9 Week 9:Conics

9.1 The Ellipse

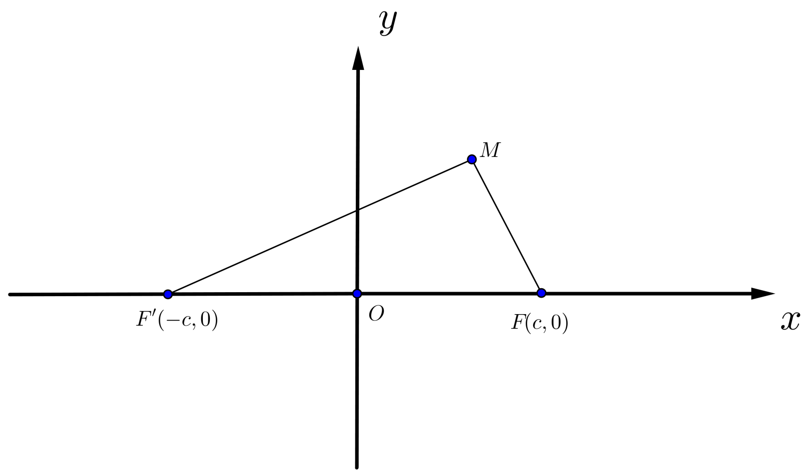
Definition 9.1. An *ellipse* is the locus of points in a plane, the sum of whose distances from two fixed points, say F and F', called *foci* is constant.

The distance between the two fixed points is called the *focal distance*

Let F and F' be the two foci of an ellipse and let $|FF'| = 2c$ be the focal distance. Suppose that the constant in the definition of the ellipse is $2a$. If M is an arbitrary point of the ellipse, it must verify the condition

$$|MF| + |MF'| = 2a.$$

One may chose a Cartesian system of coordinates centered at the midpoint of the segment $[F'F]$, so that $F(c, 0)$ and $F'(-c, 0)$.



Remark 9.1. In $\Delta MFF'$ the following inequality $|MF| + |MF'| > |FF'|$ holds. Hence $2a > 2c$. Thus, the constants a and c must verify $a > c$. $\Leftrightarrow a^2 > c^2 \Leftrightarrow \underbrace{a^2 - c^2}_{b^2} > 0$

Thus, for the generic point $M(x, y)$ of the ellipse we have succesively:

$$
\begin{aligned}
|MF| + |MF'| = 2a \quad \Leftrightarrow \quad & \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a \\
& \sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2} \\
& \cancel{x^2} - 2cx + \cancel{c^2} + \cancel{y^2} = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + \overbrace{(x+c)^2}^{\cancel{x^2}+2cx+\cancel{c^2}} + \cancel{y^2} \\
& a\sqrt{(x+c)^2 + y^2} = cx + a^2 \\
& a^2(x^2 + 2xc + c^2) + a^2y^2 = c^2x^2 + 2a^2cx + a^2 \\
& (a^2 - c^2)x^2 + a^2y^2 - a^2(a^2 - c^2) = 0.
\end{aligned}
$$

Denote $a^2 - c^2$ by b^2, as $(a > c)$. Thus $b^2x^2 + a^2y^2 - a^2b^2 = \cancel{0}$, i.e. $: a^2b^2$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{9.1}$$

Remark 9.2. The ellipse

HOMOGENEOUS OF ORDER 2

$$(E)\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad E = f^{-1}(1),\ f:\mathbb{R}^2 \to \mathbb{R},\ f(x) = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

is a regular curve and the equation of its tangent line $T_{P_0}(E)$ at some point $P_0(x_0, y_0) \in E$ is

$$T_{P_0}(E)\ \frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1. \tag{9.2}$$

Remark 9.3. The equation (9.1) is equivalent to

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}; \qquad x = \pm\frac{a}{b}\sqrt{b^2 - y^2},$$

$a^2 - x^2 \geq 0$

which means that the ellipse is symmetric with respect to both the x and the y axes. In fact, the line FF', determined by the foci of the ellipse, and the perpendicular line on the midpoint of the segment $[FF']$ are axes of symmetry for the ellipse. Their intersection point, which is the midpoint of $[FF']$, is the center of symmetry of the ellipse, or, simply, its *center*.

Remark 9.4. In order to sketch the graph of the ellipse, observe that it is enough to represent the function

$$f : [-a, a] \to \mathbb{R}, \qquad f(x) = \frac{b}{a}\sqrt{a^2 - x^2},$$

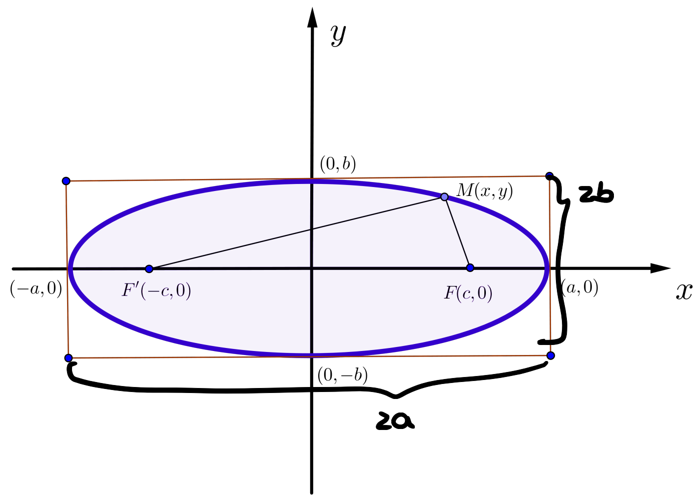
and to complete the ellipse by symmetry with respect to the x-axis.

One has

$$f'(x) = -\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}}, \qquad f''(x) = -\frac{ab}{(a^2 - x^2)\sqrt{a^2 - x^2}}.$$

x	$-a$		0		a
$f'(x)$	$\vert$	$+\ +\ +$	0	$-\ -\ -$	$\vert$
$f(x)$	0	$\nearrow$	b	$\searrow$	0
$f''(x)$	$\vert$	$-\ -\ -$	$-$	$-\ -\ -$	$\vert$

concave



9.2 The Hyperbola

Definition 9.2. The *hyperbola* is defined as the geometric locus of the points in the plane, whose absolute value of the difference of their distances to two fixed points, say F and F' is constant.

The two fixed points are called the *foci* of the hyperbola, and the distance $|FF'| = 2c$ between the foci is the *focal distance*.

Suppose that the constant in the definition is $2a$. If $M(x, y)$ is an arbitrary point of the hyperbola, then

$$||MF| - |MF'|| = 2a.$$

Choose a Cartesian system of coordinates, having the origin at the midpoint of the segment $[FF']$ and such that $F(c, 0)$, $F'(-c, 0)$.

Remark 9.5. In the triangle $\Delta MFF'$, $||MF| - |MF'|| < |FF'|$, so that $a < c$.

