MTM3691-Theory of Linear Programming

Gökhan Göksu, PhD

Week 1

Contact Information and Course Evaluation Criteria

Gökhan Göksu, PhD	
E-Mail (YTU):	gokhan.goksu@yildiz.edu.tr
E-Mail (GMail):	goekhan.goeksu@gmail.com
Website:	gokhangoksu.github.io/MTM3691
Office:	107

- Midterm I: 35 %
- Midterm II: 25 %
- Final Exam: 40 %
- Office Hours: Wednesday, 09:00-12:00

Course Content and Textbooks

- Chapter 2: Introduction to Linear Programming (LP)
  - Two-Variable LP Model
  - Characteristics of LP Model
  - Graphic Solution Method (Max/Min)
- Chapter 3: The Simplex Method
- Chapter 4: Duality and Sensitivity Analysis
- Chapter 5: Transportation Model and Various Transportation Models

H. A. Taha, Operations Research: An Introduction, 8/e, Pearson Education, 2008.

H. A. Taha, Çeviren ve Uyarlayanlar: Ş. A. Baray, Ş. Esnaf, Yöneylem Araştırması, 6. Basımdan Çeviri, Literatür Yayınevi, 2000.

What is "Linear Programming"?

- In mathematics, especially in the applied branch of operations research, linear programming problems are the optimization problems (i.e. minimization or maximization of the objective function) of a linear objective function in a way that satisfies the linear equality and/or inequality constraints.
- An optimization model is called a linear program if it has continuous variables, a single linear objective function and all its constraints consist of linear equations or inequalities.
- In other words, the single objective function and all constraints of the model should consist of weighted sums of continuous decision variables.

Terminology

- The word linear in linear programming indicates that all functions in the model must be linear.
- The word programming does not refer to "computer programming" rather it is synonymous with "planning".
- Therefore, linear programming involves planning activities that will find the optimal solution that fits a specified goal among many suitable alternatives.

History

- The study of a problem in the form of a system of linear inequalities dates back to the work of Joseph Fourier and is named the Fourier Motzkin elimination method in memory of this famous mathematician.
- While all economic planning issues become prominent "in practice" in Soviet Russia in the 1920 s, among the theoretical studies carried out to show how the entire economy could be planned theoretically, Leonid Kantorovich's contribution led to a linear programming problem being clearly revealed for the first time.
- A research group lead by George Dantzig established to examine the logistics allocation problems that arose in the United States during World War II, revealed the need to define the linear programming problem to solve such problems.

History

- To solve linear programming problems, they introduced a solution methodology called the "simplex algorithm".
- Especially since it was clearly seen that this mathematical model and solution algorithm led to the reduction of war expenses by planning costs and returns, these theoretical and practical developments remained a state secret until 1947.
- In 1947 John von Neumann, while also particularly interested in the theory of games, developed the theory of duality.
- The Nobel Prize in Economics was awarded to Kantorovic Dantzig and von Neumann in 1975 for their contributions to linear programming.

The Areas of Use

- Military Applications,
- Networks,
- Supply Chains,
- Production Planning,
- Human Resources Planning,
- Logistics,
- Agriculture,
- Transportation
- Telecommunications
- Business, Economics,
- Inventory Control,
- Energy Planning,
- Marketing,
- ...

Formulation

- Mathematically, a linear programming problem has three fundamental components.
- Objective function: An objective function, to be minimized or maximized.
  - For an *n* dimensional problem:

$$\max \; c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

    or

$$\max \; \mathbf{c}^\top \mathbf{x}$$

  - For a 2 dimensional problem:

$$\max \; c_1 x_1 + c_2 x_2$$

Formulation

- Mathematically, a linear programming problem has three fundamental components.
- Constraints: Constraints, in the form of equality or inequality.
  - For a problem with m variables and n constraints:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m
\end{aligned}
$$

or

$$\mathbf{Ax} \leq \mathbf{b}$$

Formulation

- Mathematically, a linear programming problem has three fundamental components.
- Constraints: Constraints, in the form of equality or inequality.
  - For a problem with 2 variables and 3 constraints:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 &\leq b_2 \\
a_{31}x_1 + a_{32}x_2 &\leq b_3
\end{aligned}
$$

Formulation

- Mathematically, a linear programming problem has three fundamental components.
- Nonnegativity constraints: All the variables should be greater or equal to 0.
  - For an n dimensional problem:

$$x_1 \geq 0, \; x_2 \geq 0, \; \dots, \; x_n \geq 0$$

  or

$$\mathbf{x} \geq 0$$

  - For a 2 dimensional problem:

$$x_1 \geq 0 \text{ and } x_2 \geq 0$$

Formulation

- All the components can be written in terms of vector and matrix notation as

$$\begin{aligned} \max \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b}, \ \mathbf{x} \ge 0 \end{aligned}$$

Example (The Reddy Mikks Company)

Reddy Mikks produces interior and exterior paints from two raw materials, M_1 and M_2. The basic data for the problem is presented in the table below:

	Raw material usage per ton (tons)		Daily maximum availability (tons)
	Exterior paint	Interior paint	
Raw material, M_1	6	4	24
Raw material, M_2	1	2	6
Profit per ton (1000 TL)	5	4	

Additionally, a market survey has determined that the daily interior paint demand is at most 2 tons. It was also found in the same research that the daily interior paint demand cannot exceed that for daily exterior paint demand by 1 ton. Reddy Mikks wants to determine the optimum production quantities of exterior and interior paints to maximize their daily profit.

Introduction to Linear Programming: Two-Variable LP Model

The LP model has three fundamental components:

-
-
-

The proper definition of the decision variables is an essential first step in the development of the model. Once done, the task of constructing the objective function and the constraints becomes more straightforward.

For the Reddy Mikks problem, we need to determine the daily amounts to be produced of exterior and interior paints. Thus the variables of the model are defined as

Introduction to Linear Programming: Two-Variable LP Model

After defining the decision variables, the objective function is to be defined using these definitions. The goal for interior and exterior painting will be to increase the daily profit from interior and exterior painting (i.e. to maximize profit). If we denote that z represents the daily profit, the goal will be

Next, we construct the constraints that restricts raw material usage and product demand. The raw material restrictions are expressed verbally as

$$\begin{pmatrix}\text{Usage of a raw material}\\ \text{by both paints}\end{pmatrix} \leq \begin{pmatrix}\text{Maximum raw material}\\ \text{availability}\end{pmatrix}.$$

Introduction to Linear Programming: Two-Variable LP Model

From the data given in the problem setting, we have the following:

$$\text{Usage of raw material } M_1 =$$
$$\text{Usage of raw material } M_2 =$$

Since the daily availabilities of the raw materials M_1 and M_2 are limited to 24 and 6 tons, respectively, the associated restrictions are given as

Moreover, there are further constraints to be considered.

1) Demand for interior paint is limited to maximum 2 tons per day:

2) The excess of the daily production of interior over exterior paint should not exceed 1 tons per day:

3) The **non-negativity** constraint, which is not directly included in the model and states that variables cannot take negative values, should also be added to the model:

Any values of x_1 and x_2 that satisfy all five constraints above constitute a . Otherwise, the solution is . For example, the solution $x_1 = 3$ tons per day and $x_2 = 1$ tons per day is . On the other hand the solution $x_1 = 4$ and $x_2 = 1$ is . The goal of the problem to find , or the , that maximizes the total profit.