6 Week 6:

6.1 The vector product

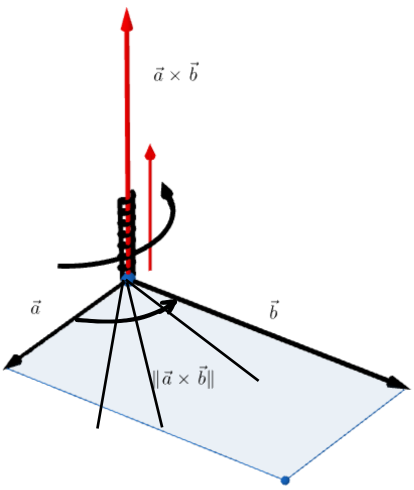
Definition 6.1. The *vector product* or the *cross product* of the vectors $\vec{a}, \vec{b} \in \mathcal{V}$ is a vector, denoted by $\vec{a} \times \vec{b}$, which is defined to be zero if $\vec{a}, \vec{b}$ are linearly dependent (collinear), and if $\vec{a}, \vec{b}$ are linearly independent (noncollinear), then it is defined by the following data:

1. $\vec{a} \times \vec{b}$ is a vector orthogonal on the two-dimensional subspace $\langle \vec{a}, \vec{b} \rangle$ of $\mathcal{V}$;

2. if $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$, then the sense of $\vec{a} \times \vec{b}$ is the one in which a right-handed screw, placed along the line passing through O orthogonal to the vectors $\vec{a}$ and $\vec{b}$, advances when it is being rotated simultaneously with the vector $\vec{a}$ from $\vec{a}$ towards $\vec{b}$ within the vector subspace $\langle \vec{a}, \vec{b} \rangle$ and the support half line of $\vec{a}$ sweeps the interior of the angle $\widehat{AOB}$ (Screw rule).

3. the *norm* (*magnitude* or *length*) of $\vec{a} \times \vec{b}$ is defined by

$$|| \vec{a} \times \vec{b} || = || \vec{a} || \cdot || \vec{b} || \sin(\vec{a}, \vec{b}). \geq 0$$



Remark 6.1. 1. *The norm (magnitude or length) of the vector $\vec{a} \times \vec{b}$ is actually the area of the parallelogram constructed on the vectors $\vec{a}, \vec{b}$.*

2. *The vectors $\vec{a}, \vec{b} \in \mathcal{V}$ are linearly dependent (collinear) if and only if $\vec{a} \times \vec{b} = \vec{0}$.*

Proposition 6.1. *The vector product has the following properties:*

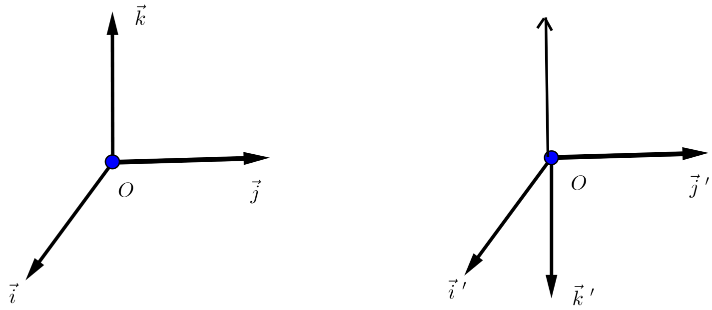
1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}, \forall\ \vec{a}, \vec{b} \in \mathcal{V}$;

2. $(\lambda \vec{a}) \times \vec{b} = \vec{a} \times (\lambda \vec{b}) = \lambda(\vec{a} \times \vec{b}), \forall \lambda \in \mathbb{R}, \vec{a}, \vec{b} \in \mathcal{V}$;

3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}, \forall\ \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$.

6.2 The vector product in terms of coordinates

If $[\vec{i}, \vec{j}, \vec{k}]$ is an orthonormal basis, observe that $\vec{i} \times \vec{j} \in \{-\vec{k}, \vec{k}\}$. We say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *direct* if $\vec{i} \times \vec{j} = \vec{k}$. If, on the contrary, $\vec{i} \times \vec{j} = -\vec{k}$, we say that the orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$ is *inverse*.



direct orthonormal basis inverse orthonormal basis

Therefore, if $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis, then $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$ and obviously $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$.

