

## 5 Week 5: Products of vectors

### 5.1 The dot product

**Definition 5.1.** The real number

$$\vec{a} \cdot \vec{b} = \begin{cases} 0 & \text{if } \vec{a} = 0 \text{ or } \vec{b} = 0 \\ \|\vec{a}\| \|\vec{b}\| \cos(\vec{a}, \vec{b}) & \text{if } \vec{a} \neq 0 \text{ and } \vec{b} \neq 0 \end{cases} \quad (5.1)$$

is called the *dot product* of the vectors  $\vec{a}, \vec{b}$ .

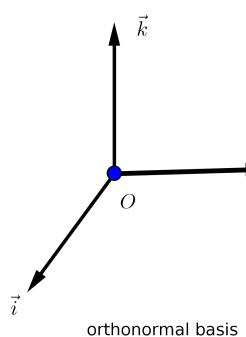
**Remark 5.1.** 1.  $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$ .

$$2. \vec{a} \cdot \vec{a} = \|\vec{a}\| \|\vec{a}\| \cos 0 = \|\vec{a}\|^2.$$

**Proposition 5.1.** The dot product has the following properties:

1.  $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}, \forall \vec{a}, \vec{b} \in \mathcal{V}$ .
2.  $\vec{a} \cdot (\lambda \vec{b}) = \lambda(\vec{a} \cdot \vec{b}), \forall \lambda \in \mathbb{R}, \vec{a}, \vec{b} \in \mathcal{V}$ .
3.  $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}, \forall \vec{a}, \vec{b}, \vec{c} \in \mathcal{V}$ .
4.  $\vec{a} \cdot \vec{a} \geq 0, \forall \vec{a} \in \mathcal{V}$ .
5.  $\vec{a} \cdot \vec{a} = 0 \Leftrightarrow \vec{a} = 0$ .

**Definition 5.2.** A basis of the vector space  $\mathcal{V}$  is said to be *orthonormal*, if  $\|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = 1$ ,  $\vec{i} \perp \vec{j}$ ,  $\vec{j} \perp \vec{k}$ ,  $\vec{k} \perp \vec{i}$  ( $\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$ ,  $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$ ). A Cartesian reference system  $R = (O, \vec{i}, \vec{j}, \vec{k})$  is said to be *orthonormal* if the basis  $[\vec{i}, \vec{j}, \vec{k}]$  is orthonormal.



**Proposition 5.2.** Let  $[\vec{i}, \vec{j}, \vec{k}]$  be an orthonormal basis and  $\vec{a}, \vec{b} \in \mathcal{V}$ . If  $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$ ,  $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$ , then

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (5.2)$$

*Proof.* Indeed,

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}) \cdot (b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}) \\ &= a_1 b_1 \vec{i} \cdot \vec{i} + a_1 b_2 \vec{i} \cdot \vec{j} + a_1 b_3 \vec{i} \cdot \vec{k} \\ &\quad + a_2 b_1 \vec{j} \cdot \vec{i} + a_2 b_2 \vec{j} \cdot \vec{j} + a_2 b_3 \vec{j} \cdot \vec{k} \\ &\quad + a_3 b_1 \vec{k} \cdot \vec{i} + a_3 b_2 \vec{k} \cdot \vec{j} + a_3 b_3 \vec{k} \cdot \vec{k} \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3. \end{aligned}$$

□

**Remark 5.2.** Let  $[\vec{i}, \vec{j}, \vec{k}]$  be an orthonormal basis and  $\vec{a}, \vec{b} \in \mathcal{V}$ . If  $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$  and  $\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$ , then

$$1. \vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2 \text{ and we conclude that } \|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

2.

$$\begin{aligned} \cos(\vec{a}, \vec{b}) &= \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|} \\ &= \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}}. \end{aligned} \quad (5.3)$$

In particular

$$\begin{aligned} \cos(\vec{a}, \vec{i}) &= \frac{a_1}{\sqrt{a_1^2 + a_2^2 + a_3^2}}; \\ \cos(\vec{a}, \vec{j}) &= \frac{a_2}{\sqrt{a_1^2 + a_2^2 + a_3^2}}; \\ \cos(\vec{a}, \vec{k}) &= \frac{a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}. \end{aligned}$$

$$3. \vec{a} \perp \vec{b} \Leftrightarrow a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$$

### 5.1.1 Applications of the dot product

#### ◊ The two dimensional setting

- **The distance between two points** Consider two points  $A(x_A, y_A), B(x_B, y_B) \in \pi$ . The norm of the vector  $\vec{AB} (x_B - x_A, y_B - y_A)$  is

$$\|\vec{AB}\| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}.$$

- **The equation of the circle**

Recall that the circle  $C(O, r)$  is the locus of points  $M$  in the plane such that  $\text{dist}(O, M) = r \iff \|\vec{OM}\| = r$ . If  $(a, b)$  are the coordinates of  $O$  and  $(x, y)$  are the coordinates of  $M$ , then

$$\begin{aligned} \|\vec{OM}\| = r &\iff \sqrt{(x - a)^2 + (y - b)^2} = r \iff (x - a)^2 + (y - b)^2 = r^2 \\ &\iff x^2 + y^2 - 2ax - 2by + c = 0, \end{aligned} \quad (5.4)$$

where  $c = a^2 + b^2 - r^2$ . Conversely, every equation of the form  $x^2 + y^2 + 2ex + 2fy + g = 0$  is the equation of the circle centered at  $(-e, -f)$  and having the radius  $r = \sqrt{e^2 + f^2 - g}$ , whenever  $e^2 + f^2 \geq g$ . One can find the equation of the circle circumscribed to the triangle  $ABC$  by imposing the requirement on the coordinates  $(x_A, y_A), (x_B, y_B)$  and  $(x_C, y_C)$  of its vertices  $A, B, C$  to verify the equation  $x^2 + y^2 + 2ex + 2fy + g = 0$ . A point  $M(x, y)$  belongs to this circumcircle if and only if

$$\begin{cases} x^2 + y^2 + 2ex + 2fy + g = 0 \\ x_A^2 + y_A^2 + 2ex_A + 2fy_A + g = 0 \\ x_B^2 + y_B^2 + 2ex_B + 2fy_B + g = 0 \\ x_C^2 + y_C^2 + 2ex_C + 2fy_C + g = 0 \end{cases} \quad (5.5)$$

One can regard the system (5.5) as linear with the unknowns  $e, g, f$ , whose compatibility is given, via the Kronecker-Capelli theorem, by

$$\left| \begin{array}{ccccc} x^2 + y^2 & x & y & 1 \\ x_A^2 + y_A^2 & x_A & y_A & 1 \\ x_B^2 + y_B^2 & x_B & y_B & 1 \\ x_C^2 + y_C^2 & x_C & y_C & 1 \end{array} \right| = 0,$$

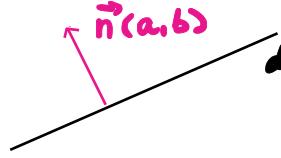
which is the equation of the circumcircle of the triangle  $ABC$ .

- **The normal vector of a line** If  $R = (O, b)$  is the orthonormal Cartesian reference system behind the equation of a line  $(d)$   $ax + by + c = 0$ , then  $\vec{n}(a, b)$  is a normal vector to the direction  $\vec{d}$  of  $d$ . Indeed, every vector of the direction  $\vec{d}$  of  $d$  has the form  $\vec{PM}$ , where  $P(x_p, y_p)$  and  $M(x, y)$  are two points on the line  $d$ . Thus,  $ax_p + by_p + c = 0 = ax_M + by_M + c$ , which shows that

$$a(x_M - x_p) + b(y_M - y_p) = 0,$$

namely

$$\vec{n} \cdot \vec{PM} = 0 \iff \vec{n} \perp \vec{PM}.$$



- **The distance from a point to a line** If  $(d)$   $ax + by + c = 0$  is a line and  $M(x_M, y_M) \in \pi$  a given point, then the distance from  $M$  to  $d$  is

$$\delta(M, d) = \frac{|ax_M + by_M + c|}{\sqrt{a^2 + b^2}}. \quad (5.6)$$

Indeed,  $\delta(M, d) = |\delta|$ , where  $\delta$  is the real scalar with the property  $\vec{PM} = \delta \frac{\vec{n}}{\|\vec{n}\|}$  and  $P(x_p, y_p)$  is the orthogonal projection of  $M(x_M, y_M)$  on  $d$ . Thus  $\vec{PM}$   $(x_M - x_p, y_M - y_p)$  and

$$\begin{aligned} \delta(M, d) &= |\delta| = \left| \vec{PM} \cdot \frac{\vec{n}}{\|\vec{n}\|} \right| = \frac{|\vec{PM} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|a(x_M - x_p) + b(y_M - y_p)|}{\sqrt{a^2 + b^2}} \\ &= \frac{|ax_M + by_M - ax_p - by_p|}{\sqrt{a^2 + b^2}} = \frac{|ax_M + by_M + c|}{\sqrt{a^2 + b^2}}. \end{aligned}$$

