MTM3502-Partial Differential Equations

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Week 9

Second Order Linear PDEs: Classification

The general form of a second order linear PDE is given as

$$\begin{aligned} &A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} \\ &+ D(x,y)u_x + E(x,y)u_y + F(x,y)u = G(x,y) \end{aligned} \tag{5.1}$$

where the coefficients A, B, C, D, E, F and G are the functions of x and y. The classification of a second order linear PDE is suggested by the classification of the quadratic equation of conic sections in analytic geometry. The PDE is said to be *hyperbolic*, *parabolic* or *elliptic* at a point (x_0, y_0) as

$$B^2(x_0,y_0) - 4A(x_0,y_0)C(x_0,y_0) \tag{5.2}$$

is positive, zero or negative.

If (5.2) is positive, zero or negative at all points, then the PDE is said to be hyperbolic, parabolic or ellyptic in a domain. In case of two independent variables, it is always possible to reduce the given equation into canonical form in a given domain, which is not possible for several independent variables. Let us consider the following transformation

$$\xi = \xi(x, y), \quad \eta = \eta(x, y) \tag{5.3}$$

with sufficiently smooth functions ξ and η. Note that, if the Jacobian

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \tag{5.4}$$

is nonzero in the region, then the transformation is well-defined and x and y can be determined uniquely from (5.3).

Then, we have

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x, \quad u_y = u_\xi \xi_y + u_\eta \eta_y, \\ u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}, \\ u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y \\ &\quad + u_\xi \xi_{xy} + u_\eta \eta_{xy}, \\ u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}, \end{aligned} \tag{5.5}$$

Substituting these values in (5.1), we obtain

$$\begin{aligned} &A^*(\xi,\eta)u_{\xi\xi} + B^*(\xi,\eta)u_{\xi\eta} + C^*(\xi,\eta)u_{\eta\eta} \\ &+ D^*(\xi,\eta)u_\xi + E^*(\xi,\eta)u_\eta + F^*(\xi,\eta)u = G^*(\xi,\eta) \end{aligned} \tag{5.6}$$

where

$$
\begin{aligned}
A^*(\xi,\eta) =& A(\xi,\eta)\xi_x^2 + B(\xi,\eta)\xi_x\xi_y + C(\xi,\eta)\xi_y^2, \\
B^*(\xi,\eta) =& 2A(\xi,\eta)\xi_x\eta_x + B(\xi,\eta)(\xi_x\eta_y + \xi_y\eta_x) \\
& + 2C(\xi,\eta)\xi_y\eta_y, \\
C^*(\xi,\eta) =& A(\xi,\eta)\eta_x^2 + B(\xi,\eta)\eta_x\eta_y + C(\xi,\eta)\eta_y^2, \\
D^*(\xi,\eta) =& A(\xi,\eta)\xi_{xx} + B(\xi,\eta)\xi_{xy} + C(\xi,\eta)\xi_{yy} \\
& + D(\xi,\eta)\xi_x + E(\xi,\eta)\xi_y, \\
E^*(\xi,\eta) =& A(\xi,\eta)\eta_{xx} + B(\xi,\eta)\eta_{xy} + C(\xi,\eta)\eta_{yy} \\
& + D(\xi,\eta)\eta_x + E(\xi,\eta)\eta_y, \\
F^*(\xi,\eta) =& F(\xi,\eta), \quad G^*(\xi,\eta) = G(\xi,\eta)
\end{aligned}
\tag{5.7}
$$

Note that, (5.6) has the same form as the original (5.1). Therefore, we can say that the nature of the equation remains invariant under such a transformation if the Jacobian does not vanish. This lies on the fact that the *discriminant* does not change under transformation

$$
\begin{aligned}
B^{*2}(\xi,\eta) - 4A^{*}(\xi,\eta)C^{*}(\xi,\eta) &= J^{2}(\xi,\eta)(B^{2}(\xi,\eta) - 4A(\xi,\eta)C(\xi,\eta)) \text{ or} \\
B^{*2}(x,y) - 4A^{*}(x,y)C^{*}(x,y) &= J^{2}(x,y)(B^{2}(x,y) - 4A(x,y)C(x,y))
\end{aligned}
\tag{5.8}
$$

Before we go further, we emphasize that we sometimes write the functions by omitting the dependence of (x, y) or (ξ, η) which can be clearly understood from the context. As an example, we may write (5.8), in short as

$$B^{*2} - 4A^*C^* = J^2(B^2 - 4AC). \tag{5.9}$$

The classification of (5.1) depends on the functions A, B and C at a given point (x, y). We, therefore, may rewrite (5.1) as

$$Au_{xx} + Bu_{xy} + Cu_{yy} = H(x, y, u, u_x, u_y) \tag{5.10}$$

and rewrite (5.6) as

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} = H^*(\xi, \eta, u, u_\xi, u_\eta). \tag{5.11}$$

In mathematics, a canonical form of a mathematical object is a standard way of presenting that object as a mathematical expression. Often, it is one which provides the simplest representation of an object and which allows it to be identified in a unique way. In most fields, a canonical form specifies a unique representation for every object, without the requirement of uniqueness of the representation.

