Statistical modelling

W2

P3

a)

i) $\sqrt{\mu_i} = \beta_0 + \beta_1 x$

$\Rightarrow \mu_i = (\beta_0 + \beta_1 x)^2$

$\Rightarrow g(\mu_i) = \sqrt{\mu_i}$

$\Rightarrow g^{-1}(x) = x^2$

ii) $\frac{1}{\mu_i^2} = \beta_0 + \beta_1 x_i$

$\Rightarrow \mu_i = \frac{1}{\sqrt{\beta_0 + \beta_1 x_i}}$

$g(\mu_i) = \frac{1}{\mu_i^2}$

$\Rightarrow g^{-1}(x) = \frac{1}{\sqrt{x}}$

(iii) $\log\left(\frac{\mu_i}{1-\mu_i}\right) = \beta_0 + \beta_1 x$

$g(\mu_i) = \log\left(\frac{\mu_i}{1-\mu_i}\right)$

$g^{-1}(x) = 1 - \frac{1}{e^{x}+1}$

e) $Y_i \sim IG(\mu, \phi)$

$$\log(\mu_i) = \beta_0 + \beta_1 \log(x_i)$$

(i) β_0, β_1, ϕ $\qquad \hat{\beta}_0 = 1,\ \hat{\beta}_1 = 0.5$

$$\hat{\phi} = 0.05$$

$$\hat{\mu}_i = e^{\hat{\beta}_0 + \hat{\beta}_1 \log(x_i)}$$

$$= e^{\hat{\beta}_0} x_i^{\hat{\beta}_1}$$

$$= e^1 5^{0.5}$$

$$\boxed{\Rightarrow \hat{\mu}_i = 6.0783}$$

(ii) $$O_i = \frac{y_i - \hat{\mu}_i}{\sqrt{var(Y_i)}}$$ $\qquad y_i = 12,\ x_i = 5$

$$= \frac{y_i - \hat{\mu}_i}{\sqrt{\phi \hat{\mu}_i^3}}$$

$$= \frac{12 - e5^{0.5}}{\sqrt{0.05 \times (e5^{0.5})^3}}$$

$$= \cancel{48.8061}$$

$$= 1.7672$$

c) $y \sim N(\mu; \sigma^2 I)$

$\mu = \cancel{\eta} = X\beta$

$$\frac{\partial l(\beta|y)}{\partial \beta} = \mu = X' D V^{-1}(y-\mu) = 0$$

$$D = \begin{pmatrix} \frac{\partial \mu_1}{\partial \eta_1} & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & \frac{\partial \mu_n}{\partial \eta_n} \end{pmatrix}$$

Since $\mu = \eta$

$\Rightarrow D = I_n$

Since $Var(Y) = \sigma^2 I$

$\Rightarrow V = \sigma^2 I \Rightarrow V^{-1} = \frac{1}{\sigma^2} I$

$$\Rightarrow \frac{\partial l(\beta|y)}{\partial \beta} = \mu = X' I \sigma^2 I (y-\mu) = 0$$

$\Leftrightarrow \sigma^2 X'(y-\mu) = 0$

$\Leftrightarrow \boxed{X'(y-\mu) = 0}$

simplified.

$$X'(Y - M) = 0$$

$$\Leftrightarrow X'(Y - X\hat{\beta}) = 0$$

$$\Leftrightarrow X'Y = X'X\hat{\beta}$$

$$\Leftrightarrow \boxed{\hat{\beta} = (X'X)^{-1}X'Y}$$