### ◊ The three dimensional setting

- **The distance between two points** Consider two points  $A(x_A, y_A, z_A), B(x_B, y_B, z_B) \in \mathcal{P}$ . The norm of the vector  $\vec{AB}$   $(x_B - x_A, y_B - y_A, z_B - z_A)$  is

$$\|\vec{AB}\| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}.$$

- **The equation of the sphere**

Recall that the sphere  $\mathcal{S}(O, r)$  is the locus of points  $M$  in space such that  $\text{dist}(O, M) = r \iff \|\vec{OM}\| = r$ . If  $(a, b, c)$  are the coordinates of  $O$  and  $(x, y, z)$  are the coordinates of  $M$ , then

$$\begin{aligned} \|\vec{OM}\| = r &\iff \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} = r \iff (x - a)^2 + (y - b)^2 + (z - c)^2 = r^2 \\ &\iff x^2 + y^2 + z^2 - 2ax - 2by - 2cz + d = 0, \end{aligned}$$

where  $d = a^2 + b^2 + c^2 - r^2$ . Conversely, every equation of the form

$$x^2 + y^2 + z^2 + 2ex + 2fy + 2gz + h = 0$$

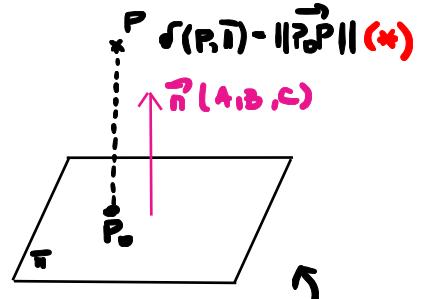
is the equation of the sphere centered at  $(-e, -g, -f)$  and having the radius  $r = \sqrt{e^2 + f^2 + g^2 - h}$ , whenever  $e^2 + f^2 + g^2 \geq h$ . One can find the equation of the sphere circumscribed to the tetrahedron  $ABCD$  by imposing the requirement on the coordinates  $(x_A, y_A, z_A)$ ,  $(x_B, y_B, z_B)$  and  $(x_C, y_C, z_C)$  and  $(x_D, y_D, z_D)$  of its vertices  $A, B, C, D$  to verify the equation  $x^2 + y^2 + z^2 - 2ax - 2by - 2cz + d = 0$ . A point  $M(x, y, z)$  belongs to this circumcircle if and only if

$$\begin{cases} x^2 + y^2 + z^2 + 2ex + 2fy + 2gz + h = 0 \\ x_A^2 + y_A^2 + z_A^2 + 2ex_A + 2fy_A + 2gz_A + h = 0 \\ x_B^2 + y_B^2 + z_B^2 + 2ex_B + 2fy_B + 2gz_B + h = 0 \\ x_C^2 + y_C^2 + z_C^2 + 2ex_C + 2fy_C + 2gz_C + h = 0 \\ x_D^2 + y_D^2 + z_D^2 + 2ex_D + 2fy_D + 2gz_D + h = 0 \end{cases} \quad (5.7)$$

One can regard the system (5.7) as linear with the unknowns  $e, g, f, h$ , whose compatibility is given, via the Kronecker-Capelli theorem, by

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ x_A^2 + y_A^2 + z_A^2 & x_A & y_A & z_A & 1 \\ x_B^2 + y_B^2 + z_B^2 & x_B & y_B & z_B & 1 \\ x_C^2 + y_C^2 + z_C^2 & x_C & y_C & z_C & 1 \\ x_D^2 + y_D^2 + z_D^2 & x_D & y_D & z_D & 1 \end{vmatrix} = 0,$$

which is the equation of the circumsphere of the tetrahedron  $ABCD$ .



- **The normal vector of a plane.** Consider the plane  $\pi : Ax + By + Cz + D = 0$  and the point  $P(x_0, y_0, z_0) \in \pi$ . The equation of  $\pi$  becomes

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \quad (5.8)$$

If  $M(x, y, z) \in \pi$ , the coordinates of  $\vec{PM}$  are  $(x - x_0, y - y_0, z - z_0)$  and the equation (5.8) tells us that  $\vec{n} \cdot \vec{PM} = 0$ , for every  $M \in \pi$ , that is  $\vec{n} \perp \vec{PM} = 0$ , for every  $M \in \pi$ , which is equivalent to  $\vec{n} \perp \vec{\pi}$ , where  $\vec{n} (A, B, C)$ . This is the reason to call  $\vec{n} (A, B, C)$  the *normal vector* of the plane  $\pi$ .

- **The distance from a point to a plane.** Consider the plane  $\pi : Ax + By + Cz + D = 0$ , a point  $P(x_P, y_P, z_P) \in \pi$  and  $M$  the orthogonal projection of  $P$  on  $\pi$ . The real number  $\delta$  given by  $\vec{MP} = \delta \cdot \vec{n}_0$  is called the *oriented distance* from  $P$  to the plane  $\pi$ , where  $\vec{n}_0 = \frac{1}{\|\vec{n}\|} \vec{n}$  is the *versor* of the normal vector  $\vec{n} (A, B, C)$ . Since  $\vec{MP} = \delta \cdot \vec{n}_0$ , it follows that  $\delta(P, M) = \|\vec{MP}\| = |\delta|$ , where  $\delta(P, M)$  stands for the distance from  $P$  to  $\pi$ . We shall show that

$$\delta = \frac{Ax_P + By_P + Cz_P + D}{\sqrt{A^2 + B^2 + C^2}}.$$

Indeed, since  $\vec{MP} = \delta \cdot \vec{n}_0$ , we get successively:

$$\begin{aligned} \delta &= \vec{n}_0 \cdot \vec{MP} = \left( \frac{1}{\|\vec{n}\|} \vec{n} \right) \cdot \vec{MP} = \frac{\vec{n} \cdot \vec{MP}}{\|\vec{n}\|} \\ &= \frac{A(x_P - x_M) + B(y_P - y_M) + C(z_P - z_M)}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{Ax_P + By_P + Cz_P - (Ax_M + By_M + Cz_M)}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{Ax_P + By_P + Cz_P + D}{\sqrt{A^2 + B^2 + C^2}}. \end{aligned}$$

Consequently, the distance from  $P$  to the plane  $\pi$  is

$$\delta(P, \pi) = \|\vec{MP}\| = |\delta| = \frac{|Ax_P + By_P + Cz_P + D|}{\sqrt{A^2 + B^2 + C^2}}. \quad (\star)$$

**Example 5.1.** Compute the distance from the point  $A(3, 1, -1)$  to the plane

$$\pi : 22x + 4y - 20z - 45 = 0.$$

**SOLUTION.**

$$\delta(A, \pi) = \frac{|22 \cdot 3 + 4 \cdot 1 - 20 \cdot (-1) - 45|}{\sqrt{22^2 + 4^2 + (-20)^2}} = \frac{45}{\sqrt{900}} = \frac{45}{30} = \frac{3}{2}.$$

## 5.2 Appendix: Orthogonal projections and reflections

### 5.2.1 The two dimensional setting

Asssume that  $R = (O, \vec{i}, \vec{j})$  is the orthonormal Cartesian system of a plane  $\pi$  behind the equation of the line  $\Delta : ax + by + c = 0$ .

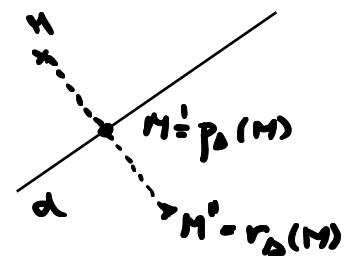
• **The orthogonal projection of a point on a line.** We define the projection of the ambient plane  $p_\Delta : \pi \rightarrow \Delta$  on  $\Delta$ , whose value  $p_\Delta$  at  $M \in \pi$  is the intersection point between  $\Delta$  and the line through  $M$  perpendicular to  $\Delta$ . Due to relations (4.24), the coordinates of  $p_\Delta(M)$ , in terms of the coordinates of  $M$  are:

$$\begin{aligned} x_M - p &= \frac{F(x_M, y_M)}{a^2 + b^2} \\ y_M - q &= \frac{F(x_M, y_M)}{a^2 + b^2}, \end{aligned}$$

where  $F(x, y) = ax + by + c$ . Consequently, the position vector of  $p_\Delta(M)$  is

$$\overrightarrow{Op_\Delta(M)} = \overrightarrow{OM} - \frac{F(M)}{a^2 + b^2} \vec{n}_\Delta,$$

where  $\vec{n}_\Delta = a \vec{i} + b \vec{j}$ .



**Proposition 5.3.** If  $R = (O, \vec{i}, \vec{j})$  is the orthonormal Cartesian reference system of the plane  $\pi$  behind the equations of the line

$$\Delta : ax + by + c = 0,$$

then

$$[p_\Delta(M)]_R = \frac{1}{a^2 + b^2} \begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix} [M]_R - \frac{c}{a^2 + b^2} [\vec{n}_\Delta]_b, \quad (5.9)$$

where  $b$  stands for the orthonormal basis  $[\vec{i}, \vec{j}]$  of  $\pi$ .

• **The reflection of the plane about a line.** We call the function  $r_\Delta : \pi \rightarrow \pi$ , whose value  $r_\Delta$  at  $M \in \pi$  is the symmetric point of  $M$  with respect to  $p_\Delta(M)$ , the *reflection of  $\pi$  about  $\Delta$* . For the position vector of  $r_\Delta(M)$  we have

$$\begin{aligned} \overrightarrow{Op_\Delta(M)} &= \frac{\overrightarrow{OM} + \overrightarrow{Or_\Delta(M)}}{2}, \text{ i.e.} \\ \overrightarrow{Or_\Delta(M)} &= 2\overrightarrow{Op_\Delta(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2 \frac{F(M)}{a^2 + b^2} \vec{n}_\Delta, \end{aligned}$$

where  $F(x, y) = ax + by + c$  and  $\vec{n}_\Delta = a \vec{i} + b \vec{j}$ . Thus, the coordinates of  $s_{\Delta,d}(M)$ , in terms of the coordinates of  $M$ , are

$$\begin{cases} x_M - 2p \frac{F(x_M, y_M)}{a^2 + b^2} \\ y_M - 2q \frac{F(x_M, y_M)}{a^2 + b^2}. \end{cases}$$

**Proposition 5.4.** If  $R = (O, \vec{i}, \vec{j})$  is the orthonormal Cartesian reference system of the plane  $\pi$  behind the equations of the line

$$\Delta : ax + by + c = 0,$$

then

$$[r_\Delta(M)]_R = \frac{1}{a^2 + b^2} \begin{pmatrix} -a^2 + b^2 & -2ab \\ -2ab & a^2 - b^2 \end{pmatrix} [M]_R - \frac{2c}{a^2 + b^2} [\vec{n}_\Delta]_b, \quad (5.10)$$

where  $b$  stands for the orthonormal basis  $[\vec{i}, \vec{j}]$  of  $\pi$ .

