Week 6: Graph Algorithms

Directed Graphs

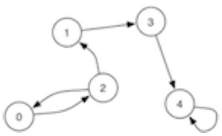
Directed Graphs (Digraphs)

In our previous discussion of graphs:

- an edge indicates a relationship between two vertices
- an edge indicates nothing more than a relationship

In many real-world applications of graphs:

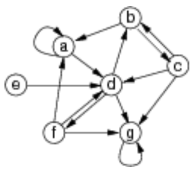
- edges are directional $(v \rightarrow w \neq w \rightarrow v)$



- edges have a *weight* (cost to go from $v \rightarrow w$)

... Directed Graphs (Digraphs)

Example digraph and adjacency matrix representation:



	a	b	c	d	e	f	g
a	1	0	0	1	0	0	0
b	1	0	1	0	0	0	0
c	0	1	0	1	0	0	1
d	0	1	0	0	0	1	1
e	0	0	0	1	0	0	0
f	1	0	0	1	0	0	1
g	0	0	0	0	0	0	1

Undirectional $\Rightarrow$ symmetric matrix
Directional $\Rightarrow$ non-symmetric matrix

Maximum #edges in a digraph with V vertices: V^2

... Directed Graphs (Digraphs)

Terminology for digraphs …

Directed path: sequence of $n \geq 2$ vertices $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_n$

- where $(v_i,v_{i+1}) \in edges(G)$ for all v_i,v_{i+1} in sequence
- if $v_1 = v_n$, we have a *directed cycle*

Reachability: w is reachable from v if $\exists$ directed path $v,\ldots,w$

Digraph Applications

Potential application areas:

Domain	**Vertex**	**Edge**
Web	web page	hyperlink
scheduling	task	precedence
chess	board position	legal move
science	journal article	citation
dynamic data	malloc'd object	pointer
programs	function	function call
`make`	file	dependency

... Digraph Applications

Problems to solve on digraphs:

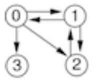
- is there a directed path from *s* to *t*? (transitive closure)
- what is the shortest path from *s* to *t*? (shortest path)
- are all vertices mutually reachable? (strong connectivity)
- how to organise a set of tasks? (topological sort)
- which web pages are "important"? (PageRank)
- how to build a web crawler? (graph traversal)

Digraph Representation

Similar set of choices as for undirectional graphs:

- array of edges (directed)
- vertex-indexed adjacency matrix (non-symmetric)
- vertex-indexed adjacency lists

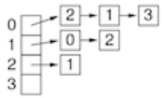
V vertices identified by *0* .. *V-1*



digraph

	0	1	2	3
0	0	1	1	1
1	1	0	1	0
2	0	1	0	0
3	0	0	0	0

adj matrix



adj lists

Reachability

Transitive Closure

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Given a digraph G it is potentially useful to know

- is vertex t reachable from vertex s?

Example applications:

- can I complete a schedule from the current state?
- is a malloc'd object being referenced by any pointer?

How to compute transitive closure?

... Transitive Closure

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One possibility:

- implement it via `hasPath(G,s,t)` (itself implemented by DFS or BFS algorithm)
- feasible if *reachable(G,s,t)* is infrequent operation

What if we have an algorithm that frequently needs to check reachability?

Would be very convenient/efficient to have:

```
reachable(G,s,t):
|  return G.tc[s][t]   // transitive closure matrix
```

Of course, if V is *very* large, then this is not feasible.

Exercise #1: Transitive Closure Matrix

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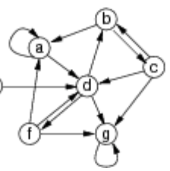
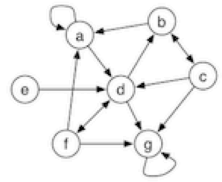
Which reachable s .. t exist in the following graph?



	a	b	c	d	e	f	g
a	1	0	0	1	0	0	0
b	1	0	1	0	0	0	0
c	0	1	0	1	0	0	1
d	0	1	0	0	0	1	1
e	0	0	0	1	0	0	0
f	1	0	0	1	0	0	1
g	0	0	0	0	0	0	1

Transitive closure of example graph:



	a	b	c	d	e	f	g
a	1	0	0	1	0	0	0
b	1	0	1	0	0	0	0
c	0	1	0	1	0	0	1
d	0	1	0	0	0	1	1
e	0	0	0	1	0	0	0
f	1	0	0	1	0	0	1
g	0	0	0	0	0	0	1

adjacency matrix

	a	b	c	d	e	f	g
a	1	1	1	1	0	1	1
b	1	1	1	1	0	1	1
c	1	1	1	1	0	1	1
d	1	1	1	1	0	1	1
e	1	1	1	1	0	1	1
f	1	1	1	1	0	1	1
g	0	0	0	0	0	0	1

reachability matrix

... Transitive Closure

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Goal: produce a matrix of reachability values

- if *tc[s][t]* is 1, then t is reachable from s
- if *tc[s][t]* is 0, then t is not reachable from s

So, how to create this matrix?

