

Utilizando transformada Z para resolver um problema de recorrência

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Abstract

Neste artigo, a transformada Z é utilizada na resolução do problema:

$$\begin{aligned} F(0) &= 0, \text{ para } n \leq 0 \\ F(n) &= \frac{1}{n^2} \sum_{0 \leq k < n} \left[\sum_{0 \leq i < k} [1 + F(k)] + 1 + \sum_{k \leq i < n} [1 + F(n-1-k)] \right], \text{ para } n > 0 \end{aligned} \quad (1)$$

1 Resolução

Simplificando-se a eq. (1):

$$\begin{aligned} n^2 F(n) &= n + \sum_{0 \leq k < n} [k + kF(k)] + \sum_{0 \leq k < n} [n - k + (n - k)F(n - 1 - k)] \\ n^2 F(n) &= n + \sum_{0 \leq k < n} [k] + \sum_{0 \leq k < n} [kF(k)] + \sum_{0 \leq k < n} [n - k] + \sum_{0 \leq k < n} [(n - k)F(n - 1 - k)] \\ n^2 F(n) &= n + \sum_{0 \leq k < n} [n] + 2 \sum_{0 \leq k < n} [kF(k)] \\ n^2 F(n) &= (n^2 + n)u(n - 1) + 2 \sum_{k \geq 0} kF(k)u(n - 1 - k) \end{aligned} \quad (2)$$

obtem-se:

$$n^2 F(n) = (n^2 + n)u(n - 1) + 2 [nF(n)] * [u(n - 1)], \quad (3)$$

onde $u(n) = 1$, para $n \geq 0$ e $u(n) = 0$, para $n < 0$.

Aplicando a transformada Z em (3) e fazendo $Y = Z\{F(n)\}$:

$$z^2 \frac{d^2}{dz^2} Y = Z \{ (n^2 + n)u(n - 1) \} - 2z \frac{d}{dz} Y Z \{ u(n - 1) \}$$

$$\begin{aligned} z^2 \frac{d^2}{dz^2} Y &= Z \{ (n^2 + n)u(n - 1) \} - 2z \left(\frac{z^{-1}}{1 - z^{-1}} \right) \frac{d}{dz} Y \\ z^2 \frac{d^2}{dz^2} Y &= Z \{ n^2 u(n - 1) \} + Z \{ nu(n - 1) \} - \left(\frac{2z}{z - 1} \right) \frac{d}{dz} Y \end{aligned}$$

$$\begin{aligned}
z^2 \frac{d^2}{dz^2} Y &= \frac{z^{-2}(1+z^{-1})}{(1-z^{-1})^3} + \frac{z^{-2}}{(1-z^{-1})^2} - \left(\frac{2z}{z-1} \right) \frac{d}{dz} Y \\
z^2 \frac{d^2}{dz^2} Y &= \frac{z(1+z^{-1})}{(z-1)^3} + \frac{1}{(z-1)^2} - \left(\frac{2z}{z-1} \right) \frac{d}{dz} Y \\
z^2 \frac{d^2}{dz^2} Y &= \frac{(z+1)}{(z-1)^3} + \frac{1}{(z-1)^2} - \left(\frac{2z}{z-1} \right) \frac{d}{dz} Y \\
z^2 \frac{d^2}{dz^2} Y &= \frac{(z+1)}{(z-1)^3} + \frac{z-1}{(z-1)^3} - \left(\frac{2z}{z-1} \right) \frac{d}{dz} Y \\
z^2 \frac{d^2}{dz^2} Y &= \frac{2z}{(z-1)^3} - \left(\frac{2z}{z-1} \right) \frac{d}{dz} Y \\
\frac{d^2}{dz^2} Y &= \frac{2z}{z^2(z-1)^3} - \left(\frac{2z}{z^2(z-1)} \right) \frac{d}{dz} Y
\end{aligned}$$

Por fim,

$$\frac{d^2}{dz^2} Y = \frac{2}{z(z-1)^3} - \left(\frac{2}{z(z-1)} \right) \frac{d}{dz} Y. \quad (4)$$

Resolvendo a equação diferencial (4), usando o *WolframAlpha*:

$$\begin{aligned}
Y(z) &= (c_1 z^2 + 2 \log(1-z)((c_1-2)z - c_1 + z^2 - 2(z-1)\log(z)) - c_1 z \\
&- c_1 - 4(z-1)Li_2(z) \\
&- 2z^2 \log(z) + 2z + 2(z-1)\log^2(1-z) \\
&+ 4z \log(z) - 7)/(z-1) + c_2
\end{aligned} \quad (5)$$

Expandindo a eq. (5):

$$\begin{aligned}
Y(z) &= (c_1 z^2)/(z-1) - \\
&(c_1 z)/(z-1) - \\
&c_1/(z-1) + \\
&(2c_1 z \log(1-z))/(z-1) - \\
&(2c_1 \log(1-z))/(z-1) + \\
&c_2 - \\
&(4z Li_2(z))/(z-1) + \\
&(4 Li_2(z))/(z-1) + \\
&(2z^2 \log(1-z))/(z-1) - \\
&(2z^2 \log(z))/(z-1) + \\
&(2z)/(z-1) - 7/(z-1) + \\
&(2z \log^2(1-z))/(z-1) - \\
&(2 \log^2(1-z))/(z-1) - \\
&(4z \log(1-z))/(z-1) - \\
&(4z \log(1-z) \log(z))/(z-1) + \\
&(4z \log(z))/(z-1) + \\
&(4 \log(1-z) \log(z))/(z-1)
\end{aligned} \quad (6)$$