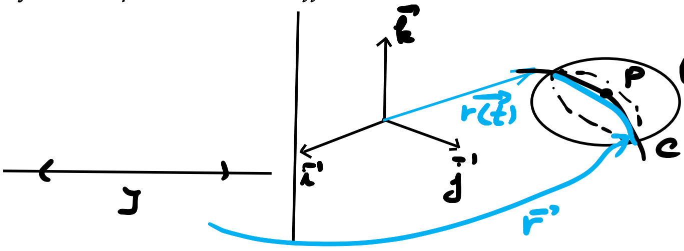
8 Week 8: Curves and surfaces

8.1 Regular curves

Definition 8.1. A subset C of $\mathbb{R}^2$ or $\mathbb{R}^3$ is said to be a *regular curve* if for every $p \in C$ there exists a neighbourhood U of p in $\mathbb{R}^2$ or $\mathbb{R}^3$ respectively and a *parametrized differentiable curve* $r : I \to U \cap C$, where $I \subseteq \mathbb{R}$ is an open set, such that

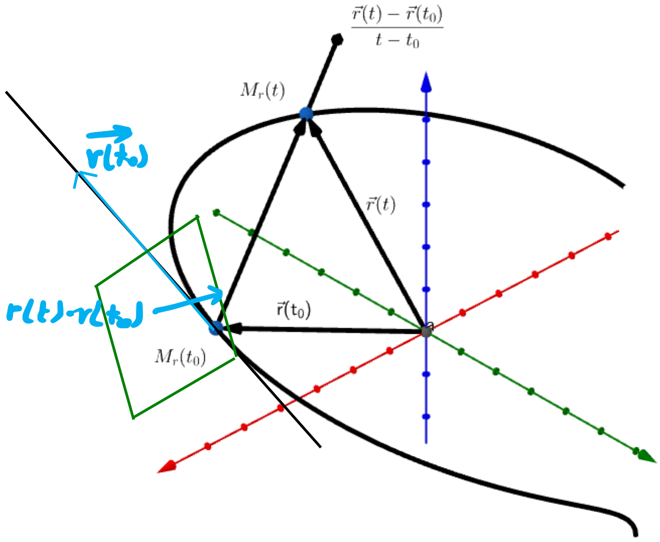
1. r is smooth;
2. $r : I \to U \cap C$ is a homeomorphism;
   r, r^{-1} are continuous
3. r is regular, i.e. $\overrightarrow{r}'(t) \neq \overrightarrow{0}$, $\forall\, t \in I$.



The parametrized differentiable curve $r : I \to U \cap C$ is called *local parametrization* or *local system of coordinates* at p and $U \cap C$ is called *coordinate neighbourhood* at p. Recall that the the tangent line of the local parametrization $r : I \to U \cap C$ at $r(t_0)$, for some $t_0 \in I$, is defined as the limit position of the line $M_r(t_0)M_r(t)$ as $t \longrightarrow t_0$. This tangent line is denoted by $(Tr)(t_0)$. A director vector of the line $M_r(t_0)M_r(t)$ is obviously

$$\frac{\overrightarrow{r}(t) - \overrightarrow{r}(t_0)}{t - t_0}, = \frac{1}{t-t_0} \cdot (\overrightarrow{r(t)} - \overrightarrow{r(t_0)})$$

which shows that $\overrightarrow{r}'(t_0)$ is a director vector of $(Tr)(t_0)$ and the direction of $(Tr)(t_0)$ is therefore $(d\overrightarrow{r})_{t_0}(\mathbb{R})$.



If $r_1 : I_1 \to U_1 \cap C$ and $r_2 : I_2 \to U_2 \cap C$ are two local parametrizations of of C at $p \in C$, then $r_1(t_1) = r_2(t_2) = p$ for some $t_1 \in I_1$ and $t_2 \in I_2$ and one can easily show that $(d\overrightarrow{r}_1)_{t_1}(\mathbb{R}) = (d\overrightarrow{r}_2)_{t_2}(\mathbb{R})$. This shows that r_1 and r_2 have the same tangent line at $r_1(t_1) = r_2(t_2) = p$.

Proposition 8.1. *The equation of the paremetrized differentiable curve $r : I \longrightarrow \mathbb{R}^2$, $r(t) = (x(t), y(t))$ at $r(t_0)$, for some* regular *point $t_0 \in I$, i.e. $\overrightarrow{r}'(t_0) \neq \overrightarrow{0}$ is*

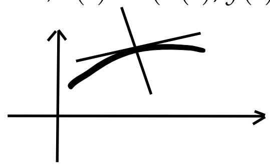
$$(Tr)(t_0) : \frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)}. \tag{8.1}$$

The equation of the normal line to r at $r(t_0)$, i.e. the line through $M_r(t_0)$ which is perpendicular to $(Tr)(t_0)$ is

$$(Nr)(t_0)\ x'(t_0)(x - x(t_0)) + y'(t_0)(y - y(t_0)) = 0. \tag{8.2}$$

$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $\vec{r}'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}$

Proposition 8.2. *The equation of the paremetrized differentiable curve $r : I \longrightarrow \mathbb{R}^3, r(t) = (x(t), y(t), z(t))$ at $r(t_0)$, for some* regular *point $t_0 \in I$, i.e. $\overrightarrow{r}'(t_0) \neq \overrightarrow{0}$ is*



$$(Tr)(t_0) : \frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)}. \tag{8.3}$$

The equation of the normal plane to r at $r(t_0)$, i.e. the plane through $M_r(t_0)$ which is perpendicular to $(Tr)(t_0)$ is

$$(Nr)(t_0)\; x'(t_0)(x - x(t_0)) + y'(t_0)(y - y(t_0)) + z'(t_0)(z - z(t_0)) = 0. \tag{8.4}$$

Remark 8.1.

1. The requirement (3) of definition (8.1), is equivalent with $(dr)_t \neq 0,\ \forall\, t \in \mathbb{R}$;
2. $U \cap C$ is the image of a regular one-to-one parametrized differentiable curve. On the other hand, there are regular one-to-one parametrized differentiable curves whose images are not parts of regular curves;
3. The role of requirement (2) in definition (8.1) is to prevent the self-intersections of the regular curves, which is not the case with the images of regular parametrized differentiable curves.
4. The requirement (3) combined with (2) ensure the existence of a unique tangent line at every point of a regular curve. The tangent line $T_p(C)$ of C at $p \in C$ is defined as the tangent line at p of a local parametrization $r : I \to U \cap C$ of C at p. The tangent line $T_p(C)$ is well-defined as the tangent at p of a local parametrization $r : I \to U \cap C$ at p is independent of r.

Definition 8.2. If $U \subseteq \mathbb{R}^2$ is an open set, $f : U \longrightarrow \mathbb{R}$ is a C^1-smooth function, then the value $a \in \mathrm{Im}(f)$ of f is said to be *regular* if $(\nabla f)(x, y) \neq 0,\ \forall (x, y) \in f^{-1}(a)$, i.e. $(df)_{(x,y)} \neq 0,\ \forall (x, y) \in f^{-1}(a)$.

Theorem 8.3. *(The preimage theorem) If $U \subseteq \mathbb{R}^2$ is an open set, $f : U \longrightarrow \mathbb{R}$ is a C^1-smooth function and $a \in Imf$ is a regular value of f, then the inverse image of a through f,*

$$f^{-1}(a) = \{(x, y) \in U | f(x, y) = a\}$$

is a planar regular curve called the regular curve of implicit ~~cartezian~~ Cartesian equation *$f(x, y) = a$.*

Definition 8.3. Let $U \subset \mathbb{R}^2$ be an open set such that $tx \in U$ for every $t \in \mathbb{R}^*_+$ and avery $x \in U$. The function $f : U \to \mathbb{R}$ is said to be *homogeneouos of order $p \in \mathbb{R}$* whenever $f(tx) = t^p f(x),\ \forall t \in \mathbb{R}^*_+, x \in U$.

For example a homogeneous polynomial function of degree $n \in \mathbb{N}$ is a a homogeneous function of order ~~p~~ n.

Example 8.1. If $f : U \to \mathbb{R}$ is a C^1-smooth homogeneous function of order $p \in \mathbb{R}^*$ and $c \in Im\, f \backslash \{0\}$, then $f^{-1}(c)$ is a regular curve.

