# MTM5101-Dynamical Systems and Chaos

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Week 11

# Time-Delay Systems (TDS)...

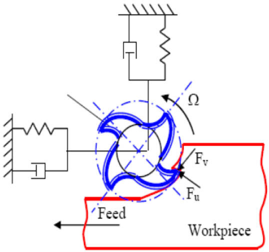



Figure: Rotating Milling Machine.

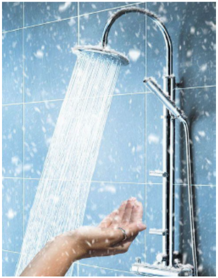


Figure: Shower.

$$\dot{x}(t) = F(x(t)) + B(\omega t)(x(t) - x(t - \delta(t)))$$

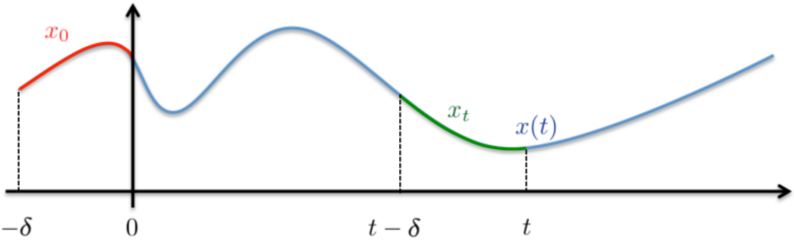
$$\dot{x}(t) = -\alpha x(t - \delta), \ \alpha > 0$$

# TDS: Notations

Consider the nonlinear TDS: $\dot{x}(t) = f(x_t, u(t))$

- State History: $x_t \in \mathcal{C}^n$ defined with the maximum delay $\delta \geq 0$ as

$$x_t(s) := x(t+s), \quad \forall s \in [-\delta, 0].$$




- $\mathcal{C}$: Set of all continuous functions $\varphi : [-\delta; 0] \to \mathbb{R}$.
- $\mathcal{U}$: Set of measurable essentially bounded signals to $\mathbb{R}^m$.
- Given $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm.
- Given any $\phi \in \mathcal{C}^n$, $\|\phi\| := \sup_{\tau \in [-\delta, 0]} |\phi(\tau)|$.
- $f : \mathcal{C}^n \times \mathbb{R}^m \to \mathbb{R}^n$, Lipschitz on bounded sets and to satisfy $f(0,0) = 0$.

# TDS: Notations

- Lyapunov-Krasovskii functional (LKF) candidate: Any functional $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$, Lipschitz on bounded sets, for which there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that
$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|), \quad \forall \phi \in \mathcal{C}^n.$$
- The LKF candidate is said to be a coercive LKF if it also satisfies
$$\underline{\alpha}(\|\phi\|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|), \quad \forall \phi \in \mathcal{C}^n.$$
- Its Driver's derivative along the solutions of $\dot{x}(t) = f(x_t, u(t))$ is then defined $\forall \phi \in \mathcal{C}^n$ and $\forall v \in \mathbb{R}^m$ as
$$D^+ V(\phi, v) := \limsup_{h \to 0^+} \frac{V(\phi^*_{h,v}) - V(\phi)}{h}.$$
where, $\forall h \in [0, \theta)$ and $\forall v \in \mathbb{R}^m$, $\phi^*_{h,v} \in \mathcal{C}^n$ is defined as
$$\phi^*_{h,v}(s) := \begin{cases} \phi(s+h), & \text{if } s \in [-\delta, -h), \\ \phi(0) + f(\phi, v)(s+h), & \text{if } s \in [-h, 0]. \end{cases}$$
- Its upper-right Dini derivative along the solutions of $\dot{x}(t) = f(x_t, u(t))$ is then defined for all $t \geq 0$ as
$$D^+ V(x_t, u(t)) := \limsup_{h \to 0^+} \frac{V(x_{t+h}) - V(x_t)}{h}.$$
- Under regularity conditions on the vector field, the Driver's derivative computed at $(x_t, u(t))$ and the upper-right Dini derivative coincides almost everywhere [Pepe, Automatica, 2007, Theorem 2].

