DATA8004: Optimization for Statistical Learning

Subject Lecturer: Man-Chung YUE

Lecture 7
Convex Optimization
Proximal gradient methods

Algorithm building block

Definition: Let $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper, closed and convex. We define the proximal operator (or proximal mapping) as

$$\mathrm{Prox}_g(x) := \operatorname*{Arg\,min}_{u \in \mathbb{R}^n} \left\{ \frac{1}{2}\|u - x\|_2^2 + g(u) \right\}.$$

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Remark:

- Since g is proper, closed and convex, it follows that for any $x \in \mathbb{R}^n$, the function
$$u \mapsto \frac{1}{2}\|u - x\|_2^2 + g(u)$$
is proper, closed and strongly convex. Thus, it has a unique minimizer thanks to Theorem 4.12. Hence, $\mathrm{Prox}_g : \mathbb{R}^n \to \mathbb{R}^n$ is well defined.
- When $g = \delta_C$ for some nonempty closed convex set C, $\mathrm{Prox}_{\delta_C} = P_C$.

Example 1

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$$\mathrm{Prox}_{\delta_C}(x) = \operatorname*{Arg\,min}_{\|u\|_1 \leq 1} \left\{ \frac{1}{2}\|u - x\|_2^2 \right\} = \operatorname*{Arg\,min}_{\|u\|_1 = 1} \left\{ \frac{1}{2}\|u\|_2^2 - u^T x \right\}.$$

Let $u = \alpha \circ v$, where $\alpha \in \{-1, 1\}^n$, $v \in \mathbb{R}^n_+$ and $\circ$ denotes entrywise product. Then

$$\min_{\|u\|_1 = 1} \left\{ \frac{1}{2}\|u\|_2^2 - u^T x \right\} = \min_{e^T v = 1, v \geq 0} \ \min_{\alpha \in \{-1,1\}^n} \left\{ \frac{1}{2}\|v\|_2^2 - v^T(\alpha \circ x) \right\}$$

$$= \min_{e^T v = 1, v \geq 0} \left\{ \frac{1}{2}\|v\|_2^2 - v^T|x| \right\} = \min_{e^T v = 1, v \geq 0} \left\{ \frac{1}{2}\|v - |x|\|_2^2 - \frac{1}{2}\|x\|_2^2 \right\}$$

with the min in α attained at $\alpha = \mathrm{sign}(x)$. Then $v = P_\Delta(|x|)$, where Δ is the unit simplex, and $|\cdot|$ is taken entrywise. Thus,

$$\mathrm{Prox}_{\delta_C}(x) = \mathrm{sign}(x) \circ P_\Delta(|x|), \quad \text{if } x \notin C.$$

See Slide 26 of Lecture 5 for P_Δ. Here, e denote the vector of ones.

Example 2

Example: Let $C = \{x \in \mathbb{R}^n : \ \|x\|_2 \leq 1\}$. Then $\text{Prox}_{\delta_C} = P_C$ and hence

$$\text{Prox}_{\delta_C}(x) = \begin{cases} x & \text{if } \|x\|_2 \leq 1, \\ \frac{x}{\|x\|_2} & \text{otherwise.} \end{cases}$$

Similarly, if $C = \{x \in \mathbb{R}^n : \ \|x\|_\infty \leq 1\}$, then for all i,

$$[\text{Prox}_{\delta_C}(x)]_i = \max\{-1, \min\{1, x_i\}\}.$$

Example 3

Example: We consider $\mathrm{Prox}_{\mu\|\cdot\|_1}$, where $\mu > 0$. Note that

$$\mathrm{Prox}_{\mu\|\cdot\|_1}(x) = \operatorname*{Arg\,min}_{u\in\mathbb{R}^n} \left\{ \sum_{i=1}^{n} \left(\frac{1}{2}(u_i - x_i)^2 + \mu|u_i| \right) \right\}$$

