MTM3691-Theory of Linear Programming

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Week 3

Course Content

- Chapter 2: Introduction to Linear Programming (LP)
- Chapter 3: The Simplex Method
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  - Computational Details of Simplex Method
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Iterative Nature of the Simplex Method

Based on the algebraic method described last week, all feasible basic solutions in a standardized LP model were reviewed in detail, one by one, and the optimum could be determined. However, this method is not an effective and successful method in terms of calculation. Simplex Method (Algorithm) is a method designed to determine the location of the optimum by focusing on a certain number of feasible basic solutions.

The simplex method always starts with a feasible basic solution (usually the origin) and then starts looking for another feasible basic solution where the objective function is further improved. Another better feasible basic solution is possible with an **increase** in the value of one of the existing non-basic variables being zero. It is possible for the value of the current zero-valued variable to become positive if one of the existing basic variables leaves the basic solution (becomes a zero-valued non-basic variable). In the simplex method, the selected zero-value variable is called **entering variable**, and the basic variable that is wanted to be left out of the solution is called **leaving variable**.

Iterative Nature of the Simplex Method

Example

Let us consider the example of last week

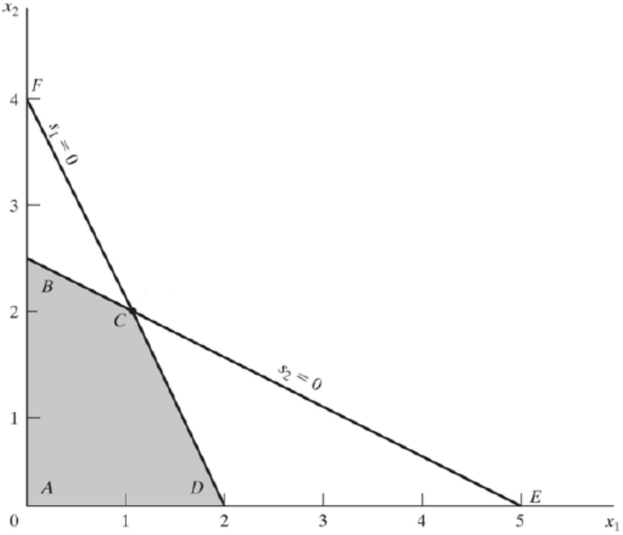
$$
\begin{aligned}
\max z &= 2x_1 + 3x_2 \\
\textit{subject to } 2x_1 + x_2 &\le 4 \\
x_1 + 2x_2 &\le 5 \\
x_1, x_2 &\ge 0
\end{aligned}
$$

The constraints of this LP can be rewritten in equality form:

$$
\begin{aligned}
2x_1 + x_2 + s_1 \qquad &= 4 \\
x_1 + 2x_2 \qquad + s_2 &= 5 \\
x_1, x_2, s_1, s_2 &\ge 0
\end{aligned}
$$

Now, let us determine the starting point!

Iterative Nature of the Simplex Method

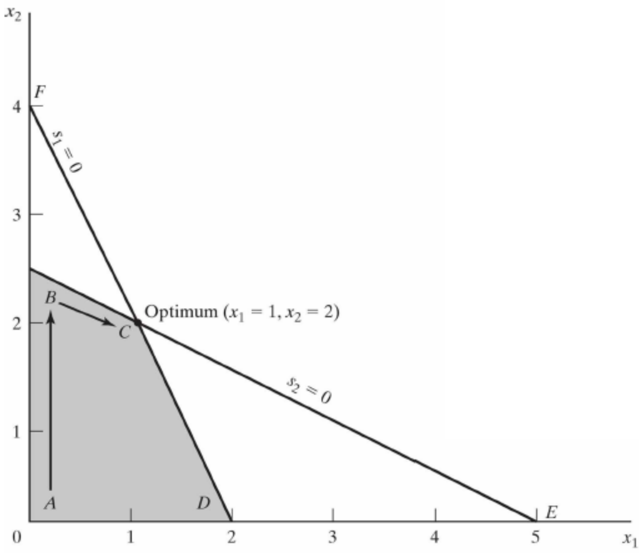


Iterative Nature of the Simplex Method

The origin $x_1 = x_2 = 0$ is chosen as the starting point. At this starting point, since our objective function z is zero; it is necessary to determine our direction by considering the best increase in the objective function (increase in maximization, decrease in minimization) by unitary increase in the non-basic x_1 and/or x_2 variables.

