Lab12-Approximation Algorithm II

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1. Let us consider the special case of the Maximum Cut problem in which the required partition of the node set must have the same cardinality. Define a polynomial-time local search algorithm for this problem and evaluate its performance ratio.

   Solution. First, we divide V into two arbitrary partition S,T with the same cardinality. Then for each pair of u, v where $u \in S$, $v \in T$, we consider swapping them. We denote the number of vertices that x is adjacent to in S by x_S and that in T by x_T. Thus, if $u_T + v_S < u_S + v_T$, we should put u into T and put v into S and get a larger cut. We repeat this procedure until we can't find any pair of u, v that can be swapped.

   For each time we repeat the procedure, the cut must be increased by at least 1. The maximum cut is at most $|E|$, so there are at most $|E|$ iterations. Our algorithm is polynomial.

   When the algorithm stops, $\forall u \in S, \forall v \in T, u_T + v_S \geq u_S + v_T$.

   So, $\sum u_T + \sum v_S \geq \sum u_S + \sum v_T$. Note that $\sum u_T = \sum v_S$.

   Thus, $\sum u_T \geq (\sum u_S + \sum v_T)/2 = |E| - \sum u_T$.

   Then, $our\ cut = \sum u_T \geq |E|/2 \geq maximum\ cut/2$. The performance ratio is 2. □

2. **Minimum Weighted Vertex Cover:** Consider the weighted version of the Minimum Vertex Cover problem in which a non-negative weight c_i is associated with each vertex v_i and we look for a vertex cover having minimum total weight.

   (a) Given a weighted graph $G = (V, E)$ with a non-negative weight c_i associated with each vertex v_i, please formulate the Minimum Weighted Vertex Cover problem as an integer linear program.

   Solution.

   $$\min \sum_{i=1}^{|V|} x_i c_i$$

   subject to

   $x_i = \{0, 1\}$,

   $\forall (v_i, v_j) \in E$, $x_i + x_j \geq 1$. □

   (b) Prove that the following algorithm finds a feasible solution of the Minimum Weighted Vertex Cover problem with value $m_{LP}(G)$ such that $m_{LP}(G)/m^*(G) \leq 2$.

   Algorithm 1: Rounding Weighted Vertex Cover

   Input: Graph $G = (V, E)$ with non-negative vertex weights;
   Output: Vertex cover V' of G;

   Let ILP_{VC} be the integer linear programming formulation of the problem;
   Let LP_{VC} be the problem obtained from ILP_{VC} by LP-relaxation;
   Let $x^*(G)$ be the optimal solution for LP_{VC};
   $V' \leftarrow \{v_i \mid x_i^*(G) \geq 0.5\}$;
   return V';

Proof. In optimal solution for LP_{VC}, for each edge (v_i, v_j), at least one of $x_i > 0.5$ and $x_j > 0.5$ should hold. Otherwise, $x_i + x_j > 1$ is impossible. So this algorithm provides a feasible solution.

For each vertex $v_i \in E$, x_i is increased by a factor of at most 2 because we only round those $x_i^*(G) \geq 0.5$ to 1. So $m_{LP}(G)/m^*(G) \leq 2$. $\square$