MTM5101-Dynamical Systems and Chaos

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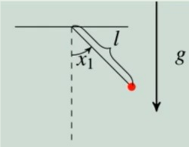
Week 8

Lyapunov Functions and Lyapunov's Stability Theorem

Example (Pendulum without Friction)



$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1$$

Recall the "spoiler" of last week, our Lyapunov function "candidate" may be

$$\begin{aligned} V(x) &= V_{\text{pot}}(x) + V_{\text{kin}}(x) \\ &= -\int_0^{x_1} -\frac{g}{\ell}\sin y\,dy + \frac{1}{2}x_2^2 = \frac{g}{\ell}(1-\cos x_1) + \frac{1}{2}x_2^2. \end{aligned}$$

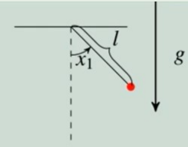
Note that, $V \in \mathcal{C}^1$. We choose

$$D = \{x \in \mathbb{R}^2 \mid |x_1| < 2\pi\} \subset \mathbb{R}^2,$$

and this implies

$$V(0) = 0 \text{ and } V(x) > 0,\ \forall x \in D \setminus \{0\}.$$

Example (Pendulum without Friction)



$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1$$

Differentiating V along the solutions of the system yields to

$$\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = \frac{g}{\ell}\sin x_1 x_2 + x_2\left(-\frac{g}{\ell}\sin x_1\right) = 0, \quad \forall x \in D$$

This makes sense, since this is a conservative system. Therefore

$$V : D = \{x \in \mathbb{R}^2 \mid |x_1| < 2\pi\} \to \mathbb{R}$$

is a Lyapunov function for $x = 0$ which tells us that $x = 0$ is a stable equilibrium point.

Example (Pendulum with Friction)

Consider, now, the system governed by pendulum with friction:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2.\end{aligned}$$

For simplicity, let us take $m = 1$.

Since a friction force is acting on this system, the system is no longer a conservative system. Friction is a dissipative force, which draws energy from the system. Let us again choose the same Lyapunov function "candidate", which we know that $V \in \mathcal{C}^1$, V is positive definite in $D = \{x \in \mathbb{R}^2 \mid |x_1| < 2\pi\}$. Now, let us check the derivative of V along the solutions of the system:

$$\dot{V}(x) = \frac{g}{\ell}\sin x_1 x_2 + x_2\left(-\frac{g}{\ell}\sin x_1 - kx_2\right) = -kx_2^2 \leq 0, \quad \forall x \in D$$

which implies that $x = 0$ is a stable equilibrium point. Moreover, we know that $x = 0$ is an asymptotically stable equilibrium point.

Example (Pendulum with Friction)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 - k\, x_2.$$

Let us replace the term $(1/2)x_2^2$ by more general quadratic form $(1/2)x^\top Px$ for some 2×2 positive definite symmetric matrix P:

$$V(x) = \frac{1}{2}x^T Px + \frac{g}{\ell}(1 - \cos x_1) = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{g}{\ell}(1 - \cos x_1)$$
$$= \frac{1}{2}p_{11}x_1^2 + p_{12}x_1x_2 + \frac{1}{2}p_{22}x_2^2 + \frac{g}{\ell}(1 - \cos x_1)$$

For the matrix P to be positive definite, the elements of P must be satisfy

$$p_{11} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0$$

The directional derivative of V along the solutions of the system yields to

$$\dot{V}(x) = \left(p_{11}x_1 + p_{12}x_2\frac{g}{\ell}\sin x_1\right)x_2 + (p_{12}x_1 + p_{22}x_2)\left(-\frac{g}{\ell}\sin x_1 - kx_2\right)$$
$$= \frac{g}{\ell}(1 - p_{22})x_2 \sin x_1 - \frac{g}{\ell}p_{12}x_1 \sin x_1 + (p_{11} - p_{12}k)x_1x_2 + (p_{12} - p_{22}k)x_2^2$$

Example (Pendulum with Friction)

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{g}{\ell}\sin x_1 - k\,x_2.\end{aligned}$$

Now, we want to choose p_{11}, p_{12} and p_{22} such that $\dot{V} < 0$. Since the cross-product terms $x_2 \sin x_1$ and $x_1 x_2$ are sign indefinite, we will cancel them by taking $p_{22} = 1$ and $p_{11} = kp_{12}$. With these choices, we have

$$p_{11}p_{22} - p_{12}^2 = p_{12}(k - p_{12}) > 0 \Rightarrow 0 < p_{12} < k \quad (\text{for } k > 0)$$

for $V(x) > 0$. Let us take $p_{12} = \frac{k}{2}$, then $\dot{V}(x)$ will be

$$\dot{V}(x) = -\frac{1}{2}\frac{g}{\ell}kx_1 \sin x_1 - kx_2^2$$

The term $x_1 \sin x_1 > 0$ for all $0 < |x_1| < \pi$. Taking $D = \{x \in \mathbb{R}^2 \mid |x_1| < \pi\}$, we see that $\dot{V}(x) < 0$ over $D \setminus \{0\}$. Thus, by Lyapunov's Direct Method, we can conclude that $x = 0$ is asymptotically stable.

