CENG 384 - Signals and Systems for Computer Engineers
Spring 2021
Homework 3

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1. (a)

$$x(t) = \frac{1}{2} + \cos \omega_0 t$$

The Euler formula provides us the representation of a periodic function in terms of harmonically related complex exponentials, as follows;

$$x(t) = \frac{1}{2} + \frac{1}{2}(e^{jw_0 t} + e^{-jw_0 t})$$

Thus, the Fourier series coefficients of the function are,

$$a_0 = a_1 = a_{-1} = \frac{1}{2}, a_k = 0 \ for \ k \neq -1, 0, 1.$$

The plot of the spectral coefficients,

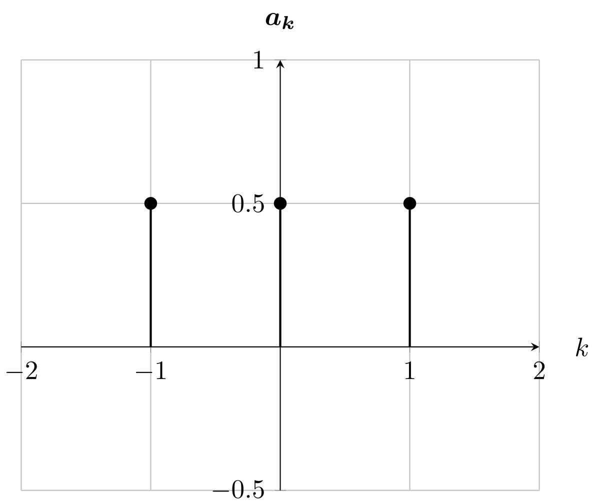

Figure 1: k vs. a_k.

(b)

$$y(t) = \frac{3}{2} + 2 \cdot \sin \omega_0 t$$

The Euler formula provides us the representation of a periodic function in terms of harmonically related complex exponentials, as follows;

$$y(t) = \frac{3}{2} + \frac{(e^{jw_0 t} - e^{-jw_0 t})}{j}$$

Thus, the Fourier series coefficients of the function are,

$$a_0 = \frac{3}{2},\ a_1 = \frac{1}{j} = -j,\ a_{-1} = \frac{-1}{j} = j,\ a_k = 0 \ for \ k \neq -1, 0, 1.$$

So the magnitude, and phase spectrums of the coefficients:

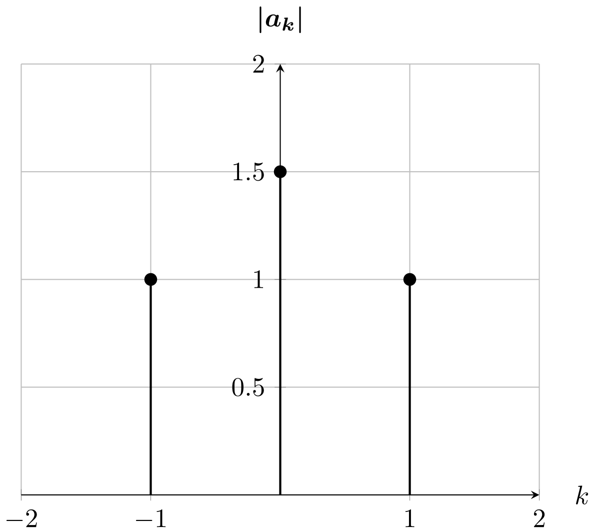

Figure 2: Magnitude Spectrum

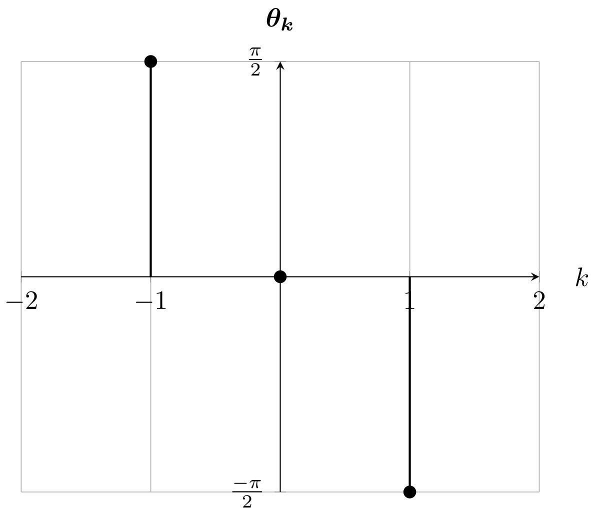

Figure 3: Phase Spectrum

(c)

