MTM4501-Operations Research

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Week 2

Course Content

- Definition of OR and Its History
- Decision Theory and Models
  - Decision Making Under Certainty
  - Decision Making Under Risk
  - Decision Making Under Uncertainty
- Network Analysis
- Inventory Management Models
- Queue Models

Decision Theory: Basic Concepts

Components of the Decision Making Problem:

- Decision Maker (Person-Group)
- Objective/Decision Criteria
- Actions/Strategies/Alternatives (a_i, $i = 1, 2, ..., m$)
- States (s_j, $j = 1, 2, ..., n$)
- Results $v(a_i, s_j)$: The value resulting from each option and event. In other words, when a_i action is selected, if s_j state is encountered, $v(a_i, s_j)$ result is obtained. In mathematical terms $v_{ij} = v(a_i, s_j)$; that is, $v(\cdot, \cdot)$ is a function of the variables a_i and s_j.

Payoff Matrix (Decision Matrix):

$$
\begin{array}{c|cccccc}
 & s_1 & s_2 & \dots & s_j & \dots & s_n \\
\hline
a_1 & v(a_1, s_1) & v(a_1, s_2) & \dots & v(a_1, s_j) & \dots & v(a_1, s_n) \\
a_2 & v(a_2, s_1) & v(a_2, s_2) & \dots & v(a_2, s_j) & \dots & v(a_2, s_n) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_i & v(a_i, s_1) & v(a_i, s_2) & \dots & v(a_i, s_j) & \dots & v(a_i, s_n) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_m & v(a_m, s_1) & v(a_m, s_2) & \dots & v(a_m, s_j) & \dots & v(a_m, s_n)
\end{array}
$$

Decision Theory: Construction of a Decision Matrix

Example

A wholesaler is considering to order apples at the beginning of the week. Apples are bought in 100 *kg boxes and sold to greengrocers. The purchasing price of apples is* 6 *TL/kg, and the selling price to greengrocers is* 10 *TL/kg. At the end of the week, the wholesaler transfers the remaining apples to marketers at a price of* 4 *TL/kg. From this wholesaler's experience from the previous weeks; he/she knows that the demand for a box of apples is at least* 0 *and at most* 4 *boxes. How many boxes of apples should this wholesaler order?*

- $0 \leq a_i \leq 4$: action a_i represents ordering i boxes of apples,
- $0 \leq s_j \leq 4$: state s_i represents demanding i boxes of apples,
- Purchasing price of a box:
- Sale price of a box to greengrocers:
- Sale price of a box to stallholders:
- Profit from a sale to greengrocers:
- Profit from a sale to stallholders:

Decision Theory: Construction of a Decision Matrix

According to this example, the decision matrix is as follows:

	s_0	s_1	s_2	s_3	s_4
a_0					
a_1					
a_2					
a_3					
a_4					

Decision Making Under Certainty

- Every parameter of the decision problem is certain.
- The Linear Programming (LP) models (include in MTM3691) are conventional examples of decision making under certainty.
- Also, Analytic Hierarchy Process (AHP) which is designed for prioritizing the alternatives is an example of decision making under certainty.

Payoff Matrix Under Certainty:

	s
a_1	$v(a_1, s)$
a_2	$v(a_2, s)$
$\vdots$	$\vdots$
a_i	$v(a_i, s)$
$\vdots$	$\vdots$
a_m	$v(a_m, s)$

- If $v(a_i, s)$ is a gain (profit), the objective will be

$$\max_{a_i} v(a_i, s),$$

- If $v(a_i, s)$ is a loss (cost), the objective will be

$$\min_{a_i} v(a_i, s)$$

Example

A wholesaler is considering to order apples at the beginning of the week. Apples are bought in 100 *kg boxes and sold to greengrocers. The purchasing price of apples is* 6 *TL/kg, and the selling price to greengrocers is* 10 *TL/kg. At the end of the week, the wholesaler transfers the remaining apples to marketers at a price of* 4 *TL/kg. From this wholesaler's experience from the previous weeks; he/she knows that the demand for a box of apples is **exactly*** 2 ***boxes**. How many boxes of apples should this wholesaler order?*

The associated payoff matrix will be

	s_2
a_0	
a_1	
a_2	
a_3	
a_4	

According to this table, the optimal solution will be .

Decision Making Under Risk

If the probability of the states can be determined according to a market research or from previous experiences, then the payoffs associated with each decision states can be described by probability distributions and the decisions made are called decision making under risk. First, the expected value of each action is determined:

$$\mathrm{E}(a_i) = \sum_{s_j} \mathrm{P}(s_j) v(a_i, s_j)$$

The option that gives the best expected value is optimal. In profit type decisions, the objective will be

$$\max_{s_i} \mathrm{E}(s_i),$$

and, in cost type decisions, the objective will be

$$\min_{s_i} \mathrm{E}(s_i),$$

Returning back to the wholesaler example, suppose that the greengrocer records 50 weeks of demands as follows:

Demands	Week	Probability
0 box	5	0.1
1 box	10	0.2
2 box	20	0.4
3 box	10	0.2
4 box	5	0.1

Decision Making Under Risk

	s_0	s_1	s_2	s_3	s_4
	0.1	**0.2**	**0.4**	**0.2**	**0.1**
a_0					
a_1					
a_2					
a_3					
a_4					

According to these demands, the expected values of the states can be calculated as follows:

$\mathrm{E}(a_0) =$

$\mathrm{E}(a_1) =$

$\mathrm{E}(a_2) =$

$\mathrm{E}(a_3) =$

$\mathrm{E}(a_4) =$

Accordingly, is optimal.

