2 Week 2: Straight lines and planes

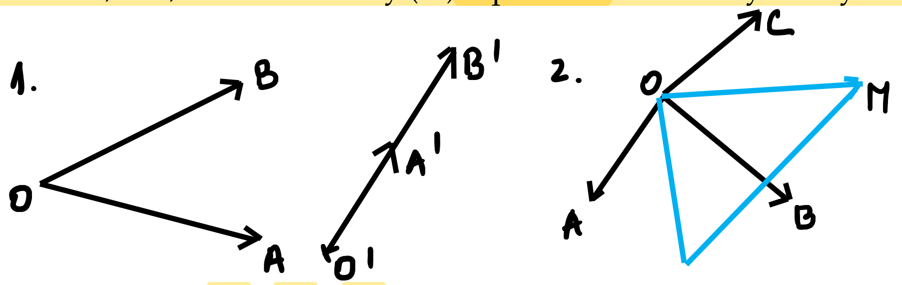
2.1 Linear dependence and linear independence of vectors

Definition 2.1. 1. The vectors $\overrightarrow{OA}, \overrightarrow{OB}$ are said to be *collinear* if the points O, A, B are collinear. Otherwise the vectors $\overrightarrow{OA}, \overrightarrow{OB}$ are said to be *noncollinear*.

2. The vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ are said to be *coplanar* if the points O, A, B, C are coplanar. Otherwise the vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ are *noncoplanar*.

Remark 2.1. 1. The vectors $\overrightarrow{OA}, \overrightarrow{OB}$ are linearly (in)dependent if and only if they are (non)collinear.

2. The vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ are linearly (in)dependent if and only if they are (non)coplanar.



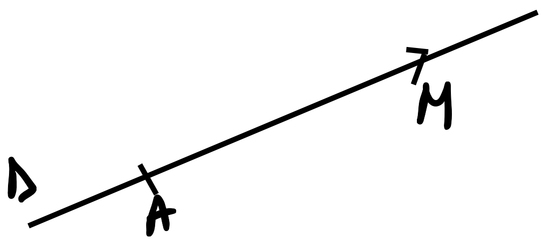
Proposition 2.1. *The vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ form a basis of $\mathcal{V}$ if and only if they are noncoplanar.*

Corollary 2.2. *The dimension of the vector space of free vectors $\mathcal{V}$ is three.*

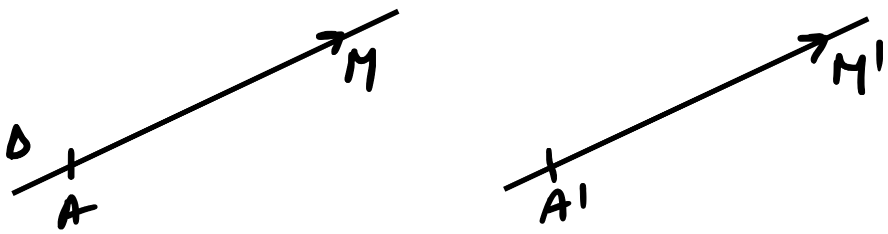
Proposition 2.3. *Let Δ be a straight line and let $A \in \Delta$ be a given point. The set*

$$\overrightarrow{\Delta} = \{\overrightarrow{AM} \mid M \in \Delta\}$$

is an one dimensional subspace of $\mathcal{V}$. It is independent on the choice of $A \in \Delta$ and is called the director subspace *of Δ or the* direction *of Δ.*



Remark 2.2. The straight lines Δ, Δ' are parallel if and only if $\overrightarrow{\Delta} = \overrightarrow{\Delta}'$



Definition 2.2. *We call* director vector *of the straigh line Δ every nonzero vector $\overrightarrow{d} \in \overrightarrow{\Delta}$.*

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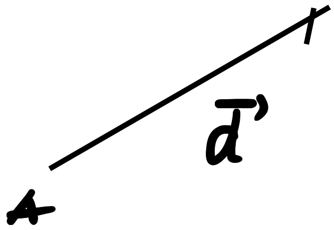
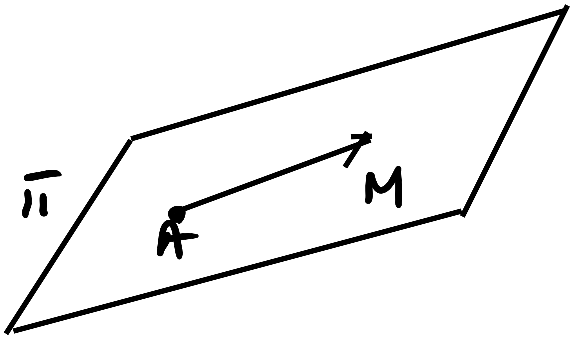
If $\overrightarrow{d} \in \mathcal{V}$ is a nonzero vector and $A \in \mathcal{P}$ is a given point, then there exits a unique straight line which passes through A and has the direction $\langle \overrightarrow{d} \rangle$. This stright line is

$$\Delta = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle \overrightarrow{d} \rangle\}.$$

Δ is called the straight line which passes through O and is parallel to the vector $\overrightarrow{d}$.



Proposition 2.4. *Let π be a plane and let $A \in \pi$ be a given point. The set $\overrightarrow{\pi} = \{\overrightarrow{AM} \in \mathcal{V} \mid M \in \pi\}$ is a two dimensional subspace of $\mathcal{V}$. It is independent on the position of A inside π and is called the* director subspace, the director plane *or the* direction *of the plane π.*

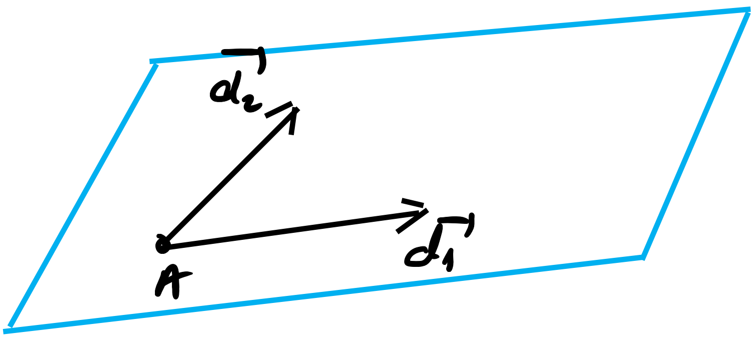


Remark 2.3. • *The planes π, π' are parallel if and only if $\overrightarrow{\pi} = \overrightarrow{\pi}'$.*

• If $\overrightarrow{d}_1, \overrightarrow{d}_2$ are two linearly independent vectors and $A \in \mathcal{P}$ is a fixed point, then there exists a unique plane through A whose direction is $\langle \overrightarrow{d}_1, \overrightarrow{d}_2 \rangle$. This plane is

$$\pi = \{M \in \mathcal{P} \mid \overrightarrow{AM} \in \langle \overrightarrow{d}_1, \overrightarrow{d}_2 \rangle\}.$$

We say that π is the plane which passes through the point A and is parallel to the vectors $\overrightarrow{d}_1$ and $\overrightarrow{d}_2$.