Observation:

$\forall i,s,t \in$ vertices(G):
 $(s,i) \in$ edges(G) and $(i,t) \in$ edges(G) $\Rightarrow$ $tc[s][t] = 1$

`tc[s][t]=1` if there is a path from s to t of length 2 ($s \to i \to t$)

... Transitive Closure

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If we implement the above as:

```
make tc[][] a copy of edges[][]
for all i∈vertices(G) do
   for all s∈vertices(G) do
      for all t∈vertices(G) do
         if tc[s][i]=1 and tc[i][t]=1 then
            tc[s][t]=1
         end if
      end for
   end for
end for
```

then we get an algorithm to convert `edges` into a *tc*

This is known as *Warshall's algorithm*

... Transitive Closure

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How it works ...

After iteration 1, `tc[s][t]` is 1 if

- either $s \to t$ exists or $s \to 0 \to t$ exists

After iteration 2, `tc[s][t]` is 1 if any of the following exist

- $s \to t$ or $s \to 0 \to t$ or $s \to 1 \to t$ or $s \to 0 \to 1 \to t$ or $s \to 1 \to 0 \to t$

Etc. … so after the V^{th} iteration, `tc[s][t]` is 1 if

- there is any directed path in the graph from s to t

Exercise #2: Transitive Closure

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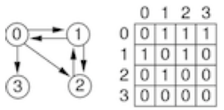
Trace Warshall's algorithm on the following graph:



digraph *adj matrix*

1st iteration `i=0`:

tc	[0]	[1]	[2]	[3]
[0]	0	1	1	1
[1]	1	1	1	1
[2]	0	1	0	0
[3]	0	0	0	0

2nd iteration `i=1`:

tc	[0]	[1]	[2]	[3]
[0]	1	1	1	1
[1]	1	1	1	1
[2]	1	1	1	1
[3]	0	0	0	0

3rd iteration `i=2`: unchanged

4th iteration `i=3`: unchanged

... Transitive Closure

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Cost analysis:

- storage: additional V^2 items (each item may be 1 bit)
- computation of transitive closure: V^3
- computation of `reachable()`: $O(1)$ after having generated `tc[][]`

Amortisation: would need many calls to `reachable()` to justify other costs

Alternative: use DFS in each call to `reachable()`
Cost analysis:

- storage: cost of queue and set during reachable
- computation of `reachable()`: cost of DFS = $O(V^2)$ (for adjacency matrix)

Digraph Traversal

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Same algorithms as for undirected graphs:

`depthFirst(v):`

1. mark `v` as visited
2. for each `(v,w)`∈*edges*(G) do
   if `w` has not been visited then
   `depthFirst(w)`

`breadth-first(v):`

1. enqueue `v`
2. while queue not empty do
   dequeue `v`
   if `v` not already visited then
   mark `v` as visited
   enqueue each vertex `w` adjacent to `v`

Example: Web Crawling

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Goal: visit every page on the web

Solution: breadth-first search with "implicit" graph

```
webCrawl(startingURL):
|  mark startingURL as alreadySeen
|  enqueue(Q,startingURL)
|  while Q is not empty do
|  |  nextPage=dequeue(Q)
|  |  visit nextPage
|  |  for each hyperLink on nextPage do
|  |  |  if hyperLink not alreadySeen then
|  |  |     mark hyperLink as alreadySeen
|  |  |     enqueue(Q,hyperLink)
|  |  |  end if
|  |  end for
```

| **end while**

`visit` scans page and collects e.g. keywords and links

Weighted Graphs

Weighted Graphs

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Graphs so far have considered

- edge = an association between two vertices/nodes
- may be a precedence in the association (directed)

Some applications require us to consider

- a *cost* or *weight* of an association
- modelled by assigning values to edges (e.g. positive reals)

Weights can be used in both directed and undirected graphs.

... Weighted Graphs

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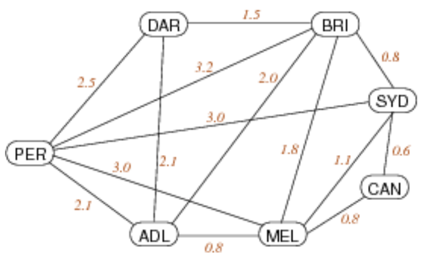
Example: major airline flight routes in Australia



Representation: edge = direct flight; weight = approx flying time (hours)

... Weighted Graphs

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Weights lead to minimisation-type questions, e.g.

1. Cheapest way to connect all vertices?

- a.k.a. *minimum spanning tree* problem
- assumes: edges are weighted and undirected

2. Cheapest way to get from *A* to *B*?

- a.k.a *shortest path* problem
- assumes: edge weights positive, directed or undirected

Exercise #3: Implementing a Route Finder

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If we represent a street map as a graph

- what are the vertices?
- what are the edges?
- are edges directional?
- what are the weights?
- are the weights fixed?

Weighted Graph Representation

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Weights can easily be added to:

- adjacency matrix representation (0/1 → int or float)
- adjacency lists representation (add int/float to list node)

Both representations work whether edges are directed or not.

... Weighted Graph Representation

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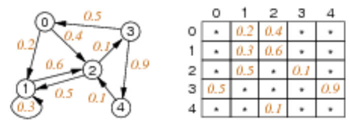
Adjacency matrix representation with weights:



Weighted Digraph   Adjacency Matrix

Note: need distinguished value to indicate "no edge".

