Seminar W12 – 9.13

Affine transformations (plane)

$\underline{y = mx + n}$ affine function

$f(x) = mx + n$

$f(x_1 + x_2) \neq f(x_1) + f(x_2)$

$\underline{y = mx}$ linear function

$\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ affine transformation:

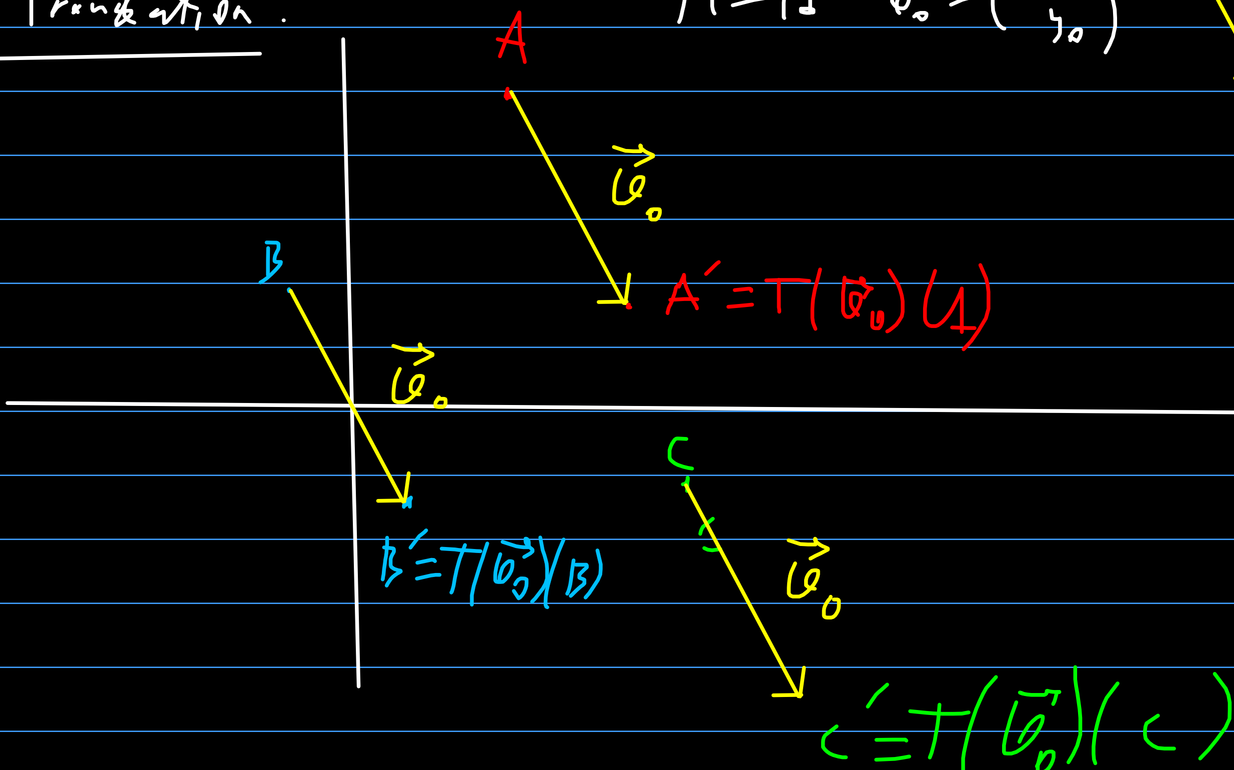
$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = \underset{\in M_{2,2}(\mathbb{R})}{\underline{M}} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \vec{\theta}_0 =$$

$$\vec{\theta}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$\hookrightarrow$ they preserve parallelism and lines (but not always distances and angles)

Translation:

$M = id \quad \vec{v_0} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$



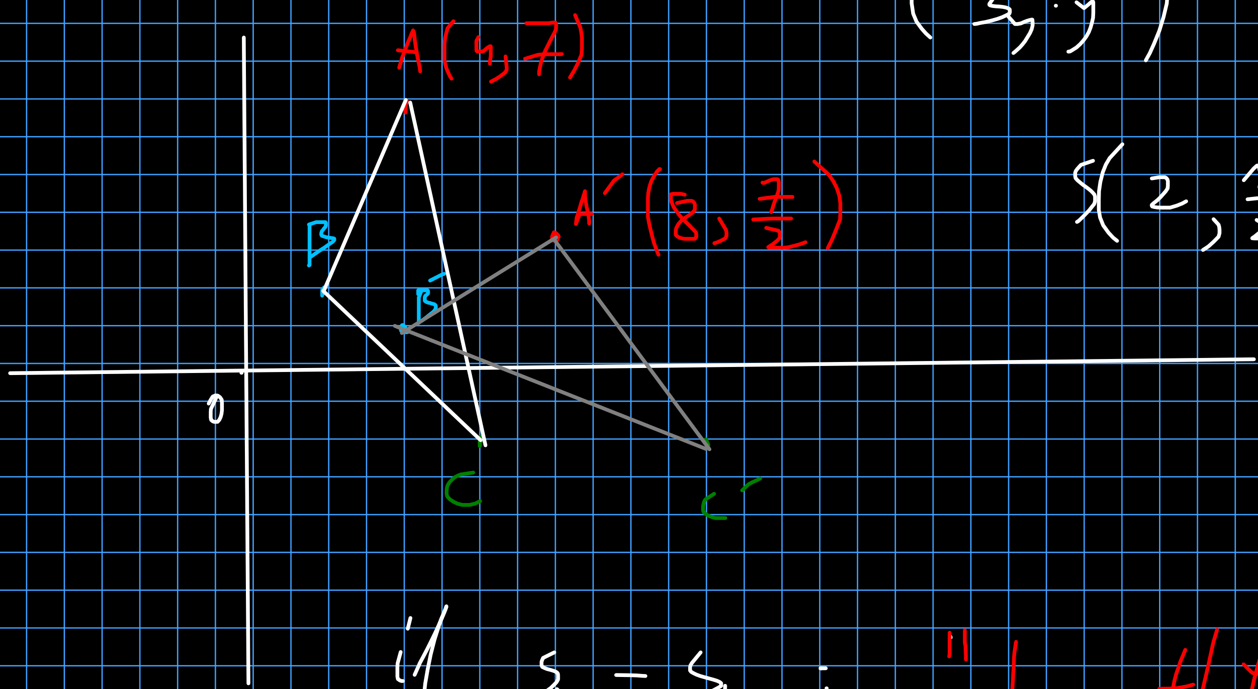
To recognize a translation: $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$

- pick a point A; if φ is a translation then $\varphi = T(\vec{AA'})$
- check this information against the other points

Scaling

$S(s_x, s_y)$, $s_x, s_y \neq 0$

$$S(s_x, s_y)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s_x \cdot x \\ s_y \cdot y \end{pmatrix}$$