Indeed, it is enough to show that c is a regular value of f. By differentiating the relation $f(tx) = t^p f(x)$ with respect to t we obtain:

$$\frac{d}{dt} f(tx) = \frac{d}{dt} t^p f(x) \Rightarrow (df)_{tx}(x) = pt^{p-1} f(x),\ \forall t \in \mathbb{R}^*_+, x \in U,$$

and the Euler's relation

$$(df)_x(x) = pf(x),\ \forall x \in U. \tag{8.5}$$

follow for $t = 1$. But for $x \in C(f)$ we have $(df)_x = 0$ and thus $(df)_x(x) = 0$, namely $f(x) = 0$. We therefore showed that $B(f) = f(C(f)) \subset \{0\}$, or, equivalently, $\mathbb{R}^* \subset \mathbb{R} \backslash B(f)$, where $C(f) \subseteq U$ stands for the closed set of critical points of f, i.e. $C(f) := \{(x, y) \in U | (df)_{(x,y)} = 0\}$. But since $c \in Im\, f \backslash \{0\}$ we deduce that c is a regular value of f and $f^{-1}(c)$ is a regular curve therefore.

1. The elipse $(E)\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a regular curve in $\mathbb{R}^2$
2. The hyperbola $(H)\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a regular curve in $\mathbb{R}^2$

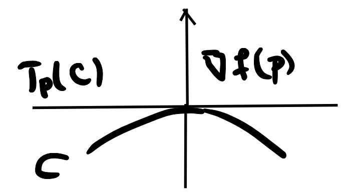
3. The parabola $(P)\ y^2 = 2px$ is a regular curve in $\mathbb{R}^2$

Proposition 8.4. *The equation of the tangent line $T_{(x_0,y_0)}(C)$ of the planar regular curve C of implicit cartezian equation $f(x,y) = a$ at the point $p = (x_0, y_0) \in C$, is*

$$T_{(x_0,y_0)}(C) : f'_x(p)(x - x_0) + f'_y(p)(y - y_0) = 0,$$ $\nabla f(p) = (f_x(p), f_y(p))$

and the equation of the normal line $N_{(x_0,y_0)}(C)$ of C at p is

$$N_{(x_0,y_0)}(C) : \frac{x - x_0}{f'_x(p)} = \frac{y - y_0}{f'_y(p)}.$$



Example 8.2. The tangent line of the general conic

$$C : \underbrace{a_{00} + 2a_{10}x + 2a_{20}y + a_{11}x^2 + 2a_{12}xy + a_{22}y^2}_{f(x,y)} = 0$$

at some of its regular point $(x_0, y_0) \in C$ is

$$a_{00} + a_{10}(x + x_0) + a_{20}(y + y_0) + a_{11}x_0x + a_{12}(xy_0 + x_0y) + a_{22}y_0y = 0 \tag{8.6}$$

and can be obtained by polarizing the conic's equation, i.e. by replacing:

1. x^2 with x_0x $\quad f_x = 2a_{10} + 2a_{11}x + 2a_{12}y$, $f(y) = 2a_{20} + 2a_{12}x + 2a_{22}y$
2. y^2 with y_0y
3. $2x$ with $x + x_0$
4. $2y$ with $y + y_0$
5. $2xy$ with $x_0y + xy_0$.

Indeed, $C = f^{-1}(0)$, where $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a second degree polynomial function given by $f(x,y) = a_{00} + 2a_{10}x + 2a_{20}y + a_{11}x^2 + 2a_{12}xy + a_{22}y^2$. Since

$$f_x = 2a_{10} + 2_{11}x + 2a_{12}y \text{ and } f_y = 2a_{20} + 2a_{12}x + 2a_{22}y,$$

it follws that

$$\begin{aligned}
T_{(x_0,y_0)}(C) \quad &: \quad (2a_{10} + 2_{11}x + 2a_{12}y)(x - x_0) + (2a_{20} + 2a_{12}x + 2a_{22}y)(y - y_0) = 0 \\
&\iff a_{10}x + a_{11}x_0x + a_{12}y_0x + a_{20}y + a_{12}x_0y + a_22y_0y = a_{10}x_0 + a_{11}x_0^2 + a_{12}y_0x_0 + a_{20}y_0 + a_{12}x_0y_0 + a_{22}y_0^2 \\
&\iff a_{10}(x + x_0) + a_{20}(y + y_0) + a_{11}x_0x + a_{12}(xy_0 + x_0y) + a_{22}y_0y = 2a_{10}x + 0 + 2a_{20}y_0 + a_{11}x_0^2 + 2a_{12}x_0y_0 + a_{22}y_0^2 \\
&\iff a_{00} + a_{10}(x + x_0) + a_{20}(y + y_0) + a_{11}x_0x + a_{12}(xy_0 + x_0y) + a_{22}y_0y = 0.
\end{aligned}$$

8.2 Parametrized differentiable surfaces

Definition 8.4. Let $U \subseteq \mathbb{R}^2$ be an open set. A smooth map $r : U \to \mathbb{R}^3$ is said to be a *parametrized differentiable surface*. The set $r(U)$ is called the *trace*, the *support*, or *the image* of r. If the differential $(dr)_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for $q \in U$, then the parametrized differentiable surface r is said to be *regular* at q. If the differential $(dr)_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for all $q \in U$, then the parametrized differentiable surface r is said to be *regular*.

Remark 8.2. Let $U \subseteq \mathbb{R}^2$ be an open set and $r : U \to \mathbb{R}^3$, $r(u,v) = (x(u,v), y(u,v), z(u,v))$ be a parametrized differentiable surface. Then r is regular at $q \in U$ if and only if

$$\vec{r_u}(q) \times \vec{r_v}(q) \neq \vec{0}.$$

$\vec{r_u}(u,v) = x_u(u,v)\vec{i} + y_u(u,v)\vec{j} + z_u(u,v)\vec{k}$
$\vec{r_v}(u,v) = x_v(u,v)\vec{i} + y_v(u,v)\vec{j} + z_v(u,v)\vec{k}$

Indeed, $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$

$$\begin{aligned}
r \text{ is regular at } q \in U &\iff (dr)_q : \mathbb{R}^2 \to \mathbb{R}^3 \text{ is one-to-one} \\
&\iff (dr)_q(e_1),\ (dr)_q(e_2) \text{ are linearly independent } (e_1 = (1,0),\ e_2 = (0,1)) \\
&\iff \vec{r_u}(q) = (d\vec{r})_q(e_1),\ \vec{r_v}(q) = (d\vec{r})_q(e_2) \text{ are linearly independent} \\
&\iff \vec{r_u}(q) \times \vec{r_v}(q) \neq \vec{0},
\end{aligned}$$

where $\vec{r} : U \longrightarrow \mathcal{V}$, $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$.

The image of a parametrized differentiable surface might have self-intersections.

8.2.1 The tangent plane and the normal line to a parametrized surface

Definition 8.5. Let $r : U \to \mathbb{R}^3$, $r(u,v) = (x(u,v), y(u,v), z(u,v))$ be a regular parametrized differentiable surface and $q = (u_0, v_0) \in U$. The plane $(Tr)(q)$ through $M_r(u_0, v_0)$, whose direction is $(d\overrightarrow{r})_q(\mathbb{R}^2)$, is called the *tangent plane* to r at $M_r(q)$ corresponding to the pair (u_0, v_0) of the parameters. The perpendicular line $(Nr)(q)$ on $(Tr)(q)$ at $M_r(q)$ is called *the normal line* to r at $M_r(q)$ corresponding to the pair (u_0, v_0) of the parameters. $M_r(q) = M_r(q_1)$

Remark 8.3. If $r : U \to \mathbb{R}^3$, $r(u,v) = (x(u,v), y(u,v), z(u,v))$ is a regular parametrized differentiable surface and $q = (u_0, v_0) \in U$, then the vectors $\overrightarrow{r_u}(q) = (d\overrightarrow{r})_q(1,0)$, $\overrightarrow{r_v}(q) = (d\overrightarrow{r})_q(0,1)$ form a basis of the two dimensional vector subspace $(d\overrightarrow{r})(\mathbb{R}^2)$ of $\mathcal{V}$ şi and $\overrightarrow{v}(q) = \overrightarrow{r_u}(q) \times \overrightarrow{r_v}(q)$ is therefore a director vector of the normal line to r at $M_r(q)$ corresponding to the pair (u_0, v_0) of the parameters.

$$\begin{aligned}
\overrightarrow{v}(q) &= \overrightarrow{r_u}(q) \times \overrightarrow{r_v}(q) = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ x_u(q) & y_u(q) & z_u(q) \\ x_v(q) & y_v(q) & z_v(q) \end{vmatrix} \\
&= \frac{\partial(y,z)}{\partial(u,v)}(q)\,\overrightarrow{i} + \frac{\partial(z,x)}{\partial(u,v)}(q)\,\overrightarrow{j} + \frac{\partial(x,y)}{\partial(u,v)}(q)\,\overrightarrow{k},
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(u,v)}(q) &= \begin{vmatrix} x_u(q) & y_u(q) \\ x_v(q) & y_v(q) \end{vmatrix}, \\
\frac{\partial(z,x)}{\partial(u,v)}(q) &= \begin{vmatrix} z_u(q) & x_u(q) \\ z_v(q) & x_v(q) \end{vmatrix}, \\
\frac{\partial(y,z)}{\partial(u,v)}(q) &= \begin{vmatrix} y_u(q) & z_u(q) \\ y_v(q) & z_v(q) \end{vmatrix}.
\end{aligned}$$

