MTM5101-Dynamical Systems and Chaos

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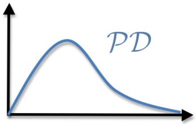
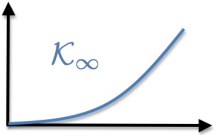
Week 10

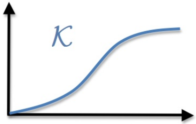
Comparison Functions



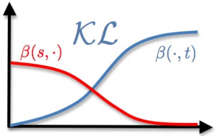
$$\begin{cases} \alpha \text{ continuous} \\ \alpha(0) = 0 \\ \alpha(s) > 0, \ \forall s > 0 \end{cases}$$



$$\begin{cases} \alpha \in \mathcal{K} \\ \lim_{s\to\infty} \alpha(s) = \infty \end{cases}$$



$$\begin{cases} \alpha \in \mathcal{PD} \\ \alpha \text{ nondecreasing} \end{cases}$$



$$\begin{cases} \beta(\cdot,t) \in \mathcal{K}, \ \forall t \geq 0 \\ \beta(s,\cdot) \text{ nonincreasing}, \ \forall s \geq 0 \\ \lim_{t\to\infty} \beta(s,t) = 0, \ \forall s \geq 0 \end{cases}$$

Comparison Functions: Examples

Example

- $\alpha(s) = \frac{1}{1+s^c}$, for any $c > 0$
- $\alpha(s) = \frac{s^c}{1+s^c}$, for any $c > 0$
- $\alpha(s) = \tan^{-1}(s)$
- $\alpha(s) = \mathrm{sat}(s) = \begin{cases} s, & \text{if } |s| \leq 1 \\ \mathrm{sgn}(s), & \text{if } |s| > 1 \end{cases}$
- $\alpha(s) = \mathrm{sat}(s; a, b) = a + \frac{2(b-a)}{\pi} \tan^{-1}(s)$
- $\alpha(s) = s^c$, for any $c > 0$
- $\alpha(s) = \min\{s, s^2\}$
- $\beta(s, r) = \frac{s}{krs+1}$
- $\beta(s, r) = s^c e^{-r}$

Equivalent Representation of Radial Unboundedness

Lemma (4.3 in [Khalil, 2002])

Let $V : D \to \mathbb{R}$ be a continuous positive definite function (may not be $\mathcal{PD}$!) defined on $D \subset \mathbb{R}^n$ that contains the origin. Let $B_r[0] \subset D$ for some $r > 0$. Then, there exist $\alpha_1, \alpha_2 \in \mathcal{K}$ defined on $[0, r]$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

for all $x \in B_r[0]$. If $D = \mathbb{R}^n$, the functions α_1 and α_2 will be defined on $[0, \infty)$ and the foregoing inequality will hold for all $x \in \mathbb{R}^n$. Moreover, if $V(x)$ is radially unbounded, then $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

Example

- $V(x) = x^\top Px \implies \lambda_{\min}(P)|x|^2 \leq x^\top Px \leq \lambda_{\max}(P)|x|^2$

- Given $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm.

Equivalent Representation of GAS

For a system with no inputs $\dot{x} = f(x)$, there is a well-known notion of global asymptotic stability (for short from now on, GAS, or "0-GAS" when referring to the system with no-inputs $\dot{x} = f(x, 0)$ associated to a given system with inputs $\dot{x} = f(x, u)$ due to Lyapunov, and usually defined in "ϵ-δ" terms. It is easy to show that the standard definition of GAS (0-GAS) in previous lecture is in fact equivalent to the existence of $\beta \in \mathcal{KL}$ satisfying the following, along the solutions of $\dot{x} = f(x)$ ($\dot{x} = f(x, 0)$)

$$|x(t, x_0)| \leq \beta(|x_0|, t), \qquad \forall x_0 \in \mathbb{R}^n, \ \forall t \geq 0.$$

Observe that, since β decreases on t, we have, in particular:

$$|x(t, x_0)| \leq \beta(|x_0|, 0), \qquad \forall x_0 \in \mathbb{R}^n, \ \forall t \geq 0.$$

which provides the stability part of the GAS definition while

$$|x(t, x_0)| \leq \beta(|x_0|, t) \underset{t \to \infty}{\to} 0, \qquad \forall x_0 \in \mathbb{R}^n,$$

which is the attractivity (convergence to the equilibrium point) part of the GAS definition.

