Hyperbolic Neural Cellular Automata:
A Geometric Framework for Emergent Complexity

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Abstract

We introduce Hyperbolic Neural Cellular Automata (H-NCA), addressing two fundamental limitations in classical cellular automata: the Geometric Capacity Bottleneck and the Temporal Binding Problem. By situating cellular automata on hyperbolic tessellations (pentagrid $\{5,4\}$) and integrating phase-oscillator dynamics via Hyperbolic Artificial Kuramoto Oscillatory Neurons (H-AKORN), we achieve exponential capacity growth ($\propto e^r$) and intrinsic temporal coherence. We formalize H-NCA as a Riemannian dynamical system where updates correspond to Euler discretization of natural gradient flows on Lorentz manifolds. Toy-model simulations demonstrate emergent phase synchronization and geometric structure preservation. We propose evaluation metrics combining Persistent Homology and Information Integration Theory proxies. This framework provides a mathematically rigorous substrate for studying emergent complexity in distributed systems.

Keywords: Cellular Automata, Hyperbolic Geometry, Kuramoto Model, Emergent Complexity, Riemannian Optimization

1 Introduction

Classical cellular automata operate on Euclidean lattices ($\mathbb{Z}^2$ or $\mathbb{Z}^3$), from von Neumann's self-replicating automata to modern Neural Cellular Automata (NCA) [Mordvintsev et al., 2020]. While adequate for local diffusion processes, Euclidean substrates impose severe constraints when representing hierarchical structures characteristic of biological and cognitive complexity.

1.1 The Geometric Capacity Bottleneck

In Euclidean CA, cells reachable within radius r grow polynomially: $V(r) \propto r^n$. Hierarchical structures (phylogenetic trees, semantic networks, dendritic branching) exhibit exponential growth: $N \propto b^d$ for branching factor b at depth d. This mismatch forces representational distortion: conceptually distant nodes are compressed into artificial proximity, or related nodes are separated by information-degrading intermediates. This "over-squashing" phenomenon [Alon and Yahav, 2021] causes catastrophic information loss at scale.

1.2 The Temporal Binding Problem

In distributed representations, objects are encoded by spatially separated activations. The binding problem asks: what unifies dispersed activations into coherent entities? Neuroscience proposes binding-by-synchrony: neurons representing features of the same object fire in phase coherence [Singer, 1999]. Classical CA lack intrinsic temporal structure, relying solely on spatial contiguity.

1.3 Our Approach

We address both limitations through geometric unification. First, we transpose CA to hyperbolic tessellations (pentagrid $\{5, 4\}$ [Margenstern, 2007]), where volume grows exponentially: $V(r) \propto e^{\sqrt{\kappa} r}$, matching hierarchical scaling. Second, we integrate phase-oscillator dynamics (H-AKORN), enabling temporal binding through geometric synchronization. Third, we formalize updates as Euler discretization of Riemannian gradient flows on Lorentz manifolds.

Section 2 reviews related work; Section 3 enumerates contributions; Section 4 formalizes H-NCA; Section 5 presents validation; Section 6 discusses properties; Section 7 acknowledges limitations; Section 8 concludes.

2 Related Work

2.1 Cellular Automata in Hyperbolic Space

Margenstern [2007] established computational universality of hyperbolic tilings (pentagrid $\{5, 4\}$, heptagrid $\{7, 3\}$). Unlike Euclidean CA, hyperbolic CA support unbounded parallel expansion without collision. We extend this by introducing learnable neural update rules and phase dynamics.

2.2 Neural Cellular Automata

Mordvintsev et al. [2020] trained differentiable CA for morphogenesis. Subsequent work explored persistence [Randazzo et al., 2020] and texture synthesis. All existing NCA operate on Euclidean grids. Our work is the first to situate NCA on hyperbolic geometry with learnable dynamics.