Let us consider the system which we analyzed in Lyapunov's indirect method (the linearization method):

$$\dot{x} = -x^3$$

Now, let us analyze the stability properties of the equilibrium point $x = 0$ by using Lyapunov's direct method. The system here may be interpreted as a mechanical system where x is the velocity and a nonlinear friction acts on the system. No potential forces act on the system, so the system energy is the kinetic energy:

$$E = E_{\text{kin}} = \frac{1}{2}v^2 = \frac{1}{2}x^2$$

So, this is one motivation for this choice of Lyapunov function candidate $V(x) = \frac{1}{2}x^2$. An another motivation is that this is a simple choice of a quadratic Lyapunov function candidate $V(x) = \frac{1}{2}x^\top Px$ where $P = I$ and since $x \in \mathbb{R}$, we have $V(x) = \frac{1}{2}x^2$.

Note that, $V \in \mathcal{C}^1$, $V(0) = 0$ and $V(x) > 0$, $\forall x \neq 0$ which implies that V is positive definite in $D \setminus \{0\} = \mathbb{R} \setminus \{0\}$. The directional derivative reads $\dot{V}(x) = x\dot{x} = -x^4 < 0$, $\forall x \neq 0$ which tells us $\dot{V}$ is negative definite in $D \setminus \{0\} = \mathbb{R} \setminus \{0\}$. By Lyapunov's Direct Method, $x = 0$ is LAS. Note that, the conditions for being strict Lyapunov function are satisfied in the whole state space $\mathbb{R}$, so it is quite natural to as the following question:

Question: Can we conclude that the origin $x = 0$ is GAS?

Let us consider the following theorem!

Lyapunov Theorem for Global Asymptotic Stability

Theorem (Lyapunov Theorem for GAS)

If

- $\exists$ *a* ***strict*** *Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}$ *for* $x = 0$ *and*
- V *is radially unbounded*

then $x = 0$ *is globally asymptotically stable (GAS).*

Definition (Radial Unboundedness)

V is **radially unbounded** if and only if $V(x) \to \infty$ as $\|x\| \to \infty$.

Example

Turning back to $V(x) = \frac{1}{2}x^2 = \frac{1}{2}\|x\|^2$, this expression tells us that V is a radially unbounded function. This shows that by Lyapunov Theorem for GAS, we can conclude that $x = 0$ is GAS for $\dot{x} = -x^3$.

Question: Why the radial unboundedness condition is necessary to conclude global asymptotic stability based on Lyapunov analysis?

Lyapunov Theorem for Global Asymptotic Stability

For continuously differentiable fcns, say $V \in \mathcal{C}^1$, the following implications hold

- positive definiteness $\Rightarrow$ level surfaces are closed for small values of c, which is required for local results
- radial unboundedness $\Rightarrow$ level surfaces are closed $\forall c$, which is required for global results

So, if the level surfaces are not closed, we may have that $\|x\| \to \infty$ even if $\dot{V} < 0$.

Example

Let us take $V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$.

Clearly, this function is positive definite. On the other hand,

- For $x_1 = 0,\ x_2 \to \infty \quad \Rightarrow \quad V(x) \to \infty$ as $\|x\| \to \infty$
- For $x_2 = 0,\ x_1 \to \infty \quad \Rightarrow \quad V(x) \to 1$ as $\|x\| \to \infty$!

So, $V(x)$ is not radially unbounded. There exist trajectories along which the time derivative of V is strictly negative, meaning that the trajectory intersects level curves corresponding to lower and lower c values, but the trajectory does not converge to the equilibrium point $x = 0$. See the figure on next slide!

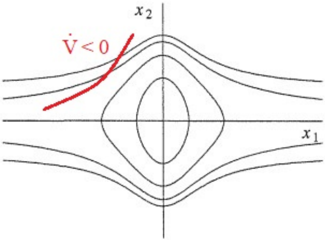


Figure: A Diverging Trajectory with $\dot{V}(x) < 0$.

