LECTURE 9 - DYNAMICAL SYSTEMS

Monday, 19 April 2021 09:57

→ Planar dynamical systems (cont)

The transformation of a planar system in polar coordinates

We consider a planar system:

$$(1)\ \dot{X}=f(X) \Leftrightarrow \begin{cases} \dot{x}=f_1(x,y) \\ \dot{y}=f_2(x,y) \end{cases} \quad \text{, where } f=\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\in C^1(\mathbb{R}^2,\mathbb{R}^2),\ X=\begin{pmatrix} x \\ y \end{pmatrix}$$

To transform (1) in polar coord. means to consider new unknowns $(\rho(t),\theta(t))$ related to the old unknowns by $x(t)=\rho(t)\cdot\cos\theta(t)$ and find a system in (ρ,θ) only, without x and y.
$y(t)=\rho(t)\cdot\sin\theta(t)$

Practically, first we write $\begin{cases} \rho^2=x^2+y^2 \\ \tan\theta=\frac{y}{x} \end{cases}$. Then we derivate w.r.t. "t": $\begin{cases} \rho\dot{\rho}=x\dot{x}+y\dot{y} \\ \frac{\dot{\theta}}{\cos^2\theta}=\frac{\dot{y}\cdot x-y\cdot\dot{x}}{x^2} \end{cases}$

→ We replace $\dot{x}=f_1(x,y)$
$\dot{y}=f_2(x,y)$ ⇒ We get: $\rho\dot{\rho}=\rho\cos\theta f_1(\rho\cos\theta,\rho\sin\theta)+\rho\sin\theta f_2(\rho\cos\theta,\rho\sin\theta)$

→ We replace $\begin{cases} x=\rho\cdot\cos\theta \\ y=\rho\sin\theta \end{cases}$ $\quad \frac{\dot{\theta}}{\cos^2\theta}=\frac{1}{\rho^2\cos^2\theta}\cdot(\rho\cos\theta f_2(\rho\cos\theta,\rho\sin\theta)-\rho\sin\theta f_1(\rho\cos\theta,\rho\sin\theta))$

Example: $\begin{cases} \dot{x}=x-y \\ \dot{y}=x+y \end{cases}$

(i) Find the flow HW $e^t\cos t$ $e^t\sin t$

(ii) Check (using the def) that the eq. point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a global repeller. HW $\begin{cases} \lim_{t\to-\infty} e^t\cos t=0 \\ \lim_{t\to-\infty} e^t\sin t=0 \end{cases}$

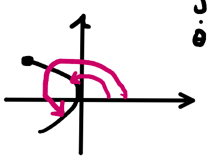
(iii) Transform to polar coordinates.

(iv) Find the shape of the orbit and represent the phase portrait.

(iii) $\begin{cases} \rho^2=x^2+y^2 \\ \tan\theta=\frac{y}{x} \end{cases} \Rightarrow \begin{cases} \rho\dot{\rho}=x\dot{x}+y\dot{y} \\ \frac{\dot{\theta}}{\cos^2\theta}=\frac{\dot{y}x-y\dot{x}}{x^2} \end{cases} \Rightarrow \begin{cases} \rho\dot{\rho}=x(x-y)+y(x+y) \\ \frac{\dot{\theta}}{\cos^2\theta}=\frac{x(x+y)-y(x-y)}{\rho^2\cos^2\theta} \end{cases} \Rightarrow$

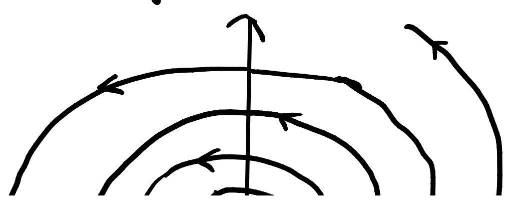
$\Rightarrow \begin{cases} \rho\dot{\rho}=x^2+y^2 \\ \dot{\theta}=\frac{x^2+y^2}{\rho^2} \end{cases} \Rightarrow \begin{cases} \dot{\rho}=\rho \\ \dot{\theta}=1 \end{cases} \Rightarrow \begin{cases} \rho=\rho_0 e^t \\ \theta=t+\theta_0 \end{cases}$, $\rho_0>0$, $\theta_0\in[0,2\pi)$, arbitrary

