$\mathcal{E} = f^{-1}(1), f: \mathbb{R}^3 \to \mathbb{R}, f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$

f is homogeneous of order 2

$\mathcal{E} = f^{-1}(1)$ - regular surface

10 Week 10: Quadrics

10.1 The ellipsoid

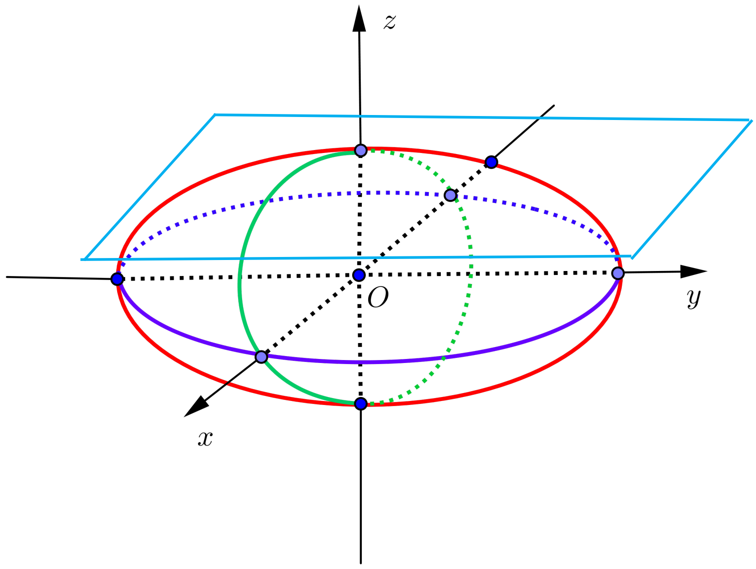
The *ellipsoid* is the quadric surface given by the equation

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*. \tag{10.1}$$

- The coordinate planes are all planes of symmetry of $\mathcal{E}$ since, for an arbitrary point $M(x, y, z) \in \mathcal{E}$, its symmetric points with respect to these planes, $M_1(-x, y, z)$, $M_2(x, -y, z)$ and $M_3(x, y, -z)$ belong to $\mathcal{E}$; therefore, the coordinate axes are axes of symmetry for $\mathcal{E}$ and the origin O is the center of symmetry of the ellipsoid (10.1);

- The traces in the coordinates planes are ellipses of equations

$$\underbrace{\begin{cases} \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \\ x = 0 \end{cases}}_{\mathcal{E}\cap(yOz)}, \quad \underbrace{\begin{cases} \frac{x^2}{a^2} + \frac{z^2}{c^2} - 1 = 0 \\ y = 0 \end{cases}}_{\mathcal{E}\cap(xOz)}, \quad \underbrace{\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \\ z = 0. \end{cases}}_{\mathcal{E}\cap(xOy)}$$



- The sections with planes parallel to xOy are given by setting $z = \lambda$ in (10.1). Then, a section is of equations $\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{\lambda^2}{c^2} \\ z = \lambda \end{cases}$. this might be negative

- If $|\lambda| < c$, the section is an ellipse

$$\begin{cases} \frac{x^2}{\left(a\sqrt{1 - \frac{\lambda^2}{c^2}}\right)^2} + \frac{y^2}{\left(b\sqrt{1 - \frac{\lambda^2}{c^2}}\right)^2} = 1 \\ z = \lambda \end{cases};$$

- If $|\lambda| = c$, the intersection is reduced to one (tangency) point $(0, 0, \lambda)$;
- If $|\lambda| > c$, the plane $z = \lambda$ does not intersect the ellipsoid $\mathcal{E}$.

The sections with planes parallel to xOz or yOz are obtained in a similar way.

10.2 The hyperboloid of one sheet

$H_1 = g^{-1}(1), g: \mathbb{R}^3 \to \mathbb{R}, g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}$

g - homogeneous of order two $\Rightarrow$ 1 - is a regular value of g $\Rightarrow$

$\Rightarrow H_1 = g^{-1}(1)$ regular surface

The surface of equation

$$\mathcal{H}_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0, \qquad a, b, c \in \mathbb{R}^*_+, \tag{10.2}$$

is called *hyperboloid of one sheet*.

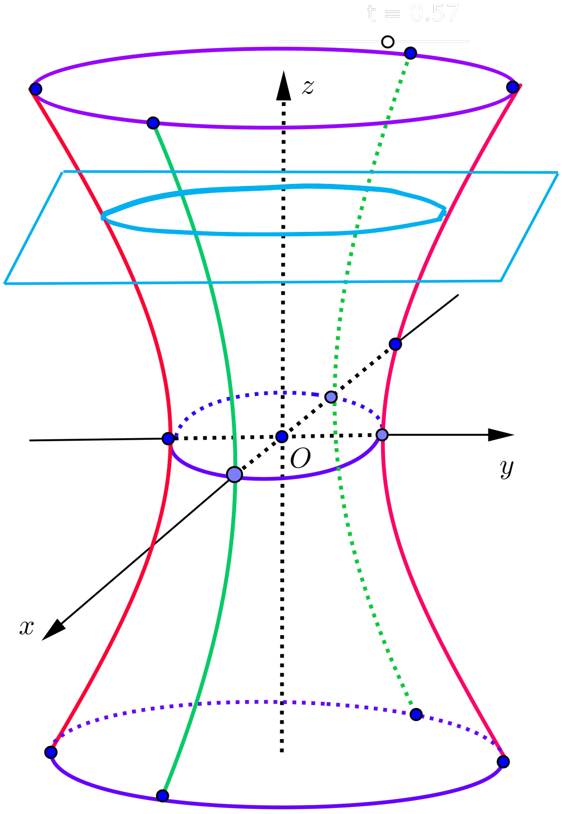
- The coordinate planes are planes of symmetry for $\mathcal{H}_1$; hence, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{H}_1$;
- The intersections with the coordinates planes are, respectively, of equations

$$\begin{cases} \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0 \\ x = 0 \\ \text{a hyperbola} \end{cases} ; \begin{cases} \frac{x^2}{a^2} - \frac{z^2}{c^2} - 1 = 0 \\ y = 0 \\ \text{a hyperbola} \end{cases} ; \begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \\ z = 0 \\ \text{an ellipse} \end{cases} ;$$