According to this objective, increasing both x_1 and x_2 improves (increases) the objective function. Since the simplex method increases one variable at each step, the variable that makes the best contribution to the objective function with a unit increase is selected first. In our example, since x_1 contributes 2 units and x_2 contributes 3 units, the iteration steps are completed by first increasing x_2, and then by increasing x_1.

Iterative Nature of the Simplex Method



Iterative Nature of the Simplex Method

The value of x_2 is increased until it reaches the corner point B. After point B, the value of x_1 was increased until the corner point C and optimum was reached. Therefore, the path followed by the simplex algorithm is $A \rightarrow B \rightarrow C$. Accessing each vertex is associated with an **iteration step**. Another important point here is that the simplex method does not follow a shortcut from the inner region of the solution space from A to C, but proceeds from the **corners** of the solution space.

In this context, we can show the transition from the graphical solution to the algebraic solution in the table below by considering the points A, B and C.

Corner Point	Basic Variables	Non-Basic (Zero) Variables
A		
B		
C		

Iterative Nature of the Simplex Method

Here, let us pay attention to the systematic change of basic variables and variables that do not form a basic solution on the path $A \to B \to C$. While moving from A to B, the variable x_2, which does not form a basic solution at A, becomes the basic variable at B and when moving from B to C, it does not create a basic solution until B. The variable x_1 becomes the basic variable at point C. In simplex method terminology, in the first step, the x_2 variable is called **entering variable**, while the s_2 variable is called **leaving variable.** Similarly, in the second step, the x_1 variable is called **entering variable**, while the s_1 variable is called **leaving variable.**

Algebraic Form of the Simplex Method

Example

$$\begin{aligned} \max z &= 2x_1 + 3x_2 \\ s.t.\ 2x_1 + x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} \max z &= 2x_1 + 3x_2 \\ s.t.\ 2x_1 + x_2 + s_1 \qquad &= 4 \\ x_1 + 2x_2 \qquad + s_2 &= 5 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

- First, let us determine a basic feasible solution. The easiest way to do this is to get $s_1 = 4$ and $s_2 = 5$ by choosing $x_1 = x_2 = 0$.
- Then, as the first iteration step of the simplex method, determining the basic variables is to obtain the objective function in terms of variables that do not form the basic solution.

Algebraic Form of the Simplex Method

Step 1:

Algebraic Form of the Simplex Method

Algebraic Form of the Simplex Method

Computational Details of Simplex Method

In this section, the computational steps of the simplex method are explained by determining

- the entering/leaving variables,
- the termination condition of the algorithm when the optimum solution is reached.

Let us recall the Reddy Mikks example.

Example

$$\begin{aligned} \max z = 5x_1 + 4x_2 & \\ s.t.\ 6x_1 + 4x_2 &\leq 24 \\ x_1 + 2x_2 &\leq 6 \\ -x_1 + x_2 &\leq 1 \\ x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Computational Details of Simplex Method

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- the entering/leaving variables,
- the termination condition of the algorithm when the optimum solution is reached.

Let us recall the Reddy Mikks example.