... Weighted Graph Representation

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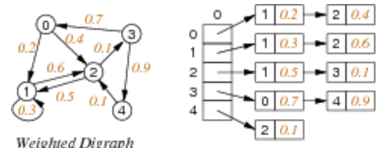
Adjacency lists representation with weights:



Weighted Digraph   Adjacency Lists

Note: if undirected, each edge appears twice with same weight

... Weighted Graph Representation

Sample adjacency matrix implementation in C requires minimal changes to previous Graph ADT:

WGraph.h

```
// edges are pairs of vertices (end-points) plus positive weight
typedef struct Edge {
   Vertex v;
   Vertex w;
   int    weight;
} Edge;

// returns weight, or 0 if vertices not adjacent
int adjacent(Graph, Vertex, Vertex);
```

... Weighted Graph Representation

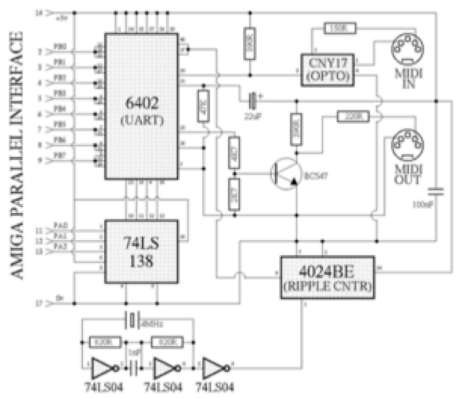
WGraph.c

```
typedef struct GraphRep {
   int **edges;  // adjacency matrix storing positive weights
                 // 0 if nodes not adjacent
   int   nV;     // #vertices
   int   nE;     // #edges
} GraphRep;


void insertEdge(Graph g, Edge e) {
   assert(g != NULL && validV(g,e.v) && validV(g,e.w));
   if (g->edges[e.v][e.w] == 0) {  // edge e not in graph
      g->edges[e.v][e.w] = e.weight;
      g->nE++;
   }
}

int adjacent(Graph g, Vertex v, Vertex w) {
   assert(g != NULL && validV(g,v) && validV(g,w));
   return g->edges[v][w];
}
```

Minimum Spanning Trees

Exercise #4: Minimising Wires in Circuits

Electronic curcuit designs often need to make the pins of several components electrically equivalent by wiring them together.



To interconnect a set of *n* pins we can use an arrangement of *n-1* wires each connecting two pins.

What kind of algorithm would …

- help us find the arrangement with the least amount of wire?

Minimum Spanning Trees

Reminder: *Spanning tree ST* of graph *G=(V,E)*

- *spanning* = all vertices, *tree* = no cycles
- *ST* is a subgraph of *G* (*G'=(V,E')* where $E' \subseteq E$)
- *ST* is *connected* and *acyclic*

Minimum spanning tree MST of graph *G*

- *MST* is a spanning tree of *G*
- sum of edge weights is no larger than any other ST

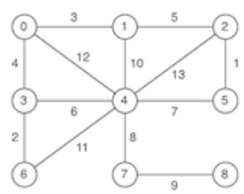
Applications: Computer networks, Electrical grids, Transportation networks …

Problem: how to (efficiently) find MST for graph *G*?

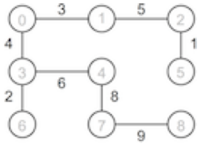
NB: MST may not be unique (e.g. all edges have same weight ⇒ every ST is MST)

... Minimum Spanning Trees

Example:



An MST …



... Minimum Spanning Trees

Brute force solution:

```
findMST(G):
|  Input  graph G
|  Output a minimum spanning tree of G
|
|  bestCost=∞
|  for all spanning trees t of G do
|  |  if cost(t)<bestCost then
|  |     bestTree=t
|  |     bestCost=cost(t)
|  |  end if
|  end for
|  return bestTree
```

Example of *generate-and-test* algorithm.

Not useful because #spanning trees is potentially large (e.g. n^{n-2} for a complete graph with n vertices)

... Minimum Spanning Trees

Simplifying assumption:

- edges in G are not directed (MST for digraphs is harder)

Kruskal's Algorithm

One approach to computing MST for graph G with V nodes:

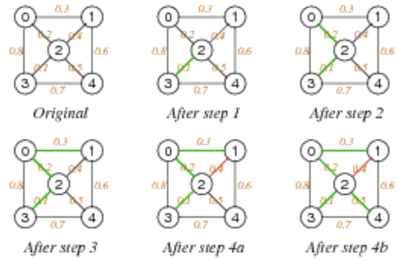
1. start with empty MST
2. consider edges in increasing weight order
   - add edge if it does not form a cycle in MST
3. repeat until *V-1* edges are added

Critical operations:

- iterating over edges in weight order
- checking for cycles in a graph

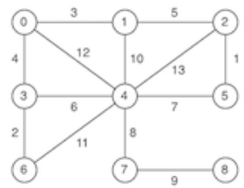
... Kruskal's Algorithm

Execution trace of Kruskal's algorithm:

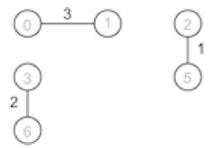


Exercise #5: Kruskal's Algorithm

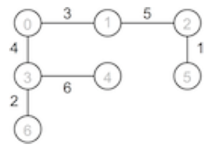
Show how Kruskal's algorithm produces an MST on:



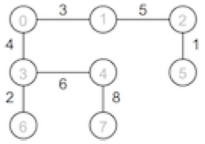
After 3rd iteration:



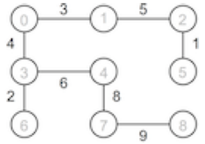
After 6th iteration:



After 7th iteration:



After 8^{th} iteration (V-1=8 edges added):



... Kruskal's Algorithm

Pseudocode:

```
KruskalMST(G):
|  Input  graph G with n nodes
|  Output a minimum spanning tree of G
|
|  MST=empty graph
|  sort edges(G) by weight
|  for each e∈sortedEdgeList do
|  |  MST = MST ∪ {e}
|  |  if MST has a cyle then
|  |     MST = MST \ {e}
|  |  end if
|  |  if MST has n-1 edges then
|  |     return MST
|  |  end if
|  end for
```


... Kruskal's Algorithm

Rough time complexity analysis …

- sorting edge list is $O(E \cdot log\ E)$
- at least V iterations over sorted edges
- on each iteration …
  - getting next lowest cost edge is $O(1)$
  - checking whether adding it forms a cycle: cost = ??

Possibilities for cycle checking:

- use DFS … too expensive?
- could use *Union-Find data structure* (see Sedgewick Ch.1)

Prim's Algorithm

Another approach to computing MST for graph $G=(V,E)$:

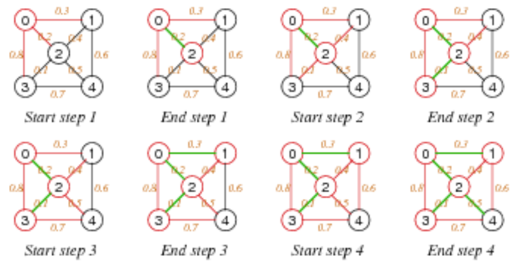
1. start from any vertex v and empty MST
2. choose edge not already in MST to add to MST
   - must be incident on a vertex s already connected to v in MST
   - must be incident on a vertex t not already connected to v in MST
   - must have minimal weight of all such edges
3. repeat until MST covers all vertices

Critical operations:

- checking for vertex being connected in a graph
- finding min weight edge in a set of edges

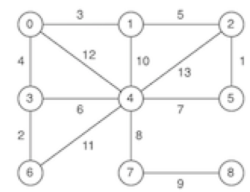
... Prim's Algorithm

Execution trace of Prim's algorithm (starting at s=0):



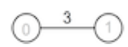
Exercise #6: Prim's Algorithm

Show how Prim's algorithm produces an MST on:

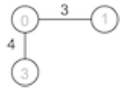


Start from vertex 0

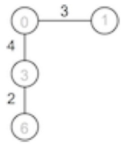
After 1^{st} iteration:



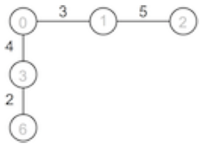
After 2^{nd} iteration:



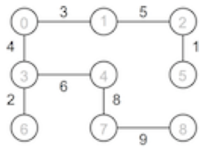
After 3rd iteration:



After 4th iteration:



After 8th iteration (all vertices covered):



... Prim's Algorithm

Pseudocode:

```
PrimMST(G):
|  Input  graph G with n nodes
|  Output a minimum spanning tree of G
|
|  MST=empty graph
|  usedV={0}
|  unusedE=edges(g)
|  while |usedV|<n do
|  |  find e=(s,t,w)∈unusedE such that {
|  |     s∈usedV, t∉usedV and w is min weight of all such edges
|  |  }
|  |  MST = MST ∪ {e}
|  |  usedV = usedV ∪ {t}
|  |  unusedE = unusedE \ {e}
|  end while
|  return MST
```

Critical operation: finding best edge

... Prim's Algorithm

Rough time complexity analysis …

- V iterations of outer loop
- in each iteration …
  - find min edge with set of edges is $O(E) \Rightarrow O(V{\cdot}E)$ overall
  - find min edge with *priority queue* is $O(\log E) \Rightarrow O(V{\cdot}\log E)$ overall

---

Sidetrack: Priority Queues

Some applications of queues require

- items processed in order of "priority"
- rather than in order of entry (FIFO — first in, first out)

Priority Queues (*PQueues*) provide this via:

- `join`: insert item into PQueue with an associated priority (replacing `enqueue`)
- `leave`: remove item with highest priority (replacing `dequeue`)

Time complexity for naive implementation of a PQueue containing N items …

- $O(1)$ for `join` $O(N)$ for `leave`

Most efficient implementation ("heap") …

- $O(\log N)$ for `join`, `leave`

Other MST Algorithms

Boruvka's algorithm … complexity $O(E{\cdot}\log V)$

- the oldest MST algorithm
- start with V separate components
- join components using min cost links
- continue until only a single component

Karger, Klein, and Tarjan … complexity $O(E)$

- based on Boruvka, but non-deterministic
- randomly selects subset of edges to consider
- for the keen, here's the paper describing the algorithm

Shortest Path

Shortest Path

Path = sequence of edges in graph G $p = (v_0,v_1), (v_1,v_2), \ldots, (v_{m-1},v_m)$

cost(path) = sum of edge weights along path

Shortest path between vertices *s* and *t*

- a simple path *p(s,t)* where *s = first(p)*, *t = last(p)*
- no other simple path *q(s,t)* has *cost(q) < cost(p)*

Assumptions: weighted digraph, no negative weights.