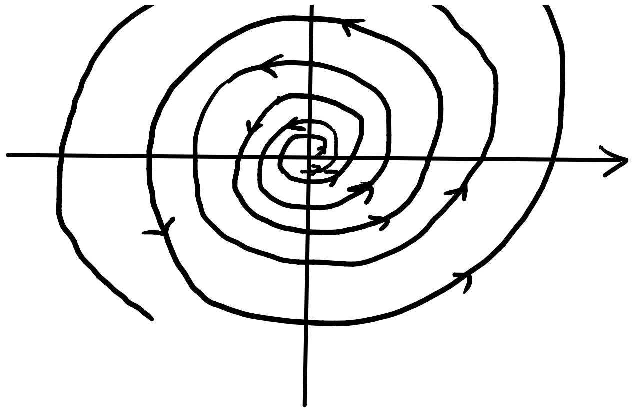
(iv) Note that $\dot{\rho}>0 \Leftrightarrow \rho\nearrow$, this means that we go further from the origin
$\dot{\theta}>0 \Leftrightarrow \theta\nearrow$, this means that it rotates counterclockwise.



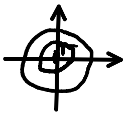
Remark: $\dot{\rho}<0 \Leftrightarrow \rho\searrow$ approaches the origin when moving along the orbit
$\dot{\theta}<0 \Leftrightarrow \theta\searrow$ clockwise rotation
$\rho=\rho_0 \Leftrightarrow$ the orbit is a circle of radius ρ_0
$\theta(t)=\theta_0 \Leftrightarrow$ the angle is constant, the orbit is a straight line through $(0,0)$ of slope $\tan\theta_0$.

For our system $\dot{\rho}>0$, $\dot{\theta}>0$





Remark: The formulation "study the stability of an equilibrium point" means to decide whether the eq. point is: attractor, repeller, stable or unstable.

Def:  Let η^* be an eq. of (1). We say that it is stable if $\forall \varepsilon>0, \exists \delta>0$ s.t whenever $\|\eta-\eta^*\|<\delta$ we have $\|\varphi(t,\eta)-\eta^*\|<\varepsilon, \forall t\in[0,\infty)$.

An eq. point is unstable when it is not stable.

Type and stability of linear planar systems

(2) $\dot{x}=Ax$, where $A\in \mathcal{M}_2(\mathbb{R}), \det A\neq 0$. Denote by $\lambda_1,\lambda_2\in\mathbb{C}$ the eigenvalues of A.

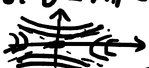
We know that $\det A=\lambda_1\cdot\lambda_2$.

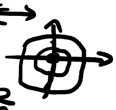
Remark: $\det A\neq 0 \Rightarrow$ the only eq. point of (2) is $0_2\in\mathbb{R}^2$

$\det A\neq 0 \Leftrightarrow \lambda_1\neq 0$ and $\lambda_2\neq 0$.

Def: We say that the eq. point $0_2\in\mathbb{R}^2$ of (2) is a:

NODE when $\lambda_1,\lambda_2\in\mathbb{R}$ and $\lambda_1\le\lambda_2<0$ or $0<\lambda_1\le\lambda_2$ 

SADLE when $\lambda_1,\lambda_2\in\mathbb{R}$ and $\lambda_1<0<\lambda_2$ 

CENTER when $\lambda_1,\lambda_2=\pm i\beta$, where $\beta\in\mathbb{R}, \beta\neq 0$ 

FOCUS when $\lambda_1,\lambda_2=\alpha\pm i\beta, \alpha\neq 0, \beta\neq 0, \alpha,\beta\in\mathbb{R}$ 

Th.1: i) If $Re(\lambda_1)<0$ and $Re(\lambda_2)<0$ then the eq. point $0_2\in\mathbb{R}^2$ of (2) is a global attractor.

ii) If $Re(\lambda_1)>0$ and $Re(\lambda_2)>0$ then the eq point $0_2\in\mathbb{R}^2$ of (2) is a global repeller.

iii) Any center is stable.

iv) Any sadle is unstable.

