Deadline: April 26, 23:59. No late submission will be accepted.
Please submit the assignment online as a single PDF file.
Show your steps clearly. A mere numerical answer will receive no scores.

1. Consider the following optimization problem.

$$\begin{array}{ll} \underset{x\in\mathbb{R}^2}{\text{Minimize}} & 3x_1 + 2x_2 \\ \text{Subject to} & x_1^2 + x_2^2 \leq 1, \\ & x_1 + x_2 \leq 0. \end{array}$$

   (a) Show that the MFCQ holds at every feasible point.

   (b) Write down the KKT conditions and find all the stationary points.

   (c) Find all global minimizers.

2. (a) Consider the following optimization problem.

$$\begin{array}{ll} \underset{x\in\mathbb{R}^2}{\text{Minimize}} & x_1^3 + x_2^3 \\ \text{Subject to} & x_1^2 + 4x_2^2 \leq 8, \\ & 2x_2 \geq x_1^2. \end{array}$$

      i. Show that the MFCQ holds at every feasible point.

      ii. Write down the KKT conditions and find all the stationary points.

   (b) Let $h(y) = \sum_{i=1}^m (y_i^4 + e^{2y_i} - 1)$ and $A \in \mathbb{R}^{(m+1)\times n}$ has full row rank, where $1 < m < n$. Let $B \in \mathbb{R}^{m\times n}$ be the matrix formed from the first m rows of A, and let a^T denote the last row of A. Let $\delta > 0$ and consider the set

$$C := \{x \in \mathbb{R}^n : \ h(Bx) \leq \delta, \ \ a^T x = 1\}.$$

   Show that the MFCQ holds at every point in C.

3. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper, closed, convex function. Show that

$$x = \text{prox}_f(x) + \text{prox}_{f^*}(x) \qquad \forall x \in \mathbb{R}^n.$$

4. Consider the following semidefinite programming problem:

$$\begin{array}{ll} \underset{X\in S^2}{\text{Minimize}} & x_{22} \\ \text{Subject to} & x_{21} = 1, \\ & X \succeq 0. \end{array}$$

   Write down its dual. Argue that the primal optimal value and dual optimal value are finite and equal, the dual optimal value is attained but the primal optimal value is not attained.

5. Explain whether the optimization problem

$$\begin{array}{ll} \text{Minimize} & x_1^3 + (x_2 + x_3)^2 \\ \text{Subject to} & x_1 \geq 0, \ x_1^2 + x_2^2 \leq x_3. \end{array}$$

   can be reformulated equivalently as an SDP problem.