MTM4501-Operations Research

Gökhan Göksu, PhD

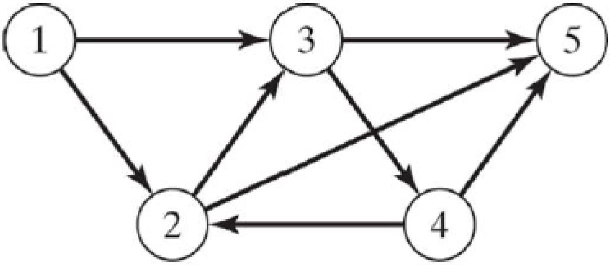
Week 2

Course Content

- Definition of OR and Its History
- Decision Theory and Models
- Network Analysis
  - Network Definitions
  - Network Models
  - Minimal Spanning Tree
- Inventory Management Models
- Queue Models

Network Definitions

The set of **nodes** connected to each other by **arcs** is called **network**. Networks are expressed with (N, A) notation. Here N stands for nodes and A stands for links.



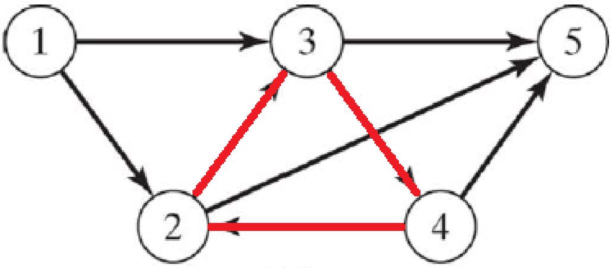
$$N = \{1, 2, 3, 4, 5\},\ A = \{(1, 2), (1, 3), (2, 3), (2, 5), (3, 4), (3, 5), (4, 2), (4, 5)\}$$

Each network has its own unique flow type (for example, oil products flow through a pipeline, traffic flows through a road network). In general, flow in a network is limited by the capacities of the arcs in the network, which may be finite or infinite.

Network Definitions

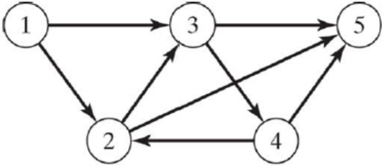
A connection that allows positive flow in one direction and zero flow in the other direction is said to be **directed** or **oriented**. All connections of **directed network** are directed.

Path is the sequence of separate branches connecting two nodes, regardless of the direction of flow in each branch. If the path connects a node to itself, it creates a **cycle** or a **loop**. For example, in the network below, the branches $(2, 3)$, $(3, 4)$, and $(4, 2)$ form a cycle.



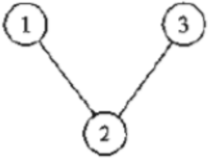
Network Definitions

Connected network is the connection of every two separate nodes by at least one path.

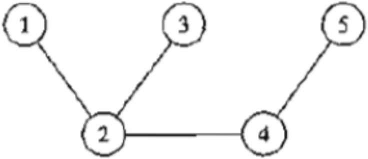


Connected network

Tree is a network that concerns only a subset of all nodes of the connected network. **Spanning tree** is the tree that connects all the nodes of the network without allowing any cycles.



Tree



Spanning tree

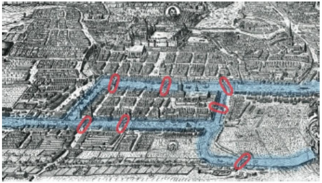
Network Models

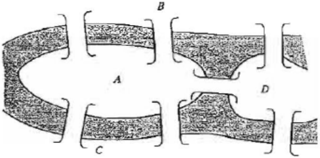
In operations research, there are many problems that can be conveniently modeled and solved as networks. To understand these, let us consider the following algorithms:

1. **Minimal spanning tree algorithm:** Considering the onshore natural gas pipeline project connecting wells in the Gulf of Mexico to inland delivery points, the purpose of the model is to minimize the construction costs of the pipeline.
2. **Shortest path algorithm:** It is the determination of the shortest route between two cities in the existing road network.
3. **Maximum flow algorithm:** It is the determination of the maximum capacity of the pipeline network that connects the coal mine to the power plant and transports the coal in water.
4. **Minimum cost capacitated network algorithm:** It is the determination of the minimum cost flow schedule of the network connected by pipeline from oil fields to refineries.
5. **Critical path (CPM) algorithm:** It is the determination of the time schedule (start and completion dates) for the activities of a construction project.

