# 3 Week 3: Cartesian equations of lines and planes

## 3.1 Cartesian and affine reference systems

If $b = [\vec{u}, \vec{v}, \vec{w}]$ is an ordered basis of $\mathcal{V}$ and $\vec{x} \in \mathcal{V}$, recall that the column vector of the coordinates of $\vec{x}$ with respect to $b$ is denoted by $[\vec{x}]_b$. In other words

$$[\vec{x}]_b = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

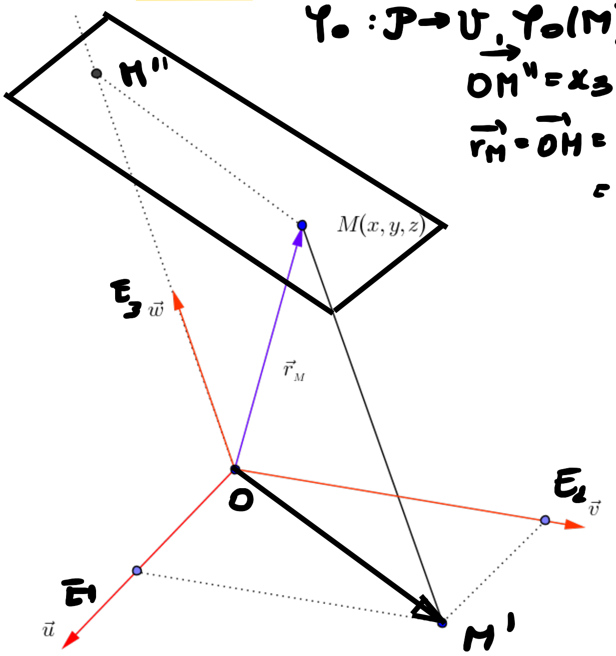
whenever $\vec{x} = x_1\vec{u} + x_2\vec{v} + x_3\vec{w}$. To emphasize the coordinates of $\vec{x}$ with respect to $b$, we shall use the notation $\vec{x}(x_1, x_2, x_3)$.

$\varphi_0 : \mathcal{P} \to \mathcal{V},\ \varphi_0(M) = \vec{OM} = \vec{r_M}$

$\vec{OM''} = x_3\vec{w}$

$\vec{r_M} = \vec{OM} = \vec{OM'} + \vec{OM''}$

$= x_1\vec{u} + x_2\vec{v} + x_3\vec{w}$




**Definition 3.1.** A *cartesian reference system* $R = (O, \vec{u}, \vec{v}, \vec{w})$ of the space $\mathcal{P}$, consists in a point $O \in \mathcal{P}$ called the *origin* of the reference system and an ordered basis $b = [\vec{u}, \vec{v}, \vec{w}]$ of the vector space $\mathcal{V}$.

Denote by $E_1, E_2, E_3$ the points for which $\vec{u} = \overrightarrow{OE_1}$, $\vec{v} = \overrightarrow{OE_2}$, $\vec{w} = \overrightarrow{OE_3}$.

**Definition 3.2.** The system of points $(O, E_1, E_2, E_3)$ is called *the affine reference system associated to the cartesian reference system* $R = (O, \vec{u}, \vec{v}, \vec{w})$.

The straight lines $OE_i$, $i \in \{1, 2, 3\}$, oriented from $O$ to $E_i$ are called *the coordinate axes*. The coordinates $x, y, z$ of the position vector $\vec{r}_M = \overrightarrow{OM}$ with respect to the basis $[\vec{u}, \vec{v}, \vec{w}]$ are called the coordinates of the point $M$ with respect to the cartesian system $R$ written $M(x, y, z)$. Also, for the column matrix of coordinates of the vector $\vec{r}_M$ we are going to use the notation $[M]_R$. In other words, if $\vec{r}_M = x\vec{u} + y\vec{v} + z\vec{w}$, then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

**Remark 3.1.** If $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ are two points, then $R=(O,\vec{u},\vec{v},\vec{w})$

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= x_B \vec{u} + y_B \vec{v} + z_B \vec{w} - (x_A \vec{u} + y_A \vec{v} + z_A \vec{w}) \\ &= (x_B - x_A) \vec{u} + (y_B - y_A) \vec{v} + (z_B - z_A) \vec{w},\end{aligned}$$

i.e. the coordinates of the vector $\overrightarrow{AB}$ are being obtained by performing the differences of the coordinates of the points $A$ and $B$.

**Remark 3.2.** If $R = (O, b)$ is a cartesian reference system, where $b = [\vec{u}, \vec{v}, \vec{w}]$ is an ordered basis of $\mathcal{V}$, recall that $\varphi_O : \mathcal{P} \longrightarrow \mathcal{V}$, $\varphi_O(M) = \overrightarrow{OM}$ is bijective and $\psi_b : \mathbb{R}^3 \longrightarrow \mathcal{V}$, $\psi_b(x, y, z) = x \vec{u} + y \vec{v} + z \vec{w}$ is a linear isomorphism. The bijection $\varphi_O$ defines a unique vector structure over $\mathcal{P}$ such that $\varphi_O$ becomes an isomorphism. This vector structure depends on the choice of $O \in \mathcal{P}$. Therefore a point $M \in \mathcal{P}$ could be identified either with its position vector $\vec{r}_M = \varphi_O(M)$, or, with the triplet $(\psi_b^{-1} \circ \varphi_O)(M) \in \mathbb{R}^3$ of its coordinates with respect to the reference system $R$. If $f : X \longrightarrow \mathbb{R}^3$ is a given application, then $\varphi_O^{-1} \circ \psi_b \circ f : X \longrightarrow \mathcal{P}$ will be denoted by $M_f$. A similar discussion can be done for a cartesian reference system $R' = (O', b')$ of a plane $\pi$, where $b' = [\vec{u}', \vec{v}']$ is an ordered basis of $\vec{\pi}$.

**Example 3.1** (Homework). Consider the tetrahedron $ABCD$, where $A(1, -1, 1)$, $B(-1, 1, -1)$, $C(2, 1, -1)$ and $D(1, 1, 2)$. Find the coordinates of:

1. the centroids $G_A$, $G_B$, $G_C$, $G_D$ of the triangles $BCD$, $ACD$, $ABD$ and $ABC$[1] respectively.
2. the midpoints $M$, $N$, $P$, $Q$, $R$ and $S$ of its edges $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[CD]$ and $[DB]$ respectively.

SOLUTION.

[1]The centroids of its faces

## 3.2 The Cylindrical Coordinate System

In order to have a valid coordinate system in the 3-dimensional case, each point of the space must be associated with a unique triple of real numbers (the coordinates of the point) and each triple of real numbers must determine a unique point, as in the case of the Cartesian system of coordinates.

Let $P(x,y,z)$ be a point in a Cartesian system of coordinates $Oxyz$ and $P'$ be the orthogonal projection of $P$ on the plane $xOy$. One can associate to the point $P$ the triple $(r,\theta,z)$, where $(r,\theta)$ are the polar coordinates of $P'$ (see Figure 1). The polar coordinates of $P'$ can be obtained by specifying the distance $\rho$ from $O$ to $P'$ and the angle $\theta$ (measured in radians), whose ”initial” side is the polar axis, i.e. the $x$-axis, and whose ”terminal” side is the ray $OP$. The *polar coordinates* of the point $P$ are $(\rho,\theta)$ (See also section (3.6.2) of the Appendix). The triple $(r,\theta,z)$ gives the *cylindrical coordinates* of the point $P$. There is the bijection

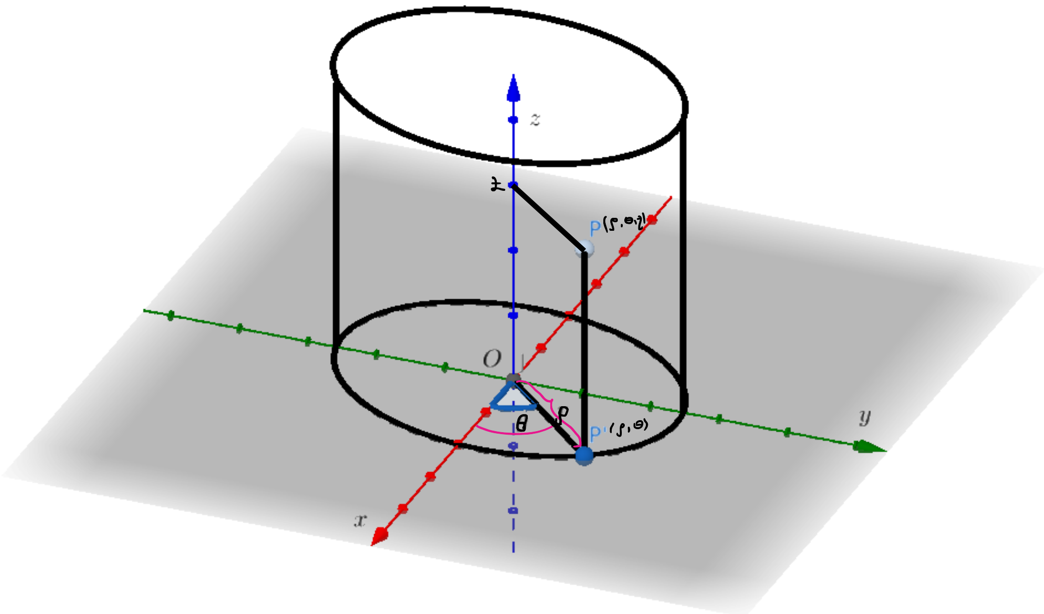



Figure 1: cylindrical coordinates

$$h_1 : \mathcal{P} \setminus \{O\} \to \mathbb{R}_+ \times [0,2\pi) \times \mathbb{R},\ P \to (r,\theta,z)$$

and one obtains a new coordinate system, named the *cylindrical coordinate system* in $\mathcal{P}$. For the conversion formulas between the cylindrical coordinates and the Cartesian coordinates we refere the reader to [1, p. 19]. Note however that once we have the cylindrical coordinates $(r,\theta,z)$ of a point $P$, then its Cartesian coordinates are

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}.$$

## 3.3 The Spherical Coordinate System

Another way to associate to each point $P$ in $\mathcal{P}$ a triple of real numbers is illustrated in Figure 2. If $P(x,y,z)$ is a point in a rectangular system of coordinates $Oxyz$ and $P'$ its orthogonal projection on $Oxy$, let $\rho$ be the length of the segment $[OP]$, $\theta$ be the oriented angle determined by $[Ox$ and $[OP'$ and $\varphi$ be the oriented angle between $[Oz$ and $[OP$. The triple