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Consider for any $s \in \mathbb{R}$ the problem $\min_{t\in\mathbb{R}}\left\{\frac{1}{2}(t-s)^2 + \mu|t|\right\}$. Let $t = \alpha v$ with $\alpha \in \{-1, 1\}$ and $v \geq 0$, then

$$\min_{t\in\mathbb{R}}\left\{\frac{1}{2}(t-s)^2 + \mu|t|\right\} = \min_{v\geq 0}\min_{\alpha\in\{-1,1\}}\left\{\frac{1}{2}[v^2 - 2\alpha vs + s^2] + \mu v\right\}$$

$$= \min_{v\geq 0}\left\{\frac{1}{2}v^2 - v|s| + \mu v + \frac{1}{2}s^2\right\} = \min_{v\geq 0}\left\{\frac{1}{2}v^2 - (|s| - \mu)v + \frac{1}{2}s^2\right\},$$

where the min in α is attained at $\alpha = \operatorname{sign}(s)$.

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where the min in α is attained at $\alpha = \mathrm{sign}(s)$. Thus, the minimizer is $t_* = \mathrm{sign}(s)\max\{|s| - \mu, 0\}$. Hence, for all i,

$$\left[\mathrm{Prox}_{\mu\|\cdot\|_1}(x)\right]_i = \mathrm{sign}(x_i) \cdot \max\{|x_i| - \mu, 0\}.$$

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Example: We consider $\mathrm{Prox}_{\mu\|\cdot\|_2}$, where $\mu > 0$. Note that

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Example 4

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Let $u = rv$ with $r \geq 0$ and $\|v\|_2 = 1$, then

$$\min_{u\in\mathbb{R}^n} \left\{ \frac{1}{2}\|u\|_2^2 - u^T x + \mu\|u\|_2 \right\} = \min_{r\geq 0} \min_{\|v\|_2=1} \left\{ \frac{1}{2}r^2 - rv^T x + \mu r \right\}$$

$$= \min_{r\geq 0} \left\{ \frac{1}{2}r^2 - r\|x\|_2 + \mu r \right\} = \min_{r\geq 0} \left\{ \frac{1}{2}r^2 - (\|x\|_2 - \mu)r \right\},$$

where the min in v is attained at $v = \mathrm{Sgn}(x)$, where

$$\mathrm{Sgn}(x) = \begin{cases} e/\|e\|_2 & \text{if } x = 0, \\ x/\|x\|_2 & \text{otherwise.} \end{cases}$$

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Let $u = rv$ with $r \geq 0$ and $\|v\|_2 = 1$, then

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where the min in v is attained at $v = \mathrm{Sgn}(x)$, where

$$\mathrm{Sgn}(x) = \begin{cases} e/\|e\|_2 & \text{if } x = 0, \\ x/\|x\|_2 & \text{otherwise.} \end{cases}$$

Thus, $\mathrm{Prox}_{\mu\|\cdot\|_2}(x) = \max\{\|x\|_2 - \mu, 0\} \cdot \mathrm{Sgn}(x)$.

Nonexpansiveness

Proposition 7.1 (Nonexpansiveness)
Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper, closed and convex. Then it holds that

$$\|\mathrm{Prox}_f(x) - \mathrm{Prox}_f(y)\|_2 \leq \|x - y\|_2$$

for all x and $y \in \mathbb{R}^n$.

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for all x and $y \in \mathbb{R}^n$.

Proof: Write $u = \mathrm{Prox}_f(x)$ and $v = \mathrm{Prox}_f(y)$. Then we see from Theorem 5.5 and Proposition 4.4(ii) that

$$x - u \in \partial f(u) \quad \text{and} \quad y - v \in \partial f(v).$$

These together with Proposition 4.4(iii) give

$$(u - v)^T[(x - u) - (y - v)] \geq 0.$$

Rearrange terms and apply the Cauchy-Schwartz inequality.