Let us determine the equation of a hyperbola. By using the definition we get $|MF| - |MF'| = \pm 2a$, namely

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a,$$

or, equivalently

$$\sqrt{(x-c)^2 + y^2} = \pm 2a + \sqrt{(x+c)^2 + y^2}.$$

We therefore have successively

$$\begin{aligned}
&\cancel{x^2} - 2cx + \cancel{c^2} + \cancel{y^2} = 4a^2 \pm 4a\sqrt{(x+c)^2 + y^2} + \cancel{x^2} + 2cx + \cancel{c^2} + \cancel{y^2}\\
&cx + a^2 = \pm a\sqrt{(x+c)^2 + y^2}\\
&c^2x^2 + 2a^2cx + a^4 = a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2\\
&(c^2 - a^2)x^2 - a^2y^2 - a^2(c^2 - a^2) = 0. \Leftrightarrow b^2x^2 - a^2y^2 - a^2b^2 = 0 \,|\, : a^2b^2
\end{aligned}$$

By using the notation $c^2 - a^2 = b^2$ ($c > a$) we obtain the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0. \tag{9.3}$$

Remark 9.6. The hyperbola

HOMOGENEOUS OF ORDER 2

$$(H)\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad H = g^{-1}(1),\ g:\mathbb{R}^2 \to \mathbb{R},\ g(x,y) = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

is a regular curve and the equation of its tangent line $T_{P_0}(H)$ at some point $P_0(x_0, y_0) \in H$ is

$$T_{P_0}(H)\ \frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1. \tag{9.4}$$

The equation (9.3) is equivalent to

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}; \qquad x = \pm\frac{a}{b}\sqrt{y^2 + b^2}.$$

$x^2 - a^2 \geq 0$

Therefore, the coordinate axes are axes of symmetry of the hyperbola and the origin is a center of symmetry equally called the *center of the hyperbola*.

Remark 9.7. To sketch the graph of the hyperbola, is it enough to represent the function

$$f : (-\infty, -a] \cup [a, \infty) \to \mathbb{R}, \qquad f(x) = \frac{b}{a}\sqrt{x^2 - a^2},$$

by taking into account that the hyperbola is symmetric with respect to the x-axis.

Since $\lim\limits_{x\to\infty} \frac{f(x)}{x} = \frac{b}{a}$ and $\lim\limits_{x\to-\infty} \frac{f(x)}{x} = -\frac{b}{a}$, it follows that $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ are asymptotes of f.

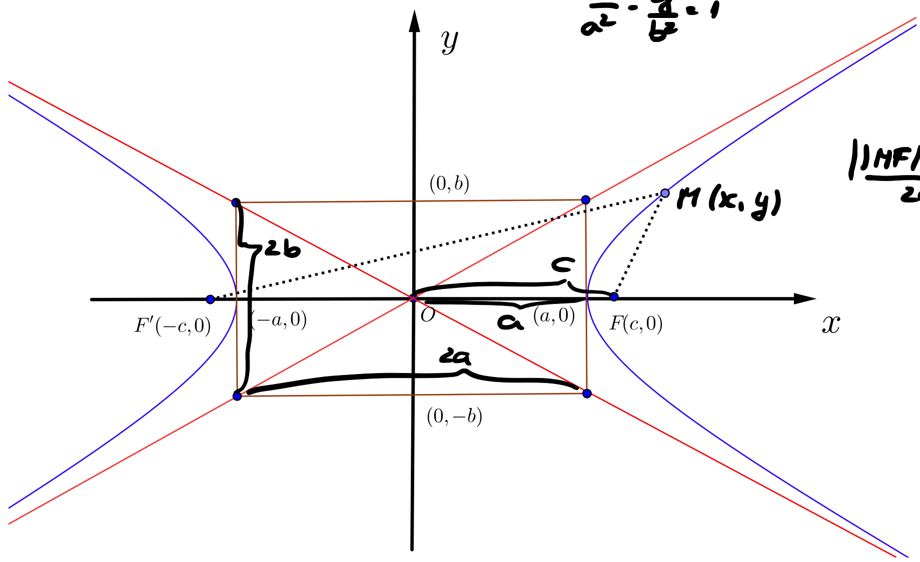
$\lim\limits_{x\to\pm\infty}\left(f(x) \pm \frac{b}{a}x\right) = 0$

One has, also

$$f'(x) = \frac{b}{a}\frac{x}{\sqrt{x^2 - a^2}}, \qquad f''(x) = -\frac{ab}{(x^2 - a^2)\sqrt{x^2 - a^2}}.$$

x	$-\infty$		$-a$		a		∞
$f'(x)$	$-$	$- - -$	$\mid$	$/\,/\,/$	$\mid$	$+ + +$	$+$
$f(x)$	∞	$\searrow$	$0\mid$	$/\,/\,/$	$\mid 0$	$\nearrow$	∞
$f''(x)$	$-$	$- - -$	$\mid$	$/\,/\,/$	$\mid$	$- - -$	$-$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



$\underbrace{||MF| - |MF'||}_{2a} < \underbrace{|FF'|}_{2c}$

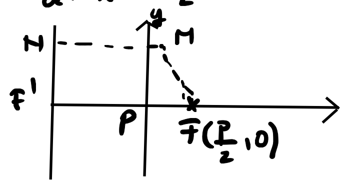
$a < c$

9.3 The Parabola

Definition 9.3. The *parabola* is a plane curve defined to be the geometric locus of the points in the plane, whose distance to a fixed line d is equal to its distance to a fixed point F.

The line d is the *director line* and the point F is the *focus*. The distance between the focus and the director line is denoted by p and represents the *parameter* of the parabola.

Consider a Cartesian system of coordinates xOy, in which $F\left(\frac{p}{2}, 0\right)$ and $d : x = -\frac{p}{2}$. If $M(x, y)$ is an arbitrary point of the parabola, then it verifies



$$|MN| = |MF|,$$

where N is the orthogonal projection of M on d.

Thus, the coordinates of a point of the parabola verify

$$\sqrt{\left(x + \frac{p}{2}\right)^2 + 0} = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2}$$

$$\left(x + \frac{p}{2}\right)^2 = \left(x - \frac{p}{2}\right)^2 = y^2$$

$$\cancel{x^2} + px + \cancel{\frac{p^2}{4}} = \cancel{x^2} - px + \cancel{\frac{p^2}{4}} + y^2,$$

and the equation of the parabola is

$$y^2 = 2px. \tag{9.5}$$

Remark 9.8. The parabola

$$(P)\, y^2 = 2px$$

is a regular curve and the equation of its tangent line $T_{Q_0}(P)$ at some point $Q_0(x_0, y_0) \in P$ is

$$T_{Q_0}(P)\; y_0 y = p(x + x_0). \tag{9.6}$$

Remark 9.9. The equation (9.5) is equivalent to $y = \pm\sqrt{2px}$, so that the parabola is symmetric with respect to the x-axis.

