Seminar 413 - 513

The projective plane:

$$\mathbb{RP}^2 = \mathbb{P}^2(\mathbb{R}) = \mathbb{P}_2(\mathbb{R}) =$$

$$= \left\{ [x:y:z] \;\middle|\; \begin{array}{l} x, y, z \in \mathbb{R} \\ (x,y,z) \neq (0,0,0) \end{array} \right\}$$

$$[x:y:z] = [\lambda x : \lambda y : \lambda z]$$

$$\forall \lambda \in \mathbb{R} \setminus \{0\}.$$

$$[2:4:6] = [1:2:3]$$

$$= \frac{\mathbb{R}^3 \setminus \{0\}}{\sim}, \quad \text{where}$$

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \iff$$

$$\iff \exists \lambda \in \mathbb{R} \setminus \{0\} : \quad \begin{array}{l} x_2 = \lambda x_1 \\ y_2 = \lambda y_1 \\ z_2 = \lambda z_1 \end{array}$$

$\mathbb{RP}^2 \simeq$ the set of lines in $\mathbb{R}^3$ that contain the origin.

$$\underbrace{\mathbb{RP}^2}_{\text{projective plane}} = \underbrace{\mathbb{RA}^2}_{\text{affine plane}} \cup \underbrace{\mathbb{R}\infty}_{\text{line at infinity}}$$

$$\mathbb{RA}^2 = \{ [x : y : z] \in \mathbb{RP}^2 \mid z \neq 0 \} =$$

$$= \left\{ \left[\overbrace{\frac{x}{z}}^{= X} : \overbrace{\frac{y}{z}}^{= Y} : 1 \right] \;\middle|\; x, y, z \in \mathbb{R},\ z \neq 0 \right\} =$$

$$= \{ [X : Y : 1] \mid X, Y \in \mathbb{R} \} \simeq$$

$$= \mathbb{R}^2$$

$$\mathbb{R}\mathbb{A}^2 \longrightarrow \mathbb{R}^2 \quad \text{bijective map}$$

$$[x : y : z] \longmapsto \left(\frac{x}{z}, \frac{y}{z}\right)$$

$$\mathbb{R}\infty = \{[x : y : z] \mid z = 0\} =$$

$$= \left\{ [x : y : 0] \;\middle|\; \begin{array}{l} x, y \in \mathbb{R} \\ (x, y) \neq (0, 0) \end{array} \right\}$$

$\hookrightarrow$ every point $[x : y : 0]$ in $\mathbb{R}\infty$ corresponds to all the parallel lines that are parallel to the vector (x, y)

Why we care:

φ_1, φ_2 affine transformations

$$\varphi_1\begin{pmatrix} x \\ y \end{pmatrix} = M_1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_1$$

$$\varphi_2\begin{pmatrix} x \\ y \end{pmatrix} = M_2 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_2$$

$$(\varphi_2 \circ \varphi_1)\begin{pmatrix} x \\ y \end{pmatrix} = \varphi_2\left(M_1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_1\right) =$$

$$= M_2 \cdot \left(M_1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_1\right) + v_2 =$$

$$= M_2 M_1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} + M_2 v_1 + v_2$$

Instead of defining an affine transformation as

$$\varphi: \mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto M \cdot \begin{pmatrix} x \\ y \end{pmatrix} + v_0$$

we define it as:

$$\varphi : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \left[\left(\begin{array}{cc|c} \multicolumn{2}{c|}{M} & u_0 \\ \hline 0 & 0 & 1 \end{array} \right) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right]$$

$$\varphi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\varphi \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

$$= \begin{bmatrix} ax + by + x_0 z \\ cx + dy + y_0 z \\ z \end{bmatrix}$$

For us, the points of interest are the ones with $z = 1$

We can define projective transformations as:

$$\psi : \mathbb{RP}^2 \to \mathbb{RP}^2$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \left[\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right]$$

Out of these transformations, the affine transformations are the ones for which $a_{31} = a_{32} = 0$ and $a_{33} \neq 0$ (for the sake of simplicity $a_{33} = 1$)

