

How small are small mutation rates?

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We consider evolutionary game dynamics in finite population of size N . When mutations are rare, the population is monomorphic most of the time. Occasionally a mutation arises. It can either reach fixation or go extinct. The evolutionary dynamics of the process under small mutation rates can be approximated by an embedded Markov chain on the pure states. Here we analyze how small the mutation rate should be to make the embedded Markov chain a good approximation by calculating the difference between the real stationary distribution and the approximated one. While for a coexistence game it is *necessary* that the mutation rate μ is less than $N^{-1/2} \exp[-N]$ to ensure that the approximation is good, for *all other* games, it is *sufficient* if the mutation rate is smaller than $(N \ln N)^{-1}$. Our results also hold for a wide class of imitation processes under arbitrary selection intensity.

Keywords: Evolutionary game theory, Mutation rates, Perturbation analysis

I. INTRODUCTION

For evolutionary dynamics in finite populations with mutations, one can think of the evolutionary dynamics on two time scales. In the short run, what is the likelihood that a single mutant or a group of mutants takes over a population? If there is a single A type individual in a population of type B , the probability of fixation of A is termed ϕ_A . This quantity has been analytically characterized in population genetics [9, 11, 22] and has more recently also been applied to evolutionary games [7, 14, 24–26, 30, 31]. On a longer time scale, one can address the average abundance of the available strategies over time [2–4, 29]. [14], following the work of [12, 13] and [21], have developed an approach to deal with this issue. For small mutation rates, the time required for a mutation to occur is much larger than that required for fixation itself. Thus there are at most two strategies in the population simultaneously most of the time. In this case the original stochastic evolutionary process can be approximated by an embedded Markov chain on those states where the population is homogeneous for one strategy. The probability of transition from one homogenous population to another is the corresponding mutation rate multiplied by the fixation probability of the mutant strategy. For simplicity, we assume that all mutation rates are identical.

In particular, when there are only 2 types of strategies, A and B , the 2×2 payoff matrix is given by

$$\begin{array}{c|cc} & A & B \\ \hline A & a & b \\ B & c & d \end{array}, \quad (1)$$

where, a is the payoff of A against A , b is the payoff of A against B , c is the payoff of B against A , and finally, d is

the payoff of B against B . In a well mixed population, an individual interacts with all other individuals with the same probability. Excluding self interactions, the average payoff for each individual of each strategy is given by

$$\pi_A(i) = a \frac{i-1}{N-1} + b \frac{N-i}{N-1} \quad \text{and} \quad (2)$$

$$\pi_B(i) = c \frac{i}{N-1} + d \frac{N-i-1}{N-1}. \quad (3)$$

Here, i is the number of individuals playing strategy A . Since often the payoff difference is of interest, we substitute $\pi_A(i) - \pi_B(i)$ by $ui + v$, where $u = \frac{a-b-c+d}{N-1}$ and $v = \frac{N(b-d)-a+d}{N-1}$.

In this case, the pure population states are ‘All play A ’ and ‘All play B ’. The transition probability from ‘All play A ’ to ‘All play B ’ is the mutation rate μ times the fixation probability of strategy B , ϕ_B . In analogy to this, the transition probability from ‘All play B ’ to ‘All play A ’ is the mutation rate μ times the fixation probability of strategy A , ϕ_A . Thus, the stationary distribution for this Markov chain is

$$\left(\frac{\phi_A}{\phi_A + \phi_B}, \frac{\phi_B}{\phi_A + \phi_B} \right) \quad (4)$$

The first element is the average proportion of time spent in state “All play A ” while the second element is the average proportion of time spent in state “All play B ”. This approach opens up a way to analytically investigate the evolutionary dynamics under mutation, selection and drift provided the mutation rate is sufficiently small [17, 18, 27, 34, 35]. However, how small do the mutation rates have to be? Numerical simulations and time scale separation analysis show that $\mu N^2 \ll 1$ ensures the validity of the approach if the game does not show any stable coexistence [1, 17, 33]. However, time scale arguments are often viewed as intuitive tools from physics and are hard to cast

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into the form of a mathematical proof. Moreover, they do not provide a precise bound for the mutation rate.

Here, by perturbation analysis, we analytically investigate how small the mutation rate must be to make this embedded Markov chain a good approximation of the original one. To this end, we use the total variation distance of probability measures to measure the quality of the approximation of the stationary distribution. For simplicity, we employ the Fermi process [5, 28, 31], a specific yet widely used imitation process. In contrast with the result by time scale separation analysis, we show that for all games except for the coexistence game, it is *sufficient* that the mutation rate is smaller than $(N \ln N)^{-1}$ to ensure that the approximation of small mutation rates is good, i.e. $\mu N \ln N \ll 1$ relaxing $\mu N^2 \ll 1$. For a coexistence game, however, it is *necessary* that the mutation rate μ is less than $N^{-1/2} \exp[-N]$. Our result is also valid for other imitation processes with continuous derivative of the imitation function [36] as well as for the Moran process with different fitness mappings [32, 36]. For any birth death processes with mutations, we also provide a numerically accessible quantity to determine how small the mutation rate should be to make the approximation good.

II. THE FERMI PROCESS WITH MUTATIONS

The Fermi process is a particular birth-death process used to model evolutionary game dynamics in a finite population. In each time step, a random individual is selected. With probability $\mu < 1/2$, a mutation or exploration event occurs and the focal individual chooses the opposite strategy. With probability $1 - \mu$, no mutation occurs. In this case, the focal individual compares its payoff to another randomly chosen individual. If the focal player is playing A and the other plays B , then the focal player adopts the strategy of the other player with probability

$$\frac{1}{1 + \exp[+\beta(\pi_A(i) - \pi_B(i))]} \quad (5)$$

where β is the intensity of selection. For small β , selection is weak and strategy changes occur almost at random. For large β , only strategies with higher payoff are adopted. Let i be the number of strategy A individuals in the population. Then the transition probabilities from i to $i \pm 1$, T_i^\pm , are given by

$$\begin{aligned} T_i^+ &= (1 - \mu) \frac{i}{N} \frac{N - i}{N} \frac{1}{1 + \exp[-\beta(\pi_A(i) - \pi_B(i))]} + \mu \frac{N - i}{N} \\ T_i^- &= (1 - \mu) \frac{i}{N} \frac{N - i}{N} \frac{1}{1 + \exp[+\beta(\pi_A(i) - \pi_B(i))]} + \mu \frac{i}{N}. \end{aligned} \quad (6)$$

The probability to stay in state i is $1 - T_i^+ - T_i^-$.