Proposition 8.5. *If $r : U \to \mathbb{R}^3$ $r(u,v) = (x(u,v), y(u,v), z(u,v))$ regular parametrized differentiable surface and $q = (u_0, v_0) \in U$, then the equation of the tangent plane to r at $M_r(q)$, corresponding to the pair (u_0, v_0) of the parameters, is*

$$\begin{vmatrix} x - x(q) & y - y(q) & z - z(q) \\ x_u(q) & y_u(q) & z_u(q) \\ x_v(q) & y_v(q) & z_v(q) \end{vmatrix} = 0,$$

i.e.

$$\frac{\partial(y,z)}{\partial(u,v)}(q)(x - x(q)) + \frac{\partial(z,x)}{\partial(u,v)}(q)(y - y(q)) + \frac{\partial(x,y)}{\partial(u,v)}(q)(z - z(q)) = 0 \tag{8.7}$$

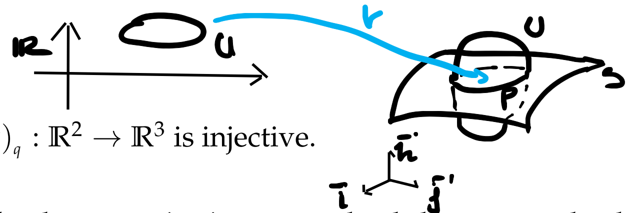
Also, the equation of the normal line to r at $M_r(q)$, corresponding to the pair (u_0, v_0) of the parameters, is:

$$\frac{x - x(q)}{\frac{\partial(y,z)}{\partial(u,v)}(q)} = \frac{y - y(q)}{\frac{\partial(z,x)}{\partial(u,v)}(q)} = \frac{z - z(q)}{\frac{\partial(x,y)}{\partial(u,v)}(q)} \tag{8.8}$$

8.3 Regular surfaces

Definition 8.6. A subset $S \subseteq \mathbb{R}^3$ is called *regular surface* if, for every point $p \in S$, there exists a neighbourhood V of p, in $\mathbb{R}^3$, and a mapping $r : U \to V \cap S$, $r(u,v) = (x(u,v), y(u,v), z(u,v))$, where $U \subseteq \mathbb{R}^2$ is an open set, with the following properties:

1. r is smooth, i.e. its coordinate functions x, y, z have arbitrary high continuous partial derivatives;



2. r is a homeomorphism;
3. For every $q \in U$, the differential $(dr)_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.

The function $r : U \to V \cap S$ is called *local parametrization* at p or *local chart* at p or *local coordinate system* at p. The neighbourhood $V \cap S$ of p in S is called *coordinate neighbourhood*. The equations

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \quad (u,v) \in U,$$

are called the *parametric equations of* of the coordinate neighbourhood $V \cap S$. The equation

$$\overrightarrow{r} = \overrightarrow{r}(u,v) \text{ where } \overrightarrow{r}(u,v) = x(u,v)\overrightarrow{i} + y(u,v)\overrightarrow{j} + z(u,v)\overrightarrow{k}$$

is called the *vector equation* of the coordinate neighbourhood $V \cap S$.

Remark 8.4. 1. Every open subset O of a regular surface $S \subseteq \mathbb{R}^3$ is a regular surface. Indeed every local parametriation $r : U \longrightarrow S \cap V$ of S at some point $p \in O$ produces a local parametrization

$$U \cap r^{-1}(O) \longrightarrow S \cap C \cap V, \; q \mapsto r(q)$$

of O at p.

2. Every regular surface can be covered by the traces of some families of local charts. Such a family of lacal charts is called an *atlas* of the surface. If the regular surface is compact, then it obviously admits finite atlases. For example the 2-sphere

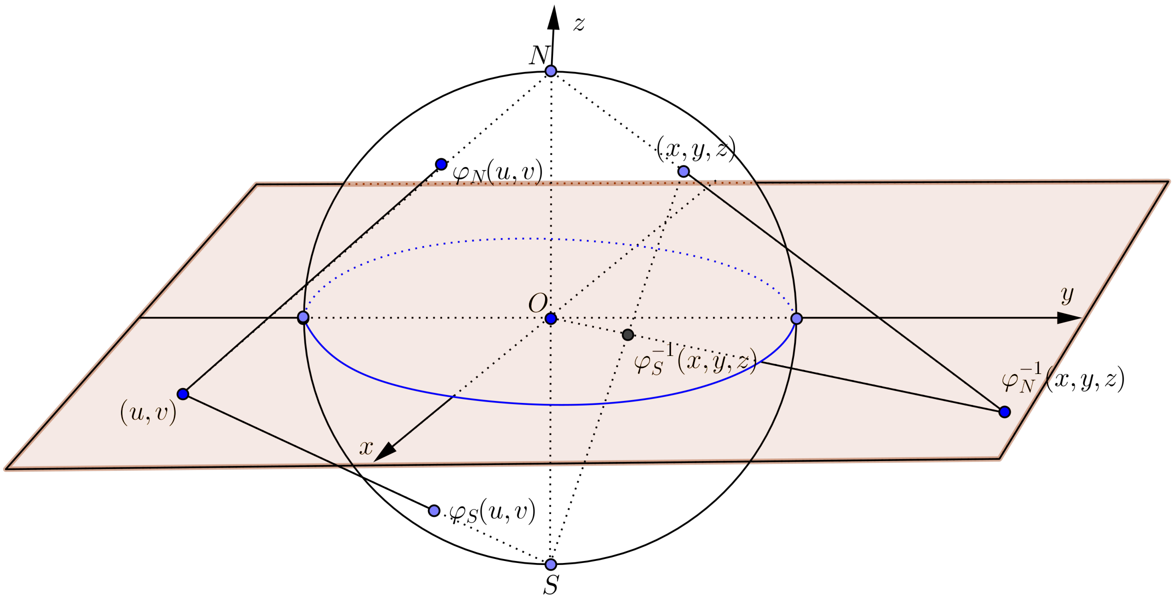
$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \; : \; x^2 + y^2 + z^2 = 1\}$$

admits an atlas with two local charts $\mathcal{A} = \{(\varphi_S, \varphi_N\}$, where

$$\varphi_S : \mathbb{R}^2 \longrightarrow S^2 \setminus \{S\}, \; \varphi_S(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2}\right)$$
$$\varphi_N : \mathbb{R}^2 \longrightarrow S^2 \setminus \{N\}, \; \varphi_N(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right)$$

and $S = (0,0,-1)$, $N = (0,0,1)$ are the south and north poles of S^2.
Note that the inverses of φ_S and φ_N are the stereogrphic projections

$$\varphi_S^{-1} : S^2 \setminus \{S\} \longrightarrow \mathbb{R}^2, \; \varphi_S^{-1}(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$
$$\varphi_N^{-1} : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2, \; \varphi_N^{-1}(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

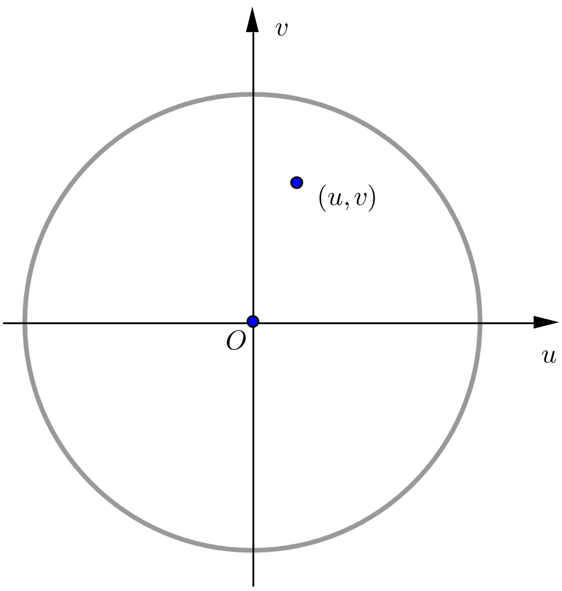


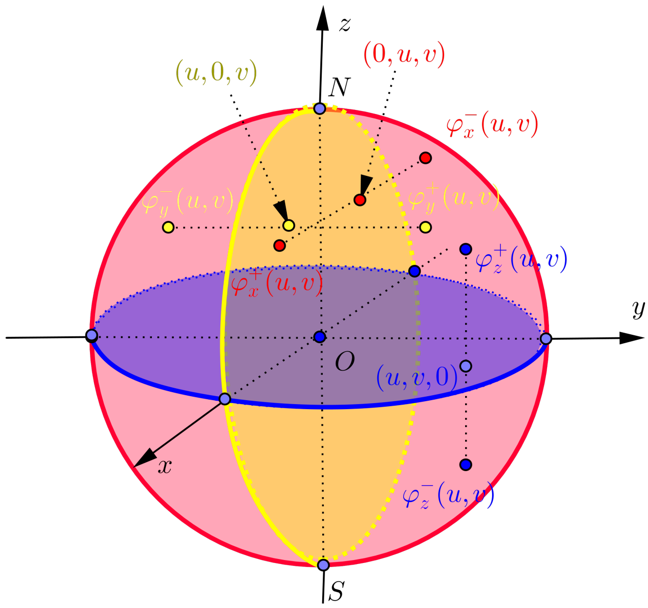
Another atlas of the sphere S^2 has 6 local charts, namely