Proposition 6.2. *If $[\vec{i}, \vec{j}, \vec{k}]$ is a direct orthonormal basis and $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$, $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$, then*

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2) \vec{i} + (a_3 b_1 - a_1 b_3) \vec{j} + (a_1 b_2 - a_2 b_1) \vec{k}, \tag{6.1}$$

or, equivalently,

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \tag{6.2}$$

Proof. Indeed,

$$\begin{aligned}
\vec{a} \times \vec{b} &= (a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}) \times (b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}) \\
&= a_1 b_1 \vec{i} \times \vec{i} + a_1 b_2 \vec{i} \times \vec{j} + a_1 b_3 \vec{i} \times \vec{k} \\
&+ a_2 b_1 \vec{j} \times \vec{i} + a_2 b_2 \vec{j} \times \vec{j} + a_2 b_3 \vec{i} \times \vec{k} \\
&+ a_3 b_1 \vec{k} \times \vec{i} + a_3 b_2 \vec{k} \times \vec{j} + a_3 b_3 \vec{k} \times \vec{k} \\
&= a_1 b_2 \vec{k} - a_1 b_3 \vec{j} - a_2 b_1 \vec{k} + a_2 b_3 \vec{i} + a_3 b_1 \vec{j} - a_3 b_2 \vec{i} \\
&= (a_2 b_3 - a_3 b_2) \vec{i} + (a_3 b_1 - a_1 b_3) \vec{j} + (a_1 b_2 - a_2 b_1) \vec{k}
\end{aligned}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

□

One can rewrite formula (6.1) in the form

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{6.3}$$

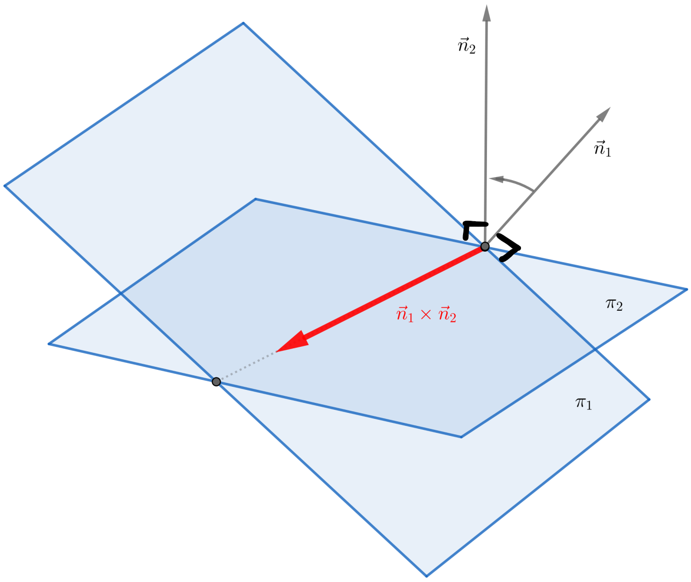
the right hand side determinant being understood in the sense of its cofactor expansion along the first line.

Remark 6.2. If $R = (O, \vec{i}, \vec{j}, \vec{k})$ is the direct Cartesian orthonormal reference system behind the equations of the line

$$(\Delta) \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_1 = 0, \end{cases}$$

then we can recover the director parameters (4.10) of Δ, in this particular case of orthonormal Cartesian reference systems, by observing that $\vec{n}_1 \times \vec{n}_2$ is a director vector of Δ, where

$$\vec{n}_1 = A_1\vec{i} + B_1\vec{j} + C_1\vec{k}$$
$$\vec{n}_2 = A_2\vec{i} + B_2\vec{j} + C_2\vec{k}.$$



Recall that

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} \vec{i} + \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} \vec{j} + \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \vec{k}.$$

Note however that the director parameters were obtained before for arbitrary Cartesian reference systems (See (4.10)).

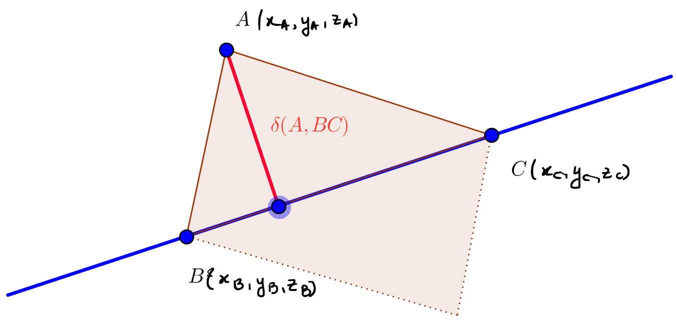
6.3 Applications of the vector product

- **The area of the triangle ABC.** $S_{ABC} = \frac{1}{2} || \overrightarrow{AB} || \cdot || \overrightarrow{AC} || \sin \widehat{BAC} = \frac{1}{2} || \overrightarrow{AB} \times \overrightarrow{AC} ||$. On the other hand

