# 4 Week 4

## 4.1 Analytic conditions of parallelism and nonparallelism

### 4.1.1 The parallelism between a line and a plane

**Proposition 4.1.** *The equation of the director subspace $\overrightarrow{\pi}$, of the plane $\pi : Ax + By + Cz + D = 0$ is $AX + BY + CZ = 0$.*

*Proof.* We first recall that

$$\overrightarrow{\pi} = \{\overrightarrow{A_0M} \mid M \in \pi\}, \tag{4.1}$$

where $A_0 \in \pi$ is an arbitrary point, and the representation (4.1) of $\overrightarrow{\pi}$ is independent on the choice of $A_0 \in \pi$. According to equation (3.8), the equation of a plane $\pi$ can be written in the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

where $A_0(x_0, y_0, z_0)$ is a point in $\pi$. In other words,

$$M(x, y, z) \in \pi \Longleftrightarrow A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

which shows that

$$\begin{aligned}
\overrightarrow{\pi} &= \{\overrightarrow{A_0M}\,(x - x_0, y - y_0, z - z_0) \mid M(x, y, z) \in \pi\} \\
&= \{\overrightarrow{A_0M}\,(x - x_0, y - y_0, z - z_0) \mid A(x - x_0) + B(y - y_0) + C(z - z_0) = 0\} \\
&= \{\overrightarrow{v}\,(X, Y, Z) \in \mathcal{V} \mid AX + BY + CZ = 0\}.
\end{aligned}$$

Thus, the equation $AX + BY + CZ = 0$ is a necessary and sufficient condition for the vector $\overrightarrow{v}\,(X, Y, Z)$ to be contained within the direction of the plane

$$\pi : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

In other words, the *equation of the director subspace* $\overrightarrow{\pi}$ is $AX + BY + CZ = 0$. □

**Corollary 4.2.** *The straight line*

$$\Delta : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

*is parallel to the plane $\pi : Ax + By + Cz + D = 0$ if and only if*

$$Ap + Bq + Cr = 0 \tag{4.2}$$

$\overrightarrow{\Delta} = \langle (p, q, r) \rangle$

*Proof.* Indeed,

$$\begin{aligned}
\Delta \| \pi &\Longleftrightarrow \overrightarrow{\Delta} \subseteq \overrightarrow{\pi} \Longleftrightarrow \langle (p, q, r) \rangle \subseteq \overrightarrow{\pi} \\
&\Longleftrightarrow \overrightarrow{d}\,(p, q, r) \in \overrightarrow{\pi} \Longleftrightarrow Ap + Bq + Cr = 0.
\end{aligned}$$

□

### 4.1.2 The intersection point of a straight line and a plane

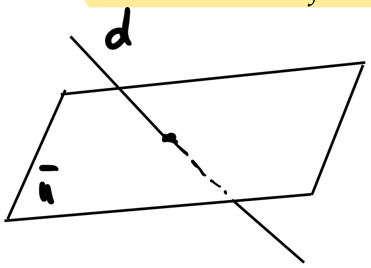
**Proposition 4.3.** *Consider a straight line*

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

*and a plane* $\pi : Ax + By + Cz + D = 0$ *which are not parallel to each other, i.e.*

$$Ap + Bq + Cr \neq 0.$$

*The coordinates of the intersection point* $d \cap \pi$ *are*




$$\begin{cases} x_0 - p\dfrac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ y_0 - q\dfrac{F(x_0, y_0, z_0)}{Ap + Bq + Cr} \\ z_0 - r\dfrac{F(x_0, y_0, z_0)}{Ap + Bq + Cr}, \end{cases} \tag{4.3}$$

*where* $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $F(x, y, z) = Ax + By + Cz + D$.

*Proof.* The parametric equations of $(d)$ are

$$(d) \quad \begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, \ t \in \mathbb{R}. \tag{4.4}$$

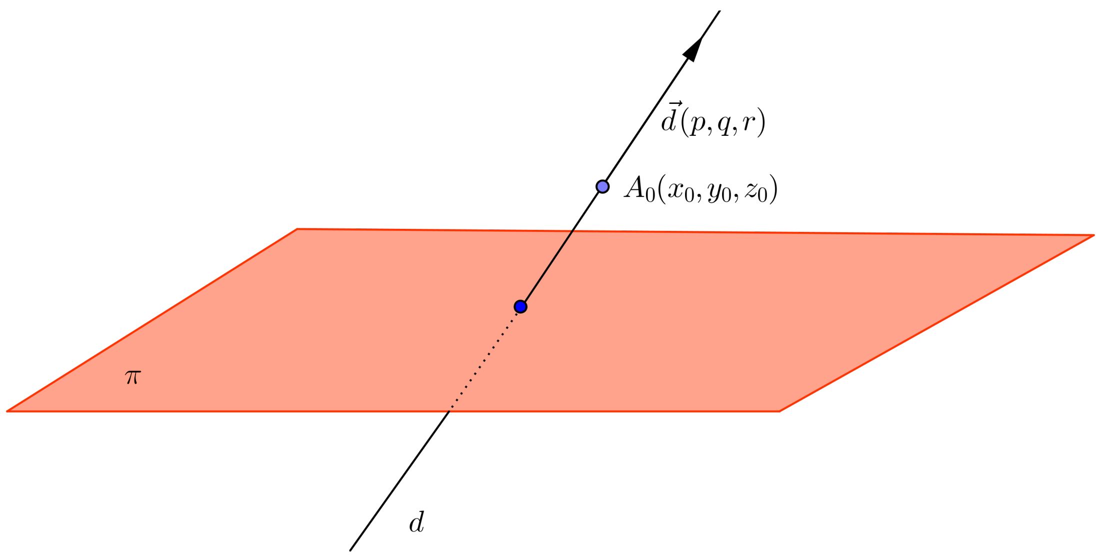
The unique value of $t \in \mathbb{R}$, which corresponds to the intersection point $d \cap \pi$, can be found by solving the equation

$$A(x_0 + pt) + B(y_0 + qt) + C(z_0 + rt) + D = 0.$$

Its unique solution is

$$t = -\frac{Ax_0 + By_0 + Cz_0 + D}{Ap + Bq + Cr} = -\frac{F(x_0, y_0, z_0)}{Ap + Bq + Cr}$$

and can be used to obtain the required coordinates (4.3) by replacing this value in (4.4). □



**Example 4.1** (Homework)**.** Decide whether the line $d$ and the plane $\pi$ are parallel or concurrent and find the coordinates of the intersection point of of $\Delta$ and $\pi$ whenever $\Delta \nparallel \pi$:

1. $d : \frac{x+2}{1} = \frac{y-1}{3} = \frac{z-3}{1}$ and $\pi : x - y + 2z = 1$.
2. $d : \frac{x-3}{1} = \frac{y+1}{-2} = \frac{z-2}{-1}$ and $\pi : 2x - y + 3z - 1 = 0$.

SOLUTION.

### 4.1.3 Parallelism of two planes

**Proposition 4.4.** *Consider the planes*

$$(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0,\ (\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0.$$

*Then* $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) \in \{1,2\}$ *and the following statemenets are equivalent*

1. $\pi_1 || \pi_2$.
2. $\dim(\vec{\pi}_1 \cap \vec{\pi}_2) = 2$, *i.e.* $\vec{\pi}_1 = \vec{\pi}_2$.
3. *rank* $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 1$.
4. *The vectors* $(A_1, B_1, C_1),\ (A_2, B_2, C_2) \in \mathbb{R}^3$ *are linearly dependent.*

**Remark 4.1.** Note that

$$\operatorname{rank}\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 1 \Leftrightarrow \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} = 0$$
$$\Leftrightarrow A_1B_2 - A_2B_1 = A_1C_2 - A_2C_1 = B_1C_2 - C_2B_1 = 0. \qquad (4.5)$$

The relations (4.5) are often written in the form

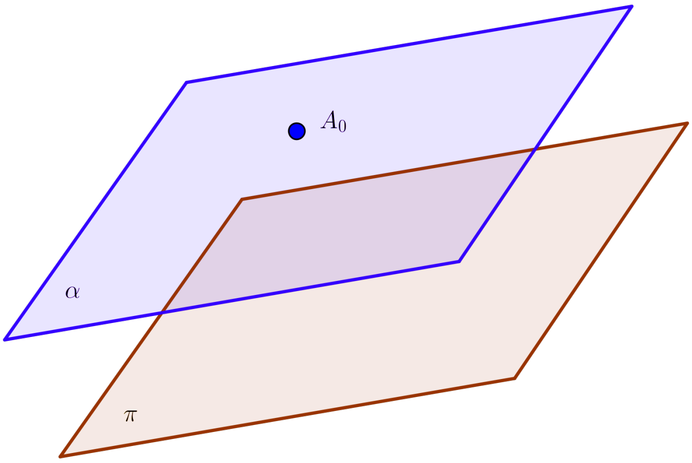
$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}, \qquad (4.6)$$

although at most two of the coefficients $A_2$, $B_2$ or $C_2$ might be zero. In fact relations (4.6) should be understood in terms of linear dependence of the vectors $(A_1, B_1, C_1),\ (A_2, B_2, C_2) \in \mathbb{R}^3$, i.e. $(A_1, B_1, C_1) = k(A_2, B_2, C_2)$, where $k \in \mathbb{R}$ is the common value of those ratios (4.6) which do not involve any zero coefficients. Let us finally mention that the equivalences (4.5) prove the equivalence (3) $\Longleftrightarrow$ (4) of Proposition 4.4.

