# Solution of Ordinary Differential Equations

UNIT 5 | [8 hrs]

**Unit 5 Solution of Ordinary Differential Equations** **7 Hrs.**

Introduction to Differential Equations, Initial Value Problem, Taylor Series Method, Picard's Method, Euler's Method and Its Accuracy, Heun's method,



Runge-Kutta Methods, Solution of Higher Order Equations, Boundary Value Problems, Shooting Method and Its Algorithm.

# Solution of Ordinary Differential Equations

Let $x$ be an independent variable and $y$ be a dependent variable. An equation with $x$, $y$ and its derivatives is called a differential equation.

Suppose the first order differential equation;

$$\frac{dy}{dx} = f(x, y) \quad \dots\dots\dots\dots \quad (i)$$

A solution to the differential equation is the value of $y$ which satisfies the differential equation.

***Initial Value Problem***

Consider the differential equation

$y' = f(x, y)$ with an initial condition $y(x_0) = y_0$.

This is the first order differential equation. Here the $y$ value at $x_0$ is given to be $y_0$. The solution $y$ at $x_0$ is given.

We must assume a small increment $h$.

$$x_1 = x_0 + h$$

$$x_2 = x_1 + h$$

$$\dots \dots \dots \dots \dots$$

$$\dots \dots \dots \dots \dots$$

$$x_{i+1} = x_i + h$$

Let us denote the $y$ values at $x_1, x_2, \dots\dots$ as $y_1, y_2, \dots\dots$ respectively.

$y_0$ is given and we must find out $y_1, y_2, \dots\dots$

The initial value $y_0$ is given. So, this differential equation is called an ***initial value problem.***

## Boundary Value Problem

Consider the following linear second order differential equation,

$$y'' + f(x)y' + g(x)y = F(x)$$

Suppose we are interested in solving this differential equation between the values $x = a$ & $z = b$. Hence $a$ & $b$ are two values such that $a < b$. Let us divide the interval $[a, b]$ into $n$ equal subintervals of length $h$ each.

Let $x_0, x_1, \dots \dots \dots, x_n$ be the pivotal points and $b$ are called the boundary points. Solving the differential equation means finding the values of $y_0, y_1, \dots \dots \dots, y_n$.

Suppose $y_0$ & $y_n$ are given. That is the solution values at the boundary points are given. Then the differential equation is called a **Boundary Value Problem**. So, the following is the general form of a boundary value problem.

$$y'' + f(x)y' + g(x)y = F(x)$$

$$y(a) = y_0 \,, \quad y(b) = y_n$$

# Boundary Value Problem

In numerical methods, a boundary value problem (BVP) refers to a type of differential equation problem that involves finding the solution to a differential equation subject to specified conditions at the boundaries of the domain. Unlike initial value problems (IVPs) that require initial conditions at a single point, boundary value problems require conditions at multiple points.

Let's consider the following second-order ordinary differential equation with boundary conditions:

$$y''(x) - 4y'(x) + 4y(x) = 0$$

with boundary conditions:

$$y(0) = 1$$
$$y(2) = 4$$

To solve this boundary value problem, we need to find the function $y(x)$ that satisfies the differential equation and the boundary conditions.

One approach to solving BVPs numerically is the shooting method. In the shooting method, we transform the BVP into an IVP by guessing an initial value for the derivative $y'(a)$ at the left boundary $x = a$ (in this case, $x = 0$), and then integrating the differential equation from $x = a$ to $x = b$ using a numerical integration method like the Runge-Kutta method. We then adjust the initial guess for $y'(a)$ iteratively until the value of $y(b)$ matches the right boundary condition $y(b) = 4$.

| Aspect | Initial Value Problem (IVP) | Boundary Value Problem (BVP) |
| --- | --- | --- |
| Definition | A differential equation with an initial condition. | A differential equation with boundary conditions. |
| Conditions | One condition at a specific point (initial point). | Multiple conditions at different boundary points. |
| Solution Type | Typically, unique solutions (may be generalizable). | Generally, not unique solutions (can have none, one, or multiple). |
| Nature of Solution | Usually requires only first-order derivatives. | Can require higher-order derivatives. |
| Existence of Solution | Often guaranteed under certain conditions (e.g., existence and uniqueness theorem). | Not always guaranteed; existence depends on the problem's nature. |
| Method of Solution | Often solved using numerical methods (e.g., Euler's method, Runge-Kutta). | May involve analytical methods or numerical techniques (e.g., Finite Element Method). |
| Example | Newton's second law of motion (acceleration as a function of time). | Heat conduction equation (temperature distribution with boundary temperatures). |

## ➢ Taylor's Series Method

$y$ is a function of $x$. It is written as $y(x)$. By Taylor's series about the point $x_0$;

$$y(x) = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + \cdots \dots \dots$$

$x_0$ & $y_0$ denote the initial value of $x$ & $y$.

***Examples***

***1. Find by Taylor's series method, the values of $y$ at $x = 0.1$ & $x = 0.2$ to fine places of decimal form.***

$$\frac{dy}{dx} = x^2 y - 1; \quad y(0) = 1$$

***Solution:***

Given,

$$\frac{dy}{dx} = x^2y - 1$$

$$y(0) = 1$$

i.e. $x_0 = 0$ & $y_0 = 1$

Here,

$$y' = x^2y - 1$$

$$y'' = x^2y' + 2xy$$

$$y''' = x^2y'' + 2xy' + 2(xy' + y) = x^2y'' + 4xy' + 2y$$

$$y^{iv} = x^2y''' + 6xy'' + 6y'$$

Now at $x_0 = 0$ & $y_0 = 1$;

$$y_0' = x_0^2y_0 - 1 = 0 - 1 = -1$$

$$y_0'' = x_0^2y_0' + 2x_0y_0 = 0 + 0 = 0$$

$$y_0''' = x_0^2y_0'' + 4x_0y_0' + 2y_0 = 0 + 0 + 2 \times 1 = 2$$

$$y_0^{iv} = x_0^2y_0''' + 6x_0y_0'' + 6y_0' = 0 + 0 + 6 \times (-1) = -6$$

Now, the Taylor's series is;

$$y(x) = y_0 + \frac{(x-x_0)}{1!}\, y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \frac{(x-x_0)^4}{4!} y_0'''' \quad \text{[Neglecting higher term]}$$

$$= 1 + \frac{x}{1!} \times (-1) + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!} \times 2 + \frac{x^4}{4!} \times (-6)$$

$$= 1 - \frac{x}{1!} + \frac{2x^3}{3!} - \frac{6x^4}{4!}$$

$$\therefore\ y(0.1) = 1 - \frac{0.1}{1!} + \frac{2\times 0.1^3}{3!} - \frac{6\times 0.1^4}{4!} = 0.900308$$

$$\therefore\ y(0.2) = 1 - \frac{0.2}{1!} + \frac{2\times 0.2^3}{3!} - \frac{6\times 0.2^4}{4!} = 0.802267$$

