

# MTM5101-Dynamical Systems and Chaos

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Week 5

# Second Order Time-Invariant Systems: Local Analysis

Given the system

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

where  $f_1, f_2 \in C^1$  and an isolated equilibrium point  $x^* = [x_1^*, x_2^*]^T$ .

- We first linearize the system about the equilibrium point:

$$\Delta \dot{x} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1^*, x_2^*) & \frac{\partial f_1}{\partial x_2}(x_1^*, x_2^*) \\ \frac{\partial f_2}{\partial x_1}(x_1^*, x_2^*) & \frac{\partial f_2}{\partial x_2}(x_1^*, x_2^*) \end{bmatrix}}_{=A} \Delta x \text{ where } \Delta x = \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}$$

- Secondly, we find the eig'val's of the resulting system matrix  $A$ : " $\lambda(A)$ ",
- Finally, we classify the equilibrium point:
  - $\lambda_1, \lambda_2 \in \mathbb{R}$ 
    - $\lambda_2 < \lambda_1 < 0 \implies$  the equilibrium point is a stable node
    - $0 < \lambda_1 < \lambda_2 \implies$  the equilibrium point is an unstable node
    - $\lambda_2 < 0 < \lambda_1 \implies$  the equilibrium point is a saddle point
  - $\lambda_1, \lambda_2 \in \mathbb{C}$ , i.e.  $\lambda_{1,2} = \alpha \pm j\beta$ ,  $\alpha \in \mathbb{R}$ ,  $\beta > 0$ 
    - $\alpha < 0 \implies$  the equilibrium point is a stable focus.
    - $\alpha > 0 \implies$  the equilibrium point is an unstable focus.
    - $\alpha = 0 \implies$  the equilibrium point is a center.

## Example

The Jacobian matrix of the function

$$f(x) = \begin{bmatrix} \frac{1}{C}[-h(x_1) + x_2] \\ \frac{1}{L}[-x_1 - Rx_2 + u] \end{bmatrix}$$

with circuit parameters  $u = 1.2 \text{ V}$ ,  $R = 1.5 \text{ k}\Omega = 1.5 \times 10^3 \text{ }\Omega$ ,  $C = 2 \text{ pF} = 2 \times 10^{-12} \text{ F}$  and  $L = 5 \text{ }\mu\text{H} = 5 \times 10^{-6} \text{ H}$  of the tunnel-diode circuit of Example 2.1 in [Khalil, 2002] is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -0,5h'(x_1) & 0,5 \\ -0,2 & -0,3 \end{bmatrix}$$

where

$$h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5,$$

$$h'(x_1) = \frac{dh}{dx_1} = 17.76 - 207.58x_1 + 688.86x_1^2 - 905.24x_1^3 + 418.6x_1^4.$$

## Example

$$f(x) = \begin{bmatrix} \frac{1}{C}[-h(x_1) + x_2] \\ \frac{1}{L}[-x_1 - Rx_2 + u] \end{bmatrix} \implies \frac{\partial f}{\partial x} = \begin{bmatrix} -0,5h'(x_1) & 0,5 \\ -0,2 & -0,3 \end{bmatrix}$$

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$$h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5,$$

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Evaluating the Jacobian matrix at the equilibrium points  $Q_1 = (0,063, 0,758)$ ,  $Q_2 = (0,285, 0,61)$ , and  $Q_3 = (0,884, 0,21)$ , respectively, yields to

$$A_1 = \begin{bmatrix} -3.598 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \lambda_{1,2}(A_1) = -3.57, -0.33 \implies Q_1 \text{ is a stable node,}$$

$$A_2 = \begin{bmatrix} 1.82 & 0.5 \\ -0,2 & -0.3 \end{bmatrix}, \lambda_{1,2}(A_2) = 1.77, -0.25 \implies Q_2 \text{ is a saddle point,}$$

$$A_3 = \begin{bmatrix} -1.427 & 0.5 \\ -0,2 & -0.3 \end{bmatrix}, \lambda_{1,2}(A_3) = -1.33, -0.4 \implies Q_3 \text{ is a stable node.}$$

## Example

The Jacobian matrix of the function

$$f(x) = \begin{bmatrix} x_2 \\ -10 \sin x_1 - x_2 \end{bmatrix}$$

of the pendulum equation of Example 2.2 in [Khalil, 2002] is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -10 \cos x_1 & -1 \end{bmatrix}$$

Evaluating the Jacobian matrix at the equilibrium points  $(0,0)$  and  $(\pi,0)$  yields, respectively, the two matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \quad \lambda_{1,2}(A_1) = -0.5 \pm j3.12$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 10 & -1 \end{bmatrix}, \quad \lambda_{1,2}(A_2) = -3.7, 2.7$$

Thus, the equilibrium point  $(0,0)$  is a stable focus and the equilibrium point  $(\pi,0)$  is a saddle point.