$f(x) = \sqrt{2px}$

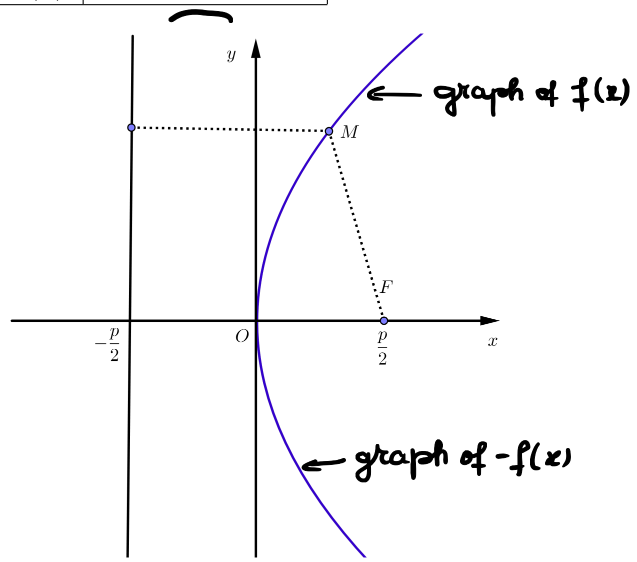
Representing the graph of the function $f : [0, \infty) \to [0, \infty)$ and using the symmetry of the curve with respect to he x-axis, one obtains the graph of the parabola.

One has

$\lim_{x \to \infty} \frac{f(x)}{x} = 0$

$$f'(x) = \frac{p}{\sqrt{2px_0}}; \qquad f''(x) = -\frac{p}{2x\sqrt{2x}}.$$

x	0		∞
$f'(x)$	$\vert$	$+\ +\ +$	$+$
$f(x)$	0	$\nearrow$	∞
$f''(x)$	$-$	$-\ -\ -$	$-$



9.4 Problems

1. Find the equations of the tangent lines to the ellipse $\mathcal{E} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$ having a given angular coefficient $m \in \mathbb{R}$. (see [1, p. 110]).

   Solution. We are looking for the lines $d : y = mx + n$, which are tangent to the ellipse, i.e. each of them has one single common point with the ellipse. Their intersection is given by the solutions of the system of equations

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0 \\ y = mx + n \end{cases},$$

   or, by replacing y in the equation of the ellipse,

$$(a^2m^2 + b^2)x^2 + 2a^2mnx + a^2(n^2 - b^2) = 0.$$

   The discriminant Δ of the last equation is given by

$$\Delta = 4[a^4m^2n^2 - a^2(a^2m^2 + b^2)(n^2 - b^2)]$$

   and the line (d) and the ellipse (E) have one single common point if and only if $a^4m^2n^2 - a^2(a^2m^2 + b^2)(n^2 - b^2) = 0$, i.e. $n = \pm\sqrt{a^2m^2 + b^2}$. The equations of the tangent lines of direction m are therefore

$$y = mx \pm \sqrt{a^2m^2 + b^2}. \tag{9.7}$$

2. **(2p)** Find the equations of the tangent lines to the ellipse $\mathcal{E} : x^2 + 4y^2 - 20 = 0$ which are orthogonal to the line $d : 2x - 2y - 13 = 0$.

   Solution.

$\mathcal{E}: x^2+4y^2-20=0 \,/ :20 \Leftrightarrow \frac{x^2}{20}+\frac{y^2}{5}=1, \ a^2=20, b^2=5$

$d: 2x-2y-13=0 \Leftrightarrow y=x-\frac{13}{2}, \ m_d=1$

$m_{T(\mathcal{E})}=\frac{-1}{m_d}=-1, \ A_0(x_0,y_0)\in\mathcal{E}$

$T_{A_0}(\mathcal{E}): \frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1 \Rightarrow \frac{x_0x}{20}+\frac{y_0y}{5}=1 \,/ \cdot 20 \Rightarrow x_0x+4y_0y=20$

$y=\frac{20-x_0x}{4y_0}=\frac{-x_0}{4y_0}x+\frac{20}{4y_0}$

$m_{T_{A_0}(\mathcal{E})}=\frac{-x_0}{4y_0}=-1 \Rightarrow x_0=4y_0$

$A_0\in\mathcal{E} \Rightarrow \frac{x_0^2}{20}+\frac{y_0^2}{5}=1 \,/ \cdot 20 \Leftrightarrow x_0^2+4y_0^2=20$

$\begin{cases} x_0=4y_0 \\ x_0^2+4y_0^2=20 \end{cases} \Leftrightarrow \begin{cases} x_0=4y_0 \\ 16y_0^2+4y_0^2=20 \end{cases} \Leftrightarrow \begin{cases} x_0=4y_0 \\ y_0^2=1 \end{cases} \Leftrightarrow \begin{cases} x_0=\pm 4 \\ y_0=\pm 1 \end{cases} \Rightarrow \frac{\pm 4x}{20}\pm\frac{y}{5}=1 \ (T_{A_0}(\mathcal{E}))$

$\Leftrightarrow \frac{x}{5}+\frac{y}{5}=\pm 1 \Leftrightarrow T_{A_0}(\mathcal{E})\ x+y=\pm 5$

3. **(2p)** Find the equations of the tangent lines to the ellipse $\mathcal{E}: \dfrac{x^2}{25} + \dfrac{y^2}{16} - 1 = 0$, passing through $P_0(10, -8)$.

Solution. $d_m : (x - x_0) \cdot m = y - y_0 \Rightarrow y + 8 = m(x - 10)$

d_m is tangent to $\mathcal{E}$ iff. the system a unique sol:

$$\begin{cases} \frac{x^2}{25} + \frac{y^2}{16} - 1 = 0 \\ y + 8 = m(x - 10) \end{cases} \Leftrightarrow \begin{cases} \left(\frac{m^2}{16} + \frac{1}{25}\right)x^2 + \left(-\frac{5m^2}{4} - m\right)x + \left(\frac{25m^2}{4} + 10m + 3\right) = 0 \\ y = m(x - 10) - 8 \end{cases}$$

The system has a unique sol. if : $\Delta = 0 \Leftrightarrow$

$$\Leftrightarrow \left(-\frac{5m^2}{4} - m\right)^2 - 4 \cdot \left(\frac{m^2}{16} + \frac{1}{25}\right) \cdot \left(\frac{25m^2}{4} + 10m + 3\right) = 0 \Leftrightarrow \frac{-3m^2}{4} + \frac{-8m}{5} - \frac{12}{25} = 0 \Leftrightarrow$$

$$\Rightarrow m_{1,2} = -\frac{16}{15} \pm \frac{4\sqrt{7}}{15} \Rightarrow d_{m_1} : y + 8 = \left(-\frac{16}{15} + \frac{4\sqrt{7}}{15}\right)(x - 10)$$

$$d_{m_2} : y + 8 = \left(-\frac{16}{15} - \frac{4\sqrt{7}}{15}\right)(x - 10)$$

$d_\infty : x = 10$

$d_\infty \cap \mathcal{E} = \{(10, y) \in \mathbb{R}^2 / 4 + \frac{y^2}{16} - 1 = 0\} \Rightarrow d_\infty$ is not tangent

4. If $M(x,y)$ is a point of the tangent line $T_{M_0}(E)$ of the ellipse $\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at one of its points $M_0(x_0, y_0) \in \mathcal{E}$, show that $\frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1$.