When the mutation rate is nonzero, this Markov process has no absorbing states. Our birth-death process satisfies the detailed balance condition

$$\psi_{j-1} T_{j-1}^+ = \psi_j T_j^- \quad \text{for} \quad 1 \leq j \leq N \quad (7)$$

where ψ_j is the probability that the system is in state j [8, 16, 20]. The stationary distribution is given by (see Appendix A)

$$\psi_j = \frac{\frac{T_0^+}{T_j^-} \prod_{i=1}^{j-1} \frac{T_i^+}{T_i^-}}{1 + \sum_{k=1}^N \frac{T_0^+}{T_k^-} \prod_{i=1}^{k-1} \frac{T_i^+}{T_i^-}}, \quad 1 \leq j \leq N. \quad (8)$$

where the empty product is one, $\prod_{i=1}^0 \frac{T_i^+}{T_i^-} = 1$. For $j = 0$, we obtain

$$\psi_0 = \frac{1}{1 + \sum_{k=1}^N \frac{T_0^+}{T_k^-} \prod_{i=1}^{k-1} \frac{T_i^+}{T_i^-}}. \quad (9)$$

For $\mu \rightarrow 0$, we have $T_0^+ = \mu = T_N^- \rightarrow 0$ and thus $\psi_0 \rightarrow (1 + \prod_{i=1}^{N-1} \frac{T_i^+}{T_i^-})^{-1}$. On the other hand, the numerators of Eq. (8) approach zero for $1 < j < N$ due to $\mu \rightarrow 0$. Thus ψ_j approach zero as $\mu \rightarrow 0$ for $1 < j < N$. Considering the normalization condition, $\sum_{j=0}^N \psi_j = 1$, we have $\psi_N \rightarrow 1 - \psi_0$. Therefore, the ratio between ψ_0 and ψ_N is $\frac{\psi_0}{\psi_N} = \prod_{i=1}^{N-1} \frac{T_i^+}{T_i^-}$. Since $\prod_{i=1}^{N-1} \frac{T_i^+}{T_i^-} = \phi_B / \phi_A$ [24], this recovers Eq. (4).

III. ESTIMATING THE ERROR IN THE APPROXIMATION OF THE STATIONARY DISTRIBUTION

For our Markov chain, all possible stationary distributions form a set S denoted by

$$S = \{(\psi_0, \psi_1, \dots, \psi_N) \mid \psi_i \geq 0, \sum_{i=0}^N \psi_i = 1\}. \quad (10)$$

We follow [10] (See also [6, 19, 23]) to define a measure for the similarity of two such distributions.

Definition: Let $z = (z_0, z_1, \dots, z_N)$ and $w = (w_0, w_1, \dots, w_N) \in S$ be two distributions. The total variation distance $d_{TV}(z, w)$ between z and w is defined by

$$d_{TV}(z, w) = \frac{1}{2} \sum_{i=0}^N |z_i - w_i| \quad (11)$$

In particular, two distributions are identical if and only if the total variation distance between them is zero. If they are maximally different, we have $d_{TV}(z, w) = 1$. We use this total variation distance as a measure for the quality of the approximation based on the embedded Markov chain described above.

As discussed above, we have from Eqs. (8) and (9)

$$\begin{aligned} \lim_{\mu \rightarrow 0} \psi_0(\mu) &= \frac{\phi_B}{\phi_A + \phi_B} \\ \lim_{\mu \rightarrow 0} \psi_i(\mu) &= 0 \quad \text{for} \quad 0 < i < N \\ \lim_{\mu \rightarrow 0} \psi_N(\mu) &= \frac{\phi_A}{\phi_A + \phi_B} \end{aligned} \quad (12)$$

This is consistent with the approach of [14], Eq. (4), which can be viewed as a zeroth order term of an approximation for small mutation rates.

Up to first order, $\psi_j(\mu)$ can be approximated by

$$\psi_j(\mu) \approx \psi_j(0) + \frac{d}{d\mu}\psi_j(0)\mu. \quad (13)$$

Our goal is to show under which circumstances the second term can be neglected compared to the first one. Based on Eqs. (8) and (9), we can address the derivative in Eq. (13) (See Appendix B 1), which involves the terms

$$\frac{d}{d\mu}\psi_0(\mu)|_{\mu=0} = -(\psi_0(0))^2 (C_1 + C_2) \quad (14)$$

$$\frac{d}{d\mu}\psi_j(\mu)|_{\mu=0} = \left(\frac{1}{T_j^-} \prod_{i=1}^{j-1} \frac{T_i^+}{T_i^-} \right) \Big|_{\mu=0} \psi_0(0), \quad 0 < j < N, \quad (15)$$

where

$$\begin{aligned} C_1 &= \left(\sum_{k=1}^{N-1} \frac{1}{T_k^-} \prod_{i=1}^{k-1} \frac{T_i^+}{T_i^-} \right) \Big|_{\mu=0} \quad (16) \\ &= \sum_{k=1}^{N-1} \frac{N^2 \{1 + \exp[(uk + v)\beta]\}}{k(N-k)} \\ &\quad \times \exp \left[\left(\frac{u}{2}(k-1)^2 + \left(\frac{u}{2} + v \right)(k-1) \right) \beta \right] \quad (17) \end{aligned}$$

and

$$\begin{aligned} C_2 &= \frac{d}{d\mu} \left(\prod_{i=1}^{N-1} \frac{T_i^+}{T_i^-} \right) \Big|_{\mu=0} \quad (18) \\ &= N \exp \left[\left(\frac{N-1}{2}(uN + 2v) \right) \beta \right] \\ &\quad \times \exp[-v\beta] (1 - \exp[(uN + 2v)\beta]) \\ &\quad \times \sum_{i=1}^{N-1} \frac{\exp[-ui\beta]}{i} \quad (19) \end{aligned}$$

Here, we have replaced $\pi_A(i) - \pi_B(i)$ by $ui + v$. The normalization of the distribution, $\sum_{j=0}^N \psi_j = 1$, is determined by the zeroth order term, cf. Eq. (12). Thus, we have $\sum_{j=0}^N \frac{d}{d\mu}\psi_j = 0$, which implies

$$\frac{d}{d\mu}\psi_N(\mu)|_{\mu=0} = - \sum_{j=0}^{N-1} \frac{d}{d\mu}\psi_j(\mu)|_{\mu=0} \quad (20)$$

We emphasize that Eqs. (14),(15),(16),(18), and (20) are valid for all the birth death processes with mutations. Eqs. (17) (19) are the special cases obtained by substituting the transition probabilities for the Fermi process, Eqs. (6).

In the following, we denote $\psi(\mu) = (\psi_0(\mu), \dots, \psi_N(\mu))$ and $\psi(0) = \lim_{\mu \rightarrow 0} \psi(\mu)$. Next, we state our main theorem.

Theorem:

Assume that the population size N is sufficiently large compared to the product of the selection intensity β and the payoff entries in Eq. (1). Evolutionary game dynamics is given by the Fermi process described above.

Given an arbitrary $\varepsilon > 0$, for all games with $a > c$ or $d > b$ there exists a $\mu^* = \varepsilon/G_1(N)$, with $G_1(N)$ of the order of $N \ln N$, such that if the mutation rate fulfills $\mu < \mu^*$, then $d_{TV}(\psi(\mu), \psi(0)) < \varepsilon$.

For games with $a < c$ and $d < b$, however, there exists $\mu^* = \varepsilon/G_2(N)$, where $G_2(N)$ is of order $\sqrt{N} \exp[N]$, such that if $d_{TV}(\psi(\mu), \psi(0)) < \varepsilon$ then $\mu < \mu^*$.