**Example 5.2.** Find the coordinates of the reflected point of  $P(-5, 13)$  with respect to the line

$$d : 2x - 3y - 3 = 0,$$

knowing that the Cartesian reference system  $R$  behind the coordinates of  $A$  and the equation of  $(d)$  is orthonormal.

HINT. According to 5.11 it follows that

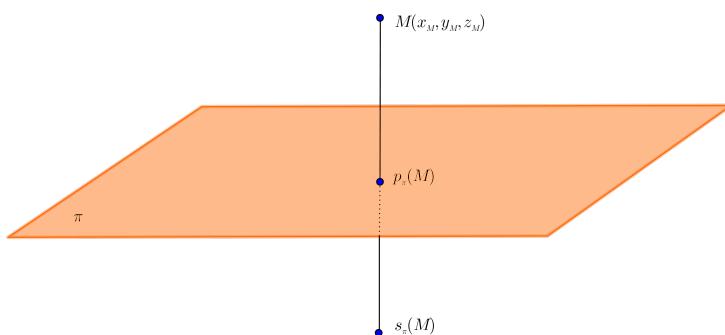
$$[r_d(P)]_R = \frac{1}{2^2 + (-3)^2} \begin{pmatrix} -2^2 + (-3)^2 & -2 \cdot 2 \cdot (-3) \\ -2 \cdot 2 \cdot (-3) & 2^2 - (-3)^2 \end{pmatrix} \begin{bmatrix} -5 \\ 13 \end{bmatrix} - \frac{2 \cdot (-3)}{2^2 + (-3)^2} \begin{bmatrix} 2 \\ -3 \end{bmatrix}. \quad (5.11)$$

## 5.2.2 The three dimensional setting

- The orthogonal projection of a point on a plane. For a given plane

$$\pi : Ax + By + Cz + D = 0$$

and a given point  $M(x_M, y_M, z_M)$ , we shall determine the coordinates of its orthogonal projection on the plane  $\pi$ , as well as the coordinates of its (orthogonal) symmetric with respect to  $\pi$ . The equation of the plane and the coordinates of  $M$  are considered with respect to some cartesian coordinate system  $R = (O, \vec{i}, \vec{j}, \vec{k})$ . In this respect we consider the orthogonal line on  $\pi$  which passes through  $M$ .



Its parametric equations are

$$\begin{cases} x = x_M + At \\ y = y_M + Bt \\ z = z_M + Ct \end{cases}, \quad t \in \mathbb{R}. \quad (5.12)$$

The orthogonal projection  $p_\pi(M)$  of  $M$  on the plane  $\pi$  is at its intersection point with the orthogonal line (5.12) and the value of  $t \in \mathbb{R}$  for which this orthogonal line (5.12) puncture the plane  $\pi$  can

be determined by imposing the condition on the point of coordinates  $(x_M + At, y_M + Bt, z_M + Ct)$  to verify the equation of the plane, namely  $A(x_M + At) + B(y_M + Bt) + C(z_M + Ct) + D = 0$ . Thus

$$t = -\frac{Ax_M + By_M + Cz_M + D}{A^2 + B^2 + C^2} = -\frac{F(x_M, y_M, z_M)}{\|\vec{n}_\pi\|^2},$$

where  $F(x, y, z) = Ax + By + Cz + D$  și  $\vec{n}_\pi = A\vec{i} + B\vec{j} + C\vec{k}$  is the normal vector of the plane  $\pi$ .

- **The orthogonal projection of the space on a plane.**

The coordinates of the orthogonal projection  $p_\pi(M)$  of  $M$  on the plane  $\pi$  are

$$\begin{cases} x_M - A \frac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ y_M - B \frac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2} \\ z_M - C \frac{F(x_M, y_M, z_M)}{A^2 + B^2 + C^2}. \end{cases}$$

Therefore, the position vector of the orthogonal projection  $p_\pi(M)$  is

$$\overrightarrow{Op_\pi(M)} = \overrightarrow{OM} - \frac{F(M)}{\|\vec{n}_\pi\|^2} \vec{n}_\pi. \quad (5.13)$$

**Proposition 5.5.** If  $R = (O, b)$  is the orthonormal Cartesian reference system behind the equation of the plane  $(\pi) Ax + By + Cz + D = 0$ , then

$$(A^2 + B^2 + C^2)[p_\pi(M)]_R = \begin{pmatrix} B^2 + C^2 & -AB & -AC \\ -AB & A^2 + C^2 & -BC \\ -AC & -BC & A^2 + B^2 \end{pmatrix} [M]_R - D[\vec{n}_\pi]_b. \quad (5.14)$$

**Remark 5.3.** The distance from the point  $M(x_M, y_M, z_M)$  to the plane  $\pi : Ax + By + Cz + D = 0$  can be equally computed by means of (5.13). Indeed,

$$\begin{aligned} \delta(M, \pi) &= \|\overrightarrow{Mp_\pi(M)}\| = \|\overrightarrow{Op_\pi(M)} - \overrightarrow{OM}\| \\ &= \left\| \frac{F(M)}{\|\vec{n}_\pi\|^2} \cdot \vec{n}_\pi \right\| = \frac{|F(M)|}{\|\vec{n}_\pi\|}. \end{aligned}$$

• **The reflection of the space about a plane.** In order to find the position vector of the orthogonally symmetric point  $r_\pi(M)$  of  $M$  w.r.t.  $\pi$ , we use the relation

$$\overrightarrow{Op_\pi(M)} = \frac{1}{2} \left( \overrightarrow{OM} + \overrightarrow{Or_\pi(M)} \right),$$

$$\begin{aligned} x' &= x_M - 2A \frac{Ax + By + Cz + D}{A^2 + B^2 + C^2} \\ y' &= y_M - 2B \frac{Ax + By + Cz + D}{A^2 + B^2 + C^2} \\ z' &= z_M - 2C \frac{Ax + By + Cz + D}{A^2 + B^2 + C^2} \end{aligned}$$

namely

$$\overrightarrow{Or_\pi(M)} = 2\overrightarrow{Op_\pi(M)} - \overrightarrow{OM} = \overrightarrow{OM} - 2 \frac{F(M)}{\|\vec{n}_\pi\|^2} \vec{n}_\pi.$$

The correspondence which associate to some point  $M$  its orthogonally symmetric point w.r.t.  $\pi$ , is called the *reflection* in the plane  $\pi$  and is denoted by  $r_\pi$ .

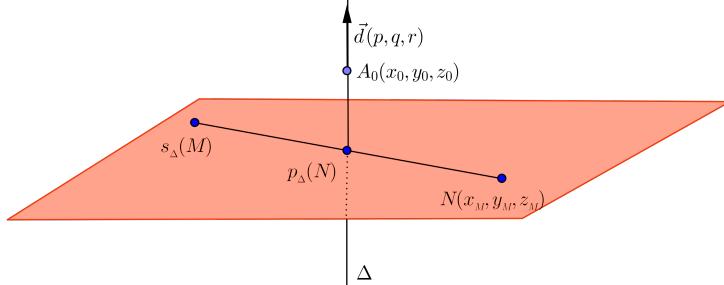
**Proposition 5.6.** If  $R = (O, b)$  is the orthonormal Cartesian reference system behind the equation of the plane  $(\pi) Ax + By + Cz + D = 0$ , then

$$(A^2 + B^2 + C^2)[r_\pi(M)]_R = \begin{pmatrix} -A^2 + B^2 + C^2 & -2AB & -2AC \\ -2AB & A^2 - B^2 + C^2 & -2BC \\ -2AC & -2BC & A^2 + B^2 - C^2 \end{pmatrix} [M]_R - 2D[\vec{n}_\pi]_b. \quad (5.15)$$

- **The orthogonal projection of the space on a line.** For a given line

$$\Delta : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

and a point  $N(x_N, y_N, z_N)$ , we shall find the coordinates of its orthogonal projection on the line  $\Delta$ , as well as the coordinates of the orthogonally symmetric point  $M$  with respect to  $\Delta$ . The equations of the line and the coordinates of the point  $N$  are considered with respect to an orthonormal coordinate system  $R = (O, \vec{i}, \vec{j}, \vec{k})$ . In this respect we consider the plane  $p(x - x_N) + q(y - y_N) + r(z - z_N) = 0$  orthogonal on the line  $\Delta$  which passes through the point  $N$ .



The parametric equations of the line  $\Delta$  are

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, t \in \mathbb{R}. \quad (5.16)$$

The orthogonal projection of  $N$  on the line  $\Delta$  is at its intersection point with the plane

$$p(x - x_N) + q(y - y_N) + r(z - z_N) = 0,$$

and the value of  $t \in \mathbb{R}$  for which the line  $\Delta$  puncture the orthogonal plane  $p(x - x_N) + q(y - y_N) + r(z - z_N) = 0$  can be found by imposing the condition on the point of coordinate  $(x_0 + pt, y_0 + qt, z_0 + rt)$  to verify the equation of the plane, namely  $p(x_0 + pt - x_N) + q(y_0 + qt - y_N) + r(z_0 + rt - z_N) = 0$ . Thus

$$t = -\frac{p(x_0 - x_N) + q(y_0 - y_N) + r(z_0 - z_N)}{p^2 + q^2 + r^2} = -\frac{G(x_0, y_0, z_0)}{\|\vec{d}_\Delta\|^2},$$

where  $G(x, y, z) = p(x - x_N) + q(y - y_N) + r(z - z_N)$  and  $\vec{d}_\Delta = p\vec{i} + q\vec{j} + r\vec{k}$  is the director vector of the line  $\Delta$ . The coordinates of the orthogonal projection  $p_\Delta(N)$  of  $N$  on the line  $\Delta$  are therefore

$$\begin{cases} x_0 - p\frac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2} \\ y_0 - q\frac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2} \\ z_0 - r\frac{G(x_0, y_0, z_0)}{p^2 + q^2 + r^2}. \end{cases}$$

Thus, the position vector of the orthogonal projection  $p_\Delta(N)$  is

$$\overrightarrow{Op_\Delta(N)} = \overrightarrow{OA_0} - \frac{\overrightarrow{G(A_0)}}{\|\vec{d}_\Delta\|^2} \vec{d}_\Delta, \quad (5.17)$$

where  $A_0(x_0, y_0, z_0) \in \Delta$ .

