Real Analysis Cheatsheet

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1 Number System

2 Topology

2.1 Finite, Countable, and Uncountable Sets

Definition 2.1 (Function (mapping), domain, range, image, inverse image, into/onto)**.**

Definition 2.2 (Cardinal number)**.** If there exists a 1-1 mapping of A onto B, we say that A and B have the same *cardinal number*, or A and B are equivalent, $A \sim B$. (reflexive, symmetric, transitive)

Definition 2.3 (Finite, infinite, countable (enumerable, denumerable), at most countable)**.**

Remark 2.4. A is infinite if A is equivalent to one of its proper subsets.

Definition 2.5 (Sequence, terms)**.** A function f defined on the set of <u>all</u> positive integers. The values x_n of f are called the *terms* of the sequence. If A is a set and if $x_n \in A$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is said to be a *sequence* in A.

Theorem 2.6. Every infinite subset of a countable set A is countable.

Corollary 2.7. No uncountable set can be a subset of a countable set.

Definition 2.8 (Collection (family))**.** Let A and Ω be sets, and suppose that with each element α of A there is associated a subset of Ω which we denote by E_α. Then $\{E_\alpha\}$ is a collection of sets.

Definition 2.9 (Intersect, disjoint)**.** If $A \cap B$ is not empty, we say that A and B intersect; otherwise they are disjoint.

Theorem 2.10. Let $\{E_n\}$, $n \in \mathbb{N}$ be a sequence of countable sets, then $S = \bigcup_{n=1}^{\infty} E_n$ is countable.

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Corollary 2.11. Suppose A is at most countable, and for every $\alpha \in A$, B_α is at most countable. Then, $T = \bigcup_{\alpha \in A} B_\alpha$ is at most countable.

Theorem 2.12. Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \ldots, a_n)$, where $a_k \in A$ $(k = 1, \ldots, n)$, and the elements $a_1, \ldots, a_n$ need not be distinct. Then B_n is countable.

Corollary 2.13. The set of all rational numbers is countable.

Theorem 2.14. Let A be the set of all sequences whose elements are digits 0 and 1. This set A is uncountable.

Proof. Cantor's diagonal process. $\square$

Corollary 2.15. The set of all real numbers is uncountable.

2.2 Metric Spaces

Definition 2.16 (Metric Space)**.** A set X, whose elements we shall call *points*, is said to be a metric space if with any two points p and q of X there is associated a real number $d(p, q)$, called the distance from p to q, s.t.

Positive definiteness $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$;

Symmetry $d(p, q) = d(q, p)$;

Triangle Inequality $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$.

Any function with these three properties is called a *distance function*, or a *metric*.

Remark 2.17. Every subset Y of a metric space X is a metric space in its own right, with the same metric.

Definition 2.18 (Segment, interval, k-cell, open(closed) ball)**.** By the segment (a, b), we mean the set of all real numbers x s.t. $a < x < b$.

By the interval $[a, b]$, we mean the set of all real numbers x s.t. $a \leq x \leq b$.

If $a_i < b_i$ for $i = 1, \ldots, k$, the set of all points $\mathbf{x} = (x_1, \ldots, x_k)$ in $\mathbb{R}^k$ whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i (1 \leq i \leq k)$ is called a *k-cell*.

If $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$, the open (or closed) ball B with center at $\mathbf{x}$ and radius r is defined to be the set of all $\mathbf{y} \in \mathbb{R}^k$ s.t. $|\mathbf{y} - \mathbf{x}| < r$ (or $|\mathbf{y} - \mathbf{x}| \leq r$).

Definition 2.19 (Convex)**.** We call a set $E \subset \mathbb{R}^k$ *convex* if $\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in E$ whenever $\mathbf{x} \in E$, $y \in E$, and $0 < \lambda < 1$.

Definition 2.20 (Neighborhood)**.** Let p be a point in matric space X. A neighborhood of p is a subset $N_r(p)$ consisting of all q s.t. $d(p,q) < r$, for some $r > 0$. The number r is called the *radius* of $N_r(p)$.

Definition 2.21 (Limit point, isolated point)**.** A point p is a *limit point* of the set E if every neighborhood of p contains a point $\underline{q \neq p}$, s.t. $q \in E$. If $p \in E$ and p is not a limit point of E, then p is called an isolated point of E.

Definition 2.22 (Closed)**.** E is closed if every limit point of E is a point of E.

Definition 2.23 (Interior point)**.** A point p is an interior point of E if there is a neighborhood N of p s.t. $N \subset E$.

