11 Week 11: Generated Surfaces

Consider the 3-dimensional Euclidean space $\mathcal{E}_3$, together with a Cartesian system of coordinates $Oxyz$. Generally, the set

$$S = \{M(x,y,z) : F(x,y,z) = 0\},$$

where $F : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a real function and D is a domain, is called *surface* of implicit equation $F(x,y,z) = 0$. For example the quadric surfaces, defined in the previous chapter for F a polynomial of degree two, are such of surfaces. On the other hand, the set

$$S_1 = \{M(x,y,z) : x = x(u,v), y = y(u,v), z = z(u,v)\},$$

where $x, y, z : D_1 \subseteq \mathbb{R}^2 \to \mathbb{R}$, is a *parameterized surface*, of parametric equations

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}, \quad (u,v) \in D_1.$$

The intersection between two surfaces is a *curve* in 3-space (remember, for instance, that the intersection between a quadric surface and a plane is a conic section, hence the conics are plane curves). Then, the set

$$C = \{M(x,y,z) : F(x,y,z) = 0, G(x,y,z) = 0\},$$

where $F, G : D \subseteq \mathbb{R}^3 \to \mathbb{R}$, is the curve of *implicit* equations

$$\begin{cases} F(x,y,z) = 0 \\ G(x,y,z) = 0 \end{cases}.$$

As before, one can parameterize the curve. The set

$$C_1 = \{M(x,y,z) : x = x(t), y = y(t), z = z(t)\},$$

where $x, y, z : I \subseteq \mathbb{R} \to \mathbb{R}$ and I is open, is called *parameterized curve* of parametric equations

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}, \quad t \in I.$$

Let be given a family of curves, depending on one single parameter λ,

$$\mathcal{C}_\lambda : \begin{cases} F_1(x,y,z;\lambda) = 0 \\ F_2(x,y,z;\lambda) = 0 \end{cases}.$$

In general, the family $\mathcal{C}_\lambda$ does not cover the entire space. By eliminating the parameter λ between the two equations of the family, one obtains the equation of the surface *generated* by the family of curves.

Suppose now that the family of curves depends on two parameters λ, μ,

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} F_1(x,y,z;\lambda,\mu) = 0 \\ F_2(x,y,z;\lambda,\mu) = 0 \end{cases},$$

and that the parameters are related through $\varphi(\lambda,\mu) = 0$ If it can be obtained an equation which does not depend on the parameters (by eliminating the parameters between the three equations), then the set of all the points which verify it is called surface *generated* by the family (or the sub-family) of curves.

11.1 Cylindrical Surfaces

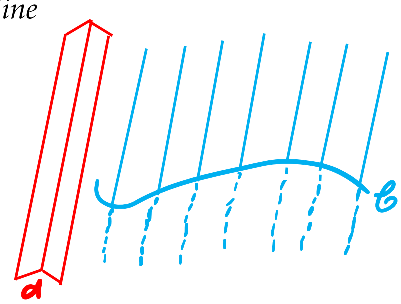
Definition 11.1. The surface generated by a variable line, called *generatrix*, which remains parallel to a fixed line d and intersects a given curve $\mathcal{C}$, is called *cylindrical surface*. The curve $\mathcal{C}$ is called the *director curve* of the cylindrical surface.

Theorem 11.1. *The cylindrical surface, with the generatrix parallel to the line*

$\pi_1(x,y,z) = A_1x + B_1y + C_1z + D_1$

$\pi_2(x,y,z) = A_2x + B_2y + C_2z + D_2$

$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases},$$



which has the director curve

$$\mathcal{C} : \begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases},$$

(d and $\mathcal{C}$ are not coplanar), is characterized by an equation of the form

$$\varphi(\pi_1, \pi_2) = 0. \tag{11.1}$$

Proof. The equations of an arbitrary line, which is parallel to

$$d : \begin{cases} \pi_1(x,y,z) = 0 \\ \pi_2(x,y,z) = 0 \end{cases}, \text{ are } d_{\lambda,\mu} : \begin{cases} \pi_1(x,y,z) = \lambda \\ \pi_2(x,y,z) = \mu \end{cases}.$$

Not every line from the family $d_{\lambda,\mu}$ intersects the curve $\mathcal{C}$. This happens only when the system of equations

$$\begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \\ \pi_1(x,y,z) = \lambda \\ \pi_2(x,y,z) = \mu \end{cases} \Leftrightarrow \begin{cases} x = x(\lambda,\mu) \\ y = y(\lambda,\mu) \\ z = z(\lambda,\mu) \end{cases}$$

is compatible. By eliminating λ and μ between four equations of the system, one obtains a *necessary condition* $\varphi(\lambda,\mu) = 0$ for the parameters λ and μ in order to nonempty intersection between the line $d_{\lambda,\mu}$. The equation of the surface can be determined now from the system

$F_1(x(\lambda,\mu), y(\lambda,\mu), z(\lambda,\mu)) = 0 \Leftrightarrow \varphi(\lambda,\mu) = 0$

$$\begin{cases} \pi_1(x,y,z) = \lambda \\ \pi_2(x,y,z) = \mu \\ \varphi(\lambda,\mu) = 0 \end{cases},$$

and it is immediate that $\varphi(\pi_1, \pi_2) = 0$. □

Remark 11.1. Any equation of the form (11.1), where π_1 and π_2 are linear function of x, y and z, represents a cylindrical surface, having the generatrices parallel to $d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$.

Example 11.1. Let us find the equation of the cylindrical surface having the generatrices parallel to

$$d : \begin{cases} x + y = 0 \\ z = 0 \end{cases}$$

and the director curve given by

$$\mathcal{C} : \begin{cases} x^2 - 2y^2 - z = 0 \\ x - 1 = 0 \end{cases}.$$

The equations of the generatrices d are

$$d_{\lambda,\mu} : \begin{cases} x+y=\lambda \\ z=\mu \end{cases}.$$

They must intersect the curve $\mathcal{C}$, i.e. the system

$$\begin{cases} x^2-2y^2-z=0 \\ x-1=0 \\ x+y=\lambda \\ z=\mu \end{cases}$$

has to be compatible. A solution of the system can be obtained using the three last equations

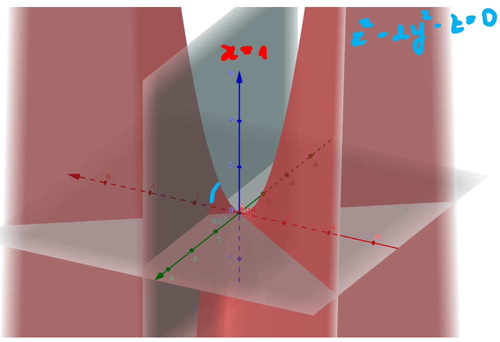
$$\begin{cases} x=1 \\ y=\lambda-1 \\ z=\mu \end{cases}$$

