Utilizando transformada Z para resolver um problema de recorrência

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Abstract

Neste artigo, a transformada Z é utilizada na resolução do problema:

$$\begin{aligned} F(0) &= 0, \text{ para } n \leq 0 \\ F(n) &= \frac{1}{n^2} \sum_{0 \leq k < n} \left[\sum_{0 \leq i < k} [1 + F(k)] + 1 + \sum_{k \leq i < n} [1 + F(n-1-k)] \right], \text{ para } n > 0 \end{aligned} \tag{1}$$

1 Resolução

Simplificando-se a eq. (1):

$$\begin{aligned} n^2F(n) &= n + \textstyle\sum_{0\leq k<n}[k + kF(k)] + \sum_{0\leq k<n}[n-k+(n-k)F(n-1-k)] \\ n^2F(n) &= n + \textstyle\sum_{0\leq k<n}[k] + \sum_{0\leq k<n}[kF(k)] + \sum_{0\leq k<n}[n-k] + \sum_{0\leq k<n}[(n-k)F(n-1-k)] \\ n^2F(n) &= n + \textstyle\sum_{0\leq k<n}[n] + 2\sum_{0\leq k<n}[kF(k)] \\ n^2F(n) &= (n^2+n)u(n-1) + 2\sum_{k\geq 0} kF(k)u(n-1-k) \end{aligned} \tag{2}$$

obtém-se:

$$n^2F(n) = (n^2+n)u(n-1) + 2\,[nF(n)] * [u(n-1)]\,, \tag{3}$$

onde $u(n) = 1,$ para $n \geq 0$ e $u(n) = 0,$ para $n < 0$.
Aplicando a transformada $\mathcal{Z}$ em (3) e fazendo $Y = \mathcal{Z}\{F(n)\}$:

$$z^2\frac{d^2}{dz^2}Y = \mathcal{Z}\left\{(n^2+n)u(n-1)\right\} - 2z\frac{d}{dz}Y\mathcal{Z}\left\{u(n-1)\right\}$$

$$\begin{aligned} z^2\frac{d^2}{dz^2}Y &= \mathcal{Z}\left\{(n^2+n)u(n-1)\right\} - 2z\left(\frac{z^{-1}}{1-z^{-1}}\right)\frac{d}{dz}Y \\ z^2\frac{d^2}{dz^2}Y &= \mathcal{Z}\left\{n^2u(n-1)\right\} + \mathcal{Z}\left\{nu(n-1)\right\} - \left(\frac{2z}{z-1}\right)\frac{d}{dz}Y \end{aligned}$$

$$
\begin{aligned}
z^2\frac{d^2}{dz^2}Y &= \frac{z^{-2}(1+z^{-1})}{(1-z^{-1})^3} + \frac{z^{-2}}{(1-z^{-1})^2} - \left(\frac{2z}{z-1}\right)\frac{d}{dz}Y \\
z^2\frac{d^2}{dz^2}Y &= \frac{z(1+z^{-1})}{(z-1)^3} + \frac{1}{(z-1)^2} - \left(\frac{2z}{z-1}\right)\frac{d}{dz}Y \\
z^2\frac{d^2}{dz^2}Y &= \frac{(z+1)}{(z-1)^3} + \frac{1}{(z-1)^2} - \left(\frac{2z}{z-1}\right)\frac{d}{dz}Y \\
z^2\frac{d^2}{dz^2}Y &= \frac{(z+1)}{(z-1)^3} + \frac{z-1}{(z-1)^3} - \left(\frac{2z}{z-1}\right)\frac{d}{dz}Y \\
z^2\frac{d^2}{dz^2}Y &= \frac{2z}{(z-1)^3} - \left(\frac{2z}{z-1}\right)\frac{d}{dz}Y \\
\frac{d^2}{dz^2}Y &= \frac{2z}{z^2(z-1)^3} - \left(\frac{2z}{z^2(z-1)}\right)\frac{d}{dz}Y
\end{aligned}
$$

Por fim,

$$
\frac{d^2}{dz^2}Y = \frac{2}{z(z-1)^3} - \left(\frac{2}{z(z-1)}\right)\frac{d}{dz}Y. \tag{4}
$$

Resolvendo a equação diferencial (4), usando o *WolframAlpha*:

$$
\begin{aligned}
&Y(z) = (c_1z^2 + 2log(1-z)((c_1-2)z - c_1 + z^2 - 2(z-1)log(z)) - c_1z \\
&-c_1 - 4(z-1)Li_2(z) \\
&-2z^2log(z) + 2z + 2(z-1)log^2(1-z) \\
&+4zlog(z) - 7)/(z-1) + c_2
\end{aligned} \tag{5}
$$

Expandindo a expressão (5):

$$
\begin{aligned}
&Y(z) = (c_1z^2)/(z-1)- \\
&(c_1z)/(z-1)- \\
&c_1/(z-1)+ \\
&(2c_1zlog(1-z))/(z-1)- \\
&(2c_1log(1-z))/(z-1)+ \\
&c_2- \\
&(4zLi_2(z))/(z-1)+ \\
&(4Li_2(z))/(z-1)+ \\
&(2z^2log(1-z))/(z-1)- \\
&(2z^2log(z))/(z-1)+ \\
&(2z)/(z-1) - 7/(z-1)+ \\
&(2zlog^2(1-z))/(z-1)- \\
&(2log^2(1-z))/(z-1)- \\
&(4zlog(1-z))/(z-1)- \\
&(4zlog(1-z)log(z))/(z-1)+ \\
&(4zlog(z))/(z-1)+ \\
&(4log(1-z)log(z))/(z-1)
\end{aligned} \tag{6}
$$