Problem setting

We consider the following optimization problem:

$$\underset{x\in\mathbb{R}^n}{\text{Minimize}} \quad f(x) + g(x), \tag{1}$$

where

- f is convex with Lipschitz continuous gradient. Specifically, there exists $L > 0$ such that

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq L\|x - y\|_2, \quad \forall x, y \in \mathbb{R}^n.$$

- g is proper, closed and convex.
- We assume that $\operatorname{Arg\,min}(f + g) \neq \emptyset$.
- We also assume that the proximal operator of g can be computed efficiently.

Example

Example: (LASSO / Compressed sensing)

Let $A \in \mathbb{R}^{m \times n}$ with $m \ll n$, $b \in \mathbb{R}^m$ and $\mu > 0$. It is common to consider the following models for sparse recovery:

$$\underset{x \in \mathbb{R}^n}{\text{Minimize}} \quad \frac{1}{2}\|Ax - b\|_2^2 + \mu\|x\|_1$$

or

$$\begin{array}{rl} \underset{x \in \mathbb{R}^n}{\text{Minimize}} & \frac{1}{2}\|Ax - b\|_2^2 \\ \text{Subject to} & \|x\|_1 \le \mu. \end{array}$$

See [Tibshirani '96] and [Foucart-Rauhut '13].

Proximal gradient algorithm

Consider (1) and recall that L is a Lipschitz continuity modulus of ∇f.

Proximal gradient algorithm: Let $x^0 \in \operatorname{dom} g$. For $k = 0, 1, \ldots$,

$$x^{k+1} = \operatorname{Prox}_{\frac{1}{L}g}\left(x^k - \frac{1}{L}\nabla f(x^k)\right).$$

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Remark:

- Note that

$$x^{k+1} = \operatorname*{Arg\,min}_{x \in \mathbb{R}^n}\left\{\nabla f(x^k)^T(x - x^k) + \frac{L}{2}\|x - x^k\|_2^2 + g(x)\right\} \quad (2)$$

- This algorithm is globally convergent. Specifically, $\{x^k\}$ converges to a global minimizer of $f + g$ from any $x^0 \in \operatorname{dom} g$. See [Lions-Mercier '79], [Tseng '91] and references therein.
- In this lecture, we follow the lines of analysis in [Tseng unpub] and [Tseng '10] to derive iteration complexity.

Taylor's inequality

Theorem 7.1 (Taylor's inequality)
Let $f \in C^1(\mathbb{R}^n)$ and suppose that there exists $L > 0$ so that

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq L\|x - y\|_2, \quad \forall x, y \in \mathbb{R}^n.$$

Then for all x and $y \in \mathbb{R}^n$, it holds that

$$f(y) \leq f(x) + \nabla f(x)^T(y - x) + \frac{L}{2}\|y - x\|_2^2.$$

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Proof sketch: For any $x, y \in \mathbb{R}^n$ and define $\psi(t) := f(x + t(y - x))$. Then $\psi(0) = f(x)$ and $\psi'(s) = (y - x)^T \nabla f(x + s(y - x))$. Hence

$$\begin{aligned}
\psi(1) = \psi(0) + \int_0^1 \psi'(s)ds &= \psi(0) + \psi'(0) + \int_0^1 (\psi'(s) - \psi'(0))ds \\
&= \psi(0) + \psi'(0) + \int_0^1 (y - x)^T[\nabla f(x + s(y - x)) - \nabla f(x)]ds \\
&\le \psi(0) + \psi'(0) + L\int_0^1 s\|y - x\|^2 ds.
\end{aligned}$$

Convergence of PG

Theorem 7.2 (PG complexity)
Consider (1) and let $\{x^k\}$ be generated by the proximal gradient algorithm on Slide 9. Then for all $k \geq 1$, it holds that

$$F(x^k) - F(\bar{x}) \leq \frac{L}{2k}\|x^0 - \bar{x}\|_2^2,$$

where $F := f + g$ and $\bar{x}$ is any element in $\operatorname{Arg\,min} F$.

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where $F := f + g$ and $\bar{x}$ is any element in $\operatorname{Arg\,min} F$.