$H_1 \cap yOz$ $\quad H_1 \cap xOz$ $\quad H_1 \cap xOy$

- The intersections with planes parallel to the coordinate planes are

$$\begin{cases} \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{\lambda^2}{a^2} \\ x = \lambda \\ \text{hyperbolas} \end{cases} ; \begin{cases} \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{\lambda^2}{b^2} \\ y = \lambda \\ \text{hyperbolas} \end{cases} ; \begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{\lambda^2}{c^2} \\ z = \lambda \\ \text{ellipses} \end{cases} ;$$



Remark: The surface $\mathcal{H}_1$ contains two families of lines, as

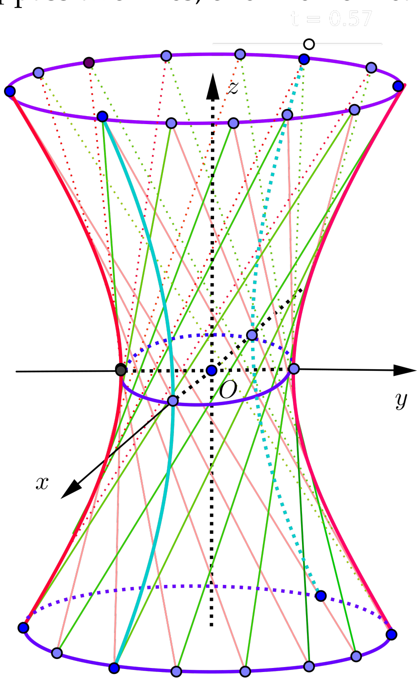
$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2} \Leftrightarrow \left(\frac{x}{a} + \frac{z}{c}\right)\left(\frac{x}{a} - \frac{z}{c}\right) = \left(1 + \frac{y}{b}\right)\left(1 - \frac{y}{b}\right).$$

The equations of the two families of lines are:

$$d_\lambda : \begin{cases} \lambda\left(\frac{x}{a} + \frac{z}{c}\right) = 1 + \frac{y}{b} \\ \frac{x}{a} - \frac{z}{c} = \lambda\left(1 - \frac{y}{b}\right) \end{cases}, \lambda \in \mathbb{R},$$

$$d'_\mu : \begin{cases} \mu\left(\frac{x}{a} + \frac{z}{c}\right) = 1 - \frac{y}{b} \\ \frac{x}{a} - \frac{z}{c} = \mu\left(1 + \frac{y}{b}\right) \end{cases}, \mu \in \mathbb{R}.$$

Through any point on $\mathcal{H}_1$ pass two lines, one line from each family.

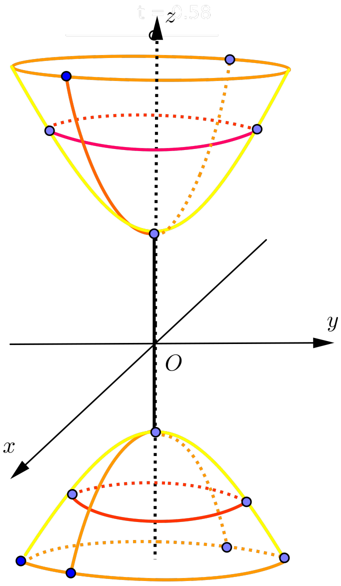


10.3 Th hyperboloid of two sheets

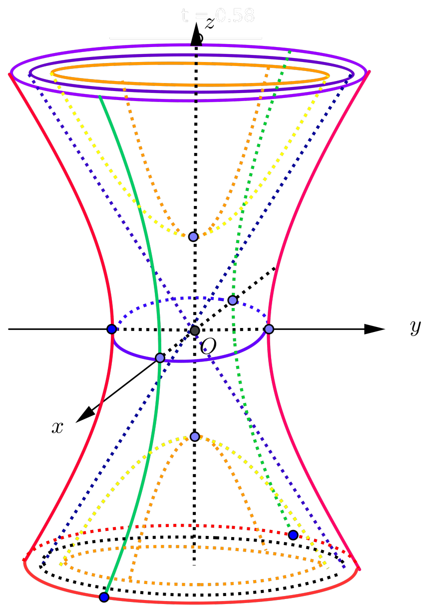
$H_2 = h^{-1}(-1)$, $h : \mathbb{R}^3 \to \mathbb{R}$, $h(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}$
h - homogeneous of order 2 $\Rightarrow -1$ - regular value of h $\Rightarrow$
$\Rightarrow H_2$ - regular surface

The *hyperboloid of two sheets* is the surface of equation

$$\mathcal{H}_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0, \qquad a, b, c \in \mathbb{R}^*_+. \tag{10.3}$$



