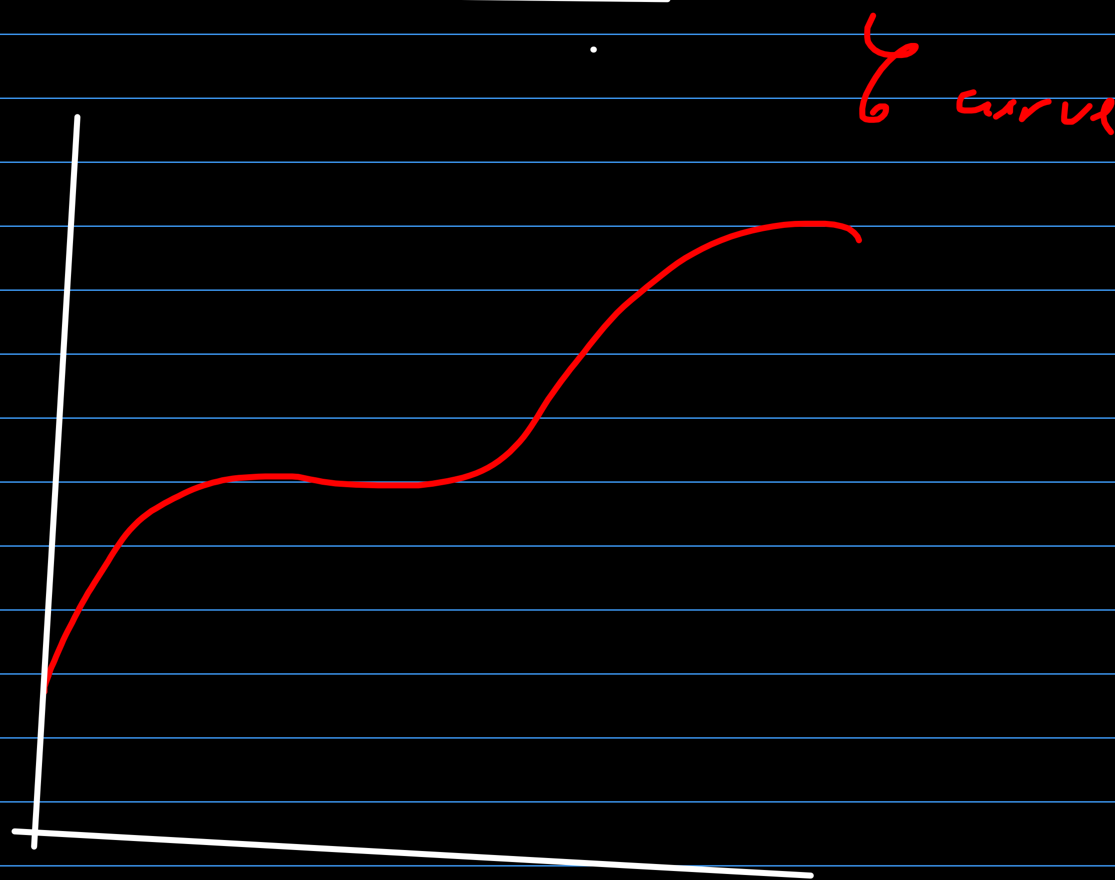
Seminar nr 8 − 913

Curves



- $\mathcal{C}$ given parametrically:

$$\mathcal{C}: \quad \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \iff \gamma = \gamma(t)$$

· $\mathcal{C}$ given implicitly:

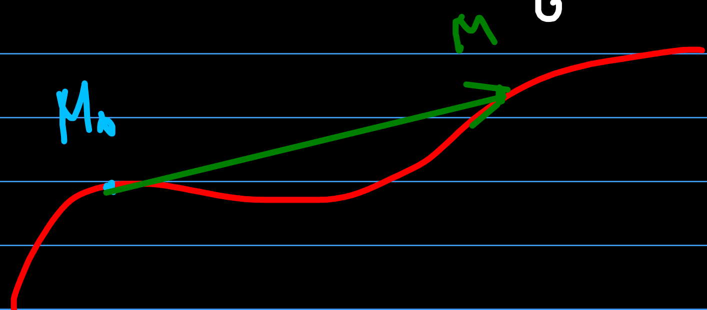
$$\mathcal{C}: \; f(x,y) = 0 \quad \text{(in 2D)}$$

$$\begin{cases} f_1(x,y,z) = 0 \\ f_2(x,y,z) = 0 \end{cases} \quad \text{(in 3D)}$$

