13 Week 13

13.1 Homogeneous coordinates

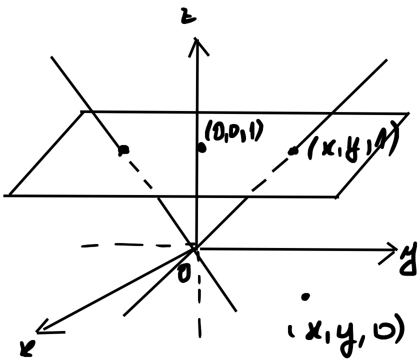
The affine transformation

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ L(x, y) = (ax + by + c, dx + ey + f)$$

can be written by using the matrix language and by equations:

1. (a) indentifying the vectors $(x, y) \in \mathbb{R}^2$ with the line matrices $[x\ y] \in \mathbb{R}^{1\times 2}$ and implicitely $\mathbb{R}^2$ with $\mathbb{R}^{1\times 2}$:
$$L[x\ y] = [x\ y] \begin{bmatrix} a & d \\ b & e \end{bmatrix} + [c\ f].$$
   (b) indentifying the vectors $(x, y) \in \mathbb{R}^2$ with the column matrices $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2\times 1}$ and implicitely $\mathbb{R}^2$ cu $\mathbb{R}^{2\times 1}$:
$$L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}.$$

2. $\begin{cases} x' = ax + by + c \\ y' = dx + ey + f. \end{cases} \Leftrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$



Observe that the representation
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$
$$\Updownarrow$$
is equivalent to
$$\begin{bmatrix} r x' \\ r y' \\ r 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r x \\ r y \\ r 1 \end{bmatrix}.$$

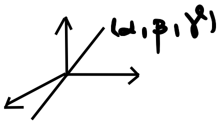
In this lesson we identify the points $(x, y) \in \mathbb{R}^2$ with the points $(x, y, 1) \in \mathbb{R}^3$ and even with the punctured lines of $\mathbb{R}^3$, (rx, ry, r), $r \in \mathbb{R}^*$. Due to technical reasons we shall actually identify the points $(x, y) \in \mathbb{R}^2$ with the punctured lines of $\mathbb{R}^3$ represented in the form

$\mathbb{R}^* = \mathbb{R} \setminus \{0\}$
$$\begin{bmatrix} rx \\ ry \\ r \end{bmatrix},\ r \in \mathbb{R}^*,$$
$\mathbb{R}^3 \setminus \{O(0,0,0)\}$

and the latter ones we shall call *homogeneous coordinates* of the point $(x, y) \in \mathbb{R}^2$. The set of homogeneous coordinates (x, y, w) will be denoted by $\mathbb{RP}^2$ and call it the real *projective plane*. The homogeneous coordinates $(x, y, w) \in \mathbb{RP}^2$, $w \neq 0$ şi $\left(\frac{x}{w}, \frac{y}{w}, 1\right)$represent the same element of $\mathbb{RP}^2$.

Remark 13.1. The projective plane $\mathbb{RP}^2$ is actually the quotient set $(\mathbb{R}^3 \setminus \{0\})/\sim$, where $'\sim'$ is the following equivalence relation on $\mathbb{R}^3 \setminus \{0\}$:

$$(x, y, w) \sim (\alpha, \beta, \gamma) \Leftrightarrow \exists r \in \mathbb{R}^* \text{ a.î. } (x, y, w) = r(\alpha, \beta, \gamma).$$



$[(x,y,z)] = \{(rx, ry, rz) \mid r \in \mathbb{R}^*\}$

Observe that the equivalence classes of the equivalence relation $\sim'$ are the punctured lines of $\mathbb{R}^3$ through the origin without the origin itself, i.e. the elements of the real projective plane $\mathbb{RP}^2$. By the column matrix

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

we also denote the equivalence class of $(x, y, w) \in \mathbb{R}^3 \setminus \{0\}$. The meaning of this notation will be understood, each time, from the context.