## Example

The system

$$\dot{x}_1 = -x_2 - \mu x_1(x_1^2 + x_2^2)$$

$$\dot{x}_2 = x_1 - \mu x_2(x_1^2 + x_2^2)$$

has an equilibrium point at the origin.

The linearized state equation at the origin has eigenvalues  $\pm j$ . Thus, the origin is a center equilibrium point for the linearized system. We can determine the qualitative behavior of the nonlinear system by representing it in the polar coordinates:

$$x_1 = r \cos \theta \quad \text{and} \quad x_2 = r \sin \theta$$

which yield

$$\dot{r} = -\mu r^3 \quad \text{and} \quad \dot{\theta} = 1$$

From these equations, it can be easily seen that the trajectories of the nonlinear system will resemble a stable focus when  $\mu > 0$  and an unstable focus when  $\mu < 0$ .

## Second Order Time-Invariant Systems: Existence of Periodic Orbits

Periodic orbits and limit cycles are **invariant sets** that you may find in phase portraits of nonlinear systems. Linear systems do not have limit cycles!

### Definition (Periodic Orbit)

If  $\exists T > 0$  such that  $x(t + T) = x(t)$ ,  $\forall t \geq 0$ , then the trajectory  $x(t)$  is called a **periodic orbit**.

### Definition (Limit Cycle)

A nontrivial (non-constant solutions) isolated (neighbor trajectories are not periodic orbits) periodic orbit (closed trajectory) is called a **limit cycle**.



Figure: Isolated vs **Not**-Isolated Periodic Orbits.

There are three existence criteria for periodic orbits:

- 1 Poincaré - Bendixson criterion
- 2 Bendixson (negative) criterion
- 3 Index method



# Second Order Time-Invariant Systems: Existence of Periodic Orbits

Before, we go further, we present some topological definitions.

## Definition (Closed set)

A set is a **closed set**, if the whole boundary is the part of the set.

## Definition (Closed ball)

$$B_r[x_0] = \{x \in X : d(x, x_0) \leq r\}$$

## Definition (Open ball)

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\}$$

## Definition (Open set)

A set is an **open set**, if the boundary is excluded from the set

## Definition (Bounded set)

A set  $M$  is a **bounded set**, if  $\exists c > 0 : d(x, y) < c, \forall x, y \in M$ .

## Second Order Time-Invariant Systems: Existence of Periodic Orbits

Now, we are ready to present the first existence criterion for periodic orbits.

### Lemma (Poincaré - Bendixson Criterion)

*Consider the system*

$$\dot{x} = f(x) \quad \text{where} \quad f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ and } f \in \mathcal{C}^1,$$

*Let  $M$  be a **closed bounded** (i.e. **compact**) subset of the plane such that*

- *$M$  contains **no** equilibrium points of the system, or it contains only **one** equilibrium point with the property that the eigenvalues of the Jacobian matrix at this point have positive real parts (unstable focus or unstable node).*
- *Every trajectory starting in  $M$  stays in  $M$  for all future time. In other words,  $M$  is a positively invariant set.*

*Then,  $M$  contains a **periodic orbit** of the system.*

In order to find an  $M$  satisfying these conditions, we will often use closed curves defined by the equation  $V(x) = c$  where  $c > 0$  and  $V$  is some continuously differentiable scalar function ( $V : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $V \in \mathcal{C}^1$ ). The choice of the function  $V$  decides which kind of curve that  $V(x) = c$  describes (eg. circle, ellipse, etc).

## Example

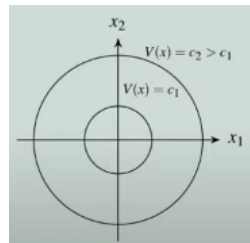
Consider the harmonic oscillator given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{=:A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies \lambda_{1,2}(A) = \pm j$$

- What is the equilibrium point of the system?  $\Rightarrow Ax = 0 \iff x = (0, 0)$
  - Which kind of equilibrium pt. is this? Center, because  $\text{Re}(\lambda_{1,2}(A)) = 0$
- 1) The first thing we do is to **choose** a  $\mathcal{C}^1$  function  $V(x) = c$  to create closed curves which we seek to trap the system trajectories within (PBC Condition 2).