The hyperboloid of two sheets



The hyperboloids of one and two sheets and their common asymptotic cone

- The coordinate planes are planes of symmetry for $\mathcal{H}_1$, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{H}_1$;
- The intersections with the coordinates planes are, respectively,

$$\left\{\begin{array}{c}\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=0\\ x=0\\ \text{a hyperbola;}\end{array}\right., \left\{\begin{array}{c}\frac{x^2}{a^2}-\frac{z^2}{c^2}+1=0\\ y=0\\ \text{a hyperbola}\end{array}\right., \left\{\begin{array}{c}\frac{x^2}{a^2}+\frac{y^2}{b^2}+1=0\\ z=0\\ \text{the empty set}\end{array}\right.;$$

$H_2 \cap yOz$ $H_2 \cap xOz$ $H_2 \cap xOy$

- The intersections with planes parallel to the coordinate planes are

$$\left\{\begin{array}{c}\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1-\frac{\lambda^2}{a^2}\\ x=\lambda \ \parallel yOz\\ \text{hyperbolas}\end{array}\right., \left\{\begin{array}{c}\frac{x^2}{a^2}-\frac{z^2}{c^2}=-1-\frac{\lambda^2}{b^2}\\ y=\lambda \ \parallel xOz\\ \text{hyperbolas}\end{array}\right., \left\{\begin{array}{c}\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1+\frac{\lambda^2}{c^2}\\ z=\lambda. \parallel xOy\end{array}\right.$$

- If $|\lambda| > c$, the section is an ellipse;
- If $|\lambda| = c$, the intersection reduces to the point of coordinates $(0, 0, \lambda)$;
- If $|\lambda| < c$, one obtains the empty set.

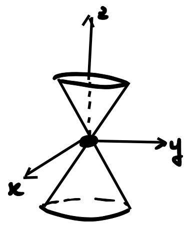
10.4 Elliptic Cones

The surface of equation

not a regular surface

$$\mathcal{C} : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \qquad a, b, c \in \mathbb{R}_+^*, \tag{10.4}$$

$\mathcal{C} \setminus \{(0,0,0)\}$ - regular surface



is called *elliptic cone*.

- The coordinate planes are planes of symmetry for $\mathcal{C}$, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{C}$;
- The intersections with the coordinates planes are

$$\begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0 \\ x = 0 \\ \text{two lines} \end{cases}, \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = 0 \\ y = 0 \\ \text{two lines} \end{cases}, \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0 \\ z = 0 \\ \text{the point } O(0,0,0). \end{cases}$$

$\mathcal{C} \cap yOz$ $\qquad$ $\mathcal{C} \cap xOz$ $\qquad$ $\mathcal{C} \cap xOy$

- The intersections with planes parallel to the coordinate planes are

$$\begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -\dfrac{\lambda^2}{a^2} \\ x = \lambda \\ \text{hyperbolas} \end{cases}; \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = -\dfrac{\lambda^2}{b^2} \\ y = \lambda \\ \text{hyperbolas.} \end{cases}, \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{\lambda^2}{c^2} \\ z = \lambda. \\ \text{ellipses} \end{cases}$$

10.5 Elliptic Paraboloids

The surface of equation

$$\mathcal{P}_e : \frac{x^2}{p} + \frac{y^2}{q} = 2z, \qquad p, q \in \mathbb{R}_+^*, \tag{10.5}$$

is called *elliptic paraboloid*.

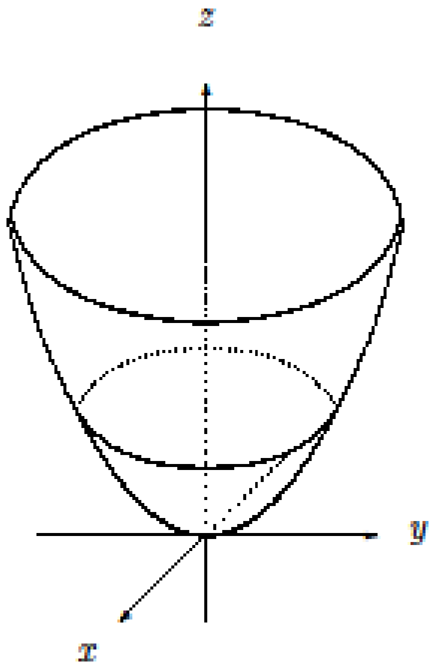
- The planes xOz and yOz are planes of symmetry;
- The traces in the coordinate planes are

$$\begin{cases} \dfrac{y^2}{q} = 2z \\ x = 0 \\ \text{a parabola} \end{cases}, \begin{cases} \dfrac{x^2}{p} = 2z \\ y = 0 \\ \text{a parabola} \end{cases}, \begin{cases} \dfrac{x^2}{p} + \dfrac{y^2}{q} = 0 \\ z = 0 \\ \text{the point } O(0,0,0). \end{cases}$$

- The intersection with the planes parallel to the coordinate planes are $\begin{cases} \dfrac{x^2}{p} + \dfrac{y^2}{q} = 2\lambda \\ z = \lambda \end{cases}$,
  - If $\lambda > 0$, the section is an ellipse;
  - If $\lambda = 0$, the intersection reduces to the origin;
  - If $\lambda < 0$, one has the empty set;

and

$$\begin{cases} \dfrac{y^2}{q} = 2z - \dfrac{\lambda^2}{p} \\ x = \lambda \\ \text{parabolas} \end{cases}; \qquad \begin{cases} \dfrac{x^2}{p} = 2z - \dfrac{\lambda^2}{q} \\ y = \lambda \\ \text{parabolas} \end{cases};$$



