

Solutions of Partial Differential Equations

UNIT 6 [5 hrs]

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Solution of Partial Differential Equation

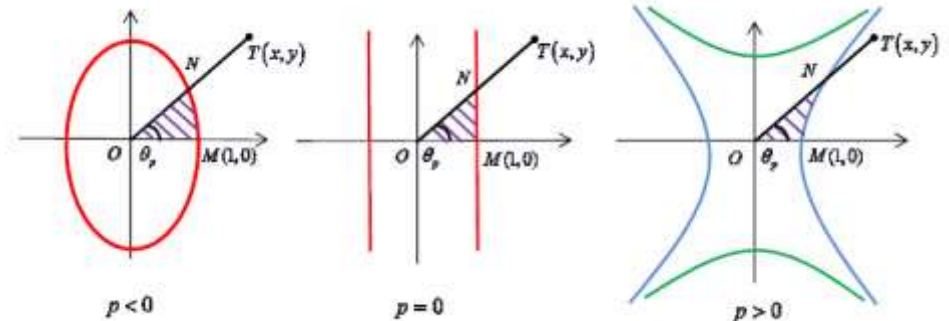
A partial differential equation is a mathematical equation that involves two or more independent variables, an unknown function (dependent on those variable), and a partial derivatives of the unknown function with respect to the independent variables.

If we represent the dependent variable as f and the two independent variables as x & y then second-order equation is given as

$$a \frac{\partial^2 f}{\partial x^2} + b \frac{\partial^2 f}{\partial x \partial y} + c \frac{\partial^2 f}{\partial y^2} = F(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$$

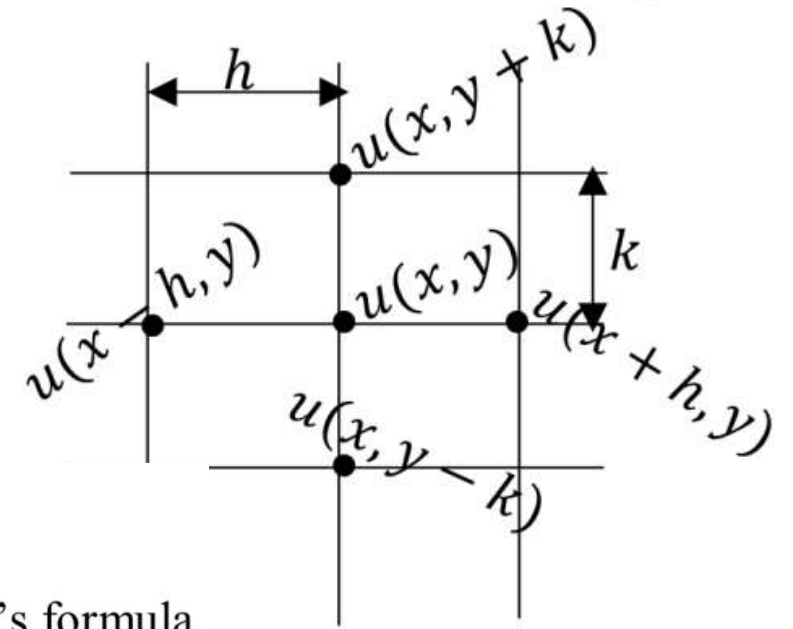
Where the coefficients a, b & c may be constants or functions of x & y . Depending upon the value of $b^2 - 4ac$, a 2nd order linear partial differential equation can be classified into three categories:

- Elliptical at point (x, y) if $b^2 - 4ac < 0$
- Parabolic if $b^2 - 4ac = 0$
- Hyperbolic if $b^2 - 4ac > 0$



Deriving Difference Equations

Consider a two dimensional solution domain as shown in figure below;



The domain is split into regular rectangle grids of width h and height k .