The linearization method to study the stability of the eq. point of nonlinear systems

(1) $\dot{x}=f(x)$, $f\in C^1(\mathbb{R}^2,\mathbb{R}^2)$

Recall that $\eta^*\in\mathbb{R}^2$ is an eq. of (1) $\Leftrightarrow f(\eta^*)=0$

We consider the linear system (3) $\dot{x}=Jf(\eta^*)x$ the linearization of (1) around the eq η^*.

For a function $f=\begin{pmatrix} f_1\\ f_2\end{pmatrix}:\mathbb{R}^2\to\mathbb{R}^2$ the Jacobian matrix of f is

$$Jf(x,y)=\begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y)\\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y)\end{pmatrix}$$

Def: We say that the eq. point η^* is hyperbolic if $Re(\lambda_1)\neq 0$ and $Re(\lambda_2)\neq 0$, where $\lambda_1,\lambda_2\in\mathbb{C}$ are the eigenvalues

$\partial x \qquad \partial y$

Def: We say that the eq. point η^* is hyperbolic if $Re(\lambda_1) \neq 0$ and $Re(\lambda_2) \neq 0$, where $\lambda_1, \lambda_2 \in \mathbb{C}$ are the eigenvalues of $Jf(\eta^*)$.

Theorem: Let η^* be a hyperbolic eq. of $\dot{x} = f(x)$. If the linear system $\dot{X} = Jf(\eta^*)X$ is an attractor/repeller then the eq η^* of $\dot{x} = f(x)$ is also an attractor/repeller. If the linear system $\dot{X} = Jf(\eta^*)X$ is a sadle then the eq. η^* of $\dot{x} = f(x)$ is unstable.

Important remark: In order to study the stability of the eq. points of a second order nonlinear scalar DE $\ddot{x} = f(x, \dot{x})$ we transform it to a planar system with unknowns x and $y = \dot{x}$. So, the planar system is

$$\begin{cases} \dot{x} = y \\ \dot{y} = f(x, y) \end{cases}.$$

First integrals of planar systems

Prop: (i) Let η^* be an eq. point $\dot{x} = f(x)$. If η^* is an attractor/repeller then there is no first integral in a neighborhood of η^*. In particular, if $\eta^* = 0_2 \in \mathbb{R}^2$ is an attractor/repeller of $\dot{X} = AX$ then this system does not have a first global integral. (ii)

Pf (i) Let η^* be an attractor of $\dot{X} = f(x)$.

$\xrightarrow{\text{Def}}$ $\exists \rho > 0$ s.t. whenever $\|\eta - \eta^*\| < \rho$ we have $\lim_{t \to \infty} \|\varphi(t, \eta) - \eta^*\| = 0$ ($\Leftrightarrow \lim_{t \to \infty} \varphi(t, \eta) = \eta^*$)

Assume, by contradiction that $\exists V \subset \mathbb{R}^2$ a neighborhood of η^* and $H : V \to \mathbb{R}$ a first integral of $\dot{x} = f(x)$.

Then, by def, $H(\varphi(t, \eta)) = H(\eta), \forall t$ s.t $\varphi(t, \eta) \in V$.