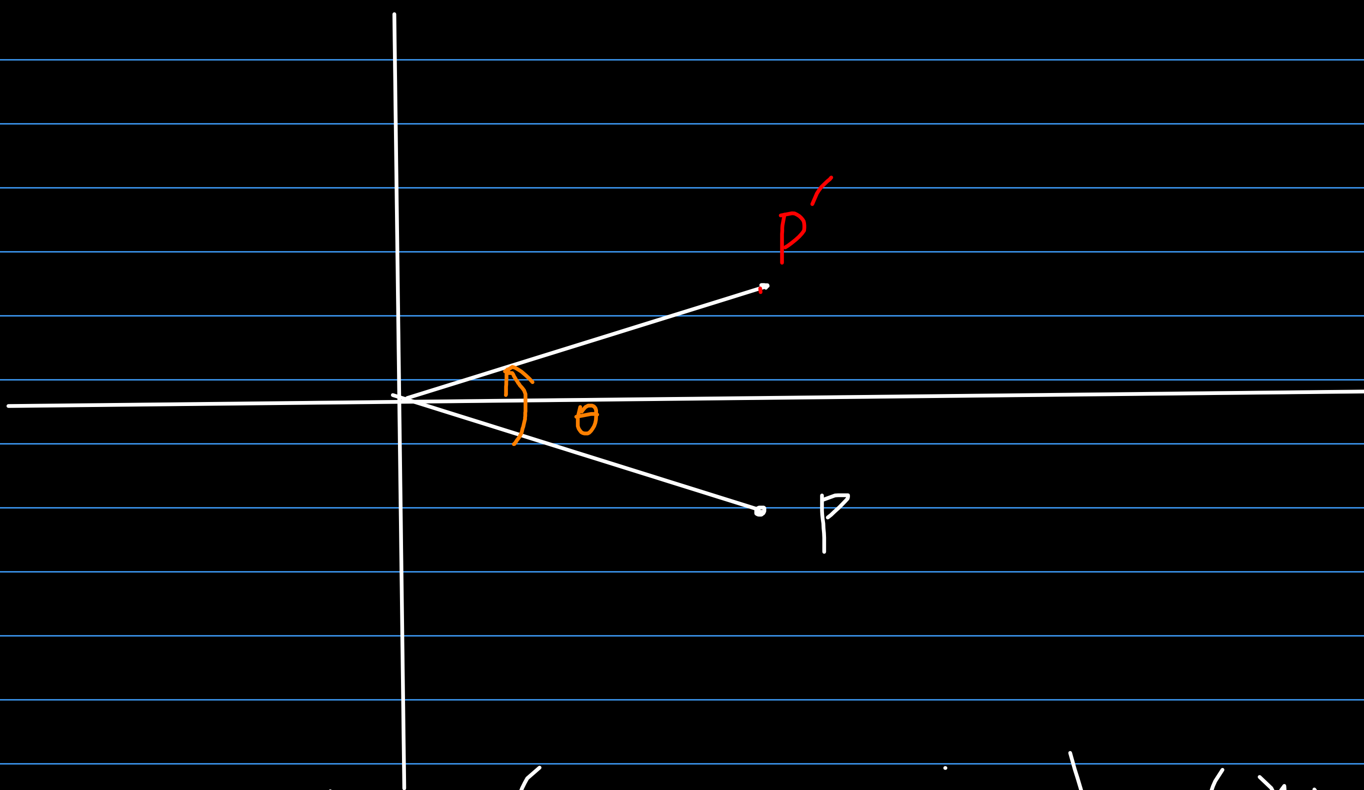
$S(2, \frac{1}{2})$

if $s_x = s_y$: "homothety"

How to recognize it: $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$

- choose A and we have $A' = \varphi(A)$
- if it is to be a scaling, then the factors should be $s_x = \frac{x_{A'}}{x_A}$, $s_y = \frac{y_{A'}}{y_A}$
- check against the other points