   Solution 1. Every director vector of the tangent line $T_{M_0}(E) : \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$ is orthogonal to the normal vector $\vec{n}\left(\frac{x_0}{a^2}, \frac{y_0}{b^2}\right)$ of the tangent line $T_{M_0}(E)$. Such an orthogonal vector is $\vec{v}\left(\frac{y_0}{b^2}, -\frac{x_0}{b^2}\right)$. Thus, the parametric equations of the tangent line are

$$T_{M_0}(E) : \begin{cases} x = x_0 + \frac{y_0}{b^2}t \\ y = y_0 - \frac{x_0}{b^2}t \end{cases}, \ t \in \mathbb{R},$$

   i.e. the coordinates of M are of this form. In order to completely solve the question , we only need to show that $\varphi \geq 1$, where $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, $\varphi(t) = \frac{\left(x_0 + \frac{y_0}{b^2}t\right)^2}{a^2} + \frac{\left(y_0 - \frac{x_0}{b^2}t\right)^2}{b^2}$. This is actually the case as

$$\begin{aligned} \varphi(t) &= \frac{x_0^2 + 2\frac{x_0 y_0}{b^2}t + \frac{y_0^2}{b^4}t^2}{a^2} + \frac{y_0^2 - 2\frac{x_0 y_0}{b^2}t + \frac{x_0^2}{a^2}t^2}{b^2} \\ &= \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{1}{a^2b^2}\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}\right)t^2 = 1 + \frac{t^2}{a^2b^2} \geq 1, \forall t \in \mathbb{R}. \end{aligned}$$

   Solution 2. Since $M(x,y) \in T_{M_0}(E) : \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$, it follows via the C-B-S inequality that

$$\begin{aligned} 1 = \frac{x_0}{a} \cdot \frac{x}{a} + \frac{y_0}{b} \cdot \frac{y}{b} &\leq \left| \frac{x_0}{a} \cdot \frac{x}{a} + \frac{y_0}{b} \cdot \frac{y}{b} \right| \\ &\leq \sqrt{\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}} \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}} = \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}. \end{aligned}$$

5. Find the equations of the tangent lines to the hyperbola $\mathcal{H} : \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ having a given angular coefficient $m \in \mathbb{R}$. (see [1, p. 115]).

   Solution. The intersection of the hyperbola $(\mathcal{H})$ with the line $(d) y = mx + n$ is given by the solution of the system

$$\begin{cases} \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0 \\ y = mx + n \end{cases}.$$

   By substituting y in the first equation, one obtains

$$(a^2m^2 - b^2)x^2 + 2a^2mnx + a^2(n^2 + b^2) = 0. \tag{9.8}$$

   - If $a^2m^2 - b^2 = 0$, (or $m = \pm\frac{b}{a}$), then the equation (9.8) becomes

$$\pm 2bnx + a(n^2 + b^2) = 0.$$

– If $n = 0$, there are no solutions (this means, geometrically, that the two asymptotes do not intersect the hyperbola);

– If $n \neq 0$, there exists a unique solution (geometrically, a line d, which is parallel to one of the asymptotes, intersects the hyperbola at exactly one point);

- If $a^2m^2 - b^2 \neq 0$, then the discriminant of the equation (9.8) is

$$\Delta = 4[a^4m^2n^2 - a^2(a^2m^2 - b^2)(n^2 + b^2)].$$

The line $d : y = mx + n$ is tangent to the hyperbola $\mathcal{H} : \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ if the discriminant Δ of the equation (9.8) is zero, i.e. $a^2m^2 - n^2 - b^2 = 0$.

– If $a^2m^2 - b^2 \geq 0$, i.e. $m \in \left(-\infty, -\frac{b}{a}\right] \cup \left[\frac{b}{a}, \infty\right)$, then $n = \pm\sqrt{a^2m^2 - b^2}$. The equations of the tangent lines to $\mathcal{H}$, having the angular coefficient m are

$$y = mx \pm \sqrt{a^2m^2 - b^2}. \tag{9.9}$$

– If $a^2m^2 - b^2 < 0$, there are no tangent lines to $\mathcal{H}$, of angular coefficient m.

6. **(2p)** Find the equations of the tangent lines to the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$ which are orthogonal to the line $d :\ 4x + 3y - 7 = 0$. $\Leftrightarrow y = -\frac{4}{3} + \frac{7}{3}$

Solution. $(d_1)\ y = mx + n;\ d_1 \perp d \Leftrightarrow m_d \cdot m_{d_1} = -1 \\ m_d = -\frac{4}{3}\Big\} \Rightarrow m_{d_1} = \frac{3}{4} \Rightarrow$

$$\Rightarrow \begin{cases} \frac{x^2}{20} - \frac{y^2}{5} = 1 \\ y = \frac{3}{4}x + n \end{cases} \Leftrightarrow \begin{cases} x^2 - 4\left(\frac{3}{4}x + n\right)^2 = 20 \\ 3x - 4y + 4n = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 - 4\left(\frac{9x^2}{16} + \frac{6xn}{4} + n^2\right) = 20 \\ 3x - 4y + 4n = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 - \frac{9x^2}{4} - 6xn - 4n^2 = 20 \\ 3x - 4y + 4n = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 5x^2 + 24nx + 16n^2 + 80 = 0 \\ 3x - 4y + 4n = 0 \end{cases}$$

$\Delta = 0 \Leftrightarrow 36n^2 - 4 \cdot \frac{5}{4}(4n^2 + 20) = 0 \Leftrightarrow 36n^2 - 20n^2 - 100 = 0 \Rightarrow$

$\Leftrightarrow 16n^2 = 100 \Leftrightarrow n = \pm\sqrt{\frac{100}{16}} = \pm\frac{10}{4} = \pm\frac{5}{2} \Rightarrow (d_1):\ 3x - 4y \pm 4 \cdot \left(\frac{5}{2}\right) = 0 \Leftrightarrow$

$\Leftrightarrow (d_1):\ 3x - 4y \pm 10 = 0$

$T_H(x_0, y_0):\ \frac{xx_0}{20} - \frac{yy_0}{5} - 1 = 0$

$\frac{yy_0}{5} = \frac{xx_0}{20} - 1$

$y = \frac{x_0 x}{4y_0} - \frac{5}{y_0}$, Assume $y_0 \neq 0 \Rightarrow m_{T_H(x_0,y_0)} = \frac{x_0}{4y_0}$

$d: y = -\frac{4}{3}x + \frac{7}{3} \Rightarrow m_d = -\frac{4}{3}$, $T_H(x_0,y_0) \perp d$ $\Big\} \Rightarrow \frac{x_0}{4y_0} \cdot \left(-\frac{4}{3}\right) = -1 \Leftrightarrow \frac{x_0}{y_0} = 3 \Rightarrow x_0 = 3y_0$

We know that: $(x_0, y_0) \in H \Rightarrow \frac{x_0^2}{20} - \frac{y_0^2}{5} - 1 = 0 \Rightarrow \frac{9y_0^2 - 4y_0^2}{20} = 1 \Rightarrow y_0 = \pm 2 \Rightarrow x_0 = \pm 6 \Rightarrow$

$\Rightarrow -\frac{x \cdot 3}{10} + \frac{2y}{5} - 1 = 0 \,/\cdot 10 \Leftrightarrow -3x + 4y - 10 = 0 \,/\cdot(-1) \Leftrightarrow 3x - 4y + 10 = 0$

$\Rightarrow \frac{3x}{10} - \frac{2y}{5} - 1 = 0 \,/\cdot 10 \Leftrightarrow 3x - 4y - 10 = 0$

7. Find the equations of the tangent lines to the parabola $\mathcal{P}: \; y^2 = 2px$ having a given angular coefficient $m \in \mathbb{R}$. (see [1, p. 119]).

