Coupled enhancer-promoter condensates

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1 Asymmetric NatComms Dimensional

$$F_{DW}(\rho_p, a, b) + \cdots = \int d^2x \left[\rho_p(\phi_p - a)^2(\phi_p - b)^2 - \chi\phi_p\phi_r + c\phi_p^2\phi_r^2 + \rho_r\phi_r^2 + \frac{\kappa}{2}(\nabla\phi_p)^2 \right]$$

With $a < b$, $a = 0.1$, $b = 0.7$, $\phi_{p,+} = 0.63$, $\phi_{p,-} = 0.13$, and $\rho_p = 1$.

2 Symmetric PFStability Dimensional

$$\begin{aligned} F_{DW}(c_0, \alpha, \beta) + \cdots = \int d^2x \Bigg[& \frac{\alpha}{4}(c - c_0)^4 + \frac{\beta}{2}(c - c_0)^2 + \frac{\lambda}{2}m^2 + \gamma mc \\ & + \frac{\kappa}{2}(\nabla c)^2 - \chi_{PD}\exp(-|\vec{r} - \vec{r}_e|^2/2\sigma^2)c \Bigg] + \frac{3k_BT}{2L_CL_P}|\vec{r}_e - \vec{r}_p|^2 \end{aligned}$$

3 Symmetric PFStability Non-Dimensional

$$\tilde{F}_{DW}(\bar{c}_1, \tilde{\beta}) + \cdots = \int d^2x \left[\frac{1}{4}(\tilde{c}_1 - \bar{c}_1)^4 + \frac{\tilde{\beta}}{2}(\tilde{c}_1 - \bar{c}_1)^2 + \frac{\tilde{\lambda}}{2}c_2^2 + \tilde{\gamma}c_1c_2 + \frac{\tilde{\kappa}}{2}|\nabla c_1|^2 - \tilde{c}\exp(-|\vec{r} - \vec{r}_0|^2/2\sigma^2)\tilde{c}_1 \right] + \frac{\tilde{k}}{2}|\vec{r}_e - \vec{r}_p|$$

4 Symmetric Binodal

$$\begin{aligned}
\frac{\partial}{\partial c}\left[\frac{\alpha}{4}(c-c_0)^4+\frac{\beta}{2}(c-c_0)^2\right] &= 0\\
\alpha(c-c_0)^3+\beta(c-c_0) &= 0\\
\alpha(c-c_0)\left(c-c_0+\sqrt{-\beta/\alpha}\right)\left(c-c_0-\sqrt{-\beta/\alpha}\right) &= 0\\
c = c_0 \pm \sqrt{-\beta/\alpha}
\end{aligned}$$

5 Dimensional conversion

$$\begin{aligned}
\rho_p(\phi_p-a)^2(\phi_p-b)^2 &= \rho_p(\phi_p^2-2a\phi_p+a^2)(\phi_p^2-2b\phi_p+b^2)\\
&= \rho_p(\phi_p^4+(-2a-2b)\phi_p^3+(a^2+b^2+4ab)\phi_p^2\\
&+(-2a^2b-2ab^2)\phi_p+a^2b^2)\\
&= \rho_p\phi_p^4+\rho_p(-2a-2b)\phi_p^3+\rho_p(a^2+b^2+4ab)\phi_p^2\\
&+\rho_p(-2a^2b-2ab^2)\phi_p+\rho_p a^2b^2
\end{aligned}$$

$$\begin{aligned}
\frac{\alpha}{4}(c-c_0)^4+\frac{\beta}{2}(c-c_0)^2 &= \frac{\alpha}{4}(c^4-4c_0c^3+6c_0^2c^2-4c_0^3c+c_0^4)+\frac{\beta}{2}(c^2-2c_0c+c_0^2)\\
&= \left(\frac{\alpha}{4}\right)c^4+\left(-\alpha c_0\right)c^3+\left(\frac{3}{2}\alpha c_0^2+\frac{\beta}{2}\right)c^2\\
&+\left(-\alpha c_0^3-\beta c_0\right)c+\left(\frac{\alpha}{4}c_0^4+\frac{\beta}{2}c_0^2\right)
\end{aligned}$$

