LECTURE 3 - DYNAMICAL SYSTEMS

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Linear differential equations with constant coefficients

1) An algorithm to find the general solution of a LHDE with CC

(1) $x^{(n)} + a_1 x^{(n-1)} + \ldots + a_n \cdot x = 0$, where $a_1, \ldots, a_n \in \mathbb{R}$

Idea: Look for solutions $x = e^{rt}$.

$x' = r \cdot e^{rt}, x'' = r^2 \cdot e^{rt}, \ldots, x^{(n)} = r^n \cdot e^{rt}$

Replace in (1) $\Rightarrow e^{rt}(r^n + a_1 \cdot r^{n-1} + \ldots + a_{n-1} \cdot r + a_n) = 0, \forall t \in \mathbb{R} \Rightarrow$

(2) $r^n + a_1 \cdot r^{n-1} + \ldots + a_{n-1} r + a_n = 0$ (n-th degree pol. eq with the coeff. as the ΔE (1))

We know that (2) has n roots in $\mathbb{C}$, counting with multiplicities.

The simplest case is when (2) has n simple real roots, denoted by $r_1, \ldots, r_n \in \mathbb{R}$.
We have that $r_m \neq r_i, m \neq i$. Then we have that $e^{r_1 t}, \ldots, e^{r_n t}$ are solutions of the ΔE (1).
We want to prove that $\{e^{r_1 t}, \ldots, e^{r_n t}\}$ are lin. indep. in $C(\mathbb{R})$.

Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ s.t. $\alpha_1 e^{r_1 t} + \ldots + \alpha_n e^{r_n t} = 0, \forall t \in \mathbb{R}$

$$\Rightarrow \begin{cases} \alpha_1 e^{r_1 t} + \ldots + \alpha_n e^{r_n t} = 0, \forall t \in \mathbb{R} \\ \alpha_1 r_1 e^{r_1 t} + \ldots + \alpha_n r_n e^{r_n t} = 0, \forall t \in \mathbb{R} \\ \ldots \\ \alpha_1 r_1^{n-1} e^{r_1 t} + \ldots + \alpha_n r_n^{n-1} e^{r_n t} = 0, \forall t \in \mathbb{R} \end{cases}$$

$$(t=0) \begin{cases} \alpha_1 + \ldots + \alpha_n = 0 \\ \alpha_1 r_1 + \ldots + \alpha_n r_n = 0 \\ \ldots \\ \alpha_1 r_1^{n-1} + \ldots + \alpha_n r_n^{n-1} = 0 \end{cases} \Rightarrow \begin{vmatrix} 1 & \ldots & 1 \\ r_1 & \ldots & r_n \\ \ldots & \ldots & \ldots \\ r_1^{n-1} & \ldots & r_n^{n-1} \end{vmatrix} = \prod_{i<m} (r_m - r_i) \neq 0 \Rightarrow$$

$\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0 \Rightarrow \{e^{r_1 t}, \ldots, e^{r_n t}\}$ are lin indep.

From the Fundamental Th. for LHDE we deduce that the gen. sol. of (1) is (in this case)
$x = c_1 \cdot e^{r_1 t} + \ldots + c_n \cdot e^{r_n t}, c_1, \ldots, c_n \in \mathbb{R}$

<u>The algorithm to find n lin. indep. sol. for eq. (1)</u>

Step 1: We write the characteristic equation

(2) $r^n + a_1 \cdot r^{n-1} + \ldots + a_{n-1} r + a_n = 0$

Step 2: We find the n roots in $\mathbb{C}$ of (2).

Step 3: We associate n functions using the rules:

→ when $r \in \mathbb{R}$ is a root of (2) of multiplicity $m \geq 1$

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$\to$ when $r \in \mathbb{R}$ is a root of (2) of multiplicity $m \geq 1$

$$\mapsto e^{rt}, te^{rt}, t^2e^{rt}, \dots, t^{m-1}e^{rt}$$

$\to$ when $r = \alpha \pm i\beta$ ($\beta \neq 0$) is a root of (2) of multiplicity $m \geq 1$

$$\mapsto e^{\alpha t}\cdot \cos \beta t, e^{\alpha t}\cdot \sin \beta t, t\cdot e^{\alpha t}\cdot \cos \beta t, t\cdot e^{\alpha t}\cdot \sin \beta t, \dots, t^{m-1}\cdot e^{\alpha t}\cdot \cos \beta t, t^{m-1}e^{\alpha t}\sin \beta t$$

Proposition: All the n functions found at Step 3 are lin. indep. sol of eq (1).

