	YTU – Faculty of Chemical and Metallurgical Enginering, Questions and Answers Sheet	NOTE CHART			
					Total
Student Name and Surname					
Students Number					
Course Name	Optimization Techniques (Midterm)			Exam Date	19/04/2024
Course Instructors	Hale Gonce Kocken Gokhan Goksu	Signature			

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1. 5p.-b. Which of the following statement(s) is/are correct?

i. $\varnothing$ and $\mathbb{R}^n$ are both compact sets.

ii. $S_1 = \{(x_1, x_2) \in \mathbb{R}^2 \ | \ a \le x_1 \le b\}$ is a compact set.

iii. $S_2 = \{x \in \mathbb{R}^n \ | \ |x^\top \cdot x - 1| < 1\}$ is not a convex set.

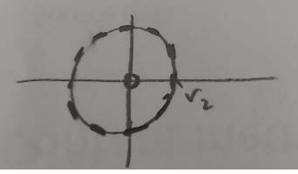
$|x^T \cdot x - 1| < 1$

$-1 < \|x\|^2 - 1 < 1$ Since $(0,0)$ is not included

$0 < \|x\|^2 < 2$ in the set, it is not convex.

Indeed,

in R^2, $0 < x^2 + y^2 < 2$



a. Only i b. Only iii c. i and iii d. ii and iii e. i, ii and iii

2. 5p-b. Which of the following statement(s) is/are correct?

i. $S_3 = \{(x_1, x_2) \in \mathbb{R}^2 \ | \ x_1^2 + x_2^2 = 1\}$ is an open set.

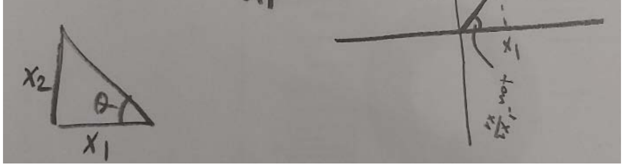
ii. $S_4 = \left\{(x_1, x_2) \in \mathbb{R}^2 \ | \ 0 < \tan^{-1}\frac{x_2}{x_1} < \frac{\pi}{2}\right\}$ is an unbounded set.

iii. $S_5 = \{x \in \mathbb{R}^n \ | \ \|x\| > 2\|x - 1\|\}$ is a closed set.

a. Only i b. Only ii c. Only iii d. i and ii e. ii and iii

$\tan^{-1}\frac{x_2}{x_1} = \theta$, $\theta \in (0, \frac{\pi}{2})$

$\tan\theta = \frac{x_2}{x_1}$



3.10p-c The critical point of the function $f(x_1, x_2, x_3, x_4) = x_1^2 + 2x_1x_2 - x_2^2 - x_3^2 - 4x_2x_4 - 3x_4^2$ is:

a. a global minimizer b. a global maximizer c. a saddle point d. a local minimizer e. a local maximizer

4. 5p-e. Let $\Omega \subset \mathbb{R}^n$ be a convex set. Which of the following statement(s) is/are correct?

i. A matrix $A \in \mathbb{R}^{n\times n}$ is negative definite if and only if its n leading principal minors Δ_i, $i = 1,2,\dots,n$ alternate in sign beginning by negative: $\Delta_1 < 0$, $\Delta_2 > 0$, $\Delta_3 < 0$, $\Delta_4 > 0$, ...

ii. Let $f : \Omega \to \mathbb{R}$ be twice partially differentiable function. Then, f is convex on Ω if and only if for each $x \in \Omega$, the Hessian $F(x)$ of f evaluated at x is a positive semidefinite matrix.

iii. A function f defined on a convex set is concave if and only if its hypograph, the set of points in $\Omega \times \mathbb{R}$ given by $\text{hyp}(f) = \{[x,\beta] \mid x \in \Omega, \beta \in \mathbb{R}, \beta \le f(x)\}$, is convex.

a. Only ii b. Only iii c. i and ii d. ii and iii e. i, ii and iii

5. 5p-d. Determine the values of a, for which the function $f(x,y,z) = x^2 + 2xz + 2ayz + z^2$ is convex?

a. $a \in \mathbb{R}$ b. $-\frac{1}{2} \le a \le \frac{1}{2}$ c. $-1 \le a \le 1$ d. $a = 0$ e. $a \ne 0$

$$\nabla f = \begin{bmatrix} 2x+2z \\ 2az \\ 2x+2ay+2z \end{bmatrix} \qquad \nabla^2 f = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 2a \\ 2 & 2a & 2 \end{bmatrix}$$

convex $\Longleftrightarrow$ pos. semi def, pos. def.

