CENG 384 - Signals and Systems for Computer Engineers
Spring 2021
Homework 1

Arda, Karaman
e2237568@ceng.metu.edu.tr

Göktürk, Fulser
e2237386@ceng.metu.edu.tr

April 19, 2021

---

1. Proof of: $\frac{de^t}{dt} = e^t$
   $[e^t]' = \lim_{h\to 0} \frac{e^{t+h}-e^t}{h}$
   $[e^t]' = e^t \cdot \lim_{h\to 0} \frac{e^h-1}{h}$
   From the definition of Euler number e is the only number for: $\lim_{h\to 0} \frac{e^h-1}{h} = 1$
   Therefore $[e^t]' = e^t \cdot 1$

2. (a) i.
   $z = x + jy,\ \overline{z} = x - jy,\ z - 3 = j - 2\overline{z}$
   $x + jy - 3 = j - 2(x - jy)$
   $3(x - 1) = yj + j$
   $3x - jy = j + 3\ so\ x = 1, y = -1.$
   Since $|z|^2 = x^2 + y^2$ and $z = 1 - j$
   $|z|^2 = 2$
   ii.

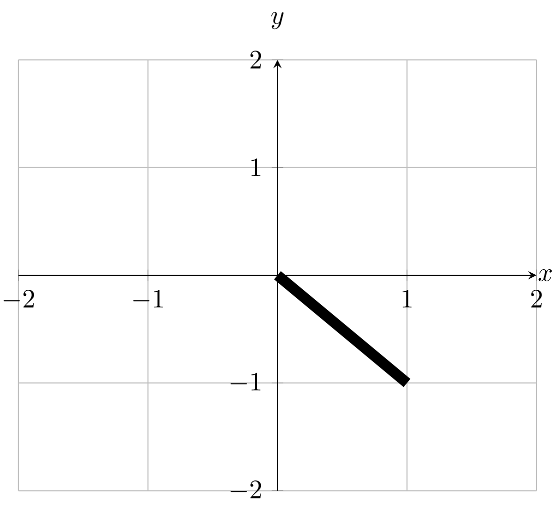

Figure 1: z on complex plane.

   (b) $z^4 = -81\ z^2 = 9i\ or\ z^2 = -9i$
   If $z^2 = 9i\ z = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i\ or\ \frac{-3\sqrt{2}}{2} + \frac{-3\sqrt{2}}{2}i;$
   it's polar form $= re^{j\theta}$
   $r = \sqrt{(\frac{3\sqrt{2}}{2})^2 + (\frac{3\sqrt{2}}{2})^2}\ or\ \sqrt{(\frac{-3\sqrt{2}}{2})^2 + (\frac{-3\sqrt{2}}{2})^2}$ both equals $r = 3$
   $\theta = \tan^{-1}(\frac{\frac{3\sqrt{2}}{2}}{\frac{3\sqrt{2}}{2}})\ or\ \tan^{-1}(\frac{\frac{-3\sqrt{2}}{2}}{\frac{-3\sqrt{2}}{2}})$ both equals $\theta = \frac{\pi}{4}$
   So it's polar form is $3e^{j\frac{\pi}{4}}$.
   If $z^2 = -9i\ z = \frac{-3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i\ or\ \frac{3\sqrt{2}}{2} + \frac{-3\sqrt{2}}{2}i;$
   it's polar form $= re^{j\theta}$
   $r = \sqrt{\frac{-3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}}\ or\ \sqrt{\frac{3\sqrt{2}}{2} + \frac{-3\sqrt{2}}{2}}$ both equals $r = 3$

$\theta = \tan^{-1}(\frac{\frac{-3\sqrt{2}}{2}}{\frac{3\sqrt{2}}{2}})$ *or* $\tan^{-1}(\frac{\frac{3\sqrt{2}}{2}}{\frac{-3\sqrt{2}}{2}})$ both equals $\theta = \frac{3\pi}{4}$
So it's polar form is $3e^{j\frac{3\pi}{4}}$.