**Example 4.2.** The equation of the plane $\alpha$ passing through the point $A_0(x_0, y_0, z_0)$, which is parallel to the plane $\pi: \ Ax + By + Cz + D = 0$ is

$$\alpha: \ A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$




### 4.1.4 Straight lines as intersections of planes

**Corollary 4.5.** *Consider the planes*

$$(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0, \ (\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0.$$

*The following statements are equivalent*

1. $\pi_1 \not\parallel \pi_2$.
2. $\dim(\overrightarrow{\pi}_1 \cap \overrightarrow{\pi}_2) = 1$.
3. *rank* $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2$.
4. *The vectors* $(A_1, B_1, C_1), \ (A_2, B_2, C_2) \in \mathbb{R}^3$ *are linearly independent.*

By using the characterization of parallelism between a line and a plane, given by Proposition 4.2, we shall find the direction of a straight line which is given as the intersection of two planes. Consider the planes $(\pi_1)\ A_1x + B_1y + C_1z + D_1 = 0$, $(\pi_2)\ A_2x + B_2y + C_2z + D_2 = 0$ such that

$$\text{rank}\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2,$$

alongside their intersection straight line $\Delta = \pi_1 \cap \pi_2$ of equations

$$(\Delta) \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_1 = 0. \end{cases}$$

Thus, $\overrightarrow{\Delta} = \overrightarrow{\pi}_1 \cap \overrightarrow{\pi}_2$ and therefore, by means of some previous Proposition, it follows that the equations of $\overrightarrow{\Delta}$ are

$$(\overrightarrow{\Delta}) \begin{cases} A_1X + B_1Y + C_1Z = 0 \\ A_2X + B_2Y + C_2Z = 0. \end{cases} \tag{4.7}$$

By solving the system (4.7) one can therefore deduce that $\overrightarrow{d}\ (p,q,r) \in \overrightarrow{\Delta} \Leftrightarrow\ \exists \lambda \in \mathbb{R}$ such that

$$(p,q,r) = \lambda \left( \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \right). \tag{4.8}$$

The relation is usually (4.8) written in the form

$$\frac{p}{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}} = \frac{q}{\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}} = \frac{r}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}. \tag{4.9}$$

Let us finaly mention that we usually choose the values

$$\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} \text{ şi } \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \tag{4.10}$$

for the director parameters $(p,q,r)$ of $\Delta$.

**Example 4.3.** Write the equations of the plane through $P(4,-3,1)$ which is parallel to the lines

$$(\Delta_1) \begin{cases} 2x \quad\quad - z + 1 = 0 \\ \quad 3y + 2z - 2 = 0. \end{cases} \text{ and } (\Delta_2) \begin{cases} x + y + z = 0 \\ 2x - y + 3z = 0. \end{cases}$$

SOLUTION. One can see the required plane as the one through $P(4,-3,1)$ which is parallel to the director vectors $\overrightarrow{d}_1\ (p_1,q_1,r_1)$ and $\overrightarrow{d}_2\ (p_2,q_2,r_2)$ of $\Delta_1$ and $\Delta_2$ respectively. One can choose

$$p_1 = \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = 3 \qquad p_2 = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 4$$
$$q_1 = \begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} = -4 \quad \text{and} \quad q_2 = \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -1$$
$$r_1 = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6 \qquad r_2 = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3.$$

Thus, the equation of the required plane is

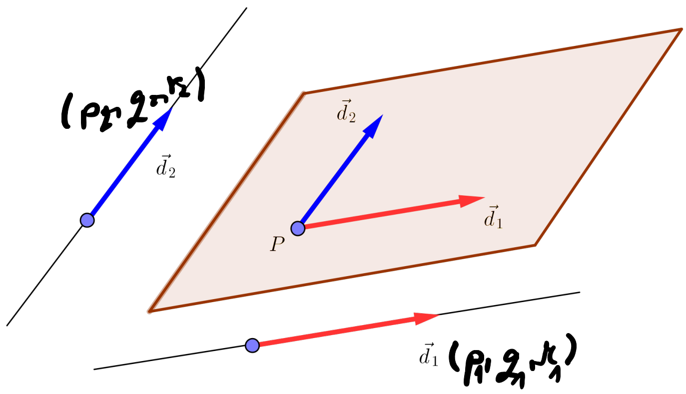



Figure 7:

$$\begin{vmatrix} x-4 & y+3 & z-1 \\ 3 & -4 & 6 \\ 4 & -1 & -3 \end{vmatrix} = 0 \iff 12(x-4) - 3(z-1) + 24(y+3) + 16(z-1) + 6(x-4) + 9(y+3) = 0$$
$$\iff 18(x-4) + 33(y+3) + 13(z-1) = 0$$
$$\iff 18x + 33y + 13z - 72 + 99 - 13 = 0$$
$$\iff 18x + 33y + 13z + 14 = 0.$$

## 4.2 Pencils of planes

**Definition 4.1.** The collection of all planes containing a given straight line

$$(\Delta)\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

is called the *pencil* or the *bundle* of planes through $\Delta$.

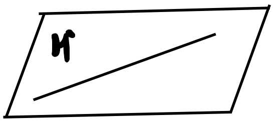
**Proposition 4.6.** *The plane $\pi$ belongs to the pencil of planes through the straight line $\Delta$ if and only if the equation of the plane $\pi$ is*

$$\lambda(A_1x + B_1y + C_1z + D_1) + \mu(A_2x + B_2y + C_2z + D_2) = 0. \tag{4.11}$$

*for some $\lambda, \mu \in \mathbb{R}$ such that $\lambda^2 + \mu^2 > 0$.*

*Proof.* Every plane in the family (4.11) obviously contains the line $\Delta$.

Conversely, assume that $\pi$ is a plane through the line $\Delta$. Consider a point $M \in \pi \setminus \Delta$ and recall that $\pi$ is completely determined by $\Delta$ and $M$. On the other hand $M$ and $\Delta$ are obviously contained in the plane $F_1(x_M, y_M, z_M)F_2(x, y, z) - F_2(x_M, y_M, z_M)F_1(x, y, z) = 0$ of the family (4.11), where $F_1, F_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $F_i(x, y, z) = A_ix + B_iy + C_iz + D_i$, for $i = 1, 2$. Thus the plane $\pi$ belongs to the family (4.11) and its equation is



$$F_1(x_M, y_M, z_M)F_2(x, y, z) - F_2(x_M, y_M, z_M)F_1(x, y, z) = 0. \quad (\bar{\pi})$$

$\lambda$ $\mu$

□

**Remark 4.2.** The family of planes $A_1x + B_1y + C_1z + D_1 + \lambda(A_2x + B_2y + C_2z + D_2) = 0$, where $\lambda$ covers the whole real line $\mathbb{R}$, is the so called *reduced pencil of planes* through $\Delta$ and it consists in all planes through $\Delta$ except the plane of equation $A_2x + B_2y + C_2z + D_2 = 0$.

**Example 4.4.** Write the equations of the plane parallel to the line $d : \ x = 2y = 3z$ passing through the line

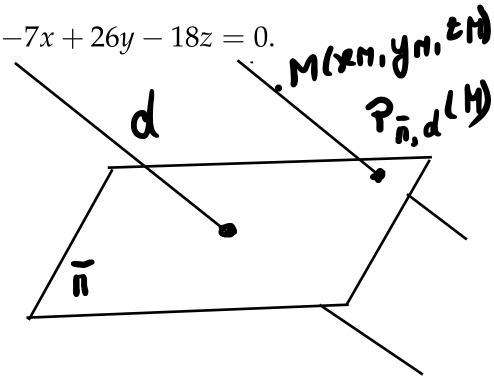
$$\Delta : \begin{cases} x + y + z = 0 \\ 2x - y + 3z = 0. \end{cases}$$

SOLUTION. Note that none of the planes $x + y + z = 0$ and $x - y + 3z = 0$, passing through $(\Delta)$, is parallel to $(d)$, as $1 \cdot 1 + 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{3} \neq 0$ and $2 \cdot 1 + (-1) \cdot \frac{1}{2} + 3 \cdot \frac{1}{3} \neq 0$. Thus, the required plane is in a reduced pencil of planes, such as the family $\pi_\lambda : x + y + z + \lambda(2x - y + 3z) = 0$, $\lambda \in \mathbb{R}$. The parallelism relation between $(d)$ and $\pi_\lambda : (2\lambda + 1)x + (1 - \lambda)y + (3\lambda + 1)z = 0$ is

$$(2\lambda + 1) \cdot 1 + (1 - \lambda) \cdot \frac{1}{2} + (3\lambda + 1) \cdot \frac{1}{3} = 0 \Longleftrightarrow 12\lambda + 6 + 3 - 3\lambda + 6\lambda + 2 = 0 \Longleftrightarrow \lambda = -\frac{11}{15}.$$

Thus, the required plane is

$$\pi_{-11/15} : \left(-2\frac{11}{15} + 1\right)x + \left(1 + \frac{11}{15}\right)y + \left(-3\frac{11}{15} + 1\right)z = 0 \Longleftrightarrow -7x + 26y - 18z = 0.$$




# Appendix

## 4.3 Projections and symmetries

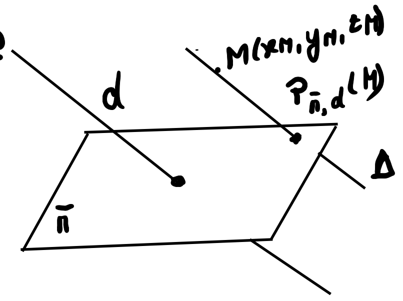
### 4.3.1 The projection on a plane parallel with a given line