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_B - x_A & y_B - x_A & z_B - z_A \\ x_C - x_A & y_C - x_A & z_C - z_A \end{vmatrix},$$

as the coordinates of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are $(x_B - x_A, y_B - x_A, z_B - z_A)$ and $(x_C - x_A, y_C - x_A, z_C - z_A)$ respectively. Thus,

$$4S^2_{ABC} = \begin{vmatrix} y_B - y_A & z_B - z_A \\ y_C - y_A & z_C - z_A \end{vmatrix}^2 + \begin{vmatrix} z_B - z_A & x_B - x_A \\ z_C - z_A & x_C - x_A \end{vmatrix}^2 + \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}^2.$$



- **The distance from one point to a straight line.**

(a) *The distance $\delta(A,BC)$ from the point $A(x_A,y_A,z_A)$ to the straight line BC*, where $B(x_B,y_B,z_B)$ and $C(x_C,y_C,z_C)$. Since

$$S_{ABC}=\frac{||\overrightarrow{BC}||\cdot\delta(A,BC)}{2}$$

it follos that

$$\delta^2(A,BC)=\frac{4S^2_{ABC}}{||\overrightarrow{BC}||^2}.$$

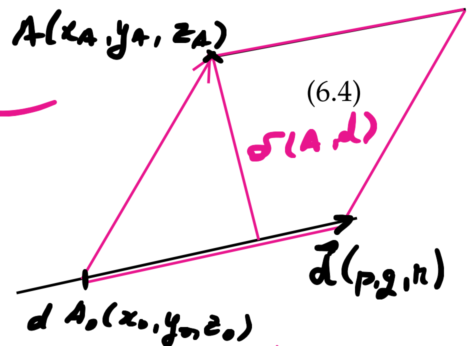
Thus, we obtain

$$\delta^2(A,BC)=\frac{\begin{vmatrix} y_B-y_A & z_B-z_A\\ y_C-y_A & z_C-z_A\end{vmatrix}^2+\begin{vmatrix} z_B-z_A & x_B-x_A\\ z_C-z_A & x_C-x_A\end{vmatrix}^2+\begin{vmatrix} x_B-x_A & y_B-y_A\\ x_C-x_A & y_C-y_A\end{vmatrix}^2}{(x_C-x_B)^2+(y_C-y_B)^2+(z_C-z_B)^2}.$$

(b) The distance from $\delta(A,d)$ from one point $A(A_A,y_A,z_A)$ to the straight line

$$d:\frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}.$$

$$\delta(A,d)=\frac{||\vec{d}\times\overrightarrow{A_0A}||}{||\vec{d}||},\tag{6.4}$$

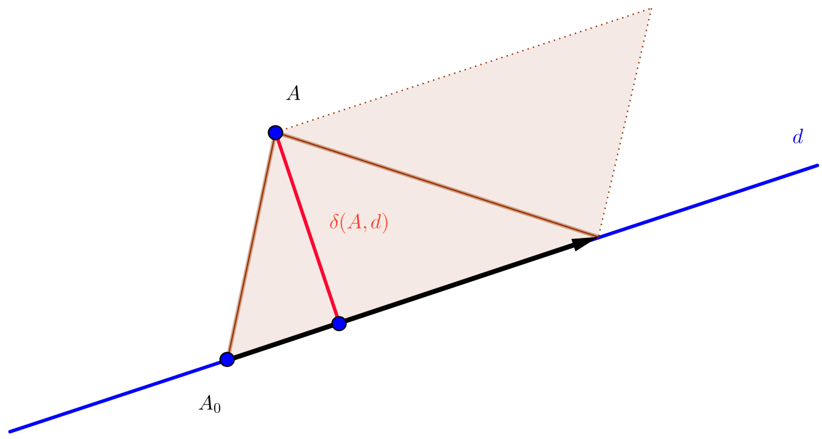


where $A_0(x_0,y_0,z_0)\in\delta$.
Since

$$\begin{aligned}\vec{d}\times\overrightarrow{A_0A} &= \begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ p&q&r\\ x_A-x_0&y_A-y_0&z_A-z_0\end{vmatrix}\\ &= \begin{vmatrix} q&r\\ y_A-y_0&z_A-z_0\end{vmatrix}\vec{i}+\begin{vmatrix} r&p\\ z_A-z_0&x_A-x_0\end{vmatrix}\vec{j}+\begin{vmatrix}p&q\\x_A-x_0&y_A-y_0\end{vmatrix}\vec{k}\end{aligned}$$

