MTM3691-Theory of Linear Programming

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Week 5

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Special Cases in the Simplex Method

This section considers four special cases that arise in the use of the simplex method:

1. Degeneracy
2. Alternative optima
3. Unbounded solutions
4. Nonexisting (or infeasible) solutions

Our interest in studying these special cases is twofold:

1. to present a *theoretical* explanation of these situations, and
2. to provide a *practical* interpretation of what these special results could mean in a real-life problem.

Special Cases in the Simplex Method: Degeneracy

In the application of the feasibility condition of the simplex method, a tie for the minimum ratio may occur and can be broken arbitrarily. When this happens, at least one basic variable will be zero in the next iteration and the new solution is said to be *degenerate*.

There is nothing alarming about a degenerate solution, with the exception of a small theoretical inconvenience, called cycling or circling, which we shall discuss shortly. From the practical standpoint, the condition reveals that the model has at least one *redundant* constraint. To provide more insight into the practical and theoretical impacts of degeneracy, a numeric example is used.

Special Cases in the Simplex Method: Degeneracy

Example

maximize $z = 3x_1 + 9x_2$

subject to

$$x_1 + 4x_2 \leq 8,$$
$$x_1 + 2x_2 \leq 4,$$
$$x_1, x_2 \geq 0$$

(slack variables: x_3, x_4)

Iteration 0 (initial)

Basic	x_1	x_2	x_3	x_4	Soln
z	-3	-9	0	0	0
x_3	1	4	1	0	8
x_4	1	2	0	1	4

x_2 enters, x_3 leaves

Iteration 1

Basic	x_1	x_2	x_3	x_4	Soln
z	$-\frac{3}{4}$	0	$\frac{9}{4}$	0	18
x_2	$\frac{1}{4}$	1	$\frac{1}{4}$	0	2
x_4	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0

x_1 enters, x_4 leaves

Iteration 2 (optimal)

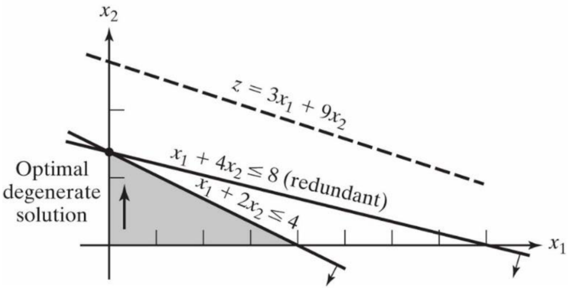
Basic	x_1	x_2	x_3	x_4	Soln
z	0	0	$\frac{3}{2}$	$\frac{3}{2}$	18
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	2
x_1	1	0	-1	2	0

Special Cases in the Simplex Method: Degeneracy

Example

$$\textbf{maximize} \quad z = 3x_1 + 9x_2$$

$$\textbf{subject to} \quad \begin{aligned} x_1 + 4x_2 &\leq 8, \\ x_1 + 2x_2 &\leq 4, \\ x_1, x_2 &\geq 0 \end{aligned}$$



Special Cases in the Simplex Method: Alternative Optima

When the objective function is parallel to a nonredundant **binding constraint** (i.e., a constraint that is satisfied as an equation at the optimal solution), the objective function can assume the same optimal value at more than one solution point, thus giving rise to alternative optima. The next example shows that there is an infinite number of such solutions. It also demonstrates the practical significance of encountering such solutions.

Example

$$
\begin{aligned}
\textbf{\textit{maximize}}\quad & z = 2x_1 + 4x_2 \\
& x_1 + 2x_2 \leq 5, \\
\textbf{\textit{subject to}}\quad & x_1 + x_2 \leq 4, \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

Special Cases in the Simplex Method: Alternative Optima

Example

maximize $z = 2x_1 + 4x_2$

subject to $x_1 + 2x_2 \leq 5,$

$x_1 + x_2 \leq 4,$

$x_1, x_2 \geq 0.$

(slack variables: x_3, x_4)

Iteration 0 (initial)

Basic	x_1	x_2	x_3	x_4	Soln
z	-2	-4	0	0	0
x_3	1	2	1	0	5
x_4	1	1	0	1	4

x_2 enters, x_3 leaves

Iteration 1 (optimum)

