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Quantum Mechanics and Linear Algebra

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1 Preamble

I have studied Quantum Mechanics and related mathematics in the past tw months under the guidance of Dr. Justin David at the Indian Institute of Science (IISc), Bangalore as a part of the Summer Research Fellowship Programme - SRFP 2011 awarded to me by the Jawaharlal Nehru Centre for Scientific and Advanced Research (JNCASR). have used the book *Introduction to Quantum Mechanics* by *Dr. J. Griffiths*. I have covered the first three chapters from that book apart from the appendix on Linear Algebra.

I have also used the first five lectures on the subject of Quantum Mechanics by Dr. V. Balakrishnan, Indian Institute of Technology, Madras (IIT-M) at the *National Program on Technology and Enhanced Learning* (NPTEL) website.

2 Introduction

Quantum Mechanics is the study of nature at the atomic and sub-atomic scale. It explains the time evolution of physical systems via a mathematical structure called the *wave function*. The wave function tells us about the probability of finding a system in a given state at a given time. It allows us to calculate the effect on the system of making measurements of properties of the system by defining the effect of those measurements on the wave function.

The first three chapters of the subject as given in the book *Introduction to Quantum Mechanics* are enlisted below :

1. The Wave Function
2. Time-Independent Schrodinger Equation
3. Formalism

The first chapter deals with an introduction to the Schrodinger Equation, The statistical interpretation of the wave function, concepts of normalization and the momentum operator.

The second chapter deals with the Time Independent formulation of the Schrodinger Equation which introduces the concept of *stationary state*. It then deals with the *idealized* cases of:

1. The Infinite Square Well
2. The Harmonic Oscillator
3. The Free Particle
4. The Delta Function Potential
5. The Finite Square Well

The third chapter deals with the Formalism involved with Quantum Mechanics. It introduces the reader to *Hilbert Space*, to the importance of understanding the roles *observables* and *operators* and the *Dirac* Notation. It also gives a generalized derivation of the Uncertainty Principle.

Apart from this I have also studied Linear Algebra.

Having explained my understanding of the first two chapters in the first month report, I would like to explain the third chapter on *formalism* and Linear Algebra.

3 Linear Algebra

A vector space consists of a set of vectors $\mid \alpha\rangle$, $\mid \beta\rangle$, $\mid \gamma\rangle$, ... together with a set of scalars $a, b, c, \ldots$), which is **closed** under two operations: vector addition and vector multiplication.

3.1 Vector Addition

The **sum** of any two vectors is another vector:

$$\mid \alpha\rangle + \mid \beta\rangle = \mid \gamma\rangle$$

Vector addition is **commutative**

$$\mid \alpha\rangle + \mid \beta\rangle = \mid \beta\rangle + \mid \beta\rangle$$

and **associative**

$$\mid \alpha\rangle + (\mid \beta\rangle + \mid \gamma\rangle) = (\mid \alpha\rangle + \mid \beta\rangle) + \mid \gamma\rangle$$

There exists a **zero** or **null** vector which returns the vector $\mid \alpha\rangle$ on addition.

3.2 Scalar Multiplication

The "product" of any scalar with any vector is another vector:

$$a \mid \alpha\rangle = \mid \gamma\rangle$$

Scalar multiplication is **distributive** with respect to vector addition:

$$a(\mid \alpha\rangle + \mid \beta\rangle) = a \mid \alpha\rangle + a \mid \beta\rangle$$

It is also **associatve** with respect to ordinary multiplication of scalars.

$$a(b \mid \alpha\rangle) = (ab) \mid \alpha\rangle$$

Multiplication by scalars 0 and 1 gives us:

$$0 \mid \alpha\rangle = |\rangle \qquad 1 \mid \alpha\rangle = \mid \alpha\rangle$$

A **linear combination** of the vectors $\mid \alpha\rangle$, $\mid \beta\rangle$, $\mid \gamma\rangle$, ... is an expression of the form

$$a \mid \alpha\rangle + b \mid \beta + c \mid \gamma\rangle + \ldots$$

A vector $\mid \lambda\rangle$ is said to be **linearly independent** of the set of vectors $\mid \alpha\rangle$, $\mid \beta\rangle$, $\mid \gamma\rangle$, ... if it cannot be written as a linear combination of them. A collection of vectors is said to **span** the space if *vector* can be written as a linear combination of the members of this set. A set of *linearly independent* vectors that spans the space is called a **basis**. The number of vectors in any basis is called the **dimension** of the space. Vectors that are both linearly independent and that span a given linear vector space form the basis of the vector set.