**2.** ***Find the solution of following differential equation using Taylor's series method.***

$$\boldsymbol{y' = (x^3 + xy^2)e^{(-x)},\ y(0) = 1 \text{ to find } y \text{ at } x = 0.1, 0.2, 0.3}$$

***Sol<sup></sup>ⁿ:***

Given,

$$y' = (x^3 + xy^2)e^{(-x)}$$

$$y(0) = 1 \text{ i.e. } x_0 = 0 \ \&\ y_0 = 1$$

Here

$$y' = (x^3 + xy^2)e^{(-x)}$$

$$\begin{aligned} y'' &= (x^3 + xy^2)(-e^{-x}) + e^{-x}(3x^2 + x2yy' + y^2) \\ &= e^{-x}(3x^2 + x2yy' + y^2 - x^3 - xy^2) \\ &= e^{-x}(3x^2 - x^3 + y^2 + 2xyy' - xy^2) \end{aligned}$$

$$\begin{aligned} y'' &= (x^3 + xy^2)(-e^{-x}) + e^{-x}(3x^2 + x2yy' + y^2) \\ &= e^{-x}(3x^2 + x2yy' + y^2 - x^3 - xy^2) \\ &= e^{-x}(3x^2 - x^3 + y^2 + 2xyy' - xy^2) \end{aligned}$$

$$\begin{aligned} y''' &= e^{-x}[6x - 3x^2 + 2yy' + 2\{x(yy'' + (y')^2) + yy'\} - (x2yy' + y^2)] - e^{-x}(3x^2 - x^3 + y^2 + 2xyy' - xy^2) \\ &= e^{-x}[6x - 3x^2 + 2yy' + 2x(yy'' + (y')^2) + 2yy' - 2xyy' - y^2 - 3x^2 + x^3 - y^2 - 2xyy' + xy^2] \\ &= e^{-x}[x^3 - 6x^2 + 6x + 4yy' + 2x(yy'' + (y')^2) - 4xyy' - 2y^2 + xy^2] \end{aligned}$$

Now at $x_0 = 0 \ \&\ y_0 = 1$;

$$y_0' = 0$$

$$y_0'' = 1(0 - 0 + 1 + 0 - 0) = 1$$

$$y_0''' = 1(0 - 0 + 0 + 0 + 0 - 0 - 2 + 0) = -2$$

Now, the Taylor's series is;

$$y(x) = y_0 + \frac{(x-x_0)}{1!}\, y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' \qquad \text{[Neglecting higher term]}$$

$$= 1 + \frac{(x-0)}{1!}\,(0) + \frac{(x-0)^2}{2!}(1) + \frac{(x-0)^3}{3!}(-2)$$

$$= 1 + \frac{x^2}{2} - \frac{x^3}{3}$$

$$\therefore y(0.1) = 1 + \frac{(0.1)^2}{2} - \frac{(0.1)^3}{3} = 1.0047$$

$$\therefore y(0.2) = 1 + \frac{(0.2)^2}{2} - \frac{(0.2)^3}{3} = 1.0173$$

$$\therefore y(0.3) = 1 + \frac{(0.3)^2}{2} - \frac{(0.3)^3}{3} = 1.036$$

***3. Use the Taylor method to solve the equation***

$$\boldsymbol{y' = x^2 + y^2}$$

***for*** $\boldsymbol{x = 0.25}$ ***and*** $\boldsymbol{x = 0.5}$ ***given*** $\boldsymbol{y(0) = 1.}$

**<u>Sol^n</u>:**

Given,

$$y' = x^2 + y^2$$

$$y(0) = 1 \text{ i.e. } x_0 = 0 \;\&\; y_0 = 1$$

Here,

$$y' = x^2 + y^2$$

$$y'' = 2x + 2yy'$$

$$y''' = 2 + 2yy'' + 2(y')^2$$

Now at $x_0 = 0 \;\&\; y_0 = 1$;

$$y_0' = 1$$

$$y_0'' = 0 + 2 = 2$$

$$y_0''' = 2 + 4 + 2 = 8$$

[Neglecting higher term]

Now, the Taylor's series is;

$$y(x) = y_0 + \frac{(x-x_0)}{1!}\, y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0'''$$

$$= 1 + \frac{(x-0)}{1!}\,(1) + \frac{(x-0)^2}{2!}(2) + \frac{(x-0)^3}{3!}(8)$$

$$= 1 + x + x^2 + \frac{8x^3}{3!}$$

$$\therefore y(0.25) = 1 + 0.25 + (0.25)^2 + \frac{8(0.25)^3}{3!} = 1.3333$$

$$\therefore y(0.25) = 1 + 0.5 + (0.5)^2 + \frac{8(0.5)^3}{3!} = 1.81667$$

## Picard's Method

Consider the differential equation;

$$\frac{dy}{dx} = f(x, y) \quad \dots\dots\dots \quad (i)$$

with given initial condition $y(x_0) = y_0$

Eq.(i) can be written as

$$dy = f(x, y)dx \quad \dots\dots\dots\dots \text{(ii)}$$

Integrating eq.(ii) from $x_0$ to $x$ w.r.to $x$.

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)dx$$

$$[y]_{y_0}^{y} = \int_{x_0}^{x} f(x, y)dx$$

$$y - y_0 = \int_{x_0}^{x} f(x, y)dx$$

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx$$

For 1st approximation we replace $y$ by $y_0$ we get,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

For 2nd approximation we replace $y$ by $y_1$ we get,

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

Similarly, for other approximation we make a general form;

$$y_i = y_0 + \int_{x_0}^{x} f(x, y_{i-1})dx$$

We continue this process until we get two successive approximation value equal.