$$z(t) = x(t) + y(t) + \cos(2\omega_0 t + \frac{\pi}{4})$$

From part a, and part b :

$$x(t) = \frac{1}{2} + \frac{1}{2}(e^{jw_0 t} + e^{-jw_0 t})$$
$$y(t) = \frac{3}{2} + \frac{(e^{jw_0 t} - e^{-jw_0 t})}{j}$$

The Euler formula provides us the representation of a periodic function in terms of harmonically related complex exponentials, as follows;

$$\frac{1}{2}(e^{j(2\omega_0 t + \frac{\pi}{4})} + e^{-j(2\omega_0 t + \frac{\pi}{4})})$$

$$\frac{1}{2}(e^{j(2\omega_0 t)} \cdot e^{\frac{j\pi}{4}} + e^{-j(2\omega_0 t)} \cdot e^{\frac{-j\pi}{4}})$$

From euler formula:

$$e^{\frac{j\pi}{4}} = \cos\frac{\pi}{4} + j\sin\frac{\pi}{4}$$

$$e^{\frac{j\pi}{4}} = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$$

So $z(t)$ equals;

$$z(t) = 2 + e^{j\omega_0 t}(\frac{1}{2} - j) + e^{-j\omega_0 t}(\frac{1}{2} + j) + e^{2j\omega_0 t}(\frac{\sqrt{2}}{4}(1+j)) + e^{-2j\omega_0 t}(\frac{\sqrt{2}}{4}(1-j))$$

Thus, the Fourier series coefficients of the function are,

$$a_0 = 2,\ a_1 = \frac{1}{2} - j,\ a_{-1} = \frac{1}{2} + j,\ a_2 = \frac{\sqrt{2}}{4}(1+j),\ a_{-2} = \frac{\sqrt{2}}{4}(1-j)\ a_k = 0\ for\ k \neq -2, -1, 0, 1, 2.$$

So the magnitude, and phase spectrums of the coefficients:

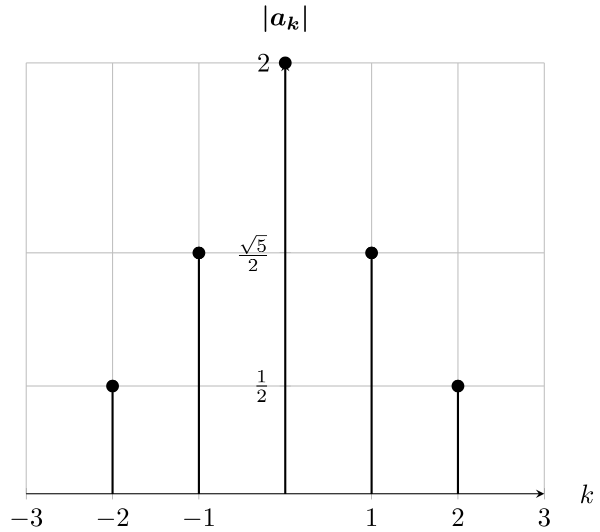

Figure 4: Magnitude Spectrum

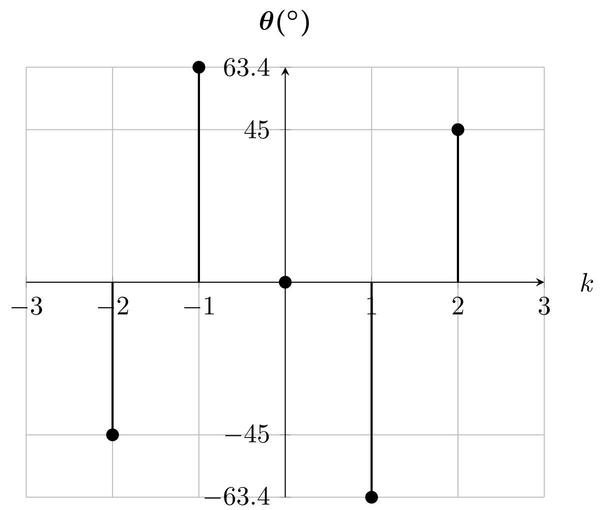

Figure 5: Phase Spectrum

2.

