CENG 384 - Signals and Systems for Computer Engineers
20202
Written Assignment 3 Solutions

June 15, 2021

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1. (a)

$$x(t) = \frac{1}{2} + \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$
$$a_0 = \frac{1}{2}, \quad a_1 = \frac{1}{2}, \quad a_{-1} = \frac{1}{2}$$

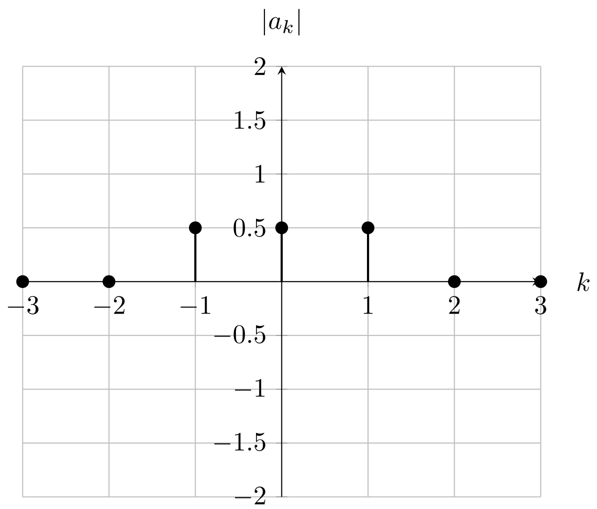


Figure 1: $|a_k|$ vs. k for $x(t)$

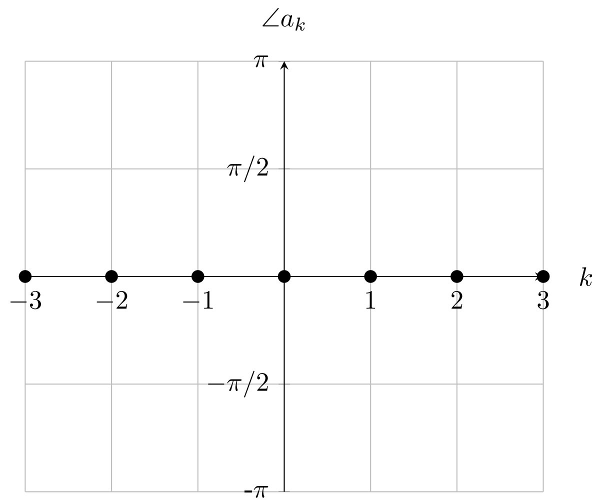


Figure 2: $\angle a_k$ vs. k for $x(t)$

(b)

$$y(t) = \frac{3}{2} + \frac{1}{j}e^{j\omega_0 t} - \frac{1}{j}e^{-j\omega_0 t}$$
$$a_0 = \frac{3}{2}, \quad a_1 = \frac{1}{j} = -j, \quad a_{-1} = \frac{-1}{j} = j$$

