MTM5101-Dynamical Systems and Chaos

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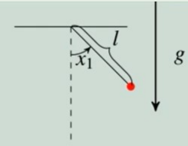
Week 8

Example (Pendulum without Friction)



$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin\, x_1$$

Recall the "spoiler" of last week, our Lyapunov function "candidate" may be

$$\begin{aligned} V(x) &= V_{\text{pot}}(x) + V_{\text{kin}}(x) \\ &= -\int_0^{x_1} -\frac{g}{\ell}\sin\, y\, dy + \frac{1}{2}x_2^2 = \frac{g}{\ell}(1-\cos\, x_1) + \frac{1}{2}x_2^2. \end{aligned}$$

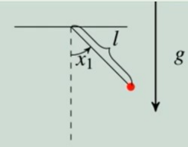
Note that, $V \in \mathcal{C}^1$. We choose

$$D = \{x \in \mathbb{R}^2 \mid |x_1| < 2\pi\} \subset \mathbb{R}^2,$$

and this implies

$$V(0) = 0 \text{ and } V(x) > 0,\ \forall x \in D \setminus \{0\}.$$

Example (Pendulum without Friction)



$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1$$

Differentiating V along the solutions of the system yields to

$$\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = \frac{g}{\ell}\sin x_1 x_2 + x_2\left(-\frac{g}{\ell}\sin x_1\right) = 0, \quad \forall x \in D$$

This makes sense, since this is a conservative system. Therefore

$$V : D = \{x \in \mathbb{R}^2 \mid |x_1| < 2\pi\} \to \mathbb{R}$$

is a Lyapunov function for $x = 0$ which tells us that $x = 0$ is a stable equilibrium point.

Example (Pendulum with Friction)

Consider, now, the system governed by pendulum with friction:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2.\end{aligned}$$

For simplicity, let us take $m = 1$.

Since a friction force is acting on this system, the system is no longer a conservative system. Friction is a dissipative force, which draws energy from the system. Let us again choose the same Lyapunov function "candidate", which we know that $V \in \mathcal{C}^1$, V is positive definite in $D = \{x \in \mathbb{R}^2 \mid |x_1| < 2\pi\}$. Now, let us check the derivative of V along the solutions of the system:

$$\dot{V}(x) = \frac{g}{\ell}\sin x_1 x_2 + x_2\left(-\frac{g}{\ell}\sin x_1 - kx_2\right) = -kx_2^2 \leq 0, \quad \forall x \in D$$

which implies that $x = 0$ is a stable equilibrium point. Moreover, we know that $x = 0$ is an asymptotically stable equilibrium point.

Example (Pendulum with Friction)

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{g}{\ell}\sin\, x_1 - k\, x_2.\end{aligned}$$

Let us replace the term $(1/2)x_2^2$ by more general quadratic form $(1/2)x^\top Px$ for some 2×2 positive definite symmetric matrix P:

$$\begin{aligned}V(x) = \frac{1}{2}x^T Px + \frac{g}{\ell}(1-\cos\, x_1) &= \frac{1}{2}\begin{bmatrix}x_1\\ x_2\end{bmatrix}^\top \begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \frac{g}{\ell}(1-\cos\, x_1)\\ &= \frac{1}{2}p_{11}x_1^2 + p_{12}x_1x_2 + \frac{1}{2}p_{22}x_2^2 + \frac{g}{\ell}(1-\cos\, x_1)\end{aligned}$$

For the matrix P to be positive definite, the elements of P must be satisfy

$$p_{11} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0$$

The directional derivative of V along the solutions of the system yields to

$$\begin{aligned}\dot{V}(x) &= \left(p_{11}x_1 + p_{12}x_2\frac{g}{\ell}\sin\, x_1\right)x_2 + (p_{12}x_1 + p_{22}x_2)\left(-\frac{g}{\ell}\sin\, x_1 - kx_2\right)\\ &= \frac{g}{\ell}(1-p_{22})x_2\sin\, x_1 - \frac{g}{\ell}p_{12}x_1\sin\, x_1 + (p_{11}-p_{12}k)x_1x_2 + (p_{12}-p_{22}k)x_2^2\end{aligned}$$