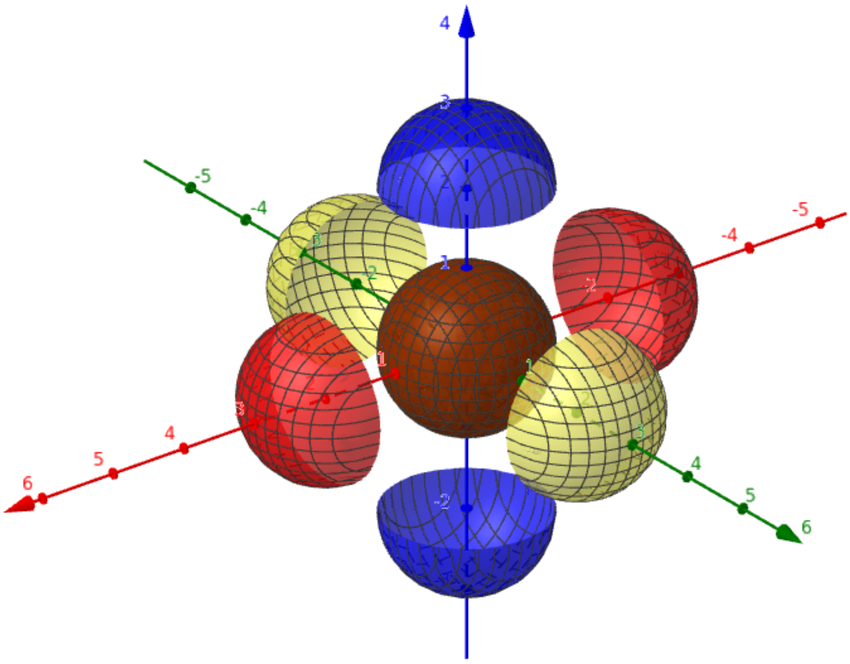
$$\mathcal{A}_1 = \{\varphi_x^{\pm}, \varphi_y^{\pm}, \varphi_z^{\pm} : B(0,1) \longrightarrow S^2\},$$

where $B(0,1)$ is the unit ball of $\mathbb{R}^2$ centered at the origin $0 \in \mathbb{R}^2$ and

$$\varphi_x^{\pm}(u, v) = (\pm\sqrt{1 - u^2 - v^2}, u, v),$$
$$\varphi_y^{\pm}(u, v) = (u, \pm\sqrt{1 - u^2 - v^2}, v),$$
$$\varphi_z^{\pm}(u, v) = (u, v, \pm\sqrt{1 - u^2 - v^2}).$$







Proposition 8.6. *If $U \subseteq \mathbb{R}^2$ is an open set and $f : U \to \mathbb{R}$ is a smooth function, then its graph $G_f = \{(x, y, f(x, y)) \mid (x, y) \in U\}$ is a regular surface.*

For example

1. The elliptic paraboloid $P_e : \dfrac{x^2}{p} + \dfrac{y^2}{q} = 2z,\ p, q > 0$ is a regular surface, as P_e is the graph of the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x, y) = \dfrac{1}{2}\left(\dfrac{x^2}{p} + \dfrac{y^2}{p}\right)$.

2. The hyperbolic paraboloid $P_h : \dfrac{x^2}{p} - \dfrac{y^2}{q} = 2z,\ p, q > 0$ is a regular surface, as P_h is the graph of the function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $g(x, y) = \dfrac{1}{2}\left(\dfrac{x^2}{p} - \dfrac{y^2}{q}\right)$.

Theorem 8.7. *(The third preimage theorem). If $U \subseteq \mathbb{R}^3$ is an open set, $f : U \to \mathbb{R}$ is a smooth functionand $a \in Im\, f$ is a regualr value of f, then*

$$f^{-1}(a) = \{(x, y, z) \in U \mid f(x, y, z) = a\}$$

is a regular surface in $\mathbb{R}^3$ called the regular surface of implicite Cartesian equation $f(x, y, z) = a$.

Proposition 8.8. *Let $U \subset \mathbb{R}^3$ be an open set such that $tx \in U$ for every $t \in \mathbb{R}_+^*$ and every $x \in U$. A function $f : U \to \mathbb{R}$ is said to be* homogeneous of order $p \in \mathbb{R}$ *if $f(tx) = t^p f(x),\ \forall t \in \mathbb{R}_+^*, x \in U$. If $f : U \to \mathbb{R}$ is a differentiable and homogeneous function of order $p \in \mathbb{R}^*$ and $c \in Im\, f \setminus \{0\}$, then $f^{-1}(c)$ is a regular surface.*

Proof. Indeed, it is enough to show that c is a regular value of f. Differentiating with respect to t, the relation $f(tx) = t^p f(x)$ we obtain

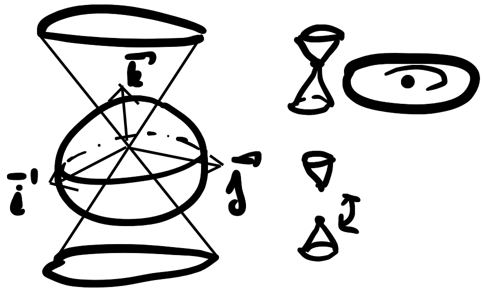
$$(df)_{tx}(x) = pt^{p-1}f(x),\ \forall t \in \mathbb{R}^*_+,\ \forall x \in U,$$

which shows, by taking $t = 1$, the Euler relation

$$(df)_x(x) = pf(x),\ \forall x \in U. \tag{8.9}$$

But for $x \in C(f)$ we have $(df)_x = 0$ and thus $(df)_x(x) = 0$, which shows that $f(x) = 0$. We therefore showed that $B(f) = f(C(f)) \subset \{0\}$, or, equivalently, $\mathbb{R}^* \subset \mathbb{R}\setminus B(f)$. But since $c \in Im\, f\setminus\{0\}$ we deduce that c is a taken regular value of f, which shows that $f^{-1}(c)$ is a regular surface. □

In particular,



1. the ellipsoid $\mathcal{E} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1,$
2. the hyperboloid of one sheet $H_1 : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1,$
3. the hyperboloid of two seets $H_2 : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1.$

are all regular surfaces. Let us finally observe that the cone $C : x^2 + y^2 - z^2 = 0$ is not a regular surface.

8.4 The tangent vector space

Let $S \subseteq \mathbb{R}^3$ be a regular surface and $p \in S$. A *tangent vector* to S at p is the tangent vector $\overrightarrow{\alpha}'(0)$ of a parametrized diferentiable curve $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = p$

Proposition 8.9. *Let $U \subseteq \mathbb{R}^2$ be an open set, let $q \in U$ and let $r : U \to S$ be a local parametrization of S. The 2-dimensional subspace $(d\overrightarrow{r})_q(\mathbb{R}^2) \subseteq \mathcal{V}$ coincides with the set of all tangent vectors to S at $r(q)$.*

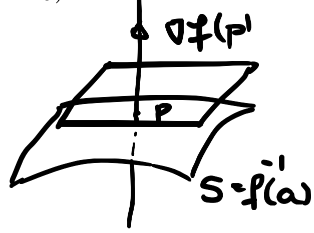
Definition 8.7. The plane through a point p of a regular surface S, whose direction is the tangent space to S at p, $\overrightarrow{T}_p(S)$, is called the *tangent plane* to S at p and is denoted by $T_p(S)$. The perpendicular line on the tangent plane of the surface S at p is called the *normal line* to the surface S at p.