The set of all *nxn* matrices forms a linear vector space. The solutions to the equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

forms a linear vector space.

Thus the concept of linear vector space is very general and hence extremely useful.

3.3 Inner Product or Scalar Product

Let us consider the set of linear vectors $\phi, \psi, \chi, \ldots$ that constitute a vector space.

Keeping ϕ as a **reference vector** we can specify certain operations on other vectors that produce scalars.

$$(\phi, \psi) = \delta_\phi[\psi]$$

where (ϕ, ψ) represents a scalar. The set of scalars formed as a result of this *operation* between ϕ and all other vectors is itself a linear vector space as it obeys the laws of **vector addition** and **scalar multiplication**. This set of quantities is denoted as $S_\phi[\psi, \chi, \ldots]$

Now, the set of vectors that operate with ϕ in order to obtain the scalar set is called the **dual** of the linear vector space. Every vector space has a *unique* dual.

Introducing the *Dirac* notation, if we define the vectors $\mid \phi\rangle$ and $\mid \psi\rangle \in V$, then the vectors $\langle\psi \mid$ and $\langle\phi \mid$ are said to $\in \tilde{V}$ which is the dual of V. The scalar associated with these two vectors is given by $\langle\phi \mid \psi\rangle$. As an example, in n dimensional Euclidean space $\tilde{V}$ is the same as V. Thus, in this special case $V = \tilde{V}$

This *method* of producing a scalar via an operation between two vectors is called the *inner product* or *scalar product*. The scalar product of ψ with itself gives us what is called the **norm** of the vector ψ.

We can define certain properties for these vectors as follows:

1. If
$$a \mid \psi\rangle = \mid \psi'\rangle,$$
then
$$\langle\psi' \mid = a^* \langle\psi \mid$$

2.
$$\langle\phi \mid \psi\rangle^* = \langle\psi \mid \phi\rangle$$

3.
$$\| \psi \| = \langle\psi \mid \psi\rangle^{\frac{1}{2}}$$

3.4 Schwarz Inequality

In the three dimensional case, we can write the Schwarz Inequality as

$$\mid \vec{a} \mid . \mid \vec{b} \mid \leqslant \mid \vec{a} \mid\mid \vec{b}$$

which simply means the *cosine* of the angle between the two vectors and is less than or equal to 1. But in generalizing this concept to finite n-dimensional space, we can re-write the Schwarz Inequality in *Dirac Notation* as follows:

$$\mid \langle\phi \mid \psi\rangle \mid^2 \leqslant \langle\phi \mid \phi\rangle\langle\psi \mid \psi\rangle$$

3.5 Orthogonal vectors Orthonormal basis

Two vectors are said to be orthogonal if their inner product is said to be zero. The inner product of a *normalized vector* with itself gives us 1. Thus the two operations above can be written very concisely with the Dirac notation using the kronecker delta function.

$$\langle\phi_i \mid \phi_j\rangle = \delta_{ij}$$

A set of vectors that satisfies the above condition is an orthonormal set/basis.

We can always construct an orthogonal basis from a given set of non-orthogonal basis vectors.

The dimensionality of a given linear vector space is equal to the number of vectors in the basis.

For a non-finite basis set, dimensionality is equal to ∞

In the case of an infinite dimensional space, the inner product need not be defined as it need not converge. Thus, a pre-requisite condition for a finite inner product is

$$\sum_{i=1}^{\infty} \| X_i \|^2 < \infty$$

This is a linear vector space of square-summable sequences denoted by l_2. Vectors in Quantum Mechanics *reside* in l_2.

3.6 Gram Schmidt Orthonormalization

This is a method of *producing* an orthonormal basis set from a given basis set of vectors. Let $v_1, v_2, v_3, \ldots$ represent a set of basis vectors. Let $u_1, u_2, u_3, \ldots$ represent the orthogonal set of vectors that can be obtained from the above set of vectors.

The steps from which we can produce $u_1, u_2, u_3, \ldots$ are enumerated below.

