# Basics of Quantum Mechanics

Quantum computing and Complex geometry, Enjoying math

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## I. QUANTUM STATE

Remind that a quantum state $\hat{\varrho} := \sum_i \rho_i |i\rangle\langle i|$ satisfies following conditions:

1. Hermiticity $\hat{\varrho} = \hat{\varrho}^\dagger$

2. Non-negativity $\hat{\varrho} \geq 0$

3. Unit trace $\mathrm{Tr}\hat{\varrho} = 1$.

These conditions imply that $\rho_i = \rho_i^*$ and $\rho_i \geq 0$ for all $i$s, and $\sum_i \rho_i = 1$.

## II. OBSERVABLE (관측량)

An observable is a dynamical variable which can be measured. For example, position, momentum, and energy are observables. In classical theory, these variables are described by real numbers.

**Postulate.**— (Born rule) In quantum theory, an observable corresponds to a Hermitian matrix called *observable operator*. A measured result is one of the eigenvalues of the observable operator, and the probability of measuring an eigenvalue is given by the projection of quantum state onto the eigenspace corresponding to the eigenvalue.

*A two-level atom.*—Consider an atom with energy levels given as $E_1$ and $E_2$. Thus, the outcomes obtained by repeatedly performing energy measurements are $E_1, E_2, \ldots, E_2, E_2$. The expectation ($k$-th moment) of energy is then given by

$$\langle E^k \rangle = E_1^k p(E_1) + E_2^k p(E_2), \tag{1}$$

where $p(E_i)$ is probability of obtaining $E_i$ as a measurement result. (Note that physical theory is for describing probability or, equivalently, all the moments of observation.)

Quantum theory assumes an observable operator for the energy observable defined by

$$\hat{E} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} = E_1|0\rangle\langle 0| + E_2|1\rangle\langle 1|. \tag{2}$$

For a quantum state $|\psi\rangle \in \mathcal{H}$, the expected value of energy becomes

$$\begin{aligned} \langle E^k \rangle &= \langle \psi | \hat{E}^k | \psi \rangle & (3) \\ &= E_1^k \langle \psi | 0 \rangle \langle 0 | \psi \rangle + E_2^k \langle \psi | 1 \rangle \langle 1 | \psi \rangle & (4) \\ &= E_1^k \left| \langle 0 | \psi \rangle \right|^2 + E_2^k \left| \langle 1 | \psi \rangle \right|^2 & (5) \\ &= E_1^k p(E_1) + E_2^k p(E_2), & (6) \end{aligned}$$

where $p(E_i)$ is probability of obtaining $E_i$ as a measurement result. We rewrite the second line as

$$E_1 \langle \psi | 0 \rangle \langle 0 | \psi \rangle + E_2 \langle \psi | 1 \rangle \langle 1 | \psi \rangle = E_1 \langle \psi | \hat{P}_0 | \psi \rangle + E_2 \langle \psi | \hat{P}_1 | \psi \rangle, \tag{7}$$

where $\hat{P}_i \in \mathcal{H}$ is called projector.

*Definition.*—Projective measurement or Von Neumann measurement is a set of projectors, e.g, $M = \{\hat{P}_0, \hat{P}_1\}$. The projectors satisfying $\hat{P}_i \geq 0$ and $\mathrm{Tr}\hat{P}_i\hat{P}_j = \delta_{ij}\ \forall i, j$, and $\sum_i \hat{P}_i = \mathbf{I}$. The projective measurement $M$ forms a *projection-valued measure*.

*Definition.*—A quantum state is a representation of our knowledge on individual outcomes in future experiments.

1. Show that $\mathrm{Tr}\left(|\psi\rangle\langle\psi|\hat{P}_i\right) = p(E_i)$.
2. Show that the sum of projectors defined in Eq. (7) is the $2 \times 2$ identity matrix $\mathbf{I}$.
3. Show that Bloch vectors of $\hat{P}_0$ and $\hat{P}_1$ are in opposite directions.
4. We can consider another energy observable defined by

$$\hat{E}' = E_+|+\rangle\langle +| + E_-|-\rangle\langle -|. \tag{8}$$

Obtain $p(E_\pm)$ and the expectation value of the observable.

5. Does the probability in quantum theory satisfy Kolmogorov's probability axioms?