Let $u(x, y)$ be the function of two independent variables x & y . Then by Taylor's formula

$$u(x + h, y) = u(x, y) + hu_x(x, y) + \frac{h^2}{2!}u_{xx}(x, y) + \frac{h^3}{3!}u_{xxx}(x, y) + \cdots \dots \dots (i)$$

$$u(x - h, y) = u(x, y) - hu_x(x, y) + \frac{h^2}{2!}u_{xx}(x, y) - \frac{h^3}{3!}u_{xxx}(x, y) + \cdots \dots \dots (ii)$$

$$u(x, y + k) = u(x, y) + ku_y(x, y) + \frac{k^2}{2!}u_{yy}(x, y) + \frac{k^3}{3!}u_{yyy}(x, y) + \cdots \dots \dots (iii)$$

$$u(x, y - k) = u(x, y) - ku_y(x, y) + \frac{k^2}{2!}u_{yy}(x, y) - \frac{k^3}{3!}u_{yyy}(x, y) + \cdots \dots \dots (iv)$$

Subtracting eq.(ii) by (i) & neglecting higher power of h we get;

$$u(x + h, y) - u(x - h, y) = 2hu_x(x, y)$$

$$\therefore u_x(x, y) = \frac{u(x + h, y) - u(x - h, y)}{2h}$$

Subtracting eq.(iv) by (iii) & neglecting higher power of k we get;

$$u(x, y + k) - u(x, y - k) = 2ku_y(x, y)$$

$$\therefore u_y(x, y) = \frac{u(x, y + k) - u(x, y - k)}{2k}$$

Adding eq.(i) and (ii) and neglecting higher of power of h we get;

$$u(x + h, y) + u(x - h, y) = 2u(x, y) + h^2 u_{xx}(x, y)$$

$$\therefore u_{xx}(x, y) = \frac{1}{h^2} [u(x + h, y) - 2u(x, y) + u(x - h, y)] \text{---(a)}$$

Adding eq.(iii) and (iv) and neglecting higher of power of k we get;

$$u(x, y + k) + u(x, y - k) = 2u(x, y) + k^2 u_{yy}(x, y)$$

$$\therefore u_{yy}(x, y) = \frac{1}{k^2} [u(x, y + k) - 2u(x, y) + u(x, y - k)] \text{---(b)}$$

Also,

$$u_{xy}(x, y) = \frac{u(x + h, y + k) - u(x + h, y - k) - u(x - h, y + k) + u(x - h, y - k)}{4hk}$$

Laplace's Equation

The equation $u_{xx} + u_{yy} = 0$ is the Laplace equation, then from above eq.(a) & (b) we have,

$$\frac{1}{h^2} [u(x+h, y) - 2u(x, y) + u(x-h, y)] + \frac{1}{k^2} [u(x, y+k) - 2u(x, y) + u(x, y-k)] = 0$$

If $h = k$ we get,

$$u(x+h, y) + u(x, y+k) + u(x-h, y) + u(x, y-k) - 4u(x, y) = 0$$

$$\therefore u(x, y) = \frac{1}{4} [u(x+h, y) + u(x, y+k) + u(x-h, y) + u(x, y-k)]$$

This is the difference equation for Laplace's equation.

Poisson's Equation

The equation $u_{xx} + u_{yy} = g(x, y)$ is the given Poisson's equation, then from above eq.(a) & (b) we have,

$$\frac{1}{h^2} [u(x+h, y) - 2u(x, y) + u(x-h, y)] + \frac{1}{k^2} [u(x, y+k) - 2u(x, y) + u(x, y-k)] = g(x, y)$$

If $h = k$ we get,

$$u(x+h, y) + u(x, y+k) + u(x-h, y) + u(x, y-k) - 4u(x, y) = h^2 g(x, y)$$

This is the difference equation for Poisson's equation.

