LECTURE 5 - DYNAMICAL SYSTEMS

Monday, 22 March 2021 10:04

Linear differential systems (linear diff. eq. in $\mathbb{R}^n$)

(1) $X' = A(t)X + F(t)$, where $A \in C(J, \mathcal{M}_n(\mathbb{R}))$, $F \in C(J, \mathbb{R}^n)$

$J \subset \mathbb{R}$ nonempty open interval, $n \in \mathbb{N}^*$

Def: A function $\varphi : J \to \mathbb{R}^n$ is a solution of system (1) if $\varphi \in C^1(J, \mathbb{R}^n)$ and $\varphi'(t) = A(t)\varphi(t) + F(t)$, $\forall t \in J$

Remarks: Let $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ --- \\ x_n(t) \end{pmatrix}$ $A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ --- \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{pmatrix}$ $F(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$

(1) $\Leftrightarrow$ $x_1' = a_{11}(t)\cdot x_1 + a_{12}(t)\cdot x_2 + \dots + a_{1n}(t)x_n + f_1(t)$

$x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n + f_n(t)$

$A(t)$ is the matrix of the system; $F(t)$ is the non-homog. part

$X' = A(t)X$ is the homog. part. When $A(t)$ is const (does not depend on t) we say that the system has constant coeff.

Def: A function $U : J \to \mathcal{M}_n(\mathbb{R})$ is said to be a matrix solution of $X' = A(t)X$ if U is C^1 and $U'(t) = A(t)U(t)$, $\forall t \in J$

Remark: $U(t)$ is a matrix solution $\Leftrightarrow$ each column of $U(t)$ is a solution

The fundamental theorems

→ **The existence and uniqueness theorem**: Let $t_0 \in J$ be fixed and $\eta \in \mathbb{R}^n$. We have that the IVP

$\begin{cases} X' = A(t)X + F(t) \\ X(t_0) = \eta \end{cases}$ has a unique solution.

→ **The fundamental theorem for LHS**: The set of solutions of the system $X' = A(t)X$ is a linear space of dim n.

Hence, there exist $X_1, \dots, X_n$ n linearly indep. sol of $X' = A(t)X$ and the general sol. is $X = c_1X_1 + c_2X_2 + \dots + c_nX_n$, $c_1, c_2, \dots, c_n \in \mathbb{R}$.

Moreover, denoting $U(t) = (X_1(t)\ X_2(t) \dots X_n(t))$ we can write the gen sol as

$$X = U(t)C, \quad C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n \text{ arbitrary}$$

Remark: A matrix sol. whose columns are lin. indep is called a fundamental matrix sol. It can be proved that $U(t)$ is a fundam. matrix sol $\Leftrightarrow$ $U(t)$ is a matrix sol s.t. $\det U(t) \neq 0$, $\forall t \in J$

Proof: Define $\mathcal{L}(X)(t) = X'(t) - A(t)X(t)$, $\forall t \in J$, $\forall X \in C^1(J, \mathbb{R}^n)$ we obtain $\mathcal{L} : C^1(J, \mathbb{R}^n) \to C(J, \mathbb{R}^n)$ map between 2 linear spaces (over $\mathbb{R}$)

$\mathcal{L}(X) = 0 \Leftrightarrow X' - A(t)X = 0$

$\mathcal{L}$ is linear $\Leftrightarrow$ $\mathcal{L}(\alpha_1X_1 + \alpha_2X_2) = \alpha_1\mathcal{L}(X_1) + \alpha_2\mathcal{L}(X_2)$, $\forall \alpha_1, \alpha_2 \in \mathbb{R}$, $\forall X_1, X_2 \in C^1(J, \mathbb{R}^n)$ $\Leftrightarrow$

$\Leftrightarrow$ $(\alpha_1X_1 + \alpha_2X_2)' - A(t)(\alpha_1X_1 + \alpha_2X_2) = \alpha_1(X_1' - A(t)X_1) + \alpha_2(X_2' - A(t)X_2)$, $\forall \dots$ TRUE

We have that $\ker\mathcal{L}$ is the set of sol. of $X' = A(t)X$. We know that $\ker\mathcal{L}$ is a linear space.

$T : \ker\mathcal{L} \to \mathbb{R}^n$ $T(X) = X(t_0)$, where $t_0 \in J$ is fixed

T is linear $\Leftrightarrow$ $T(\alpha_1X_1 + \alpha_2X_2) = \alpha_1T(X_1) + \alpha_2T(X_2)$, $\forall \dots$ $\Leftrightarrow$ $(\alpha_1X_1 + \alpha_2X_2)(t_0) = \alpha_1\cdot X_1(t_0) + \alpha_2X_2(t_0)$, ... TRUE

T is bijective $\Leftrightarrow$ $\forall \eta \in \mathbb{R}^n$ $\exists!$ sol. in $\ker\mathcal{L}$ of $T(X) = \eta$

$\Leftrightarrow$ $\forall \eta \in \mathbb{R}^n$ $\exists!$ sol of $\begin{cases} X' = A(t)X \\ X(t_0) = \eta \end{cases}$

TRUE by the $\exists!$ th.

