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MATH.APP.460-2021-2022-1 Numerical Analysis (English).
Sampsa Pursiainen (Juho Kanniainen, Jere Mäkinen). **Exercise session 4, Wed 6.4.2022**. Calculate (or try to calculate) the tasks below before the session. Prepare to present orally in the session the solution (or solution trial) of each task marked as done. Alternatively the solutions can be handed in as a single PDF document via a submit link on the course's moodle page. The solutions can be computer assisted and include scripts.

1. Convert the higher-order ordinary differential equation to a first-order system of equations.
   (a) $y'' - ty = 0$ (Airy's equation)
   (b) $y'' - 2ty' + 2y = 0$ (Hermite's equation)
   (c) $y'' - ty' - y = 0$

2. (a) Show that $y(t) = (e^t + e^{-t} - t^2)/2 - 1$ is the solution of the initial value problem $y''' - y' = t$, with $y(0) = y'(0) = y''(0) = 0$.
   (b) Convert the differential equation to a system of three first-order equations.
   (c) Use Euler's Method with step size $h = 1/4$ to approximate the solution on $[0, 1]$.

3. Apply Euler's Method with step sizes $h = 0.1$ and $h = 0.01$ to the following initial value problems. Plot the approximate solutions and the correct solution on $[0, 1]$, and find the global truncation error at $t = 1$.
   (a)
$$\begin{cases} y_1' &= y_1 + y_2 \\ y_2' &= -y_1 + y_2 \\ y_1(0) &= 1 \\ y_2(0) &= 0 \end{cases}$$
   Exact solution is $y_1(t) = e^t \cos t$, $y_2(t) = -e^t \sin t$.
   (b)
$$\begin{cases} y_1' &= -y_2 \\ y_2' &= y_1 \\ y_1(0) &= 1 \\ y_2(0) &= 0 \end{cases}$$
   Exact solution is $y_1(t) = \cos t$, $y_2(t) = \sin t$.
   (c)
$$\begin{cases} y_1' &= y_1 + 3y_2 \\ y_2' &= 2y_1 + 2y_2 \\ y_1(0) &= 5 \\ y_2(0) &= 0 \end{cases}$$

Exact solution is $y_1(t) = 3e^{-t} + 2e^{4t}$, $y_2(t) = -2e^{-t} + 2e^{4t}$

4. Write a program that implements RK23, and apply to approximating the solutions of the following initial value problems with a relative tolerance of 10^{-8} on $[0, 1]$. Ask the program to stop exactly at the endpoint $t = 1$. Report the maximum step size used and the number of steps. Plot the results with the exact solution.

   (a) $y' = t + y$

   (b) $y' = t - y$

   (c) $y' = 4t - 2y$

5. The Lorenz equations

$$\frac{dx}{dt} = p(y - x)$$
$$\frac{dy}{dt} = rx - y - xz$$
$$\frac{dz}{dt} = xy - qz$$

   are a well known example of a system with chaotic behavior for certain values of the parameters. The system was studied by Lorenz in connection with the problem of finding the effect of heating a horizontal fluid layer from below. Plot the famous "butterfly-shaped" Lorenz attractor by solving the Lorenz equations using the following parameters and initial value.

$$p = 10, \quad q = 8/3, \quad r = 28, \quad [x\ y\ z] = [1\ 1\ 2]$$