1. $u_1 = v_1$
2. $u_2 = v_2 - proj_{u_1}(v_2)$
3. $u_3 = v_3 - proj_{u1}(v_3), \ldots$

The method highlighted above proceeds as follows: to compute u_i, it projects v_i orthogonally onto the subspace U generated by $u_1, \ldots u_{i\hat{a}1}$, which is the same as the subspace generated by $v_1, \ldots v_{i-1}$. The vector u_i is then defined to be the difference between v_i and this projection, guaranteed to be orthogonal to all of the vectors in the subspace U.

The terms $u_1, u_2, u_3, \ldots$ from the orthogonalized basis. The *orthonormalized* basis can be obtained by dividing each of the u_i's by the magnitude of that particular vector.

3.7 The projection operator

The projection operator is denoted by

$$\hat{P} \equiv \mid \alpha\rangle\langle\alpha \mid$$

which picks out the portion of any other vector that *lies along* $\mid \alpha\rangle$:

$$\hat{P} = \langle\alpha \mid \beta \mid \alpha\rangle$$

If $\mid e_n\rangle$ is a discrete orthonormal basis, then:

$$\sum_n \mid e_n\rangle\langle e_n \mid \alpha\rangle = \mid \alpha\rangle$$

which leads us to the statement of **completeness**

$$\sum_n \mid e_n\rangle\langle e_n \mid = 1$$

The projection operator is **idempotent**. Thus $\hat{P}^2 = \hat{P}$. This tells us that any projection operator always has the two eigenvalues 0 and 1.

Any matrix can represented by a linear combination of elementary matrices. The elementary matrices are essentially projection operators. Thus, any matrix can be equivalently represented by linear combination of projection operators constructed from a set of basis vectors. Using this concept, we can obtain any element of a given nxn matrix. If a_{ij} represents the ij^{th} element of an nxn matrix:

$$\langle\phi_i \mid A \mid \phi_j\rangle = a_{ij}.$$

It is clear from the above equation that the diagonal elements are represented by projection operators.

3.8 Functions in a linear vector space

We defined l_2 to be a space of square summable sequences. Now, for a function space, as it does not have discretion of components, we replace the summation by an integral. Thus,

$$\int_{i=1}^{\infty} \mid f(x) \mid^2 dx < \infty$$

This *function* space is called L_2. The function must attain the value of 0 at either of the ∞'s. Though $f(x)$ is allowed to have discontinuities, it should be *integrable* throughout.

L_2 is *self-dual*. Now the question that naturally arises is how do we find the dual of $f(x)$? Surprisingly, it can be done by the *fourier transform* of $f(x)$! If we denote the dual of $f(x)$ by $f(\tilde{k})$, then Parseval's theorem tells us that,

$$\int_{-\infty}^{\infty} \mid f(\tilde{k}) \mid^2 dk = \int_{-\infty}^{\infty} \mid f(x) \mid^2 dx < \infty$$

Thus Parseval's theorem assures us that the norm of the function $f(\tilde{k})$ in dual space is also a finite number and is in fact preserved and equal to the norm of $f(x)$! Thus, surely then we can construct $f(k)$ via a Fourier Transform!

In fact, a fourier transform in itself is simply a shift of basis to a periodic one for function spaces. On a similar note, we can also extent this concept to the Fourier series. Essentially, any periodic function can be written as a linear combination of sines and cosines. Thus we can say any periodic function $f(x)$ is a function in function space with components $a_n \cos nx$ and $b_n \sin nx$. By suitable combinations, we can in fact write the function $f(x)$ as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where c_n is a suitable combination of a_n's and b_n's. If $f(x)$ is not periodic then it can be written as a fourier integral rather than series!

3.9 Orthonormality and Completeness Relations in function space

In the square summable space of l_2, the orthonormality and completeness relations can be written as follows:

1. Orthonormality

$$\sum_{n,m} \langle \phi_n \mid \phi_m \rangle = \delta_{nm}$$

This can be written in function space as

$$\int_a^b dx f_n^*(x) f_m(x) = \delta_{mn}$$

2. Completeness. In vector space,

$$\sum_n \mid \phi_n\rangle\langle\phi_m \mid = 1$$

This can be written in function space as

$$\sum_n f_n^*(x) f_n(x') = \delta(x - x')$$

3.10 Change of Basis

Sometimes, owing to the pure convenience of manipulation of the mathematical symbols, it might be necessary to switch from one basis set to another. Let a basis set labelled by $\mid \phi_i\rangle$ span a linear vector space. In that case a vector $\mid \psi_n\rangle$ belonging to the same linear vector space can be written as:

$$\mid \psi_n\rangle = \sum_n c_n \mid \phi_n\rangle$$

c_n can be written as

$$\langle\phi_m \mid \psi_n\rangle = c_m$$

As the basis for a given linear vector space is not unique, we can have another basis set on $\mid \chi_i\rangle$ to describe the components $\mid \psi\rangle$

$$\mid \psi\rangle = \sum_i d_i \mid \chi_i\rangle, \text{or} \quad d_j = \langle\chi_j \mid \psi_n\rangle$$