Note: From now on, unless written explicitly, the solutions $x(t, x_0)$ or $x(t, x_0, u)$ for $\dot{x} = f(x)$ and $\dot{x} = f(x, u)$, respectively, will be written in short as $x(t)$ to avoid cumbersome notation!

Theorem (4.8 in [Khalil, 2002])

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ and $D \subset \mathbb{R}^n$ be a domain containing the origin. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

$$W_1(x) \leq V(x) \leq W_2(x)$$
$$\frac{\partial V}{\partial x} f(x) \leq 0$$

for all $t \geq 0$ and $x \in D$, where W_1 and W_2 are continuous positive definite functions on D. Then, $x = 0$ is stable.

Equivalent Lyapunov Theorem for AS

Theorem (4.9 in [Khalil, 2002])

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ and $D \subset \mathbb{R}^n$ be a domain containing the origin. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

$$W_1(x) \leq V(x) \leq W_2(x)$$
$$\frac{\partial V}{\partial x} f(x) \leq -W_3(x)$$

for all $t \geq 0$ and $x \in D$, where W_1, W_2 and W_3 are continuous positive definite functions on D. Then, $x = 0$ is asymptotically stable. Moreover, if r and c are chosen such that $B_r[0] = \{x \in D \mid |x| \leq r\}$ and $c < \min_{|x|=r} W_1(x)$, then every trajectory starting in $\{x \in B_r[0] \mid W_2(x) \leq c\}$ satisfies

$$|x(t)| \leq \beta(|x(0)|, t), \quad \forall t \geq 0$$

for some $\beta \in \mathcal{KL}$.

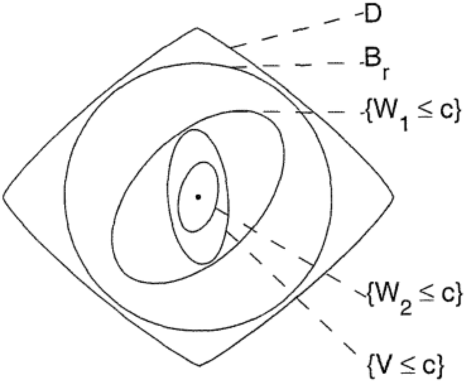


Figure: Geometric representation of sets in Theorem 4.9.

Equivalent Lyapunov Theorem for GAS

Theorem (4.9 in [Khalil, 2002])

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ ($D = \mathbb{R}^n$!) be a continuously differentiable function such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\frac{\partial V}{\partial x} f(x) \leq -W_3(x)$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$, where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and W_3 is continuous positive definite function on $\mathbb{R}^n$. Then, $x = 0$ is GAS.

Let us remember the Lyapunov theorem for ES/GES shown last week:

Theorem (4.10 in [Khalil, 2002])

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

$$k_1|x|^a \leq V(x) \leq k_2|x|^a$$
$$\frac{\partial V}{\partial x} f(x) \leq -k_3|x|^a$$

for all $t \geq 0$ and $x \in D$, where k_1, k_2, k_3 and a are positive constants. Then, $x = 0$ is ES. If $D = \mathbb{R}^n$, then $x = 0$ is GES.