Consider a straight line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

$(d)\ \dfrac{x-x_0}{p} = \dfrac{y-y_0}{q} = \dfrac{z-z_0}{r}$




$(\Delta) \begin{cases} x = x_M + pt \\ y = y_M + qt \quad , t \in \mathbb{R} \\ z = z_M + rt \end{cases}$

$F(x, y, z) = Ax + By + Cz + D$

$\pi: Ax + By + Cz + D = 0 \Rightarrow A\cdot(x_M + pt) + B(y_M + qt) + C(z_M + rt) + D = 0$

$\Leftrightarrow \underbrace{Ax_M + By_M + Cz_M + D}_{F(x_M, y_M, z_M)} + t(\underbrace{Ap + Bq + Cr}_{\neq 0}) = 0 \Leftrightarrow t = -\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr}$

and a plane $\pi : Ax + By + Cz + D = 0$ which are not parallel to each other, i.e.

$$Ap + Bq + Cr \neq 0.$$

For these given data we may define the projection $p_{\pi,d} : \mathcal{P} \longrightarrow \pi$ of $\mathcal{P}$ on $\pi$ parallel to $d$, whose value $p_{\pi,d}(M)$ at $M \in \mathcal{P}$ is the intersection point between $\pi$ and the line through $M$ which is parallel to $d$. Due to relations (4.3), the coordinates of $p_{\pi,d}(M)$, in terms of the coordinates of $M$, are

$$\begin{cases} x_M - p\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ y_M - q\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr} \\ z_M - r\dfrac{F(x_M, y_M, z_M)}{Ap + Bq + Cr}, \end{cases} \tag{4.12}$$

where $F(x, y, z) = Ax + By + Cz + D$.

Consequently, the position vector of $p_{\pi,d}(M)$ is

$$\overrightarrow{Op_{\pi,d}(M)} = \overrightarrow{OM} - \frac{F(M)}{Ap + Bq + Cr}\overrightarrow{d}\ . \tag{4.13}$$

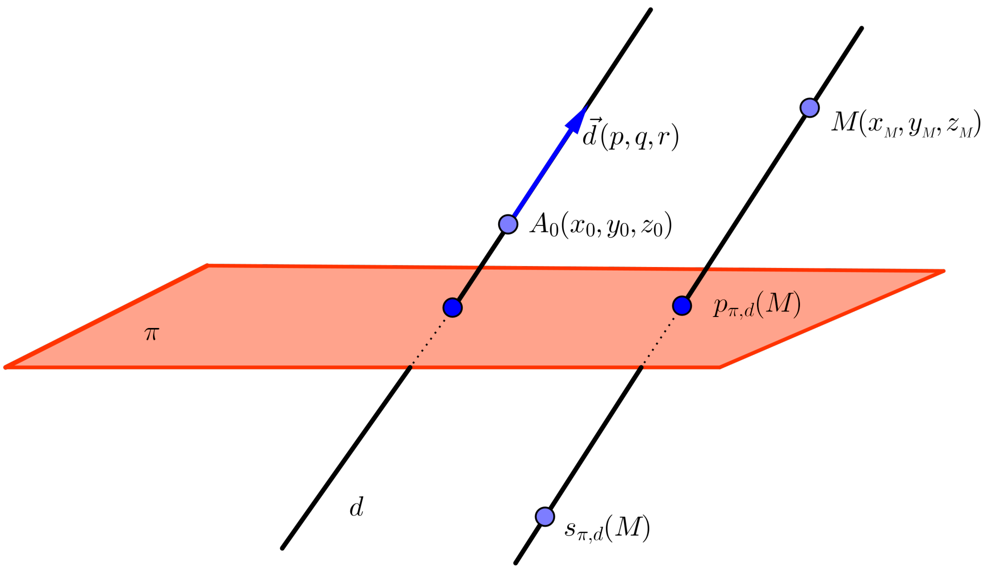
**Proposition 4.7.** *If $R = (O, b)$ is the Cartesian reference system behind the equations of the line*

$$(d)\ \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

*and the plane $(\pi)\ Ax + By + Cz + D = 0$, concurrent with $(d)$, then*

$$[p_{\pi,d}(M)]_R = \frac{1}{Ap + Bq + Cr}\begin{pmatrix} Bq + Cr & -Bp & -Cp \\ -Aq & Ap + Cr & -Cq \\ -Ar & -Br & Ap + Bq \end{pmatrix}[M]_R - \frac{D}{Ap + Bq + Cr}[\overrightarrow{d}]_b,$$

*where $\overrightarrow{d}\ (p, q, r)$ stands for the director vector of the line $(d)$.*




### 4.3.2 The symmetry with respect to a plane parallel with a given line

We call the function $s_{\pi,d} : \mathcal{P} \longrightarrow \mathcal{P}$, whose value $s_{\pi,d}(M)$ at $M \in \mathcal{P}$ is the symmetric point of $M$ with respect to $p_{\pi,d}(M)$ *the symmetry of $\mathcal{P}$ with respect to $\pi$ parallel to $d$*. The direction of $d$ is equally called the *direction* of the symmetry and $\pi$ is called the *axis* of the symmetry. For the position vector of $s_{\pi,d}(M)$ we have

$$\overrightarrow{Op_{\pi,d}(M)}=\frac{\overrightarrow{OM}+\overrightarrow{Os_{\pi,d}(M)}}{2}, \text{ i.e.} \tag{4.14}$$

$$\overrightarrow{Os_{\pi,d}(M)}=2\,\overrightarrow{Op_{\pi,d}(M)}-\overrightarrow{OM}=\overrightarrow{OM}-2\frac{F(M)}{Ap+Bq+Cr}\,\overrightarrow{d}\,. \tag{4.15}$$

**Proposition 4.8.** *If $R=(O,b)$ is the Cartesian reference system behind the equations of the line*

$$(d)\ \frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}$$

*and the plane $(\pi)\ Ax+By+Cz+D=0$, concurrent with $(d)$, then*

$$(Ap+Bq+Cr)[s_{\pi,d}(M)]_R=\begin{pmatrix} -Ap+Bq+Cr & -2Bp & -2Cp \\ -2Aq & Ap-Bq+Cr & -2Cq \\ -2Ar & -2Br & Ap+Bq-Cr \end{pmatrix}[M]_R-2D[\overrightarrow{d}]_b, \tag{4.16}$$

*where $\overrightarrow{d}\,(p,q,r)$ stands for the director vector of the line $(d)$.*

### 4.3.3 The projection on a straight line parallel with a given plane

Consider a straight line

$$d:\frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}$$

and a plane $\pi: Ax+By+Cz+D=0$ which are not parallel to each other, i.e.

$$Ap+Bq+Cr\neq 0.$$

For these given data we may define the projection $p_{d,\pi}:\mathcal{P}\longrightarrow d$ of $\mathcal{P}$ on $d$, whose value $p_{d,\pi}(M)$ at $M\in\mathcal{P}$ is the intersection point between $d$ and the plane through $M$ which is parallel to $\pi$. Due to relations (4.3), the coordinates of $p_{d,\pi}(M)$, in terms of the coordinates of $M$, are

$$\begin{cases} x_0-p\dfrac{G_M(x_0,y_0,z_0)}{Ap+Bq+Cr} \\ y_0-q\dfrac{G_M(x_0,y_0,z_0)}{Ap+Bq+Cr} \\ z_0-r\dfrac{G_M(x_0,y_0,z_0)}{Ap+Bq+Cr}, \end{cases} \tag{4.17}$$

where $G_M(x,y,z)=A(x-x_M)+B(y-y_M)+C(z-z_M)$. Consequently, the position vector of $p_{d,\pi}(M)$ is

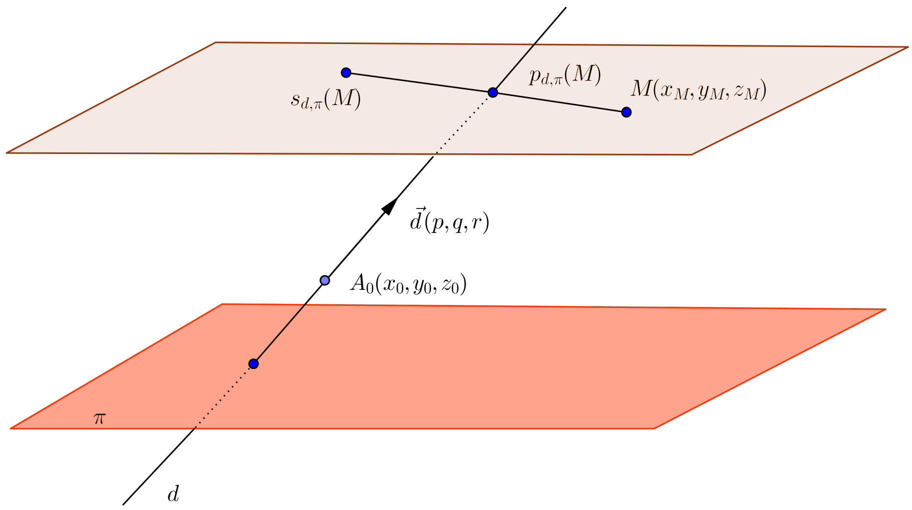
$$\overrightarrow{Op_{d,\pi}(M)}=\overrightarrow{OA_0}-\frac{G_M(A_0)}{Ap+Bq+Cr}\,\overrightarrow{d}, \text{ where } A_0(x_0,y_0,z_0). \tag{4.18}$$