Example

$$\begin{aligned}
\max z &= 5x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 \\
s.t.\ 6x_1 + 4x_2 + s_1 &= 24 \quad (\textit{Raw material } M_1) \\
x_1 + 2x_2 + s_2 &= 6 \quad (\textit{Raw material } M_2) \\
-x_1 + x_2 + s_3 &= 1 \quad (\textit{Market limit}) \\
x_2 + s_4 &= 2 \quad (\textit{Demand limit}) \\
x_1, x_2, s_1, s_2, s_3, s_4 &\geq 0
\end{aligned}$$

- s_1, s_2, s_3 and s_4 are called slack variables.
- Rewrite the objective function in **residual form**: $z - 5x_1 - 4x_2 = 0$

Computational Details of Simplex Method

Example

$$
\begin{aligned}
\max z &= 5x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 \\
\text{s.t. } 6x_1 + 4x_2 + s_1 &= 24 \quad (\textit{Raw material } M_1) \\
x_1 + 2x_2 + s_2 &= 6 \quad (\textit{Raw material } M_2) \\
-x_1 + x_2 + s_3 &= 1 \quad (\textit{Market limit}) \\
x_2 + s_4 &= 2 \quad (\textit{Demand limit}) \\
x_1, x_2, s_1, s_2, s_3, s_4 &\geq 0
\end{aligned}
$$

- **Residual form:** $z - 5x_1 - 4x_2 = 0$
- Starting simplex tableau:

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution	
z	1	-5	-4	0	0	0	0	0	z-row
s_1	0	6	4	1	0	0	0	24	s_1-row
s_2	0	1	2	0	1	0	0	6	s_2-row
s_3	0	-1	1	0	0	1	0	1	s_3-row
s_4	0	0	1	0	0	0	1	2	s_4-row

Computational Details of Simplex Method

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution	
z	1	-5	-4	0	0	0	0	0	z-row
s_1	0	6	4	1	0	0	0	24	s_1-row
s_2	0	1	2	0	1	0	0	6	s_2-row
s_3	0	-1	1	0	0	1	0	1	s_3-row
s_4	0	0	1	0	0	0	1	2	s_4-row

This tableau includes the basic and non-basic variables, as well as the solution in the corresponding iteration step. As mentioned before, the simplex iteration starts from the origin, that is, $(x_1, x_2) = (0, 0)$. Accordingly, in the initial step, the basic variables are $(s_1, s_2, s_3, s_4) = (24, 6, 1, 2)$ and the objective function value is $z = 0$.

Computational Details of Simplex Method

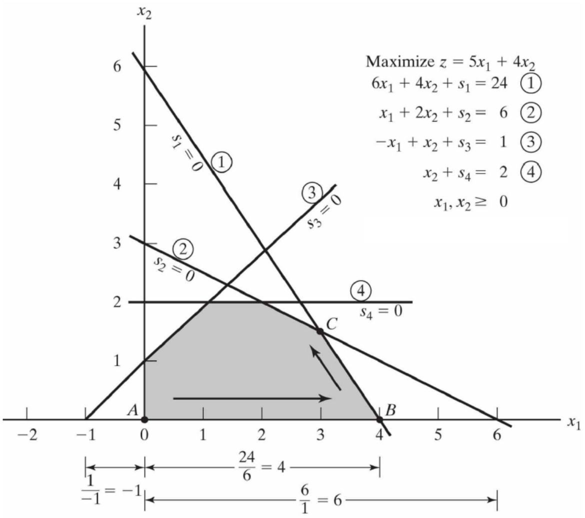
The fact that the objective function is directly proportional to x_1 and x_2 shows that the objective function can be improved when these variables are increased. In this context, considering $z = 5x_1 + 4x_2$, **the biggest increase** is seen to be provided by x_1, so x_1 is chosen as **entering variable**. This rule is also called **optimality condition**.

To determine the leaving variable, non-negative ratios are calculated with the coefficients of the input variable x_1 for the right side (solution column) of the equations.

Basic	x_1	Solution	Ratio
s_1	6	24	$x_1 = 24/6 = 4 \leftarrow$min
s_2	1	6	$x_1 = 6/1 = 6$
s_3	-1	1	$x_1 = 1/-1 = -1$ (ignore)
s_4	0	2	$x_1 = 2/0$ (ignore)

Then s_1, which gives **the smallest ratio**, is selected as the leaving variable.