10.6 Hyperbolic Paraboloids

The *hyperbolic paraboloid* is the surface given by the equation

$$\mathcal{P}_h : \frac{x^2}{p} - \frac{y^2}{q} = 2z, \qquad p, q > 0. \tag{10.6}$$

- The planes xOz and yOz are planes of symmetry;
- The traces in the coordinate planes are, respectively,

$$\begin{cases} -\dfrac{y^2}{q} = 2z \\ x = 0 \\ \text{a parabola} \end{cases}; \quad \begin{cases} \dfrac{x^2}{p} = 2z \\ y = 0 \\ \text{a parabola} \end{cases}; \quad \begin{cases} \dfrac{x^2}{p} - \dfrac{y^2}{q} = 0 \\ z = 0 \\ \text{two lines.} \end{cases};$$

- The intersection with the planes parallel to the coordinate planes are

$$\begin{cases} \dfrac{y^2}{q} = -2z + \dfrac{\lambda^2}{p} \\ x = \lambda \\ \text{parabolas} \end{cases}; \quad \begin{cases} \dfrac{x^2}{p} = 2z + \dfrac{\lambda^2}{q} \\ y = \lambda \\ \text{parabolas.} \end{cases}$$

$$\begin{cases} \dfrac{x^2}{p} - \dfrac{y^2}{q} = 2\lambda \\ z = \lambda \\ \text{hyperbolas} \end{cases}$$

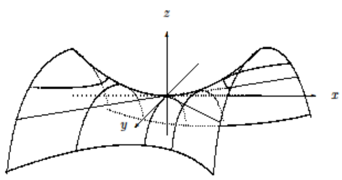
Remark: The hyperbolic paraboloid contains two families of lines. Since

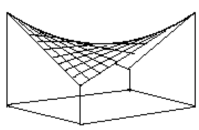
$$\left(\frac{x}{\sqrt{p}}+\frac{y}{\sqrt{q}}\right)\left(\frac{x}{\sqrt{p}}-\frac{y}{\sqrt{q}}\right)=2z,$$

then the two families are, respectively, of equations

$$d_\lambda : \begin{cases} \dfrac{x}{\sqrt{p}}-\dfrac{y}{\sqrt{q}}=\lambda \\ \lambda\left(\dfrac{x}{\sqrt{p}}+\dfrac{y}{\sqrt{q}}\right)=2z \end{cases}, \lambda \in \mathbb{R} \text{ and}$$

$$d'_\mu : \begin{cases} \dfrac{x}{\sqrt{p}}+\dfrac{y}{\sqrt{q}}=\mu \\ \mu\left(\dfrac{x}{\sqrt{p}}-\dfrac{y}{\sqrt{q}}\right)=2z \end{cases}, \mu \in \mathbb{R}.$$





10.7 Singular Quadrics

Elliptic Cylinder, Hyperbolic Cylinder, Parabolic Cylinder

- The *elliptic cylinder* is the surface of equation

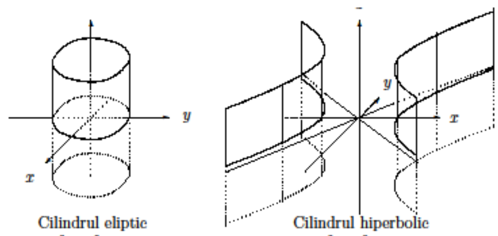
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0, \quad a,b>0 \text{ or } \frac{x^2}{a^2}+\frac{z^2}{c^2}-1=0, \frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0. \tag{10.7}$$

- The *hyperbolic cylinder* is the surface of equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0, \quad a,b>0 \text{ or } \frac{x^2}{a^2}-\frac{z^2}{c^2}-1=0, \frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0. \tag{10.8}$$

- The *parabolic cylinder* is the surface of equation

$$y^2=2px, \quad p>0, \quad \text{(or an alternative equation)}. \tag{10.9}$$



Cilindrul eliptic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Cilindrul hiperbolic $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Cilindrul parabolic $y^2=2px$

Conul $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$

10.8 Problems

1. Find the intersection points of the ellipsoid

$$\frac{x^2}{16}+\frac{y^2}{12}+\frac{z^2}{4}=1$$

with the line

$$\frac{x-4}{2}=\frac{y+6}{-3}=\frac{z+2}{-2}$$

and write the equations of the tangent planes as well as the equations of the normal lines to the ellipsoid at the intersection points.

Solution.

$$\ell:\begin{cases} x=4+2t \\ y=-6-3t \\ z=-2-2t \end{cases} \Rightarrow \ell\cap\mathcal{E}: \frac{(4+2t)^2}{16}+\frac{(-6-3t)^2}{12}+\frac{(-2-2t)^2}{4}=1 \Leftrightarrow$$

$$\Leftrightarrow \frac{4t^2+16t+16}{16}+\frac{9t^2+36t+36}{12}+\frac{4t^2+8t+4}{4}=1 \Leftrightarrow \frac{t^2}{4}+t+1+\frac{3t^2}{4}+3t+3+t^2+2t+1=1 \Rightarrow$$

$$\Rightarrow 2t^2+6t+4=0/:2 \Leftrightarrow t^2+3t+2=0 \Rightarrow t_1=-2,\ t_2=-1$$

For $t=-2 \Rightarrow A(0,0,2)$

$t=-1 \Rightarrow B(2,-3,0)$

The tangent plane to the surface at (x_0,y_0,z_0): $f'_x(x_0,y_0,z_0)(x-x_0)+f'_y(x_0,y_0,z_0)(y-y_0)+f'_z(x_0,y_0,z_0)(z-z_0)=0$

