear systems, whose linearization around the eq. point $(0,0)$ is $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$, thus $(0,0)$ is not hyperbolic, but the eq. $(0,0)$ was a global attractor, a global repeller and stable.
For sure, the LH does not work for nonhyp. eq. p.

Ex: $\begin{cases} \dot{x} = -y + x(1-x^2-y^2) \\ \dot{y} = x + y(1-x^2-y^2) \end{cases}$ a) Check that: $\varphi(t,1,0) = (\cos t, \sin t)$, $\forall t \in \mathbb{R}$

b) Pass to polar coord. and rep the phase portrait. Reading the phase portrait specify the stability of the eq. $(0,0)$. Is there an attractor?

a) $\varphi(t,\eta)$ is, by def, the unique sol of the ivp: $\begin{cases} \dot{X} = f(X) \\ X(0) = \eta \end{cases}$, $\forall \eta \in \mathbb{R}^2$

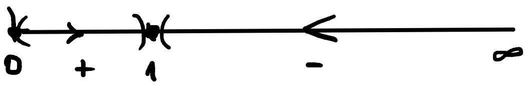
We have to check that the unique sol. of the ivp

$\begin{cases} \dot{x} = -y + x(1-x^2-y^2) \\ \dot{y} = x + y(1-x^2-y^2) \\ x(0) = 1 \\ y(0) = 0 \end{cases}$ is $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$

$(\cos t)' = -\sin t + \cos t(1-\cos^2 t - \sin^2 t)$
$(\sin t)' = \cos t + \sin t(1-\cos^2 t - \sin^2 t)$, $\forall t \in \mathbb{R}$
$\cos t/_{t=0} = 1$ TRUE
$\sin t/_{t=0} = 0$

b) $\begin{cases} \rho\dot{\rho} = x\dot{x} + y\dot{y} \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{\dot{y}x - y\dot{x}}{x^2} \end{cases}$ $\Rightarrow \begin{cases} \rho\dot{\rho} = -xy + x^2(1-x^2-y^2) + xy + y^2(1-x^2-y^2) \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{x^2 + \cancel{xy(1-x^2-y^2)} + y^2 - \cancel{xy(1-x^2-y^2)}}{x^2} \end{cases}$ $\Rightarrow \begin{cases} \rho\dot{\rho} = (1-x^2-y^2)(x^2+y^2) = \rho^2(1-\rho^2) \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{\rho^2}{\rho^2 \cdot \cos^2\theta} \end{cases}$, $\Rightarrow$

$\Rightarrow \begin{cases} \dot{\rho} = \rho(1-\rho^2) \\ \dot{\theta} = 1 \end{cases}$

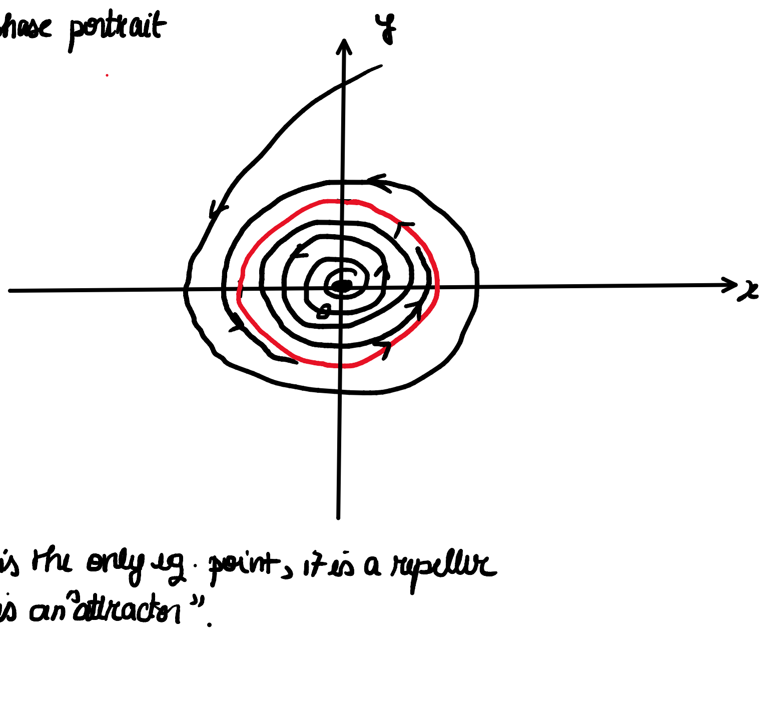


$\rho(1-\rho^2) = 0 \rightarrow \rho = 0$
$\rightarrow \rho = \pm 1$

$\underset{>0}{\rho}(1-\rho)\underset{>0}{(1+\rho)}$

$\dot{\theta} > 0$ $\theta \nearrow$ $\longrightarrow$

$\rightarrow$ the phase portrait



$(0,0)$, $\varphi(t,1,0) = (\cos t, \sin t)$
$\gamma_{(1,0)} \begin{cases} x = \cos t \\ y = \sin t \end{cases}$, $t \in \mathbb{R}$
circle (trig.) $\rho = 1$

$(0,0)$ is the only eq. point, it is a repeller
$\rho = 1$ is an "attractor".