How many sides does a circle have ?

Introduction

This essay discusses the question - How many sides does a circle have ?

This question follows from the property that the circle is a closed surface. But the point of difference between the circle and other closed surfaces with sides is that the circle has no distinguishable side. For that matter, the phrase - ' side of a circle' is debatable.

Through this essay I would like to prove that the circle is a regular polygon with 'n' sides such that n tends to infinity.

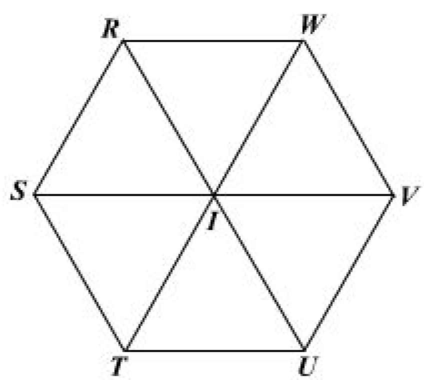
Brief idea of regular polygons, some of their properties and relation to the circle.

The simplest regular polygon is the equilateral triangle. The sum of the angles is π.

The next regular polygon is the square. The sum of the angles is 2π.

In a similar manner, the next polygon is the regular pentagon. The sum of all angles in 3π.

So, we can generalize the statement that the sum of all angles for a 'n' sided polygon is $(n-2)\pi$.



Let us now take a polygon, the regular hexagon as an example.

Using the above figure, we can deduce a few simple properties without formal proofs. These properties hold for any regular polygon. The above shown regular hexagon, as stated is purely an example to visually confirm the properties stated below.

1) There exists a centre for each and every regular polygon. In the above case it is I.

2) The distance of the centre from each and every vertex of the regular polygon is a constant. Let the constant be 'r'.
In the above diagram, RI = SI = TI = UI = VI = WI = r.

3) Any regular polygon, can be divided into triangles such that the centre of the polygon and two consecutive vertices form the vertices of the triangle. All such triangles formed are symmetric in nature.
In the above diagram, $\Delta RSI \simeq \Delta IST \simeq \Delta ITU \simeq \Delta IUV \simeq \Delta IVW \simeq \Delta IWR$.

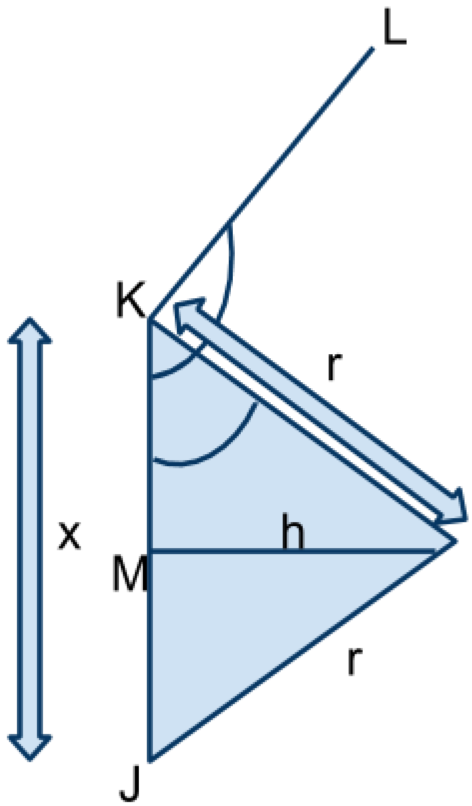
Proof

As proved above, the sum of all angles for a regular polygon is (n-2)π. It is to be noted that all angles formed at the vertices are of equal measure.Thus, the value of each angle in a regular polygon is (n-2)π/n, where n denotes the number of sides that form the regular polygon.

Let the value of each angle of a regular polygon be Θ.

$$\Theta = (n-2)\pi/n (Equation\ 1)$$

As per statement 3 above, let us consider a particular triangle that is formed by a side of the regular polygon with the centre of the polygon.