Note that $G_M(A_0)=A(x_0-x_M)+B(y_0-y_M)+C(z_0-z_M)=F(A_0)-F(M)$, where $F(x,y,z)=Ax+By+Cz+D$. Consequently the coordinates of $p_{d,\pi}(M)$, in terms of the coordinates of $M$, are

$$\begin{cases} x_0+p\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr} \\ y_0+q\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr} \\ z_0+r\dfrac{F(M)-F(A_0)}{Ap+Bq+Cr}, \end{cases} \tag{4.19}$$

and the position vector of $p_{d,\pi}(M)$ is

$$\overrightarrow{Op_{d,\pi}(M)}=\overrightarrow{OA_0}+\frac{F(M)-F(A_0)}{Ap+Bq+Cr}\vec{d}, \text{ where } A_0(x_0,y_0,z_0). \tag{4.20}$$




### 4.3.4 The symmetry with respect to a line parallel with a plane

We call the function $s_{d,\pi}:\mathcal{P}\longrightarrow\mathcal{P}$, whose value $s_{d,\pi}(M)$ at $M\in\mathcal{P}$ is the symmetric point of $M$ with respect to $p_{d,\pi}(M)$, *the symmetry of $\mathcal{P}$ with respect to $d$ parallel to $\pi$*. The direction of $\pi$ is equally called the *direction* of the symmetry and $d$ is called the *axis* of the symmetry. For the position vector of $s_{d,\pi}(M)$ we have

$$\overrightarrow{Op_{d,\pi}(M)}=\frac{\overrightarrow{OM}+\overrightarrow{Os_{d,\pi}(M)}}{2}, \text{ i.e.} \tag{4.21}$$

$$\begin{aligned}\overrightarrow{Os_{d,\pi}(M)} &= 2\,\overrightarrow{Op_{d,\pi}(M)}-\overrightarrow{OM}\\ &= 2\,\overrightarrow{OA_0}-\overrightarrow{OM}+2\frac{F(M)-F(A_0)}{Ap+Bq+Cr}\vec{d}\,.\end{aligned} \tag{4.22}$$

## 4.4 Projections and symmetries in the two dimensional setting

### 4.4.1 The intersection point of two concurrent lines

Consider two lines

$$d:\frac{x-x_0}{p}=\frac{y-y_0}{q}$$

şi $\Delta: ax+by+c=0$ which are not parallel to each other, i.e.

$$ap+bq\neq 0.$$

The parametric equations of $d$ are:

$$\begin{cases} x=x_0+pt\\ y=y_0+qt\end{cases}, t\in\mathbb{R} \tag{4.23}$$

The value of $t\in\mathbb{R}$ for which this line (4.23) punctures the line $\Delta$ can be determined by imposing the condition on the point of coordinates

$$(x_0+pt, y_0+qt)$$

to verify the equationof the line $\Delta$, namely

$$a(x_0 + pt) + b(y_0 + qt) + c = 0.$$

Thus

$$t = -\frac{ax_0 + by_0 + c}{ap + bq} = -\frac{F(x_0, y_0)}{ap + bq},$$

where $F(x, y) = ax + by + c$.

The coordinates of the intersection point $d \cap \Delta$ are:

$$\begin{aligned} & x_0 - p\frac{F(x_0, y_0)}{ap + bq} \\ & y_0 - q\frac{F(x_0, y_0)}{ap + bq}. \end{aligned} \tag{4.24}$$

### 4.4.2 The projection on a line parallel with another given line

Consider two straight non-parallel lines

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q}$$

and $\Delta : ax + by + c = 0$ which are not parallel to each other, i.e. $ap + bq \neq 0$. For these given data we may define the projection $p_{\Delta,d} : \pi \to \Delta$ of $\pi$ on $\Delta$ parallel cu $d$, whose value $p_{\Delta,d}$ at $M \in \pi$ is the intersection point between $\Delta$ and the line through $M$ which is parallelt to $d$. Due to relations (4.24), the coordinates of $p_{\Delta,d}(M)$, in terms of the coordinates of $M$ are:

$$\begin{aligned} & x_M - p\frac{F(x_M, y_M)}{ap + bq} \\ & y_M - q\frac{F(x_M, y_M)}{ap + bq}, \end{aligned}$$

where $F(x, y) = ax + by + c$.

Consequently, the position vector of $p_{\Delta,d}(M)$ is

$$\overrightarrow{Op_{\Delta,d}(M)} = \overrightarrow{OM} - \frac{F(M)}{ap + bq}\overrightarrow{d},$$

where $\overrightarrow{d} = p\overrightarrow{e} + g\overrightarrow{f}$.

**Proposition 4.9.** *If $R$ is the Cartesian reference system of the plane $\pi$ behind the equations of the concurrent lines*

$$\Delta : ax + by + c = 0 \text{ and } d : \frac{x - x_0}{p} = \frac{y - y_0}{q},$$

*then*

$$[p_{\Delta,d}(M)]_R = \frac{1}{ap + bq}\begin{pmatrix} bq & -bp \\ -aq & ap \end{pmatrix}[M]_R - \frac{c}{ap + bq}[\overrightarrow{d}]_b. \tag{4.25}$$

### 4.4.3 The symmetry with respect to a line parallel with another line

We call the function $s_{\Delta,d} : \pi \to \pi$, whose value $s_{\Delta,d}$ at $M \in \pi$ is the symmetric point of $M$ with respect to $p_{\Delta,d}(M)$, the *symmetry of $\pi$ with respect to $\Delta$ parallel to $d$*. The direction of $d$ is equally called the direction of the symmetry and $\pi$ is called the *axis of the symmetry*. For the position vector of $s_{\Delta,d}(M)$ we have

$$\overrightarrow{Op_{\Delta,d}(M)} = \frac{\overrightarrow{OM} + \overrightarrow{Os_{\Delta,d}(M)}}{2}, \text{ i.e.}$$

$$\overrightarrow{Os_{\Delta,d}(M)} = 2\overrightarrow{Op_{\Delta,d}(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2\frac{F(M)}{ap+bq}\overrightarrow{d},$$

where $F(x,y) = ax + by + c$. Thus, the coordinates of $s_{\Delta,d}(M)$, in terms of the coordinates of $M$, are

$$\begin{cases} x_M - 2p\dfrac{F(x_M, y_M)}{ap+bq} \\ y_M - 2q\dfrac{F(x_M, y_M)}{ap+bq}. \end{cases}$$

**Proposition 4.10.** *If $R$ is the Cartesian reference system of the plane $\pi$ behind the equations of the concurrent lines*

$$\Delta : ax + by + c = 0 \text{ and } d : \frac{x - x_0}{p} = \frac{y - y_0}{q},$$

*then*

$$[s_{\Delta,d}(M)]_R = \frac{1}{ap+bq}\begin{pmatrix} -ap+bq & -2bp \\ -2aq & ap-bq \end{pmatrix}[M]_R - \frac{2c}{ap+bq}[\overrightarrow{d}]_b. \tag{4.26}$$

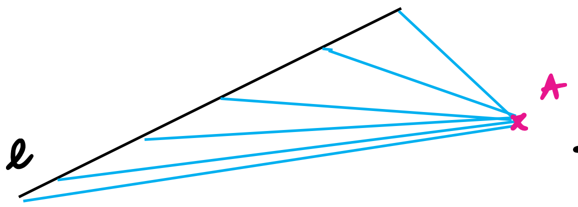
## 4.5 Problems

1. Write the equation of the plane determined by the line

$$(d)\ \begin{cases} x - 2y + 3z = 0 \\ 2x + z - 3 = 0 \end{cases}$$

and the point $A(-1,2,6)$.

SOLUTION.




$\pi_{\alpha,\beta}: \alpha(x-2y+3z)+\beta(2x+z-3)=0$

$\pi_{\alpha\beta}: x(\alpha+2\beta)+y(-2\alpha)+z(3\alpha+\beta)-3\beta=0$

$A \in \pi_{\alpha\beta} \Rightarrow -1(\alpha+2\beta)-4\alpha+6(3\alpha+\beta)-3\beta=0 \Leftrightarrow$

$\Leftrightarrow 13\alpha+\beta=0 \Leftrightarrow \beta=-13\alpha \Rightarrow$ The planes that we want:

$\pi_{\alpha,-13\alpha}: -25\alpha x-2\alpha y-10\alpha z+39\alpha=0 \Rightarrow$

$\Rightarrow \alpha(-25x-2y-10z+39)=0$ / $\alpha \neq 0$ $\Rightarrow$ We have a unique plane:

$\pi_{1,-13}: -25x-2y-10z+39=0$

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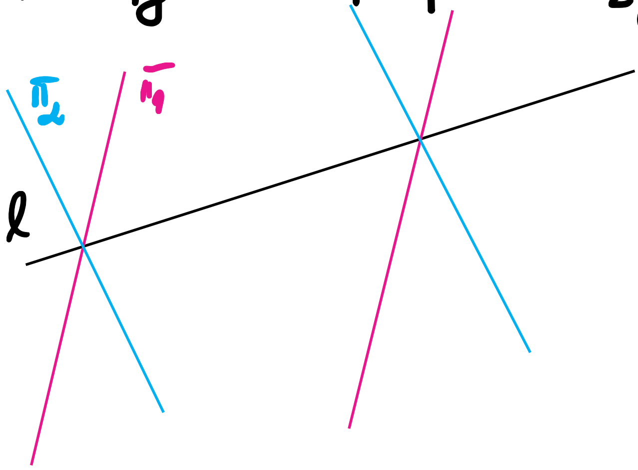
# Pencils of planes

$$\ell : \begin{cases} \pi_1 : A_1x + B_1y + C_1z + D_1 = 0 \\ \pi_2 : A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

$$\pi_{\alpha,\beta} : \alpha(A_1x + B_1y + C_1z + D_1) + \beta(A_2x + B_2y + C_2z + D_2) = 0$$