Proof: Let $x \in \operatorname{dom} g$. Note from Taylor's inequality that we have for all $k \geq 0$

$$F(x^{k+1}) \leq f(x^k) + \nabla f(x^k)^T(x^{k+1} - x^k) + \frac{L}{2}\|x^{k+1} - x^k\|_2^2 + g(x^{k+1})$$

$$\leq f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{L}{2}\|x - x^k\|_2^2 + g(x) - \frac{L}{2}\|x^{k+1} - x\|_2^2,$$

where the 2nd inequality follows from (2) and Theorem 4.12.

Convergence of PG cont.

Proof of Theorem 7.2 cont.: Setting $x = x^k$, we get

$$F(x^{k+1}) \le f(x^k) + g(x^k) - \frac{L}{2}\|x^{k+1} - x^k\|_2^2 \le F(x^k)$$

showing that $\{F(x^k)\}$ is nonincreasing.

Now, pick any $\bar{x} \in \operatorname{Arg\,min} F$ and let $x = \bar{x}$, then

$$\begin{aligned} F(x^{k+1}) &\le f(x^k) + \nabla f(x^k)^T(\bar{x} - x^k) + \frac{L}{2}\|\bar{x} - x^k\|_2^2 + g(\bar{x}) - \frac{L}{2}\|x^{k+1} - \bar{x}\|_2^2 \\ &\le f(\bar{x}) + g(\bar{x}) + \frac{L}{2}\|\bar{x} - x^k\|_2^2 - \frac{L}{2}\|x^{k+1} - \bar{x}\|_2^2. \end{aligned}$$

Convergence of PG cont.

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showing that $\{F(x^k)\}$ is nonincreasing.

Now, pick any $\bar{x} \in \operatorname{Arg\,min} F$ and let $x = \bar{x}$, then

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$$\le f(\bar{x}) + g(\bar{x}) + \frac{L}{2}\|\bar{x} - x^k\|_2^2 - \frac{L}{2}\|x^{k+1} - \bar{x}\|_2^2.$$

Hence,

$$(k+1)[F(x^{k+1}) - F(\bar{x})] \le \textstyle\sum_{i=0}^{k}(F(x^{i+1}) - F(\bar{x}))$$
$$\le \frac{L}{2}\sum_{i=0}^{k}[\|\bar{x} - x^i\|_2^2 - \|x^{i+1} - \bar{x}\|_2^2] \le \frac{L}{2}\|\bar{x} - x^0\|_2^2.$$

Remarks on PG

Remark:

- The efficiency of PG relies heavily on the efficiency for computing proximal mapping. Each iteration involves one evaluation of ∇f and one proximal mapping computation.
- When $g = 0$, the PG is just the steepest descent with constant stepsize in Lecture 2. This suggests that PG may require lots of iterations in practice.
- PG requires explicit knowledge of L, which can be restrictive in applications. There are variants using line-search strategies to replace the constant L.
- In this lecture, we look at another kind of acceleration strategy based on Nesterov's extrapolation techniques; see [Nesterov '83] and [Nesterov '06]. Our discussions follow closely [Tseng unpub] and [Tseng '10].

Accelerated PG

Consider (1) and recall that L is a Lipschitz continuity modulus of ∇f.

Accelerated PG: Let $\theta_0 = \theta_{-1} = 1$, $x^0 = x^{-1} \in \operatorname{dom} g$. For $k \geq 0$, compute

$$\begin{cases} y^k &= x^k + \theta_k(\theta_{k-1}^{-1} - 1)(x^k - x^{k-1}), \\ x^{k+1} &= \operatorname{Prox}_{\frac{1}{L}g}\left(y^k - \frac{1}{L}\nabla f(y^k)\right), \end{cases}$$

and choose $\theta_{k+1} \in (0, 1]$ so that

$$\frac{1-\theta_{k+1}}{\theta_{k+1}^2} \leq \frac{1}{\theta_k^2}. \tag{3}$$

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- Notice that $\theta_k \equiv 1$ verifies (3): This choice gives back PG!