$\mathcal{C}$ curve

The tangent at $\mathcal{C}$ in the point M_0 is a line that contains M_0 and has direction given by

$$\vec{u} = \lim_{\substack{M \to M_0 \\ M \in \mathcal{C}}} \frac{\overrightarrow{M_0 M}}{\|\overrightarrow{M_0 M}\|}$$



$$\boxed{\bullet \text{ if } \mathscr{C} \text{ is given as } \quad \gamma = \gamma(t) = (x(t), y(t), z(t))}$$

$$\Rightarrow T_{\mathscr{C}}(t=t_0) : \frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)}$$

$$\lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} = \gamma'(t_0)$$

<u>The normal line (for the 2D case)</u>

$$\Rightarrow N_{\mathscr{C}}(t=t_0) :$$

$$y'(t_0) \cdot (x - x(t_0)) + x'(t_0) \cdot (y - y(t_0)) = 0$$

The normal plane (for the 3D case)

$N_{\mathcal{C}}(t=t_0)$:

$$x'(t_0)\cdot(x - x(t_0)) + y'(t_0)\cdot(y - y(t_0)) + z'(t_0)\cdot(z - z(t_0)) = 0$$

· if $\mathcal{C}$ is given implicitly: (planar curve)

$$f(x,y) = 0$$

$\Rightarrow$ $T_{\mathcal{C}}(x_0, y_0)$: $f'_x(x_0, y_0)\cdot(x - x_0) + f'_y(x_0, y_0)\cdot(y - y_0) = 0$

$N_{\mathcal{C}}(x_0, y_0)$: $\dfrac{x - x_0}{f'_x(x_0, y_0)} = \dfrac{y - y_0}{f'_y(x_0, y_0)}$

81. Show that the angle between the tangent of the circular helix

$$\gamma : \begin{cases} x = a\cos t \\ y = a\sin t \quad , t \in \mathbb{R} \\ z = bt \end{cases}$$

and the z-axis is constant

$$T_\gamma : \frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)}$$

$x'(t_0) = -a\sin t_0$

$y'(t_0) = a\cos t_0$

$z'(t_0) = b$

$$\Rightarrow T_6 : \frac{x - a\cos t_0}{-a\sin t_0} = \frac{y - a\sin t_0}{a\cos t_0} = \frac{z - bt_0}{b}$$

$$\Rightarrow \overrightarrow{u}_{T_6} \left(-a\sin t_0, a\cos t_0, b\right)$$

$Oz: x = y = 0$

$$\Rightarrow \overrightarrow{u}_{Oz} = (0, 0, 1)$$

$$\Rightarrow \cos \widehat{(T_6, Oz)} = \frac{\overrightarrow{u}_{T_6} \cdot \overrightarrow{u}_{Oz}}{\|\overrightarrow{u}_{T_6}\| \cdot \|\overrightarrow{u}_{Oz}\|} =$$

$$= \frac{b}{\sqrt{a^2\sin^2 t_0 + a^2\cos^2 t_0 + b^2}} =$$

$$= \frac{b}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow m(\widehat{T_{t_0}, Oz}) = \arccos \frac{b}{\sqrt{a^2+b^2}}$$

does not depend on t_0 $\Rightarrow$

$\Rightarrow$ it is constant.

8.8. Write the equations of the tangent line and the normal plane for the following curves

$$\begin{cases} x = e^t \cos 3t \\ y = e^t \sin 3t \\ z = e^{-2t} \end{cases}$$ at

the points corresponding to the values $t=0$ and $t = \frac{\pi}{4}$ of the param

