MATLAB Tutorial Session #2
MTM4502-Optimization Techniques
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Task 1 (Lagrange Multipliers)

As you know, *the Method of Lagrange Multipliers* is used to find the extreme values of a function whose domain is constrained to lie within some particular subset of the plane.

The Method of Lagrange Multipliers

Suppose that $f(x, y)$ and $g(x, y)$ are differentiable functions. To find the local extrema (minimum and/or maximum values) of f subject to the constraint $g(x, y) = 0$, find the values of x, y and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \text{ and } g(x, y) = 0$$

where $\nabla = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}^T$.

Example

Write a MATLAB program to find the minimum of the function

$$f(x, y) = 3x^2 - 4xy + 4y^2 - 9x - 2y + \frac{51}{4}$$

on the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$

Here, there is a simple visualization of this example.

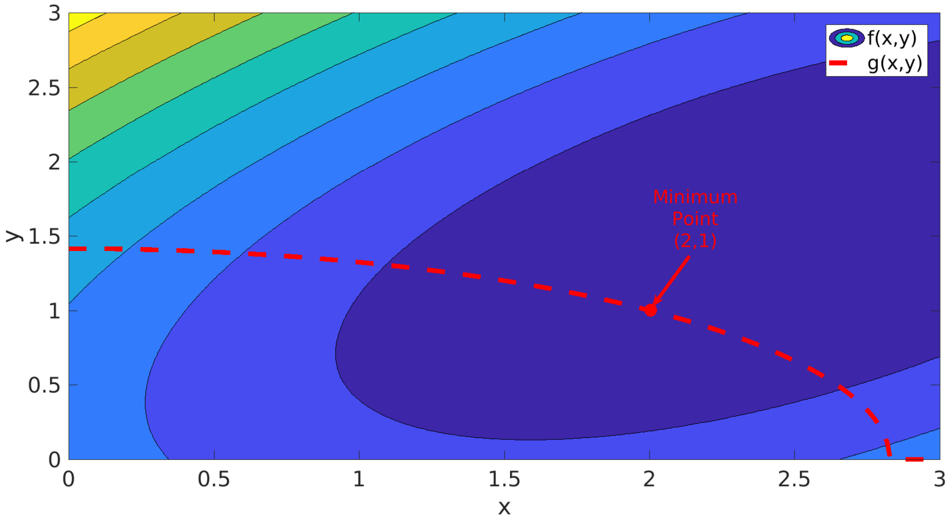

Figure 1: A Simple Visualization of the Minimization Problem.

Task 2 (Solving Linear System of Equations using CVX)

Consider the following system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{200\times 1000}$, $\mathbf{x} \in \mathbb{R}^{1000\times 1}$ and $\mathbf{b} \in \mathbb{R}^{200\times 1}$. Finding $\mathbf{x}$ for given $\mathbf{A}$ and $\mathbf{b}$ is named as an $ill-posed\ problem$, since you may have no solution or infinitely many solutions. In other words, you may have no unique solution.

However, it is not the end of the world. There are some ways to regularize ill-posed problems by converting them to convert them into convex optimization problems. In this task, you will learn some of them.

Generate matrix the $\mathbf{A}$ and the vector $\mathbf{b}$ with independent and identically distributed random Gaussian entries (with zero mean and unity variance).

To solve the optimization problems below you take the advantage of CVX developed by the research group of Prof. Stephen Boyd [1].

Subtask 1

Solve the equation (1) by solving the following optimization problem.

$$\begin{array}{ll} \textit{minimize} & \|\mathbf{x}\|_2 \\ \textit{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \end{array} \tag{2}$$

Subtask 2 (Tykhonov Regularization)

Solve the equation (1) by solving the following optimization problem

$$\textit{minimize} \quad \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 + \lambda\|\mathbf{x}\|_2 \tag{3}$$

Take λ values as 0.3, 0.7, 1 and execute your script for each of them to see the results.

Subtask 3 (Reduced Tykhonov Regularization)

Solve the equation (1) by solving the following optimization problem

$$\begin{array}{ll} \textit{minimize} & \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 + \lambda\|\mathbf{x}\|_2 \\ \textit{subject to} & x_i = 0,\ \textit{where}\ i \in \mathcal{I} \end{array} \tag{4}$$

Here $\mathcal{I}$ is the set of indices representing the 20 zero entries which are randomly chosen from the set $\{1, 2, ..., 1000\}$. Take λ values as 0.3, 0.7, 1 and execute your script for each of them to see the results.

Subtask 4

Execute the optimization problems in subtask 1, 2 and 3 by using the norms $\|\cdot\|_p$ with $p = 1$ and $p = \infty$. Make explanations for these subtasks.

References

[1] http://cvxr.com/cvx/

Historical Remark-Andrey Nikolayevich Tykhonov

Andrey Nikolayevich Tykhonov (October 30, 1906 – October 7, 1993) was a Soviet and Russian mathematician and geophysicist known for important contributions to topology, functional analysis, mathematical physics, and ill-posed problems. He was also one of the inventors of the magnetotellurics method in geophysics.

Born near Smolensk, he studied at the Moscow State University where he received a Ph.D. in 1927 under the direction of Pavel Sergeevich Alexandrov. In 1933 he was appointed as a professor at Moscow State University. He became a corresponding member of the USSR Academy of Sciences on 29 January 1939 and a full member of the USSR Academy of Sciences on 1 July 1966.

Tykhonov worked in a number of different fields in mathematics. He made important contributions to topology, functional analysis, mathematical physics, and certain classes of ill-posed problems. He is best known for his work on topology, including the metrization theorem he proved in 1926, and the Tykhonov's theorem, which states that every product of arbitrarily many compact topological spaces is again compact. In his honor, completely regular topological spaces are also named *Tykhonov spaces*.

In mathematical physics, he proved the fundamental uniqueness theorems for the heat equation and studied Volterra integral equations.

He founded the theory of asymptotic analysis for differential equations with small parameter in the leading derivative.

Tykhonov regularization, is the most commonly used method of regularization of ill-posed problems. In statistics, the method is known as ridge regression, in machine learning it is known as weight decay, and with multiple independent discoveries, it is also variously known as the Tykhonov–Miller method, the Phillips–Twomey method, the constrained linear inversion method, and the method of linear regularization. It is related to the Levenberg–Marquardt algorithm for non-linear least-squares problems.

The finite-dimensional case was expounded by Arthur E. Hoerl, who took a statistical approach and by Manus Foster, who interpreted this method as a Wiener–Kolmogorov (Kriging) filter.

References

[2] https://en.wikipedia.org/wiki/Andrey_Nikolayevich_Tikhonov

[3] https://en.wikipedia.org/wiki/Tikhonov_regularization