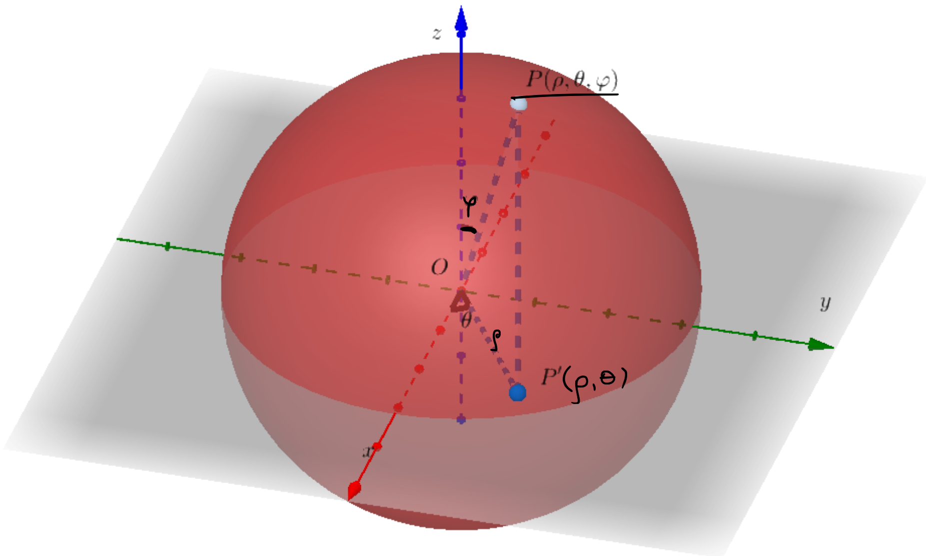



Figure 2: spherical coordinates

$(\rho,\theta,\varphi)$ gives the *spherical coordinates* of the point $P$. This way, one obtains the bijection

$$h_2 : \mathcal{P} \setminus \{O\} \to \mathbb{R}_+ \times [0, 2\pi) \times [0, \pi], P \to (\rho, \theta, \varphi),$$

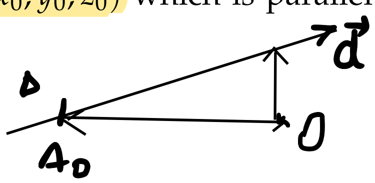
which defines a new coordinate system in $\mathcal{P}$, called the *spherical coordinate system* . For the conversion formulas between the spherical coordinate system and the Cartesian coordinate system we refere the reader to [1, p. 20]. Note however that once we have the cylindrical coordinates $(\rho,\theta,\varphi)$ of a point $P$, then its Cartesian coordinates are

$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}.$$

## 3.4 The cartesian equations of the straight lines

Let $\Delta$ be the straight line passing through the point $A_0(x_0, y_0, z_0)$ which is parallel to the vector $\vec{d}\,(p, q, r)$. Its vector equation is

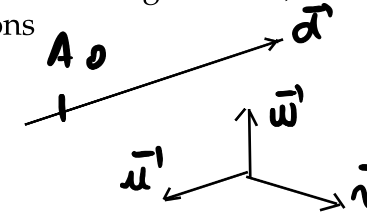
$$\vec{r}_M = \vec{r}_{A_0} + \lambda\,\vec{d},\ \lambda \in \mathbb{R}. \tag{3.1}$$




Denoting by $x, y, z$ the coordinates of the generic point $M$ of the straight line $\Delta$, its vector equation (3.1) is equivalent to the following system of relations

$$\lambda = \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}$$

$$\begin{cases} x = x_0 + \lambda p \\ y = y_0 + \lambda q \\ z = z_0 + \lambda r \end{cases},\ \lambda \in \mathbb{R} \tag{3.2}$$



Indeed, the vector equation of $\Delta$ can be written, in terms of the coordinates of the vectors $\vec{r}_M$, $\vec{r}_{A_0}$ and $\vec{d}$, as follows:

$$\overbrace{x\vec{u}+y\vec{v}+z\vec{w}}^{\vec{r}_M}=\overbrace{x_0\vec{u}+y_0\vec{v}+z_0\vec{w}}^{\vec{r}_{A_0}}+\lambda\overbrace{(p\vec{u}+q\vec{v}+r\vec{w})}^{\vec{d}}$$
$$\Longleftrightarrow x\vec{u}+y\vec{v}+z\vec{w}=(x_0+p\lambda)\vec{u}+(y_0+q\lambda)\vec{v}+(z_0+r\lambda)\vec{w},\ \lambda\in\mathbb{R}$$

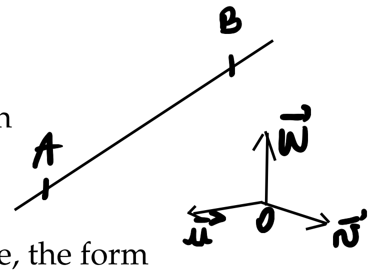
which is obviously equivalent to (3.2). The relations (3.2) are called the *parametric equations* of the straight line $\Delta$ and they are equivalent to the following relations

$$\frac{x-x_0}{p}=\frac{y-y_0}{q}=\frac{z-z_0}{r}. \tag{3.3}$$

If $r=0$, for instance, the canonical equations of the straight line $\Delta$ are

$\vec{d}=\vec{AB}$

$$\frac{x-x_0}{p}=\frac{y-y_0}{q}\ \wedge\ z=z_0.$$




If $A(x_A,y_A,z_A)$, $B(x_B,y_B,z_B)$ are different points of the line $\Delta$, then

$$\overrightarrow{AB}\,(x_B-x_A,y_B-y_A,z_B-z_A)$$

is a director vector of $\Delta$, its canonical equations having, in this case, the form

$$\frac{x-x_A}{x_B-x_A}=\frac{y-y_A}{y_B-y_A}=\frac{z-z_A}{z_B-z_A}. \tag{3.4}$$

**Example 3.2.** Consider the tetrahedrom $ABCD$, where $A(1,-1,1)$, $B(-1,1,-1)$, $C(2,1,-1)$ and $D(1,1,2)$, as well as the centroids $G_A$, $G_B$, $G_C$, $G_D$ of the triangles $BCD$, $ACD$, $ABD$ and $ABC$[2] respectively. Show that the medians $AG_A$, $BG_B$, $CG_C$ and $DG_D$ are concurrent and find the coordinates of their intersection point.

SOLUTION. One can easily see that the coordinates of the centroids $G_A$, $G_B$, $G_C$, $G_D$ are $(2/3,1,0)$, $(4/3,1/3,2/3)$, $(1/3,1/3,2/3)$ and $(2/3,1/3,-1/3)$ respectively. The equations of the medians $AG_A$ and $BG_B$ are

$$(AG_A)\ \frac{x-1}{2/3-1}=\frac{y+1}{1-(-1)}=\frac{z-1}{0-1}\Longleftrightarrow\frac{x-1}{-1/3}=\frac{y+1}{2}=\frac{z-1}{-1}$$
$$(BG_B)\ \frac{x+1}{4/3+1}=\frac{y-1}{1/3-1}=\frac{z+1}{2/3+1}\Longleftrightarrow\frac{x+1}{7/3}=\frac{y-1}{-2/3}=\frac{z+1}{5/3}.$$

Thus, the director space of the median $AG_A$ is $\left\langle\left(-\frac{1}{3},2,-1\right)\right\rangle=\langle(-1,6,-3)\rangle$ and the director space of the median $BG_B$ is $\left\langle\left(\frac{7}{3},-\frac{2}{3},\frac{5}{3}\right)\right\rangle=\langle(7,-2,5)\rangle$. Consequently, the parametric equations of the medians $AG_A$ and $BG_B$ are

$$(AG_A)\begin{cases}x=1-t\\y=-1+6t\\z=1-3t\end{cases},\ t\in\mathbb{R}\text{ and }(BG_B)\begin{cases}x=-1+7s\\y=1-2s\\z=-1+5s\end{cases},\ s\in\mathbb{R}.$$

Thus, the two medians $AG_A$ and $BG_B$ are concurrent if and only if there exist $s,t\in\mathbb{R}$ such that

$$\begin{cases}1-t=-1+7s\\-1+6t=1-2s\\1-3t=-1+5s\end{cases}\Longleftrightarrow\begin{cases}7s+t=2\\2s+6t=2\\5s+3t=2\end{cases}\Longleftrightarrow\begin{cases}7s+t=2\\s+3t=1\\5s+3t=2.\end{cases}$$

[2] The centroids of its faces

This system is compatible and has the unique solution $s = t = \frac{1}{4}$, which shows that the two medians $AG_A$ and $BG_B$ are concurrent and

$$AG_A \cap BG_B = \left\{ G\left(\frac{3}{4}, \frac{1}{2}, \frac{1}{4}\right) \right\}.$$

One can similarly show that $BG_B \cap CG_C = CG_C \cap AG_A = \left\{ G\left(\frac{3}{4}, \frac{1}{2}, \frac{1}{4}\right) \right\}$.

**Example 3.3** (Homework)**.** Consider the tetrahedrom $ABCD$, where $A(1, -1, 1)$, $B(-1, 1, -1)$, $C(2, 1, -1)$ and $D(1, 1, 2)$, as well as the midpoints $M$, $N$, $P$, $Q$, $R$ and $S$ of its edges $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[CD]$ and $[DB]$ respectively. Show that the lines $MR$, $PQ$ and $NS$ are concurrent and find the coordinates of their intersection point.

SOLUTION.

## 3.5 The cartesian equations of the planes

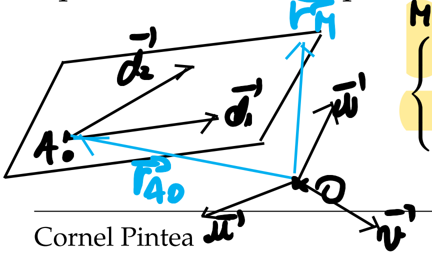
Let $A_0(x_0, y_0, z_0) \in \mathcal{P}$ and $\overrightarrow{d}_1 (p_1, q_1, r_1)$, $\overrightarrow{d}_2 (p_2, q_2, r_2) \in \mathcal{V}$ be linearly independent vectors, that is

$\vec{d_1}, \vec{d_2}$ are lin. indep $\Leftrightarrow$ $\text{rank} \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{pmatrix} = 2.$

The vector equation of the plane $\pi$ passing through $A_0$ which is parallel to the vectors $\overrightarrow{d}_1 (p_1, q_1, r_1)$, $\overrightarrow{d}_2 (p_2, q_2, r_2)$ is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + \lambda_1 \overrightarrow{d}_1 + \lambda_2 \overrightarrow{d}_2, \ \lambda_1, \lambda_2 \in \mathbb{R}. \tag{3.5}$$

If we denote by $x, y, z$ the coordinates of the generic point $M$ of the plane $\pi$, then the vector equation (3.5) is the equivalent to the following system of relations




$M(x, y, z) \in \pi$ iff the system is compatible $\Rightarrow$ rank $\underbrace{\mathcal{M}}_{2}$ = rank $\overline{\mathcal{M}}$

$$\begin{cases} x = x_0 + \lambda_1 p_1 + \lambda_2 p_2 \\ y = y_0 + \lambda_1 q_1 + \lambda_2 q_2 \\ z = z_0 + \lambda_1 r_1 + \lambda_2 r_2 \end{cases}, \ \lambda_1, \lambda_2 \in \mathbb{R}. \tag{3.6}$$

$\Rightarrow$ rank $\overline{\mathcal{M}}^* = 2$ - (3.7)