$$\rho_p = \frac{\alpha}{4} \tag{1}$$

$$\rho_p(-2a-2b) = -\alpha c_0 \tag{2}$$

$$\rho_p(a^2+b^2+4ab) = \frac{3}{2}\alpha c_0^2+\frac{\beta}{2} \tag{3}$$

$$\rho_p(-2a^2b-2ab^2) = (-\alpha c_0^3-\beta c_0) \tag{4}$$

$$\rho_p a^2b^2 = \left(\frac{\alpha}{4}c_0^4+\frac{\beta}{2}c_0^2\right) \tag{5}$$

Substitute Equation 1 to Equation 2.

$$\frac{\alpha}{4}(-2a-2b) = -\alpha c_0$$

$$c_0 = \frac{a+b}{2} \tag{6}$$

Substitute Equation 6 to Equation 4.

$$\rho_p 2ab(a+b) = \frac{a+b}{2}\left[\alpha\left(\frac{a+b}{2}\right)^2+\beta\right]$$

$$16\rho_p ab = \alpha(a^2+2ab+b^2)+4\beta \tag{7}$$

Substitute Equation 7 to Equation 3

$$\rho_p(a^2+b^2+4ab) = \frac{3}{2}\alpha\left(\frac{a+b}{2}\right)^2 + \frac{\beta}{2}$$

$$8\rho_p(a^2+b^2+4ab) = 3\alpha\left(a^2+2ab+b^2\right) + 4\beta \tag{8}$$

Subtract Equation 7 and Equation 8.

$$8\rho_p a^2 + 8\rho_p b^2 + 16\rho_p ab = 2\alpha(a^2+2ab+b^2)$$

$$\alpha = \frac{8\rho_p a^2 + 8\rho_p b^2 + 16\rho_p ab}{2(a^2+2ab+b^2)} = \frac{8\rho_p(a+b)^2}{2(a+b)^2} = 4\rho_p$$

This is consistent with Equation 1. Substitute to Equation 7.

$$16\rho_p ab = 4\rho_p(a^2+2ab+b^2) + 4\beta$$

$$-4\rho_p(a^2-2ab+b^2) = 4\beta$$

$$\beta = -\rho_p(a-b)^2 \tag{9}$$

We have used Equation 1, Equation 2, Equation 4. Check with Equation 3, Equation 5.

For Equation 3:

$$LHS = \rho_p(a^2+b^2+4ab)$$

$$\begin{aligned} RHS &= \frac{3}{2}\alpha c_0^2 + \frac{\beta}{2} = \frac{3}{2}(4\rho_p)\left(\frac{a+b}{2}\right)^2 - \frac{\rho_p(a-b)^2}{2} \\ &= \frac{3}{2}\rho_p(a+b)^2 - \frac{\rho_p}{2}(a-b)^2 = \rho_p(a^2+4ab+b^2) \end{aligned}$$

$LHS = RHS$ consistent with Equation 3.

For Equation 5:

$$LHS = \rho_p a^2 b^2$$

$$\begin{aligned} RHS &= \left(\frac{\alpha}{4}c_0^4 + \frac{\beta}{2}c_0^2\right) = \rho_p\frac{(a+b)^4}{16} - \frac{\rho_p}{2}(a-b)^2\frac{(a+b)^2}{4} \\ &= \frac{\rho_p}{16}(a+b)^2\left[(a+b)^2 - 2(a-b)^2\right] \end{aligned}$$

$$LHS \neq RHS$$

Equations differ by a scalar, which is not important in free energy gradients.

$$LHS - RHS = \rho_p \left[a^2 b^2 - \frac{1}{16}(a+b)^2 \left\{ (a+b)^2 - 2(a-b)^2 \right\} \right]$$

Hence,

$$\begin{cases} \alpha = 4\rho_p \\ \beta = -\rho_p (a-b)^2 \\ c_0 = 0.5(a+b) \end{cases}$$

Likewise,

$$\begin{cases} \rho_p = 0.25\alpha \\ a = c_0 - \sqrt{-\beta/\alpha} \\ b = c_0 + \sqrt{-\beta/\alpha} \end{cases}$$