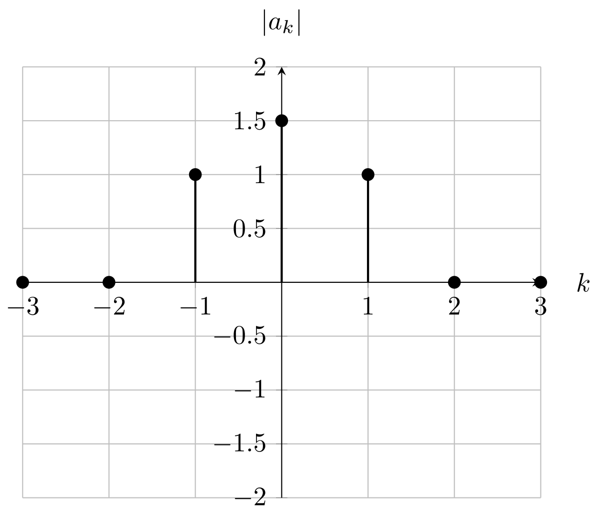


Figure 3: $|a_k|$ vs. k for $x(t)$

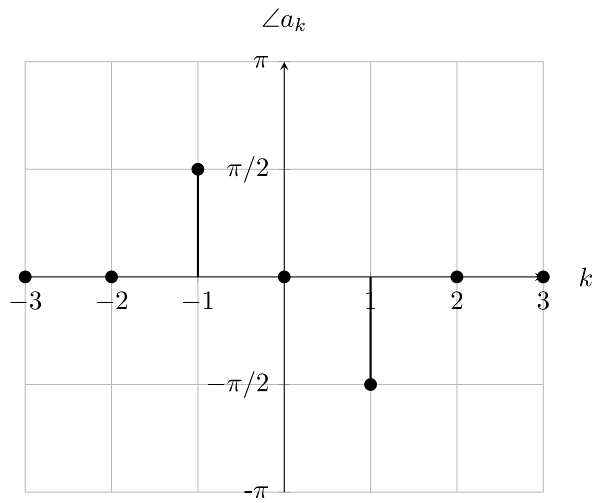


Figure 4: $\angle a_k$ vs. k for $x(t)$

(c)

$$
\begin{aligned}
z(t) &= 2+\cos\omega_0 t+2\sin\omega_0 t+\cos(2\omega_0 t+\frac{\pi}{4})\\
&= 2+\left(\frac{1}{2}+\frac{1}{j}\right)e^{j\omega_0 t}+\left(\frac{1}{2}-\frac{1}{j}\right)e^{-j\omega_0 t}\\
&\quad+\left(\frac{1}{2}e^{j\frac{\pi}{4}}\right)e^{j2\omega_0 t}+\left(\frac{1}{2}e^{-j\frac{\pi}{4}}\right)e^{-j2\omega_0 t}
\end{aligned}
$$

$$
a_0=2,\quad a_1=\left(\frac{1}{2}+\frac{1}{j}\right)=\left(\frac{1}{2}-j\right),\quad a_{-1}=\left(\frac{1}{2}-\frac{1}{j}\right)=\left(\frac{1}{2}+j\right),\quad a_2=\frac{\sqrt{2}}{4}(1+j),\quad a_{-2}=\frac{\sqrt{2}}{4}(1-j)
$$

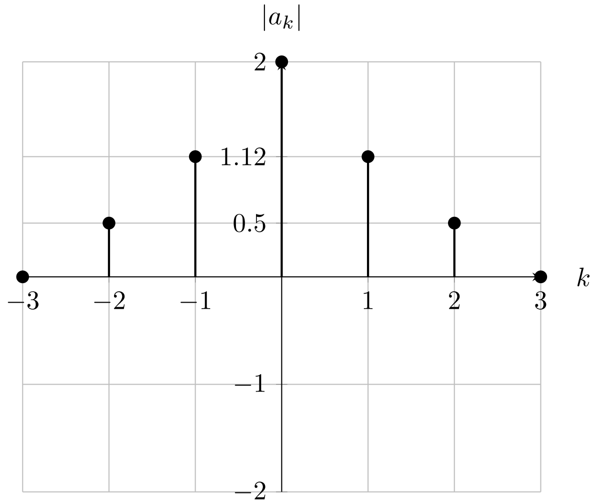


Figure 5: $|a_k|$ vs. k for $x(t)$

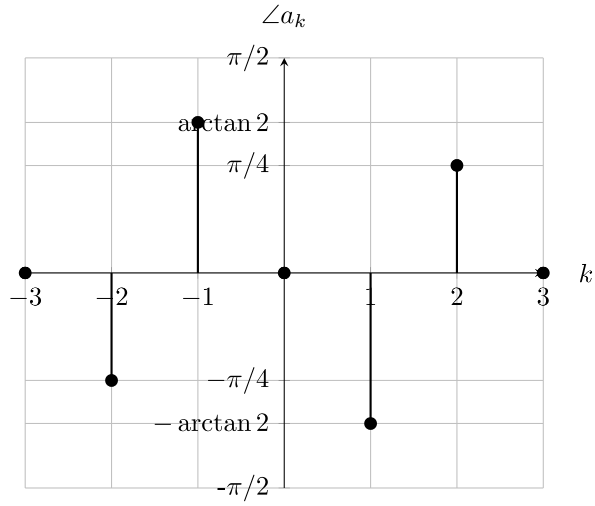


Figure 6: $\angle a_k$ vs. k for $x(t)$

2.