$x(0)=1 \quad , \quad y(0)=0 \, , \quad z(0)=1$

$x'(t) = e^{t}\cos 3t - 3e^{t}\sin 3t$

$\Rightarrow x'(0) = 1$

$y'(t) = e^{t}\sin(3t) + 3\cdot e^{t}\cos 3t$

$\Rightarrow y'(0) = 3$

$z'(t) = -2e^{-2t}$

$\Rightarrow z'(0) = -2$

$$\Rightarrow T_{\varphi}(t=0): \frac{x - x(0)}{x'(0)} = \frac{y - y(0)}{y'(0)} = \frac{z - z(0)}{z'(0)}$$

$$\Rightarrow T_{\varphi}(t=0) = \frac{x-1}{1} = \frac{y}{3} = \frac{z-1}{-2}$$

$$\Rightarrow N_{\varphi}(t=0): x'(0)\cdot(x - x(0)) + $$
$$+ y'(0)\cdot(y - y(0)) + z'(0)\cdot(z - z(0)) = 0$$

$$\Rightarrow (x-1) + 3\cdot y + (-2)\cdot(z-1) = 0$$

8.? Write the equation of the tangent line and the normal line at the point $P(0,1)$ of the curve:

$$\mathcal{C}: \quad x^3 + x^2 y - y + 1 = 0$$

$$f(x,y) = x^3 + x^2 y - y + 1$$

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy$$

$$\frac{\partial f}{\partial y} = x^2 - 1$$

$$\frac{\partial f}{\partial x}(0,1) = 0 \quad , \quad \frac{\partial f}{\partial y}(0,1) = -1$$

$$T_{\mathcal{C}}(x_0, y_0): \frac{\partial f}{\partial x}(x_0, y_0)\cdot(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)\cdot(y - y_0)$$

$$= 0$$

$$\Rightarrow T_{\mathcal{C}}(x_0, y_0): (-1)\cdot(y-1) = 0 \Leftrightarrow$$

$$\Leftrightarrow y = 1$$

$$N_{\mathcal{C}}(x_0, y_0): \quad x = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 0 + t\cdot 0 \\ y = 1 + t\cdot(-1) \end{cases}$$

Surfaces

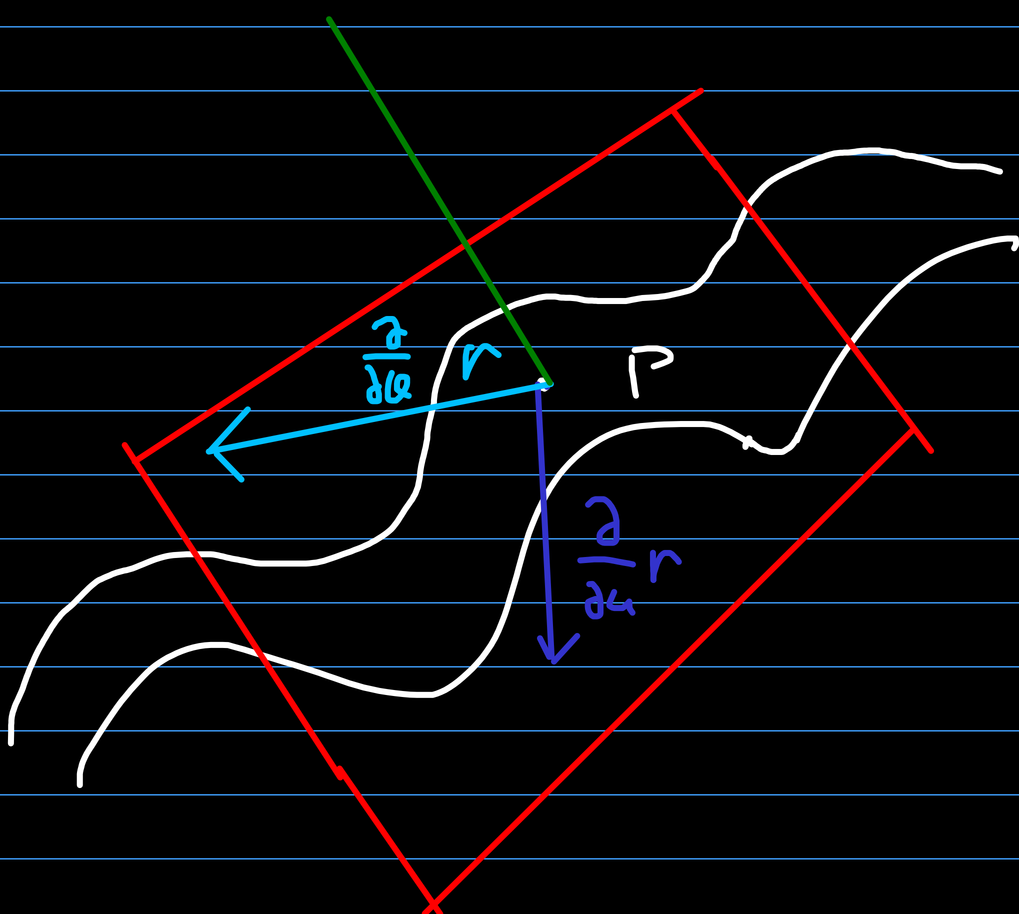
Parametric case:

