Localization of Rings

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Multiplicatively Closed Subset

Definition. Let $(R, +, \cdot)$ be a ring with unity 1_R. Then $S \subseteq R$ is **multiplicatively closed subset** if and only if

1. $1_R \in S$;
2. $x, y \in S \implies x \cdot y \in S$.

Example. Recall that

A prime ideal $P \subsetneq R$ is an ideal such that:

$$\forall r, s \in R : [rs \in P \implies r \in P \vee s \in P] .$$

Let R be a ring and $P \lhd R$ be a prime ideal of R. Consider

$$S := R \setminus P = \{r \in R : r \notin P\} .$$

We claim that S is a multiplicatively closed subset of R:

1. $(1 \in S)$ Assume that $1 \in P$ then
$$r \cdot 1 = r \in P$$
for any $r \in R$. This means that every element $r \in R$ is also in P; i.e., $P = R$. It contradicts the fact the $P \subsetneq R$. We conclude that $1 \notin P$. Thus
$$1 \notin P \implies 1 \in R \setminus P = S.$$

2. $(x, y \in S \implies xy \in S)$ Let $s_1, s_2 \in S = R \setminus P$. By the definition of S, we have
$$s_1 \notin P \quad \text{and} \quad s_2 \notin P.$$
From the definition of a prime ideal, we know that
$$[rs \in P \Rightarrow r \in P \vee s \in P] \equiv [r \notin P \wedge s \notin P \Rightarrow rs \notin P].$$
Thus $s_1 s_2 \notin P$, and so
$$s_1 s_2 \notin P \implies s_1 s_2 \in R \setminus P = S.$$

Localization of Rings

Definition. Let R be a commutative ring, and let $S \subseteq R$ be a *multiplicatively closed subset* of R. Then

$$(r_1, s_1) \sim (r_2, s_2) \iff \exists t \in S \quad \text{s.t.} \quad t(r_1 s_2 - r_2 s_1) = 0$$

is an equivalence relation on $R \times S$. The **localization of** R **at** S is a set of all equivalence classes

$$S^{-1}R := \left\{ \left[(r, s) \right] : r \in R, s \in S \right\} = \left\{ \frac{r}{s} : r \in R, s \in S \right\}$$

It is a ring with the addition and multiplication

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} := \frac{r_1 s_2 + r_2 s_1}{s_1 s_2} \quad \text{and} \quad \frac{r_1}{s_1}\frac{r_2}{s_2} := \frac{r_1 r_2}{s_1 s_2}.$$

Note that the additive identity is $\frac{0}{1}$ and multiplicative identity $\frac{1}{1}$.

Example. The localization of $\mathbb{Z}$ at $\mathbb{Z}^*(= \mathbb{Z} \setminus \{0\})$ is a ring

$$\mathbb{Q} = (\mathbb{Z}^*)^{-1}\mathbb{Z} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}^* \right\}$$

with

$$\frac{a}{b} \sim \frac{c}{d} \iff ad = bc.$$

Note. Consider $R = \mathbb{Z}/6\mathbb{Z}$ and

$$S = \{2^0, 2^1, 2^2, 2^3, \dots\} = \{1, 2, 4\} \subseteq \mathbb{Z}/6\mathbb{Z}.$$

Then, for any $r \in R$, we have

$$\frac{r}{2} = \frac{4}{4} \cdot \frac{r}{2} = \frac{4 \cdot r}{4 \cdot 2} = \frac{4 \cdot r}{2} = \frac{2}{2} \cdot \frac{2 \cdot r}{1} = \frac{2 \cdot r}{1} \quad \text{in} \quad S^{-1}R.$$

This implies

$$\frac{r}{2^j} = \frac{2^j \cdot r}{1} \quad \text{for} \quad j = 0, 1, 2, \cdots,$$

so that $\frac{1}{1} = \frac{1}{1}$, $\frac{1}{2} = \frac{2}{1}$ and $\frac{1}{4} = \frac{4}{1}$; i.e., every fraction of $S^{-1}R$ has the form $\frac{r}{1}$. Moreover,

$$\frac{3}{1} = \frac{3 \cdot 2}{1 \cdot 2} = \frac{0}{2} = \frac{0}{1} \implies \frac{4}{1} = \frac{1}{1} \quad \text{and} \quad \frac{5}{1} = \frac{2}{1}.$$

Thus, we have

$$S^{-1}R = \left\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}\right\} \simeq \mathbb{Z}/3\mathbb{Z}.$$

Issue with the Condition "$r_1s_2 - r_2s_1 = 0$" in $\mathbb{Z}/6\mathbb{Z}$

- Consider $\frac{4}{1} = \frac{1}{1}$.
$$r_1s_2 - r_2s_1 = 4 \cdot 1 - 1 \cdot 1 = 3.$$
We would need some $t \in S$ such that $t \cdot 3 = 0 \in \mathbb{Z}/6\mathbb{Z}$. Thus, we choose $t = 2 \in S = \{1, 2, 4\}$.

- Consider $\frac{5}{1} = \frac{2}{1}$.
$$5 \cdot 1 - 2 \cdot 1 = 3.$$
Thus, we choose $2 \in S$.

- Consider
$$\frac{3}{2} = \frac{1}{2} \cdot \frac{3}{1} = \frac{2}{1} \cdot \frac{3}{1} = \frac{3 \cdot 2}{1} = \frac{0}{1}.$$
That is, $\frac{3}{2} = \frac{0}{1}$. Then
$$3 \cdot 1 - 0 \cdot 2 = 3.$$
Thus, we choose $2 \in S$.