***1. Obtain a solution up to the fifth approximation of the equation*** $\frac{dy}{dx} = y + x$ ***such that*** $y(0) = 1$ ***using Picard's process of successive approximation.***

***Solution:***

Here,

$$\frac{dy}{dx} = y + x$$

$$y(0) = 1$$

i.e. $x_0 = 0$ & $y_0 = 1$

Using Picard's formula, we have;

$$y_i = y_0 + \int_{x_0}^{x} f(x, y_{i-1})dx$$

1st approximation;

$$y_1 = y_0 + \int_0^x f(x, y_0)dx$$
$$y_1 = 1 + \int_0^x (y_0 + x)dx$$
$$y_1 = 1 + \int_0^x (y_0 + x)dx$$
$$y_1 = 1 + \int_0^x (1 + x)dx$$
$$y_1 = 1 + x + \frac{x^2}{2}$$

2nd approximation;

$$y_2 = y_0 + \int_0^x f(x, y_1)dx$$
$$y_2 = 1 + \int_0^x (y_1 + x)dx$$
$$y_2 = 1 + \int_0^x (1 + 2x + \frac{x^2}{2})dx$$
$$y_2 = 1 + x + x^2 + \frac{x^3}{6}$$

3rd approximation;

$$y_3 = y_0 + \int_0^x f(x, y_2)dx$$
$$y_3 = 1 + \int_0^x (y_2 + x)dx$$
$$y_3 = 1 + \int_0^x (1 + 2x + x^2 + \frac{x^3}{6})dx$$
$$y_3 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

4th approximation;

$$y_4 = y_0 + \int_0^x f(x, y_3)dx$$
$$y_4 = 1 + \int_0^x (y_3 + x)dx$$
$$y_4 = 1 + \int_0^x (1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24})dx$$
$$y_4 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

5th approximation;

$$y_5 = y_0 + \int_0^x f(x, y_4)dx$$
$$y_4 = 1 + \int_0^x (y_4 + x)dx$$
$$y_4 = 1 + \int_0^x (1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120})dx$$
$$y_4 = 1 + \int_0^x (1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120})dx$$
$$y_4 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}$$

**2.** ***Use Picard's method, estimate y(0.1) of the following equation;***

$$\boldsymbol{y'(x) = x^2 + y^2, \quad y(0) = 0}$$

***<u>Solution:</u>***

Here,

$y'(x) = x^2 + y^2$

$y(0) = 0$

i.e. $x_0 = 0$ & $y_0 = 0$

Using Picard's formula, we have;

$$y_i = y_0 + \int_{x_0}^{x} f(x, y_{i-1})dx$$

First approximation;

$$y_1 = y_0 + \int_0^x f(x, y_0)dx$$

$$y_1 = 0 + \int_0^x x^2 dx = \frac{x^3}{3}$$

$At\ x = 0.1,$

$$y_1 = 0.00033$$

Second approximation,

$$y_2 = y_0 + \int_0^x f(x, y_1)dx$$

$$y_2 = 0 + \int_0^x f(x, y_1)dx = \frac{x^3}{3} + \frac{x^7}{63}$$

$At\ x = 0.1,$

$$y_2 = 0.00033$$

Here, $y_1 = y_2$ up to 5 decimal places.

$\therefore y(0.1) = 0.00033$

## ➢ **Euler's Method**

In Euler's method, the slope at $(x_i, y_i)$ is used to estimate the value of $y(x_{i+1})$ as below;

$$y(x_{i+1}) = y(x_i) + m_1 h \;;\; m_1 = f(x_i, y_i)$$

Choosing smaller values of $h$ leads to more accurate results and more computation time.

***Algorithm:***

1. *Define* $f(x, y)$.
2. *Read* $x_0, y_0, h$ *and* $xn$ *where* $x_0$ & $y_0$ *are initial conditions, h is the interval and xp is the required value.*
3. $n = \frac{xp - x_0}{h}$
4. *Start loop from* $i = 1$ *to* $n$
5. $y = y_0 + h * f(x_0, y_0)$
   $x = x + h$
6. *Print values of* $y_0$ & $x_0$.
7. *Check if* $x < xp$
   *assign* $x_0 = x$ *and* $y_0 = y$
   *else*
   *goto 8.*
8. *End loop* i
9. *Stop*

## ***Examples***

***1. Given*** $\boldsymbol{y' = xy}$***,*** $\boldsymbol{y(1) = 1}$***. Find*** $\boldsymbol{y(2)}$ ***with*** $\boldsymbol{h = 0.25}$***.***

***Solution:***

Here,

$y' = f(x, y) = xy$

$y(1) = 1$ **i.e.** $x_0 = 1$ & $y_0 = 1$

Then,

$y(1) = y_0 = 1$

$$y(1.25) = y_1 = y_0 + hf(x_0, y_0) = y_0 + h(x_0 * y_0) = 1 + 0.25 * (1 * 1) = 1.25$$

$$y(1.5) = y_2 = y_1 + hf(x_1, y_1) = 1.25 + 0.25 * (1.25 * 1.25) = 1.64$$

$$y(1.75) = y_3 = y_2 + hf(x_2, y_2) = 1.64 + 0.25 * (1.5 * 1.64) = 2.26$$

$$y(2) = y_4 = y_3 + hf(x_3, y_3) = 2.26 + 0.25 * (1.75 * 2.26) = 3.25$$

Hence,

$y(2) = 3.25$

**2.** ***Given the equation*** $\boldsymbol{y' = 2x^3 - 3xy,\ y(1) = 2.}$ ***Find*** $\boldsymbol{y(2.5)}$ ***with*** $\boldsymbol{h = 0.5.}$

***<u>Solution:</u>***

Here,

$$y' = f(x,y) = 2x^3 - 3xy$$

$y(1) = 2$ **i.e.** $x_0 = 1$ & $y_0 = 2$

Then,

$$y(1) = y_0 = 2$$

$$y(1.5) = y_1 = y_0 + hf(x_0, y_0) = 2 + 0.5[2 - 3(1)(2)] = 0$$

$$y(2) = y_2 = y_1 + hf(x_1, y_1) = 0 + 0.5[2 * 1.5^3 - 3 * 1.5 * 0] = 3.375$$

$$y(2.5) = y_3 = y_2 + hf(x_2, y_2) = 3.375 + 0.5[2 * 2^3 - 3 * 2 * 3.375] = 1.25$$

Hence,

$$y(2.5) = 1.25$$

## ➢ **Heun's Method**

- This method is also called ***second order Runge-Kutta method*** or ***Modified Euler's method.***

In Heun's method, we use the average of the slopes computed at the beginning and at the end of the interval.