$$
\begin{aligned}
A_0 &= \frac{2}{T}\int_0^T x(t)dt=\frac{2}{T}\int_0^{T_1} M dt\\
&= \frac{2}{T}\left(Mt\Big|_0^{T_1}\right)\\
&= \frac{2MT_1}{T}
\end{aligned}
$$

$$
\begin{aligned}
A_k &= \frac{2}{T}\int_0^T x(t)\cos k\omega_0 t dt=\frac{2}{T}\int_0^{T_1} M\cos k\omega_0 t dt\\
&= \frac{2M}{T}\left[\frac{1}{k\omega_0}\sin k\omega_0 t\right]_0^{T_1}\\
&= \frac{2M}{T}\frac{T}{2\pi k}\sin\frac{2\pi k T_1}{T}\\
&= \frac{M}{k\pi}\sin\frac{2\pi k T_1}{T}
\end{aligned}
$$

$$
\begin{aligned}
B_k &= \frac{2}{T}\int_0^T x(t)\sin k\omega_0 t dt=\frac{2}{T}\int_0^{T_1} M\sin k\omega_0 t dt\\
&= \frac{2M}{T}\left[-\frac{1}{k\omega_0}\cos k\omega_0 t\right]_0^{T_1}\\
&= \frac{2M}{T}\frac{T}{2\pi k}\left[1-\cos\frac{2\pi k T_1}{T}\right]\\
&= \frac{M}{k\pi}\left[1-\cos\frac{2\pi k T_1}{T}\right]
\end{aligned}
$$

3. (a) Here is the figure for $x(t)$:

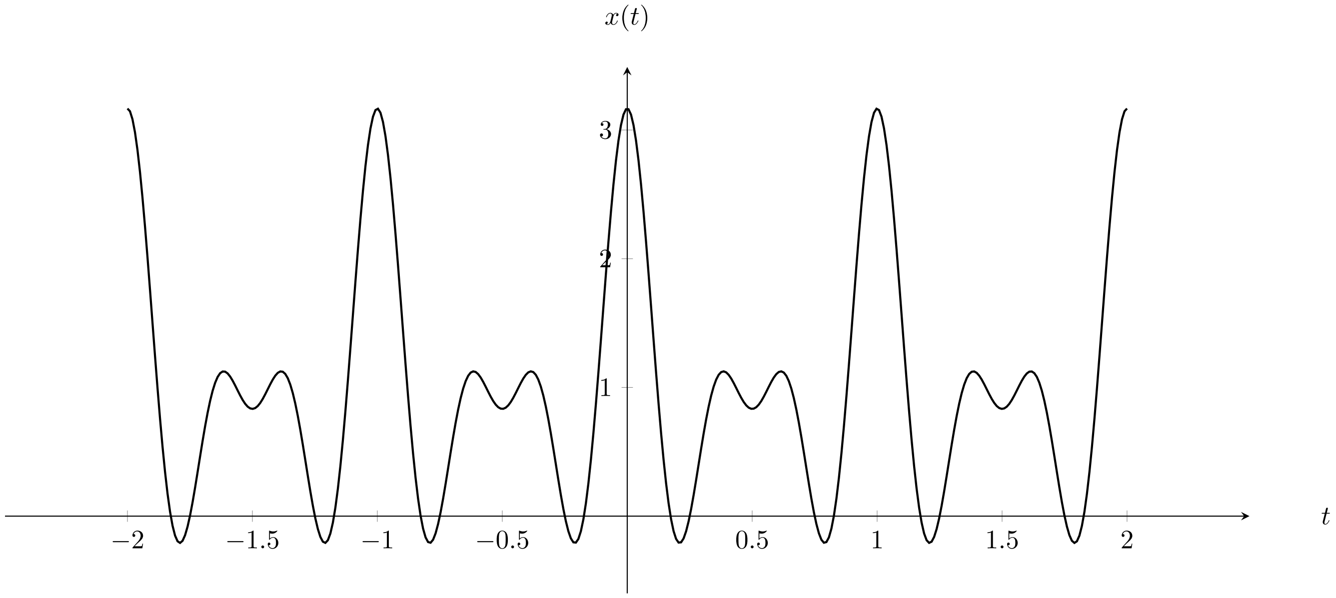


Figure 7: $x(t)$ vs. t

(b)