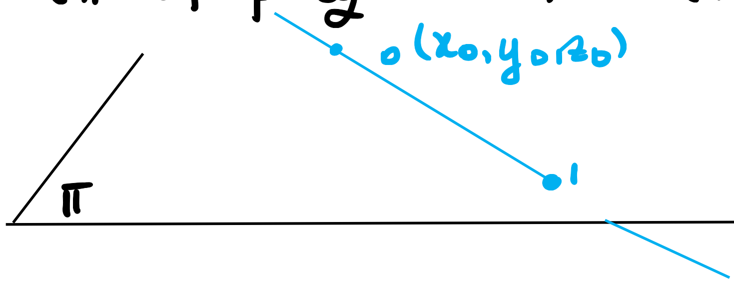

$\pi : Ax + By + Cz + D = 0 \Rightarrow \vec{n_\pi}(A, B, C)$ normal vector of the plane $\pi$

$(\forall \vec{v} \parallel \pi : \vec{n_\pi} \perp \vec{v} \ (\Leftrightarrow \vec{n_\pi} \cdot \vec{v} = 0))$


- if $\ell$ - line, $\ell : \begin{cases} x = x_0 + \lambda p \\ y = y_0 + \lambda q \\ z = z_0 + \lambda r \end{cases}$

$\ell \parallel \pi \Leftrightarrow Ap + Bq + Cr = 0 \Leftrightarrow \vec{n_\pi} \cdot \vec{\ell} = 0$

$\ell \nparallel \pi$ (if $Ap + Bq + Cr \neq 0$), then $\exists M : \{M\} = \ell \cap \pi$




$$\begin{cases} x_M = x_0 - \underbrace{\dfrac{Ax_0 + By_0 + Cz_0 + D}{Ap + Bq + Cr}}_{\lambda_M} \cdot p \\ y_M = y_0 - \dfrac{Ax_0 + By_0 + Cz_0 + D}{Ap + Bq + Cr} \cdot q \\ z_M = z_0 - \dfrac{Ax_0 + By_0 + Cz_0 + D}{Ap + Bq + Cr} \cdot r \end{cases}$$

Ex: $\pi: x + 2y - 5z = 0$

$$l: \frac{x-2}{3} = \frac{y+1}{7} = \frac{z}{-2}$$

Find the intersection point $\{M\} = l \cap \pi$ (without using the formula above)

$$l: \begin{cases} x = 3t+2 \\ y = 7t-1 \\ z = -2t \end{cases}$$

$$M: \begin{cases} l \\ \pi \end{cases} \Leftrightarrow \begin{cases} x = 3t+2 \\ y = 7t-1 \\ z = -2t \\ x + 2y - 5z = 0 \end{cases} \Leftrightarrow \begin{cases} x = 3t+2 \\ y = 7t-1 \\ z = -2t \\ 3t+2+2(7t-1)-5(-2t) = 0 \end{cases} \Leftrightarrow \begin{cases} x = 3t+2 \\ y = 7t-1 \\ z = -2t \\ 27t = 0 \end{cases} \Leftrightarrow$$

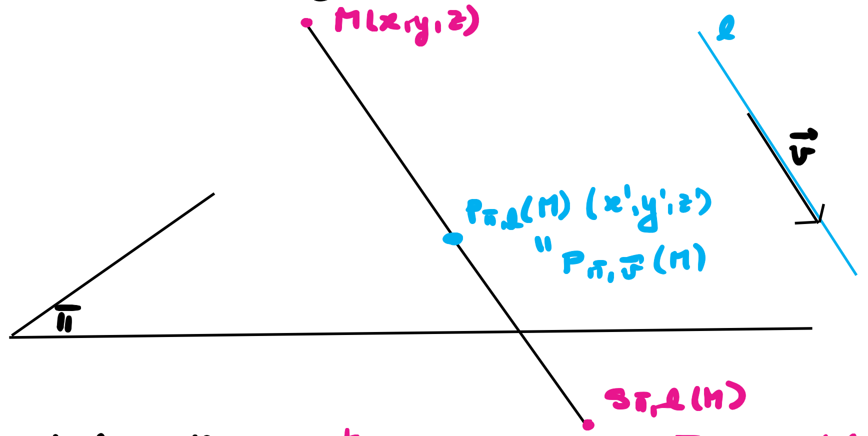
$$\Leftrightarrow \begin{cases} x = 2 \\ y = -1 \\ z = 0 \end{cases}$$

## Projections and reflections

$\pi: Ax + By + Cz + D = 0$

$$l: \begin{cases} x = x_0 + \lambda p \\ y = y_0 + \lambda q \\ z = z_0 + \lambda r \end{cases}$$

$l \nparallel \pi$ (i.e. $Ap + Bq + Cr \neq 0$)




We have the projection on to the plane $\pi$, parallel with the line $l$.

$P_{\pi,l}: \mathbb{R}^3 \to \pi$
$(x,y,z) \mapsto (x',y',z')$

$$\begin{cases} x' = x - \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot p \\[2ex] y' = y - \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot q \\[2ex] z' = z - \dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot r \end{cases}$$

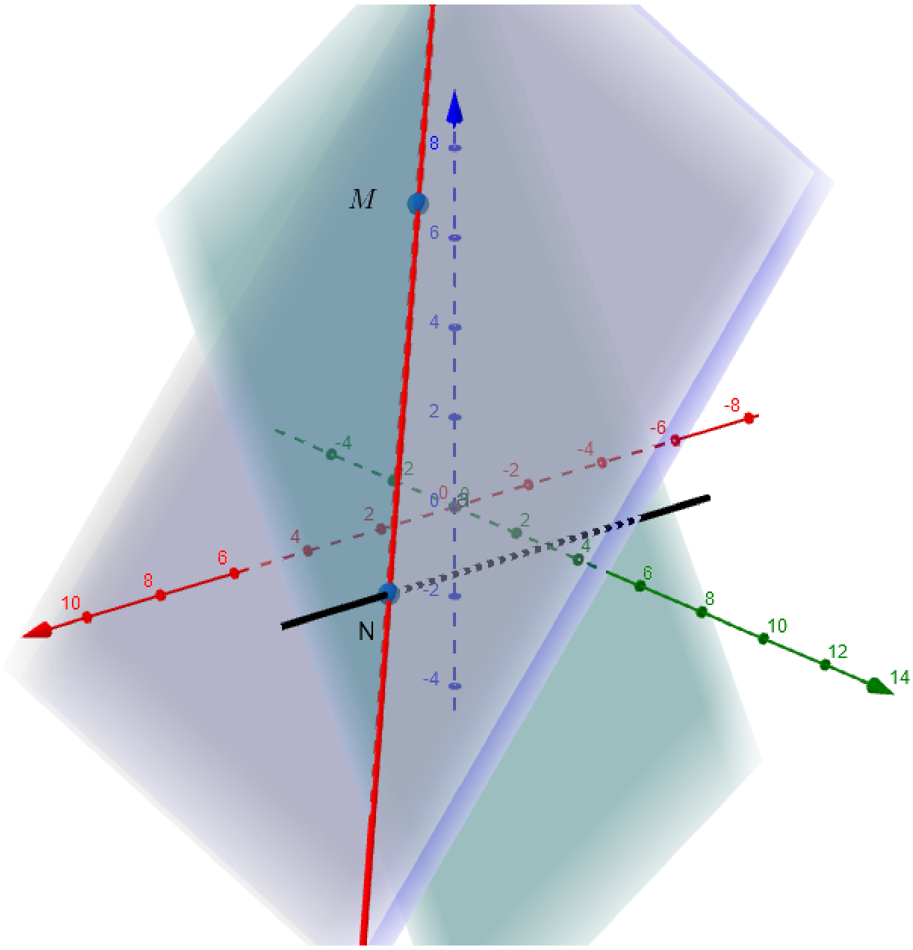
$S_{\pi,l}: \mathbb{R}^3 \to \mathbb{R}^3$
$(x,y,z) \mapsto (x'',y'',z'')$

$$\begin{cases} x'' = x - 2\dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot p \\[2ex] y'' = y - 2\dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot q \\[2ex] z'' = z - 2\dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \cdot r \end{cases}$$

2. Write the equation of the line which passes through the point $M(1,0,7)$, is parallel to the plane $(\pi)\ 3x - y + 2z - 15 = 0$ and intersects the line

$$(d)\ \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

SOLUTION 1. The equation of the plane $\alpha$ passing through the point $M(1,0,7)$, which is parallel to the plane $(\pi)\ 3x - y + 2z - 15 = 0$, is $(\alpha)\ 3(x-1) - (y-0) + 2(z-7) = 0$, i.e. $(\alpha)\ 3x - y + 2z - 17 = 0$.




The parametric equations of the line $d$ are

$$\begin{cases} x = 1 + 4t \\ y = 3 + 2t \\ z = t \end{cases}, \ t \in \mathbb{R}.$$

The coordinates of the intersection point $N$ between the line $(d)$ and the plane $\alpha$ can be obtained by solving the equation $3((1+4t) - (3+2t) + 2t - 17 = 0$. The required line is $MN$.