- **The reflection of the space about a line.** In order to find the position vector of the orthogonally symmetric point  $r_\Delta(N)$  of  $N$  with respect to the line  $\Delta$  we use the relation

$$\overrightarrow{Op_\Delta(N)} = \frac{1}{2} \left( \overrightarrow{ON} + \overrightarrow{Or_\Delta(N)} \right)$$

i.e.

$$\overrightarrow{Os_{\Delta}(N)} = 2 \overrightarrow{Op_{\Delta}(N)} - \overrightarrow{ON} = 2 \overrightarrow{OA_0} - 2 \frac{\overrightarrow{G(A_0)}}{\|\overrightarrow{d_{\Delta}}\|^2} \overrightarrow{d_{\Delta}} - \overrightarrow{ON}.$$

The correspondence which associate to some point  $M$  its orthogonally symmetric point w.r.t.  $\delta$ , is called the *reflection* in the line  $\delta$  and is denoted by  $r_{\delta}$ .

### 5.3 Problems

1. (2p) Consider the triangle  $ABC$  and the midpoint  $A'$  of the side  $[BC]$ . Show that

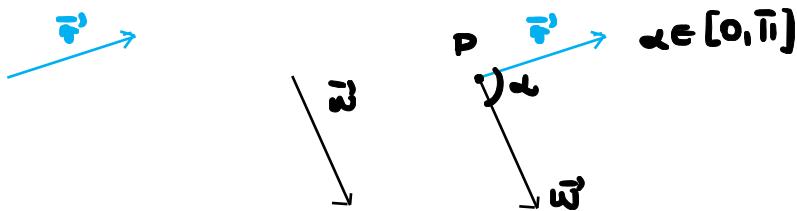
$$4 \overrightarrow{AA'}^2 - \overrightarrow{BC}^2 = 4 \overrightarrow{AB} \cdot \overrightarrow{AC}.$$

*Solution.*

$$\begin{aligned}
 & A' - \text{midpoint of } [BC] \Rightarrow \overrightarrow{AA'} = \frac{\overrightarrow{AB} + \overrightarrow{AC}}{2} \quad \left. \begin{aligned} \overrightarrow{AA'}^2 &= \frac{(\overrightarrow{AB} + \overrightarrow{AC})^2}{4} \\ \overrightarrow{BC}^2 &= \overrightarrow{AC}^2 - \overrightarrow{AB}^2 \end{aligned} \right\} \Rightarrow \\
 & \overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} \Rightarrow \overrightarrow{BC} \cdot \overrightarrow{BC} = (\overrightarrow{AC} - \overrightarrow{AB}) \cdot (\overrightarrow{AC} - \overrightarrow{AB}) \Rightarrow \overrightarrow{BC}^2 = (\overrightarrow{AC} - \overrightarrow{AB})^2 \\
 & \overline{4 \overrightarrow{AA'}^2 - \overrightarrow{BC}^2} = \cancel{4 \frac{(\overrightarrow{AB} + \overrightarrow{AC})^2}{4}} - (\overrightarrow{AC} - \overrightarrow{AB})^2 \\
 & = \overrightarrow{AB}^2 + 2\overrightarrow{AB} \cdot \overrightarrow{AC} + \overrightarrow{AC}^2 - \cancel{\overrightarrow{AC}^2} + 2\overrightarrow{AC} \cdot \overrightarrow{AB} - \cancel{\overrightarrow{AB}^2} \\
 & = \underline{\underline{4 \overrightarrow{AB} \cdot \overrightarrow{AC}}}
 \end{aligned}$$

### Dot product (Scalar product)

$$\vec{v}, \vec{w} \in \mathbb{V}_n, \vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos(\vec{v}, \vec{w})$$



If we fix a ref system that is orthonormal, then:

$$\vec{v}(a_1, b_1, c_1), \vec{w}(a_2, b_2, c_2) \Rightarrow \vec{v} \cdot \vec{w} = a_1 a_2 + b_1 b_2 + c_1 c_2$$

$\mathcal{R} = (0, [\vec{i}, \vec{j}, \vec{k}])$ , orthonormal = orthogonal + normed

$$\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0 \quad \|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = \|\vec{e}'\| = 1$$

### The distance from a point to a plane

$$\text{II : } Ax + By + Cz + D = 0, \text{ dist}(P, II) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

$$P(x_0, y_0, z_0)$$

### The distance from a point to a line in 2D

$$\text{L : } Ax + By + C = 0, \text{ dist}(P, L) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$P(x_0, y_0)$$

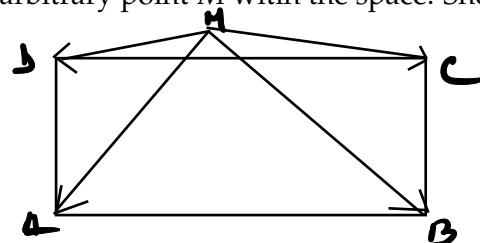
2. (2p) Consider the rectangle  $ABCD$  and the arbitrary point  $M$  within the space. Show that

$$(a) \overrightarrow{MA} \cdot \overrightarrow{MC} = \overrightarrow{MB} \cdot \overrightarrow{MD}.$$

$$(b) \overrightarrow{MA}^2 + \overrightarrow{MC}^2 = \overrightarrow{MB}^2 + \overrightarrow{MD}^2.$$

*Solution.*

$ABCD$  - Rectangle



$$a) \overrightarrow{MB} = \overrightarrow{MA} + \overrightarrow{AB} \quad \therefore \overrightarrow{MB} \cdot \overrightarrow{MD} = (\overrightarrow{MA} + \overrightarrow{AB})(\overrightarrow{MC} + \overrightarrow{CD}) =$$

$$\overrightarrow{MD} = \overrightarrow{MC} + \overrightarrow{CD}$$

$$= \overrightarrow{MA} \cdot \overrightarrow{MC} + \overrightarrow{MA} \cdot \overrightarrow{CD} + \overrightarrow{AB} \cdot \overrightarrow{MC} + \overrightarrow{AB} \cdot \overrightarrow{CD} \quad \cancel{\overrightarrow{AB} = -\overrightarrow{CD}}$$

$$\therefore \overrightarrow{MB} \cdot \overrightarrow{MD} = \overrightarrow{MA} \cdot \overrightarrow{MC} + \overrightarrow{CD}(\overrightarrow{MA} - \overrightarrow{MC} - \overrightarrow{CD}) = \overrightarrow{MA} \cdot \overrightarrow{MC} + \overrightarrow{CD} \cdot \overrightarrow{DA} =$$

$$\cancel{\overrightarrow{CD} + \overrightarrow{MA} + \overrightarrow{DC}} = \overrightarrow{MA} \cdot \overrightarrow{MC} + \overrightarrow{AB} \cdot \overrightarrow{CD} =$$

$$0, \overrightarrow{AB} \perp \overrightarrow{CD}$$

$$\therefore \overrightarrow{MB} \cdot \overrightarrow{MD} = \overrightarrow{MA} \cdot \overrightarrow{MC}$$

$$b) \overrightarrow{MB}^2 + \overrightarrow{MD}^2 = (\overrightarrow{MA} + \overrightarrow{AB})^2 + (\overrightarrow{MC} + \overrightarrow{CD})^2 = \overrightarrow{MA}^2 + 2\overrightarrow{MA} \cdot \overrightarrow{AB} + \overrightarrow{AB}^2 + \overrightarrow{MC}^2 + 2\overrightarrow{MC} \cdot \overrightarrow{CD} + \overrightarrow{CD}^2 =$$

$$= \overrightarrow{MA}^2 + \overrightarrow{MC}^2 + 2\overrightarrow{MA} \cdot \overrightarrow{AB} + 2\overrightarrow{MC} \cdot \overrightarrow{CD} + \overrightarrow{AB}^2 + \overrightarrow{CD}^2 =$$

$$= \overrightarrow{MA}^2 + \overrightarrow{MC}^2 + 2\overrightarrow{MA} \cdot \overrightarrow{AB} - 2\overrightarrow{MC} \cdot \overrightarrow{AB} + 2\overrightarrow{AB}^2 =$$

$$= \overrightarrow{MA}^2 + \overrightarrow{MC}^2 + 2\overrightarrow{AB}(\overrightarrow{MA} - \overrightarrow{MC} + \overrightarrow{AB}) =$$

$$= \overrightarrow{MA}^2 + \overrightarrow{MC}^2 + 2\overrightarrow{AB}(\cancel{\overrightarrow{CA} + \overrightarrow{AB}}) = \overrightarrow{MA}^2 + \overrightarrow{MC}^2 + 0 = \overrightarrow{MA}^2 + \overrightarrow{MC}^2$$

$$\overrightarrow{AB} \perp \overrightarrow{CD}$$

3. (3p) Find the angle between:

(a) the straight lines

$$(d_1) \begin{cases} x + 2y + z - 1 = 0 \\ x - 2y + z + 1 = 0 \end{cases} \quad (d_2) \begin{cases} x - y - z - 1 = 0 \\ x - y + 2z + 1 = 0 \end{cases}$$

(b) the planes

$$\pi_1 : x + 3y + 2z + 1 = 0 \text{ and } \pi_2 : 3x + 2y - z = 6.$$

(c) the plane  $xOy$  and the straight line  $M_1M_2$ , where  $M_1(1, 2, 3)$  and  $M_2(-2, 1, 4)$ .