Considering a second order linear PDE, reducing a PDE into a canonical form means that the second order terms of the PDE is represented in terms of either $u_{\xi\eta}$ or $u_{\xi\xi}$ and $u_{\eta\eta}$ to represent the cases $B^{*2} - 4A^*C^* > 0$, $B^{*2} - 4A^*C^* = 0$ and $B^{*2} - 4A^*C^* < 0$. We, therefore, analyze the canonical forms in three subsections.

Second Order Linear PDEs: Canonical Forms of Hyperbolic PDEs

We, firstly, consider the case that $B^2 - 4AC > 0$. Note that, under coordinate change with a non-vanishing Jacobian, the determinant yields to $B^{*2} - 4A^*C^* > 0$ (See (5.9)). We have two cases to obtain canonical forms of hyperbolic PDEs:

- $A^* = C^* = 0$ and $B^* \neq 0$,
- $C^* = -A^* \neq 0$ and $B^* = 0$.

Considering the case that $A^* = C^* = 0$ and $B^* \neq 0$, we are able to write the PDE in terms of $u_{\xi\eta}$:

$$\begin{aligned} A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \\ &\implies A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0, \\ C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0 \\ &\implies A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0. \end{aligned} \tag{5.12}$$

Along the curves $\xi =$ constant and $\eta =$ constant, we have $\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}$ and $\frac{dy}{dx} = -\frac{\eta_x}{\eta_y}$ (note that $d\xi = \xi_x dx + \xi_y dy = 0$ and $d\eta = \eta_x dx + \eta_y dy = 0$). From this fact and the roots of (5.12), we have

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{5.13}$$

which are called as *characteristic equations*. These equations are ODEs for families of curves in the xy-plane along which $\xi =$ constant and $\eta =$ constant and the integrals of these equations are called the *characteristic curves*. The solutions, therefore, may be written as

$$\phi_1(x, y) = c_1 \text{ and } \phi_2(x, y) = c_2, \text{ for } c_1, \ c_2 \text{ are constants.} \tag{5.14}$$

Hence the transformations

$$\xi = \phi_1(x, y) \text{ and } \eta = \phi_2(x, y) \tag{5.15}$$

will transform (5.10) into a canonical form.

Since $B^2 - 4AC > 0$, the integration of (5.13) yield to two real and distinct families of characteristics. The equation (5.11) reduces to

$$u_{\xi\eta} = H_1 \tag{5.16}$$

where $H_1 = \frac{H^*}{B^*}$ (note that $B^* \neq 0$). This form is called the *first canonical form of the hyperbolic PDEs*.

Similarly, for $C^* = -A^* \neq 0$ and $B^* = 0$, we are able to write the PDE in terms of $u_{\xi\xi}$ and $u_{\eta\eta}$

$$u_{\xi\xi} - u_{\eta\eta} = H_2 \tag{5.17}$$

which is called the *second canonical form of the hyperbolic PDEs* where $H_2 = \frac{H^*}{A^*}$ (note that $A^* \neq 0$).

Example 18

Find canonical form of the PDE

$$y^2 u_{xx} - x^2 u_{yy} = 0. \tag{5.18}$$

Multiplying both sides with -4, we obtain

$$\begin{aligned} & -4u_{\xi\eta} = \frac{2\eta}{\eta^2-\xi^2}u_\xi - \frac{2\xi}{\eta^2-\xi^2}u_\eta \\ \Longrightarrow \quad & -4u_{\xi\eta} = \left(\ln(\eta^2-\xi^2)\right)_\eta u_\xi + \left(\ln(\eta^2-\xi^2)\right)_\xi u_\eta. \end{aligned} \tag{6.1}$$

Without loss of generality, let us choose

$$-2u_{\xi\eta} = \left(\ln(\eta^2-\xi^2)\right)_\eta u_\xi, \quad (u_1 = u_\xi), \tag{6.2a}$$

$$-2u_{\xi\eta} = \left(\ln(\eta^2-\xi^2)\right)_\xi u_\eta, \quad (u_2 = u_\eta), \tag{6.2b}$$