Let us start with  $V(x) = x_1^2 + x_2^2$

$$M = \{x \in \mathbb{R}^2 \mid V(x) \leq c\}$$



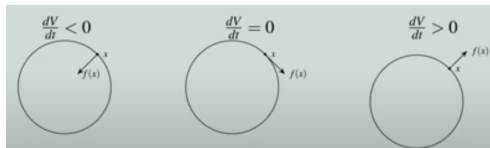
## Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{=:A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies \lambda_{1,2}(A) = \pm j$$

2) Calculate the directional derivative (Lie derivative)

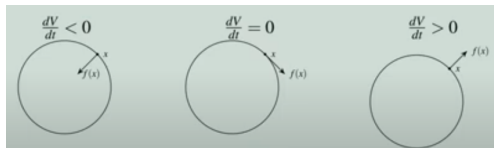
$$\begin{aligned} L_f V &= \frac{dV}{dt} = \left\langle \frac{\partial V}{\partial x}, \frac{\partial x}{\partial t} \right\rangle = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = 2x_1 x_2 - 2x_1 x_2 = 0 \end{aligned}$$

Specifically, we analyze how the function  $V$  varies with time when the system is on the curve that we have chosen to define border of  $M$  ( $\equiv \partial M$ ):



## Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{=:A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies \lambda_{1,2}(A) = \pm j$$



2) In the border of  $M$ , i.e. on  $\partial M$  we have

$$\nabla V \cdot f \Big|_{V(x)=c} = \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} \Big|_{V(x)=c} = 0 \Big|_{V(x)=c} = 0,$$

which implies that the trajectories do not leave the set  $\{x \in \mathbb{R}^2 \mid V(x) \leq c\}$  (PBC Condition 2). But, since this set contains an equilibrium point which is not an unstable focus or node, PBC Condition 1 is not satisfied.

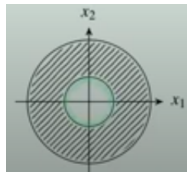
## Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{=:A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies \lambda_{1,2}(A) = \pm j$$

2) Let us try an another set for  $M = \{x \in \mathbb{R}^2 \mid c_1 \leq V(x) \leq c_2\}$ . Since

$$\nabla V \cdot f \Big|_{V(x)=c_1} \geq 0,$$

$$\nabla V \cdot f \Big|_{V(x)=c_2} \leq 0,$$



the trajectories are trapped in  $M$  (PBC Condition 1) and there is no equilibrium point, the existence of a periodic orbit in  $M$  is guaranteed by invoking Poincaré - Bendixson Lemma.

**Warning:** Poincaré - Bendixson Lemma ensures existence, not uniqueness!

## Lemma (Bendixson (Negative) Criterion)

*Consider the system*

$$\dot{x} = f(x) \quad \text{where} \quad f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ and } f \in \mathcal{C}^1,$$

*If, on a simply connected region  $D$  on the plane, the expression  $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$  is **not identically zero** and **does not change sign**, then the system has **no** periodic orbits lying entirely in  $D$ .*

Let us recall the definition of a **simply connected region**.

## Definition (Simply Connected Region)

A region  $D$  is called a **simply connected region**, if for every simply closed curve  $C$  in  $D$ , the inner region of  $C$  is a subset of  $D$  (a curve which does not intersect itself).

### Theorem (The Index Theorem)

*Let  $C$  be a closed curve of the plane and let the indices of the equilibrium points in  $C$  are as the following:*

- *The index of a node, a focus or a center is  $+1$ .*
- *The index of a saddle is  $-1$ .*

*If the sum of the indices of the equilibrium points in  $C$  is equal to 1, then there exists a periodic orbit  $\gamma \in C$ .*

The index method is generally useful in ruling out the existence of periodic orbits in certain regions of the plane. As a consequence, if there is no equilibrium point in a closed curve of the plane, then the sum of the indices of the equilibrium points in that closed curve will be different than 1 (note that  $p \Rightarrow q \iff q' \Rightarrow p'$ ).



## Theorem (The Index Theorem)

*Let  $C$  be a closed curve of the plane and let the indices of the equilibrium points in  $C$  be as the following:*

- *The index of a node, a focus or a center is  $+1$ .*
- *The index of a saddle is  $-1$ .*

*If the sum of the indices of the equilibrium points in  $C$  is equal to 1, then there exists a periodic orbit  $\gamma \in C$ .*

For instance, if  $\sum_i I_i = 0$  where  $I_i$  denotes the index of the equilibrium point  $i$ , it is guaranteed that there does not exist a periodic orbit in the corresponding closed curve (recall again that  $p \Rightarrow q \iff q' \Rightarrow p'$ ). So, in order to have a periodic orbit, there should exist at least one equilibrium point.

