

Utilizando transformada Z para resolver um problema de recorrência

Péricles Lopes Machado

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Abstract

Neste artigo, a transformada Z é utilizada na resolução do problema:

$$\begin{aligned} F(0) &= 0, \text{ para } n \leq 0 \\ F(n) &= \frac{1}{n^2} \sum_{0 \leq k < n} \left[\sum_{0 \leq i < k} [1 + F(k)] + 1 + \sum_{k \leq i < n} [1 + F(n-1-k)] \right], \text{ para } n > 0 \end{aligned} \quad (1)$$

1 Resolução

Simplificando-se a eq. (1):

$$\begin{aligned} n^2 F(n) &= n + \sum_{0 \leq k < n} [k + kF(k)] + \sum_{0 \leq k < n} [n-k + (n-k)F(n-1-k)] \\ n^2 F(n) &= n + \sum_{0 \leq k < n} [k] + \sum_{0 \leq k < n} [kF(k)] + \sum_{0 \leq k < n} [n-k] + \sum_{0 \leq k < n} [(n-k)F(n-1-k)] \\ n^2 F(n) &= n + \sum_{0 \leq k < n} [n] + 2 \sum_{0 \leq k < n} [kF(k)] \\ n^2 F(n) &= (n^2 + n) + 2 \sum_{k \geq 0} kF(k)u(n-1-k) \end{aligned} \quad (2)$$

obtem-se:

$$n^2 F(n) = (n^2 + n) + 2 [nF(n)] * [u(n-1)], \quad (3)$$

onde $u(n) = 1$, para $n \geq 0$ e $u(n) = 0$, para $n < 0$.

Aplicando a transformada Z em (3) e fazendo $Y = \mathcal{Z}\{F(n)\}$:

$$\begin{aligned} z^2 \frac{d^2}{dz^2} Y &= \mathcal{Z}\{n^2 + n\} - 2z \frac{d}{dz} Y \mathcal{Z}\{u(n-1)\} \\ z^2 \frac{d^2}{dz^2} Y &= \mathcal{Z}\{n^2 + n\} - 2z \left(\frac{z^{-1}}{1-z^{-1}} \right) \frac{d}{dz} Y \\ z^2 \frac{d^2}{dz^2} Y &= \mathcal{Z}\{n^2\} + \mathcal{Z}\{n\} - \left(\frac{2z}{z-1} \right) \frac{d}{dz} Y \end{aligned}$$

$$\begin{aligned}
z^2 \frac{d^2}{dz^2} Y &= \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3} + \frac{z^{-1}}{(1-z^{-1})^2} - \left(\frac{2z}{z-1} \right) \frac{d}{dz} Y \\
z^2 \frac{d^2}{dz^2} Y &= \frac{z^2(1+z^{-1})}{(z-1)^3} + \frac{z}{(z-1)^2} - \left(\frac{2z}{z-1} \right) \frac{d}{dz} Y \\
z^2 \frac{d^2}{dz^2} Y &= \frac{z(z+1)}{(z-1)^3} + \frac{z}{(z-1)^2} - \left(\frac{2z}{z-1} \right) \frac{d}{dz} Y \\
z^2 \frac{d^2}{dz^2} Y &= \frac{z(z+1)}{(z-1)^3} + \frac{z(z-1)}{(z-1)^3} - \left(\frac{2z}{z-1} \right) \frac{d}{dz} Y \\
z^2 \frac{d^2}{dz^2} Y &= \frac{2z^2}{(z-1)^3} - \left(\frac{2z}{z-1} \right) \frac{d}{dz} Y \\
\frac{d^2}{dz^2} Y &= \frac{2}{(z-1)^3} - \left(\frac{2}{z(z-1)} \right) \frac{d}{dz} Y
\end{aligned}$$

Por fim,

$$\frac{d^2}{dz^2} Y = \frac{2}{(z-1)^3} - \left(\frac{2}{z(z-1)} \right) \frac{d}{dz} Y. \quad (4)$$