13-1. Find the concatenation (product) of an anticlockwise rotation about the origin through an angle of $\frac{3\pi}{2}$, followed by a scaling by a factor of 3 units in the x-direction and 2 units in the y-direction

$$S(3,2) \circ R_{\frac{3\pi}{2}}$$

$$[R_{\frac{3\pi}{2}}] = \begin{pmatrix} \cos\frac{3\pi}{2} & -\sin\frac{3\pi}{2} & 0 \\ \sin\frac{3\pi}{2} & \cos\frac{3\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[S(3,2)] = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[S(3,2) \circ R_{\frac{3\pi}{2}}] = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \left(S(3,2)\circ R_{\frac{3\pi}{2}}\right)\cdot\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \left[\begin{pmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}\right] =$$

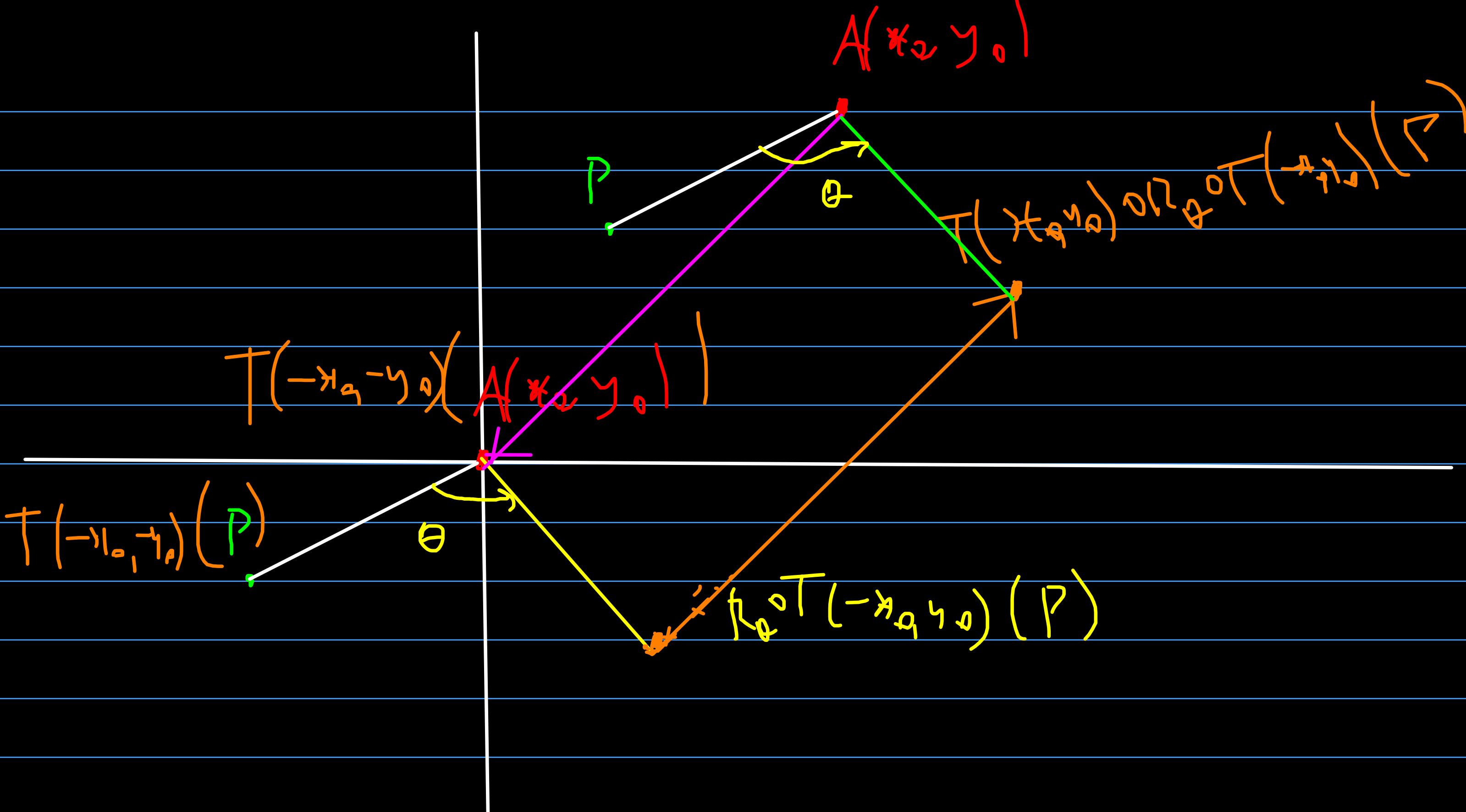
$$= \begin{bmatrix} 3y \\ -2x \\ 1 \end{bmatrix}$$