So, take ρ small enough s.t $B_\rho(\eta^*) \subset V$, we have $\lim_{t \to \infty} H(\varphi(t, \eta)) = H(\eta), \forall \eta$ s.t $\|\eta - \eta^*\| < \rho$

$\xrightarrow{H \text{ is cont.}}$ $H(\lim_{t \to \infty} \varphi(t, \eta)) = H(\eta), \forall \eta$ s.t $\|\eta - \eta^*\| < \rho \Rightarrow H(\eta^*) = H(\eta), \forall \eta \in B_\rho(\eta^*) \Rightarrow H$ is const. in $B_\rho(\eta^*)$

contrad. with the def. of first integral (it must not be locally constant)

(ii) η^* is an attractor of $\dot{X} = AX \Rightarrow \eta^*$ is a global attractor $\Rightarrow \lim_{t \to \infty} \varphi(t, \eta) = \eta^*, \forall \eta \in \mathbb{R}^2$. Assume, by contrad, that $H : \mathbb{R}^2 \to \mathbb{R}$ is a global f I $\Rightarrow H(\varphi(t, \eta)) = H(\eta), \forall t \in \mathbb{R}, \forall \eta \in \mathbb{R}^2 \Rightarrow \lim_{t \to \infty} H(\varphi(t, \eta)) = H(\eta), \forall \eta \in \mathbb{R}^2 \Rightarrow$

$\xrightarrow{H \text{ cont}}$ $H(\eta^*) = H(\eta)), \forall \eta \in \mathbb{R}^2$; i.e H is const on $\mathbb{R}^2$, contrad.

Prop (How to check that H is a first integral without having the flow)

Let $U \subset \mathbb{R}^2$ be open, connected, nonempty and $H \in C^1(U)$ be a non-locally const. function.

We have that H is a first integral in U of $\begin{cases} \dot{x} = f_1(x, y) \\ \dot{y} = f_2(x, y) \end{cases} \Leftrightarrow \frac{\partial H}{\partial x}(x, y) \cdot f_1(x, y) + \frac{\partial H}{\partial y}(x, y) f_2(x, y) = 0, \forall (x, y) \in U$

Pf: $H(\varphi(t, \eta)) = H(\eta), \forall t, \forall \eta \in U$ s.t $\varphi(t, \eta) \in U$

$\Leftrightarrow \frac{d}{dt} H(\varphi(t, \eta)) = 0$ — " — , $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$

$\Leftrightarrow \frac{\partial H}{\partial x}(\varphi(t, \eta)) \cdot \dot{\varphi}_1(t, \eta) + \frac{\partial H}{\partial y}(\varphi(t, \eta)) \dot{\varphi}_2(t, \eta) = 0$ — " —

$\Leftrightarrow \frac{\partial H}{\partial x}(\underbrace{\varphi(t, \eta)}_{\text{arb.}}) \cdot f_1(\varphi(t, \eta)) + \frac{\partial H}{\partial y}(\varphi(t, \eta)) \cdot f_2(\varphi(t, \eta)) = 0$ — " —

$\Leftrightarrow \frac{\partial H}{\partial x}(x, y) \cdot f_1(x, y) + \frac{\partial H}{\partial y}(x, y) \cdot f_2(x, y) = 0, \forall (x, y) \in V$

A method to find a first integral of some system $\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$

Step 1: Write $\frac{dy}{dx} = \frac{f_2(x,y)}{f_1(x,y)}$

Step 2: Integrate the above DE and write its general sol as

$$H(x,y) = c, \ c \in \mathbb{R}$$

Step 3: Find a domain U for the function H and check that this is a first integral of (1)

A method to integrate the DE found at Step 1, in case that it is a separable DE, i.e. if it has the form:

$$\frac{dy}{dx} = g_1(x) \cdot g_2(y)$$

$\frac{dy}{g_2(y)} = g_1(x)dx \Rightarrow \int \frac{dy}{g_2(y)} = \int g_1(x)\,dx \qquad G_2(y) = G_1(x) + c, \ c \in \mathbb{R}$

Now, if it is possible, simplify the previous relation, if not $H(x,y) = G_2(y) - G_1(x)$.