Using Heun's method, we can estimate the value of $y(x_{i+1})$ as below;

$$y(x_{i+1}) = y(x_i) + \frac{h}{2}(m_1 + m_2) \qquad // \; y(x_{i+1}) = y(x_i + h)$$

Where, $m_1 = f(x_i, y_i)$

$$m_2 = f(x_i + h, y_i + m_1 \times h)$$

***Algorithm:***

1. *Define* $f(x, y)$.
2. *Read* $x_0, y_0, h$ *and* $n$
3. *For i=0 to n-1 do*
4. $x_{i+1} = x_i + h$
5. $m_1 = f(x_i, y_i)$
6. $m_2 = f(x_i + h, y_i + m_1 \times h)$
7. $y_{i+1} = y_i + \frac{h}{2}(m_1 + m_2)$
8. *Print* $x_{i+1}$ , $y_{i+1}$
9. *Next i*
10. *End*

***1.* Use the Heun's method to estimate $y(0.4)$ when $y'(x) = x^2 + y^2$ with $y(0) = 0$. Assume $h = 0.2$.**

Here,

$y'(x) = f(x, y) = x^2 + y^2$

$y(0) = 0$ **i.e.** $x_0 = 0$ & $y_0 = 0$

$h = 0.2$

From Heun's method, we have;

1st iteration:

$$m_1 = f(x_0, y_0) = x_0^2 + y_0^2 = 0 + 0 = 0$$

$$m_2 = f(x_0 + h, y_0 + m_1 * h) = f(0 + 0.2, 0 + 0 * 0.2) = f(0.2, 0) = 0.2^2 + 0^2 = 0.04$$

$$\therefore y(x_0 + h) = y(x_0) + \frac{h}{2}(m_1 + m_2)$$

$$y(0 + 0.2) = y(0.2) = y(0) + \frac{0.2}{2}(m_1 + m_2) = 0 + \frac{0.2}{2}(0 + 0.04) = 0.004$$

$$\therefore y(0.2) = 0.004$$

2nd iteration:

Here,

$x_1 = 0.2$ & $y_1 = 0.004$

$$m_1 = f(x_1, y_1) = x_1^2 + y_1^2 = 0.2^2 + 0.004^2 = 0.040016$$

$$m_2 = f(x_1 + h, y_1 + m_1 * h) = f(0.4, 0.012) = 0.4^2 + 0.012^2 = 0.160144$$

$$\therefore y(x_1 + h) = y(x_1) + \frac{h}{2}(m_1 + m_2)$$

$$y(0.2 + 0.2) = y(0.4) = y(0.2) + \frac{0.2}{2}(0.040016 + 0.160144) = 0.004 + 0.02 = 0.024$$

$$\boldsymbol{\therefore y(0.4) = 0.024}$$

**2.** ***Apply Runge Kutta method of 2nd order to find an approximate value of $y$ when $x = 0.2$ given that $\frac{dy}{dx} = x + y$ and $y(0) = 1$.***

Here,

$\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{x} + \mathrm{y}$

$\mathrm{y}(0) = 1$ i.e. $x_0 = 0$ & $y_0 = 1$

let us assume $h = 0.2$

$$m_1 = f(x_0, y_0) = f(0, 1) = 0 + 1 = 1$$

$$m_2 = f(x_0 + h, y_0 + m_1 \times h) = f(0 + 0.2, 1 + 1 \times 0.2) = f(0.2, 1.2) = 1.4$$

Then,

$$y(0.2) = y(0) + \frac{0.2}{2}(m_1 + m_2) = 1 + \frac{0.2}{2}(1 + 1.4) = 1.24$$

$$\therefore y(0.2) = 1.24$$

## Fourth Order Runge-Kutta (R-K) Method

$$y_{i+1} = y_i + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

Where,

$$m_1 = f(x_i, y_i)$$
$$m_2 = f(x_i + \frac{h}{2}, y_i + \frac{m_1 h}{2})$$
$$m_3 = f(x_i + \frac{h}{2}, y_i + \frac{m_2 h}{2})$$
$$m_4 = f(x_i + h, y_i + m_3 h)$$

$$y_1 = y_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

Where,

$$m_1 = f(x_0, y_0)$$
$$m_2 = f(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2})$$
$$m_3 = f(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2})$$
$$m_4 = f(x_0 + h, y_0 + m_3 h)$$

Similarly, for second interval

$$y_2 = y_1 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

Where,

$$m_1 = f(x_1, y_1)$$
$$m_2 = f(x_1 + \frac{h}{2}, y_1 + \frac{m_1 h}{2})$$
$$m_3 = f(x_1 + \frac{h}{2}, y_1 + \frac{m_2 h}{2})$$
$$m_4 = f(x_1 + h, y_1 + m_3 h)$$

***Algorithm:***

1. *Define* $f(x, y)$.
2. *Read* $x_0, y_0, h$ *and* $n$
3. *For i=0 to n-1 do*
4. $x_{i+1} = x_i + h$
5. $m_1 = f(x_i, y_i)$
6. $m_2 = f(x_i + \frac{h}{2}, y_i + \frac{m_1 h}{2})$
7. $m_3 = f(x_i + \frac{h}{2}, y_i + \frac{m_2 h}{2})$
8. $m_4 = f(x_i + h, y_i + m_3 h)$
9. $y_{i+1} = y_i + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$
10. *Print* $x_{i+1}$ , $y_{i+1}$
11. *Next i*
12. *End*

**1.** ***Apply Runge Kutta method of 4th order to find an approximate value of*** $\boldsymbol{y}$ ***when*** $\boldsymbol{x = 0.2}$ ***given that*** $\boldsymbol{\frac{dy}{dx} = x + y}$ ***and*** $\boldsymbol{y(0) = 1}$.