Computational Details of Simplex Method



Computational Details of Simplex Method

Since the calculated ratios indicate the point where the constraints intersect the axis of the entering variable (x_1 in this step), they determine the exiting variable and the new value of the entering variable. As can be seen from the figure, in order to reach the corner point B, the x_1 axis must be increased by the smallest of the intersection points, that is, by 4 units. x_1 should not be increased further as increases beyond point B will give an inappropriate solution. At point B, the basic variable s_1 associated with the first constraint becomes the variable that takes the value of zero. This step associated with the ratio calculation is called **feasibility condition** because it ensures the suitability of the new solution.

This change in point B, that is, the entry of x_1 into the simplex table and the exit of s_1, changes the basic and non-basic solution variables as follows:

- Variables that do not form a fundamental solution at B: (s_1, x_2)
- Basic variables at point B: (x_1, s_2, s_3, s_4)

In the simplex table, the change step is done by **Gauss-Jordan row operations**. The input variable column is called **pivot column**, and the output variable row is called **pivot row**. The intersection of the pivot column and the pivot row is called **pivot element**.

Computational Details of Simplex Method

The initial simplex table with the pivot row and column highlighted is as follows:

			Entering $\downarrow$							
	Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution	
	z	1	-5	-4	0	0	0	0	0	
Leaving $\leftarrow$	s_1	0	6	4	1	0	0	0	24	Pivot Row
	s_2	0	1	2	0	1	0	0	6	
	s_3	0	-1	1	0	0	1	0	1	
	s_4	0	0	1	0	0	0	1	2	
			Pivot Column	1	0	0	0	1	2	

Computational Details of Simplex Method

Gauss-Jordan row operations, where the new basic solution is produced, are done in two ways:

- **Pivot row:**
  - The variable that comes out in the base column is replaced with the variable that enters.
  - (New pivot row)=(Previous pivot row)/(Pivot element)
- **All other lines including z:**
  - (New row)=(Previous row)-(Corresponding pivot column coefficient)$\times$(New pivot row)

Computational Details of Simplex Method

In our example, these calculations are reflected in the table as follows:

- In the basic column x_1 and s_1 are swapped:

$$\begin{aligned}
(\text{New } x_1 \text{ row}) &= (\text{Current } s_1 \text{ row})\,/6 \\
&= \frac{1}{6}\begin{pmatrix} 0 & 6 & 4 & 1 & 0 & 0 & 0 & 24 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & \frac{2}{3} & \frac{1}{6} & 0 & 0 & 0 & 4 \end{pmatrix}
\end{aligned}$$

- New z row:

$$\begin{aligned}
(\text{New } z \text{ row}) &= (\text{Current } z \text{ row}) - (-5) \times (\text{New } x_1 \text{ row}) \\
&= \begin{pmatrix} 1 & -5 & -4 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&\quad - (-5) \times \begin{pmatrix} 0 & 1 & \frac{2}{3} & \frac{1}{6} & 0 & 0 & 0 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & -\frac{2}{3} & \frac{5}{6} & 0 & 0 & 0 & 20 \end{pmatrix}
\end{aligned}$$

- New s_2 row:

$$\begin{aligned}
(\text{New } s_2 \text{ row}) &= (\text{Current } s_2 \text{ row}) - (1) \times (\text{New } x_1 \text{ row}) \\
&= \begin{pmatrix} 0 & 1 & 2 & 0 & 1 & 0 & 0 & 6 \end{pmatrix} \\
&\quad - (1) \times \begin{pmatrix} 0 & 1 & \frac{2}{3} & \frac{1}{6} & 0 & 0 & 0 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & \frac{4}{3} & -\frac{1}{6} & 1 & 0 & 0 & 2 \end{pmatrix}
\end{aligned}$$