Example (Pendulum with Friction)

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{g}{\ell}\sin x_1 - k\,x_2.\end{aligned}$$

Now, we want to choose p_{11}, p_{12} and p_{22} such that $\dot{V} < 0$. Since the cross-product terms $x_2 \sin x_1$ and $x_1 x_2$ are sign indefinite, we will cancel them by taking $p_{22} = 1$ and $p_{11} = kp_{12}$. With these choices, we have

$$p_{11}p_{22} - p_{12}^2 = p_{12}(k - p_{12}) > 0 \Rightarrow 0 < p_{12} < k \quad (\text{for } k > 0)$$

for $V(x) > 0$. Let us take $p_{12} = \frac{k}{2}$, then $\dot{V}(x)$ will be

$$\dot{V}(x) = -\frac{1}{2}\frac{g}{\ell}kx_1 \sin x_1 - kx_2^2$$

The term $x_1 \sin x_1 > 0$ for all $0 < |x_1| < \pi$. Taking $D = \{x \in \mathbb{R}^2 \mid |x_1| < \pi\}$, we see that $\dot{V}(x) < 0$ over $D \setminus \{0\}$. Thus, by Lyapunov's Direct Method, we can conclude that $x = 0$ is asymptotically stable.

Let us consider the system which we analyzed in Lyapunov's indirect method (the linearization method):

$$\dot{x} = -x^3$$

Now, let us analyze the stability properties of the equilibrium point $x = 0$ by using Lyapunov's direct method. The system here may be interpreted as a mechanical system where x is the velocity and a nonlinear friction acts on the system. No potential forces act on the system, so the system energy is the kinetic energy:

$$E = E_{\text{kin}} = \frac{1}{2}v^2 = \frac{1}{2}x^2$$

So, this is one motivation for this choice of Lyapunov function candidate $V(x) = \frac{1}{2}x^2$. An another motivation is that this is a simple choice of a quadratic Lyapunov function candidate $V(x) = \frac{1}{2}x^\top Px$ where $P = I$ and since $x \in \mathbb{R}$, we have $V(x) = \frac{1}{2}x^2$.

Note that, $V \in \mathcal{C}^1$, $V(0) = 0$ and $V(x) > 0$, $\forall x \neq 0$ which implies that V is positive definite in $D \setminus \{0\} = \mathbb{R} \setminus \{0\}$. The directional derivative reads $\dot{V}(x) = x\dot{x} = -x^4 < 0$, $\forall x \neq 0$ which tells us $\dot{V}$ is negative definite in $D \setminus \{0\} = \mathbb{R} \setminus \{0\}$. By Lyapunov's Direct Method, $x = 0$ is LAS. Note that, the conditions for being strict Lyapunov function are satisfied in the whole state space $\mathbb{R}$, so it is quite natural to as the following question:

Question: Can we conclude that the origin $x = 0$ is GAS?

Let us consider the following theorem!

Lyapunov Theorem for Global Asymptotic Stability

Theorem (Lyapunov Theorem for GAS)

If

- $\exists$ *a* ***strict*** *Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}$ *for* $x = 0$ *and*
- V *is radially unbounded*

then $x = 0$ *is globally asymptotically stable (GAS).*

Definition (Radial Unboundedness)

V is **radially unbounded** if and only if $V(x) \to \infty$ as $\|x\| \to \infty$.

Example

Turning back to $V(x) = \frac{1}{2}x^2 = \frac{1}{2}\|x\|^2$, this expression tells us that V is a radially unbounded function. This shows that by Lyapunov Theorem for GAS, we can conclude that $x = 0$ is GAS for $\dot{x} = -x^3$.

Question: Why the radial unboundedness condition is necessary to conclude global asymptotic stability based on Lyapunov analysis?

Lyapunov Theorem for Global Asymptotic Stability

For continuously differentiable fcns, say $V \in \mathcal{C}^1$, the following implications hold

- positive definiteness $\Rightarrow$ level surfaces are closed for small values of c, which is required for local results
- radial unboundedness $\Rightarrow$ level surfaces are closed $\forall c$, which is required for global results

So, if the level surfaces are not closed, we may have that $\|x\| \to \infty$ even if $\dot{V} < 0$.