Here,

$$\frac{dy}{dx} = f(x,y) = x + y$$

$y(0) = 1$ $i.e.$ $x_0 = 0$ & $y_0 = 1$

let us assume $h = 0.2$

Now, from Runge-Kutta method, we have,

$$m_1 = f(x_0, y_0) = x_0 + y_0 = 0 + 1 = 1$$

$$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}\right) = f\left(0 + \frac{0.2}{2}, 1 + \frac{1 \times 0.2}{2}\right) = f(0.1, 1.1) = 1.2$$

$$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2}\right) = f\left(0 + \frac{0.2}{2}, 1 + \frac{1.2 \times 0.2}{2}\right) = f(0.1, 1.12) = 1.22$$

$$m_4 = f(x_0 + h, y_0 + m_3 h) = f(0 + 0.2, 1 + 1.22 \times 0.2) = f(0.2, 1.244) = 1.444$$

Hence,

$$y_1 = y(x_0 + h) = y_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

$$\therefore y(0.2) = 1 + \frac{0.2}{6}(1 + 2 \times 1.2 + 2 \times 1.22 + 1.444)$$

$$= 1.2428$$

**2.** ***Obtain y(1.5) from given differential equation using Runge-Kutta 4th order method.***

$$\frac{dy}{dx} + 2x^2y = 1 \text{ and } y(1) = 0. \qquad [\text{take } h=0.25]$$

Here,

$f(x, y) = \frac{dy}{dx} = 1 - 2x^2y$

$y(1) = 0$ i.e. $x_0 = 1$ & $y_0 = 0$

$h = 0.25$

Hence,

$$y_1 = y(x_0 + h) = y_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

$$\therefore y(1.25) = 0 + \frac{0.25}{6}(1 + 2 \times 0.684 + 2 \times 0.784 + 0.388)$$
$$= 0.18$$

Now, from Runge-Kutta method, we have,

<u>1st iteration</u>

$$m_1 = f(x_0, y_0) = 1 - 2x_0^2y_0 = 1 - 2 \times 1^2 \times 0 = 1 - 0 = 1$$

$$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1h}{2}\right) = f\left(1 + \frac{0.25}{2}, 0 + \frac{1\times0.25}{2}\right) = f(1.125, 0.125) = 0.684$$

$$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2h}{2}\right) = f\left(1 + \frac{0.25}{2}, 0 + \frac{0.684\times0.25}{2}\right) = f(1.125, 0.0855) = 0.784$$

$$m_4 = f(x_0 + h, y_0 + m_3h) = f(1 + 0.25, 0 + 0.784 \times 0.25) = f(1.25, 0.196) = 0.388$$

2nd iteration

Now,

$x_1 = 1.25$ & $y_1 = 0.18$

$m_1 = f(x_1, y_1) = 1 - 2x_1^2 y_1 = 1 - 2 \times 1.25^2 \times 0.18 = 1 - 0.5625 = 0.437$

$m_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{m_1 h}{2}\right) = f\left(1.25 + \frac{0.25}{2}, 0.18 + \frac{0.437 \times 0.25}{2}\right) = f(1.375, 0.235) = 0.111$

$m_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{m_2 h}{2}\right) = f\left(1.25 + \frac{0.25}{2}, 0.18 + \frac{0.111 \times 0.25}{2}\right) = f(1.375, 0.194) = 0.266$

$m_4 = f(x_1 + h, y_1 + m_3 h) = f(1.25 + 0.25, 0.18 + 0.266 \times 0.25) = f(1.5, 0.246) =$ -0.107

Hence,

$y_2 = y(x_1 + h) = y_1 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$

$\therefore y(1.5) = 0.18 + \frac{0.25}{6}(0.437 + 2 \times 0.111 + 2 \times 0.266 + (-0.107))$

$= 0.2251$

$\therefore \boldsymbol{y(1.5) = 0.2251}$

## Solving Higher Order Differential Equation

A high order differential equation is in the form

$$\frac{d^m y}{dx^m} = f\left(x, y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}, \dots \dots \dots, \frac{d^{m-1} y}{dx^{m-1}}\right)$$

with $m$ initial condition given as;

$y(x_0) = a_1$

$y'(x_0) = a_2$

.        .

.        .

.        .

$y^{m-1}(x_0) = a_m$

Let us denote,

$y = y_1$

$\frac{dy}{dx} = y_2$

$\frac{d^2 y}{dx^2} = y_3$

.        .

.        .

.        .

$\frac{d^{m-1} y}{dx^{m-1}} = y_m$

Then we can write,

$\frac{dy_1}{dx} = y_2$ with $y_1(x_0) = y_{10} = a_1$

$\frac{dy_2}{dx} = y_3$ with $y_2(x_0) = y_{20} = a_2$

.

.

.

$\frac{dy^{m-1}}{dx} = y_m$ with $y_{m-1}(x_0) = y_{(m-1)0} = a_{m-1}$

$\frac{dy^m}{dx} = \frac{d^m y}{dx^m} = F(x, y_1, y_2, \dots \dots \dots, y_m)$ with $y_m(x_0) = y_{m0} = a_m$

This system is similar to the system of first order equation.

Hence, we can solve this by any procedure applied for first order equation.

## **Representation of Higher Order Equation into Simultaneous Equation**

Consider the seconder order differential equation

$$y'' = f(x, y, y')$$

$$y(x_0) = y_0, y'(x_0) = y'_0$$

This can be converted into a system of Simultaneous equations.

Put $z = y'$

Therefore the equation becomes $z' = f(x, y, z)$

$$y(x_0) = y_0$$

That is we have,

$$y' = z$$

$$z' = f(x, y, z)$$

$$y(x_0) = y_0, z(x_0) = y_0$$

This is a set of Simultaneous equations and hence can be solved. Any higher order equation can thus be transformed into simultaneous equation.