From (6.2a), we have

$$\begin{aligned}
\frac{du_1}{u_1} &= -\frac{\left(\ln(\eta^2-\xi^2)\right)_\eta}{2}d\eta \\
\implies \quad \ln u_1 &= -\frac{\ln(\eta^2-\xi^2)}{2} + \ln\varphi(\xi) \\
\implies \quad u_1 = u_\eta &= \frac{\varphi(\xi)}{\sqrt{\eta^2-\xi^2}} \\
\implies \quad u &= \varphi(\xi)\tanh^{-1}\left(\frac{\eta}{\sqrt{\eta^2-\xi^2}}\right) + \varphi_1(\xi)
\end{aligned} \tag{6.3}$$

Similarly, from (6.2b), we have

$$\begin{aligned}
u_2 = u_\xi &= \frac{\phi(\eta)}{\sqrt{\eta^2-\xi^2}} \\
\implies \quad u &= \phi(\eta)\tan^{-1}\left(\frac{\xi}{\sqrt{\eta^2-\xi^2}}\right) + \phi_1(\eta)
\end{aligned} \tag{6.4}$$

Solving (6.3) and (6.4) together, we obtain

$$u(\xi,\eta) = c_1 \tanh^{-1}\left(\frac{\eta}{\sqrt{\eta^2-\xi^2}}\right)\tan^{-1}\left(\frac{\xi}{\sqrt{\eta^2-\xi^2}}\right) + c_2$$
$$\Longrightarrow \quad u(x,y) = c_1 \tanh^{-1}\left(\frac{y^2+x^2}{2xy}\right)\tan^{-1}\left(\frac{y^2-x^2}{2xy}\right) + c_2 \tag{6.5}$$

which is a solution to (5.18) where c_1 and c_2 are arbitrary constants.

The general form of a second order linear PDE was given as

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G. \tag{5.1}$$

If A, B, C, D, E and F are real constants and G is a continuous function of x and y, then (5.1) is called a second order linear PDE with constant coefficients. This week, we will consider PDEs of these particular type and present an operator factorization technique for these type of PDEs.

In order to restate (5.1) in operator form, we define the following partial differential operators

$$\begin{gathered} D_x = \frac{\partial}{\partial x}, D_y = \frac{\partial}{\partial y}, \\ D_x^2 = \frac{\partial^2}{\partial x^2}, D_y^2 = \frac{\partial^2}{\partial y^2}, D_x D_y = \frac{\partial^2}{\partial x \partial y}. \end{gathered} \tag{7.1}$$

Then, (5.1) will be restated as

$$Lu = G(x, y) \tag{7.2}$$

where

$$L = AD_x^2 + BD_xD_y + CD_y^2 + DD_x + ED_y + F \tag{7.3}$$

is the partial differential operator for (5.1). The operator L is a linear operator which satisfy the following property, for real constants c_1 and c_2 and partially second order differentiable functions u_1 and u_2,

$$L(c_1u_1 + c_2u_2) = c_1Lu_1 + c_2Lu_2. \tag{7.3}$$

Suppose that u_1 and u_2 are the solutions to the homogeneous PDE

$$Lu = 0 \implies Lu_1 = 0 \text{ and } Lu_2 = 0. \tag{7.4}$$

Then, from linearity of L, the linear combination of u_1 and u_2, which is $c_1 u_1 + c_2 u_2$ for nonzero reals c_1 and c_2, is also a solution of (7.4). This property is called as the *superposition principle*. The general solution of (7.4) is called the *homogeneous solution* and denoted as u_h whereas the particular solution to (7.2) is called the *particular solution* and denoted as u_p. Thus, the general solution of (7.2) have the following form

$$u = u_h + u_p. \tag{7.5}$$

Suppose that the operator L can be factorized as

$$L = L_1 L_2 = (\alpha_1 D_x + \beta_1 D_y + \gamma_1)(\alpha_2 D_x + \beta_2 D_y + \gamma_2) \tag{7.6}$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$, for $i = 1, 2$. If $u = u(x,t)$ is a solution of $L_2 u = 0$, then $Lu = L_1 L_2 u = L_1 \cdot 0 = 0$ so that it is also a solution for $Lu = 0$ (the same can also shown for $L_1 u = 0$, without loss of generality). Now, let us consider

$$L_1 u = (\alpha_1 D_x + \beta_1 D_y + \gamma_1) u = \alpha_1 p + \beta_1 q + \gamma_1 u = 0. \tag{7.7}$$