Proposition 8.10. *If $V \subseteq \mathbb{R}^3$ is an open set, $f : V \to \mathbb{R}$ is a smooth function, $a \in Im\, f$ is a regular value of f and $p \in f^{-1}(a)$, then the equation of the tangent plane to the regular surface $S = f^{-1}(a)$, of implicit equation $f(x, y, z) = a$, at some point $p \in S$ is:*

$$f_x(p)(x - x_0) + f_y(p)(y - y_0) + f_z(p)(z - z_0) = 0. \tag{8.10}$$

and the equation of the normal line to S at p is:

$$\frac{x - x_0}{f_x(p)} = \frac{y - y_0}{f_y(p)} = \frac{z - z_0}{f_z(p)} \tag{8.11}$$



For example the tanget plane of the quadric

$$(Q)\ a_{00} + 2a_{10}x + 2a_{20}y + 2a_{30}z + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + a_{11}x^2 + a_{22}y^2 + a_{33}z^2 = 0$$

at some of its point $A_0(x_0, y_0, z_0) \in Q$ is

$$T_{A_0}(Q)\ a_{00} + a_{10}(x + x_0) + a_{20}(y + y_0) + a_{30}(z + z_0) + a_{12}(x_0y + xy_0) + a_{13}(z_0x + zx_0) + 2a_{23}(y_0z + yz_0) \\ + a_{11}x_0x + a_{22}y_0y + a_{33}z_0z = 0.$$

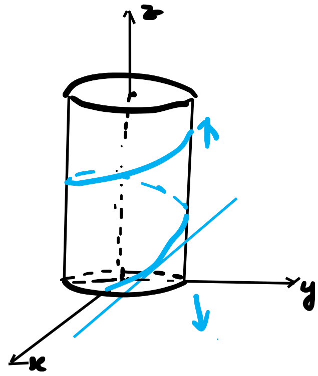
and can be obtained by polarizing the quadric's equation, i.e. by replacing

1. x^2 with x_0x
2. y^2 with y_0y
3. z^2 with z_0z
4. $2x$ with $x+x_0$
5. $2y$ with $y+y_0$
6. $2z$ with $z+z_0$
7. $2xy$ with x_0y+xy_0
8. $2yz$ with y_0z+yz_0
9. $2zx$ with z_0x+zx_0.

8.5 Problems

1. **(2p)** Show that the angle between the tangent of the circular helix

$$\begin{cases} x = a\cos t \\ y = a\sin t \\ z = bt \end{cases}$$



$x^2+y^2=a^2$

and the z-axis is constant.

Solution.

$\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j} + bt\vec{k}$

$\vec{r}'(t) = -a\sin t\,\vec{i} + a\cos t\,\vec{j} + b\vec{k}$

$\cos(\widehat{\vec{k},\vec{r}'(t)}) = \dfrac{\vec{k}\cdot\vec{r}'(t)}{\|\vec{k}\|\cdot\|\vec{r}'(t)\|} = \dfrac{b}{1\cdot\|\vec{r}'(t)\|} = \dfrac{b}{\|\vec{r}'(t)\|} = \dfrac{b}{\sqrt{a^2+b^2}}$ - const $\Rightarrow$

$\|\vec{r}'(t)\|^2 = (-a\sin(t))^2 + (a\cos t)^2 + b^2 = a^2\sin^2(t) + a^2\cos^2 t + b^2 = a^2+b^2$

$\Rightarrow m(\vec{k},\vec{r}'(t))$ - const.

$T_G(t=t_0): \dfrac{x-x(t_0)}{x'(t_0)} = \dfrac{y-y(t_0)}{y'(t_0)} = \dfrac{z-z(t_0)}{z'(t_0)} \Leftrightarrow \dfrac{x-a\cos(t_0)}{-a\sin(t_0)} = \dfrac{y-a\sin(t_0)}{a\cos(t_0)} = \dfrac{z-bt_0}{b} \Rightarrow$

$\Rightarrow \vec{v}_{T_G}(t=t_0): (-a\sin(t_0), a\cos(t_0), b)$

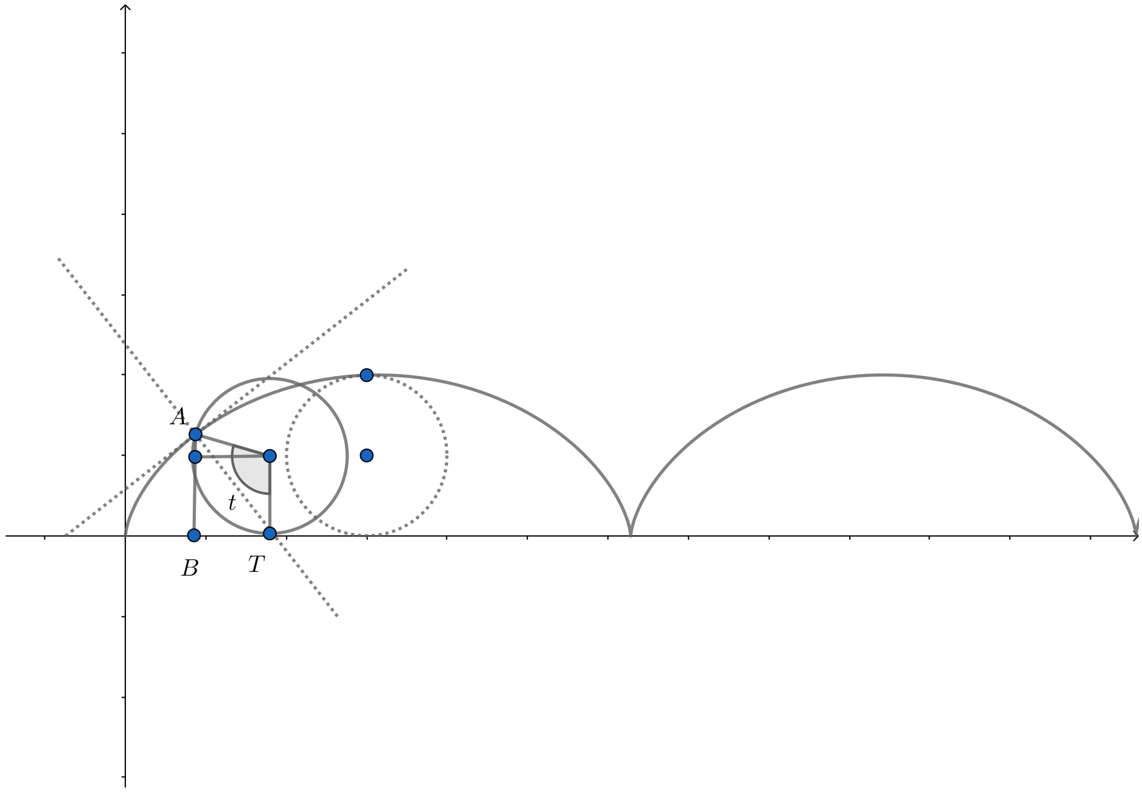
$\vec{v}_{Oz}(0,0,1)$

$m(\widehat{T_G, Oz}) = \arccos\dfrac{\vec{v}_{T_G}\cdot\vec{v}_{Oz}}{\|\vec{v}_{T_G}\|\,\underbrace{\|\vec{v}_{Oz}\|}_{1}} = \arccos\dfrac{b}{\sqrt{a^2\sin^2(t_0)+a^2\cos^2(t_0)+b^2}} = \arccos\dfrac{b}{\sqrt{a^2+b^2}}$, does not depend on $t_0 \Rightarrow$ const

2. **(3p)** A *cycloid* is the curve traced by a chosen point on the circumference of a circle which rools along a stright line without slipping. Show that the parametric equations of the are:

$$\begin{cases} x = r(t - \sin t) \\ y = r(1 - \cos t) \end{cases}, t \in \mathbb{R}.$$

Solution.



3. **(2p)** Show that the normal line to the cycloid at a certain point passes through the tangency point between the generating circle and the line alnog which the generating circle rools on.

   Solution.

4. An *epicycloid* is a plane curve traced by a chosen point on the circumference of a circle which rools without slipping around a fixed circle. Find the equations of the epicicloid.

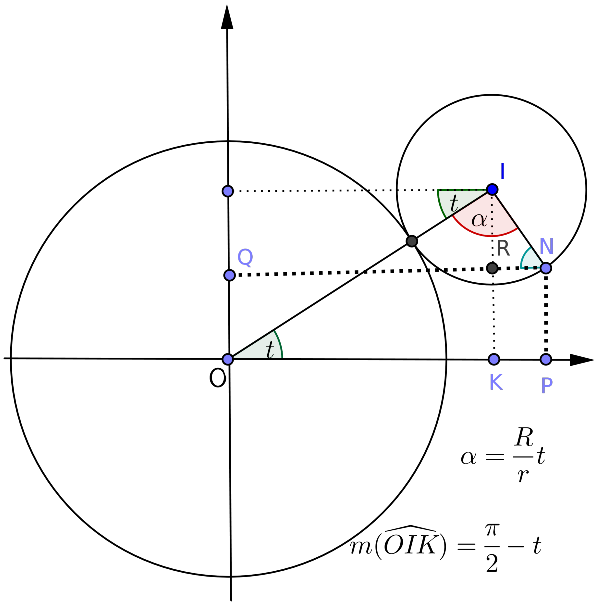
   Solution. The equations of the epicicloid are

$$\begin{cases} x = (R+r)\cos t - r\cos\left(\dfrac{R+r}{r}t\right) \\ y = (R+r)\sin t - r\sin\left(\dfrac{R+r}{r}t\right) \end{cases}, t \in \mathbb{R},$$

   or

$$\begin{cases} x = r(k+1)\cos t - r\cos(k+1)t \\ y = r(k+1)\sin t - r\sin(k+1)t \end{cases}, t \in \mathbb{R},$$

   where $k = \dfrac{R}{r}$. If k is an integer, then the epicycloid is a closed curve.



$$\alpha = \frac{R}{r}t$$

$$m(\widehat{OIK}) = \frac{\pi}{2} - t$$

$$m(\widehat{NIR}) = \alpha - m(\widehat{OIK}) = -\frac{\pi}{2} + \left(\frac{R}{r} + 1\right)t$$

$$m(\widehat{INR}) = \frac{\pi}{2} - m(\widehat{NIR}) = \pi - \frac{R+r}{r}t$$

$$IR = r\sin\frac{R+r}{r}t, RN = -r\cos\frac{R+r}{2}t$$

5. A *hypocycloid* is a plane curve traced by a chosen point on a small circle that rools without slipping within a larger circle. Find the equations of the hypocycloid.