Basic	x_1	x_2	x_3	x_4	Soln
z	0	0	2	0	10
x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{5}{2}$
x_4	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	$\frac{3}{2}$

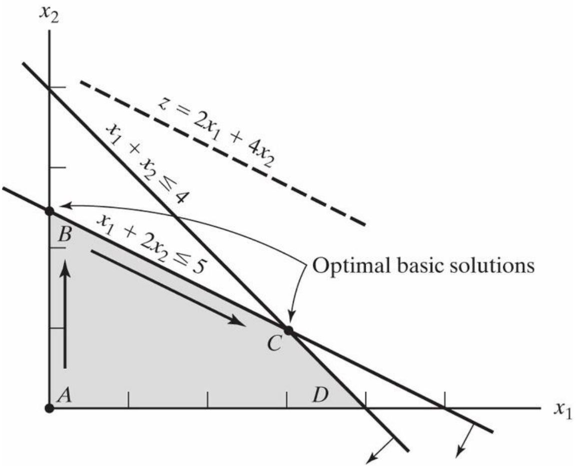
x_1 enters, x_4 leaves

Iteration 2 (alternative optimum)

Basic	x_1	x_2	x_3	x_4	Soln
z	0	0	2	0	10
x_2	0	1	1	-1	1
x_1	1	0	-1	2	3

Special Cases in the Simplex Method: Alternative Optima

The following figure demonstrates how alternative optima can arise in the LP model when the objective function is parallel to a binding constraint. Any point on the line segment BC represents an alternative optimum with the same objective value $z = 10$.



Special Cases in the Simplex Method: Alternative Optima

Iteration 1 gives the optimum solution $x_1 = 0$, $x_2 = \frac{5}{2}$ and $z = 10$, which coincides with point B in the figure. How do we know from this tableau that alternative optima exist? Look at the z-equation coefficients of the *nonbasic* variables in iteration 1. The coefficient of nonbasic x_1 is zero, indicating that x_1 can enter the basic solution without changing the value of z, but causing a change in the values of the variables. Iteration 2 does just that—letting x_1 enter the basic solution and forcing x_4 to leave. The new solution point occurs at C $(x_1 = 3,\ x_2 = 1,\ z = 10)$.

Special Cases in the Simplex Method: Alternative Optima

The simplex method determines only the two corner points B and C. Mathematically, we can determine all the points (x_1, x_2) on the line segment BC as a nonnegative weighted average of points B and C. Thus, given

$$B: \ x_1 = 0, \ x_2 = \tfrac{5}{2} \qquad C: \ x_1 = 3, \ x_2 = 1,$$

then all the points on the line segment BC are given by

$$\begin{aligned} \hat{x}_1 =& \alpha(0) + (1-\alpha)(3) = 3 - 3\alpha, \\ \hat{x}_2 =& \alpha\left(\tfrac{5}{2}\right) + (1-\alpha)(1) = 1 + \tfrac{3}{2}\alpha, \quad 0 \le \alpha \le 1. \end{aligned}$$

When $\alpha = 0$, $(\hat{x}_1, \hat{x}_2) = (3, 1)$, which is point C. When $\alpha = 1$, $(\hat{x}_1, \hat{x}_2) = (0, \frac{5}{2})$, which is point B. For values of α between 0 and 1, $(\hat{x}_1, \hat{x}_2)$ lies between B and C.

Special Cases in the Simplex Method: Alternative Optima

Remarks. In practice, alternative optima are useful because we can choose from many solutions without experiencing deterioration in the objective value. For instance, in the present example, the solution at B shows that activity 2 only is at a positive level, whereas at C both activities are positive. If the example represents a product-mix situation, there may be advantages in producing two products rather than one to meet market competition. In this case, the solution at C may be more appealing.

Special Cases in the Simplex Method: Unbounded Solution

In some LP models, the values of the variables may be increased indefinitely without violating any of the constraints—meaning that the solution space is unbounded in at least one variable. As a result, the objective value may increase (maximization case) or decrease (minimization case) indefinitely. In this case, both the solution space and the optimum objective value are unbounded.

Unboundedness points to the possibility that the model is poorly constructed. The most likely irregularity in such models is that one or more nonredundant constraints have not been accounted for, and the parameters (constants) of some constraints may not have been estimated correctly.