$$x(t) = 1 + \frac{1}{4}(\underbrace{e^{j2\pi t} + e^{-j2\pi t}}_{T=1}) + \frac{1}{2}(\underbrace{e^{j4\pi t} + e^{-j4\pi t}}_{T=1/2}) + \frac{1}{3}(\underbrace{e^{j6\pi t} + e^{-j6\pi t}}_{T=1/3})$$

$$T_{overall} = 1 \quad \text{so} \quad \omega_0 = 2\pi$$

$$a_0 = 1, \quad a_1 = a_{-1} = \frac{1}{4}, \quad a_2 = a_{-2} = \frac{1}{2} \quad a_3 = a_{-3} = \frac{1}{3}$$

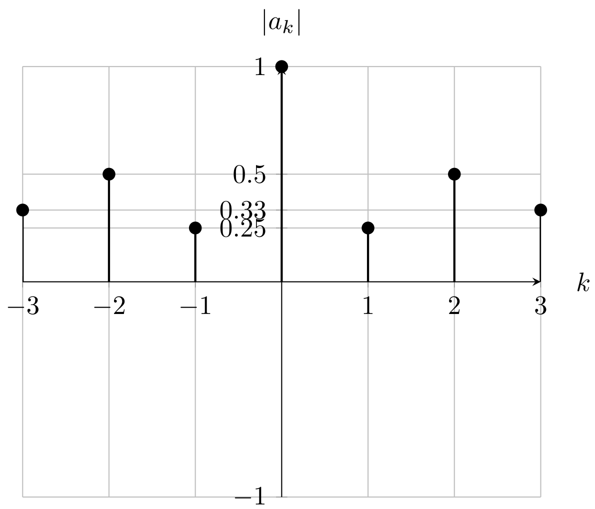


Figure 8: $|a_k|$ vs. k for $x(t)$

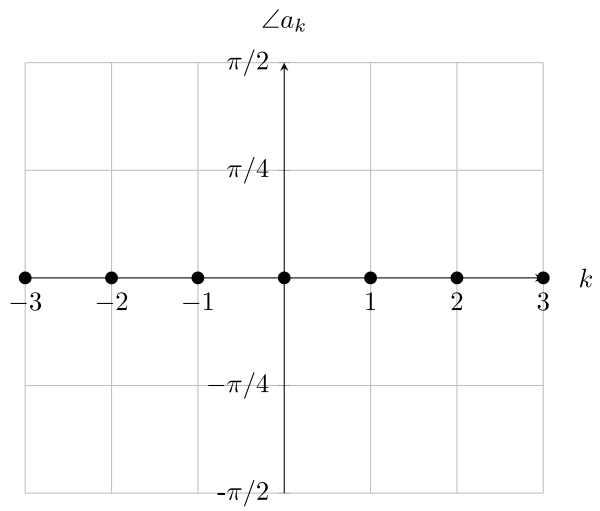


Figure 9: $\angle a_k$ vs. k for $x(t)$

(c) Although the fundamental period of $x(t)$ is 1, it is a superposition of signals some of which have fundamental periods smaller than 1. a_1, a_{-1}, a_2, a_{-2}, a_3 and a_{-3} correspond to these signals. As they are not zero, they indicate signals with angular frequencies that are twice or three times the angular frequency of $x(t)$, or equivalently, signals with fundamental periods that are a half or a third of the fundamental period of $x(t)$.