- Car trailer system
  - Video: https://www.youtube.com/watch?v=4jk9H5AB4lM
- Aircraft
  - Video: https://www.youtube.com/watch?v=4UfmsqtTGa0
- Car active suspension system
  - Video: https://www.youtube.com/watch?v=kRt7H0k8A4k
- Building
  - Video: https://www.youtube.com/shorts/rJ72LruGgyU

For linear systems $\dot{x} = Ax + Bu$:

- $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$
- If A is a Hurwitz matrix ($Re(\lambda_i(A)) < 0$ for all $i = 1, ..., n$), then the linear system is 0-GAS.
- Such a 0-GAS linear system automatically satisfies all reasonable "input-to-state stability" properties [Sontag, 1990][1]:
  - Bounded inputs $\Rightarrow$ bounded state (BIBS) trajectories
  - Converging inputs $\Rightarrow$ converging state (CICS) trajectories

This is generally not the case for nonlinear systems $\dot{x} = f(x, u)$!

Example

Consider the scalar system ($n = 1$) with a single input ($m = 1$)
$\dot{x} = -x + (x^2 + 1)u$:

- The system is clearly 0-GAS, since it reduces to $\dot{x} = -x$ when $u \equiv 0$.
- However, for $u = (2t + 2)^{-1/2}$ and $x_0 = \sqrt{2}$, the system produces unbounded and even diverging state trajectory $x(t) = (2t + 2)^{1/2}$!

[1] Mathematical Control Theory: Deterministic Finite Dimensional Systems

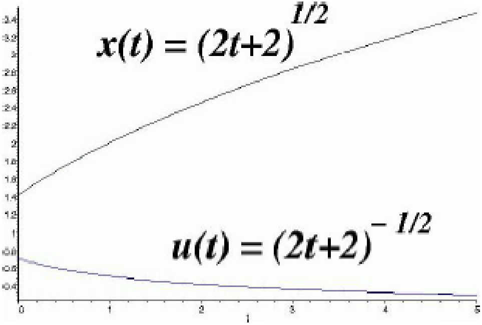


Figure: Diverging state for converging input.

Recall the solution of the linear system $\dot{x} = Ax + Bu$ can be written as:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

If A is Hurwitz, there exists some $k, \lambda > 0$ such that $\|e^{At}\| \leq ke^{-\lambda t}$ which, in turn, gives the following state estimate

$$\begin{aligned}
|x(t)| \leq & k|x(0)|e^{-\lambda t} + \int_0^t ke^{-\lambda(t-\tau)}\|B\||u(\tau)|d\tau \\
\leq & k|x(0)|e^{-\lambda t} + k\|B\| \sup_{\tau\in[0,t]} |u(\tau)|\, e^{-\lambda t} \int_0^t e^{\lambda\tau}d\tau \\
= & k|x(0)|e^{-\lambda t} + k\|B\| \sup_{\tau\in[0,t]} |u(\tau)|\, e^{-\lambda t} \left(\frac{e^{\lambda t} - 1}{\lambda}\right) \\
\leq & k|x(0)|e^{-\lambda t} + \frac{k\|B\|}{\lambda} \sup_{\tau\in[0,t]} |u(\tau)| \left(1 - e^{-\lambda t}\right) \\
\leq & \bar{k}|x(0)|e^{-\lambda t} + \bar{k} \sup_{\tau\in[0,t]} |u(\tau)|
\end{aligned}$$

where $\bar{k} = k \cdot \max\{1, \frac{\|B\|}{\lambda}\}$

Motivated with this estimation, for linear systems, three most typical ways of defining "input-to-state stability" in terms of operators $\{L^2, L^\infty\} \to \{L^2, L^\infty\}$ are as follows:

- "$L^\infty \to L^\infty$": $c|x(t)| \leq |x_0|e^{-\lambda t} + \sup_{\tau\in[0,t]} |u(\tau)|$
- "$L^2 \to L^\infty$": $c|x(t)| \leq |x_0|e^{-\lambda t} + \int_0^t |u(\tau)|^2 d\tau$
- "$L^2 \to L^2$": $c \int_0^t |x(\tau)|^2 d\tau \leq |x_0| + \int_0^t |u(\tau)|^2 d\tau$

The missing case "$L^\infty \to L^2$" is less interesting, being too restrictive, for practical reasons! Concerning the nonlinear system $\dot{x} = f(x, u)$, in general, under "some" nonlinear coordinate change (see [Sontag, 2004]), we arrive to the following three concepts (or "estimates") for nonlinear systems:

- "$L^\infty \to L^\infty$": $\alpha(|x(t)|) \leq \beta(|x_0|, t) + \sup_{\tau\in[0,t]} \gamma(|u(\tau)|)$
- "$L^2 \to L^\infty$": $\alpha(|x(t)|) \leq \beta(|x_0|, t) + \int_0^t \gamma(|u(\tau)|)d\tau$
- "$L^2 \to L^2$": $\int_0^t \alpha(|x(\tau)|)d\tau \leq \alpha_0(|x_0|) + \int_0^t \gamma(|u(\tau)|)d\tau$

Here, the functions (which measure the impacts of the state or input) are $\alpha, \alpha_0, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$. The "$L^\infty \to L^\infty$" (and "$L^2 \to L^2$" as well) estimate leads us to the first concept that of *input-to-state stability (ISS)* whereas "$L^2 \to L^\infty$" estimate leads us to the second concept that of *integral input-to-state stability (iISS)*.

ISS and iISS Notions

Definition: Input-to-State Stability (ISS) [Sontag, IEEE TAC, 1989]

The system $\dot{x} = f(x, u)$ is ISS if there exist $\beta \in \mathcal{KL}$ and $\nu \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathcal{U}$,

$$|x(t; x_0, u)| \leq \beta(|x_0|, t) + \nu\left(\|u\|\right), \quad \forall t \geq 0.$$

$\rightarrow$ Vanishing transients "proportional" to initial state's norm
$\rightarrow$ Steady-state error "proportional" to input amplitude.

Definition: Integral Input-to-State Stability (iISS) [Sontag, SCL, 1998]

The system $\dot{x} = f(x, u)$ is iISS if there exist $\beta \in \mathcal{KL}$ and $\nu_1, \nu_2 \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathcal{U}$,

$$|x(t; x_0, u)| \leq \beta(|x_0|, t) + \nu_1\left(\int_0^t \nu_2(|u(s)|)ds\right), \quad \forall t \geq 0.$$

$\rightarrow$ Measures the impact of input energy.

ISS and iISS: Central tools in nonlinear analysis and control:

- Theoretical contributions to: output feedback, optimal control, hybrid systems, predictive control, chaotic systems. . .
- Applications in: robotics, production lines, transportation, bio-chemical networks, control under communication constraints, neuroscience, . . .

ISS	**iISS**
$\dot{x} = f(x, 0)$ is GAS	$\dot{x} = f(x, 0)$ is GAS
Bounded input $\Rightarrow$ Bounded state	Bounded energy input $\Rightarrow$ Bounded, converging state
Converging input $\Rightarrow$ Converging state	Converging input $\not\Rightarrow$ Converging state
Cascade: ISS + ISS $\Rightarrow$ ISS	Cascade: iISS + iISS $\not\Rightarrow$ iISS

In practice, some systems do exhibit robustness for inputs under a certain threshold, but diverge for larger ones.

Strong iISS: halfway between ISS and iISS.

- Part of the success of ISS and iISS is due to their Lyapunov characterizations
- Lyapunov function candidate (LFC):
  - $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuously differentiable
  - $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$
  - $V(x) \to \infty$ whenever $|x| \to \infty$.

Theorem: ISS and iISS Characterization [Sontag, Wang, SCL, 1995] & [Angeli et al., IEEE TAC, 2000]

The system $\dot{x} = f(x, u)$ is ISS (resp. iISS) if and only if there exist a LFC V, $\gamma \in \mathcal{K}_\infty$, and $\alpha \in \mathcal{K}_\infty$ (resp. $\alpha \in \mathcal{PD}$) such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$

$$\frac{\partial V}{\partial x} f(x, u) \leq -\alpha(|x|) + \gamma(|u|). \qquad \text{(D-}\mathcal{K}_\infty/\mathcal{PD})$$

ISS Characterization in Implication Form

Theorem: ISS Characterization in Implication Form [Sontag, Wang, SCL, 1995]