Remark 2.4. Let $\Delta \subset \mathcal{P}$ be a straight line and $\pi \subset \mathcal{P}$ be given plane.

1. If $A \in \Delta$ is a given point, then $\varphi_O(\Delta) = \overrightarrow{r}_A + \overrightarrow{\Delta}$.

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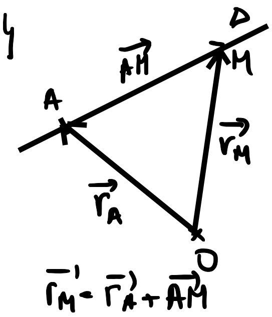
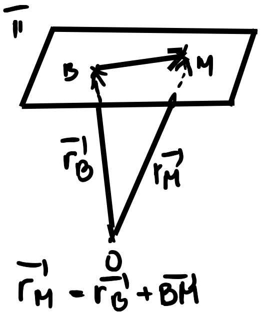
2. If $B \in \Delta$ is a given point, then $\varphi_O(\pi) = \vec{r}_B + \vec{\pi}$.

$(O \in \mathcal{P}),\ \varphi_O : \mathcal{P} \to \mathcal{V}$

$2)\ \varphi_O(\Delta) = \{\varphi_O(M) / M \in \Delta\} = \{\vec{OM} / M \in \Delta\}$
$= \{\vec{OA} + \vec{AM} \mid M \in \Delta\}$
$= \vec{r}_A + \{\vec{AM} \mid M \in \Delta\} = \vec{r}_A + \vec{\Delta}$

Similarly, $\varphi_O(\pi) = \vec{r}_B + \vec{\pi}$





Generally speaking, a subset X of a vector space is called *linear variety* if either $X = \emptyset$ or there exists $a \in V$ and a vector subspace U of V, such that $X = a + U$.

$$\dim(X) = \begin{cases} -1 & \text{dacă } X = \emptyset \\ \dim(U) & \text{dacă } X = a + U, \end{cases}$$

Proposition 2.5. *The bijection φ_O transforms the straight lines and the planes of the affine space $\mathcal{P}$ into the one and two dimnensional linear varieties of the vector space $\mathcal{V}$ respectively.*

2.2 The vector ecuations of the straight lines and planes

Proposition 2.6. *Let Δ be a straight line, let π be a plane, $\{\vec{d}\}$ be a basis of $\vec{\Delta}$ and let $[\vec{d}_1, \vec{d}_2]$ be an ordered basis of $\vec{\pi}$.*

1. *The points $M \in \Delta$ are characterized by the* vector equation of Δ

$$\vec{r}_M = \vec{r}_A + \lambda \vec{d},\ \lambda \in \mathbb{R} \tag{2.1}$$

where $A \in \Delta$ is a given point.

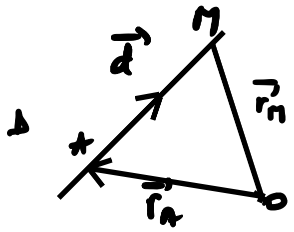
2. *The points $M \in \pi$ are characterized by the* vector equation of π

$$\vec{r}_M = \vec{r}_A + \lambda_1 \vec{d}_1 + \lambda_2 \vec{d}_2,\ \lambda_1, \lambda_2 \in \mathbb{R}, \tag{2.2}$$

where $A \in \pi$ is a given point.

PROOF. $\varphi_O(\Delta) = \vec{r}_A + \vec{\Delta}$

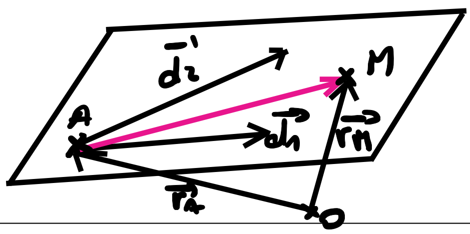
$M \in \Delta \Leftrightarrow \varphi_O(M) \in \vec{r}_A + \vec{\Delta} \Leftrightarrow \vec{r}_M \in \vec{r}_A + \langle\vec{d}\rangle \Leftrightarrow \exists\, \lambda \in \mathbb{R}$ s.t. $\vec{r}_M = \vec{r}_A + \lambda \vec{d}$



$\varphi_O(\pi) = \vec{r}_A + \vec{\pi},\ A \in \pi$

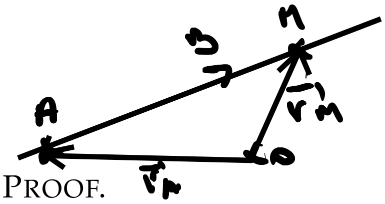
$M \in \pi \Leftrightarrow \varphi_O(M) \in \varphi_O(\pi) = \vec{r}_A + \vec{\pi} \Leftrightarrow \vec{r}_M \in \vec{r}_A + \langle \vec{d}_1, \vec{d}_2 \rangle \Leftrightarrow$

$\Leftrightarrow \exists\, \lambda_1, \lambda_2 \in \mathbb{R}$ s.t. $\vec{r}_M = \vec{r}_A + \lambda_1 \vec{d}_1 + \lambda_2 \vec{d}_2$



□

Corollary 2.7. *If $A, B \in \mathcal{P}$ are different points, then the vector equation of the line AB is*



$$\vec{r}_M = (1-\lambda)\,\vec{r}_A + \lambda\,\vec{r}_B,\ \lambda \in \mathbb{R}. \tag{2.3}$$

PROOF.

