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**MATH.APP.460-2021-2022-1 Numerical Analysis (English).**
**Sampsa Pursiainen** (Juho Kanniainen, Jere Mäkinen). **Exercise session 5, Wed 20.4.2022**. Calculate (or try to calculate) the tasks below before the session. Prepare to present orally in the session the solution (or solution trial) of each task marked as done. Alternatively the solutions can be handed in as a single PDF document via a submit link on the course's moodle page. The solutions can be computer assisted and include scripts.

1. The three-component gravity force field $\vec{f}$ of the Earth–Moon system is approximately of the form

$$\vec{f} = -c_1 \frac{x - \vec{p}}{\|x - \vec{p}\|^3} - c_2 \frac{x - \vec{q}}{\|x - \vec{q}\|^3},$$

where the location of the Earth is $\vec{p} = (0, 0, 5)$ and Moon is $\vec{q} = (0, 0, 50)$ and the scaling constants are $c_1 = 100$ and $c_2 = 0.05$. Calculate the trajectory of a spacecraft with mass $m = 1$ moving freely in the gravity field according to $\vec{f} = m\vec{a}$, where $\vec{a}$ is the acceleration, and leaving from the surface of the Earth (radius $r_1 = 2$) with velocity $9 \leq v \leq 11$. Use the numerical ODY solver of your choice.

   (a) Can you make the spacecraft to land on the Moon or to orbit it?

   (b) If you add a breaking and/or accelerating force acting along the trajectory for a given period time, can the spacecraft be made to orbit the Moon a few times and then return back on the surface of the Earth?

   (c) How does the time-step of the solver affect the accuracy of the solution and is it critical for the success of the computation?

2. Prove that the function $u(x,t) = e^{-\pi t} \sin \pi x$, is a solution of the heat equation $\pi u_t = u_{xx}$ with the specified initial boundary conditions:

$$\begin{cases} u(x, 0) = \sin \pi x \text{ for } 0 \leq x \leq 1 \\ u(0, t) = 0 \text{ for } 0 \leq t \leq 1 \\ u(1, t) = 0 \text{ for } 0 \leq t \leq 1 \end{cases}$$

3. Solve the equation $u_t = 2u_{xx}$ for $0 \leq x \leq 1, 0 \leq t \leq 1$, with the initial and boundary conditions that follow, using the Forward Difference Method with step sizes $h = 0.1$ and $k = 0.002$. Plot the approximate solution. Compare with the exact solution from the Exercise 2. What happens if $k > 0.003$ is used?

$$\begin{cases} u(x, 0) = 2 \cosh x \text{ for } 0 \leq x \leq 1 \\ u(0, t) = 2e^{2t} \text{ for } 0 \leq t \leq 1 \\ u(1, t) = (e^2 + 1)e^{2t-1} \text{ for } 0 \leq t \leq 1 \end{cases}$$

4. Consider equation $\pi u_t = u_{xx}$ for $0 \le x \le 1$, $0 \le t \le 1$ with the initial and boundary conditions that follow. Set step size $h = 0.1$. For what step sizes k is the Forward Difference Method stable? Apply the Forward Difference Method with step sizes $h = 0.1$, $k = 0.01$ and compare with the exact solution from Exercise **??**.

$$\begin{cases} u(x,0) = \sin \pi x \text{ for } 0 \le x \le 1 \\ u(0,t) = 0 \text{ for } 0 \le t \le 1 \\ u(1,t) = 0 \text{ for } 0 \le t \le 1 \end{cases}$$

5. Find the best rank-one approximation of the matrix $\begin{bmatrix} 0 & 1 \\ 1 & \frac{3}{2} \end{bmatrix}$.