it follows that

$$\delta(A,d)=\frac{\sqrt{\begin{vmatrix} q&r\\ y_A-y_0&z_A-z_0\end{vmatrix}^2+\begin{vmatrix} r&p\\ z_A-z_0&x_A-x_0\end{vmatrix}^2+\begin{vmatrix}p&q\\x_A-x_0&y_A-y_0\end{vmatrix}^2}}{\sqrt{p^2+q^2+r^2}}.$$



6.4 The double vector (cross) product

The *double vector (cross) product* of the vectors $\vec{a}, \vec{b}, \vec{c}$ is the vector $\vec{a} \times (\vec{b} \times \vec{c})$ $\underset{\text{usually}}{\neq} \underbrace{(\vec{a} \times \vec{b}) \times \vec{c}}_{\in\langle \vec{a}, \vec{b}\rangle}$

Proposition 6.3.

$$\vec{a} \times (\vec{b} \times \vec{c}) = \underbrace{(\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{a} \cdot \vec{b})\, \vec{c}}_{\in\langle \vec{b}, \vec{c}\rangle} = \begin{vmatrix} \vec{b} & \vec{c} \\ \vec{a} \cdot \vec{b} & \vec{a} \cdot \vec{c} \end{vmatrix}, \quad \forall\ \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}. \tag{6.5}$$

Proof. (Sketch) If the vectors $\vec{b}$ and $\vec{c}$ are linearly dependent, then both sides are obviously zero. Otherwise one can choose an orthonormal basis $[\vec{i}, \vec{j}, \vec{k}]$, related to the vectors $\vec{a}, \vec{b}$ and $\vec{c}$, such that

$\vec{i} = \frac{\vec{b}}{\|\vec{b}\|}$

$$\vec{b} = b_1\, \vec{i}, \vec{c} = c_1\, \vec{i} + c_2\, \vec{j}, \vec{a} = a_1\, \vec{i} + a_2\, \vec{j} + a_3\, \vec{k}.$$

For example one can choose $\vec{i}$ to be $\vec{b}\ / \|\ \vec{b}\ \|$ and $\vec{j}$ a unit vector in the subspace $\langle \vec{b}, \vec{c} \rangle$ which is perpendicular on $\vec{b}$. Finally, one can choose $\vec{k} = \vec{i} \times \vec{j}$. By computing the two sides of the equality 6.5, in terms of coordinates and the vectors $\vec{i}, \vec{j}, \vec{k}$, one gets the same result. □

Corollary 6.4. *1.* $(\vec{a} \times \vec{b}) \times \vec{c} = \underbrace{(\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{b} \cdot \vec{c})\, \vec{a}}_{\in\langle \vec{a}, \vec{b}\rangle} = \begin{vmatrix} \vec{b} & \vec{a} \\ \vec{c} \cdot \vec{b} & \vec{c} \cdot \vec{a} \end{vmatrix}, \forall\ \vec{a}, \vec{b}, \vec{c} \in \mathcal{V};$

2. $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}$, $\forall\ \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$ *(Jacobi's identity).*

Proof. While the first identity follows immediately via 6.5, for the Jacobi's identity we get successively:

$$\begin{aligned} &\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) \\ &= (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{a} \cdot \vec{b})\, \vec{c} + (\vec{b} \cdot \vec{a})\, \vec{c} - (\vec{b} \cdot \vec{c})\, \vec{a} + (\vec{c} \cdot \vec{b})\, \vec{a} - (\vec{c} \cdot \vec{a})\, \vec{b} = \vec{0}. \end{aligned}$$

□

6.5 Problems

1. **(2p)** Show that $\|\ \vec{a} \times \vec{b}\ \| \leq \|\ \vec{a}\ \| \cdot \|\ \vec{b}\ \|$, $\forall\ \vec{a}, \vec{b}, \in \mathcal{V}$.

   Solution.

