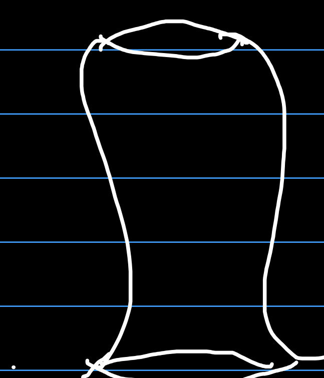
Seminar W11 - 913

Generated surfaces

- → Ruled surfaces
  - → conical
  - → cylindrical
  - → conoidal
- → Revolution surfaces



Example 1.2 : Conical surface with vertex $V(1,1,1)$ whose director curve is $\mathcal{C}: \begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases}$.

Step 1 : Write the eqn. of the generatrix

the moving line that traces the surface

$$V \in d_{\lambda,\mu} : \quad d_{\lambda,\mu} : \frac{x-1}{a} = \frac{y-1}{b} = \frac{z-1}{c}$$

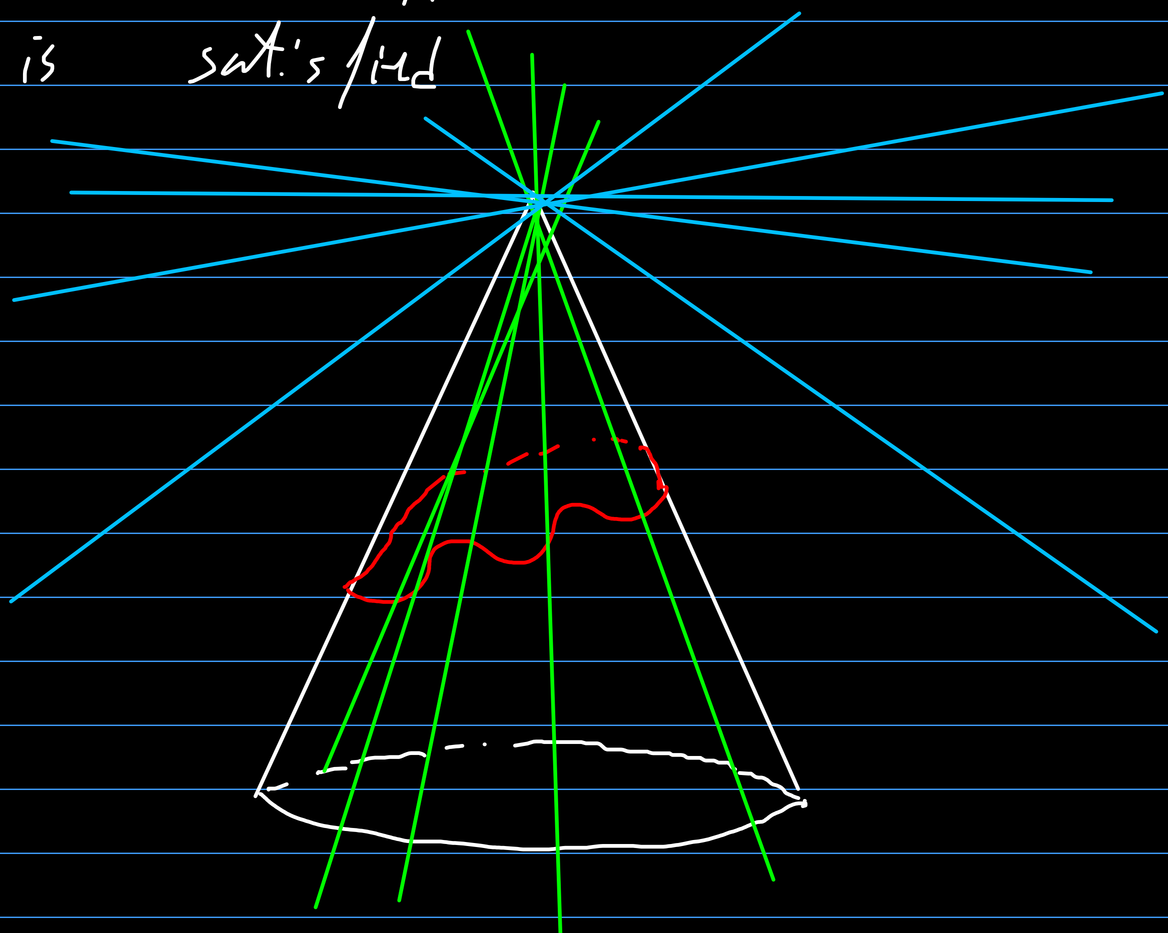
$$\Leftrightarrow d_{\lambda,\mu} : \begin{cases} x-1 = \underbrace{\frac{a}{b}}_{=:\lambda} \cdot (y-1) \\ x-1 = \underbrace{\frac{a}{c}}_{=\mu} \cdot (z-1) \end{cases}.$$

$$\Leftrightarrow \boxed{d_{\lambda,\mu} : \begin{cases} x-1 = \lambda \cdot (y-1) \\ x-1 = \mu (z-1) \end{cases}}$$