Rotations (around the origin) R_θ



$$R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

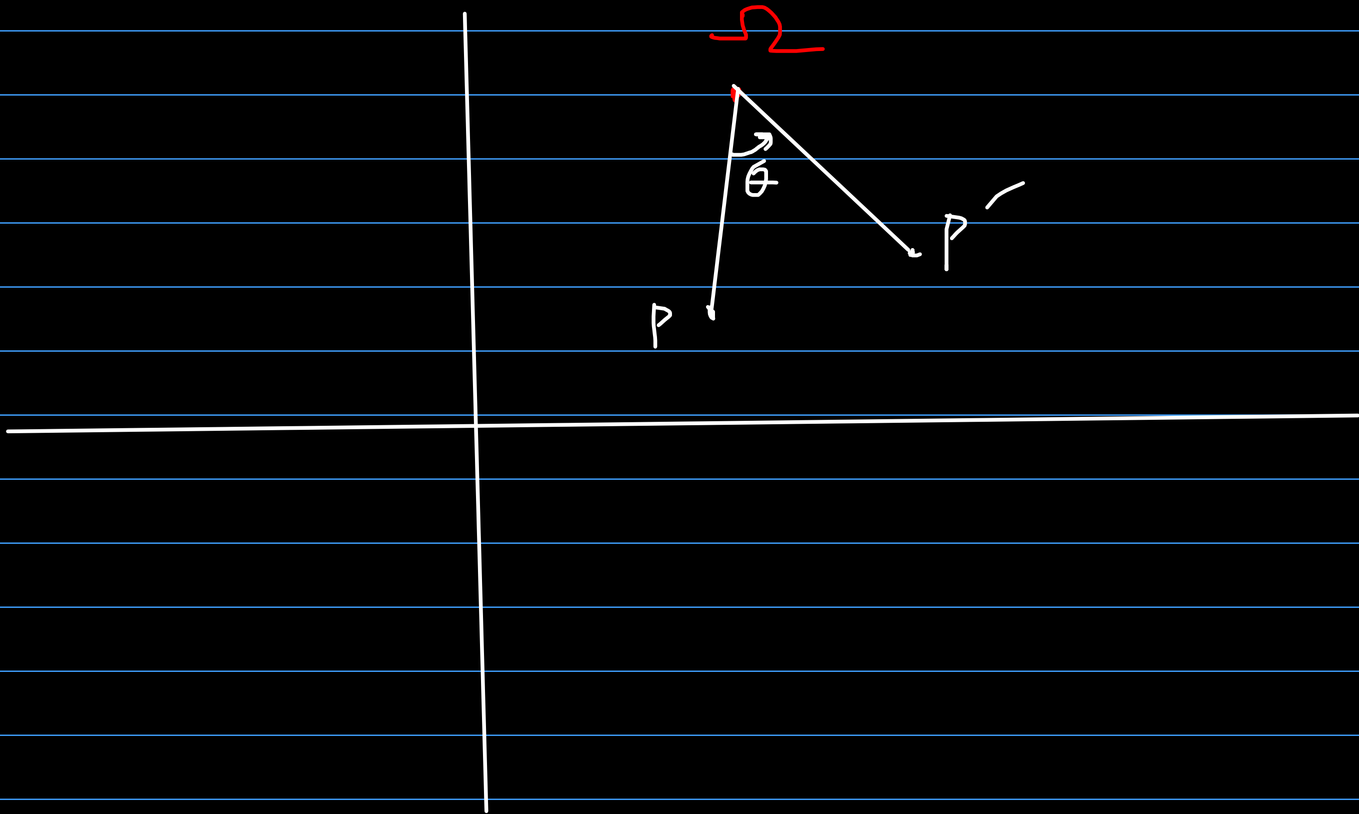
How do we recognize them? $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$