Step 4: Write the general sol using the Fundam. Th. for LHDE.

Examples: Find the general sol. of

(1) $x'' - x = 0$

(2) $x'' + 4x = 0$

(3) $x'' + 2x' + x = 0$

(4) $x'' + x' + x = 0$

(1) $r^2 - 1 = 0 \Rightarrow \left\langle \begin{array}{l} r_1 = 1 \mapsto e^t \\ r_2 = -1 \mapsto e^{-t} \end{array} \right. \Rightarrow x = c_1 e^t + c_2 \cdot e^{-t}, c_1, c_2 \in \mathbb{R}$

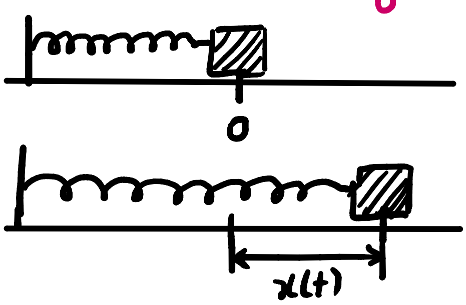
(2) $r^2 + 4 = 0 \Rightarrow r_{1,2} = \pm 2i \mapsto \cos(2t), \sin(2t) \Rightarrow x = c_1 \cos(2t) + c_2 \cdot \sin(2t), c_1, c_2 \in \mathbb{R}$

(3) $r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow \underbrace{r_{1,2} = -1}_{\text{double root}} \mapsto e^{-t}, t\cdot e^{-t} \Rightarrow x = c_1 e^{-t} + c_2 \cdot t \cdot e^{-t}, c_1, c_2 \in \mathbb{R}$

(4) $r^2 + r + 1 = 0 \quad r_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$\mapsto e^{-\frac{1}{2}t}\cdot \cos \frac{\sqrt{3}}{2}t, e^{-\frac{1}{2}t}\sin \frac{\sqrt{3}}{2}t \Rightarrow x = c_1 \cdot e^{-\frac{1}{2}t}\cdot \cos \frac{\sqrt{3}}{2}t + c_2 \cdot e^{-\frac{1}{2}t}\sin \frac{\sqrt{3}}{2}t, c_1, c_2 \in \mathbb{R}$

APPLICATION: the spring-mass system



the equilibrium position (rest position, no movement)

$x(t)$ is the displacement of the mass wrt the eq. position at time t

$x'(t)$ the velocity

$x''(t)$ the acceleration

Newton's second law: $F = m \cdot a$

The force :- the elastic (restoring) force, which according to the Hooke's law is prop. to the displacement : $F_e = -k \cdot x(t), k>0$

- the friction (damping) force, which we approximate as being prop to the velocity : $F_d = -\nu x'(t), \nu > 0$
- an external force : $F_{ext} = f(t)$

So $F = F_{ext} + F_e + F_d = f(t) - kx - \nu x'$

$F = m \cdot a \Leftrightarrow -kx - \nu x' + f(t) = mx'' \Leftrightarrow x'' + \frac{\nu}{m} \cdot x' + \frac{k}{m} x = f(t)$

$\frac{\nu}{m} \geq 0, \frac{k}{m} \geq 0$ — second order Ln-HDE with const coeff.

Case 1: Motion without damping, without external force.

(3) $x'' + \frac{k}{m} \cdot x = 0$

Case 2: Motion with damping, without external force.

(4) $x'' + \frac{\nu}{m} \cdot x' + \frac{k}{m} x = 0, \nu > 0, k > 0, m > 0$

Case 3: Motion without damping, with external force $f(t) = A \cos \omega t, A > 0, \omega > 0$.