$f(x,y,z)=\frac{x^2}{16}+\frac{y^2}{12}+\frac{z^2}{4} \Rightarrow \frac{\partial f}{\partial x}=\frac{x}{8},\ \frac{\partial f}{\partial y}=\frac{y}{6},\ \frac{\partial f}{\partial z}=\frac{z}{2}$

For $A(0,0,2)$: $\frac{0}{8}(x-0)+\frac{0}{6}(y-0)+\frac{2}{2}(z-2)=0 \Rightarrow z-2=0 \Leftrightarrow z=2$

$B(2,-3,0)$: $\frac{2}{8}(x-2)+\frac{-3}{6}(y+3)+\frac{0}{2}(z-0)=0 \Leftrightarrow \frac{1}{4}x-\frac{1}{2}y-2=0$

The eq of the normal line in a point (x_0,y_0,z_0): $(x,y,z)=(x_0,y_0,z_0)+t(f'_x(x_0,y_0,z_0),f'_y(x_0,y_0,z_0),f'_z(x_0,y_0,z_0))$

For $A(0,0,2)$: $\vec{r}_{line_1}=\langle 0,0,2\rangle+t\langle 0,0,1\rangle=\langle 0,0,2+t\rangle$

$B(2,-3,0)$: $\vec{r}_{line_2}=\langle 2,-3,0\rangle+t\langle \frac{1}{4},-\frac{1}{2},0\rangle=\langle 2+\frac{1}{4}t,-3-\frac{1}{2}t,0\rangle$

2. Find the rectilinear generatrices of the quadric $4x^2 - 9y^2 = 36z$ which passes through the point $P(3\sqrt{2}, 2, 1)$.

Solution.

$$(2x-3y)(2x+3y)=36z \Rightarrow d_\lambda : \begin{cases} 2x-3y = 36\lambda \\ (2x+3y)\cdot\lambda = z \end{cases}$$

$$P \in d_\lambda \Rightarrow \begin{cases} 6\sqrt{2}-6 = 36\lambda \\ (6\sqrt{2}+6)\cdot\lambda = 1 \end{cases} \Rightarrow \begin{cases} \lambda = \dfrac{6\sqrt{2}-6}{36} = \dfrac{\sqrt{2}-1}{6} \\ \lambda = \dfrac{1}{6\sqrt{2}+6} = \dfrac{1}{6}\cdot\dfrac{\sqrt{2}-1}{2-1} = \dfrac{\sqrt{2}-1}{6} \end{cases} \Rightarrow \lambda = \dfrac{\sqrt{2}-1}{6} \Rightarrow d_{\frac{\sqrt{2}-1}{6}} \begin{cases} 2x-3y = 6(\sqrt{2}-1) \\ (2x+2y)\dfrac{\sqrt{2}-1}{6} = z \end{cases}$$

$$d'_\mu : \begin{cases} (2x-3y)\mu = z \\ 2x+3y = 36\mu \end{cases}$$

$$P \in d'_\mu \Rightarrow \begin{cases} (6\sqrt{2}-6)\mu = 1 \\ 6\sqrt{2}+6 = 36\mu \end{cases} \Rightarrow \begin{cases} \mu = \dfrac{1}{6(\sqrt{2}-1)} = \dfrac{\sqrt{2}+1}{6} \\ \mu = \dfrac{6(\sqrt{2}+1)}{36} = \dfrac{\sqrt{2}+1}{6} \end{cases} \Rightarrow \mu = \dfrac{\sqrt{2}+1}{6} \Rightarrow d'_{\frac{\sqrt{2}+1}{6}} \text{ is another solution}$$

3. Find the rectilinear generatrices of the hyperboloid of one sheet

$$(\mathcal{H}_1)\ \frac{x^2}{36}+\frac{y^2}{9}-\frac{z^2}{4}=1$$

which are parallel to the plane $(\pi)\ x+y+z=0$.

Solution.

$$d_\lambda : \begin{cases} \lambda\left(\frac{x}{6}+\frac{z}{2}\right)=1+\frac{y}{3} \\ \frac{x}{6}-\frac{z}{2}=\lambda\left(1-\frac{y}{3}\right) \end{cases} \Leftrightarrow \begin{cases} \frac{\lambda x}{6}-\frac{y}{3}+\frac{\lambda z}{2}=1 \\ \frac{x}{6}+\frac{\lambda y}{3}-\frac{z}{2}=\lambda \end{cases}$$

$$d'_\mu : \begin{cases} \mu\cdot\left(\frac{x}{6}+\frac{z}{2}\right)=1-\frac{y}{3} \\ \frac{x}{6}-\frac{z}{2}=\mu\left(1+\frac{y}{3}\right) \end{cases} \Leftrightarrow \begin{cases} \frac{\mu x}{6}+\frac{y}{3}+\frac{\mu z}{2}=1 \\ \frac{x}{6}-\frac{\mu y}{3}-\frac{z}{2}=\mu \end{cases}$$

$$\vec{d_\lambda}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\lambda}{6} & -\frac{1}{3} & \frac{\lambda}{2} \\ \frac{1}{6} & \frac{\lambda}{3} & -\frac{1}{2} \end{vmatrix}=\frac{1-\lambda^2}{6}\vec{i}+\frac{\lambda}{6}\vec{j}+\frac{1+\lambda^2}{18}\vec{k} \Rightarrow \vec{d_\lambda}\left(\frac{1-\lambda^2}{6},\frac{\lambda}{6},\frac{1+\lambda^2}{18}\right)$$

$$\vec{d'_\mu}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\mu}{6} & \frac{1}{3} & \frac{\mu}{2} \\ \frac{1}{6} & -\frac{\mu}{3} & -\frac{1}{2} \end{vmatrix}=\frac{\mu^2-1}{6}\vec{i}+\frac{\mu}{6}\vec{j}+\frac{1-\mu^2}{18}\vec{k} \Rightarrow \vec{d'_\mu}\left(\frac{\mu^2-1}{6},\frac{\mu}{6},\frac{1-\mu^2}{18}\right)$$

