LECTURE 4 - DYNAMICAL SYSTEMS

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1) LHDE with constant coefficients
2) The complex exponential
3) The matrix exponential

1) (1) $x''+a_1x'+a_2x=0$, $a_1,a_2\in\mathbb{R}$

(2) $r^2+a_1r+a_2=0$ $\quad p(r)=r^2+a_1r+a_2$ - the characteristic polynomial of (1)

$p'(r)=2r+a_1$

Properties: Let $\alpha\in\mathbb{R}$.

(i) $e^{\alpha t}$ is a sol. of (1) $\Leftrightarrow$ α is a root of p

(ii) $te^{\alpha t}$ is a sol. of (1) $\Leftrightarrow$ α is a double root of p

Dem: $x=te^{\alpha t}$, $x'=e^{\alpha t}+\alpha te^{\alpha t}$, $x''=\alpha e^{\alpha t}+\alpha e^{\alpha t}+\alpha^2te^{\alpha t}$

$te^{\alpha t}$ is a sol of (1) $\Leftrightarrow 2\alpha e^{\alpha t}+\alpha^2te^{\alpha t}+a_1e^{\alpha t}+a_1\alpha te^{\alpha t}+a_2te^{\alpha t}=0, \forall t\in\mathbb{R}$

$\Leftrightarrow (\alpha^2+a_1\alpha+a_2)te^{\alpha t}+(2\alpha+a_1)e^{\alpha t}=0, \forall t\in\mathbb{R}$

$\Leftrightarrow \begin{cases}\alpha^2+a_1\alpha+a_2=0\\ 2\alpha+a_1=0\end{cases} \Leftrightarrow \begin{cases}p(\alpha)=0\\ p'(\alpha)=0\end{cases} \Leftrightarrow \alpha$ is a double root of p

(iii) $e^{\alpha t}\cdot\cos\beta t$ is a sol of (1) $\Leftrightarrow (\alpha+i\beta)$ is a root of p

New ideas

Def: Let $z\in\mathbb{C}$. e^z or $\exp(z)$ is, by def., the sum of the series $1+\frac{1}{1!}z+\frac{1}{2!}z^2+\ldots+\frac{1}{n!}z^n+\ldots$ (3).

Remark: It is known that (3) is convergent $\forall z\in\mathbb{C}$, thus $\exists\, e^z, \forall z\in\mathbb{C}$.

We prove this now for $z=i\beta$, $\beta\in\mathbb{R}^*$

$$i^k=\begin{cases}1, & k=0\\ i, & k=1\\ -1, & k=2\\ -i, & k=3\end{cases}$$

$1+\frac{1}{1!}(i\beta)+\frac{1}{2!}(i\beta)^2+\ldots+\frac{1}{n!}(i\beta)^n+\ldots=$

$=\left(1-\frac{1}{2!}\beta^2+\frac{1}{4!}\beta^4-\frac{1}{6!}\beta^6+\ldots\right)+i\left(\frac{1}{1!}\beta-\frac{1}{3!}\beta^3+\frac{1}{5!}\beta^5+\ldots\right)$

$=\cos\beta+i\sin\beta \Rightarrow e^{i\beta}=\cos\beta+i\sin\beta$. For $\beta=\pi\Rightarrow e^{i\pi}=-1$

$e^{i\pi}+1=0$

Euler's formula: $e^{\alpha+i\beta}=e^{\alpha}(\cos\beta+i\sin\beta)$, $\alpha,\beta\in\mathbb{R}$

Consider now $t\mapsto e^{zt}$, where $z\in\mathbb{C}$ fixed

$\mathbb{R}\mapsto\mathbb{C}$

$\gamma:\mathbb{R}\to\mathbb{C}$, $\gamma(t)=u(t)+iv(t) \overset{def}{\Rightarrow} \gamma'(t)=u'(t)+iv'(t)$

Take $\gamma(t)=e^{zt}$, $z=\alpha+i\beta$ $\Rightarrow u(t)=e^{\alpha t}\cdot\cos\beta t$, $v(t)=e^{\alpha t}\sin\beta t$