$\vec{r}_M = \vec{r}_A + \lambda \vec{d}$

We can choose $\vec{d} = \vec{AB} = \vec{OB} - \vec{OA} = \vec{r}_B - \vec{r}_A$

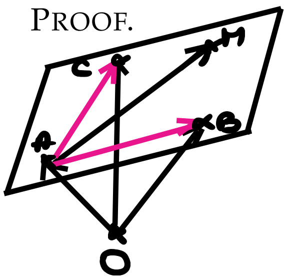
$\vec{r}_M = \vec{r}_A + \lambda(\vec{r}_B - \vec{r}_A) \Leftrightarrow \vec{r}_M = \vec{r}_A + \lambda \vec{r}_B - \lambda \vec{r}_A \Leftrightarrow \vec{r}_M = (1-\lambda)\vec{r}_A + \lambda \vec{r}_B$

□

Corollary 2.8. *If A, B, $C \in \mathcal{P}$ are three noncolinear points, then the vector equation of the plane (ABC) is*

$$\vec{r}_M = (1-\lambda_1-\lambda_2)\,\vec{r}_A + \lambda_1\,\vec{r}_B + \lambda_2\,\vec{r}_C,\ \lambda_1,\ \lambda_2 \in \mathbb{R}. \tag{2.4}$$

PROOF.



$\vec{r}_M = \vec{r}_A + \lambda_1 \vec{d}_1 + \lambda_2 \vec{d}_2$ (1)

$\vec{d}_1, \vec{d}_2$ are lin. indep and $\vec{\pi} = \langle \vec{d}_1, \vec{d}_2 \rangle$

We can choose $\vec{d}_1 = \vec{AB} = \vec{r}_B - \vec{r}_A$

$\vec{d}_2 = \vec{AC} = \vec{r}_C - \vec{r}_A$

(1) $\Leftrightarrow \vec{r}_M = (1-\lambda_1-\lambda_2)\vec{r}_A + \lambda_1 \vec{r}_B + \lambda_2 \vec{r}_C$

□

Example 2.1. Consider the points C' and B' on the sides AB and AC of the triangle ABC such that $\vec{AC'} = \lambda\ \vec{BC'}$, $\vec{AB'} = \mu\ \vec{CB'}$. The lines BB' and CC' meet at M. If $P \in \mathcal{P}$ is a given point and $\vec{r}_A = \vec{PA}$, $\vec{r}_B = \vec{PB}$, $\vec{r}_C = \vec{PC}$ are the position vectors, with respect to P, of the vertices A, B, C respectively, show that

$$\vec{r}_M = \frac{\vec{r}_A - \lambda\,\vec{r}_B - \mu\,\vec{r}_C}{1-\lambda-\mu}. \tag{2.5}$$

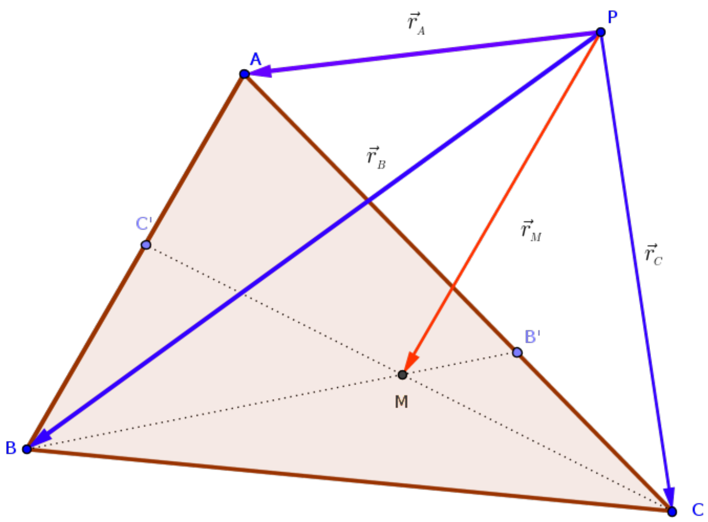
SOLUTION: $BB' \cap CC' = \{M\}$

$BB' : \vec{r}_X = (1-s)\vec{r}_B + s\vec{r}_{B'},\ s \in \mathbb{R}$

$\vec{AB'} = \mu \vec{CB'} \Leftrightarrow \vec{r}_{B'} - \vec{r}_A = \mu(\vec{r}_{B'} - \vec{r}_C) \Leftrightarrow (1-\mu)\vec{r}_{B'} = \vec{r}_A - \mu \vec{r}_C$

$\Leftrightarrow \vec{r}_{B'} = \dfrac{\vec{r}_A - \mu \vec{r}_C}{1-\mu}$

$BB' : \vec{r}_X = (1-s)\vec{r}_B + s\vec{r}_{B'} = (1-s)\vec{r}_B + s \cdot \dfrac{\vec{r}_A - \mu \vec{r}_C}{1-\mu},\ s \in \mathbb{R}$



$*$ BB': $\vec{r_x} = \frac{s}{1-\mu}\vec{r_A} + (1-s)\vec{r_B} - \frac{\mu s}{1-\mu}\vec{r_C}$

CC': $\vec{r_y} = (1-t)\vec{r_C} + t\cdot\vec{r_{C'}},\ t\in\mathbb{R}$

$\overrightarrow{AC'} = \lambda\overrightarrow{BC'} \Leftrightarrow \vec{r_{C'}} - \vec{r_A} = \lambda(\vec{r_{C'}} - \vec{r_B}) \Leftrightarrow (1-\lambda)\vec{r_{C'}} = \vec{r_A} - \lambda\vec{r_B} \Leftrightarrow$

$\vec{r_{C'}} = \frac{\vec{r_A} - \lambda\vec{r_B}}{1-\lambda} \Rightarrow \vec{r_y} = (1-t)\vec{r_C} + \frac{t\vec{r_A}}{1-\lambda} - \frac{t\lambda\vec{r_B}}{1-\lambda} \Leftrightarrow$

