

12 Week 12. Transformations

12.1 Transformations of the plane

Definition 12.1. An *affine transformation* of the plane is a perturbation by a translation of a linear transformation, i.e.

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, L(x, y) = (ax + by + c, dx + ey + f), \quad (12.1)$$

for some constant real numbers a, b, c, d, e, f .

By using the matrix language, the action of the map L can be written in the form

$$(x, y) \in \mathbb{R}^2, [x y] \in \mathbb{R}^{1 \times 2} \quad L(x, y) = [x y] \begin{bmatrix} a & d \\ b & e \end{bmatrix} + [c f].$$

The affine transformation L can be also identified with the map $L^c : \mathbb{R}^{2 \times 1} \longrightarrow \mathbb{R}^{2 \times 1}$ given by

$$(x, y) \in \mathbb{R}^2, [x y] \in \mathbb{R}^{2 \times 1} \quad L^c \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} ax + by + c \\ dx + ey + f \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

$$= [L] \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}, \text{ where } [L] = \begin{bmatrix} a & b \\ d & e \end{bmatrix}.$$

Lemma 12.1. If $(aB - bA)^2 + (dB - eA)^2 > 0$, then the affine transformation (14.1) maps the line $\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix} = 0$ to the line $\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix} = 0$

$$aB - bA \neq 0 \text{ or } dB - eA \neq 0$$

$$(d) Ax + By + C = 0 \quad A \neq 0 \text{ or } B \neq 0$$

to the line

$$(eA - dB)x + (aB - bA)y + (bf - ce)A - (af - cd)B + (ae - bd)C = 0.$$

If $aB - bA = dB - eA = 0$, then $ae - bd = 0$ and $L|_d$ is the constant map $\left(\frac{cB - bC}{B}, \frac{fB - eC}{B}\right)$.

Definition 12.2. An affine transformation (14.1) is said to be *singular* if

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix} = 0 \text{ i.e. } ae - bd = 0.$$

and non-singular otherwise.

12.1.1 Translations

Note that the affine transformation L is nonsingular if and only if it is invertible. In such a case the inverse L^{-1} is a non-singular affine transformation and $[L^{-1}] = [L]^{-1}$.

Definition 12.3. The *translation* of vector $(h, k) \in \mathbb{R}^2$ is the affine transformation

$$T(h, k) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, [T(h, k)](x, y) = (x + h, y + k). \quad \begin{cases} x' = x + h \\ y' = y + k \end{cases}$$

Thus

$$[T(h, k)^c] \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x + h \\ y + k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix},$$

i.e.

$$[T(h, k)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad \begin{aligned} T(h, k) \circ T(l, m) &= T(h+l, k+m) \\ T(0, 0) &= \text{id}_{\mathbb{R}^2} \end{aligned}$$

Note that the translation $T(h, k)$ is non-singular (invertible) and $(T(h, k))^{-1} = T(-h, -k)$.

12.1.2 Scaling about the origin

Definition 12.4. The *scaling about the origin* by non-zero scaling factors $(s_x, s_y) \in \mathbb{R}^2$ is the affine transformation

$$S(s_x, s_y) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, [S(s_x, s_y)](x, y) = (s_x \cdot x, s_y \cdot y).$$

Thus

$$[S(s_x, s_y)^c] \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} s_x \cdot x \\ s_y \cdot y \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

i.e. $S(s_x, s_y) \circ S(s'_x, s'_y) = S(s_x s'_x, s_y s'_y)$
 $S(1, 1) = \text{id}_{\mathbb{R}^2}$

$$[S(s_x, s_y)] = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}. \quad \begin{cases} x' = s_x \cdot x \\ y' = s_y \cdot y \end{cases}$$

Note that the scaling about the origin by non-zero scaling factors $(s_x, s_y) \in \mathbb{R}^2$ is non-singular (invertible) and $(S(s_x, s_y))^{-1} = S(s_x^{-1}, s_y^{-1})$.