- check if $\mathrm{Fix}(\varphi) = \{0\}$

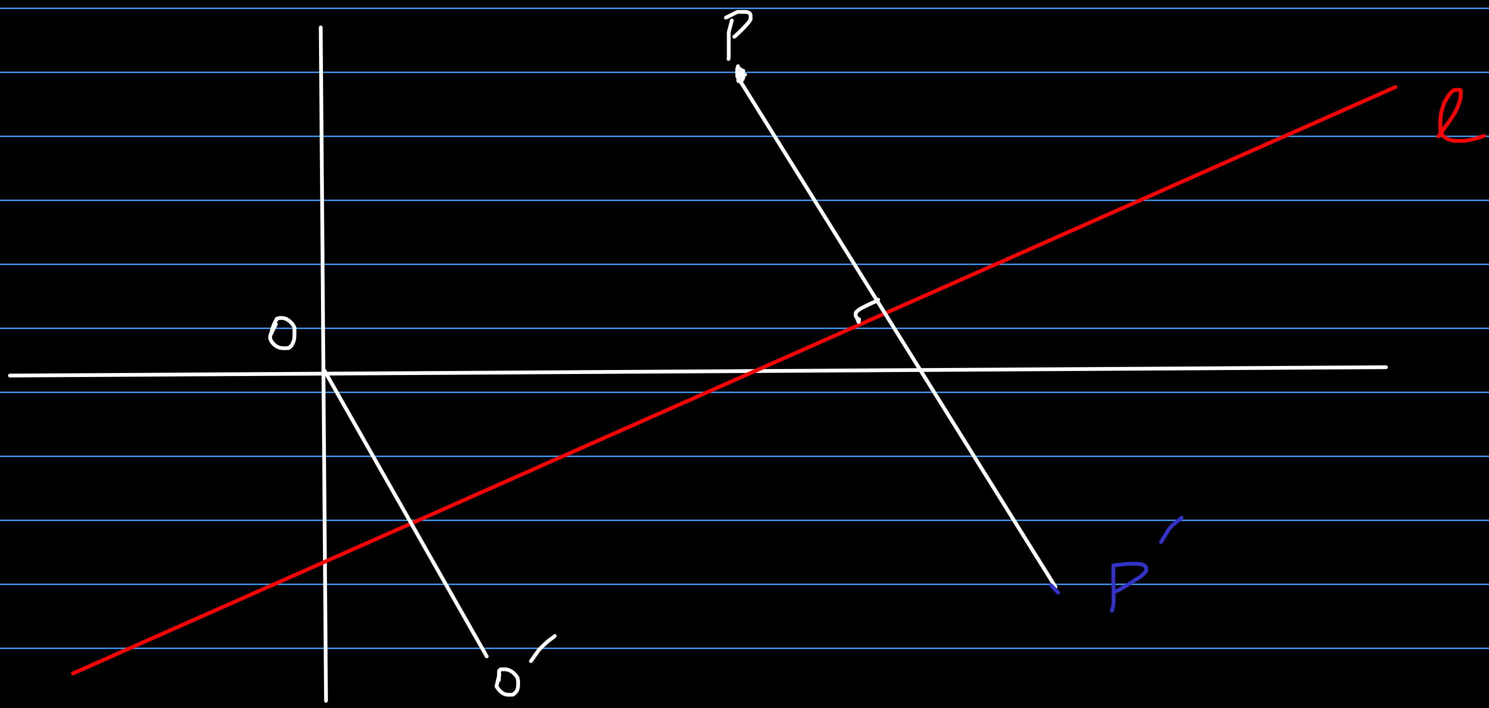
$$\left[\begin{array}{l} f : A \to B \\ \mathrm{Fix}(f) = \{x \in A \mid f(x) = x\} \end{array}\right]$$