Definition 13.1. A *projective transformation* of the projective plane $\mathbb{RP}^2$ is a transformation

$L\left(\lambda\begin{pmatrix}x\\y\\z\end{pmatrix}\right)=\lambda L\begin{pmatrix}x\\y\\z\end{pmatrix}$ $L:\mathbb{R}^3\to\mathbb{R}^3$

$$L:\mathbb{RP}^2\longrightarrow\mathbb{RP}^2,\ L\begin{bmatrix}x\\y\\w\end{bmatrix}=\begin{bmatrix}a&b&c\\d&e&f\\g&h&k\end{bmatrix}\begin{bmatrix}x\\y\\w\end{bmatrix}=\begin{bmatrix}ax+by+cw\\dx+ey+fw\\gx+hy+kw\end{bmatrix},\tag{13.1}$$

where $a,b,c,d,e,f,g,h,k\in\mathbb{R}$. Note that

$$\begin{bmatrix}a&b&c\\d&e&f\\g&h&k\end{bmatrix}$$

is called the *homogeneous transformation matrix* of L. Note that every nonsingular 3×3 matrix defines a projective transformation. Also, if λ is a nonzero real scalar, then the nonsingular matrices

$$\begin{bmatrix}a&b&c\\d&e&f\\g&h&k\end{bmatrix}\text{ and }\lambda\cdot\begin{bmatrix}a&b&c\\d&e&f\\g&h&k\end{bmatrix}$$

define the same projective transformation $\mathbb{RP}^2$. Therefore, the homogeneous transformation matrix of a projective transformation is an entire class, appearing as an element, of a projective space rather than one single matrix.

Observe that a projective transformation (14.2) is well defined since

$$L\begin{bmatrix}rx\\ry\\rw\end{bmatrix}=\begin{bmatrix}arx+bry+crw\\drx+ery+frw\\grx+hry+krw\end{bmatrix}=\begin{bmatrix}r(ax+by+cw)\\r(dx+ey+fw)\\r(gx+hy+kw)\end{bmatrix}.$$

If $g=h=0$ and $k\neq 0$, then the projective transformation (14.2) is said to be *affine*. The restriction of the affine transformation (14.2), which corresponds to the situation $g=h=0$ and $k=1$, to the subspace $w=1$, has the form

$$L\begin{bmatrix}x\\y\\1\end{bmatrix}=\begin{bmatrix}a&b&c\\d&e&f\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\1\end{bmatrix}=\begin{bmatrix}ax+by+cw\\dx+ey+fw\\1\end{bmatrix},\tag{13.2}$$

i.e. $L^c\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}a&b\\d&e\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}c\\f\end{bmatrix}$

$$\begin{cases}x'=ax+by+c\\y'=dx+ey+f.\end{cases}\tag{13.3}$$

Remark 13.2. If $L_1,L_2:\mathbb{RP}^2\longrightarrow\mathbb{RP}^2$ are two projective applications, then their product (concatenation) transformation $L_1\circ L_2$ is also a projective transformation and its homogeneous transformation matrix is the product of the homogeneous transformation matrices of L_1 and L_2 .

Indeed, if

$$L_1\begin{bmatrix}x\\y\\w\end{bmatrix}=\begin{bmatrix}a_1&b_1&c_1\\d_1&e_1&f_1\\g_1&h_1&k_1\end{bmatrix}\begin{bmatrix}x\\y\\w\end{bmatrix}$$

and

$$L_2\begin{bmatrix}x\\y\\w\end{bmatrix}=\begin{bmatrix}a_2&b_2&c_2\\d_2&e_2&f_2\\g_2&h_2&k_2\end{bmatrix}\begin{bmatrix}x\\y\\w\end{bmatrix}$$

then

$$(L_1\circ L_2)\begin{bmatrix}x\\y\\w\end{bmatrix}=\left(\begin{bmatrix}a_1&b_1&c_1\\d_1&e_1&f_1\\g_1&h_1&k_1\end{bmatrix}\begin{bmatrix}a_2&b_2&c_2\\d_2&e_2&f_2\\g_2&h_2&k_2\end{bmatrix}\right)\begin{bmatrix}x\\y\\w\end{bmatrix}$$

Remark 13.3. If $L_1,L_2:\mathbb{RP}^2\longrightarrow\mathbb{RP}^2$ are two affine applications, then their product $L_1\circ L_2$ is also an affine transformation.