(d)

$$\begin{aligned} H(j\omega) &= \int_0^\infty e^{-2\tau} e^{-j\omega\tau} d\tau \\ &= -\frac{1}{2+j\omega} e^{-2\tau} e^{-j\omega\tau} \Big|_0^\infty \\ &= \frac{1}{2+j\omega} \end{aligned}$$

We know from chapter 3.8

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

Therefore, using the equations for $H(j\omega)$ and $y(t)$, together with the fact that $\omega_0 = 2\pi$, we get

$$y(t) = \sum_{k=-3}^{3} b_k e^{jk2\pi t},$$

with $b_k = a_k H(jk2\pi)$, so that

$$b_0 = \frac{1}{2},$$

$$b_1 = \frac{1}{4}\left(\frac{1}{2+j2\pi}\right), \quad b_{-1} = \frac{1}{4}\left(\frac{1}{2-j2\pi}\right),$$

$$b_2 = \frac{1}{2}\left(\frac{1}{2+j4\pi}\right), \quad b_{-2} = \frac{1}{2}\left(\frac{1}{2-j4\pi}\right),$$

$$b_3 = \frac{1}{3}\left(\frac{1}{2+j6\pi}\right), \quad b_{-3} = \frac{1}{3}\left(\frac{1}{2-j6\pi}\right).$$

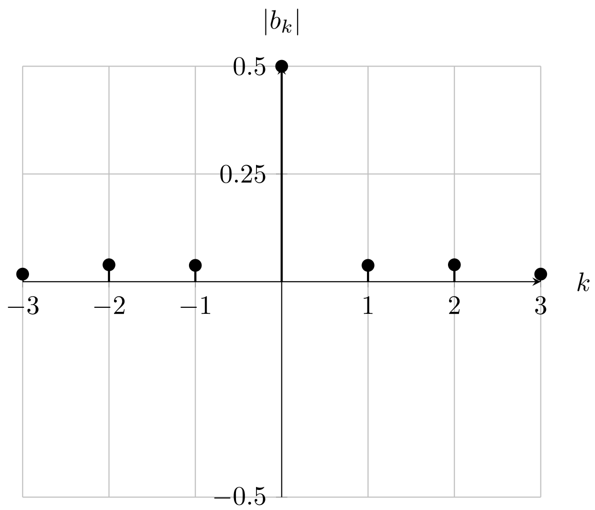


Figure 10: $|b_k|$ vs. k for $y(t)$

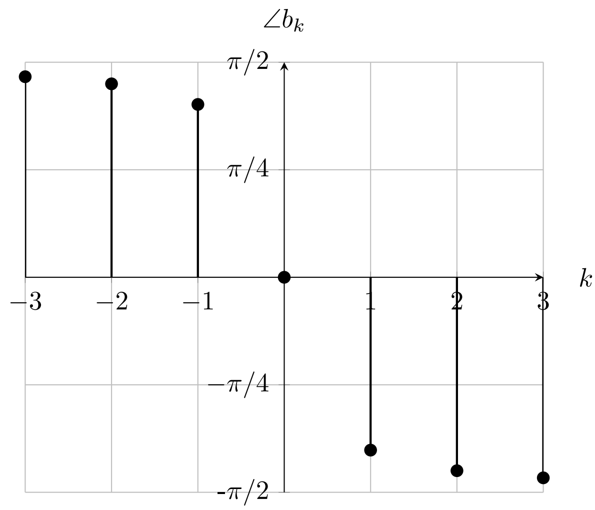


Figure 11: $\angle b_k$ vs. k for $y(t)$

4. (a) Time shifting and time reversal do not change the fundamental period. We also use the linearity property here.

$$x(t-3) \overset{\text{FS}}{\longleftrightarrow} a_k e^{-jk(2\pi/T)3}$$

and

$$x(-t) \overset{\text{FS}}{\longleftrightarrow} a_{-k}.$$

So

$$\frac{1}{3}x(t-3) - \frac{2}{7}x(-t) \overset{\text{FS}}{\longleftrightarrow} \frac{1}{3}a_k e^{-3jk(2\pi/T)} - \frac{2}{7}a_{-k}.$$

(b)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Differentiation property is the result of taking the first derivative:

$$\frac{dx(t)}{dt} = jk\omega_0 \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Take the second derivative:

$$\frac{d^2x(t)}{dt^2} = j^2k^2\omega_0^2 \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Finally, take the third derivative:

$$\frac{d^3x(t)}{dt^3} = j^3k^3\omega_0^3 \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Therefore, we obtain the following result:

$$\frac{d^3x(t)}{dt^3} \overset{FS}{\longleftrightarrow} -jk^3\omega_0^3 a_k = -jk^3\left(\frac{2\pi}{T}\right)^3 a_k.$$