$\Leftrightarrow \vec{r_y} = \frac{t}{1-\lambda}\vec{r_A} - \frac{t\lambda}{1-\lambda}\vec{r_B} + (1-t)\vec{r_C},\ t\in\mathbb{R}$ $*$

$$\{M\} = BB'\cap CC' \Rightarrow \vec{r_M} = \frac{s}{1-\mu}\vec{r_A} + (1-s)\vec{r_B} - \frac{\mu s}{1-\mu}\vec{r_C},\quad s,t\in\mathbb{R}$$
$$= \frac{t}{1-\lambda}\vec{r_A} - \frac{t\lambda}{1-\lambda}\vec{r_B} + (1-t)\vec{r_C}$$

We are looking for $s,t\in\mathbb{R}$ s.t.

$$\begin{cases} \frac{s}{1-\mu} = \frac{t}{1-\lambda} \Leftrightarrow t = \frac{(1-\lambda)s}{1-\mu} \\ 1-s = -\frac{\lambda t}{1-\lambda} \Rightarrow 1-s = -\frac{\lambda s}{1-\mu} \\ \frac{-\mu s}{1-\mu} = 1-t \quad (*) \end{cases}$$

check $(*)$ $-\frac{\mu}{\cancel{1-\mu}}\cdot\frac{\cancel{1-\mu}}{1-\mu-\lambda} = 1-\frac{1-\lambda}{1-\mu-\lambda}$

$\Rightarrow \frac{-\mu}{1-\mu-\lambda} = \frac{-\mu}{1-\mu-\lambda}$, true

$$s\left(\frac{\lambda}{1-\mu} - 1\right) = -1 \Rightarrow s = \frac{1}{1-\frac{\lambda}{1-\mu}} = \frac{1-\mu}{1-\mu-\lambda} \Rightarrow t = \frac{1-\lambda}{1-\mu-\lambda} \Rightarrow$$

$$\Rightarrow \vec{r_M} = \frac{1}{1-\mu-\lambda}\vec{r_A} - \frac{\lambda}{1-\mu-\lambda}\vec{r_B} - \frac{\mu}{1-\mu-\lambda}\vec{r_C} \Rightarrow$$

$$\Rightarrow \vec{r_M} = \frac{\vec{r_A} - \lambda\vec{r_B} - \mu\vec{r_C}}{1-\lambda-\mu}$$

□

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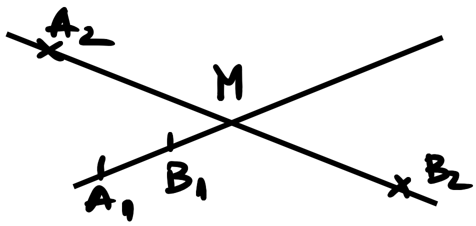
l - line in the Euclidian plane

We fix a reference system

Then $\forall M \in l\ \exists!\ \lambda \in \mathbb{R}$ s.t.: $\vec{r_M} = \lambda \vec{r_A} + (1-\lambda)\vec{r_B}$

If $M \in [AB]$ and $\frac{AM}{MB} = \alpha \Rightarrow \vec{r_M} = \frac{\alpha}{\alpha+1}\vec{r_B} + \frac{1}{\alpha+1}\vec{r_A}$

Let $\{M\} = l_1 \cap l_2$, $A_1, B_1 \in l_1$, $A_2, B_2 \in l_2$



Template for proofs

Step 1: We write the fact that $M \in l_1$ and $M \in l_2$ by using the vector eq. $\Rightarrow \exists\ \lambda, \mu \in \mathbb{R}$: $\vec{r_M} = \lambda \vec{r_{A_1}} + (1-\lambda)\vec{r_{B_1}}$ (1)

$$\vec{r_M} = \mu \vec{r_{A_2}} + (1-\mu)\vec{r_{B_2}} \quad (2)$$

Step 2: We find two vectors $\vec{v}, \vec{w}$ that are always linearly independent.

Step 3: We write $\vec{r_{A_1}}, \vec{r_{B_1}}, \vec{r_{A_2}}, \vec{r_{B_2}}$ in terms of $\vec{v}, \vec{w}$.

Step 4: You have obtained from (1) and (2) that:

$$\alpha(\lambda,\mu)\vec{v} + \beta(\lambda,\mu)\vec{w} = \vec{0}$$

Step 5: $\vec{v}, \vec{w}$ lin. indep $\Rightarrow \begin{cases} \alpha(\lambda,\mu) = 0 \\ \beta(\lambda,\mu) = 0 \end{cases}$. Solve the system to get λ (and μ).

Step 6: Replace λ (or μ) in (1) or (2). $\Rightarrow$ Find $\vec{r_M}$ in terms of $\vec{v}$ and $\vec{w}$.

1) a) $\vec{r_{A_1}} = \frac{1}{2}(\vec{r_B} + \vec{r_C})$

$\vec{r_{B_1}} = \frac{1}{2}(\vec{r_A} + \vec{r_C})$

$\vec{r_G} = \lambda \vec{r_A} + (1-\lambda)\vec{r_{A_1}}$

$\vec{r_G} = \mu \vec{r_B} + (1-\mu)\vec{r_{B_1}}$

$\Rightarrow \lambda \vec{r_A} + (1-\lambda)\frac{1}{2}(\vec{r_B} + \vec{r_C}) = \mu \vec{r_B} + (1-\mu)\frac{1}{2}(\vec{r_A} + \vec{r_C})$

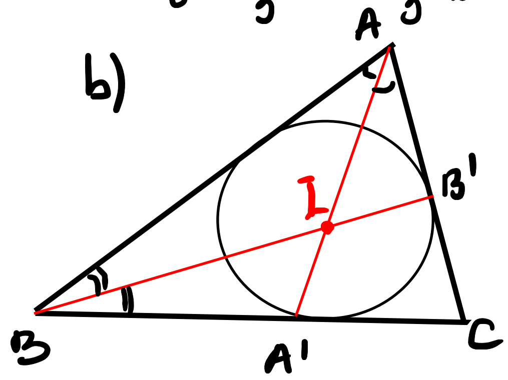
$\Leftrightarrow \vec{r_A}\left(\lambda - \frac{1}{2}(1-\mu)\right) + \vec{r_B}\left((1-\lambda)\frac{1}{2} - \mu\right) +$
$+ \vec{r_C}\left((1-\lambda)\frac{1}{2} - (1-\mu)\frac{1}{2}\right) = \vec{0}$ $\Rightarrow$