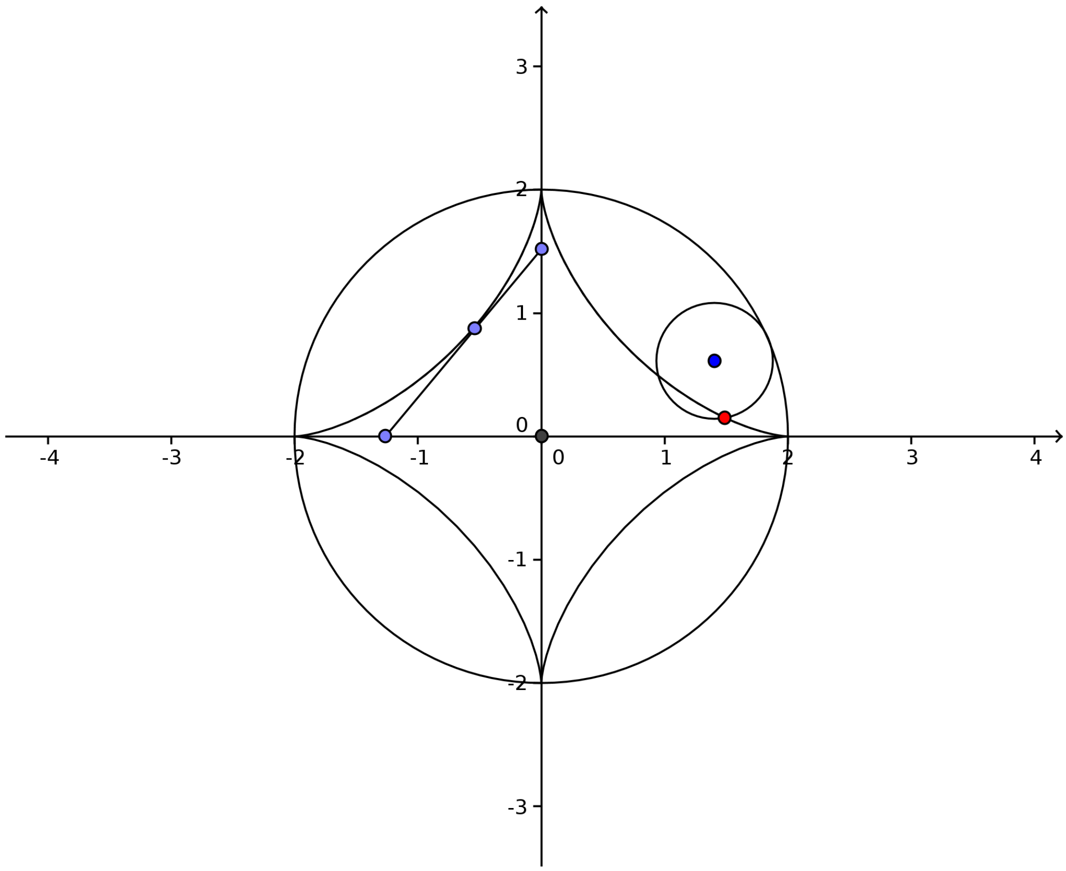
   Answer: The equations of the hypocycloid are:

$$\begin{cases} x = (R-r)\cos t + r\cos\left(\dfrac{R-r}{r}t\right) \\ y = (R-r)\sin t - r\sin\left(\dfrac{R-r}{r}t\right) \end{cases}, t \in \mathbb{R},$$

   or

$$\begin{cases} x = r(k-1)\cos t + r\cos(k-1)t \\ y = r(k-1)\sin t - r\sin(k-1)t \end{cases}, t \in \mathbb{R},$$

   where $k = \dfrac{R}{r}$. If k is an integer, then the hypocycloid is a closed curve. In particular, for $k = 4$ the hypocicloid is called *astroid*

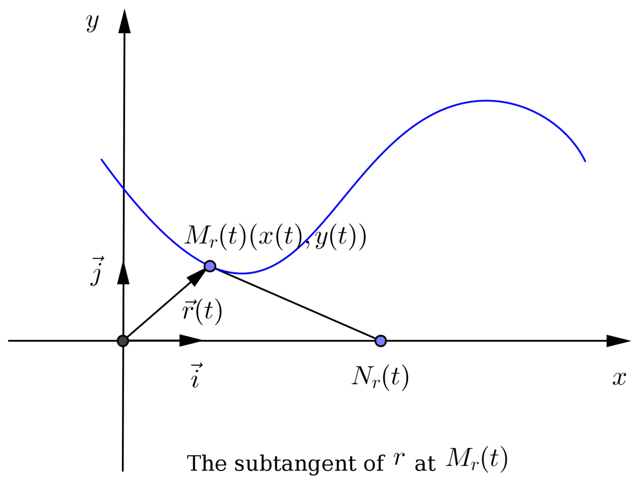


6. The *subtangent* of a planar parametrized differentiable curve is the segment which unify the tangency point between the tangent and the curve with the intersection point between the tangent and the x-axis. Show that the length of the subtangent of the planar parametrized differentiable curve

$$r : (0, \pi) \to \mathbb{R}^2,\ r(t) = a\big(\ln\tan(t/2) + \cos t, \sin t\big),$$

called the *tractrix* is constant and equal to a.

Solution.



The parametric equations of the tractrix are

$$\begin{cases} x = a\log\tan(t/2) + a\cos t \\ y = a\sin t \end{cases},\ t \in (0, \pi)$$

and its vector equation is

$$\overrightarrow{r}(t) = (a\ln\tan(t/2) + a\cos t)\,\overrightarrow{i} + (a\sin t)\,\overrightarrow{j}.$$

and its tangent vector

$$\begin{aligned}\overrightarrow{r}'(t) &= \left(a\frac{1}{\tan(t/2)}\,\frac{1}{\cos^2(t/2)}\,\frac{1}{2} - a\sin t\right)\overrightarrow{i} + (a\cos t)\,\overrightarrow{j}\\ &= \left(\frac{a}{\sin t} - a\sin t\right)\overrightarrow{i} + (a\cos t)\,\overrightarrow{j}\\ &= \frac{a\cos^2 t}{\sin t}\,\overrightarrow{i} + (a\cos t)\,\overrightarrow{j} = a\cos t(\cot t\,\overrightarrow{i} + \overrightarrow{j}).\end{aligned}$$

Thus, the equation of the tangent line to the tractrix at the regular points $M_r(t)$, i.e. $t \in (0,\pi)\setminus\{0\}$ is

$$(T_r)(t) : \frac{X - x(t)}{x'(t)} = \frac{Y - y(t)}{y'(t)} \Longleftrightarrow \frac{X - a\log\tan(t/2) - a\cos t}{a\cos t\cot t} = \frac{Y - a\sin}{a\cos t}. \tag{8.12}$$