$\pi$ - the plane through $A_0$ which is parallel to $\vec{d_1}$ and $\vec{d_2}$

Indeed, the vector equation of $\pi$ can be written, in terms of the coordinates of the vectors $\vec{r}_M$, $\vec{r}_{A_0}$, $\vec{d}_1$ and $\vec{d}_2$, as follows:

$$\overbrace{x\vec{u}+y\vec{v}+z\vec{w}}^{\vec{r}_M}=\overbrace{x_0\vec{u}+y_0\vec{v}+z_0\vec{w}}^{\vec{r}_{A_0}}+\lambda_1\overbrace{(p_1\vec{u}+q_1\vec{v}+r_1\vec{w})}^{\vec{d}_1}+\lambda_2\overbrace{(p_2\vec{u}+q_2\vec{v}+r_2\vec{w})}^{\vec{d}_2}$$
$$\Longleftrightarrow x\vec{u}+y\vec{v}+z\vec{w}=(x_0+\lambda_1p_1+\lambda_2p_2)\vec{u}+(y_0+\lambda_1q_1+\lambda_2q_2)\vec{v}+(z_0+\lambda_1r_1+\lambda_2r_2)\vec{w},$$
$$\lambda_1,\lambda_2\in\mathbb{R},$$

which is obviously equivalent to (3.6). The relations (3.6) characterize the points of the plane $\pi$ and are called the *parametric equations* of the plane $\pi$. More precisely, the compatibility of the linear system (3.6) with the unknowns $\lambda_1,\lambda_2$ is a necessary and sufficient condition for the point $M(x,y,z)$ to be contained within the plane $\pi$. On the other hand the compatibility of the linear system (3.6) is equivalent to

$$M=\begin{pmatrix}p_1 & p_2\\ q_1 & q_2\\ r_1 & r_2\end{pmatrix}\qquad \bar{M}^t \quad M^t=\begin{matrix}p_1 & p_2 & x-x_0\\ q_1 & q_2 & y-y_0\\ r_1 & r_2 & z-z_0\end{matrix}$$

$$\begin{vmatrix}x-x_0 & y-y_0 & z-z_0\\ p_1 & q_1 & r_1\\ p_2 & q_2 & r_2\end{vmatrix}=0, \tag{3.7}$$

which expresses the equality between the rank of the coefficient matrix of the system and the rank of the extended matrix of the system. The equation (3.7) is a characterization of the points of the plane $\pi$ in terms of the Cartesian coordinates of the generic point $M$ and is called the *cartesian equation* of the plane $\pi$. On can put the equation (3.7) in the form

$$A=\begin{pmatrix}q_1 & r_1\\ q_2 & r_2\end{pmatrix}$$

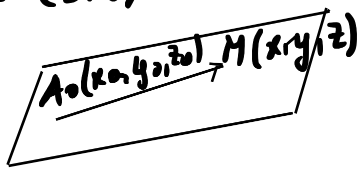
$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0 \text{ or} \tag{3.8}$$

$$B=\begin{pmatrix}r_1 & p_1\\ r_2 & p_2\end{pmatrix}\quad C=\begin{pmatrix}p_1 & q_1\\ p_2 & q_2\end{pmatrix}$$

$$Ax+By+Cz+D=0, \tag{3.9}$$

where the coefficients $A,B,C$ satisfy the relation $A^2+B^2+C^2>0$. It is also easy to show that every equation of the form (3.9) represents the equation of a plane. Indeed, if $A\neq 0$, then the equation (3.9) is equivalent to

$$\begin{vmatrix}x+\frac{D}{A} & y & z\\ B & -A & 0\\ C & 0 & -A\end{vmatrix}=0. \Leftrightarrow (3.9)$$




We observe that one can put the equation (3.8) in the form

$$AX+BY+CZ=0 \tag{3.10}$$

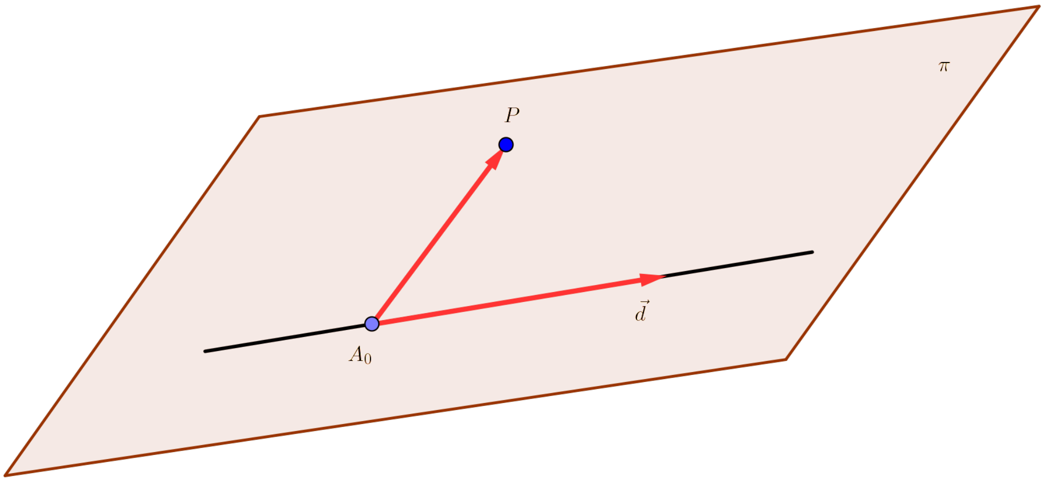
$\overrightarrow{A_0M}(x-x_0, y-y_0, z-z_0)$

where $X=x-x_0$, $Y=y-y_0$, $Z=z-z_0$ are the coordinates of the vector $\overrightarrow{A_0M}$.

**Example 3.4.** Write the equation of the plane determined by the point $P(-1,1,2)$ and the line $(\Delta)\ \frac{x-1}{3}=\frac{y}{2}=\frac{z+1}{-1}$.

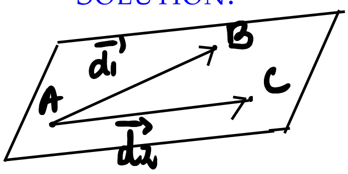
SOLUTION. Note that $P\notin\Delta$, as $\frac{-1-1}{3}\neq\frac{1}{2}\neq -3=\frac{2+1}{-1}$, i.e. the point $P$ and the line $\Delta$ determine, indeed, a plane, say $\pi$. One can regard $\pi$ as the plane through the point $A_0(1,0,-1)$ which is parallel to the vectors $\overrightarrow{A_0P}\ (-1-1,1-0,2-(-1))=\overrightarrow{A_0P}\ (-2,1,3)$ and $\vec{d}\ (3,2,-1)$. Thus, the equation of $\pi$ is

$$\begin{vmatrix}x-1 & y & z+1\\ -2 & 1 & 3\\ 3 & 2 & -1\end{vmatrix}=0\Longleftrightarrow x-y+z=0.$$




**Example 3.5** (Homework). Generalize Example 3.4: Write the equation of the plane determined by the line $(\Delta)$ $\frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r}$ and the point $M(x_M, y_M, z_M) \notin \Delta$.

SOLUTION.




$\vec{d_1} = \vec{AB}(x_B - x_A, y_B - y_A, z_B - z_A)$, replace in (3.7) $p_1, q_1, r_1$,
$\vec{d_2} = \vec{AC}(x_C - x_A, y_C - y_A, z_C - z_A)$ $p_2, q_2, r_2$

**Remark 3.3.** If $A(x_A, y_A, z_A), B(x_B, y_B, z_B), C(x_C, y_C, z_C)$ are noncollinear points, then the plane $(ABC)$ determined by the three points can be viewed as the plane passing through the point $A$ which is parallel to the vectors $\vec{d}_1 = \overrightarrow{AB}$, $\vec{d}_2 = \overrightarrow{AC}$. The coordinates of the vectors $\vec{d}_1$ şi $\vec{d}_2$ are

$$(x_B - x_A, y_B - y_A, z_B - z_A) \text{ and } (x_C - x_A, y_C - y_A, z_C - z_A) \text{ respectively.}$$

Thus, the equation of the plane $(ABC)$ is

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ x_B - x_A & y_B - y_A & z_B - z_A \\ x_C - x_A & y_C - y_A & z_C - z_A \end{vmatrix} = 0, \tag{3.11}$$

or, echivalently

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0. \tag{3.12}$$

Thus, four points $A(x_A, y_A, z_A), B(x_B, y_B, z_B), C(x_C, y_C, z_C)$ and $D(x_D, y_D, z_D)$ are coplanar if and ony if

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0. \tag{3.13}$$

**Example 3.6** (Homework)**.** Write the equation of the plane determined by the points $M_1(3,-2,1)$, $M_2(5,4,1)$ and $M_3(-1,-2,3)$.

SOLUTION.

$$\begin{vmatrix} x & y & z & 1 \\ 3 & -2 & 1 & 1 \\ 5 & 4 & 1 & 1 \\ -1 & -2 & 3 & 1 \end{vmatrix} = 0$$

**Remark 3.4.** If $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$ are three points ($abc \neq 0$), then for the equation of the plane $(ABC)$ we have successively:

$$\begin{vmatrix} x & y & z & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x & y & z-c & 1 \\ a & 0 & -c & 1 \\ 0 & b & -c & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x & y & z-c \\ a & 0 & -c \\ 0 & b & -c \end{vmatrix} = 0$$

$$\Longleftrightarrow ab(z-c) + bcx + acy = 0 \Longleftrightarrow bcx + acy + abz = abc$$

$$\Longleftrightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \tag{3.14}$$

The equation (3.14) of the plane $(ABC)$ is said to be in *intercept form* and the $x, y, z$-intercepts of the plane $(ABC)$ are $a, b, c$ respectively.

**Example 3.7** (Homework)**.** Write the equation of the plane $(\pi)$ $3x - 4y + 6z - 24 = 0$ in intercept form.

SOLUTION.

$$3x - 4y + 6z - 24 = 0 \,/ +24 \Leftrightarrow 3x - 4y + 6z = 24 \,/ :24 \Rightarrow$$

$$\Leftrightarrow \frac{3x}{24} - \frac{4y}{24} + \frac{6z}{24} = 1 \Leftrightarrow \frac{x}{8} - \frac{y}{6} + \frac{z}{4} = 1$$

## 3.6 Appendix: The Cartesian equations of lines in the two dimensional setting

### 3.6.1 Cartesian and affine reference systems

If $b = [\overrightarrow{e}, \overrightarrow{f}]$ is an ordered basis of the director subspace $\overrightarrow{\pi}$ of the plane $\pi$ and $\overrightarrow{x} \in \overrightarrow{\pi}$, recall that the column vector of $\overrightarrow{x}$ with respect to $b$ is being denoted by $[\overrightarrow{x}]_b$. In other words

$$[\overrightarrow{x}]_b = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

whenever $\overrightarrow{x}= x_1 \overrightarrow{e} + x_2 \overrightarrow{f}$.

**Definition 3.3.** A *cartesian reference system* of the plane $\pi$, is a system $R = (O, \overrightarrow{e}, \overrightarrow{f})$, where $O$ is a point from $\pi$ called the *origin* of the reference system and $b = [\overrightarrow{e}, \overrightarrow{f}]$ is a basis of the vector space $\overrightarrow{\pi}$.

Denote by $E, F$ the points for which $\overrightarrow{e} = \overrightarrow{OE}$, $\overrightarrow{f} = \overrightarrow{OF}$.

**Definition 3.4.** The system of points $(O, E, F)$ is called *the affine reference system associated to the cartesian reference system* $R = (O, \overrightarrow{e}, \overrightarrow{f})$.