(- check if $OP = OP'$, $\forall P \in \mathbb{R}^2$)

- check if $m(\widehat{POP'}) = \theta$



Reflections (w.r.t a line) $\quad r_\ell$



$\ell : ax + by + c = 0$

$$r_\ell\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{a^2+b^2}\begin{pmatrix} b^2-a^2 & -2ab\\ -2ab & a^2-b^2\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix} +$$

$$- \frac{2c}{a^2+b^2}\cdot\begin{pmatrix}a\\b\end{pmatrix}$$

$r_x := r_{Ox}$, $r_y := r_{Oy}$

$$r_x\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\-y\end{pmatrix}$$

$$r_y\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-x\\y\end{pmatrix}$$

r_ℓ linear transformation $\iff O \in \ell$

To recognize them:

- check if $\mathrm{Fix}(\varphi) = \ell$

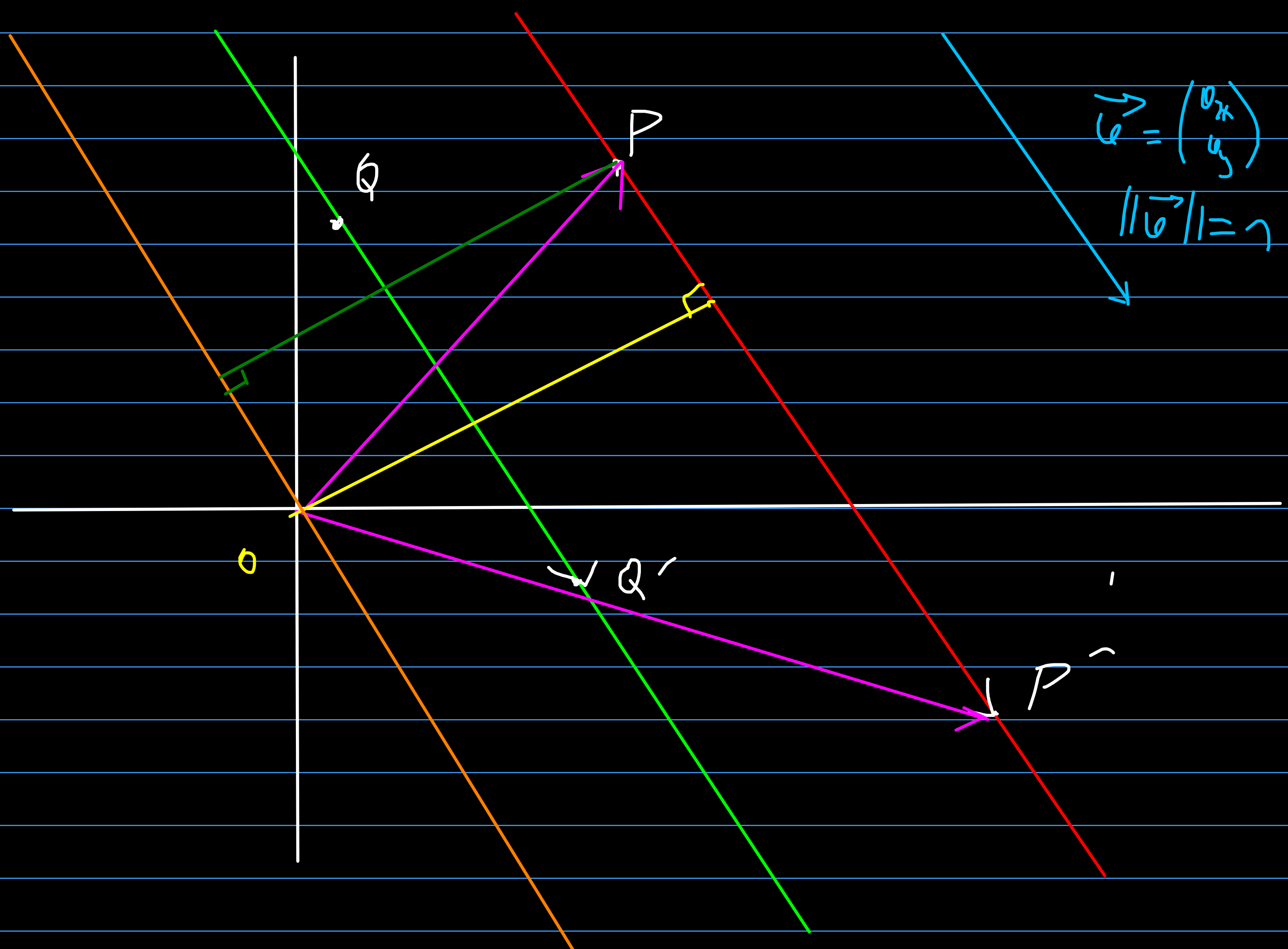
Yes — No

No: not a refl.

Yes: might be r_ℓ or a shear

- check if $\forall P$: ℓ is the perpendicular bisector of PP'

Shears : $sh(\vec{v}, r)$



$$sh(\vec{u}, r)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} + \underline{r \cdot \delta(O, \ell_P) \cdot \vec{u}}$$

$P(x,y)$

ℓ_P = the line through P whose direction is $\vec{u}$

$\ell : ax + by + c = 0$

$$\delta(A, \ell) = \frac{a x_A + b y_A + c}{\sqrt{a^2 + b^2}}$$

$\hookrightarrow$ <u>oriented distance</u>

$$sh(u, r)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} - r \cdot \delta(P, \ell_O) \cdot \vec{u}$$

$$sh(u, r)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix} 1 - r u_x u_y & r u_x^2 \\ -r u_y^2 & 1 + r u_x u_y \end{pmatrix} \cdot \begin{pmatrix}x\\y\end{pmatrix}$$

To recognize a shear: $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$

- pick a point A, $A' = \varphi(A)$
  if φ is a shear:
  $\overrightarrow{AA'} \parallel \overrightarrow{e}$
- if not $\Rightarrow$ not a shear

12.1. Find the image of the triangle ABC through the reflection with regards to the line $d: x-y=2$

$A(-1,2)$, $B(-3,-1)$, $C(3,3)$

$c=-2$, $a=1$, $b=-1$

$$[r_d] = \frac{1}{a^2+b^2} \cdot \begin{pmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{pmatrix} =$$

$$= \frac{1}{2} \cdot \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$[\bar{v}_0) = \frac{-2c}{a^2+b^2} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \frac{4}{2} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$r_\ell\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}2\\-2\end{pmatrix}=$$

$$=\begin{pmatrix}y+2\\x-2\end{pmatrix}$$

$$r_\ell(A)=r_\ell(-1,2)=(4,-3)$$

$$r_\ell(B)=r_\ell(-3,-1)=(1,-4)$$

$$r_\ell(C)=r_\ell(3,3)=(5,1)$$

11.2. Find the image of the triangle ABC through the clockwise rotation of angle $\frac{\pi}{6}$ where $A(6,4)$, $B(6,2)$, $C(10,6$