The system $\dot{x} = f(x, u)$ is ISS if and only if there exist a LFC V, $\chi \in \mathcal{K}$, and $\tilde{\alpha} \in \mathcal{K}$ such that for any $x \in \mathbb{R}^n$ and any $u \in \mathbb{R}^m$

$$|x| \geq \chi(|u|) \quad \Longrightarrow \quad \frac{\partial V}{\partial x} f(x, u) \leq -\tilde{\alpha}(|x|) \tag{D-IF}$$

Remark

Clearly, (D-$\mathcal{K}_\infty$) implies (D-IF). Assume now that (D-IF) holds with some $\tilde{\alpha} \in \mathcal{K}$ and $\chi \in \mathcal{K}$. Without loss of generality, one can assume that $\tilde{\alpha} \in \mathcal{K}_\infty$ (see [Lin, Sontag, Wang, SCL, 1995, Remark 4.1]). Let

$$\gamma(r) = \max\{0, \hat{\gamma}(r)\}$$

where

$$\hat{\gamma}(r) = \max\left\{\frac{\partial V}{\partial x} f(x, u) + \tilde{\alpha}(\chi(|u|)) \;:\; |u| \leq r, |x| \leq \chi(r)\right\}.$$

Then $\gamma \in \mathcal{C}$, $\gamma(0) = 0$ and, therefore, $\gamma \in \mathcal{K}_\infty$ by definition. (D-$\mathcal{K}_\infty$) holds because $\gamma(r) \geq \sup_{|u|=r} \frac{\partial V}{\partial x} f(x, u) + \tilde{\alpha}(|x|)$ (consider the two seperate cases $|x| \geq \chi(|u|)$ and $|x| \leq \chi(|u|)$).

Definition: Strong iISS [Chaillet et. al., IEEE TAC, 2014]

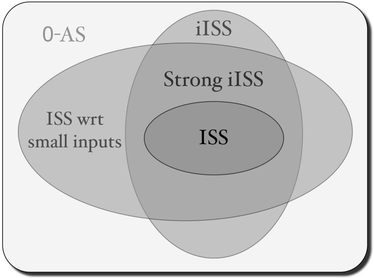
The system $\dot{x} = f(x, u)$ is Strongly iISS if it is:

- iISS
- ISS with respect to small inputs

i.e., if there exist $\beta \in \mathcal{KL}$, $\nu_1, \nu_2, \nu \in \mathcal{K}_\infty$ and input threshold $R > 0$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathcal{U}$,

$$|x(t; x_0, u)| \leq \beta(|x_0|, t) + \nu_1 \left(\int_0^t \nu_2(|u(s)|) ds \right)$$

$$\|u\| \leq R \quad \Rightarrow \quad |x(t; x_0, u)| \leq \beta(|x_0|, t) + \nu(\|u\|) \,.$$



- For all $u \in \mathcal{U}$, the solution exists at all times
- $\int_0^t \nu_2(|u(s)|) ds < \infty \Rightarrow$ bounded and converging state
- Converging input $\Rightarrow$ converging state
- $\|u\| \leq R \Rightarrow$ bounded state.

Halfway Between ISS and iISS: Strong iISS Property

Theorem: $\mathcal{K}$ dissipation rate $\Rightarrow$ Strong iISS [Chaillet et. al., IEEE TAC, 2014]

If there exists a LFC $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$\frac{\partial V}{\partial x} f(x,u) \leq -\alpha(|x|) + \gamma(|u|).$$

where $\alpha \in \mathcal{K}$ and $\gamma \in \mathcal{K}_\infty$, then the system $\dot{x} = f(x,u)$ is Strongly iISS with input threshold $R = \gamma^{-1} \circ \alpha(\infty)$.

