DYNAMICAL SYSTEMS - HOMEWORK 1
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1.2.1) How many solutions have each of the following problems:

a) $x''+t^2x=0$, $x(0)=0$

b) $x''+t^2x=0, x(0)=0, x'(0)=0$

c) $x''+t^2x=0, x(0)=0, x'(0)=0, x''(0)=1$?

a) From the fundamental theorem for LHDEs $\Rightarrow$ the set of solutions of an n^{th} order LHDE is a linear space of dimension n. For the second order LHDE there ~~are~~ is a set of solutions of dimension 2, so there are 2 linearly independent solutions (x_1 and x_2), that give the general solution:

$$x=c_1\cdot x_1+c_2\cdot x_2, \text{ with } c_1,c_2\in\mathbb{R}$$

b) We have the following IVP: $\begin{cases} x''+t^2x=0 \\ x(0)=0 \\ x'(0)=0 \end{cases}$

Applying the existence and uniqueness theorem for the Initial Value Problem, for this IVP there is a unique solution.

c) $\begin{cases} x''+t^2x=0 \\ x(0)=0 \\ x'(0)=0 \\ x''(0)=1 \end{cases}$

For $t=0$, the LHDE will be: $\underbrace{x''(0)}_{1}+0^2\cdot\underbrace{x(0)}_{0}=0 \Leftrightarrow 1=0$, false $\Rightarrow$

$\Rightarrow$ For this equation there are no solutions

1.2.5) a) Find a particular solution of the form $x_p = a\cdot e^t (a\in\mathbb{R})$ for the equation $x' - 2x = e^t$.

b) Find a particular solution of the form $x_p = be^{-t} (b\in\mathbb{R})$ for the equation $x' - 2x = e^{-t}$.

c) Using the Superposition Principle, and a) and b), find a particular solution for the equation $x' - 2x = 5e^t - 3e^{-t}$.

d) Find the general solution of $x' - 2x = 5e^t - 3e^{-t}$.

a) $\left.\begin{array}{l} x_p = a\cdot e^t,\ a\in\mathbb{R} \text{ - particular solution of } x' - 2x = e^t \\ x_p' = a\cdot e^t \end{array}\right\} \Rightarrow a\cdot e^t - 2ae^t = e^t \,/ : e^t \neq 0 \iff$

$\iff a - 2a = 1 \iff -a = 1 \,/ \cdot(-1)\neq 0 \iff a = -1 \in\mathbb{R} \Rightarrow$ A particular solution of $x' - 2x = e^t$ is: $x_p = -e^t$.

b) $\left.\begin{array}{l} x_p = be^{-t},\ b\in\mathbb{R} \text{ - particular solution of } x' - 2x = e^{-t} \\ x_p' = -be^{-t} \end{array}\right\} \Rightarrow -be^{-t} - 2be^{-t} = e^{-t} \,/ : e^{-t} \neq 0 \iff$

$\iff -b - 2b = 1 \iff -3b = 1 \,/ : (-3)\neq 0 \iff b = -\frac{1}{3} \in\mathbb{R} \Rightarrow$ A particular solution of $x' - 2x = e^{-t}$ is: $x_p = -\frac{1}{3}e^{-t}$.

c) $x' - 2x = 5e^t - 3e^{-t}$. Let $f = x' - 2x = \mathcal{L}(x)$

$\left.\begin{array}{l} \alpha_1 = 5,\ f_1 = e^t \\ \alpha_2 = -3,\ f_2 = e^{-t} \end{array}\right\}$ s.t. $f = \alpha_1 f_1 + \alpha_2 f_2 = 5e^t - 3e^{-t}$

$\left.\begin{array}{l} \text{From a)} \Rightarrow \mathcal{L}(x) = f_1 \text{ has the particular solution } x_{p_1} = -e^t \\ \text{From b)} \Rightarrow \mathcal{L}(x) = f_2 \text{ has the particular solution } x_{p_2} = -\frac{1}{3}e^{-t} \end{array}\right\} \Rightarrow$

$\Rightarrow$ From the Superposition Principle, $x_p = \alpha_1 x_{p_1} + \alpha_2 x_{p_2} \Rightarrow$

$\Rightarrow x_p = 5\cdot(-e^t) + \left(-\frac{1}{3}\right)e^{-t}\cdot(-3) = -5e^t + e^{-t}$

d) From c), we know that a particular solution of $x' - 2x = 5e^t - 3e^{-t}$ is: $x_p = -5e^t - e^{-t}$.

The general solution is: $x = x_h + x_p$, where x_p - a particular solution, x_h - the solution of the LHDE: $x' - 2x = 0$

We compute x_h.

$x' - 2x = 0 \Rightarrow$ The characteristic eq.: $r - 2 = 0 \Rightarrow r = 2 \in\mathbb{R}$, unique solution of multiplicity 1 (1 root) $\Rightarrow \vdash e^{2t} \Rightarrow x_h = c\cdot e^{2t}, c\in\mathbb{R} \Rightarrow$

$\Rightarrow$ The general solution of $x' - 2x = 5e^t - 3e^{-t}$ is:

$$x = c\cdot e^{2t} + (-5e^t - e^{-t}),\ c\in\mathbb{R} \iff$$

$$\iff x = c\cdot e^{2t} - 5e^t - e^{-t},\ c\in\mathbb{R}$$

4.3.4) Find the general solution of $x'-x=e^{t-1}$. Justify the result in two ways.

I. Using the integrating factor method.

We consider $\mu(t)=e^{-t}$ and then we multiply the initial LN-HDE.

$x'-x=e^{t-1} /\cdot e^{-t} \Leftrightarrow x'\cdot e^{-t}-x\cdot e^{-t}=e^{t-1}\cdot e^{-t} \Leftrightarrow x'\cdot e^{-t}+x\cdot(e^{-t})'=e^{-1} \Leftrightarrow$

$\Leftrightarrow (x\cdot e^{-t})'=\frac{1}{e} / \int$ -integrate "t" $\Leftrightarrow x\cdot e^{-t}=\int \frac{1}{e}\,dt+c \Leftrightarrow x\cdot e^{-t}=\frac{1}{e}\cdot t+c / \cdot e^{t} \Leftrightarrow$

$\Leftrightarrow x=e^{t-1}\cdot t+c\cdot e^{t}, c\in\mathbb{R}$ - the general solution of the Ln-HDE $x'-x=e^{t-1}$

II. Using Lagrange method to find a particular solution and the general solution of the LHDE associated to our LN-HDE.

The LHDE associated to $x'-x=e^{t-1}$ is $x'-x=0$

The characteristic equation: $r-1=0 /+1 \Rightarrow r=1\in\mathbb{R}$, root of multiplicity $1 \Rightarrow$

$\Rightarrow x=e^{t} \Rightarrow$ the general solution is: $x=c\cdot e^{t}, c\in\mathbb{R}$

The solution of the Ln-HDE can be written as: $x=x_h+x_p$, where:

- x_h is the general solution of the LHDE: $x_h=c\cdot e^{t}, c\in\mathbb{R} \Rightarrow A(t)=t$
- x_p - a particular solution

Applying Lagrange, we are looking for $x_p=\varphi(t)\cdot e^{A(t)}$, where $\varphi(t)$ is a function.

$a(t)=-1, A(t)=t \Rightarrow A'(t)=1 \Rightarrow A'(t)=-a(t)$

We replace ~~x_p in t~~ with x_p in the Ln-HDE:

~~$\varphi(t)$~~ $(\varphi(t)\cdot e^{t})'-\varphi(t)\cdot e^{t}=e^{t-1} \Leftrightarrow \varphi'(t)\cdot e^{t}+\cancel{\varphi(t)\cdot e^{t}}-\cancel{\varphi(t)\cdot e^{t}}=e^{t-1} \Leftrightarrow$

$\Leftrightarrow \varphi'(t)\cdot e^{t}=e^{t-1} /\cdot e^{-t} \Rightarrow \varphi'(t)=\frac{1}{e} \Rightarrow \varphi(t)=\int_0^t \frac{1}{e}\,ds \Rightarrow \varphi(t)=\frac{1}{e}t \Rightarrow$

$\Rightarrow x_p=\frac{1}{e}t\cdot e^{t}=t\cdot e^{t-1}$

So, the general solution will be $x=c\cdot e^{t}+t\cdot e^{t-1}, c\in\mathbb{R}$, the same as in the first way.

1.6.1) Let $A \in \mathcal{M}_2(\mathbb{R})$. Using both methods that we learned, the characteristic eq. method and reduction to second order eq, find the general solution of the system $x' = Ax$ in each of the following situations. Also find ~~the general solution of the sys~~ a fundamental matrix solution and, finally, find e^{tA}, the principal matrix solution.

j) $A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ k) $A = \begin{pmatrix} 0 & 4 \\ 5 & 1 \end{pmatrix}$ m) $A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

j) λ-eigenvalue of $A \Leftrightarrow \det(A - \lambda I_2) = 0 \Leftrightarrow$

$\Leftrightarrow \begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Leftrightarrow (2-\lambda)^2 = 0 \Leftrightarrow \lambda_{1,2} = 2 \in \mathbb{R}$, so the eigenvalues of A are $\lambda_1 = \lambda_2 = 2$

$u_1 = ?$, an eigenvector corresponding to $\lambda_1 = 2 \Leftrightarrow u_1 \in \mathbb{R}^2, u_1 \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, Au_1 = 2u_1, u_1 = \begin{pmatrix} a \\ b \end{pmatrix}$

$Au_1 = 2u_1 \Leftrightarrow \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \cdot \begin{pmatrix} a \\ b \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2a \\ a+2b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix} \Leftrightarrow \begin{cases} 2a = 2a, \text{true} \\ a+2b = 2b \,/ -2b \end{cases} \Leftrightarrow \begin{cases} 2a = 2a \\ a = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow a = 0, b \in \mathbb{R}$. Choose $a = 0, b = 1 \Rightarrow u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$u_2 = ?$, an eigenvector corresponding to $\lambda_2 = 2 \Leftrightarrow u_2 \in \mathbb{R}^2, u_2 \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, Au_2 = 2u_2, u_2 = \begin{pmatrix} a \\ b \end{pmatrix}$

$Au_2 = 2u_2 \Leftrightarrow \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{cases} a = 0 \\ b \in \mathbb{R} \end{cases}$. However we choose b, u_1 and u_2 will not be linearly independent: $\begin{vmatrix} 0 & 0 \\ 1 & b \end{vmatrix} = b \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0 \Rightarrow u_1$ and u_2 are linearly independent, so A is not diagonalizable

$\Rightarrow$ For this matrix we can't apply the characteristic equation method.
So, we are going to use the reduction to a second order differential equation method.

