Numerical Methods of Thermo-Fluid Dynamics I

Practical No. 2 - Group 9

Numerical Solution of Boundary Layer Equation

Abhik Sarkar (23149662), Gokhul Shriram Ravi(23212226)

Table of Contents

List of Figures

1 Problem

Given the boundary layer equations over a flat plate:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{Re}\frac{\partial^2 u}{\partial y^2} \tag{2}$$

In the above equation, the Reynolds number is derived as $Re = u_\infty L/v$, where v is the kinematic viscosity. L is the length of the plate, and u_∞ is the free-stream velocity of the outer flow.
Here, the velocity components u and v are dimensionless, $u = \bar{u}/u_\infty$ and $v = \bar{v}/u_\infty$, where $\bar{u}$ and $\bar{v}$ are the dimensional velocities. Lengths have been made dimensionless by using the plate length, $x = \bar{x}/L$. The boundary conditions are as follows:

$y = 0$	$u = v = 0$	[no-slip condition]
$y \to \infty$	$u = 1$	[Free outflow $\bar{u} \to u_\infty$]

2 Discretization

2.1 Discretization of Continuity Equation

To discretize the continuity equation given in Equation 1, we will use the below suitable scheme:

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{2}\left(\frac{u_{i,j} - u_{i-1,j}}{\Delta x} + \frac{u_{i,j-1} - u_{i-1,j-1}}{\Delta x}\right) \tag{3}$$

Now, for $\frac{\partial v}{\partial y}$, we will use the backward difference scheme:

$$\left(\frac{\partial v}{\partial y}\right)_{i,j} = \left(\frac{v_{i,j} - v_{i,j-1}}{\Delta y}\right) \tag{4}$$

By substituting the value of Equation 3 and Equation 4 in Equation 1, we get,

$$\left(\frac{u_{i,j} - u_{i-1,j} + u_{i,j-1} - u_{i-1,j-1}}{2\Delta x}\right) + \left(\frac{v_{i,j} - v_{i,j-1}}{\Delta y}\right) = 0 \tag{5}$$

After rearranging the Equation 5 for $v_{i,j}$, we get,

$$v_{i,j} = v_{i,j-1} - \frac{\Delta y}{2\Delta x}(u_{i,j} - u_{i-1,j} + u_{i,j-1} - u_{i-1,j-1}) \tag{6}$$

As a result, the above discretization is first order accurate in Δx and Δy since it uses both central difference and backward difference methods.

2.2 Discretization of Momentum Equation

Since only left boundary conditions are provided, we will apply the forward difference method for $\frac{\partial u}{\partial x}$ in order to discretize the momentum equation given in Equation 2.

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} \tag{7}$$

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} \tag{8}$$

By taking its second order derivative, we get,

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} \tag{9}$$

Now, by substituting the value of Equation 7, 8 and 9 in Equation 2, we get,

$$u_{i,j}\left(\frac{u_{i+1,j} - u_{i,j}}{\Delta x}\right) + v_{i,j}\left(\frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right) = \frac{1}{Re}\left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2}\right) \tag{10}$$

Therefore, after rearranging the Equation 10 for u, we get,

$$u_{i+1,j} = u_{i,j} + \frac{\Delta x}{Re}\left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{u_{i,j}(\Delta y)^2}\right) - v_{i,j}\frac{\Delta x}{u_{i,j}}\left(\frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right) \tag{11}$$

The discretization of u presented in Equation 11 is a first-order accurate scheme. This accuracy is attributed to the use of both forward difference, as seen in Equation 7, and central difference methods, as observed in Equation 8 and Equation 9. As a result, the overall discretization method is explicit, indicating that it directly calculates the future value of $u_{i+1,j}$ based on current values.
While explicit methods are straightforward, it is important to note that their stability is subject to the Courant-Friedrichs-Lewy (CFL) criterion, which places constraints on the time step to prevent numerical instability. For the stability of this scheme, it is required that the spatial step $(\overline{\Delta x})$ satisfy the below condition:

$$\Delta x \leq \frac{\frac{1}{2}u_{i,j}(\Delta y)^2}{\nu} \tag{12}$$

where ν is the kinematic viscosity. This criterion ensures that the spatial step is chosen appropriately relative to the velocity $(u_{i,j})$ and the grid spacing in the y-direction (Δy) to prevent numerical instability. It's crucial to adhere to this stability criterion when selecting the discretization parameters for accurate and stable simulations.

3 MATLAB Code

3.1 Results and velocity over the entire domain

In this section, the results and the plots for the velocity profile over the entire domain are discussed. First, the contour plot for the u component of the velocity is shown below.