Decision Making Under Uncertainty: The Pessimistic Criterion

Based on this example, let us evaluate this decision making problem according to various decision-making criteria.

- **The Pessimistic Criterion (MaxiMin/MiniMax):** The decision maker is pessimistic. No matter which option he/she chooses, he/she expects the worst-case scenario to happen. Then the action that will give the best of the worst outcomes is optimal. If $v(a_i, s_j)$ is the cost, the **minimax** criterion should be evaluated:

$$\min_{a_i} \left\{ \max_{s_j} v(a_i, s_j) \right\}$$

If $v(a_i, s_j)$ is a gain, the **maximin** criterion should be evaluated:

$$\max_{a_i} \left\{ \min_{s_j} v(a_i, s_j) \right\}$$

	s_0	s_1	s_2	s_3	s_4	P
a_0						
a_1						
a_2						
a_3						
a_4						

According to this evaluation, the action is optimal.

Decision Making Under Uncertainty: The Optimistic Criterion

- **Optimistic Criterion (MaxiMax/MiniMin):** The decision maker is optimistic. Whichever option he/she chooses, he/she expects the event that will give the best result to happen. The option that will give the best of the best results is optimal. If $v(a_i, s_j)$ is a cost, the **minimin** criterion should be evaluated:

$$\min_{a_i} \left\{ \min_{s_j} v(a_i, s_j) \right\}$$

If $v(a_i, s_j)$ is the gain, the **maximax** criterion should be evaluated:

$$\max_{a_i} \left\{ \max_{s_j} v(a_i, s_j) \right\}$$

	s_0	s_1	s_2	s_3	s_4	O
a_0						
a_1						
a_2						
a_3						
a_4						

According to this evaluation, is optimal.

Decision Making Under Uncertainty: Equiprobability Criterion

- **Equiprobability (Laplace) Criterion:** The decision maker expects the states to occur with equal probabilities. The action that will give the best expected value is optimal. If $v(a_i, s_j)$ is a cost, the objective should be

$$\min_{a_i} \left\{ \sum_{j=1}^{n} \mathrm{P}(s_j) v(a_i, s_j) \right\} = \min_{a_i} \left\{ \frac{1}{n} \sum_{j=1}^{n} v(a_i, s_j) \right\}$$

If $v(a_i, s_j)$ is a gain, the objective should be

$$\max_{a_i} \left\{ \sum_{j=1}^{n} \mathrm{P}(s_j) v(a_i, s_j) \right\} = \max_{a_i} \left\{ \frac{1}{n} \sum_{j=1}^{n} v(a_i, s_j) \right\}$$

	s_0	s_1	s_2	s_3	s_4	E
a_0						
a_1						
a_2						
a_3						
a_4						

According to this evaluation, is optimal.

Decision Making Under Uncertainty: Hurwicz Criterion

- **Hurwicz Criterion:** The decision maker defines $\alpha \in (0,1)$ optimism coefficient and $(1-\alpha)$ pessimism coefficient. If $v(a_i, s_j)$ is a cost, the objective should be

$$\min_{a_i} \left\{ \alpha \min_{s_j} v(a_i, s_j) + (1-\alpha) \max_{s_j} v(a_i, s_j) \right\}$$

If $v(a_i, s_j)$ is a gain, the objective should be

$$\max_{a_i} \left\{ \alpha \max_{s_j} v(a_i, s_j) + (1-\alpha) \min_{s_j} v(a_i, s_j) \right\}$$

	s_0	s_1	s_2	s_3	s_4	$H(\alpha = 0.7)$
a_0						
a_1						
a_2						
a_3						
a_4						

According to the degree of optimism $\alpha = 0.7$, the action is optimal.

Decision Making Under Uncertainty: Savage Regret Criterion

- **Savage Regret Criterion:** Rather than cost/profit values in the decision matrix, the new regret values for the regret matrix are defined:

$$p(a_i, s_j) = \begin{cases} v(a_i, s_j) - \min_{a_k} v(a_k, s_j), & \text{if } v(a_i, s_j) \text{ is a cost,} \\ \max_{a_k} v(a_k, s_j) - v(a_i, s_j), & \text{if } v(a_i, s_j) \text{ is a profit,} \end{cases}$$

	s_0	s_1	s_2	s_3	s_4
a_0					
a_1					
a_2					
a_3					
a_4					

According to this decision matrix, the regret matrix can be created as follows:

	s_0	s_1	s_2	s_3	s_4	S
a_0						
a_1						
a_2						
a_3						
a_4						

Since the regret matrix is always a cost type matrix, the pessimism criterion should be applied. Accordingly, is optimal.