$$\mathcal{S}: \quad r = r(u,v) \Leftrightarrow \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

Ex.: S sphere (centered in O with radius)

$$\begin{cases} x = \cos u \cdot \cos v \\ y = \cos u \cdot \sin v \\ z = \sin u \end{cases}$$

$$u \in [-\pi, \pi], \quad v \in [0, 2\pi]$$



$T_y\ (u = u_0,\ v = v_0)$:

$$\begin{vmatrix} x - x(u_0, v_0) & y - y(u_0, v_0) & z - z(u_0, v_0) \\ \frac{\partial}{\partial u} x(u_0, v_0) & \frac{\partial}{\partial u} y(u_0, v_0) & \frac{\partial}{\partial u} z(u_0, v_0) \\ \frac{\partial}{\partial v} x(u_0, v_0) & \frac{\partial}{\partial v} y(u_0, v_0) & \frac{\partial}{\partial v} z(u_0, v_0) \end{vmatrix}$$

$$\vec{n} = \frac{\partial}{\partial u} r \times \frac{\partial}{\partial v} r =$$

$$= (A, B, C)$$

$$\Rightarrow T_y\ (u_0, v_0) : A(x - x(u_0, v_0)) +$$

$$+ B \cdot (y - y(u_0, v_0)) + C(z - z(u_0, v_0)) =$$

$$= 0$$

The normal line of the surface y at $P =$

= the line that contains p and is perpendicular to the tangent plane

$\Rightarrow N_{\mathcal{Y}}(u_0, v_0)$:

$$\frac{x - x(u_0, v_0)}{A} = \frac{y - y(u_0, v_0)}{B} = \frac{z - z(u_0, v_0)}{C}$$

<u>The implicit case:</u>

$$\mathcal{Y} : \quad f(x, y, z) = 0$$

$\Rightarrow T_{\mathcal{Y}}(x_0, y_0, z_0)$:

$$f'_x(x_0, y_0, z_0) \cdot (x - x_0) + f'_y(x_0, y_0, z_0) \cdot (y - y_0) + f'_z(x_0, y_0, z_0) \cdot (z - z_0) = 0$$

$N_y(x_0, y_0, z_0)$:

$$\frac{x - x_0}{f'_x} = \frac{y - y_0}{f'_y} = \frac{z - z_0}{f'_z}$$

8.4. Write the equations of the tangent planes of the hyperboloid of one sheet

$$\mathcal{H}: x^2 + y^2 - z^2 = 1$$

at the points of the form $(x_0, y_0, 0)$ and show that they are parallel to the z-axis.

$$f(x,y,z) = x^2 + y^2 - z^2 = 1$$

$$\frac{\partial f}{\partial x} = 2x \Rightarrow \frac{\partial f}{\partial x}(x_0, y_0, 0) = 2x_0$$

$$\frac{\partial f}{\partial y} = 2y \Rightarrow \frac{\partial f}{\partial y}(x_0, y_0, 0) = 2y_0$$

$$\frac{\partial f}{\partial z} = -2z \Rightarrow \frac{\partial f}{\partial z}(x_0, y_0, 0) = 0$$