   Solution. The intersection beteen the porabla (P) and the line $(d)y = mx + n$ is given by the solution of the system

$$\begin{cases} y^2 = 2px \\ y = mx + n. \end{cases}$$

   This leads to a second degree equation

$$m^2x^2 + 2(mn - p)x + n^2 = 0,$$

   having the discriminant

$$\Delta = 4p(2mn - p) \tag{9.10}$$

   The line $d: y = mx + n$ (with $m \neq 0$) is tangent to the parabola $\mathcal{P}: y^2 = 2px$ if the discriminant Δ which appears in (9.10) is zero, i.e. $2mn = p$. Then, the equation of the tangent line to $\mathcal{P}$, having the angular coefficient m, is

$$y = mx + \frac{p}{2m}. \tag{9.11}$$

8. **(2p)** Find the equation of the tangent line to the parabola $\mathcal{P}: \; y^2 - 8x = 0$, parallel to $d: 2x + 2y - 3 = 0$.

   Solution. $y^2 - 8x = 0 \Leftrightarrow y^2 = 8x \Rightarrow p = 4$

$T_P(x_0, y_0): yy_0 = p(x + x_0) \to$ tangent line to P in (x_0, y_0)

If $y_0 \neq 0 \Rightarrow m_{T_P(x_0,y_0)} = \frac{4}{y_0} \overset{T_P(x_0,y_0) \parallel d}{\Rightarrow} m_d = -1 \Rightarrow y_0 = -4$

$(x_0, y_0) \in P: y_0^2 = 8x_0 \Rightarrow 16 = 8x_0 \Rightarrow x_0 = 2 \Rightarrow T_P(2,-4): x + y + 2 = 0$

$y = 0: x_0 = 0 \Rightarrow T_P(0,0): x = 0 \Rightarrow m_{T_P(0,0)} = \infty \Rightarrow T_P(0,0) \nparallel d$

9. **(2p)** Find the equation of the tangent line to the parabola $\mathcal{P}: \; y^2 - 36x = 0$, passing through $P(2, 9)$.

   Solution. A line that passes through $P(2,9)$ has the form: $d_m: y - 9 = \overset{\nearrow \text{slope}}{m}(x-2)$, $m \in \mathbb{R} \cup \{\infty\}$.

We can consider for $x = 2$ that $m = \infty$

d_m is tangent to $\mathcal{P} \Leftrightarrow$ the following system has a unique solution: $\begin{cases} y^2 - 36x = 0 \\ y - 9 = m(x-2) \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} y = m(x-2) + 9 \\ (m(x-2)+9)^2 - 36x = 0 \end{cases} \Leftrightarrow \begin{cases} y = m(x-2)+9 \\ m^2(x-2)^2 + 2m(x-2)\cdot 9 + 81 - 36x = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} y = m(x-2) + 9 \\ m^2x^2 - 4m^2x + 4m^2 + 18mx - 36m + 81 - 36x = 0 \end{cases} \Leftrightarrow \begin{cases} y = m(x-2)+9 \\ m^2x^2 + (-4m^2 + 18m - 36)x + (4m^2 - 36m + 81) = 0 \end{cases}$

The sol. of the system correspond to the solutions in x of the equation:

$m^2x^2 + (-4m^2 + 18m - 36)x + (4m^2 - 36m + 81) = 0$, which has a unique sol. iff $\Delta = 0$

$\Delta = (-4m^2 + 18m - 36)^2 - 4m^2(4m^2 - 36m + 81) = \cancel{16m^4} + \cancel{324m^2} + 1296 - \cancel{144m^3} + 288m^2 - 1296m - \cancel{16m^4} + \cancel{144m^3} - \cancel{324m^2}$

$= 288m^2 - 1296m + 1296 = 144(2m^2 - 9m + 9)$

$\Delta = 0 \Leftrightarrow 2m^2 - 9m + 9 = 0$

$\Delta' = (-9)^2 - 4\cdot 2\cdot 9 = 81 - 72 = 9 > 0$

$\nearrow m_1 = \frac{9+3}{4} = \frac{12}{4} = 3 \Rightarrow d_{m_1}: y - 9 = 3(x-2) \Leftrightarrow y - 9 = 3x - 6 \Leftrightarrow 3x - y + 3 = 0$

$\searrow m_2 = \frac{9-3}{4} = \frac{6}{4} = \frac{3}{2} \Rightarrow d_{m_2}: y - 9 = \frac{3}{2}(x-2) \,/\, 2 \Leftrightarrow 2y - 18 = 3x - 6 \Leftrightarrow 3x - 2y + 12 = 0$

For $m = \infty \Rightarrow d_\infty: x = 2$

$d_\infty \cap \mathcal{P} = \{(2,y) \in \mathbb{R}^2 \,/\, y^2 - 72 = 0\} = \{(2, 6\sqrt{2}), (2, -6\sqrt{2})\} \Rightarrow d_\infty$ is not tangent to $\mathcal{P}$

10. **(3p)** Find the locus of the orthogonal projections of the center $O(0,0)$ of the ellipse

$$E:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

on its tangents.

Solution.

11. **(3p)** Find the locus of the orthogonal projections of the center $O(0,0)$ of the hyperbola

$$H : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

on its tangents.

Solution.

let $d: y = mx + n$ be a line

- we compute the intersection between the line d and the hyperbola

$$\begin{cases} \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \\ y = mx + n \mid ()^2 \end{cases} \Rightarrow \begin{cases} b^2x^2 - a^2y^2 = a^2b^2 \\ y^2 = m^2x^2 + 2mnx + n^2 \end{cases}$$

we replace y in the first eq: $b^2x^2 - a^2(m^2x^2 + 2mnx + n^2) = a^2b^2 \Leftrightarrow$

$\Leftrightarrow x^2(b^2 - a^2m^2) - 2a^2mnx - a^2(n^2 + b^2) = 0$

d is tangent to $H \Leftrightarrow \Delta = 0$

$\Delta = 4a^4m^2n^2 + 4a^2(n^2+b^2)(b^2 - a^2m^2)$

$\Delta = 4(a^2m^2n^2 + a^2b^2 - a^4b^2m^2)$

$\Delta = 0 \Rightarrow a^2m^2n^2 + a^2b^2 - a^4b^2m^2 = 0 \Rightarrow$

$\Rightarrow n = \pm\sqrt{a^2m^2 - b^2} \Rightarrow (t): y = mx \pm \sqrt{a^2m^2 - b^2}$

let d' be a line that goes through $O(0;0)$, $d' \perp t$

$$\left.\begin{array}{l} d' \perp t \Rightarrow m_{d'} \cdot m_t = -1 \\ m_t = m \end{array}\right\} \Rightarrow m_{d'} = \frac{-1}{m} \Rightarrow (d'): y = \frac{-1}{m}x$$