Computational Details of Simplex Method

- New s_3 row:

$$
\begin{aligned}
(\text{New } s_3 \text{ row}) &= (\text{Current } s_3 \text{ row}) - (-1) \times (\text{New } x_1 \text{ row}) \\
&= \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \\
&\quad - (-1) \times \begin{pmatrix} 0 & 1 & \frac{2}{3} & \frac{1}{6} & 0 & 0 & 0 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & \frac{5}{3} & \frac{1}{6} & 0 & 1 & 0 & 5 \end{pmatrix}
\end{aligned}
$$

- New s_4 row:

$$
\begin{aligned}
(\text{New } s_4 \text{ row}) &= (\text{Current } s_4 \text{ row}) - (0) \times (\text{New } x_1 \text{ row}) \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \\
&\quad - (0) \times \begin{pmatrix} 0 & 1 & \frac{2}{3} & \frac{1}{6} & 0 & 0 & 0 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}
\end{aligned}
$$

Computational Details of Simplex Method

The new tableau for the new basic solution (x_1, s_2, s_3, s_4) will be as follows:

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	-2/3	5/6	0	0	0	20
x_1	0	1	2/3	1/6	0	0	0	4
s_2	0	0	4/3	-1/6	1	0	0	2
s_3	0	0	5/3	1/6	0	1	0	5
s_4	0	0	1	0	0	0	1	2

When the variables x_2 and s_1 that do not form a new basic solution, are selected as zero, the solution column will be the new basic solution ($x_1 = 4$, $s_2 = 2$, $s_3 = 5$, $s_4 = 2$). The objective function value corresponding to the new basic solution is $z = 20$.

Computational Details of Simplex Method

Let us determine the entering and leaving variables for the next iteration step. According to the optimality condition, x_2 is selected as the entering variable. The following table is evaluated for the feasibility condition.

Basic	x_2	Solution	Ratio
x_1	2/3	4	$x_2 = 4/(2/3) = 6$
s_2	4/3	2	$x_2 = 2/(4/3) = 1.5 \leftarrow$min
s_3	5/3	5	$x_2 = 5/(5/3) = 3$
s_4	1	2	$x_2 = 2/1 = 2$

s_2 is determined as the leaving variable. Accordingly, the new pivot row and pivot column are determined.

				$\downarrow$					
	Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
	z	1	0	-2/3	5/6	0	0	0	20
	x_1	0	1	2/3	1/6	0	0	0	4
$\leftarrow$	s_2	0	0	4/3	-1/6	1	0	0	2
	s_3	0	0	5/3	1/6	0	1	0	5
	s_4	0	0	1	0	0	0	1	2

Computational Details of Simplex Method

While x_2 is entering the basic solution instead of s_2, the following Gauss-Jordan row operations are applied:

- New x_2 pivot row:

$$(\text{New } x_2 \text{ row}) = (\text{Current } s_2 \text{ row}) / (4/3)$$

- New z row:

$$(\text{New } z \text{ row}) = (\text{Current } z \text{ row}) - (-2/3) \times (\text{New } x_2 \text{ row})$$

- New s_2 row:

$$(\text{New } x_1 \text{ row}) = (\text{Current } x_1 \text{ row}) - (2/3) \times (\text{New } x_2 \text{ row})$$

- New s_3 row:

$$(\text{New } s_3 \text{ row}) = (\text{Current } s_3 \text{ row}) - (5/3) \times (\text{New } x_2 \text{ row})$$

- New s_4 row:

$$(\text{New } s_4 \text{ row}) = (\text{Current } s_4 \text{ row}) - (1) \times (\text{New } x_2 \text{ row})$$

Computational Details of Simplex Method

	Basic	z	x_1	$\downarrow$ x_2	s_1	s_2	s_3	s_4	Solution
	z	1	0	-2/3	5/6	0	0	0	20
	x_1	0	1	2/3	1/6	0	0	0	4
$\leftarrow$	s_2	0	0	4/3	-1/6	1	0	0	2
	s_3	0	0	5/3	1/6	0	1	0	5
	s_4	0	0	1	0	0	0	1	2

The simplex tableau after Gauss-Jordan row operations is formed as follows:

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	0	3/4	1/2	0	0	21
x_1	0	1	0	1/4	-1/2	0	0	3
x_2	0	0	1	-1/8	3/4	0	0	3/2
s_3	0	0	0	3/8	-5/4	1	0	5/2
s_4	0	0	0	1/8	-3/4	0	1	1/2

According to the optimality condition, none of the coefficients in the z-row are negative anymore. Therefore, the tableau is optimal.