From the Lagrange auxiliary equation, we have

$$\frac{dx}{\alpha_1} = \frac{dy}{\beta_1} = \frac{du}{-\gamma_1 u}. \tag{7.8}$$

Now, we consider three cases:

- Suppose $\alpha_1 \neq 0$ holds. From (7.8), we have

$$\begin{aligned} \frac{dx}{\alpha_1} = \frac{dy}{\beta_1} &\implies \beta_1 x - \alpha_1 y = c_1 \\ -\frac{\gamma_1}{\alpha_1} dx = \frac{du}{u} &\implies u = c_2 e^{-\frac{\gamma_1}{\alpha_1} x}. \end{aligned} \tag{7.9}$$

  Therefore, the general solution of (7.7) will be

$$u_{h1} = e^{-\frac{\gamma_1}{\alpha_1} x} \varphi(\beta_1 x - \alpha_1 y), \tag{7.10}$$

  where φ is an arbitrary function.

From the Lagrange auxiliary equation, we have

$$\frac{dx}{\alpha_1} = \frac{dy}{\beta_1} = \frac{du}{-\gamma_1 u}. \tag{7.8}$$

Now, we consider three cases:

- Suppose $\beta_1 \neq 0$ holds. From (7.8), the general solution of (7.7) will be

$$u_{h1} = e^{-\frac{\gamma_1}{\beta_1} y} \tilde{\varphi}(\beta_1 x - \alpha_1 y), \tag{7.11}$$

  where $\tilde{\varphi}$ is an arbitrary function.
- In the general case that both $\alpha_1 \neq 0$ and $\beta_1 \neq 0$ hold, either (7.10) or (7.11) may be chosen for general solution.

Similarly, the general solution of

$$\begin{aligned} L_2 u =& (\alpha_2 D_x + \beta_2 D_y + \gamma_2) u \\ =& \alpha_2 p + \beta_2 q + \gamma_2 u = 0 \end{aligned} \tag{7.12}$$

will be either

$$u_{h2} = e^{-\frac{\gamma_2}{\alpha_2} x} \phi(\beta_2 x - \alpha_2 y), \tag{7.13}$$

for $\alpha_2 \neq 0$ or

$$u_{h2} = e^{-\frac{\gamma_2}{\beta_2} y} \tilde{\phi}(\beta_2 x - \alpha_2 y), \tag{7.14}$$

for $\beta_2 \neq 0$. As a result, the general homogeneous solution of $Lu = 0$ will be

$$\begin{aligned} u_h =& u_{h1} + u_{h2} \\ =& e^{-\frac{\gamma_1}{\alpha_1} x} \varphi(\beta_1 x - \alpha_1 y) + e^{-\frac{\gamma_2}{\alpha_2} x} \phi(\beta_2 x - \alpha_2 y). \end{aligned} \tag{7.15}$$

In the case that $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$, the homogeneous solutions of $Lu = 0$ will be

$$u_h = e^{-\frac{\gamma}{\alpha}x}\left(x\varphi(\beta x - \alpha y) + \phi(\beta x - \alpha y)\right). \tag{7.16}$$

for $\alpha \neq 0$ and

$$u_h = e^{-\frac{\gamma}{\beta}y}\left(y\varphi(\beta x - \alpha y) + \phi(\beta x - \alpha y)\right). \tag{7.17}$$

for $\beta \neq 0$.

The particular solution requires that we make an “initial assumption" about the form of the particular solution $u_p(x, y)$, but with the coefficients left unspecified. For the particular solution, the type of $G(x, y)$ in (7.2) determines the types of the particular solution candidate. Here are some of the following particular solution candidates for various types.

Table: Particular Solution Candidates for Various Types of $G(x, y)$.

$G(x, y)$	Particular Solution Candidate $u_p(x, y)$
Trigonometric	Trigonometric
Polynomial	Polynomial
Exponential	Exponential

Example 21

Find the homogeneous and particular solution of

$$u_{xx} - u_{yy} - u_x + u_y = 2\cos(3x + 2y). \tag{7.18}$$

Example 22

Find the general solution of

$$4u_{xx} - 4u_{xy} + u_{yy} + 4u_x - 2u_y = 0. \tag{7.29}$$

Example 23

Find the general solution of

$$(2D_x^2 - 5D_xD_y + 2D_y^2)u = 5e^{x+3y}. \tag{7.32}$$

where D_x^2, D_xD_y and D_y^2 are defined as in (7.1)