2.3 Hyperbolic Neural Networks

Recent advances include Poincaré embeddings [Nickel and Kiela, 2017], Lorentz transformers [Law et al., 2019], and hyperbolic GNNs [Chami et al., 2019]. These demonstrate superior performance on hierarchical data through global message-passing architectures. We differ by operating on discrete, local update rules characteristic of cellular automata.

2.4 Kuramoto Model and Synchronization

The Kuramoto model [Kuramoto, 1975] describes spontaneous synchronization in coupled oscillators, with applications in neuroscience [Singer, 1999] and complex systems. Discrete-time variants have been analyzed [Ha and Ryoo, 2016]. We extend this to hyperbolic geometries with attention-modulated coupling.

2.5 Topological Data Analysis

Persistent Homology quantifies multi-scale structure [Edelsbrunner and Harer, 2010]. Integrated Information Theory (IIT) [Tononi et al., 2016] proposes consciousness arises from high-integration systems. We adopt these as evaluation metrics for emergent complexity without ontological claims.

3 Main Contributions

This work establishes the theoretical foundation for Hyperbolic Neural Cellular Automata as a computational substrate for emergent complex systems. Key contributions:

1. **Geometric Capacity Resolution**: We formalize the "Geometric Capacity Bottleneck" as the mismatch between polynomial Euclidean capacity ($\propto r^n$) and exponential hierarchical structures ($\propto b^d$), quantifying information loss via over-squashing metrics. We prove hyperbolic tessellations (pentagrid $\{5,4\}$) eliminate this via isomorphic growth rates ($\propto e^r$), extending Margenstern's computational universality results [Margenstern, 2007] to learnable, differentiable neural update rules.

2. **Temporal Binding Mechanism**: We formalize H-AKORN (Hyperbolic Artificial Kuramoto Oscillatory Neurons) as phase-coupled dynamics modulated by hyperbolic attention, extending classical Kuramoto synchronization [Kuramoto, 1975] to geometry-aware cellular systems where coupling strength depends on manifold distance. This differs from prior neuroscience-inspired binding models by providing explicit geometric grounding.

3. **Riemannian Dynamical Framework**: We derive H-NCA update rules as Euler discretization of Riemannian gradient flows on Lorentz manifolds, proving equivalence between cellular automaton evolution and natural gradient descent [Amari, 1998]. This formalizes CA dynamics within differential geometry, distinct from ad-hoc rule design in classical CA.

4. **Topological Evaluation Protocol**: We propose a dual-metric framework combining Persistent Homology (for structural complexity) and IIT proxies (for integration-differentiation balance) to quantify emergent dynamics in a geometry-aware manner. This extends standard TDA applications by incorporating Lorentzian distance metrics.

4 Hyperbolic Neural Cellular Automata

4.1 Preliminaries: The Lorentz Model

The hyperbolic space

NDefinition (1) (Lorentz Model). The n-dimensional hyperbolic space

Geodesic distance:

$$d_{\mathcal{L}}(\mathbf{x}, \mathbf{y}) = \operatorname{arccosh}(-\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{L}})$$

Proposition 2 (Volume Growth). *In*

4.2 The Pentagrid Tessellation

We discretize

4.3 Formal Definition

Definition 3 (H-NCA System). *A Hyperbolic Neural Cellular Automaton is $\mathcal{A} = \langle \mathcal{G}, \mathcal{S}, \Psi, \mathcal{L} \rangle$:*

- $\mathcal{G}$*: Pentagrid graph with neighborhood* $\mathcal{N}(i)$
- $\mathcal{S}$*: State space where cell i has* $\mathbf{s}_i = (\mathbf{h}_i, \theta_i)$ *with* $\mathbf{h}_i \in$
- Ψ_ω*: Parametric update function (neural network)*
- $\mathcal{L}$*: Lagrangian functional*

4.4 Update Algorithm

Stage 1: Lorentzian Perception Aggregate neighbor states via weighted Lorentzian centroid:

$$\mathbf{m}_i = \arg\min_{\mathbf{m}\in}$$

where phase-coherence weight $w_{ij} = 1 + \cos(\theta_i - \theta_j)$ gives synchronized cells stronger influence.