*Solution.*

$$(a) d_1: \begin{cases} 2x + 2z = 0 \\ x - 2y + z + 1 = 0 \end{cases} \Rightarrow \begin{cases} x = -z \\ -x - 2y + z + 1 = 0 \end{cases} \Rightarrow \begin{cases} x = -z \\ 2y = 1 \\ x = -z \end{cases} \Rightarrow \begin{cases} y = \frac{1}{2} \\ x = -z \end{cases}$$

$$\therefore d_1: \begin{cases} x = t \\ y = \frac{1}{2} + 0 \cdot t \\ z = -t \end{cases} \Rightarrow \vec{d}_1(1, 0, -1)$$

$$d_2: \begin{cases} x - y - z - 1 = 0 \\ x - y + 2z + 1 = 0 \end{cases} \Rightarrow \begin{cases} -3z - 2 = 0 \\ x - y + 2z + 1 = 0 \end{cases} \Rightarrow \begin{cases} z = -\frac{2}{3} \\ x - y - \frac{5}{3} + 1 = 0 \\ x - y = \frac{2}{3} \end{cases}$$

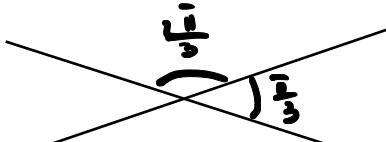
$$\therefore \begin{cases} x = y + \frac{1}{3} \\ z = -\frac{2}{3} \end{cases} \text{ and } d_2: \begin{cases} x = t \\ y = t - \frac{1}{3} \\ z = -\frac{2}{3} + t \end{cases} \Rightarrow \vec{d}_2(1, 1, 0)$$

$$\vec{d}_1 \cdot \vec{d}_2 = (1, 0, -1) \cdot (1, 1, 0) = 1 \cdot 1 + 0 \cdot 1 + (-1) \cdot 0 = 1$$

$$\|\vec{d}_1\| = \sqrt{1 + 0 + 1} = \sqrt{2}$$

$$\|\vec{d}_2\| = \sqrt{1 + 1 + 0} = \sqrt{2}$$

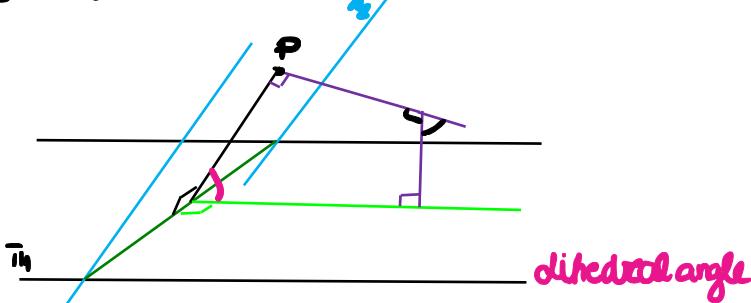
$$\therefore \cos(\vec{d}_1, \vec{d}_2) = \frac{\vec{d}_1 \cdot \vec{d}_2}{\|\vec{d}_1\| \cdot \|\vec{d}_2\|} = \frac{1}{2} \Rightarrow m(\widehat{d_1, d_2}) = \frac{\pi}{3}$$

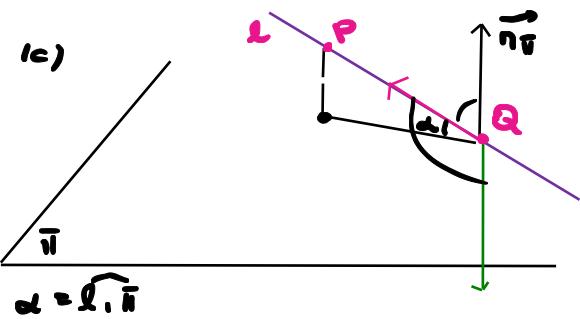


$$(b) \bar{\pi}_1: x + 3y + 2z + 1 = 0 \quad \text{and} \quad \bar{n}_{\bar{\pi}_1}(1, 3, 2) = \cos(\widehat{\bar{n}_{\bar{\pi}_1}, \bar{n}_{\bar{\pi}_2}}) = \frac{\bar{n}_{\bar{\pi}_1} \cdot \bar{n}_{\bar{\pi}_2}}{\|\bar{n}_{\bar{\pi}_1}\| \cdot \|\bar{n}_{\bar{\pi}_2}\|} =$$

$$\bar{\pi}_2: 3x + 2y - z = 0 \quad \bar{n}_{\bar{\pi}_2}(3, 2, -1)$$

$$= \frac{1 \cdot 3 + 3 \cdot 2 + 2 \cdot (-1)}{\sqrt{1+9+4} \cdot \sqrt{9+4+1}} = \frac{8}{14} = \frac{4}{7} \Rightarrow m(\widehat{\bar{\pi}_1, \bar{\pi}_2}) = \frac{\pi}{3}$$





$$\cos(\overrightarrow{n_{xy}}, \overrightarrow{M_1 M_2}) = \frac{0 \cdot (-3) + 0 \cdot (-4) + 1 \cdot 1}{\sqrt{9+1+1} \cdot \sqrt{11}} = \frac{1}{\sqrt{11}}$$

$\arccos\left(\frac{1}{\sqrt{11}}\right) \in [0; \frac{\pi}{2}]$

$$\Rightarrow m(M_1 M_2, xOy) = \frac{\pi}{2} - \arccos\left(\frac{1}{\sqrt{11}}\right)$$

$$M_1 M_2 : \frac{x-1}{-2-1} = \frac{y-2}{1-2} = \frac{z-3}{4-3} \Leftrightarrow \frac{x-1}{-3} = \frac{y-2}{-1} = \frac{z-3}{1} \Leftrightarrow \overrightarrow{M_1 M_2}(-3, -1, 1) \parallel$$

$$\Rightarrow \text{plane } xOy \Rightarrow z=0 \Rightarrow \overrightarrow{n_{xy}}(0, 0, 1)$$

4. (3p) Consider the noncoplanar vectors  $\vec{OA} (1, -1, -2)$ ,  $\vec{OB} (1, 0, -1)$ ,  $\vec{OC} (2, 2, -1)$  related to an orthonormal basis  $\vec{i}, \vec{j}, \vec{k}$ . Let  $H$  be the foot of the perpendicular through  $O$  on the plane  $ABC$ . Determine the components of the vectors  $\vec{OH}$ .

*Solution.*

We know that  $\mathbf{0}(0,0,0) \cdot \mathbf{x}_0 = 0, \mathbf{y}_0 = 0, \mathbf{z}_0 = 0 \Rightarrow A(1-1, -2), B(1, 0, -1), C(2+2, -1)$

A, B, C - noncollinear points  $\rightarrow$  the plane (ABC):

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{vmatrix} x & y & z & 1 \\ 1 & -1 & -2 & 1 \\ 1 & 0 & -1 & 1 \\ 2 & 2 & -1 & 1 \end{vmatrix} = 0 \Leftrightarrow x \cdot \begin{vmatrix} -1 & -2 & 1 \\ 0 & -1 & 1 \\ 2 & -1 & 1 \end{vmatrix} - y \cdot \begin{vmatrix} 1 & -2 & 1 \\ 1 & 0 & 1 \\ 2 & -1 & 1 \end{vmatrix} + z \cdot \begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & 2 & -1 \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow x(-x-4+2-1) - y(-x-x-4+2+1+2) + z(x-x-2+1) - (-x+2+x-1) = 0 \Leftrightarrow$$

$$\Leftrightarrow 2x + y - 2 + 1 = 0 \rightarrow \text{the eq of the plane (ABC)}$$

For the plane (ABC) and point  $O(0,0,0)$  we shall det. the coord. of its orthogonal projection on the plane. The eq. of the plane and the coord. of O are considered with respect to the cartesian coordinate system  $R = (O, \vec{i}, \vec{j}, \vec{k})$ . In this respect, we consider the orthogonal line on (ABC) which passes through  $O(\vec{OH})$ .

Param. eq of  $\vec{OH}$ :

$$\begin{cases} x = x_0 + At \\ y = y_0 + Bt, t \in \mathbb{R} \\ z = z_0 + Ct \end{cases}$$

$$\Leftrightarrow t = -\frac{Ax_0 + By_0 + Cz_0 + D}{A^2 + B^2 + C^2} = -\frac{-2-0+10-1 \cdot 0+1}{4+0+1} = \frac{1}{6} \Leftrightarrow \begin{cases} x = 0 + (-2) \cdot \frac{1}{6} \\ y = 0 + 1 \cdot \frac{1}{6} \\ z = 0 - 1 \cdot \frac{1}{6} \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \frac{1}{6} \\ y = \frac{1}{6} \\ z = -\frac{1}{6} \end{cases} \rightarrow \vec{OH} \left( \frac{1}{6}, \frac{1}{6}, -\frac{1}{6} \right)$$

• The orthogonal projection of a point on a plane. For a given plane

$$\pi : Ax + By + Cz + D = 0$$

and a given point  $M(x_M, y_M, z_M)$ , we shall determine the coordinates of its orthogonal projection on the plane  $\pi$ , as well as the coordinates of its (orthogonal) symmetric with respect to  $\pi$ . The equation of the plane and the coordinates of  $M$  are considered with respect to some cartesian coordinate system  $R = (O, \vec{i}, \vec{j}, \vec{k})$ . In this respect we consider the orthogonal line on  $\pi$  which passes through  $M$ .

