

# MATH FOUNDATION OF DATA SCIENCE HOMEWORK 2

>>NAME:  
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**Instructions:** Please complete the homework in LaTeX, produce a pdf, and submit it on time as indicated on Moodle.

## 1 Probability Theory Basics

**Q1 [5 points]** Use the axioms of probability to prove that

$$\forall A \in \mathcal{F}, P(A^c) = 1 - P(A).$$

[Solution:](#)

**Q2 [5 points]** Suppose events  $A, B, C$  are mutually independent. Prove that  $A \cup B$  and  $C$  are independent.

[Solution:](#)

**Q3 [10 points]** Suppose a sequence of events  $\{A_n : n \geq 1\}$  is a monotonically increasing sequence, i.e.,  $A_n \subseteq A_{n+1}, \forall n \geq 1$ . Define  $\lim_{n \rightarrow \infty} A_n = \cup_{i=1}^{\infty} A_i$ . Prove the continuity theorem for probability using the axioms:

$$\lim_{n \rightarrow \infty} P(A_n) = P(\lim_{n \rightarrow \infty} A_n).$$

**Solution:**

**Q4 [10 points]** The moment generating function (MGF) of a random variable  $X$  is defined as:

$$\psi(\lambda) := \mathbb{E}e^{\lambda X} = \int_{-\infty}^{+\infty} e^{\lambda x} dF_X(x)$$

if the right-hand side exists. Prove that if the  $k$ -th moment of  $X$  exists, then

$$\mathbb{E}X^k = \psi^{(k)}(0).$$

**Solution:**

## 2 Statistical Inference and Learning

**Q5 [10 points]** Let  $X_1, \dots, X_n$  be a sample from a geometric distribution with unknown parameter  $\theta$ , i.e., its density is  $p(x; \theta) = (1 - \theta)^{x-1} \theta$ . What is the maximum likelihood estimator (MLE) of  $\theta$ ?

**Solution:**

**Q6 [10 points]** Let  $X$  be a continuous random variable with the following PDF:

$$p(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Also, suppose given  $X$ ,  $Y$  has distribution:

$$P(y|x) = x(1-x)^{y-1}, \quad \text{for } y = 1, 2, 3, \dots \quad (2)$$

What is the MAP estimate of  $X$  given  $Y = 2$ ?

**Solution:**

**Q7 [10 points]** The infection rate of some bacteria in the whole population is 0.02. The result of testing is either positive or negative, but the result may be inaccurate. Suppose:

$$\begin{aligned}P(\text{negative}|\text{infected}) &= 0.01, \\P(\text{negative}|\text{noninfected}) &= 0.97.\end{aligned}$$

If one is tested positive, what is the probability of him/her being infected? (The solution should be in fraction form, i.e.,  $p/q$  for two integers  $p, q$ .)

**Solution:**

**Q8 [10 points]** For linear regression, consider the following statistical model for the data  $(x, y)$ :

$$y = \theta^\top x + \epsilon$$

where  $\epsilon$  has a Laplace distribution

$$p(\epsilon) = \frac{1}{2b} \exp\left(-\frac{|\epsilon|}{b}\right)$$

for some  $b > 0$ . Also assume a Laplace prior for  $\theta$ : each dimension  $\theta_i$  is independent and

$$p(\theta_i) = \frac{1}{2b_\theta} \exp\left(-\frac{|\theta_i|}{b_\theta}\right)$$

for some  $b_\theta > 0$ .

Given a set of data points  $\{(x_i, y_i)\}_{i=1}^n$ , use MAP to derive the training objective for  $\theta$ .

**Solution:**

**Q9 [10 points]** Let  $X_1, \dots, X_n$  be a random sample from an exponential population with pdf:

$$p(x|\theta) = \begin{cases} e^{-(x-\theta)} & x \geq \theta \\ 0 & x < \theta, \end{cases}$$

where  $-\infty < \theta < \infty$ . Consider testing  $H_0 : \theta \leq \theta_0$  versus  $H_1 : \theta > \theta_0$ , where  $\theta_0$  is a fixed value. Compute the likelihood ratio test statistic  $\lambda(x_1, \dots, x_n)$ .

**Solution:**

### 3 Information Theory

**Q10 [10 points]** The conditional mutual information of random variables  $X$  and  $Y$  given  $Z$  is defined by

$$I(X; Y|Z) = H(X|Z) - H(X|Y, Z) = \mathbb{E}_{p(x,y,z)} \log \frac{p(X, Y|Z)}{p(X|Z)p(Y|Z)}.$$

For joint probability mass functions  $p(x, y)$  and  $q(x, y)$ , the conditional relative entropy  $D(p(y|x)||q(y|x))$  is the average of the relative entropies between the conditional probability mass functions  $p(y|x)$  and  $q(y|x)$  averaged over the probability mass function  $p(x)$ . More precisely,

$$D(p(y|x); q(y|x)) = \sum_x p(x) \sum_y p(y|x) \log \frac{p(y|x)}{q(y|x)} = \mathbb{E}_{p(x,y)} \log \frac{p(Y|X)}{q(Y|X)}.$$

Prove:

(1) Chain rule for entropy: let  $X_1, \dots, X_n$  be drawn according to  $p(x_1, x_2, \dots, x_n)$ . Then

$$H(X_1, X_2, \dots, X_n) = \sum_{i=1}^n H(X_i | X_{i-1}, \dots, X_1).$$

(2) Chain rule for mutual information:

$$I(X_1, X_2, \dots, X_n; Y) = \sum_{i=1}^n I(X_i; Y | X_{i-1}, \dots, X_1).$$

(3) Chain rule for relative entropy:

$$D(p(x, y) || q(x, y)) = D(p(x) || q(x)) + D(p(y|x) || q(y|x)).$$

**Solution:**

## 4 Concentration

**Q11 [10 points]** Imagine we have an algorithm for solving some decision problem (e.g. is a given number  $p$  a prime?). Suppose the algorithm makes a decision at random and returns the correct answer with probability  $1/2 + \delta$  with some  $\delta > 0$ , which is just a bit better than a random guess. To improve the performance, we run the algorithm  $N$  times and take the majority vote. Show that, for any  $\epsilon \in (0, 1)$ , the answer is correct with probability at least  $1 - \delta$ , as long as

$$N \geq \frac{1}{2\delta^2} \ln \left( \frac{1}{\epsilon} \right).$$

**Solution:**