LECTURE 11 - DYNAMICAL SYSTEMS

Monday, 10 May 2021 09:56

I. The direction field associated to a differential eq. II. Numerical methods

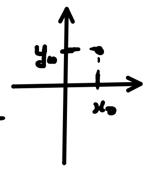
(1) $y'(x) = g(x, y(x)),\ g \in C^1(\mathbb{R}^2)$ — scalar differential eq.

(2) $\begin{cases} \dot{x}(t) = f_1(x(t), y(t)) \\ \dot{y}(t) = f_2(x(t), y(t)) \end{cases}$ $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ → planar autonomous differential system

Remark: We have existence and uniqueness of the solution for any IVP.

(1) Any solution curve (graph of a given solution) is represented in $\mathbb{R}^2$.

(2) Any orbit is represented in $\mathbb{R}^2$.



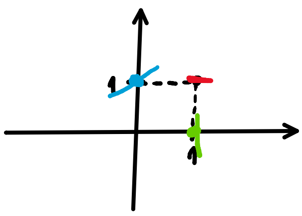
Def.: The direction field is a collection of vectors in $\mathbb{R}^2$. The vector through a given point $(x_0, y_0) \in \mathbb{R}^2$

→ has the slope $m = g(x_0, y_0)$ in the case of eq (1)

→ is parallel to the vector $(f_1(x_0,y_0), f_2(x_0,y_0))$ / has the slope $m = \dfrac{f_2(x_0,y_0)}{f_1(x_0,y_0)}$

Ex: $y' = 1 - \dfrac{x}{y^2}$ (3). Plot the vectors of the direction fields corresponding to the points $(1,1), (0,1), (1,0)$.

$g(x,y) = 1 - \dfrac{x}{y^2}$, $m_1 = g(1,1) = 0$ $m_2 = g(0,1) = 1$ $m_3 = g(1,0) = \infty$



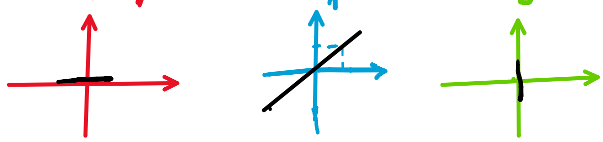
$m_1 = 0 \Rightarrow$ horizontal $m_2 = 1 \Rightarrow$ parallel to $y = x$

$m_3 = \infty \Rightarrow$ vertical

(4) $\begin{cases} \dot{x} = y^2 \\ \dot{y} = -x + y^2 \end{cases}$ Plot the vectors of the dir. fields corresp to $(1,1), (0,1), (1,0)$.

$f(x,y) = \begin{pmatrix} y^2 \\ -x+y^2 \end{pmatrix}$ $f(1,1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $f(0,1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $f(1,0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$m_1 = \dfrac{0}{1} = 0$ $m_2 = \dfrac{1}{1} = 1$ $m_3 = \dfrac{-1}{0}$



→ the slope of the direction field is $m(x,y) = \dfrac{-x+y^2}{y^2} = 1 - \dfrac{x}{y^2} = g(x,y)$. (3) and (4) have the same direction field

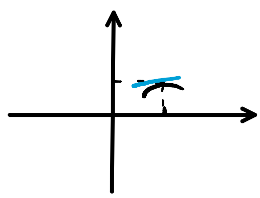
Prop: Let $(x_0, y_0) \in \mathbb{R}^2$ be fixed, arbitrary. The slope of the direction field of (1) / system (2) is equal to the slope of the solution curve of (1) / orbit of (2), that passes through (x_0, y_0) (in other words, the direction field is tangent to the solution curve of (1) / orbit of (2) in any point).

Pf.: I. $y' = g(x,y)$ The sol. curve that passes through (x_0, y_0) is denoted by $\varphi(x)$. Then

$$\begin{cases} \varphi'(x) = g(x, \varphi(x)),\ \forall x \in V(x_0) \\ \varphi(x_0) = y_0 \end{cases} \quad (5)$$

→ the slope of the d.f. is $m_0 = g(x_0, y_0)$ (def.)