So, T is an izomorphism of linear spaces $\Rightarrow$ the dim of $\ker\mathcal{L}$ is eq. to the dim. of $\mathbb{R}^n$ which is n. □

The fundamental th. for linear non-hom. systems

So, T is an izomorphism of linear spaces $\Rightarrow$ the dim of Ker $\mathcal{L}$ is eq. to the dim . of ...

The fundamental th. for linear non-hom. systems

The gen sol of $X' = A(t)X + F(t)$ is $X = X_h + X_p$ where X_h = gen sol of $X' = A(t)X$ and X_p sat $X_p' = A(t)X_p + F(t)$ (i.e. X_p is a part sol of the Ln-HS)

Proof: $X' = A(t)X + F(t) \Rightarrow \mathcal{L}(X) = F$ whose set of sol is Ker $\mathcal{L} + \{X_p\}$. $\square$

The Lagrange method to find X_p

We want to find X_p s.t. $X_p' = A(t)X_p + F(t)$ once we know that the gen sol of $X' = A(t)X$ is a fundamental matrix sol, that is $U'(t) = A(t)U(t)$ and $\det U(t) \neq 0, \forall t \in \mathcal{J}$

idea of Lagrange: find $\varphi : \mathcal{J} \to \mathbb{R}^n$ s.t. $X_p = U(t)\varphi(t) \Rightarrow \underbrace{U'(t)}_{= A(t)U(t)}\varphi(t) + U(t)\varphi'(t) = A(t)U(t)\varphi(t) + F(t), \forall t \in \mathcal{J}$

$\Leftrightarrow U(t)\varphi'(t) = F(t), \forall t \in \mathcal{J} \Leftrightarrow \varphi'(t) = U(t)^{-1}F(t), \forall t \in \mathcal{J} \Leftrightarrow \varphi(t) = \int_{t_0}^{t} U(s)^{-1}F(s)\,ds\ \forall t \in \mathcal{J}$, for $t_0 \in \mathcal{J}$ fixed

$\Leftrightarrow X_p = U(t)\int_{t_0}^{t} U(s)^{-1}F(s)\,ds$. $\square$

Linear homogeneous systems with constant coefficients

(2) $X' = AX$, $A \in \mathcal{M}_n(\mathbb{R})$

Def: Denote $E : \mathbb{R} \to \mathcal{M}_n(\mathbb{R})$ of class C^1 s.t. $E(t)$ is a matrix sol of (2), that is $E'(t) = AE(t), \forall t \in \mathbb{R}$ and $E(0) = \mathcal{I}_n$ (the identity). This $E(t)$ is called the principal matrix sol of (2).

Remark: It is known $E(t)$ is a fundamental matrix sol.

From Lecture 4 we know that $\frac{d}{dt}e^{tA} = Ae^{tA}$ and $e^{tA}\big|_{t=0} = \mathcal{I}_n$. Then, $E(t) = e^{tA}$, or, in other words, e^{tA} is the principal fundamental matrix sol. of $X' = AX$. This also means that the gen sol of $X' = AX$ is $X = e^{tA}\cdot c$, $c \in \mathbb{R}^n$ and the unique sol of the IVP $\begin{cases} X' = AX \\ X(0) = \eta \end{cases}$ is $\varphi(t) = e^{tA}\eta, \forall t \in \mathbb{R}$

Similar matrices

Def: Let $A, B \in \mathcal{M}_n(\mathbb{R})$. We say that A and B are similar if there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{R})$ s.t. $A = PBP^{-1}$.

Property: Let A and B be similar. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A and $u \in \mathbb{C}^n$ a corresp. eigenvector. Then λ is also an eigenvalue of B and $P^{-1}u$ is a corresp. eigenvector of B.

Proof: $\xrightarrow{hyp} Au = \lambda u \Rightarrow PBP^{-1}u = \lambda u \Rightarrow BP^{-1}u = \lambda P^{-1}u$ $\square$

$u \neq 0$, P invertible $\Rightarrow P^{-1}u \neq 0$

Property: Let A and B be similar matrices. Then

(i) $A^k = PB^kP^{-1}, \forall k \in \mathbb{N}^*$

(ii) $e^{tA} = Pe^{tB}P^{-1}, \forall t \in \mathbb{R}$

Pf (i): by induc. $k = 1$ $A = PBP^{-1}$ true by hyp

$A^k = PB^kP^{-1} \overset{?}{\Rightarrow} A^{k+1} = PB^{k+1}P^{-1}$

$A^{k+1} = A^k \cdot A = (PB^kP^{-1})(PBP^{-1}) = PB^k(P^{-1}P)BP^{-1} = PB^{k+1}P^{-1}$ $\square$

(ii) $e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!}(tA)^k = \sum_{k=0}^{\infty} \frac{t^k}{k!}PB^kP^{-1} \overset{\mathcal{J}}{=} P\left(\sum_{k=0}^{\infty} \frac{1}{k!}(tB)^k\right)P^{-1} = P\cdot e^{tB}P^{-1}$ $\square$