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos k\omega t + B_k \sin k\omega t$$

$$A_0 = \frac{M \cdot T_1}{T}$$

$$A_K = \frac{2}{T} \cdot \int_0^T x(t) cos k\omega_0 t dt$$

$$A_K = \frac{2}{T} \cdot \int_0^{T_1} M cos k\omega_0 t dt + = \int_{T_1}^{T} 0 cos k\omega_0 t dt$$

$$A_K = \frac{2M}{T} \cdot [\frac{sin k\omega_0 t}{k\omega_0}] + 0$$

$$A_K = \frac{2M}{T} \cdot [\frac{sin k 2\frac{\pi}{T} T_1}{k\frac{2\pi}{T}}] =$$

$$A_K = \frac{M}{k\pi} \cdot sin 2\pi k \frac{T_1}{T}$$

$$A_K = \frac{M}{K\pi} sin(2\pi K \frac{T_1}{T})$$

For B_K:

$$B_K = \frac{2}{T} \cdot \int_0^T x(t) sin k\omega_0 t dt$$

$$= \frac{2}{T} \cdot \int_0^{T_1} M sin k\omega_0 t dt$$

$$= \frac{2M}{T} \cdot [\frac{-cos k\omega_0 t}{k\omega_0 t}]$$

$$= \frac{2M}{T} \cdot [\frac{-cos k\frac{2\pi}{T} T_1}{K\frac{2\pi}{T}} + \frac{1}{K\frac{2\pi}{T}}]$$

$$B_K = \frac{M}{K\pi} \cdot [1 - cos(2\pi K \frac{T_1}{T})]$$

3. (a) q3a.jpeg

   (b)

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$

The Euler formula provides us the representation of a periodic function in terms of harmonically related complex exponentials, as follows;

$$x(t) = 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) + \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t})$$

Thus, the Fourier series coefficients of the function are,

$$a_0 = 1,\ a_2 = \frac{1}{4},\ a_4 = \frac{1}{2},\ a_6 = \frac{1}{3},\ a_k = 0\ for\ k \neq 0, 2, 4, 6.$$

The plot of the spectral coefficients,

   (c) Recall that, each spectral coefficient, a_k shows the amount of the corresponding harmonic frequency in the signal, $x(t)$. For small periods, the signal, $x(t)$ has relatively less low frequency components, compared to the other signals. As we increase the period of the signal, the low frequency components increase and the rate of change of the spectral coefficients decrease. Investigation of the behaviour of the spectral coefficients show the amount of each frequency component relative to each other, which makes the signal. This is why we call the plot, $a_k vs. k$ as spectrum, meaning the band of frequencies, in $x(t)$.

   (d)

4. (a)

   (b)

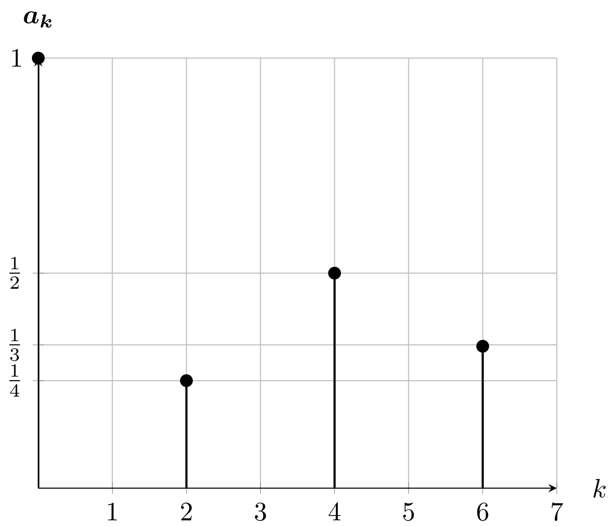

Figure 6: k vs. a_k.

