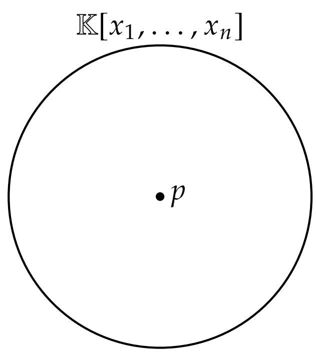
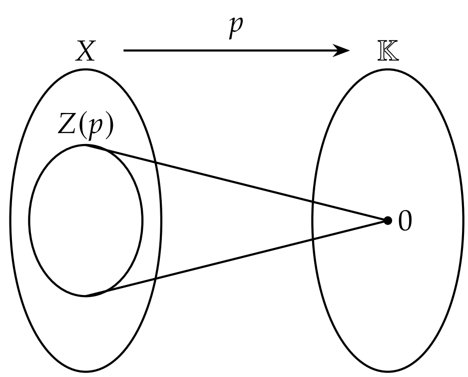
# Topology for Polynomial Rings

August 24, 2024

- $\mathbb{C}$: Field of Complex Numbers
- $\mathbb{K}$: Field
- $\mathbf{x} := (x_1, \ldots, x_n)$
- $X := \mathbb{K}^n$

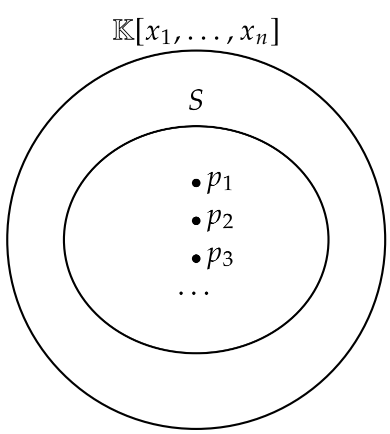
| **Polynomial Ring (Set)** | $\mathbb{C}[x]$ | $\mathbb{C}[x,y]$ | $\mathbb{K}[x_1,\ldots,x_n]$ |
|---|---|---|---|
| **Element** | $f \in \mathbb{C}[x]$ | $g \in \mathbb{C}[x,y]$ | $p \in \mathbb{K}[x_1,\ldots,x_n]$ |
| | $\begin{array}{rccc} f : & \mathbb{C} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & f(z) \end{array}$ | $\begin{array}{rccc} g : & \mathbb{C}^2 & \longrightarrow & \mathbb{C} \\ & (\alpha,\beta) & \longmapsto & g(\alpha,\beta) \end{array}$ | $\begin{array}{rccc} p : & X & \longrightarrow & \mathbb{K} \\ & \mathbf{x} & \longmapsto & p(\mathbf{x}) \end{array}$ |
| **Zero Set** | $Z(f) = \{z \in \mathbb{C} : f(z) = 0\}$ | $Z(g) = \{(\alpha,\beta) \in \mathbb{C}^2 : f(\alpha,\beta) = 0\}$ | $Z(p) = \{\mathbf{x} \in X : p(\mathbf{x}) = 0\}$ |

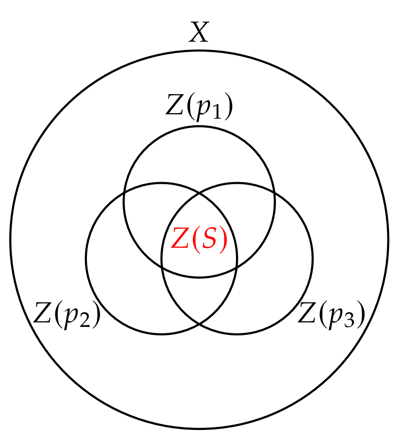



For any set of polynomials $S \subseteq \mathbb{K}[x_1,\ldots,x_n]$, the **zero set** $Z(S)$ is defined as:

$$\boxed{Z(S) := \{\mathbf{x} \in X : p(\mathbf{x}) = 0 \text{ for all } p \in S\}}.$$