For the proof of this *Theorem*, we have to infer when the total variation between the distribution with and without mutation is smaller than ε . By Eqs. (11) and (13), we have

$$d_{TV}(\psi(\mu), \psi(0)) = \frac{1}{2} \left[\sum_{i=0}^N |\psi'_i(0)| \right] \mu \quad (21)$$

Replacing $\psi'_N(0)$ by Eq. (20) leads to

$$\begin{aligned} d_{TV}(\psi(\mu), \psi(0)) &= \frac{1}{2} \left[\sum_{i=0}^{N-1} |\psi'_i(0)| + \left| \sum_{i=0}^{N-1} \psi'_i(0) \right| \right] \mu \\ &\leq \frac{1}{2} \left[\sum_{i=0}^{N-1} |\psi'_i(0)| + \sum_{i=0}^{N-1} |\psi'_i(0)| \right] \mu \\ &= \sum_{i=0}^{N-1} |\psi'_i(0)| \mu \quad (22) \end{aligned}$$

First, note that $\psi'_i(0) > 0$ for $i = 1, \dots, N-1$, cf. Eq. (15). Thus, we have

$$\begin{aligned} \sum_{i=1}^{N-1} |\psi'_i(0)| &= \sum_{i=1}^{N-1} \psi'_i(0) \\ &= \left(\sum_{i=1}^{N-1} \frac{1}{T_i^-} \prod_{k=1}^{i-1} \frac{T_k^+}{T_k^-} \right) \Big|_{\mu=0} \psi_0(0) \\ &= C_1 \psi_0(0). \quad (23) \end{aligned}$$

On the other hand, we have

$$|\psi'_0(0)| = (\psi_0(0))^2 |C_1 + C_2| \quad (24)$$

Taking Eq. (23) and (24) into Expression.(22) as well as considering $\psi_0(0) \leq 1$ leads to

$$\begin{aligned} d_{TV}(\psi(\mu), \psi(0)) &= (|C_1 + C_2| \psi_0(0) + C_1) \psi_0(0) \mu \\ &\leq (|C_1 + C_2| + C_1) \mu \quad (25) \end{aligned}$$

C_1 is positive as seen directly from the definition Eq. (16). For C_2 , it is positive when $uN + 2v < 0$ and it is negative otherwise. However, for the game fulfilling $uN + 2v > 0$,

we can look at a transformed game in which A and B are exchanged. This leads to $\tilde{u} = u$ and $\tilde{v} = (N(c - a) - d + a)/(N - 1)$. Using $v + \tilde{v} = -\tilde{u}N$ leads to $\tilde{u}N + 2\tilde{v} < 0$. Since the exchange of strategies does not affect our general result, we thus always consider the game satisfying $uN + 2v < 0$. In this case, both C_1 and C_2 are positive, yielding

$$d_{TV}(\psi(\mu), \psi(0)) \leq (2C_1 + C_2)\mu \quad (26)$$

Thus, the scaling of $2C_1 + C_2$ with N allows us to assess how the total variation distance scales with N . For games except for the coexistence game, we can derive an upper bound for the mutation rate: $2C_1 + C_2$ is smaller than a quantity $G_1(N)$ of order $N \ln N$ for large N (See Appendix B 2 a and Appendix B 2 b). Hence, we have $d_{TV}(\psi(\mu), \psi(0)) \leq G_1(N)\mu$. For any $\varepsilon > 0$, we define $u^* = \varepsilon/G_1(N)$ and whenever $u < u^*$, the error we are making when considering the stationary distribution without mutations instead of the one with mutations is smaller than ε , $d_{TV}(\psi(\mu), \psi(0)) < \varepsilon$. Since we can specify an upper bound for u^* , the condition is sufficient.

For the coexistence game ($a < c$ and $b > d$), we only find a lower bound: $2C_1 + C_2$ is greater than a quantity $G_2(N)$ of order $\sqrt{N} \exp[N]$ for large N (See Appendix B 2 c). For any $\varepsilon > 0$, we can define $u^* = \varepsilon/G_2(N)$. Only when $u < u^*$, the error of our approximation is small, $d_{TV}(\psi(\mu), \psi(0)) < \varepsilon$.

This completes the proof of our *Theorem*.

Corollary: As per the above *Theorem*, for games with $a > c$ or $d > b$, i.e. 2×2 games except for the coexistence game, if μ is smaller than the error ε times $(N \ln N)^{-1}$, then $d_{TV}(\psi(\mu), \psi(0)) < \varepsilon$. Thus $\mu < \varepsilon (N \ln N)^{-1}$ is a sufficient condition to ensure that the embedded Markov chain is a good approximation of the original one. In analogy to this, for coexistence game, the *Theorem* implies that $\mu < \varepsilon \exp[-N]N^{-1/2}$ is only a necessary condition.

In the following we investigate what the mutation rate should be for neutral evolution, $\beta = 0$. In this case, the selection is absent and the strategies evolve due to mutation and neutral drift. Eq. (24) still holds since we do not employ β to obtain Eq. (24). In this case, we have $C_2 = 0$ and

$$\begin{aligned} C_1 &= \sum_{i=1}^{N-1} \frac{2N^2}{i(N-i)} \\ &= 2N \sum_{i=1}^{N-1} \left(\frac{1}{i} + \frac{1}{N-i} \right) \\ &= 4NH_{N-1} \end{aligned} \quad (27)$$

where $H_{N-1} = \sum_{i=1}^{N-1} 1/i$ is the Harmonic number, which is of order $\ln N$ for large N . Therefore $2C_1 + C_2$ is of order $N \ln N$ for large N . From Eq. (26), we have that mutation rate of order $(N \ln N)^{-1}$ is sufficient to make the approximation good.

Finally, we address the validity of our approach for other processes. The Fermi process is a special imitation process whose imitation function is the Fermi function, Eq. (5).

For a general imitation function, we show the *Theorem* is also valid, provided the first order derivative of the imitation function is continuous (See Appendix C).

IV. DISCUSSION AND CONCLUSION

We have investigated how small the mutation rate should be to make the stationary distribution obtained with a mutation rate going to zero a good approximation of the “real” stochastic process with nonzero mutation rate. For a non-coexistence game, it is *sufficient* that the mutation rate is smaller than a quantity of the order of $(N \ln N)^{-1}$. For a coexistence game, however, it is *necessary* that the mutation rate μ is less than a quantity of the order of $(\sqrt{N})^{-1} \exp[-N]$. These results are valid for any nonzero selection intensity. When the selection intensity vanishes, the mutation rate μ of order $(N \ln N)^{-1}$ is *sufficient* to make the approximation good. Therefore, we can say that the order of μ , which makes the approximation good, does not change compared to the neutral case provided the game allows no coexistence.

For a non-coexistence game, our results can also be interpreted in a time scale separation analysis framework: For large population size N , the conditional fixation time is of order $N \ln N$ [1]. On an average the time between two mutations is $1/\mu$. The embedded Markov chain is valid as long as there are at most two strategies in the population, i.e. $1/\mu \gg N \ln N$. Thus time scale separation analysis also yields $\mu \ll (N \ln N)^{-1}$.

To formulate the problem mathematically, we introduced the total variation distance to measure how “good” the embedded Markov chain is compared to the original one. We can also introduce other measures of distances. A natural question arises: How much does the definition of the distance influence the results? In analogy to Eq. (11), the distance between z and w induced by the p -Norm is given by

$$\|z - w\|_p = \left(\sum_{i=0}^N |z_i - w_i|^p \right)^{\frac{1}{p}}, \quad p \geq 1 \quad (28)$$

In particular, we have $\|\psi(\mu) - \psi(0)\|_1 = 2d_{TV}(\psi(\mu), \psi(0))$ by the definition of the total variation distance. Since $\|\psi(\mu) - \psi(0)\|_p \leq \|\psi(\mu) - \psi(0)\|_1$ for $p > 1$ as well as $d_{TV}(\psi(\mu), \psi(0)) \leq G_1(N)\mu$ for a non-coexistence game, we have $\|\psi(\mu) - \psi(0)\|_p \leq 2G_1(N)\mu$ for $p > 1$. By identical arguments, the *Theorem* is also valid for non-coexistence games under this definition of distance. For a coexistence game, however, we have $\|\psi(\mu) - \psi(0)\|_p \geq \|\psi(\mu) - \psi(0)\|_1/(N+1)$ for $p > 1$. In analogy to the above discussion, the *Theorem* should be reformed by replacing $G_2(N) = \sqrt{N} \exp[N]$ by $\exp[N]/\sqrt{N}$. Therefore, our *Theorem* is robust with respect to the definition of distance for a non-coexistence game while needs reformation for a coexistence game. But the reformed theory illustrates that the critical mutation rate

for a coexistence game decreases more rapidly compared to that of the non-coexistence games. Thus the results are qualitatively robust with respect to the definition of the distance.