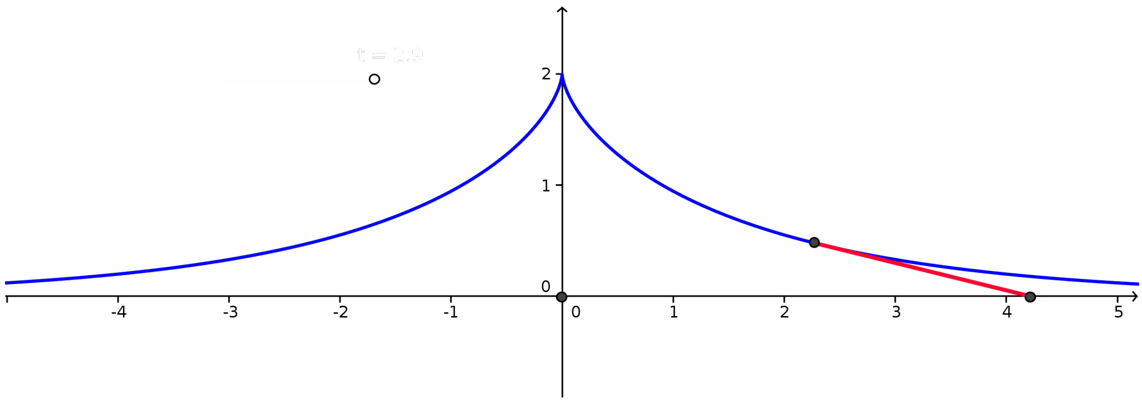
The coordinates of the intersection point $N_r(t)$ of the tangent $T_r(t)$ to the tractrix at $M_r(t)$ with the x-axis can be obtained by taking $Y = 0$ in (8.12), which implies $X = a\log\tan(t/2)$, i.e. $N_r(t)(a\log\tan(t/2), 0)$. The distance between

$$M_r(t)\,(a\log\tan(t/2) + a\cos t, a\sin t)\ \text{ and } N_r(t)(a\log\tan(t/2), 0)$$

is

$$\sqrt{(a\log\tan(t/2) + a\cos t - a\cos t)^2 + (a\sin t - 0)^2} = \sqrt{a^2} = |a| = a.$$

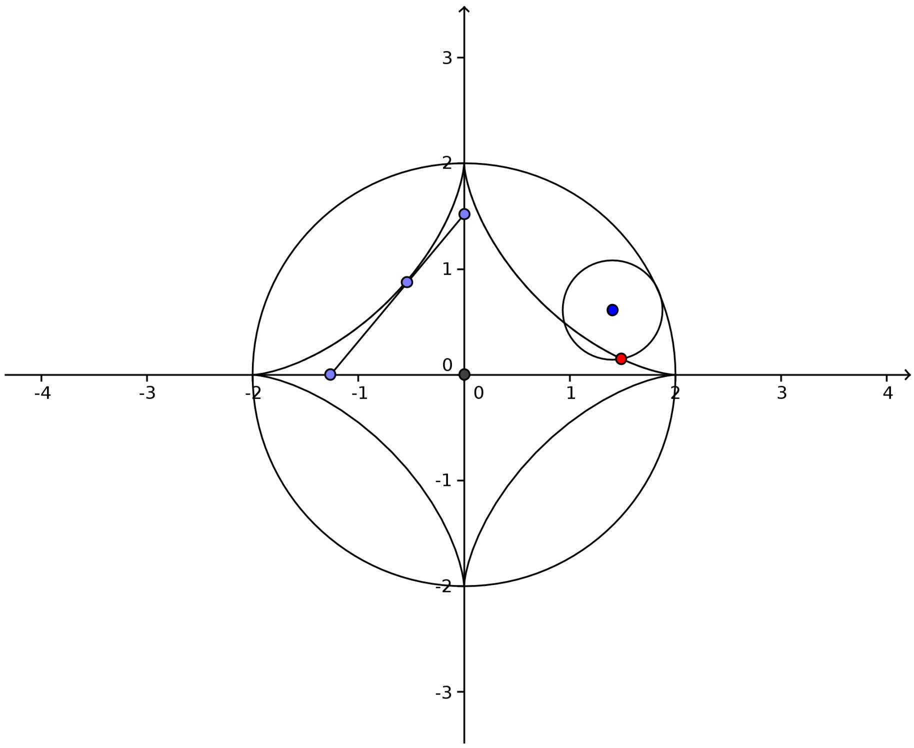
Note that $t = \pi/2$ is the only singular point of $\overrightarrow{r}$. Since $\overrightarrow{r}''(\pi/2) = a\,\overrightarrow{j}$, it follows that $t = \pi/2$ is a singular point of order two for $\overrightarrow{r}$, i.e. $\overrightarrow{r}''(\pi/2)$ is a director vector of the tangent line of r at $t = \pi/2$. In other words the y-axis is the tangent line to r at $t = \pi/2$. Note that $M_r(\pi/2)(0,a)$ and $N_r(\pi/2)$ is the origin $O(0,0)$. Thus the distance between $M_r(\pi/2)(0,a)$ and $N_r(\pi/2)$ is a.



7. **(2p)** Show that the tangents of the astroid

$$\begin{cases} x = r\cos^3 t\\ y = r\sin^3 t\end{cases}$$

determines on the coordinate axes segments of constant length.



Solution.

8. Write the equations of the tangent line and the normal plane for the following curves, whenever these associated objects are well-determined:

   (a) **(2p)**

   $$\begin{cases} x = e^t \cos 3t \\ y = e^t \sin 3t \\ z = e^{-2t} \end{cases} \quad \text{at the point corresponding to the value } t = 0 \text{ of the parameter}$$

   (b) **(2p)**

   $$\begin{cases} x = e^t \cos 3t \\ y = e^t \sin 3t \\ z = e^{-2t} \end{cases} \quad \text{at the point corresponding to the value } t = \frac{\pi}{4} \text{ of the parameter}$$

Solution.

a) $\frac{x-x(0)}{x'(0)} = \frac{y-y(0)}{y'(0)} = \frac{z-z(0)}{z'(0)} \Rightarrow \frac{x-1}{1} = \frac{y}{3} = \frac{z-1}{-2}$ (eq. of the tangent line)

$x(0) = 1,\ y(0) = 0,\ z(0) = 1$

$x'(t) = e^t \cdot (\cos 3t - 3\sin 3t),\ y'(t) = e^t(\sin 3t + 3\cos 3t),\ z'(t) = -2e^{-2t} \Rightarrow$ $x'(0) = 1$, $y'(0) = 3$, $z'(0) = -2$

$x'(0)\cdot(x - x(0)) + y'(0)\cdot(y - y(0)) + z'(0)(z - z(0)) = 0 \Leftrightarrow x - 1 + 3y - 2(z-1) = 0 \Leftrightarrow x + 3y - 2z + 1 = 0$
(eq. of the normal plane)

9. **(2p)** Write the equations of the tangent planes of the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ at the points of the form $(x_0, y_0, 0)$ and show that these are parallel to the z-axis.

Solution. $f : \mathbb{R}^3 \to \mathbb{R},\ f(x,y,z) = x^2+y^2-z^2-1$

$(TH)_{(x_0,y_0,0)}\ f'_x(x_0,y_0,0)(x-x_0) + f'_y(x_0,y_0,0)(y-y_0) = 0 \Leftrightarrow 2x_0(x-x_0) + 2y_0(y-y_0) = 0 \,/ :2$

$\Leftrightarrow x_0(x-x_0) + y_0(y-y_0) = 0$

$f'_x = 2x,\ f'_y = 2y,\ f'_z = -2z$

$\vec{n}(x_0, y_0, 0),\ \vec{k}(0,0,1)$

$\vec{n} \cdot \vec{k} = x_0 \cdot 0 + y_0 \cdot 0 + 0 \cdot 1 = 0 \Leftrightarrow \vec{n} \perp \vec{k} \Leftrightarrow (TH)_{(x_0,y_0,0)} \parallel Oz$

Extra: $\mathcal{S}$ sphere : $\begin{cases} x = \cos u \cdot \cos v \\ y = \cos u \cdot \sin v \\ z = \sin u \end{cases}$. Write the eq. of the tangent plane to $\mathcal{S}$ at the point $(0,0,1)$ (i.e the point corresponding to $u = \frac{\pi}{2}, v = 0$)

$\frac{\partial}{\partial u} r = (-\sin u \cos v, -\sin u \cdot \sin v, \cos u)$

$\frac{\partial}{\partial v} r = (-\cos u \sin v, \cos u \cdot \cos v, 0)$

$T_{\mathcal{S}} : \begin{vmatrix} x & y & z-1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$

(the parametrization is wrong!)

10. **(2p)** Show that the trace of the parametrized differentiable curve $\alpha : \mathbb{R} \to \mathbb{R}^3$, $\alpha(t) = (e^t \cos t, e^t \sin t, 2t)$ is contained in the regular surface of equation $z = \ln(x^2 + y^2)$ and write the equation of the tangent plane of the surface at the points $\alpha(t)$, $t \in \mathbb{R}$.