(c) Magnitude $= \frac{\sqrt{(\frac{1}{2})^2+(\frac{1}{2})^2)}\cdot\sqrt{(1)^2+(-1)^2}}{\sqrt{(1)^2+(-\sqrt{3})^2}} = \frac{1}{2}$
Angle $= \angle(\frac{(\frac{1}{2}+\frac{1}{2}j)\cdot(1-j)}{(1-\sqrt{3}j)}) = \angle(\frac{1}{2}+\frac{1}{2}j) + \angle(1-j) - \angle(1-\sqrt{3}j) = \frac{\pi}{3}$

(d) $z = \frac{-3}{j} \cdot e^{j\frac{\pi}{2}}$
$e^{j\frac{\pi}{2}} = j$ so $z = \frac{-3j}{j}$
$z = -3$
It's polar form $= 3e^{j\pi}$

3. Below is the signal for $y(t) = 2 \cdot x(\frac{1}{2}t + 3)$

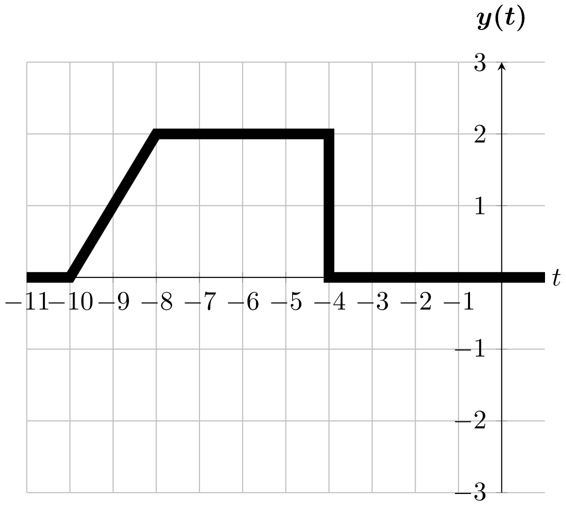

Figure 2: t vs. $x(t)$.

4. (a) Below is the signal for $x[-n] + x[2n+1]$

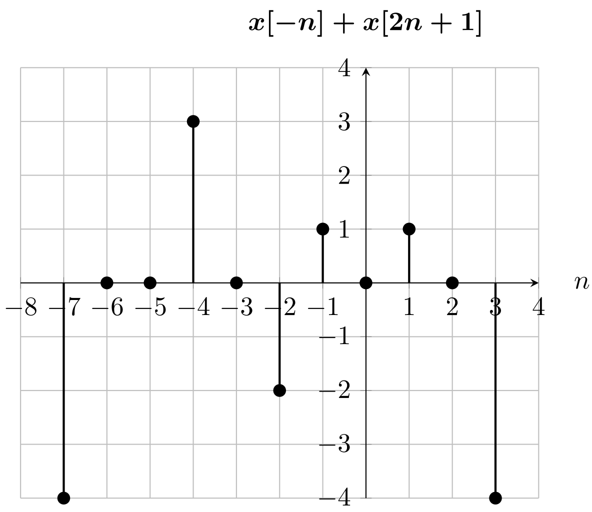

Figure 3: n vs. $x[-n] + x[2n+1]$.

(b) $x[-n] + x[2n+1]$ in terms of the unit impulse function is:

$$-4\delta(n+7) + 3\delta(n+4) - 2\delta(n+2) + \delta(n+1) + \delta(n-1) - 4\delta(n-3)$$

5. (a) $x(t) = 3\cos(7\pi t - \frac{4\pi}{5}) = x(t+T)$
$\omega = \frac{2\pi}{T}$
$\omega = 7\pi$
$7\pi = \frac{2\pi}{T}$
$T = \frac{2}{7}$
Since T equals a finite and nonzero real value, $x(t)$ is periodic and $T = \frac{2}{7}$.

(b) $x[n] = \sin[4n - \frac{\pi}{2}] = x[n + N_0]$
$\Omega = \frac{2\pi}{N_0}$
$\Omega = 4$
$4N_0 = 2\pi m$ where m is an integer.
$N_0 = \frac{\pi m}{2}$
There is no integer that satisfies this equation so it is not periodic.

