Real Analysis Cheatsheet

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1 Number System

2 Topology

2.1 Finite, Countable, and Uncountable Sets

Definition 2.1 (Function (mapping), domain, range, image, inverse image, into/onto)**.**

Definition 2.2 (Cardinal number)**.** If there exists a 1-1 mapping of A onto B, we say that A and B have the same *cardinal number*, or A and B are equivalent, $A \sim B$. (reflexive, symmetric, transitive)

Definition 2.3 (Finite, infinite, countable (enumerable, denumerable), at most countable)**.**

Remark 2.4. A is infinite if A is equivalent to one of its proper subsets.

Definition 2.5 (Sequence, terms)**.** A function f defined on the set of <u>all</u> positive integers. The values x_n of f are called the *terms* of the sequence. If A is a set and if $x_n \in A$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is said to be a *sequence* in A.

Theorem 2.6. Every infinite subset of a countable set A is countable.

Corollary 2.7. No uncountable set can be a subset of a countable set.

Definition 2.8 (Collection (family))**.** Let A and Ω be sets, and suppose that with each element α of A there is associated a subset of Ω which we denote by E_α. Then $\{E_\alpha\}$ is a collection of sets.

Definition 2.9 (Intersect, disjoint)**.** If $A \cap B$ is not empty, we say that A and B intersect; otherwise they are disjoint.

Theorem 2.10. Let $\{E_n\}$, $n \in \mathbb{N}$ be a sequence of countable sets, then $S = \bigcup_{n=1}^{\infty} E_n$ is countable.

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Corollary 2.11. Suppose A is at most countable, and for every $\alpha \in A$, B_α is at most countable. Then, $T = \bigcup_{\alpha\in A} B_\alpha$ is at most countable.

Theorem 2.12. Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \dots, a_n)$, where $a_k \in A$ $(k = 1, \dots, n)$, and the elements $a_1, \dots, a_n$ need not be distinct. Then B_n is countable.

Corollary 2.13. The set of all rational numbers is countable.

Theorem 2.14. Let A be the set of all sequences whose elements are digits 0 and 1. This set A is uncountable.

Proof. Cantor's diagonal process. □

Corollary 2.15. The set of all real numbers is uncountable.

2.2 Metric Spaces

Definition 2.16 (Metric Space)**.** A set X, whose elements we shall call *points*, is said to be a metric space if with any two points p and q of X there is associated a real number $d(p, q)$, called the distance from p to q, s.t.

Positive definiteness $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$;

Symmetry $d(p, q) = d(q, p)$;

Triangle Inequality $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$.

Any function with these three properties is called a *distance function*, or a *metric*.

Remark 2.17. Every subset Y of a metric space X is a metric space in its own right, with the same metric.

Definition 2.18 (Segment, interval, k-cell, open(closed) ball)**.** By the segment (a, b), we mean the set of all real numbers x s.t. $a < x < b$.

By the interval $[a, b]$, we mean the set of all real numbers x s.t. $a \leq x \leq b$.

If $a_i < b_i$ for $i = 1, \dots, k$, the set of all points $\mathbf{x} = (x_1, \dots, x_k)$ in $\mathbb{R}^k$ whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i (1 \leq i \leq k)$ is called a *k-cell*.

If $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$, the open (or closed) ball B with center at $\mathbf{x}$ and radius r is defined to be the set of all $\mathbf{y} \in \mathbb{R}^k$ s.t. $|\mathbf{y} - \mathbf{x}| < r$ (or $|\mathbf{y} - \mathbf{x}| \leq r$).

Definition 2.19 (Convex)**.** We call a set $E \subset \mathbb{R}^k$ *convex* if $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in E$ whenever $\mathbf{x} \in E$, $y \in E$, and $0 < \lambda < 1$.

Definition 2.20 (Neighborhood)**.** Let p be a point in matric space X. A neighborhood of p is a subset $N_r(p)$ consisting of all q s.t. $d(p,q) < r$, for some $r > 0$. The number r is called the *radius* of $N_r(p)$.

Definition 2.21 (Limit point, isolated point)**.** A point p is a *limit point* of the set E if every neighborhood of p contains a point $\underline{q \neq p}$, s.t. $q \in E$. If $p \in E$ and p is not a limit point of E, then p is called an isolated point of E.

Definition 2.22 (Closed)**.** E is closed if every limit point of E is a point of E.

Definition 2.23 (Interior point)**.** A point p is an interior point of E if there is a neighborhood N of p s.t. $N \subset E$.

Definition 2.24 (Open)**.** E is open if every point of E is an interior point of E.

Definition 2.25 (Complement)**.** Let E be a set in metric space X. The complement of E (denoted by E^c) is the set of all points $p \in X$ s.t. $p \notin E$.