The straight lines $OE$, $OF$, oriented from $O$ to $E$ and from $O$ to $F$ respectively, are called *the coordinate axes*. The coordinates $x, y$ of the position vector $\overrightarrow{r}_M = \overrightarrow{OM}$ with respect to the basis $[\overrightarrow{e}, \overrightarrow{f}]$ are called the coordinates of the point $M$ with respect to the cartesian system $R$ written $M(x, y)$. Also, for the column matrix of coordinates of the vector $\overrightarrow{r}_M$ we are going to use the notation $[M]_R$. In other words, if $\overrightarrow{r}_M = x \overrightarrow{e} + y \overrightarrow{f}$, then

$$[M]_R = [\overrightarrow{OM}]_b = \begin{pmatrix} x \\ y \end{pmatrix}.$$

**Remark 3.5.** If $A(x_A, y_A)$, $B(x_B, y_B)$ are two points, then

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} = x_B \overrightarrow{e} + y_B \overrightarrow{f} - (x_A \overrightarrow{e} + y_A \overrightarrow{f}) \\ &= (x_B - x_A) \overrightarrow{e} + (y_B - y_A) \overrightarrow{f}, \end{aligned}$$

i.e. the coordinates of the vector $\overrightarrow{AB}$ are being obtained by performing the differences of the coordinates of the points $A$ and $B$.

### 3.6.2 The Polar Coordinate System [1, p. 17]

As an alternative to a Cartesian coordinate system (RS) one considers in the plane $\pi$ a fixed point $O$, called *pole* and a half-line directed to the right of $O$, called *polar axis* (see Figure 3). By specifying the distance $\rho$ from $O$ to a point $P$ and an angle $\theta$ (measured in radians), whose

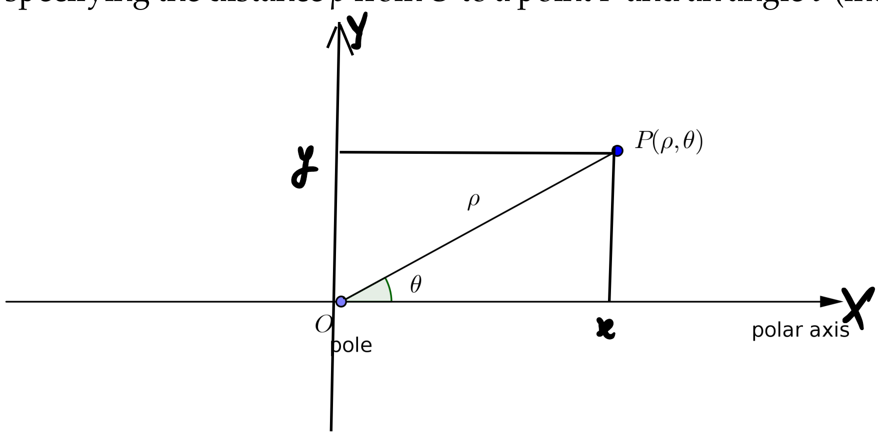



Figure 3: The pole and the polar axis related to a polar coordinate system

"initial" side is the polar axis and whose "terminal" side is the ray $OP$, the *polar coordinates* of the point $P$ are $(\rho, \theta)$. One obtains a bijection

$$\pi \setminus \{O\} \to \mathbb{R}_+ \times [0, 2\pi), \quad P \to (\rho, \theta)$$

which associates to any point $P$ in $\pi \setminus \{O\}$ the pair $(\rho, \theta)$ (suppose that $O(0,0)$). The positive real number $\rho$ is called the *polar ray* of $P$ and $\theta$ is called the *polar angle* of $P$.

Consider the Cartesian coordinate system in $\pi$, whose origin $O$ is the pole and whose positive half-axis $Ox$ is the polar axis (see Figure 4). The following transformation formulas give the connection between the coordinates of an arbitrary point in the two systems of coordinates.

Let $P$ be a given point of polar coordinates $(\rho, \theta)$. Its Cartesian coordinates are

$$\begin{cases} x_P = \rho \cos\theta \\ y_P = \rho \sin\theta \end{cases} . \tag{3.15}$$

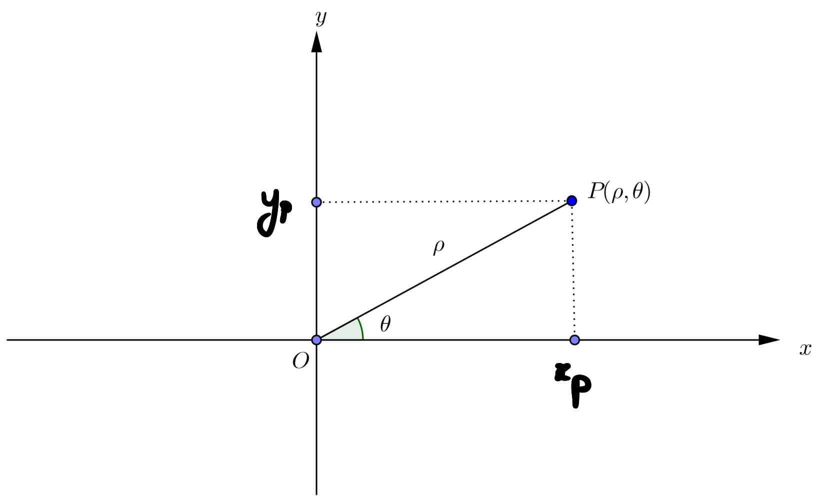



Figure 4: Polar coordinates

Let $P$ be a point of Cartesian coordinates $(x, y)$. It is clear that the polar ray of $P$ is given by the formula

$$\rho = \sqrt{x^2 + y^2}. \tag{3.16}$$

In order to obtain the polar angle of $P$, it must be considered the quadrant where $P$ is situated. One obtains the following formulas:

*Case* 1. If $x \neq 0$, then using $\tan\theta = \frac{y}{x}$, one has

$$\theta = \arctan\frac{y}{x} + k\pi, \quad \text{where} \quad k = \begin{cases} 0 \text{ if } P \in I \cup ]Ox \\ 1 \text{ if } P \in II \cup III \cup ]Ox' \\ 2 \text{ if } P \in IV; \end{cases};$$

*Case* 2. If $x = 0$ and $y \neq 0$, then

$$\theta = \begin{cases} \frac{\pi}{2} \text{ when } P \in ]Oy \\ \frac{3\pi}{2} \text{ when } P \in ]Oy'; \end{cases}$$

*Case* 3. If $x = 0$ and $y = 0$, then $\theta = 0$.

### 3.6.3 Parametric and Cartesian equations of Lines

Let $\Delta$ be a line passing through the point $A_0(x_0, y_0) \in \pi$ which is parallel to the vector $\overrightarrow{d}\,(p,q)$. Its vector equation is

$$\overrightarrow{r}_M = \overrightarrow{r}_{A_0} + t\,\overrightarrow{d},\ t \in \mathbb{R}. \tag{3.17}$$

If $(x, y)$ are the coordinates of a generic point $M \in \Delta$, then its vector equation (3.17) is equivalent to the following system

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \end{cases},\ t \in \mathbb{R}. \tag{3.18}$$

The relations are called the *parametric equations* of the line $\Delta$ and they are equivalent to the following equation

$(Ax + By + C = 0)$

$\Leftrightarrow qx - py - qx_0 + py_0 = 0$

$$\frac{x - x_0}{p} = \frac{y - y_0}{q}, \Leftrightarrow q(x - x_0) = p(y - y_0) \tag{3.19}$$

called the *canonical equation*of $\Delta$. If $q = 0$, then the equation (3.19) becomes $y = y_0$.

If $A(x_A, y_A)$ are two different points of the plane $\pi$, then $\overrightarrow{AB}\ (x_B - x_A, y_B - y_A)$ is a director vector of the line $AB$ and the canonical equation of the line $AB$ is

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A}. \tag{3.20}$$

The equation (3.20) is equivalent to

$$\begin{vmatrix} x - x_A & y - y_A \\ x_B - x_A & y_B - y_A \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x - x_A & y - y_A & 1 \\ x_B - x_A & y_B - y_A & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_B & y_B & 1 \end{vmatrix} = 0.$$

Thus, three poins $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0. \tag{3.21}$$

### 3.6.4 General Equations of Lines

We can put the equation (3.19) in the form

$$ax + by + c = 0, \qquad \text{with} \quad a^2 + b^2 > 0, \tag{3.22}$$

which means that any line from $\pi$ is characterized by a first degree equation. Conversely, such of an equation represents a line, since the formula (3.22) is equivalent to

$$\frac{x + \frac{c}{a}}{-\frac{b}{a}} = \frac{y}{1}$$

and this is the *symmetric equation* of the line passing through $P_0\left(-\frac{c}{a}, 0\right)$ and parallel to $\overline{v}\left(-\frac{b}{a}, 1\right)$. The equation (3.22) is called *general equation* of the line.

**Remark 3.6.** The lines

$$(d)\ ax+by+c=0 \text{ and } (\Delta)\ \frac{x-x_0}{p}=\frac{x-x_0}{q}$$

are parallel if and only if $ap+bq=0$. Indeed, we have:

$$\begin{aligned} d\|\Delta \ &\Longleftrightarrow \vec{d}=\vec{\Delta} \Longleftrightarrow \langle \vec{u}\ (p,q)\rangle = \langle \vec{v}\ \left(-\tfrac{b}{a},1\right)\rangle \Longleftrightarrow \exists t\in\mathbb{R} \text{ s.t. } \vec{u}\ (p,q)=t\ \vec{v}\ \left(-\tfrac{b}{a},1\right) \\ &\Longleftrightarrow \exists t\in\mathbb{R} \text{ s.t. } \ =-t\tfrac{b}{a} \text{ and } q=t \Longleftrightarrow ap+bq=0. \end{aligned}$$

### 3.6.5 Reduced Equations of Lines

Consider a line given by its general equation $Ax+By+C=0$, where at least one of the coefficients $A$ and $B$ is nonzero. One may suppose that $B\neq 0$, so that the equation can be divided by $B$. One obtains

$$y=mx+n \tag{3.23}$$

which is said to be the *reduced equation* of the line.

*Remark*: If $B=0$, (3.22) becomes $Ax+C=0$, or $x=-\dfrac{C}{A}$, a line parallel to $Oy$. (In the same way, if $A=0$, one obtains the equation of a line parallel to $Ox$).

Let $d$ be a line of equation $y=mx+n$ in a Cartesian system of coordinates and suppose that the line is not parallel to $Oy$. Let $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ be two different points on $d$ and $\varphi$ be the angle determined by $d$ and $Ox$ (see Figure 5); $\varphi\in[0,\pi]\setminus\{\pi/2\}$. The points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ belong to $d$, hence

$$\begin{cases} y_1=mx_1+n \\ y_2=mx_2+n, \end{cases}$$

and $x_2\neq x_1$, since $d$ is not parallel to $Oy$. Then,

$$m=\frac{y_2-y_1}{x_2-x_1}=\tan\varphi. \tag{3.24}$$

The number $m=\tan\varphi$ is called the *angular coefficient* of the line $d$. It is immediate that the equation of the line passing through the point $P_0(x_0,y_0)$ and of the given angular coefficient $m$ is

$$y-y_0=m(x-x_0). \tag{3.25}$$

### 3.6.6 Intersection of Two Lines

Let $d_1: a_1x+b_1y+c_1=0$ and $d_2: a_2x+b_2y+c_2=0$ be two lines in $\mathcal{E}_2$. The solution of the system of equation

$$\begin{cases} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{cases}$$

will give the set of the intersection points of $d_1$ and $d_2$.