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Remark:

- Notice that $\theta_k \equiv 1$ verifies (3): This choice gives back PG! Accelerated PG typically chooses $\theta_k = O(1/k)$. We will stick to this latter choice.
- A common choice of θ_k is $\frac{2}{k+2}$. Another popular choice is

$$\theta_{k+1} = \frac{1}{2}\left(\sqrt{\theta_k^4 + 4\theta_k^2} - \theta_k^2\right); \tag{4}$$

  one can show using (4) and induction that $\theta_k \leq \frac{2}{k+2}$. Exercise!

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The update of y^k is an *extrapolation* step as $\theta_k(\theta_{k-1}^{-1} - 1) \geq 0$.

Accelerated PG

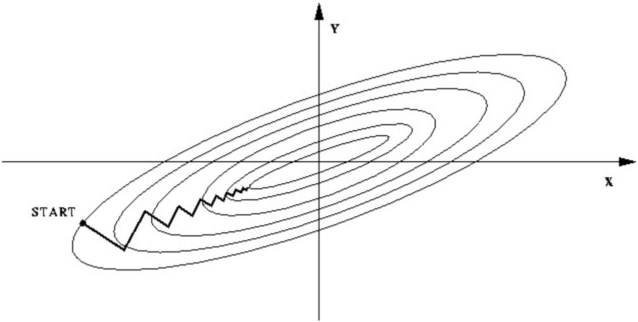

Figure: Trajectory of gradient descent on quadratic problems

Accelerated PG

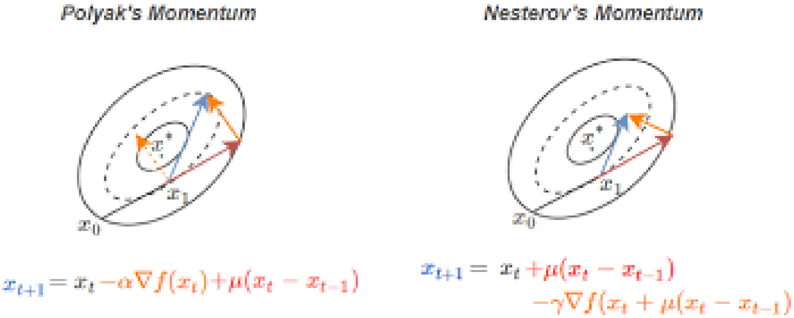

Figure: Correcting gradient direction by momentum

Accelerated PG

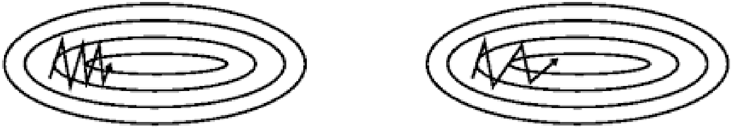

Figure: GD with and without momentum

Convergence of accelerated PG

Theorem 7.3 (accelerated PG complexity)
Consider (1) and let $\{x^k\}$ be generated by the accelerated proximal gradient algorithm on Slide 14. Then for all $k \geq 1$, it holds that

$$F(x^k) - F(\bar{x}) \leq \frac{L\theta_{k-1}^2}{2}\|x^0 - \bar{x}\|_2^2,$$

where $F := f + g$ and $\bar{x}$ is any element in $\operatorname{Arg\,min} F$.

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where $F := f + g$ and $\bar{x}$ is any element in $\operatorname{Arg\,min} F$.

Proof: Let $y \in \operatorname{dom} g$. Note from Taylor's inequality that for all $k \geq 0$

$$F(x^{k+1}) \leq f(y^k) + \nabla f(y^k)^T(x^{k+1} - y^k) + \frac{L}{2}\|x^{k+1} - y^k\|_2^2 + g(x^{k+1})$$

$$\leq f(y^k) + \nabla f(y^k)^T(y - y^k) + \frac{L}{2}\|y - y^k\|_2^2 + g(y) - \frac{L}{2}\|x^{k+1} - y\|_2^2,$$

where the 2nd inequality follows from (2) and Theorem 4.12.