Definition 2.24 (Open)**.** E is open if every point of E is an interior point of E.

Definition 2.25 (Complement)**.** Let E be a set in metric space X. The complement of E (denoted by E^c) is the set of all points $p \in X$ s.t. $p \notin E$.

Definition 2.26 (Perfect)**.** E is perfect if E is closed and if every point of E is a limit point of E.

Definition 2.27 (Bounded)**.** Let E be a set in metric space X. E is bounded if there is a real number M and a point $q \in X$ s.t. $d(p,q) < M$ for all $p \in E$.

Definition 2.28 (Dense)**.** E is dense in the metric space X if every point of X is a point of E or a limit point E (or both).

Theorem 2.29. Every neighborhood is an open set.

Theorem 2.30. If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

Corollary 2.31. A finite set has no limit points.

Theorem 2.32. Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α. Then

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c).$$

Theorem 2.33. A set E is open if and only if its complement is closed.

Corollary 2.34. A set F is closed if and only if its complement is open.

Remark 2.35. Prove the complement is open when it's hard to prove the primal is closed.

Theorem 2.36. Let $\{E_\alpha\}$ be a collection of open sets E_α, and $\{F_\alpha\}$, a collection of closed sets F_α.

(a) $\bigcup_\alpha E_\alpha$ is open, and when $\{E_\alpha\}$ is finite, $\bigcap_\alpha E_\alpha$ is also open.

(b) $\bigcap_\alpha F_\alpha$ is closed, and when $\{F_\alpha\}$ is finite, $\bigcup_\alpha F_\alpha$ is also closed.

Remark 2.37. Examples when finiteness is violated are $(\frac{1}{n}, \frac{1}{n}), n \in \mathbb{N}$ and $[\frac{1}{n}, 1], n \in \mathbb{N}$.

Definition 2.38 (Closure)**.** If X is a metric space and $E \subset X$, then the closure of E is $\bar{E} = E \cup E'$, where E' denotes the set of all the limit points of E <u>in X</u>.

Theorem 2.39. If X is a metric space and $E \subset X$, then

(a) $\bar{E}$ is closed.

(b) $E = \bar{E}$ iff E is closed.

(c) $\bar{E} \subset F$ for every closed set $F \in X$ s.t. $E \subset F$.

By (a) and (c), $\bar{E}$ is the smallest closed subset of X that contains E.

Theorem 2.40. Let E be a <u>nonempty</u> set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \bar{E}$. Hence, $y \in E$ if E is closed.

Theorem 2.41. Suppose $Y \subset X$. A subset E of Y is open relative to Y iff $E = Y \cap G$ for some open subset $G \subset X$.

2.3 Compact Sets

Definition 2.42 (Open cover)**.** The open cover of a set E in a metric space X is a collection $\{G_\alpha\}$ of open subsets $G_\alpha \in X$ s.t. $E \subset \bigcup_\alpha G_\alpha$.

Definition 2.43 (Compact)**.** A subset K of X is compact iff there exists a finite subcover for every open cover of K.

Remark 2.44. Every finite set is compact.

Theorem 2.45. Suppose $K \subset Y \subset X$. Then K is compact relative to X iff K is compact relative to Y.

Theorem 2.46. Compact subsets of metric spaces are closed.

Theorem 2.47. Closed subsets of compact sets are compact.

Theorem 2.48. The intersection of a compact set and a closed set is compact.

Theorem 2.49 (Finite Intersection Property)**.** If $\{K_\alpha\}$ is a collection of <u>compact</u> subsets of a metric space s.t. the intersection of any finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap_\alpha K_\alpha$ is nonempty.

Corollary 2.50. If $\{K_n\}$ is a sequence of nonempty compact sets s.t. $K_n \supset K_{n+1}, n \in \mathbb{N}$, then $\bigcap_1^\infty K_n$ is not empty.

Theorem 2.51. The set K is compact iff every infinite subset of K has a limit point in K.

Proof. Countable bases. □

Corollary 2.52 (Weierstrass Theorem)**.** Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Theorem 2.53. Every k-cell is compact.

Theorem 2.54 (Heine-Bored Theorem)**.** A subset of $\underline{\mathbb{R}^k}$ is compact iff it's closed and bounded.

Remark 2.55. A subset of a general metric space does not necessarily have the above property. For instance, in the metric space of bounded real sequences and maximum difference in all coordinates as the metric, the subset with only one coordinate being 1 and others being 0 is closed, bounded, but not compact.

Theorem 2.56. Any nonempty perfect subset of $\mathbb{R}^k$ is uncountable.