$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \Rightarrow \begin{matrix} a_{12} = 0 \\ a_{21} = 1 \neq 0 \end{matrix} \Rightarrow$ we are going to have the coupled system:

$\begin{cases} x' = 2x + 0y \\ y' = x + 2y \end{cases} \Leftrightarrow \begin{cases} x' = 2x \\ y' = x + 2y \end{cases} \Leftrightarrow \begin{cases} x' = 2x \\ x = y' - 2y \end{cases}$

~~$x' = 2x \Rightarrow x'' = 2x'$~~

$x = y' - 2y \Rightarrow x' = y'' - 2y' \Rightarrow 2x = y'' - 2y' \Rightarrow 2y' - 4y = y'' - 2y' \Rightarrow$

$\Rightarrow y'' - 4y' + 4y = 0 \Rightarrow$ the charact. eq: $r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow$

$\Rightarrow r_{1,2} = 2 \in \mathbb{R} \mapsto e^{2t}, te^{2t} \Rightarrow y = c_1 e^{2t} + c_2 \cdot te^{2t}, c_1, c_2 \in \mathbb{R} \Rightarrow$

$\Rightarrow x = y' - 2y \Rightarrow x = \cancel{2c_1 e^{2t}} + \cancel{2tc_2 e^{2t}} + c_2 e^{2t} \cancel{- 2c_1 e^{2t}} \cancel{- 2c_2 te^{2t}}$
$= c_2 e^{2t}$, $c_2 \in \mathbb{R}$

$\Rightarrow \begin{cases} x = c_2 e^{2t} \\ y = c_1 e^{2t} + c_2 te^{2t} \end{cases}$, $c_1, c_2 \in \mathbb{R}$ - the general solution of the system $AX = X'$

Denote $U = \begin{pmatrix} 0 & e^{2t} \\ e^{2t} & te^{2t} \end{pmatrix} \Rightarrow \det U = 0 - (e^{2t})^2 = -e^{4t} \neq 0, \forall t \in \mathbb{R} \Rightarrow U$ is a fundamental matrix solution

We know that e^{tA} is the unique solution of the IVP: $\begin{cases} X' = AX \\ X(0) = I_2 \end{cases}$

$\Rightarrow$ the first column of e^{tA} is the solution of the IVP: $\begin{cases} X' = AX \\ X(0) = \begin{pmatrix}1\\0\end{pmatrix} \end{cases}$ (1)

$\Rightarrow$ the ~~first~~ second column of e^{tA} is the solution of the IVP: $\begin{cases} X' = AX \\ X(0) = \begin{pmatrix}0\\1\end{pmatrix} \end{cases}$ (2)

(1): $\begin{cases} x(0) = c_2 = 1 \\ y(0) = c_1 = 0 \end{cases}$ $\Rightarrow$ the sol. of (1) is $X_1 = \begin{pmatrix} e^{2t} \\ te^{2t} \end{pmatrix}$ $\Rightarrow$

(2): $\begin{cases} x(0) = 0 \\ y(0) = 1 \end{cases}$ $\Rightarrow \begin{cases} c_2 = 0 \\ c_1 = 1 \end{cases}$ $\Rightarrow$ the sol. of (2) is $X_2 = \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$

$\Rightarrow e^{tA} = (X_1 \ X_2) = \begin{pmatrix} e^{2t} & 0 \\ te^{2t} & e^{2t} \end{pmatrix}$ - the principal matrix solution

k) I. The characteristic eq. method

λ - eigenvalue of $A \Leftrightarrow \det(A - \lambda I_2) = 0 \Leftrightarrow \begin{vmatrix} -\lambda & 4 \\ 5 & 1-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda(\lambda - 1) - 20 = 0$

$\Leftrightarrow \lambda^2 - \lambda - 20 = 0 \Leftrightarrow \lambda^2 + 4\lambda - 5\lambda - 20 = 0 \Leftrightarrow (\lambda+4)(\lambda-5) = 0$ $\nearrow \lambda_1 = -4$ $\searrow \lambda_2 = 5$

$u_1 = ?$ an eigenvector corresponding to $\lambda_1 = -4$

$u_1 \in \mathbb{R}^2, u_1 = \begin{pmatrix}a\\b\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}, Au_1 = \lambda_1 u_1 \Leftrightarrow \begin{pmatrix} 0 & 4 \\ 5 & 1 \end{pmatrix} \cdot \begin{pmatrix}a\\b\end{pmatrix} = -4\begin{pmatrix}a\\b\end{pmatrix} \Leftrightarrow$

$\Leftrightarrow \begin{pmatrix} 4b \\ 5a+b \end{pmatrix} = \begin{pmatrix} -4a \\ -4b \end{pmatrix} \Leftrightarrow \begin{cases} 4b = -4a \\ 5a + b = -4b \end{cases} \Leftrightarrow \begin{cases} b = -a \\ 5b = -5a \end{cases} \Leftrightarrow \begin{cases} b = -a \\ b = -a \end{cases}$

Choose $a = 1, b = -1 \Rightarrow u_2 = \begin{pmatrix}1\\-1\end{pmatrix}$

$u_2 = ?$ an ~~eigenvalue~~ eigenvector corresponding to $\lambda_2 = 5$

$u_2 \in \mathbb{R}^2, u_2 = \begin{pmatrix}c\\d\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}, Au_2 = \lambda_2 u_2 \Leftrightarrow \begin{pmatrix} 0 & 4 \\ 5 & 1 \end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix} = 5\begin{pmatrix}c\\d\end{pmatrix} \Leftrightarrow \begin{pmatrix} 4d \\ 5c+d \end{pmatrix} = \begin{pmatrix} 5c \\ 5d \end{pmatrix} \Leftrightarrow$

$\Leftrightarrow \begin{cases} 4d = 5c \\ 5c = 4d \end{cases}$

Choose $c = 4, d = 5 \Rightarrow u_2 = \begin{pmatrix}4\\5\end{pmatrix}$

So, we have for A the eigenvalues $\lambda_1 = -4, \lambda_2 = 5$, real numbers and two eigenvectors $u_1 = \begin{pmatrix}1\\-1\end{pmatrix}$ and $u_2 = \begin{pmatrix}4\\5\end{pmatrix}$.

Since $\begin{vmatrix} 1 & 4 \\ -1 & 5 \end{vmatrix} = 5 + 4 = 9 \neq 0 \Rightarrow u_1$ and u_2 are linearly independant

$\Rightarrow$ A is diagonalizable $\Rightarrow$ we can apply the characteristic equation method

$\Rightarrow$ The solutions of A are: $e^{-4t}\begin{pmatrix}1\\-1\end{pmatrix}$ and $e^{5t}\begin{pmatrix}4\\5\end{pmatrix}$.

$\Rightarrow$ the general solution is: $X = c_1 e^{-4t}\begin{pmatrix}1\\-1\end{pmatrix} + c_2 e^{5t}\begin{pmatrix}4\\5\end{pmatrix}$, $c_1, c_2 \in \mathbb{R}$

$X' = AX \Leftrightarrow \begin{cases} x' = 4y \\ y' = 5x + 4 \end{cases}$ $\Rightarrow$ the general solution: $\begin{cases} x = c_1 e^{-4t} + 4c_2 e^{5t} \\ y = -c_1 e^{-4t} + 5c_2 e^{5t} \end{cases}$, $c_1, c_2 \in \mathbb{R}$

$\Delta = diag(\lambda_1, \lambda_2) = \begin{pmatrix} -4 & 0 \\ 0 & 5 \end{pmatrix}$, $\det \Delta \neq 0, \forall t \in \mathbb{R} \Rightarrow \Delta$ is a fundamental solution

$P = (u_1\ u_2) = \begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix}$, $\det(P) = +9 \Rightarrow P^{-1} = \frac{1}{9}\begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix}$

$\Rightarrow e^{tA} = P\, diag(e^{\lambda_1 t}, e^{\lambda_2 t}) P^{-1} = \begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} e^{-4t} & 0 \\ 0 & e^{5t} \end{pmatrix} \cdot \begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix} \cdot \frac{1}{9} = \begin{pmatrix} e^{-4t} & 4e^{5t} \\ -e^{-4t} & 5e^{5t} \end{pmatrix} \begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix} \frac{1}{9} =$

$= \frac{1}{9} \cdot \begin{pmatrix} 5e^{-4t} + 4e^{5t} & -4e^{-4t} + 4e^{5t} \\ -5e^{-4t} + 5e^{5t} & 4e^{-4t} + 5e^{5t} \end{pmatrix}$ - the principal matrix solution

II. Reduction to a second order differential equation method

$A = \begin{pmatrix} 0 & 4 \\ 5 & 1 \end{pmatrix} \Rightarrow \begin{matrix} a_{12} = 4 \neq 0 \\ a_{21} = 5 \neq 0 \end{matrix} \Rightarrow$ we have the following coupled system:

$\begin{cases} x' = 4y \\ y' = 5x + y \end{cases} \Leftrightarrow \begin{cases} y = \frac{1}{4}x' \\ y' = 5x + y \end{cases} \Rightarrow \left(\frac{1}{4}x'\right)' = 5x + \frac{1}{4}x' \Leftrightarrow \frac{1}{4}x'' = 5x + \frac{1}{4}x' \,/\cdot 4 \Leftrightarrow$

$\Leftrightarrow x'' - x' - 20x = 0 \Leftrightarrow$

$\Leftrightarrow$ we have the following charact. eq: $r^2 - r - 20 = 0$ $\begin{matrix} \nearrow r_1 = -4 \\ \searrow r_2 = 5 \end{matrix}$ $\Rightarrow$

$\Rightarrow \begin{matrix} \nearrow r_1 = -4 \mapsto e^{-4t} \\ \searrow r_2 = 5 \mapsto e^{5t} \end{matrix}$, sol. $\Rightarrow x = c_1 e^{-4t} + e^{5t} \cdot c_2$, $c_1, c_2 \in \mathbb{R} \Rightarrow$

$\Rightarrow y = \frac{1}{4}(c_1 e^{-4t} + c_2 e^{5t})' = \frac{1}{4}(-4c_1 e^{-4t} + 5c_2 e^{5t}) = \frac{5}{4}c_2 e^{5t} - c_1 e^{-4t}$, $c_1, c_2 \in \mathbb{R} \Rightarrow$

$\Rightarrow \begin{cases} x = c_1 e^{-4t} + c_2 e^{5t} \\ y = -c_1 e^{-4t} + \frac{5}{4}c_2 e^{5t} \end{cases}$, $c_1, c_2 \in \mathbb{R}$ - general solution of the system $X' = AX$.