5. Let's first name the coefficients for each signal.

$$x[n] \overset{FS}{\longleftrightarrow} a_k$$
$$y[n] \overset{FS}{\longleftrightarrow} b_k$$
$$x[n]y[n] \overset{FS}{\longleftrightarrow} d_k$$

(a)

$$x[n] = \frac{1}{2j}(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}), \quad \omega_0 = \frac{\pi}{2}$$
$$a_1 = \frac{1}{2j} = -\frac{j}{2}, \quad a_{-1} = -\frac{1}{2j} = \frac{j}{2}.$$

(b)

$$y[n] = 1 + \frac{1}{2}(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}), \quad \omega_0 = \frac{\pi}{2}$$
$$b_0 = 1, \quad b_1 = b_{-1} = \frac{1}{2}.$$

(c) Multiplication property:

$$x[n]y[n] \overset{FS}{\longleftrightarrow} \sum_{l=<N>} a_l b_{k-l}$$

$$\begin{aligned}
d_k &= \sum_{l=0}^{3} a_l b_{k-l}, \quad \text{since} \quad N = 4 \\
&= \underbrace{\cancel{a_0 b_k}}_{\cancel{a_0}=0} + a_1 b_{k-1} + \underbrace{\cancel{a_2 b_{k-2}}}_{\cancel{a_2}=0} + a_3 b_{k-3} \\
&= a_1 b_{k-1} + \underbrace{a_3 b_{k-3}}_{a_3 = a_{-1}} \\
&= a_1 b_{k-1} + a_{-1} b_{k-3}
\end{aligned}$$

$$\begin{aligned}
d_0 &= a_1 b_{-1} + a_{-1} b_{-3} \\
&= a_1 b_{-1} + a_{-1} b_1 \\
&= \frac{1}{4j} - \frac{1}{4j} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
d_2 = d_{-2} &= a_1 b_1 + a_{-1} b_{-1} \\
&= \frac{1}{4j} - \frac{1}{4j} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
d_1 &= a_1 b_0 + \underbrace{\cancel{a_{-1} b_{-2}}}_{\cancel{b_{-2}}=0} \\
&= \frac{1}{2j} = -\frac{j}{2}
\end{aligned}$$

$$
\begin{aligned}
d_{-1} &= \underbrace{a_1 \cancel{b_{-2}}}_{\cancel{b_{-2}=0}} + \underbrace{a_{-1} b_{-4}}_{b_{-4}=b_0} \\
&= a_{-1} b_0 \\
&= -\frac{1}{2j} = \frac{j}{2}
\end{aligned}
$$

(d)

$$
\begin{aligned}
x[n]y[n] &= (\sin\frac{\pi}{2}n)(1+\cos\frac{\pi}{2}n) \\
&= \sin\frac{\pi}{2}n + \frac{1}{2}\underbrace{\cancel{\sin \pi n}}_{\cancel{\text{always } 0}} \\
&= \sin\frac{\pi}{2}n \\
&= \frac{1}{2j}(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n})
\end{aligned}
$$

$$
d_1 = \frac{1}{2j} = -\frac{j}{2}, \quad d_{-1} = -\frac{1}{2j} = \frac{j}{2}.
$$

As you can see the results are the same as the ones found in part c.

6. By Euler's Equation we have a_k as

$$
a_k = \frac{1}{2}\left(e^{jk\frac{\pi}{6}} + e^{-jk\frac{\pi}{6}}\right) + \frac{1}{2j}\left(e^{jk\frac{5\pi}{6}} - e^{-jk\frac{5\pi}{6}}\right).
$$

And we know from the analysis equation

$$
a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\omega_0 n}.
$$

Here, by inspection, we see that $N = 12$ and $\omega_0 = \frac{\pi}{6}$. Therefore we got

$$
a_k = \frac{1}{12}\sum x[n]e^{-jk\frac{\pi}{6}n}.
$$

Using these equations we will now analyze a_1 to specify $x[n]$:

$$
a_1 = \frac{1}{2}\left(e^{j\frac{\pi}{6}} + e^{-j\frac{\pi}{6}}\right) + \frac{1}{2j}\left(e^{j\frac{5\pi}{6}} - e^{-j\frac{5\pi}{6}}\right) = \frac{1}{12}x[1]e^{-j\frac{\pi}{6}} + \frac{1}{12}x[-1]e^{j\frac{\pi}{6}} + \frac{1}{12}x[5]e^{-j\frac{5\pi}{6}} + \frac{1}{12}x[-5]e^{j\frac{5\pi}{6}}
$$