1) If $\dfrac{a_1}{a_2}\neq\dfrac{b_1}{b_2}$, the system has a unique solution $(x_0,y_0)$ and the lines have a unique intersection point $P_0(x_0,y_0)$. They are *secant*.

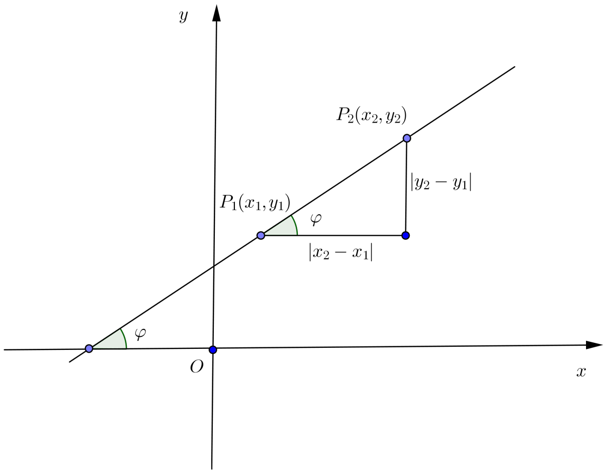



Figure 5:

2) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$, the system is not compatible, and the lines have no points in common. They are *parallel*.

3) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$, the system has an infinity of solutions, and the lines coincide. They are *identical*.

If $d_i : a_i x + b_i y + c_i = 0$, $i = \overline{1,3}$ are three lines in $\mathcal{E}_2$, then they are concurrent if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{3.26}$$

### 3.6.7 Bundles of Lines ([1])

The set of all the lines passing through a given point $P_0$ is said to be a *bundle* of lines. The point $P_0$ is called the *vertex* of the bundle.

If the point $P_0$ is of coordinates $P_0(x_0, y_0)$, then the equation of the bundle of vertex $P_0$ is

$$r(x - x_0) + s(y - y_0) = 0, \qquad (r, s) \in \mathbb{R}^2 \setminus \{(0,0)\}. \tag{3.27}$$

*Remark*: Tthe *reduced bundle* of line through $P_0$ is,

$$y - y_0 = m(x - x_0), \qquad m \in \mathbb{R}, \tag{3.28}$$

and covers the bundle of lines through $P_0$, except the line $x = x_0$. Similarly, the family of lines

$$x - x_0 = k(y - y_0), \qquad k \in \mathbb{R}, \tag{3.29}$$

covers the bundle of lines through $P_0$, exceptthe line $y = y_0$.

If the point $P_0$ is given as the intersection of two lines, then its coordinates are the solution of the system

$$\begin{cases} d_1 : a_1 x + b_1 y + c_1 = 0 \\ d_2 : a_2 x + b_2 y + c_2 = 0 \end{cases},$$

assumed to be compatible. The equation of the bundle of lines through $P_0$ is

$$r(a_1x+b_1y+c_1)+s(a_2x+b_2y+c_2)=0, \qquad (r,s)\in\mathbb{R}^2\setminus\{(0,0)\}. \tag{3.30}$$

*Remark*: As before, if $r\neq 0$ (or $s\neq 0$), one obtains the reduced equation of the bundle, containing all the lines through $P_0$, except $d_1$ (respectively $d_2$).

### 3.6.8 The Angle of Two Lines ([1])

Let $d_1$ and $d_2$ be two concurrent lines, given by their reduced equations:

$$d_1: y=m_1x+n_1 \quad \text{and} \quad d_2: y=m_2x+n_2.$$

The angular coefficients of $d_1$ and $d_2$ are $m_1=\tan\varphi_1$ and $m_2=\tan\varphi_2$ (see Figure 6). One may suppose that $\varphi_1\neq\frac{\pi}{2}$, $\varphi_2\neq\frac{\pi}{2}$, $\varphi_2\geq\varphi_1$, such that $\varphi=\varphi_2-\varphi_1\in[0,\pi]\setminus\{\frac{\pi}{2}\}$.

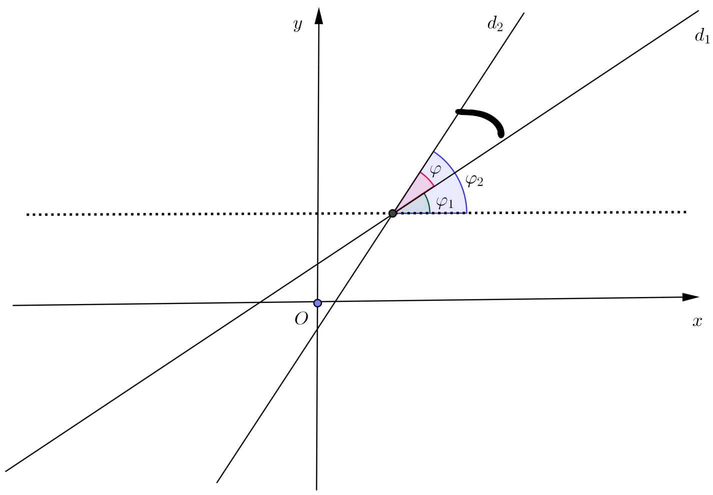



Figure 6:

The angle determined by $d_1$ and $d_2$ is given by

$$\tan\varphi=\tan(\varphi_2-\varphi_1)=\frac{\tan\varphi_2-\tan\varphi_1}{1+\tan\varphi_1\tan\varphi_2},$$

hence

$$\tan\varphi=\frac{m_2-m_1}{1+m_1m_2}. \tag{3.31}$$

1) The lines $d_1$ and $d_2$ are parallel if and only if $\tan\varphi=0$, therefore

$$d_1\parallel d_2 \iff m_1=m_2. \tag{3.32}$$

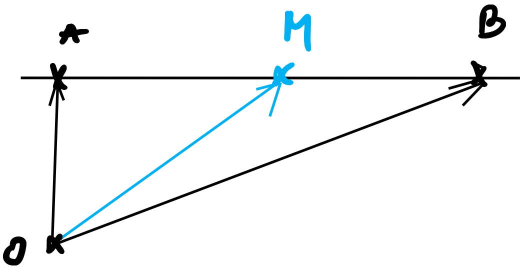
2) The lines $d_1$ and $d_2$ are orthogonal if and only if they determine an angle of $\frac{\pi}{2}$, hence

$$d_1\perp d_2 \iff m_1m_2+1=0. \tag{3.33}$$

# Seminar 3 - Micu (10/03/2021)

$l$ - line

$A, B \in l$, $A \neq B$




Vector eq:

$\vec{r_M} = \lambda \vec{r_A} + (1-\lambda)\vec{r_B}$, $\lambda \in \mathbb{R}$

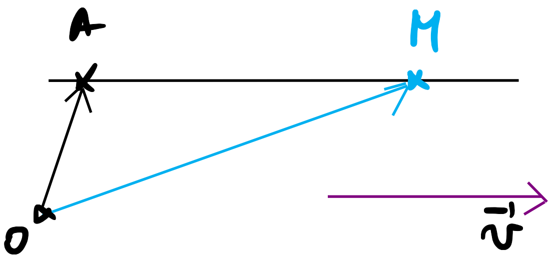
Param. eq:

$$\begin{cases} x = \lambda x_A + (1-\lambda) x_B \\ y = \lambda y_A + (1-\lambda) y_B \\ z = \lambda z_A + (1-\lambda) z_B \end{cases}$$

Canonical eq:

$$(\lambda =) \frac{x - x_B}{x_A - x_B} = \frac{y - y_B}{y_A - y_B} = \frac{z - z_B}{z_A - z_B}$$

$A \in l$, $\vec{v} \parallel l$




$\vec{r_M} = \vec{r_A} + t \cdot \vec{v}$

$t \in \mathbb{R}$

$$\begin{cases} x = x_A + t \cdot x_{\vec{v}} \\ y = y_A + t \cdot y_{\vec{v}} \\ z = z_A + t \cdot z_{\vec{v}} \end{cases}$$

$$(t =) \frac{x - x_A}{x_{\vec{v}}} = \frac{y - y_B}{y_{\vec{v}}} = \frac{z - z_B}{z_{\vec{v}}}$$

! The transition param → can. needs to be done with care in the case where the denominators might be zero.

E.g. if $x_{\vec{v}} = 0$, and $y_{\vec{v}} \neq 0$, $z_{\vec{v}} \neq 0 \Rightarrow \begin{cases} x = x_A \\ \frac{y - y_A}{y_{\vec{v}}} = \frac{z - z_A}{z_{\vec{v}}} \end{cases}$

if $x_{\vec{v}} = 0$, $y_{\vec{v}} = 0$ and $z_{\vec{v}} \neq 0 \Rightarrow \begin{cases} x = x_A \\ y = y_A \end{cases}$

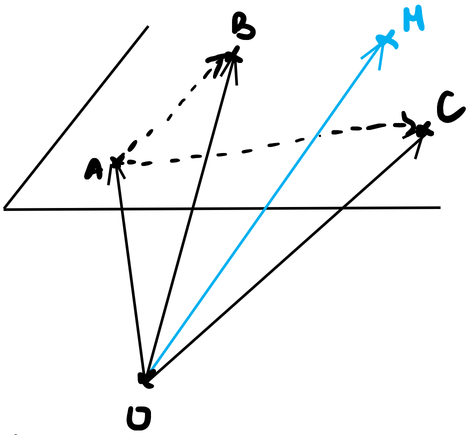
Implicit form: $\begin{cases} A_1 x + B_1 y + C_1 z + D_1 = 0 \\ A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases}$

Explicit form: → in the 2D case: $y = mx + n$

↳ slope

# Planes - $\Pi$ plane

$A, B, C \in \Pi$ non-collinear




Vector eq:

$$\vec{r_M} = (1-\lambda-\mu)\vec{r_A} + \lambda\vec{r_B} + \mu\vec{r_C}$$
$$\lambda, \mu \in \mathbb{R}$$

Parametric eq:

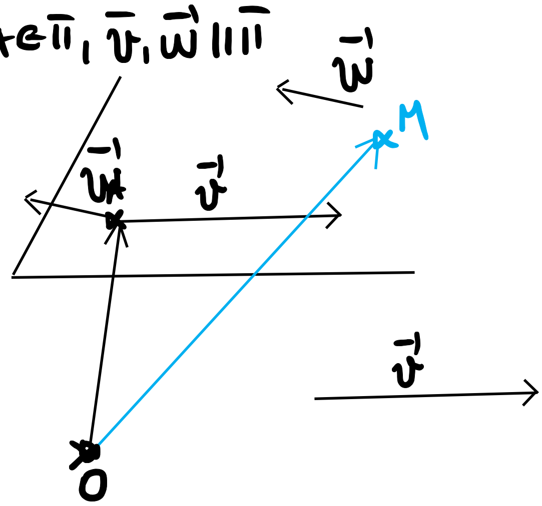
$$\begin{cases} x = (1-\lambda-\mu)x_A + \lambda x_B + \mu x_C \\ y = (1-\lambda-\mu)y_A + \lambda y_B + \mu y_C \\ z = (1-\lambda-\mu)z_A + \lambda z_B + \mu z_C \end{cases}$$

Canonical eq:

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ x_B - x_A & y_B - y_A & z_B - z_A \\ x_C - x_A & y_C - y_A & z_C - z_A \end{vmatrix} = 0$$

$$\Leftrightarrow \begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0$$

$A \in \Pi$, $\vec{v}, \vec{w} \parallel \Pi$



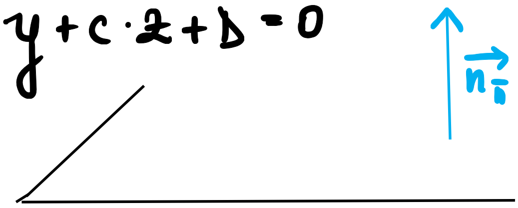

$$\vec{r_M} = \vec{r_A} + \alpha\vec{v} + \beta\vec{w}, \quad \alpha, \beta \in \mathbb{R}$$

$$\begin{cases} x = x_A + \alpha x_{\vec{v}} + \beta x_{\vec{w}} \\ y = y_A + \alpha y_{\vec{v}} + \beta y_{\vec{w}} \\ z = z_A + \alpha z_{\vec{v}} + \beta z_{\vec{w}} \end{cases}$$

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ x_{\vec{v}} & y_{\vec{v}} & z_{\vec{v}} \\ x_{\vec{w}} & y_{\vec{w}} & z_{\vec{w}} \end{vmatrix} = 0$$

Implicit eq: $A \cdot x + B y + C \cdot z + D = 0$

$\hookrightarrow \vec{n_\Pi}(A, B, C)$



### 3.7 Problems

1. Write the equation of the plane which passes through $M_0(1,-2,3)$ and is parallel to the vectors $\overrightarrow{v}_1\,(1,-1,0)$ and $\overrightarrow{v}_2\,(-3,2,4)$.