(c) $x[n] = 2\cos[\frac{7\pi}{5}n] + 7\sin[\frac{5\pi}{2}n - \frac{\pi}{3}]$
$\frac{7\pi}{5}N_1 = 2\pi m_1 \quad \frac{5\pi}{2}N_2 = 2\pi m_2$
$N_1 = \frac{10m_1}{7} \quad N_2 = \frac{4m_2}{5}$
$N_1 = 10 \quad N_2 = 4$
Least Common multiplier of N_1 and N_2 is 20. So period of the signal is 20.

6. (a) Functions whose graphs are symmetric about the y-axis are called even functions.
$x(t) \neq x(-t)$
Functions with a graph that is symmetric about the origin is called an odd function.
$x(t) \neq -x(-t)$
Since $x(t)$ does not satisfy these conditions, signal $x(t)$ is neither odd nor even.

(b) We've already found whether the signal is even or odd. Now, in order to find even and odd decompositions of $x(t)$, we have:

$$x(t) = \text{Ev}\{\text{x(t)}\} + \text{Odd}\{\text{x(t)}\}$$
$$x(t) = \frac{1}{2}\{x(t) - x(-t)\} + \frac{1}{2}\{x(t) + x(-t)\}$$

So Ev{x(t)} can be drawn as:

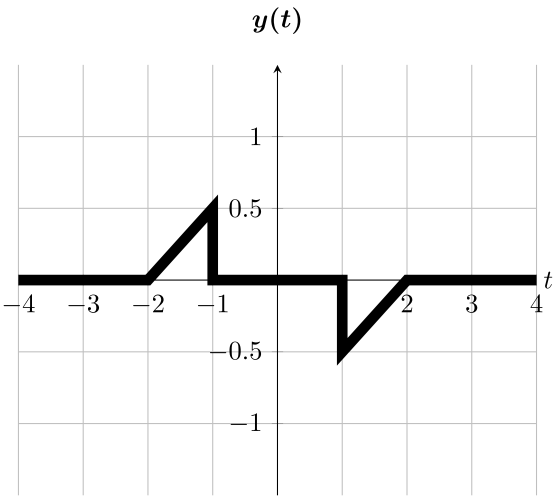

Figure 4: t vs. $x(t)$.

and Odd{x(t)} can be drawn as:

7. (a) $x(t)$ in terms of the unit step function is:
$-3\mu(t-2) + 5\mu(t-3) - 3\mu(t-5)$

(b) Note that, $\frac{d(u(t))}{dt} = \delta(t)$ So, $\frac{d}{dt}x(t) = -3\delta(t-2) + 5\delta(t-3) - 3\delta(t-5)$

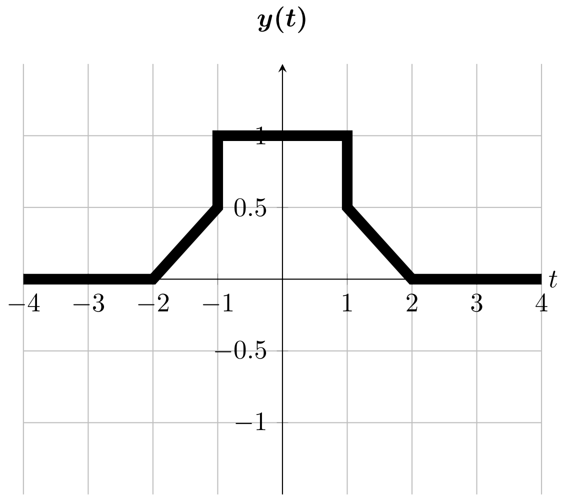

Figure 5: t vs. $x(t)$.