Examples

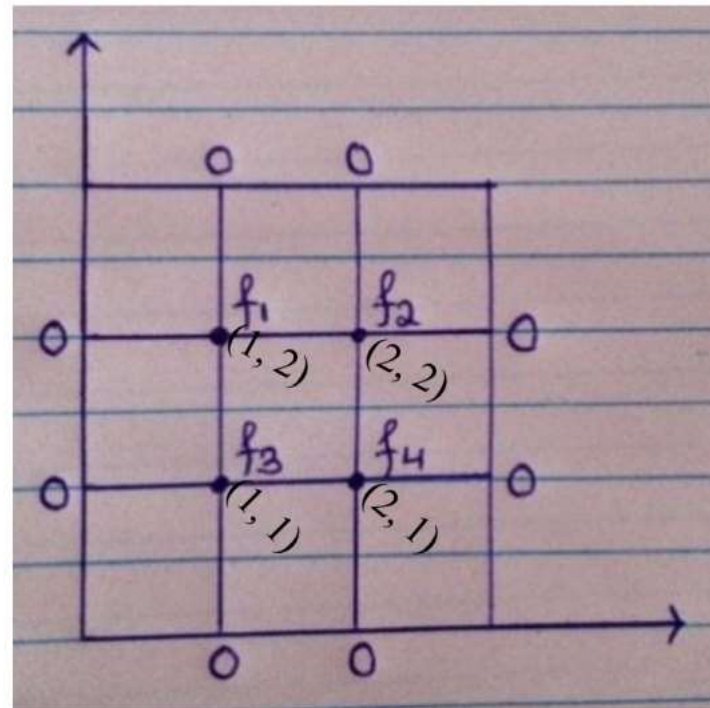
1. Solve the Poisson's equation $\nabla^2 f = 2x^2y^2$ over the square domain $0 \leq x \leq 3$ and $0 \leq y \leq 3$ with $f = 0$ on the boundary and $h=1$.

Solution:

Given Poisson's eqn. is $\nabla^2 f = 2x^2y^2$; $0 \leq x \leq 3, 0 \leq y \leq 3$

$h = 1$

Let's divide the domain into grids of 3×3 with $f = 0$ at the boundary as below



Now, from the difference equation for the Poisson's equation,

At f_1

$$0 + 0 + f_2 + f_3 - 4f_1 = 1^2 \times 2 \times 1^2 \times 2^2$$

$$\text{Or, } f_2 + f_3 - 4f_1 = 8 \dots\dots\dots (i)$$

At f_2

$$0 + 0 + f_1 + f_4 - 4f_2 = 1^2 \times 2 \times 2^2 \times 2^2$$

$$\text{Or, } f_1 + f_4 - 4f_2 = 32 \dots\dots\dots (ii)$$

At f_3

$$0 + 0 + f_1 + f_4 - 4f_3 = 1^2 \times 2 \times 2^2 \times 1^2$$

$$\text{Or, } f_1 + f_4 - 4f_3 = 2 \dots\dots\dots (iii)$$

At f_4

$$0 + 0 + f_2 + f_3 - 4f_4 = 1^2 \times 2 \times 1^2 \times 1^2$$

$$\text{Or, } f_2 + f_3 - 4f_4 = 8 \dots\dots\dots (iv)$$

Solving these equations,

Using eq.(ii) in (iii)

$$32 - f_4 + 4f_2 + f_4 - 4f_3 = 2$$

$$4f_2 - 4f_3 = -30 \dots\dots\dots (a)$$

Using eq.(ii) in (iv)

$$f_2 + f_3 - 4(32 - f_1 + 4f_2) = 8$$

$$f_2 + f_3 - 128 + 4f_1 - 16f_2 = 8$$

$$-15f_2 + f_3 + 4f_1 = 136 \dots\dots\dots (b)$$

& we have eq. (i)

$$f_2 + f_3 - 4f_1 = 8 \dots\dots\dots (i)$$

Solving equation (a), (b) & (i) we get,

$$f_2 = -\frac{43}{4}$$

$$f_3 = -\frac{13}{4}$$

$$f_1 = -\frac{11}{2}$$

Using these values in eq. (ii)

$$f_4 = -\frac{11}{2}$$

$$\therefore f_1 = -\frac{11}{2}, f_2 = -\frac{43}{4}, f_3 = -\frac{13}{4} \text{ \& } f_4 = -\frac{11}{2}$$

2. The steady state two dimensional heat-flow in a metal plate is defined by $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$.

Given the boundary conditions as shown in figure below, find the temperatures at interior points T_1 , T_2 , T_3 and T_4 .