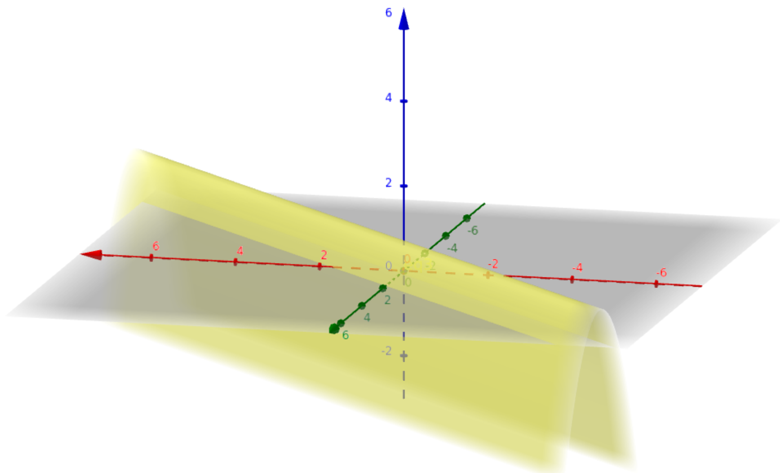
and, replacing in the first one, one obtains the compatibility condition

$$2(\lambda-1)^2+\mu-1=0.$$

Thus, the equation of the required cylindrical surface is

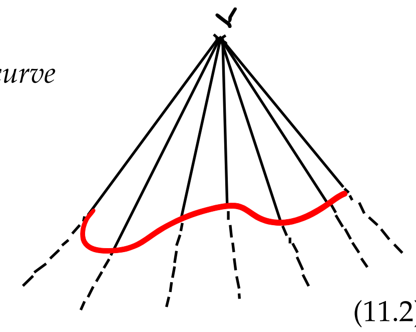
$$2(x+y-1)^2+z-1=0.$$





11.2 Conical Surfaces

Definition 11.2. The surface generated by a variable line, called *generatrix*, which passes through a fixed point V and intersects a given curve $\mathcal{C}$, is called *conical surface*. The point V is called the *vertex* of the surface and the curve $\mathcal{C}$ *director curve*.



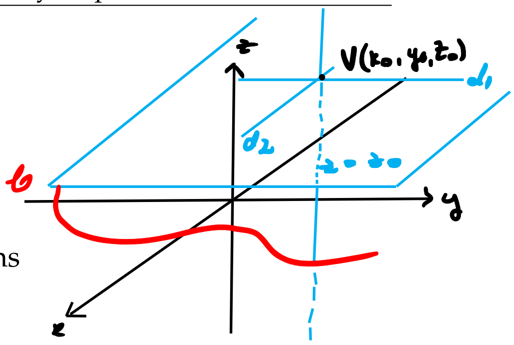
Theorem 11.2. *The conical surface, of vertex $V(x_0,y_0,z_0)$ and director curve*

$$\mathcal{C} : \begin{cases} F_1(x,y,z)=0 \\ F_2(x,y,z)=0 \end{cases},$$

(V and $\mathcal{C}$ are not coplanar), is characterized by an equation of the form

$$\varphi\left(\frac{x-x_0}{z-z_0}, \frac{y-y_0}{z-z_0}\right)=0. \tag{11.2}$$

$d_1 \begin{cases} z - z_0 = 0 \\ x - x_0 = 0 \end{cases}$ $d_2 \begin{cases} y - y_0 = 0 \\ z - z_0 = 0 \end{cases}$



Proof. The equations of an arbitrary line through $V(x_0, y_0, z_0)$ are

$$d_{\lambda\mu} : \begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \end{cases}.$$

A generatrix has to intersect the curve $\mathcal{C}$, hence the system of equations

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases} \Rightarrow \begin{cases} x = x(\lambda,\mu) \\ y = y(\lambda,\mu) \\ z = z(\lambda,\mu) \end{cases}$$

must be compatible. This happens for some values of the parameters λ and μ, which verify a *compatibility condition*

$$\varphi(\lambda, \mu) = 0$$

obtained by eliminating x, y and z in the the previous system of equations. In these conditions, the equation of the conical surface rises from the system

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ \varphi(\lambda, \mu) = 0 \end{cases},$$

i.e.

$$\varphi\left(\frac{x - x_0}{z - z_0}, \frac{y - y_0}{z - z_0}\right) = 0.$$

□

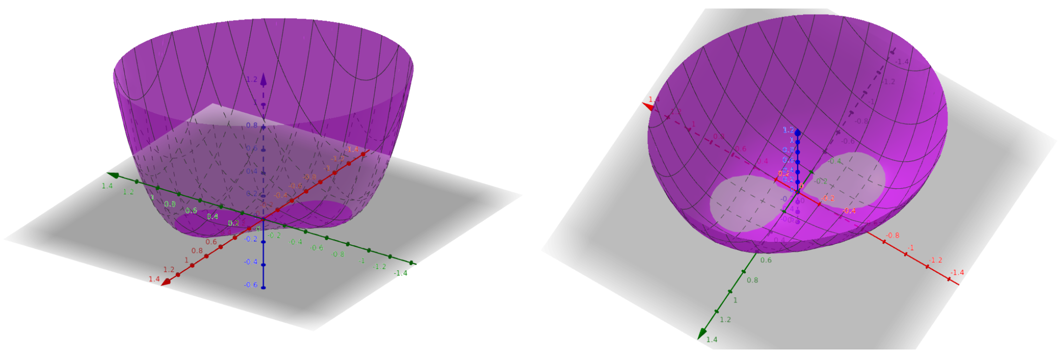
Remark 11.2. If φ is a polynomial function, then the equation (11.2) can be written in the form

$$\phi(x - x_0, y - y_0, z - z_0) = 0,$$

where ϕ is homogeneous with respect to $x - x_0$, $y - y_0$ and $z - z_0$. If φ is polynomial and V is the origin of the system of coordinates, then the equation of the conical surface is $\phi(x, y, z) = 0$, with ϕ a homogeneous polynomial. Conversely, an algebraic homogeneous equation in x, y and z represents a conical surface with the vertex at the origin.

