CENG 384 - Signals and Systems for Computer Engineers
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Homework 4

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1. (a)

$$g(t) = 4x(t) - 5y(t) + \int (x(t) - 6y(t))dt$$
$$\text{and } y(t) \int g(t)dt = \int [4x(t) - 5y(t) + \int (x(t) - 6y(t)dt]dt$$

Differentiate it once

$$\frac{dy(t)}{dt} = 4x(t) - 5y(t) + \int (x(t) - 6y(t))dt$$

Differentiate it once more

$$\frac{d^2y(t)}{dt^2} = \frac{4dx(t)}{dt} - \frac{5dy(t)}{dt} + x(t) - 6y(t)$$

So differential equation is;

$$\frac{d^2y(t)}{dt^2} + \frac{5dy(t)}{dt} + 6y(t) = \frac{4dx(t)}{dt} + x(t)$$

(b)

$$(j\omega)^2Y(j\omega) + 5(j\omega)Y(j\omega) + 6Y(j\omega) = 4(j\omega)X(j\omega) + X(j\omega)$$

So Frequency Response is;

$$Y(j\omega)[6 + 5j\omega - \omega^2] = X(j\omega)[1 + 4j\omega]$$

(c)

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{1 + 4j\omega}{6 - \omega^2 + 5j\omega}$$

By using Partial Fraction;

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{7}{2 + j\omega} - \frac{3}{3 + j\omega}$$

By applying inverse Fourier transform;

$$h(t) = (7e^{-2t} - 3e^{-3t})u(t)$$

(d)

$$x(t) = \frac{1}{4}e^{\frac{-t}{4}}u(t)$$

By applying Fourier Transform;

$$X(j\omega) = \frac{1}{4}(\frac{1}{\frac{1}{4}+j\omega})$$
$$\frac{Y(j\omega)}{X(j\omega)} = (\frac{7}{2+j\omega} - \frac{3}{3+j\omega})$$
$$Y(j\omega) = (\frac{7}{2+j\omega} - \frac{3}{3+j\omega}) \cdot \frac{1}{4}(\frac{1}{\frac{1}{4}+j\omega})$$
$$Y(j\omega) = \frac{1}{4}[(\frac{7}{2+j\omega}) \cdot (\frac{1}{\frac{1}{4}+j\omega}) - (\frac{3}{3+j\omega}) \cdot (\frac{1}{\frac{1}{4}+j\omega})]$$
$$Y(j\omega) = \frac{1}{4}[\frac{4}{\frac{1}{4}+j\omega} - \frac{4}{2+j\omega} - \frac{1.090}{\frac{1}{4}+j\omega} + \frac{1.090}{3+j\omega}]$$
$$Y(j\omega) = \frac{1}{4}[\frac{2.9}{\frac{1}{4}+j\omega} - \frac{4}{2+j\omega} + \frac{1.09}{3+j\omega}]$$

By applying inverse Fourier Transform;

$$y(t) = \frac{1}{4}(2.9e^{\frac{-t}{4}} - 4e^{-2t} + 1.09e^{-3t})u(t)$$
$$y(t) = (0.72e^{\frac{-t}{4}} - e^{-2t} + 0.27e^{-3t})u(t)$$

2. (a)

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega+4}{(j\omega)^2+5j\omega+6}$$
$$Y(j\omega)\cdot(j\omega)^2 + Y(j\omega)\cdot 5j\omega + Y(j\omega)\cdot 6 = X(j\omega)\cdot j\omega + X(j\omega)\cdot 4$$

By applying inverse Fourier transform;

$$\frac{d^2(y)}{dt^2} + 5\cdot\frac{d(y)}{dt} + 6\cdot y(t) = \frac{d(x)}{dt} + x(t)\cdot 4$$

So the answer becomes;

$$y\prime\prime + 5\cdot y\prime + 6\cdot y = x\prime + 4\cdot x$$

(b)

$$H(j\omega) = \frac{j\omega+4}{(j\omega)^2+5j\omega+6}$$
$$H(j\omega) = \frac{j\omega+4}{(j\omega+2)\cdot(j\omega+3)}$$
$$H(j\omega) = \frac{2}{(j\omega+2)} - \frac{1}{(j\omega+3)}$$

By applying inverse Fourier transform;

$$h(t) = 2e^{-2t}\mu(t) + e^{-3t}\mu(t)$$
So the answer becomes; $h(t) = (2e^{-2t} + e^{-3t})\cdot\mu(t)$

(c)

$$Y(j\omega) = X(j\omega)\cdot H(j\omega)$$
$X(j\omega) = e^{-4t}\mu(t) - t\cdot e^{-4t}\mu(t)$ is given.