$u'(t)=\alpha e^{\alpha t}\cos\beta t-\beta e^{\alpha t}\sin\beta t$, $v'(t)=\alpha e^{\alpha t}\sin\beta t+\beta e^{\alpha t}\cos\beta t$

$u'(t) = \alpha e^{\alpha t}\cos\beta t - \beta e^{\alpha t}\sin\beta t$, $v'(t) = \alpha e^{\alpha t}\sin\beta t + \beta e^{\alpha t}\cos\beta t$

$\Rightarrow \gamma'(t) = u'(t) + iv'(t) = \alpha e^{(\alpha+i\beta)t} + i\beta e^{(\alpha+i\beta)t} = (\alpha+i\beta)e^{(\alpha+i\beta)t} = z e^{zt}$

Thus $\frac{d}{dt}(e^{zt}) = ze^{zt}$

One can also prove that $e^{z_1+z_2} = e^{z_1}\cdot e^{z_2}$, $\forall z_1, z_2 \in \mathbb{C}$

Remark: e^{zt} verifies a LHDE with CC $\Leftrightarrow$ z is a root of the charact eq.

Property: If $\gamma: \mathbb{R} \to \mathbb{C}$, $\gamma = u + iv$, is a function that verifies a LHDE then u and v are sol. of the LHDE.

Prove for eq. (1): γ verifies (1) $\Leftrightarrow \gamma'' + a_1\gamma' + a_2\gamma = 0 \Leftrightarrow (u+iv)'' + a_1(u+iv)' + a_2(u+iv) = 0 \Leftrightarrow$

$\Leftrightarrow \underbrace{(u'' + a_1u' + a_2u)}_{\text{real}} + i\underbrace{(v'' + a_1v' + a_2v)}_{\text{real}} = 0 \Leftrightarrow \begin{cases} u'' + a_1u' + a_2u = 0 \\ v'' + a_1v' + a_2v = 0 \end{cases} \Leftrightarrow$ u and v are sol of (1).

Remark: $(\alpha + i\beta)$ is root of $p \Leftrightarrow \alpha \pm i\beta$ roots of $p \Leftrightarrow$

$\Leftrightarrow e^{(\alpha+i\beta)t}$ and $e^{(\alpha-i\beta)t}$ verify (1) $\Leftrightarrow \begin{cases} e^{(\alpha+i\beta)t} = e^{\alpha t}\cos\beta t + ie^{\alpha t}\sin\beta t \\ e^{(\alpha-i\beta)t} = e^{\alpha t}\cos\beta t - ie^{\alpha t}\sin\beta t \end{cases}$

$e^{\alpha t}\cos\beta t$ and $e^{\alpha t}\sin\beta t$ are solutions of (1)

3) The matrix exponential.

Let $A \in \mathcal{M}_n(\mathbb{R})$.

Def.: e^A or $\exp(A)$ is the sum of the matrix series $I_n + \frac{1}{1!}A + \frac{1}{2!}A^2 + \dots + \frac{1}{n!}A^n + \dots$ (convergent, $\forall A \in \mathcal{M}_n(\mathbb{R})$)

Here $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$, the identity matrix. Remark that $e^A \in \mathcal{M}_n(\mathbb{R})$.

Particular cases

$e^{O_n} = I_n$, $e^{I_n} = I_n\left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \dots\right) = eI_n$ $(A^k = I_n, \forall k)$

$e^{tI_n} = I_n\left(1 + \frac{1}{1!}t + \frac{1}{2!}t^2 + \dots + \frac{1}{n!}t^n + \dots\right) = e^tI_n$ $(A^k = t^kI_n, \forall k)$

$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$ $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ $\quad e^{tA} = ?$

$A^k = \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{pmatrix}$ $\quad e^{tA} = I_n + \frac{1}{1!}tA + \frac{1}{2!}t^2A^2 + \dots + \frac{1}{n!}t^nA^n + \dots$