$$[R_{\frac{\pi}{6}}] = \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$R_{\frac{\pi}{6}}(A) = R_{\frac{\pi}{6}}(6,4) = (3\sqrt{3}-2,\ 3+2\sqrt{3})$$

$$R_{\frac{\pi}{6}}(B) = R_{\frac{\pi}{6}}(6,2) = (3\sqrt{3}-1,\ 3+\sqrt{3})$$

$$R_{\frac{\pi}{6}}(C) = R_{\frac{\pi}{6}}(10,6) = (5\sqrt{3}-3,\ 5+3\sqrt{3})$$

11.3. ABCD quadrilateral

$A(1,1)$, $B(3,1)$, $C(2,2)$, $D(\frac{3}{2},3)$

Find the image of ABCD through the transformations:

a) $T(1,2)$, $S(2,\frac{5}{2})$, r_x

b) r_y, $R_{-\frac{\pi}{2}}$, $R_{\frac{\pi}{2}}$

c) $sh\left(\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \frac{3}{2}\right)$

$$T(1,2)(1,1) = (2,3)$$

$$T(1,2)(3,1) = (4,3)$$

$$T(1,2)(2,2) = (3,4)$$

$$T(1,2)\left(\tfrac{3}{2},3\right) = \left(\tfrac{5}{2},4\right)$$

$$S\left(2,\tfrac{5}{2}\right)(1,1) = \left(2,\tfrac{5}{2}\right)$$

$$S\left(2,\tfrac{5}{2}\right)(3,1) = \left(6,\tfrac{5}{2}\right)$$

$$S\left(2,\tfrac{5}{2}\right)(2,2) = (4,5)$$

$$S\left(2,\tfrac{5}{2}\right)\left(\tfrac{3}{2},3\right) = \left(3,\tfrac{15}{2}\right)$$

$$r_x(1,1) = (1,-1)$$

$$r_x(3,1) = (3,-1)$$

$$r_x(2,2) = (2,-2)$$

$$r_x\left(\tfrac{3}{2},3\right) = \left(\tfrac{3}{2},-3\right)$$

b) $r_y(1,1) = (-1,1)$

$$r_y(3,-1) = (-3,1)$$

$$r_y(2,2) = (-2,2)$$

$$r_y\left(\frac{3}{2},3\right) = \left(-\frac{3}{2},3\right)$$

$$\left[R_{-\frac{\pi}{2}}\right] = \begin{pmatrix} \cos\left(-\frac{\pi}{2}\right) & -\sin\left(-\frac{\pi}{2}\right) \\ \sin\left(-\frac{\pi}{2}\right) & \cos\left(-\frac{\pi}{2}\right) \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$R_{-\frac{\pi}{2}}(1,1) = (1,-1)$$

$$R_{-\frac{\pi}{2}}(3,1) = (1,-3)$$

$$R_{-\frac{\pi}{2}}(2,2) = (2,-2)$$

$$R_{-\frac{\pi}{2}}\left(\frac{3}{2},3\right) = \left(3,-\frac{3}{2}\right)$$

$$[R_{\frac{\pi}{2}}] = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$R_{\frac{\pi}{2}}(A) = -R_{-\frac{\pi}{2}}(A) = (-1, 1)$$

$$sh\left(\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \frac{3}{2}\right)$$

$$[sh(\vec{v}, r)] = \begin{pmatrix} 1 - r u_x u_y & r u_x^2 \\ -r u_y^2 & 1 + r u_x u_y \end{pmatrix} =$$

$$= \begin{pmatrix} 1 - \frac{3}{2} \cdot \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} & \frac{3}{2} \cdot \frac{4}{5} \\ -\frac{3}{2} \cdot \frac{1}{5} & 1 + \frac{3}{2} \cdot \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{2}{5} & \frac{6}{5} \\ -\frac{3}{10} & \frac{8}{5} \end{pmatrix}$$

$sh(\vec{v}, r)(1,1) = \left(\frac{8}{5}, \frac{13}{10}\right)$

$sh(\vec{v}, r)(3,1) = \left(\frac{12}{5}, \frac{7}{10}\right)$

$sh(\vec{v}, r)(2,2) = \left(\frac{16}{5}, \frac{26}{10}\right)$

$sh(\vec{v}, r)\left(\frac{3}{2}, 3\right) = \left(3, \frac{55}{20}\right)$

Ex.: f affine transf. so that

$f(A) = A'$, $f(B) = B'$

$A(1,2)$, $B(2,3)$

$A'(-1,-1)$, $B'(-2,-3)$

Decide which of the following transformations is a valid candidate for f.