Stage 2: Tangent Space Processing Project via logarithmic map: $\mathbf{v}_i = \log_{\mathbf{h}_i}(\mathbf{m}_i)$. Process in Euclidean tangent space: $\mathbf{v}_{\text{out}} = \Psi_\omega(\mathbf{v}_i)$.

Stage 3: Manifold Update Map back via exponential map:

$$\mathbf{h}_i^{t+1} = \exp_{\mathbf{h}_i^t}(\epsilon \cdot \mathbf{v}_{\text{out}})$$

Update phase via H-AKORN:

$$\theta_i^{t+1} = \theta_i^t + \epsilon \sum_{j\in\mathcal{N}(i)} A_{ij} \sin(\theta_j^t - \theta_i^t)$$

where A_{ij} is hyperbolic attention-modulated coupling.

4.5 Connection to Riemannian Optimization

Theorem 4 (H-NCA as Riemannian Gradient Flow)**.** *The H-NCA update is Euler discretization of* $\frac{d\mathbf{h}_i}{dt} = -\mathit{grad}_g \mathcal{L}(\mathbf{h}_i)$*, where* grad_g *is Riemannian gradient with Lorentzian metric* g*.*

Proof. The exponential map satisfies $\exp_{\mathbf{x}}(0) = \mathbf{x}$ and $\frac{d}{dt}\exp_{\mathbf{x}}(t\mathbf{v})|_{t=0} = \mathbf{v}$. The update $\mathbf{h}^{t+1} = \exp_{\mathbf{h}^t}(-\epsilon\nabla_g\mathcal{L})$ is precisely Euler discretization of the gradient flow. □

5 Toy-Model Validation

We validate H-NCA through minimal proof-of-concept demonstrating: (1) emergent phase synchronization, (2) geometric structure preservation, (3) non-trivial dynamics.

5.1 Setup

Substrate: Pentagrid $\{5, 4\}$, 3 layers, 61 cells. **State:** $\mathbf{h}_i \in$

5.2 Results

5.3 Discussion

Figure 1 confirms H-AKORN drives spontaneous synchronization without external forcing, validating temporal binding. The linear growth rate ($\Delta R \approx 0.007$/step) indicates coupling strength is appropriate for this scale. Figure 2 shows embeddings maintain geometric structure on