A - origin of the system: $\frac{(1-\lambda)}{2}\vec{AB} + \frac{(1-\lambda)}{2}\vec{AC} = \mu\vec{AB} + \frac{(1-\mu)}{2}\vec{AC}$

$\begin{cases} \frac{(1-\lambda)}{2} = \frac{(1-\mu)}{2} \\ \frac{(1-\lambda)}{2} = \mu \end{cases} \Rightarrow \begin{cases} 1-\lambda = 1-\mu \\ 1-\lambda = 2\mu \end{cases} \Rightarrow \begin{cases} \lambda = \mu \\ 3\lambda = 1 \end{cases} \Rightarrow \lambda = \mu = \frac{1}{3} \Rightarrow$

$\Rightarrow \vec{r_G} = \frac{1}{3}\vec{r_A} + \frac{1}{3}\vec{r_B} + \frac{1}{3}\vec{r_C}$

b)



$BC = a$
$CA = b$
$AB = c$

$\frac{BA'}{A'C} = \frac{\frac{1}{2}BA' \cdot d(A,BC)}{\frac{1}{2}A'C \cdot d(A,BC)} = \frac{\mathcal{A}_{\Delta ABA'}}{\mathcal{A}_{\Delta ACA'}} =$

$= \frac{\cancel{\frac{1}{2}} AB \cancel{AA'} \cancel{\sin(\sphericalangle BAA')}}{\cancel{\frac{1}{2}} CA \cdot \cancel{AA'} \cdot \cancel{\sin(\sphericalangle CAA')}} = \frac{AB}{CA} = \frac{c}{b} \Rightarrow$

$\Rightarrow \frac{BA'}{A'C} = \frac{c}{b} \overset{k}{} \Rightarrow \vec{r_{A'}} = \frac{1}{1+k}\vec{r_B} + \frac{k}{1+k}\vec{r_C} = \frac{1}{1+\frac{c}{b}}\vec{r_B} + \frac{\frac{c}{b}}{1+\frac{c}{b}}\vec{r_C} =$

$= \frac{b}{b+c}\vec{r_B} + \frac{c}{b+c}\vec{r_C}$

AA' : $\lambda\vec{r_A} + (1-\lambda)\vec{r_{A'}} = \lambda\vec{r_A} + (1-\lambda)\left(\frac{b}{b+c}\vec{r_B} + \frac{c}{b+c}\vec{r_C}\right), \lambda \in \mathbb{R}$

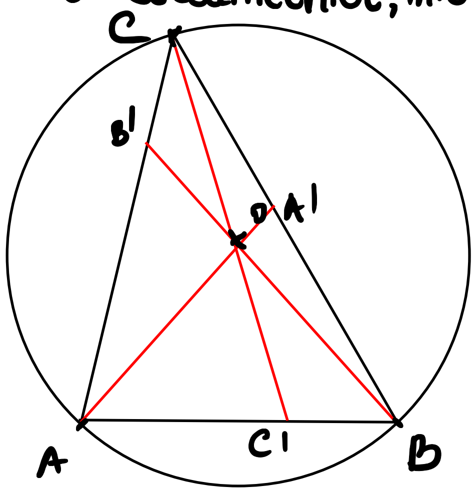
$\frac{B'C}{B'A} = \frac{BC}{BA} = \frac{a}{c} \Rightarrow \vec{r_{B'}} = \frac{c}{a+c}\vec{r_C} + \frac{a}{a+c}\cdot\vec{r_A}$

BB', $\mu\vec{r_B} + (1-\mu)\vec{r_{B'}} = \mu\vec{r_B} + (1-\mu)\left(\frac{c}{a+c}\vec{r_C} + \frac{a}{a+c}\cdot\vec{r_A}\right)$

$\{I\} = AA' \cap BB' \Rightarrow \vec{r_I} = \lambda\vec{r_A} + (1-\lambda)\left(\frac{b}{b+c}\vec{r_B} + \frac{c}{b+c}\vec{r_C}\right)$, for some $\lambda, \beta \in \mathbb{R}$
$\mu\vec{r_B} + (1-\mu)\left(\frac{c}{a+c}\vec{r_C} + \frac{a}{a+c}\cdot\vec{r_A}\right)$

Solving for λ, β, you will get the conclusion

c) O - circumcenter, intersection of the perpendicular bisectors



$\vec{r_O} = ?$

Find the vector form of the lines AA' and BB'.

- To find $\vec{r_{A'}}$ it suffices to find $\frac{A'B}{A'C}$.

$$\frac{A'B}{A'C} = \frac{A_{\Delta BAA'}}{A_{\Delta CAA'}} = \frac{BA \cdot \sin(\measuredangle BAO)}{AC \cdot \sin(\measuredangle CAO)}$$

$\measuredangle AOB = 2\measuredangle C$

One can use sin Th. in ΔBOA: $\frac{BA}{\sin \measuredangle BOA} = \frac{OB = R}{\sin \measuredangle BAO}$ =)

=) $BA \cdot \sin \measuredangle BAO = R \cdot \sin 2\measuredangle C$

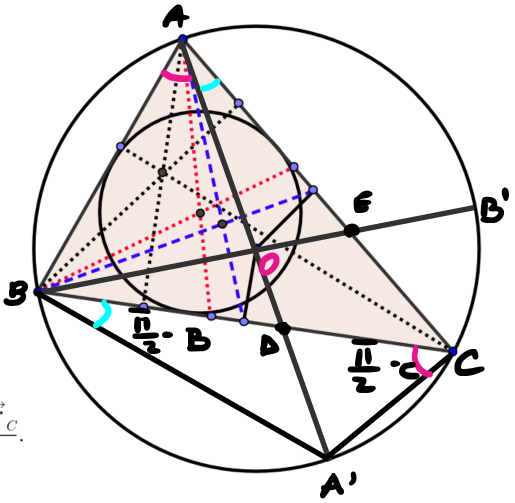
in ΔBOC, get BC

ΔAOC, get AC

2.3 Problems

1. ([4, Problem 17, p. 5]) Consider the triangle ABC, its centroid G, its orthocenter H, its incenter I and its circumcenter O. If $P \in \mathcal{P}$ is a given point and $\vec{r}_A = \overrightarrow{PA}$, $\vec{r}_B = \overrightarrow{PB}$, $\vec{r}_C = \overrightarrow{PC}$ are the position vectors with respect to P of the vertices A, B, C respectively, show that:

$a = BC$, $b = AC$, $c = AB$



a) $\vec{r}_G := \overrightarrow{PG} = \dfrac{\vec{r}_A + \vec{r}_B + \vec{r}_C}{3}$.

b) $\vec{r}_I := \overrightarrow{PI} = \dfrac{a\,\vec{r}_A + b\,\vec{r}_B + c\,\vec{r}_C}{a+b+c}$.

c) $\vec{r}_H := \overrightarrow{PH} = \dfrac{(\tan A)\,\vec{r}_A + (\tan B)\,\vec{r}_B + (\tan C)\,\vec{r}_C}{\tan A + \tan B + \tan C}$.

d) $\vec{r}_O := \overrightarrow{PO} = \dfrac{(\sin 2A)\,\vec{r}_A + (\sin 2B)\,\vec{r}_B + (\sin 2C)\,\vec{r}_C}{\sin 2A + \sin 2B + \sin 2C}$.

We fix a reference system.

Solution. $AA' \cap BC = \{D\}$, $BB' \cap AC = \{E\}$. Show that $\dfrac{BD}{DC} = \dfrac{\sin 2C}{\sin 2B}$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$, R - radius of the circum. circle

Using the sin th in ΔABD: $\dfrac{BD}{\sin \widehat{BAD}} = \dfrac{AD}{\sin B} = \dfrac{c}{\sin \widehat{ADB}}$ $\Rightarrow$

ΔADC: $\dfrac{CD}{\sin \widehat{CAD}} = \dfrac{b}{\sin \widehat{ADC}} = \dfrac{AD}{\sin C}$

$\Rightarrow \dfrac{BD}{CD} \cdot \dfrac{\sin \widehat{CAD}}{\sin \widehat{BAD}} = \dfrac{\frac{AD}{\sin B}}{\frac{AD}{\sin C}} = \dfrac{\sin C}{\sin B} \Rightarrow \dfrac{BD}{CD} \cdot \dfrac{\cos B}{\cos C} = \dfrac{\sin C}{\sin B}$ (1)

$\widehat{BAA'} \equiv \widehat{BCA'}$ (they corresp. to $\overset{\frown}{BA'}$) and $\widehat{CAA'} \equiv \widehat{CBA'}$ (they both corresp to $\overset{\frown}{CA'}$)

$m(\widehat{ACA'}) = \frac{\pi}{2} \Rightarrow m(\widehat{BCA'}) = \frac{\pi}{2} - m(\hat{C})$

$m(\widehat{ABA'}) = \frac{\pi}{2} \Rightarrow m(\widehat{CBA'}) = \frac{\pi}{2} - m(\hat{B})$

$\Rightarrow \sin(\widehat{CAD}) = \sin\left(\frac{\pi}{2} - B\right) = \cos B$

$\sin(\widehat{BAD}) = \sin\left(\frac{\pi}{2} - C\right) = \cos C$

(1) $\Rightarrow \dfrac{BD}{CD} = \dfrac{\cos C \cdot \sin C}{\cos B \cdot \sin B} = \dfrac{\sin 2C}{\sin 2B}$, $\dfrac{AE}{EC} = \dfrac{\sin 2C}{\sin 2A}$

d) $\sin 2A =: \alpha$, $\sin 2B =: \beta$, $\sin 2C =: \gamma$

$\vec{r}_O = \lambda \vec{r}_D + (1-\lambda)\vec{r}_A = \mu \vec{r}_E + (1-\mu)\vec{r}_B$, $\lambda, \mu \in \mathbb{R}$ and $\dfrac{BD}{DC} = \dfrac{\gamma}{\beta} \Rightarrow \vec{r}_D = \dfrac{\vec{r}_B + \frac{\gamma}{\beta}\vec{r}_C}{1 + \frac{\gamma}{\beta}} = \dfrac{\beta \vec{r}_B + \gamma \vec{r}_C}{\beta + \gamma}$

$$\frac{AE}{EC}=\frac{\gamma}{\alpha}\Rightarrow \vec{r_E}=\frac{\alpha\vec{r_A}+\gamma\vec{r_C}}{\alpha+\gamma}$$

$$\Rightarrow (1-\lambda)\vec{r_A}+\frac{\lambda\beta}{\beta+\gamma}\vec{r_B}+\frac{\lambda\gamma}{\beta+\gamma}\vec{r_C}=(1-\mu)\vec{r_B}+\frac{\mu\alpha}{\alpha+\gamma}\vec{r_A}+\frac{\mu\gamma}{\alpha+\gamma}\vec{r_C}$$

$$\Rightarrow \left(1-\lambda-\frac{\mu\alpha}{\alpha+\gamma}\right)\vec{r_A}+\left(\frac{\lambda\beta}{\beta+\gamma}-1+\mu\right)\vec{r_B}+\left(\frac{\lambda\gamma}{\beta+\gamma}-\frac{\mu\gamma}{\alpha+\gamma}\right)\vec{r_C}=\vec{0}\quad (1)$$

$\vec{AB}$ and $\vec{AC}$ are linearly indep., because if not, then ΔABC would be degenerate.

We choose $\vec{v}=\vec{AB}$ and $\vec{w}=\vec{AC}$

$\vec{r_B}=\vec{r_A}+\vec{v}$, $\vec{r_C}=\vec{r_A}+\vec{w}$

$$(1)\Rightarrow \left(1-\lambda-\frac{\mu\alpha}{\alpha+\gamma}\right)\vec{r_A}+\left(\frac{\lambda\beta}{\beta+\gamma}-1+\mu\right)(\vec{r_A}+\vec{v})+\left(\frac{\lambda\gamma}{\beta+\gamma}-\frac{\mu\gamma}{\alpha+\gamma}\right)(\vec{r_A}+\vec{w})=\vec{0}$$

$$\underbrace{\left(\cancel{1}-\lambda-\frac{\alpha\mu}{\alpha+\gamma}+\frac{\lambda\beta}{\beta+\gamma}-\cancel{1}+\mu+\frac{\lambda\gamma}{\beta+\gamma}-\frac{\mu\gamma}{\alpha+\gamma}\right)}_{0}\vec{r_A}+\left(\frac{\lambda\beta}{\beta+\gamma}-1+\mu\right)\vec{v}+\left(\frac{\lambda\gamma}{\beta+\gamma}-\frac{\mu\gamma}{\alpha+\gamma}\right)\vec{w}=\vec{0}$$

