CSE 321 - Homework 1

1) (a) is $(2^n+n^3) \in O(4^n)$?

Assume
for all $n \geq n_0$

$(2^n+n^3) <= c \cdot 4^n$

$\frac{2^n+n^3}{4^n} <= c \Rightarrow \frac{2^n}{4^n} + \frac{n^3}{4^n} \leq c \Rightarrow \frac{1}{2^n} + \frac{n^3}{4^n} <= c$

Left side goes to zero while n is getting larger and larger. c is a constant. ⇒ so after a value n_0, this is always true.

$c=100 \quad n_0=1$

True ✓

Assume
(b) for all $n \geq n_0$

$\sqrt{10n^2+7n+3} \geq c \cdot n$

$\sqrt{\frac{10n^2+7n+3}{4n}} \geq c \Rightarrow \sqrt{10n+7+3/n} \geq c$ → const.

$3/n$ → goes to 0 when n gets larger.
7 → const
$10n$ → gets larger and larger

After a specific value n_0, this is always true.

$c=1, n_0=10$

True ✓

(c) Assume for all $n \geq n_0$

$0 \leq n^2+n < c \cdot n^2 \Rightarrow 0 \leq 1+1/n < c$ → This should be true for any real number c but it is not for $c \geq 2$

This is false ✗

(d) Assume for all $n \geq n_0$

$3\log_2^2 n \geq \log_2 n^2 \cdot c \Rightarrow \frac{3\log_2 n \cdot \cancel{\log_2 n}}{2\cancel{\log_2 n}} \geq c$

$3\log_2 n$ → goes to inf. when n gets larger
c → const.

True
$c=1, n_0=2$

and

$3\log_2^2 n \leq \log_2 n^2 \cdot c$ → This is false.

False ✗

Assume

(e) Assume for all $n > n_0$

$0 \leq (n^3+1)^6 \leq n^3 \cdot c$

$\Rightarrow \dfrac{n^{18} + \dots + 1}{n^3} \leq c \Rightarrow n^{15} + \dots + 1/n^3 \leq c$ $>$ const

Left side goes to infinity when n gets larger. So after a certain point, left-hand side is always larger then constant c.

False ✗

(2)

a) $2n\log(n+2)^2 + (n+2)^2\log\frac{n}{2}$

$$\Theta(n\log^2 n) + \Theta(n^2\log n) \Rightarrow \lim_{n\to\infty} \frac{n\log^2 n}{n^2\log n} = \lim_{n\to\infty}\frac{\log n}{n} = 0$$ so $n^2\log n$ is more precise

$\Theta(n^2\log n)$

b) $0.001n^4 + 3n^3 + 1$

↳ by the formal definition of theta notation, constants are neglicable when n goes to infinity and most significant term is n^4. $\Theta(n^4)$

(3)

a) Compare $\log n, n^{\log n}, n^{1.5}$

since it is asymptotic notation we can assume this is $\ln(n)$

L'Hospital

$$\lim_{n\to\infty}\frac{\log n}{n^{\log n}} = \lim_{n\to\infty}\frac{\frac{d}{dn}\log n}{\frac{d}{dn}n^{\log n}} = \lim_{n\to\infty}\frac{1/n}{\underbrace{2\log n . n^{\log n}}_{A}} = \lim_{n\to\infty}\frac{1}{2\log n . n^{\log n}} = 0$$

$\boxed{\text{so } n^{\log n} > \log n}$

↳ $y = n^{\log n}$

$\log y = \log(n^{\log n})$

$= \log n . \log n = \log^2 n$

$\Rightarrow \frac{y'}{y} = \frac{2\log n}{n} \Rightarrow y' = \frac{2\log n . y}{n}$

$= \frac{2\log n . n^{\log n}}{n}$

$$\lim_{n\to\infty}\frac{n^{1.5}}{n^{\log n}} = \lim_{n\to\infty}\frac{1}{n^{\log n - 1.5}} = 0$$

$\boxed{\text{so } n^{\log n} > n^{1.5}}$

L'hopital

$$\lim_{n\to\infty}\frac{\log n}{n^{1.5}} = \lim_{n\to\infty}\frac{\frac{d}{dn}\log n}{\frac{d}{dn}n^{1.5}} = \lim_{n\to\infty}\frac{1/n}{1.5n^{0.5}} = \lim_{n\to\infty}\frac{1}{1.5n^{1.5}} = 0$$