A straight line $d: \frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r} \parallel$ plane $: Ax+By+Cz+D=0 \Leftrightarrow Ap+Bq+Cr=0$

$$d_\lambda \parallel \pi \Leftrightarrow \frac{1-\lambda^2}{6}+\frac{\lambda}{6}+\frac{1+\lambda^2}{18}=0 \Leftrightarrow 3-3\lambda^2+3\lambda+1+\lambda^2=0 \Rightarrow \lambda_{1,2}=\frac{-3\pm\sqrt{41}}{-4}$$

$$d_\mu \parallel \pi \Leftrightarrow \frac{\mu^2-1}{6}+\frac{\mu}{6}+\frac{1-\mu^2}{18}=0 \Leftrightarrow 2\mu^2+3\mu-4=0 \Rightarrow \mu_{1,2}=\frac{-3\pm\sqrt{41}}{-4}$$

$$d_{\lambda_1} : \begin{cases} \frac{-3+\sqrt{41}}{-4}\left(\frac{x}{6}+\frac{z}{2}\right)=1+\frac{y}{3} \\ \frac{x}{6}-\frac{z}{2}=\frac{-3+\sqrt{41}}{-4}\left(1-\frac{y}{3}\right) \end{cases} \qquad d_{\lambda_2} : \begin{cases} \frac{-3-\sqrt{41}}{-4}\left(\frac{x}{6}+\frac{z}{2}\right)=1+\frac{y}{3} \\ \frac{x}{6}-\frac{z}{2}=\frac{-3-\sqrt{41}}{-4}\left(1-\frac{y}{3}\right) \end{cases}$$

$$d_{\mu_1} : \begin{cases} \frac{-3+\sqrt{41}}{-4}\left(\frac{x}{6}+\frac{z}{2}\right)=1-\frac{y}{3} \\ \frac{x}{6}-\frac{z}{2}=\frac{-3+\sqrt{41}}{-4}\left(1+\frac{y}{3}\right) \end{cases} \qquad d_{\mu_2} : \begin{cases} \frac{-3-\sqrt{41}}{-4}\left(\frac{x}{6}+\frac{z}{2}\right)=1-\frac{y}{3} \\ \frac{x}{6}-\frac{z}{2}=\frac{-3-\sqrt{41}}{-4}\left(1+\frac{y}{3}\right) \end{cases}$$

4. Find the locus of points on the hyperbolic paraboloid $(\mathcal{P}_h)\ y^2 - z^2 = 2x$ through which the rectilinear generatrices are perpendicular.

   Solution.

$$d_\lambda : \begin{cases} y - z = \lambda \\ \lambda(y+z) = 2x \end{cases}, \lambda \in \mathbb{R} \qquad d_\mu : \begin{cases} y + z = \mu \\ \mu \cdot (y - z) = 2x \end{cases}$$

$$d_\lambda : \begin{cases} y = z + \lambda \\ \lambda(y+z) = 2x \end{cases} \Leftrightarrow \begin{cases} y = z + \lambda \\ \lambda(2z+\lambda) = 2x \end{cases} \Leftrightarrow \begin{cases} y = z + \lambda \\ x = \lambda z + \frac{\lambda^2}{2} \end{cases} \Leftrightarrow \frac{x - \frac{\lambda^2}{2}}{\lambda} = y - \lambda = z \Leftrightarrow \frac{x - \frac{\lambda^2}{2}}{\lambda} = \frac{y-\lambda}{1} = \frac{z-0}{1} \Rightarrow \vec{d_\lambda} = (\lambda, 1, 1)$$

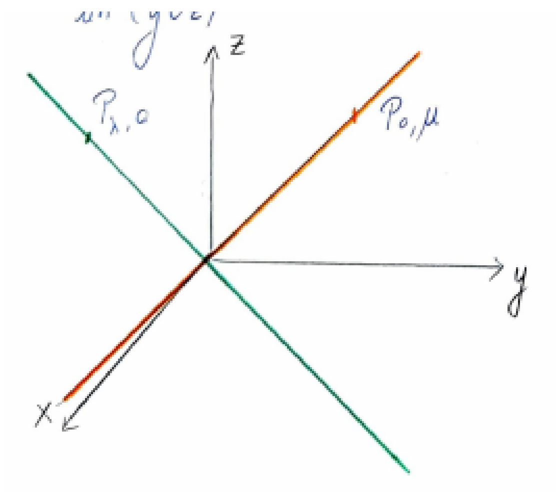
$$d_\mu : \begin{cases} y = \mu - z \\ \mu(y - z) = 2x \end{cases} \Leftrightarrow \begin{cases} y = \mu - z \\ \mu(\mu - 2z) = 2x \end{cases} \Leftrightarrow \begin{cases} y = \mu - z \\ x = -\mu z + \frac{\mu^2}{2} \end{cases} \Leftrightarrow \frac{x - \frac{\mu^2}{2}}{-\mu} = \mu - y = z \Leftrightarrow \frac{x - \frac{\mu^2}{2}}{-\mu} = \frac{y-\mu}{-1} = \frac{z-0}{1} \Rightarrow \vec{d_\mu} = (-\mu, -1, 1)$$