$$\left(S(3,2)\circ R_{\frac{3\pi}{2}}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ -2 & 0 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} =$$

$$= \begin{pmatrix} 3y \\ -2x \end{pmatrix}$$

13.3. $R_\theta(x_0, y_0) = T(x_0, y_0) \circ R_\theta \circ T(-x_0, -y_0)$

the rotation by an angle θ around a point $A(x_0, y_0)$



$$[R_\theta(x_0, y_0)] = \begin{pmatrix} \cos\theta & -\sin\theta & \alpha_0 \\ \sin\theta & \cos\theta & \beta_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\alpha_0 = -x_0 \cos\theta + y_0 \sin\theta + x_0$$

$$\beta_0 = -x_0 \sin\theta - y_0 \cos\theta + y_0$$

13.* Let l_1, l_2 be parallel lines. Show that $r_{l_1} \circ r_{l_2}$ is a translation (and find the vector of this translation).

$$l_1: \quad ax + by + c_1 = 0$$

$$l_2: \quad ax + by + c_2 = 0$$

$$c_1, c_2 \in \mathbb{R}$$

$$r_l \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2+b^2} \cdot \begin{pmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} +$$

$$+ \frac{-2c}{a^2+b^2} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

$$[r_{l_2}] = \begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & \frac{-2ab}{a^2+b^2} & \frac{-2c_2a}{a^2+b^2} \\ \frac{-2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & \frac{-2c_2b}{a^2+b^2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$[r_{l_1}] = \begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & \frac{-2ab}{a^2+b^2} & \frac{-2c_1a}{a^2+b^2} \\ \frac{-2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & \frac{-2c_1b}{a^2+b^2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$[r_{l_2}] = \begin{pmatrix} b^2-a^2 & -2ab & -2c_2a \\ -2ab & a^2-b^2 & -2c_2b \\ 0 & 0 & a^2+b^2 \end{pmatrix}$$

$$[r_{l_1}] = \begin{pmatrix} b^2-a^2 & -2ab & -2c_1a \\ -2ab & a^2-b^2 & -2c_1b \\ 0 & 0 & a^2+b^2 \end{pmatrix}$$

$$[r_{l_1} o r_{l_2}] = \begin{pmatrix} b^2-a^2 & -2ab & -2c_1a \\ -2ab & a^2-b^2 & -2c_1b \\ 0 & 0 & a^2+b^2 \end{pmatrix}\begin{pmatrix} b^2-a^2 & -2ab & -2c_2a \\ -2ab & a^2-b^2 & -2c_2b \\ 0 & 0 & a^2+b^2 \end{pmatrix}$$

$$= C$$

$$c_{11} = (b^2-a^2)^2 + (2ab)^2 = b^4 - 2a^2b^2 + a^4 +$$
$$+ 4a^2b^2 = (b^2+a^2)^2$$

$$c_{12} = -2ab(b^2-a^2) - 2ab(a^2-b^2) =$$
$$= -2ab^3 + 2a^3b - 2a^3b + 2ab^3 = 0$$

$$c_{13} = -2c_2a(b^2-a^2) + 4ab^2c_2 - 2c_1a(a^2+b^2)$$

$$c_{21} = -2ab(b^2-a^2) - 2ab(a^2-b^2) = 0$$

$$c_{22} = 4a^2b^2 + (a^2-b^2)^2 = (a^2+b^2)^2$$

$$c_{23} = 4a^2b\,c_2 - 2c_2b(a^2-b^2) - 2c_1b(a^2+b^2)$$

$$c_{31} = 0 \quad , \qquad c_{32} = 0 \ , \ c_{33} = (a^2+b^2)^2$$

$$[v_{\ell_1} \circ v_{\ell_2}] = \begin{pmatrix} (a^2+b^2)^2 & 0 & c_{13} \\ 0 & (a^2+b^2)^2 & c_{23} \\ 0 & 0 & (a^2+b^2)^2 \end{pmatrix} \simeq$$

$$= \begin{pmatrix} 1 & 0 & \frac{c_{13}}{(a^2+b^2)^2} \\ 0 & 1 & \frac{c_{23}}{(a^2+b^2)^2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow r_{\ell_1} \circ r_{\ell_2} = T\left(\frac{c_{13}}{(a^2+b^2)^2}, \frac{c_{23}}{(a^2+b^2)^2} \right)$$