Example 11.2. Let us determine the equation of the conical surface, having the vertex $V(1, 1, 1)$ and the director curve

$$\mathcal{C} : \begin{cases} (x^2 + y^2)^2 - xy = 0 \\ z = 0 \end{cases}.$$



The family of lines passing through V has the equations

$$d_{\lambda\mu} : \begin{cases} x - 1 = \lambda(z - 1) \\ y - 1 = \mu(z - 1) \end{cases}.$$

The system of equations

$$\begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0 \\ x-1 = \lambda(z-1) \\ y-1 = \mu(z-1) \end{cases}$$

must be compatible. A solution is

$$\Rightarrow \begin{cases} x = 1-\lambda \\ y = 1-\mu \\ z = 0 \end{cases},$$

and, replaced in the first equation of the system, gives the compatibility condition

$$[(1-\lambda)^2 + (1-\mu)^2]^2 - (1-\lambda)(1-\mu) = 0.$$

The equation of the conical surface is obtained by eliminating the parameters λ and μ in

$$\begin{cases} x-1 = \lambda(z-1) \\ y-1 = \mu(z-1) \\ ((1-\lambda)^2 + (1-\mu)^2)^2 - (1-\lambda)(1-\mu) = 0 \end{cases}.$$

Expressing $\lambda = \dfrac{x-1}{z-1}$ and $\mu = \dfrac{y-1}{z-1}$ and replacing in the compatibility condition, one obtains

$$\left[\left(\frac{z-x}{z-1}\right)^2 + \left(\frac{z-y}{z-1}\right)^2\right]^2 - \left(\frac{z-x}{z-1}\right)\left(\frac{z-y}{z-1}\right) = 0,$$

or

$$[(z-x)^2 + (z-y)^2]^2 - (z-x)(z-y)(z-1)^2 = 0.$$

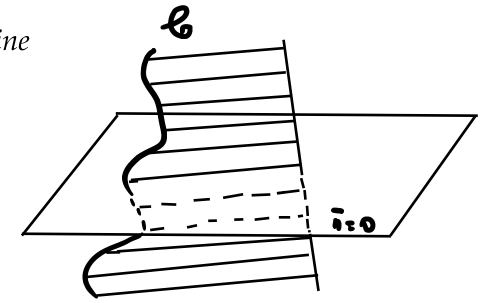
11.3 Conoidal Surfaces

Definition 11.3. The surface generated by a variable line, which intersects a given line d and a given curve $\mathcal{C}$, and remains parallel to a given plane π, is called *conoidal surface*. The curve $\mathcal{C}$ is the *director curve* and the plane π is the *director plane* of the conoidal surface.

Theorem 11.3. *The conoidal surface whose generatrix intersects the line*

$\pi_1(x,y,z) = A_1x + B_1y + C_1z + D_1$
$\pi_2(x,y,z) = A_2x + B_2y + C_2z + D_2$

$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$$



and the curve

$$\mathcal{C} : \begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases}$$

and has the director plane $\pi = 0$, (π is not parallel to d and that $\mathcal{C}$ is not contained into π), is characterized by an equation of the form

$\pi(x,y,z) = Ax + By + Cz + D$

$$\varphi\left(\pi, \frac{\pi_1}{\pi_2}\right) = 0. \tag{11.3}$$

Proof. An arbitrary generatrix of the conoidal surface is contained into a plane parallel to π and, on the other hand, comes from the bundle of planes containing d. Then, the equations of a generatrix are

$$d_{\lambda\mu} : \begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \end{cases}.$$

Again, the generatrix must intersect the director curve, hence the system of equations

$$\left\{\begin{array}{c} \pi = \lambda \\ \pi_1 = \mu\pi_2 \\ F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{array}\right. \quad \Leftrightarrow \left\{\begin{array}{l} x = x(\lambda,\mu) \\ y = y(\lambda,\mu) \\ z = z(\lambda,\mu) \end{array}\right.$$

has to be compatible. This leads to a compatibility condition

$$\varphi(\lambda,\mu) = 0,$$

and the equation of the conoidal surface is obtained from

$$\left\{\begin{array}{c} \pi = \lambda \\ \pi_1 = \mu\pi_2 \\ \varphi(\lambda,\mu) = 0 \end{array}\right. .$$

By expressing λ and μ, one obtains (11.3). □

Example 11.3. Let us find the equation of the conoidal surface, whose generatrices are parallel to xOy and intersect Oz and the curve

$$\left\{\begin{array}{l} y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{array}\right. .$$

The equations of xOy and Oz are, respectively,

$$xOy : z = 0, \quad \text{and} \quad Oz : \left\{\begin{array}{c} x = 0 \\ \cancel{z = 0} \\ y = 0 \end{array}\right. ,$$

so that the equations of the generatrix are

$$d_{\lambda,\mu} : \left\{\begin{array}{c} x = \lambda y \\ z = \mu \end{array}\right. .$$

From the compatibility of the system of equations

$$\left\{\begin{array}{c} x = \lambda y \\ z = \mu \\ y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{array}\right. , \Leftrightarrow \left\{\begin{array}{l} x = \lambda y \\ z = \mu \\ y^2 = 2\mu - 2 \\ \lambda^2 y^2 - 2\mu + 1 = 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x = \lambda y \\ z = \mu \\ y^2 = 2\mu - 2 \end{array}\right.$$

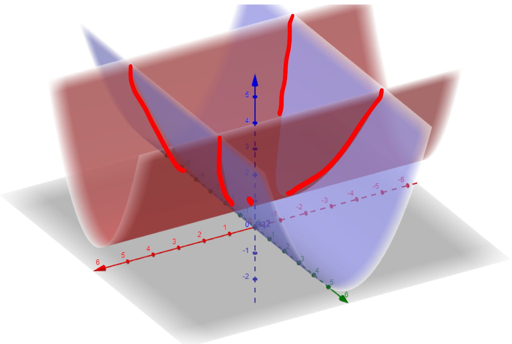
one obtains the compatibility condition

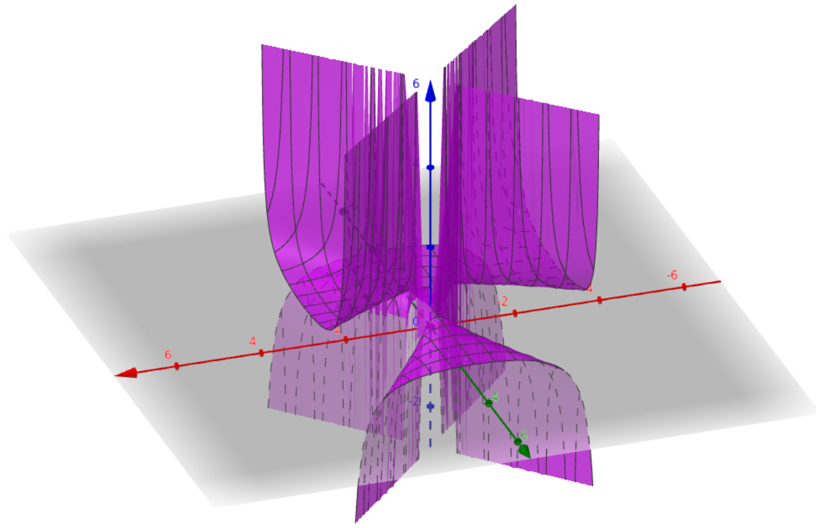
$$\lambda^2(2\mu - 2) - 2\mu + 1 = 0$$

$$2\lambda^2\mu - 2\lambda^2 - 2\mu + 1 = 0, \quad \Leftrightarrow 2\lambda^2\mu - 2\lambda^2 - 2\mu + 1 = 0$$

and, replacing $\lambda = \dfrac{x}{y}$ and $\mu = z$, the equation of the conoidal surface is

