Basics of Quantum Mechanics

Quantum computing and Complex geometry, Enjoying math

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I. ALGEBRAIC PROPERTIES OF PAULI MATRICES

Definition.—Hermitian operators $\mathbf{I}$, $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ are given by

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

where $\mathbf{I}$ is called identity operator, and $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ are called Pauli operators (or matrices). Pauli matrices form a basis for the real vector space of 2×2 Hermitian matrices.

1. Show that the identity and Pauli operators are Hermitian.

2. Show that the identity and Pauli operators are unitary.

3. Show that $I\,|\psi\rangle = |\psi\rangle$

4. Calculate $\hat{\sigma}_x\,|+\rangle$ and $\hat{\sigma}_x\,|-\rangle$.

5. Calculate $\hat{\sigma}_y\,|+i\rangle$ and $\hat{\sigma}_y\,|-i\rangle$.

6. Calculate $\hat{\sigma}_z\,|0\rangle$ and $\hat{\sigma}_z\,|1\rangle$.

7. Show that an exponential of anti-Hermitian operator $i\hat{G}$, $e^{i\hat{G}}$, is a unitary operator. Use $(\hat{A}^n)^\dagger = (\hat{A}^\dagger)^n$ and $e^{\hat{A}} := \mathbf{I} + \hat{A} + \frac{1}{2!}\hat{A}^2 + \frac{1}{3!}\hat{A}^3 \cdots$. $\hat{G}$ is called *generator*.

Definition (Bloch representation).—An arbitrary Hermitian operator can be written by a linear combination of the identity and Pauli operators with real-valued coefficients. In other words, a given Hermitian operator $\hat{H}$ can be read

$$\hat{H} = c_0\mathbf{I} + c_x\hat{\sigma}_x + c_y\hat{\sigma}_y + c_z\hat{\sigma}_z. \tag{2}$$

1. Show that Pauli operators are traceless, i.e., $\mathrm{Tr}\hat{\sigma}_i = 0$ for all $i \in \{x, y, z\}$.

2. Calculate $\mathrm{Tr}\hat{\sigma}_i\hat{\sigma}_j$ for $i, j \in \{x, y, z\}$. Note that $\mathrm{Tr}\hat{A}\hat{B} = \mathrm{Tr}\hat{B}\hat{A}$.

3. Calculate $\mathrm{Tr}\hat{H}\mathbf{I}$, $\mathrm{Tr}\hat{H}\hat{\sigma}_x$, $\mathrm{Tr}\hat{H}\hat{\sigma}_y$, and $\mathrm{Tr}\hat{H}\hat{\sigma}_z$.

II. DENSITY MATRIX

A density matrix $\hat{\varrho}$ of a quantum state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ is defined by

$$\hat{\varrho} := |\psi\rangle\langle\psi| = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \tag{3}$$

1. Show that the density matrix $\hat{\varrho}$ is a Hermitian operator.

2. Show that

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

3. Show that the Bloch representation of $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ is given by

$$|0\rangle\langle 0| = \frac{1}{2}\left(\mathbf{I} + \hat{\sigma}_x\right) \quad \text{and} \quad |1\rangle\langle 1| = \frac{1}{2}\left(\mathbf{I} - \hat{\sigma}_x\right). \tag{5}$$

Remark.—A density matrix of qubit state $|\psi\rangle = \sin\frac{\theta}{2}|0\rangle + e^{i\phi}\cos\frac{\theta}{2}|1\rangle$ is given by

$$\begin{aligned} |\psi\rangle\langle\psi| &= \frac{1}{2}\left(\mathbf{I} + \sin\theta\cos\phi\hat{\sigma}_x + \sin\theta\sin\phi\hat{\sigma}_y + \cos\theta\hat{\sigma}_z\right) \\ &= \frac{1}{2}\left(\mathbf{I} + \vec{r}\cdot\vec{\sigma}\right), \end{aligned} \tag{6}$$

where $\vec{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ and the vector $\vec{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is called *Bloch vector*. It is worth to note that $|\psi\rangle\langle\psi| \geq 0$ *if and only if* $|\vec{r}| \leq 1$.