Corollary 2.57. Any segment or interval is uncountable. $\mathbb{R}$ is uncountable.

Remark 2.58 (The Cantor set)**.** An uncountable subset of $\mathbb{R}^1$ which contains no segment.

Definition 2.59 (Separated)**.** Two subsets A and B of a metric space X are said to be separated if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty.

Remark 2.60. Separate $\Rightarrow$ disjoint. Disjoint $\not\Rightarrow$ separated.

Definition 2.61 (Connected sets)**.** A set $E \subset X$ is said to be connnected if E is not a union of two nonempty separated sets.

Theorem 2.62. A subset E of the real line $\mathbb{R}^1$ is connected iff it holds that, if $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.

Theorem 2.63. Properties of connnected sets.

(a) If A is connected and $A \subset B \subset \bar{A}$, then B is also connected.

(b) The union of a collection of connected sets with some point in common to all of them is connected.

(c) If A and B are separated sets in X and $A \cup B = X$, and if $C \subset X$ is connected, then $C \subset A$ or $C \subset B$.

(d) A is connected iff for every pair of open sets U, V, s.t. $A \subset U \cup V$ and $U \cap V \cap A = \emptyset$, then $A \subset U$ or $A \subset V$.

(e) A is connected $\Rightarrow \bar{A}$ is connected.

Definition 2.64 (Algebraic). A complex number z is said to be algebraic if there are integers $a_0, \dots, a_n$, not all zero, s.t. $a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$.

Definition 2.65 (Separable). A metric space is called separable if it contains a countable dense subset.

Definition 2.66 (Base). A collection $\{V_\alpha\}$ of open subsets of X is said to be a base for X if for every $x \in X$ and every open set $G \subset X$ s.t. $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Theorem 2.67. Every separable metric space has a countable base (and vice versa).

3 Numerical Sequences and Series

3.1 Convergent Sequences

Definition 3.1 (Converge). A sequence $\{p_n\}$ in a metric space X is said to converge in X if there is a point $p \in X$ with the following property: For any $\epsilon > 0$, there is an integer $N > 0$ s.t. $n \geq N$ implies that $d(p_n, p) < \epsilon$. We write $\lim_{n\to\infty} p_n = p$. If $\{p_n\}$ does not converge, it is said to diverge.

Definition 3.2 (Range, Bounded). The set of all points $p_n (n = 1, 2, 3, \dots)$ is the range of $\{p_n\}$. The sequence $\{s_n\}$ is said to be bounded if its range is bounded.

Theorem 3.3. Let $\{p_n\}$ be a sequence in a metric space X.

(a) $\{p_n\}$ converges to $p \in X$ iff every neighborhood of p contains p_n for all but finitely many n.

(b) If $p, q \in X$, and $\{p_n\}$ converges to p and converges to q, then $p = q$.

(c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

(d) If $E \subset X$ and p is a limit point of E, then there exists a sequence $\{p_n\}$ in E s.t. $p = \lim_{n\to\infty} p_n$.

Theorem 3.4. Suppose $\mathbf{x}_n \in \mathbb{R}^k, n \in \mathbb{N}$ and $\mathbf{x}_n = (\alpha_{1,n}, \dots, \alpha_{k,n})$. Then $\{\mathbf{x}_n\}$ converges to $\mathbf{x} = (\alpha_1, \dots, \alpha_k)$ iff $\lim_{n\to\infty} \alpha_{j,n} = \alpha_j$. for $1 \leq j \leq k$.

Theorem 3.5. Suppose $\{\mathbf{x}_n\}$, $\{\mathbf{y}_n\}$ are sequences in $\mathbb{R}^k$, $\{\beta_n\}$ is a sequence of real numbers, and $\mathbf{x}_n \to \mathbf{x}$, $\mathbf{y}_n \to \mathbf{y}$, $\beta_n \to \beta$. Then

$$\lim_{n\to\infty} (\mathbf{x}_n + \mathbf{y}_n) = \mathbf{x} + \mathbf{y}, \quad \lim_{n\to\infty} (\mathbf{x}_n \cdot \mathbf{y}_n) = \mathbf{x} \cdot \mathbf{y}, \quad \lim_{n\to\infty} \beta_n \mathbf{x}_n = \beta \mathbf{x}.$$

For $k = 2$, $\lim_{n\to\infty} \frac{1}{\mathbf{x}_n} = \frac{1}{\mathbf{x}}$, provided $\mathbf{x}_n \neq 0$ for $n \in \mathbb{N}$, and $s \neq 0$.