Now, how do we shift from 'c_m's to the 'd_j's ? This can be done in two different *ways*:

(a) - We can write $\mid \chi_i\rangle$ in terms of $\mid \phi_m\rangle$

$$\mid \chi_i\rangle = \sum_m h_{mi} \mid \phi_m\rangle$$

We can plug in this form of $\mid \chi_i\rangle$ into the formulae above to yield;

$$\mid \psi\rangle = \sum_{i,m} h_{mi} d_i \mid \phi_m\rangle$$

We can simply equate the two formulae for $d_i \mid \psi\rangle$ and thus, obtain

$$c_n = \sum_i h_{ni} d_i$$

(b) - Into the above formula we can substitute for the quantities to yield

$$\langle\phi_m \mid \psi\rangle = \sum_i \langle\phi_n \mid \chi_i\rangle\langle\chi_i \mid \psi\rangle$$

Rearranging a bit, we can obtain:

$$\langle\phi_m \mid \psi\rangle = \langle\phi_n \sum_i \mid \chi_i\rangle\langle\chi_i \mid \psi\rangle$$

This gives us the *unit operator* of χ_i inserted between the basis vector and the vector ψ. Simply speaking, it is the unit operator of the basis we want to change to.

4 The Uncertainty Princple

For any observable A, we have

$$\sigma_A^2 = \langle (\hat{A} - \langle A \rangle)\Psi \mid (\hat{A} - \langle A \rangle)\Psi \rangle = \langle f \mid f \rangle$$

where $f \equiv (\hat{A} - \langle A \rangle)\Psi$. Likewise, for any *observable B*,

$$\sigma_B^2 = \langle (\hat{B} - \langle B \rangle)\Psi \mid (\hat{B} - \langle B \rangle)\Psi \rangle = \langle g \mid g \rangle$$

where $g \equiv (\hat{B} - \langle B \rangle)\Psi$. Now, Invoking the Scwharz inequality,

$$\sigma_A^2 \sigma_B^2 = \langle f \mid f \rangle \langle g \mid g \rangle$$

Any complex number z can be written as

$$\mid z \mid^2 = \left[\frac{z - z^*}{2i} \right]^2$$

Therefore, letting $z = \langle f \mid g \rangle$,

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{\langle f \mid g \rangle - \langle g \mid f \rangle}{2i} \right)^2$$

But as f can be written in terms of $\hat{A}$ and g can be written in terms of $\hat{B}$, we can evaluate $\langle f \mid g \rangle$ to be

$$\langle f \mid g \rangle = \langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \, \langle \hat{A} \rangle$$

Similary,

$$\langle g \mid f \rangle = \langle \hat{B}\hat{A} \rangle - \langle \hat{A} \rangle \, \langle \hat{A} \rangle.$$

So,

$$\langle f \mid g \rangle - \langle g \mid f \rangle = \langle \hat{A}\hat{B} \rangle - \langle \hat{B}\hat{A} \rangle$$

where we can write

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

Conclusion:

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{\langle [\hat{A}\hat{B}] \rangle}{2i} \right)^2$$

This is the **generalized uncertainty principle**. The *original* uncertainty principle as we know it is obtained for the observables $\hat{x}$ and $\hat{p}$.

$$[\hat{x}, \hat{p}] = i\hbar.$$

And for these pair of observables, we get

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

Thus we can see that there is an *uncertainty principle* for every pair of observers that do not commute. We call them **incompatible observers**. Incompatible observers do not commute entirely and hence do not have a shared set of eigenfunctions. Contrastingly, as commuting matrices can be diagonalized by the same similarity transformation, they share a common set of eigenfunctions.