2. **(3p)** Let $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ be pairwise noncollinear vectors. Show that the necesssary and sufficient condition for the existence of a triangle ABC with the properties $\overrightarrow{BC}=\overrightarrow{a}$, $\overrightarrow{CA}=\overrightarrow{b}$, $\overrightarrow{AB}=\overrightarrow{c}$ is

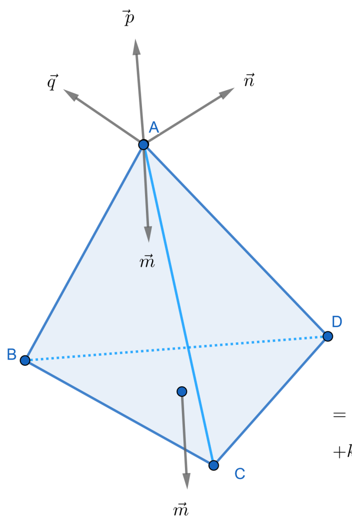
$$\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{b} \times \overrightarrow{c} = \overrightarrow{c} \times \overrightarrow{a}.$$

From the equalities of the norms deduce the low of sines.

Solution.

3. **(3p)** Show that the sum of some outer-pointing vectors perpendicular on the faces of a tetrahedron which are proportional to the areas of the faces is the zero vector.

Solution.



The proportionality of $\vec{m}, \vec{n}, \vec{p}, \vec{q}$ with the areas of the corresponding faces of the tetrahedron show that

$\vec{m} = k\overrightarrow{BD} \times \overrightarrow{BC}$, $\vec{n} = k\overrightarrow{AC} \times \overrightarrow{AD}$

$\vec{p} = k\overrightarrow{AD} \times \overrightarrow{AB}$, $\vec{q} = k\overrightarrow{AB} \times \overrightarrow{AC}$

Thus, $\vec{m} + \vec{n} + \vec{p} + \vec{q}$

$= k\overrightarrow{BD} \times \overrightarrow{BC} + k\overrightarrow{AC} \times \overrightarrow{AD} +$

$+ k\overrightarrow{AD} \times \overrightarrow{AB} + k\overrightarrow{AB} \times \overrightarrow{AC}$

$= k(\overrightarrow{AD} - \overrightarrow{AB}) \times (\overrightarrow{AC} - \overrightarrow{AB}) + k\overrightarrow{AC} \times \overrightarrow{AD}$

$+ k\overrightarrow{AD} \times \overrightarrow{AB} + k\overrightarrow{AB} \times \overrightarrow{AC} =$

$= k\overrightarrow{AD} \times \overrightarrow{AC} - k\overrightarrow{AD} \times \overrightarrow{AB} - k\overrightarrow{AB} \times \overrightarrow{AC} + k\overrightarrow{AB} \times \overrightarrow{AB} =$

$+ k\overrightarrow{AC} \times \overrightarrow{AD} + k\overrightarrow{AD} \times \overrightarrow{AB} + k\overrightarrow{AB} \times \overrightarrow{AC} = \vec{0}.$

4. **(2p)** Find the distance from the point $P(1, 2, -1)$ to the straight line (d) $x = y = z$.

Solution. $\vec{v_\ell}(1,1,1), A_0(1,1,1)\in \ell, \vec{PA_0}(0,-1,2)$

$$\vec{PA_0}\times\vec{v_\ell} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -\vec{i}+2\vec{j}+\vec{k}-2\vec{i} = -3\vec{i}+2\vec{j}+\vec{k} \Rightarrow \|\vec{PA_0}\times\vec{v_\ell}\| = \sqrt{9+4+1}=\sqrt{14}$$

$$\|\vec{v_\ell}\| = \sqrt{3}$$

$$\Rightarrow \text{dist}(P,\ell) = \frac{\|\vec{PA_0}\times\vec{v_\ell}\|}{\|\vec{v_\ell}\|} = \frac{\sqrt{14}}{\sqrt{3}}$$

5. **(3p)** Find the area of the triangle ABC and the lengths of its hights, where $A(-1,1,2)$, $B(2,-1,1)$ and $C(2,-3,-2)$.