Resolvendo a equação diferencial (4), usando o *WolframAlpha*:

$$\begin{aligned}
Y(z) &= (c_1 z^2 + \\
&2 \log(1-z)((c_1-2)z - c_1 + z^2 - 2(z-1)\log(z)) - \\
&c_1 z - c_1 + 2(-2(z-1)Li_2(z) + z^2(-\log(z)) + z + \\
&(z-1)\log^2(1-z) + \\
&2z\log(z-3))/(z-1) + \\
&c_2
\end{aligned} \quad (5)$$

Expandindo a eq. (5):

$$\begin{aligned}
Y(z) &= \\
&\frac{c_1 z^2}{z-1} - \frac{c_1 z}{z-1} - \frac{c_1}{z-1} + \\
&\frac{2c_1 z \log(1-z)}{z-1} - \frac{2c_1 \log(1-z)}{z-1} + \\
&c_2 - \\
&\frac{4z Li_2(z)}{z-1} + \frac{4Li_2(z)}{z-1} + \\
&\frac{2z^2 \log(1-z)}{z-1} - \frac{2z^2 \log(z)}{z-1} + \\
&\frac{2z}{z-1} - \\
&\frac{z-1}{6} - \\
&\frac{z-1}{z-1} + \\
&\frac{2z \log^2(1-z)}{4z \log(1-z)} - \frac{2 \log^2(1-z)}{z-1} - \\
&\frac{z-1}{4z \log(1-z)} - \\
&\frac{z-1}{4z \log(1-z) \log(z)} + \\
&\frac{z-1}{4 \log(1-z) \log(z)} + \\
&\frac{4z \log(z)}{z-1}
\end{aligned} \quad (6)$$

Simplificando a eq. (6):

$$\begin{aligned}
Y(z) = & c_1 z - \frac{c_1}{z-1} + c_2 - \\
& 4Li_2(z) \left[\frac{z-1}{z-1} \right] + \\
& \frac{2z^2}{z-1} [\log(1-z) - \log(z)] + \\
& \frac{2z}{z-1} - \\
& \frac{z-1}{6} + \\
& \frac{z-1}{z-1} + \\
& 4\log(1-z)\log(z) \left[\frac{z+1}{z-1} \right] + \\
& 2\log^2(1-z) - \\
& 4\log(1-z) \left[\frac{z}{z-1} \right] - \\
& 4\log(z) \left[\frac{z}{z-1} \right] + \\
& 2c_1 \log(1-z)
\end{aligned}$$

Reorganizando:

$$\begin{aligned}
Y(z) = & c_1 z - \frac{c_1}{z-1} + c_2 - \\
& 4Li_2(z) + \\
& \frac{2z^2 - 4z}{z-1} [\log(1-z) - \log(z)] + \\
& \frac{2z-6}{z-1} + \\
& \frac{z-1}{z-1} + \\
& 4\log(1-z)\log(z) \left[\frac{z+1}{z-1} \right] + \\
& 2\log^2(1-z) + 2c_1 \log(1-z)
\end{aligned} \tag{7}$$

Aplicando a transformada inversa \mathcal{Z}^{-1} na eq. (7):

$$\begin{aligned}
F(n) = & \mathcal{Z}^{-1} \{c_1 z\} - \mathcal{Z}^{-1} \left\{ \frac{c_1}{z-1} \right\} + \mathcal{Z}^{-1} \{c_2\} - \\
& \mathcal{Z}^{-1} \{4Li_2(z)\} + \\
& \mathcal{Z}^{-1} \left\{ \frac{2z^2 - 4z}{z-1} [\log(1-z) - \log(z)] \right\} + \\
& \mathcal{Z}^{-1} \left\{ \frac{2z-6}{z-1} \right\} + \\
& \mathcal{Z}^{-1} \left\{ 4\log(1-z)\log(z) \left[\frac{z+1}{z-1} \right] \right\} + \\
& \mathcal{Z}^{-1} \{2\log^2(1-z)\} + \\
& \mathcal{Z}^{-1} \{2c_1 \log(1-z)\}
\end{aligned}$$

