Stat

W1

Exercise 3.

(a) $m_{112} = Y_i = \beta_0 + \beta_j + a_l + \varepsilon_i \qquad \varepsilon_i \sim N(0, \sigma^2)$

$$\beta = \begin{bmatrix} \beta_0 \cancel{1} \\ \beta_2 \\ \beta_3 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \qquad X = \begin{array}{c} a\,a \\ a\,b \\ a\,c \\ b\,a \\ b\,b \\ a\,c \\ \\ \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

(b) M: $y \sim N(X\beta, \sigma^2 I)$

$$E(X\hat{\beta}) = E(X(X'X)^{-1}X'Y)$$
$$= X(X'X)^{-1}X'E(Y)$$
$$= X(X'X)^{-1}X'X\beta$$
$$= XI\beta$$
$$= X\beta$$

$$\mathrm{cov}(X\hat{\beta}) = \mathrm{cov}(X(X'X)^{-1}X'Y)$$
$$= X(X'X)^{-1}X'\,\mathrm{cov}(Y)\,X(X'X)^{-1}X'$$
$$= X(X'X)^{-1}X'\,\sigma^2 I\,X(X'X)^{-1}X'$$
$$= \sigma^2 X(X'X)^{-1}X'X(X'X)^{-1}X'$$
$$= \sigma^2 X(X'X)^{-1}X'$$

(c)

$$Y_f \sim N(x_f'\beta, \underbrace{\sigma^2}_{\sigma_1^2})$$

$$\hat{Y}_f = \hat{\mu}_f = x_f'\hat{\beta} \sim N(x_f'\beta, \underbrace{\sigma^2 x_f'(X'X)^{-1}x_f}_{\sigma_2^2})$$

$$(\hat{\beta} \sim N(\beta, \sigma^2(X'X)^{-1}))$$

$$\Rightarrow Y_f - \hat{Y}_f \sim N(x_f'\beta - x_f'\beta, \sigma_1^2 + \sigma_2^2)$$

(Since y and Y_f are independent $\Rightarrow$ Y_f and $\hat{Y}_f$ are independent).

$$\Leftrightarrow Y_f - \hat{Y}_f \sim N(0, \sigma^2 + \sigma^2 x_f'(X'X)^{-1}x_f)$$

$$\Leftrightarrow Y_f - \hat{Y}_f \sim N(0, \sigma^2(1 + x_f'(X'X)^{-1}x_f))$$

$$\Rightarrow Var(e_f) = \sigma^2(1 + x_f'(X'X)^{-1}x_f)$$