SOLUTION 2. The required line can be equally regarded as the intersection line between the plane $\alpha$ (passing through the point $M(1,0,7)$, which is parallel to the plane $(\pi)$) and the plane determined by the given line $(d)$ and the point $M$. While the equation $3x - y + 2z - 17 = 0$ of $\alpha$ was already used above, the equation of the plane determined by the line $(d)$ and the point $M$ can be determined via the pencil of planes through

$$(d) \begin{cases} \dfrac{x-1}{4} = \dfrac{y-3}{2} \\ \dfrac{y-3}{2} = \dfrac{z}{1} \end{cases} \Leftrightarrow (d) \begin{cases} x - 2y + 5 = 0 \\ y - 2z - 3 = 0. \end{cases}$$

Note that none of the planes $x - 2y + 5 = 0$ or $y - 2z - 3 = 0$ passes through $M$, which means that the plane determined by $d$ and $M$ is in the reduced pencil of planes

$$(\pi_\lambda)\ x - 2y + 5 = 0 + \lambda(y - 2z - 3) = 0.$$

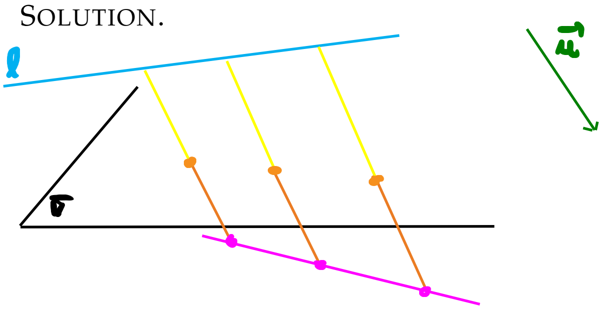
The plane determined by $d$ and $M$ can be found by imposing on the coordinates of $M$ to verify the equation of $\pi_\lambda$.

3. Write the equations of the projection of the line

$$(d)\ \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$$

on the plane $\pi : x + 2y - z = 0$ parallel to the direction $\overrightarrow{u}\,(1, 1, -2)$. Write the equations of the symmetry of the line $d$ with respect to the plane $\pi$ parallel to the direction $\overrightarrow{u}\,(1, 1, -2)$.

SOLUTION.




$$l: \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ x + y - z + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = z - 1 \\ z = z \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = t \\ z = t + 1 \end{cases}$$

$$\begin{cases} x'' = x - 2\dfrac{Ax + By + Cz + D}{Ap + Bq + Cr} \cdot p \\ y'' = y - 2\dfrac{Ax + By + Cz + D}{Ap + Bq + Cr} \cdot q \\ z'' = z - 2\dfrac{Ax + By + Cz + D}{Ap + Bq + Cr} \cdot r \end{cases}$$

$\vec{u} = (p, q, r) = (1, 1, -2)$

$(A, B, C, D) = (1, 2, -1, 0)$

$$\frac{Ax + By + Cz + D}{Ap + Bq + Cr} = \frac{1\cdot 0 + 2t + (-1)(t+1)}{1\cdot 1 + 1\cdot 2 + (-1)(-2)} = \frac{t-1}{5} \Rightarrow \begin{cases} x'' = 0 - 2\dfrac{t-1}{5}\cdot 1 = -\dfrac{2}{5}t + \dfrac{2}{5} \\ y'' = t - 2\dfrac{t-1}{5}\cdot 1 = \dfrac{3}{5}t + \dfrac{2}{5} \\ z'' = t + 1 - 2\cdot\dfrac{t-1}{5}(-2) = \dfrac{9t}{5} + \dfrac{1}{5} \end{cases}$$

$\hookrightarrow$ eq. of a line

4. Prove Proposition 4.7  $R=(O,b)$ - Cartesian reference system behind

SOLUTION.

line $(d): \frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}$  plane $(\pi): Ax+By+Cz+D$, denote $M(x_M,y_M,z_M)$
$p_{\pi,d}(M)(x'_M,y'_M,z'_M)$

We start from $\overrightarrow{O\,p_{\pi,d}M}=\overrightarrow{OM}-\frac{F(M)}{Ap+Bq+Cr}\vec{d}\Rightarrow$

$\Rightarrow$ Written in $R=(O,b)\Rightarrow \begin{pmatrix}x'_M\\y'_M\\z'_M\end{pmatrix}=\begin{pmatrix}x_M\\y_M\\z_M\end{pmatrix}-\frac{Ax_M+By_M+Cz_M+D}{Ap+Bq+Cr}\cdot\begin{pmatrix}p\\q\\r\end{pmatrix}\Leftrightarrow$

$\Leftrightarrow \begin{pmatrix}x'_M\\y'_M\\z'_M\end{pmatrix}=\frac{Ap+Bq+Cr}{Ap+Bq+Cr}\begin{pmatrix}x_M\\y_M\\z_M\end{pmatrix}-\frac{Ax_M+By_M+Cz_M}{Ap+Bq+Cr}\begin{pmatrix}p\\q\\r\end{pmatrix}-\frac{D}{Ap+Bq+Cr}\cdot\begin{pmatrix}p\\q\\r\end{pmatrix}\Leftrightarrow$

$\Leftrightarrow \begin{pmatrix}x'_M\\y'_M\\z'_M\end{pmatrix}=\frac{1}{Ap+Bq+Cr}\cdot\begin{pmatrix}\cancel{Apx_M}+Bqx_M+Crx_M-\cancel{Apx_M}-Bpy_M-Cz_Mp\\ Apy_M+\cancel{Bqy_M}+Cry_M-Aqx_M-\cancel{Bqy_M}-Cz_Mq\\ Apz_M+Bqz_M+\cancel{Crz_M}-Arx_M-Bry_M-\cancel{Crz_M}\end{pmatrix}-\frac{D}{Ap+Bq+Cr}\cdot\begin{pmatrix}p\\q\\r\end{pmatrix}\Leftrightarrow$

$\Leftrightarrow \underbrace{\begin{pmatrix}x'_M\\y'_M\\z'_M\end{pmatrix}}_{[p_{\pi,d}(M)]_R}=\frac{1}{Ap+Bq+Cr}\cdot\begin{pmatrix}Bq+Cr & -Bp & -Cp\\ -Aq & Ap+Cr & -Cq\\ -Ar & -Br & Ap+Bq\end{pmatrix}\cdot\underbrace{\begin{pmatrix}x_M\\y_M\\z_M\end{pmatrix}}_{[M]_R}-\frac{D}{Ap+Bq+Cr}\cdot\underbrace{\begin{pmatrix}p\\q\\r\end{pmatrix}}_{[\vec{d}]_R}$

$*\ [M]_R=[\overrightarrow{OM}]_R=\begin{pmatrix}x_M-x_0\\y_M-y_0\\z_M-z_0\end{pmatrix}=\begin{pmatrix}x_M\\y_M\\z_M\end{pmatrix}$

5. Prove Proposition 4.8

SOLUTION.

$$S_{\pi,d}(M)\begin{cases} x'' = x - 2p\dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \\ y'' = y - 2q\dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \\ z'' = z - 2r\dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \end{cases}$$

$$[S_{\pi,d}(M)]_R = \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} x - 2p\dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \\ y - 2q\dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \\ z - 2r\dfrac{Ax+By+Cz+D}{Ap+Bq+Cr} \end{bmatrix} =$$

$$= \frac{1}{Ap+Bq+Cr}\cdot\begin{bmatrix} Apx+Bqx+Crx-2pAx-2pBy-2pCz-2pD \\ Apy+Bqy+Cry-2pAx-2pBy-2pCz-2qD \\ Apz+Bqz+Crz-2pAx-2pBy-2pCz-2rD \end{bmatrix} =$$

$$= \frac{1}{Ap+Bq+Cr}\cdot\begin{bmatrix} (-Ap+Bq+Cr)x-2pBy-2pCz-2pD \\ -2qAx+(Ap-Bq+Cr)y-2qCz-2qD \\ -2rAx-2rBy+(Ap+Bq-Cr)z-2rD \end{bmatrix} =$$

$$= \frac{1}{Ap+Bq+Cr}\cdot\left(\begin{bmatrix} -Ap+Bq+Cr & -2pB & -2pC \\ -2Aq & Ap-Bq+Cr & -2qC \\ -2Ar & -2rB & Ap+Bq+Cr \end{bmatrix}\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{[M]_R} - 2D\underbrace{\begin{bmatrix} p \\ q \\ r \end{bmatrix}}_{[\vec{d}]_b}\right)$$

6. Show that two different parallel lines are either projected onto parallel lines or on two points by a projection $p_{\pi,d}$, where

$$\pi : Ax + By + Cz + D = 0, \;\; d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and $\pi \not\parallel d$.

SOLUTION.