Using the difference equation for the Laplace equation,

At point T_1

$$T_1 = \frac{1}{4}(100 + 100 + T_2 + T_3)$$

$$\text{Or, } 4T_1 - T_2 - T_3 = 200 \dots\dots\dots (i)$$

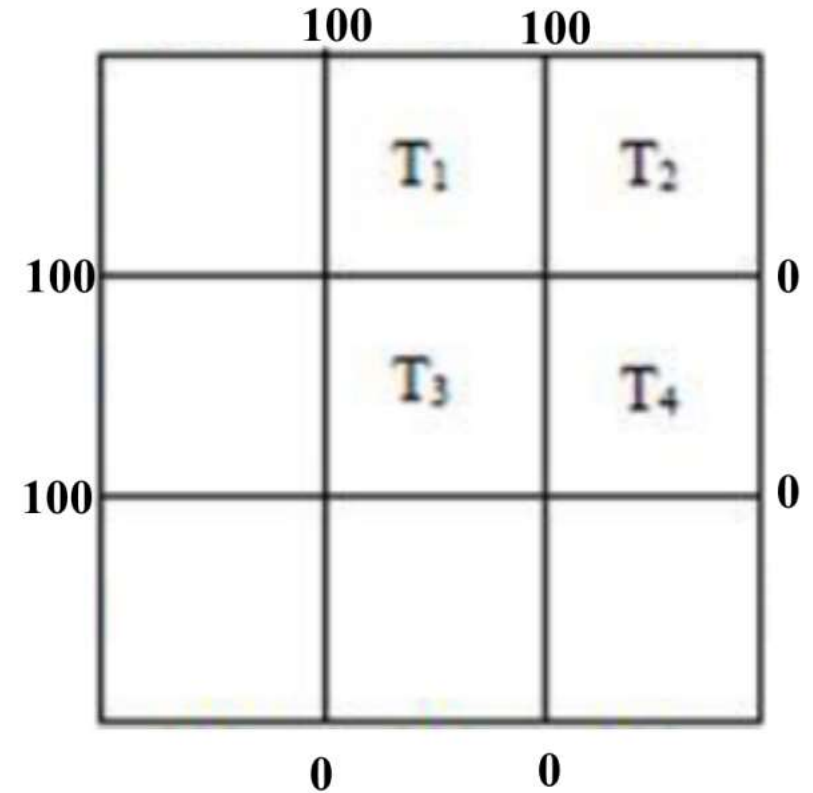
At point T_2

$$T_2 = \frac{1}{4}(100 + 0 + T_1 + T_4)$$

$$\text{Or, } 4T_2 - T_1 - T_4 = 100 \dots\dots\dots (ii)$$

At point T_3

$$T_3 = \frac{1}{4}(100 + 0 + T_1 + T_4)$$



$$\text{Or, } 4T_3 - T_1 - T_4 = 100 \dots\dots\dots (\text{iii})$$

At point T₄

$$T_4 = \frac{1}{4}(0 + 0 + T_2 + T_3)$$

$$\text{Or, } 4T_4 - T_2 - T_3 = 0 \dots\dots\dots (\text{iv})$$

Solving equation (i), (ii), (iii) & (iv) we get;