5 Eigenfunctions of Hermitian Operators

Hermitian Operators are simply those operators that satisfy the following inner product equation

$$\langle \Psi \mid \hat{Q}\Psi \rangle = \langle \hat{Q}\Psi \mid \Psi \rangle$$

or

$$\langle f \mid \hat{Q}f \rangle = \langle \hat{Q}f \mid f \rangle$$

Observables in quantum mechanics are represented by Hermitian operators. That is because:

(a) The expectation values of hermitian operators are real.

(b) They give rise to *real* eigenvalues.

For any observable Q:

$$\hat{Q}\Psi = q\Psi$$

where Ψ is an eigenfunction of Q. Quite simply, it means that it is possible to prepare a state of a system such that any measurement of Q would return the same value q. The state in which this is possible is called the **eigenfunction** and is represented by Ψ. This state is also called a **determinate state**.

Now, I would like to state 2 theorems, without trivial proofs. The normalizable eigenfunctions of a hermitian operator have *real* eigenvalues. Eigenfunctions belonging to distinct eigenvalues are *orthogonal*

6 Generalized Statistical Interpretation

If you measure any observable $Q(x,p)$ on a particle in the state $\Psi(x,t)$, you are certain to get *one of the eigenvalues* of the hermitian operator $\hat{Q}(x, \frac{-i}{dx})$ If the spectrum $\hat{Q}$ is discrete, the ability of getting a particular eigenvalue q_n associated with the orthonormal eigenfunction $f(x)$ is

$$\mid c_n \mid^2, \text{ where } c_n = \langle f_n \mid \Psi \rangle$$

If the spectrum is continuous, with real eigenvalues $q(z)$ and associated orthonormalized eigenfunctions $f_z(x)$, the probability of getting a result in the range dz

$$\mid c(z) \mid^2 dz \text{ where } c(z) = \langle f_z \mid \Psi \rangle$$

The eigenfunctions of an observable operator are complete. Thus, the wave functions can be written as a linear combination of them:

$$\Psi(x,t) = \sum_n c_n f_n(x)$$

We can obtain c_n to be as follows, exploiting the orthonormality of the eigenfunctions.

$$c_n = \langle f_n \mid \Psi \rangle = \int f_n(x)^* \Psi(x,t) dx$$

Exploiting the normalization of the wave function $\langle \Psi \mid \Psi \rangle = 1$, we can show that

$$\sum_n \mid c_n \mid^2 = 1$$

Similarly the expectation value of Q should be the sum over all the possible outcomes of the eigenvalue times the probability of getting that eigenvalue:

$$\langle Q \rangle = \sum_n q_n \mid c_n \mid^2$$

7 Miscellaneous Problems

In this section, I would like to highlight few of the problems I have solved in the topics on quantum mechanics in the second half of my intern-ship.

7.1 Particle on a disc

Consider a particle of mass m subject to no potentials constrained to move on a disc of radius a. Derive the wave function for the particle and energy eigenstates.

Solution : This is a 2 dimensional problem unlike the one's we have dealt with so far. We consider $\Psi = \Psi(r, \theta, t)$. Thus, the laplacian in the Hamiltonian operator should be written in terms of polar co-ordinates.

$$\bigtriangledown^2 = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

We assume the solution $\Psi(x, t)$ to be of the separable form as functions of r and θ.

$$\Psi = R(r)\Theta(\theta)$$

Thus, the Schrodinger equation becomes

$$-\frac{\hbar^2}{2m} \bigtriangledown^2 \Psi(r, \theta) = -\left(\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) (R(r)\Theta(\theta)) = E * R(r)\Theta(\theta)$$

This equation can be solved by diving both sides by $R(r)\Theta(\theta)$ and making the simplifying substitution

$$k^2 = \frac{2mE}{\hbar^2}$$

After simplification, we arrive at the form

$$\frac{1}{R(r) r^{-2}} \left[R^{''}(r) + \frac{1}{r} R^{'}(r) + k^2 R(r) \right] + \frac{\Theta^{''}(theta)}{\Theta(\theta)} = 0$$

This can be written as

$$\frac{1}{R(r) r^{-2}} \left[R^{''}(r) + \frac{1}{r} R^{'}(r) + k^2 R(r) \right] = -\frac{\Theta^{''}(theta)}{\Theta(\theta)} = l^2$$