**Q.** ***Solve the following differential equation for y(0.5) using Heun's method.***

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2xy = 1 \textit{ with } y(0) = 1 \textit{ and } y'(0) = 1 \ .$$

Here,

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2xy = 1$$

$$x_0 = 0 \;\&\; y_0 = 1, y'(0) = 1 = z_0$$

Put $\frac{dy}{dx} = z$ & differentiating w.r.to $x$ we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$

Equation assumes the form:

$$\frac{dz}{dx} + 3z + 2xy = 1$$

We have system of equations,

$$y' = \frac{dy}{dx} = z = f(x, y, z) \qquad \text{[let slope} = m_i\text{]}$$

$$\frac{dz}{dx} = 1 - 2xy - 3z = g(x, y, z) \qquad \text{[let slope} = l_i\text{]}$$

$let\ h = 0.5$

Now,

$$m_1 = f(x_0, y_0, z_0) = f(0,1,1) = 1$$

$$l_1 = g(x_0, y_0, z_0) = g(0,1,1) = 1 - 2 \times 0 \times 1 - 3 \times 1 = -2$$

Similarly,

$$m_2 = f(x_0 + h, y_0 + hm_1, z_0 + hl_1) = f(0 + 0.5, 1 + 0.5 \times 1, 1 + 0.5 \times (-2))$$

$$= f(0.5, 1.5, 0) = 0$$

$$l_2 = g(0.5, 1.5, 0) = 1 - 2 \times 0.5 \times 1.5 - 3 \times 0 = -0.5$$

$$\therefore y(0.5) = y_0 + \frac{h}{2}(m_1 + m_2)$$

$$= 1 + \frac{0.5}{2}(1 + 0)$$

$$= 1.25$$

$$\therefore y'(0.5) = y'(0) + \frac{h}{2}(l_1 + l_2)$$

$$= 1 + \frac{0.5}{2}(-2 - 0.5)$$

$$= 0.375$$

***Q. Solve the following differential equation to find $y(0.1)$ using 4th order Runge-Kutta method.***

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1 \textit{ with } y(0) = 1 \textit{ and } y'(0) = 0$$

Here,

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$$

$$x_0 = 0 \ \&\ y_0 = 1, y'(0) = 0 = z_0$$

Put $\frac{dy}{dx} = z$ & differentiating w.r.to $x$ we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$

Equation assumes the form:

$$\frac{dz}{dx} - x^2z - 2xy = 1$$

We have system of equations,

$$y' = \frac{dy}{dx} = z = f(x, y, z) \qquad \text{[let slope} = m_i]$$

$$\frac{dz}{dx} = 1 + 2xy + x^2z = g(x, y, z) \qquad \text{[let slope} = l_i]$$

We have,

$y(x_0 + h) = y_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$ ……….. (i)

$x_0 + h = 0.1 \Rightarrow h = 0.1$

$m_1 = f(x_0, y_0, z_0) = f(0, 1, 0) = 0$

$l_1 = g(x_0, y_0, z_0) = g(0, 1, 0) = 1 + 2 \times 0 \times 1 + 0^2 \times 0 = 1$

$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}, z_0 + \frac{l_1 h}{2}\right) = f(0.05, 1, 0.05) = 0.05$

$l_2 = g\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}, z_0 + \frac{l_1 h}{2}\right) = g(0.05, 1, 0.05) = 1.10$

$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2}, z_0 + \frac{l_2 h}{2}\right) = f(0.05, 1.0025, 0.055) = 0.055$

$l_3 = g\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2}, z_0 + \frac{l_2 h}{2}\right) = g(0.05, 1.0025, 0.055) = 1.1$

$m_4 = f(x_0 + h, y_0 + m_3 h, z_0 + l_3 h) = f(0.1, 1.005, 0.11) = 0.11$

$l_4 = g(x_0 + h, y_0 + m_3 h, z_0 + l_3 h) = g(0.1, 1.005, 0.11) = 1.202$

$m_1, m_2, m_3$ & $m_4$ values substituted in eq. (i)

$$y(0.1) = 1 + \frac{0.1}{6}(0 + 2 \times 0.05 + 2 \times 0.055 + 0.11)$$
$$= 1.0053$$

**Boundary Value Problem**

Consider the following linear second order differential equation,

$$y'' + f(x)y' + g(x)y = F(x)$$

Suppose we are interested in solving this differential equation between the values $x = a$ & $z = b$. Hence $a$ & $b$ are two values such that $a < b$. Let us divide the interval $[a, b]$ into $n$ equal subintervals of length $h$ each.

Let $x_0, x_1, \dots \dots \dots, x_n$ be the pivotal points and $b$ are called the boundary points. Solving the differential equation means finding the values of $y_0, y_1, \dots \dots \dots, y_n$.

Suppose $y_0$ & $y_n$ are given. That is the solution values at the boundary points are given. Then the differential equation is called a **Boundary Value Problem**. So, the following is the general form of a boundary value problem.

$$y'' + f(x)y' + g(x)y = F(x)$$

$$y(a) = y_0\ , \quad y(b) = y_n$$

# Shooting Method

obtained. We hav
is achieved.

**Algorithm**

Start

Read Boundary conditions, say xa, xb, ya & yb

Read the point at which solution is needed, say xp

Read accuracy limit, say E

Convert higher order differential equation to system of d

Read value of h

Approximate first approximation as below:

Set x=xa y=ya

$g_1$=(yb-ya)/(xb-xa)

Calculate y(xb) by using Euler's method

Set $v_1$=y

If(y<yb)

$g_2=2g_1$

else

$g_2=g_1/2$

Calculate y(xb) by using Euler's method

Set $v_2$=y

Compute new values of y(xb) as below

Compute $g_3 = g_2 - \dfrac{v_2 - y_b}{v_2 - v_1}(g_2 - g_1)$

Find $y(x_b)$ by using Euler's method

Compute error

if(error<E)

Display solution

Go to step 9

Else

Set $v_1=v_2$ $v_2$=y(xb)

Set $g_1=g_2$ $g_2=g_3$

Go to step 8

Terminate

**Example**

Solve the ordinary differential equation given below by using shooting method with Euler's method. And calculate the value of y(1.5) by using h=0.5.