$\pi \not\parallel d \Rightarrow Ap + Bq + Cr \neq 0$

Let:

$$d_1 : \begin{cases} x = x_1 + t v_x \\ y = y_1 + t v_y \\ z = z_1 + t v_z \end{cases} \qquad d_2 : \begin{cases} x = x_2 + s v_x \\ y = y_2 + s v_y \\ z = z_2 + s v_z \end{cases}$$

$F(x,y,z) := Ax + By + Cz + D$, $\vec{d} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

$$\forall P(x,y,z) \in d_1 \Rightarrow p_{\pi,d}(P) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \frac{F(x,y,z)}{Ap + Bq + Cr} \cdot \vec{d}$$

$$= \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t\vec{v} - \frac{A(x_1 + t v_x) + B(y_1 + t v_y) + C(z_1 + t v_z) + D}{Ap + Bq + Cr} \cdot \vec{d}$$

$$= \underbrace{\begin{pmatrix} x_1 - p\,\dfrac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \\ y_1 - q\,\dfrac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \\ z_1 - r\,\dfrac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \end{pmatrix}}_{\text{eq of } d_1',\; P_1' \in d_1'} + t\left(\vec{v} - \frac{Av_x + Bv_y + Cv_z}{Ap + Bq + Cr} \cdot \vec{d}\right)$$

director vector: $\vec{w} = \vec{v} - \dfrac{Av_x + Bv_y + Cv_z}{Ap + Bq + Cr} \cdot \vec{d}$

Analog:

eq of line $d_2'$: $\begin{pmatrix} x_2 - p\,\dfrac{Ax_2 + By_2 + Cz_2 + D}{Ap + Bq + Cr} \\ y_2 - q\,\dfrac{Ax_2 + By_2 + Cz_2 + D}{Ap + Bq + Cr} \\ z_2 - r\,\dfrac{Ax_2 + By_2 + Cz_2 + D}{Ap + Bq + Cr} \end{pmatrix}$, $P_2' \in d_2'$, $\vec{w} = \vec{v} - \dfrac{Av_x + Bv_y + Cv_z}{Ap + Bq + Cr} \cdot \vec{d}$ $\Bigg\} \Rightarrow$

$\Rightarrow d_1$ and $d_2$ are projected onto the parallel lines $d_1'$ and $d_2'$ if $w$ is nonzero or onto the points $P_1'$ and $P_2'$ if $w$ is the zero vector.

$d_1$ and $d_2$ - projected onto points $\Rightarrow \vec{v} = \dfrac{Av_x + Bv_y + Cv_z}{Ap + Bq + Cr}\,\vec{d} \Leftrightarrow$

$\Leftrightarrow \vec{v} = \dfrac{\vec{n_\pi} \cdot \vec{v}}{\vec{n_\pi} \cdot \vec{d}} \cdot \vec{d} \Leftrightarrow \vec{v} \parallel \vec{d}$

7. Show that two different parallel lines are mapped onto parallel lines by a symmetry $s_{\pi,d}$, where

$$\pi : Ax + By + Cz + D = 0, \;\; d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and $\pi \nparallel d$.

SOLUTION.

$\pi \nparallel d \Rightarrow Ap + Bq + Cr \neq 0$

Let: $d_1 : \begin{cases} x = x_1 + t v_x \\ y = y_1 + t v_y \\ z = z_1 + t v_z \end{cases}$ $\quad d_2 : \begin{cases} x = x_2 + s v_x \\ y = y_2 + s v_y \\ z = z_2 + s v_z \end{cases}$

$F(x,y,z) := Ax + By + Cz + D$, $\vec{d} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

$\forall P(x,y,z) \in d_1$:

$$s_{\pi,d}(P) = 2p_{\pi,d}(P) - \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - 2\frac{F(x,y,z)}{Ap+Bq+Cr} \cdot \vec{d}$$

$$s_{\pi,d}(P) = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t\vec{v} - 2 \cdot \frac{A(x_1 + t v_x) + B(y_1 + t v_y) + C(z_1 + t v_z) + D}{Ap + Bq + Cr} \cdot \vec{d}$$

coordinate vector

$$= \underbrace{\begin{pmatrix} x_1 - 2p\frac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \\ y_1 - 2q\frac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \\ z_1 - 2r\frac{Ax_1 + By_1 + Cz_1 + D}{Ap + Bq + Cr} \end{pmatrix}} + t\left(\vec{v} - 2\frac{Av_x + Bv_y + Cv_z}{Ap + Bq + Cr} \cdot \vec{d}\right)$$

param. eq. of a line $d_1''$ that contain the point $P_1''$ $\Rightarrow \vec{w} = \vec{v} - 2\frac{Av_x + Bv_y + Cv_z}{Ap + Bq + Cr} \cdot \vec{d}$ - director vector

coordinate vector

Similarly, $d_2''$, $P_2'' \in d_2''$ $\begin{pmatrix} x_2 - 2p\frac{Ax_2 + By_2 + Cz_2 + D}{Ap + Bq + Cr} \\ y_2 - 2q\frac{Ax_2 + By_2 + Cz_2 + D}{Ap + Bq + Cr} \\ z_2 - 2r\frac{Ax_2 + By_2 + Cz_2 + D}{Ap + Bq + Cr} \end{pmatrix}$, $\vec{w} = \vec{v} - 2\frac{Av_x + Bv_y + Cv_z}{Ap + Bq + Cr} \cdot \vec{d}$ - director vector

The director vector is the same $\Rightarrow d_1'' \parallel d_2'' \Leftrightarrow w$ is nonzero

If $w$ is zero $\Rightarrow \vec{v} = 2\frac{Av_x + Bv_y + Cv_z}{Ap + Bq + Cr} \cdot \vec{d} \Rightarrow \vec{v}' = 2\frac{\vec{n_\pi}' \cdot \vec{v}'}{\vec{n_\pi}' \cdot \vec{d}'} \cdot \vec{d}'$

$\vec{v}' \parallel \vec{d}'$

$\Rightarrow \cos(\widehat{\vec{n_\pi}', \vec{v}'}) = \cos(\widehat{\vec{n_\pi}', \vec{d}'}) \Rightarrow \frac{\vec{n_\pi}' \cdot \vec{v}'}{\vec{n_\pi}' \cdot \vec{d}'} = \frac{\|\vec{v}'\|}{\|\vec{d}'\|} \Rightarrow \vec{v}' = 2 \cdot \frac{\|\vec{v}'\|}{\|\vec{d}'\|} \cdot \vec{d}' \Rightarrow$

$\Rightarrow \|\vec{v}'\| = 2\frac{\|\vec{v}'\|}{\cancel{\|\vec{d}'\|}} \cdot \cancel{\|\vec{d}'\|} \Rightarrow \|\vec{v}'\| = 2\|\vec{v}'\| \Rightarrow \vec{v}' = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ - the lines $d_1$ and $d_2$ would be just points, contra

$\Rightarrow$ the reflections of $d_1$ and $d_2$ are parallel

8. Assume that $R = (O, b)$ ($b = [\vec{u}, \vec{v}, \vec{w}]$) is the Cartesian reference system behind the equations of a plane $\pi : Ax + By + Cz + D = 0$ and a line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}.$$

If $\pi \not\parallel d$, show that

(a) $\overrightarrow{p_{\pi,d}(M)p_{\pi,d}(N)} = p(\overrightarrow{MN})$, for all $M, N \in \mathcal{V}$, where $p : \mathcal{V} \longrightarrow \mathcal{V}$ is the linear transformation whose matrix representation is

$$[p]_b = \frac{1}{Ap + Bq + Cr} \begin{pmatrix} Bq + Cr & -Bp & -Cp \\ -Aq & Ap + Cr & -Cq \\ -Ar & -Br & Ap + Bq \end{pmatrix}.$$

SOLUTION.

$$d : \begin{cases} x = x_0 + tp \\ y = y_0 + tq \\ z = z_0 + tr \end{cases}, F(x,y,z) = Ax + By + Cz + D$$

$$\vec{r}_{p_{\pi,d}(M)} = \vec{r}_M - \frac{F(M)}{Ap+Bq+Cr} \cdot \vec{d}$$

$$\overrightarrow{p_{\pi,d}(M)\, p_{\pi,d}(N)} = \vec{r}_{p_{\pi,d}(N)} - \vec{r}_{p_{\pi,d}(M)} = \vec{r}_N - \frac{F(N)}{Ap+Bq+Cr}\vec{d} - \vec{r}_M + \frac{F(M)}{Ap+Bq+Cr} \cdot \vec{d} =$$

$$= \overrightarrow{MN} - \frac{1}{Ap+Bq+Cr} \cdot \vec{d}\,(F(N) - F(M))$$

$$F(N) - F(M) = (Ax_N + By_N + Cz_N + D) - (Ax_M + By_M + Cz_M + D) =$$
$$= A(x_N - x_M) + B(y_N - y_M) + C(z_N - z_M) = \vec{n}_\pi \cdot \overrightarrow{MN}$$

$$\overrightarrow{p_{\pi,d}(M)\, p_{\pi,d}(N)} = \overrightarrow{MN} - \frac{\vec{n}_\pi \cdot \overrightarrow{MN}}{\vec{n}_\pi \cdot \vec{d}} \cdot \vec{d} = f(\overrightarrow{MN})$$

$$f : \mathcal{V} \to \mathcal{V} \quad (*)$$
$$\vec{v} \mapsto \vec{v} - \frac{\vec{n}_\pi \cdot \vec{v}}{\vec{n}_\pi \cdot \vec{d}}\, \vec{d}$$

$$\Rightarrow f(e_1) = e_1 - \frac{\vec{n}_\pi \cdot \vec{e}_1}{\vec{n}_\pi \cdot \vec{d}} \cdot \vec{d} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{A}{Ap+Bq+Cr} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} \Rightarrow$$

$$\Rightarrow [f(e_1)]_b = \begin{pmatrix} 1 - \frac{Ap}{Ap+Bq+Cr} \\ -\frac{Aq}{Ap+Bq+Cr} \\ -\frac{Ar}{Ap+Bq+Cr} \end{pmatrix}, \quad [f(e_2)]_b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{B}{Ap+Bq+Cr} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} \Rightarrow$$

$$[f(e_3)]_b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{C}{Ap+Bq+Cr} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$\Rightarrow [f]_b = \left( [f(e_1)]_b \mid [f(e_2)]_b \mid [f(e_3)]_b \right)$. In $(*)$ we should have proven that $f$ is, indeed, a linear map