Finding shortest path between two given nodes known as *source-target* SP problem

Variations: *single-source* SP, *all-pairs* SP

Applications: navigation, routing in data networks, …

Single-source Shortest Path (SSSP)

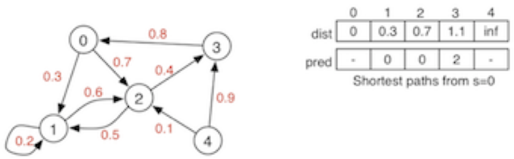
53/86

Given: weighted digraph *G*, source vertex *s*

Result: shortest paths from *s* to all other vertices

- `dist[]` *V*-indexed array of cost of shortest path from *s*
- `pred[]` *V*-indexed array of predecessor in shortest path from *s*

Example:



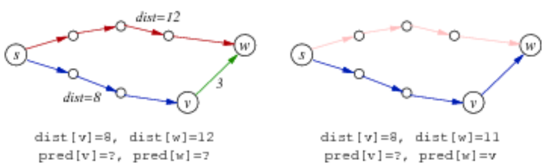
Edge Relaxation

54/86

Assume: `dist[]` and `pred[]` as above   (but containing data for shortest paths *discovered so far*)

`dist[v]` is length of shortest known path from *s* to *v*
`dist[w]` is length of shortest known path from *s* to *w*

Relaxation updates data for *w* if we find a shorter path from *s* to *w*:



Relaxation along edge *e*=`(v,w,weight)`:

- if **`dist[v]+weight < dist[w]`** then
  update `dist[w]:=dist[v]+weight` and `pred[w]:=v`

Dijkstra's Algorithm

55/86

One approach to solving single-source shortest path problem …

Data: *G*, *s*, `dist[]`, `pred[]` and

- *vSet*: set of vertices whose shortest path from *s* is unknown

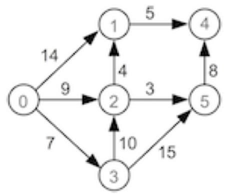
Algorithm:

```
dist[]  // array of cost of shortest path from s
pred[]  // array of predecessor in shortest path from s

dijkstraSSSP(G,source):
|  Input graph G, source node
|
|  initialise dist[] to all ∞, except dist[source]=0
|  initialise pred[] to all -1
|  vSet=all vertices of G
|  while vSet≠∅ do
|  |  find s∈vSet with minimum dist[s]
|  |  for each (s,t,w)∈edges(G) do
|  |     relax along (s,t,w)
|  |  end for
|  |  vSet=vSet\{s}
|  end while
```

Exercise #7: Dijkstra's Algorithm

56/86

Show how Dijkstra's algorithm runs on (source node = 0):



---

	[0]	[1]	[2]	[3]	[4]	[5]
dist	0	∞	∞	∞	∞	∞
pred	–	–	–	–	–	–

dist	0	14	9	7	∞	∞
pred	–	0	0	0	–	–

dist	0	14	9	7	∞	22
pred	–	0	0	0	–	3

dist	0	13	9	7	∞	12
pred	–	2	0	0	–	2

dist	0	13	9	7	20	12
pred	–	2	0	0	5	2

dist	0	13	9	7	18	12
pred	–	2	0	0	1	2

... Dijkstra's Algorithm

Why Dijkstra's algorithm is correct:

Hypothesis.
(a) For visited *s* … *dist[s]* is shortest distance from source
(b) For unvisited *t* … *dist[t]* is shortest distance from source *via visited nodes*

Proof.
Base case: no visited nodes, *dist[source]=0*, *dist[s]=∞* for all other nodes

Induction step:

1. If *s* is unvisited node with minimum *dist[s]*, then *dist[s]* is shortest distance from source to *s*:
   - if ∃ shorter path via only visited nodes, then *dist[s]* would have been updated when processing the predecessor of *s* on this path
   - if ∃ shorter path via an unvisited node *u*, then *dist[u]<dist[s]*, which is impossible if *s* has min distance of all unvisited nodes
2. This implies that (a) holds for *s* after processing *s*
3. (b) still holds for all unvisited nodes *t* after processing *s*:
   - if ∃ shorter path via *s* we would have just updated *dist[t]*
   - if ∃ shorter path without *s* we would have found it previously

... Dijkstra's Algorithm

Time complexity analysis …

Each edge needs to be considered once ⇒ $O(E)$.

Outer loop has $O(V)$ iterations.

Implementing "**find** `s∈vSet` **with** `minimum dist[s]`"

1. try all `s∈vSet` ⇒ cost = $O(V)$ ⇒ overall cost = $O(E + V^2) = O(V^2)$
2. using a PQueue to implement extracting minimum
   - can improve overall cost to $O(E + V \cdot log\ V)$ (for best-known implementation)

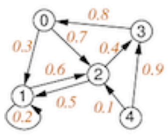
All-pair Shortest Path (APSP)

Given: weighted digraph *G*

Result: shortest paths between all pairs of vertices

- `dist[][]` $V \times V$-indexed matrix of cost of shortest path from v_{row} to v_{col}
- `path[][]` $V \times V$-indexed matrix of next node in shortest path from v_{row} to v_{col}

Example:



Weighted Digraph

dist

V	0	1	2	3	4
0	0	0.3	0.7	1.1	inf
1	1.8	0	0.6	1.0	inf
2	1.2	0.5	0	0.4	inf
3	0.8	1.1	1.5	0	inf
4	1.3	0.6	0.1	0.5	0

path

V	0	1	2	3	4
0	–	1	2	2	–
1	2	–	2	2	–
2	3	1	–	3	–
3	0	0	0	–	–
4	2	2	2	2	–

Shortest paths between all vertices

Floyd's Algorithm

One approach to solving all-pair shortest path problem…

Data: *G*, `dist[][]`, `path[][]` Algorithm:

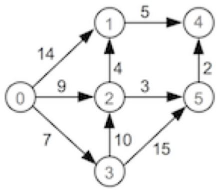
```
dist[][]  // array of cost of shortest path from s to t
path[][]  // array of next node after s on shortest path from s to t

floydAPSP(G):
|  Input graph G
|
|  initialise dist[s][t]=0 for each s=t
|                       =w for each (s,t,w)∈edges(G)
|                       =∞ otherwise
|  initialise path[s][t]=t for each (s,t,w)∈edges(G)
|                       =-1 otherwise
|  for all i∈vertices(G) do
|  |  for all s∈vertices(G) do
|  |  |  for all t∈vertices(G) do
|  |  |  |  if dist[s][i]+dist[i][t] < dist[s][t] then
```

```
|    |    |    |       dist[s][t]=dist[s][i]+dist[i][t]
|    |    |    |       path[s][t]=path[s][i]
|    |    |    |    end if
|    |    |    end for
|    |    end for
|    end for
```

Exercise #8: Floyd's Algorithm

62/86

Show how Floyd's algorithm runs on:



After 1st iteration `i=0`: unchanged

After 2nd iteration `i=1`:

dist	[0]	[1]	[2]	[3]	[4]	[5]	path	[0]	[1]	[2]	[3]	[4]	[5]
[0]	0	14	9	7	19	∞	[0]	–	1	2	3	1	–
[1]	∞	0	∞	∞	5	∞	[1]	–	–	–	–	4	–
[2]	∞	4	0	∞	9	3	[2]	–	1	–	–	1	5
[3]	∞	∞	10	0	∞	15	[3]	–	–	2	–	–	5
[4]	∞	∞	∞	∞	0	∞	[4]	–	–	–	–	–	–
[5]	∞	∞	∞	∞	2	0	[5]	–	–	–	–	4	–

After 3rd iteration `i=2`:

dist	[0]	[1]	[2]	[3]	[4]	[5]	path	[0]	[1]	[2]	[3]	[4]	[5]
[0]	0	13	9	7	18	12	[0]	–	2	2	3	2	2
[1]	∞	0	∞	∞	5	∞	[1]	–	–	–	–	4	–
[2]	∞	4	0	∞	9	3	[2]	–	1	–	–	1	5
[3]	∞	14	10	0	19	13	[3]	–	2	2	–	2	2
[4]	∞	∞	∞	∞	0	∞	[4]	–	–	–	–	–	–
[5]	∞	∞	∞	∞	2	0	[5]	–	–	–	–	4	–

After 4th iteration `i=3`: unchanged

After 5th iteration `i=4`: unchanged

After 6th iteration `i=5`:

dist	[0]	[1]	[2]	[3]	[4]	[5]	path	[0]	[1]	[2]	[3]	[4]	[5]
[0]	0	13	9	7	14	12	[0]	–	2	2	3	2	2
[1]	∞	0	∞	∞	5	∞	[1]	–	–	–	–	4	–
[2]	∞	4	0	∞	5	3	[2]	–	1	–	–	5	5
[3]	∞	14	10	0	15	13	[3]	–	2	2	–	2	2
[4]	∞	∞	∞	∞	0	∞	[4]	–	–	–	–	–	–
[5]	∞	∞	∞	∞	2	0	[5]	–	–	–	–	4	–

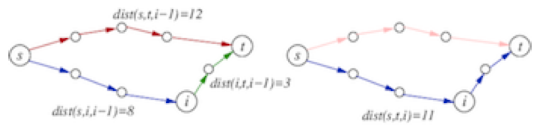
... Floyd's Algorithm

64/86

Why Floyd's algorithm is correct:

A shortest path from *s* to *t* using only nodes from $\{0,\ldots,i\}$ is the shorter of

- a shortest path from *s* to *t* using only nodes from $\{0,\ldots,i-1\}$
- a shortest path from *s* to *i* using only nodes from $\{0,\ldots,i-1\}$
  plus a shortest path from *i* to *t* using only nodes from $\{0,\ldots,i-1\}$



Also known as Floyd-Warshall algorithm   (can you see why?)

... Floyd's Algorithm

65/86

Cost analysis …

- initialising `dist[][],path[][]` $\Rightarrow O(E)$
- *V* iterations to update `dist[][],path[][]` $\Rightarrow O(V^3)$

Time complexity of Floyd's algorithm: $O(V^3)$ (same as Warshall's algorithm for transitive closure)

Network Flow

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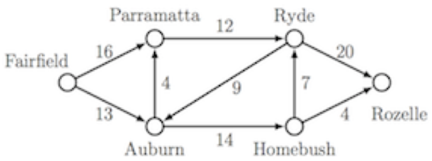
Exercise #9: Merchandise Distribution

Lucky Cricket Company …

- produces cricket balls in Fairfield
- has a warehouse in Rozelle that stocks them
- ships them from factory to warehouse by leasing space on trucks with limited capacity:



What kind of algorithm would …

- help us find the maximum number of crates that can be shipped from Fairfield to Rozelle per day?