$$\frac{d^2y}{dx^2} = 6x \qquad y(1) = 2 \qquad y(2) = 9$$

Let $\frac{dy}{dx} = z$

Then

$$\frac{dz}{dx} = 6x$$

This gives us two first order differential equations

$$\frac{dy}{dx} = z \qquad y(1) = 2$$

$$\frac{dz}{dx} = 6x \qquad z(1) = unknown$$

Let us assume

$$z(1) = \frac{y(2) - y(1)}{2-1} = 7$$

Now, set up the initial value problem as

$$\frac{dy}{dx} = z \qquad y(1) = 2$$

$$\frac{dz}{dx} = 6x \qquad z(1) = 7$$

Where,

$$f_1(x, y, z) = z$$

$$f_2(x, y, z) = 6x$$

From Euler's method, we know that

$$y_{i+1} = y_i + f_1(x_i, y_i, z_i)h$$

$$z_{i+1} = z_i + f_2(x_i, y_i, z_i)h$$

**Calculate First Approximation**

**Iteration 1**

$$x_0 = 1 \quad y_0 = 2 \quad z_0 = 7$$

$$y_1 = y_0 + f_1(x_0, y_0, z_0)h$$

$$= 2 + f_1(1,2,7)h = 5.5$$

$$z_1 = z_0 + f_2(x_0, y_0, z_0)h$$

$$= 7 + f_2(1,2,7)h = 10$$

**Iteration 2**

$$x_1 = 1.5 \quad y_1 = 5.5 \quad z_1 = 10$$

$$y_2 = y_1 + f_1(x_1, y_1, z_1)h$$

$$= 5.5 + f_1(1.5, 5.5, 10)h = 10.5$$

$$z_2 = z_1 + f_2(x_1, y_1, z_1)h$$

$$= 10 + f_2(1.5, 5.5, 10)h = 14.5$$

Thus, $y(2) = 10.5$

The given value of this boundary condition is: $y(2) = 9$.

Since, predicted value of $y(2)$ is higher than actual value

Let us assume

$$z(1) = \frac{1}{2} \times \frac{y(2) - y(1)}{2 - 1} = 3.5$$

## Calculation of Second Approximation

**Iteration 1**

$$x_0 = 1 \quad y_0 = 2 \quad z_0 = 3.5$$

$$y_1 = y_0 + f_1(x_0, y_0, z_0)h$$

$$= 2 + f_1(1,2,3.5)h = 3.75$$

$$z_1 = z_0 + f_2(x_0, y_0, z_0)h$$

$$= 3.5 + f_2(1,2,3.5)h = 6.5$$

**Iteration 2**

$$x_1 = 1.5 \quad y_1 = 3.75 \quad z_1 = 6.5$$

$$y_2 = y_1 + f_1(x_1, y_1, z_1)h$$

$$= 3.75 + f_1(1.5,3.75,6.5)h = 7$$

$$z_2 = z_1 + f_2(x_1, y_1, z_1)h$$

$$= 6.5 + f_2(1.5,3.75,6.5)h = 11$$

Thus, $y(2)=7$

Since, Predicted values $y(1.5)$ is lower than actual value

Use linear interpolation on the previous guesses to obtain new guess as below:

$$g_3 = g_2 - \frac{v_2 - v}{v_2 - v_1}(g_2 - g_1)$$

$$= 3.5 - \frac{7-9}{7-10.5}(3.5-7) = 3.5 + 0.57 \times 3.5 \approx 5.5$$

Thus, new guess ($g_3$)=5.5.

**Calculation of Third Approximation**

**Iteration 1**

$$x_0 = 1 \quad y_0 = 2 \quad z_0 = 5.5$$

$$y_1 = y_0 + f_1(x_0, y_0, z_0)h$$

$$= 2 + f_1(1,2,5.5)h = 4.75$$

$$z_1 = z_0 + f_2(x_0, y_0, z_0)h$$

$$= 5.5 + f_2(1,2,5.5)h = 8.5$$

**Iteration 2**

$$x_1 = 1.5 \quad y_1 = 4.75 \quad z_1 = 8.5$$

$$y_2 = y_1 + f_1(x_1, y_1, z_1)h$$

$$= 4.75 \cancel{= 0} + f_1(1.5,4.75,8.5)h = 9$$

$$z_2 = z_1 + f_2(x_1, y_1, z_1)h$$

$$= 8.5 + f_2(1.5,4.75,8.5)h = 13$$

Thus, *y(2)=9*

And the given value of this boundary condition is: *y(2)=9.*

Thus, we can use third approximation to obtain value *y(1.5)*

$\Rightarrow$ *y(1.5)=4.75*

Solve the ordinary differential equation given below by using shooting method with Euler's method. And calculate the value of y(3) and y(6) by using h=3.

$$\frac{d^2y}{dx^2} - 2y = 72x - 8x^2 \qquad y(0) = 0 \qquad y(9) = 0$$

**Solution:**

Let $\frac{dy}{dx} = z$

Then

$$\frac{dz}{dx} - 2y = 72x - 8x^2$$

This gives us two first order differential equations

$$\frac{dy}{dx} = z \qquad y(0) = 0$$

$$\frac{dz}{dx} = 2y + 72x - 8x^2 \qquad z(0) = unknown$$

Let us assume

$$z(0) = \frac{y(9) - y(0)}{9 - 0} = 0$$

Now, set up the initial value problem as

$$\frac{dy}{dx} = z \qquad y(0) = 0$$

$$\frac{dz}{dx} = 2y + 72x - 8x^2 \qquad z(0) = 0$$

Where,

$$f_1(x, y, z) = z$$

$$f_2(x, y, z) = 2y + 72x - 8x^2$$

From Euler's method, we know that

$$y_{i+1} = y_i + f_1(x_i, y_i, z_i)h$$

$$z_{i+1} = z_i + f_2(x_i, y_i, z_i)h$$

**Calculate First Approximation**

**Iteration 1**

$$x_0 = 0 \quad y_0 = 0 \quad z_0 = 0$$
$$y_1 = y_0 + f_1(x_0, y_0, z_0)h$$
$$= 0 + f_1(0,0,0)h = 0$$
$$z_1 = z_0 + f_2(x_0, y_0, z_0)h$$
$$= 0 + f_2(0,0,0)h = 0$$

**Iteration 2**

$$x_1 = 3 \quad y_1 = 0 \quad z_1 = 0$$
$$y_2 = y_1 + f_1(x_1, y_1, z_1)h$$
$$= 0 + f_1(3,0,0)h = 0$$
$$z_2 = z_1 + f_2(x_1, y_1, z_1)h$$
$$= 0 + f_2(3,0,0)h = 432$$

**Iteration 3**

$$x_2 = 6 \quad y_2 = 0 \quad z_2 = 432$$
$$y_3 = y_2 + f_1(x_2, y_2, z_2)h$$
$$= 0 + f_1(6,0,432)h = 1296$$
$$z_3 = z_2 + f_2(x_2, y_2, z_2)h$$
$$= 0 + f_2(6,0,432)h = 432$$