Example

Let us take $V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$.

Clearly, this function is positive definite. On the other hand,

- For $x_1 = 0$, $x_2 \to \infty \quad \Rightarrow \quad V(x) \to \infty$ as $\|x\| \to \infty$
- For $x_2 = 0$, $x_1 \to \infty \quad \Rightarrow \quad V(x) \to 1$ as $\|x\| \to \infty$!

So, $V(x)$ is not radially unbounded. There exist trajectories along which the time derivative of V is strictly negative, meaning that the trajectory intersects level curves corresponding to lower and lower c values, but the trajectory does not converge to the equilibrium point $x = 0$. See the figure on next slide!

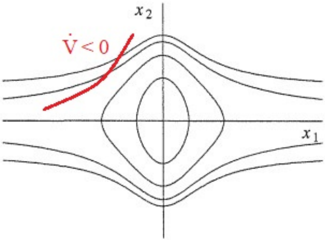


Figure: A Diverging Trajectory with $\dot{V}(x) < 0$.

Although the value of the V function decreases along the trajectory, the trajectory is allowed to slip away from the origin since the level curves are not closed.

See also KYP Lectures (L.4.4-10:57): `https://youtu.be/mIkgW_gUKjo?list=PLdeo5-jZaFjNPRGbKxWXrwnkNvjOkP_j8&t=657`

Lyapunov Theorem for Global Exponential Stability

We also have a Lyapunov theorem for exponential stability. We still consider the same system as before

$$\dot{x} = f(x)$$

where $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz and $x = 0 \in D$ is an equilibrium point of the system.

Theorem (Exponential Stability)

If there exists a function $V : D \to \mathbb{R}$ and constants $a, k_1, k_2, k_3 > 0$ such that

i) $V \in C^1$

ii) $k_1 \|x\|^a \leq V(x) \leq k_2 \|x\|^a, \forall x \in D$ $(V(x) \to \infty$ *as* $\|x\| \to \infty)$

iii) $\dot{V}(x) \leq -k_3 \|x\|^a, \quad \forall x \in D$

Then, $x = 0$ is ***exponentially stable (ES)****.*

Remark (Global Exponential Stability)

If the conditions in Exponential Stability Theorem are satisfied with $D = \mathbb{R}^n$, then $x = 0$ is globally exponentially stable (GES). The condition (ii) implies radial unboundedness condition. Hence, there is no need to impose radial unboundedness condition for GES.

Some further remarks:

- The Exponential Stability Theorem is also called Barbashin-Krasovskii Theorem.
- $\|\cdot\|$ can be any p-norm on the vector state space.
- This condition is stricter than the Asymptotic Stability Theorem because ES is stricter than AS.

Global Exponential Stability Convergence Rate: If the equilibrium point $x = 0$ of $\dot{x} = f(x)$ is globally exponentially stable, then the solution of the system satisfies

$$\|x(t)\| \leq \left(\frac{k_2}{k_1}\right)^{\frac{1}{a}} \|x(0)\| e^{-\frac{k_3}{k_2 a} t}, \quad \forall t \geq 0, \quad \|x(0)\| < c$$

where $c > 0$.

Example

Let us analyze the stability properties of the equilibrium point(s) of the system

$$\dot{x} = -x - x^3$$

by using Lyapunov direct method which we analyzed in Lyapunov's indirect method (the linearization method, with $a = 1$).

Note that,

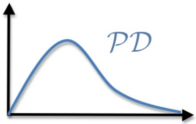
$$\dot{x} = -x - x^3 = -x(1 + x^2) = 0$$

so that $x = 0$ is the only equilibrium point. As shown before, $V(x) = \frac{1}{2}x^2 = \frac{1}{2}\|x\|^2$ is a Lyapunov function candidate for all $x \in \mathbb{R}$ and $V \in \mathcal{C}^1$. (ii) of Exponential Stability Theorem is also satisfied with $k_1 = k_2 = \frac{1}{2}$, $a = 2$. The directional derivative of V along this system reads