LPMs

$\Delta_1 = 2$

$\Delta_2 = 0 \Rightarrow$ ~~pos def~~

$\Delta_3 = -2a \begin{vmatrix} 2 & 2 \\ 0 & 2a \end{vmatrix} = -2a(4a) = -8a^2$

PMs

1st: $2, 0, 2 \ge 0$

2nd: $\begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix}, \begin{vmatrix} 0 & 2a \\ 2a & 2 \end{vmatrix} \ge 0$

$0, \quad 0, \quad -4a^2$

3rd: $-8a^2 \ge 0$

$-4a^2 \ge 0$

$a^2 \le 0$

$\boxed{a = 0}$

$-8a^2 \ge 0$ ✓

6. 10p. Consider the function $f = x^3 - 3x^2 + 2xy - y^2$ and its critical point(s). Which of the following statement(s) is/are correct?

i. f has two critical points.

ii. f has a saddle point at $(0,0)$ and a local max at $\left(\frac{4}{3}, \frac{4}{3}\right)$.

iii. f has a local max at $(0,0)$ and a local min at $\left(\frac{4}{3}, \frac{4}{3}\right)$.

iv. f has one critical point and it is a local max.

a. Only i b. Only ii c. i and ii d. i and iii e. ii and iv

CEVAP a

$f(x,y) = x^3 - 3x^2 + 2xy - y^2$

$\nabla f = \begin{bmatrix} 3x^2 - 6x + 2y \\ 2x - 2y \end{bmatrix} \qquad \nabla^2 f = \begin{bmatrix} 6x - 6 & 2 \\ 2 & -2 \end{bmatrix}$

$\left.\begin{array}{l} 3x^2 - 6x + 2y = 0 \\ x = y \end{array}\right\rangle \quad \begin{array}{l} 3x^2 - 6x + 2x = 0 \\ 3x^2 - 4x = 0 \\ x(3x-4) = 0 \\ x = 0 \qquad x = 4/3 \end{array}$

$A(0,0) \qquad B\left(\frac{4}{3}, \frac{4}{3}\right)$

$\nabla^2 f(A) = \begin{bmatrix} -6 & 2 \\ 2 & -2 \end{bmatrix} \quad \begin{array}{l} \Delta_1 = -6 < 0 \\ \Delta_2 = 12 - 4 > 0 \end{array} \quad$ neg. def. max.

$A(0,0)$

$\nabla^2 f(B) = \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix} \quad \left.\begin{array}{l} \Delta_1 = 0 > 0 \\ \Delta_2 = -4 - 4 < 0 \end{array}\right) \quad$ Saddle. (indef.)

$\left(\frac{4}{3}, \frac{4}{3}\right)$

7. 10p. Consider the function $f = 3x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_1 + x_2$ and the starting point $[2,0]^T$. Then, the next point obtained by Newton's Algorithm is:

a. $\left[\frac{11}{10}, \frac{3}{2}\right]^T$ b. $\left[\frac{13}{10}, \frac{17}{10}\right]^T$ c. $\left[\frac{3}{2}, \frac{19}{10}\right]^T$ d. $\left[\frac{3}{10}, -\frac{1}{10}\right]^T$ e. $\left[\frac{17}{10}, \frac{21}{10}\right]^T$

Cevap d

$f = 3x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_1 + x_2 \qquad x^{(0)} = [2,0]^T$

$\nabla f = \begin{bmatrix} 6x_1 - 2x_2 - 2 \\ -2x_1 + 4x_2 + 1 \end{bmatrix} \qquad \nabla^2 f = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \quad \begin{bmatrix} 6 & -2 & 1 & 0 \\ -2 & 4 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/3 & 1/6 & 0 \\ 0 & 10/3 & 1/3 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1/3 & 1/6 & 0 \\ 0 & 1 & 1/10 & 3/10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{6} + \frac{1}{30} & 1/10 \\ 0 & 1 & 1/10 & 3/10 \end{bmatrix} \Rightarrow (\nabla^2 f)^{-1} = \begin{bmatrix} 1/5 & 1/10 \\ 1/10 & 3/10 \end{bmatrix}$