By applying inverse Fourier transform;

$$X(j\omega) = \frac{1}{(j\omega+4)} - \frac{1}{(j\omega+4)^2}$$
$$X(j\omega) = \frac{j\omega+4-1}{(j\omega+4)^2}$$

By combining the part above;

$$Y(j\omega) = \frac{j\omega+3}{(j\omega+4)^2} \cdot \frac{j\omega+4}{(j\omega+2)\cdot(j\omega+3)}$$
$$Y(j\omega) = \frac{1}{(j\omega+4)\cdot(j\omega+2)}$$

(d)

$$Y(j\omega) = \frac{1}{(j\omega+4)\cdot(j\omega+2)} = \frac{1}{2}\cdot[\frac{1}{(j\omega+2)} - \frac{1}{(j\omega+4)}]$$

By applying inverse Fourier transform;

$$y(t) = \frac{1}{2}\cdot e^{-2t}\mu(t) - \frac{1}{2}\cdot e^{-4t}\mu(t)$$
$$y(t) = \frac{1}{2}\cdot[e^{-2t} - e^{-4t}]\cdot\mu(t)$$

3. (a)

$$f(t) = e^{-|t|}$$

The Fourier transform of signal f(t):

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$
$$F(\omega) = \int_{-\infty}^{\infty} e^{-|t|}e^{-j\omega t}dt$$
$$F(\omega) = \int_{-\infty}^{0} e^{t}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-t}e^{-j\omega t}dt$$
$$F(\omega) = \int_{-\infty}^{0} e^{t(1-j\omega)}dt + \int_{0}^{\infty} e^{-t(1+j\omega)}dt$$
$$F(\omega) = [\frac{1}{1-j\omega}e^{t(1-j\omega)}]_{-\infty}^{0} - [\frac{1}{(1+j\omega)}e^{-t(1+j\omega)}]_{0}^{\infty}$$
$$F(\omega) = [\frac{1}{1-j\omega} - 0] - [0 - \frac{1}{1+j\omega}]$$
$$F(\omega) = [\frac{1}{1-j\omega}] + [\frac{1}{1+j\omega}]$$

So the Fourier transform of $f(t) = e^{-|t|}$

$$F(\omega) = \frac{2}{1+\omega^2}$$

(b) Fourier transform of

$$f(t) = e^{-|t|} \text{ is}$$
$$F(\omega) = \frac{2}{1+\omega^2}$$

Fourier transform of

$$x(t) = te^{-|t|}$$
$$x(t) = tf(t) \text{ is}$$

By using multiplication of t with f(t) property;

$$X(\omega) = j\frac{dF(\omega)}{d\omega}$$
$$X(\omega) = j[-\frac{4\omega}{(1+\omega^2)^2}]$$

So the Fourier Transform of $x(t) = te^{-|t|}$

$$X(\omega) = -j[\frac{4\omega}{(1+\omega^2)^2}]$$

(c) Let;

$$g(t) = \frac{4t}{(1+t^2)^2}$$

The Fourier transform of $x(t) = te^{-|t|}$ is

$$X(\omega) = -j[\frac{4\omega}{(1+\omega^2)^2}]$$
$$x(t) \leftrightarrow X(\omega)$$
$$te^{-|t|} \leftrightarrow -j[\frac{4\omega}{(1+\omega^2)^2}]$$

By using the duality property of Fourier transform;

$$\text{If } x(t) \leftrightarrow X(\omega) \text{ then}$$
$$X(t) \leftrightarrow 2\pi x(-\omega)$$
$$-j[\frac{4t}{(1+t^2)^2}] \leftrightarrow 2\pi(-\omega)e^{-|(-\omega)|}$$
$$-j[\frac{4t}{(1+t^2)^2}] \leftrightarrow -2\pi(\omega)e^{-|\omega|}$$
$$j[\frac{4t}{(1+t^2)^2}] \leftrightarrow 2\pi(\omega)e^{-|\omega|}$$

Dividing both sides by j;

$$\frac{4t}{(1+t^2)^2} \leftrightarrow \frac{1}{j}2\pi\omega e^{-|w|}$$
$$\frac{4t}{(1+t^2)^2} \leftrightarrow -j\cdot 2\pi\omega e^{-|w|}$$