(If so, find an example
If not, show why.)

- translation
- scaling
- reflection
- shear

$A(1,2)$, $B(2,3)$

$A'(-1,-1)$, $B'(-2,-3)$

? translation

$$\left.\begin{array}{l} \overrightarrow{AA'}\ (-2,-3) \\ \overrightarrow{BB'}\ (-4,-6) \end{array}\right\} \Rightarrow \text{not a translation}$$

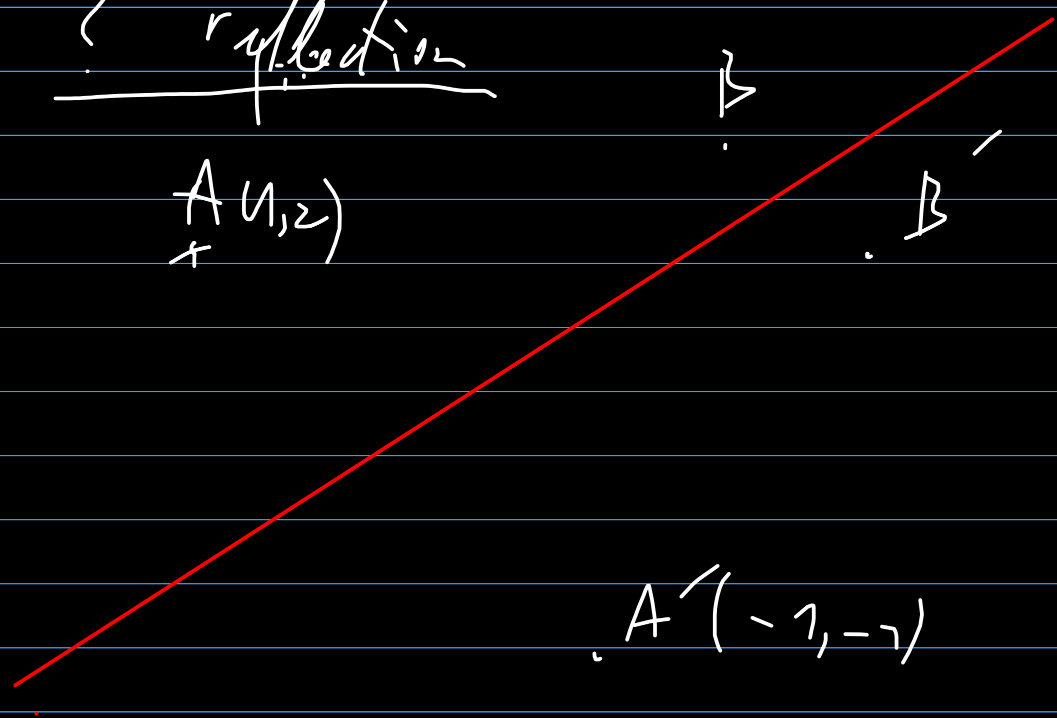
? Scaling

$$\left.\begin{array}{ll} \dfrac{x_A}{x_{A'}} = -1, & \dfrac{y_A}{y_{A'}} = -\dfrac{1}{2} \\[2ex] \dfrac{x_B}{x_{B'}} = -1 & \dfrac{y_B}{y_{B'}} = -1 \end{array}\right\} \Rightarrow \text{not a Scaling}$$

? shear

$\vec{AA'} = 2 \cdot \vec{BB'}$ $\Rightarrow$ it could be a shear $\Rightarrow$ $\vec{u} = \dfrac{\vec{AA'}}{\|\vec{AA'}\|} = \left(\dfrac{2}{\sqrt{13}}, \dfrac{-3}{\sqrt{13}}\right)$

? reflection



$\Rightarrow$ if it were a reflection, then if M is the midpoint of AA' and N is the midpoint of BB', then $MN \perp AA'$ and $MN \perp BB'$