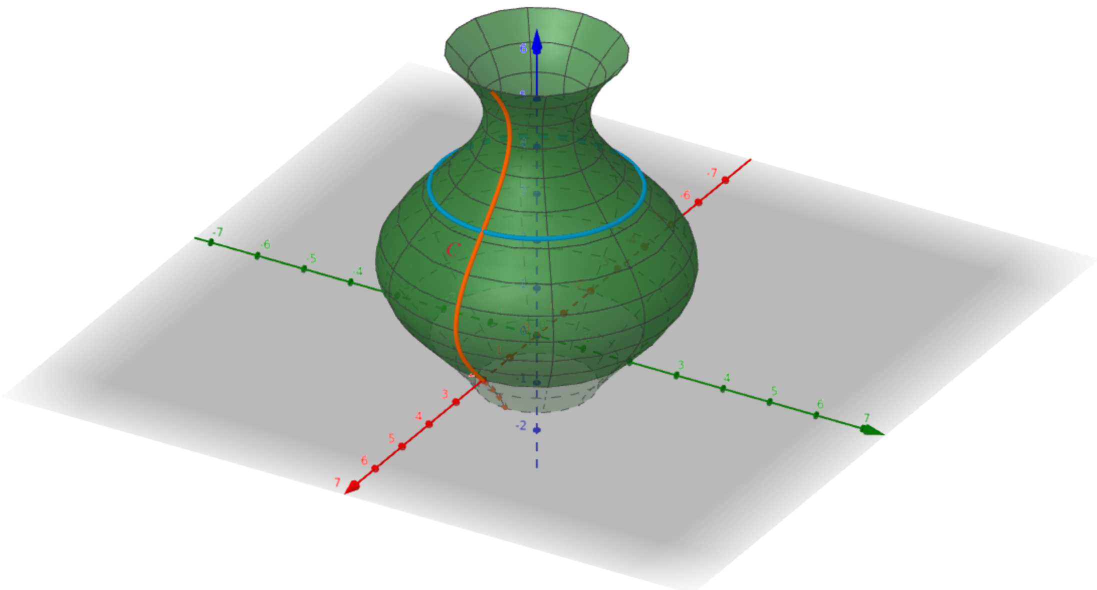
$$2x^2z - 2y^2z - 2x^2 + y^2 = 0. \tag{11.4}$$





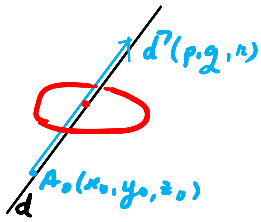
11.4 Revolution Surfaces

Definition 11.4. The surface generated by rotating of a given curve $\mathcal{C}$ around a given line d is said to be a *revolution surface*.



Theorem 11.4. *The equation of the revolution surface generated by the curve*

$$\mathcal{C} : \begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases},$$



in its rotation around the line

$$d : \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{r},$$

is of the form

$$\varphi((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2, px + qy + rz) = 0. \tag{11.5}$$

Proof. An arbitrary point on the curve $\mathcal{C}$ will describe, in its rotation around d, a circle situated into a plane orthogonal on d and having the center on the line d. This circle can be seen as the intersection between a sphere, having the center on d and of variable radius, and a plane, orthogonal on d, so that its equations are

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \lambda. \\ px + qy + rz = \mu \end{cases}$$

The circle has to intersect the curve $\mathcal{C}$, therefore the system

$$\left.\begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \\ (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \lambda \\ px + qy + rz = \mu \end{cases}\right\} \Rightarrow \begin{cases} x = x(\lambda, \mu) \\ y = y(\lambda, \mu) \\ z = z(\lambda, \mu) \end{cases}$$

must be compatible. One obtains the compatibility condition

$$\varphi(\lambda, \mu) = 0,$$

which, after replacing the parameters, gives the equation of the surface (11.5). □

11.5 Problems

1. Find the equation of the cylindrical surface whose director curve is the planar curve

$$(C) \begin{cases} y^2 + z^2 = x \\ x = 2z \end{cases}$$

and the generatrix is perpendicular to the plane of the director curve.

Solution.

The plane of the curve $\mathcal{C}$ is : $\pi : x = 2z$

We know that $d_{\lambda,\mu} \perp \pi$

Step 1: $d_{\lambda,\mu} : \begin{cases} \frac{x - x_0}{1} = \frac{z - z_0}{-2} \\ y = y_0 \end{cases} \Leftrightarrow \begin{cases} -2x - z = \underbrace{-2x_0 - z_0}_{\lambda} \\ y = y_0 = \mu \end{cases}$

Step 2: $\begin{cases} -2x - z = \lambda \\ y = \mu \\ y^2 + z^2 = x \\ x = 2z \end{cases} \Leftrightarrow \begin{cases} y = \mu \\ -5z = \lambda \\ x = 2z \\ y^2 + z^2 = x \end{cases} \Rightarrow \begin{cases} y = \mu \\ z = \frac{\lambda}{-5} \\ x = \frac{-2\lambda}{5} \\ y^2 + z^2 = x \end{cases}$

$\Rightarrow$ the compatibility cond: $\mu^2 + \frac{\lambda^2}{25} + \frac{2\lambda}{5} = 0 \Rightarrow$ The eq.: $25y^2 + (-2x - z)^2 + 10(-2x - z) = 0$

2. A disk of radius 1 is centered at the point $A(1,0,2)$ and is parallel to the plane yOz. A source of light is placed at the point $P(0,0,3)$. Characterize analitically the shadow of the disk rushed over the plane xOy.

   Solution. Consider the conical surface of vertex P whose director curve is the circle of radius 1 which is centered at the point $A(1,0,2)$ and is parallel to the plane yOz. The shadow of the disk rushed over the plane xOy is the convex component of the complement, in the plane xOy, of the intersection curve between the plane xOy and the described conical surface.