3.2 Subsequences

Definition 3.6 (Subsequence)**.** Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers, s.t. $n_1 < n_2 < n_3 < \cdots$. Then the sequence $\{p_{n_1}\}$ is called a subsequence of $\{p_n\}$. If $\{p_{n_1}\}$ converges, its limit is called a subsequential limit of $\{p_n\}$.

Theorem 3.7.

(a) $\{p_n\}$ converges to p iff every subsequence of $\{p_n\}$ converges to p.

(b) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X.

(c) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

Theorem 3.8. The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.

3.3 Cauchy Sequences

Definition 3.9 (Cauchy sequence)**.** A sequence $\{p_n\}$ in a metric space X is said to be a Cauchy sequence if for every $\epsilon > 0$, there is an integer N s.t. $d(p_n, p_m) < \epsilon$ if $n \geq N$ and $m \geq N$.

Definition 3.10 (Diameter)**.** Let E be a non-empty subset of metric space X, and let S be the set of all $d(p,q)$ with $p \in E$ and $q \in E$. The sup of S is called the diameter of E.

Proposition 3.11. If $\{p_n\}$ is a sequence in X and E_n consists of the points $p_N, p_{N+1}, \ldots$, then $\{p_n\}$ is a Cauchy sequence iff $\lim_{N\to\infty} \text{diam} E_N = 0$.

Theorem 3.12. $\text{diam}\bar{E} = \text{diam}E$.

Theorem 3.13. If K_n is a sequence of compact sets in X s.t. $K_n \subset K_{n+1}$ $(n = 1, 2, \ldots)$ and if $\lim_{n\to\infty} \text{diam}K_n = 0$, then $\bigcap_1^\infty K_n$ consists of exactly one point.

Theorem 3.14. Convergent $\Rightarrow$ Cauchy.

Theorem 3.15. X compact, $\{p_n\}$ Cauchy in $X \Rightarrow \{p_n\}$ converges to some point of X.

Theorem 3.16 (Cauchy criterion)**.** In $\mathbb{R}^k$, Cauchy $\Rightarrow$ convergent.

Definition 3.17 (Complete)**.** A metric space in which every Cauchy sequence converges is said to be complete.

Theorem 3.18. Compact $\Rightarrow$ complete.

Theorem 3.19. Every closed subset E of a complete metric space X is complete.

Definition 3.20 (Monotonically increasing/decreasing)**.** $s_n \leq s_{n+1}, (n = 1, 2, \dots) \Rightarrow$ monotonically increasing. $s_n \geq s_{n+1}, (n = 1, 2, \dots) \Rightarrow$ monotonically decreasing.

Theorem 3.21. Monotonically $\Rightarrow \{s_n\}$ converges iff bounded.

3.4 Upper and lower limits

Definition 3.22. $s_n \to +\infty$: for every real M there is an integer M s.t. $n \geq N$ implies $s_n \geq M$. Similarly, $s_n \to +\infty$: for every real M there is an integer M s.t. $n \geq N$ implies $s_n \geq M$.

Definition 3.23 (Upper and lower limits)**.** Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of numbers x in the extended real number system s.t. $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$. This set E contains all subsequential limits plus possibly $+\infty, -\infty$. Put $s^* = \sup E$, and $s_* = \inf E$. The numbers s^*, s_* are called the upper and lower limits of $\{s_n\}$. Denote $\limsup_{n\to\infty} s_n = s^*$, $\liminf_{n\to\infty} s_n = s_*$.

Theorem 3.24. (a) $s^* \in E$. (b) $x > s^* \Rightarrow$ there is an integer N s.t. $n \geq N$ implies $s_n < x$.

Theorem 3.25. $s_n \leq t_n$ for $n \geq N \Rightarrow \liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n$, $\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$.

Theorem 3.26. $\{x_n\}$ is bounded and $\limsup_{n\to\infty} x_n = M \Leftrightarrow$

(1) $\forall \epsilon > 0, \exists N$ s.t. $x_n < \underline{M + \epsilon}, \forall n > N$.

(2) $\forall \epsilon > 0, \forall n \in \mathbb{N}, \exists k > n$ s.t. $x_k > M - \epsilon$.

Corollary 3.27. $\{x_n\}$ is bounded and $M = \limsup_{n\to\infty} x_n \Rightarrow \exists x_{n_k} \to M$.

Corollary 3.28. $y_n \to y > 0 \Rightarrow \limsup_{n\to\infty} x_n y_n = y \limsup_{n\to\infty} x_n$.

