Exam-secondary date

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Ex.1) Study the solvability of the BVP: $x''+9x=2, x(0)=x(3\pi)=0$. Here, as usual, the unknown is the function $t\in\mathbb{R}\mapsto x(t)\in\mathbb{R}$.

$x''+9x=2 \Rightarrow x_p=\frac{2}{9}$

We need a general solution of the LHDE: $x''+9x=0 \Rightarrow$

$\Rightarrow$ The char. eq: $\lambda^2+9=0 \Rightarrow \lambda^2=-9 \Rightarrow \lambda_{1,2}=\pm 3i \mapsto e^0\cos 3t, e^0\sin 3t \rightsquigarrow \cos 3t, \sin 3t \Rightarrow$

$\alpha=0, \beta=3$

$\Rightarrow x_h=c_1\cdot\cos 3t+c_2\cdot\sin 3t$

$x_p=\frac{2}{9}$

$\Rightarrow x=c_1\cdot\cos 3t+c_2\cdot\sin 3t+\frac{2}{9}, \forall t\in\mathbb{R}$

$$\begin{cases} x(0)=0 \\ x(3\pi)=0\end{cases} \Leftrightarrow \begin{cases} c_1\cdot\overbrace{\cos 0}^{1}+c_2\cdot\overbrace{\sin 0}^{0}+\frac{2}{9}=0 \\ c_1\cdot\underbrace{\underbrace{\cos 9\pi}_{\cos\pi}}_{-1}+c_2\cdot\underbrace{\underbrace{\sin 9\pi}_{\sin\pi}}_{0}+\frac{2}{9}=0\end{cases} \Leftrightarrow \begin{cases} c_1=-\frac{2}{9} \\ c_1=\frac{2}{9}\end{cases}$$

, so the system has no solution

Ex.2) Study the solvability of the following problem: $x_{k+2}=e^{x_{k+1}}-5x_k+7, k\geq 0, x_0=0, x_1=0, x_2=8$. Here, as usual, the unknown is the seq. $k\in\mathbb{N}\mapsto x_k\in\mathbb{R}$

$x_{k+2}=e^{x_{k+1}}-5x_k+7 \Rightarrow x_{k+2}-e^{x_{k+1}}+5x_k=7$

For $k=0 \Rightarrow x_2-e^{x_1}+5x_0=7 \Rightarrow 8-e^0+5\cdot 0=7 \Rightarrow 7=7$, "true" $\Rightarrow$ The problem is solvable.

Ex.3) Find the general solution of: $\begin{cases} x'=-7x \\ y'=x+7y\end{cases}$, using the characteristic eq. method for systems.

$A=\begin{pmatrix} -7 & 0 \\ 1 & 7\end{pmatrix} \Rightarrow \det(A-\lambda I_2)=0 \Leftrightarrow \begin{vmatrix} -7-\lambda & 0 \\ 1 & 7-\lambda\end{vmatrix}=0 \Leftrightarrow -(7+\lambda)(7-\lambda)=0 \Rightarrow \lambda^2=49 \Rightarrow \lambda_1=-7$
$\lambda_2=7$

$u_1=?$ an eigenvector corresp. to $\lambda_1=-7 \Leftrightarrow u_1\in\mathbb{R}^2, u_1\neq\begin{pmatrix}0\\0\end{pmatrix}, Au_1=-7u_1, u_1=\begin{pmatrix}a\\b\end{pmatrix}$

$Au_1=-7u_1 \Leftrightarrow \begin{pmatrix} -7 & 0 \\ 1 & 7\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}-7a\\-7b\end{pmatrix} \Leftrightarrow \begin{cases} -7a=-7a \\ a+7b=-7b\end{cases} \Leftrightarrow \begin{cases} a\in\mathbb{R} \\ b=-\frac{a}{14}\end{cases}$. Choose $a=14 \Rightarrow b=-1 \Rightarrow u_1=\begin{pmatrix}14\\-1\end{pmatrix}$

$Au_2=7u_2 \Leftrightarrow \begin{pmatrix} -7 & 0 \\ 1 & 7\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}7a\\7b\end{pmatrix} \Leftrightarrow \begin{cases} -7a=7a \\ a+7b=7b\end{cases} \Leftrightarrow \begin{cases} 14a=0 \\ a=0\end{cases} \Rightarrow$ Choose $a=0, b=1 \Rightarrow u_2=\begin{pmatrix}0\\1\end{pmatrix}$

So: - eigenvalues: $\lambda_1=-7, \lambda_2=7$
- eigenvectors: $u_1=\begin{pmatrix}14\\-1\end{pmatrix}, u_2=\begin{pmatrix}0\\1\end{pmatrix}$; $\begin{vmatrix} 14 & 0 \\ -1 & 1\end{vmatrix}=14\neq 0 \Rightarrow u_1, u_2$ - lin. indep.