   HINT.
$$\begin{vmatrix} x-0 & y+2 & z-3 \\ 1 & -1 & 0 \\ -3 & 2 & 4 \end{vmatrix} = 0.$$

$\Rightarrow -4x + 2(z-3) - 3(z-3) - 4(y+2) = 0 \Rightarrow$

$\Rightarrow -4x - 4y - 8 - z + 3 = 0 \Rightarrow -4x - 4y - z - 5 = 0$

2. Write the equation of the line which passes through $A(1,-2,6)$ and is parallel to

   (a) The $x$-axis;

   (b) The line $(d_1)\ \dfrac{x-1}{2} = \dfrac{y+5}{-3} = \dfrac{z-1}{4}$.

   (c) The vector $\overrightarrow{v}\,(1,0,2)$.

   SOLUTION.

Param. Canonical

a) $d \parallel Ox \Rightarrow d \parallel \vec{v}(1,0,0) \Rightarrow \begin{cases} x = 1+t \\ y = -2+0t \\ z = 6+0t \end{cases}$ $\begin{cases} y = -2 \\ z = 6 \end{cases}$

b) $d: \begin{cases} x = 1+2t \\ y = -2+(-3)t \\ z = 6+4t \end{cases}$

c) $d: \begin{cases} \dfrac{x-1}{1} = \dfrac{z-6}{2} \\ y = -2 \end{cases}$

3. Write the equation of the plane which contains the line

$$(d_1)\ \frac{x-3}{2} = \frac{y+4}{1} = \frac{z-2}{-3}$$

and is parallel to the line

$$(d_2)\ \frac{x+5}{2} = \frac{y-2}{2} = \frac{z-1}{2}.$$

HINT.

$$\begin{vmatrix} x-3 & y+4 & z-2 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{vmatrix} = 0.$$

$A(3,-4,2) \in d_1 \in \pi \Rightarrow A \in \pi$

$$\left.\begin{array}{l} d_1 \in \pi \Rightarrow \vec{d_1} \parallel \pi \\ d_2 \parallel \pi \Rightarrow \vec{d_2} \parallel \pi \end{array}\right\} \Rightarrow \begin{vmatrix} x-3 & y+4 & z-2 \\ 2 & 1 & -3 \\ 2 & 2 & 2 \end{vmatrix} = 0$$

$\Rightarrow 2(x-3)+4(z-2)-6(y+4)-2(z-2)+6(x-3)-4(y+4)=0$

$\Rightarrow 8(x-3)-10(y+4)+2(z-2)=0$

$\Rightarrow 8x-24-10y-40+2z-4=0$

$\Rightarrow 8x-10y+2z-68=0$

4. Consider the points $A(\alpha, 0, 0)$, $B(0, \beta, 0)$ and $C(0, 0, \gamma)$ such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{a} \text{ where } a \text{ is a constatnt.}$$

Show that the plane $(A, B, C)$ passes through a fixed point.

SOLUTION. The equation of the plane $(ABC)$ can be written in intercept form, namely

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1.$$

The given relation shows that the point $P(a, a, a) \in (ABC)$ whenever $\alpha, \beta, \gamma$ verifies the given relation.

5. Write the equation of the line which passes through the point $M(1, 0, 7)$, is parallel to the plane $(\pi)$ $3x - y + 2z - 15 = 0$ and intersects the line

$$(d)\ \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

*Solution.*

6. Write the equation of the plane which passes through $M_0(1,-2,3)$ and cuts the positive coordinate axes through equal intercepts.

   SOLUTION. The general equation of such a plane is $x+y+z=a$. In this particular case $a=1+(-2)+3=2$ and the equation of the required plane is $x+y+z=2$.

7. Write the equation of the plane which passes through $A(1,2,1)$ and is parallel to the straight lines

$$(d_1)\begin{cases} x+2y-z+1=0 \\ x-y+z-1=0 \end{cases} \qquad (d_2)\begin{cases} 2x-y+z=1 \\ x-y+z=0. \end{cases}$$

   SOLUTION. We need to find some director parameters of the lines $(d_1)$ and $(d_2)$. In this respect we may solve the two systems. The general solution of the first system is

$$\begin{cases} x=-\frac{1}{3}t+\frac{1}{3} \\ y=\frac{2}{3}t-\frac{2}{3} \\ z=t \end{cases},\ t\in\mathbb{R}$$

   and the general solution of the second system is

$$\begin{cases} x=1 \\ y=t+1 \\ z=t \end{cases},\ t\in\mathbb{R}$$

   and these are the parametric equations of the lines $(d_1)$ and $(d_2)$. Thus, the direction of the line $(d_1)$ is the 1-dimensional subspace

$$\left\langle\left(-\frac{1}{3},\frac{2}{3},1\right)\right\rangle=\langle(-1,2,3)\rangle,$$

   and the direction of the line $(d_2)$ is the 1-dimensional subspace $\langle(0,1,1)\rangle$.

   Consequently, some director parameters of the line $(d_1)$ are $p_1=-1, q_1=2, r_1=3$ and some director parameters of the line $(d_2)$ are $p_2=0, q_2=r_2=1$. Finaly, the equation of the required plane is

$$\begin{vmatrix} x-1 & y-2 & z-1 \\ -1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix}=0.$$

   The computation of the determinant is left to the reader.

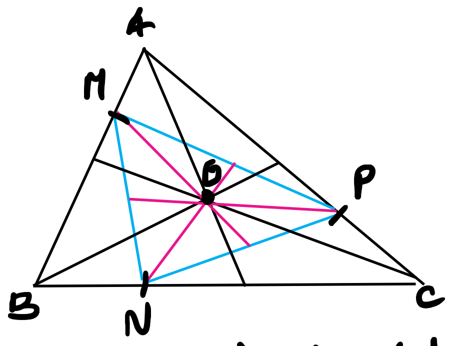
   **A few questions in the two dimensional setting ([1])**

8. The sides $[BC]$, $[CA]$, $[AB]$ of the triangle $\Delta ABC$ are divided by the points $M$, $N$ respectively $P$ into the same ratio $k$. Prove that the triangles $\Delta ABC$ and $\Delta MNP$ have the same center of gravity.

   SOLUTION.

Let $G$ - c.g. for $\Delta ABC$ (1) : $\vec{r_G} = \frac{1}{3}(\vec{r_A}+\vec{r_B}+\vec{r_C})$




$$\left.\begin{array}{l} M\in[AB],\ \frac{AM}{MB}=k \Rightarrow \vec{r}_M=\frac{1}{1+k}\vec{r}_A+\frac{k}{1+k}\vec{r}_B \\ N\in[BC],\ \frac{BN}{NC}=k \Rightarrow \vec{r}_N=\frac{1}{1+k}\vec{r}_B+\frac{k}{1+k}\vec{r}_C \\ P\in[CA],\ \frac{CP}{CA}=k \Rightarrow \vec{r}_P=\frac{1}{1+k}\vec{r}_C+\frac{k}{1+k}\vec{r}_A \end{array}\right\} \Rightarrow$$

$$\Rightarrow \vec{r}_M+\vec{r}_N+\vec{r}_P=\left(\frac{1}{1+k}+\frac{k}{1+k}\right)\vec{r}_A+\left(\frac{1}{1+k}+\frac{k}{1+k}\right)\vec{r}_B+\left(\frac{1}{1+k}+\frac{k}{1+k}\right)\vec{r}_C \Rightarrow$$

$$\Rightarrow \vec{r}_M+\vec{r}_N+\vec{r}_P=\vec{r}_A+\vec{r}_B+\vec{r}_C \,/\cdot\frac{1}{3} \Rightarrow \frac{1}{3}(\vec{r}_M+\vec{r}_N+\vec{r}_P)=\underbrace{\frac{1}{3}(\vec{r}_A+\vec{r}_B+\vec{r}_C)}_{\vec{r}_G} \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow \frac{1}{3}(\vec{r}_M+\vec{r}_N+\vec{r}_P)=\vec{r}_G \\ \Delta MNP-\text{triangle} \end{array}\right\} \Rightarrow G-\text{c.g. for } \Delta MNP \ (2)$$

For (1), (2) $\Rightarrow$ $\Delta ABC$, $\Delta MNP$ - have the same center of gravity

9. Sketch the graph of $x^2 - 4xy + 3y^2 = 0$.

SOLUTION.