$U = \begin{pmatrix} e^{-4t} & e^{5t} \\ -e^{-4t} & \frac{5}{4}e^{5t} \end{pmatrix} \Rightarrow \det U = \frac{5}{4}e^{t} + e^{t} = \frac{9}{4}e^{t} \neq 0 \Rightarrow U$ is a fundamental matrix solution $, \forall t \in \mathbb{R}$

e^{tA} is the unique solution of the IVP: $\begin{cases} X' = AX \\ X(0) = I_2 \end{cases} \Rightarrow$

$\Rightarrow$ the first column of $\cancel{A}$ e^{tA} is the solution of the IVP: $\begin{cases} X' = AX \\ X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$ (3)

$\Rightarrow$ the ~~fit~~ second column of e^{tA} is the solution of the IVP: $\begin{cases} X' = AX \\ X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$ (4)

(3): $\begin{cases} X(0) = c_1 + c_2 = 1 \\ Y(0) = -c_1 + \frac{5}{4}c_2 = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 + c_2 = 1 \\ c_1 = \frac{5}{4}c_2 \end{cases} \Leftrightarrow \begin{cases} c_1 = \frac{5}{4}c_2 \\ \frac{5}{4}c_2 + c_2 = 1 \end{cases} \Leftrightarrow \begin{cases} c_1 = \frac{5}{4}c_2 \\ \frac{9}{4}c_2 = 1 \end{cases} \Rightarrow \begin{cases} c_2 = \frac{4}{9} \\ c_1 = \frac{5}{9} \end{cases} \Rightarrow$

$\Rightarrow X_1 = \begin{pmatrix} \frac{5}{9}e^{-4t} + \frac{4}{9}e^{5t} \\ -\frac{5}{9}e^{-4t} + \frac{5}{9}e^{5t} \end{pmatrix}$

$(4): \begin{cases} X(0)=c_1+c_2=0 \\ Y(0)=-c_1+\frac{5}{4}c_2=1 \end{cases} \Leftrightarrow \begin{cases} c_2=-c_1 \\ -c_1-\frac{5}{4}c_1=1 \end{cases} \Leftrightarrow \begin{cases} c_2=-c_1 \\ -\frac{9}{4}c_1=1 \end{cases} \Leftrightarrow \begin{cases} c_2=-c_1 \\ c_1=-\frac{4}{9} \end{cases} \Leftrightarrow \begin{cases} c_1=-\frac{4}{9} \\ c_2=\frac{4}{9} \end{cases} \Rightarrow$

$\Rightarrow X_2=\begin{pmatrix} -\frac{4}{9}e^{-4t}+\frac{4}{9}e^{5t} \\ \frac{4}{9}e^{-4t}+\frac{5}{9}e^{5t} \end{pmatrix}$

$\Rightarrow e^{tA}=(X_1\ \ X_2)=\begin{pmatrix} \frac{5}{9}e^{-4t}+\frac{4}{9}e^{5t} & -\frac{4}{9}e^{-4t}+\frac{4}{9}e^{5t} \\ -\frac{5}{9}e^{-4t}+\frac{5}{9}e^{5t} & \frac{4}{9}e^{-4t}+\frac{5}{9}e^{5t} \end{pmatrix}$ - principal matrix solution

m) $A=\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

λ - eigenvalue of $A \Leftrightarrow \det(A-\lambda I_2)=0 \Leftrightarrow \begin{vmatrix} -\lambda & -2 \\ 2 & -\lambda \end{vmatrix}=0 \Leftrightarrow \lambda^2+4=0 \Leftrightarrow$

$\Leftrightarrow \lambda_{1,2}=\pm 2i \Rightarrow$ the eigenvalues are not real numbers $\Rightarrow A$ is not diagonalizable

$\Rightarrow$ We can use only the reduction to a second order differential equation.

$A=\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \Rightarrow a_{12}=-2\neq 0$, $a_{21}=2\neq 0$ $\Rightarrow$ we have the following coupled system:

$\begin{cases} x'=-2y \\ y'=2x \end{cases}$

$x'=-2y \Rightarrow x''=-2y'$, $y'=2x$ $\Rightarrow x''=-4x \Rightarrow x''+4x=0 \Rightarrow$ the charact. eq.: $r^2+4=0 \Leftrightarrow$

$\Leftrightarrow r_{1,2}=\pm 2i \in \mathbb{C} \Rightarrow$ the root is a root of multiplicity 1

$r_{1,2}=\pm 2i \mapsto \cos(2t), \sin(2t) \Rightarrow x=c_1\cos(2t)+c_2\sin(2t)$, $c_1,c_2\in\mathbb{R}$

$\Rightarrow y=-\frac{1}{2}x'=-\frac{1}{2}(-2c_1\sin(2t)+2c_2\cos(2t)) \Rightarrow y=c_1\sin(2t)-c_2\cos(2t)$, $c_1,c_2\in\mathbb{R}$ $\Rightarrow$

$\Rightarrow \begin{cases} x=c_1\cos(2t)+c_2\sin(2t) \\ y=c_1\sin(2t)-c_2\cos(2t) \end{cases}$, $c_1,c_2\in\mathbb{R}$ - the general solution of the system

Denote $U=\begin{pmatrix} \cos(2t) & \sin(2t) \\ \sin(2t) & -\cos(2t) \end{pmatrix} \Rightarrow \det U=-\cos^2(2t)-\sin^2(2t)=-1\neq 0, \forall t\in\mathbb{R} \Rightarrow$

$\Rightarrow U$ is a fundamental matrix solution

e^{tA} is the unique sol. of the IVP: $\begin{cases} X'=AX \\ X(0)=I_2 \end{cases}$ ↗ first column of e^{tA}: $\begin{cases} X'=AX \\ X(0)=\begin{pmatrix}1\\0\end{pmatrix} \end{cases}$ (5)

↘ second column of e^{tA}: $\begin{cases} X'=AX \\ X(0)=\begin{pmatrix}0\\1\end{pmatrix} \end{cases}$ (6)

$(5): \begin{cases} X(0)=c_1=1 \\ Y(0)=-c_2=0 \end{cases} \Rightarrow \begin{cases} c_1=1 \\ c_2=0 \end{cases} \Rightarrow$ sol: $X_1=\begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$

$(6): \begin{cases} X(0)=c_1=0 \\ Y(0)=-c_2=1 \end{cases} \Rightarrow \begin{cases} c_1=0 \\ c_2=-1 \end{cases} \Rightarrow$ sol: $X_2=\begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix}$

$\Rightarrow e^{tA}=(X_1\ X_2)=\begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix}$ - principal matrix solution

1.7.5) Let $t \in \mathbb{R}$. Using the Euler's formula compute $e^{it}, e^{i\pi}, e^{i\pi/2}, e^{(-1+i)t}$.

$e^{it} = ?$

For $e^{it} \Rightarrow \begin{cases} \alpha = 0 \in \mathbb{R} \\ \beta = t \in \mathbb{R} \end{cases} \xRightarrow[\text{formula}]{\text{Euler's}} e^{it} = \underbrace{e^0}_{1}(\cos t + i\sin t) \Rightarrow e^{it} = \cos t + i\sin t, t \in \mathbb{R}$

$e^{i\pi} = ?$

For $e^{i\pi} \Rightarrow \begin{cases} \alpha = 0 \in \mathbb{R} \\ \beta = \pi \in \mathbb{R} \end{cases} \xRightarrow[\text{formula}]{\text{Euler's}} e^{i\pi} = \underbrace{e^0}_{1}(\underbrace{\cos \pi}_{-1} + i\underbrace{\sin \pi}_{0}) \Rightarrow e^{i\pi} = -1$

$e^{i\pi/2} = ?$

For $e^{i\pi/2} \Rightarrow \begin{cases} \alpha = 0 \in \mathbb{R} \\ \beta = \frac{\pi}{2} \in \mathbb{R} \end{cases} \xRightarrow[\text{formula}]{\text{Euler's}} e^{i\pi/2} = \underbrace{e^0}_{1}(\underbrace{\cos \frac{\pi}{2}}_{0} + i\underbrace{\sin \frac{\pi}{2}}_{1}) \Rightarrow e^{i\pi/2} = i$

$e^{(-1+i)t} = ?$

For $e^{(-1+i)t} = e^{-t+it} \Rightarrow \begin{cases} \alpha = -t \in \mathbb{R} \\ \beta = t \in \mathbb{R} \end{cases} \xRightarrow[\text{formula}]{\text{Euler's}} e^{(-1+i)t} = e^{-t}(\cos(+t) + i\sin(+t)), t \in \mathbb{R}$

Remark: Euler's formula $\rightarrow$ $e^{\alpha + i\beta} = e^{\alpha}(\cos \beta + i\sin \beta), \alpha, \beta \in \mathbb{R}$

1.7.8) Find the linear homogeneous differential equation of minimal order that has as solution the function $1+t(1+e^{-t})$.

