Localization of Rings

August 30, 2024

Multiplicatively Closed Subset

Definition. Let $(R, +, \cdot)$ be a ring with unity 1_R. Then S is **multiplicatively closed subset** if and only if

1. $1_R \in S$;
2. $x, y \in S \implies x \cdot y \in S$.

Localization of Rings

Definition. Let R be a commutative ring, and let $S \subseteq R$ be a *multiplicatively closed subset* of R. Then

$$(r_1, s_1) \sim (r_2, s_2) \iff \exists t \in S \quad \text{s.t.} \quad t(r_1 s_2 - r_2 s_1) = 0$$

is an equivalence relation on $R \times S$. The **localization of** R **at** S is a set of all equivalence classes

$$S^{-1}R := \left\{ \big[(r, s)\big] : r \in R, s \in S \right\} = \left\{ \frac{r}{s} : r \in R, s \in S \right\}$$

It is a ring with the addition and multiplication

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} := \frac{r_1 s_2 + r_2 s_1}{s_1 s_2} \quad \text{and} \quad \frac{r_1}{s_1}\frac{r_2}{s_2} := \frac{r_1 r_2}{s_1 s_2}.$$

Note that the additive identity is $\frac{0}{1}$ and multiplicative identity $\frac{1}{1}$.

Example. The localization of $\mathbb{Z}$ at $\mathbb{Z}^*(= \mathbb{Z} \setminus \{0\})$ is a ring

$$\mathbb{Q} = (\mathbb{Z}^*)^{-1}\mathbb{Z} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}^* \right\}$$

with

$$\frac{a}{b} \sim \frac{c}{d} \iff ad = bc.$$