- we compute the intersection of d' and t; the locus is $\{M\} = d' \cap t$

$$\begin{cases} y = \frac{-1}{m}x \\ y = mx \pm \sqrt{a^2m^2 - b^2} \end{cases} \Rightarrow \begin{cases} m = \frac{-x}{y} \\ y = mx \pm \sqrt{a^2m^2 - b^2} \end{cases} \Rightarrow y = -\frac{x^2}{y} \pm \sqrt{a^2\frac{x^2}{y^2} - b^2} \Big| \cdot y \Rightarrow$$

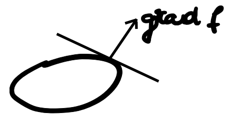
$\Rightarrow y^2 + x^2 = \pm\sqrt{a^2x^2 - b^2y^2} \mid ()^2 \Rightarrow (x^2+y^2)^2 = a^2x^2 - b^2y^2$ - The locus - except the origin

12. Show that a ray of light through a focus of an ellipse reflects to a ray that passes through the other focus (optical property of the ellipse).

Solution. Let $F_1(-c,0)$, $F_2(c,0)$ be the foci of the ellipse $\mathcal{E}:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Recall that the gradient $\text{grad}(f)(x_0,y_0)=(f_x(x_0,y_0),f_y(x_0,y_0))$ is a normal vector of the ellipse $\mathcal{E}$ to its point $M_0(x_0,y_0)$, where

$$f(x,y)=\delta(F_1,M)+\delta(F_2,M)=\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}$$

$(E)\ f(x,y)=2a$



and $M(x,y)$. Note that

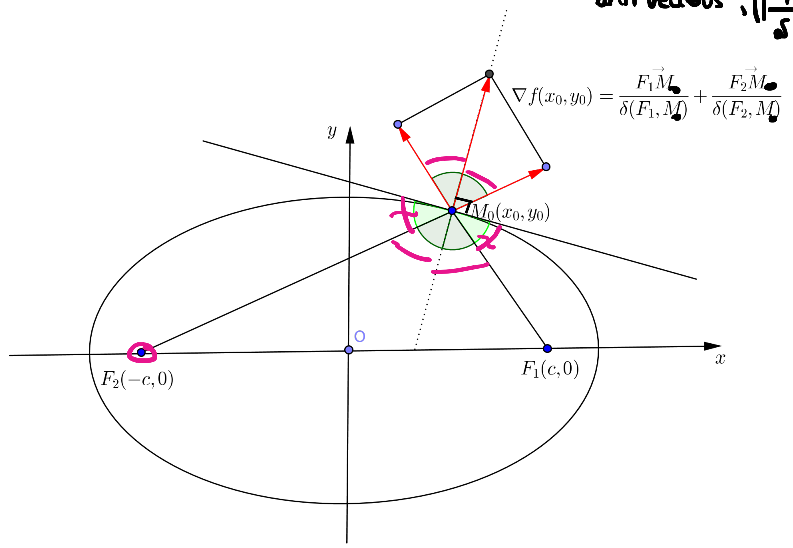
$$f_x(x_0,y_0)=\frac{x_0-c}{\delta(F_1,M_0)}+\frac{x_0+c}{\delta(F_2,M_0)} \text{ and } f_y(x_0,y_0)=\frac{y_0}{\delta(F_1,M_0)}+\frac{y_0}{\delta(F_2,M_0)},$$

and shows that

$$\begin{aligned}\text{grad}(f) &= (f_x(x_0,y_0),f_y(x_0,y_0))=\left(\frac{x_0-c}{\delta(F_1,M_0)}+\frac{x_0+c}{\delta(F_2,M_0)},\frac{y_0}{\delta(F_1,M_0)}+\frac{y_0}{\delta(F_2,M_0)}\right)\\ &= \frac{(x_0-c,y_0)}{\delta(F_1,M_0)}+\frac{(x_0+c,y_0)}{\delta(F_2,M_0)}=\frac{\overrightarrow{F_1M_0}}{\delta(F_1,M_0)}+\frac{\overrightarrow{F_2M_0}}{\delta(F_2,M_0)}.\end{aligned}$$

$\frac{v}{t}=t^{-1}v$

unit vectors, $\left\|\frac{\overrightarrow{F_1M_0}}{\delta(F_1,M_0)}\right\|=\frac{1}{\delta(F_1,M_0)}\cdot\|\overrightarrow{F_1M_0}\|=1$



The versors $\frac{\overrightarrow{F_1M_0}}{\delta(F_1,M_0)}$ and $\frac{\overrightarrow{F_2M_0}}{\delta(F_2,M_0)}$ point towards the exterior of the ellipse $\mathcal{E}$ and their sum make obviously equal angles with the directions of the vectors $\overrightarrow{F_1M_0}$ and $\overrightarrow{F_2M_0}$ and (the sum) is also orthogonal to the tangent $T_{M_0}(\mathcal{E})$ of the ellipse at $M_0(x_0,y_0)$. This shows that the angle between the ray F_1M and the tangent $T_{M_0}(\mathcal{E})$ equals the angle between the ray F_2M and the tangent $T_{M_0}(\mathcal{E})$.

13. Show that a ray of light through a focus of a hyperbola reflects to a ray that passes through the other focus (optical property of the hyperbola).

Solution. Let $F_1(-c,0)$, $F_2(c,0)$ be the foci of the hyperbola $\mathcal{E}:\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Recall that the gradient $\text{grad}(f)(x_0,y_0)=(f_x(x_0,y_0),f_y(x_0,y_0))$ is a normal vector of the hyperbola $\mathcal{H}$ to its point $M_0(x_0,y_0)$, where

$$f(x,y)=\delta(F_2,M)-\delta(F_1,M)=\sqrt{(x-c)^2+y^2}-\sqrt{(x+c)^2+y^2} \tag{9.12}$$

on the left hand side branch of $\mathcal{H}$ and

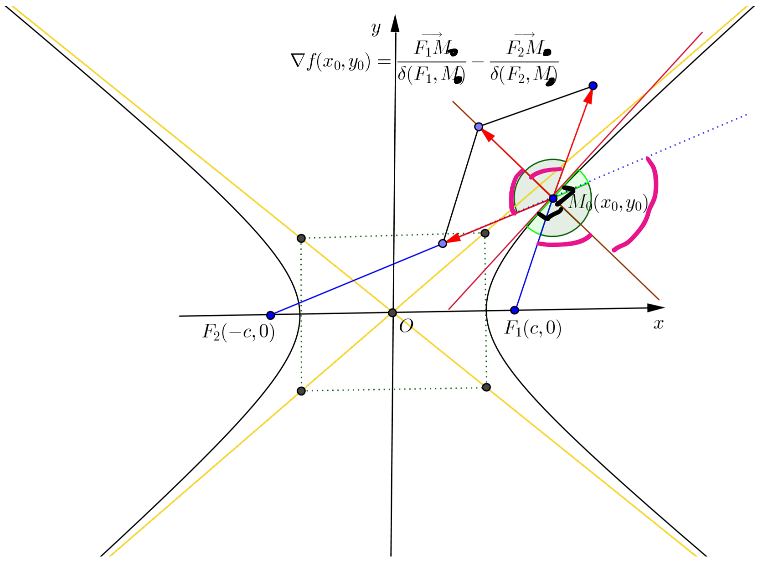
$$f(x,y) = \delta(F_1, M) - \delta(F_2, M) = \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} \tag{9.13}$$