We have shown that the *Theorem* is not only valid for the Fermi process, but also for a general imitation process with continuous derivative of the imitation function [36]. By definition an imitation process involves an imitator and a role model and the strategy of the role model can be adopted by the imitator. Individuals are more likely to imitate those with higher fitness. This has been termed as ‘monotonicity’ in [15]. In addition, the *Theorem* is also valid for the Moran process with continuous differentiable fitness mappings. The proof is quite similar to that of the general imitation process and thus we do not show it in the appendix. For the Moran process, the monotonicity of the payoff to fitness mapping is also needed. This ensures that individuals with higher payoff have more chance to reproduce.

Since the proof of the *Theorem* depends only on C_1 and C_2 as defined in Eqs. (16) and (18) and the triangle inequality used in Eq. (22), Eq. (25) is valid for general evolutionary processes that can be described by a birth death process with mutations. The Moran processes with different fitness functions are of this kind [36]. Therefore, for any such process, given the error bound ε , the critical mutation bound that ensures that the approximation by the embedded Markov chain is good, i.e., $d_{TV}(\psi(\mu), \psi(0)) \leq \varepsilon$, is $\varepsilon / (|C_1 + C_2| + C_1)$. In other words, the numerical value of $|C_1 + C_2| + C_1$ is sufficient to determine the critical mutation bound. Considering that $|C_1 + C_2| + C_1$ is numerically accessible, it paves the way to determine the critical mutation bound. This mutation bound for the Fermi process is given in Appendix B 3

In contrast with the 2×2 games, it would be challenging to address what the mutation rate has to be for more than two strategies. For multi-strategy games it is difficult to obtain the exact stationary distribution. However, when there are at most two strategies in the population, then pairwise competition between all strategies is the main force of selection, therefore, our results for 2×2 can still shed light on how small the mutation rate should be. In fact, for $n \times n$ games, we optimistically speculate that our *Theorem* is also valid, whenever there are no stable internal equilibria in the simplex and the sub-simplices.

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Appendix A: The stationary distribution

Here, we recall the calculation of the stationary distribution ψ_j for a one dimensional birth-death process without absorbing states [8, 16, 20]. The stationary distribution fulfills the detailed balance condition $\psi_{j-1}T_{j-1}^+ = \psi_jT_j^-$. We rearrange this to

$$\psi_j = \frac{T_{j-1}^+}{T_j^-} \psi_{j-1}. \quad (\text{A1})$$

Therefore

$$\begin{aligned} \psi_1 &= \frac{T_0^+}{T_1^-} \psi_0 \\ \psi_2 &= \frac{T_1^+}{T_2^-} \psi_1 = \frac{T_0^+ T_1^+}{T_2^- T_1^-} \psi_0 \\ \psi_3 &= \frac{T_2^+}{T_3^-} \psi_2 = \frac{T_0^+ T_1^+ T_2^+}{T_3^- T_2^- T_1^-} \psi_0. \end{aligned} \quad (\text{A2})$$

In general, we have

$$\psi_j = \frac{T_0^+}{T_j^-} \prod_{i=1}^{j-1} \frac{T_i^+}{T_i^-} \psi_0, \quad 1 \leq j \leq N. \quad (\text{A3})$$

On the other hand, $\sum_{j=0}^N \psi_j = 1$. Thus, we have

$$1 = \sum_{j=0}^N \psi_j = \psi_0 \left(1 + \sum_{j=1}^N \frac{T_0^+}{T_j^-} \prod_{i=1}^{j-1} \frac{T_i^+}{T_i^-} \right) \quad (\text{A4})$$

and hence

$$\psi_0 = \frac{1}{1 + \sum_{j=1}^N \frac{T_0^+}{T_j^-} \prod_{i=1}^{j-1} \frac{T_i^+}{T_i^-}}. \quad (\text{A5})$$

Therefore, by Eq.(A2)

$$\psi_j = \frac{\frac{T_0^+}{T_j^-} \prod_{i=1}^{j-1} \frac{T_i^+}{T_i^-}}{1 + \sum_{k=1}^N \frac{T_0^+}{T_k^-} \prod_{i=1}^{k-1} \frac{T_i^+}{T_i^-}}, \quad 1 \leq j \leq N. \quad (\text{A6})$$

Appendix B: Estimating the critical mutation rate for the Fermi process

In this section, we consider the first order term of the Taylor approximation of the stationary distribution for small mutation rates. This provides part of the proof of the *Theorem* in the main text.

1. The first order term of the stationary distribution in the mutation rate

We calculate the first order expansion of the stationary distribution at state 0 under small mutation. Since $T_N^- =$

$\mu = T_0^+$, we have

$$\psi_0(\mu) = \frac{1}{1 + \mu \left(\sum_{k=1}^{N-1} \frac{1}{T_k^-} \prod_{i=1}^{k-1} \frac{T_i^+}{T_i^-} \right) + \prod_{i=1}^{N-1} \frac{T_i^+}{T_i^-}} \quad (\text{B1})$$

Thus, $\frac{d}{d\mu} \psi_0|_{\mu=0}$ is given by

$$\begin{aligned} \frac{d}{d\mu} \psi_0|_{\mu=0} = & -\psi_0^2(0) \left[\underbrace{\left(\sum_{k=1}^{N-1} \frac{1}{T_k^-} \prod_{i=1}^{k-1} \frac{T_i^+}{T_i^-} \right)}_{C_1} \right]_{\mu=0} \\ & + \underbrace{\frac{d}{d\mu} \left(\prod_{i=1}^{N-1} \frac{T_i^+}{T_i^-} \right)}_{C_2} \Big|_{\mu=0} \end{aligned} \quad (\text{B2})$$

This equation is valid for all evolutionary birth-death processes. Substituting Eq.(6) into C_1 yields

$$\begin{aligned} C_1 &= \sum_{k=1}^{N-1} \frac{N^2 \{1 + \exp[(uk + v)\beta]\}}{k(N - k)} \prod_{i=1}^{k-1} \exp[(ui + v)\beta] \\ &= \sum_{k=1}^{N-1} \frac{N^2 \{1 + \exp[(uk + v)\beta]\}}{k(N - k)} \exp \left[\sum_{i=1}^{k-1} (ui + v)\beta \right] \\ &= \sum_{k=1}^{N-1} \frac{N^2 \{1 + \exp[(uk + v)\beta]\}}{k(N - k)} \exp \left[\left(\frac{u}{2}k + v \right) (k - 1)\beta \right] \end{aligned} \quad (\text{B3})$$