$1+t(1+e^{-t}) = 1+t+t\cdot e^{-t}$ is a solution $\Rightarrow 1, t$ and $t\cdot e^{-t}$ are solutions $\Rightarrow$

$\Rightarrow 1, t, e^{-t}$ and $t\cdot e^{-t}$ are solutions $\Rightarrow e^{0t}, t\cdot e^{0t}, e^{-t}, t\cdot e^{-t}$ are solutions $\Rightarrow$

$$\Rightarrow \left.\begin{array}{l} r_1 = 0 \text{ - double real root} \\ r_2 = -1 \text{ - double real root} \end{array}\right\} \Rightarrow (r-0)^2(r-(-1))^2 = 0 \Leftrightarrow r^2(r+1)^2 = 0 \Leftrightarrow$$

$\Leftrightarrow r^2(r^2+2r+1) = 0 \Leftrightarrow r^4+2r^3+r^2 = 0$ - characteristic equation $\Rightarrow$

$\Rightarrow$ the searched LHDE is: $x^{(4)}+2x^{(3)}+x'' = 0$, ~~tot~~ which is an equation of 4th order. Its general solution is: $x = c_1+c_2t+c_3e^{-t}+c_4te^{-t}, c_1, c_2, c_3, c_4 \in \mathbb{R}$

1.7.9) Let $k, \eta \in \mathbb{R}$ be fixed. Find the solution of the IVP: $x' = k(21-x), x(0) = \eta$.

IVP: $\begin{cases} x' = k(21-x) \\ x(0) = \eta \end{cases}$

$x' = k(21-x) \Leftrightarrow x' = 21k - xk \Leftrightarrow$ ~~$x' + kx - 21k = 0 \Rightarrow$ the ce~~

~~$\Rightarrow$ the characteristic equation: r~~

$\Leftrightarrow x' + xk = 21k$. We can easily observe that a particular solution is $x_p = 21$: $21' + 21k = 21k \Leftrightarrow 0 + 21k = 21k \Leftrightarrow 21k = 21k$, true.

We have to find now the general solution of the associated LHDE:

$x' + xk = 0 \Rightarrow$ The characteristic eq: $r + k = 0 \Rightarrow r = -k \in \mathbb{R}$, root of multiplicity $1 \Rightarrow r = -k \mapsto e^{-kt} \Rightarrow x_h = c \cdot e^{-kt}, c \in \mathbb{R}$

The general solution for the Ln-HDE: $x' + xk = 21k$ is:

$$x = x_h + x_p = c \cdot e^{-kt} + 21, c \in \mathbb{R}$$

From the IVP $\Rightarrow x(0) = \eta$ $\Bigg\} \Rightarrow c \cdot \underbrace{e^{-k \cdot 0}}_{1} + 21 = \eta \,\Big|_{-21} \Rightarrow c = \eta - 21 \in \mathbb{R} \Rightarrow$

$\Rightarrow$ The general solution of the IVP is: $x = (\eta - 21) \cdot e^{-kt} + 21, \eta, k \in \mathbb{R}$, fixed

1.7.10) We consider the equation $x''-x=te^{-2t}$.

a) Find a particular solution of the form $x_p(t)=(at+b)e^{-2t}$, where $a,b\in\mathbb{R}$.

b) Find the general solution

c) Find the solution that satisfies the initial conditions $x(0)=0, x'(0)=0$.

a) $x_p=(at+b)e^{-2t}$ - particular solution $\Rightarrow$

$\Rightarrow \left((at+b)e^{-2t}\right)''-(at+b)e^{-2t}=t\cdot e^{-2t} \quad (*)$

$\left((at+b)e^{-2t}\right)'=(at+b)'\cdot e^{-2t}+(at+b)\cdot(e^{-2t})'=a\cdot e^{-2t}+(at+b)\cdot(-2)e^{-2t}=$

$=a\cdot e^{-2t}-2(at+b)e^{-2t}=(a-2b-2at)e^{-2t}$

$\left((at+b)e^{-2t}\right)''=\left((a-2b-2at)e^{-2t}\right)'=(a-2b-2at)'e^{-2t}+(a-2b-2at)(e^{-2t})'=$

$=-2ae^{-2t}-2(a-2b-2at)e^{-2t}=(-2a-2a+4b+4at)e^{-2t}=(-4a+4b+4at)e^{-2t}$

$(*)\Rightarrow(-4a+4b+4at)e^{-2t}-(at+b)e^{-2t}=t\cdot e^{-2t} \Leftrightarrow$

$\Leftrightarrow(-4a+4b+4at-at-b)e^{-2t}=t\cdot e^{-2t}\Leftrightarrow$

$\Leftrightarrow(-4a+3b+3at)e^{-2t}=t\cdot e^{-2t} \;/ : e^{-2t}\neq 0 \;\Leftrightarrow$

$\Leftrightarrow -4a+3b+3at=t,\ \forall t\in\mathbb{R}\Rightarrow\begin{cases}-4a+3b=0\\3a=1\end{cases}\Leftrightarrow\begin{cases}a=\frac{1}{3}\\-\frac{4}{3}+3b=0\end{cases}\Leftrightarrow$

$\Leftrightarrow\begin{cases}a=\frac{1}{3}\in\mathbb{R}\\b=\frac{4}{9}\in\mathbb{R}\end{cases}\Rightarrow x_p(t)=\left(\frac{1}{3}t+\frac{4}{9}\right)e^{-2t}$ - particular solution

b) From a) $\Rightarrow x_p(t)=\left(\frac{1}{3}t+\frac{4}{9}\right)e^{-2t}$ is a particular solution. We shall find the general solution of the LHDE: $x''-x=0\Rightarrow$ the characteristic eq: $r^2-1=0\Rightarrow$

$\Rightarrow(r-1)(r+1)=0 \to r_1=1\mapsto e^t$, sol; $r_2=-1\mapsto e^{-t}$, sol $\Rightarrow$ the general sol: $x_h=c_1e^t+c_2e^{-t}, c_1,c_2\in\mathbb{R}$

$\Rightarrow$ The general solution of the LN-HDE is: $x=x_h+x_p\Rightarrow$

$\Rightarrow x=c_1e^t+c_2\cdot e^{-t}+\left(\frac{1}{3}t+\frac{4}{9}\right)e^{-2t}, c_1,c_2\in\mathbb{R}$

c) $x(t)=c_1e^t+c_2\cdot e^{-t}+\left(\frac{1}{3}t+\frac{4}{9}\right)e^{-2t}, c_1,c_2\in\mathbb{R}\Rightarrow$

$\Rightarrow x'(t)=c_1e^t-c_2e^{-t}+\left(\frac{1}{3}-2\cdot\frac{4}{9}-2\cdot\frac{1}{3}t\right)e^{-2t}, c_1,c_2\in\mathbb{R}\Rightarrow$

$\Rightarrow x'(t)=c_1e^t-c_2e^{-t}+\left(-\frac{5}{9}-\frac{2}{3}t\right)e^{-2t}, c_1,c_2\in\mathbb{R}$

$\begin{cases}x(0)=0\\x'(0)=0\end{cases}\Rightarrow\begin{cases}c_1+c_2+\frac{4}{9}=0\\c_1-c_2-\frac{5}{9}=0\end{cases}\Rightarrow\begin{cases}c_1+c_2=-\frac{4}{9}\\c_1-c_2=\frac{5}{9}\end{cases}\oplus\Rightarrow 2c_1=\frac{1}{9}\Rightarrow c_1=\frac{1}{18}\Rightarrow$

$\Rightarrow\frac{1}{18}-c_2=\frac{5}{9}\Rightarrow c_2=\frac{1}{18}-{}^{2)}\frac{5}{9}=\frac{1-10}{18}=\frac{-9}{18}=\frac{-1}{2}\Rightarrow$

$\Rightarrow x=\frac{1}{18}e^t-\frac{1}{2}e^{-t}+\left(\frac{1}{3}t+\frac{4}{9}\right)e^{-2t}$

1.7.12) Let $\mathcal{L} : C^2(\mathbb{R}) \to C(\mathbb{R})$ be defined by $\mathcal{L}x = x'' - 2x' + x$ for any $x \in C^2(\mathbb{R})$.

a) Prove that $\mathcal{L}$ is a linear map. What is the dimension of its kernel?

b) Find the general solution of the eq. $x'' - 2x' + x = \cos 2t$ knowing that it has a particular solution of the form $a\cos 2t + b\sin 2t$, for some $a, b \in \mathbb{R}$.

c) Let $f_1(t) = e^{2t}$ and $f_2(t) = e^{-2t}$ for all $t \in \mathbb{R}$. Find a particular solution of the equation $\mathcal{L}x = 3f_1 + 5f_2$.

a) $\mathcal{L}$ is a linear map $\Leftrightarrow \mathcal{L}(\alpha x + \beta y) = \alpha \mathcal{L}(x) + \beta \cdot \mathcal{L}(y), \forall x, y \in C^2(\mathbb{R})$

$\forall \alpha, \beta \in \mathbb{R}$

$$\mathcal{L}(\alpha x + \beta y) = (\alpha x + \beta y)'' - 2(\alpha x + \beta y)' + (\alpha x + \beta y) =$$
$$= \alpha x'' + \beta y'' - 2\alpha x' - 2\beta y' + \alpha x + \beta y =$$
$$= \alpha(x'' - 2x' + x) + \beta(y'' - 2y' + y) =$$
$$= \alpha \cdot \mathcal{L}(x) + \beta \cdot \mathcal{L}(y), \forall x, y \in C^2(\mathbb{R}) \forall \alpha, \beta \in \mathbb{R} \Rightarrow$$

$\Rightarrow \mathcal{L}$ is a linear map ~~generated by~~

The kernel of $\mathcal{L}$ is the set of solutions for the LHDE: $x'' - 2x' + x = 0$

$x'' - 2x' + x = 0 \Rightarrow$ the characteristic eq: $r^2 - 2r + 1 = 0 \Leftrightarrow (r-1)^2 = 0 \Rightarrow$

$\Rightarrow r_1 = r_2 = 1 \Rightarrow 1$ is a real root of multiplicity 2 $\Rightarrow$

$\Rightarrow 1 \mapsto e^t, te^t \Rightarrow$ The solutions are e^t and $t \cdot e^t$ and the dimension of the kernel is 2, because e^t and $t \cdot e^t$ are linearly independent solutions.

b) The solution of $x'' - 2x' + x = \cos 2t$ is $x = x_h + x_p$, where x_h is the general solution of the LHDE $x'' - 2x' + x = 0$ and x_p is a particular solution.