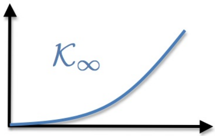
$$\dot{V}(x) = x\dot{x} = -x^2 - x^4 \leq -x^2 = -\|x\|^2$$

which tells us that (iii) of Exponential Stability Theorem is satisfied with $k_3 = 1$, $a = 2$. Note that $D = \mathbb{R}$, so that $x = 0$ is GES. The solution of this system satisfies the following GES convergence rate

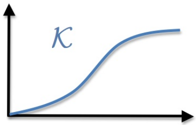
$$\|x(t)\| \leq \|x(0)\| e^{-t}, \quad \forall t \geq 0.$$



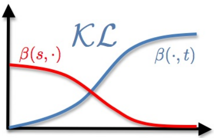
$$\begin{cases} \alpha \text{ continuous} \\ \alpha(0) = 0 \\ \alpha(s) > 0, \ \forall s > 0 \end{cases}$$



$$\begin{cases} \alpha \in \mathcal{K} \\ \lim_{s\to\infty} \alpha(s) = \infty \end{cases}$$



$$\begin{cases} \alpha \in \mathcal{PD} \\ \alpha \text{ increasing} \end{cases}$$



$$\begin{cases} \beta(\cdot,t) \in \mathcal{K}, \ \forall t \geq 0 \\ \beta(s,\cdot) \text{ nonincreasing}, \ \forall s \geq 0 \\ \lim_{t\to\infty} \beta(s,t) = 0, \ \forall s \geq 0 \end{cases}$$

Example

- $\alpha(s) = \frac{1}{1+s^c}$, for any $c > 0$
- $\alpha(s) = \frac{s^c}{1+s^c}$, for any $c > 0$
- $\alpha(s) = \tan^{-1}(s)$
- $\alpha(s) = \mathrm{sat}(s) = \begin{cases} s, & \text{if } |s| \leq 1 \\ \mathrm{sgn}(s), & \text{if } |s| > 1 \end{cases}$
- $\alpha(s) = \mathrm{sat}(s; a, b) = a + \frac{2(b-a)}{\pi} \tan^{-1}(s)$
- $\alpha(s) = s^c$, for any $c > 0$
- $\alpha(s) = \min\{s, s^2\}$
- $\beta(s, r) = \frac{s}{krs+1}$
- $\beta(s, r) = s^c e^{-r}$

Equivalent Representation of Radial Unboundedness

Lemma (4.3 in [Khalil, 2002])

Let $V : D \to \mathbb{R}$ be a continuous positive definite function (may not be $\mathcal{PD}$!) defined on $D \subset \mathbb{R}^n$ that contains the origin. Let $B_r[0] \subset D$ for some $r > 0$. Then, there exist $\alpha_1, \alpha_2 \in \mathcal{K}$ defined on $[0, r]$ such that

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|)$$

for all $x \in B_r[0]$. If $D = \mathbb{R}^n$, the functions α_1 and α_2 will be defined on $[0, \infty)$ and the foregoing inequality will hold for all $x \in \mathbb{R}^n$. Moreover, if $V(x)$ is radially unbounded, then $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

Example

- $V(x) = x^\top P x \implies \lambda_{\min}(P)|x|^2 \le x^\top P x \le \lambda_{\max}(P)|x|^2$

- Given $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm.

Equivalent Representation of GAS

For a system with no inputs $\dot{x} = f(x)$, there is a well-known notion of global asymptotic stability (for short from now on, GAS, or "0-GAS" when referring to the system with no-inputs $\dot{x} = f(x, 0)$ associated to a given system with inputs $\dot{x} = f(x, u)$ due to Lyapunov, and usually defined in "ϵ-δ" terms. It is easy to show that the standard definition of GAS (0-GAS) in previous lecture is in fact equivalent to the existence of $\beta \in \mathcal{KL}$ satisfying the following, along the solutions of $\dot{x} = f(x)$ ($\dot{x} = f(x, 0)$)

$$|x(t, x_0)| \leq \beta(|x_0|, t), \qquad \forall x_0 \in \mathbb{R}^n, \ \forall t \geq 0.$$

Observe that, since β decreases on t, we have, in particular:

$$|x(t, x_0)| \leq \beta(|x_0|, 0), \qquad \forall x_0 \in \mathbb{R}^n, \ \forall t \geq 0.$$

which provides the stability part of the GAS definition while

$$|x(t, x_0)| \leq \beta(|x_0|, t) \underset{t \to \infty}{\to} 0, \qquad \forall x_0 \in \mathbb{R}^n,$$

which is the attractivity (convergence to the equilibrium point) part of the GAS definition.