Definition 2.26 (Perfect)**.** E is perfect if E is closed and if every point of E is a limit point of E.

Definition 2.27 (Bounded)**.** Let E be a set in metric space X. E is bounded if there is a real number M and a point $q \in X$ s.t. $d(p,q) < M$ for all $p \in E$.

Definition 2.28 (Dense)**.** E is dense in the metric space X if every point of X is a point of E or a limit point E (or both).

Theorem 2.29. Every neighborhood is an open set.

Theorem 2.30. If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

Corollary 2.31. A finite set has no limit points.

Theorem 2.32. Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α. Then

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c).$$

Theorem 2.33. A set E is open if and only if its complement is closed.

Corollary 2.34. A set F is closed if and only if its complement is open.

Remark 2.35. Prove the complement is open when it's hard to prove the primal is closed.

Theorem 2.36. Let $\{E_\alpha\}$ be a collection of open sets E_α, and $\{F_\alpha\}$, a collection of closed sets F_α.

(a) $\bigcup_\alpha E_\alpha$ is open, and when $\{E_\alpha\}$ is finite, $\bigcap_\alpha E_\alpha$ is also open.

(b) $\bigcap_\alpha F_\alpha$ is closed, and when $\{F_\alpha\}$ is finite, $\bigcup_\alpha F_\alpha$ is also closed.

Remark 2.37. Examples when finiteness is violated are $(\frac{1}{n}, \frac{1}{n}), n \in \mathbb{N}$ and $[\frac{1}{n}, 1], n \in \mathbb{N}$.

Definition 2.38 (Closure)**.** If X is a metric space and $E \subset X$, then the closure of E is $\bar{E} = E \cup E'$, where E' denotes the set of all the limit points of E <u>in X</u>.

Theorem 2.39. If X is a metric space and $E \subset X$, then

(a) $\bar{E}$ is closed.

(b) $E = \bar{E}$ iff. E is closed.

(c) $\bar{E} \subset F$ for every closed set $F \in X$ s.t. $E \subset F$.

By (a) and (c), $\bar{E}$ is the smallest closed subset of X that contains E.

Theorem 2.40. Let E be a <u>nonempty</u> set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \bar{E}$. Hence, $y \in E$ if E is closed.

Theorem 2.41. Suppose $Y \subset X$. A subset E of Y is open relative to Y iff. $E = Y \cap G$ for some open subset $G \subset X$.

2.3 Compact Sets

Definition 2.42 (Open cover)**.** The open cover of a set E in a metric space X is a collection $\{G_\alpha\}$ of open subsets $G_\alpha \in X$ s.t. $E \subset \bigcup_\alpha G_\alpha$.

Definition 2.43 (Compact)**.** A subset K of X is compact iff. there exists a finite subcover for every open cover of K.

Remark 2.44. Every finite set is compact.

Theorem 2.45. Suppose $K \subset Y \subset X$. Then K is compact relative to X iff. K is compact relative to Y.

Theorem 2.46. Compact subsets of metric spaces are closed.

Theorem 2.47. Closed subsets of compact sets are compact.

Theorem 2.48. The intersection of a compact set and a closed set is compact.

Theorem 2.49 (Finite Intersection Property)**.** If $\{K_\alpha\}$ is a collection of <u>compact</u> subsets of a metric space s.t. the intersection of any finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap_\alpha K_\alpha$ is nonempty.

Corollary 2.50. If $\{K_n\}$ is a sequence of nonempty compact sets s.t. $K_n \supset K_{n+1}, n \in \mathbb{N}$, then $\bigcap_1^\infty K_n$ is not empty.

Theorem 2.51. The set K is compact iff. every infinite subset of K has a limit point in K.

Proof. Countable bases. $\square$

Corollary 2.52 (Weierstrass Theorem)**.** Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Theorem 2.53. Every k-cell is compact.

Theorem 2.54 (Heine-Bored Theorem)**.** A subset of $\underline{\mathbb{R}^k}$ is compact iff. it's closed and bounded.

Remark 2.55. A subset of a general metric space does not necessarily have the above property. For instance, in the metric space of bounded real sequences and maximum difference in all coordinates as the metric, the subset with only one coordinate being 1 and others being 0 is closed, bounded, but not compact.

Theorem 2.56. Any nonempty perfect subset of $\mathbb{R}^k$ is uncountable.

Corollary 2.57. Any segment or interval is uncountable. $\mathbb{R}$ is uncountable.

Remark 2.58 (The Cantor set)**.** An uncountable subset of $\mathbb{R}^1$ which contains no segment.

Definition 2.59 (Separated)**.** Two subsets A and B of a metric space X are said to be separated if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty.