$$\left.\begin{array}{l}\Rightarrow\left(\frac{\lambda\beta}{\beta+\gamma}-1+\mu\right)\vec{v}+\left(\frac{\lambda\gamma}{\beta+\gamma}-\frac{\mu\gamma}{\alpha+\gamma}\right)\vec{w}=\vec{0}\\ \vec{v},\vec{w}-\text{lin. indep}\end{array}\right\}\Rightarrow\begin{cases}\frac{\lambda\beta}{\beta+\gamma}-1+\mu=0\\ \frac{\lambda\gamma}{\beta+\gamma}-\frac{\mu\gamma}{\alpha+\gamma}=0\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}\mu=1-\frac{\lambda\beta}{\beta+\gamma}\\ \frac{\lambda\gamma}{\beta+\gamma}+\frac{\lambda\beta\gamma}{(\beta+\gamma)(\alpha+\gamma)}-\frac{\gamma}{\alpha+\gamma}=0\end{cases}\quad\Rightarrow\lambda=\frac{\frac{\gamma}{\alpha+\gamma}}{\frac{\gamma}{\beta+\gamma}+\frac{\gamma\beta}{(\beta+\gamma)(\alpha+\gamma)}}=\frac{\beta+\gamma}{(\alpha+\gamma)+\beta}=\frac{\beta+\gamma}{\beta+\gamma+\alpha}$$

$\Rightarrow \vec{r_O}=\frac{\alpha}{\alpha+\beta+\gamma}\vec{r_A}+\frac{\beta}{\alpha+\beta+\gamma}\vec{r_B}+\frac{\gamma}{\alpha+\beta+\gamma}\vec{r_C}$, replace α,β,γ with $\sin 2A, \sin 2B, \sin 2C \Rightarrow$

$\Rightarrow$ conclusion

2. Consider the angle BOB' and the points $A\in[OB]$, $A'\in[OB']$. Show that

$$\overrightarrow{OM}=m\frac{1-n}{1-mn}\,\overrightarrow{OA}\;+\;n\frac{1-m}{1-mn}\,\overrightarrow{OA'}$$

$$\overrightarrow{ON}=m\frac{n-1}{n-m}\,\overrightarrow{OA}\;+\;n\frac{m-1}{m-n}\,\overrightarrow{OA'}.$$

where $\{M\}=AB'\cap A'B$, $\{N\}=AA'\cap BB'$, $\vec{u}=\overrightarrow{OA}$, $\vec{v}=\overrightarrow{OA'}$, $\overrightarrow{OB}=m\,\overrightarrow{OA}$ and $\overrightarrow{OB'}=n\,\overrightarrow{OA'}$.

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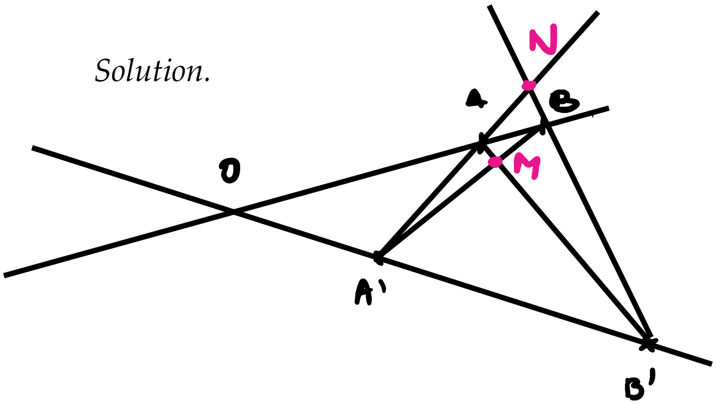
Solution.



$$\overrightarrow{OM}=\lambda\overrightarrow{OA}+(1-\lambda)\overrightarrow{OB'}=\lambda\cdot\vec{u}+(1-\lambda)n\vec{v} \Rightarrow$$
$$=\mu\overrightarrow{OB}+(1-\mu)\overrightarrow{OA'}=\mu m\cdot\vec{u}+(1-\mu)\vec{v}$$

$$\Rightarrow \left.\begin{array}{l}(\lambda-\mu m)\vec{u}+((1-\lambda)n-1+\mu)\vec{v}=\vec{0}\\ \vec{u},\vec{v}\text{ - lin indep}\end{array}\right\} \Rightarrow \begin{cases}\lambda-\mu m=0\\ (1-\lambda)n-1+\mu=0\end{cases}\Rightarrow$$

$$\Rightarrow \mu=\frac{1-n}{1-nm}\Rightarrow \overrightarrow{OM}=\frac{1-n}{1-nm}\cdot m\cdot\vec{u}+\left(1-\frac{1-n}{1-nm}\right)\vec{v}=$$
$$=\frac{1-n}{1-nm}\cdot m\cdot\vec{u}-\frac{n(1-m)}{1-nm}\vec{v}$$

Same thing for $\overrightarrow{ON}$.

AB': A point $M\in AB'$ iff $\overrightarrow{OM}=\lambda\overrightarrow{OA}+(1-\lambda)\overrightarrow{OB'}$, $\lambda\in\mathbb{R}$

$$\left.\begin{array}{l}AA': N\in AA' \text{ iff } \overrightarrow{ON}=\beta\overrightarrow{OA}+(1-\beta)\cdot\overrightarrow{OA'},\ \beta\in\mathbb{R}\\ BB': N\in BB' \text{ iff } \overrightarrow{ON}=\gamma\overrightarrow{OB}+(1-\gamma)\overrightarrow{OB'},\ \gamma\in\mathbb{R}\end{array}\right\}\Rightarrow$$

$$\beta\vec{u}+(1-\beta)\vec{v}=\gamma m\vec{u}+(1-\gamma)n\vec{v}\Rightarrow(\beta-\gamma m)\vec{u}+(1-\beta-(1-\gamma)n)\vec{v}=\vec{0}$$

$$\vec{u},\vec{v}\text{ - lin. indep}\Rightarrow\begin{cases}\beta-\gamma m=0 \Rightarrow \beta=\gamma m\\ 1-\beta-(1-\gamma)n=0\end{cases}\Rightarrow 1-\gamma m-(1-\gamma)n=0\Rightarrow$$

$$\Rightarrow 1-\gamma m-n+\gamma n=0\Rightarrow\gamma(n-m)=n-1\Rightarrow\gamma=\frac{n-1}{n-m}\Rightarrow$$

$$\beta=m\cdot\frac{n-1}{n-m}\Rightarrow\overrightarrow{ON}=m\cdot\frac{n-1}{n-m}\vec{u}+n\cdot\frac{-m+1}{n-m}\vec{v}=$$
$$=m\cdot\frac{n-1}{n-m}\vec{u}+n\cdot\frac{m-1}{m-n}\vec{v}$$

