Mathematical Notations for Graph Theory Course (Fall 2019)

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Abstract

Food for thought for eager learners:

- *"You'll need to set small, specific goals to master a skill, but first you'll want to be sure of the basics."* Source: *Learn better* by Ulrich Boser.
- *"Learning is an iterative process that requires that you revisit what you have learnt."* Source: *Make it stick* by Henry L. Roediger III and Mark A. McDaniel.
- *"The good news is that we now know of simple and practical strategies that anybody can use; at any point in life, to learn better and remember longer: various forms of retrieval practice, such as low-stakes quizzing and self-testing, spacing out practice, interleaving the practice of different but related topics or skills, trying to solve a problem before being taught the solution, distilling the underlying principles or rules that differentiate types of problems, and so on."* Source: *Make it stick* by Henry L. Roediger III and Mark A. McDaniel.

Graph morphospace to compare shortest path routing-based and diffusion-based graph efficiency measures

Check the following reference: (Goni et al., 2013).

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Table 1: *Major mathematical notations used in graph theory (lectures 5 and 6 on graph information flow efficiency).*

Mathematical notation	Definition
N	number of nodes
$L = \frac{1}{N}\sum_i l_i = \frac{1}{N(N-1)}\sum_{I\neq j} l_{ij}$	characteristic path length (Watts and Strogatz, 1998)
$L' = N(N-1)[\sum_{i\neq j}\frac{1}{l_{ij}}]^{-1}$	harmonic mean (Newman, 2003)
$E_{glob} = \frac{1}{L'} = \frac{1}{N(N-1)}\sum_{i\neq j}\frac{1}{l_{ij}}$	global efficiency (Latora and Marchiori, 2001)
$E_{nodal}(j) = \frac{1}{N-1}\sum_i \frac{1}{l_{ij}}$	nodal efficiency (Latora and Marchiori, 2001)
$E_{local}(i) = \frac{1}{N_{G_i}(N_{G_i}-1)}\sum_{i,h\in G_i}\frac{1}{l_{jh}}$	local efficiency (Latora and Marchiori, 2001)
$\Omega_{vu} = \{u, a_1, \ldots, a_K, v\}$	ordered sequence of nodes to go from node u to v
$S(\Omega_{vu}) = -log_2(P(\Omega_{vu}))$	search information on Ω_{vu} (Sneppen et al., 2005)
$P(\Omega_{vu}) \neq P(\Omega_{uv})$	asymmetric search information (Sneppen et al., 2005)
$S(\Omega_{vu}) = \frac{-log_2(P(\Omega_{vu}))-log_2(P(\Omega_{uv}))}{2}$	symmetric search information (Goñi et al., 2014)
$P(\Omega_{vu}) = \frac{w_{a_1 u}}{s_u}\times\frac{w_{a_2 a_1}}{s_{a_1}}\times\cdots\times\frac{w_{v a_K}}{s_{a_K}}$	probability of a random walker on Ω_{vu} going from u to v
$s_j = \sum_{i\neq j} w_{ij}$	strength of node j (weighted graph)
$m_{ij} = \frac{\sum_{k\neq i,j}(w_{ik}+w_{jk})\mathbf{1}_{w_{ik}}\mathbf{1}_{w_{jk}}}{\sum_{k\neq j} w_{ik}+\sum_{k\neq i} w_{jk}}$	matching index between nodes i and j to compute path transitivity (Goñi et al., 2014)
$M(\Omega_{vu}) = \frac{2}{\lvert\Omega_{vu}\rvert\lvert\Omega_{vu}-1\rvert}\sum_{i>j\in\Omega_{vu}} m_{ij}$	path transitivity of path Ω_{vu} (Goñi et al., 2014)
$U = WS^{-1}$	$U \in \mathbb{R}^{N\times N}$ is the product of the weighted graph adjacency matrix W
	and the reciprocal of the diagonal node strength matrix S
$U_{ij} = w_{ij}/s_j$	element (i,j) of matrix U
$\pi_{ij} = 1 - \sum_{n=1}^{N}[U_j^H]_{ni}$	probability that a walker is at node j starting from i in H steps
$\Pi^H = \sum_{i\neq j}\frac{1}{N-1}\pi_{ij}$	shortest path probability of a random walker on a whole graph
$< X_i > = \sum_{t=0}^{\infty} P(X_{ij} > t)$	
$= \sum_{t=0}^{\infty}\sum_{n=1}^{N}[U_j^t]_{ni}$	
$\sum_{n=1}^{N}[(1-U_j)^{-1}]_{ni}$	the number of hops that a random walker takes to walk from node i to node j (Wang and Pei, 2008)
$P(X_{ij} > t) = \sum_{n=1}^{N}[U_j^t]_{ni}$	probability of a random walker requiring more than t hops to arrive at node j
	it is the sum of the probabilities of the walker still being at one of the nodes other than node j after exactly t hops
$E_{diff} = \frac{1}{N(N-1)}\sum_{i\neq j}\frac{1}{< X_{ij} >}$	diffusion efficiency of a graph (Goni et al., 2013)
$Com_{ij} = \sum_{n=0}^{\infty}\frac{[A^n]_{ij}}{n!} = e^A$	communicability between nodes i and j (Estrada and Hatano, 2008)
$S^{-1/2}WS^{-1/2}$	reduced adjancenct matrix
$[(S^{-1/2}WS^{-1/2})^n]_{ij}$	reduced adjacency matrix
$w_{ij}/\sqrt{s_i}\sqrt{s_j}$	element (i,j) of the reduced adjacency matrix
$\Lambda = S - W = \begin{cases} s_i, \ i = j \\ -w_{ij}, i \neq j \end{cases}$	graph Laplacian
$\Lambda' = 1 - \underbrace{WS^{-1}}_{U} = \begin{cases} 1, \ i = j \\ -w_{ij}/s_j, i \neq j \end{cases}$	normalized graph Laplacian
$\Lambda'_{sym} = \mathbf{1} - S^{-1/2}WS^{-1/2}$	symmetric normalized graph Laplacian

Estrada, E., Hatano, N., 2008. Communicability in complex networks. Physical Review E 77, 036111.

Goni, J., Avena-Koenigsberger, A., de Mendizabal, N.V., van den Heuvel, M.P., Betzel, R.F., Sporns, O., 2013. Exploring the morphospace of communication efficiency in complex networks. PLoS One 8, e58070.

Goñi, J., van den Heuvel, M.P., Avena-Koenigsberger, A., de Mendizabal, N.V., Betzel, R.F., Griffa, A., Hagmann, P., Corominas-Murtra, B., Thiran, J.P., Sporns, O., 2014. Resting-brain functional connectivity predicted by analytic measures of network communication. Proceedings of the National Academy of Sciences 111, 833–838.

Latora, V., Marchiori, M., 2001. Efficient behavior of small-world networks. Physical review letters 87, 198701.

Newman, M.E., 2003. The structure and function of complex networks. SIAM review 45, 167–256.

Sneppen, K., Trusina, A., Rosvall, M., 2005. Hide-and-seek on complex networks. EPL (Europhysics Letters) 69, 853.

Wang, S.P., Pei, W.J., 2008. First passage time of multiple brownian particles on networks with applications. Physica A: Statistical Mechanics and its Applications 387, 4699–4708.

Watts, D.J., Strogatz, S.H., 1998. Collective dynamics of ‘small-world’ networks. nature 393, 440.