Note: From now on, unless written explicitly, the solutions $x(t, x_0)$ or $x(t, x_0, u)$ for $\dot{x} = f(x)$ and $\dot{x} = f(x, u)$, respectively, will be written in short as $x(t)$ to avoid cumbersome notation!

Equivalent Lyapunov Theorem for Stability

Theorem (4.8 in [Khalil, 2002])

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ and $D \subset \mathbb{R}^n$ be a domain containing the origin. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

$$W_1(x) \le V(x) \le W_2(x)$$
$$\frac{\partial V}{\partial x} f(x) \le 0$$

for all $t \ge 0$ and $x \in D$, where W_1 and W_2 are continuous positive definite functions on D. Then, $x = 0$ is stable.

Theorem (4.9 in [Khalil, 2002])

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ and $D \subset \mathbb{R}^n$ be a domain containing the origin. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

$$W_1(x) \leq V(x) \leq W_2(x)$$

$$\frac{\partial V}{\partial x} f(x) \leq -W_3(x)$$

for all $t \geq 0$ and $x \in D$, where W_1, W_2 and W_3 are continuous positive definite functions on D. Then, $x = 0$ is asymptotically stable. Moreover, if r and c are chosen such that $B_r[0] = \{x \in D \mid |x| \leq r\}$ and $c < \min_{|x|=r} W_1(x)$, then every trajectory starting in $\{x \in B_r[0] \mid W_2(x) \leq c\}$ satisfies

$$|x(t)| \leq \beta(|x(0)|, t), \quad \forall t \geq 0$$

for some $\beta \in \mathcal{KL}$.

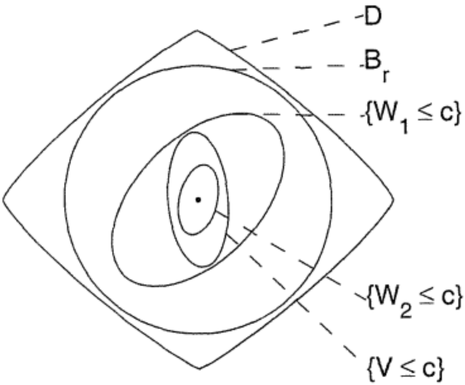


Figure: Geometric representation of sets in Theorem 4.9.

Equivalent Lyapunov Theorem for GAS

Theorem (4.9 in [Khalil, 2002])

Let $x = 0$ *be an equilibrium point for* $\dot{x} = f(x)$. *Let* $V : \mathbb{R}^n \to \mathbb{R}$ *($D = \mathbb{R}^n$!) be a continuously differentiable function such that*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\frac{\partial V}{\partial x} f(x) \leq -W_3(x)$$

for all $t \geq 0$ *and* $x \in \mathbb{R}^n$, *where* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *and* W_3 *is continuous positive definite function on* $\mathbb{R}^n$. *Then,* $x = 0$ *is GAS.*

Let us remember the Lyapunov theorem for ES/GES shown last week:

Theorem (4.10 in [Khalil, 2002])

Let $x = 0$ *be an equilibrium point for* $\dot{x} = f(x)$. *Let* $V : D \to \mathbb{R}$ *be a continuously differentiable function such that*

$$k_1|x|^a \leq V(x) \leq k_2|x|^a$$
$$\frac{\partial V}{\partial x} f(x) \leq -k_3|x|^a$$

for all $t \geq 0$ *and* $x \in D$, *where* k_1, k_2, k_3 *and* a *are positive constants. Then,* $x = 0$ *is ES. If* $D = \mathbb{R}^n$, *then* $x = 0$ *is GES.*