on the right hand side branch of $\mathcal{H}$ and $M(x,y)$. We shall only use the version (9.12) of f, as judgement for the version (9.13) works in a similar way. Note that

$$f_x(x_0,y_0) = \frac{x_0 - c}{\delta(F_1, M_0)} - \frac{x_0 + c}{\delta(F_2, M_0)} \text{ and } f_y(x_0,y_0) = \frac{y_0}{\delta(F_1, M_0)} - \frac{y_0}{\delta(F_2, M_0)},$$

and shows that

$$\begin{aligned} \operatorname{grad}(f) &= (f_x(x_0,y_0), f_y(x_0,y_0)) = \left(\frac{x_0 - c}{\delta(F_1, M_0)} - \frac{x_0 + c}{\delta(F_2, M_0)}, \frac{y_0}{\delta(F_1, M_0)} - \frac{y_0}{\delta(F_2, M_0)} \right) \\ &= \frac{(x_0 - c, y_0)}{\delta(F_1, M_0)} - \frac{(x_0 + c, y_0)}{\delta(F_2, M_0)} = \frac{\overrightarrow{F_1 M_0}}{\delta(F_1, M_0)} - \frac{\overrightarrow{F_2 M_0}}{\delta(F_2, M_0)}. = \frac{\overrightarrow{F_1M_0}}{\delta(F_1,M_0)} + \frac{-\overrightarrow{F_2M_0}}{\delta(F_2M_0)} \end{aligned}$$



The versors $\dfrac{\overrightarrow{F_1 M_0}}{\delta(F_1, M_0)}$ and $-\dfrac{\overrightarrow{F_2 M_0}}{\delta(F_2, M_0)}$ point towards the 'exterior' of the hyperbola $\mathcal{H}$[3] and their sum make obviously equal angles with the directions of the vectors $\overrightarrow{F_1 M_0}$ and $\overrightarrow{F_2 M_0}$ and (the sum) is also orthogonal to the tangent $T_{M_0}(\mathcal{H})$ of the hyperbola at $M_0(x_0,y_0)$. This shows that the angle between the ray $F_1 M$ and the tangent $T_{M_0}(\mathcal{H})$ equals the angle between the ray $F_2 M$ and the tangent $T_{M_0}(\mathcal{H})$.

14. Show that a ray of light through a focus of a parabola reflects to a ray parallel to the axis of the parabola (optical property of the parabola).

Solution. Let $F(\frac{p}{2}, 0)$ be the focus of the parabola $\mathcal{P} : y^2 = 2px$ and $d : x = -\frac{p}{2}$ be its director line. Recall that the gradient $\operatorname{grad}(f)(x_0,y_0) = (f_x(x_0,y_0), f_y(x_0,y_0))$ is a normal vector of parabola $\mathcal{P}$ to its point $M_0(x_0,y_0)$, where

$$f(x,y) = \delta(F, M) - \delta(M, d) = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} - \left(x + \frac{p}{2}\right)$$

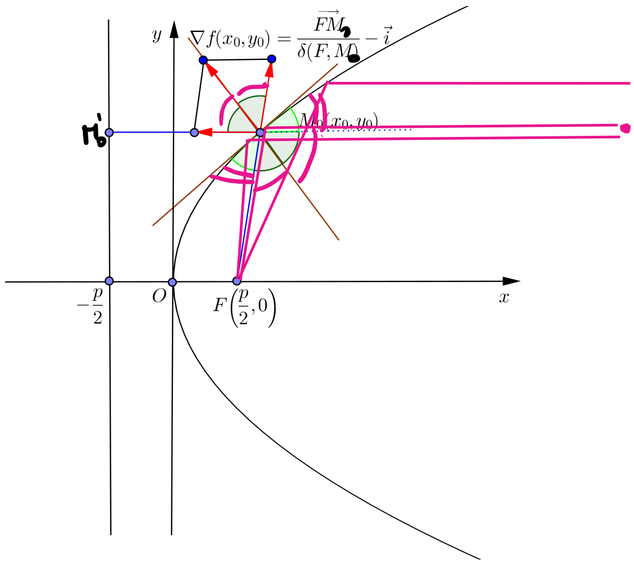
and $M(x,y)$. Note that

$$f_x(x_0,y_0) = \frac{x_0 - \frac{p}{2}}{\delta(F, M_0)} - 1 \text{ and } f_y(x_0,y_0) = \frac{y_0}{\delta(F, M_0)},$$

[3]The exterior of a hyperbola is the nonconvex component of its complement

and shows that

$$\begin{aligned}\operatorname{grad}(f) &= (f_x(x_0,y_0), f_y(x_0,y_0)) = \left(\frac{x_0-\frac{p}{2}}{\delta(F,M_0)} - 1, \frac{y_0}{\delta(F,M_0)}\right)\\ &= \left(\frac{x_0-\frac{p}{2}}{\delta(F,M_0)}, \frac{y_0}{\delta(F,M_0)}\right) - (1,0) = \frac{\overrightarrow{FM_0}}{\delta(F,M_0)} - \overrightarrow{i}.\end{aligned}$$



The versors $\dfrac{\overrightarrow{FM_0}}{\delta(F,M_0)}$ and $-\overrightarrow{i}$ point towards the ’exterior’ of the parabola $\mathcal{P}$[4] and their sum make obviously equal angles with the directions of the vectors $\overrightarrow{FM_0}$ and $\overrightarrow{i}$ and (the sum) is also orthogonal to the tangent line $T_{M_0}(\mathcal{P})$ of the parabola at $M_0(x_0,y_0)$. This shows that the angle between the ray FM and the tangent line $T_{M_0}(\mathcal{P})$ equals the angle between Ox and the tangent $T_{M_0}(\mathcal{E})$.

References

[1] Andrica, D., Ţopan, L., Analytic geometry, Cluj University Press, 2004.

[2] Galbură Gh., Radó, F., Geometrie, Editura didactică şi pedagogică-Bucureşti, 1979.

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[4] Radó, F., Orban, B., Groze, V., Vasiu, A., Culegere de Probleme de Geometrie, Lit. Univ. ”Babeş-Bolyai”, Cluj-Napoca, 1979.

[5] Tabachnikov, S., Geometry and Billiards, American Mathematical Society, 2005.