Next, we address C_2 . Let $g(\mu) = \prod_{i=1}^{N-1} \frac{T_i^+}{T_i^-}$, therefore $\ln g(\mu) = \sum_{i=1}^{N-1} (\ln T_i^+ - \ln T_i^-)$. The derivative of this quantity is given by $\frac{d}{d\mu} \ln g(\mu) = \frac{g'(\mu)}{g(\mu)}$, which results in $C_2 = \frac{d}{d\mu} g(\mu)|_{\mu=0} = g(0) \frac{d}{d\mu} \ln g(\mu)|_{\mu=0}$. On the other hand, $\frac{d}{d\mu} \ln g(\mu) = \sum_{i=1}^{N-1} \left(\frac{T_i^{+'}}{T_i^+} - \frac{T_i^{-'}}{T_i^-} \right)$. Therefore,

$$C_2 = \prod_{i=1}^{N-1} \left(\frac{T_i^+}{T_i^-} \right) \Big|_{\mu=0} \sum_{i=1}^{N-1} \left(\frac{T_i^{+'}}{T_i^+} - \frac{T_i^{-'}}{T_i^-} \right) \Big|_{\mu=0}. \quad (\text{B4})$$

By Eq.(6), we have $T_i^{+'}|_{\mu=0} = \frac{N-i}{N} - T_i^+|_{\mu=0}$ and $T_i^{-'}|_{\mu=0} = \frac{i}{N} - T_i^-|_{\mu=0}$. Substituting these expressions into Eq.(B4) yields

$$\begin{aligned} C_2 &= \prod_{i=1}^{N-1} \left(\frac{T_i^+}{T_i^-} \right) \Big|_{\mu=0} \sum_{i=1}^{N-1} \left(\frac{N-i}{NT_i^+} - \frac{i}{NT_i^-} \right) \Big|_{\mu=0} \quad (\text{B5}) \\ &= \exp \left[\sum_{i=1}^{N-1} (ui + v)\beta \right] \\ &\quad \times \left[\sum_{i=1}^{N-1} \frac{N \{1 + \exp[-\beta(ui + v)]\}}{i} \right. \\ &\quad \left. - \sum_{i=1}^{N-1} \frac{N \{1 + \exp[\beta(ui + v)]\}}{N - i} \right] \\ &= \exp \left[\left(u \frac{(N-1)N}{2} + v(N-1) \right) \beta \right] \\ &\quad \times \left[\sum_{i=1}^{N-1} \frac{N \{1 + \exp[-\beta(ui + v)]\}}{i} \right. \\ &\quad \left. - \sum_{i=1}^{N-1} \frac{N \{1 + \exp[\beta(u(N-i) + v)]\}}{i} \right] \end{aligned} \quad (\text{B6})$$

where we have exchanged the summation variable in the second sum, $i \leftrightarrow N - i$. Next, we can drop common terms in the two sums and arrive at

$$\begin{aligned} C_2 &= N \exp \left[\left(\frac{N-1}{2} (uN + 2v) \right) \beta \right] \\ &\quad \times \exp[-v\beta] (1 - \exp[(uN + 2v)\beta]) \\ &\quad \times \sum_{i=1}^{N-1} \frac{\exp[-ui\beta]}{i} \end{aligned} \quad (\text{B7})$$

2. Scaling of the first order term with N

Next, we estimate the order of $2C_1 + C_2$. To facilitate the calculation, we classify the 2×2 games by the payoff difference parameters, u and v

Classification of the game	
Neither u nor v is zero	(i) $u < 0$ and $v < 0$
	(ii) $u < 0$ and $v > 0$, coexistence game
	(iii) $u < 0$ and $v > 0$, non-coexistence game
	(iv) $u > 0$ and $v > 0$
	(v) $u > 0$ and $v < 0$, coordination game
	(vi) $u > 0$ and $v < 0$, non-coordination game
Either u or v is zero	(vii) $u = 0$ and $v < 0$
	(viii) $u = 0$ and $v > 0$
	(ix) $u > 0$ and $v = 0$
	(x) $u < 0$ and $v = 0$
Both u and v are zero	(xi) $u = 0$ and $v = 0$

With this classification, we have to prove that for case (ii), i.e. the coexistence game, $2C_1 + C_2$ is greater than $G_2(N)$ which is of order $\sqrt{N} \exp[N]$, whereas for all the other cases, $2C_1 + C_2$ is less than $G_1(N)$ which is of order $N \ln N$. We only show the calculations for case (i) (ii) and (v), for the rest of the cases it can be proved by identical techniques. For case (xi) though, it is identical with the case without selection intensity. Further, without loss of generality, we assume that the payoff entries are of order 1. Thus u is of the order of $1/N$ and v as well as $\lambda = uN + 2v < 0$ are of order 1 when N is large. On the other hand, for large N , $\lambda < 0$ is equivalent to the risk dominance condition of strategy B . Also, since β can be absorbed into the payoff entries in the transition probabilities, we let β be one for simplicity.

a. Dominance of strategy B with $u < 0$ and $v < 0$

For C_1 , we have

$$\begin{aligned} C_1 &= \sum_{i=1}^{N-1} \frac{N^2 (1 + \exp[ui + v])}{i(N-i)} \\ &\quad \exp\left[\frac{u}{2}(i-1)^2 + \left(\frac{u}{2} + v\right)(i-1)\right] \\ &< \sum_{i=1}^{N-1} \frac{2N^2}{i(N-i)} \exp\left[\frac{u}{2}(i-1)^2 + \left(\frac{u}{2} + v\right)(i-1)\right] \\ &< 2N^2 \sum_{i=1}^{N-1} \frac{1}{i(N-i)} \\ &= 2N \left(\sum_{i=1}^{N-1} \frac{1}{i} + \sum_{i=1}^{N-1} \frac{1}{N-i} \right) \\ &= 4NH_{N-1} \end{aligned} \quad (\text{B8})$$

The Harmonic number H_{N-1} is of order $\ln N$ for large N , thus C_1 is smaller than a quantity of order $N \ln N$.

For C_2 , we have (with $\lambda = uN + 2v < 0$)

$$\begin{aligned} C_2 &= N \exp\left[\frac{N-1}{2}\lambda - v\right] (1 - \exp[\lambda]) \sum_{k=1}^{N-1} \frac{\exp[-uk]}{k} \\ &\leq N \exp\left[\frac{N-1}{2}\lambda - v\right] (1 - \exp[\lambda]) \exp[-u(N-1)] \sum_{k=1}^{N-1} \frac{1}{k} \\ &= N \exp\left[\frac{N-1}{2}\lambda - v\right] (1 - \exp[\lambda]) \exp[-u(N-1)] H_{N-1} \end{aligned}$$

$u < 0$ is of order $1/N$ and $\lambda < 0$ is of order 1. Thus, $N \exp\left[\frac{N-1}{2}\lambda - v\right] (1 - \exp[\lambda]) \exp[-u(N-1)] H_{N-1}$ is of order $N \ln N \exp[-N]$, which is much smaller than $N \ln N$. Thus, C_2 can be neglected compared to C_1 ; $G_1(N) = 8NH_{N-1} > 2C_1 + C_2$ scales at most with $N \ln N$.

b. Coordination game with $u > 0$ and $v < 0$

To estimate the order of C_1 , let

$$F(i) = \frac{u}{2}i^2 + \left(\frac{u}{2} + v\right)i, \quad (\text{B10})$$

We have $F(0) = 0$ and $F(N-1) = (N-1)(uN+2v)/2 = (N-1)\lambda/2 < 0$. On the other hand, $F''(i) = \frac{u}{2}$. Since $u > 0$, $F(i)$ is a convex function which implies

$$\begin{aligned} F(i) &= F\left(\frac{i}{N-1}(N-1) + \left(1 - \frac{i}{N-1}\right)0\right) \\ &\leq \frac{i}{N-1}F(N-1) + \left(1 - \frac{i}{N-1}\right)F(0) \\ &\leq 0, \end{aligned} \quad (\text{B11})$$

where equality holds for $i = 0$ only. Therefore for C_1 , we have

$$\begin{aligned} C_1 &= \sum_{i=1}^{N-1} \frac{N^2 (1 + \exp[ui + v])}{i(N-i)} \\ &\quad \exp\left[\frac{u}{2}(i-1)^2 + \left(\frac{u}{2} + v\right)(i-1)\right] \end{aligned} \quad (\text{B12})$$

$$\begin{aligned} &< \sum_{i=1}^{N-1} \frac{N^2 (1 + \exp[ui + v])}{i(N-i)} \\ &< \sum_{i=1}^{N-1} \frac{N^2 (1 + \exp[u(N-1) + v])}{i(N-i)} \\ &= N (1 + \exp[u(N-1) + v]) \left(\sum_{i=1}^{N-1} \frac{1}{i} + \sum_{i=1}^{N-1} \frac{1}{N-i} \right) \\ &= 2 (1 + \exp[u(N-1) + v]) NH_{N-1} \end{aligned} \quad (\text{B13})$$

Considering that u and v are of order $1/N$ and 1. C_1 is less than a quantity of order $N \ln N$.