$\boxed{\text{so } n^{1.5} > \log n}$

Growing fastest to slowest

$n^{\log n}, n^{1.5}, \log n$

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③ b) compare $n!, 2^n, n^2$

$$\lim_{n\to\infty} \frac{n!}{2^n} \Rightarrow \lim_{n\to\infty} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{2^n} = \lim_{n\to\infty} \sqrt{2\pi n}\cdot\left(\frac{n}{2e}\right)^n = \infty \quad \boxed{\text{So, } n! > 2^n}$$

$$\lim_{n\to\infty} \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{\frac{d}{dn}2^n}{\frac{d}{dn}n^2} = \lim_{n\to\infty} \frac{2^n \ln 2}{2n} = \lim_{n\to\infty} \frac{\frac{d}{dn}2^n \ln 2}{\frac{d}{dn}2n} = \lim_{n\to\infty} \frac{2^n \ln 2 . \ln 2}{2}$$

$$= \lim_{n\to\infty} \frac{2^n . \ln^2 2}{2} = \infty \quad \boxed{\text{So, } 2^n > n^2}$$

Grow

fastest $n!$, 2^n, n^2 slowest

c) compare $n\log n, \sqrt{n}$

$$\lim_{n\to\infty} \frac{n^{0.5}}{n\log n} = \frac{1}{n^{0.5}\log n} = 0$$

so $n\log n$ grows faster then $n^{0.5}$

d) compare $n2^n, 3^n$

$$\lim_{n\to\infty} \frac{n2^n}{3^n} = \lim_{n\to\infty} \frac{n}{\left(\frac{3}{2}\right)^n} = \lim_{n\to\infty} \frac{\frac{d}{dn}n}{\frac{d}{dn}\left(\frac{3}{2}\right)^n} = \lim_{n\to\infty} \frac{1}{\left(\frac{3}{2}\right)^n . \ln\left(\frac{3}{2}\right)} = 0$$

So, 3^n grows faster then $n \times 2^n$

e) $\sqrt{n+10}$, n^3

$$\lim_{n\to\infty} \frac{(n+10)^{0.5}}{n^3} = \frac{\frac{d}{dn}(n+10)^{0.5}}{\frac{d}{dn}n^3} = \lim_{n\to\infty} \frac{0.5\times(n+10)^{-0.5}}{3n^2} = \frac{0.5}{3n^2\times(n+10)^{0.5}} = 0$$

constant

n^3 grows faster then $\sqrt{n+10}$

4)

a) Basic operation is comparison.

b) $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n-1} 1 = \sum_{i=1}^{n-1} (n-1)-(i+1)+1 = \sum_{i=1}^{n-1} n-i+1 = n+(n-1)+\dots 1 = \frac{n.(n+1)}{2}$

c) $\lim_{n\to\infty} \frac{\frac{n^2+n}{2}}{n^2} = \lim_{n\to\infty} \frac{\left(\frac{n^2+n}{2}\right)'}{(n^2)'} = \lim_{n\to\infty} \frac{\left(\frac{2n+1}{2}\right)}{(2n)} = \frac{1}{2} \to \text{const}$

So $\frac{n^2-n}{2}$ grows same with n^2. $\Theta(n^2)$

5)

a) Basic operation is matrix multiplication.
(consists of assignment, addition and multiplication)

b) $\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} 1 = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} (n-1) = \sum_{i=0}^{n-1} (n-1)(n-1) = (n-1)(n-1)(n-1) = (n-1)^3$

$= n^3-3n^2+3n-1$

→ L'hopital

c) $\lim_{n\to\infty} \frac{n^3-3n^2+3n-1}{n^3} = \frac{3n^2-6n+3}{3n^2} = \frac{6n-6}{6n} = \frac{6}{6} \to \text{const}$ $\Theta(n^3)$

6) algorithm(A[0...n-1], desired)

```
index=0;
for(i=0 to n-1)
    for(j=0 to n-1)
        if(A[i]*A[j]==desired)
            C[index]=pair(A[i],A[j]) //C is an array of pairs
            index++
```

$\Theta(n^2)$ $\Theta(n)$ const const Const

$\Theta(n^2)$