Theorem 3.29. $\inf\{x_n\} \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq \sup\{x_n\}$.

3.5 Series

Definition 3.30 (Series, partial sums)**.** Series $\sum_{n=1}^{\infty} a_n$. Partial sums $s_n = \sum_{k=1}^{n} a_k$.

Theorem 3.31 (Cauchy criterion)**.** $\sum a_n$ converges iff $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|\sum_{k=n}^{m} a_k| \leq \epsilon$ if $m \geq n \geq N$.

Theorem 3.32. $\sum a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$.

Theorem 3.33. A series of nonnegative terms converges iff its partial sums form a bounded sequence.

Theorem 3.34. $\sum c_n$ converges, $|a_n| \leq c_n, \forall n \geq N \Rightarrow \sum a_n$ converges. $\sum d_n$ diverges, $a_n \geq d_n \geq 0, \forall n \geq N_0 \Rightarrow \sum a_n$ diverges.

3.6 Series of nonnegative terms

Theorem 3.35. If $0 \leq x \leq 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. If $x \geq 1$, the series diverges.

Theorem 3.36. $\underline{a_1 \geq a_2 \geq \cdots \geq 0 \Rightarrow \sum_{n=1}^{\infty} a_n}$ converges iff $\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + \ldots$ converges.

Theorem 3.37. $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Theorem 3.38. $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p > 1$, and diverges if $p \leq 1$.

3.7 The number e

Definition 3.39. $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Theorem 3.40. $\lim_{n\to\infty}(1 + \frac{1}{n})^n = e$.

Remark 3.41. $0 < e - s_n < \frac{1}{n!n}$.

Theorem 3.42. e is irrational.

Theorem 3.43 (Root test)**.** Given $\sum a_n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \Rightarrow$

(a) if $\alpha < 1$, $\sum a_n$ converges;

(b) if $\alpha > 1$, $\sum a_n$ diverges;

(c) if $\alpha = 1$, the test gives no information.

Theorem 3.44 (Ratio test)**.** The series $\sum a_n$

(a) converges if $\limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$,

(b) diverges if $|\frac{a_{n+1}}{a_n}| \geq 1, \forall n \geq N$.

Theorem 3.45. For any sequence $\{c_n\}$ of positive numbers, $\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n}$, $\limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$.

Remark 3.46. $\log n \leq \frac{n^p}{p}, \forall n > 0, p > 0$.

3.8 Power Series

Definition 3.47 (Power series)**.** $\sum_{n=0}^{\infty} c_n z^n$, c_n are called the coefficients of the series; z is a complex number.

Theorem 3.48. Given $\sum c_n z^n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$, $R = \frac{1}{\alpha}$ (If $\alpha = 0$, $R = +\infty$; if $\alpha = +\infty$, $R = 0$). Then $\sum c_n z^n$ converges if $|z| < R$, and diverges if $|z| > R$.

Theorem 3.49 (Partial summation formula)**.** Given $\{a_n\}, \{b_n\}$, put $A_n = \sum_{k=0}^{n} a_k$ if $n \geq 0$; put $A_{-1} = 0$. Then if $0 \leq p \leq q$, we have $\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$.

Theorem 3.50. Suppose

(a) the partial sums A_n of $\sum a_n$ form a bounded sequence;

(b) $b_0 \geq b_1 \geq b_2 \geq \ldots$;

(c) $\lim_{n\to\infty} b_n = 0$.

Then $\sum a_n b_n$ converges.

Theorem 3.51 (Leibnitz)**.** Suppose

(a) $|c_1| \geq |c_2| \geq |c_3| \geq \ldots$;

(b) $c_{2m-1} \geq 0, c_{2m} \leq 0$ $(m = 1, 2, \ldots)$ (alternating series);

(c) $\lim_{n\to\infty} c_n = 0$.

Then $\sum c_n$ converges.

3.9 Absolute convergence

Definition 3.52 (Converge absolutely)**.** The series $\sum a_n$ is said to converge absolutely if the series $\sum |a_n|$ converges. If $\sum a_n$ converges, but $\sum |a_n|$ diverges, then $\sum a_n$ is said to converge non-absolutely.

Theorem 3.53. Converge absolutely $\Rightarrow$ converge.

3.10 Addition and Multiplication of Series

Theorem 3.54. If $\sum a_n = A$, and $\sum b_n = B$, then $\sum(a_n + b_n) = A + B$, and $\sum ca_n = cA$.

Theorem 3.55. If $\sum a_n$ converges and $\{b_n\}$ is monotonic and bounded, then $\sum a_n b_n$ converges.