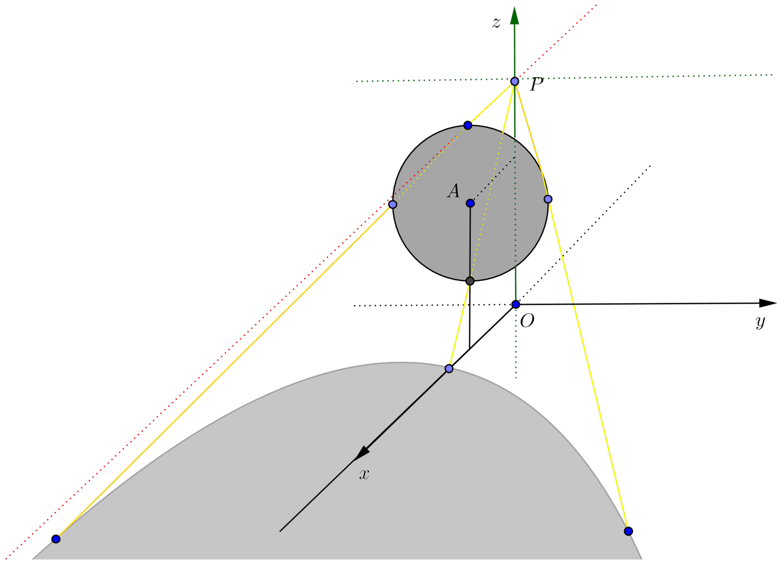
   In order to find the equation of the conical surface we consider the lines

$$(Oz)\ \begin{cases} x=0 \\ y=0 \end{cases} \text{ and } (d)\ \begin{cases} x=0 \\ z=3 \end{cases}$$

   as well as the family of lines

$$(\Delta_{\lambda\mu})\ \begin{cases} y-\lambda x=0 \\ z-3-\mu x=0 \end{cases}$$

depending on the parameters λ and μ of the reduced pencils of lines $x - \lambda y = 0$ through Oz and $z - 3 - \mu z = 0$ through d.



The circle C which borders the disk is given by the equations

$$(C)\ \begin{cases} (x-1)^2 + y^2 + (z-2)^2 = 1 \\ x = 1. \end{cases}$$

The intersection point of the line $\Delta_{\lambda\mu}$ with the plane of the circle is described by the system

$$\begin{cases} x = 1 \\ y - \lambda x = 0 \\ z - 3 - \mu x = 0 \end{cases}$$

which has the solution

$$(\Delta_{\lambda\mu} \cap (x = 1))\ \begin{cases} x = 1 \\ y = \lambda \\ z = 3 + \mu. \end{cases} \tag{11.6}$$

By imposing the condition on the intersection point (11.6) to belong the other surface which defines C, namely the sphere $(x-1)^2 + y^2 + (z-2)^2 = 1$, we obtain the relation $\lambda^2 + (\mu+1)^2 = 1$, between λ and μ, in order to have concurrence between $\Delta_{\lambda\mu}$ and C. The equation of the conical surface is

$$\left(\frac{y}{x}\right)^2 + \left(\frac{z-3}{x} + 1\right)^2 = 1, \text{ or } y^2 + (x + z - 3)^2 = x^2.$$

The latter equation is equivalent with

$$y^2 + z^2 + 2xz - 6x - 6z + 9 = 0.$$

Its intersection curve with the plane xOy is the parabola

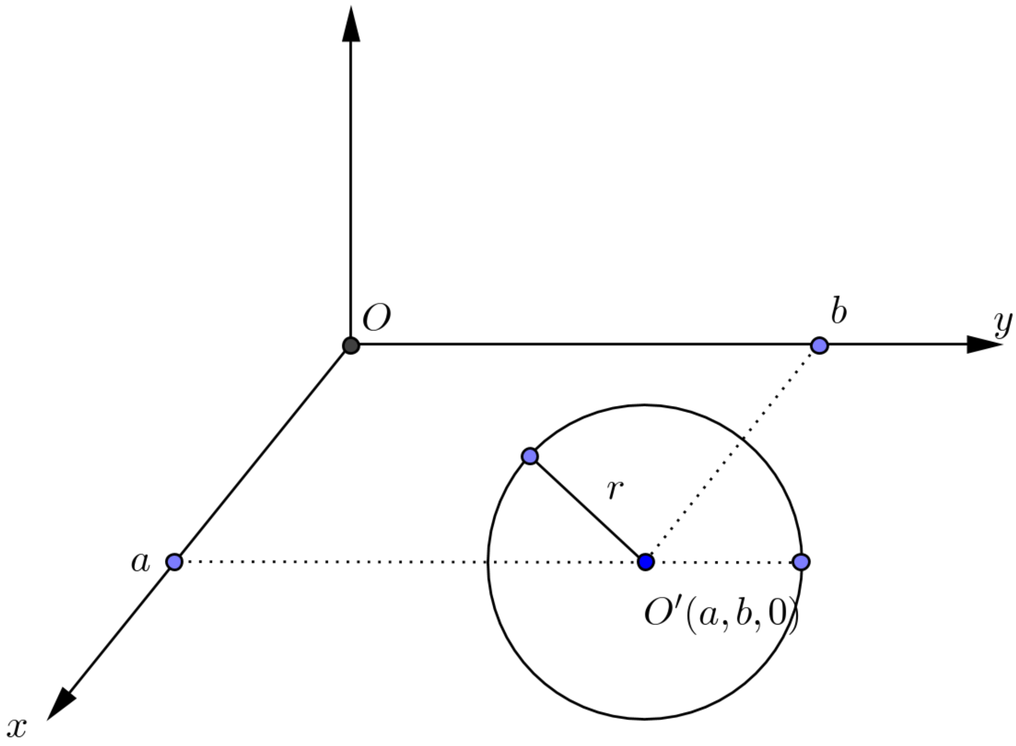
$$(\mathcal{P}\ \begin{cases} z = 0 \\ y^2 - 6x + 9 = 0. \end{cases}$$

The convex compoennt of the compelment $xOy \setminus \mathcal{P}$ coincides with the required shadow and is characterized by the following system

$$\begin{cases} y^2 - 6x + 9 & \leq 0 \\ z = 0. \end{cases}$$

3. Consider a circle and a line parallel with the plane of the circle. Find the equation of the conoidal surface generated by a variable line which intersects the line (d) and the circle (C) and remains orthogonal to (d). (The Willis conoid)

   Solution. We have the freedom to choose the orthonormal Cartesian reference system, as the involved geometrical objects are not given through any equations. In this respect we are going to consider the given line d as the z-axis and the xOy-plane as the orthogonal plane on d through the center O' of the given circle, where O is the intersection point of this plane with the line d, i.e. the z-axis. Within this plane we shall choose the y-axis to be the one through the point O which is parallel to the plane of the circle and the remaining x-axis is now completely determined.



   The equations of $d = Oz$ with respect to this reference system are

$$(d = Oz)\ \begin{cases} x = 0 \\ y = 0 \end{cases}$$

   and the equations of the given circle are

$$(C)\ \begin{cases} (x-a)^2 + (y-b)^2 + z^2 = r^2 \\ x = a \end{cases},$$

   where $(a, b, 0)$ are the coordinates of the center O' and r is the radius of the given circle.

