MATH.APP.460-2021-2022-1 Numerical Analysis (English).
Sampsa Pursiainen (Juho Kanniainen, Jere Mäkinen). **Final exam 1, Fri 29.4.2022**. This remote examination is the final exam of the course MATH.APP.460-2021-2022-1 Numerical Analysis (English). Answer to the tasks below. The answers can include mathematical equations and analysis but also computer scripts or pseudo code. Hand in your answers as a single PDF file via the upload link provided on the moodle page by Friday, April 29th, 2022, 23:55.

1. Provide a brief explanation for the following concepts of numerical analysis, and answer to the questions with a mathematical justification.
   (a) *Runge phenomenon.* How to reduce it?
   (b) *Runge-Kutta pair.* How to adapt the size of the time step with such a pair?
   (c) *Stability of a finite difference method solution for the heat equation.* What is a necessary condition for the spatial and temporal step-sizes?
2. (a) Write a Matlab code (or a pseudocode) to solve a linear system $Ax = b$ with respect to x using the Gaussian elimination method with partial pivoting.
   (b) Solve the system by finding the LU factorization and then carrying out the two-step back substitution.
$$\begin{bmatrix} 3 & 7 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \end{bmatrix}$$
3. (a) What is the difference between Jacobi and Gauss-Seidel method?
   (b) Does Jacobi iteration converge, when applied to solve the system $Ax = b$ with respect to x, when
$$A = \begin{bmatrix} 5 & 1 & -1 \\ 3 & 6 & 2 \\ 2 & -6 & 9 \end{bmatrix}$$
   and $b = [1, 1, 1]^T$?
   (c) Does Gauss-Seidel method converge when applied to this system?
4. Prove beginning from Taylor's approximation $f(x) = f(w) + f'(w)(x - w) + f''(c_x)(x - w)^2$ of $\int_a^b f(x)dx$ that the composite Midpoint Rule applied to integrate the real-valued integrable function $f(x)$ satisfies the formula
$$\int_a^b f(x)dx = h \sum_{i=1}^{m} f(w_i) + \frac{(b-a)h^2}{24} f''(c).$$
   in any closed and bounded interval of the real line for a given subinterval length h.

5. (a) Use two-point forward-difference formula to approximate $f'(1)$, and find the approximation error, where $f(x) = \ln x$, for $h = 0.1$.

   (b) Use three-point centered-difference formula to approximate $f'(0)$ where $f(x) = e^x$, for $h = 0.1$.

6. Assume that $f(t, y)$ has a Lipschitz constant $L = 1$ for the variable y and that the value y_i of the solution of the initial value problem

$$\begin{cases} y' = f(t, y) \\ y(0) = y_0 \\ t \text{ in } [0, T] \end{cases}$$

at t_i is approximated by w_i from a one-step ODE solver with local truncation error $e_i \leq Ch^{k+1}$, for $C = 1$ and $k \geq 0$. Then, for each $0 < t_i < T$, the solver has global truncation error

$$g_i = |w_i - y_i| \leq \frac{Ch^k}{L}\left(e^{Lt_i} - 1\right)$$

. Estimate the global truncation error at $T = 10$ for

   (a) Euler's method with $e_i \leq Ch^2$

   (b) Midpoint method with $e_i \leq Ch^3$,

if at $T = 1$ both methods have the same local truncation error $g_i = 0.1$ and $h = 0.1$.