6 Non-dimensional conversion

Use the following characteristic scales

$$[C] = (c_0/\bar{c})$$

$$[T] = \frac{1}{k_d}$$

$$[L] = \sqrt{M_m \lambda / k_d}$$

$$[E] = [\alpha][C]^4[L^2]$$

This is a functional derivative

$$\partial_t c = M_c \nabla^2 \left(\frac{\delta F}{\delta c} \right) = M_c \nabla^2 \left(\alpha (c - c_0)^3 + \beta (c - c_0) + \gamma m - k \nabla^2 c \right)$$

$$\partial_t m = M_m \lambda \nabla^2 m + k_p(\vec{r}) c - k_d m$$

$$\tilde{c} = \frac{c}{c_0/\bar{c}} \quad \tilde{m} = \frac{m}{c_0/\bar{c}} \quad \tilde{t} = k_d t \quad k = \frac{k_p}{k_d} \quad \tilde{r} = \frac{r}{l_{\text{RNA}}} = \frac{r}{\sqrt{M_m \lambda / k_d}}$$

$$\partial_{\tilde{t}} \tilde{c} = \frac{M_c}{k_d} \nabla^2 \left(\alpha (c_0/\bar{c})^2 (\tilde{c} - \bar{c})^3 + \beta (\tilde{c} - \bar{c}) + \gamma \tilde{m} - \kappa \nabla^2 \tilde{c} \right)$$

$$\partial_{\tilde{t}} \tilde{c} = \frac{M_c \alpha (c_0/\bar{c})^2}{M_m \lambda} \tilde{\nabla}^2 \left((\tilde{c} - \bar{c})^3 + \beta \frac{1}{\alpha (c_0/\bar{c})^2} (\tilde{c} - \bar{c}) + \gamma \frac{1}{\alpha (c_0/\bar{c})^2} \tilde{m} - \kappa \frac{1}{\alpha (c_0/\bar{c})^2} \frac{k_d}{M_m} \tilde{\nabla}^2 \tilde{c} \right)$$

$$\partial_{\tilde{t}} \tilde{c} = M \tilde{\nabla}^2 \left((\tilde{c} - \bar{c})^3 + \tilde{\beta} (\tilde{c} - \bar{c}) + \tilde{\gamma} \tilde{m} - \tilde{\kappa} \nabla^2 \tilde{c} \right)$$

$$M = \frac{M_c \alpha (c_0/\bar{c})^2}{M_m \lambda} \quad \tilde{\beta} = \beta \frac{1}{\alpha (c_0/\bar{c})^2} \quad \tilde{\gamma} = \gamma \frac{1}{\alpha (c_0/\bar{c})^2} \quad \tilde{\kappa} = \frac{\kappa}{\alpha (c_0/\bar{c})^2} \frac{k_d}{M_m}$$

$$\partial_{\tilde{t}} \tilde{m} = \frac{M_m \lambda}{k_d} \tilde{\nabla}^2 \tilde{m} + k \tilde{c} - \tilde{m}$$

$$\partial_{\tilde{t}}\tilde{m} = \tilde{\nabla}^2\tilde{m} + k\tilde{c} - \tilde{m}$$

7 Enhancer dynamics

Notice the change in notation from the dimensional to dimensionless equation. I'm shifting the notation form Pradeep's notes to the documentation in the PFStability code.

$$F_{DW}(c_0, \alpha, \beta) = \int d^2x \bigg[\frac{\alpha}{4}(c - c_0)^4 + \frac{\beta}{2}(c - c_0)^2 + \frac{\lambda}{2}m^2 + \gamma mc$$
$$+ \frac{\kappa}{2}(\nabla c)^2 - \chi_{PD} \exp(-|\vec{r} - \vec{r}_e|^2/2\sigma^2)c \bigg] + \frac{3k_BT}{2L_CL_P}|\vec{r}_e - \vec{r}_p|^2$$

