Basics of Quantum Mechanics

Quantum computing and Complex geometry, Enjoying math

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I. QUANTUM STATE

Quantum states $|0\rangle$ and $|1\rangle$ are represented by

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1}$$

1. Show that each of quantum states $|0\rangle$ and $|1\rangle$ is a unit length vector.

2. Show that the quantum states $|0\rangle$ and $|1\rangle$ are orthogonal each other.

3. Represent a quantum state $|\psi\rangle = (\alpha, \beta)^\intercal$ in terms of $|0\rangle$ and $|1\rangle$, where $\alpha, \beta \in \mathbb{C}$ satisfying $|\alpha|^2 + |\beta|^2 = 1$, and $\intercal$ is the transpose operation.

4. Calculate $p_0 = |\langle 0|\psi\rangle|^2$ and $p_1 = |\langle 1|\psi\rangle|^2$.

5. Do the same things for quantum states given by

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2}$$

6. Do the same things for quantum states given by

$$|+i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad |-i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{3}$$

The pairs of quantum states $\{|0\rangle, |1\rangle\}$, $\{|+\rangle, |-\rangle\}$, and $\{|+i\rangle, |-i\rangle\}$ correspond to the three axes of Bloch sphere, $\pm x$, $\pm y$, and $\pm z$, respectively. See Fig. 1.

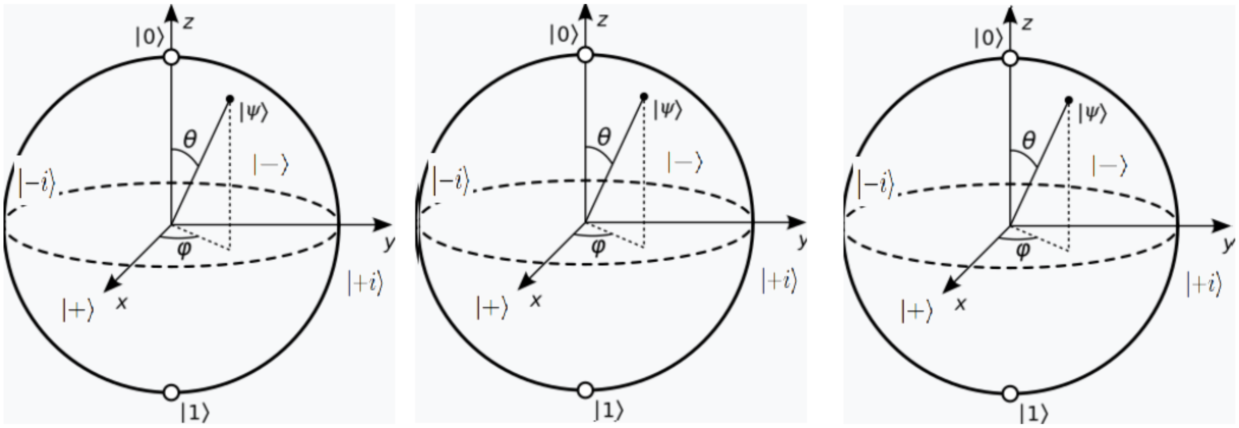

FIG. 1. An illustration of Bloch spheres from Wikipedia.

II. UNITARY OPERATION AND QUANTUM GATES

A. Unitary operator

Definition.—A Hermitian operator is a matrix satisfying $H = H^\dagger$, where $\dagger$ is called the dagger operation defined by $\hat{A}^\dagger := (\hat{A}^*)^\intercal = (\hat{A}^\intercal)^*$.

Definition.—A unitary operator is a matrix satisfying $UU^\dagger = U^\dagger U = \mathbf{I}$.

Consider quantum gates X, Y, Z, H, and CX:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \text{and} \quad CX = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

1. Show that the quantum gates are unitary.

2. Calculate $X|0\rangle$, $Y|0\rangle$, $Z|0\rangle$, $X|1\rangle$, $Y|1\rangle$, and $Z|1\rangle$.

3. Calculate $X|\pm\rangle$, $Y|\pm\rangle$, and $Z|\pm\rangle$.

4. Calculate $X|\pm i\rangle$, $Y|\pm i\rangle$, and $Z|\pm i\rangle$.

5. Find points corresponding to the quantum states obtained from 2–4 on the Bloch sphere.

6. Show that an exponential of anti-Hermitian operator $i\hat{G}$, $e^{i\hat{G}}$, is a unitary operator. Use $(\hat{A}^n)^\dagger = (\hat{A}^\dagger)^n$ and $e^{\hat{A}} := \mathbf{I} + \hat{A} + \frac{1}{2!}\hat{A}^2 + \frac{1}{3!}\hat{A}^3 \cdots$. The operator $\hat{G}$ is called *generator*.

A unitary operator $U_{\vec{n}}(\theta)$ can be represented by

$$U_{\vec{n}}(\theta) := \exp\left[i\frac{\theta}{2}\left(n_x X + n_y Y + n_z Z\right)\right] = I\cos\left(\frac{\theta}{2}\right) + i\vec{n}\cdot\vec{\sigma}\sin\left(\frac{\theta}{2}\right), \tag{4}$$

where $\vec{n} := (n_x, n_y, n_z)$ and $\vec{\sigma} := (X, Y, Z)$.

1. Obtain $e^{i\frac{\pi}{2}X}\left|0\right\rangle$, $e^{i\frac{\pi}{2}Y}\left|0\right\rangle$, $e^{i\frac{\pi}{2}Z}\left|0\right\rangle$, $e^{i\frac{\pi}{2}X}\left|1\right\rangle$, $e^{i\frac{\pi}{2}Y}\left|1\right\rangle$, and $e^{i\frac{\pi}{2}Z}\left|1\right\rangle$.

2. Represent $U_{\vec{n}}(\theta)$ as the form of matrix.

3. Discuss the relation between the unitary operation and rotation matrix.

4. Draw a point corresponding to $U_{\vec{n}}(\theta)\left|0\right\rangle$ on the Bloch sphere.

Remark.—For a qubit state, the unitary operator $U_{\vec{n}}(\theta)$ is a *universal quantum gate.*

Consider a set of gates for universal quantum computation:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \quad S = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}, \quad T = \begin{pmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{pmatrix}, \quad \text{and} \quad CX = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}. \tag{5}$$

Any quantum gate can be decomposed into a circuit of H, S, T, and CX.

Definition.—A quantum circuit consisting of H, S, and CX is called *Clifford circuit.*

Definition.—Two independent quantum states $\left|\psi\right\rangle$ and $\left|\phi\right\rangle$ can be represented as a quantum state $\left|\Psi\right\rangle = \left|\psi\right\rangle \otimes \left|\phi\right\rangle$, where $\otimes$ is a tensor product. For given two quantum states $(a, b)^\intercal$ and $(c, d)^\intercal$, their tensor product is given by

$$\begin{pmatrix}a\\ b\end{pmatrix} \otimes \begin{pmatrix}c\\ d\end{pmatrix} = \begin{pmatrix}ac\\ ad\\ bc\\ bd\end{pmatrix}. \tag{6}$$

1. Calculate $H\left|0\right\rangle$ and $H\left|1\right\rangle$.

2. Calculate $CX\left|0\right\rangle\left|0\right\rangle$ and $CX\left|+\right\rangle\left|0\right\rangle$.

3. Represent the state $CX\left|+\right\rangle\left|0\right\rangle$ as a tensor product of two quantum states. Is it possible or not?