Flow Networks

68/86

Flow network …

- weighetd graph $G=(V,E)$
- distinct nodes $s \in V$ (*source*), $t \in V$ (*sink*)

Edge weights denote *capacities*

Applications:

- Distribution networks, e.g.
  - source: oil field
  - sink: refinery
  - edges: pipes
- Traffic flow

... Flow Networks

69/86

Flow in a network $G=(V,E)$ … nonnegative $f(v,w)$ for all vertices $v,w \in V$ such that

- $f(v,w) \leq capacity$ for each edge $e=(v,w,capacity) \in E$
- $f(v,w)=0$ if no edge between v and w
- total flow *into* a vertex = total flow *out of* a vertex:

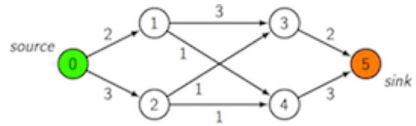
$$\sum_{x \in V} f(x, v) = \sum_{y \in V} f(v, y) \quad \text{for all } v \in V \setminus \{s,t\}$$

Maximum flow … no other flow from s to t has larger value

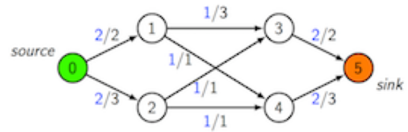
... Flow Networks

70/86

Example:



A (maximum) flow …



Augmenting Paths

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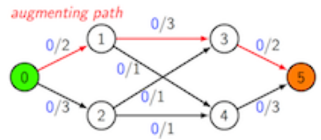
Assume … $f(v,w)$ contains current flow

Augmenting path: any path from source s to sink t that can currently take more flow

Example:



Residual Network

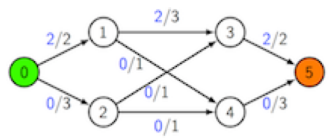
72/86

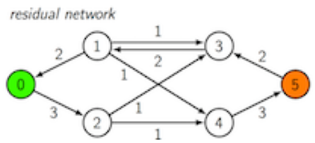
Assume … flow network $G=(V,E)$ and flow $f(v,w)$

Residual network (V,E'):

- same vertex set V
- for each edge $v \xrightarrow{c} w \in E$ …
  - $f(v,w) < c$ ⇒ add edge $(v \xrightarrow{c-f(v,w)} w)$ to E'
  - $f(v,w) > 0$ ⇒ add edge $(v \xleftarrow{f(v,w)} w)$ to E'

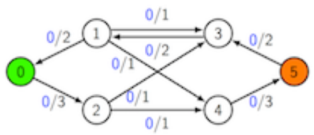
Example:





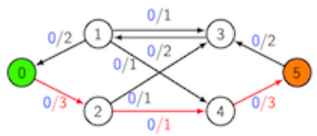
Exercise #10: Augmenting Paths and Residual Networks

Find an augmenting path in:



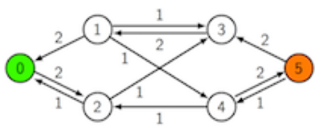
and show the residual network after augmenting the flow

1. Augmenting path:



   maximum additional flow = 1

2. Residual network:



Can you find a further augmenting path in the new residual network?

Edmonds-Karp Algorithm

One approach to solving maximum flow problem …

maxflow(G):

1. Find a shortest augmenting path
2. Update `flow[][]` so as to represent residual graph
3. Repeat until no augmenting path can be found

... Edmonds-Karp Algorithm

Algorithm:

```
flow[][]  // V×V array of current flow
visited[] /* array of predecessor nodes on shortest path
             from source to sink in residual network */

maxflow(G):
|  Input  flow network G with source s and sink t
|  Output maximum flow value
|
|  initialise flow[v][w]=0 for all vertices v, w
|  maxflow=0
|  while ∃shortest augmenting path visited[] from s to t do
|  |  df = maximum additional flow via visited[]
|  |  // adjust flow so as to represent residual graph
|  |  v=t
|  |  while v≠s do
|  |  |  flow[visited[v]][v] = flow[visited[v]][v] + df;
|  |  |  flow[v][visited[v]] = flow[v][visited[v]] - df;
|  |  |  v=visited[v]
|  |  end while
|  |  maxflow=maxflow+df
|  end while
|  return maxflow
```

Shortest augmenting path can be found by standard BFS

... Edmonds-Karp Algorithm

Time complexity analysis …

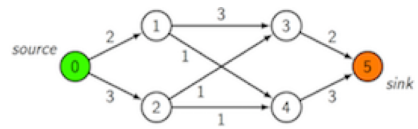
- *Theorem*. The number of augmenting paths needed is at most $V·E/2$.
  ⇒ Outer loop has $O(V·E)$ iterations.
- Finding augmenting path ⇒ $O(E)$ (consider only vertices connected to source and sink ⇒ $O(V+E)=O(E)$)

Overall cost of Edmonds-Karp algorithm: $O(V·E^2)$

Note: Edmonds-Karp algorithm is an implementation of general *Ford-Fulkerson method*

