Deadline: 23:59, March 5, 2025

1. Let $\phi \in C^2(\mathbb{R}^n)$ and suppose that there exists $L > 0$ such that $\|\nabla^2\phi(y)\|_2 \le L$ for all $y \in \mathbb{R}^n$. Show that
$$\|\nabla\phi(u) - \nabla\phi(v)\|_2 \le L\|u - v\|_2 \ \ \forall u,\ v \in \mathbb{R}^n.$$

2. Consider the function $f(x) = (x_1 + x_2^2)^2$, the point $x^* = \begin{bmatrix}1 & 0\end{bmatrix}^T$ and the direction $d^* = \begin{bmatrix}-1 & 1\end{bmatrix}^T$. Show that d^* is a descent direction of f at x^*, and find all stepsizes that satisfy the *exact line search* criterion at x^* along d^*.

3. Consider the function $f(x) = e^{-x}$. Consider an iterate of the following form
$$x_{k+1} = x_k + \alpha_k d_k,$$
where $d_k = -f'(x_k)$ and α_k is obtained via Armijo line search by backtracking with $\bar{\alpha}_k \equiv 1$ and $\sigma = 0.1$. Start with $x_0 = 0$.
   (a) Show that $e^{-y} \le 1 - 0.1y$ whenever $y \in [0, 1]$.
   (b) Show that $\alpha_0 = 1$ and $x_1 = 1$.
   (c) Show that, for all $k \ge 0$, it holds that $x_{k+1} > 0$ and $\alpha_k = 1$.

4. Let $Q \succ 0$ and $b \in \mathbb{R}^n$. Define
$$f(x) = \frac{1}{2}x^TQx - b^Tx.$$
   (a) Suppose that $\bar{x}$ is not a stationary point of f and suppose the steepest descent method with exact line search is applied to minimizing f starting from $\bar{x}$.
   Show that the stepsize that satisfies the *exact line search* criterion at $\bar{x}$ along $-\nabla f(\bar{x})$ is given by
$$\frac{\|\nabla f(\bar{x})\|^2}{[\nabla f(\bar{x})]^TQ\nabla f(\bar{x})}.$$
   (b) Suppose that x^* is the unique minimizer of f and let v be any eigenvector of Q.
      i. Let $x^0 = x^* + v$. Show that x^0 is not a stationary point of f.
      ii. Show that the steepest descent method with exact line search initialized at $x^0 = x^* + v$ gives $x^1 = x^*$.

5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by
$$f(x) = (x_1 + x_2 - 1)_+^2 + \frac{1}{2}\|x\|_2^2,$$
where
$$t_+ := \begin{cases} t & \text{if } t \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$
   (a) Compute $\nabla f(x)$.
   (b) Consider an iterate of the following form:
$$x^{k+1} = x^k + \alpha_k d^k.$$
   Let d^k be the steepest descent direction and α_k be chosen to satisfy the Wolfe's condition. Suppose that x^k is nonstationary for all k.
      i. Show that the sequence $\{x^k\}$ is bounded.
      ii. Show that any accumulation point of $\{x^k\}$ is stationary.

6. Let $f(x) = h(Ax) + \mu\|x\|^2$, where $h(y) = \sum_{i=1}^m \ln(1 + e^{-y_i})$, $A \in \mathbb{R}^{m\times n}$, and $\mu > 0$.
   (a) By considering $\|\nabla^2 h(y)\|_2$ and using Question 1, show that for any u and $v \in \mathbb{R}^m$, it holds that
$$\|\nabla h(u) - \nabla h(v)\|_2 \le \frac{1}{4}\|u - v\|_2.$$

(b) Show that at any nonstationary point, the Newton direction $-[\nabla^2 f(x)]^{-1}\nabla f(x)$ is a descent direction.

(c) Consider an iterate of the following form:

$$x^{k+1} = x^k + \alpha_k d^k.$$

Let d^k be the Newton direction and α_k be chosen to satisfy the Wolfe's condition. Suppose that x^k is nonstationary for all k.

i. Show that the sequence $\{x^k\}$ is bounded.

ii. Show that any accumulation point of $\{x^k\}$ is stationary.