Convergence of accelerated PG cont.

Proof of Theorem 7.3 cont.: Using convexity of f, we further have

$$F(x^{k+1}) \leq F(y) + \frac{L}{2}\|y - y^k\|_2^2 - \frac{L}{2}\|x^{k+1} - y\|_2^2.$$

Convergence of accelerated PG cont.

Proof of Theorem 7.3 cont.: Using convexity of f, we further have

$$F(x^{k+1}) \leq F(y) + \frac{L}{2}\|y - y^k\|_2^2 - \frac{L}{2}\|x^{k+1} - y\|_2^2.$$

Now, let $\bar{x} \in \operatorname{Arg\,min} F$ and set $y = (1-\theta_k)x^k + \theta_k \bar{x}$: This is a convex combination because $\theta_k \in [0, 1]$. Hence,

$$\begin{aligned}
F(x^{k+1}) \leq\ & F((1-\theta_k)x^k + \theta_k\bar{x}) + \frac{L}{2}\|(1-\theta_k)x^k + \theta_k\bar{x} - y^k\|_2^2 \\
& - \frac{L}{2}\|(1-\theta_k)x^k + \theta_k\bar{x} - x^{k+1}\|_2^2 \\
=\ & F((1-\theta_k)x^k + \theta_k\bar{x}) + \frac{L\theta_k^2}{2}\|\bar{x} + (\theta_k^{-1} - 1)x^k - \theta_k^{-1}y^k\|_2^2 \\
& - \frac{L\theta_k^2}{2}\|\bar{x} + (\theta_k^{-1} - 1)x^k - \theta_k^{-1}x^{k+1}\|_2^2
\end{aligned}$$

Convergence of accelerated PG cont.

Proof of Theorem 7.3 cont.: Here comes the magic

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$$\begin{aligned}
z^k &:= -(\theta_k^{-1} - 1)x^k + \theta_k^{-1} y^k \\
&= -(\theta_k^{-1} - 1)x^k + \theta_k^{-1} x^k + (\theta_{k-1}^{-1} - 1)(x^k - x^{k-1}) \\
&= -(\theta_{k-1}^{-1} - 1)x^{k-1} + \theta_{k-1}^{-1} x^k.
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Proof of Theorem 7.3 cont.: Here comes the magic — Observe that y^k is defined in such a way so that

$$\begin{aligned} z^k &:= -(\theta_k^{-1} - 1)x^k + \theta_k^{-1} y^k \\ &= -(\theta_k^{-1} - 1)x^k + \theta_k^{-1} x^k + (\theta_{k-1}^{-1} - 1)(x^k - x^{k-1}) \\ &= -(\theta_{k-1}^{-1} - 1)x^{k-1} + \theta_{k-1}^{-1} x^k. \end{aligned}$$

Thus, we have further that

$$\begin{aligned} F(x^{k+1}) &\leq F((1-\theta_k)x^k + \theta_k \bar{x}) + \frac{L\theta_k^2}{2}\|\bar{x} - z^k\|_2^2 - \frac{L\theta_k^2}{2}\|\bar{x} - z^{k+1}\|_2^2 \\ &\leq (1-\theta_k)F(x^k) + \theta_k F(\bar{x}) + \frac{L\theta_k^2}{2}\|\bar{x} - z^k\|_2^2 - \frac{L\theta_k^2}{2}\|\bar{x} - z^{k+1}\|_2^2. \end{aligned}$$

Convergence of accelerated PG cont.