From a) $\Rightarrow x_h = c_1 e^t + c_2 \cdot t \cdot e^t, c_1, c_2 \in \mathbb{R}$

$x_p = a\cos 2t + b\sin 2t$ - particular solution $\Rightarrow$

$\Rightarrow x_p'' - 2x_p' + x_p = \cos 2t \;(*)$

$x_p = a\cos 2t + b\sin 2t \Rightarrow x_p' = -2a\sin 2t + 2b\cos 2t$

$x_p'' = -4a\cos 2t - 4b\sin 2t$

$(*) \Rightarrow -4a\cos 2t - 4b\sin 2t + 4a\sin 2t - 4b\cos 2t + a\cos 2t + b\sin 2t = \cos 2t$

$\Leftrightarrow \cos 2t(-4a - 4b + a) + \sin 2t(-4b + 4a + b) = \cos 2t, \forall t \in \mathbb{R} \Leftrightarrow$

$\Leftrightarrow \begin{cases} -3a - 4b = 1 \;/\cdot 4 \\ 4a - 3b = 0 \;/\cdot 3 \end{cases} \Leftrightarrow \begin{cases} -12a - 16b = 4 \\ 12a - 9b = 0 \end{cases} \Leftrightarrow -25b = 4 \Rightarrow b = \frac{4}{-25} \Rightarrow$

$\Rightarrow 4a + 3 \cdot \frac{4}{25} = 0 \Leftrightarrow a + \frac{3}{25} = 0 \Rightarrow a = -\frac{3}{25} \Rightarrow x_p = \frac{-3}{25}\cos 2t - \frac{4}{25}\sin 2t \Rightarrow$

$\Rightarrow$ ~~x_h~~ the general solution is $x = c_1 e^t + c_2 e^t \cdot t - \frac{3}{25}\cos 2t - \frac{4}{25}\sin 2t, c_1, c_2 \in \mathbb{R}$

c) To find ~~the pa~~ a particular solution for $\mathcal{L}x = 3f_1+5f_2$ we need to find some particular solutions for $\mathcal{L}x = f_1$ and $\mathcal{L}x = f_2$.

$\mathcal{L}x = f_1 \Leftrightarrow x''-2x'+x = e^{2t}$

The general solution of the LHDE is: $x_h = c_1 e^t + c_2\, t\cdot e^t,\ c_1, c_2 \in \mathbb{R}$

We are looking for a particular sol of the form: $x_{p_1} = \varphi_1(t)x_1(t) + \varphi_2(t)x_2(t)$, where φ_1 and φ_2 satisfy $\varphi_1'x_1 + \varphi_2'x_2 = 0$ and $x_1(t) = e^t,\ x_2(t) = t\cdot e^t$ (1)

$x_{p_1}' = \varphi_1'x_1 + \varphi_1x_1' + \varphi_2'x_2 + \varphi_2x_2'$

$= \underbrace{(\varphi_1'x_1 + \varphi_2'x_2)}_{0} + \varphi_1x_1' + \varphi_2x_2' = \varphi_1x_1' + \varphi_2x_2'$ $\Rightarrow$

$x_{p_1}'' = \varphi_1'x_1' + \varphi_1x_1'' + \varphi_2'x_2' + \varphi_2x_2''$

$\Rightarrow x_{p_1}'' - 2x_{p_1}' + x_{p_1} = e^{2t} \Rightarrow \varphi_1'x_1' + \varphi_1x_1'' + \varphi_2'x_2' + \varphi_2x_2'' - 2\varphi_1x_1' - 2\varphi_2x_2' + \varphi_1x_1 + \varphi_2x_2 = e^{2t}$

$\Leftrightarrow \varphi_1(x_1'' - 2x_1' + x_1) + \varphi_2(x_2'' - 2x_2' + x_2) + \varphi_1'x_1' + \varphi_2'x_2' = e^{2t}$ $\Rightarrow$

But x_1 and x_2 are solutions, so $\begin{cases} x_1'' - 2x_1' + x_1 = 0 \\ x_2'' - 2x_2' + x_2 = 0 \end{cases}$

$\Rightarrow \varphi_1'x_1' + \varphi_2'x_2' = e^{2t}$

We have the following system: $\begin{cases} \varphi_1'(t)e^t + \varphi_2'(t)\cdot t\cdot e^t = 0 \\ \varphi_1'(t)e^t + \varphi_2'(t)(e^t + t\cdot e^t) = e^{2t} \end{cases}$ $(\Leftrightarrow)$

$(\Leftrightarrow) \begin{cases} \varphi_1'(t)e^t + \varphi_2'(t)\cdot t\cdot e^t = 0 \;/:e^t \\ \varphi_1'(t)e^t + \varphi_2'(t)\cdot(1+t)\cdot e^t = e^{2t} \;/:e^t \end{cases} (\Leftrightarrow) \begin{cases} \varphi_1'(t) + \varphi_2'(t)\cdot t = 0 \;/\cdot(-1)<0 \\ \varphi_1'(t) + \varphi_2'(t)(1+t) = e^t \end{cases} (\Leftrightarrow)$

$(\Leftrightarrow) \begin{cases} -\varphi_1'(t) - \varphi_2'(t)\cdot t = 0 \\ \varphi_1'(t) + \varphi_2'(t)(1+t) = e^t \end{cases}$ $(+) \Rightarrow \varphi_2'(t)(1+t-\cancel{t}) = e^t \Rightarrow \varphi_2'(t) = e^t$

$\begin{cases} \varphi_1'(t) + \varphi_2'(t)\cdot t = 0 \\ \varphi_2'(t) = e^t \end{cases} \Rightarrow \begin{cases} \varphi_1'(t) = -t\cdot e^t \\ \varphi_2'(t) = e^t \end{cases} \Rightarrow \begin{cases} \varphi_1(t) = \int -te^t\,dt = -(t-1)e^t \\ \varphi_2(t) = \int e^t\,dt = e^t \end{cases} \Rightarrow$

$\Rightarrow$ A part sol is: $x_{p_1} = -(t-1)e^t e^t + e^t\cdot t\cdot e^t = (1-t)e^{2t} + te^{2t} = e^{2t}$

$\mathcal{L}x = f_2 \Leftrightarrow x'' - 2x' + x = e^{-2t}$

The general solution of the LHDE is: $x_h = c_1\underbrace{e^t}_{x_1} + c_2\underbrace{t\,e^t}_{x_2},\ c_1, c_2 \in \mathbb{R}$

$x_{p_2} =$ ~~$\varphi_1 e$~~ $\varphi_1(t)\cdot e^t + \varphi_2(t)\cdot t\cdot e^t$, where φ_1 and φ_2 satisfy $\varphi_1'(t)\cdot e^t + \varphi_2'(t)\,te^t = 0$.

$x_{p_2}' =$ ~~$\varphi_1'(t)$~~ $\varphi_1(t)\cdot e^t + \varphi_2(t)\cdot e^t(1+t)$

$x_{p_2}'' = \varphi_1'(t)\cdot e^t + \varphi_1(t)\cdot e^t + \varphi_2'(t)\cdot e^t(1+t) + \varphi_2(t)\cdot e^t(2+t)$ $\Rightarrow$

$\Rightarrow x_{p_2}'' - 2x_{p_2}' + x_{p_2} = e^{-2t} \Rightarrow \varphi_1'(t)\cdot e^t + \varphi_1(t)\cdot e^t + \varphi_2'(t)\cdot e^t(1+t) + \varphi_2(t)\cdot e^t(2+t) - 2\varphi_1(t)e^t -$

$-2\varphi_2(t)e^t(1+t) + \varphi_1(t)\cdot e^t + \varphi_2(t)\cdot t\cdot e^t = e^{-2t} \Rightarrow$

$\Rightarrow \varphi_1(t)(\underbrace{e^t - 2e^t + e^t}_{0}) + \varphi_2(t)(\underbrace{e^t(2+t) - 2e^t(1+t) + t\cdot e^t}_{0}) + \varphi_1'(t)e^t + \varphi_2'(t)e^t(1+t) = e^{-2t}$

$\Rightarrow \varphi_1'(t)e^t + \varphi_2'(t)e^t(1+t) = e^{-2t}$

$\Rightarrow \begin{cases} \varphi_1'(t)e^t + \varphi_2'(t)\cdot e^t\cdot t = 0 \\ \varphi_1'(t)e^t + \varphi_2'(t)\cdot e^t(1+t) = e^{-2t} \end{cases} \Leftrightarrow \begin{cases} \varphi_1'(t) + \varphi_2'(t)\cdot t = 0 \\ \varphi_1'(t) + \varphi_2'(t)(1+t) = e^{-3t} \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} -\varphi_1'(t) - \varphi_2'(t)\cdot t = 0 \\ \varphi_1'(t) + \varphi_2'(t)(1+t) = e^{-3t} \end{cases} \oplus \Rightarrow \varphi_2'(t) = e^{-3t} \Rightarrow \varphi_1'(t) = -\varphi_2'(t)\cdot t$

$= -t\cdot e^{-3t}$

$\begin{cases} \varphi_1'(t) = -t\,e^{-3t} \\ \varphi_2'(t) = e^{-3t} \end{cases} \Rightarrow \begin{cases} \varphi_1(t) = \int -te^{-3t}dt = \frac{(3t+1)e^{-3t}}{9} \\ \varphi_2(t) = \int e^{-3t}dt = -\frac{e^{-3t}}{3} \end{cases} \Rightarrow$

$\Rightarrow x_{p_2} = \frac{(3t+1)e^{-3t}}{9}e^t - \cancel{3}^{\,9}\frac{e^{-3t}}{3}\cdot t\cdot e^t = \frac{(3t+1)e^{-2t} - 3te^{-2t}}{9} = \frac{e^{-2t}}{9}$.