5. (a) The Discrete Fourier Series pair is:

$$x[n] = \sum_{k=<n>} a_k \cdot e^{jk\frac{2\pi}{N}}$$

$$a_k = \frac{1}{N} \sum_{k=<n>} x[n] \cdot e^{-jk\frac{2\pi}{N}}$$

Given $x[n] = sin(\frac{\pi}{2} \cdot n)$, here $N = 4$.

$$x[n] = \frac{e^{j\frac{\pi}{4}n} - e^{j\frac{\pi}{4}n}}{2j}$$

$$a_1 = \frac{1}{2j}e^j$$

$$a_{-1} = -\frac{1}{2j} \cdot e^{-j} \quad \text{and the rest is zero.}$$

(b) The Discrete Fourier Series pair is:

$$x[n] = \sum_{k=<n>} a_k \cdot e^{jk\frac{2\pi}{N}}$$

$$a_k = \frac{1}{N} \sum_{k=<n>} x[n] \cdot e^{-jk\frac{2\pi}{N}}$$

Given $y[n] = 1 + cos(\frac{\pi}{2} \cdot n)$, here $N = 4$.

$$y[n] = 1 + \frac{e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}}{2}$$

$$b_0 = 1$$

$$b_1 = \frac{1}{2}$$

$$b_{-1} = \frac{1}{2} \quad \text{and the rest is zero.}$$

(c)

$$z[n] = x[n] \cdot y[n]$$

From multiplication property, the Fourier coefficients of $z[n]$ convolution of Fourier series coefficients of $x[n]$ and $y[n]$.

$$c_0 = a_0 \cdot b_0 + a_1 \cdot b_{-1} + a_2 \cdot b_{-2} + a_3 \cdot b_{-3} + a_4 \cdot b_{-4} + a_5 \cdot b_{-5}$$

$$c_0 = \frac{1}{4j}$$

We have non-zero values at a_0, a_1, b_0, b_{-1}.

(d)

$$
\begin{aligned}
z[n] &= x[n] \cdot y[n] \\
&= sin(\frac{\pi}{2}n) \cdot [1 + cos(\frac{\pi}{2}n)] \\
&= sin(\frac{\pi}{2}n) + sin(\frac{\pi}{2}n) \cdot cos(\frac{\pi}{2}n) \\
** \, sin(\alpha) \cdot cos(\beta) &= \frac{1}{2} \cdot [sin(\alpha+\beta) + sin(\alpha-\beta)] \\
z[n] &= sin(\frac{\pi}{2}n) + \frac{1}{2} \cdot [sin(\pi n) + sin(0)] \\
z[n] &= sin(\frac{\pi}{2}n) \\
z[n] &= \frac{e^{j\frac{\pi}{2}n} - e^{j\frac{\pi}{2}n}}{2j}
\end{aligned}
$$

From the part a) :

$$
\begin{aligned}
c_1 &= \frac{1}{2j} e^{j} \\
c_{-1} &= -\frac{1}{2j} \cdot e^{-j}
\end{aligned}
$$

6.

$$\omega_0 = \frac{\pi}{6}, \; N = \frac{2\pi}{\omega_0}$$

$$N = 12$$

$$a_k = \frac{1}{12} \sum_{n=0}^{11} x[n] e^{-jk(\frac{\pi}{6})n}$$

$$a_k = \cos\frac{k\pi}{6} + \sin\frac{5k\pi}{6}$$

$$a_k = \frac{1}{2}e^{j\frac{k\pi}{6}} + \frac{1}{2}e^{-j\frac{k\pi}{6}} + \frac{1}{2j}e^{j\frac{5k\pi}{6}} - \frac{1}{2j}e^{-j\frac{5k\pi}{6}}$$

Hence by comparing equations we can write x[n].

$$x[n] = 6\delta[n-1] + 6\delta[n-11] - 6j\delta[n-5] + 6j\delta[n-7], \; 0 \le n \le 11$$