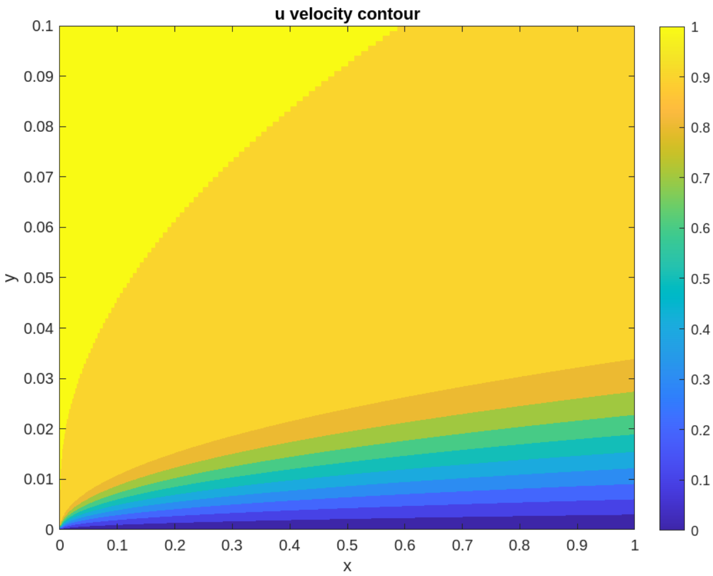

Fig. 1: Contour plot for u

The computational domain extends from $x = 0$ to $x = L = 1m$ and from $y = 0$ to $y = 2\delta = 0.1$. There is a distinct velocity gradient in the areas close to the plate which slowly fades as the distance from the plate increases. From the plot, we can clearly see the boundary layer separation; the flow above the boundary layer exhibiting free stream velocity as expected. The formation of the boundary layer can be attributed to the viscous forces that come into play in fluid flows- the velocity near the plate equalling zero and consistently increases as the distance from the plate increases.

The contour plot for the v component of velocity is shown below.

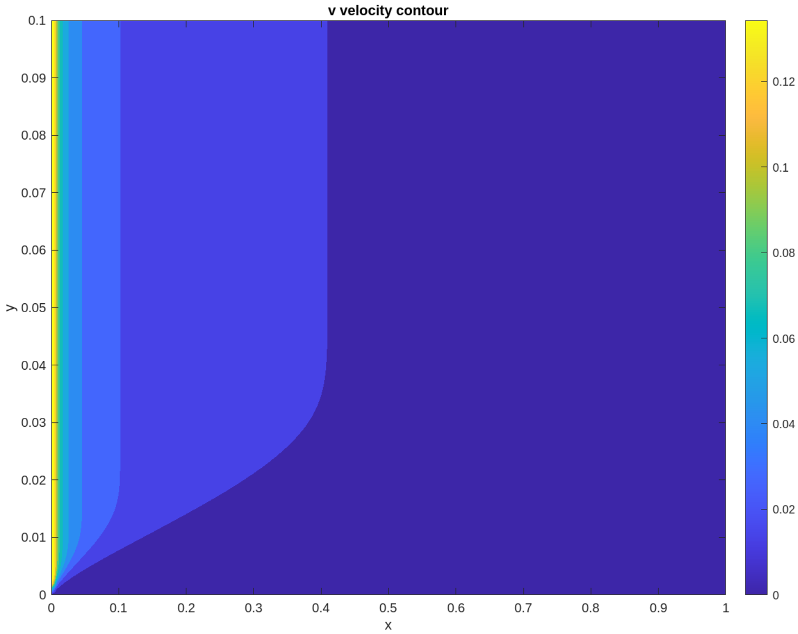

Fig. 2: Contour plot for v

From the above plot, we can see the stark difference between the contour plots for u and v, with regards to the magnitude of the velocity components. A velocity gradient in the v component occurs due to the viscous drag. The magnitude of the gradient is however small compared to the u velocity gradient, but is nevertheless interesting to note.

4 Comparison of study between Numerical solution and Blasius equation

The velocity profiles are retrieved at $x = 0.0005$ and $x = 0.5$ by solving both the Boundary layer equation and by solving the Blasius equation. The following plots were obtained.