$G_f \cap Ox \Rightarrow y=0 \Rightarrow x=0 \Rightarrow O(0,0) \in G_f$

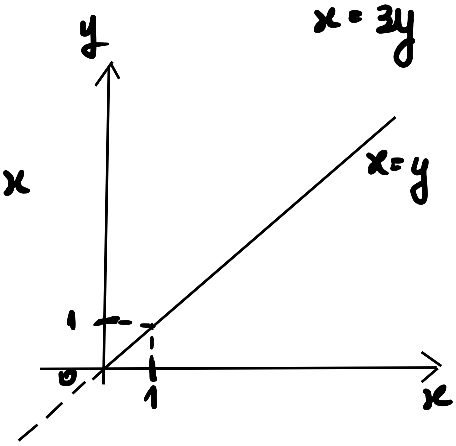
$G_f \cap Oy \Rightarrow x=0 \Rightarrow y=0$

$x^2-4xy+3y^2=0 \Leftrightarrow x^2-xy-3xy+3y^2=0 \Rightarrow x(x-y)-3y(x-y)=0 \Rightarrow$

$\Rightarrow (x-y)(x-3y)=0 \Rightarrow x=y$ or $x=3y$

I. $x=y$

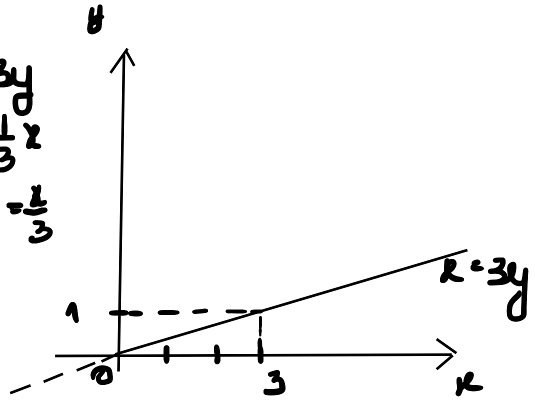
$f(x)=x$



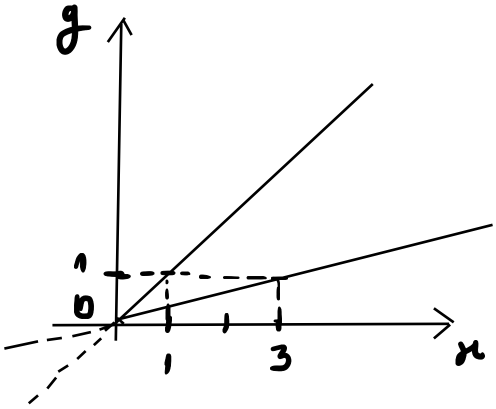

II. $x=3y$

$y=\frac{1}{3}x$

$f(x)=\frac{x}{3}$




$G_f$ for $x^2-4xy+3y^2=0$



10. Find the equation of the line passing through the intersection point of the lines

$$d_1 : 2x - 5y - 1 = 0, \quad d_2 : x + 4y - 7 = 0$$

and through a point $M$ which divides the segment $[AB]$, $A(4, -3)$, $B(-1, 2)$, into the ratio $k = \frac{2}{3}$.

SOLUTION.

Let $\{N\} = d_1 \cap d_2$

$M \in (AB), \frac{AM}{MB} = \frac{2}{3}$

$$\begin{cases} N \in d_1 \\ N \in d_2 \end{cases} \Rightarrow \begin{cases} 2x_N - 5y_N - 1 = 0 \\ x_N + 4y_N - 7 = 0 \;/\cdot(-2) \end{cases} \overset{\oplus}{\Rightarrow} -13y_N + 13 = 0 \Rightarrow y_N = 1 \Rightarrow$$

$$\Rightarrow x_N = 7 - 4y_N = 7 - 4 = 3 \Rightarrow N(3,1)$$

$$M \in (AB), \frac{AM}{MB} = \frac{2}{3} \Rightarrow \frac{AM}{AM+MB} = \frac{2}{2+3} \Rightarrow \frac{AM}{AB} = \frac{2}{5} \Rightarrow AM = \frac{2}{5}AB$$

$$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} = \sqrt{(-1-4)^2 + (2+3)^2} = \sqrt{5^2 + 5^2} = \sqrt{2 \cdot 5^2} = 5\sqrt{2}$$

$$\Rightarrow AM = \frac{2}{\cancel{5}} \cdot \cancel{5}\sqrt{2} = 2\sqrt{2} \Rightarrow \sqrt{(x_M - x_A)^2 + (y_M - x_A)^2} = 2\sqrt{2} \;/^2 \Leftrightarrow$$

$$\Leftrightarrow (x_M - 4)^2 + (y_M + 3)^2 = 8 \quad (1)$$

$$AB : \frac{y - y_A}{y_B - y_A} = \frac{x - x_A}{x_B - x_A} \Rightarrow AB: \frac{y+3}{2+3} = \frac{x-4}{-1-4} \Leftrightarrow AB: \frac{y+3}{5} = \frac{x-4}{-5} \;/\cdot 5 \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow AB: y + 3 = -x + 4 \Rightarrow AB: x + y - 1 = 0 \\ M \in AB \end{array}\right\} \Rightarrow x_M + y_M - 1 = 0 \Rightarrow$$

$$\Rightarrow y_M = 1 - x_M \overset{(1)}{\Rightarrow} (x_M - 4)^2 + (1 - x_M + 3)^2 = 8 \Leftrightarrow$$

$$\Leftrightarrow (x_M - 4)^2 + (4 - x_M)^2 = 8 \Leftrightarrow 2(x_M - 4)^2 = 8 \Leftrightarrow (x_M - 4)^2 = 4 \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow x_M - 4 \in \{-2; 2\} \Rightarrow x_M \in \{2; 6\} \\ M \in (AB), \text{ so } x_M < x_A \\ \text{inside the line } (x_B < x_A \Rightarrow x_B < x_M < x_A) \end{array}\right\} \Rightarrow$$

$$\Rightarrow x_M = 2 \Rightarrow y_M = 1 - 2 = -1 \Rightarrow M(2; -1)$$

$$\begin{matrix} N(3,1)\in d \\ M(2,-1)\in d \end{matrix} \Rightarrow d: \frac{y-y_N}{y_M-y_N}=\frac{x-x_N}{x_M-x_N} \Rightarrow d: \frac{y-1}{-1-1}=\frac{x-3}{2-3} \Leftrightarrow$$

$$\Leftrightarrow d: \frac{y-1}{-2}=\frac{x-3}{-1} \Big/ \cdot(-2) \Rightarrow d: y-1=2x-6 \Rightarrow d: 2x-y-5=0$$

11. Let $A$ be a mobile point on the $Ox$ axis and $B$ a mobile point on $Oy$, so that $\dfrac{1}{OA}+\dfrac{1}{OB}=k$ (constant). Prove that the lines $AB$ passes through a fixed point.

SOLUTION.

$A\in(Ox)$, $B\in(Oy)$, $\dfrac{1}{OA}+\dfrac{1}{OB}=k$

Let $a\in\mathbb{R}, b\in\mathbb{R}$ s.t. $A(a,0)$ and $B(0,b)$. So we have: $\dfrac{1}{a}+\dfrac{1}{b}=k$, $a\neq 0, b\neq 0$

Consider the vector eq. of line $AB$ def. by $A(a,0)$ and vector $\vec{AB}(-a,b)$ (since $\vec{AB}=\vec{AO}+\vec{OB}$)

$\vec{r}=\vec{r_A}+\lambda\vec{AB} \Rightarrow \begin{cases} x=a-\lambda a \\ y=\lambda b\end{cases} \Rightarrow \begin{cases} x=(1-\lambda)a \\ y=\lambda b\end{cases}$

Since $\dfrac{1}{a}+\dfrac{1}{b}=k \Rightarrow \dfrac{1}{ka}+\dfrac{1}{kb}=1 \Rightarrow$

$\Rightarrow \dfrac{\frac{1}{k}}{a}=1-\dfrac{\frac{1}{k}}{b}=t \Rightarrow \begin{cases}\frac{1}{k}=t\cdot a \\ \frac{1}{k}=(1-t)b\end{cases} \Rightarrow$ the point $K\left(\frac{1}{k},\frac{1}{k}\right)$ belongs to the line $AB$ for any choice of $a$ and $b \Rightarrow$

$\Rightarrow AB$ passes through the fixed point $K\left(\frac{1}{k},\frac{1}{k}\right)$

12. Find the equation of the line passing through the intersection point of

$$d_1 : 3x - 2y + 5 = 0, \quad d_2 : 4x + 3y - 1 = 0$$

and crossing the positive half axis of $Oy$ at the point $A$ with $OA = 3$.

SOLUTION.

Let $P(x_P, y_P)$ be such that $d_1 \cap d_2 = \{P\} \Rightarrow \begin{cases} 3x_P - 2y_P + 5 = 0 \,/\cdot 3 \\ 4x_P + 3y_P - 1 = 0 \,/\cdot 2 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} 9x_P - 6y_P + 15 = 0 \\ 8x_P + 6y_P - 2 = 0 \end{cases} \Rightarrow 17x_P + 13 = 0 \Rightarrow x_P = \frac{-13}{17} \Rightarrow$

$\Rightarrow y_P = \frac{3x_P + 5}{2} \Rightarrow y_P = \frac{3 \cdot \left(-\frac{13}{17}\right) + 5}{2} = \frac{-39 + 85}{17 \cdot 2} = \frac{46}{17 \cdot 2} = \frac{23}{17} \Rightarrow$

$\Rightarrow P\left(\frac{-13}{17}, \frac{23}{17}\right)$

$\left.\begin{array}{l} OA = 3 \\ A \in \text{positive half axis of } Oy \end{array}\right\} \Rightarrow A(0,3)$

$\Rightarrow d$ passes through $P$ and $A \Rightarrow m_d = \frac{y_P - y_A}{x_P - x_A} = \frac{\frac{23}{17} - 3}{-\frac{13}{17} - 0} \Rightarrow$

$\Rightarrow m_d = \frac{23 - 3 \cdot 17}{-13} \Rightarrow m_d = \frac{28}{13} \Big| \Rightarrow$

$A \in d$

$\Rightarrow d : y - y_A = m_d(x - x_A) \Rightarrow d : y - 3 = \frac{28}{13}(x - 0) \,/\cdot 13 \Rightarrow$

$\Rightarrow d : 13y - 39 = 28x \Rightarrow d : 28x - 13y + 39 = 0$

13. Find the parametric equations of the line through $P_1$ and $P_2$, when

    (a) $P_1(3,-2)$, $P_2(5,1)$;

    (b) $P_1(4,1)$, $P_2(4,3)$.

    SOLUTION.

(a) $\vec{r} = \vec{u} + \lambda\vec{v}$, $\lambda \in \mathbb{R}$

$\vec{u} = \overrightarrow{OP_1} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\vec{v} = \overrightarrow{P_1P_2} = \begin{pmatrix} 5-3 \\ 1-(-2) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\Rightarrow \vec{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow$

$\Rightarrow \begin{cases} x = 3 + 2\lambda \\ y = -2 + 3\lambda \end{cases}$ - parametric eq

(b) $\vec{r} = \vec{u} + \lambda\vec{v}$, $\lambda \in \mathbb{R}$

$\vec{u}' = \overrightarrow{OP_1} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\vec{v}' = \overrightarrow{P_1P_2} = \begin{pmatrix} 4-4 \\ 1-3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Rightarrow$ param eq

$\Rightarrow \vec{r}' = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Rightarrow \begin{cases} x = 4 \\ y = 1 + 2\lambda \end{cases}$

14. Find the parametric equations of the line through $P(-5, 2)$ and parallel to $\overline{v}(2, 3)$.

SOLUTION.

The vector eq. of this line is eq. to $\vec{r_M} = \vec{r_P} + \lambda \vec{v}$, $\lambda \in \mathbb{R}$

$\Rightarrow \begin{cases} x = -5 + \lambda \cdot 2 \\ y = 2 + \lambda \cdot 3 \end{cases}, \lambda \in \mathbb{R}$ - parametric eq.

15. Show that the equations

$$x = 3 - t, y = 1 + 2t \qquad \text{and} \qquad x = -1 + 3t, y = 9 - 6t$$

represent the same line.

SOLUTION.