$\Rightarrow e^{-7t}\begin{pmatrix}14\\-1\end{pmatrix}$ and $e^{7t}\begin{pmatrix}0\\1\end{pmatrix}$ are lin. indep. sol of $A\cdot X=X' \Rightarrow$ gen. sol: $X=c_1e^{-7t}\begin{pmatrix}14\\-1\end{pmatrix}+c_2e^{7t}\begin{pmatrix}0\\1\end{pmatrix}, c_1, c_2\in\mathbb{R} \Rightarrow$

$\Leftrightarrow \begin{cases} x(t)=c_1e^{-7t}\cdot 14 \\ y(t)=c_1e^{-7t}\cdot(-1)+c_2\cdot e^{7t}\end{cases}$

Ex.4) We consider a non-linear planar system: $\begin{cases}\dot{x}=x+1 \\ \dot{y}=-3y-xy\end{cases} \Rightarrow \begin{matrix} f_1(x,y)=x+1 \\ f_2(x,y)=-3y-xy\end{matrix} \Rightarrow Jf(x,y)=\begin{pmatrix} 1 & 0 \\ -y & -x-3\end{pmatrix}$

a) Find its eq. point and the lin. system around it.

$\begin{cases} x+1=0 \\ -3y-xy=0\end{cases} \Leftrightarrow \begin{cases} x=-1 \\ -3y+y=0\end{cases} \Leftrightarrow \begin{cases} x=-1 \\ y=0\end{cases} \Rightarrow \eta=(-1,0) \Rightarrow Jf(-1,0)=\begin{pmatrix} 1 & 0 \\ 0 & -2\end{pmatrix} \Rightarrow \dot{X}=Jf(-1,0)\cdot X^{(?)}$

$\Leftrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} x' = x \\ y' = -2y \end{cases}$ → lin. system

b) Find a first integral.

$$\frac{dy}{dx} = \frac{-3y - xy}{x+1} \Leftrightarrow \frac{dy}{dx} = \frac{-y(3+x)}{x+1} \Leftrightarrow \frac{dy}{dx} = \frac{(3+x)}{x+1} \cdot (-y) \Leftrightarrow \frac{dy}{y} = \frac{-(3+x)}{x+1} \cdot dx \Leftrightarrow$$

$$\Leftrightarrow \int \frac{1}{y} dy = -\int \frac{3+x}{x+1} dx \Leftrightarrow \ln|y| = -\int \left(1 + \frac{2}{x+1}\right) dx \Leftrightarrow \ln|y| = -x - 2\ln|x+1| + C \Leftrightarrow$$

$$\Leftrightarrow H(x,y) = \ln|y| + 2\ln|x+1| - x \Leftrightarrow H: (-1,\infty) \times (0,\infty) \to \mathbb{R}, H(x,y) = \ln y + 2\ln(x+1) + x$$

Check: $\frac{\partial H}{\partial x} f_1(x,y) + \frac{\partial H}{\partial y} f_2(x,y) = 0 \Leftrightarrow \left(\frac{2}{x+1} + 1\right)(x+1) + \frac{1}{y}(-3y - xy) = 0 \Leftrightarrow$

$\Leftrightarrow 2 + (x+1) - 3 - x = 0 \Leftrightarrow \cancel{2 + x + 1 - 3 - x} = 0 \Leftrightarrow 0 = 0$ „true" $\Rightarrow H: (-1,\infty) \times (0,\infty) \to \mathbb{R}, H(x,y) = \ln y + 2\ln(x+1) + x$ is a first integral

c) The lin. system: $\begin{cases} x' = x \\ y' = -2y \end{cases}$

$$\frac{dy}{dx} = \frac{-2y}{x} \Leftrightarrow -\frac{1}{2} \cdot \frac{1}{y} dy = \frac{1}{x} dx \Leftrightarrow -\frac{1}{2} \ln|y| = \ln|x| + c \Leftrightarrow \ln|x| + \frac{1}{2}\ln|y| = c \,/\cdot 2 \Leftrightarrow$$

$$\Leftrightarrow 2\ln|x| + \ln|y| = c \Leftrightarrow \ln(|x^2 y|) = c \Leftrightarrow |x^2 y| = c_2,\ x^2 y = H(x,y), H: \mathbb{R}^2 \to \mathbb{R}$$

$\frac{\partial H}{\partial x} \cdot x + \frac{\partial H}{\partial y} \cdot (-2y) = 2xy \cdot x + x^2 \cdot (-2y) = 2x^2 y - 2x^2 y = 0$, true $\Rightarrow H: \mathbb{R}^2 \to \mathbb{R}, H(x,y) = x^2 y$, global first integral

d) Phase portrait:

$x^2 y = c \Rightarrow y = c \cdot \frac{1}{x^2}$