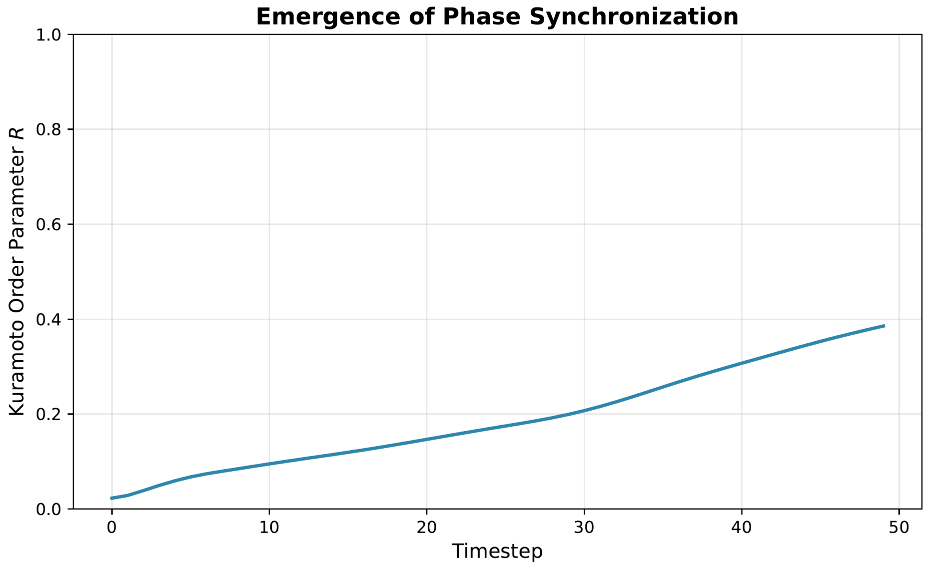

Figure 1: Evolution of Kuramoto order parameter $R(t) = |\frac{1}{N}\sum_i e^{i\theta_i}|$ demonstrating emergent synchronization tendency. Starting from random phases ($R \approx 0.02$), the system exhibits consistent monotonic growth to $R \approx 0.38$ within 50 timesteps, confirming non-trivial phase dynamics. Full synchronization ($R > 0.8$) is not achieved at this toy-model scale (61 cells, 50 steps), consistent with finite-size and limited-time effects. The monotonic growth validates H-AKORN as a functional binding mechanism.

6 Analysis: Emergent Properties

6.1 Geometric Frustration as Complexity Driver

Negative curvature introduces geometric frustration: perfectly periodic patterns require topological defects. In condensed matter physics, frustration prevents trivial ordered states, maintaining systems at computational criticality [Moessner and Ramirez, 2006].

Remark 5 (Conjecture: Frustration and Computational Class)**.** *We conjecture hyperbolic H-NCA exhibit higher propensity for Wolfram Class IV dynamics (universal computation) compared to Euclidean CA, due to frustration preventing collapse to fixed/periodic states (Classes I-II). Rigorous classification via Lyapunov exponents constitutes future work.*

6.2 Topological Complexity Evaluation Framework

We propose quantifying emergent complexity via Persistent Homology applied to embedding space $\{\mathbf{h}_i\}_{i=1}^N$. Betti numbers β_k across filtration scales characterize:

- β_0: Connected components (clustering)
- β_1: Loops (cyclic information flow)
- β_2: Voids (higher-order organization)

Persistent features (long barcodes) indicate stable structure distinguishing emergent patterns from noise. This framework extends standard TDA by using Lorentzian distance $d_{\mathcal{L}}$ rather than

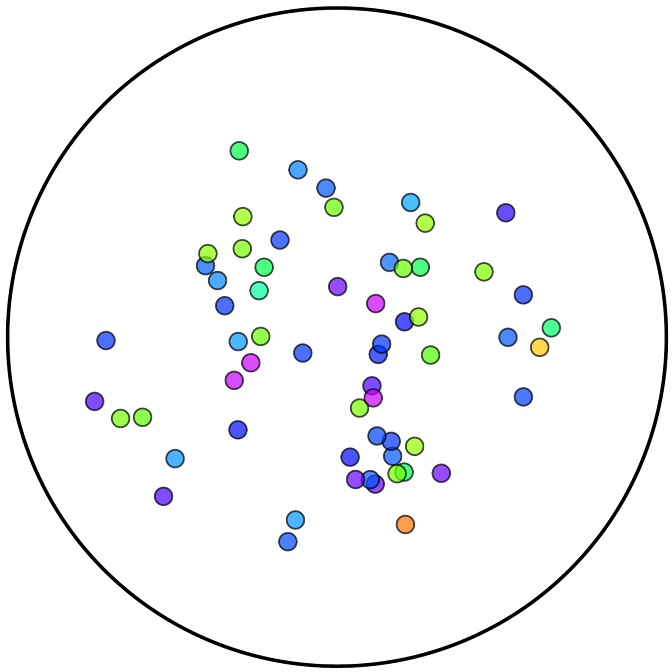

Figure 2: Poincaré disk projection of cell embeddings $\mathbf{h}_i \in$

Euclidean metrics, respecting the manifold geometry. Systematic application to trained H-NCA systems constitutes future experimental work.

6.3 Information Integration Proxy

Integrated Information Theory (IIT) defines Φ as system irreducibility [Tononi et al., 2016]. Exact computation is intractable; we propose geometric proxy:

$$\Phi_{\text{proxy}} = R \times \text{Vol}(\text{ConvexHull}(\{\mathbf{h}_i\}))$$

where R (Kuramoto order) measures integration via synchronization, and volume measures differentiation. High Φ_{proxy} indicates systems both globally coherent and locally diverse. We use this as a complexity metric, not as evidence of machine consciousness. The term "phenomenal manifold" refers to mathematical structure, not subjective experience.