7. (a) To find Fourier series coefficients of $x[n]$ we will use following formula for 1 period:

$$a_k = \tfrac{1}{N} \textstyle\sum_{n=<N>} x[n] e^{-jkw_0 n}$$

Now we need N and $w_0 = \frac{2\pi}{N}$. From the graph of $x[n]$ we can deduce that $N = 4$ and $w_0 = \frac{\pi}{2}$. To find the Fourier Series coefficients:

$$
\begin{aligned}
a_k &= \frac{1}{4} \sum_{n=0}^{3} x[n] e^{-jk\frac{\pi}{2}n} \\
&= \frac{1}{4}(x[0] + x[1]e^{-jk\frac{\pi}{2}} + x[2]e^{-2jk\frac{\pi}{2}} + x[3]e^{-3jk\frac{\pi}{2}}) \\
&= 0 + \frac{1}{4}(cos(k\frac{\pi}{2}) - jsin(k\frac{\pi}{2})) + \frac{1}{2}(cos(k\pi) - jsin(k\pi)) + \frac{1}{4}(cos(3k\frac{\pi}{2}) - jsin(3k\frac{\pi}{2}))
\end{aligned}
$$

From above:

$$
\begin{aligned}
a_0 &= \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1 \\
a_1 &= -\frac{j}{4} - \frac{1}{2} + \frac{j}{4} = -\frac{1}{2} \\
a_2 &= -\frac{1}{4} + \frac{1}{2} - \frac{1}{4} = 0 \\
a_3 &= \frac{j}{4} - \frac{1}{2} - \frac{j}{4} = -\frac{1}{2}
\end{aligned}
$$

In conclusion we can say: $a_n = a_{n+4} = a_{n-4}$, the other coefficients can be found using this.
We can plot the magnitude spectrum of the coefficients:

And the phase spectrum of the coefficients:

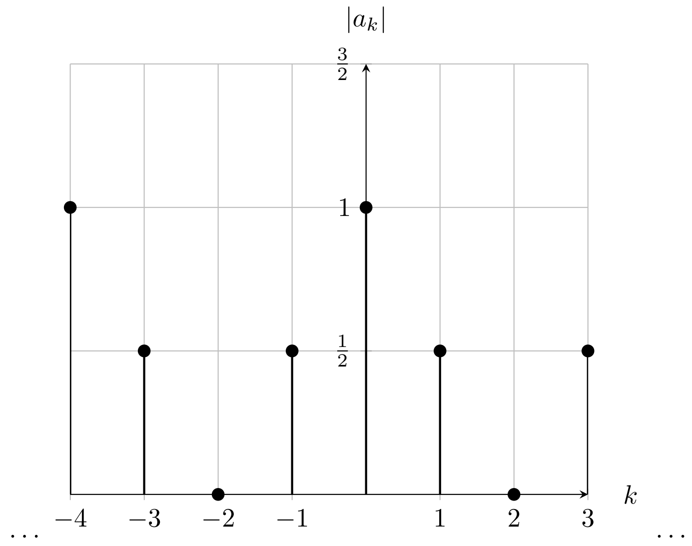

Figure 7: k vs. $|a_k|$.

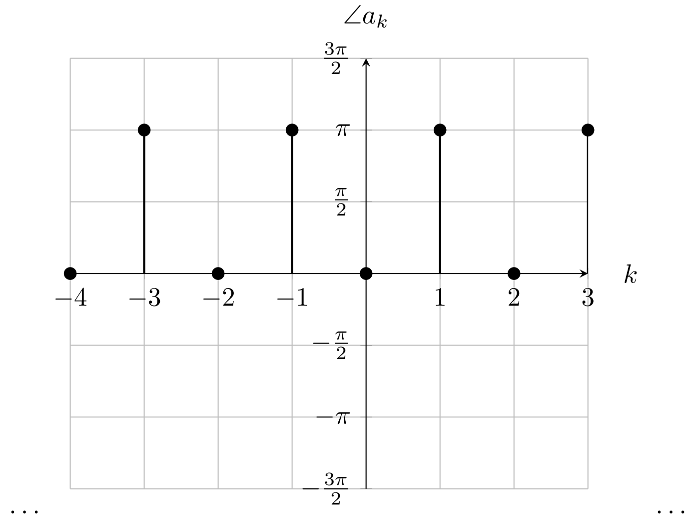

Figure 8: k vs. $\angle a_k$.