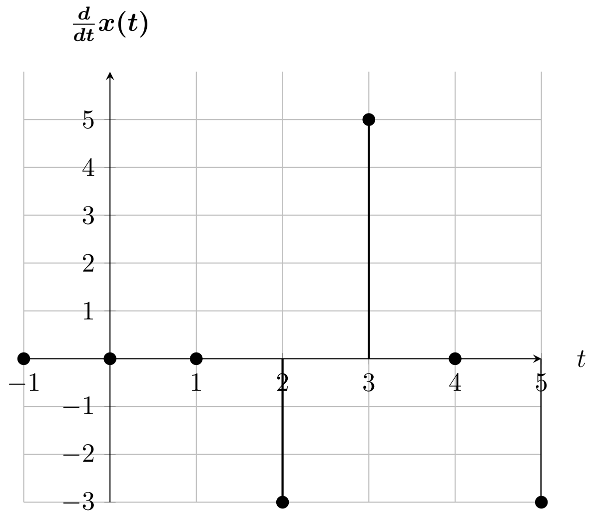

Figure 6: t vs. $\frac{d}{dt}x(t)$.

8. (a) To begin with ;
   i. Memoryless: y[n] depends on only x[n] for $\forall$.

   ii. Linearity: T$(\alpha x_1[u] + \beta x_2[u]) = \alpha T(x_1[u]) + \beta T(x_2[u])$

   iii. Time-invariance: $T(x[u]) = y[u] \iff T(x[u - n_0) = y[u - n_0]$

   iv. Causality: if $x_1[n] = x_2[n]$ for $n \leq n_0$, then $y_1[n] = y_2[n]$ for $n \leq n_0$.

   v. Stability: (BIBO) if —x[n]— $\leq \beta_x \leq \infty \Rightarrow$ —y[n]— $\leq \beta_y \leq \infty$

   i. The system has memory. Proof:
   $y[n] = x[3n - 5]$
   For $n = 1$ $y[1] = x[-2]$.

   ii. The system is stable since its amplitude is bounded and it does not varies due to input.
   iii. The system depends on only the present outputs, so this system is causal.
   iv. For a system to be linear, it needs to hold superposition property. Let x_1 and x_2 be two input signals:

$$y_1[n] = x_1[3n - 5]$$
$$y_2[n] = x_2[3n - 5]$$

   When we add these up and multiply by some constants a_1 and a_2, we will have a y_3 as:

$$\begin{aligned} y_3[n] &= a_1 \cdot y_1[n] + a_2 \cdot y_2[n] \\ &= a_1 \cdot x_1[3n - 5] + a_2 \cdot x_2[3n - 5] \end{aligned}$$

and on the other hand when we first perform addition and multiplication then put the signal as input to the system we will have a $y_3^{'}$ such as:

$$
\begin{aligned}
x_3[n] &= a_1 \cdot x_1[3n-5] + a_2 \cdot x_2[3n-5] \\
y_3^{'}[n] &= x_3[n] \\
&= a_1 \cdot x_1[3n-5] + a_2 \cdot x_2[3n-5]
\end{aligned}
$$

Since $y_3 = y_3'$ superposition property holds and system is linear.

v. The system is invertible. Proof:

$$
\begin{aligned}
x[n] &= h^{-1}(y[n]) \\
x[n] &= y[\frac{n+5}{3}]
\end{aligned}
$$

vi. Checking the time in-variance:

$$
\begin{gathered}
\text{Let } x_1[3n-5] = x[3n-n_0-5] \\
y[n] = x_1[3n-5] \\
\text{So } y[n] = x[3n-n_0-5]
\end{gathered}
$$

On the other hand we have:

$$
\begin{aligned}
y^{'}[n] &= y[n-n_0] \\
&= x[3n-3n_0-5]
\end{aligned}
$$

Since $y[n] \neq y^{'}[n]$ system is time variant.

(b) i. The system has memory. Proof:
$y(t) = x(3t-5)$
For $n = 1$ $y(1) = x(-2)$.
So it has to remember past values.Therefore system has memory.

ii. The system is stable since its amplitude is bounded and it does not varies due to input.

iii. The system depends on the present or past outputs, so this system is causal.