Its parametric equations are

$$\begin{cases} x = x_M + At \\ y = y_M + Bt \\ z = z_M + Ct \end{cases}, t \in \mathbb{R} \quad (5.12)$$

The orthogonal projection  $p_\pi(M)$  of  $M$  on the plane  $\pi$  is at its intersection point with the orthogonal line (5.12) and the value of  $t \in \mathbb{R}$  for which this orthogonal line (5.12) puncture the plane  $\pi$  can

be determined by imposing the condition on the point of coordinates  $(x_M + At, y_M + Bt, z_M + Ct)$  to verify the equation of the plane, namely  $A(x_M + At) + B(y_M + Bt) + C(z_M + Ct) + D = 0$ . Thus

$$t = -\frac{Ax_M + By_M + Cz_M + D}{A^2 + B^2 + C^2} = -\frac{F(x_M, y_M, z_M)}{\|\vec{n}_\pi\|^2},$$

where  $F(x, y, z) = Ax + By + Cz + D$  și  $\vec{n}_\pi = A\vec{i} + B\vec{j} + C\vec{k}$  is the normal vector of the plane  $\pi$ .

5. (2p) Find the points on the z-axis which are equidistant with respect to the planes

$$\pi_1 : 12x + 9y - 20z - 19 = 0 \text{ and } \pi_2 : 16x + 12y + 15z - 9 = 0.$$

*Solution.*

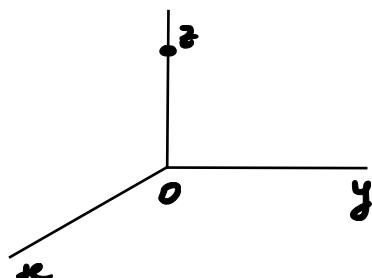
$$P(x_0, y_0, z_0)$$

$$\text{dist}(P, \pi_1) = \frac{|12x_0 + 9y_0 - 20z_0 - 19|}{\sqrt{12^2 + 9^2 + 20^2}} = \frac{|12x_0 + 9y_0 - 20z_0 - 19|}{25}$$

$$\text{dist}(P, \pi_2) = \frac{|16x_0 + 12y_0 + 15z_0 - 9|}{\sqrt{16^2 + 12^2 + 15^2}} = \frac{|16x_0 + 12y_0 + 15z_0 - 9|}{25}$$

$$P \in O \Rightarrow x_0 = y_0 = 0 \Rightarrow \text{dist}(P, \pi_1) = \frac{|-20z_0 - 19|}{25}$$

$$\text{dist}(P, \pi_2) = \frac{|15z_0 - 9|}{25}$$



$$\text{dist}(P, \pi_1) = \text{dist}(P, \pi_2) \Rightarrow |-20z_0 - 19| = |15z_0 - 9|$$

$$\text{Case I} : -20z_0 - 19 = 15z_0 - 9$$

$$\Rightarrow z_0 = \frac{-10}{35} = \frac{-2}{7}$$

$$\therefore P(0, 0, -\frac{2}{7})$$

$\swarrow$  geometric

$$\text{Case II} : -20z_0 - 19 = -15z_0 + 9$$

$$\Rightarrow z_0 = \frac{28}{5}$$

$$\therefore P(0, 0, \frac{28}{5})$$

The locus of points equidistant to two non-parallel planes consists of two perpendicular planes called the *director planes*.

6. (2p) Consider two planes

$$(\pi_1) A_1x + B_1y + C_1z + D_1 = 0$$

$$(\pi_2) A_2x + B_2y + C_2z + D_2 = 0$$

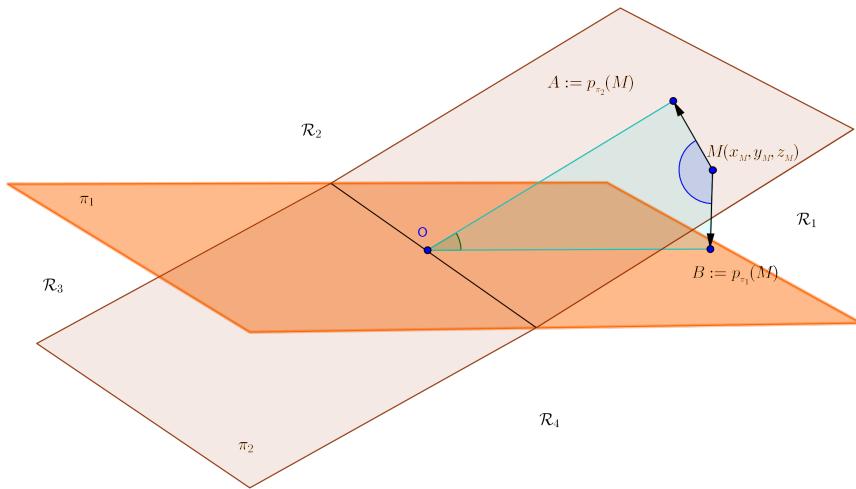
which are not parallel and not perpendicular as well. The two planes  $\pi_1, \pi_2$  devide the space into four regions  $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$  and  $\mathcal{R}_4$ , two of which, say  $\mathcal{R}_1$  and  $\mathcal{R}_3$ , correspond to the acute dihedral angle of the two planes. Show that  $M(x, y, z) \in \mathcal{R}_1 \cup \mathcal{R}_3$ , if and only if

$$F_1(x, y, z) \cdot F_2(x, y, z) \underbrace{(A_1A_2 + B_1B_2 + C_1C_2)}_{\text{acute region}} < 0,$$

where  $F_1(x, y, z) = A_1x + B_1y + C_1z + D_1$  and  $F_2(x, y, z) = A_2x + B_2y + C_2z + D_2$ .

*Hint.* The non-parallelism relation between the two planes is equivalent with the condition

$$\text{rank} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2.$$



The point  $M$  belongs to the union  $\mathcal{R}_1 \cup \mathcal{R}_3$  if and only if the angle of the vectors  $\overrightarrow{Mp_{\pi_1}(M)}$  and  $\overrightarrow{Mp_{\pi_2}(M)}$  is at least  $90^\circ$ , as the quadrilateral  $OAMB$  is inscriptible. More formally

$$\begin{aligned} M(x, y, z) \in \mathcal{R}_1 \cup \mathcal{R}_3 &\Leftrightarrow m(\overrightarrow{Mp_{\pi_1}(M)}, \overrightarrow{Mp_{\pi_2}(M)}) > 90^\circ \\ &\Leftrightarrow \overrightarrow{Mp_{\pi_1}(M)} \cdot \overrightarrow{Mp_{\pi_2}(M)} < 0, \end{aligned}$$

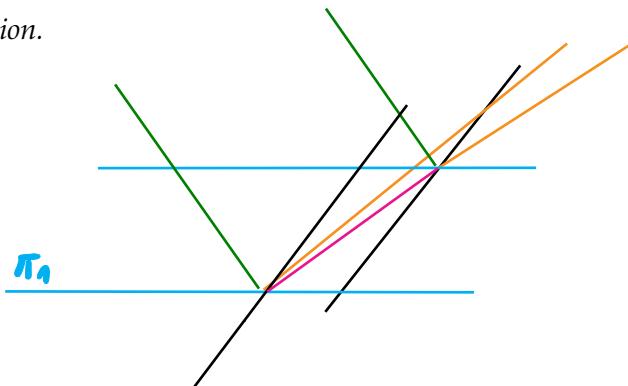
where  $p_{\pi_1}(M), p_{\pi_2}(M)$  are the orthogonal projections of  $M$  on the planes  $\pi_1$  and  $\pi_2$  respectively.

*Solution.*

$F_1$  $F_2$ 

7. (3p) Consider the planes  $(\pi_1)$   $2x + y - 3z - 5 = 0$ ,  $(\pi_2)$   $x + 3y + 2z + 1 = 0$ . Find the equations of the bisector planes of the dihedral angles formed by the planes  $\pi_1$  and  $\pi_2$  and select the one contained into the acute regions of the dihedral angles formed by the two planes.

*Solution.*



$$\left. \begin{aligned} \text{dist } P(x,y,z) &= \text{dist } (P, \pi_1) = \frac{|2x+y-3z-5|}{\sqrt{14}} \\ \text{dist } (P, \pi_2) &= \frac{|x+3y+2z+1|}{\sqrt{14}} \\ \text{dist } (P, \pi_1) &= \text{dist } (P, \pi_2) \end{aligned} \right\} \Rightarrow |2x+y-3z-5| = |x+3y+2z+1|$$

$$\text{Case I: } 2x+y-3z-5 = x+3y+2z+1 \Rightarrow \pi_3: x-2y-5z-6=0$$

$$\text{Case II: } 2x+y-3z-5 = -x-3y-2z-1 \Rightarrow \pi_4: 3x+4y-z-4=0$$

Let  $M(0, -3, 0) \in \pi_4$ . We check if  $M$  belongs to the acute region.

$$F_1(M) = F_1(0, -3, 0) = -8$$

$$F_2(M) = F_2(0, -3, 0) = -8$$

$$A_1 A_2 + B_1 B_2 + C_1 C_2 = \vec{n}_{\pi_1} \cdot \vec{n}_{\pi_2} = (2, 1, -3) \cdot (1, 3, 2) = 2 + 3 - 6 = -1$$

$$\Rightarrow F_1(M) \cdot F_2(M) \cdot (\vec{n}_{\pi_1} \cdot \vec{n}_{\pi_2}) = -64 < 0 \Rightarrow M \in \text{acute region} \Rightarrow \pi_4 \subset \text{acute regions}$$

8. (3p) Let  $a, b$  be two real numbers such that  $a^2 \neq b^2$ . Consider the planes:

$$(\alpha_1) ax + by - (a+b)z = 0$$

$$(\alpha_2) ax - by - (a-b)z = 0$$

and the quadric  $(C)$ :  $a^2x^2 - b^2y^2 + (a^2 - b^2)z^2 - 2a^2xz + 2b^2yz - a^2b^2 = 0$ . If  $a^2 < b^2$ , show that the quadric  $C$  is contained in the acute regions of the dihedral angles formed by the two planes. If, on the contrary,  $a^2 > b^2$ , show that the quadric  $C$  is contained in the obtuse regions of the dihedral angles formed by the two planes.