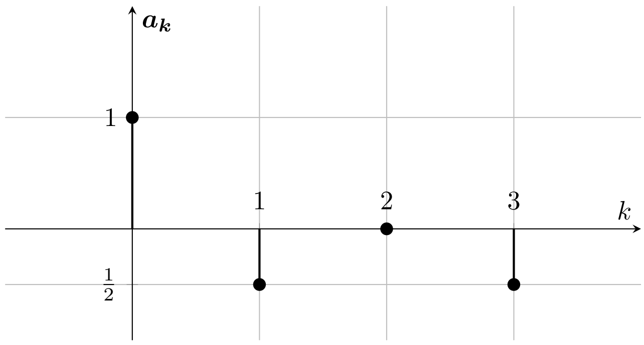
$$
x[1] = 6, \quad x[-1] = x[11] = 6, \quad x[5] = 6j, \quad x[-5] = x[7] = -6j
$$

So for $0 \le n \le 11$, we have $x[n]$ as

$$
x[n] = 6\delta[n-1] + 6j\delta[n-5] - 6j\delta[n-7] + 6\delta[n-11].
$$

7. (a) $N = 4 \quad w_0 = \frac{2\pi}{4} = \frac{\pi}{2} \quad$ so $\quad a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jkw_0 n}$

$$
\begin{aligned}
a_0 &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^0 = \frac{1}{4}[0+1+2+1] = 1 \\
a_1 &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\frac{\pi}{2}n} = -\frac{1}{2} \\
a_2 &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\pi n} = 0 \\
a_3 &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\frac{3\pi}{2}n} = -\frac{1}{2}
\end{aligned}
$$



$a_k = a_{k+N}$ so for $k > 3$, a_k will repeat with $N = 4$

The magnitude of spectral coefficients:

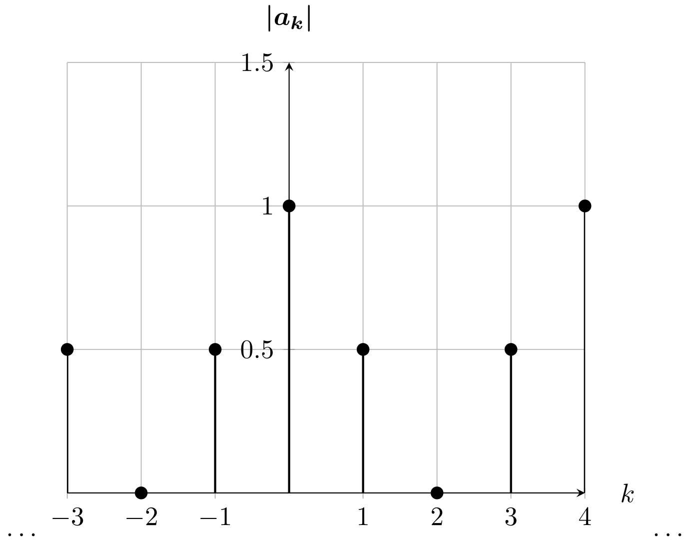


Figure 12: k vs. $|a_k|$.

Phase of the spectral coefficients:

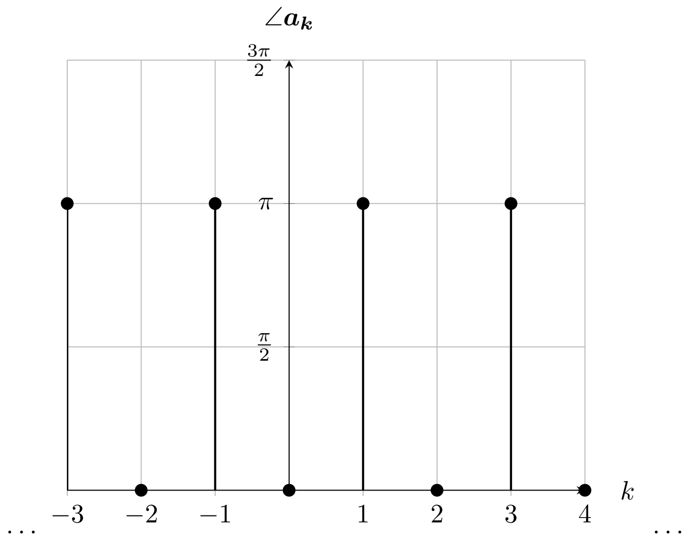


Figure 13: k vs. $\angle a_k$.

(b) i.

$$y[n] = x[n] - \sum_{k=-\infty}^{\infty} \delta[n - 3 + N \cdot k] \quad k \in Z, \;\; N = 4$$

ii.

$$a_0 = \frac{1}{4}\sum_{n=0}^{3} y[n]e^0 = \frac{1}{4}[0+1+2+0] = \frac{3}{4}$$

$$a_1 = \frac{1}{4}\sum_{n=0}^{3} y[n]e^{-j\frac{\pi}{2}n} = \frac{-j}{4} - \frac{1}{2}$$

$$a_2 = \frac{1}{4}\sum_{n=0}^{3} y[n]e^{-j\pi n} = \frac{1}{4}$$

$$a_3 = \frac{1}{4}\sum_{n=0}^{3} y[n]e^{-j\frac{3\pi}{2}n} = \frac{j}{4} - \frac{1}{2}$$

The magnitude of spectral coefficients:

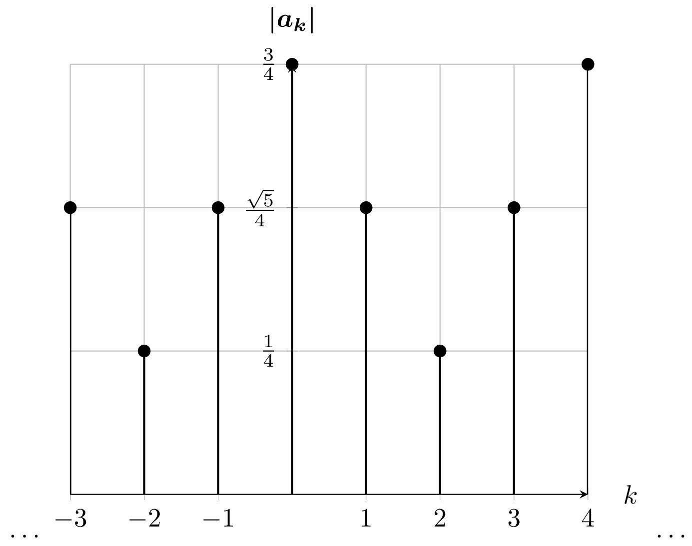


Figure 14: k vs. $|a_k|$.

Phase of spectral coefficients:

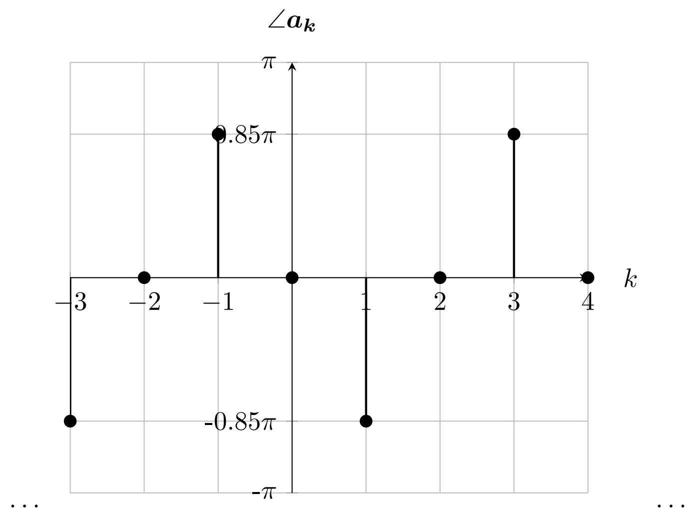


Figure 15: k vs. $\angle a_k$.