Thus, *y(9)=1296*

The given value of this boundary condition is: *y(9)=0.*

Since, predicted value of *y (9)* is much higher than actual value

Assume that z (0)= -10

**Calculation of Second Approximation**

**Iteration 1**

$$x_0 = 0 \quad y_0 = 0 \quad z_0 = -10$$
$$y_1 = y_0 + f_1(x_0, y_0, z_0)h$$
$$= 0 + f_1(0,0,-10)h = -30$$
$$z_1 = z_0 + f_2(x_0, y_0, z_0)h$$
$$= 0 + f_2(0,0,-10)h = 0$$

**Iteration 2**

$$x_1 = 3 \quad y_1 = -30 \quad z_1 = 0$$
$$y_2 = y_1 + f_1(x_1, y_1, z_1)h$$
$$= 0 + f_1(3,-30,0)h = 0$$
$$z_2 = z_1 + f_2(x_1, y_1, z_1)h$$
$$= 0 + f_2(3,-30,0)h = 252$$

**Iteration 3**

$$x_2 = 6 \quad y_2 = 0 \quad z_2 = 252$$
$$y_3 = y_2 + f_1(x_2, y_2, z_2)h$$
$$= 0 + f_1(6,0,252)h = 756$$
$$z_3 = z_2 + f_2(x_2, y_2, z_2)h$$
$$= 0 + f_2(6,0,252)h = 432$$

Thus, *y(9)=756*

Since, Predicted values y(9) is much higher than actual value

Use linear interpolation on the previous guesses to obtain new guess as below:

$$g_3 = g_2 - \frac{v_2 - v}{v_2 - v_1}(g_2 - g_1)$$

$$= -10 - \frac{756-0}{756-1296}(-10-0) = -10 - 1.4\times 10 = -24$$

Thus, new guess ($g_3$) = – 24

**Calculation of Third Approximation**

**Iteration 1**

$$x_0 = 0 \quad y_0 = 0 \quad z_0 = -24$$

$$y_1 = y_0 + f_1(x_0, y_0, z_0)h$$

$$= 0 + f_1(0,0,-24)h = -72$$

$$z_1 = z_0 + f_2(x_0, y_0, z_0)h$$

$$= 0 + f_2(0,0,-24)h = 0$$

**Iteration 2**

$$x_1 = 3 \quad y_1 = -72 \quad z_1 = 0$$

$$y_2 = y_1 + f_1(x_1, y_1, z_1)h$$

$$= 0 + f_1(3,-72,0)h = 0$$

$$z_2 = z_1 + f_2(x_1, y_1, z_1)h$$

$$= 0 + f_2(3,-72,0)h = 0$$

**Iteration 3**

$$x_2 = 6 \quad y_2 = 0 \quad z_2 = 0$$

$$y_3 = y_2 + f_1(x_2, y_2, z_2)h$$

$$= 0 + f_1(6,0,0)h = 0$$

$$z_3 = z_2 + f_2(x_2, y_2, z_2)h$$

$$= 0 + f_2(6,0,0)h = 432$$

Thus, *y(9)=0*

Thus, *y(9)=0*

And the given value of this boundary condition is: *y(9)=0*.

Thus, we can use third approximation to obtain value *y(3)* and *y(6)*

=> *y(3)=-72* *y(6)=0*

# C program for boundary value problem using shooting method

```
#include <stdio.h>
#include <math.h>

#define MAX_ITERATIONS 1000
#define EPSILON 1e-6

// Define the function f(x, y, y')
double f(double x, double y, double yp) {
   // Replace this with your ODE function, for example:
   // return x * yp - y;
   // or any other ODE you want to solve.
}

// Shooting method to solve the boundary value problem
double shooting_method(double a, double b, double A, double
B) {
   double ya, yb, ypa, ypb, ym, yp, y;

   // Initial guesses for the derivatives at the boundaries
   double yp_low = 0.0;
   double yp_high = 1.0;
```

```
// Bisection method to find the correct initial condition
  for (int i = 0; i < MAX_ITERATIONS; i++) {
     y = ya = A;
     yp = ypa = yp_low;
     double mid = (yp_low + yp_high) / 2.0;

     // Numerical integration using Euler's method
     double h = (b - a) / 1000;
     for (double x = a; x < b; x += h) {
        ypb = yp;
        yp = yp + h * f(x, y, yp);
        y = y + h * ypb;
     }
     yb = y;
```

```c
 // Check if the solution is close enough to the boundary condition B
        if (fabs(yb - B) < EPSILON) {
            return yb;
        }

        // Adjust the interval for bisection
        if (yb > B) {
            yp_high = mid;
        } else {
            yp_low = mid;
        }
    }

    // If the maximum number of iterations is reached, return an error value
    return NAN;
}

int main() {
    // Define the boundary conditions and the interval [a, b]
    double a = 0.0;
    double b = 1.0;
    double A = 0.0;
    double B = 1.0;

    // Solve the boundary value problem using the shooting method
    double yb = shooting_method(a, b, A, B);

    if (!isnan(yb)) {
        printf("The value of y(%lf) is approximately %lf\n", b, yb);
    } else {
        printf("Failed to converge to a solution.\n");
    }

    return 0;
}
```

# Homework 8

Course)

(b) Solve the following differential equation within $1 \le x \le 2$ using Runge-Kutta $4^{th}$ order method. (5)

$$\frac{dy}{dx} + 3x - 4y = 2, \text{ with } y(1) = 1. \text{ (Take } h = 0.25\text{)}.$$

5. How can you solve higher order differential equation? Explain. Solve the following differential within $0 \le x \le 1$ using Heun's method.(3 + 5) asked in 2069

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2xy = 1 \text{ with } y(0)=1 \text{ and } y'(0) = 1 \text{ (take h = 0.5)}$$

# Homework 9

12. How boundary value problems differs from initial value problems? Discuss shooting method for solving boundary value problem. asked in 2077

Solve the following boundary value problem using shooting method.(8)

$$\frac{d^2y}{dx^2} - 2x^2y = 1, \text{ with } y(0) = 1 \text{ and } y(1) = 1 \text{ [Take h = 0.5]}.$$