Deadline: April 12, 23:59. No late submission will be accepted.
Please submit the assignment online as a single PDF file.
Show your steps clearly. A mere numerical answer will receive no scores.

1. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper, closed and convex, and $\lambda > 0$. Consider the inf-projection defined by

$$e_\lambda f(x) := \inf_{y \in \mathbb{R}^n} \left\{ \frac{1}{2\lambda} \|y - x\|_2^2 + f(y) \right\}.$$

   (a) Argue that $e_\lambda f$ is convex and finite-valued, and the infimum is attainable for each $x \in \mathbb{R}^n$.
   (b) Show that $(e_1 f)^*(u) = \frac{1}{2}\|u\|_2^2 + f^*(u)$.
   (c) Show that

$$\partial(e_1 f)(x) = \{x - \hat{x}\},$$

   where $\hat{x}$ is a point attaining the infimum in $e_1 f(x)$.

   Remark: Indeed, it holds that $e_\lambda f \in C^1(\mathbb{R}^n)$ with gradient at x given by $\frac{1}{\lambda}(x - \mathrm{Prox}_{\lambda f}(x))$. Moreover, for each $x \in \mathbb{R}^n$, it holds that $\lim_{\lambda \downarrow 0} e_\lambda f(x) = f(x)$ (pointwise convergence). The function $e_\lambda f$ is called the Moreau envelope and provides a general strategy for building smooth approximations to a proper closed convex function.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(t) := \ln(1 + e^t)$
   (a) Show that f is convex.
   (b) Show that

$$f^*(u) = \begin{cases} u \ln u + (1-u)\ln(1-u) & \text{if } 0 < u < 1, \\ 0 & \text{if } u = 0 \text{ or } 1, \\ \infty & \text{otherwise.} \end{cases}$$

   With the interpretation of ln as an extended-real-valued function with $\ln(t) = -\infty$ whenever $t \le 0$, and with the convention $0 \ln 0 = 0$, the above formula is usually simply written as $f^*(u) = u \ln u + (1-u)\ln(1-u)$.

3. Consider the following function

$$F(x) = \sum_{i=1}^m \ln(1 + e^{a_i^T x}) + \mu \|x\|_1,$$

   where $a_i \in \mathbb{R}^n$ for all i and $\mu > 0$. Let $A \in \mathbb{R}^{m \times n}$ be the matrix whose ith row is a_i^T and let $\|A\|_2 = \sqrt{2}$. Assume the knowledge from Q2.
   (a) Show that F has a minimizer.
   (b) Write down the Fenchel dual problem. Determine whether the strong duality holds and whether the dual problem has an optimal solution with justification.

4. Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and let $x \in C$. Define

$$C_x := \{d \in \mathbb{R}^n : \ x + td \in C \quad \forall t > 0\}.$$

   (a) Show that C_x is nonempty, closed and convex.
   (b) Show that C_x is independent of the choice of $x \in C$, i.e., $C_x = C_y$ for any $x, y \in C$.

   The set C_x is called the recession cone / asymptotic cone of the nonempty closed convex set C, and is usually denoted by C_∞ or C^∞ or 0^+C.

5. Let $\Pi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$ be defined as $\Pi(x, y, z) = (x, z)$. Let $F : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ be a proper function and let $v(x) := \inf_{y \in \mathbb{R}^m} F(x, y)$ be its inf-projection.
   For an extended-real-valued function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$, we define its strict epigraph as

$$\mathrm{epi}_S h := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : \ h(x) < r\}.$$

   (a) Show that $\mathrm{epi}_S v = \Pi(\mathrm{epi}_S F)$.
   (b) Show that $\Pi(\mathrm{epi}\, F) \subseteq \mathrm{epi}\, v \subseteq \overline{\Pi(\mathrm{epi}\, F)}$.
   (c) Suppose that $\Pi(\mathrm{epi}\, F)$ is closed. Show that $\mathrm{epi}\, v = \Pi(\mathrm{epi}\, F)$ and v is closed.