For C_2 , since $u > 0$, we have

$$\begin{aligned} C_2 &= N \exp\left[\frac{N-1}{2}\lambda - v\right] (1 - \exp[\lambda]) \sum_{k=1}^{N-1} \frac{\exp[-uk]}{k} \\ &\leq N \exp\left[\frac{N-1}{2}\lambda - v\right] (1 - \exp[\lambda]) \exp[-u] \sum_{k=1}^{N-1} \frac{1}{k} \\ &= N \exp\left[\frac{N-1}{2}\lambda - v\right] (1 - \exp[\lambda]) \exp[-u] H_{N-1} \end{aligned} \quad (\text{B9})$$

$$(\text{B14})$$

In analogy to the order analysis for Eq. (B9), C_2 is much smaller than C_1 . Hence, $2C_1 + C_2$ scales with N as $N \ln N$. Thus our quantity $G_1(N)$ in the proof is $4(1 + \exp[u(N-1) + v]) NH_{N-1}$.

c. *Coexistence of strategy A and B with $u < 0$ and $v > 0$*

We show that for a coexistence game, $2C_1 + C_2$ is greater than a quantity of order $\sqrt{N} \exp(N)$. For C_1 , we have

$$\begin{aligned} C_1 &= \sum_{i=1}^{N-1} \frac{N^2 (1 + \exp[ui + v])}{i(N-i)} \exp \left[\frac{u}{2} (i-1)^2 + \left(\frac{u}{2} + v \right) (i-1) \right] \\ &> 4 \sum_{i=1}^{N-1} \exp[ui + v] \exp \left[\frac{u}{2} (i-1)^2 + \left(\frac{u}{2} + v \right) (i-1) \right] \\ &= 4 \sum_{i=1}^{N-1} \exp \left[\frac{u}{2} i^2 + \left(\frac{u}{2} + v \right) i \right] \\ &= 4 \exp \left[-\frac{u}{2} \left(\frac{1}{2} + \frac{v}{u} \right)^2 \right] \sum_{i=1}^{N-1} \exp \left[\frac{u}{2} \left(i + \frac{1}{2} + \frac{v}{u} \right)^2 \right] \end{aligned}$$

When the population size N is large, we can set $x = i/(N-1)$ and approximate the sum in the above equation by an integral,

$$(N-1) \int_0^1 \exp \left[-\frac{1}{2} \left(\sqrt{-u} \left((N-1)x + \frac{1}{2} + \frac{v}{u} \right) \right)^2 \right] dx \quad (\text{B17})$$

Let $t = \sqrt{-u}[(N-1)x + \frac{1}{2} + \frac{v}{u}]$, then the above integral is

$$\begin{aligned} &\frac{1}{N-1} \sqrt{-\frac{2\pi}{u}} \\ &\times \left[\Phi \left(\sqrt{-u} \left(N - \frac{1}{2} + \frac{v}{u} \right) \right) - \Phi \left(\sqrt{-u} \left(\frac{1}{2} + \frac{v}{u} \right) \right) \right] \quad (\text{B18}) \end{aligned}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$ is the cumulative distribution function of the Gaussian distribution. For a coexistence game, $ui + v = 0$ has a solution i between 1 and $N-1$. Thus $-\frac{v}{u} \leq N-1$. With this, we have $0 < N - \frac{1}{2} + \frac{v}{u}$. Hence, $\sqrt{-u} (N - \frac{1}{2} + \frac{v}{u})$ is of order $+\sqrt{N}$ and approaches $+\infty$ as the population size N goes to infinity. Thus, $\Phi(\sqrt{-u} (N - \frac{1}{2} + \frac{v}{u}))$ approaches 1 as N approaches infinity. Similarly, a coexistence game implies $0 < -\frac{v}{u}$ and thus $\sqrt{-u} (\frac{1}{2} + \frac{v}{u})$ scales as $-\sqrt{N}$. Therefore, the second term $\Phi(\sqrt{-u} (\frac{1}{2} + \frac{v}{u}))$ approaches 0 as N approaches infinity. This means that the sum in Eq. (B16) is larger than $\sqrt{-\frac{2\pi}{u}}$ for large N , yielding a lower bound for C_1 ,

$$C_1 > 4 \sqrt{-\frac{2\pi}{u}} \exp \left[-\frac{u}{2} \left(\frac{1}{2} + \frac{v}{u} \right)^2 \right]. \quad (\text{B19})$$

Now, $u < 0$ scales as $1/N$, whereas v becomes independent of N for large N . Hence, C_1 scales as $\sqrt{N} \exp[N]$, i.e. it increases more than exponentially with N . For C_2 , the order estimation is identical to Eq.(B9), C_2 becomes infinitely small for large N . Therefore, $2C_1 + C_2$ scales

as $\sqrt{N} \exp[N]$ and the mutation rate has to go to zero rapidly to ensure that the approximation remains good when the population size is increased. Thus $G_2(N)$ is $8 \sqrt{-\frac{2\pi}{u}} \exp \left[-\frac{u}{2} \left(\frac{1}{2} + \frac{v}{u} \right)^2 \right]$.

3. A numerically accessible bound for the mutation rate

In this part of the Appendix, we show, for a given non-coexistence game, how the critical mutation rate depends on the payoff entries. By the proof provided above, this mutation rate is $(2C_1)^{-1}\varepsilon$ for large population size, where ε is the given tolerance of the error. Thus we only need to derive the relationship between C_1 and the payoff entries.