Exercise #11: Edmonds-Karp Algorithm

Show how Edmonds-Karp algorithm runs on:



flow	[0]	[1]	[2]	[3]	[4]	[5]	c-f	[0]	[1]	[2]	[3]	[4]	[5]
[0]	0	0	0	0	0	0	[0]	–	2	3	–	–	–
[1]	0	0	0	0	0	0	[1]	–	–	–	3	1	–
[2]	0	0	0	0	0	0	[2]	–	–	–	1	1	–
[3]	0	0	0	0	0	0	[3]	–	–	–	–	–	2
[4]	0	0	0	0	0	0	[4]	–	–	–	–	–	3
[5]	0	0	0	0	0	0	[5]	–	–	–	–	–	–

augmenting path: 0-1-3-5, `df`: 2

flow	[0]	[1]	[2]	[3]	[4]	[5]	c-f	[0]	[1]	[2]	[3]	[4]	[5]
[0]	0	2	0	0	0	0	[0]	–	0	3	–	–	–
[1]	-2	0	0	2	0	0	[1]	2	–	–	1	1	–
[2]	0	0	0	0	0	0	[2]	–	–	–	1	1	–
[3]	0	-2	0	0	0	2	[3]	–	2	–	–	–	0
[4]	0	0	0	0	0	0	[4]	–	–	–	–	–	3
[5]	0	0	0	-2	0	0	[5]	–	–	–	2	–	–

augmenting path: 0-2-4-5, `df`: 1

flow	[0]	[1]	[2]	[3]	[4]	[5]	c-f	[0]	[1]	[2]	[3]	[4]	[5]
[0]	0	2	1	0	0	0	[0]	–	0	2	–	–	–
[1]	-2	0	0	2	0	0	[1]	2	–	–	1	1	–
[2]	-1	0	0	0	1	0	[2]	1	–	–	1	0	–
[3]	0	-2	0	0	0	2	[3]	–	2	–	–	–	0
[4]	0	0	-1	0	0	1	[4]	–	–	1	–	–	2
[5]	0	0	0	-2	-1	0	[5]	–	–	–	2	1	–

augmenting path: 0-2-3-1-4-5, `df`: 1

flow	[0]	[1]	[2]	[3]	[4]	[5]	c-f	[0]	[1]	[2]	[3]	[4]	[5]
[0]	0	2	2	0	0	0	[0]	–	0	1	–	–	–
[1]	-2	0	0	1	1	0	[1]	2	–	–	2	0	–
[2]	-2	0	0	1	1	0	[2]	2	–	–	0	0	–
[3]	0	-1	-1	0	0	2	[3]	–	1	1	–	–	0
[4]	0	-1	-1	0	0	2	[4]	–	1	1	–	–	1
[5]	0	0	0	-2	-2	0	[5]	–	–	–	2	2	–

Digraph Applications

PageRank

Goal: determine which Web pages are "important"

Approach: ignore page contents; focus on hyperlinks

- treat Web as graph: page = vertex, hyperlink = directed edge
- pages with many incoming hyperlinks are important
- need to compute "incoming degree" for vertices

Problem: the Web is a *very* large graph

- approx. 10^{14} pages, 10^{15} hyperlinks

Assume for the moment that we could build a graph …

Most frequent operation in algorithm "Does edge (v,w) exist?"

... PageRank

Simple PageRank algorithm:

```
PageRank(myPage):
|  rank=0
|  for each page in the Web do
|  |  if linkExists(page,myPage) then
|  |     rank=rank+1
|  |  end if
|  end for
```

Note: requires *inbound* link check

... PageRank

V = # pages in Web, E = # hyperlinks in Web

Costs for computing PageRank for each representation:

Representation	linkExists(v,w)	Cost
Adjacency *matrix*	`edge[v][w]`	*1*

Adjacency *lists* `inLL(list[v],w)` $\cong E/V$

Not feasible …

- adjacency matrix … $V \cong 10^{14} \Rightarrow$ matrix has 10^{28} cells
- adjacency list … V lists, each with $\cong$10 hyperlinks $\Rightarrow 10^{15}$ list nodes

So how to really do it?

... PageRank

Approach: the random web surfer

- if we randomly follow links in the web …
- … more likely to re-discover pages with many inbound links

```
curr=random page, prev=null
for a long time do
|  if curr not in array ranked[] then
|     rank[curr]=0
|  end if
|  rank[curr]=rank[curr]+1
|  if random(0,100)<85 then               // with 85% chance ...
|     prev=curr
|     curr=choose hyperlink from curr  // ... crawl on
|  else
|     curr=random page                  // avoid getting stuck
|     prev=null
|  end if
end for
```

Could be accomplished while we crawl web to build search index

---

Exercise #12: Implementing Facebook

Facebook could be considered as a giant "social graph"

- what are the vertices?
- what are the edges?
- are edges directional?

What kind of algorithm would …

- help us find people that you might like to "befriend"?

Summary

- Digraphs, weighted graphs: representations, applications
- Reachability
  - Warshall
- Minimum Spanning Tree (MST)
  - Kruskal, Prim
- Shortest path problems
  - Dijkstra (single source SPP)
  - Floyd (all-pair SSP)
- Flow networks
  - Edmonds-Karp (maximum flow)

- Suggested reading (Sedgewick):
  - digraphs … Ch. 19.1-19.3
  - weighted graphs … Ch. 20-20.1
  - MST … Ch. 20.2-20.4
  - SSP … Ch. 21-21.3
  - network flows … Ch. 22.1-22.2

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