# GAS/0-GAS Characterization for TDS

## Definition (0-GAS)

The TDS $\dot{x}(t) = f(x_t, u(t))$ is said to be globally asymptotically stable in the absence of inputs (0-GAS) (or the input-free system $\dot{x}(t) = f(x_t, 0)$ is GAS) if there exists $\beta \in \mathcal{KL}$ such that, the solution of the input-free system $\dot{x}(t) = f(x_t, 0)$ satisfies

$$|x(t)| \leq \beta(\|x_0\|, t), \qquad \forall t \geq 0.$$

## Proposition (0-GAS characterization, [Hale, 1977, Corollary 3.1., p. 119)])

*The TDS is 0-GAS if and only if there exist a LKF $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$ and a function $\alpha \in \mathcal{PD}$ such that, for all $\phi \in \mathcal{C}^n$,*

$$D^+ V(\phi) \leq -\alpha(|\phi(0)|).$$

## Proposition (0-GAS characterization, [Chaillet, G, Pepe, IEEE TAC, 2022])

*The TDS is 0-GAS if and only if there exist a LKF $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$ and a function $\sigma \in \mathcal{KL}$ such that, for all $\phi \in \mathcal{C}^n$,*

$$D^+ V(\phi) \leq -\sigma(|\phi(0)|, \|\phi\|).$$

# ISS/iISS for TDS

## Definition (ISS, [Pepe, Jiang, SCL, 2006])

The system is ISS if there exist $\nu \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that, for any $x_0 \in \mathcal{C}^n$ and any $u \in \mathcal{U}$,

$$|x(t)| \leq \beta(\|x_0\|, t) + \nu(\|u\|), \quad \forall t \geq 0.$$

## Definition (iISS, [Pepe, Jiang, SCL, 2006])

The TDS is said to be iISS if there exists $\beta \in \mathcal{KL}$ and $\nu, \sigma \in \mathcal{K}_\infty$ such that, for any $x_0 \in \mathcal{C}^n$ and any $u \in \mathcal{U}$, its solution satisfies

$$|x(t)| \leq \beta(\|x_0\|, t) + \nu\left(\int_0^t \sigma(|u(s)|)ds\right), \quad \forall t \geq 0.$$

- Forward completeness [Hale, 1977, Theorem 3.2, p. 43]
- Asymptotic stability in the absence of inputs (0-GAS)

# LKF Characterization for ISS/iISS

## Proposition (ISS LKF, Necessity: [Pepe, Karafyllis, IJC, 2013], Sufficiency: [Pepe, Jiang, SCL, 2006])

*The TDS is ISS if and only if there exists a LKF candidate* $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$, $\alpha \in \mathcal{K}_\infty$ *and* $\gamma \in \mathcal{K}_\infty$, *such that the following holds:*

$$D^+ V(x_t, u(t)) \leq -\alpha(V(x_t)) + \gamma(|u(t)|), \quad \forall t \geq 0.$$

$\to$ Finite-dimensional case: [Sontag, IEEE TAC, 1989].

## Proposition (iISS LKF, Necessity: [Lin, Wang, CDC, 2018], Sufficiency: [Pepe, Jiang, SCL, 2006])

*The TDS is iISS if and only if there exists a LKF candidate* $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$, $\alpha \in \mathcal{PD}$ *and* $\gamma \in \mathcal{K}_\infty$, *such that the following holds:*

$$D^+ V(x_t, u(t)) \leq -\alpha(V(x_t)) + \gamma(|u(t)|), \quad \forall t \geq 0.$$

$\to$ Finite-dimensional case: [Angeli et al., IEEE TAC, 2000].

# Robustness Properties

## Definition (BEBS, BECS)

The TDS $\dot{x}(t) = f(x_t, u(t))$ is said to have the bounded energy-bounded state (BEBS) property, if there exists $\zeta \in \mathcal{K}_\infty$ such that its solution satisfies

$$\int_0^\infty \zeta(|u(s)|)ds < \infty \quad \Rightarrow \quad \sup_{t \geq 0} |x(t)| < \infty.$$

It is said to have the bounded energy-converging state (BECS) property if there exists $\zeta \in \mathcal{K}_\infty$ such that, its solution satisfies

$$\int_0^\infty \zeta(|u(s)|)ds < \infty \quad \Rightarrow \quad \lim_{t \to \infty} |x(t)| = 0.$$

## Definition (UBEBS)

If the system $\dot{x}(t) = f(x_t, u(t))$ is said to have the uniform bounded energy-bounded state (UBEBS) property if there exist $\alpha, \xi, \zeta \in \mathcal{K}_\infty$ and $c \geq 0$ such that, $\forall x_0 \in \mathcal{C}^n$ and $\forall u \in \mathcal{U}$, its solution satisfies

$$\alpha(|x(t)|) \leq \xi(\|x_0\|) + \int_0^t \zeta(|u(s)|)ds + c, \quad \forall t \geq 0.$$

# Soln Characterizations and Zero-Output Dissipativity of iISS TDS

## Proposition (iISS⇔0-GAS+UBEBS, [Chaillet, G, Pepe, IEEE TAC, 2022])

*The TDS $\dot{x}(t) = f(x_t, u(t))$ is iISS if and only if it is 0-GAS and owns the UBEBS property.*