Definition 3.56 (Product)**.** The product of $\sum a_n$ and $\sum b_n$ is $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ for $n = 0, 1, 2, \ldots$

Theorem 3.57. The product of two convergent series (to A and B) converges (to AB) if at least one of the two series converges absolutely.

Theorem 3.58. $\sum a_n \to A, \sum b_n \to B, \sum c_n \to C$, $c_n = \sum_{k=0}^{n} a_k b_{n-k} \Rightarrow C = AB$.

Theorem 3.59. The product of two absolutely convergent series converges absolutely.

3.11 Rearrangements

Definition 3.60 (Rearrangement)**.** Let $\{k_n\}$, $n = 1, 2, \ldots$, be a sequence in which every positive integer appears once and only once. Putting $a'_n = a_{k_n}$, $(n = 1, 2, \ldots)$, we say $\sum a'_n$ is a rearrangement of $\sum a_n$.

Theorem 3.61. Let $\sum a_n$ be a series of real numbers which converges non-absolutely. Suppose $-\infty \leq \alpha \leq \beta \leq \infty$. Then, there exists a rearrangement $\sum a'_n$ with partial sums s'_n s.t. $\liminf_{n\to\infty} s'_n = \alpha$, $\limsup_{n\to\infty} s'_n = \beta$.

Theorem 3.62. If $\sum a_n$ is a series of complex numbers which converges absolutely, then every rearrangement of $\sum a_n$ converges, and they all converge to the same sum.

4 Continuity

4.1 Limits of functions

Definition 4.1 (Limits of functions)**.** Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y, and p is a <u>limit point of E</u>. We write $f(x) \to q$ as $x \to p$, or $\lim_{x\to p} f(x) = q$ if there is a point $q \in Y$ with the following property: For every $\epsilon > 0$, there exists $\delta > 0$ s.t. $d_Y(f(x), q) < \epsilon$ for all $x \in E$ for which $0 < d_X(x, p) < \delta$.

Theorem 4.2. $\lim_{x\to p} f(x) = q$ iff $\lim_{n\to\infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E s.t. $\underline{p_n \neq p}$, $\lim_{n\to\infty} p_n = p$.

Corollary 4.3. If f has a limit at p, this limit is unique.

Definition 4.4 ($f + g$, $f - g$, fg, f/g, $\mathbf{f} + \mathbf{g}$, $\mathbf{f} \cdot \mathbf{g}$, $\lambda\mathbf{f}$)**.**

Theorem 4.5. $\lim_{x\to p} f(x) = A, \lim_{x\to p} g(x) = B \Rightarrow$

(a) $\lim_{x\to p}(f + g)(x) = A + B$;

(b) $\lim_{x\to p}(fg)(x) = AB$;

(c) $\lim_{x\to p}(\frac{f}{g})(x) = \frac{A}{B}$, if $B \neq 0$.

Remark 4.6. $\lim_{x\to p}(\mathbf{f} + \mathbf{g})(x) = \mathbf{A} + \mathbf{B}$, $\lim_{x\to p}(\mathbf{f} \cdot \mathbf{g})(x) = \mathbf{A} \cdot \mathbf{B}$,

4.2 Continuous Functions

Definition 4.7 (Continuous)**.** Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y. Then f is said to be continuous <u>at p</u> if $\forall\epsilon > 0$, $\exists\delta > 0$ s.t. $d_Y(f(x), f(p)) < \epsilon$, $\forall x \in E$ for which $d_X(x, p) < \delta$.

Remark 4.8. Every function f is continuous at isolated points.

Theorem 4.9. If $p \in E$ is a limit point of E, then f is continuous at p iff $\lim_{x \to p} f(x) = f(p)$.

Theorem 4.10. If f is continuous at p and g is continuous at $f(p)$, then $f \circ g$ is continuous at p.

Theorem 4.11. $f : X \to Y$ is continuous on X iff $f^{-1}(V)$ is open in X for every open set $V \subset Y$ ($f^{-1}(C)$ is closed in X for every closed set C in Y).

Remark 4.12. $f^{-1}(E^c) = [f^{-1}(E)]^c$.

Theorem 4.13. Let f and g be complex continuous functions on a metric space X. Then $f + g$, fg, and f/g are continuous on X.

Theorem 4.14. $\mathbf{f}(x) = (f_1(x), \ldots, f_k(x))$ is continuous iff $f_1, \ldots, f_k$ is continuous.