→ the slope of φ in (x_0, y_0) is $\varphi'(x_0)$



$(5) \Rightarrow \varphi'(x_0) = g(x_0, \varphi(x_0)) = g(x_0, y_0)$

II. $\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$ Take $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ be the sol. of the system whose orbit passes through (x_0, y_0) $\Rightarrow$

$$\Rightarrow \begin{cases} \dot{\varphi}_1(t) = f_1(\varphi_1(t), \varphi_2(t)) \\ \dot{\varphi}_2(t) = f_2(\varphi_1(t), \varphi_2(t)) \\ \varphi_1(0) = x_0 \\ \varphi_2(0) = y_0 \end{cases}, \forall t \in V(0)$$

The corresponding orbit is $\underbrace{\{(\varphi_1(t), \varphi_2(t)) : t \in V(0)\}}_{\gamma}$ and the vector $(\dot{\varphi}_1(t), \dot{\varphi}_2(t))$ is tangent to γ in the point $(\varphi_1(t_0), \varphi_2(t_0)), \forall t_0 \in V(0)$

Take $t_0 = 0$: The vector $(\dot{\varphi}_1(0), \dot{\varphi}_2(0))$ is tangent to γ in (x_0, y_0)

But $\dot{\varphi}_1(0) = f_1(x_0, y_0)$
$\dot{\varphi}_2(0) = f_2(x_0, y_0)$ $\Rightarrow$ The vector $\underbrace{(f_1(x_0,y_0), f_2(x_0,y_0))}_{\text{by def. this gives the direction field}}$ is tangent to the orbit

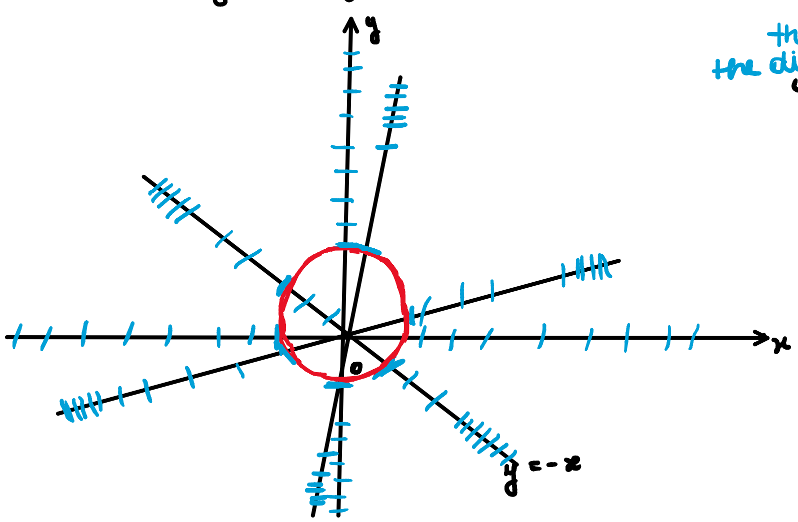
Ex: $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ $\quad \frac{dy}{dx} = -\frac{x}{y}$ (same dir. field)

Rep. the dir field and "guess" the shape of the orbits of the system.

Def: The m-isocline of eq. (1)/system (2) is $\{(x,y) \in \mathbb{R}^2 : g(x,y) = m\}$ / $\{(x,y) \in \mathbb{R}^2 : \frac{f_2(x,y)}{f_1(x,y)} = m\}$.
(The slope of the dir. field is the same for any point in an isocline).

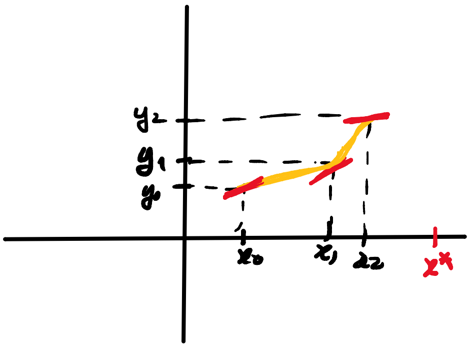
The 1-isocline: $-\frac{x}{y} = 1 \Leftrightarrow y = -x$

m-isocline: $-\frac{x}{y} = m \Leftrightarrow$ the slope of the dir. field in any point of the line
$\Leftrightarrow y = \underbrace{-\frac{1}{m}}_{\text{the slope of the line}} x$ (line)



An orbit is orthogonal to any line that passes through the origin. So, an orbit must be a circle centered in the origin.