Hence, The Fourier Transform of

$$g(t) = \frac{4t}{(1+t^2)^2} \text{ is}$$
$$G(\omega) = -j\cdot 2\pi\omega e^{-|\omega|}$$

4. (a) The difference equation can be denoted as;

$$2x(n) + \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) = y(n)$$

(b) In order to find the frequency response of the system;

$$2X(e^j\omega) + \frac{3}{4}e^{-j\omega}Y(e^{j\omega}) - \frac{1}{8}e^{-2j\omega}Y(e^{j\omega}) = Y(e^{-j\omega})$$
$$Y(e^{-j\omega})\cdot[1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}] = 2X(e^{j\omega})$$
$$\text{We know that ;}$$
$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$
$$H(e^{j\omega}) = \frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega}}$$

(c) In order to find the impulse response of the system;

$$H(e^{j\omega}) = \frac{2}{(1-\frac{1}{2}e^{-j\omega})\cdot(1-\frac{1}{4}e^{-j\omega})}$$
$$H(e^{j\omega}) = \frac{A}{(1-\frac{1}{2}e^{-j\omega})} + \frac{B}{(1-\frac{1}{4}e^{-j\omega})}$$
$$\ldots$$
$$H(e^{j\omega}) = \frac{4}{(1-\frac{1}{2}e^{-j\omega})} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})}$$

Therefore the answer becomes;

$$f(n) = 4(\frac{1}{2})^n\mu(n) - 2(\frac{1}{4})^n\mu(n)$$

(d) Given;

$$x(n) = (\frac{1}{4})^n\mu(n)$$
$$x(n) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

So $Y(e^{j\omega})$ becomes;

$$Y(e^{j\omega}) = X(e^{j\omega})\cdot H(e^{j\omega})$$
$$Y(e^{j\omega}) = \frac{2}{(1-\frac{1}{2}e^{-j\omega})\cdot((1-\frac{1}{4}e^{-j\omega})^2)}$$

By applying the partial fraction expansion;

$$Y(e^{j\omega}) = \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B_0}{1-\frac{1}{4}e^{-j\omega}} + \frac{B_1}{(1-\frac{1}{4}e^{-j\omega})^2}$$
$$\ldots$$
$$B_1 = -2$$
$$B_0 = 4$$
$$A = 8$$

So the equation becomes;

$$Y(e^{j\omega}) = \frac{8}{1-\frac{1}{2}e^{-j\omega}} + \frac{4}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2}$$
$$y(n) = 8(\frac{1}{2})^n\mu(n) + 4(\frac{1}{4})^n\mu(n) - 2(n+1)(\frac{1}{4})^n\mu(n)$$

5. When two systems are said to be connected in parallel the frequency response of the combined system becomes;

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$
$$\text{So } H_1 \text{ becomes,}$$
$$H_1(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

So the we can write the equation as,

$$\frac{5e^{-j\omega}-12}{e^{-2j\omega}-7e^{-j\omega}+12} = \frac{1}{1-\frac{1}{3}e^{-j\omega}} + H_2(e^{j\omega})$$

So to find H_2;

$$\frac{\frac{5}{12}e^{-j\omega}-1}{(1-\frac{1}{3}e^{-j\omega})\cdot(1-\frac{1}{4}e^{-j\omega})} = \frac{1}{1-\frac{1}{3}e^{-j\omega}} + H_2(e^{j\omega})$$
$$\frac{\frac{5}{12}e^{-j\omega}-1}{(1-\frac{1}{3}e^{-j\omega})\cdot(1-\frac{1}{4}e^{-j\omega})} = \frac{1}{1-\frac{1}{3}e^{-j\omega}} + \frac{A}{1-\frac{1}{4}e^{-j\omega}}$$

So we can deduct that $H_2(e^{j\omega}) = \frac{A}{1-\frac{1}{4}e^{-j\omega}}$.
So by solving the mathematical equations, we were left with this;

$$\frac{5}{12}e^{-j\omega} - 1 = (1 + A) + (-\frac{1}{4} - \frac{A}{3})e^{-j\omega}$$

by equating the $(e^{-j\omega})$ terms,

$$\frac{5}{12} = -\frac{1}{4} - \frac{A}{3}$$

$$A = -2$$

So the answer becomes,

$$h_2(n) = -2(\frac{1}{4})^n \mu(n)$$

6. (a) To determine the difference equation, we represent $H(e^{j\omega})$ as $\frac{Ye^{j\omega}}{X(e^{j\omega})}$.