$x^{(1)} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/5 & 1/10 \\ 1/10 & 3/10 \end{bmatrix} \begin{bmatrix} 10 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 - \frac{3}{10} \\ 1 - \frac{9}{10} \end{bmatrix} = \begin{pmatrix} 3/10 \\ -1/10 \end{pmatrix}$

$\nabla f(x^{(1)}) = \begin{bmatrix} \frac{18}{10} + \frac{2}{10} - 2 \\ -\frac{6}{10} - \frac{4}{10} + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x^{(1)} = x^*$

8. Given the function $f(x,y) = x^2 + xy + y^2 - 2x$.

i. 5p. Which of the following is $\underset{\alpha}{\operatorname{argmin}}\, f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$?

a. -1/2 b. 1/2 c. 1/3 d. -1/3 e. None

ii. 5p. Given initial point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, find the next point using the method of steepest descent.

a. $\begin{bmatrix} 1 \\ -1/3 \end{bmatrix}$ b. $\begin{bmatrix} 3/2 \\ -1 \end{bmatrix}$ c. $\begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$ d. $\begin{bmatrix} 1 \\ 1/3 \end{bmatrix}$ e. None

$$f = x^2 + xy + y^2 - 2x$$

$$x^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1+\alpha \\ -2\alpha \end{pmatrix}$$

$$f(x^1) = (1+\alpha)^2 + (1+\alpha)(-2\alpha) + 4\alpha^2 - 2(1+\alpha) = \phi(\alpha)$$

$$= 1 + \cancel{2\alpha} + \alpha^2 - \cancel{2\alpha} - 2\alpha^2 + 4\alpha^2 - 2 - 2\alpha$$

$$= 3\alpha^2 - 2\alpha - 1 = \phi(\alpha)$$

$$\phi'(\alpha) = 6\alpha - 2 = 0$$

$$\alpha = 1/3$$

$$\nabla f = \begin{pmatrix} 2x+y-2 \\ x+2y \end{pmatrix} \qquad \nabla^2 f = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{bmatrix} 2-2 \\ 1+0 \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x^{next} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \alpha \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\alpha \end{pmatrix}$$

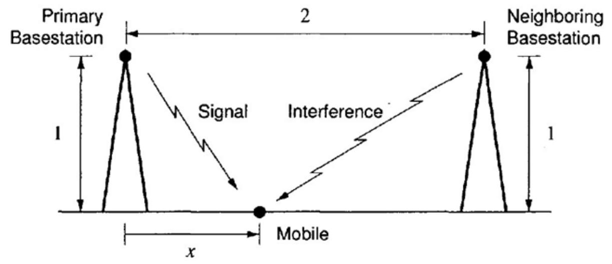
$$f(x^{next}) = 1 - \alpha + \alpha^2 - 2 = \phi(\alpha)$$

$$\phi'(\alpha) = 2\alpha - 1 = 0$$

$$\alpha = +1/2.$$

$$x^{next} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$$

9. (15p) The figure shows a simplified model of a cellular wireless system. A "mobile" user is located at position x. See the figure below.



There are two basestation antennas, one for the primary basestation and another for the neighboring basestation. Both antennas are transmitting signals to the mobile user, at equal power. However, the power of the received signal as measured by the mobile is the reciprocal of the squared distance from the associated antenna (primary or neighboring basestation). We are interested in finding the position of the mobile that maximizes the signal-to-interference ratio, which is the ratio of the received signal power from the primary basestation to the received signal power from the neighboring basestation.

The squared distance from the mobile to the primary antenna is $1+x^2$, while the squared distance from the mobile to the neighboring antenna is $1+(2-x)^2$. Therefore, the signal-to-interference ratio is

$$f(x)=\frac{1+x^2}{1+(2-x)^2}$$

Considering the objective function above, determine the optimal position of the mobile user that maximizes the signal-to-interference ratio.

$$f'(x)=\frac{4(x^2-2x-1)}{1-(2-x)^2}=\frac{-4x^2+8x+4}{(x^2-4x+5)^2}$$

5

$$x^2-2x-1=0$$

$$\Delta=4-4.(-1)=8$$

$$x_{1,2}=\frac{2\mp2\sqrt{2}}{2}=1\mp\sqrt{2}$$

5

$$1-(2-x)^2=0$$

$$2-x=1 \qquad 2-x=-1$$

$$\boxed{x=1} \qquad \boxed{x=3}$$

f' undefined.