$$A_{ABC} = \frac{1}{2}\|\vec{AB}\times\vec{AC}\|,\ \vec{AB}(3,-2,-1),\ \vec{AC}(3,-4,-4),\ \vec{BC}(0,-2,-3)$$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & -1 \\ 3 & -4 & -4 \end{vmatrix} = 4\vec{i}+9\vec{j}-6\vec{k}$$

$$\|\vec{AB}\times\vec{AC}\| = \sqrt{16+81+36} = \sqrt{133} \Rightarrow A_{ABC} = \frac{\sqrt{133}}{2}$$

$$h_A = \frac{2A_{ABC}}{\|\vec{BC}\|} = \frac{2\frac{\sqrt{133}}{2}}{\sqrt{13}} = \frac{\sqrt{133}}{\sqrt{13}}$$

$$h_B = \frac{2A_{ABC}}{\|\vec{AC}\|} = \frac{\sqrt{133}}{\sqrt{41}}$$

$$h_C = \frac{2A_{ABC}}{\|\vec{AB}\|} = \frac{\sqrt{133}}{\sqrt{14}}$$

6. **(3p)** Let d_1, d_2, d_3, d_4 be pairwise skew straight lines. Assuming that $d_{12}\perp d_{34}$ and $d_{13}\perp d_{24}$, show that $d_{14}\perp d_{23}$, where d_{ik} is the common perpendicular of the lines d_i and d_k.

Solution. A director vector of the common perpendicular d_{ij} is $\vec{d}_i\times\vec{d}_j$, where $\vec{d}_r$ stands for a director vector of d_r. Therefore we have successively:

$$\begin{aligned} d_{12}\perp d_{34} &\Leftrightarrow \vec{d}_1\times\vec{d}_2 \perp \vec{d}_3\times\vec{d}_4 \Leftrightarrow (\vec{d}_1\times\vec{d}_2)\cdot(\vec{d}_3\times\vec{d}_4)=0 \\ &\Leftrightarrow \begin{vmatrix} \vec{d}_1\cdot\vec{d}_3 & \vec{d}_1\cdot\vec{d}_4 \\ \vec{d}_2\cdot\vec{d}_3 & \vec{d}_2\cdot\vec{d}_4 \end{vmatrix} = 0 \Leftrightarrow (\vec{d}_1\cdot\vec{d}_3)(\vec{d}_2\cdot\vec{d}_4) = (\vec{d}_1\cdot\vec{d}_4)(\vec{d}_2\cdot\vec{d}_3). \end{aligned}$$

Similalry

$$\begin{aligned} d_{13}\perp d_{24} &\Leftrightarrow \vec{d}_1\times\vec{d}_3 \perp \vec{d}_2\times\vec{d}_4 \Leftrightarrow (\vec{d}_1\times\vec{d}_3)\cdot(\vec{d}_2\times\vec{d}_4)=0 \\ &\Leftrightarrow \begin{vmatrix} \vec{d}_1\cdot\vec{d}_2 & \vec{d}_1\cdot\vec{d}_4 \\ \vec{d}_3\cdot\vec{d}_2 & \vec{d}_3\cdot\vec{d}_4 \end{vmatrix} = 0 \Leftrightarrow (\vec{d}_1\cdot\vec{d}_2)(\vec{d}_3\cdot\vec{d}_4) = (\vec{d}_1\cdot\vec{d}_4)(\vec{d}_3\cdot\vec{d}_2). \end{aligned}$$

Therefore we have

$$(\vec{d}_1 \cdot \vec{d}_3)(\vec{d}_2 \cdot \vec{d}_4) = (\vec{d}_1 \cdot \vec{d}_4)(\vec{d}_2 \cdot \vec{d}_3) = (\vec{d}_1 \cdot \vec{d}_2)(\vec{d}_3 \cdot \vec{d}_4),$$

which shows that

$$(\vec{d}_1 \cdot \vec{d}_3)(\vec{d}_2 \cdot \vec{d}_4) - (\vec{d}_1 \cdot \vec{d}_2)(\vec{d}_3 \cdot \vec{d}_4) = 0 \Leftrightarrow \begin{vmatrix} \vec{d}_1 \cdot \vec{d}_2 & \vec{d}_1 \cdot \vec{d}_3 \\ \vec{d}_4 \cdot \vec{d}_2 & \vec{d}_4 \cdot \vec{d}_3 \end{vmatrix} = 0 \Leftrightarrow d_{14} \perp d_{23}.$$

Seminar W6 - Nicu - 31/03/2020

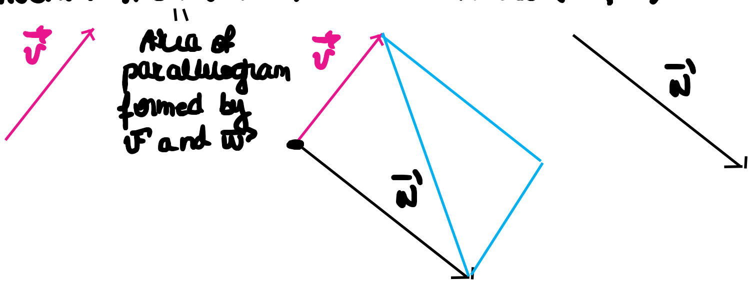
Cross product (vector product)