Theorem 4.15. If $\mathbf{f}$ and $\mathbf{g}$ are continuous mappings of X into $\mathbb{R}^k$, then $\mathbf{f}+\mathbf{g}$ and $\mathbf{f}\cdot\mathbf{g}$ are continuous on X.

Theorem 4.16. If $f : X \to Y$ is a continuous mapping, then $f(\bar{E}) \in \overline{f(E)}, \forall E \in X$.

Theorem 4.17. If f is a continuous real function on X, then the zero set $Z(f)$ is closed.

Theorem 4.18. If $f, g : X \to Y$ are continuous mappings and E is a dense subset of X, then $f(E)$ is dense in $f(X)$. If $g(p) = g(E), \forall p \in E$, then $g(p) = f(p), \forall p \in X$.

4.3 Continuity and Compactness

Definition 4.19 (Bounded)**.** A mapping $\mathbf{f}$ of a set E into $\mathbb{R}^k$ is said to be bounded if $\exists M$ s.t. $|\mathbf{f}(x)| \leq M, \forall x \in E$.

Theorem 4.20. f : compact $X \to Y$ is continuous $\Rightarrow f(X)$ is compact.

Remark 4.21. $E \subset Y \Rightarrow f(f^{-1}(E)) \subset E$; $E \subset X \Rightarrow f^{-1}(f(E)) \supset E$;

Theorem 4.22. $\mathbf{f}$ is a continuous mapping of a compact metric space X into $\mathbb{R}^k \Rightarrow \mathbf{f}(X)$ is closed and bounded, $\mathbf{f}$ is bounded.

Theorem 4.23. Suppose f is a continuous real function on a compact metric space X, and $M = \sup_{p \in X} f(p)$, $m = \inf_{p \in X} f(p)$. $\Rightarrow \exists p, q \in X$ s.t. $f(p) = M, f(q) = m$.

Theorem 4.24. Suppose f is a continuous 1-1 mapping of a <u>compact</u> metric space X <u>onto</u> a metric space Y. Then the inverse mapping f^{-1} defined on Y by $f^{-1}(f(x)) = x, (x \in X)$ is a continuous mapping of Y onto X.

Definition 4.25 (Uniformly continuous)**.** Let f be a mapping of a metric space X into a metric space Y. We say f is uniformly continuous on X if $\forall \epsilon > 0, \exists \delta > 0$ s.t. $d_Y(f(p), f(q)) < \epsilon, \forall p, q \in X$ for which $d_X(p, q) < \delta$.

Theorem 4.26. f : compact $X \to Y$ is continuous $\Rightarrow f$ is uniformly continuous on X.

Theorem 4.27. If f is a real uniformly continuous function on the bounded set $E \subset \mathbb{R}^1$, then f is bounded on E.

4.4 Continuity and connectedness

Theorem 4.28. If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then $f(E)$ is connected.

Theorem 4.29. Let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$, and if c is a number s.t. $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ s.t. $f(x) = c$.

Remark 4.30. The converse does not hold.

4.5 Discontinuity

Definition 4.31. Let f be defined on (a, b). Consider any point x s.t. $a \leq x < b$. We write $f(x+) = q$ if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x, b) s.t. $t_n \to x$. To obtain the definition of $f(x-)$, for $a < x \leq b$, we restrict ourselves to sequences $\{t_n\}$ in (a, x).

Definition 4.32 (The first kind (simple) discontinuity)**.** Let f be defined on (a, b). We say f have a discontinuity or the first kind, or a simple discontinuity at x if $f(x+)$ and $f(x-)$ exist. Otherwise the discontinuity is said to be of the second kind.

4.6 Monotonic Functions

Definition 4.33. Let f be real on (a, b). Then f is said to be monotonically increasing on (a, b) if $a < x < y < b$ implies $f(x) \leq f(y)$. If the last inequality is reversed, we obtain the definition of a monotonically decreasing function.

Theorem 4.34. Let f be monotonically increasing on (a, b). Then $f(x+)$ and $f(x-)$ exist at every point of x of (a, b), and $\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t)$. Furthermore, if $a < x < y < b$, then $f(x+) \leq f(y-)$.

Corollary 4.35. Monotonic functions have no discontinuities of the second kind.

Theorem 4.36. Let f be monotonic on (a, b). Then the set of points (a, b) at which f is discontinuous is at most countable.

4.7 Infinite Limits and Limits at Infinity

Definition 4.37 (The neighborhood of $+\infty$ and $-\infty$)**.** For any real c, the set of real numbers x s.t. $x > c$ is called a neighborhood of $+\infty$ and is written $(c, +\infty)$. Similarly, the set $(-\infty, c)$ is a neighborhood of $-\infty$.