(5) $x'' + \frac{k}{m} x = A \cdot \cos \omega t, k > 0, m > 0, A > 0, \omega > 0.$

Case 1: $x'' + \frac{k}{m} x = 0, \quad \omega_0 = \sqrt{\frac{k}{m}} > 0$

$\Rightarrow x'' + \omega_0^2 x = 0 \Rightarrow r^2 + \omega_0^2 = 0 \Leftrightarrow r_{1,2} = \pm i\omega_0 \mapsto \cos \omega_0 t, \sin \omega_0 t \Rightarrow$

$\Rightarrow x = c_1 \cos \omega_0 t + c_2 \cdot \sin \omega_0 t, \; c_1, c_2 \in \mathbb{R}$

Any sol. is periodic with the main period $T = \frac{2\pi}{\omega_0}$

bounded

oscillatory around 0 with const. amplitude

Case 2: $x'' + \frac{\nu}{m} x' + \frac{k}{m} \cdot x = 0$

$r^2 + \frac{\nu}{m} r + \frac{k}{m} = 0, \Delta = \frac{\nu^2}{m^2} - 4\frac{k}{m} = \frac{\nu^2 - 4km}{m^2}$

I. $\nu > \sqrt{4km} \; (\Leftrightarrow \Delta > 0) \quad r_1 \cdot r_2 = \frac{k}{m} > 0, r_1 + r_2 = -\frac{\nu}{m} < 0$

$r_1 = -\frac{\nu}{2m} - \frac{\sqrt{\nu^2 - 4km}}{2m} < 0$

$\mapsto e^{r_1 t}, e^{r_2 t} \Rightarrow x = c_1 \cdot e^{r_1 t} + c_2 \cdot e^{r_2 t}, c_1, c_2 \in \mathbb{R}$

$$r_1 = -\frac{v}{2m} - \frac{\sqrt{v^2-4km}}{2m} < 0$$

$$r_2 = -\frac{v}{2m} + \frac{\sqrt{v^2-4km}}{2m} < 0$$

$\Rightarrow e^{r_1 t}, e^{r_2 t} \Rightarrow x = c_1 \cdot e^{r_1 t} + c_2 \cdot e^{r_2 t}, \; c_1, c_2 \in \mathbb{R}$

Any sol. is exponentially decreasing to 0 as $t \to \infty$

This case is OVERDAMPING, $v > \sqrt{4km}$

↳ big

II. $v = \sqrt{4km}$ CRITICAL DAMPING

$r_{1,2} = -\frac{v}{2m}$ double root $\mapsto e^{-\frac{v}{2m}t}, \; t \cdot e^{-\frac{v}{2m}t} \Rightarrow$

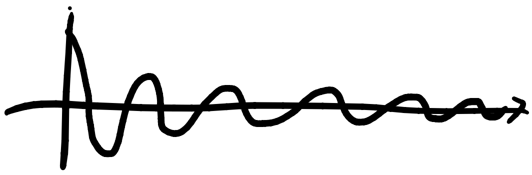
$\Rightarrow x = c_1 \cdot e^{-\frac{v}{2m}t} + c_2 \cdot t \cdot e^{-\frac{v}{2m}t}, \; c_1, c_2 \in \mathbb{R}$

Any sol. is exp. decreasing to 0 as $t \to \infty$

III. $v < \sqrt{4km}$ UNDERDAMPING

$r_{1,2} = -\frac{v}{2m} \pm i \underbrace{\frac{\sqrt{4km - v^2}}{2m}}_{\beta} \mapsto e^{-\frac{v}{2m}t} \cos \beta t, \; e^{-\frac{v}{2m}t} \sin \beta t \Rightarrow$

$\Rightarrow x = e^{-\frac{v}{2m}t}(c_1 \cos \beta t + c_2 \sin \beta t)$ any sol. oscillates around 0 with exp. decreasing amplitude



Case 3: $x'' + \frac{k}{m} x = A \cdot \cos \omega t, \; \omega_0 = \sqrt{\frac{k}{m}}$

$x'' + \omega_0^2 x = A \cdot \cos \omega t \quad , x = x_h + x_p$

$x'' + \omega_0^2 x = 0 \Rightarrow x_h = c_1 \cos \omega_0 t + c_2 \cdot \sin \omega_0 t$

When $\omega \neq \omega_0 \Rightarrow x_p = \frac{A}{\omega_0^2 - \omega^2} \cdot \cos \omega t$

$\omega = \omega_0 \Rightarrow x_p = \frac{A}{2\omega_0} \cdot t \cdot \sin \omega_0 t$ — unbounded fct osc with amplitude that increases to ∞

Resonance (the ext. frequency = the initial frequency)

YT: The Tacoma Bridge disaster