*Solution.*

We need to check the conditions for a point of the quadric (any point) that is:  $P(x,y,z) \in \mathbb{R}^3$  if  $a^2x^2 - b^2y^2 + (a^2 - b^2)z^2 - 2a^2xz + 2b^2yz - a^2b^2 = 0$

$$\Leftrightarrow a^2x^2 - b^2y^2 + (a^2 - b^2)z^2 - 2a^2xz + 2b^2yz = a^2b^2 \quad \forall (x,y,z) \in \mathbb{R}^3$$

\* for being contained in the acute / obtuse regions

For this we use what we proved at exercise (c):

$$P(x,y,z) \in R_1 \cup R_2 \Leftrightarrow F_1(x,y,z) \cdot F_2(x,y,z) \cdot (A_1A_2 + B_1B_2 + C_1C_2) < 0$$

(acute regions)

$$\Leftrightarrow P(x,y,z) \in R_2 \cup R_6 \Leftrightarrow F_1(x,y,z) \cdot F_2(x,y,z) \cdot (A_1A_2 + B_1B_2 + C_1C_2) > 0$$

(obtuse regions)

where  $F_1(x,y,z) = A_1x + B_1y + C_1z + D_1$   
 $F_2(x,y,z) = A_2x + B_2y + C_2z + D_2$

From our case

$$P(x,y,z) \in R_1 \cup R_3 \Leftrightarrow [ax + by - (a+b)z][ax - by - (a-b)z] \cdot [a^2 - b^2 + (a^2 - b^2)] < 0$$

$$\Leftrightarrow [(ax - a^2) + (by - b^2)][(ax - a^2) - (by - b^2)] \cdot [a^2 - b^2 + (a^2 - b^2)] < 0$$

$$\Leftrightarrow [(ax - a^2)^2 - (by - b^2)^2] \cdot (2a^2 - 2b^2) < 0$$

$$\Leftrightarrow (a^2x^2 + a^2z^2 - 2a^2xz - b^2y^2 - b^2z^2 + 2b^2yz) \cdot 2(a^2 - b^2) < 0$$

$$\Leftrightarrow \underbrace{(a^2x^2 - b^2y^2 + (a^2 - b^2)z^2 - 2a^2xz + 2b^2yz)}_{= 0} \cdot 2(a^2 - b^2) < 0$$

because we check (1)

thus for a point  $P(x,y,z) \in C$   $\Leftrightarrow a^2b^2 \cdot 2(a^2 - b^2) < 0$

$a^2b^2 > 0, \forall a,b \in \mathbb{R}$

$2 > 0$

$$\Leftrightarrow a^2 - b^2 < 0$$

Therefore: • if  $a^2 < b^2$ :  $F_1(x,y,z) \cdot F_2(x,y,z) \cdot (A_1A_2 + B_1B_2 + C_1C_2) < 0$   
 $\Rightarrow \forall P(x,y,z) \in C \rightarrow P(x,y,z) \in R_1 \cup R_3$   
 $\Rightarrow$  the quadric  $C$  is contained in the acute regions

• otherwise  
if  $a^2 > b^2$ :  $F_1(x,y,z) \cdot F_2(x,y,z) \cdot (A_1A_2 + B_1B_2 + C_1C_2) > 0$   
 $\Rightarrow \forall P(x,y,z) \in C \rightarrow P(x,y,z) \in R_2 \cup R_6$   
 $\Rightarrow$  the quadric  $C$  is contained in the obtuse regions

9. If two pairs of opposite edges of the tetrahedron  $ABCD$  are perpendicular ( $AB \perp CD$ ,  $AD \perp BC$ ), show that

- (a) The third pair of opposite edges are perpendicular too ( $AC \perp BD$ ).
- (b)  $AB^2 + CD^2 = AC^2 + BD^2 = BC^2 + AD^2$ .
- (c) The heights of the tetrahedron are concurrent.  
(Such a tetrahedron is said to be orthocentric)

*Solution.* Denote by  $\vec{AB} = \vec{b}$ ,  $\vec{AC} = \vec{c}$  and  $\vec{AD} = \vec{d}$ .

$$(a) AB \perp CD \implies \vec{b}(\vec{d} - \vec{c}) = 0 \implies \vec{b}\vec{d} = \vec{b}\vec{c} = k$$

$$AD \perp BC \implies \vec{d}(\vec{c} - \vec{b}) = 0 \implies \vec{c}\vec{d} = \vec{b}\vec{d} = k,$$

$$\text{deci } \vec{c}\vec{b} = \vec{c}\vec{d} \implies \vec{c}(\vec{b} - \vec{d}) = 0 \implies AC \perp BD.$$

$$(b) AB^2 + CD^2 = \vec{b}^2 + (\vec{d} - \vec{c})^2 = \vec{b}^2 + \vec{d}^2 + \vec{c}^2 - 2k;$$

$$AC^2 + BD^2 = \vec{c}^2 + (\vec{d} - \vec{b})^2 = \vec{b}^2 + \vec{c}^2 + \vec{d}^2 - 2k;$$

$$BC^2 + AD^2 = \vec{d}^2 + (\vec{c} - \vec{b})^2 = \vec{b}^2 + \vec{c}^2 + \vec{d}^2 - 2k.$$

- (c) We shall show that there exists a point  $H$  such that  $AH \perp (DBC)$ ,  $BH \perp (ACD)$ ,  $CH \perp (ABD)$ ,  $DH \perp (ABC)$ . Let  $\vec{h} = \vec{AH} = m\vec{a} + n\vec{b} + p\vec{c}$ . Writing the conditions  $\vec{AH} \perp \vec{BC}$ ,  $\vec{CD}$ ;  $\vec{BH} \perp \vec{AC}$ ,  $\vec{AD}$ ;  $\vec{CH} \perp \vec{AB}$ ,  $\vec{AD}$ ;  $\vec{DH} \perp \vec{AB}$ ,  $\vec{AC}$  we obtain a consistent system with one single solution:

$$\begin{cases} b^2m + kn + kp = k \\ km + c^2n + kp = k \\ km + kn + d^2p = k. \end{cases} \quad (5.18)$$

Indeed the matrix of the system is

$$A = \begin{pmatrix} b^2 & k & k \\ k & c^2 & k \\ k & k & d^2 \end{pmatrix}$$

and for its determinant we have successively

$$\begin{aligned} \det(A) &= \begin{vmatrix} b^2 & k & k \\ k & c^2 & k \\ k & k & d^2 \end{vmatrix} = \begin{vmatrix} b \cdot b & b \cdot c & b \cdot c \\ c \cdot b & c \cdot c & c \cdot d \\ d \cdot b & d \cdot c & d \cdot d \end{vmatrix} \\ &= \begin{vmatrix} b_1^2 + b_2^2 + b_3^2 & b_1c_1 + b_2c_2 + b_3c_3 & b_1d_1 + b_2d_2 + b_3d_3 \\ c_1b_1 + c_2b_2 + c_3b_3 & c_1^2 + c_2^2 + c_3^2 & c_1d_1 + c_2d_2 + c_3d_3 \\ d_1b_1 + d_2b_2 + d_3b_3 & d_1c_1 + d_2c_2 + d_3c_3 & d_1^2 + d_2^2 + d_3^2 \end{vmatrix} \\ &= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} \cdot \begin{vmatrix} b_1 & c_1 & d_1 \\ b_1 & c_2 & d_2 \\ b_1 & c_3 & d_3 \end{vmatrix} = (\vec{b}, \vec{c}, \vec{d}) \cdot \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = (\vec{b}, \vec{c}, \vec{d})^2. \end{aligned}$$

The linear independence of the vectors  $\vec{b}, \vec{c}, \vec{d}$  ensure that  $(\vec{b}, \vec{c}, \vec{d}) \neq 0$  and shows that the linear system (5.18) is consistent and has one single solution.

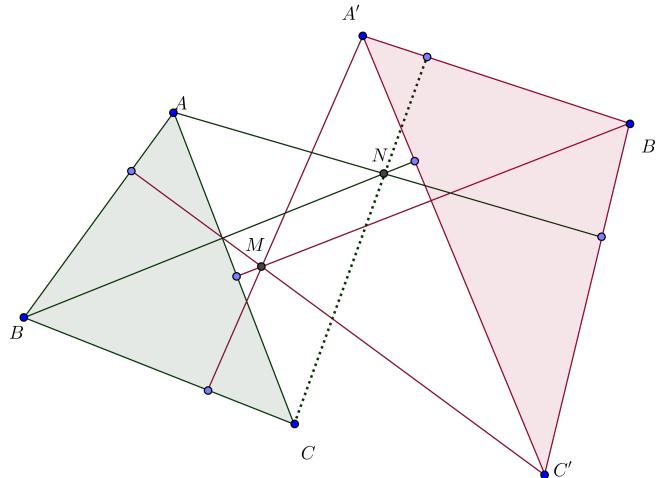
10. Two triangles  $ABC$  și  $A'B'C'$  are said to be *orthologic* if they are in the same plane and the perpendicular lines from the vertices  $A', B', C'$  on the sides  $BC, CA, AB$  are concurrent. Show

that, in this case, the perpendicular lines from the vertices  $A, B, C$  on the sides  $B'C', C'A', A'B'$  are concurrent too.

*Solution* Due to the given hypothesis, we have

$$\vec{MA}' \cdot \vec{BC} = \vec{MB}' \cdot \vec{CA} = \vec{MC}' \cdot \vec{AB} = 0 \quad (5.19)$$

We now consider the perpendicular lines from the vertices  $A$  and  $B$  on the edges  $B'C'$  and  $C'A'$  and denote by  $N$  their intersection point.