Computational Details of Simplex Method

The optimum solution from the simplex tableau can be summarized as follows:

Decision Variable	Optimum Value	Recommendation
x_1	3	Produce 3 tons of exterior paint daily.
x_2	3/2	Produce 1.5 tons of interior paint daily.
z	21	Daily profit is 21000 TL.

The resulting solution also provides information about the status of the resources. If all activities (variables) of the model use that resource completely, that resource is called **scarce resource**. Otherwise, it is called **abundant resource**. This information can be obtained by checking the relevant constraints specifying the sources of the slack variables in the optimum table. If the slack value is zero, it means that the resource is fully used and is a scarce resource. Otherwise, if the slack variable is positive, it means that this resource is an abundant resource.

Resource	Slack Value	Status
Row material M_1	$s_1 = 0$	Scarce
Row material M_2	$s_2 = 0$	Scarce
Market limit	$s_3 = \frac{5}{2}$	Abundant
Demand limit	$s_4 = \frac{1}{2}$	Abundant

Computational Details of Simplex Method

Remarks: The simplex tableau also provides a wealth of additional information, such as:

- **Sensitivity analysis:** It is the analysis of the conditions under which the current solution remains unchanged.
- **Post-optimal analysis:** It is the analysis related to finding a new optimal solution by changing the data.

Simplex Method (Summary)

So far, the maximization problem has been discussed. In the optimality condition of the minimization problem, on the contrary to the maximization problem, the variable with the largest positive coefficient that does not form a basic solution is selected as the entering variable. This is due to the fact that $\max z$ and $\min -z$ problems are equivalent. In the feasibility condition, the rule for selecting the leaving variable does not change. According to this,

- **Optimality condition:** In the maximization (minimization) problem, the one with the largest negative (positive) coefficient in absolute value among the variables in the z-line that do not form a basic solution is selected as the entering variable. In case of a tie, a random selection can be made. Optimum is achieved in the iteration step where the z-row coefficients of the variables that do not form a basic solution are not all negative (positive).
- **Feasibility condition:** In both maximization and minimization problems, the leaving variable is determined by choosing the smallest of the non-negative ratios. In case of a tie, random selection can be made again.

Simplex Method (Summary)

- **Gauss-Jordan row operations:**
  - **Pivot row:**
    - In the basic column, the leaving variable and the entering variable are replaced.
    - (New pivot row)=(Current pivot row)/(Pivot element)
  - **All other rows, including z:**
    - (New row)=(Current row)-(Pivot column coefficient)$\times$(New pivot row)

The steps of the simplex method are

- **Step 1:** The starting basic feasible solution is determined.
- **Step 2:** The entering variable is determined according to the optimality condition. If there is no new entering variable, the final solution is optimal. Otherwise, proceed to **Step 3**.
- **Step 3:** The leaving variable is determined according to the feasibility condition.
- **Step 4:** The new basic solution is calculated according to Gauss-Jordan row operations and proceed to **Step 2**.

Simplex Method (Examples)
Example

$$
\begin{aligned}
\max z &= 6x_1 + 5x_2 \\
\textit{subject to } x_1 + x_2 &\leq 5 \\
3x_1 + 2x_2 &\leq 12 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Simplex Method (Examples)
Example

$$\begin{aligned} \min z &= 2x_1 + 7x_2 - 2x_3 \\ \textit{subject to } x_1 + x_2 + x_3 &\leq 1 \\ -4x_1 - 2x_2 + 3x_3 &\leq 2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$