$$d_\lambda \perp d_\mu \Leftrightarrow \vec{d_\lambda} \cdot \vec{d_\mu} = 0 \Leftrightarrow -\lambda\mu - 1 + 1 = 0 \Leftrightarrow \lambda\mu = 0 \Leftrightarrow \lambda = 0 \text{ or } \mu = 0$$

$$\text{I. } \lambda = 0 : P_{0,\mu} : \begin{cases} d_0 \\ d'_\mu \end{cases} \Leftrightarrow \begin{cases} \frac{y-0}{1} = \frac{z-0}{1} \text{ and } x = \frac{\lambda^2}{2} = 0 \\ \frac{x - \frac{\mu^2}{2}}{-\mu} = \frac{y-\mu}{-1} = \frac{z-0}{1} \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = z \\ \frac{x - \frac{\mu^2}{2}}{-\mu} = \frac{y-\mu}{-1} = z \end{cases} \Leftrightarrow y = \frac{\mu}{2} \Rightarrow P_{0,\mu}\left(0, \frac{\mu}{2}, \frac{\mu}{2}\right) \Rightarrow$$

$\Rightarrow$ this part of the locus is a line, the first bisector in (yOz).

$$\text{II. } \mu = 0 : P_{\lambda,0} : \begin{cases} d_\lambda \\ d'_0 \end{cases} \Leftrightarrow P_{\lambda,0}\left(0, \frac{\lambda}{2}, -\frac{\lambda}{2}\right) \Rightarrow$$ the locus is the second bisector in (yOz)



5. Compute the distance from $O(0,0,0)$ to the tangent plane $T_M(\mathcal{E})$ of the ellipsoid

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at some of its point $M(x, y, z) \in \mathcal{E}$.

Solution.

The equation of the tangent plane of the ellipsoid at some point $M(x_M, y_M, z_M)$ is:

$$T_M(\mathcal{E}): f'_x(M)(x-x_M) + f'_y(M)(y-y_M) + f'_z(M)(z-z_M) = 0,$$

where $f'_x(M) = \frac{2x_M}{a^2}$, $f'_y(M) = \frac{2y_M}{b^2}$, $f'_z(M) = \frac{2z_M}{c^2}$, $M(x_M, y_M, z_M)$

$$\Rightarrow T_M(\mathcal{E}): \frac{2x_M}{a^2}(x-x_M) + \frac{2y_M}{b^2}(y-y_M) + \frac{2z_M}{c^2}(z-z_M) = 0 \Leftrightarrow$$

$$\Leftrightarrow T_M(\mathcal{E}): \underbrace{\frac{2x_M}{a^2}}_{A} x + \underbrace{\frac{2y_M}{b^2}}_{B} y + \underbrace{\frac{2z_M}{c^2}}_{C} z \underbrace{- \left(\frac{2x_M^2}{a^2} + \frac{2y_M^2}{b^2} + \frac{2z_M^2}{c^2} \right)}_{D} = 0$$

The distance from $O(0,0,0)$ to the plane $T_M(\mathcal{E})$ is:

$$\delta(O, T) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2+B^2+C^2}} = \frac{\left| - \left(\frac{2x_M^2}{a^2} + \frac{2y_M^2}{b^2} + \frac{2z_M^2}{c^2} \right) \right|}{\sqrt{\frac{4x_M^2}{a^4} + \frac{4y_M^2}{b^2} + \frac{4z_M^2}{c^2}}} =$$

$$= \frac{\cancel{2}\left(\frac{x_M^2}{a^2} + \frac{y_M^2}{b^2} + \frac{z_M^2}{c^2} \right)}{\cancel{2}\sqrt{\frac{x_M^2}{a^4} + \frac{y_M^2}{b^2} + \frac{z_M^2}{c^2}}} = \frac{\frac{x_M^2}{a^2} + \frac{y_M^2}{b^2} + \frac{z_M^2}{c^2}}{\sqrt{\frac{x_M^2}{a^4} + \frac{y_M^2}{b^4} + \frac{z_M^2}{c^4}}}$$

Also, $M(x_M, y_M, z_M) \in \mathcal{E} \Leftrightarrow \frac{x_M^2}{a^2} + \frac{y_M^2}{b^2} + \frac{z_M^2}{c^2} = 1$

Hence, $\delta(O, T) = \dfrac{1}{\sqrt{\frac{x_M^2}{a^4} + \frac{y_M^2}{b^4} + \frac{z_M^2}{c^4}}}$.

6. Show that the intersection between a straight line d and the sphere $S(O,r)$ is a singleton if and only if $\text{dist}(O,d) = r$.

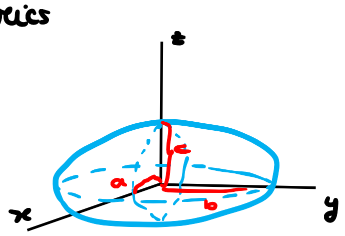
   Solution.