First Network Example: The Bridges of Königsberg

The Prussian city of Königsberg (now Kaliningrad in Russia) was founded in 1254, consisting of four districts (A, B, C and D pictured) connected by seven bridges around the Pregolya River. There was a problem floating around among the residents of the city as to whether a round trip of four sections could be made **by crossing each bridge exactly once**. There is no limit placed on the number of times any of the four sections can be visited.

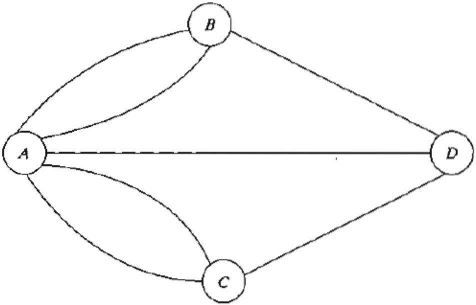




Bridges of Königsberg

In the mid-eighteenth century, the famous mathematician Leonhard Euler developed a special “road-building" argument to prove that making such a journey was impossible. Later, in the early nineteenth century, the same problem was solved by representing it as a network where each of the four segments (A, B, C, and D) was a node and each bridge was a link connecting these nodes.

First Network Example: The Bridges of Königsberg



In the network-based solution, the desired round trip (starting and ending in one part of the city) is impossible because there are four nodes and each is associated with an odd number of connections that do not allow different entry and exit to each part of the city (hence the use of different bridges). As a general solution, there is a cycle if all nodes have an even number of connections or if exactly two nodes have an odd number of connections. Otherwise, there is no such tour. This example shows how solving this problem is made easier using network representation.

First Network Example: The Bridges of Königsberg

SOLVTIO PROBLEMATIS
AD
GEOMETRIAM SITVS
PERTINENTIS.

AVCTORE
Leonh. Eulero.

§. 1.

Tabula VIII. PRaeter illam Geometriae partem, quae circa quantitates verſatur, et omni tempore ſummo ſtudio eſt exculta, alterius partis etiamnum admodum ignotae primus mentionem fecit *Leibnitzius*, quam Geometriam ſitus vocauit. Iſta pars ab ipſo in ſolo ſitu determinando, ſituſque proprietatibus eruendis occupata eſſe ſtatuitur; in quo negotio neque ad quantitates reſpiciendum, neque calculo quantitatum vtendum ſit. Cuiusmodi autem problemata ad hanc ſitus Geometriam pertineant, et quali methodo in iis reſoluendis vti oporteat, non ſatis eſt definitum. Quamobrem, cum nuper problematis cuiusdam mentio eſſet facta, quod quidem ad geometriam pertinere videbatur, at ita erat comparatum, vt neque determinationem quantitatum requireret, neque ſolutionem calculi quantitatum ope admitteret, id ad geometriam ſitus referre haud dubitaui: praeſertim quod in eius ſolutione ſolus ſitus in conſiderationem veniat, calculus vero nullius prorſus ſit vſus. Methodum ergo meam quam ad huius generis proble-

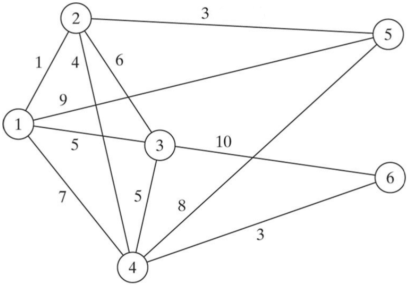
mata

Minimal Spanning Tree

The minimal spanning tree algorithm deals with associating the nodes of the network with each other, using the shortest connection of the branches, directly or indirectly. A typical application of this is the construction of roads connecting one or more cities. The most economical design of the road system means minimizing the total length of roads, and the result is achieved by adapting the minimal spanning tree algorithm.