Solution. $\begin{cases} x = e^t\cos t \\ y = e^t \sin t,\ t \in \mathbb{R} \\ z = 2t \end{cases}$ $\ln(x^2+y^2) = \ln(e^{2t}\cos^2 t + e^{2t}\sin^2 t) = \ln(e^{2t}(\underbrace{\cos^2 t + \sin^2 t}_{1})) = \ln(e^{2t}) = 2t = z$

$\alpha(x,y) = \ln(x^2+y^2)$

$S: z = \ln(x^2+y^2)$, $M_0(x_0, y_0, z_0) \in S$ $T_{M_0}(S)$

$S: g^{-1}(0)$, $g(x,y,z) = \ln(x^2+y^2) - z$

$T_{M_0}(S)$ $g_x(M_0)(x-x_0) + g_y(M_0)(y-y_0) + g_z(M_0)(z-z_0) = 0$

$g_x = \frac{2x}{x^2+y^2}$, $g_y = \frac{2y}{x^2+y^2}$, $g_z = -1$

$g_x(\alpha(t)) = \frac{2x}{e^{2t}} = \frac{2e^t\cos t}{e^{2t}} = 2 \cdot e^{-t}\cos t$

$g_y(\alpha(t)) = 2e^{-t}\sin t$

$g_z(\alpha(t)) = -1$

$2e^{-t}\cos t(x - e^t\cos t) + 2e^{-t}\sin t(y - e^t\sin t) + 2t - z = 0$

$(2e^{-t}\cos t)x - \underline{2\cos^2 t} + (2e^{-t}\sin t)y - \underline{2\sin^2 t} + 2t - z = 0$

$2e^{-t}\cos t\, x + 2e^{-t}\sin t\, y + 2t - 2 = z$

11. **(3p)** Show that the tangent planes of the surface of equation $z = xf(\frac{y}{x})$, where f is a differentiable function, are passing through the origin.

    Solution.

12. **(3p)** Show that the set $S = \{(x, y, z) \in \mathbf{R}^3 \,|\, xyz = a^3\},\ a \neq 0$ is a regular surface and the its tangent plane at an arbitrary point $p \in S$ determines on the coordinate axes three points which form, together with the origin a tetrahedron of constant volume (independent of p).

Solution.

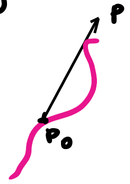
Seminar 916 - 14/04/2021

Curves: → given parametrically: $\mathcal{C}: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$ (3D)

→ given implicitly: - planar: $f(x,y) = 0$

- spatial: $\begin{cases} f_1(x,y,z) = 0 \\ f_2(x,y,z) = 0 \end{cases}$

The tangent to the curve $\mathcal{C}$ at the point P_0 is a line that contains P_0 and has a direction specified by the vector:



$$\vec{v} = \lim_{\substack{P \to P_0 \\ P \in \mathcal{C}}} \frac{\overrightarrow{P_0P}}{\|\overrightarrow{P_0P}\|}$$

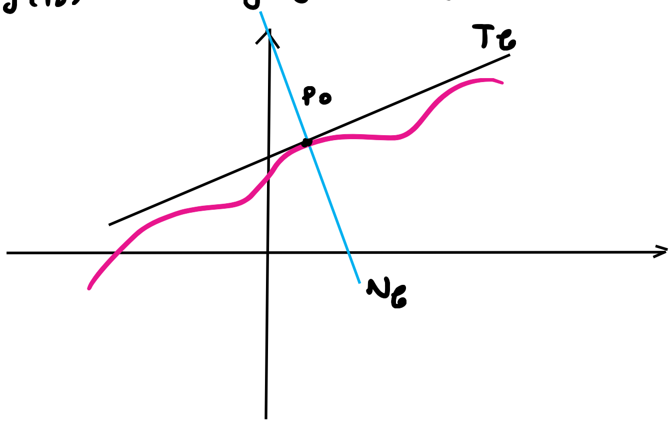
If $\mathcal{C}$ is given parametrically by $\mathcal{C}: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$, then the tangent line to $\mathcal{C}$ is the point $P_0(t = t_0)$ is:

$$T_{\mathcal{C}}(t = t_0): \frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)} \ (3D)$$

→ if $\mathcal{C}$ is in 2D, normal line = line ⊥ tangent and contains the point P_0

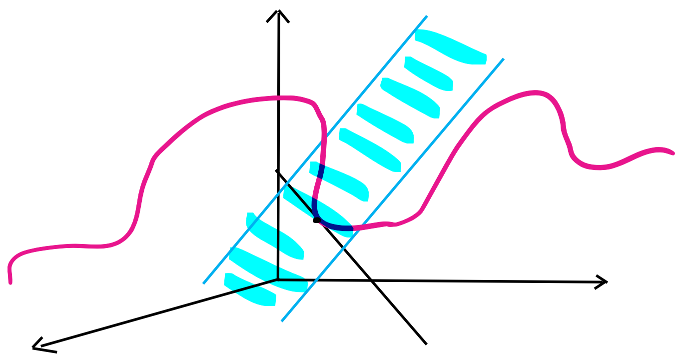
$$m_{T_{\mathcal{C}}} = \frac{y'(t_0)}{x'(t_0)} \Rightarrow m_{N_{\mathcal{C}}} = \frac{-x'(t_0)}{y'(t_0)}$$

$$N_{\mathcal{C}}(t = t_0): -\frac{x'(t_0)}{y'(t_0)}(x - x(t_0)) = y - y(t_0) \Leftrightarrow y'(t_0)(y - y(t_0)) + x'(t_0)(x - x(t_0)) = 0$$



→ if $\mathcal{C}$ is in 3D, normal plane = plane that is perpendicular to the tangent and contains P_0

$$N_{\mathcal{C}}(t = t_0): x'(t_0)\cdot(x - x(t_0)) + y'(t_0)(y - y(t_0)) + z'(t_0)\cdot(z - z(t_0)) = 0$$



If $\mathcal{C}$ is given implicitly (and is planar) then: $\mathcal{C}: f(x,y) = 0$

$$T_{\mathcal{C}}(x_0, y_0) = f'_x(x_0,y_0)\cdot(x - x_0) + f'_y(x_0,y_0)\cdot(y - y_0) = 0$$

$$N_{\mathcal{C}}(x_0, y_0): \frac{x - x_0}{f'_x(x_0,y_0)} = \frac{y - y_0}{f'_y(x_0,y_0)}$$

Extra: Find the eq. of the tangent line and normal line to the curve $\mathcal{C}$ at the point $P_0(1,0)$.

$$\mathcal{C}: x^3 - x^2y + y^4 - 1 = 0$$

$f = x^3 - x^2y + y^4 - 1$

$\frac{\partial f}{\partial x}(x,y) = 3x^2 - 2xy$ $\Rightarrow T_{\mathcal{C}}(x_0,y_0): \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) = 0 \Rightarrow$

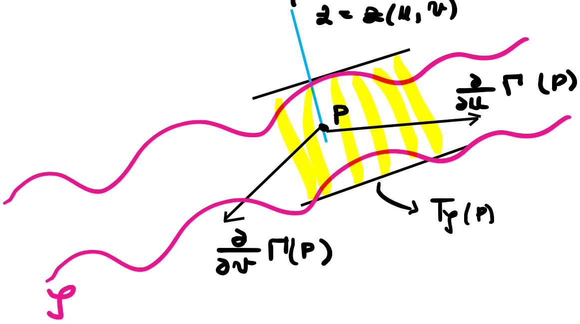
$\frac{\partial f}{\partial y}(x,y) = -x^2 + 4y^3$ $\Rightarrow T_{\mathcal{C}}(1,0): 3(x-1) - y = 0$

$N_{\mathcal{C}}(1,0): \frac{x-1}{3} = \frac{y}{-1}$

Surfaces $\rightarrow$ given parametrically: $\mathcal{S}: \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$

$\Gamma: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$(u,v) \mapsto (x,y,z)$



The tangent plane to $\mathcal{S}$ in $P(u_0, v_0)$:

$$T_{\mathcal{S}}(u=u_0, v=v_0): \begin{vmatrix} x - x(u_0,v_0) & y - y(u_0,v_0) & z - z(u_0,v_0) \\ \frac{\partial x}{\partial u}(u_0,v_0) & \frac{\partial y}{\partial u}(u_0,v_0) & \frac{\partial z}{\partial u}(u_0,v_0) \\ \frac{\partial x}{\partial v}(u_0,v_0) & \frac{\partial y}{\partial v}(u_0,v_0) & \frac{\partial z}{\partial v}(u_0,v_0) \end{vmatrix} = 0$$

Normal line: $N_{\mathcal{S}}(u=u_0, v=v_0): \frac{x - x(u_0,v_0)}{\frac{\partial(y,z)}{\partial(u,v)}} = \frac{y - y(u_0,v_0)}{\frac{\partial(z,x)}{\partial(u,v)}} = \frac{z - z(u_0,v_0)}{\frac{\partial(x,y)}{\partial(u,v)}}$

$\rightarrow$ if the surface is given implicitly: $\mathcal{S}: f(x,y,z) = 0$

$T_{\mathcal{S}}(x_0,y_0,z_0): f'_x(x_0,y_0,z_0)\cdot(x-x_0) + f'_y(x_0,y_0,z_0)\cdot(y-y_0) + f'_z(x_0,y_0,z_0)\cdot(z-z_0) = 0$

$N_{\mathcal{S}}(x_0,y_0,z_0): \frac{x-x_0}{f'_x(x_0,y_0,z_0)} = \frac{y-y_0}{f'_y(x_0,y_0,z_0)} = \frac{z-z_0}{f'_z(x_0,y_0,z_0)}$