## Lemma (UBEBS with $c = 0$, [Chaillet, G, Pepe, IEEE TAC, 2022])

*If the system $\dot{x}(t) = f(x_t, u(t))$ is 0-GAS, then the following properties are equivalent:*

- *The system satisfies the UBEBS estimate.*
- *The system satisfies the UBEBS estimate with $c = 0$.*

## Proposition (iISS⇔0-GAS+zero-output dissipativity, [Chaillet, G, Pepe, IEEE TAC, 2022])

*The TDS $\dot{x}(t) = f(x_t, u(t))$ is iISS if and only if it is 0-GAS and there exists a LKF $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$ and $\mu \in \mathcal{K}_\infty$ such that*

$$D^+ V(\phi, v) \leq \mu(|v|), \quad \forall \phi \in \mathcal{C}^n, \forall v \in \mathbb{R}^m.$$

# iISS LKFs

## Definition (iISS LKF)

A LKF $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$ is said to be:

- an iISS LKF with point-wise dissipation rate for $\dot{x}(t) = f(x_t, u(t))$ if $\exists \alpha \in \mathcal{PD}$ and $\gamma \in \mathcal{K}_\infty$ such that, $\forall \phi \in \mathcal{C}^n$ and $\forall v \in \mathbb{R}^m$,
$$D^+ V(\phi, v) \leq -\alpha(|\phi(0)|) + \gamma(|v|).$$
- an iISS LKF with LKF-wise dissipation rate for $\dot{x}(t) = f(x_t, u(t))$ if $\exists \alpha \in \mathcal{PD}$ and $\gamma \in \mathcal{K}_\infty$ such that, $\forall \phi \in \mathcal{C}^n$ and $\forall v \in \mathbb{R}^m$,
$$D^+ V(\phi, v) \leq -\alpha(V(\phi)) + \gamma(|v|).$$
- an iISS LKF with history-wise dissipation rate for $\dot{x}(t) = f(x_t, u(t))$ if $\exists \alpha \in \mathcal{PD}$ and $\gamma \in \mathcal{K}_\infty$ such that, $\forall \phi \in \mathcal{C}^n$ and $\forall v \in \mathbb{R}^m$,
$$D^+ V(\phi, v) \leq -\alpha(\|\phi\|) + \gamma(|v|).$$
- an iISS LKF with $\mathcal{KL}$ dissipation rate for $\dot{x}(t) = f(x_t, u(t))$ if $\exists \sigma \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that, $\forall \phi \in \mathcal{C}^n$ and $\forall v \in \mathbb{R}^m$,
$$D^+ V(\phi, v) \leq -\sigma(|\phi(0)|, \|\phi\|) + \gamma(|v|).$$

- $\alpha$ and $\sigma$ are called dissipation rate,
- $\gamma$ is called supply rate.

# New LKF Characterizations for iISS TDS

## Theorem (iISS LKF Characterizations, [Chaillet, G, Pepe, IEEE TAC, 2022])

*The following statements are equivalent for the TDS $\dot{x}(t) = f(x_t, u(t))$:*

(i) *The TDS admits a coercive iISS LKF with history-wise dissipation.*

(ii) *The TDS admits an iISS LKF with LKF-wise dissipation.*

(iii) *The TDS admits an iISS LKF with history-wise dissipation.*

(iv) *The TDS admits an iISS LKF with $\mathcal{KL}$ dissipation.*

(v) *The TDS is iISS.*

*Moreover, the TDS is iISS if*

(vi) *The TDS admits an iISS LKF with point-wise dissipation.*

# New LKF Characterizations for iISS TDS

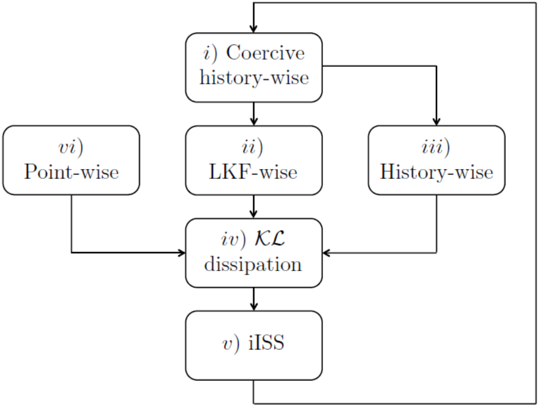



Figure: Proof Strategy.