Example

Midwest TV Cable Company is considering to provide cable broadcasting services to 5 new residential areas. Potential cable connections (in km) between the five regions are given below. It is desired to determine the most economical cable network.



Minimal Spanning Tree

The steps of the procedure are given as follows. Let $N = \{1, 2, ..., n\}$ be the set of nodes of the network and define

$$C_k = \text{Set of nodes that have been permanently connected at iteration } k$$

$$\overline{C}_k = \text{Set of nodes as yet to be connected permanently after iteration } k$$

- **Step 0:** Set $C_0 = \emptyset$ and $\overline{C}_0 = N$.
- **Step 1:** Start with any node i in the unconnected set $\overline{C}_0$ and set $C_1 = \{i\}$, which renders $\overline{C}_1 = N \setminus \{i\}$. Set $k = 2$.
- **General step k:** Select a node j^* in the unconnected set $\overline{C}_{k-1}$ that yields the shortest arc to a node in the connected set C_{k-1}.
  Link j^* permanently to C_{k-1} and remove it from $\overline{C}_{k-1}$, i.e.

  $$C_k = C_{k-1} \cup \{j^*\}, \quad \overline{C}_k = \overline{C}_{k-1} \setminus \{j^*\}$$

  If the set of unconnected nodes $\overline{C}_k$ is empty, then stop. Otherwise, set $k = k + 1$ and repeat the step.

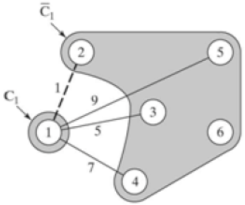
Minimal Spanning Tree

In our example, let us start the algorithm with node 1 (it can also be started with an another node):

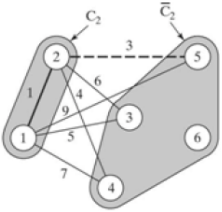
$$C_1 = \{1\}, \quad \overline{C}_1 = \{2, 3, 4, 5, 6\}$$

Iterations of the algorithm are given below. Here, thin arcs give all candidate connections between C and $\overline{C}$. Bold arcs indicate permanent connections between nodes in the set of connected nodes C, and dashed arcs indicate new (permanent) link added in each iteration. For example, in iteration 1, among all candidate links, link $(1, 2)$ is the shortest link (=1km) from node 1 to node 2, 3, 4, 5, and 6 of the set of unconnected nodes $\overline{C}_1$. Therefore, the connection $(1, 2)$ is made permanent and the following result is obtained for $j^* = 2$:

$$C_2 = \{1, 2\}, \quad \overline{C}_2 = \{3, 4, 5, 6\}$$

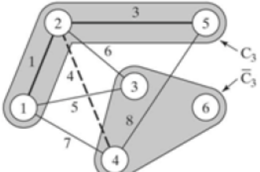


iteration 1

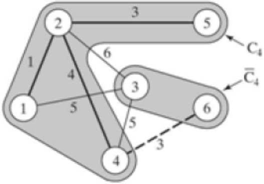


iteration 2

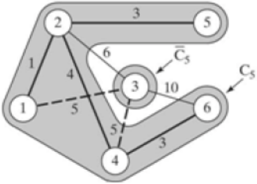
Minimal Spanning Tree



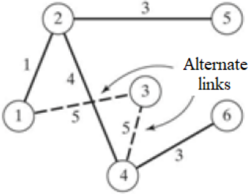
iteration 3



iteration 4



iteration 5



iteration 6
(minimal spanning tree)

The solution found with the minimal spanning tree algorithm is shown in iteration 6. The resulting minimum cable length to provide the desired cable broadcast service is $1 + 3 + 4 + 3 + 5 = 16$ km.