13.4. $P(x_0, y_0)$, $Q(x_1, y_1)$, $P \neq Q$

$\theta \in \mathbb{R}$

Show that $R_{-\theta}(x_1, y_1) \circ R_\theta(x_0, y_0)$ is a translation.

$$[R_\theta(x_0, y_0)] = \begin{pmatrix} \cos\theta & -\sin\theta & \alpha_0 \\ \sin\theta & \cos\theta & \beta_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[R_{-\theta}(x_1, y_1)] = \begin{pmatrix} \cos\theta & \sin\theta & \alpha_1 \\ -\sin\theta & \cos\theta & \beta_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[R_{-\theta}(x_1, y_1) \cdot R_\theta(x_0, y_0)] =$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & \alpha_1 \\ -\sin\theta & \cos\theta & \beta_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -\sin\theta & \alpha_0 \\ \sin\theta & \cos\theta & \beta_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= M$$

$$m_{11} = \cos^2\theta + \sin^2\theta = 1$$

$$m_{12} = -\cos\theta \sin\theta + \cos\theta \sin\theta = 0$$

$$m_{13} = \alpha_0 \cos\theta + \beta_0 \sin\theta + \alpha_1$$

$$m_{21} = -\cos\theta \sin\theta + \cos\theta \cdot \sin\theta = 0$$

$$m_{22} = \sin^2\theta + \cos^2\theta = 1$$

$$m_{23} = -\alpha_0 \sin\theta + \beta_0 \cdot \cos\theta + \beta_1$$

$$m_{31} = 0, \quad m_{32} = 0, \quad m_{33} = 1$$

$$\Rightarrow \quad [R_{-\theta}(x_1, y_1) \circ R_\theta(x_0, y_0)] =$$

$$= \begin{pmatrix} 1 & 0 & m_{13} \\ 0 & 1 & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow R_{-\theta}(x_1, y_1) \circ R_\theta(x_0, y_0) \text{ is } T(m_{13}, m_{23})$$

$$m_{13} = \alpha_0 \cos\theta + \beta_0 \sin\theta + \alpha_1$$

$$m_{23} = -\alpha_0 \sin\theta + \beta_0 \cdot \cos\theta + \beta_1$$

$$\alpha_0 = -x_0 \cdot \cos\theta + y_0 \sin\theta + x_0$$

$$\beta_0 = -x_0 \sin\theta - y_0 \cdot \cos\theta + y_0$$

$$\alpha_1 = -x_1 \cdot \cos\theta - y_1 \sin\theta + x_1$$

$$\beta_1 = x_1 \cdot \sin\theta - y_1 \cos\theta + y_1$$

$$m_{13} = -x_0 \cos^2\theta + y_0 \sin\theta \cos\theta +$$

$$+ x_0 \cdot \cos\theta - x_0 \sin^2\theta - y_0 \sin\theta \cos\theta +$$

$$+ y_0 \sin\theta - x_1 \cos\theta - y_1 \sin\theta + x_1 =$$

$$= (x_1 - x_0) + \cos\theta (x_0 - x_1) + (y_0 - y_1) \sin\theta$$

$$m_{21} = x_0 \sin\theta \cos\theta - y_0 \sin^2\theta - x_0 \sin\theta$$

$$- x_0 \sin\theta \cdot \cos\theta - y_0 \cos^2\theta + y_0 \cos\theta$$

$$+ x_1 \sin\theta - y_1 \cos\theta + y_1$$

$$= (y_1 - y_0) + \cos\theta (y_0 - y_1) + \sin\theta (x_1 - x_0)$$

$\Rightarrow$ The translation vector is:

$$\begin{pmatrix} (x_1 - x_0)(1-\cos\theta) + (y_0 - y_1)\cdot\sin\theta \\ (y_1 - y_0)(1-\cos\theta) + (x_0 - x_1)\cdot\sin\theta \end{pmatrix}$$