Consider

$$\cancel{\mathcal{T} := \left\{Z(p) \subseteq X : p \in \mathbb{K}[x_1, \ldots, x_n]\right\} \subseteq 2^X}$$

$$\mathcal{T} := \left\{Z(S) \subseteq X : S \subseteq \mathbb{K}[x_1, \ldots, x_n]\right\} \subseteq 2^X$$

We claim that $\mathcal{T}$ is a topology on $X$:

(i) **(Whole Space and Empty Set)**

- Let $S = \emptyset$. By the definition,

$$Z(\emptyset) = \left\{\mathbf{x} \in X : p(\mathbf{x}) = 0 \text{ for all } p \in \emptyset\right\}.$$

  Therefore, $Z(\emptyset) = X \in \mathcal{T}$ since the condition is vacuously true for every point $\mathbf{x} \in X$.

- Let $S = \{1\}$, where $1 \in \mathbb{K}[x_1, \ldots, x_n]$. Then

$$Z(\{1\}) = \{\mathbf{x} \in X : 1 = 0\} = \emptyset \in \mathcal{T}.$$

(ii) **(Arbitrary Intersections)** Consider $\left\{Z(S_i)\right\}_{i \in \Lambda} \subseteq \mathcal{T}$, where each $Z(S_i)$ is the zero set of some set of polynomials $S_i \subseteq \mathbb{K}[x_1, \ldots, x_n]$. Then

$$\begin{aligned}
\bigcap_{i \in \Lambda} Z(S_i) &= \left\{\mathbf{x} \in X : \mathbf{x} \in Z(S_i) \text{ for all } i \in \Lambda\right\} \\
&= \left\{\mathbf{x} \in X : p(\mathbf{x}) = 0 \text{ for all } p \in S_i \text{ and for all } i \in \Lambda\right\} \\
&= \left\{\mathbf{x} \in X : p(\mathbf{x}) = 0 \text{ for all } p \in \bigcup_{i \in \Lambda} S_i\right\} \\
&= Z\left(\bigcup_{i \in \lambda} S_i\right)
\end{aligned}$$

Let $S := \bigcup_{i \in \Lambda} S_i \subseteq \mathbb{K}[x_1, \ldots, x_n]$. Then

$$\bigcap_{i \in \Lambda} Z(S_i) = Z\left(\bigcup_{i \in \Lambda} S_i\right) = Z(S) \in \mathcal{T}.$$

(iii) **(Finite Unions)** Consider two zero sets $Z(S_1)$ and $Z(S_2)$, where $S_1, S_2 \subseteq \mathbb{K}[x_1, \ldots, x_n]$. Then

$$\begin{aligned} Z(S_1) \cup Z(S_2) &= \left\{\mathbf{x} \in X : \mathbf{x} \in Z(S_1) \text{ or } \mathbf{x} \in Z(S_2)\right\} \\ &= \left\{\mathbf{x} \in X : p(\mathbf{x}) = 0 \text{ for all } p \in S_1 \text{ or } q(\mathbf{x}) = 0 \text{ for all } q \in S_2\right\} \\ &= \left\{\mathbf{x} \in X : (p \cdot q)(\mathbf{x}) = 0 \text{ for all } p \in S_1 \text{ and for all } q \in S_2\right\} \end{aligned}$$

Let

$$S := \left\{p \cdot q \in \mathbb{K}[x_1, \ldots, x_n] : p \in S_1 \text{ and } q \in S_2\right\} \subseteq \mathbb{K}[x_1, \ldots, x_n].$$

Then

$$\begin{aligned} Z(S_1) \cup Z(S_2) &= \left\{\mathbf{x} \in X : (p \cdot q)(\mathbf{x}) = 0 \text{ for all } p \in S_1 \text{ and for all } q \in S_2\right\} \\ &= \left\{\mathbf{x} \in X : (p \cdot q)(\mathbf{x}) = 0 \text{ for all } p \cdot q \in S\right\} \\ &= Z(S) \in \mathcal{T}. \end{aligned}$$

Hence it is proved

□