   We next consider the family lines

$$d_{\lambda\mu}\ \begin{cases} y = \lambda x \\ z = \mu \end{cases}$$

   which are all parallel to the xOy plane, i.e. perpendicular to $d = Oz$. Out of this family we need to select those lines having some nonempty intersection with the circle C. In this respect we solve the system

$$\begin{cases} x = a \\ y = \lambda x \\ z = \mu \end{cases}$$

whose solution

$$\begin{cases} x = a \\ y = \lambda a \\ z = \mu \end{cases}$$

represent the coordinates of the intersection point between the line $d_{\lambda\mu}$ and the plane $x = a$ of the circle. The connection relation between the parameters λ and μ in order for the line $d_{\lambda\mu}$ to intersect the circle is $(\lambda a - b)^2 + \mu^2 = r^2$. Thus, the remaining steps towards the equation of the Willis connoid are

$$\left(\frac{y}{x}a - b\right)^2 + z^2 = r^2 \Longrightarrow (ya - b)^2 + x^2z^2 = r^2x^2.$$

4. Find the equation of the revolution surface generated by the rotation of a variable line through a fixed line.

   Solution. We are going to splite the solution in two casses depending on whether the two given lines are coplanar or not. The coplanar case will be splited again in two subcasses depending on whether the two lines are parallel or concurrent. In order to simplify the calculations, a suitable orthonormal coordinate system will be considered in each situation.

   (a) *The two lines are coplanar.*

      i. *The two lines are parallel.* We choose the coordinate reference system to have the fixed line as the z-axis and a common perpendicular line as the x-axis. The y-axis is now completely determined. The equations of the two lines are

$$(d_1 = Oz) \begin{cases} x = 0 \\ y = 0 \end{cases} \quad (d_2) \begin{cases} x = a \\ y = 0. \end{cases}$$

Consider the family of circles

$$(C_{\lambda\mu}) \begin{cases} x^2 + y^2 + z^2 = \lambda \\ z = \mu \end{cases}$$

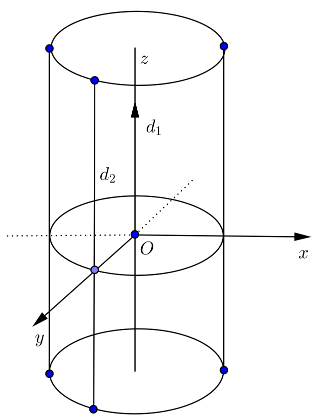
and determine the its intersection point with the halfplane of xOz in which the line d_2 lies, by solving the system

$$\begin{cases} x^2 + y^2 + z^2 = \lambda \\ z = \mu \\ y = 0 \end{cases}$$

for x, y and z in terms of λ and μ. This solution is

$$x = \sqrt{\lambda - \mu^2},\ y = 0,\ z = \mu$$

and this intersection point meets the line d_2 when $x = a$, namely $\lambda - \mu^2 = a^2$. Thus, the equation of the revolution surface, in this case, is $x^2 + y^2 = a^2$ and represents a circular cylinder.



ii. The two lines are concurrent. We choose the coordinate reference system to have the fixed line as the z-axis and their common perpendicular line as the x-axis. The y-axis is now completely determined. The equations of the two lines are

$$(d_1 = Oz) \begin{cases} x = 0 \\ y = 0 \end{cases} \quad (d_2) \begin{cases} x = 0 \\ z = my. \end{cases}$$

Consider the family of circles

$$(C_{\lambda\mu}) \begin{cases} x^2 + y^2 + z^2 = \lambda \\ z = \mu \end{cases}$$

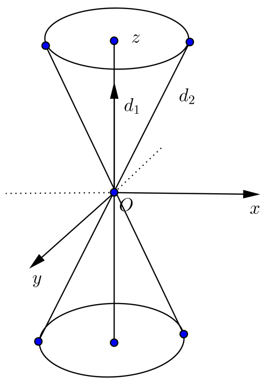
and determine the its intersection points with the coordinate plane yOz, by solving the system

$$\begin{cases} x = 0 \\ x^2 + y^2 + z^2 = \lambda \\ z = \mu \end{cases}$$

for x, y and z in terms of λ and μ. This solution is

$$x = 0,\ y = \pm\sqrt{\lambda - \mu^2},\ z = \mu$$

and this intersection points meets the line d_2 when $z = my$, namely $\mu^2 = \lambda - \mu^2$. Thus, the equation of the revolution surface, in this case, is $x^2 + y^2 = z^2$ and represents a circular cone.



(b) *The two lines are non-coplanar*. We choose the coordinate reference system to have the fixed line as the z-axis and their common perpendicular line as the x-axis. The y-axis is now completely determined. The equations of the two lines are

$$(d_1 = Oz) \begin{cases} x = 0 \\ y = 0 \end{cases} \quad (d_2) \begin{cases} x = a \\ z = my. \end{cases}$$

Consider the family of circles

$$(C_{\lambda\mu}) \begin{cases} x^2 + y^2 + z^2 = \lambda \\ z = \mu \end{cases}$$

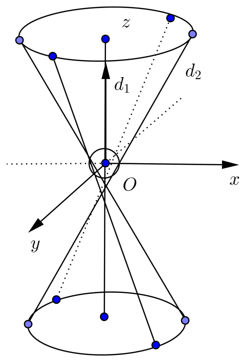
and determine the its intersection points with the plane $x = a$, by solving the system

$$\begin{cases} x = a \\ x^2 + y^2 + z^2 = \lambda \\ z = \mu \end{cases}$$

for x, y and z in terms of λ and μ. This solution is

$$x = a,\ y = \pm\sqrt{\lambda - \mu^2 - a^2},\ z = \mu$$

and this intersection points meets the line d_2 when $z = my$, namely $\mu^2 = \lambda - \mu^2 - a^2$. Thus, the equation of the revolution surface, in this case, is $x^2 + y^2 - z^2 = a^2$ and represents a circular hyperboloiud of one sheet.