$\Rightarrow$ A part sol: $x_{p_2} = \frac{e^{-2t}}{9}$

For $\mathcal{L}x = f_1$, we have $x_{p_1} = e^{2t}$
For $\mathcal{L}x = f_2$, we have $x_{p_2} = \frac{e^{-2t}}{9}$
$\alpha_1 = 3, \alpha_2 = 5$
$\Rightarrow$ Superposition Principle

$\Rightarrow$ A particular solution for $\mathcal{L}x = 3f_1 + 5f_2$ is $x_p = 3x_{p_1} + 5x_{p_2} \Leftrightarrow$

$\Leftrightarrow x_p = 3\cdot e^{2t} + \frac{5}{9}e^{-2t}$

1.7.19) We consider the differential equation:

$$x'' + 4x = \cos 2t$$

a) Find a solution of the form $x_p = t(a\cos 2t + b\sin 2t)$, with $a, b \in \mathbb{R}$.

b) Find its general solution.

c) Describe the motion of a spring-mass system governed by this equation.

a) $x_p = t(a\cos 2t + b\sin 2t), a, b \in \mathbb{R} \Rightarrow x_p' = (a\cos 2t + b\sin 2t) + t(-2a\sin 2t + 2b\cos 2t)$

$\Rightarrow x_p' = (a\cos 2t + b\sin 2t) + 2t(-a\sin 2t + b\cos 2t)$

$\Rightarrow x_p'' = (-2a\sin 2t + 2b\cos 2t) + 2(-a\sin 2t + b\cos 2t) + 2t(-2a\cos 2t - 2b\sin 2t)$

$= -4a\sin 2t + 4b\cos 2t - 4t(a\cos 2t + b\sin 2t)$

x_p - solution $\Rightarrow x_p'' + 4x_p = \cos 2t \Leftrightarrow -4a\sin 2t + 4b\cos 2t \cancel{-4t(a\cos 2t + b\sin 2t)} +$
$\cancel{+4t(a\cos 2t + b\sin 2t)} = \cos 2t \Leftrightarrow -4a\sin 2t + 4b\cos 2t = \cos 2t, \forall t \in \mathbb{R} \Rightarrow$

$\Rightarrow \begin{cases} -4a = 0 \\ 4b = 1 \end{cases} \Rightarrow \begin{cases} a = 0 \in \mathbb{R} \\ b = \frac{1}{4} \in \mathbb{R} \end{cases} \Rightarrow x_p = t \cdot \frac{1}{4}\sin 2t$ - solution

b) From a) we have the particular solution: $x_p = \frac{1}{4} t\sin 2t$

The general solution is: $x = x_h + x_p$, where x_h is the general solution of the LHDE: $x'' + 4x = 0 \Rightarrow$ the characteristic eq: $r^2 + 4 = 0 \Rightarrow r^2 = -4 \Rightarrow r_{1,2} = \pm 2i \in \mathbb{C}$

$r_{1,2} = \pm 2i \in \mathbb{C}$, multiplicity $1 \mapsto \cos 2t, \sin 2t \Rightarrow x_h = c_1 \cdot \cos 2t + c_2 \sin 2t, c_1, c_2 \in \mathbb{R}$

$\Rightarrow$ The general solution of $x'' + 4x = \cos 2t$ is:

$$x = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{4} t \cdot \sin 2t, c_1, c_2 \in \mathbb{R}$$

c) In the case of a spring-mass system we can see that the equation has the form: $x'' + \frac{k}{m}x = A\cos \omega t$, $k > 0, m > 0, A > 0, \omega > 0$, which corresponds to a motion without damping, with the external force: $A\cos \omega t = f(t)$

From $x'' + 4x = \cos 2t \Rightarrow \frac{k}{m} = 4, A = 1, \omega = 2 \Rightarrow \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{4} = 2$

Because $\omega = \omega_0$ and $x_p = \frac{1}{2\omega_0} t\sin \omega_0 t$ (unbounded, oscillatory), we have that the motion of the spring-mass system governed by the given equation is oscillatory, and the oscillations occur with an amplitude that increases to ∞. This phenomenon is called resonance.

$x = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{1}{2\omega_0} t \sin \omega_0 t, c_1, c_2 \in \mathbb{R}$, all the solutions are unbounded

1.7.24) We say that a differential equation exibits resonance when all its solutions are unbounded. For what values of the mass m will $mx''+25x=12\cos(36\pi t)$ exhibit resonance?

$$mx''+25x=12\cos(36\pi t) \Big/ : m>0 \Rightarrow x''+\frac{25}{m}\cdot x=\frac{12}{m}\cdot\cos(36\pi t) \Rightarrow$$

$$\Rightarrow \frac{k}{m}=\frac{25}{m},\ A=\frac{12}{m},\ \omega=36\pi \Rightarrow \omega_0=\sqrt{\frac{k}{m}}=\sqrt{\frac{25}{m}}$$

For the differential equation to exhibit resonance, we need to impose the following condition: $\omega=\omega_0$

$$\omega=\omega_0 \Leftrightarrow 36\pi=\sqrt{\frac{25}{m}} \Big/^2 \Leftrightarrow \cancel{m}\ (36\pi)^2=\frac{25}{m} \Rightarrow m=\frac{25}{(36\pi)^2}$$

1.7.25) Find the general solution of $\ddot{\theta}+\dot{\theta}+\theta=0$. Prove that $\lim\limits_{t\to\infty}\theta(t)=0$ for any solution θ of this differential equation.

$\ddot{\theta}+\dot{\theta}+\theta=0 \Rightarrow$ the characteristic eq: $x^2+x+1=0$

$\left.\begin{array}{l} a=1 \\ b=1 \\ c=1 \end{array}\right\} \Rightarrow \Delta=b^2-4ac=1^2-4\cdot1\cdot1=-3<0 \Rightarrow x_{1,2}=\frac{-b\pm i\sqrt{-\Delta}}{2a} \Rightarrow$

$\Rightarrow x_{1,2}=\frac{-1\pm i\sqrt{3}}{2}=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}\in\mathbb{C}$, roots of multiplicity 1 $\Rightarrow$

$\Rightarrow x_{1,2}\mapsto e^{-\frac{1}{2}t}\cdot\cos\left(\frac{\sqrt{3}}{2}t\right), e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$, solutions $\Rightarrow$

$\Rightarrow$ The general solution is: $\theta(t)=c_1\cdot e^{-\frac{1}{2}t}\cdot\cos\left(\frac{\sqrt{3}}{2}t\right)+c_2\cdot e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right), c_1,c_2\in\mathbb{R}$

$\lim\limits_{t\to\infty}\theta(t)=\lim\limits_{t\to\infty}\left[c_1\cdot e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)+c_2\cdot e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$

$=\lim\limits_{t\to\infty}\left(c_1\cdot e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)\right)+\lim\limits_{t\to\infty}\left(c_2\cdot e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$

$=c_1\underbrace{\lim\limits_{t\to\infty}\left(e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)\right)}_{l_1}+c_2\underbrace{\lim\limits_{t\to\infty}\left(e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)}_{l_2}$ (*)

$\forall t\in\mathbb{R}\Rightarrow -1\leq\cos\left(\frac{\sqrt{3}}{2}t\right)\leq 1 /\cdot e^{-\frac{1}{2}t}>0 \Leftrightarrow -e^{-\frac{1}{2}t}\leq e^{-\frac{1}{2}t}\cdot\cos\left(\frac{\sqrt{3}}{2}t\right)\leq e^{-\frac{1}{2}t}, \forall t\in\mathbb{R}$

$\lim\limits_{t\to\infty}e^{-\frac{1}{2}t}=e^{-\frac{1}{2}\infty}=e^{-\infty}=0=\lim\limits_{t\to\infty}-e^{-\frac{1}{2}t}$ $\xRightarrow{\text{Squeeze Theorem}}$

$\Rightarrow \lim\limits_{t\to\infty}e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)=0 \Leftrightarrow l_1=0$ (1)

$\forall t\in\mathbb{R}\Rightarrow -1\leq\sin\left(\frac{\sqrt{3}}{2}t\right)\leq 1 /\cdot e^{-\frac{1}{2}t}>0 \Leftrightarrow -e^{-\frac{1}{2}t}\leq e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)\leq e^{-\frac{1}{2}t}, \forall t\in\mathbb{R}$

$\lim\limits_{t\to\infty}-e^{-\frac{1}{2}t}=0=\lim\limits_{t}e^{-\frac{1}{2}t}$ $\xRightarrow{\text{Squeeze Theorem}}$

$\Rightarrow \lim\limits_{t\to\infty}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)=0 \Leftrightarrow l_2=0$ (2)

From (*), (1), (2) $\Rightarrow \lim\limits_{t\to\infty}\theta(t)=c_1\cdot 0+c_2\cdot 0=0$, ~~for any~~ $\Rightarrow$

$\Rightarrow \lim\limits_{t\to\infty}\theta(t)=0$ for any solution θ of the given equation

1.7.29) We consider the differential equation

$$t^2x''+2tx'-2x=0,\ t\in(0;\infty)$$

a) Find the solutions of the form $x(t)=t^{\kappa}$ where $\kappa\in\mathbb{R}$ has to be determined.

b) Specify its type and find its general solution.