Proof of Theorem 7.3 cont.: Rearranging terms, we have for all $k \geq 0$

$$F(x^{k+1}) - F(\bar{x}) \leq (1-\theta_k)[F(x^k) - F(\bar{x})] + \frac{L\theta_k^2}{2}\|\bar{x} - z^k\|_2^2 - \frac{L\theta_k^2}{2}\|\bar{x} - z^{k+1}\|_2^2.$$

Hence

$$\begin{aligned}
&\frac{(1-\theta_{k+1})}{\theta_{k+1}^2}[F(x^{k+1}) - F(\bar{x})] + \frac{L}{2}\|\bar{x} - z^{k+1}\|_2^2 \\
&\leq \frac{1}{\theta_k^2}[F(x^{k+1}) - F(\bar{x})] + \frac{L}{2}\|\bar{x} - z^{k+1}\|_2^2 \\
&\leq \frac{(1-\theta_k)}{\theta_k^2}[F(x^k) - F(\bar{x})] + \frac{L}{2}\|\bar{x} - z^k\|_2^2 \\
&\leq \cdots \leq \frac{1-\theta_0}{\theta_0^2}[F(x^0) - F(\bar{x})] + \frac{L}{2}\|\bar{x} - x^0\|_2^2 = \frac{L}{2}\|\bar{x} - x^0\|_2^2,
\end{aligned}$$

since $z^0 = x^0$ and $\theta_0 = 1$.

Remark on accelerated PG

Remark: With θ_k chosen to be $O(\frac{1}{k})$:

- APG is in general not a descent algorithm!
- This class of method is commonly known as optimal methods: According to Section 2.1.2 in [Nesterov '06], there exists a convex function f with Lipschitz gradient and $\operatorname{Arg\,min} f \neq \emptyset$ such that for any first-order method that generates iterates as

$$x^k \in x^0 + \operatorname{span}\{\nabla f(x^0), \nabla f(x^1), \ldots, \nabla f(x^{k-1})\}, \quad k \geq 1,$$

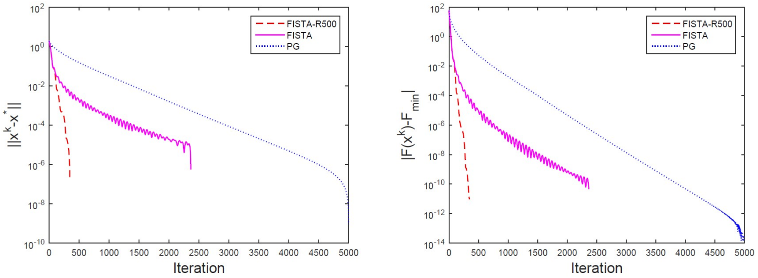
  it holds that for any $\bar{x} \in \operatorname{Arg\,min} f$,

$$f(x^k) - f(\bar{x}) \geq \frac{3L\|x^0 - \bar{x}\|_2^2}{32(k+1)^2}$$

  whenever $1 \leq k \leq \frac{1}{2}(n-1)$.
- APG motivated researches on extrapolation, and many variants have since been proposed. See [Becker-Candès-Grant '11] and the associated software TFOCS.

Restart strategy

- The optimality result suggests that Nesterov's extrapolation techniques is only optimal when k is not too large.
- This suggests restarting the θ_k from time to time, i.e., set $\theta_{k-1} = \theta_k = 1$ after certain number of iterations and/or some criterion is satisfied; see [O'Donoghue-Candès '15].
- The restart strategy has been implemented in TFOCS.



The pictures are taken from [Wen-Chen-Pong '17].

Case study

As an example, we illustrate how to design termination criterion for the following convex optimization problem; see for example [Wen-Chen-Pong '17].

Consider

$$\underset{x \in \mathbb{R}^n}{\text{Minimize}} \quad F(x) := \frac{1}{2}\|Ax - b\|_2^2 + \mu\|x\|_1, \tag{5}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $\mu > 0$.

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- If we set $g(x) = \mu\|x\|_1$ and $h(y) = \frac{1}{2}\|y-b\|_2^2$, the above problem is $\inf_{x\in\mathbb{R}^n}\{h(Ax)+g(x)\}$. Notice that $\operatorname{dom} g = \mathbb{R}^n$ and $\operatorname{dom} h = \mathbb{R}^m$. Thus, Theorem 5.4 states that (5) has the same optimal value as Notice that the $-$ve sign is moved from h^* to g^*

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$$\sup_{u\in\mathbb{R}^m}\{-g^*(-A^Tu) - h^*(u)\}.$$

Moreover, this dual problem also has an optimal solution.