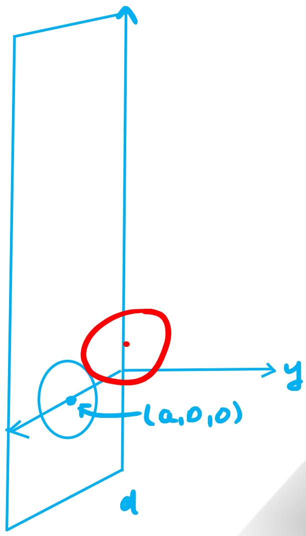


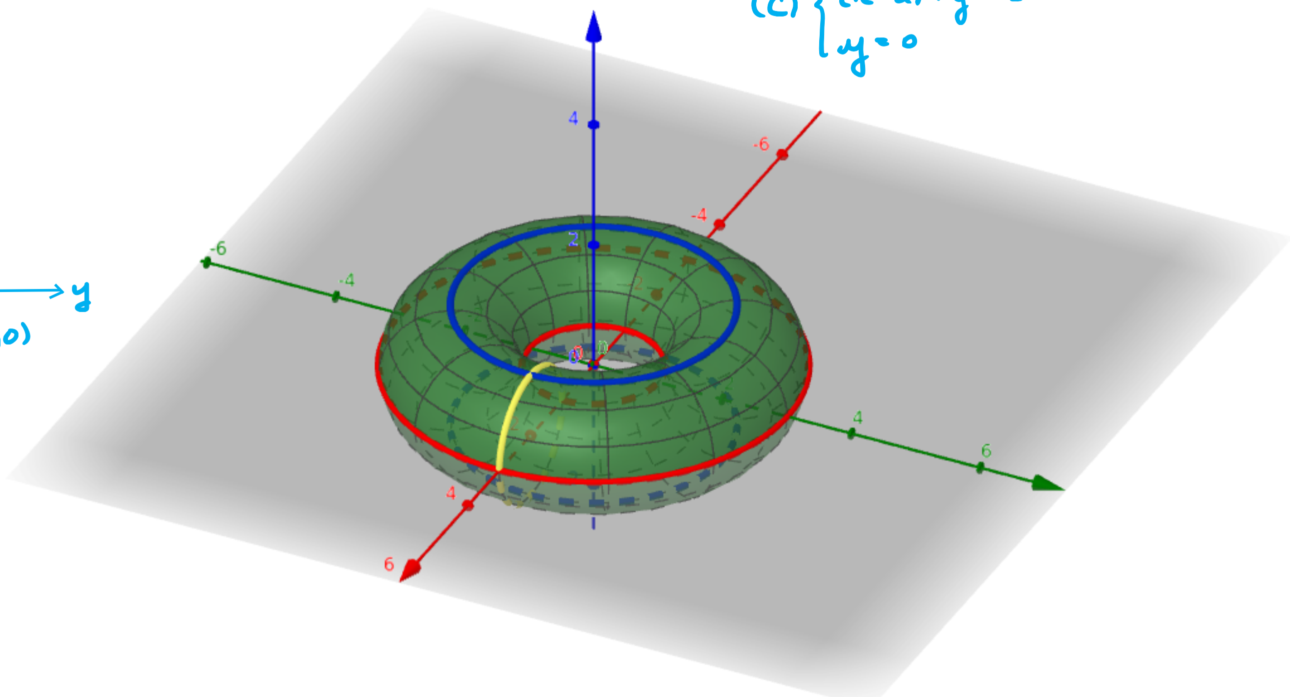
5. The *torus* is the revolution surface obtained by the rotation of a circle C about a fixed line (d) within the plane of the circle such that $d \cap C = \emptyset$. Find the equation of the torus[5]

$a > r > 0$

$(d)\begin{cases} x = 0 \\ y = 0 \end{cases}$

$(C)\begin{cases} (x-a)^2 + y^2 + z^2 = r^2 \\ y = 0 \end{cases}$





Solution.

[5]The torus is a regular surface.

$$(d)\begin{cases} x = 0 \\ y = 0 \end{cases} \qquad (C)\begin{cases} (x-a)^2+y^2+z^2 = r^2 \\ y = 0 \end{cases} \Leftrightarrow \begin{cases} (x-a)^2+z^2 = r^2 \\ y = 0 \end{cases}$$

$$C_{\lambda,\mu}\begin{cases} x^2+y^2+z^2 = \lambda \\ z = \mu \end{cases}$$

$$\begin{cases} x^2+y^2+z^2 = \lambda \\ z = \mu \\ (x-a)^2+z^2 = r^2 \\ y = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 = \lambda - \mu^2 \\ z = \mu \\ y = 0 \\ (x-a)^2+z^2 = r^2 \end{cases} \Leftrightarrow \begin{cases} x = \pm\sqrt{\lambda-\mu^2} \\ y = 0 \\ z = \mu \\ (x-a)^2+z^2 = r^2 \end{cases}$$

$$\Rightarrow \left(\sqrt{\lambda-\mu^2}-a\right)^2+\mu^2 = r^2 \Leftrightarrow \left(\sqrt{x^2+y^2+\cancel{z^2-z^2}}-a\right)^2+z^2 = r^2 \Leftrightarrow \left(\sqrt{x^2+y^2}-a\right)^2+z^2 = r^2 \text{ (eq. of the torus)}$$

Seminar 11 - 916 - 12/05/2021

Generated surfaces: → ruled surfaces (generated by a line) → conicals, cylindricals, conoidal surfaces
→ revolution surfaces (generated by a curve rotating around a line)

Example 11.2: Determine the eq. of the conical surface having the vertex $V(1,1,1)$ and the director curve:

$$\mathcal{C}: \begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases}$$

Step 1: We find all the lines that satisfy cond. 1 $\Rightarrow$ the generatrices

↳ conical: $d_{\lambda,\mu} \ni V$ (point)

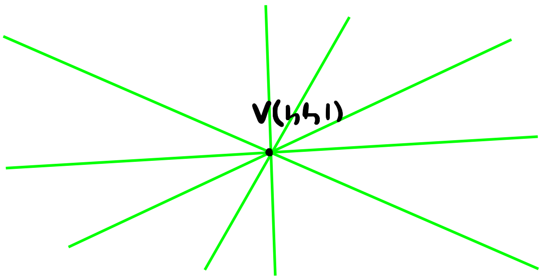
cylindrical: $d_{\lambda,\mu} \parallel l$ (line)

conoidal: $d_{\lambda\mu} \parallel \pi$ (plane)

$d_{\lambda,\mu} \cap l \neq \emptyset$, (line)

In our example: $V(1,1,1)$

$$d_{\lambda,\mu}: \frac{x-1}{a} = \frac{y-1}{b} = \frac{z-1}{c} \Leftrightarrow d_{\lambda,\mu}: \begin{cases} b(x-1) = a(y-1) \\ c(x-1) = a(z-1) \end{cases} \Leftrightarrow d_{\lambda,\mu}: \begin{cases} x-1 = \overbrace{\tfrac{a}{b}}^{\lambda}(y-1) \\ x-1 = \underbrace{\tfrac{a}{c}}_{\mu}(z-1) \end{cases}$$



Step 2: Out of the generatrices in step 1, we will now select only the ones that satisfy

cond. 2: $d_{\lambda,\mu} \cap \mathcal{C} \neq \emptyset$
↳ director curve

The generatrices that we choose will correspond to values of the parameters λ and μ s.t the system is compatible

$$\begin{cases} d_{\lambda,\mu}: \begin{cases} x-1 = \lambda(y-1) \\ x-1 = \mu(z-1) \end{cases} \\ \mathcal{C}: \begin{cases} (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases} \end{cases} \Leftrightarrow \begin{cases} x-1 = \lambda(y-1) \\ x-1 = \mu(z-1) \\ (x^2+y^2)^2 - xy = 0 \\ z = 0 \end{cases} \Leftrightarrow \begin{cases} z = 0 \\ x-1 = \lambda(y-1) \\ x-1 = -\mu \\ (x^2+y^2)^2 - xy = 0 \end{cases} \Leftrightarrow \begin{cases} z = 0 \\ x = 1-\mu \\ -\mu = \lambda(y-1) \\ (x^2+y^2)^2 - xy = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z = 0 \\ x = 1-\mu \\ y = -\frac{\mu}{\lambda} + 1 \\ (x^2+y^2)^2 - xy = 0 \end{cases}$$