$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{1!}t\lambda_1 & 0 & \cdots & 0 \\ 0 & \frac{1}{1!}t\lambda_2 & \cdots & 0 \\ 0 & 0 & \cdots & \frac{1}{1!}t\lambda_n \end{pmatrix} + \dots + \begin{pmatrix} \frac{1}{n!}t^n\lambda_1^n & 0 & \cdots & 0 \\ 0 & \frac{1}{n!}t^n\lambda_2^n & \cdots & 0 \\ 0 & 0 & \cdots & \frac{1}{n!}t^n\lambda_n^n \end{pmatrix} + \dots =$

$= \operatorname{diag}(e^{t\lambda_1}, e^{t\lambda_2}, \dots, e^{t\lambda_n}) \Leftrightarrow e^{t\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}} = \begin{pmatrix} e^{t\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{t\lambda_2} & \cdots & 0 \\ 0 & 0 & \cdots & e^{t\lambda_n} \end{pmatrix}$

$$\begin{pmatrix} 0 & e^{t\lambda_2} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & e^{t\lambda_n} \end{pmatrix}$$

$$e^{t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \qquad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathcal{I}_2$$

$$A^3 = -\mathcal{I}_2 A = -A, \; A^4 = -A^2 = \mathcal{I}_2$$

$$= \mathcal{I}_2 + \frac{1}{1!}tA + \frac{1}{2!}t^2A^2 + \frac{1}{3}t^3A^3 + \frac{1}{4}t^4A^4 + \dots + \frac{1}{n!}t^nA^n + \dots =$$

$$= \begin{pmatrix} 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \dots & -\frac{1}{1!}t + \frac{1}{3!}t^3 - \frac{1}{5!}t^5 + \dots \\ \frac{1}{1!}t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \dots & 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \dots \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \Rightarrow$$

$$\Rightarrow e^{t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \forall t \in \mathbb{R}$$

Property: $\frac{d}{dt}(e^{tA}) = A \cdot e^{tA}, \forall t \in \mathbb{R}$, $e^{A_1 + A_2} = e^{A_1} \cdot e^{A_2}$, iff $A_1 \cdot A_2 = A_2 \cdot A_1$

Comments: We know that, for $\lambda \in \mathbb{R}^*$, the function $e^{\lambda t}$ is the unique sol of the IVP $\begin{cases} x' = \lambda x \\ x(0) = 1 \end{cases}$. But, we also have that, for $A \in \mathcal{M}_n(\mathbb{R})$, the function e^{tA} is the unique sol of the IVP $\begin{cases} X' = AX \\ X(0) = \mathcal{I}_n \end{cases}$

Explanation: $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$ $X = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix}$, with columns $X_1, X_2, \dots, X_n$

Note that $X' = AX \Leftrightarrow X_i' = AX_i, 1 \le i \le n$

$\mathcal{I}_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 1 \end{pmatrix}$, with columns $e_1, e_2, \dots, e_n$ $\qquad X(0) = \mathcal{I}_n \Leftrightarrow X_i(0) = e_i, 1 \le i \le n$

So $\begin{cases} X' = AX \\ X(0) = \mathcal{I}_n \end{cases} \Leftrightarrow \begin{cases} X_i' = AX_i \\ X_i(0) = e_i \end{cases}, 1 \le i \le n$

$$\Leftrightarrow \begin{cases} x_{1i}' = a_{11}x_{1i} + a_{12}x_{2i} + \dots + a_{1n}x_{ni} & x_{1i}(0) = 0 \\ x_{2i}' = a_{21}x_{1i} + a_{22}x_{2i} + \dots + a_{2n}x_{ni} & x_{2i}(0) = 0 \\ \dots & x_{ii}(0) = 1 \\ x_{ni}' = a_{n1}x_{1i} + a_{n2}x_{2i} + \dots + a_{nn}x_{ni} & x_{ni}(0) = 0 \end{cases}$$

We proved that $e^{t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$. We know that this is the unique sol

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of $\begin{cases} X' = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} X \\ X(0) = I_n \end{cases} \Leftrightarrow \begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$ is the unique sol of $\begin{cases} x_1' = -x_2, & x_1(0) = 1 \\ x_2' = x_1, & x_2(0) = 0 \end{cases}$

and $\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}$ is the unique sol of $\begin{cases} x_1' = -x_2, & x_1(0) = 0 \\ x_2' = x_1, & x_2(0) = 1 \end{cases}$