The following examples show how unboundedness, in both the solution space and the objective value, can be recognized in the simplex tableau.

Special Cases in the Simplex Method: Unbounded Solution

Example

maximize $z = 2x_1 + x_2$

subject to $x_1 - x_2 \leq 10,$

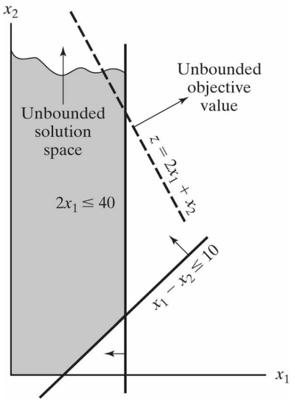
$2x_1 \leq 40,$

$x_1, x_2 \geq 0.$

(slack variables: x_3, x_4)

Starting Iteration

Basic	x_1	x_2	x_3	x_4	Soln
z	-2	-1	0	0	0
x_3	1	-1	1	0	10
x_4	2	0	0	1	40



Special Cases in the Simplex Method: Unbounded Solution

In the starting tableau, both x_1 and x_2 have negative z-row coefficients, so either can improve the solution. Because x_1 has the most negative coefficient, it is normally selected as the entering variable. However, all constraint coefficients under x_2 (i.e., the denominators in the minimum–ratio feasibility test) are negative or zero. This means there is no leaving variable and x_2 can be increased indefinitely without violating any constraint.

Since each unit increase in x_2 raises z by 1, an infinite increase in x_2 leads to an infinite increase in z. Thus, the problem has no bounded solution: the feasible region is unbounded in the direction of x_2, and the objective value z can be increased indefinitely.

Special Cases in the Simplex Method: Infeasible Solution

LP models with inconsistent constraints have no feasible solution. This situation can never occur if all the constraints are of the type $\leq$ with nonnegative right–hand sides because the slacks provide a feasible solution. For other types of constraints, we use artificial variables. Although the artificial variables are penalized in the objective function to force them to zero at the optimum, this can occur only if the model has a feasible space. Otherwise, at least one artificial variable will be positive in the optimum iteration. From the practical standpoint, an infeasible space points to the possibility that the model is not formulated correctly.

Example

$$\begin{aligned} \textbf{\textit{maximize}} \quad & z = 3x_1 + 2x_2 \\ & 2x_1 + x_2 \leq 2, \\ \textbf{\textit{subject to}} \quad & 3x_1 + 4x_2 \geq 12, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Special Cases in the Simplex Method: Infeasible Solution

Example

maximize $z = 3x_1 + 2x_2$

subject to

$$2x_1 + x_2 \leq 2,$$
$$3x_1 + 4x_2 \geq 12,$$
$$x_1, x_2 \geq 0.$$

Using the penalty $M = 100$ for the artificial variable R, the following tableaux provide the simplex iterations of the model.

Iteration 0 (initial)

Basic	x_1	x_2	x_4	x_3	R	Soln
z	-303	-402	100	0	0	-1200
x_3	2	1	0	1	0	2
R	3	4	-1	0	1	12

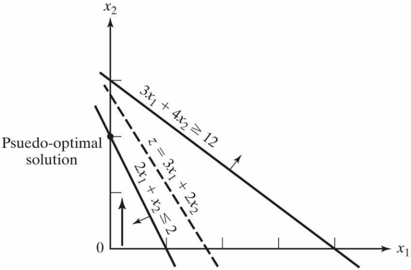
x_2 enters, x_3 leaves

Iteration 1 (pseudo–optimum)

Basic	x_1	x_2	x_4	x_3	R	Soln
z	501	0	100	402	0	-396
x_2	2	1	0	1	0	2
R	-5	0	-1	-4	1	4

Special Cases in the Simplex Method: Infeasible Solution

Optimum iteration 1 shows that the artificial variable R is *positive* $(= 4)$, which indicates that the problem is infeasible. The following figure demonstrates the infeasible solution space.



By allowing the artificial variable to be positive, the simplex method, in essence, has reversed the direction of the inequality from $3x_1 + 4x_2 \geq 12$ to $3x_1 + 4x_2 \leq 12$. The result is what we may call a pseudo-optimal solution.