Utilizando transformada Z para resolver um problema de recorrência

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Abstract

Neste artigo, a transformada Z é utilizada na resolução do problema:

$$
\begin{aligned}
F(0) &= 0,\ \text{para } n \leq 0 \\
F(n) &= \frac{1}{n^2} \sum_{0 \leq k < n} \left[\sum_{0 \leq i < k} [1 + F(k)] + 1 + \sum_{k \leq i < n} [1 + F(n-1-k)] \right],\ \text{para } n > 0
\end{aligned} \tag{1}
$$

1 Resolução

Simplificando-se a eq. (1):

$$
\begin{aligned}
n^2 F(n) &= n + \textstyle\sum_{0 \leq k < n} [k + kF(k)] + \sum_{0 \leq k < n} [n - k + (n-k) F(n-1-k)] \\
n^2 F(n) &= n + \textstyle\sum_{0 \leq k < n} [k] + \sum_{0 \leq k < n} [kF(k)] + \sum_{0 \leq k < n} [n-k] + \sum_{0 \leq k < n} [(n-k) F(n-1-k)] \\
n^2 F(n) &= n + \textstyle\sum_{0 \leq k < n} [n] + 2 \sum_{0 \leq k < n} [kF(k)] \\
n^2 F(n) &= (n^2 + n) + 2 \sum_{k \geq 0} kF(k) u(n-1-k)
\end{aligned} \tag{2}
$$

obtém-se:

$$
n^2 F(n) = (n^2 + n) + 2 [nF(n)] * [u(n-1)], \tag{3}
$$

onde $u(n) = 1,$ para $n \geq 0$ e $u(n) = 0,$ para $n < 0$.
Aplicando a transformada $\mathcal{Z}$ em (3) e fazendo $Y = \mathcal{Z}\{F(n)\}$:

$$
z^2 \frac{d^2}{dz^2} Y = \mathcal{Z} \left\{ n^2 + n \right\} - 2z \frac{d}{dz} Y \mathcal{Z} \left\{ u(n-1) \right\}
$$

$$
\begin{aligned}
z^2 \frac{d^2}{dz^2} Y &= \mathcal{Z} \left\{ n^2 + n \right\} - 2z \left(\frac{z^{-1}}{1 - z^{-1}} \right) \frac{d}{dz} Y \\
z^2 \frac{d^2}{dz^2} Y &= \mathcal{Z} \left\{ n^2 \right\} + \mathcal{Z} \left\{ n \right\} - \left(\frac{2z}{z-1} \right) \frac{d}{dz} Y
\end{aligned}
$$

$$
\begin{aligned}
z^2\frac{d^2}{dz^2}Y &= \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3} + \frac{z^{-1}}{(1-z^{-1})^2} - \left(\frac{2z}{z-1}\right)\frac{d}{dz}Y\\
z^2\frac{d^2}{dz^2}Y &= \frac{z^2(1+z^{-1})}{(z-1)^3} + \frac{z}{(z-1)^2} - \left(\frac{2z}{z-1}\right)\frac{d}{dz}Y\\
z^2\frac{d^2}{dz^2}Y &= \frac{z(z+1)}{(z-1)^3} + \frac{z}{(z-1)^2} - \left(\frac{2z}{z-1}\right)\frac{d}{dz}Y\\
z^2\frac{d^2}{dz^2}Y &= \frac{z(z+1)}{(z-1)^3} + \frac{z(z-1)}{(z-1)^3} - \left(\frac{2z}{z-1}\right)\frac{d}{dz}Y\\
z^2\frac{d^2}{dz^2}Y &= \frac{2z^2}{(z-1)^3} - \left(\frac{2z}{z-1}\right)\frac{d}{dz}Y\\
\frac{d^2}{dz^2}Y &= \frac{2}{(z-1)^3} - \left(\frac{2}{z(z-1)}\right)\frac{d}{dz}Y
\end{aligned}
$$

Por fim,

$$
\frac{d^2}{dz^2}Y = \frac{2}{(z-1)^3} - \left(\frac{2}{z(z-1)}\right)\frac{d}{dz}Y. \tag{4}
$$

Resolvendo a equação diferencial (4), usando o *WolframAlpha*:

$$
\begin{aligned}
&Y(z) = (c_1z^2+\\
&2\log(1-z)((c_1-2)z - c_1 + z^2 - 2(z-1)\log(z))-\\
&c_1z - c_1 + 2(-2(z-1)Li_2(z) + z^2(-\log(z)) + z+\\
&(z-1)\log^2(1-z)+\\
&2z\log(z) - 3))/(z-1)+\\
&c_2
\end{aligned} \tag{5}
$$

Expandindo a eq. (5):

$$
\begin{aligned}
&Y(z) =\\
&\frac{c_1z^2}{z-1} - \frac{c_1z}{z-1} - \frac{c_1}{z-1}+\\
&\frac{2c_1z\log(1-z)}{z-1} - \frac{2c_1\log(1-z)}{z-1}+\\
&c_2-\\
&\frac{4zLi_2(z)}{z-1} + \frac{4Li_2(z)}{z-1}+\\
&\frac{2z^2\log(1-z)}{z-1} - \frac{2z^2\log(z)}{z-1}+\\
&\frac{2z}{z-1}-\\
&\frac{6}{z-1}+\\
&\frac{2z\log^2(1-z)}{z-1} - \frac{2\log^2(1-z)}{z-1}-\\
&\frac{4z\log(1-z)}{z-1}-\\
&\frac{4z\log(1-z)\log(z)}{z-1}+\\
&\frac{4\log(1-z)\log(z)}{z-1}+\\
&\frac{4z\log(z)}{z-1}
\end{aligned} \tag{6}
$$

Simplificando a eq. (6):

$$
\begin{aligned}
&Y(z) = \\
&c1z - \frac{c_1}{z-1} + 2c_1 \log(1-z) + c_2 - \\
&4Li_2(z)\frac{z+1}{z-1} + \\
&\frac{2z^2}{z-1}\left[\log(1-z) - \log(z)\right] + \\
&\frac{2z}{z-1} - \\
&\frac{6}{z-1} + \\
&2\log^2(1-z) - \frac{4z\log(1-z)}{z-1} - \\
&4\log(1-z)\log(z)\left[\frac{z+1}{z-1}\right] + \\
&\frac{4z\log(z)}{z-1}
\end{aligned}
$$