Although the value of the V function decreases along the trajectory, the trajectory is allowed to slip away from the origin since the level curves are not closed.

See also KYP Lectures (L.4.4-10:57): `https://youtu.be/mIkgW_gUKjo?list=PLdeo5-jZaFjNPRGbKxWXrwnkNvjOkP_j8&t=657`

Lyapunov Theorem for Global Exponential Stability

We also have a Lyapunov theorem for exponential stability. We still consider the same system as before

$$\dot{x} = f(x)$$

where $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz and $x = 0 \in D$ is an equilibrium point of the system.

Theorem (Exponential Stability)

If there exists a function $V : D \to \mathbb{R}$ and constants $a, k_1, k_2, k_3 > 0$ such that

i) $V \in C^1$

ii) $k_1 \|x\|^a \leq V(x) \leq k_2 \|x\|^a, \forall x \in D$ *($V(x) \to \infty$ as $\|x\| \to \infty$)*

iii) $\dot{V}(x) \leq -k_3 \|x\|^a, \quad \forall x \in D$

*Then, $x = 0$ is **exponentially stable (ES)**.*

Remark (Global Exponential Stability)

If the conditions in Exponential Stability Theorem are satisfied with $D = \mathbb{R}^n$, then $x = 0$ is globally exponentially stable (GES). The condition (ii) implies radial unboundedness condition. Hence, there is no need to impose radial unboundedness condition for GES.

Lyapunov Theorem for Global Exponential Stability

Some further remarks:

- The Exponential Stability Theorem is also called Barbashin-Krasovskii Theorem.
- $\|\cdot\|$ can be any p-norm on the vector state space.
- This condition is stricter than the Asymptotic Stability Theorem because ES is stricter than AS.

Global Exponential Stability Convergence Rate: If the equilibrium point $x = 0$ of $\dot{x} = f(x)$ is globally exponentially stable, then the solution of the system satisfies

$$\|x(t)\| \leq \left(\frac{k_2}{k_1}\right)^{\frac{1}{a}} \|x(0)\| e^{-\frac{k_3}{k_2 a} t}, \quad \forall t \geq 0, \quad \|x(0)\| < c$$

where $c > 0$.

Example

Let us analyze the stability properties of the equilibrium point(s) of the system

$$\dot{x} = -x - x^3$$

by using Lyapunov direct method which we analyzed in Lyapunov's indirect method (the linearization method, with $a = 1$).

Note that,

$$\dot{x} = -x - x^3 = -x(1 + x^2) = 0$$

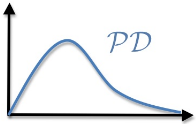
so that $x = 0$ is the only equilibrium point. As shown before, $V(x) = \frac{1}{2}x^2 = \frac{1}{2}\|x\|^2$ is a Lyapunov function candidate for all $x \in \mathbb{R}$ and $V \in \mathcal{C}^1$. (ii) of Exponential Stability Theorem is also satisfied with $k_1 = k_2 = \frac{1}{2}$, $a = 2$. The directional derivative of V along this system reads

$$\dot{V}(x) = x\dot{x} = -x^2 - x^4 \leq -x^2 = -\|x\|^2$$

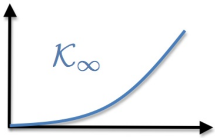
which tells us that (iii) of Exponential Stability Theorem is satisfied with $k_3 = 1$, $a = 2$. Note that $D = \mathbb{R}$, so that $x = 0$ is GES. The solution of this system satisfies the following GES convergence rate

$$\|x(t)\| \leq \|x(0)\|e^{-t}, \quad \forall t \geq 0.$$

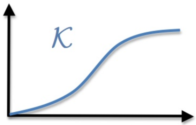
Comparison Functions



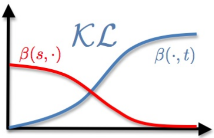
$$\begin{cases} \alpha \text{ continuous} \\ \alpha(0) = 0 \\ \alpha(s) > 0,\ \forall s > 0 \end{cases}$$



$$\begin{cases} \alpha \in \mathcal{K} \\ \lim_{s\to\infty} \alpha(s) = \infty \end{cases}$$



$$\begin{cases} \alpha \in \mathcal{PD} \\ \alpha \text{ increasing} \end{cases}$$



$$\begin{cases} \beta(\cdot,t) \in \mathcal{K},\ \forall t \geq 0 \\ \beta(s,\cdot) \text{ nonincreasing},\ \forall s \geq 0 \\ \lim_{t\to\infty} \beta(s,t) = 0,\ \forall s \geq 0 \end{cases}$$