c) Find the solution of the IVP

$$t^2x''+2tx'-2x=0,\ x(1)=0,\ x'(1)=1$$

a) $x(t)=t^{\kappa}$ solution

$$\left.\begin{array}{l} x'(t)=(t^{\kappa})'=\kappa\cdot t^{\kappa-1} \\ x''(t)=(\kappa\cdot t^{\kappa-1})'=\kappa(\kappa-1)t^{\kappa-2} \end{array}\right\} \Rightarrow x''(t)\cdot t^2+2tx'(t)-2\cdot x(t)=0 \Leftrightarrow$$

$$\Rightarrow \kappa(\kappa-1)t^{\kappa-2}\cdot t^2+2t\cdot\kappa\cdot t^{\kappa-1}-2t^{\kappa}=0 \Rightarrow \kappa(\kappa-1)t^{\kappa}+2\kappa t^{\kappa}-2t^{\kappa}=0 \Leftrightarrow$$

$$\Leftrightarrow t^{\kappa}(\kappa^2-\kappa+2\kappa-2)=0 \Leftrightarrow t^{\kappa}(\kappa^2+\kappa-2)=0 \quad \left.\begin{array}{l} \\ t\in(0;\infty)\Rightarrow t^{\kappa}\neq 0\end{array}\right\} \Rightarrow \kappa^2+\kappa-2=0 \Leftrightarrow$$

$$\Leftrightarrow \cancel{\kappa^2-1+2\kappa-2}\ \kappa^2-\kappa+2\kappa-2=0 \Leftrightarrow \kappa(\kappa-1)+2(\kappa-1)=0 \Leftrightarrow (\kappa+2)(\kappa-1)=0 \Rightarrow$$

$$\Rightarrow \left[\begin{array}{l} \kappa_1=-2 \Rightarrow x(t)=t^{-2} \\ \kappa_2=1 \Rightarrow x(t)=t \end{array}\right. \Rightarrow \text{The solutions of the form } x(t)=t^{\kappa} \text{ are } t^{-2} \text{ and } t.$$

b) The given differential equation can be written as:

$$t^2\left(x''+2\frac{1}{t}x'-2\cdot\frac{1}{t^2}x\right)=0,\ t\in(0;\infty) \,/ :t^2\neq 0 \Leftrightarrow$$

$\Leftrightarrow x''+\frac{2}{t}x'-\frac{2}{t^2}x=0,\ t\in(0;\infty)$, which is a second order linear homogeneous ~~equation~~ differential equation with non-constant coefficients, thus there exist x_1 and x_2 linearly independent solutions, because the set of solutions is a linear space of dimension 2.

From a) $\Rightarrow x_1(t)=t^{-2}$ and $x_2(t)=t$ are solutions of the LHDE and this solutions are linearly independent $\Rightarrow$

$\Rightarrow$ the general solution is: $x=c_1x_1(t)+c_2x_2(t),\ c_1,c_2\in\mathbb{R} \Rightarrow$

$\Rightarrow x=c_1t^{-2}+c_2\cdot t,\ c_1,c_2\in\mathbb{R}$

c) We know that the solution of the LHDE ~~are~~ is:

$$x=c_1t^{-2}+c_2t,\ c_1,c_2\in\mathbb{R} \Rightarrow x'=-2c_1t^{-3}+c_2,\ c_1,c_2\in\mathbb{R}$$

From the IVP: $\begin{cases} x(1)=0 \\ x'(1)=1 \end{cases} \Leftrightarrow \begin{cases} c_1\cdot 1^{-2}+c_2\cdot 1=0 \\ -2c_1 1^{-3}+c_2=1 \end{cases} \Leftrightarrow \begin{cases} c_1+c_2=0 \\ -2c_1+c_2=1 \end{cases} \Leftrightarrow \begin{cases} c_2=-c_1 \\ -2c_1+c_2=1 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} c_2=-c_1 \\ -3c_1=1 \,/ :-3<0 \end{cases} \Leftrightarrow \left.\begin{cases} c_1=-\frac{1}{3}\in\mathbb{R} \\ c_2=\frac{1}{3}\in\mathbb{R} \end{cases}\right\} \Rightarrow$ The solution of the IVP is:

$$x=-\frac{1}{3}t^{-2}+\frac{1}{3}t$$

1.7.3 4) We use the notation

$$L(x) = x'' + 25x$$

i) Find the solution of the IVP; Represent this integral curve and describe its long-term behaviour.

$$L(x) = 0,\ x(0) = 0,\ x'(0) = 1$$

ii) Let $\varphi_1(t) = t\cos(5t)$ and $\varphi_2(t) = t\sin(5t)$ for all $t \in \mathbb{R}$. Compute $L(5)$, $L(\varphi_1)$, $L(\varphi_2)$.

iii) Find a constant solution for $L(x) = 5$.

iv) Find the general solution of the differential eq: $L(x) = 25 - 25\sin(5t)$.

i) IVP: $\begin{cases} x'' + 25x = 0 \\ x(0) = 0 \\ x'(0) = 1 \end{cases}$

$x'' + 25x = 0 \Rightarrow$ The characteristic equation is: $r^2 + 25 = 0 \Rightarrow$

$\Rightarrow r^2 = -25 \Rightarrow r_{1,2} = \pm 5i \in \mathbb{C} \mapsto \cos(5t), \sin(5t) \Rightarrow$

$\Rightarrow$ the general solution is: $x = c_1\cos(5t) + c_2\sin(5t),\ c_1, c_2 \in \mathbb{R}$

$x(0) = c_1 \cdot \cos(5\cdot 0) + c_2 \cdot \sin(5\cdot 0) \Rightarrow x(0) = c_1$

$x' = -5c_1\sin(5t) + 5c_2\cos(5t) \Rightarrow x'(0) = 5c_1\sin 0 + 5c_2\cos 0 = 5c_2$

$\begin{cases} x(0) = 0 \\ x'(0) = 1 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ 5c_2 = 1 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = \frac{1}{5} \end{cases} \Rightarrow x = \frac{1}{5}\sin(5t)$ - the solution of the IVP

$\forall t \in \mathbb{R} \Rightarrow -1 \leq \sin(5t) \leq 1 \,/\, \frac{1}{5} > 0 \Rightarrow -\frac{1}{5} \leq \frac{1}{5}\sin(5t) \leq \frac{1}{5} \Rightarrow \left|\frac{1}{5}\sin(5t)\right| \leq \frac{1}{5} \Rightarrow$

$\Rightarrow |x| \leq \frac{1}{5} \Rightarrow$ the amplitude is $\frac{1}{5}$

The integral curve is oscillatory, with the amplitude being $\frac{1}{5}$.

We know that the sine function has $T_0 = 2\pi k, k \in \mathbb{Z}$ as a period.

~~$\Rightarrow \sin t$~~ We should ~~ex~~ find T for $\frac{1}{5}\sin(5t)$, where T is the period.

$x(t) = \frac{1}{5}\sin(5t)$

$x(t+T) = \frac{1}{5}\sin(5t + 5T)$

T - period

$\Rightarrow \sin(5t) = \sin(5t + 5T)$

But for $\sin(u), u \in \mathbb{R}$, the period is $T_0 = 2\pi k, k \in \mathbb{Z}$

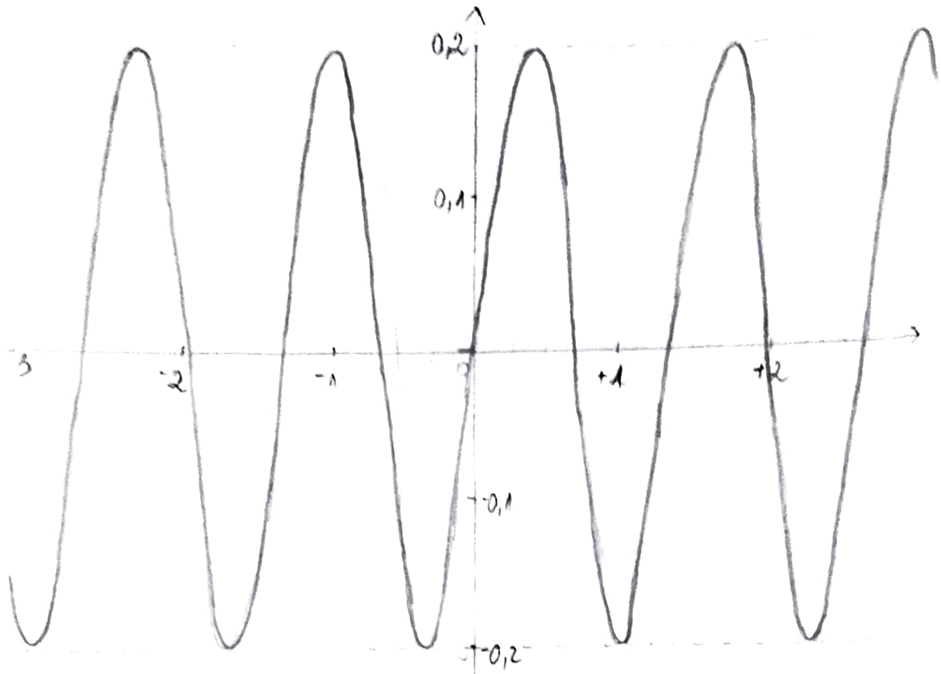
$\Rightarrow 5T = T_0 \Rightarrow$

$\Rightarrow 5T = 2\pi k \Rightarrow T = \frac{2\pi k}{5}, k \in \mathbb{Z}$ - period for the solution of the IVP

In conclusion, the solution of the IVP is:

- oscillatory around 0 (because sine is oscillatory around 0)
- periodic with the general period: $T = \frac{2\pi k}{5}, k \in \mathbb{Z}$
- with constant amplitude $A = \frac{1}{5}$
- bounded between $-\frac{1}{5}$ and $\frac{1}{5}$. In long-term behaviour, the function oscillates between $-\frac{1}{5}$ and $\frac{1}{5}$ and its values are repeating after $T = \frac{2\pi k}{5}$ ~~time~~, as period.