$\vec{v}, \vec{w}$ vectors

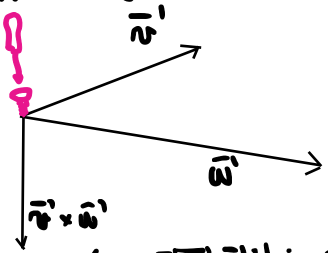
- if $\vec{v}, \vec{w}$ lin dep $\Rightarrow \vec{v} \times \vec{w} = \vec{0}$
- if $\vec{v}, \vec{w}$ lin. indep $\Rightarrow \vec{v} \times \vec{w} \in \mathcal{V}$

$\rightarrow$ direction: perp to $\vec{v}$ and $\vec{w}$, it is actually perp to $\langle \vec{v}, \vec{w} \rangle$

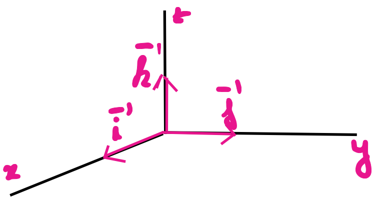
$\rightarrow$ norm: $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \sin(\widehat{\vec{v}, \vec{w}})$



$\rightarrow$ orientation: the screw rule



If the ref. system $(O, [\vec{i}, \vec{j}, \vec{k}])$ is orthonormal and direct (for us all the time)
$\hookrightarrow \vec{i} \times \vec{j} = \vec{k}$



, then the cross product is computed as follows:

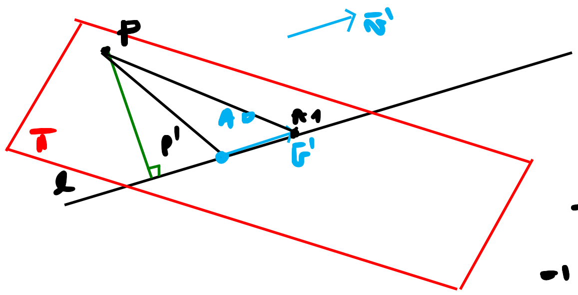
$\vec{v}(a_1, b_1, c_1)$, $\vec{w}(a_2, b_2, c_2)$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (b_1c_2 - b_2c_1)\vec{i} - (a_1c_2 - a_2c_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k} =$$
$$= (b_1c_2 - b_2c_1, a_1c_2 - a_2c_1, a_1b_2 - a_2b_1)$$

$\rightarrow$ the cross product is anti-commutative, bilinear

$\forall \alpha, \beta \in \mathbb{R}, \forall v_1, v_2, w \in \mathcal{V}: (\alpha\vec{v}_1 + \beta\vec{v}_2) \times \vec{w} = \alpha\vec{v}_1 \times \vec{w} + \beta \cdot \vec{v}_2 \times \vec{w}$

The distance from a point to a line in 3D



π plane so that $\pi \perp \ell$, $\pi \ni P$

$\{P'\} = \ell \cap \pi \Rightarrow \ell \perp PP' \Rightarrow PP'$ is the perpendicular from P to the line ℓ

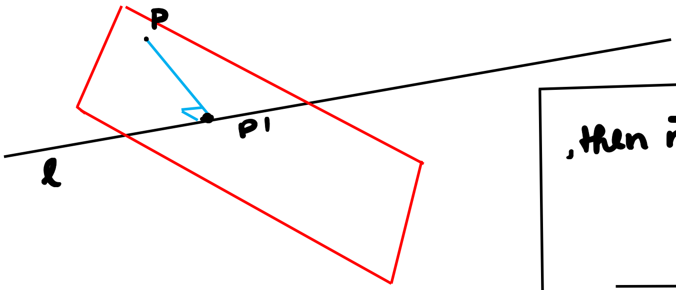
Take $A_0 \in \ell$, $\vec{v} \parallel \ell \Rightarrow \exists A_1 \in \ell$ so that: $\overrightarrow{A_0A_1} = \vec{v} \Rightarrow$