Proof: $f: V \to V$

$$\vec{v} \mapsto \vec{v} - \frac{\vec{n_\pi} \cdot \vec{v}}{\vec{n_\pi} \cdot \vec{l}} \vec{l}$$

Let $\alpha, \beta \in \mathbb{R}, \vec{v}, \vec{w} \in V$

$$f(\alpha\vec{v} + \beta\vec{w}) = \alpha\vec{v} + \beta\vec{w} - \frac{\vec{n_\pi}(\alpha\vec{v} + \beta\vec{w})}{\vec{n_\pi} \cdot \vec{l}} \cdot \vec{l} =$$

$$= \alpha\vec{v} + \beta\vec{w} - \frac{\alpha(\vec{n_\pi} \cdot \vec{v}) + \beta(\vec{n_\pi} \cdot \vec{w})}{\vec{n_\pi} \cdot \vec{l}} \vec{l} =$$

$$= \left(\alpha\vec{v} - \frac{\alpha \vec{n_\pi} \cdot \vec{v}}{\vec{n_\pi} \cdot \vec{l}} \cdot \vec{l}\right) + \left(\beta\vec{w} - \frac{\beta \vec{n_\pi} \cdot \vec{w}}{\vec{n_\pi} \cdot \vec{l}} \cdot \vec{l}\right) =$$

$$= \alpha f(\vec{v}) + \beta \cdot f(\vec{w})$$

(b) $\overrightarrow{s_{\pi,d}(M)s_{\pi,d}(N)} = s(\overrightarrow{MN})$, for all $M, N \in \mathcal{V}$, where $s : \mathcal{V} \longrightarrow \mathcal{V}$ is the linear transformation whose matrix representation is

$$[s]_b = \frac{1}{Ap+Bq+Cr}\begin{pmatrix} -Ap+Bq+Cr & -2Bp & -2Cp \\ -2Aq & Ap-Bq+Cr & -2Cq \\ -2Ar & -2Br & Ap+Bq-Cr \end{pmatrix}.$$

SOLUTION.

Not: $F(x,y,z) := Ax+By+Cz+D$, $\vec{d} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$$[\overrightarrow{s_{\pi,d}(M)s_{\pi,d}(N)}]_b = [\overrightarrow{r_{s_{\pi,d}(N)}}]_b - [\overrightarrow{r_{s_{\pi,d}(M)}}]_b =$$

$$= \left(\begin{pmatrix} x_N \\ y_N \\ z_N \end{pmatrix} - 2\frac{F(x_N,y_N,z_N)}{Ap+Bq+Cr}\cdot\vec{d}\right) - \left(\begin{pmatrix} x_M \\ y_M \\ z_M \end{pmatrix} - 2\frac{F(x_M,y_M,z_M)}{Ap+Bq+Cr}\cdot\vec{d}\right) =$$

$$= \begin{pmatrix} x_N - x_M \\ y_N - y_M \\ z_N - z_M \end{pmatrix} - \frac{2}{Ap+Bq+Cr}\cdot(F(x_N,y_N,z_N) - F(x_M,y_M,z_M))\cdot\vec{d} =$$

$$= [\overrightarrow{MN}]_b - \frac{2}{Ap+Bq+Cr}\left(A(x_N-x_M)+B(y_N-y_M)+C(z_N-z_M)\right)\cdot\vec{v} =$$

$$= [\overrightarrow{MN}]_b - \frac{2}{Ap+Bq+Cr}\begin{pmatrix} Ap(x_N-x_M)+Bp(y_N-y_M)+Cp(z_N-z_M) \\ Aq(x_N-x_M)+Bq(y_N-y_M)+Cq(z_N-z_M) \\ Ar(x_N-x_M)+Br(y_N-y_M)+Cr(z_N-z_M) \end{pmatrix} =$$

$$= [\overrightarrow{MN}]_b - \frac{2}{Ap+Bq+Cr}\begin{pmatrix} Ap & Bp & Cp \\ Aq & Bq & Cq \\ Ar & Br & Cr \end{pmatrix}\cdot[\overrightarrow{MN}]_b =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}[\overrightarrow{MN}]_b - \frac{2}{Ap+Bq+Cr}\begin{pmatrix} Ap & Bp & Cp \\ Aq & Bq & Cq \\ Ar & Br & Cr \end{pmatrix}\cdot[\overrightarrow{MN}]_b =$$

$$= \frac{1}{Ap+Bq+Cr}\left(\begin{pmatrix} Ap+Bq+Cr & 0 & 0 \\ 0 & Ap+Bq+Cr & 0 \\ 0 & 0 & Ap+Bq+Cr \end{pmatrix} - \begin{pmatrix} 2Ap & 2Bp & 2Cp \\ 2Aq & 2Bq & 2Cq \\ 2Ar & 2Br & 2Cr \end{pmatrix}\right)\cdot[\overrightarrow{MN}]_b =$$

$$= \frac{1}{Ap+Bq+Cr}\begin{pmatrix} -Ap+Bq+Cr & -2Bp & -2Cp \\ -2Aq & Ap-Bq+Cr & -2Cq \\ -2Ar & -2Br & Ap+Bq-Cr \end{pmatrix}\cdot[\overrightarrow{MN}]_b =$$

$$= [s]_b\cdot[\overrightarrow{MN}]_b =$$

$$= [s(\overrightarrow{MN})]_b$$

We have thus shown that $\overrightarrow{s_{\pi,d}(M)s_{\pi,d}(N)} = s(\overrightarrow{MN})$.

9. Consider a plane $\pi : \overset{F(x,y,z)}{Ax + By + Cz + D} = 0$ and a line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}.$$

If $\pi \not\parallel d$, show that

(a) $p_{\pi,d} \circ p_{\pi,d} = p_{\pi,d}$.

(b) $s_{\pi,d} \circ s_{\pi,d} = id_{\mathcal{P}}$.

SOLUTION.

a) $M \in \mathcal{P}$, $M(x_M, y_M, z_M) \Rightarrow x_{p_{\pi,d}(M)} = x_M - p\cdot \frac{F(x_M,y_M,z_M)}{Ap+Bq+Cr}$

$y_{p_{\pi,d}(M)} = y_M - q\cdot \frac{F(x_M,y_M,z_M)}{Ap+Bq+Cr}$

$z_{p_{\pi,d}(M)} = z_M - r\cdot \frac{F(x_M,y_M,z_M)}{Ap+Bq+Cr}$

$x_{p_{\pi,d}(p_{\pi,d}(M))} = x_{p_{\pi,d}(M)} - p\cdot \frac{F(x_{p_{\pi,d}(M)}, y_{p_{\pi,d}(M)}, z_{p_{\pi,d}(M)})}{Ap+Bq+Cr}$

$= x_M - \frac{p}{Ap+Bq+Cr}\cdot\left(\underbrace{F(x_M,y_M,z_M) + F(x_{p_{\pi,d}(M)}, y_{p_{\pi,d}(M)}, z_{p_{\pi,d}(M)})}\right)$

$= A(x_M + x_{p_{\pi,d}(M)}) + B(y_M + y_{p_{\pi,d}(M)}) + C(z_M + z_{p_{\pi,d}(M)}) + 2D =$

$= 2(Ax_M + By_M + Cz_M + D) - \cancel{(Ap+Bq+Cr)}\frac{F(x_M,y_M,z_M)}{\cancel{Ap+Bq+Cr}} = F(x_M,y_M,z_M) \Rightarrow$

$\Rightarrow x_{p_{\pi,d}(p_{\pi,d}(M))} = x_M - \frac{p\cdot F(x_M,y_M,z_M)}{Ap+Bq+Cr} \Rightarrow x_{p_{\pi,d}(p_{\pi,d}(M))} = x_{p_{\pi,d}(M)}$

Similarly: $y_{p_{\pi,d}(p_{\pi,d}(M))} = y_{p_{\pi,d}(M)}$ $\Rightarrow p_{\pi,d}(p_{\pi,d}(M)) = p_{\pi,d}(M) \Rightarrow$

$z_{p_{\pi,d}(p_{\pi,d}(M))} = z_{p_{\pi,d}(M)}$

$\Rightarrow p_{\pi,d} \circ p_{\pi,d} = p_{\pi,d}$

b) $M \in \mathcal{P} \Rightarrow \overrightarrow{Os_{\pi,d}(M)} = 2\overrightarrow{Op_{\pi,d}(M)} - \overrightarrow{OM} \Rightarrow$

$\Rightarrow \overrightarrow{Os_{\pi,d}(s_{\pi,d}(M))} = 2\overrightarrow{Op_{\pi,d}(s_{\pi,d}(M))} - \overrightarrow{Os_{\pi,d}(M)} =$

$= 2\left(\overrightarrow{Op_{\pi,d}(s_{\pi,d}(M))} - \overrightarrow{Op_{\pi,d}(M)}\right) + \overrightarrow{OM}$

$Ms_{\pi,d}(M) \parallel d \Rightarrow p_{\pi,d}(M) = p_{\pi,d}(s_{\pi,d}(M))$, so $\overrightarrow{Os_{\pi,d}(s_{\pi,d}(M))} = \vec{O} \Rightarrow$

$\Rightarrow s_{\pi,d} \circ s_{\pi,d} = id_{\mathcal{P}}$

10. Prove Proposition 4.9.

    SOLUTION.

11. Prove Proposition 4.10.

    SOLUTION.