[4]The exterior of a parabola is the nonconvex component of its complement

$D \subseteq \mathbb{R}^2$ open set, $F : D \to \mathbb{R}$

$\mathcal{L} \in \operatorname{Im} F$ a regular value of $F \Leftrightarrow (\operatorname{grad} F)(x,y) \neq 0, \forall (x,y) \in F^{-1}(\mathcal{L})$
$(\nabla F)(x,y) \neq 0$

$C := F^{-1}(\mathcal{L})$ is a regular curve $\quad C : F(x,y) = \mathcal{L}$

?$(\operatorname{grad} F)(x_0,y_0) \perp T_{(x_0,y_0)}(C)$, $(x_0,y_0) \in C$

consider a local parametrization $r : (-\varepsilon,\varepsilon) \to C$, a parametrized diff curve s.t. $r(0) = (x_0,y_0)$

$r'(0) = u \in T_{(x_0,y_0)}(C)$

$r(t) \in F^{-1}(\mathcal{L}) \Leftrightarrow F(r(t)) = \mathcal{L}, \forall t \in (-\varepsilon,\varepsilon)$

$\vec{r}(t) = (x(t), y(t))$, $r'(t) = (x'(t), y'(t))$

$F(r(t)) = \mathcal{L} \Leftrightarrow F(x(t),y(t)) = \mathcal{L}, \forall t \in (-\varepsilon,\varepsilon) \Rightarrow \dfrac{d}{dt} F(x(t),y(t)) = 0 \Leftrightarrow$

$\Leftrightarrow F_x(x(t),y(t)) x'(t) + F_y(x(t),y(t)) \cdot y'(t) = 0 \Leftrightarrow (\operatorname{grad} F)(x(t),y(t)) \cdot x'(t) = 0, \forall t \in (-\varepsilon,\varepsilon) \Leftrightarrow$

$\Leftrightarrow (\operatorname{grad} F)(r(t)) \cdot r'(t) = 0, \forall t \in (-\varepsilon,\varepsilon) \Leftrightarrow r'(t) \perp (\operatorname{grad} F)(r(t)), \forall t \in (-\varepsilon,\varepsilon)$

$t = 0 \Rightarrow (\operatorname{grad} F)(r(0)) \perp r'(0) \Leftrightarrow (\operatorname{grad} F)(x_0,y_0) \perp u$, thus $(\operatorname{grad} F)(x_0,y_0) \perp T_{(x_0,y_0)}(C)$

Seminar 9 - 916 - 21/04/2021

Conics

$l: ax+by+c=0$ line

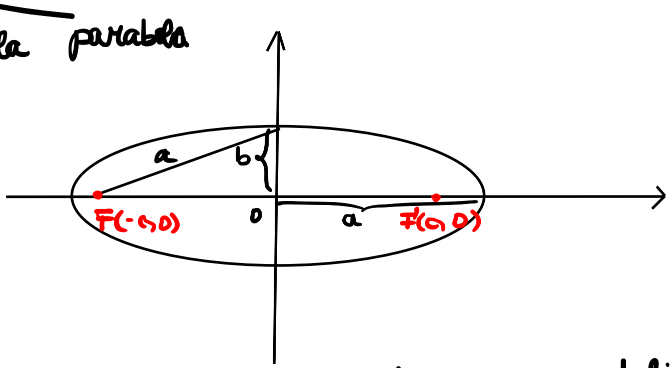
$\mathcal{C}: a_{11}x^2+2a_{12}xy+a_{22}y^2+2a_{10}x+2a_{01}y+a_{00}=0$

↳ conics (nondegenerate)

ellipse hyperbola parabola

Ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

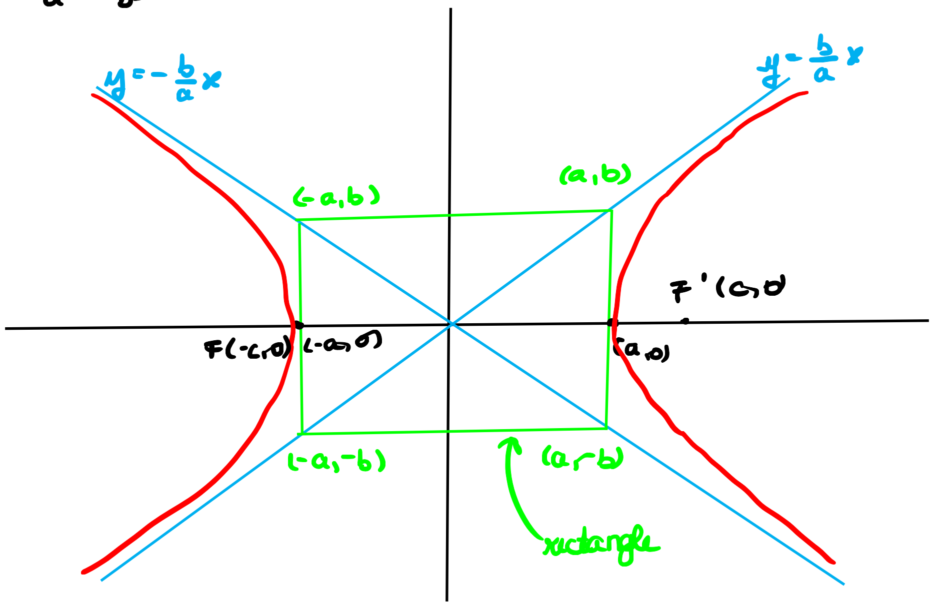
$c=\sqrt{a^2-b^2}$



→ locus of points M in the plane s.t. $MF+MF'=2a$, where F, F' - fixed points called the foci of the ellipse

$T_{\mathcal{E}}(x_0,y_0): \frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$

Hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $c=\sqrt{a^2+b^2}$

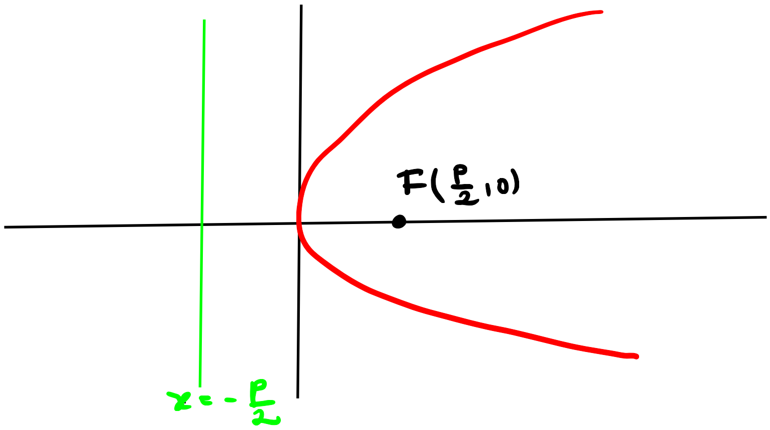


→ locus of points M in the plane for which $|MF-MF'|=2a$, F, F' - fixed points called the foci of the hyperbola

→ The hyperb. has the oblical asymptotes: $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$

$T_{\mathcal{H}}(x_0,y_0): \frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1$

Parabola: $\mathcal{P}: y^2=2px$



→ locus of points M in the plane that are equidistant to a line d are called the director line (directrix) and a point F, called the focus

$T_P(x_0, y_0): yy_0 = p(x + x_0)$