The representation of the integral curve:



ii) $\mathcal{L}(x) = x'' + 25x \Rightarrow \mathcal{L}(5) = \underset{0}{5''} + 25\cdot 5 \Rightarrow \mathcal{L}(5) = 125$

$\varphi_1(t) = t\cos(5t) \Rightarrow \mathcal{L}(\varphi_1) = (t\cos(5t))'' + 25t\cos(5t)$

$(t\cos(5t))' = \cancel{\cos(5t) + 5\sin(5t)}\ \cos(5t) - 5t\sin(5t)$

$(t\cos(5t))'' = (\cos(5t) - 5t\sin(5t))' = -5\sin(5t) - 5(\sin(5t) + t\cdot 5\cdot\cos(5t)) =$

$= -5\sin(5t) - 5\sin(5t) - 25t\cos(5t) = -10\sin(5t) - 25t\cos(5t)$

$\Rightarrow \mathcal{L}(\varphi_1) = -10\sin(5t) \cancel{- 25t\cos(5t)} + \cancel{25\cos(5t)} = -10\sin(5t)$

$\varphi_2(t) = t\sin(5t) \Rightarrow \mathcal{L}(\varphi_2) = (t\sin(5t))'' + 25t\sin(5t)$

$(t\sin(5t))' = \sin(5t) + 5t\cos(5t)$

$(t\sin(5t))'' = (\sin(5t) + 5t\cos(5t))' = 5\cos(5t) + 5(\cos(5t) - 5t\sin(5t)) =$

$= 5\cos(5t) + 5\cos(5t) - 25t\sin(5t) = 10\cos(5t) - 25t\sin(5t)$

$\Rightarrow \mathcal{L}(\varphi_2) = 10\cos(5t) \cancel{- 25t\sin(5t)} + \cancel{25t\sin(5t)} = 10\cos(5t)$

iii) $\mathcal{L}(x) = 5 \Leftrightarrow x'' + 25x = 5$

~~We can se~~ Let $x_p = c \in \mathbb{R}$ ~~a~~ be a constant solution $\Rightarrow$

$\Rightarrow \underset{0}{c''} + 25c = 5 \Rightarrow 25c = 5 /:25 \Rightarrow c = \frac{5}{25} = \frac{1}{5} \in \mathbb{R} \Rightarrow x_p = \frac{1}{5} \in \mathbb{R}$ - constant solution

iv) The general solution of $\mathcal{L}(x) = 25 - 25\sin(5t)$ is:

$x = x_h + x_p$, where x_h - general sol. of the LHDE $\mathcal{L}(x) = 0$

x_p - a particular solution

~~Fo~~ From i) $\Rightarrow$ for $\mathcal{L}(x) = 0$, the general solution is: ~~x_h~~

$x_h = c_1\cdot\cos(5t) + c_2\sin(5t),\ c_1, c_2 \in \mathbb{R}$

$\mathcal{L}(x) = 25 - 25\sin(5t) = \cancel{25\ 5\ 2} = 5\cdot 5 - 5\cdot 5\sin(5t)$

Denote $f_1 = 5, \alpha_1 = 5$

$f_2 = -5\sin(5t), \alpha_2 = 5$

A particular solution for $\mathcal{L}(x) = 5$ is, from iii) $\Rightarrow x_p = \frac{1}{5}$

We need a particular solution for $\mathcal{L}(x) = -5\sin(5t)$

$\left.\begin{array}{l}\mathcal{L}(x) = -5\sin(5t) \,/\cdot 2>0 \Rightarrow 2\mathcal{L}(x) = -10\sin(5t) \\ \text{ii)} \Rightarrow \mathcal{L}(t\cos(5t)) = -10\sin(5t)\end{array}\right\} \Rightarrow$

$\Rightarrow 2\mathcal{L}(x) = \mathcal{L}(t\cdot\cos(5t))$ (*)

$\mathcal{L}$ - linear map $\Leftarrow \mathcal{L}(\alpha x + \beta y) = \alpha\mathcal{L}(x) + \beta\mathcal{L}(y), \forall \alpha, \beta \in \mathbb{R}, \forall x, y \in C'(\mathbb{R})$

$\mathcal{L}(\alpha x + \beta y) = (\alpha x + \beta y)'' + 25(\alpha x + \beta y) = (\alpha x)'' + (\beta y)'' + 25\alpha x + 25\beta y =$

$= \alpha x'' + \beta y'' + 25\alpha x + 25\beta y =$

$= (\alpha x'' + 25\alpha x) + (\beta y'' + 25\beta y) =$

$= \alpha\cdot\mathcal{L}(x) + \beta\cdot\mathcal{L}(y), \forall \alpha, \beta \in \mathbb{R}, \forall x, y \in C'(\mathbb{R}) \Rightarrow$

$\Rightarrow \mathcal{L}$ is a linear map $\Rightarrow 2\mathcal{L}(x) = \mathcal{L}(2x)$

(*) $\Rightarrow \mathcal{L}(2x) = \mathcal{L}(t\cdot\cos(5t)) \Rightarrow$ A particular sol: $2x_p = t\cdot\cos(5t) \Rightarrow x_p = \frac{1}{2}t\cos(5t)$

$\left.\begin{array}{l}\mathcal{L}(x) = 5; \alpha_1 = 5 \Rightarrow x_{p_1} = \frac{1}{5} \\ \mathcal{L}(x) = -5\sin(5t), \alpha_2 = 5 \Rightarrow x_{p_2} = \frac{1}{2}t\cdot\cos(5t)\end{array}\right\}$ superposition Principle

A particular solution for $\mathcal{L}(x) = 25 - 25\sin(5t)$ is:

$$x_p = \frac{1}{\cancel{5}}\cdot\cancel{5} + 5\cdot\frac{t}{2}\cos(5t) = 1 + \frac{5t}{2}\cos(5t)$$

$\Rightarrow$ The general solution of $\mathcal{L}(x) = 25 - 25\sin(5t)$ is:

$$x = c_1\cdot\cos(5t) + c_2\sin(5t) + 1 + \frac{5t}{2}\cos(5t), \quad c_1, c_2 \in \mathbb{R}$$

1.7.35) We consider the differential equation

$$x' + \frac{1}{t^2}x = 0,\ t\in(-\infty,0).$$

a) Check that $x = e^{1/t}$ is a solution of this DE.

b) Find the solution of the IVP $x' + \frac{1}{t^2}x = 0, x(-1) = 1$

c) Find the general solution of $x' + \frac{1}{t^2}x = 1 + \frac{1}{t}, t\in(-\infty;0)$.

a) $x = e^{1/t}$ - solution $\Rightarrow (e^{1/t})' + \frac{1}{t^2}\cdot e^{1/t} = 0 \Leftrightarrow$

$\Leftrightarrow \left(\frac{1}{t}\right)'\cdot e^{1/t} + \frac{1}{t^2}\cdot e^{1/t} = 0 \Leftrightarrow \left(-\frac{1}{t^2}\right)\cdot e^{-1/t} + \frac{1}{t^2}\cdot e^{-1/t} = 0 \Leftrightarrow 0 = 0$, true $\Rightarrow$

$\Rightarrow x = e^{1/t}$ is a solution of the DE

b) The given DE is a linear homogeneous DE of first order, so its set of solutions is a linear space of dimension 1. Thus, there exists x_1 - solution of the DE. From a) $\Rightarrow x_1 = e^{1/t}$. So, the general solution is:

$x = c\cdot e^{1/t}, c\in\mathbb{R}$

$$\text{IVP: }\begin{cases} x' + \frac{1}{t^2}x = 0 \\ x(-1) = 1 \end{cases} \Leftrightarrow \begin{cases} x = c\cdot e^{1/t}, c\in\mathbb{R} \\ x(-1) = 1 \end{cases}$$

$x(-1) = c\cdot e^{1/-1} = c\cdot e^{-1} = \frac{c}{e}$

$\Rightarrow \frac{c}{e} = 1 \Rightarrow c = e\in\mathbb{R} \Rightarrow$ The solution of the IVP is: $x = e\cdot e^{1/t}$

c) The general solution of $x' + \frac{1}{t^2}x = 1 + \frac{1}{t}, t\in(-\infty;0)$ is:

$x = x_h + x_p$, where x_h - general sol. of $x' + \frac{1}{t^2}x = 0$

x_p - particular solution

From b) $\Rightarrow x_h = c\cdot e^{1/t}, c\in\mathbb{R} \Rightarrow A(t) = \frac{1}{t}$

Applying Lagrange, we are looking for $x_p = \varphi(t)\cdot e^{A(t)}$, $\varphi(t)$ - function

$a(t) = \frac{1}{t^2}, A(t) = \frac{1}{t} \Rightarrow A'(t) = -\frac{1}{t^2} = -a(t)$

We replace with x_p in the Ln-HDE:

$(\varphi(t)\cdot e^{1/t})' + \frac{1}{t^2}\cdot\varphi(t)\cdot e^{1/t} = 1 + \frac{1}{t} \Leftrightarrow \varphi'(t)\cdot e^{1/t} + \cancel{\varphi(t)\cdot\left(-\frac{1}{t^2}\right)e^{1/t}} + \cancel{\frac{1}{t^2}\varphi(t)\cdot e^{1/t}} = 1 + \frac{1}{t}$

$\Leftrightarrow \varphi'(t)\cdot e^{1/t} = 1 + \frac{1}{t} \Rightarrow \varphi'(t) = e^{-1/t}\left(1 + \frac{1}{t}\right) \Rightarrow$

$\Rightarrow \varphi(t) = \int_0^t e^{-1/s}\left(1 + \frac{1}{s}\right)ds = \int_0^t \left(e^{-1/s} + \frac{e^{-1/s}}{s}\right)ds = \int_0^t \frac{e^{-1/s}}{s}ds + \underbrace{\int_0^t e^{-1/s}ds}_{y}\ (*)$

$y = \int_0^t 1\cdot e^{-1/s}ds = \int_0^t (s)'\cdot e^{-1/s}ds = t e^{-1/t} - \int_0^t s\cdot\left(-\frac{1}{s}\right)'\cdot e^{-1/s}ds =$

$= \cancel{t e^{-1}}\, t\cdot e^{-1/t} - \int_0^t \cancel{s}\cdot\frac{1}{s^{\cancel{2}}}\cdot e^{-1/s}ds = t\cdot e^{-1/t} - \int_0^t \frac{e^{-1/s}}{s}ds$

$(*) \Rightarrow \varphi(t) = \cancel{\int_0^t \frac{e^{-1/s}}{s}ds} + t\cdot e^{-1/t} - \cancel{\int_0^t \frac{e^{-1/s}}{s}ds} = t\cdot e^{-1/t} \Rightarrow x_p = t\cdot e^{-1/t}\cdot e^{1/t} \Rightarrow$

$\Rightarrow x_p = t$ - particular sol $\Rightarrow$ the general sol is: $x = c\cdot e^{1/t} + t$