Step 2: Find a compatibility condition between λ and μ that ensures that

$$d_{\lambda,\mu} \cap \mathcal{C} \neq \emptyset$$

is satisfied



Step 1 = blue + green

Step 2 = green

In step 2, in order to ensure that the lines $d_{\lambda,\mu}$ are the ones that intersect $\mathcal{C}$, we solve the system:

$$\begin{cases} d_{\lambda,\mu}: \begin{cases} x-1 = \lambda\cdot(y-1) \\ x-1 = \mu(z-1) \end{cases} \\ \\ \mathcal{C}: \begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x-1 = \lambda\cdot(y-1) \\ x-1 = \mu\cdot(z-1) \\ z = 0 \\ (x^2+y^2)^2 - xy = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z = 0 \\ x - 1 = \lambda \cdot (y-1) \\ x - 1 = -\mu \\ (x^2+y^2)^2 - xy = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z = 0 \\ x = 1 - \mu \\ -\mu = \lambda \cdot (y-1) \\ (x^2+y^2)^2 - xy = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z = 0 \\ x = 1 - \mu \\ y = 1 - \frac{\mu}{\lambda} \\ (x^2+y^2)^2 - xy = 0 \end{cases}$$

By replacing the values of x, y, z in terms of λ, μ, we get the compatibility condition

$$\left((1-\mu)^2+\left(1-\frac{\mu}{\lambda}\right)^2\right)^2-(1-\mu)\cdot\left(1-\frac{\mu}{\lambda}\right)=0$$

<u>Step 3:</u> Replace the values of λ and μ in the compatibility condition with their <u>initial</u> expression in x, y, z.

In our case, the expressions are

$$\lambda=\frac{x-1}{y-1} \qquad \mu=\frac{x-1}{z-1}$$

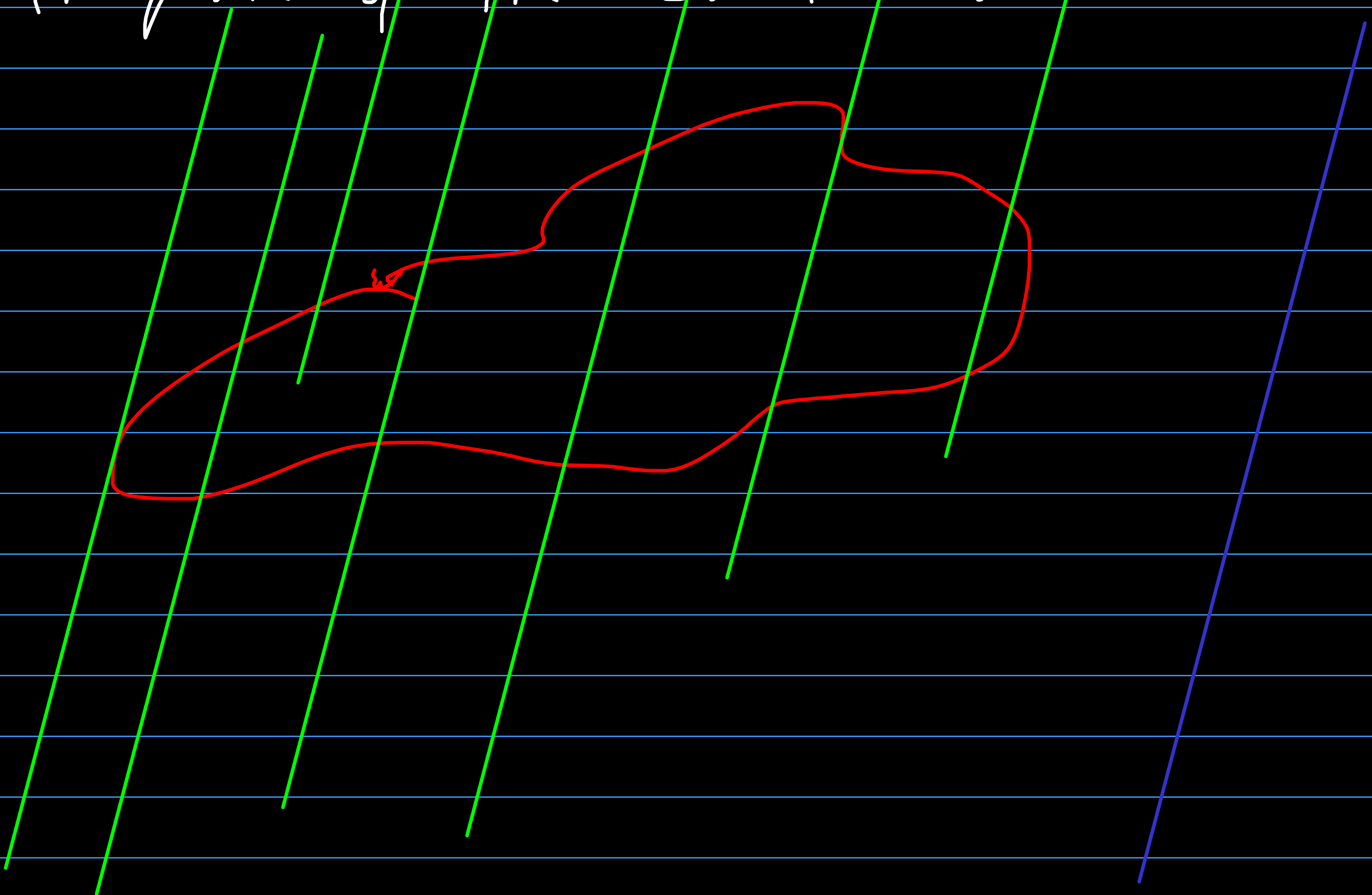
Therefore the implicit equation of the conical surface is:

$$\left(\left(1-\frac{x-1}{z-1}\right)^2+\left(1-\frac{y-1}{z-1}\right)^2\right)^2-\left(1-\frac{x-1}{z-1}\right)\cdot\left(1-\frac{y-1}{z-1}\right)=0$$

11.1. Find the equation of the cylindrical surface whose director curve is the planar curve

$$\mathcal{C}: \begin{cases} y^2 + z^2 = x \\ x = 2z \end{cases}$$

and the generatrix is perpendicular to the plane of the director curve



In the case of a cylindrical surface the condition on the generatrices $d_{\lambda,\mu}$ is that they must all have the same direction (i.e. they must all be parallel to a line)

The plane that the curve $\mathcal{C}$ is contained in is $\Pi: x = 2z$

We are told that $d_{\lambda,\mu} \perp \Pi$

$$\Rightarrow d_{\lambda,\mu}: \begin{cases} \dfrac{x - x_0}{1} = \dfrac{z - z_0}{-2} \\ y - y_0 = 0 \end{cases}$$

this is the general equation of a line that is perpendicular to Π:

$$d_{\lambda,\mu}: \begin{cases} -2x - z = -2x_0 - z_0 \\ y = y_0 \end{cases}$$

$=: \lambda$

$= \mu$

$\Rightarrow d_{\lambda,\mu} : \begin{cases} -2x - z = \lambda \\ y = \mu \end{cases}$

Step 2:

$$\begin{cases} -2x - z = \lambda \\ y = \mu \\ y^2 + z^2 = x \\ x = 2z \end{cases} \Leftrightarrow \begin{cases} x = 2z \\ -5z = \lambda \\ y = \mu \\ y^2 + z^2 = 2z \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 2z \\ z = -\frac{\lambda}{5} \\ y = \mu \\ y^2 + z^2 = 2z \end{cases}$$

$\Rightarrow$ The compatibility condition is:

$$\mu^2 + \frac{\lambda^2}{25} + \frac{2\lambda}{5} = 0$$

Step 3: $\lambda = -2x - z$

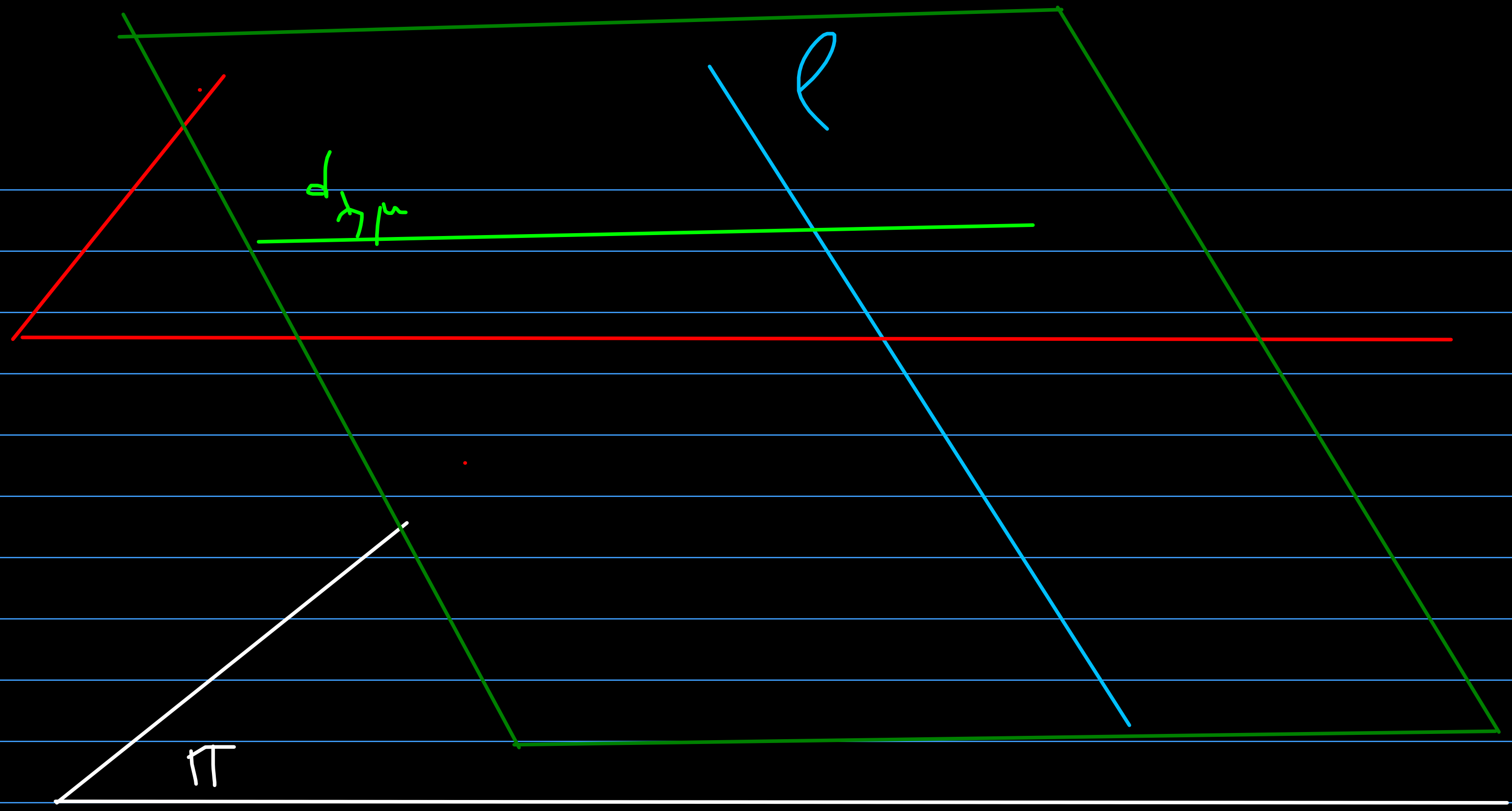
$\mu = y$

=> the equation of the cylindrical surface is:

$$y^2 + \frac{(-2x-z)^2}{25} + \frac{-4x-2z}{5} = 0$$

Example 11.3 Find the equation of the conoidal surface, whose generatrices are parallel to xOy and intersect Oz and the curve

$$\begin{cases} y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases}$$



The generatrices of such a conoid are lines that are parallel to π and contain ℓ.

Every generatrix of the conoid is an intersection between a red plane (parallel to π) and a dark green plane (contains ℓ)

$\Pi: \ Ax + By + Cz + D = 0$

$$\ell: \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

$$d_{\lambda,\mu}: \begin{cases} A_1x + B_1y + C_1z + D_1 = \lambda(A_2x + B_2y + C_2z + D_2) \\ Ax + By + Cz + D = \mu \end{cases}$$

Example 11.3 Find the equation of the conoidal surface, whose generatrices are parallel to xOy and intersect Oz and the curve

$$\begin{cases} y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases}$$

$\Pi = xOy: \quad z = 0$

$$\ell = Oz: \begin{cases} y = 0 \\ x = 0 \end{cases}$$

Step 1 :

$$d_{\lambda,\mu} : \begin{cases} z = \mu \\ x = \lambda y \end{cases}$$

Step 2 :

$$\begin{cases} z = \mu \\ x = \lambda y \\ y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases} \iff$$

$$\iff \begin{cases} z = \mu \\ x = \lambda y \\ y^2 - 2\mu + 2 = 0 \\ \lambda^2 y^2 - 2\mu + 1 = 0 \end{cases} \iff$$

$$\Leftrightarrow \begin{cases} z = \mu \\ x = \lambda y \\ y^2 = 2\mu - 2 \\ \lambda^2 \cdot (2\mu - 2) - 2\mu + 1 = 0 \end{cases}$$