		$1-\sqrt{2}$		$1+\sqrt{2}$	
f'	−		+		—
	↘	min	↗	max	↘

or

$$f''(x)=\frac{4(2x-2).(1-(2-x)^2)-4(x^2-2x-1).(+2(2-x))}{[1-(2-x)^2]^2}$$

$$f''(x)=\frac{8(x-3)(x^2-3)}{(x^2-4x+5)^3}$$

$$f''(1+\sqrt{2})<0$$

5

$\Rightarrow x=1+\sqrt{2}$

is the max.

$$f(1+\sqrt{2}) = \frac{1+(1+\sqrt{2})^2}{1+(2-1-\sqrt{2})^2} = \frac{1+(1+\sqrt{2})^2}{1+(1-\sqrt{2})^2} = \frac{4+2\sqrt{2}}{4-2\sqrt{2}}$$

$$= \frac{16+16\sqrt{2}+8}{16-8} = 3+2\sqrt{2}$$

$$f(1-\sqrt{2}) = \frac{1+(1-\sqrt{2})^2}{1+(2-1+\sqrt{2})^2} = \frac{1+(1-\sqrt{2})^2}{1+(1+\sqrt{2})^2} = \frac{4-2\sqrt{2}}{4+2\sqrt{2}}.$$

$$= 3-2\sqrt{2}$$

$$f(1) = \frac{1+1}{1+1} = 1$$

$$f(3) = \frac{1+9}{1+1} = 5$$

10. Consider a quadratic function with

$$Q = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

a. 5p. For the matrix Q, construct a set of Q-conjugate vectors with a given vector $d^{(0)} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\top$.

b. 10p. Find $x^{(3)}$ by basic conjugate direction algorithm with an initial point $x^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$ and the Q-conjugate vectors that you find in part (a) for

c. 10p. Find $x^{(2)}$ by conjugate gradient algorithm with an initial point $x^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$.

$$d^{(0)T} Q d^{(1)} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = d_2 = 0.$$

$$d^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$d^{(0)T} Q d^{(2)} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$= d_2 = 0.$$

$$d^{(1)T} Q d^{(2)} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$= d_1 + d_3 = 0$$

$$d^{(1)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

b)

Given a starting point $x^{(0)}$, and Q-conjugate directions $d^{(0)}, d^{(1)}, \ldots, d^{(n-1)}$; for $k \geq 0$,

$$\begin{aligned} g^{(k)} &= \nabla f(x^{(k)}) = Qx^{(k)} - b, \\ \alpha_k &= -\frac{g^{(k)T} d^{(k)}}{d^{(k)T} Q d^{(k)}}, \\ x^{(k+1)} &= x^{(k)} + \alpha_k d^{(k)}. \end{aligned}$$

x0 = 0 0 0

g0 = -1 1 0

d0 = 0 1 0

d1 = 1 0 0

d2 = 1 0 -1

alpha0 = -1

x1 = 0 -1 0

g1 = -1 0 0

alpha1 = 1

x2 = 1 -1 0

g2 = 0 0 1

alpha2 = 1

x3 = 2 -1 -1

g3 = 0 0 0

c)

1. Set $k := 0$; select the initial point $x^{(0)}$.
2. $g^{(0)} = \nabla f(x^{(0)})$. If $g^{(0)} = 0$, stop, else set $d^{(0)} = -g^{(0)}$.
3. $\alpha_k = -\frac{g^{(k)T} d^{(k)}}{d^{(k)T} Q d^{(k)}}$.
4. $x^{(k+1)} = x^{(k)} + \alpha_k d^{(k)}$.
5. $g^{(k+1)} = \nabla f(x^{(k+1)})$. If $g^{(k+1)} = 0$, stop.
6. $\beta_k = \frac{g^{(k+1)T} Q d^{(k)}}{d^{(k)T} Q d^{(k)}}$.
7. $d^{(k+1)} = -g^{(k+1)} + \beta_k d^{(k)}$.
8. Set $k := k + 1$; go to step 3.

g0 = -1 1 0

d0 = 1 -1 0

alpha0 = 1

x1 = 1 -1 0

g1 = 0 0 1

beta0 = 0.5000

d1 = 0.5000 -0.5000 -1.0000

alpha1 = 0.6667

x2 = 1.3333 -1.3333 -0.6667 = 4/3 -1/3 -2/3