13.2 Transformations of the plane in homogeneous coordinates

In this section we shall identify an affine transformation of $\mathbb{RP}^2$ with its homogeneous transformation matrix

13.3 Translations and scalings

• The homogeneous transformation matrix of the translation $T(h,k)$ is

$$T(h,k) = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}.$$

• The homogeneous transformation matrix of the scaling $S(s_x, s_y)$ is

$$S(s_x, s_y) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

13.4 Reflections

• The homogeneous transformation matrix of reflection r_x about the x-axis is

$$r_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

• The homogeneous transformation matrix of reflection r_y about the y-axis is

$$r_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

• The homogeneous transformation matrix of reflection r_l about the line $l : ax + by + c = 0$ is

$$r_l = \begin{bmatrix} \frac{b^2 - a^2}{a^2 + b^2} & -\frac{2ab}{a^2 + b^2} & -\frac{2ac}{a^2 + b^2} \\ -\frac{2ab}{a^2 + b^2} & \frac{a^2 - b^2}{a^2 + b^2} & -\frac{2bc}{a^2 + b^2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Since in homogeneous coordinates multiplication by a factor does not affect the result, the above matrix can be multiplied by a factor $a^2 + b^2$ to give the homogeneous matrix of a general reflection

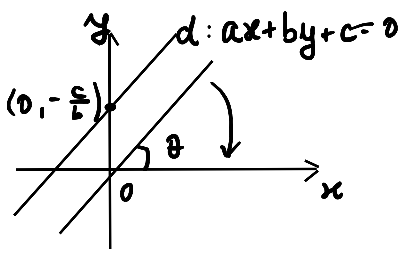
$$\begin{bmatrix} b^2 - a^2 & -2ab & -2ac \\ -2ab & a^2 - b^2 & -2bc \\ 0 & 0 & a^2 + b^2 \end{bmatrix}.$$

Example 13.1. Consider a line (d) $ax + by + c$ whose slope is $\mathrm{tg}\theta = -\frac{a}{b}$. By using the observation that the reflection r_d in the line d is the following concatenation (product)

$$r_d = T(0, -c/b) \circ R_\theta \circ r_x \circ R_{-\theta} \circ T(0, c/b),$$

one can show that the homogeneous transformation matrix of r_d is

$$\begin{bmatrix} \frac{b^2 - a^2}{a^2 + b^2} & -\frac{2ab}{a^2 + b^2} & -\frac{2ac}{a^2 + b^2} \\ -\frac{2ab}{a^2 + b^2} & \frac{a^2 - b^2}{a^2 + b^2} & -\frac{2bc}{a^2 + b^2} \\ 0 & 0 & 1 \end{bmatrix}.$$



Solution. The homogeneous matrix of the concatenation

$$T(0,-c/b)\circ R_\theta\circ r_x\circ R_{-\theta}\circ T(0,c/b)$$

is

$$\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & -c/b\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & c/b\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta-\sin^2\theta & 2\sin\theta\cos\theta & 2\frac{c}{b}\sin\theta\cos\theta\\ 2\sin\theta\cos\theta & \sin^2\theta-\cos^2\theta & \frac{c}{b}\left(\sin^2\theta-\cos^2\theta-1\right)\\ 0 & 0 & 1\end{bmatrix}. \tag{13.4}$$

Since $\mathrm{tg}\theta=-\frac{a}{b}$, it follows that $\frac{a^2}{b^2}=\frac{\sin^2\theta}{\cos^2\theta}=\frac{\sin^2\theta}{1-\sin^2\theta}=\frac{1-\cos^2\theta}{\cos^2\theta}$, namely

$$\sin^2\theta=\frac{a^2}{a^2+b^2}\ \text{and}\ \cos^2\theta=\frac{b^2}{a^2+b^2}.$$

Thus

$$\sin\theta=\pm\frac{a}{\sqrt{a^2+b^2}}\ \text{and}\ \cos\theta=\mp\frac{b}{\sqrt{a^2+b^2}},\ \text{as}\ \frac{\sin\theta}{\cos\theta}=\mathrm{tg}\theta=-\frac{a}{b}.$$

Therefore $\sin\theta\cos\theta=-\frac{ab}{a^2+b^2}$ and the matrix (13.4) becomes

$$\begin{bmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} & -\frac{c}{b}\frac{2ab}{a^2+b^2}\\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & \frac{c}{b}\left(\frac{a^2-b^2}{a^2+b^2}-1\right)\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} & -\frac{2ac}{a^2+b^2}\\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & -\frac{2bc}{a^2+b^2}\\ 0 & 0 & 1\end{bmatrix}.$$

13.5 Rotations

The homogeneous transformation matrix of the rotation R_θ about the origin through an angle θ is

$$R_\theta=\begin{bmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Example 13.2. The homogeneous transformation matrix of the product (concatenation) $T(h,k)\circ R_\theta$ is the product

$$\begin{bmatrix} 1 & 0 & h\\ 0 & 1 & k\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix} \cos\theta & -\sin\theta & h\\ \sin\theta & \cos\theta & k\\ 0 & 0 & 1\end{bmatrix}.$$