# New LKF Characterizations for iISS TDS

## Proof (Sketch).

- (iv)⇒(v): iISS LKF with $\mathcal{KL}$ dissipation $\Rightarrow$ 0-GAS+zero-output dissipativity $\Rightarrow$ iISS.
- (v)⇒(i): iISS $\Rightarrow$
  - $\exists$ coercive LKF $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$, $\nu \in \mathcal{K}_\infty$ with $D^+V(\phi, v) \leq \nu(|v|)$, $\forall \phi \in \mathcal{C}^n$, $v \in \mathbb{R}^m$ (Lin, Wang, CDC, 2018).
  - $\exists$ coercive LKF $V_1 : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$, $\pi \in \mathcal{K} \cap \mathcal{C}^1$ with $\pi'(s) > 0$, $\forall s \geq 0$, $\alpha \in \mathcal{PD}$, $\gamma \in \mathcal{K}_\infty$ such that $W_1 := \pi \circ V_1$ satisfies $D^+W_1(\phi, v) \leq -\alpha(\|\phi\|) + \gamma(|v|)$.

  $\mathcal{V} := V + W_1$ is a coercive iISS LKF with history-wise dissipation.
- (i)⇒(iii): Trivial as any coercive LKF is a LKF.

# New LKF Characterizations for iISS TDS

## Proof (Sketch-Continued).

Fact [Angeli et. al, IEEE TAC, 2000]: $\forall \alpha \in \mathcal{PD}$, $\exists \mu \in \mathcal{K}_\infty, \ell \in \mathcal{L}$ such that $\alpha(s) \geq \mu(s)\ell(s)$, $\forall s \geq 0$.

- (i)⇒(ii): $V$ is coercive history-wise LKF $\Rightarrow \exists V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$, $\alpha \in \mathcal{PD}$, $\underline{\alpha}, \overline{\alpha}, \gamma \in \mathcal{K}_\infty$ such that

$$\begin{aligned} \underline{\alpha}(\|\phi\|) \leq & V(\phi) \leq \overline{\alpha}(\|\phi\|) \\ D^+V(\phi, v) \leq & -\alpha(\|\phi\|) + \gamma(|v|) \\ \leq & -\mu(\|\phi\|)\ell(\|\phi\|) + \gamma(|v|) \\ \leq & -\mu \circ \overline{\alpha}^{-1}(V(\phi))\ell \circ \underline{\alpha}^{-1}(V(\phi)) + \gamma(|v|) \end{aligned}$$

  $\Rightarrow$ $V$ is iISS LKF with LKF-wise dissipation.

- (ii)⇒(iv): $V$ is iISS LKF with LKF-wise dissipation $\Rightarrow$ Fact $\Rightarrow$ $V$ is iISS LKF with $\mathcal{KL}$ dissipation rate.
- (iii)⇒(iv): $V$ is iISS LKF with history-wise dissipation rate $\Rightarrow$ Fact $\Rightarrow$ $V$ is iISS LKF with $\mathcal{KL}$ dissipation rate.
- (vi)⇒(iv): Implication follows by using the fact and observing $\alpha(|\phi(0)|) \geq \mu(|\phi(0)|)\ell(|\phi(0)|) \geq \mu(|\phi(0)|)\ell(\|\phi\|)$ for any $\alpha \in \mathcal{PD}$, $\mu \in \mathcal{K}_\infty$, $\ell \in \mathcal{L}$. □

## Example

Consider the following TDS:

$$\dot{x}(t) = -\frac{|x(t)|}{1 + \|x_t\|^2} + u(t).$$

Consider the LKF (proposed in [Pepe, Jiang, SCL, 2006]) defined as

$$W(\phi) := \sup_{s\in[-\delta,0]} e^s Q(\phi(s)), \quad \forall \phi \in \mathcal{C},$$

where the function $Q \ : \ \mathbb{R} \to \mathbb{R}^+$ is defined as

$$Q(x) = \begin{cases} \frac{1}{2}x^2, & if |x| \leq 1, \\ |x| - \frac{1}{2}, & if |x| > 1. \end{cases}$$

After cumbersome calculation, it is possible to get

$$D^+W(\phi, v) \leq \begin{cases} -W(\phi), & \text{if } W > Q(\phi(0)), \\ \max\left\{-W, Q'(\phi(0))\left(\frac{-\phi(0)}{1+\|\phi\|^2} + v\right)\right\}, & \text{if } W = Q(\phi(0)). \end{cases}$$

which then also implies $D^+W(\phi, v) \leq -\alpha(W) + |v|$ where $\alpha(s) = \frac{s}{1+\underline{\alpha}^{-1}(s)^2}$ again after some calculation.

### Example

Consider the following TDS:

$$\dot{x}(t) = -\frac{|x(t)|}{1+\|x_t\|^2} + u(t).$$

On contrary $V(\phi) = |\phi(0)|$ satisfies, $\forall \phi \in \mathcal{C}$ and $\forall v \in \mathbb{R}$

$$D^+V(\phi, v) \leq -\frac{|x(t)|}{1+\|x_t\|^2} + |v|,$$

does the same job with $\mathcal{KL}$ dissipation.