Seminar 10 - 28/04/2021 - 916

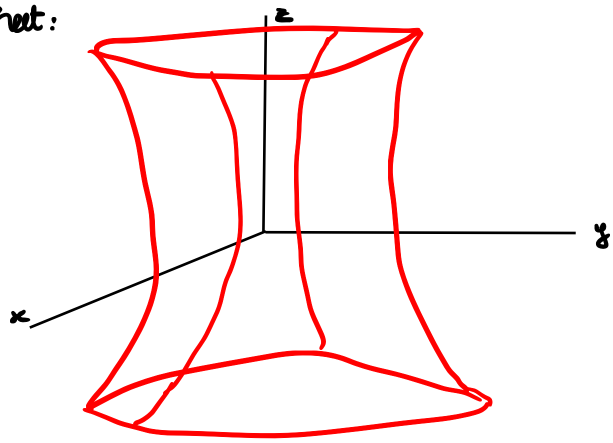
$\mathcal{E} : f(x,y)=0,\ f\in\mathbb{R}[x,y] \to$ conics

$S : f(x,y,z)=0,\ f\in\mathbb{R}[x,y,z] \to$ quadrics

→ Ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

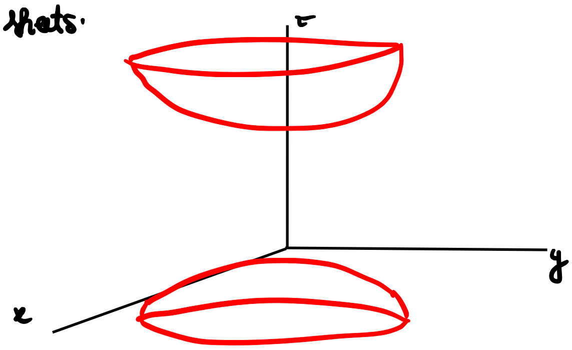


→ Hyperboloid of one sheet:



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

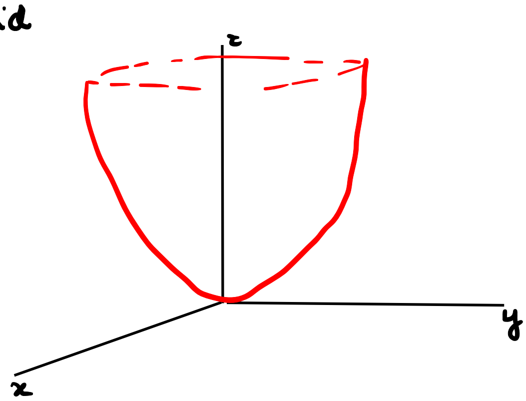
→ Hyperboloid of two sheets:



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$

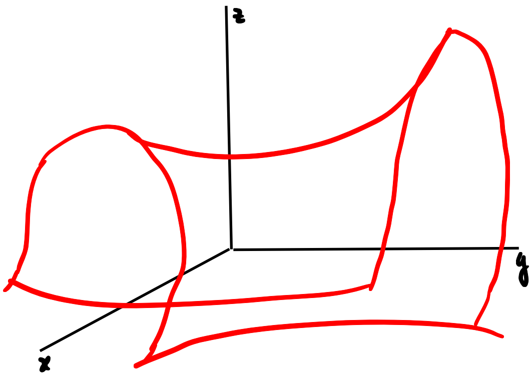
$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

→ Elliptic paraboloid



$$\frac{x^2}{p}+\frac{y^2}{q}=2z,\ p,q>0$$

→ Hyperbolic parabolloid



$$\frac{x^2}{p} - \frac{y^2}{q} = 2z$$

Rectilinear generatrices → only exist on hyperboloids of one sheet and hyperbolic paraboloids

$\exists f, \mathcal{S} : f(x,y,z) = 0 \Rightarrow T_{\mathcal{S}}(x_0, y_0, z_0) : f'_x(x_0,y_0,z_0)\cdot(x-x_0) + f'_y(x_0,y_0,z_0)\cdot(y-y_0) + f'_z(x_0,y_0,z_0)(z-z_0) = 0$

↳ tangent plane

$$N_{\mathcal{S}}(x_0,y_0,z_0) : \frac{x-x_0}{f'_x(x_0,y_0,z_0)} = \frac{y-y_0}{f'_y(x_0,y_0,z_0)} = \frac{z-z_0}{f'_z(x_0,y_0,z_0)}$$

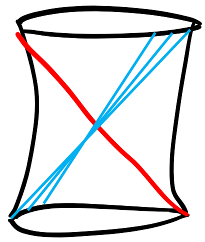
Rectilinear generatrices

→ $\mathcal{H} : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \Rightarrow \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2} \rightsquigarrow \left(\frac{x}{a} - \frac{z}{c}\right)\left(\frac{x}{a} + \frac{z}{c}\right) = \left(1 - \frac{y}{b}\right)\left(1 + \frac{y}{b}\right)$

$$d_\lambda : \begin{cases} \frac{x}{a} - \frac{z}{c} = \lambda\left(1 - \frac{y}{b}\right) \\ \lambda\left(\frac{x}{a} + \frac{z}{c}\right) = 1 + \frac{y}{b} \end{cases}$$

(every point on every line d_λ is a point on the hyperboloid)

$$d'_\mu : \begin{cases} \frac{x}{a} - \frac{z}{c} = \mu\left(1 + \frac{y}{b}\right) \\ \mu\left(\frac{x}{a} + \frac{z}{c}\right) = 1 - \frac{y}{b} \end{cases}$$



→ hyperbolic parabolloid :

$$\frac{x^2}{p} - \frac{y^2}{q} = 2z \Rightarrow \left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}\right)\left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right) = 2z \Rightarrow$$

→ $d_\lambda : \begin{cases} \frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}} = 2\lambda \\ \lambda\left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right) = z \end{cases}$ $\quad d'_\mu : \begin{cases} \mu\left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}\right) = z \\ \frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}} = 2\mu \end{cases}$