$\begin{cases} x = 3-t \\ y = 1+2t \end{cases} \Rightarrow \begin{cases} t = 3-x \\ t = \frac{y-1}{2} \end{cases} \Rightarrow 3-x = \frac{y-1}{2} \Big/ \cdot 2 \Rightarrow 6-2x = y-1 \Rightarrow$

$\Rightarrow 2x+y-7=0$ (eq. $d_1$)

$\begin{cases} x = -1+3t \\ y = 9-6t \end{cases} \Rightarrow \begin{cases} t = \frac{x+1}{3} \\ t = \frac{9-y}{6} \end{cases} \Rightarrow \frac{x+1}{3} = \frac{9-y}{6} \Big/ \cdot 6 \Rightarrow 2x+2 = 9-y \Rightarrow$

$\Rightarrow 2x+y-7=0$ (eq. $d_2$)

We see that $d_1$ and $d_2$ have identical eq, so the eq rep the same line.

16. Find the vector equation of the line $P_1P_2$, where

    (a) $P_1(2,-1)$, $P_2(-5,3)$;

    (b) $P_1(0,3)$, $P_2(4,3)$.

SOLUTION.

(a) $\overrightarrow{P_1P_2}(x_{P_2}-x_{P_1}, y_{P_2}-y_{P_1}) \Rightarrow \overrightarrow{P_1P_2}=(-5-2, 3+1) \Rightarrow \left.\begin{array}{l}\overrightarrow{P_1P_2}(-7,4)\in d\\ P_1(2,-1)\in d\end{array}\right\} \Rightarrow$

$\Rightarrow \vec{r}=\vec{r}_{P_1}+\lambda\overrightarrow{P_1P_2} \Rightarrow \vec{r}=(2,-1)+\lambda(-7,4) \Rightarrow \vec{r}=(2-7\lambda, -1+4\lambda) \Rightarrow$

$\Rightarrow \vec{r}=(2-7\lambda)\vec{i}+(-1+4\lambda)\vec{j}$

(b) $\overrightarrow{P_1P_2}(4-0, 3-3) \Rightarrow \left.\begin{array}{l}\overrightarrow{P_1P_2}(4,0)\in d\\ P_1(0,3)\in d\end{array}\right\} \Rightarrow$

$\Rightarrow \vec{r}=\vec{r}_{P_1}+\lambda\overrightarrow{P_1P_2} \Rightarrow \vec{r}=(0,3)+\lambda(4,0) \Rightarrow \vec{r}=(4\lambda, 3) \Rightarrow$

$\Rightarrow \vec{r}=4\lambda\vec{i}+3\vec{j}$

17. Given the line $d: 2x+3y+4=0$, find the equation of a line $d_1$ through the point $M_0(2,1)$, in the following situations:

   (a) $d_1$ is parallel with $d$;

   (b) $d_1$ is orthogonal on $d$;

   (c) the angle determined by $d$ and $d_1$ is $\varphi = \frac{\pi}{4}$.

   SOLUTION.

(a) $d_1 \parallel d \Rightarrow m_{d_1} = m_d \Rightarrow m_{d_1} = -\frac{2}{3}$

$$\left.\begin{array}{l} d_1: y = m_{d_1}\cdot x + n \\ M_0(2,1)\in d_1 \end{array}\right\} \Rightarrow y_{M_0} = m_{d_1}\cdot x_{M_0} + n \Rightarrow n = y_{M_0} - m_{d_1}\cdot x_{M_0} \Rightarrow$$

$$\Rightarrow n = 1 - \left(-\frac{2}{3}\right)\cdot 2 \Rightarrow n = {}^{3)}1 + \frac{4}{3} \Rightarrow n = \frac{7}{3} \Rightarrow$$

$$\Rightarrow d_1: y = -\frac{2}{3}x + \frac{7}{3} \,\Big/ \cdot 3 \Rightarrow d_1: 3y = -2x + 7 \Rightarrow d_1: 2x + 3y - 7 = 0$$

(b) $d_1 \perp d \Rightarrow m_{d_1}\cdot m_d = -1 \Rightarrow m_{d_1} = \frac{-1}{m_d} \Rightarrow m_{d_1} = \frac{-1}{-\frac{2}{3}} = \frac{3}{2}$

$$\left.\begin{array}{l} d_1: y = m_{d_1}x + n \\ M_0(2,1)\in d_1 \end{array}\right\} \Rightarrow y_{M_0} = m_{d_1}x_{M_0} + n \Rightarrow n = y_{M_0} - m_{d_1}\cdot x_{M_0} \Rightarrow$$

$$\Rightarrow n = 1 - \frac{3}{\cancel{2}}\cdot \cancel{2} \Rightarrow n = -2 \Rightarrow$$

$$\Rightarrow d_1: y = \frac{3}{2}x - 2 \,\Big/ \cdot 2 \Rightarrow d_1: 2y = 3x - 4 \Rightarrow d_1: 3x - 2y - 4 = 0$$

(c)
$$\left.\begin{array}{l} (\widehat{d,d_1}) = \frac{\pi}{4} \\ \text{Not}: (\widehat{d,d_1}) = \varphi \end{array}\right\} \Rightarrow \tan\varphi = \tan\frac{\pi}{4} = 1 \Rightarrow$$

$$\Rightarrow \tan\varphi = \pm\frac{m_{d_1} - m_d}{1 + m_{d_1} m_d}$$

Case 1: $1 = \dfrac{m_{d_1} + \frac{2}{3}}{1 + m_{d_1}\cdot\frac{2}{3}} \Rightarrow 1 = \dfrac{3m_{d_1} + 2}{3 + 2m_{d_1}} \Rightarrow 3 + 2m_{d_1} = 3m_{d_1} + 2 \Rightarrow m_{d_1} = 1 \Rightarrow$

$$\Rightarrow \left.\begin{array}{l} d_1: y = m_{d_1}x + n \\ M_0(2,1)\in d_1 \end{array}\right\} \Rightarrow 1 = 1\cdot 2 + n \Rightarrow n = -1 \Rightarrow d_1: y = x - 1$$

Case 2: $1 = -\dfrac{3m_{d_1} + 2}{3 + 2m_{d_1}} \Rightarrow 3 + 2m_{d_1} = -3m_d - 2 \Rightarrow 5m_{d_1} = -5 \Rightarrow m_{d_1} = -1 \Rightarrow$

$$\Rightarrow \left.\begin{array}{l} d_1: y = m_{d_1}x + n \\ M_0(2,1)\in d_1 \end{array}\right\} \Rightarrow 1 = -2 + n \Rightarrow n = 3 \Rightarrow d_1: y = -x + 3$$

18. The vertices of the triangle $\Delta ABC$ are the intersection points of the lines

$$d_1 : 4x + 3y - 5 = 0, \quad d_2 : x - 3y + 10 = 0, \quad d_3 : x - 2 = 0.$$

   (a) Find the coordinates of $A, B, C$.
   (b) Find the equations of the median lines of the triangle.
   (c) Find the equations of the heights of the triangle.

   SOLUTION.

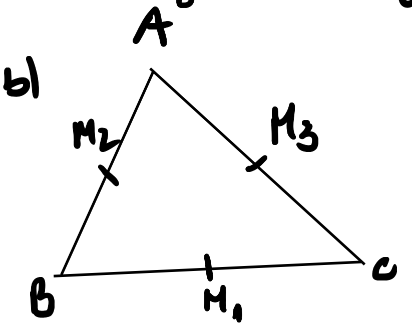
a) $\{A\} = d_2 \cap d_3$, $\{B\} = d_3 \cap d_1$, $\{C\} = d_1 \cap d_2$

$A: \begin{cases} x-2=0 \\ x-3y+10=0 \end{cases} \Rightarrow \begin{cases} x=2 \\ 3y=12 \end{cases} \Rightarrow \begin{cases} x=2 \\ y=4 \end{cases} \Rightarrow A(2,4)$

$B: \begin{cases} 4x+3y-5=0 \\ x-2=0 \end{cases} \Rightarrow \begin{cases} 3y=5-8 \\ x=2 \end{cases} \Rightarrow \begin{cases} y=-1 \\ x=2 \end{cases} \Rightarrow B(2,-1)$

$C: \begin{cases} 4x+3y-5=0 \\ x-3y+10=0 \end{cases} \Rightarrow \begin{cases} 5x+5=0 \\ x-3y+10=0 \end{cases} \Rightarrow \begin{cases} x=-1 \\ -1-3y+10=0 \end{cases} \Rightarrow \begin{cases} x=-1 \\ y=3 \end{cases} \Rightarrow C(-1,3)$

b)




$M_1\left(\frac{2+(-1)}{2}, \frac{-1+3}{2}\right) \Rightarrow M_1\left(\frac{1}{2}, 1\right)$

$M_2\left(\frac{2+(-1)}{2}, \frac{4+3}{2}\right) \Rightarrow M_2\left(\frac{1}{2}, \frac{7}{2}\right)$

$M_3\left(\frac{2+2}{2}, \frac{4+(-1)}{2}\right) \Rightarrow M_3\left(2, \frac{3}{2}\right)$

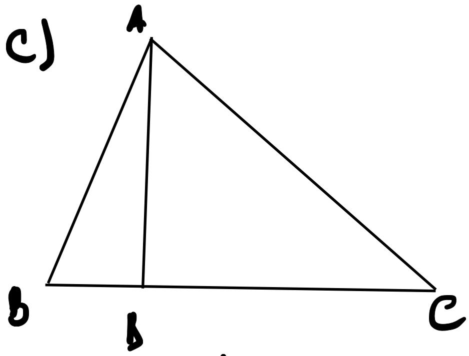
$m_{AM_1} = \frac{y_A - y_{M_1}}{x_A - x_{M_1}} = \frac{4-1}{2-\frac{1}{2}} = \frac{3}{\frac{3}{2}} = 2 \Rightarrow AM: y - y_{M_1} = m_{AM_1}(x - x_{M_1}) \Rightarrow$

$\Rightarrow AM_1: y-1 = 2\left(x-\frac{1}{2}\right)$

$BM_2: \frac{y-y_B}{y_{M_2}-y_B} = \frac{x-x_B}{x_{M_2}-x_B} \Rightarrow \frac{y+1}{\frac{7}{2}+1} = \frac{x-2}{\frac{1}{2}-2}$

$CM_3: \frac{y-y_C}{y_{M_3}-y_C} = \frac{x-x_C}{x_{M_3}-x_C} \Rightarrow \frac{y-3}{\frac{3}{2}-3} = \frac{x+1}{2+1}$

c)



$AD \perp BC \Rightarrow m_{AD} \cdot m_{BC} = -1$

$m_{BC} = \frac{y_B - y_C}{x_B - x_C} = \frac{2+1}{-1-3} = \frac{-3}{4}$

$\Rightarrow m_{AD} = \frac{4}{3} \Rightarrow$

$\Rightarrow AD: y - y_A = m_{AD}(x - x_A) \Rightarrow$

$\Rightarrow y - 4 = \frac{4}{3}(x-2)$

For the others, similarly

1) $M_0(-1,2,0)$, $\vec{v_1}(1,2,3)$, $\vec{v_2}(0,-1,6)$

$$\begin{vmatrix} x+1 & y-2 & z \\ 1 & 2 & 3 \\ 0 & -1 & 6 \end{vmatrix} = 0 \Rightarrow 12(x+1) - z + 3(x+1) - 6(y-2) = 0 \Leftrightarrow$$

$$\Leftrightarrow 15(x+1) - 6(y-2) - z = 0 \Leftrightarrow$$

$$\Leftrightarrow 15x - 6y - z + 27 = 0$$

To go to parametric: $\begin{cases} x = \lambda \\ y = \mu \\ z = 15\lambda - 6\mu + 27 \end{cases}$