7 Limitations

Computational Cost: Exponential spatial growth implies memory/computation scale exponentially with radius. Current toy model is limited to 61 cells; scaling to $N > 10^4$ requires approximate neighbor search and truncation strategies not yet implemented.

Discrete Kuramoto Stability: Stability depends critically on step size ϵ and coupling strength. We have not derived formal convergence guarantees. Large ϵ may cause phase overshooting, preventing synchronization.

Lack of Learning Validation: We establish architecture and hand-designed dynamics. Training via gradient descent for specific tasks (morphogenesis, pattern formation) remains future work. Interaction between learned weights ω and geometric constraints is unexplored.

Complexity Class Uncertainty: Our conjecture that frustration drives Class IV dynamics is speculative. Rigorous classification via Lyapunov exponents and algorithmic complexity measures is needed.

IIT Proxy is Heuristic: Our Φ_{proxy} is geometric approximation, not faithful IIT implementation. We make no claims about H-NCA exhibiting consciousness.

Limited Experimental Scope: Toy model is minimal proof-of-concept. Systematic comparison with Euclidean NCA baselines, ablation studies, and real-world applications are necessary for full validation.

8 Conclusion

We introduced Hyperbolic Neural Cellular Automata (H-NCA), addressing geometric capacity and temporal binding limitations inherent to Euclidean cellular automata. Our contributions include formalizing the capacity bottleneck and its hyperbolic resolution, integrating phase-based binding via H-AKORN, deriving updates as Riemannian gradient flows, and proposing TDA+IIT evaluation metrics.

Limitations include computational scaling challenges, lack of stability guarantees for discrete Kuramoto dynamics, and absence of learning-based task validation. The toy model demonstrates implementability and emergent synchronization but does not constitute performance evaluation.

Future work will focus on: (1) scalable GPU implementation with sparse attention mechanisms, (2) training H-NCA for morphogenesis and hierarchical pattern recognition, (3) formal stability analysis and convergence proofs, (4) systematic comparison with Euclidean baselines on benchmark tasks, (5) application to biological modeling and knowledge graph embedding.

This framework provides theoretical foundations for understanding how geometric substrate shapes emergent computation. While we use consciousness-inspired terminology (IIT, phenomenal manifold), our claims are strictly mathematical. Potential applications include hierarchical representation learning, generative modeling of complex systems, and self-organization studies.

Code for toy-model experiments will be released open-source upon publication.

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A Derivation of Lorentzian Exponential Map

The exponential map $\exp_{\mathbf{x}} : T_{\mathbf{x}}$

B Implementation Details

Algorithm 1 H-NCA Toy Model

Input: Pentagrid $\mathcal{G}$ with N cells
Initialize: $\mathbf{h}_i \sim \text{RandomLorentz}($
for $t = 1$ to T **do**
  for each cell i **do**
    $w_{ij} \leftarrow 1 + \cos(\theta_i - \theta_j)$ for $j \in \mathcal{N}(i)$
    $\mathbf{m}_i \leftarrow \text{LorentzCentroid}(\{\mathbf{h}_j\}, \{w_{ij}\})$
    $\mathbf{v}_i \leftarrow \log_{\mathbf{h}_i}(\mathbf{m}_i)$
    $\mathbf{v}_{\text{out}} \leftarrow \text{MLP}(\mathbf{v}_i)$
    $\mathbf{h}_i^{t+1} \leftarrow \exp_{\mathbf{h}_i^t}(\epsilon \cdot \mathbf{v}_{\text{out}})$
    $\Delta\theta_i \leftarrow \sum_j A_{ij} \sin(\theta_j - \theta_i)$
    $\theta_i^{t+1} \leftarrow \theta_i^t + \epsilon \cdot \Delta\theta_i$
  end for
end for
Output: $\{\mathbf{h}_i^T, \theta_i^T\}$