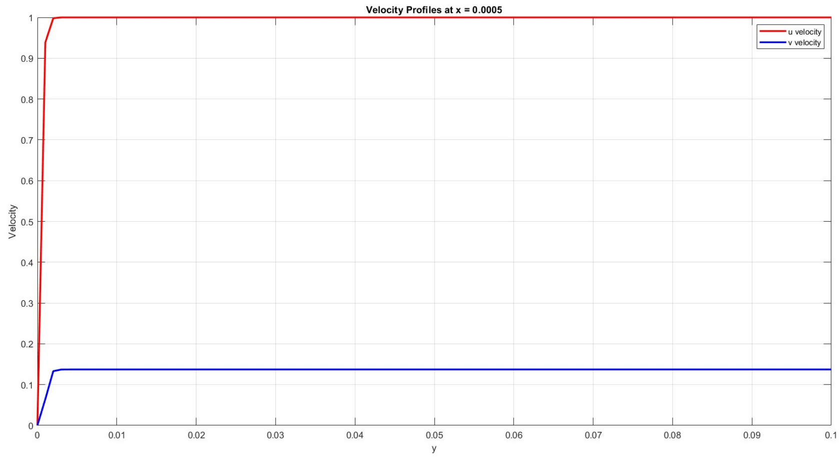

Fig. 3: Velocity profile at $x = 0.0005$ by boundary layer equation

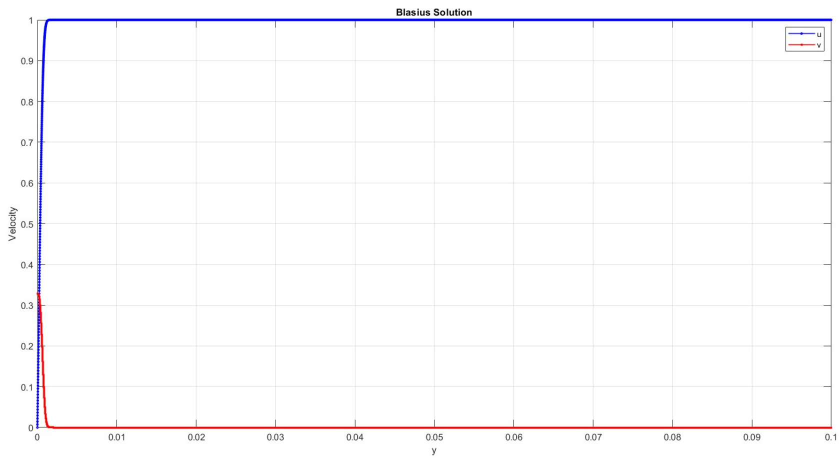

Fig. 4: Velocity profile at $x = 0.0005$ by blasius equation

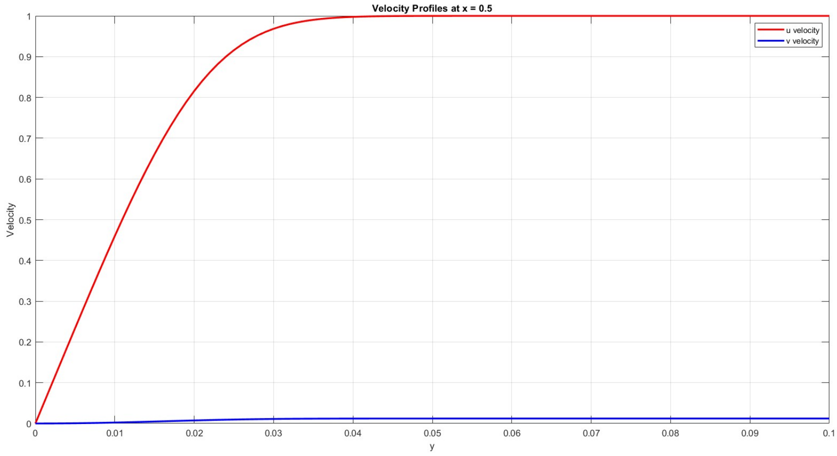

Fig. 5: Velocity profile at $x = 0.5$ by boundary layer equation

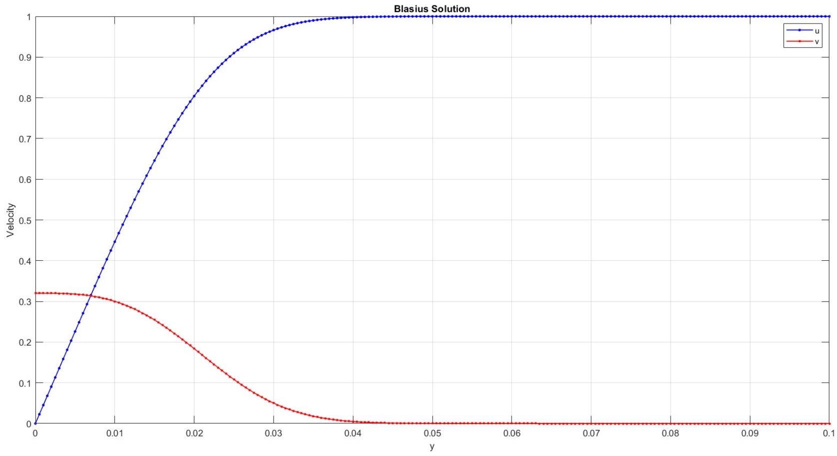

Fig. 6: Velocity profile at $x = 0.5$ by blasius equation

From these plots we can say that the behaviour of both the numerical solution and the Blasius solution are very similar. It is interesting to note that, the velocity profiles obtained from both the numerical method and the Blasius equations are almost an exact match for the $x = 0.5$, however the same cannot be said for the velocity profiles at $x = 0.0005$. In the case of velocity profiles for $x = 0.0005$ there are slight discrepancies and the values obtained from the numerical methods and solving the Blasius equations do not match exactly. This might be due to the grid size chosen for the numerical method, and could be reduced by using a finer grid size. But this would mean increased computational cost and time.