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-j2\omega}}$$

$$Y(e^{j\omega}) - \frac{1}{6}e^{-j\omega}Y(e^{j\omega}) - \frac{1}{6}e^{-j2\omega}Y(e^{j\omega}) = X(e^{j\omega})$$

By applying Inverse Discrete-Time Fourier transform on both sides;

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$$

So the difference equation is

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$$

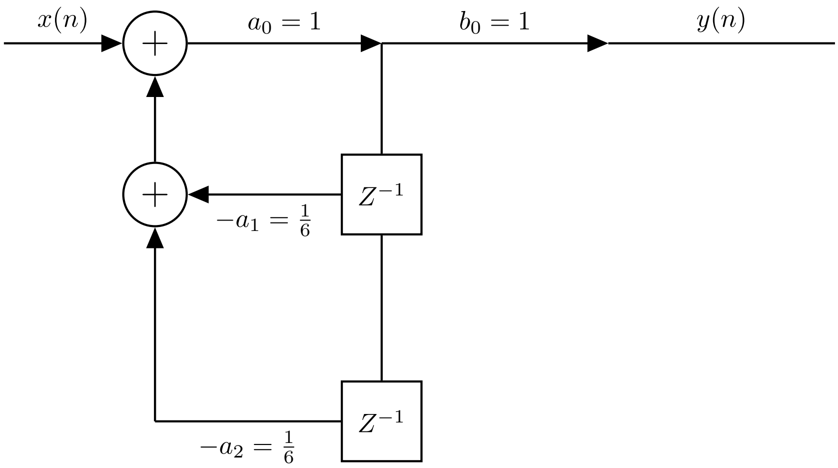
(b) The difference equation by part a;

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$$

By comparing with linear constant coefficient difference equation;

$$a_0y(n) + a_1y(n-1) + a_2y(n-2) + ... + a_Ny(n-N) = b_0x(n) + b_1x(n-1) + ... + b_MX(n-M)$$

Here $a_0 = 1, a_1 = \frac{-1}{6}, a_2 = \frac{-1}{6}, b_0 = 1$
So the block diagram by using this form;



(c) Given;

$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-j2\omega}}$$
$$H(e^{j\omega}) = \frac{6}{6 - e^{-j\omega} - e^{-j2\omega}}$$
$$H(e^{j\omega}) = \frac{6}{6 - 3e^{-j\omega} + 2e^{-j\omega} - e^{-j2\omega}}$$
$$H(e^{j\omega}) = \frac{6}{3(2 - e^{-j\omega}) + e^{-j\omega}(2 - e^{-j\omega})}$$
$$H(e^{j\omega}) = \frac{6}{(3 + e^{-j\omega})(2 - e^{-j\omega})}$$

By applying Partial Fraction;

$$\frac{6}{(3 + e^{-j\omega})(2 - e^{-j\omega})} = \frac{A}{3 + e^{-j\omega}} + \frac{B}{2 - e^{-j\omega}}$$
$$A = \frac{6}{5}$$
$$B = \frac{6}{5}$$
$$H(e^{j\omega}) = \frac{\frac{6}{5}}{3 + e^{-j\omega}} + \frac{\frac{6}{5}}{2 - e^{-j\omega}}$$
$$H(e^{j\omega}) = \frac{\frac{6}{5}}{3(1 + \frac{1}{3}e^{-j\omega})} + \frac{\frac{6}{5}}{2(1 - \frac{1}{2}e^{-j\omega})}$$
$$H(e^{j\omega}) = \frac{\frac{2}{5}}{1 + \frac{1}{3}e^{-j\omega}} + \frac{\frac{3}{5}}{1 - \frac{1}{2}e^{-j\omega}}$$

By applying inverse Discrete-Time Fourier transform on both sides;

$$h(n) = \frac{2}{5} \cdot (\frac{-1}{3})^n u(n) + \frac{3}{5} \cdot (\frac{1}{2}^n) u(n).$$

So Impulse Response is;

$$h(n) = \frac{2}{5} \cdot (\frac{-1}{3})^n u(n) + \frac{3}{5} \cdot (\frac{1}{2}^n) u(n).$$