$\Rightarrow PP'$ height in $\Delta PA_0A_1 \Rightarrow$

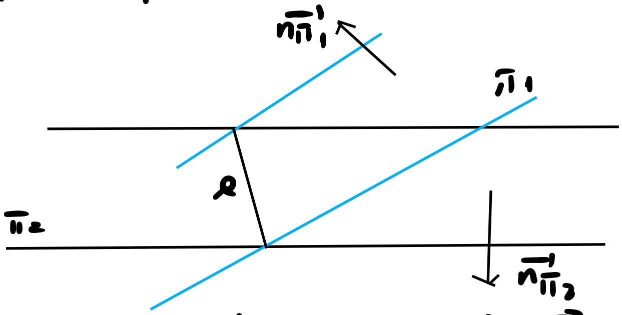
$$\Rightarrow PP' = \frac{2 \mathcal{A}_{PA_0A_1}}{A_0A_1} = \frac{\|\overrightarrow{PA_0} \times \overrightarrow{A_0A_1}\|}{\|\overrightarrow{A_0A_1}\|} = \frac{\|\overrightarrow{PA_0} \times \vec{v}\|}{\|\vec{v}\|}$$

Extra: Consider the line:

$$l: \begin{cases} \pi_1: x+2y-8z+5=0 \\ \pi_2: 2x+y+z+1=0 \end{cases}$$, and the point $P(1,2,3)$. Find the eq of the perp from the point P onto the line l.



If $l: \begin{cases} \pi_1: A_1x+B_1y+C_1z+D_1=0 \\ \pi_2: A_2x+B_2y+C_2z+D_2=0 \end{cases}$, then $\vec{n_{\pi_1}} \times \vec{n_{\pi_2}}$ is a director vector of the line l.



$$\left.\begin{array}{l} \vec{n_{\pi_1}} \perp \pi_1 \Rightarrow \vec{n_{\pi_1}} \perp l \\ \vec{n_{\pi_2}} \perp \pi_2 \Rightarrow \vec{n_{\pi_2}} \perp l \end{array}\right\} \Rightarrow l \parallel \vec{n_{\pi_1}} \times \vec{n_{\pi_2}}$$

$\vec{n_{\pi_1}}(1,2,-8)$, $\vec{n_{\pi_2}}(2,1,1) \Rightarrow$

$\Rightarrow \vec{v_l} = \vec{n_{\pi_1}} \times \vec{n_{\pi_2}} =$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -8 \\ 2 & 1 & 1 \end{vmatrix} = 2\vec{i} + \vec{k} - 16\vec{j} - 4\vec{k} + 8\vec{i} - \vec{j} =$$

$$= 10\vec{i} - 3\vec{k} - 17\vec{j}$$

$\Rightarrow \vec{v_l}(10,-17,-3)$

We now write the eq of the plane π that is perp to l and contains P.

$\vec{v_l} \perp \pi \Rightarrow \vec{v_l} \parallel \vec{n_\pi} \Rightarrow \pi: 10x - 17y - 3z + D = 0$

$P \in \pi \Rightarrow 10 \cdot 1 - 17 \cdot 2 - 3 \cdot 3 + D = 0 \Rightarrow D = 34 + 9 - 10 \Rightarrow D = 33 \Rightarrow$

$\Rightarrow \pi: 10x - 17y - 3z + 33 = 0$

$$P' = \pi \cap l: \begin{cases} 10x - 17y - 3z + 33 = 0 \\ x + 2y - 8z + 5 = 0 \\ 2x + y + z + 1 = 0 \end{cases} \Rightarrow \begin{pmatrix} 10 & -17 & -3 & -33 \\ 1 & 2 & -8 & -5 \\ 2 & 1 & 1 & -1 \end{pmatrix} \overset{L_1 \leftrightarrow L_2}{\sim}$$

$$\sim \begin{pmatrix} 1 & 2 & -8 & -5 \\ 10 & -17 & -3 & -33 \\ 2 & 1 & 1 & -1 \end{pmatrix} \underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_2 \leftarrow L_2 - 10L_1}{\sim}} \begin{pmatrix} 1 & 2 & -8 & -5 \\ 0 & -37 & 77 & 17 \\ 0 & -3 & 17 & 9 \end{pmatrix} \overset{L_3 \leftarrow L_3 - \frac{3}{37}L_2}{\sim}$$

$$\sim \begin{pmatrix} 1 & 2 & -8 & -5 \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix} \rightarrow \begin{array}{l} x_{P'} = \ldots \\ y_{P'} = \ldots \\ z_{P'} = \ldots \end{array}$$

$$\Rightarrow \frac{x - x_P}{x_{P'} - x_P} = \frac{y - y_P}{y_{P'} - y_P} = \frac{z - z_P}{z_{P'} - z_P}$$