Case study cont.

Notice that

$$g^*(v) = \sup_{y \in \mathbb{R}^n} \left\{ y^T v - \mu \|y\|_1 \right\} = \delta_{\|\cdot\|_\infty \le \mu}(v),$$

$$h^*(w) = \sup_{x \in \mathbb{R}^m} \left\{ w^T x - \frac{1}{2}\|x - b\|_2^2 \right\} = \frac{1}{2}\|w\|_2^2 + b^T w.$$

Hence, the dual problem is given by

$$\begin{array}{ll} \underset{u \in \mathbb{R}^m}{\text{Maximize}} & D(u) := -\frac{1}{2}\|u\|_2^2 - b^T u \\ \text{Subject to} & \|A^T u\|_\infty \le \mu. \end{array} \tag{6}$$

Case study cont.

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- Note that we always have $F(x) \ge \inf F \ge D(u)$ whenever u satisfies $\|A^T u\|_\infty \le \mu$.
- Suppose we apply PG / APG to minimize F in (5). Then $F(x^k) \ge \inf F$ for all k and $F(x^k) \to \inf F$. Can we construct (approximately) feasible $\{u^k\}$ suitably so that $D(u^k) \to \inf F$?

Case study cont.

Let $\bar{x}$ solve (5). Then Theorem 5.5 and Proposition 4.4(ii) give

$$0 \in \partial(h \circ A + g)(\overline{x}) = A^T \nabla h(A\overline{x}) + \partial g(\overline{x}).$$

Using Young's inequality and Theorem 5.1(iii), we have

$$\begin{aligned} & -A^T \nabla h(A\overline{x}) \in \partial g(\overline{x}) \\ \iff\ & \overline{x} \in \partial g^*(-A^T\overline{y}) \ \text{ and } \ \overline{y} = \nabla h(A\overline{x}) \\ \iff\ & \overline{x} \in \partial g^*(-A^T\overline{y}) \ \text{ and } \ A\overline{x} \in \partial h^*(\overline{y}). \end{aligned}$$

Case study cont.

Let $\bar{x}$ solve (5). Then Theorem 5.5 and Proposition 4.4(ii) give

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The last relation above gives

$$\begin{aligned} 0 &\in \partial h^*(\overline{y}) - A\overline{x} \subseteq \partial h^*(\overline{y}) - A\partial g^*(-A^T \overline{y}) \\ &\subseteq \partial[h^* + g^* \circ (-A^T)](\overline{y}) \end{aligned}$$

Thus, $\overline{y} := \nabla h(A\overline{x})$ solves the dual problem.

Case study cont.

Thus:

- Suppose $\{x^k\}$ converges to / clusters at a minimizer $\overline{x}$ of (5).
- Now, $\nabla h(A\overline{x})$ solves (6). Hence, $\{\nabla h(Ax^k)\}$ will be a sequence that converges to / clusters at a solution of (6).

Case study cont.

Thus:

- Suppose $\{x^k\}$ converges to / clusters at a minimizer $\overline{x}$ of (5).
- Now, $\nabla h(A\overline{x})$ solves (6). Hence, $\{\nabla h(Ax^k)\}$ will be a sequence that converges to / clusters at a solution of (6).

Next, we define:

$$u^k = \min\left\{1, \frac{\mu}{\|A^T\nabla h(Ax^k)\|_\infty}\right\}\nabla h(Ax^k).$$

Then:

(a) $u^k \to \nabla h(A\overline{x})$; (b) $\|A^T u^k\|_\infty \le \mu$ for all k.

Thus, $\{u^k\}$ is a maximizing sequence of (6). Hence,

$$\inf F \leftarrow F(x^k) \ge \inf F \ge D(u^k) \to \inf F.$$

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