Equivalently, we can state the following:

Corollary: Non-vanishing dissipation rate $\Rightarrow$ Strong iISS [Chaillet et. al., IEEE TAC, 2014]

If there exists a LFC $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$\frac{\partial V}{\partial x} f(x,u) \leq -W(x) + \gamma(|u|).$$

where $\gamma \in \mathcal{K}_\infty$ and W is continuous positive definite satisfying $W_\infty := \liminf_{|x|\to\infty} W(x) > 0$, then the system $\dot{x} = f(x,u)$ is Strongly iISS with input threshold $R = \gamma^{-1}(W_\infty)$.

However, the converse does not hold:

Counter-example: Strong iISS $\not\Rightarrow$ $\mathcal{K}$ dissipation rate [Chaillet et. al., IEEE TAC, 2014]

The scalar system

$$\dot{x} = -\frac{x}{1+x^2}\left[1 - |x|(u^2 - |u|)\right],$$

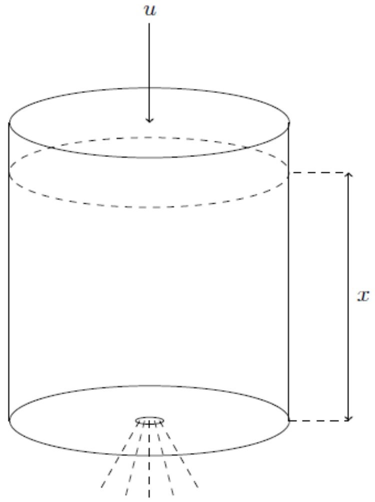
is Strongly iISS. However, for all $\alpha \in \mathcal{K}$ and $\gamma \in \mathcal{K}_\infty$ no differentiable function $V : \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfies

$$\frac{\partial V}{\partial x}(x) f(x,u) \leq -\alpha(|x|) + \gamma(|u|).$$

- iISS: $V_1(x) = \frac{1}{2}\ln(1+x^2)$ gives $\dot{V}_1 \leq -x^2/(1+x^2)^2 + u^2 + |u|$
- ISS wrt $|u| < 1$: $V_2(x) = x^4/4$ gives $\dot{V}_2 \leq -x^4/(1+x^2)$.

ISS, iISS and Strong iISS Example

Consider the tank with a flat bottom and not necessarily constant cross-section [Daskovskiy, IFAC POL, 2019].



ISS, iISS and Strong iISS Example

By Toricelli's law, the level x of the liquid changes with time due to the inflow and outflow can be described by the following differential equation:

$$\dot{x} = -\frac{a(x)\mu\sqrt{2gx}}{A(x)} + \frac{u}{A(x)}$$

where

$A \ : \ [0, \infty) \to (0, \infty)$ is the cross-section area of the tank at the height$x$

$a \ : \ [0, \infty) \to [0, \infty)$ is the area of the hole, that in general may also depend on x

u is the rate of inflow to the tank

Let us consider the following three cases:

- Suppose that $a(x) = a = (\mu\sqrt{2g})^{-1}$ and $A(x) = 1 + \sqrt[4]{x}$. The ISS property can be verified by considering $V(x) = |x| = x$. For this V, we have

  $$\dot{V} = -\frac{\sqrt{x}}{1 + \sqrt[4]{x}} + \frac{u}{1 + \sqrt[4]{x}}$$

  Now, observe that we have

  $$|x| \geq 4u^2 \quad \Longrightarrow \quad \dot{V} \leq -\frac{\sqrt{x}}{2\left(1 + \sqrt[4]{x}\right)}$$

  and since the function $\tilde{\alpha}(s) = \frac{\sqrt{s}}{2\left(1+\sqrt[4]{s}\right)}$ and $\chi(s) = 4s^2$ are two class $\mathcal{K}_\infty$ functions, we can conclude that the system is ISS.

Let us consider the following three cases (continued):

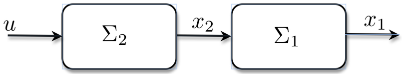
- Suppose that $a(x) = \frac{1}{1+x}$ and $A(x) = A = \mu\sqrt{2g}$. The iISS property can be verified by considering $V(x) = |x| = x$. For this V, we have

$$\dot{V} \leq -\frac{\sqrt{x}}{1+x} + \frac{u}{A}$$

  Since the function $\alpha(s) = \frac{\sqrt{s}}{1+s}$ is a class $\mathcal{PD}$ function whereas $\gamma(s) = \frac{s}{A}$ is a class $\mathcal{K}_\infty$ function, we can conclude that the system is iISS.