5 Differentiation

5.1 The derivative of a real function

Definition 5.1. Let $f : [a, b] \to \mathbb{R}$. For any $x \in [a, b]$, form the quotient $\phi(t) = \frac{f(t)-f(x)}{t-x}$ $(a < t < b, t \neq x)$, and define $f'(x) = \lim_{t\to x} \phi(t)$.

Theorem 5.2. If $f : [a, b] \to \mathbb{R}$ is differentiable at a point $x \in [a, b]$, then f is continuous at x.

Theorem 5.3. Suppose $f, g : [a, b] \to \mathbb{R}$ are differentiable at $x \in [a, b]$. Then $f + g$, fg, and f/g are differentiable at x, and $(f + g)'(x) = f'(x) + g'(x)$, $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$, and $(f/g)'(x) = [g(x)f'(x) - g'(x)f(x)]/g^2(x)$ with $g(x) \neq 0$ assumed.

Theorem 5.4. Suppose $f : [a, b] \to \mathbb{R}$ is continuous, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f, and g is differentiable at $f(x)$. If $h(t) = g(f(t))$ $a \leq t \leq b$, then h is differentiable at $f(x)$, and $h'(x) = g'(f(x))f'(x)$.

Example 5.5. Let f be defined by $f(x) = x \sin \frac{1}{x}$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$. Then $f'(0)$ does not exist.

Example 5.6. Let f be defined by $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$. Then $f'(0) = 0$, but f' is not a continuous function.

5.2 Mean value theorems

Definition 5.7 (Local maximum)**.** We say $f : X \to \mathbb{R}$ has a local maximum at $p \in X$ if $\exists \delta > 0$ s.t. $f(q) \leq f(p), \forall q \in N_\delta(x)$.

Theorem 5.8. If $f : [a, b] \to \mathbb{R}$ has a local maximum at $x \in (a, b)$, and $f'(x)$ exists, then $f'(x) = 0$.

Theorem 5.9 (Generalized mean value theorem)**.** If f and g are continuous real functions on $[a, b]$ and are differentiable in (a, b), then $\exists x \in (a, b)$ s.t. $[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$.

Theorem 5.10 (Mean value theorem)**.** If f is a continuous real function on $[a, b]$ which is differentiable in (a, b), then $\exists x \in (a, b)$ s.t. $f(b) - f(a) = (b - a)f'(x)$.

Theorem 5.11. Suppose f is differentiable in (a, b).

(a) If $f'(x) \geq 0 \ \forall x \in (a, b)$, then f is monotonically increasing.

(b) If $f'(x) = 0 \ \forall x \in (a, b)$, then f is constant.

(a) If $f'(x) \leq 0 \ \forall x \in (a, b)$, then f is monotonically decreasing.

5.3 The continuity of derivatives

Theorem 5.12. Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ s.t. $f'(x) = \lambda$.

Corollary 5.13. If f is differentiable on $[a, b]$, then f' have no simple discontinuity on $[a, b]$.

5.4 L'Hospital's Rule

Theorem 5.14 (L'Hospital's rule)**.** Suppose f and g are real and differentiable in (a, b), and $g'(x) \neq 0, \forall x \in (a, b)$, where $-\infty \leq a < b \leq +\infty$. Suppose $f'(x)/g'(x) \to A$ as $x \to a$. If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ or if $g(x) \to +\infty$ as $x \to a$, then $f(x)/g(x) \to A$ as $x \to a$.

5.5 Taylor's Theorem

Theorem 5.15 (Taylor's Theorem)**.** Suppose f is a real function on $[a, b]$, $n \in \mathbb{Z}_+$, f^{n-1} is continuous on $[a, b]$, $f^{(n)}(t)$ exists for any $t \in (a, b)$. Let α, β be distinct points of $[a, b]$. Then $\exists x \in (a, b)$ s.t. $f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$.

5.6 Uniformly Differentiable

Definition 5.16 (Uniformly differentiable)**.** We say that $f : [a, b] \to \mathbb{R}$ is uniformly differentiable if $\exists \delta > 0$ s.t. $|\frac{f(t)-f(x)}{t-x} - f'(x)| < \epsilon$ whenever $0 < |t - x| < \delta, a \leq x \leq b, a \leq t \leq b$.

Theorem 5.17. $f : [a, b] \to \mathbb{R}$ is uniformly differentiable if f' is continuous on $[a, b]$.