For coordination games, Eq. (B13) provides such a relationship. For dominance games, however, it is not straightforward from Eq. (B8). But based on Eq. (B8), we have

$$C_1 < \sum_{i=1}^{N-1} \frac{2N^2}{i(N-i)} \exp \left[\frac{u}{2} (i-1)^2 + \left(\frac{u}{2} + v \right) (i-1) \right] \quad (\text{B20})$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} C_1 &< \left(\sum_{i=1}^{N-1} \frac{2N^2}{i(N-i)} \right)^{\frac{1}{2}} \\ &\quad \left(\sum_{i=1}^{N-1} \underbrace{\exp \left[\frac{u}{2} (i-1)^2 + \left(\frac{u}{2} + v \right) (i-1) \right]}_{R(i-1)} \right)^{\frac{1}{2}} \quad (\text{B21}) \end{aligned}$$

By using $\sum_{i=1}^{N-1} R(i-1) = \sum_{i=0}^{N-2} R(i) = \sum_{i=1}^{N-1} R(i) + R(0) - R(N-1)$, the above inequality can be rewritten as

$$\begin{aligned} C_1 &< \left(\sum_{i=1}^{N-1} \frac{2N^2}{i(N-i)} \right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{i=1}^{N-1} \exp \left[\frac{u}{2} i^2 + \left(\frac{u}{2} + v \right) i \right] \right. \\ &\quad \left. + 1 - \exp \left[(N-1) \frac{Nu + 2v}{2} \right] \right)^{\frac{1}{2}} \quad (\text{B22}) \end{aligned}$$

The first factor of the r.h.s of the inequality, by Eq. (27), scales as $2\sqrt{N \ln N}$. The second factor is similar to the expressions obtained by Eqs. (B15) (B16) (B17) (B18). It can be approximated by the square root of

$$\begin{aligned} &\exp \left[-\frac{u}{2} \left(\frac{1}{2} + \frac{v}{u} \right)^2 \right] \sqrt{-\frac{2\pi}{u}} \\ &\left[\Phi \left(\sqrt{-u} \left(N - \frac{1}{2} + \frac{v}{u} \right) \right) - \Phi \left(\sqrt{-u} \left(\frac{1}{2} + \frac{v}{u} \right) \right) + 1 \right] \end{aligned}$$

for large N where $\Phi(x)$ is the standard Gaussian distribution function. Thus

$$C_1 < 2\sqrt{N \ln N} \sqrt{-\frac{2\pi}{u}} \exp \left[-\frac{u}{4} \left(\frac{1}{2} + \frac{v}{u} \right)^2 \right] \\ \times \sqrt{\Phi \left(\sqrt{-u} \left(N - \frac{1}{2} + \frac{v}{u} \right) \right) - \Phi \left(\sqrt{-u} \left(\frac{1}{2} + \frac{v}{u} \right) \right)} + 1 \quad (\text{B23})$$

This allows to estimate a numerical value for the critical mutation bound for given payoff entries of a non-coexistence game and error tolerance without the need to evaluate sums. If the system is not too large such that sums can be evaluated numerically, the first line of Eq. (B13) gives a more precise estimate.

Appendix C: Estimating the critical mutation rate for general imitation processes

For the general imitation process with mutations, an individual is picked up from the well mixed population of size N . With probability $1 - \mu$, imitation occurs: The focal individual imitates another random individual with a probability $g(\beta \Delta \pi_i)$, where $\Delta \pi_i = \pi_A - \pi_B$ and β is the selection intensity. Here $g(x)$ is an increasing function. This implies that the more successful the opponent is, the more likely the focal individual imitates it. With probability $\mu < 1/2$, mutation or exploration occurs: The focal individual switches to the opposite strategy.

In analogy to the transition probabilities given by Eq. (6), we have

$$T_i^+ = (1 - \mu) \frac{i}{N} \frac{N - i}{N} g(+\beta \Delta \pi_i) + \mu \frac{N - i}{N} \\ T_i^- = (1 - \mu) \frac{i}{N} \frac{N - i}{N} g(-\beta \Delta \pi_i) + \mu \frac{i}{N}. \quad (\text{C1})$$

and $1 - T_i^+ - T_i^-$. In this Appendix, we show that the *Theorem* is also valid for a wide class of imitation processes. The only technical requirement is that the imitation function is increasing and that its derivative is continuous.

1. The form of the first order term

For the general imitation process with mutations, we still have C_1 and C_2 defined in Eqs. (16) and (18). For C_1 , we obtain

$$C_1 = \left(\sum_{i=1}^{N-1} \frac{1}{T_i^-} \prod_{k=1}^{i-1} \frac{T_k^+}{T_k^-} \right) \Big|_{\mu=0} \\ = \sum_{i=1}^{N-1} \frac{N^2}{i(N-i)} \frac{1}{g(-\beta \Delta \pi_i)} \prod_{k=1}^{i-1} \frac{g(+\beta \Delta \pi_k)}{g(-\beta \Delta \pi_k)} \quad (\text{C2})$$

By making use of the identity $x = \exp[\ln x]$ for $x = \prod_{k=1}^{i-1} \frac{g(+\beta \Delta \pi_k)}{g(-\beta \Delta \pi_k)}$, we arrive at

$$C_1 = \sum_{i=1}^{N-1} \frac{N^2}{i(N-i)} \frac{1}{g(-\beta \Delta \pi_i)} \exp \left[\sum_{k=1}^{i-1} \ln \left[\frac{g(+\beta \Delta \pi_k)}{g(-\beta \Delta \pi_k)} \right] \right] \quad (\text{C3})$$

For C_2 , note that the derivation of Eq. (B5) is independent of the imitation function given, thus it is valid for all imitation processes. We have

$$C_2 = \prod_{k=1}^{N-1} \left(\frac{T_k^+}{T_k^-} \right) \Big|_{\mu=0} \sum_{k=1}^{N-1} \left(\frac{N-k}{NT_k^+} - \frac{k}{NT_k^-} \right) \Big|_{\mu=0} \\ = \prod_{k=1}^{N-1} \frac{g(+\beta \Delta \pi_k)}{g(-\beta \Delta \pi_k)} \sum_{k=1}^{N-1} \frac{N}{N-k} \left(\frac{1}{g(+\beta \Delta \pi_{N-k})} - \frac{1}{g(-\beta \Delta \pi_k)} \right) \\ = \exp \left[\sum_{k=1}^{N-1} \ln \left[\frac{g(+\beta \Delta \pi_k)}{g(-\beta \Delta \pi_k)} \right] \right] \\ \times \sum_{k=1}^{N-1} \frac{N}{N-k} \left(\frac{1}{g(+\beta \Delta \pi_{N-k})} - \frac{1}{g(-\beta \Delta \pi_k)} \right) \quad (\text{C4})$$

For C_2 , if $\frac{1}{g(+\beta \Delta \pi_{N-k})} - \frac{1}{g(-\beta \Delta \pi_k)}$ is non-negative for all the k , then C_2 is non-negative. Since $g(x)$ is an increasing function, this is equivalent to $\Delta \pi_{N-k} \leq -\Delta \pi_k$, i.e., $uN + 2v \leq 0$. If this is not the case, we can exchange strategy A and B , as described in the main text. This yields a transformed game which fulfills $\tilde{u}N + 2\tilde{v} \leq 0$ without influencing the main results. Therefore, we always consider the case for $uN + 2v \leq 0$, such that both C_1 and C_2 are non-negative.

2. Scaling of the first order term with N

To estimate the order of $2C_1 + C_2$, we absorb the selection intensity β into the payoff difference term in analogy to the proof above, i.e. we formally set $\beta = 1$. The quantity u is of order $1/N$ and v is of order 1. Without loss of generality (see above), $uN + 2v \leq 0$ is also assumed to ensure $C_2 > 0$.