$$\tilde{F}(\tilde{c}_1, \tilde{c}_2, \vec{r}_e) = \frac{F}{\alpha(c_0/\bar{c})^4(M_m\lambda/k_d)} = \int d^2\tilde{x} \bigg[\frac{1}{4}(\tilde{c}_1 - \bar{c}_1)^4 + \frac{\tilde{\beta}}{2}(\tilde{c}_1 - \bar{c}_1)^2 + \frac{\tilde{\lambda}}{2}c_2^2 + \tilde{\gamma}c_1c_2 + \frac{\tilde{\kappa}}{2}|\nabla c_1|^2$$
$$- \tilde{c} \exp(-|\vec{r} - \vec{r}_e|^2/2\sigma^2)\tilde{c}_1 \bigg] + \frac{\tilde{k}}{2}|\vec{r}_e - \vec{r}_p|^2$$

Recall our characteristic concentration, time, length, and energy scales. Note that $\bar{c}$ is dimensionless.

$$[\bar{c}] = 1$$

$$[C] = (c_0/\bar{c})$$
$$[T] = \frac{1}{k_d}$$
$$[L] = \sqrt{M_m\lambda/k_d}$$
$$[E] = [\alpha][C]^4[L^2]$$

$$F = [E]\tilde{F} = \alpha(c_0/\bar{c})^4(M_m\lambda/k_d)\tilde{F}$$

$$[\chi_{PD}] = \frac{[E]}{[L]^2[C]} = [\alpha][C^3] = \alpha(c_0/\bar{c})^3$$

$$\chi_{PD} = [\chi_{PD}]\tilde{c} = \alpha(c_0/\bar{c})^3\tilde{c}$$

$$[M_D] = \frac{[\partial_{\vec{r}_e}]}{[\nabla_{\vec{x}_e}F]} = \frac{[L]/[T]}{[E]/[L]} = \frac{[L]^2}{[E][T]} = \frac{[L]^2}{[\alpha][C]^4[L^2][T]} = \frac{1}{[\alpha][C]^4[T]} = \frac{k_d}{\alpha(c_0/\bar{c})^4}$$

$$M_D = [M_D]\tilde{M}_D = \frac{k_d}{\alpha(c_0/\bar{c})^4}\tilde{M}_D$$

This is a partial derivative (gradient)

$$\partial_t \vec{r}_e = -M_D \vec{\nabla}_{\vec{r}_e} F = M_D \int d^2x \left[\chi_{PD} c_1 \left(\frac{\vec{r} - \vec{r}_e}{\sigma^2} \right) \exp\left(-\frac{|\vec{r} - \vec{r}_e|^2}{2\sigma^2} \right) \right] - M_D \frac{3k_B T}{L_P L_C} (\vec{r}_e - \vec{r}_p)$$

$$k_d \left(\sqrt{M_m \lambda / k_d} \right) \partial_{\tilde{t}} \vec{\tilde{r}}_e = M_D \int d^2\tilde{x} (M_m \lambda / k_d) \Bigg[\chi_{PD} c_1 \Bigg(\frac{1}{\sqrt{M_m \lambda / k_d}} \frac{\vec{\tilde{r}} - \vec{\tilde{r}}_e}{\tilde{\sigma}^2} \Bigg) \exp\Bigg(- \frac{|\vec{\tilde{r}} - \vec{\tilde{r}}_e|^2}{2\tilde{\sigma}^2} \Bigg) \Bigg] - M_D \frac{3k_B T}{L_P L_C} \left(\sqrt{M_m \lambda / k_d} \right) (\vec{\tilde{r}}_e - \vec{\tilde{r}}_p)$$

$$\partial_{\tilde{t}} \vec{\tilde{r}}_e = \frac{M_D}{k_d} \int d^2\tilde{x} \left[\chi_{PD} c_1 \left(\frac{\vec{\tilde{r}} - \vec{\tilde{r}}_e}{\tilde{\sigma}^2} \right) \exp\left(-\frac{|\vec{\tilde{r}} - \vec{\tilde{r}}_e|^2}{2\tilde{\sigma}^2} \right) \right] - \frac{M_D}{k_d} \frac{3k_B T}{L_P L_C} (\vec{\tilde{r}}_e - \vec{\tilde{r}}_p)$$