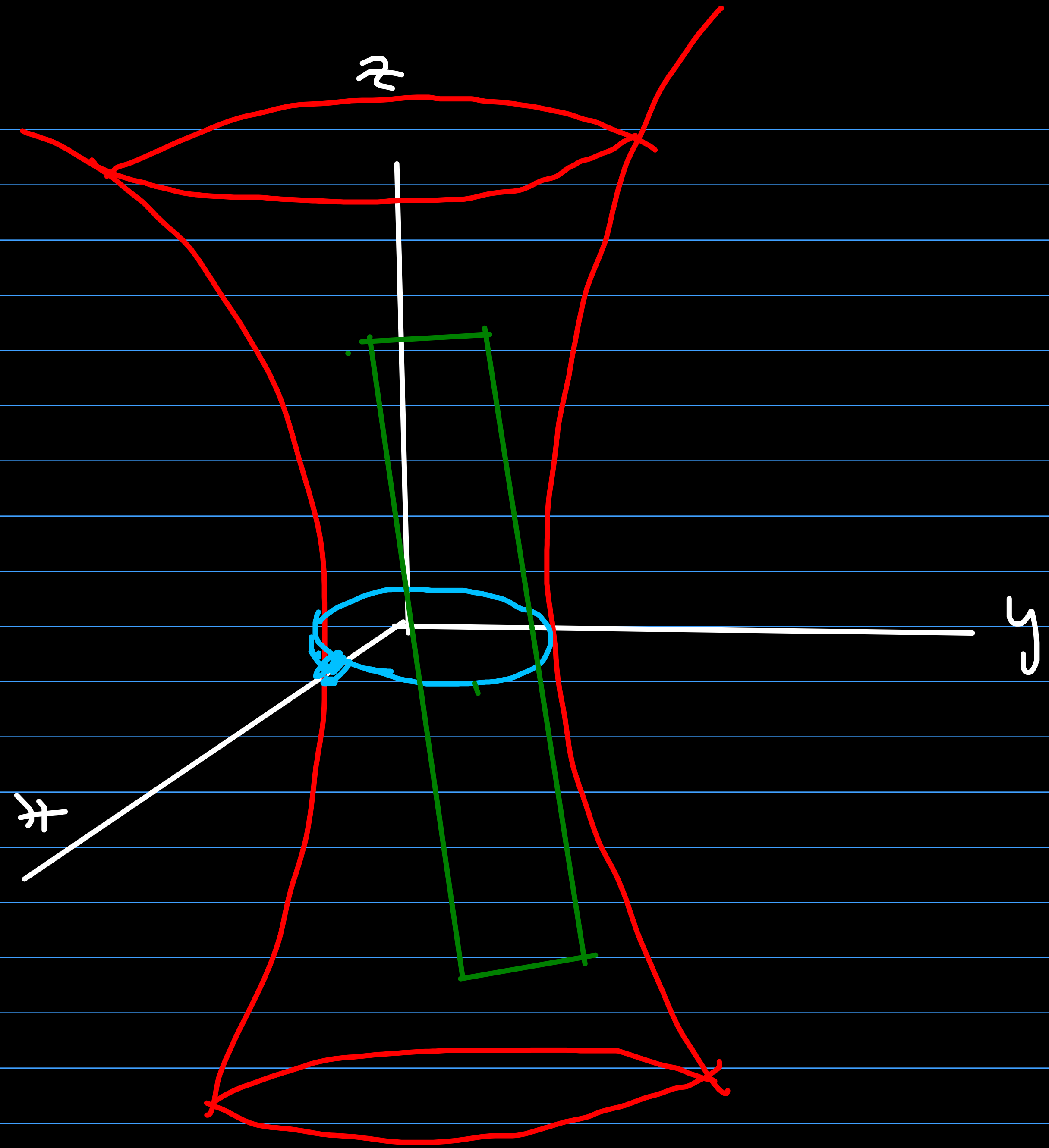
$$\Rightarrow T_{\mathcal{H}} : \quad 2x_0(x - x_0) + 2y_0(y - y_0) = 0$$