Introduction to numerical methods



$y'(x) = g(x, y(x))$

$m_0 = g(x_0, y_0)$

$m_1 = g(x_1, y_1)$

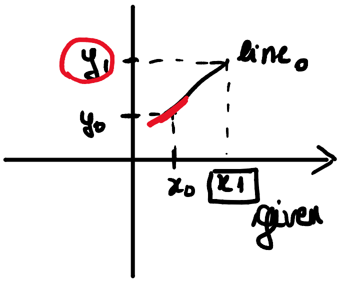
$m_2 = g(x_2, y_2)$

$\boxed{x_1}$ given

$y_1 = ?$

(x_0, y_0) given $\quad m_0 = f(x_0, y_0)$

$\downarrow \dfrac{y_1 - y_0}{x_1 - x_0}$



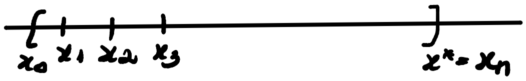
$\Rightarrow \dfrac{y_1 - y_0}{x_1 - x_0} = g(x_0, y_0) \Rightarrow$

$\Rightarrow y_1 = y_0 + g(x_0, y_0)(x_1 - x_0)$

Euler's numerical procedure

$\begin{cases} y' = g(x, y) \\ y(x_0) = y_0 \end{cases}$ Denote by $\varphi : [x_0, x^*] \to \mathbb{R}$ its unique solution

Fix $n \geq 1$, $n \in \mathbb{N}$
the number of steps



Fix $\{x_1, x_2, \dots, x_n = x^*\}$ in the interval $[x_0, x^*]$

These points $x_0 < x_1 < x_2 < \dots < x_n = x^*$ form a partition of the interval $[x_0, x^*]$.

Our aim is to find $y_1, y_2, \dots, y_n \in \mathbb{R}$ s.t. $y_i \approx \varphi(x_i)$ ("good" approx. of the exact value of the exact sol).

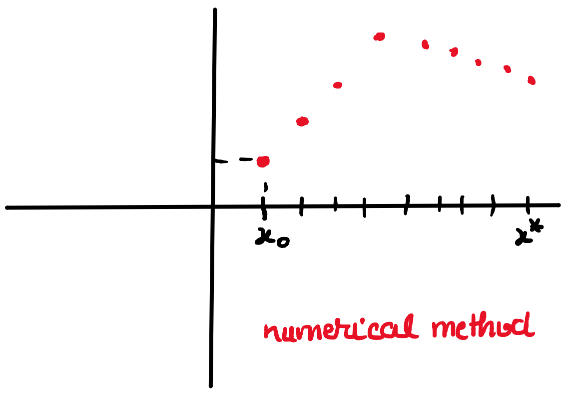
The basic formula is:

$y_{k+1} = y_k + (x_{k+1} - x_k)\, g(x_k, y_k)$, $k = \overline{0, n-1}$

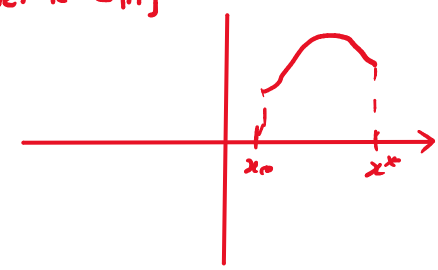
Usually we take $x_{k+1} - x_k = h$, $k = \overline{0, n-1}$, h - a constant step-size

Euler's formula with constant step-size $h > 0$:

$y_{k+1} = y_k + h\, g(x_k, y_k)$, $k = \overline{0, n-1}$



$\{(x_k, y_k) : k = \overline{0, n}\}$



numerical method

interpolation method