The above diagram represents a section of a regular polygon. I is the centre of the polygon. JK and KL are two consecutive sides. The polygon angle, which is the angle between two consecutive sides given by equation 1 and represented by the symbol Θ is given by angle LKJ.

As per known properties of a regular polygon, IK and IJ, the distances of the vertices from the centre are a constant measured by the value 'r'.

The length of the side of the regular polygon is given by the letter x. In the above figure, KJ = KL = x.

For the 'n' sided regular polygon, the circumference is given by 'n'x. If the circumference is represented by 'C',

$$C = nx \ (Equation\ 2)$$

By symmetry, in the above diagram, LKJ = 2IKJ = Θ. Thus, angle IKJ = Θ/2.

Let IM be the perpendicular from the center of polygon I to the side KJ. Let its measure be 'h'. By property of symmetry for isosceles triangles,
ΔIMJ $\simeq$ Δ IMK.

By using basic trigonometric properties for ΔIMK,

$$cos(\Theta/2) = x/2r (Equation\ 3)$$

Substituting the value of Θ from equation 1 in equation 3, we get,

$$cos((n-2)\pi/2n) \;=\; x/2r\,.$$

This on simplification can be written as,

$$cos(\pi/2 \;-\pi/n) \;=\; x/2r\,.$$

By using the property that cos(π /2-u)=sin(u),

$$sin(\pi/n) = \; x/2r\,.$$

On rearrangement of terms, we get,

$$x = 2rsin(\pi/n)(Equation4)$$

This value of x can be substituted in equation 2 to yield

$$C = 2rnsin(\pi/n)(Equation5)$$

This equation is of importance because it gives the circumference of a 'n' sided polygon, as a function of the number of sides.

The idea to be implemented is that a circle is a polygon with infinite number of sides. So, in equation 5 let us, take the limit of n to ∞

$$C = 2r\lim_{n\to\infty} nsin(\pi/n)(Equation6)$$

The above equation cannot be simplified as such as the limit cannot be directly applied. Hence a substitution can be introduced to bring about a simplification in equation 6.

The simplification is as follows -> let 1/n = y.

As $n \to \infty$, $1/n = y \to 0$. Thus $\lim_{n\to\infty} f(n) = \lim_{y\to 0} f(1/y)$.

This idea can be used to simplify equation 6. Thus, substituting 1/n as y and changing the limit from n to y, we get,

$$C \;=\; 2r\lim_{y\to 0} sin(\pi y)/y$$

Multiplying the numerators and denominators of the above equation by π ,

$$C = 2r\lim_{y\to 0} sin(\pi y) * \pi/\pi * y(Equation7)$$

A theorem based on the limits of functions states that

$$\lim_{u\to 0} sin(u)/u = 1$$

This can be used to evaluate equation 7 as πy as the independent variable of the sine function shows this property. Thus, reducing the $\lim_{y\to 0} sin(\pi y)/\pi y$ to 0, we get,

$$C = 2\pi r$$

Thus, we see that circle is indeed a polygon of sorts, with the number of sides, tending to infinity.

Conclusion

The circle is a polygon, as this has been proved above. This result is actually based on the following imagination. Consider any regular polygon, let us say a hexagon. It has 6 sides. If we double the number of sides, retaining the distance between the centre of the polygon to the vertices of the polygon a constant, we get a dodecagon. Consequently, the measure of the side would decrease. Thus the distance between two successive vertices would decrease. As the number of sides of the regular polygon tends to ∞, the vertices would get infinitesimally close. Hence any two successive points will be at the same distance from the centre of the polygon, which is nothing but the property of a circle - locus of a point such that as it moves, it continues to be at equal distance to a particular point.

Equation 6 in the presentation above, is of prime importance as it highlights the the circumference of any regular polygon. Integrating 6 with respect to n will give us the area of the regular polygon. Once the integration is done, if we apply the limit of infinity to the number of sides 'n', we will get the area as πr^2 which is yet another confirmation of the fact that the circle is indeed a polygon of a special kind.

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