12.1.3 Reflections

Definition 12.5. The *reflections about the x-axis* and the *y-axis* respectively are the affine transformation

$$r_x, r_y : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, r_x(x, y) = (x, -y), r_y = (-x, y).$$

Thus

$r_x \circ r_y = \text{id}_{\mathbb{R}^2}$
 $r_y \circ r_x = \text{id}_{\mathbb{R}^2}$

$$[r_x^c] \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

i.e.

$$[r_x] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \text{ Similarly } [r_y] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$r_x: \begin{cases} x' = x \\ y' = -y \end{cases}$
 $r_y: \begin{cases} x' = -x \\ y' = y \end{cases}$

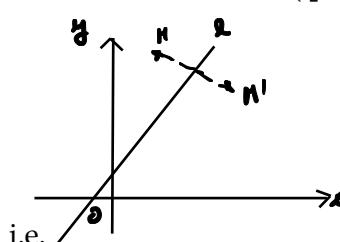
Note that $r_x = S(-1, 1)$ and $r_y = S(1, -1)$. Thus the two reflections are non-singular (invertible) and $r_x^{-1} = r_x, r_y^{-1} = r_y$.

Definition 12.6. The *reflection $r_l : \mathbb{R}^2 \rightarrow \mathbb{R}^2$* about the line l maps a given point M to the point M' defined by the property that l is the perpendicular bisector of the segment MM' . One can show that the action of the reflection about the line $l : ax + by + c = 0$ is

$$r_l(x, y) = \left(\frac{b^2 - a^2}{a^2 + b^2} x - \frac{2ab}{a^2 + b^2} y - \frac{2ac}{a^2 + b^2}, -\frac{2ab}{a^2 + b^2} x + \frac{a^2 - b^2}{a^2 + b^2} y - \frac{2bc}{a^2 + b^2} \right).$$

Thus

$$\begin{aligned} [r_l^c] \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) &= \begin{bmatrix} \frac{b^2 - a^2}{a^2 + b^2} x - \frac{2ab}{a^2 + b^2} y - \frac{2ac}{a^2 + b^2} \\ -\frac{2ab}{a^2 + b^2} x + \frac{a^2 - b^2}{a^2 + b^2} y - \frac{2bc}{a^2 + b^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{b^2 - a^2}{a^2 + b^2} & -\frac{2ab}{a^2 + b^2} \\ -\frac{2ab}{a^2 + b^2} & \frac{a^2 - b^2}{a^2 + b^2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{2ac}{a^2 + b^2} \\ \frac{2bc}{a^2 + b^2} \end{bmatrix}, \end{aligned}$$



i.e.

$$[r_l] = \frac{1}{a^2 + b^2} \begin{bmatrix} b^2 - a^2 & -2ab \\ -2ab & a^2 - b^2 \end{bmatrix}.$$

Note that the reflection r_l is non-singular (invertible) and $r_l^{-1} = r_l$.

12.1.4 Rotations

Definition 12.7. The rotation $R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ about the origin through an angle θ maps a point $M(x, y)$ into a point $M'(x', y')$ with the properties that the segments $[OM]$ and $[OM']$ are congruent and the $m(\widehat{MOM'}) = \theta$. If $\theta > 0$ the rotation is supposed to be *anticlockwise* and for $\theta < 0$ the rotation is *clockwise*. If $(x, y) = (r \cos \varphi, r \sin \varphi)$, then the coordinates of the rotated point are $(r \cos(\theta + \varphi), r \sin(\theta + \varphi)) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$, i.e.

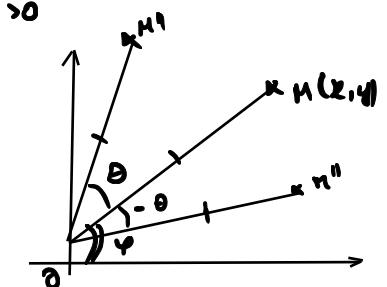
$$R_\theta(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta). \quad \theta > 0$$

Thus

$$\begin{aligned} [R_\theta^c] \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) &= \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \end{aligned}$$

i.e.

$$[R_\theta] = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$



$$\begin{aligned} \left\{ \begin{array}{l} x = r \cos \varphi \quad x = \|OM\| \\ y = r \sin \varphi \end{array} \right. \\ \left\{ \begin{array}{l} x' = r \cos(\theta + \varphi) \\ y' = r \sin(\theta + \varphi) \end{array} \right. \\ \therefore \left\{ \begin{array}{l} x' = r \cos \theta \cos \varphi - r \sin \theta \sin \varphi \\ y' = r \sin \theta \cos \varphi + r \cos \theta \sin \varphi \end{array} \right. \end{aligned}$$