Conversely, in case that  $\sum_i I_i = -1 + 1 + 1 = 1$ , it is possible to have a periodic orbit in the corresponding closed curve but the index theorem does not guarantee the existence of the periodic orbit.

Starting from next week, we will start introducing

- Lyapunov stability properties for TI (autonomous) systems
  - Stability
  - Asymptotic stability
  - Exponential stability
  - Local versus global
- Lyapunov stability analysis
  - Lyapunov's indirect method
  - Lyapunov's direct method

# Lyapunov Stability: The Control Problem

- Physical system (a process)
  - Inputs
  - Outputs
- Step of specifications
  - Keep the temperature at a constant value
  - Keep the traffic density at a critical value
  - Keep the concentration of a substance at a certain value
  - Keep the speed of a car at a constant value (cruise control)
- Control law should make the closed-loop system (CLS) according to the following specifications
  - Regulation problem
  - Tracking problem

# Lyapunov Stability: Regulation Problem

- $x_{\text{ref}} = \text{const}$
- Ship, car parking, etc.
- Find  $u(t) = \gamma(t, x(t))$  such that the closed-loop system (CLS)  
 $\dot{x}(t) = f_p(t, x(t), \gamma(t, x(t))) = \tilde{f}(t, x(t))$  has a desired behavior.
- Desired CLS behavior:
  - $x_{\text{ref}}$  an equilibrium point, i.e.  $\tilde{f}(t, x_{\text{ref}}) = 0$
  - Convergence:  $\lim_{t \rightarrow \infty} x(t) = x_{\text{ref}}$
  - Start close  $\implies$  stay close

$\left. \begin{array}{l} \text{Asymptotic} \\ \text{stability} \end{array} \right\} \equiv$

## Asymptotic Stabilization Problem

*Find  $u(t) = \gamma(t, x(t))$  such that  $x_{\text{ref}}$  is an asymptotically stable equilibrium point of  $\dot{x}(t) = f_p(t, x(t), \gamma(t, x(t))) = \tilde{f}(t, x(t))$ .*

- Coordinate transformation:

$$e(t) = x(t) - x_{\text{ref}} \quad (x = x_{\text{ref}} \iff e = 0),$$

$$\dot{e}(t) = \dot{x}(t) - \dot{x}_{\text{ref}} = \dot{x}(t) = \tilde{f}(t, e(t) + x_{\text{ref}}) = f(t, e(t)) \rightarrow x_{\text{ref}} \text{ goes as a parameter in the function}$$

## Asymptotic Stabilization Problem

*Find  $u(t) = \gamma(t, e(t))$ ,  $e(t) = x(t) - x_{\text{ref}}$  such that  $e = 0$  is an asymptotically stable equilibrium point of  $\dot{e}(t) = f(t, e(t))$ .*

# Lyapunov Stability: Tracking Problem

- $x_{\text{ref}}(t)$
- Spray painting a car, ship
- Desired CLS behavior:
  - on trajectory  $\implies$  stay on trajectory
  - convergence to trajectory
  - start close  $\implies$  stay close
- Coordinate transformation:

$$\begin{aligned}e(t) &= x(t) - x_{\text{ref}}(t), \\ \dot{e}(t) &= \dot{x}(t) - \dot{x}_{\text{ref}}(t) = \underbrace{f_p(t, e(t) + x_{\text{ref}}(t), u(t)) - \dot{x}_{\text{ref}}(t)}_{:= \bar{f}_p(t, e(t), u(t))}\end{aligned}$$

## Asymptotic Stabilization Problem

*Find  $u(t) = \gamma(t, e(t))$ ,  $e(t) = x(t) - x_{\text{ref}}(t)$  such that  $e = 0$  is an asymptotically stable equilibrium point of  $\dot{e}(t) = \bar{f}_p(t, e(t), \gamma(t, e(t))) = f(t, e(t))$ .*

## Asymptotic Stabilization Problem (Regulation Problem)

*Find  $u(t) = \gamma(t, e(t))$ ,  $e(t) = x(t) - x_{ref}$  such that  $e = 0$  is an asymptotically stable equilibrium point of  $\dot{e}(t) = f(t, e(t))$ .*

## Asymptotic Stabilization Problem (Tracking Problem)

*Find  $u(t) = \gamma(t, e(t))$ ,  $e(t) = x(t) - x_{ref}(t)$  such that  $e = 0$  is an asymptotically stable equilibrium point of  $\dot{e}(t) = f(t, e(t))$ .*

- Therefore, from now on, for simplicity, we will
  - either consider  $\dot{x} = f(t, x)$ ,  $x = 0$  equilibrium point
  - or consider  $\dot{x} = f(x)$ ,  $x = 0$  equilibrium point.