Now, the equation for $\Theta(\theta)$ can be solved trivially to give us the solution

$$\Theta(\theta) = e^{\pm il\theta}$$

By applying the condition that $\Theta(\theta + 2\pi) = \Theta(\theta)$ we can show that l is an integer. The other equation is of the form of Bessel's equation which essentially demands a series solution in the form of Bessel's functions. The general solution for Bessel's equation can be written as:

$$R(r) = c_1 J_l(kr) + c_2 Y_l(kr)$$

where $J_l(kr)$ and $Y_l(kr)$ are the Bessel functions of the first and second kind. At the point $r = 0$, the wavefunction is defined. But the $Y_l(kr)$ section of the equation diverges and fails to make sense. Thus, we can set $c_2 = 0$ and thereby conclude that the solution for $R(r)$ is only a combination of the bessel equation of the first kind. The boundary condition that has to be applied to the solution is

$$R(r = a) = 0$$

Applying this boundary condition we get

$$c_1 J_l(ka) = 0.$$

This condition tells us that ka belongs to some zero of the Bessel Function. Though this value need not be an integer, we know from the condition that the values of k come out to be the *zeros* of the Bessel function of the first kind.

Essentially the $Y_l(kr)$ exists only when there is no condition of quantization.

7.2 The Transfer Matrix 'S'

Consider a particle placed before five finite square potential barriers of magnitude $+V_l0$ with incident and reflected amplitudes to be A and B respectively. Find the incident and reflected amplitude after it *passes* through the potential barriers. **Solution :** This problem can be solved very easily by employing the transfer matrix S which gives us the outgoing amplitudes in terms of the incoming amplitudes.

$$S = \begin{matrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{matrix}$$

The above matrix is denoted by **S** and $\begin{matrix} F \\ G \end{matrix}$ be the set of outgoing amplitudes is denoted by $H^{'}$ and if H denotes the incoming amplitudes:

$$H^{'} = SH$$

This gives the outgoing amplitudes for the first barrier. As all the barriers are of identical potential V_0,

$$H^{'} = S^5 H$$

7.3 Find the eigenfunctions and eigenvalues of the position operator

Solution : Let $g_y(x)$ be the eigenfunction and y the eigenvalue:

$$xg_y(x) = yg_y(x)$$

Here y is a fixed number (for any given eigenfunction), but x is a continuous variable. What function of x has the property that multiplying it by x is the same as multiplying it by a constant y? Obviously, the value has to be zero, except at $x = y$. This tells us that it is nothing but the Dirac delta function!

$$g_y(x) = \delta(x - y)$$

It is evident that the eigenfunctions are not square-integrable, but again they admit Dirac *orthonormality*:

$$\int_{-\infty}^{\infty} g^*_{y'}(x)dx = \int_{-\infty}^{\infty} \delta(x - y^{'})\delta(x - y)dx = \delta(y - y^{'})$$

then

$$\langle g_{y'} \mid g_y \rangle = \delta(y - y^{'})$$

These eigenfunctions are also complete:

$$f(x) = \int_{-\infty}^{\infty} c(y)g_y(x)dy = \int_{-\infty}^{\infty} c(y)\delta(x - y)dy$$

7.4 Find the eigenfunctions and eigenvalues of the momentum operator

Solution : If $f_p(x)$ is the eigenfunction and p the eigenvalue:

$$\frac{\hbar}{i}\frac{d}{dx}f_p(x) = pf_p(x)$$

The general solution is

$$f_p(x) = Ae^{\frac{ipx}{\hbar}}$$

This is not square integrable for *any* complex value of p. Thus. the momentum operator has no eigenfunctions in Hilbert Space. Yet, if we restrict ourselves to real eigenvalues, we can prove that these functions despite not essentially being in l_2 project a certain form of *orthonormality*. As any two eigenfunctions of the momentum operator are orthogonal,

$$\int_{-\infty}^{\infty} f_{p'}(x)f_p(x)dx = \mid A \mid^2 \int_{-\infty}^{\infty} e^{i\frac{(p-p^{'})x}{\hbar}}dx = \mid A \mid^2 2\pi\hbar\delta(p - p^{'}).$$

Choosing A conveniently, we get

$$\langle f_{p'} \mid f_p \rangle = \delta(p - p^{'})$$

The eigenfunctions are also *complete*, with the sum replaced by an integral:

$$f(x) = \int_{-\infty}^{\infty} c(p)f_p(x)dp = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} c(p)e^{\frac{ipx}{\hbar}}dp$$