Stats Modelling

$\boxed{W4}$ 3

a) $Y_i \sim Cat(\theta_{i1}, \theta_{i2}, \theta_{i3})$

$$\log\left(\frac{\theta_{i2}}{\theta_{i1}}\right) = x_i' \beta_2$$

$$\cancel{\log} \Rightarrow \theta_{i2} = \theta_{i1} e^{x_i' \beta_2}$$

$$\log\left(\frac{\theta_{i3}}{\theta_{i1}}\right) = x_i' \beta_3$$

$$\Rightarrow \theta_{i3} = \theta_{i1} e^{x_i' \beta_3}$$

$$\theta_{i1} + \theta_{i2} + \theta_{i3} = 1$$

$$\Leftrightarrow \theta_{i1} + \theta_{i1} e^{x_i' \beta_2} + \theta_{i1} e^{x_i' \beta_3} = 1$$

$$\Leftrightarrow \theta_{i1} = \frac{1}{1 + e^{x_i' \beta_2} + e^{x_i' \beta_3}}$$

$$\theta_{i2} = \frac{e^{x_i' \beta_2}}{1 + e^{x_i' \beta_2} + e^{x_i' \beta_3}}$$

$$\theta_{i3} = \frac{e^{x_i' \beta_3}}{1 + e^{x_i' \beta_2} + e^{x_i' \beta_3}}$$

b) rv Y_i

$$\cancel{\log(B} \quad \text{logit}(P(Y_i \leq k)) = B_{0k} + B_1 x_{i1} \qquad k = 1,2.$$

$$\Rightarrow \quad P(Y_i \leq k) = \frac{e^{B_{0k} + B_1 x_{i1}}}{1 + e^{B_{0k} + B_1 x_{i1}}}$$

$$P(Y_i = 1) = P(Y_i \leq 1) = \frac{e^{B_{01} + B_1 x_{i1}}}{1 + e^{B_{01} + B_1 x_{i1}}}$$

$$P(Y_i \leq 2) = \frac{e^{B_{02} + B_1 x_{i1}}}{1 + e^{B_{02} + B_1 x_{i1}}}$$

$$\Rightarrow \quad P(Y_i = 2) = P(Y_i \leq 2) - P(Y_i \leq 1)$$

$$= \frac{e^{B_{02} + B_1 x_{i1}}}{1 + e^{B_{02} + B_1 x_{i1}}} - \frac{e^{B_{01} + B_1 x_{i1}}}{1 + e^{B_{01} + B_1 x_{i1}}}$$

$$P(Y_i = 3) = 1 - P(Y_i = 1) - P(Y_i = 2)$$

$$= 1 - \frac{e^{B_{01} + B_1 x_{i1}}}{1 + e^{B_{01} + B_1 x_{i1}}} - \frac{e^{B_{02} + B_1 x_{i1}}}{1 + e^{B_{02} + B_1 x_{i1}}} + \frac{e^{B_{01} + B_1 x_{i1}}}{1 + e^{B_{01} + B_1 x_{i1}}}$$

$$= 1 - \frac{e^{B_{02} + B_1 x_{i1}}}{1 + e^{B_{02} + B_1 x_{i1}}}$$

c) $$\frac{\partial l(\beta,\phi)}{\partial \beta_j} = \sum_{i=1}^{n} \frac{y_i - \mu_i}{Var(Y_i)} x_{ij} \left(\frac{\partial \mu_i}{\partial \eta_i}\right) = 0$$

$Y_i \sim Ber(\mu_i)$

$\Rightarrow Var(Y_i) = \mu_i(1-\mu_i)$

$logit(\mu_i) = \eta_i = \beta_0 \Rightarrow \mu_i = \dfrac{e^{\beta_0}}{1+e^{\beta_0}}$

$$\Rightarrow \left(\frac{\partial \mu_i}{\partial \eta_i}\right) = \mu_i(1-\mu_i)$$

$$\Rightarrow \frac{\partial l(\beta,\phi)}{\partial \beta_j} = \sum_{i=1}^{n} \frac{y_i - \mu_i}{\mu_i(1-\mu_i)} x_{ij}\, \mu_i(1-\mu_i) =$$

$$= \sum_{i=1}^{n} (y_i - \mu_i)\, \underbrace{x_{ij}}_{1}$$

$$= \sum_{i=1}^{n} (y_i - \mu_i)$$

$$\Rightarrow \mu_i = \bar{y}$$

$$\Rightarrow \frac{e^{\beta_0}}{1+e^{\beta_0}} = \bar{y}$$

$$\Rightarrow \beta_0 = logit(\bar{y})$$

$$\left(\text{With } \bar{y} = \frac{1}{n}\sum y_i\right)$$