iv. For a system to be linear, it needs to hold superposition property. Let x_1 and x_2 be two input signals:

$$
\begin{aligned}
y_1(t) &= x_1(3t-5) \\
y_2(t) &= x_2(3t-5)
\end{aligned}
$$

When we add these up and multiply by some constants a_1 and a_2, we will have a y_3 as:

$$
\begin{aligned}
y_3(t) &= a_1 \cdot y_1(t) + a_2 \cdot y_2(t) \\
&= a_1 \cdot x_1(3t-5) + a_2 \cdot x_2(3t-5)
\end{aligned}
$$

and on the other hand when we first perform addition and multiplication then put the signal as input to the system we will have a $y_3^{'}$ such as:

$$
\begin{aligned}
x_3(t) &= a_1 \cdot x_1(3t-5) + a_2 \cdot x_2(3t-5) \\
y_3^{'}(t) &= x_3(t) \\
&= a_1 \cdot x_1(3t-5) + a_2 \cdot x_2(3t-5)
\end{aligned}
$$

Since $y_3 = y_3'$ superposition property holds and system is linear.

v. The system is invertible. Proof:

$$
\begin{aligned}
x(t) &= h^{-1}(y(n)) \\
x(t) &= y(\frac{t+5}{3})
\end{aligned}
$$

vi. Checking the time in-variance:

$$
\begin{gathered}
\text{Let } x_1(t) = x(3t-t_0-5) \\
y(t) = x_1(3t-5) \\
\text{So } y(t) = x(3t-t_0-5)
\end{gathered}
$$

On the other hand we have:

$$
\begin{aligned}
y^{'}(t) &= y(t-t_0) \\
&= x(3t-3t_0-5)
\end{aligned}
$$

Since $y(t) \neq y^{'}(t)$ system is time variant.

(c) i. The system has memory. Proof:
$y(t) = tx(t-1)$
For $n = 1$ $y(1) = x(0)$.
So it has to remember past values.Therefore system has memory.

ii. The system is not stable since its amplitude is not bounded and it varies due to input.

iii. The system depends on the present or past outputs, so this system is causal.

iv. For a system to be linear, it needs to hold superposition property. Let x_1 and x_2 be two input signals:

$$y_1(t) = t \cdot x_1(t-1)$$
$$y_2(t) = t \cdot x_2(t-1)$$

When we add these up and multiply by some constants a_1 and a_2, we will have a y_3 as:

$$\begin{aligned} y_3(t) &= a_1 \cdot y_1(t) + a_2 \cdot y_2(t) \\ &= a_1 \cdot t \cdot x_1(t-1) + a_2 \cdot t \cdot x_2(t-1) \end{aligned}$$

and on the other hand when we first perform addition and multiplication then put the signal as input to the system we will have a $y_3^{'}$ such as:

$$\begin{aligned} x_3(t) &= a_1 \cdot x_1(t-1) + a_2 \cdot x_2(t-1) \\ y_3^{'}(t) &= t \cdot x_3(t-1) \\ &= a_1 \cdot t \cdot x_1(t-1) + a_2 \cdot t \cdot x_2(t-1) \end{aligned}$$

Since $y_3 = y_3'$ superposition property holds and system is linear.

v. The system is not invertible. Proof:

$$\begin{aligned} x(t) &\neq h^{-1}(y(t)) \\ x(t) &\neq \frac{y(t+1}{t} \\ &For\ example\ for\ t = 2; \\ &x(2) = \frac{y(3)}{2}\ but\ y(3) = 3 \cdot x(2). \end{aligned}$$

Therefore the system is not invertible.

vi. Checking the time in-variance:

$$\text{Let } x_1(t-1) = x(t-t_0-1)$$
$$y(t) = t \cdot x_1(t-1)$$
$$\text{So } y(t) = t \cdot x(t-t_0-1)$$

On the other hand we have:

$$\begin{aligned} y^{'}(t) &= y(t-t_0) \\ &= (t-t_0) \cdot x(t-t_0-1) \end{aligned}$$

Since $y(t) \neq y^{'}(t)$ system is time variant.

(d) i. The system has memory. Proof:
$y[n] = \sum_{k=1}^{\infty} x[n-k]$
For $n = 1$ $y[1] = x[0] + x[-1] + \ldots$
So it has to remember past values.Therefore system has memory.

ii. The system is not stable since its amplitude is not bounded and it varies due to input.

iii. The system depends on the present or past outputs, so this system is causal.