$\Rightarrow$ compatibility condition:

$$\lambda^2 \cdot (2\mu - 2) - 2\mu + 1 = 0$$

<u>Step 3</u>: $\mu = z$, $\lambda = \frac{x}{y}$

$\Rightarrow$ eqn. of the surface:

$$\frac{x^2}{y^2} \cdot (2z - 2) - 2z + 1 = 0$$

$$\Rightarrow 2x^2 z - 2x^2 - 2y^2 z + y^2 = 0$$

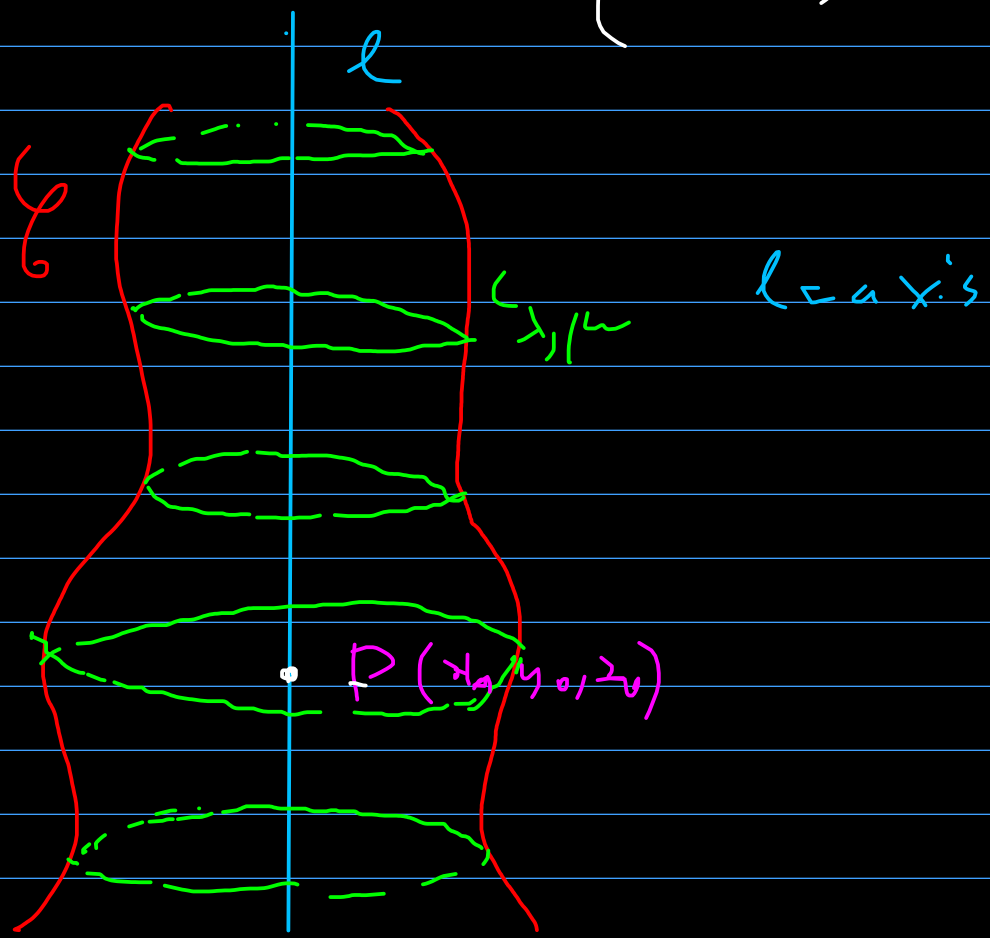
Revolution surfaces

Curve $\mathcal{C}$ (director curve).

axis ℓ, $\vec{\ell}(a, b, c)$

a revolution surface will be built using generating circles

Step 1: $C_{\lambda,\mu}$: $\begin{cases} (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = \lambda \\ ax + by + cz = \mu \end{cases}$



11.5.

We will apply all this to the following particular case

$$\mathcal{C} : \begin{cases} x^2 + y^2 = 1 \\ z = 0 \end{cases}$$

$$\ell = Oy$$