(b) i. We need to add a negative impulse (i.e $-\delta(n)$) at every $n+1$ th point to ensure periodicity of $y[n]$. So we would end up something such as:

$$y[n] = x[n] - \sum_{k=-\infty}^{\infty} \delta(n+1-4k)$$

ii. To find Fourier series coefficients of $y[n]$:

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jkw_0 n}$$

And to be able to find them using this formula we need N and $w_0 = \frac{2\pi}{N}$. From the graph of $y[n]$ we can see that $N = 4$ and $w_0 = \frac{\pi}{2}$. To find the Fourier Series coefficients:

$$\begin{aligned} a_k &= \frac{1}{4} \sum_{n=0}^{3} x[n] e^{-jk\frac{\pi}{2}n} \\ &= \frac{1}{4}(x[0] + x[1]e^{-jk\frac{\pi}{2}} + x[2]e^{-2jk\frac{\pi}{2}} + x[3]) \\ &= 0 + \frac{1}{4}(cos(k\frac{\pi}{2}) - jsin(k\frac{\pi}{2})) + \frac{1}{2}(cos(k\pi) - jsin(k\pi)) + 0 \end{aligned}$$

From above:

$$\begin{aligned} a_0 &= \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \\ a_1 &= -\frac{j}{4} - \frac{1}{2} = -\frac{1}{4}(j+2) \\ a_2 &= -\frac{1}{4} + \frac{1}{2} = \frac{1}{4} \\ a_3 &= \frac{j}{4} - \frac{1}{2} = \frac{1}{4}(j-2) \end{aligned}$$

From periodicity we can say: $a_n = a_{n+4} = a_{n-4}$, so other coefficients can be found from this.

The magnitude spectrum of the coefficients: The phase spectrum of the coefficients:

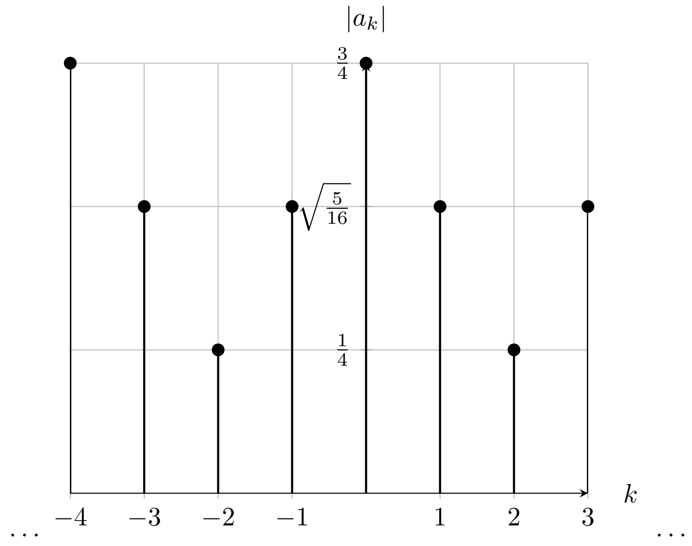

Figure 9: k vs. $|a_k|$.

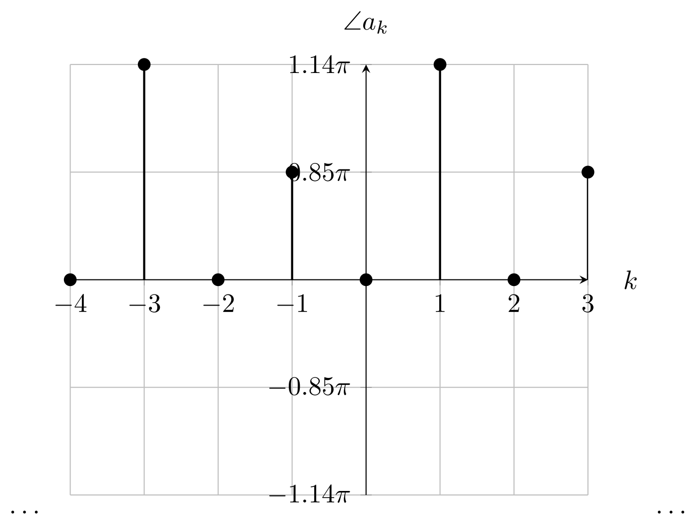

Figure 10: k vs. $\angle a_k$.