In order to find the homogeneous transformation matrix of the inverse transformation

$$(T(h,k)\circ R_\theta)^{-1}=R_\theta^{-1}\circ T(h,k)^{-1}=R_{-\theta}\circ T(-h,-k)$$

of the product (concatenation) homogeneous transformation $T(h,k)\circ R_\theta$ we can either multiply the homogeneous transformation matrices of the inverse transformations $R_\theta^{-1}=R_\theta^{-1}$ and $T(h,k)^{-1}=$

$T(-h,-k)$ or use the next proposition. The product of the homogeneous transformation matrices of the inverse transformations $R_\theta^{-1} = R_\theta^{-1}$ and $T(h,k)^{-1} = T(-h,-k)$ is

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix} =$$
$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix} =$$
$$= \begin{bmatrix} \cos\theta & \sin\theta & -h\cos\theta - k\sin\theta \\ -\sin\theta & \cos\theta & h\sin\theta - k\cos\theta \\ 0 & 0 & 1 \end{bmatrix}.$$

Proposition 13.1. *A homogeneous transformation L is invertible if and only if its homogeneous transformation matrix, say T, is invertible and T^{-1} is the transformation matrix of L^{-1}.*

Proof. Suppose that L has an inverse L^{-1} with transformation matrix T_1. The product transformation $L \circ L^{-1} = id$ has the transformation matrix $TT_1 = I_3$. Similarly, $L^{-1} \circ L = I_3$ has the transformation matrix $T_1 T = I_3$. Thus $T_1 = T^{-1}$. Conversely, assume that T has an inverse T^{-1}, and let L_1 be the homogeneous transformation defined by T^{-1}. Since $TT^{-1} = I_3$ and $T^{-1}T = I_3$, it follows that $L \circ L_1 = I$ and $L_1 \circ L = I$. Hence L_1 is the inverse transformation of L.

Example 13.3. The homogeneous transformation matrix of inverse

$$(T(h,k) \circ R_\theta)^{-1} = R_\theta^{-1} \circ T(h,k)^{-1} = R_{-\theta} \circ T(-h,-k)$$

of the product (concatenation) homogeneous transformation $T(h,k) \circ R_\theta$ is the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta & h \\ \sin\theta & \cos\theta & k \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & -h\cos\theta - k\sin\theta \\ -\sin\theta & \cos\theta & h\sin\theta - k\cos\theta \\ 0 & 0 & 1 \end{bmatrix}.$$

13.6 Shears

The homogeneous transformation matrix of the shear is

$$[Sh(v,r)] = \begin{bmatrix} 1 - rv_1v_2 & rv_1^2 & 0 \\ -rv_2^2 & 1 + rv_1v_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

13.7 Problems

1. Find the concatenation (product) of an anticlockwise rotation about the origin through an angle of $\frac{3\pi}{2}$ followed by a scaling by a factor of 3 units in the x-direction and 2 units in the y-direction. (Hint: $S(3,2)R_{3\pi/2}$)

   Solution

2. Find the homogeneous matrix of the product (concatenation) $S(3,2) \circ R_{\frac{3\pi}{2}}$.

   Solution

3. Find the equations of the rotation $R_\theta(x_0, y_0)$ about the point $M_0(x_0, y_0)$ through an angle θ.

   Solution The homogeneous transformation matrix of the rotation $R_\theta(x_0, y_0)$ about the point $M_0(x_0, y_0)$ through an angle θ is

$$\begin{aligned} R_\theta(x_0, y_0) &= T(x_0, y_0) R_\theta T(-x_0, -y_0) \\ &= \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta & -x_0\cos\theta + y_0\sin\theta + x_0 \\ \sin\theta & \cos\theta & -x_0\sin\theta - y_0\cos\theta + y_0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

   Thus, the equations of the required rotation are:

$$\begin{cases} x' &= x\cos\theta - y\sin\theta - x_0\cos\theta + y_0\sin\theta + x_0 \\ y' &= x\sin\theta + y\cos\theta - x_0\sin\theta - y_0\cos\theta + y_0. \end{cases}.$$

4. Show that the concatenation (product) of two rotations, the first through an angle θ about a point $P(x_0, y_0)$ and the second about a point $Q(x_1, y_1)$ (distinct from P) through an angle $-\theta$ is a translation.

   Solution