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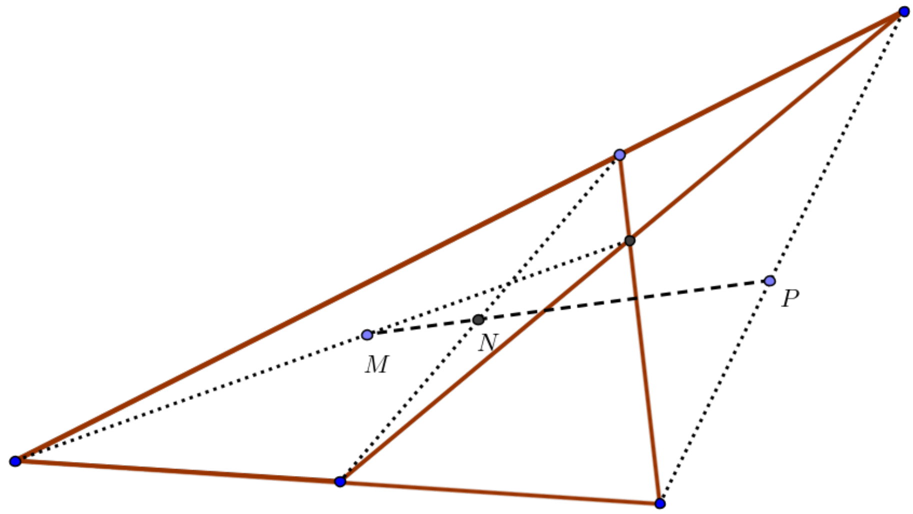
3. Show that the midpoints of the diagonals of a complet quadrilateral are collinear (Newton's theorem).



Solution.

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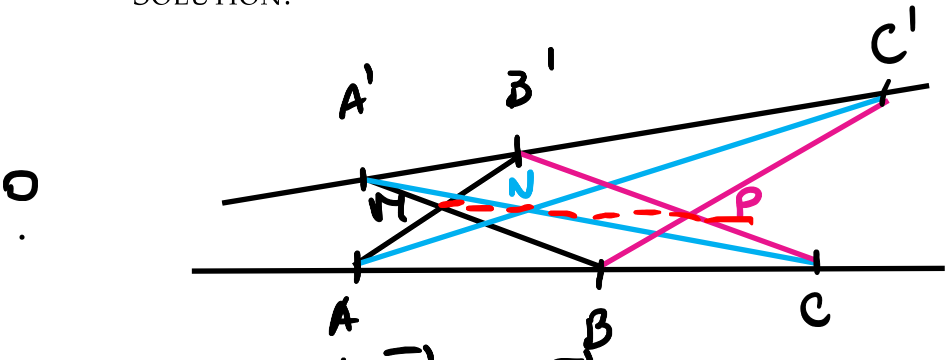
4. Let d, d' be concurrent straight lines and $A, B, C \in d$, $A', B', C' \in d'$. If the following relations $AB' \not\parallel A'B$, $AC' \not\parallel A'C$, $BC' \not\parallel B'C$ hold, show that the points $\{M\} := AB' \cap A'B$, $\{N\} := AC' \cap A'C$, $\{P\} := BC' \cap B'C$ are collinear (Pappus' theorem).

SOLUTION.



Find $k \in \mathbb{R}$ s.t. $\overrightarrow{MN} = k \cdot \overrightarrow{NP}$

$\overline{v} = \overrightarrow{OA}$, $\overline{w} = \overrightarrow{OA'}$

A, B, C are collinear $\Rightarrow$ $\overrightarrow{OB} = \alpha \cdot \overline{v}$, $\alpha \in \mathbb{R}$

$\overrightarrow{OC} = \beta \cdot \overline{v}$, $\beta \in \mathbb{R}$

A', B', C' $\,$ " $\Rightarrow$ $\overrightarrow{OB'} = l \cdot \overline{w}$, $l \in \mathbb{R}$

$\overrightarrow{OC'} = g \cdot \overline{w}$, $g \in \mathbb{R}$

① Write the vector eq. of: AB' and $A'B$, $M \in AB' \cap A'B$

solving a system of 2 eq, we express $\overrightarrow{OM}$! See problem 2

② Do the same thing for $\overrightarrow{ON}$ and $\overrightarrow{OP}$

③ Write $\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$ and everything in terms of $\overline{v}, \overline{w}$ (L.I)

$\overrightarrow{NP} = \overrightarrow{OP} - \overrightarrow{ON}$

Say $\overrightarrow{MN} = \alpha_1 \overline{v} + \beta_1 \overline{w}$ $\Rightarrow$ Check if: $\dfrac{\alpha_1}{\alpha_2} = \dfrac{\beta_1}{\beta_2}$

$\overrightarrow{NP} = \alpha_2 \overline{v} + \beta_2 \overline{w}$

5. Let d, d' be two straight lines and $A, B, C \in d$, $A', B', C' \in d'$ three points on each line such that $AB' \| BA'$, $AC' \| CA'$. Show that $BC' \| CB'$ (the affine Pappus' theorem).

   SOLUTION.

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6. Let us consider two triangles ABC and $A'B'C'$ such that the lines AA', BB', CC' are concurrent at a point O and $AB \nparallel A'B'$, $BC \nparallel B'C'$ and $CA \nparallel C'A'$. Show that the points $\{M\} = AB \cap A'B'$, $\{N\} = BC \cap B'C'$ and $\{P\} = CA \cap C'A'$ are collinear (Desargues).

   SOLUTION.

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