$$T_1 = 75$$

$$T_2 = 50$$

$$T_3 = 50$$

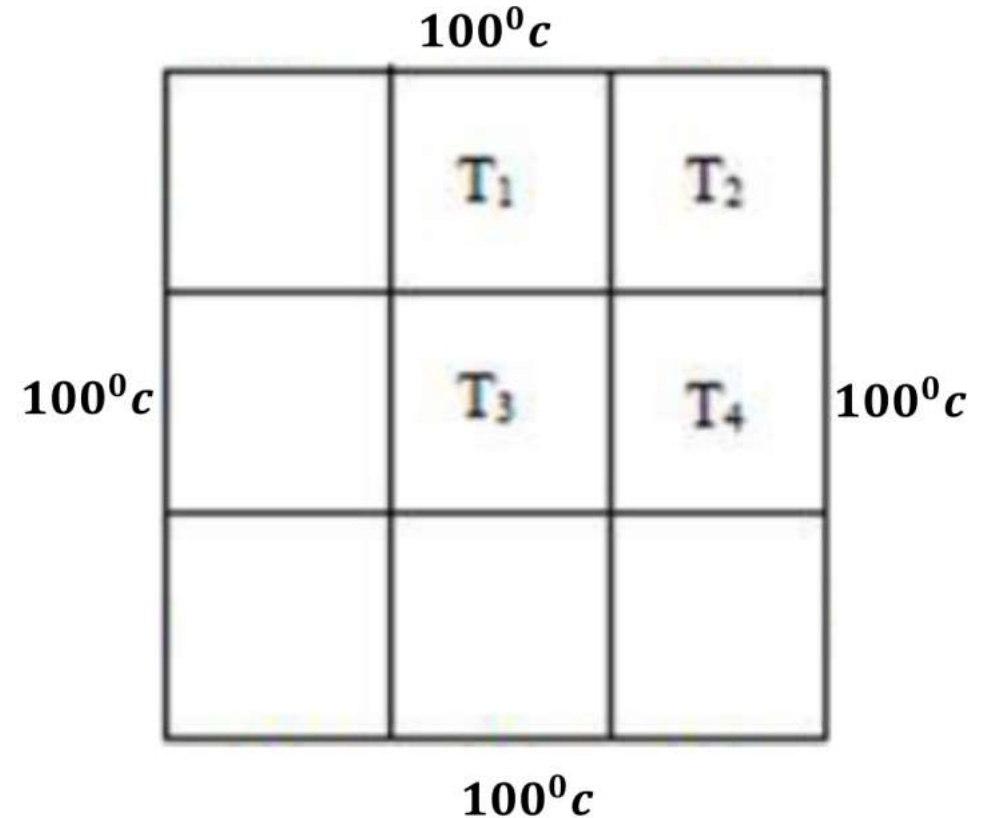
$$T_4 = 25$$

3. The steady-state two dimensional heat flow in a metal plate is defined by $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$.

Consider a metal plate of size $30\text{cm} \times 30\text{cm}$, the boundaries of which are held at 100°C . Calculate the temperature at interior points of the plate. Assume the grid size of $10\text{cm} \times 10\text{cm}$.

Solution:

Let T_1, T_2, T_3 & T_4 are the values of interior grid point.



Using the difference equation for the Laplace equation,

At point T_1

$$T_1 = \frac{1}{4}(100 + 100 + T_2 + T_3)$$

$$\text{Or, } 4T_1 - T_2 - T_3 = 200 \dots\dots\dots (i)$$

At point T_2

$$T_2 = \frac{1}{4}(100 + 100 + T_1 + T_4)$$

$$\text{Or, } 4T_2 - T_1 - T_4 = 200 \dots\dots\dots (ii)$$

At point T_3

$$T_3 = \frac{1}{4}(100 + 100 + T_1 + T_4)$$

$$\text{Or, } 4T_3 - T_1 - T_4 = 200 \dots\dots\dots (iii)$$

At point T_4

$$T_4 = \frac{1}{4}(100 + 100 + T_2 + T_3)$$

$$\text{Or, } 4T_4 - T_2 - T_3 = 200 \dots\dots\dots (iv)$$

Solving equation (i), (ii), (iii) & (iv) we get;

$$T_1 = 100$$

$$T_2 = 100$$

$$T_3 = 100$$

$$T_4 = 100$$

Homework 10

6. Derive a difference equation to represent a Poisson's equation. Solve the Poisson's equation $\nabla^2 f = 2x^2y^2$ over the domain $0 \leq x \leq 3, 0 \leq y \leq 3$ with $f = 0$ on the boundary and $h = 1$. (3 + 5) asked in 2066

6. a) How can you obtain numerical solution of a partial differential equation? Explain. (3) asked in 2069

b) The steady-state two-dimensional heat-flow in a metal plate is defined by $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$. Given the boundary conditions as shown in figure below, find the temperature at interior points T_1, T_2, T_3 and T_4 . (5)