For the coordination game, $u > 0$ and $v < 0$, we only need to prove that $2C_1 + C_2$ is less than a quantity of order $N \ln N$. For C_1 , we have

$$C_1 = \sum_{i=1}^{N-1} \frac{N^2}{i(N-i)} \frac{1}{g(-\Delta \pi_i)} \exp \left[\sum_{k=1}^{i-1} \ln \left[\frac{g(\Delta \pi_k)}{g(-\Delta \pi_k)} \right] \right] \\ < \sum_{i=1}^{N-1} \frac{N^2}{i(N-i)} \frac{1}{g(-\Delta \pi_N)} \exp \left[\sum_{k=1}^{i-1} \ln \left[\frac{g(\Delta \pi_k)}{g(-\Delta \pi_k)} \right] \right] \quad (\text{C5})$$

By Lagrange mean value theorem, for every $1 \leq k \leq N-1$

there exists $\xi_k \in [0, 1]$, s.t.

$$\begin{aligned} & \ln [g(\Delta\pi_k)] - \ln [g(-\Delta\pi_k)] \\ &= uN \frac{g'(uN\xi_k + v)}{g(uN\xi_k + v)} (\Delta\pi_k - (-\Delta\pi_k)) \\ &\leq 2uN \frac{M}{g(v)} \Delta\pi_k \end{aligned} \quad (C6)$$

where M is the maximum of $g'(x)$ for $x \in [v, uN + v]$. Since v and $uN + v$ are of order 1, M only depends on the imitation function and payoff entries rather than the population size N for large N . Thus we can consider it to be of order 1 in what concerns the scaling of N . On the other hand, since $g'(x)$ is continuous as we assume, there exists $y^* \in [0, 1]$ such that $M = g'(y^*) > 0$. Therefore, $uN \frac{M}{g(v)} > 0$ becomes independent of N for large N . This implies

$$C_1 < \sum_{i=1}^{N-1} \frac{N^2}{i(N-i)} \frac{1}{g(-\Delta\pi_N)} \exp \left[2uN \frac{M}{g(v)} \sum_{k=1}^{i-1} \Delta\pi_k \right] \quad (C7)$$

Therefore, it degenerates to Eq. (B12) for coordination game of the Fermi process. Following the proof therein, finally we arrived at

$$C_1 < 2 \frac{1}{g(-\Delta\pi_N)} NH_{N-1}. \quad (C8)$$

Since $g(-\Delta\pi_N) = g(-uN - v)$ is only dependent on the imitation function and the payoff entries, it is independent of N . Thus, C_1 is smaller than a quantity of order $N \ln N$.

Next, we consider C_2 . We have

$$\begin{aligned} C_2 &= \exp \left[\underbrace{\sum_{k=1}^{N-1} \ln \left[\frac{g(\Delta\pi_k)}{g(-\Delta\pi_k)} \right]}_{D_1} \right] \\ &\quad \underbrace{\sum_{k=1}^{N-1} \frac{N}{N-k} \left(\frac{1}{g(\Delta\pi_{N-k})} - \frac{1}{g(-\Delta\pi_k)} \right)}_{D_2}, \end{aligned} \quad (C9)$$

which is a product of $\exp[D_1]$ and D_2 . For D_1 , we have

$$\begin{aligned} D_1 &= \sum_{k=1}^{N-1} \ln \left[\frac{g(\Delta\pi_k)}{g(-\Delta\pi_k)} \right] \\ &= \sum_{k=1}^{N-1} \ln [g(\Delta\pi_k)] - \sum_{k=1}^{N-1} \ln [g(-\Delta\pi_k)] \\ &= \sum_{k=1}^{N-1} \ln [g(\Delta\pi_k)] - \sum_{k=1}^{N-1} \ln [g(-\Delta\pi_{N-k})] \\ &= \sum_{k=1}^{N-1} (\ln [g(\Delta\pi_k)] - \ln [g(-\Delta\pi_{N-k})]) \end{aligned} \quad (C10)$$

Again, by Lagrange mean value theorem, for every $1 \leq k \leq N-1$, there exists $\zeta_k \in [0, 1]$, s.t.

$$\begin{aligned} & \ln [g(\Delta\pi_k)] - \ln [g(-\Delta\pi_{N-k})] \\ &= uN \frac{g'(uN\zeta_k + v)}{g(uN\zeta_k + v)} (\Delta\pi_k - (-\Delta\pi_{N-k})) \\ &\leq uN \frac{M}{g(v)} (uN + 2v) \end{aligned} \quad (C11)$$

where $M > 0$ is the maximum of $g'(x)$ on $[v, uN + v]$ as defined above. Thus we have

$$\sum_{k=1}^{N-1} \ln \left[\frac{g(\Delta\pi_k)}{g(-\Delta\pi_k)} \right] < u(N-1)N \frac{M}{g(v)} (uN + 2v) \quad (C12)$$

Remembering that $uN + 2v$ is negative and of order 1, $u(N-1)N \frac{M}{g(v)} (uN + 2v)$ is smaller than zero and of order N for large N . Therefore, $\exp \left[\sum_{k=1}^{N-1} \ln \left[\frac{g(\Delta\pi_k)}{g(-\Delta\pi_k)} \right] \right]$ is smaller than a quantity of order $\exp[-N]$.

For D_2 , since $u > 0$, $\Delta\pi_k$ is increasing with k . In addition, $g(x)$ is increasing, we have

$$\begin{aligned} \frac{1}{g(+\Delta\pi_{N-k})} - \frac{1}{g(-\Delta\pi_k)} &= \frac{g(-\Delta\pi_k) - g(+\Delta\pi_{N-k})}{g(-\Delta\pi_k)g(+\Delta\pi_{N-k})} \\ &< \frac{g(-\Delta\pi_k) - g(+\Delta\pi_{N-k})}{g(-\Delta\pi_N)g(+\Delta\pi_0)} \end{aligned} \quad (C13)$$

By Lagrange mean value theorem, there exists $\eta_k \in [0, 1]$ s.t.

$$\begin{aligned} & g(-\Delta\pi_k) - g(+\Delta\pi_{N-k}) \\ &= g'(-\Delta\pi_k + \eta_k(-\Delta\pi_k - \Delta\pi_{N-k})) (-\Delta\pi_k - \Delta\pi_{N-k}) \\ &< -H(uN + 2v) \end{aligned} \quad (C14)$$

where $H > 0$ is the maximum of $g'(x)$ on $[-v, +v]$, where $\Delta\pi_k$ and $-\Delta\pi_k$ lie. In analogy to previous discussion, it is independent of N when N is large. Thus we have

$$\frac{1}{g(+\Delta\pi_{N-k})} - \frac{1}{g(-\Delta\pi_k)} < \frac{-H(uN + 2v)}{g(-uN - v)g(v)} \quad (C15)$$

Further, we have

$$\begin{aligned} & \sum_{k=1}^{N-1} \frac{N}{N-k} \left(\frac{1}{g(+\Delta\pi_{N-k})} - \frac{1}{g(-\Delta\pi_k)} \right) \\ &< \left(\frac{-H(uN + 2v)}{g(-uN - v)g(v)} \right) \sum_{k=1}^{N-1} \frac{N}{N-k} \\ &= \left(\frac{-H(uN + 2v)}{g(-uN - v)g(v)} \right) NH_{N-1} \end{aligned} \quad (C16)$$

Note that $\frac{-H(uN + 2v)}{g(-uN - v)g(v)}$ positive and independent of N for large N . D_2 is smaller than a quantity of order $N \ln N$. Finally, $C_2 = \exp[D_1]D_2$ is of order $N \ln N \exp[-N]$; it becomes infinitely small for large N . This means that the

scaling of $2C_1 + C_2$ is determined by the scaling of C_1 and thus the critical mutation rate scales as $N \ln N$.

For the coexistence game and dominant game, the procedure of the proof for general imitation function is also identical to that of the coordination game: For C_1 , we make use of Lagrange mean value theorem to establish a relationship

between $\ln \left[\frac{g(\Delta\pi_k)}{g(-\Delta\pi_k)} \right]$ and $\Delta\pi_k$, then it can be deduced by the proof the corresponding game for the Fermi process. For C_2 , for all games, it is infinitely small for large population size. The proof is identical with that of the coordination game for general imitation function.

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