Remark 2.60. Separate $\Rightarrow$ disjoint. Disjoint $\not\Rightarrow$ separated.

Definition 2.61 (Connected sets)**.** A set $E \subset X$ is said to be connnected if E is not a union of two nonempty separated sets.

Theorem 2.62. A subset E of the real line $\mathbb{R}^1$ is connected iff. it holds that, if $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.

Theorem 2.63. Properties of connnected sets.

(a) If A is connected and $A \subset B \subset \bar{A}$, then B is also connected.

(b) The union of a collection of connected sets with some point in common to all of them is connected.

(c) If A and B are separated sets in X and $A \cup B = X$, and if $C \subset X$ is connected, then $C \subset A$ or $C \subset B$.

(d) A is connected iff. for every pair of open sets U, V, s.t. $A \subset U \cup V$ and $U \cap V \cap A = \emptyset$, then $A \subset U$ or $A \subset V$.

(e) A is connected $\Rightarrow \bar{A}$ is connected.

Definition 2.64 (Algebraic)**.** A complex number z is said to be algebraic if there are integers $a_0, \dots, a_n$, not all zero, s.t.

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Definition 2.65 (Separable)**.** A metric space is called separable if it contains a countable dense subset.

Definition 2.66 (Base)**.** A collection $\{V_\alpha\}$ of open subsets of X is said to be a base for X if for every $x \in X$ and every open set $G \subset X$ s.t. $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Theorem 2.67. Every separable metric space has a countable base (and vice versa).

3 Numerical Sequences and Series

3.1 Convergent Sequences

Definition 3.1 (Converge)**.** A sequence $\{p_n\}$ in a metric space X is said to converge in X if there is a point $p \in X$ with the following property: For any $\epsilon > 0$, there is an integer $N > 0$ s.t. $n \geq N$ implies that $d(p_n, p) < \epsilon$. We write $\lim_{n\to\infty} p_n = p$. If $\{p_n\}$ does not converge, it is said to diverge.

Definition 3.2 (Range, Bounded)**.** The set of all points $p_n (n = 1, 2, 3, \dots)$ is the range of $\{p_n\}$. The sequence $\{s_n\}$ is said to be bounded if its range is bounded.

Theorem 3.3. Let $\{p_n\}$ be a sequence in a metric space X.

(a) $\{p_n\}$ converges to $p \in X$ iff. every neighborhood of p contains p_n for all but finitely many n.

(b) If $p, q \in X$, and $\{p_n\}$ converges to p and converges to q, then $p = q$.

(c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

(d) If $E \subset X$ and p is a limit point of E, then there exists a sequence $\{p_n\}$ in E s.t. $p = \lim_{n\to\infty} p_n$.

Theorem 3.4. Suppose $\mathbf{x}_n \in \mathbb{R}^k, n \in \mathbb{N}$ and $\mathbf{x}_n = (\alpha_{1,n}, \dots, \alpha_{k,n})$. Then $\{\mathbf{x}_n\}$ converges to $\mathbf{x} = (\alpha_1, \dots, \alpha_k)$ iff. $\lim_{n\to\infty} \alpha_{j,n} = \alpha_j$. for $1 \leq j \leq k$.

Theorem 3.5. Suppose $\{\mathbf{x}_n\}$, $\{\mathbf{y}_n\}$ are sequences in $\mathbb{R}^k$, $\{\beta_n\}$ is a sequence of real numbers, and $\mathbf{x}_n \to \mathbf{x}$, $\mathbf{y}_n \to \mathbf{y}$, $\beta_n \to \beta$. Then

$$\lim_{n\to\infty}(\mathbf{x}_n + \mathbf{y}_n) = \mathbf{x} + \mathbf{y}, \quad \lim_{n\to\infty}(\mathbf{x}_n \cdot \mathbf{y}_n) = \mathbf{x} \cdot \mathbf{y}, \quad \lim_{n\to\infty} \beta_n \mathbf{x}_n = \beta \mathbf{x}.$$

For $k = 2$, $\lim_{n\to\infty} \frac{1}{\mathbf{x}_n} = \frac{1}{\mathbf{x}}$, provided $\mathbf{x}_n \neq 0$ for $n \in \mathbb{N}$, and $s \neq 0$.

3.2 Subsequences

Definition 3.6 (Subsequence)**.** Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers, s.t. $n_1 < n_2 < n_3 < \cdots$. Then the sequence $\{p_{n_1}\}$ is called a subsequence of $\{p_n\}$. If $\{p_{n_1}\}$ converges, its limit is called a subsequential limit of $\{p_n\}$.

Theorem 3.7.

(a) $\{p_n\}$ converges to p iff. every subsequence of $\{p_n\}$ converges to p.

(b) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X.

(c) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

Theorem 3.8. The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.