Simplificando:

$$\begin{aligned}
F(n) = & \\
& \mathcal{Z}^{-1} \left\{ \frac{c_1 z - c_1 - c_1 z^{-1}}{1 - z^{-1}} \right\} + \\
& \mathcal{Z}^{-1} \left\{ \frac{2 - 6z^{-1}}{1 - z^{-1}} \right\} + \\
& c_2 \delta(n) - \\
& \mathcal{Z}^{-1} \{4Li_2(z)\} + \\
& \mathcal{Z}^{-1} \left\{ \frac{2z^2 - 4z}{z - 1} [\log(1 - z) - \log(z)] \right\} + \\
& \mathcal{Z}^{-1} \left\{ 4 \log(1 - z) \log(z) \left[\frac{z + 1}{z - 1} \right] \right\} + \\
& \mathcal{Z}^{-1} \{2 \log^2(1 - z)\} + \\
& \mathcal{Z}^{-1} \{2c_1 \log(1 - z)\}
\end{aligned}$$

Mais simplificações:

$$\begin{aligned}
F(n) = & \\
& \mathcal{Z}^{-1} \left\{ \frac{c_1 z}{1 - z^{-1}} \right\} + \mathcal{Z}^{-1} \left\{ \frac{-c_1}{1 - z^{-1}} \right\} + \mathcal{Z}^{-1} \left\{ \frac{-c_1 z^{-1}}{1 - z^{-1}} \right\} + \\
& \mathcal{Z}^{-1} \left\{ \frac{2 - 6z^{-1}}{1 - z^{-1}} \right\} + \\
& c_2 \delta(n) - \\
& \mathcal{Z}^{-1} \{4Li_2(z)\} + \\
& \mathcal{Z}^{-1} \left\{ \frac{2z^2 - 4z}{z - 1} [\log(1 - z) - \log(z)] \right\} + \\
& \mathcal{Z}^{-1} \left\{ 4 \log(1 - z) \log(z) \left[\frac{z + 1}{z - 1} \right] \right\} + \\
& \mathcal{Z}^{-1} \{2 \log^2(1 - z)\} + \\
& \mathcal{Z}^{-1} \{2c_1 \log(1 - z)\}
\end{aligned}$$

Resolvendo algumas inversas:

$$\begin{aligned}
F(n) = & \\
& 2u(n) - 6u(n - 1) + \\
& c_1 u(n + 1) - c_1 u(n) - c_1 u(n - 1) + \\
& c_2 \delta(n) - \\
& \mathcal{Z}^{-1} \{4Li_2(z)\} + \\
& \mathcal{Z}^{-1} \left\{ \frac{2z^2 - 4z}{z - 1} \right\} * \mathcal{Z}^{-1} \{\log(1 - z) - \log(z)\} + \\
& 4\mathcal{Z}^{-1} \left\{ \frac{z + 1}{z - 1} \right\} * \mathcal{Z}^{-1} \{\log(1 - z) \log(z)\} + \\
& \mathcal{Z}^{-1} \{2 \log^2(1 - z)\} + \\
& \mathcal{Z}^{-1} \{2c_1 \log(1 - z)\}
\end{aligned}$$

$$\begin{aligned}
F(n) = & \\
& 2u(n) - 6u(n-1) + \\
& c_1 u(n+1) - c_1 u(n) - c_1 u(n-1) + \\
& c_2 \delta(n) - \\
& \mathcal{Z}^{-1} \{4Li_2(z)\} + \\
& \{2u(n+1) - 4u(n)\} * \mathcal{Z}^{-1} \{\log(1-z) - \log(z)\} + \\
& 4\{u[n] + u[n-1]\} * \mathcal{Z}^{-1} \{\log(1-z) \log(z)\} + \\
& 2\mathcal{Z}^{-1} \{\log(1-z)\} * \mathcal{Z}^{-1} \{\log(1-z)\} + \\
& 2c_1 \mathcal{Z}^{-1} \{\log(1-z)\}
\end{aligned}$$