12.1.5 Shears

Definition 12.8. Given a fixed direction in the plane specified by a unit vector $v = (v_1, v_2)$, consider the lines Δ with direction v and the oriented distance d from the origin. The *shear* about the origin of factor r in the direction v is defined to be the transformation which maps a point $M(x, y)$ on Δ to the point $M' = M + rdv$. The equation of the line through M of direction v is $v_2X - v_1Y + (v_1y - v_2x) = 0$. The oriented distance from the origin to this line is $v_1y - v_2x$. Thus the action of the shear $Sh(v, r) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ about the origin of factor r in the direction v is

$$\begin{aligned} Sh(v, r)(x, y) &= (x, y) + rd(v_1, v_2) \\ &= (x, y) + (r(v_1y - v_2x)v_1, r(v_1y - v_2x)v_2) \\ &= (x, y) + (-rv_1v_2x + rv_1^2y, -rv_2^2x + rv_1v_2y) \\ &= ((1 - rv_1v_2)x + rv_1^2y, -rv_2^2x + (1 + rv_1v_2)y) \end{aligned}$$

Thus

$$\begin{aligned} [Sh(v, r)^c] \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) &= \begin{bmatrix} (1 - rv_1v_2)x + rv_1^2y \\ -rv_2^2x + (1 + rv_1v_2)y \end{bmatrix} \\ &= \begin{bmatrix} 1 - rv_1v_2 & rv_1^2 \\ -rv_2^2 & 1 + rv_1v_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \end{aligned}$$

$$\text{i.e. } [Sh(v, r)] = \begin{bmatrix} 1 - rv_1v_2 & rv_1^2 \\ -rv_2^2 & 1 + rv_1v_2 \end{bmatrix}. \quad v_1y - v_2x$$

the oriented
distance from origin to Δ

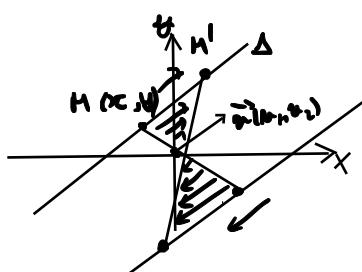
$$\cdot |v_1y - v_2x| = ||v_1 \quad v_2||$$

$$\Delta: \frac{x - x_0}{v_1} = \frac{y - y_0}{v_2}$$

$$\Delta: v_2X - v_1Y + v_1y - v_2x = 0$$

$$= \frac{|v_1y - v_2x|}{\sqrt{v_1^2 + v_2^2}},$$

$$M(x, y) \mapsto M' = M + rdv$$



12.2 Problems

1. Find the image of the triangle ABC through the reflection in the line (d) $x - y = 2$, where $A(-1, 2)$, $B(-2, -1)$ and $C(3, 3)$.

Solution.

2. Find the image of the triangle ABC through the clockwise rotation of angle 30° , where $A(6, 4)$, $B(6, 2)$ and $C(10, 6)$.

Solution.

3. Consider a quadrilateral with vertices $A(1,1)$, $B(3,1)$, $C(2,2)$, and $D(1.5,3)$. Find the image quadrilaterals through the translation $T(1,2)$, the scaling $S(2,2.5)$, the reflections about the x and y -axes, the clockwise and anticlockwise rotations through the angle $\pi/2$ and the shear $Sh\left(\left(2/\sqrt{5}, 1/\sqrt{5}\right), 1.5\right)$.

Solution.

4. Let $M(x, y)$ be a mobile point on the ellipse $(\mathcal{E}) \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that the locus of centroids of the triangles MFF' , where F and F' are the foci of the ellipse, is the image through a scaling of equal factors (a homothety) of the given ellipse \mathcal{E} . Find the equation of the locus..

Solution.

5. Consider the line (d) $ax + by + c = 0$ and the points $A, B \notin d$. Find the coordinates of the point $M \in d$ such that $\text{dist}(A, B) + \text{dist}(M, B)$ is minimal.

Solution.