- (Exercise) Show that the system is Strong iISS when $a(x) = \frac{1}{\sqrt{1+x}}$ and $A(x) = A = \mu\sqrt{2g}$.

Cascades of ISS or iISS Systems



$$
\begin{aligned}
\Sigma_1: \quad \dot{x}_1 &= f_1(x_1, x_2) \\
\Sigma_2: \quad \dot{x}_2 &= f_2(x_2, u)
\end{aligned}
$$

- ISS is naturally preserved in cascade [Sontag, EJC, 1995]
- iISS is not preserved by cascade [Panteley, Loría, Automatica, 2001] & [Arcak et al., SICON, 2002].

Theorem [Chaillet, Angeli, SCL, 2008]

Let V_1 and V_2 be two Lyapunov functional candidates. Assume that there exist $\gamma_1, \gamma_2 \in \mathcal{K}$, and $\alpha_1, \alpha_2 \in \mathcal{PD}$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and all $u \in \mathbb{R}^m$,

$$
\begin{aligned}
\frac{\partial V_1}{\partial x_1} f_1(x_1, x_2) &\leq -\alpha_1(|x_1|) + \gamma_1(|x_2|) \\
\frac{\partial V_2}{\partial x_2} f_2(x_2, u) &\leq -\alpha_2(|x_2|) + \gamma_2(|u|) \,.
\end{aligned}
$$

If $\gamma_1(s) = \mathcal{O}_{s \to 0^+}(\alpha_2(s))$, then the cascade is iISS.

- $q_2(s) = \mathcal{O}_{s \to 0^+}(q_1(s))$: Given $q_1, q_2 \in \mathcal{PD}$, we say that q_1 *has greater growth* than q_2 around zero if $\exists\, k \geq 0$ such that $\limsup_{s \to 0^+} q_2(s)/q_1(s) \leq k$.

Cascades of Strong iISS Systems

$$\xrightarrow{u} \boxed{\Sigma_2} \xrightarrow{x_2} \boxed{\Sigma_1} \xrightarrow{x_1}$$

$$\begin{aligned} \Sigma_1 : \quad \dot{x}_1 &= f_1(x_1, x_2) \\ \Sigma_2 : \quad \dot{x}_2 &= f_2(x_2, u) \end{aligned}$$

Theorem: Strong iISS is preserved under cascade [Chaillet et. al., Automatica, 2014]

If the systems $\dot{x}_1 = f_1(x_1, u_1)$ and $\dot{x}_2 = f_2(x_2, u_2)$ are Strongly iISS, then the cascade (2) is Strongly iISS.

Corollary: iISS + Strong iISS $\Rightarrow$ iISS [Chaillet et. al., Automatica, 2014]

If $\dot{x}_1 = f_1(x_1, u_1)$ is Strongly iISS and $\dot{x}_2 = f_2(x_2, u_2)$ is iISS, then (2) is iISS.

Corollary: GAS + Strong iISS $\Rightarrow$ GAS [Chaillet et. al., Automatica, 2014]

If $\dot{x}_1 = f_1(x_1, u_1)$ is Strongly iISS and $\dot{x}_2 = f_2(x_2)$ is GAS, then

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_2) \end{aligned} \quad \text{is GAS.}$$

ISS/iISS Systems: Summary

	ISS	**Strong iISS**	**iISS**
0-GAS	✓	✓	✓
Forward completeness $\forall u \in \mathcal{U}$	✓	✓	✓
Bounded input-Bounded state	✓	For $\|u\| < R$	☹
Converging input-Converging state	✓	✓	☹
Preservation under cascade	✓	✓	Growth rate
Lyapunov characterization	$\alpha \in \mathcal{K}_\infty$	Open question	$\alpha \in \mathcal{PD}$