Thus

$$\vec{NA} \cdot \vec{B'C'} = \vec{NB} \cdot \vec{C'A'} = 0.$$

By using the relations (5.19) we obtain

$$\begin{aligned} & \vec{MA}' \cdot \vec{BC} + \vec{MB}' \cdot \vec{CA} + \vec{MC}' \cdot \vec{AB} = 0 \\ \Leftrightarrow & \vec{MA}' \cdot (\vec{NC} - \vec{NB}) + \vec{MB}' \cdot (\vec{NA} - \vec{NC}) + \vec{MC}' \cdot (\vec{NB} - \vec{NA}) = 0 \\ \Leftrightarrow & (\vec{MB}' - \vec{MC}') \cdot \vec{NA} + (\vec{MC}' - \vec{MA}') \cdot \vec{NB} + (\vec{MA}' - \vec{MB}') \cdot \vec{NC} = 0 \\ \Leftrightarrow & \vec{C'B'} \cdot \vec{NA} + \vec{A'C'} \cdot \vec{NB} + \vec{B'A'} \cdot \vec{NC} = 0 \\ \Leftrightarrow & \vec{B'A'} \cdot \vec{NC} = 0 \Leftrightarrow NC \perp A'B'. \end{aligned}$$

11. (2p) Find the orthogonal projection

(a) of the point  $A(1, 2, 1)$  on the plane  $\pi : x + y + 3z + 5 = 0$ .

(b) of the point  $B(5, 0, -2)$  on the straight line  $(d)$   $\frac{x-2}{3} = \frac{y-1}{2} = \frac{z-3}{4}$ .

*Solution.*

$$(a) \vec{r}_{P_{\pi}, v(A)} = \vec{r}_A - \frac{\vec{n}(A)}{\|\vec{n}\|} \cdot \vec{v}$$

Since the projection is orthogonal, we have  $\vec{v} \cdot \vec{n} = 0$

$$\therefore \vec{r}_{P_{\pi}(A)} = \vec{r}_A - \frac{\vec{n}(A)}{\|\vec{n}\|^2} \cdot \vec{n} \quad \therefore \vec{r}_{P_{\pi}(A)} = \left(\begin{array}{l} 1 \\ 2 \\ 1 \end{array}\right) - \frac{1+2+3+5}{1^2+1^2+3^2} \cdot \left(\begin{array}{l} 1 \\ 1 \\ 3 \end{array}\right) =$$

$$= \left(\begin{array}{l} 1 \\ 1 \\ 1 \end{array}\right) - \frac{11}{11} \cdot \left(\begin{array}{l} 1 \\ 1 \\ 3 \end{array}\right) = \left(\begin{array}{l} 0 \\ 0 \\ -2 \end{array}\right)$$

b) orthogonal projection of  $B(5, 0, -2)$  on a straight line (a):  $\frac{x-2}{3} = \frac{y-1}{2} = \frac{z+2}{4}$

$$G(x, y, z) = p(x - x_0) + g(y - y_0) + h(z - z_0) \Rightarrow G(x_0, y_0, z_0) = 3(2-5) + 2(-1) + 4(-3+2) =$$

$$= 3 \cdot (-3) + 2 + 4 \cdot 5 = -9 + 2 + 20 = 13$$

$\Leftrightarrow$  the coordinates of the projection are

$$\begin{cases} x_0 - p \frac{G(x_0, y_0, z_0)}{p^2 + g^2 + h^2} \\ y_0 - g \frac{G(x_0, y_0, z_0)}{p^2 + g^2 + h^2} \\ z_0 - h \frac{G(x_0, y_0, z_0)}{p^2 + g^2 + h^2} \end{cases} \Rightarrow \begin{cases} 2 - \frac{13}{29} \cdot 3 \\ 1 - \frac{13}{29} \cdot 2 \\ 3 - \frac{13}{29} \cdot 4 \end{cases} = \begin{cases} \frac{58-39}{29} \\ \frac{29-26}{29} \\ \frac{87-52}{29} \end{cases} = \begin{cases} \frac{19}{29} \\ \frac{3}{29} \\ \frac{35}{29} \end{cases}$$

$$\Rightarrow P_{dl}(B) \left( \frac{19}{29}, \frac{3}{29}, \frac{35}{29} \right)$$

### A few questions in the two dimensional setting

12. (3p) Find the coordinates of the point  $P$  on the line  $d : 2x - y - 5 = 0$  for which the sum  $AP + PB$  is minimum, when  $A(-7, 1)$  and  $B(-5, 5)$ .

13. (2p) Find the coordinates of the circumcenter (the center of the circumscribed circle) of the triangle determined by the lines  $\underline{d_1 : 4x - y + 2 = 0}$ ,  $\underline{d_2 : x - 4y - 8 = 0}$  and  $\underline{d_3 : x + 4y - 8 = 0}$ .

*Solution.*

$$\begin{aligned} d_1 \cap d_2 = \{A\} &\rightarrow \begin{cases} 4x - y + 2 = 0 \\ x - 4y - 8 = 0 \end{cases} \quad (\Rightarrow) \quad \begin{cases} 4x - y + 2 = 0 \\ -4x + 16y + 32 = 0 \end{cases} \Rightarrow 15y = -34 \Rightarrow y = \frac{-34}{15}, \\ -4x + \frac{34}{15} + 2 = 0 &\Rightarrow 4x = \frac{64}{15} \quad (\Rightarrow) \quad x = \frac{16}{15} \Rightarrow A\left(-\frac{16}{15}, -\frac{34}{15}\right) \end{aligned}$$

$$d_2 \cap d_3 = \{B\} \rightarrow \begin{cases} x - 4y - 8 = 0 \\ x + 4y - 8 = 0 \end{cases} \quad (\Rightarrow) \quad 2x = 16 \quad (\Rightarrow) \quad x = 8 \Rightarrow 4y = 8 - x \Rightarrow y = 0 \Rightarrow B(8, 0)$$

$$d_1 \cap d_3 = \{C\} \rightarrow \begin{cases} 4x - y + 2 = 0 \\ x + 4y - 8 = 0 \end{cases} \quad (\Rightarrow) \quad 16x - 4y + 8 = 0 \quad (\Rightarrow) \quad 12x = 0 \Rightarrow x = 0 \Rightarrow y = 2 \Rightarrow C(0, 2)$$

$$O(x, y) \rightarrow |OA| = |OB| = |OC|$$

$$d_1^2 = \left(x + \frac{16}{15}\right)^2 + \left(y + \frac{34}{15}\right)^2; d_2^2 = (x - 8)^2 + (y - 0)^2 = (x - 8)^2 + y^2$$

$$d_1^2 = d_2^2 \rightarrow \left(x + \frac{16}{15}\right)^2 + \left(y + \frac{34}{15}\right)^2 = (x - 8)^2 + y^2 \rightarrow x = \frac{191}{60} - \frac{1}{4}y \quad (\Rightarrow) \quad -4y - 4x = 191 - 15$$

$$d_3^2 = (x - 0)^2 + (y - 2)^2 = x^2 + (y - 2)^2$$

$$d_2^2 = d_3^2 \rightarrow (x - 8)^2 + y^2 = x^2 + (y - 2)^2 \rightarrow 16x - 4y - 60 = 0 \rightarrow 4x - y - 15 = 0$$

$$\Rightarrow x = \frac{34}{15}, y = -\frac{14}{15}$$

14. (3p) Given the bundle of lines of equations  $(1-t)x + (2-t)y + t - 3 = 0$ ,  $t \in \mathbb{R}$  and  $x + y - 1 = 0$ , find:

- the coordinates of the vertex of the bundle;
- the equation of the line in the bundle which cuts  $Ox$  and  $Oy$  in  $M$  respectively  $N$ , such that  $OM^2 \cdot ON^2 = 4(OM^2 + ON^2)$ .

*Solution.*

15. (2p) Let  $\mathcal{B}$  be the bundle of lines of vertex  $M_0(5, 0)$ . An arbitrary line from  $\mathcal{B}$  intersects the lines  $d_1 : y - 2 = 0$  and  $d_2 : y - 3 = 0$  in  $M_1$  respectively  $M_2$ . Prove that the line passing through  $M_1$  and parallel to  $OM_2$  passes through a fixed point.

16. (3p) The vertices of the quadrilateral  $ABCD$  are  $A(4, 3)$ ,  $B(5, -4)$ ,  $C(-1, -3)$  and  $D((-3, -1))$ .

- (a) Find the coordinates of the intersection points  $\{E\} = AB \cap CD$  and  $\{F\} = BC \cap AD$ ;
- (b) Prove that the midpoints of the segments  $[AC]$ ,  $[BD]$  and  $[EF]$  are collinear.

*Solution.*

17. (3p) Let  $M$  be a point whose coordinates satisfy

$$\frac{4x + 2y + 8}{3x - y + 1} = \frac{5}{2}.$$

- (a) Prove that  $M$  belongs to a fixed line  $(d)$ ;
- (b) Find the minimum of  $x^2 + y^2$ , when  $M \in d \setminus \{M_0(-1, -2)\}$ .

*Solution.*

(a)  $2(4x + 2y + 8) = 5(3x - y + 1) \Rightarrow 8x + 4y + 16 = 15x - 5y + 5 \Rightarrow$   
 $-7x - 9y - 11 = 0 \text{ - line } l$

(b)  $x^2 + y^2 \geq 2xy \text{ with equality if } x = y$

We will look at the point on the line that satisfies this, which is  $Q(a, a)$ ,

$$\Rightarrow 7a - 9a - 11 = 0 \Rightarrow a = -\frac{11}{2} \Rightarrow \min(x^2 + y^2) = 2 \cdot \frac{121}{4} = \frac{121}{2}$$

$$(x, y) \neq (-1, -2)$$

18. (3p) Find the locus of the points whose distances to two orthogonal lines have a constant ratio.

*Solution.*