$$\Leftrightarrow \quad x_0(x - x_0) + y_0(y - y_0) = 0$$

$$\overrightarrow{n_{T_{\mathcal{H}}}}(x_0, y_0, 0)$$

$$\overrightarrow{u_{Oz}}(0, 0, 1)$$

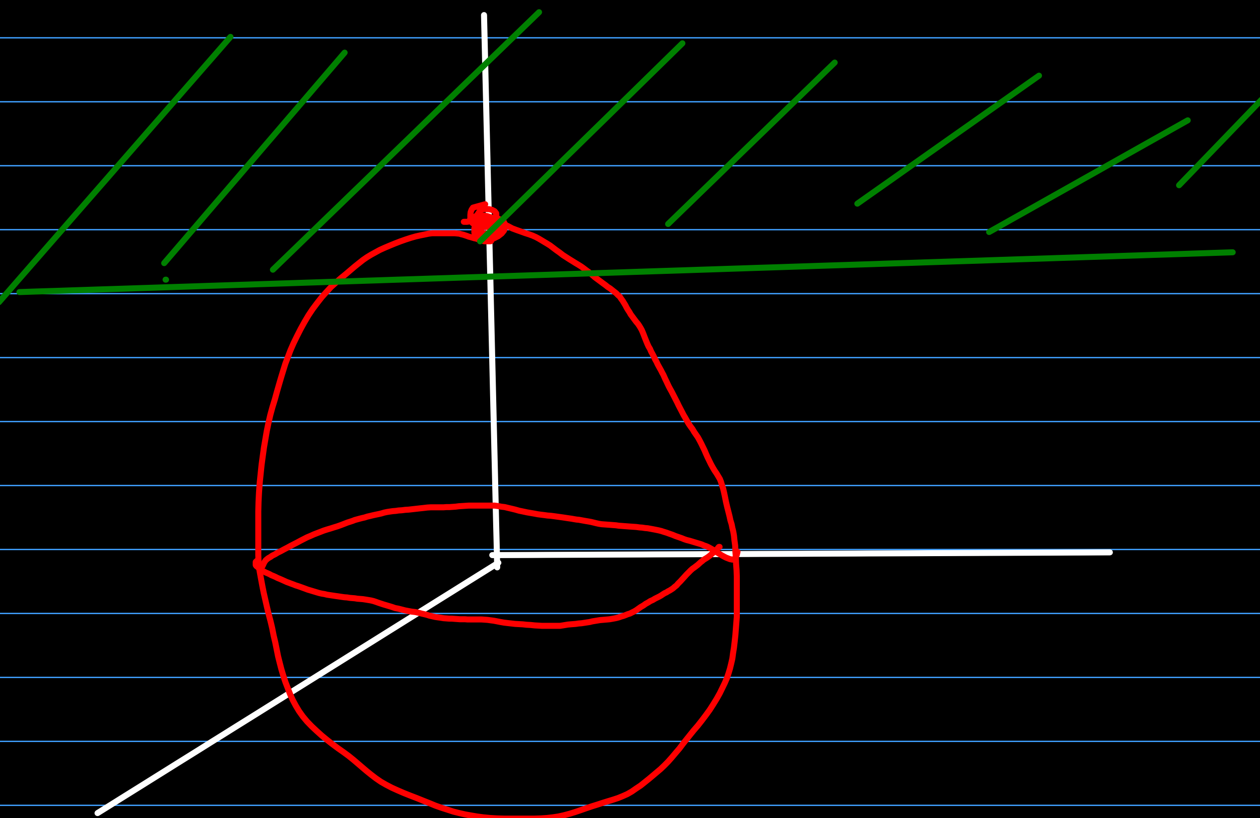
$$\overrightarrow{n_{T_{\mathcal{H}}}} \cdot \overrightarrow{u_{Oz}} = 0 \Rightarrow T_{\mathcal{H}} \parallel Oz$$



8.??

$$
\gamma : \begin{cases} x = \cos u \cos v \\ y = \cos u \sin v \\ z = \sin u \end{cases}
$$

Find the tangent plane and the normal line at the point $P(u=0, v=\frac{\pi}{2})$



$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left(\cos u \cos v\right) =$$

$$= -\sin u \cos v$$

$$\frac{\partial x}{\partial v} = -\cos u \sin v$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\left(\cos u \ \sin v \right) =$$

$$= -\sin u \ \sin v$$

$$\frac{\partial y}{\partial v} = \cos u \ \cos v$$

$$\frac{\partial z}{\partial u} = 1 \qquad \frac{\partial z}{\partial v} = \cos v$$

$$\Rightarrow \begin{vmatrix} x & y & z-1 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad y - z + 1 = 0$$