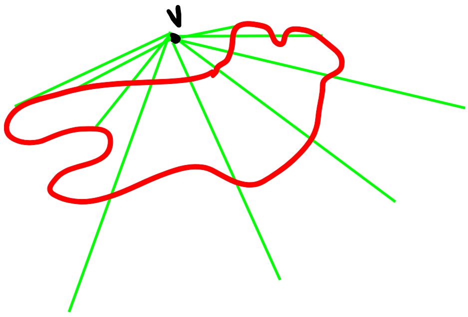
We now obtain the compatibility cond.: $\left((1-\mu)^2 + \left(1-\frac{\mu}{\lambda}\right)^2\right)^2 - (1-\mu)\left(1-\frac{\mu}{\lambda}\right) = 0$

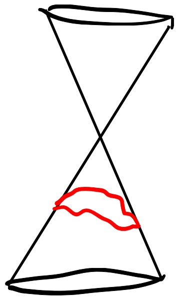
Step 3: We replace λ and μ by their expressions in x, y, z from the beginning (when they are introduced)

$\lambda = \frac{x-1}{y-1}$

$\mu = \frac{x-1}{z-1}$

Therefore, the eq. of the conical surface is: $\left(\left(1-\frac{x-1}{z-1}\right)^2 + \left(1-\frac{x-1}{y-1}\right)^2\right)^2 - \left(1-\frac{x-1}{z-1}\right)\left(1-\frac{y-1}{z-1}\right) = 0$



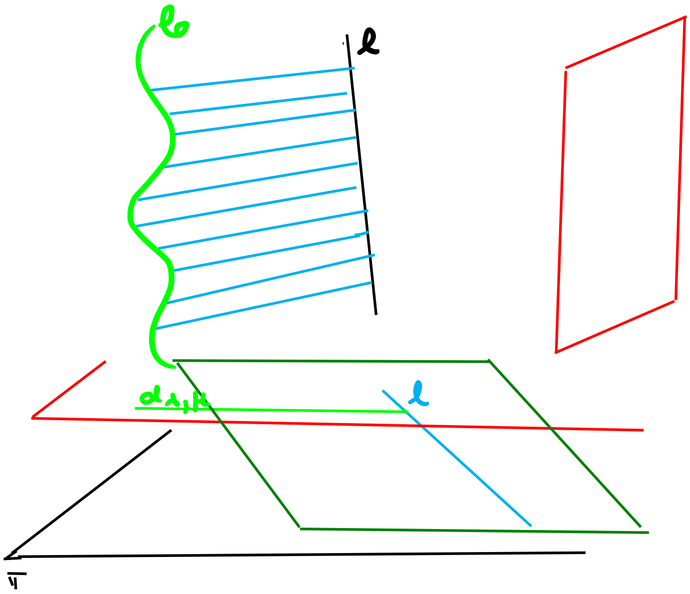


Conoidal surfaces

Cond. 1: $d_{\lambda,\mu} \parallel \pi$, π plane

$d_{\lambda,\mu} \cap \ell \neq \emptyset$, ℓ line

Cond 2: $d_{\lambda,\mu} \cap \mathcal{C} \neq \emptyset$



$d_{\lambda,\mu}$: $\begin{cases} \text{a plane parallel to } \pi \\ \text{a plane that contains } \ell \end{cases}$ → $d_{\lambda,\mu}$: $\begin{cases} \text{a plane parallel to } \pi \\ \text{a plane from the (reduced) pencil of planes of } \ell \end{cases}$

$\pi: Ax + By + Cz + D = 0$

$\ell: \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$ → $d_{\lambda,\mu}: \begin{cases} Ax + By + Cz + D = \lambda \\ A_1x + B_1y + C_1z + D_1 + \mu(A_2x + B_2y + C_2z + D_2) = 0 \end{cases}$

Example 11.3: Find the eq. of the conoidal surface whose generatrices are parallel to xOy and intersect Oz and have the director curve: $\mathcal{C}: \begin{cases} y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases}$

$\pi = xOy: z = 0$

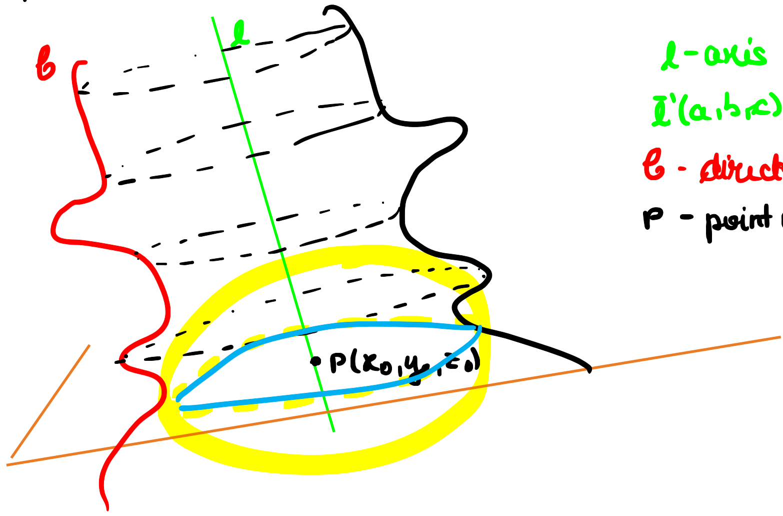
$\ell = Oz: \begin{cases} x = 0 \\ y = 0 \end{cases}$ → $d_{\lambda,\mu} \begin{cases} z = \lambda \\ x = \mu \cdot y \end{cases}$

$\begin{cases} z = \lambda \\ x = \mu \cdot y \\ y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases}$ ⟺ $\begin{cases} z = \lambda \\ x = \mu \cdot y \\ y^2 - 2\lambda + 2 = 0 \\ \mu^2 y^2 - 2\lambda + 1 = 0 \end{cases}$ ⟺ $\begin{cases} z = \lambda \\ x = \mu y \\ y^2 = 2\lambda - 2 \\ \mu^2(2\lambda - 2) - 2\lambda + 1 = 0 \end{cases}$

→ compatibility cond: $\mu^2(2\lambda - 2) - 2\lambda + 1 = 0$ → the eq. of the surface: $\frac{x^2}{y^2} \cdot (2z - 2) - 2z + 1 = 0$

$\lambda = z, \mu = \frac{x}{y}$

Revolution surfaces



l - axis

$\bar{l}(a, b, c)$

$\mathcal{C}$ - director curve

P - point on the axis

Step 1: Write all the possible circles (no generating lines $\Rightarrow$ generating circles) whose center lies on the axis.

$$C_{\lambda,\mu} : \begin{cases} (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = \lambda \\ ax + by + cz = \mu \end{cases}$$

Step 2 and 3 are the same