$$\partial_{\tilde{t}} \vec{\tilde{r}}_e = \left\{ \frac{M_D \alpha (c_0/\bar{c})^4}{k_d} \right\} \int d^2\tilde{x} \left[\left\{ \frac{\chi_{PD}}{\alpha (c_0/\bar{c})^3} \right\} \left\{ \frac{c_1}{(c_0/\bar{c})} \right\} \left(\frac{\vec{\tilde{r}} - \vec{\tilde{r}}_e}{\tilde{\sigma}^2} \right) \exp\left(-\frac{|\vec{\tilde{r}} - \vec{\tilde{r}}_e|^2}{2\tilde{\sigma}^2} \right) \right] - \left\{ \frac{M_D \alpha (c_0/\bar{c})^4}{k_d} \right\} \left\{ \frac{3k_B T}{L_P L_C} \frac{1}{\alpha (c_0/\bar{c})^4} \right\} (\vec{\tilde{r}}_e - \vec{\tilde{r}}_p)$$

$$\partial_{\tilde{t}} \vec{\tilde{r}}_e = \tilde{M}_D \int d^2\tilde{x} \left[\tilde{c} \tilde{c}_1 \left(\frac{\vec{\tilde{r}} - \vec{\tilde{r}}_e}{\tilde{\sigma}^2} \right) \exp\left(-\frac{|\vec{\tilde{r}} - \vec{\tilde{r}}_e|^2}{2\tilde{\sigma}^2} \right) \right] - \tilde{M}_D \tilde{k} (\vec{\tilde{r}}_e - \vec{\tilde{r}}_p)$$

$$\tilde{M}_D = \frac{M_D \alpha (c_0/\bar{c})^4}{k_d}$$

$$\tilde{k} = \frac{3k_B T}{L_P L_C} \frac{1}{\alpha (c_0/\bar{c})^4}$$

$$\tilde{c} = \frac{\chi_{PD}}{\alpha (c_0/\bar{c})^3}$$

8 Protein/RNA Coupled Dynamics

$$\tilde{F}_{DW}(\bar{c}_1, \tilde{\beta}) + \cdots = \int d^2x \left[\frac{1}{4} (\tilde{c}_1 - \bar{c}_1)^4 + \frac{\tilde{\beta}}{2} (\tilde{c}_1 - \bar{c}_1)^2 + \frac{\tilde{\lambda}}{2} c_2^2 + \tilde{\gamma} c_1 c_2 + \frac{\tilde{\kappa}}{2} |\nabla c_1|^2 - \tilde{c} \exp(-|\vec{r} - \vec{r}_0|^2 / 2\sigma^2) \tilde{c}_1 \right] + \frac{\tilde{k}}{2} |\vec{r}_e - \vec{r}_p|$$

$$\partial_{\tilde{t}} \tilde{c}_1 = M \tilde{\nabla}^2 \left((\tilde{c}_1 - \bar{c})^3 + \tilde{\beta} (\tilde{c}_1 - \bar{c}) + \tilde{\gamma} \tilde{c}_2 - \tilde{\kappa} \nabla^2 \tilde{c}_1 - \exp(-|\vec{r} - \vec{r}_0|^2 / 2\sigma^2) \tilde{c}_1 \right)$$

$$\partial_{\tilde{t}} \tilde{c}_2 = \tilde{\nabla}^2 \tilde{c}_2 + k \tilde{c}_1 - \tilde{c}_2$$

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$$\frac{\delta F_G}{\delta \vec{x}_e} = -M_D \int d^2x \left[\chi_{PD} c_1 \vec{\nabla}_{\vec{x}_e} \exp\left(-\frac{|\vec{r} - \vec{r}_e|^2}{2\sigma^2} \right) \right] = -M_D \oint dS \chi_{PD} c_1 \exp\left(-\frac{|\vec{r} - \vec{r}_e|^2}{2\sigma^2} \right) + M_D \int d^2x \exp\left(-\frac{|\vec{r} - \vec{r}_e|^2}{2\sigma^2}$$